A Performance Analysis of Optical Filters in Direct Detection Receivers

by

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Abstract

The presence of conventional electronic repeaters in optical communication networks restricts the rate of transmission to only a small fraction of the theoretical capacity of single mode fiber. The use of optical amplifiers in an all-optical network is expected to increase the accessible bandwidth to a level close to the theoretical fiber bandwidth.

Optical amplifiers generate additive quantum noise that mixes with the transmitted signal during the detection process resulting in a larger bit error rate. The introduction of predetection optical filters improves the receiver's performance by rejecting some of the amplifier noise power. We investigate the efficiency of Mach-Zehnder filters and Fabry-Perot interferometers in recovering rectangular pulses of monochromatic light mixed with optical noise.

The performance of direct detection receivers used in combination with these two filters is evaluated for ASK and FSK systems. The analysis takes into account the effects of shot noise, intersymbol interference and electronic thermal noise generated by the post detection circuitry. Chernoff bounds are used to approximate the bit error rate.

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Contents

1 Introduction .......................................................... 7
  1.1 Background ......................................................... 7
  1.2 Thesis Overview ................................................... 9

2 Mach-Zehnder optical filters ......................................... 11
  2.1 Structure of a single stage Mach-Zehnder filter .............. 11
  2.2 Structure of a multistage Mach-Zehnder filter .............. 12
      2.2.1 Multistage Mach-Zehnder filter with 2 branches per stage ... 12
      2.2.2 Multistage Mach-Zehnder filter with b branches per stage (b≥2) 18
      2.2.3 ISI introduced by Mach-Zehnder filters .................. 30
  2.3 Frequency response of a single stage Mach-Zehnder filter . 31
  2.4 Frequency response of a multistage Mach-Zehnder filter . 33
      2.4.1 Multistage Mach-Zehnder filter with 2 branches per stage ... 33
      2.4.2 Multistage Mach-Zehnder filter with b branches per stage (b≥2) 39
  2.5 Concluding remarks about Mach-Zehnder chains .............. 49

3 Fabry-Perot Interferometers ........................................ 53
  3.1 Structure of Fabry-Perot interferometers ...................... 53
  3.2 ISI introduced by Fabry-Perot filters ......................... 58
  3.3 Frequency response of Fabry-Perot interferometers .......... 58
  3.4 Symmetric Fabry-Perot filters ................................ 66
CONTENTS

4 Photodetection of Gaussian optical fields 69
  4.1 Structure of receivers for ASK and FSK systems ............... 69
  4.2 Sources of Noise ........................................... 71
    4.2.1 The optical amplifier noise ................................ 71
    4.2.2 The electronic thermal noise ............................. 72
    4.2.3 The shot noise ........................................... 73
  4.3 Derivation of count statistics ................................. 77
    4.3.1 Field mode decomposition ................................ 78
    4.3.2 Derivation of the received optical energy statistics ...... 80
    4.3.3 The electron count density function and its S-transform . 83
  4.4 The bit error rate for ASK and FSK receivers .................. 92
    4.4.1 The Chernoff bound ..................................... 93
    4.4.2 Bit error rate for ASK receivers .......................... 94
    4.4.3 Bit error rate for FSK receivers .......................... 95
    4.4.4 Exact bit error rate for FSK receivers ................. 96

5 Performance of Mach-Zehnder filters 99
  5.1 Solving the integral equation ................................. 99
    5.1.1 The filtered incident signal pulse $m(t)$ .................. 100
    5.1.2 The filtered optical noise correlation function .......... 105
    5.1.3 Derivation of the eigenfunctions and eigenvalues ....... 106
    5.1.4 Derivation of the signal modes .......................... 111
    5.1.5 Derivation of the ISI modes .............................. 113
  5.2 The average noise energy ..................................... 114
  5.3 The total signal energy ...................................... 116
  5.4 The total ISI energy ........................................ 118
  5.5 The total noise-noise and signal-noise beat variances ....... 121
  5.6 Performance of Mach-Zehnder filters in ASK receivers ........ 124
    5.6.1 A performance analysis including ISI .................... 124
CONTENTS

5.6.2 A performance analysis ignoring ISI ........................................ 131
5.7 Performance of Mach-Zehnder filters in FSK receivers ................. 135
5.7.1 A performance analysis including ISI ...................................... 135
5.7.2 A performance analysis ignoring ISI ...................................... 142
5.8 Accuracy of Chernoff bounds .................................................. 146

6 Performance of Fabry-Perot interferometers .................................. 148
6.1 Solving the integral equation .................................................. 148
6.1.1 The filtered incident signal pulse $m(t)$ ................................ 149
6.1.2 The filtered optical noise correlation function ....................... 151
6.1.3 Derivation of the eigenfunctions and eigenvalues .................... 152
6.1.4 Derivation of the signal modes ........................................... 155
6.1.5 Derivation of the ISI modes .............................................. 156
6.2 The average noise energy ...................................................... 157
6.3 The total signal energy .......................................................... 159
6.4 The total ISI energy ............................................................. 161
6.5 The total noise-noise and signal-noise beat variances ................. 163
6.6 Performance of Fabry-Perot filters in ASK receivers ................... 166
6.6.1 A performance analysis including ISI .................................... 166
6.6.2 A performance analysis ignoring ISI .................................... 173
6.7 Performance of Fabry-Perot filters in FSK receivers ................... 177
6.7.1 A performance analysis including ISI .................................... 177
6.7.2 A performance analysis ignoring ISI .................................... 184
6.8 Accuracy of Chernoff bounds ................................................ 188

7 Conclusions ............................................................................. 190
7.1 Summary .............................................................................. 190
7.2 Directions for further research .............................................. 195
List of Figures

2-1 Single-stage Mach-Zehnder filter .......................... 11
2-2 Mach-Zehnder filter with n stages (2 branches per stage) .... 12
2-3 Incoming rectangular pulse $s(t)$ ............................ 13
2-4 Output of the first stage ........................................ 14
2-5 Output of the second stage ...................................... 14
2-6 Output $m(t)$ of a filter with an infinitely large number of stages 15
2-7 Mach-Zehnder filter with n stages (2 branches per stage and $a \geq 2$) 15
2-8 Output of the first stage ($a \geq 2$) .......................... 16
2-9 Output of the second stage ($a \geq 2$) ........................ 16
2-10 Output $m(t)$ of a filter with an infinitely large number of stages ($a \geq 2$) 17
2-11 Mach-Zehnder filter with n stages (b branches per stage) .... 18
2-12 Incoming rectangular pulse $s(t)$ ............................ 19
2-13 Output $m(t)$ of a filter with an infinitely large number of stages 20
2-14 Mach-Zehnder filter with n stages (3 branches per stage) .... 20
2-15 Output of the first stage ........................................ 21
2-16 Output of the second stage ...................................... 21
2-17 Mach-Zehnder filter with n stages (4 branches per stage) .... 22
2-18 Output of the first stage ........................................ 22
2-19 Output of the second stage ...................................... 23
2-20 Mach-Zehnder filter with n stages (b branches per stage $a \geq b$) 24
2-21 Output $m(t)$ of a filter with an infinitely large number of stages ($a \geq b$) 25
LIST OF FIGURES

2.22 Mach-Zehnder filter with n stages (3 branches per stage and \( a \geq 3 \)) ........................................ 26
2.23 Output of the first stage .................................................. 26
2.24 Output of the second stage ................................................. 27
2.25 Output \( m(t) \) of a filter with an infinitely large number of stages \((a \geq 4)\) ................................................. 27
2.26 Mach-Zehnder filter with n stages (4 branches per stage \( a \geq 4 \)) .................................................. 28
2.27 Output of the first stage .................................................. 28
2.28 Output of the second stage ................................................. 29
2.29 Output \( m(t) \) of a filter with an infinitely large number of stages \((a \geq 4)\) ................................................. 29
2.30 Triangular outputs (case where \( a = b \)) ................................................. 30
2.31 Trapezoidal outputs (case where \( a > b \)) ................................................. 31
2.32 Frequency response of a single stage Mach-Zehnder filter (\( \nu \) is expressed in units of \( \frac{1}{\Delta f} \)) ................................................. 32
2.33 Frequency response when \( n = 1, a = 2, \) and \( b = 2 \) (\( \nu \) is expressed in units of \( \frac{1}{\Delta f} \)) ................................................. 35
2.34 Frequency response when \( n = 2, a = 2, \) and \( b = 2 \) (\( \nu \) is expressed in units of \( \frac{1}{\Delta f} \)) ................................................. 35
2.35 Frequency response when \( n = 3, a = 2, \) and \( b = 2 \) (\( \nu \) is expressed in units of \( \frac{1}{\Delta f} \)) ................................................. 35
2.36 Frequency response when \( n = 1, a = 3, \) and \( b = 2 \) (\( \nu \) is expressed in units of \( \frac{1}{\Delta f} \)) ................................................. 36
2.37 Frequency response when \( n = 2, a = 3, \) and \( b = 2 \) (\( \nu \) is expressed in units of \( \frac{1}{\Delta f} \)) ................................................. 36
2.38 Frequency response when \( n = 3, a = 3, \) and \( b = 2 \) (\( \nu \) is expressed in units of \( \frac{1}{\Delta f} \)) ................................................. 36
2.39 Frequency response when \( n = 1, a = 3, \) and \( b = 3 \) (\( \nu \) is expressed in units of \( \frac{1}{\Delta f} \)) ................................................. 41
2.40 Frequency response when \( n = 2, a = 3, \) and \( b = 3 \) (\( \nu \) is expressed in units of \( \frac{1}{\Delta f} \)) ................................................. 41
2-41 Frequency response when $n = 3$, $a = 3$, and $b = 3$ ($\nu$ is expressed in units of $\frac{1}{\pi}$) .......................................................... 41
2-42 Frequency response when $n = 1$, $a = 4$, and $b = 3$ ($\nu$ is expressed in units of $\frac{1}{\pi}$) .......................................................... 42
2-43 Frequency response when $n = 2$, $a = 4$, and $b = 3$ ($\nu$ is expressed in units of $\frac{1}{\pi}$) .......................................................... 42
2-44 Frequency response when $n = 3$, $a = 4$, and $b = 3$ ($\nu$ is expressed in units of $\frac{1}{\pi}$) .......................................................... 42
2-45 Frequency response when $n = 1$, $a = 4$, and $b = 4$ ($\nu$ is expressed in units of $\frac{1}{\pi}$) .......................................................... 46
2-46 Frequency response when $n = 2$, $a = 4$, and $b = 4$ ($\nu$ is expressed in units of $\frac{1}{\pi}$) .......................................................... 46
2-47 Frequency response when $n = 3$, $a = 4$, and $b = 4$ ($\nu$ is expressed in units of $\frac{1}{\pi}$) .......................................................... 46
2-48 Frequency response when $n = 1$, $a = 5$, and $b = 4$ ($\nu$ is expressed in units of $\frac{1}{\pi}$) .......................................................... 47
2-49 Frequency response when $n = 2$, $a = 5$, and $b = 4$ ($\nu$ is expressed in units of $\frac{1}{\pi}$) .......................................................... 47
2-50 Frequency response when $n = 3$, $a = 5$, and $b = 4$ ($\nu$ is expressed in units of $\frac{1}{\pi}$) .......................................................... 47
2-51 Incoming rectangular pulse $s(t)$ .................................................. 49
2-52 Output $m(t)$ of a matched Mach-Zehnder filter (case where $T = T'$) .......................................................... 51
2-53 Output $m(t)$ of a Mach-Zehnder filter (case where $T > T'$) .......................................................... 51
3-1 Fabry-Perot filter .......................................................... 54
3-2 Incoming rectangular pulse $s(t)$ .................................................. 55
3-3 Output of a Fabry-Perot filter .......................................................... 56
3-4 Output $m(t)$ of a Fabry-Perot filter operating under ideal conditions .......................................................... 57
LIST OF FIGURES

3-5 Frequency response of a filter with finesse $F$ (expressed in units of $\frac{\omega_p}{\Delta \nu}$) ................................................................. 61
3-6 Incoming rectangular pulse $s(t)$ .................................................. 67
3-7 Output $m(t)$ of a Fabry-Perot filter .............................................. 68
4-1 Receiver used for ASK systems ....................................................... 70
4-2 Receiver used for FSK systems ....................................................... 70
4-3 Schematic of the photodiode model ................................................ 73
5-1 Incoming rectangular pulse $s(t)$ ...................................................... 100
5-2 Filtered waveform corresponding to an "OFF" "OFF" "OFF" input ........... 101
5-3 Filtered waveform corresponding to an "ON" "OFF" "OFF" input ........... 101
5-4 Filtered waveform corresponding to an "OFF" "OFF" "ON" input .......... 102
5-5 Filtered waveform corresponding to an "ON" "OFF" "ON" input .......... 102
5-6 Filtered waveform corresponding to an "OFF" "ON" "OFF" input .......... 103
5-7 Filtered waveform corresponding to an "ON" "ON" "OFF" input .......... 103
5-8 Filtered waveform corresponding to an "OFF" "ON" "ON" input .......... 104
5-9 Filtered waveform corresponding to an "ON" "ON" "ON" input .......... 104
5-10 Filtered signal pulse envelope $m'(t)$ ............................................ 112
5-11 ISI envelope $m'_{ISI}(t)$ .............................................................. 113
5-12 Average total noise energy $\sum_{i=0}^{\infty} \lambda_i$ (expressed in units of $N_0$) ................................. 115
5-13 Sum of the first 10 eigenvalues $\sum_{i=0}^{9} \lambda_i$ (expressed in units of $N_0$) ........................................ 115
5-14 Total signal energy received $\sum_{i=0}^{\infty} ||m_i||^2$ (expressed in units of $E$) ..................................................... 117
5-15 Sum of the first 10 signal modes $\sum_{i=0}^{9} ||m_i||^2$ (expressed in units of $E$) ..................................................... 117
5-16 Total ISI energy received $\sum_{i=0}^{\infty} ||m_{ISI_i}||^2$ (expressed in units of $E$) ..................................................... 119
5-17 Sum of the first 10 ISI modes $\sum_{i=0}^{9} ||m_{ISI_i}||^2$ (expressed in units of $E$) ..................................................... 119
5-18 Total noise-noise beat variance $\sum_{i=0}^{\infty} \lambda_i^2$ (expressed in units of $N_0 \times N_0$) ...................... 122
5-19 Sum of the first 10 noise-noise beat terms $\sum_{i=0}^{9} \lambda_i^2$ (expressed in units of $N_0 \times N_0$) ...................... 122
LIST OF FIGURES

5-20 Total signal-noise beat variance $\sum_{i=0}^{\infty} 2\lambda_i \|m_i\|^2$ (expressed in units of $E \times N_0$) ................................................................. 123

5-21 Sum of the first 10 signal-noise beat terms $\sum_{i=0}^{9} 2\lambda_i \|m_i\|^2$ (expressed in units of $E \times N_0$) ................................................................. 123

5-22 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($N_0 = \infty$, ISI is not ignored) .................................................. 124

5-23 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 0 \frac{pA}{Hz^{1/2}}$, $N_0 = 1$, ISI is not ignored) .......... 127

5-24 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 0 \frac{pA}{Hz^{1/2}}$, $N_0 = 10$, ISI is not ignored) .......... 127

5-25 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 0 \frac{pA}{Hz^{1/2}}$, $N_0 = 50$, ISI is not ignored) .......... 127

5-26 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 1 \frac{pA}{Hz^{1/2}}$, $N_0 = 10$, ISI is not ignored) .......... 128

5-27 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 1 \frac{pA}{Hz^{1/2}}$, $N_0 = 50$, ISI is not ignored) .......... 128

5-28 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 1 \frac{pA}{Hz^{1/2}}$, $N_0 = 100$, ISI is not ignored) .......... 128

5-29 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 3 \frac{pA}{Hz^{1/2}}$, $N_0 = 10$, ISI is not ignored) .......... 129

5-30 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 3 \frac{pA}{Hz^{1/2}}$, $N_0 = 100$, ISI is not ignored) .......... 129

5-31 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 3 \frac{pA}{Hz^{1/2}}$, $N_0 = 1000$, ISI is not ignored) .......... 129

5-32 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 10 \frac{pA}{Hz^{1/2}}$, $N_0 = 10$, ISI is not ignored) .......... 130

5-33 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 10 \frac{pA}{Hz^{1/2}}$, $N_0 = 100$, ISI is not ignored) .......... 130
5-34 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
$(i_{th} = 10 \frac{P_A}{H_{2/73}^2}, N_0 = 1000, \text{ISI is not ignored})$ .................................................. 130
5-35 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
$(N_0 = \infty, \text{ISI is ignored})$ .................................................................................. 131
5-36 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
$(i_{th} = 0 \frac{P_A}{H_{2/73}^2}, N_0 = 1, \text{ISI is ignored})$ .................................................. 133
5-37 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
$(i_{th} = 0 \frac{P_A}{H_{2/73}^2}, N_0 = 10, \text{ISI is ignored})$ .................................................. 133
5-38 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
$(i_{th} = 10 \frac{P_A}{H_{2/73}^2}, N_0 = 10, \text{ISI is ignored})$ .................................................. 134
5-39 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
$(i_{th} = 10 \frac{P_A}{H_{2/73}^2}, N_0 = 1000, \text{ISI is ignored})$ .......................................... 134
5-40 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers
$(N_0 = \infty, \text{ISI is not ignored})$ .................................................................................. 137
5-41 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers
$(i_{th} = 0 \frac{P_A}{H_{2/73}^2}, N_0 = 1, \text{ISI is not ignored})$ .................................................. 138
5-42 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers
$(i_{th} = 0 \frac{P_A}{H_{2/73}^2}, N_0 = 10, \text{ISI is not ignored})$ .................................................. 138
5-43 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers
$(i_{th} = 0 \frac{P_A}{H_{2/73}^2}, N_0 = 50, \text{ISI is not ignored})$ .................................................. 138
5-44 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers
$(i_{th} = 1 \frac{P_A}{H_{2/73}^2}, N_0 = 10, \text{ISI is not ignored})$ .................................................. 139
5-45 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers
$(i_{th} = 1 \frac{P_A}{H_{2/73}^2}, N_0 = 50, \text{ISI is not ignored})$ .................................................. 139
5-46 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers
$(i_{th} = 1 \frac{P_A}{H_{2/73}^2}, N_0 = 100, \text{ISI is not ignored})$ .......................................... 139
5-47 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers
$(i_{th} = 3 \frac{P_A}{H_{2/73}^2}, N_0 = 10, \text{ISI is not ignored})$ .................................................. 140
5-48 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

$(i_{th} = 3 \frac{P_A}{H^2 q}, N_0 = 100, \text{ISI is not ignored})$ ........................................... 140

5-49 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

$(i_{th} = 10 \frac{P_A}{H^2 q}, N_0 = 10, \text{ISI is not ignored})$ ........................................... 141

5-50 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

$(i_{th} = 10 \frac{P_A}{H^2 q}, N_0 = 100, \text{ISI is not ignored})$ ........................................... 141

5-51 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

$(i_{th} = 10 \frac{P_A}{H^2 q}, N_0 = 1000, \text{ISI is not ignored})$ ........................................... 141

5-52 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

$(N_0 = \infty, \text{ISI is ignored})$ ........................................... 142

5-53 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

$(i_{th} = 0 \frac{P_A}{H^2 q}, N_0 = 1, \text{ISI is ignored})$ ........................................... 144

5-54 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

$(i_{th} = 0 \frac{P_A}{H^2 q}, N_0 = 10, \text{ISI is ignored})$ ........................................... 144

5-55 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

$(i_{th} = 10 \frac{P_A}{H^2 q}, N_0 = 10, \text{ISI is ignored})$ ........................................... 145

5-56 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

$(i_{th} = 10 \frac{P_A}{H^2 q}, N_0 = 1000, \text{ISI is ignored})$ ........................................... 145

5-57 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

$(i_{th} = 0 \frac{P_A}{H^2 q}, N_0 = 1, \text{ISI is ignored})$ ........................................... 147

5-58 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

$(i_{th} = 0 \frac{P_A}{H^2 q}, N_0 = 10, \text{ISI is ignored})$ ........................................... 147

5-59 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

$(i_{th} = 0 \frac{P_A}{H^2 q}, N_0 = \infty, \text{ISI is ignored})$ ........................................... 147

6-1 Incoming rectangular pulse $s(t)$ ........................................... 149

6-2 Output $m(t)$ of a Fabry-Perot filter ........................................... 149
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6-3</td>
<td>Filtered waveform envelope corresponding to an “OFF” “ON” input</td>
<td>150</td>
</tr>
<tr>
<td>6-4</td>
<td>Filtered waveform envelope corresponding to an “ON” “OFF” input</td>
<td>150</td>
</tr>
<tr>
<td>6-5</td>
<td>Filtered signal pulse envelope $m'(t)$</td>
<td>156</td>
</tr>
<tr>
<td>6-6</td>
<td>ISI envelope $m'_{ISI}(t)$</td>
<td>157</td>
</tr>
<tr>
<td>6-7</td>
<td>Average total noise energy $\sum_{i=0}^{\infty} \lambda_i$ (expressed in units of $N_0$)</td>
<td>158</td>
</tr>
<tr>
<td>6-8</td>
<td>Sum of the first 20 eigenvalues $\sum_{i=0}^{19} \lambda_i$ (expressed in units of $N_0$)</td>
<td>158</td>
</tr>
<tr>
<td>6-9</td>
<td>Total signal energy received $\sum_{i=0}^{\infty}</td>
<td></td>
</tr>
<tr>
<td>6-10</td>
<td>Sum of the first 20 signal modes $\sum_{i=0}^{19}</td>
<td></td>
</tr>
<tr>
<td>6-11</td>
<td>Total ISI energy received $\sum_{i=0}^{\infty}</td>
<td></td>
</tr>
<tr>
<td>6-12</td>
<td>Sum of the first 20 ISI modes $\sum_{i=0}^{19}</td>
<td></td>
</tr>
<tr>
<td>6-13</td>
<td>Total noise-noise beat variance $\sum_{i=0}^{\infty} \lambda_i^2$ (expressed in units of $N_0 \times N_0$)</td>
<td>164</td>
</tr>
<tr>
<td>6-14</td>
<td>Sum of the first 20 noise-noise beat terms $\sum_{i=0}^{19} \lambda_i^2$ (expressed in units of $N_0 \times N_0$)</td>
<td>164</td>
</tr>
<tr>
<td>6-15</td>
<td>Total signal-noise beat variance $\sum_{i=0}^{\infty} 2\lambda_i</td>
<td></td>
</tr>
<tr>
<td>6-16</td>
<td>Sum of the first 20 signal-noise beat terms $\sum_{i=0}^{19} 2\lambda_i</td>
<td></td>
</tr>
<tr>
<td>6-17</td>
<td>Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($N_0 = \infty$, ISI is not ignored)</td>
<td>167</td>
</tr>
<tr>
<td>6-18</td>
<td>Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 0$ $\frac{pA}{Hz^{1/2}}$, $N_0 = 1$, ISI is not ignored)</td>
<td>169</td>
</tr>
<tr>
<td>6-19</td>
<td>Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 0$ $\frac{pA}{Hz^{1/2}}$, $N_0 = 10$, ISI is not ignored)</td>
<td>169</td>
</tr>
<tr>
<td>6-20</td>
<td>Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 0$ $\frac{pA}{Hz^{1/2}}$, $N_0 = 50$, ISI is not ignored)</td>
<td>169</td>
</tr>
<tr>
<td>6-21</td>
<td>Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 1$ $\frac{pA}{Hz^{1/2}}$, $N_0 = 10$, ISI is not ignored)</td>
<td>170</td>
</tr>
</tbody>
</table>
LIST OF FIGURES

6-22 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
   ($i_{th} = 1 \frac{pA}{H z^{1/3}}$, $N_0 = 50$, ISI is not ignored) ........................................... 170
6-23 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
   ($i_{th} = 1 \frac{pA}{H z^{1/3}}$, $N_0 = 100$, ISI is not ignored) ........................................... 170
6-24 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
   ($i_{th} = 3 \frac{pA}{H z^{1/3}}$, $N_0 = 10$, ISI is not ignored) ........................................... 171
6-25 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
   ($i_{th} = 3 \frac{pA}{H z^{1/3}}$, $N_0 = 100$, ISI is not ignored) ........................................... 171
6-26 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
   ($i_{th} = 3 \frac{pA}{H z^{1/3}}$, $N_0 = 1000$, ISI is not ignored) ........................................... 171
6-27 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
   ($i_{th} = 10 \frac{pA}{H z^{1/3}}$, $N_0 = 10$, ISI is not ignored) ........................................... 172
6-28 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
   ($i_{th} = 10 \frac{pA}{H z^{1/3}}$, $N_0 = 100$, ISI is not ignored) ........................................... 172
6-29 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
   ($i_{th} = 10 \frac{pA}{H z^{1/3}}$, $N_0 = 1000$, ISI is not ignored) ........................................... 172
6-30 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
   ($N_0 = \infty$, ISI is ignored) ........................................... 173
6-31 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
   ($i_{th} = 0 \frac{pA}{H z^{1/3}}$, $N_0 = 1$, ISI is ignored) ........................................... 175
6-32 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
   ($i_{th} = 0 \frac{pA}{H z^{1/3}}$, $N_0 = 10$, ISI is ignored) ........................................... 175
6-33 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
   ($i_{th} = 10 \frac{pA}{H z^{1/3}}$, $N_0 = 10$, ISI is ignored) ........................................... 176
6-34 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
   ($i_{th} = 10 \frac{pA}{H z^{1/3}}$, $N_0 = 1000$, ISI is ignored) ........................................... 176
6-35 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers
   ($N_0 = \infty$, ISI is not ignored) ........................................... 178
LIST OF FIGURES

6-36 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

\[ (i_{th} = 0, \frac{P_A}{H_{2/1/3}}, N_0 = 1, \text{ISI is not ignored}) \] .......................... 180

6-37 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

\[ (i_{th} = 0, \frac{P_A}{H_{2/1/3}}, N_0 = 10, \text{ISI is not ignored}) \] .......................... 180

6-38 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

\[ (i_{th} = 0, \frac{P_A}{H_{2/1/3}}, N_0 = 50, \text{ISI is not ignored}) \] .......................... 180

6-39 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

\[ (i_{th} = 1, \frac{P_A}{H_{2/1/3}}, N_0 = 10, \text{ISI is not ignored}) \] .......................... 181

6-40 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

\[ (i_{th} = 1, \frac{P_A}{H_{2/1/3}}, N_0 = 50, \text{ISI is not ignored}) \] .......................... 181

6-41 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

\[ (i_{th} = 1, \frac{P_A}{H_{2/1/3}}, N_0 = 100, \text{ISI is not ignored}) \] .......................... 181

6-42 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

\[ (i_{th} = 3, \frac{P_A}{H_{2/1/3}}, N_0 = 10, \text{ISI is not ignored}) \] .......................... 182

6-43 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

\[ (i_{th} = 3, \frac{P_A}{H_{2/1/3}}, N_0 = 100, \text{ISI is not ignored}) \] .......................... 182

6-44 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

\[ (i_{th} = 3, \frac{P_A}{H_{2/1/3}}, N_0 = 1000, \text{ISI is not ignored}) \] .......................... 182

6-45 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

\[ (i_{th} = 10, \frac{P_A}{H_{2/1/3}}, N_0 = 10, \text{ISI is not ignored}) \] .......................... 183

6-46 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

\[ (i_{th} = 10, \frac{P_A}{H_{2/1/3}}, N_0 = 100, \text{ISI is not ignored}) \] .......................... 183

6-47 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

\[ (i_{th} = 10, \frac{P_A}{H_{2/1/3}}, N_0 = 1000, \text{ISI is not ignored}) \] .......................... 183

6-48 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

\[ (N_0 = \infty, \text{ISI is ignored}) \] .......................... 184

6-49 Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers

\[ (i_{th} = 0, \frac{P_A}{H_{2/1/3}}, N_0 = 1, \text{ISI is ignored}) \] .......................... 186
<table>
<thead>
<tr>
<th>FIGURE</th>
<th>DESCRIPTION</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>6-50</td>
<td>Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0\frac{p_A}{H^2}, N_0 = 10$, ISI is ignored)</td>
<td>186</td>
</tr>
<tr>
<td>6-51</td>
<td>Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 10\frac{p_A}{H^2}, N_0 = 10$, ISI is ignored)</td>
<td>187</td>
</tr>
<tr>
<td>6-52</td>
<td>Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 10\frac{p_A}{H^2}, N_0 = 1000$, ISI is ignored)</td>
<td>187</td>
</tr>
<tr>
<td>6-53</td>
<td>Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0\frac{p_A}{H^2}, N_0 = 1$, ISI is ignored)</td>
<td>189</td>
</tr>
<tr>
<td>6-54</td>
<td>Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0\frac{p_A}{H^2}, N_0 = 10$, ISI is ignored)</td>
<td>189</td>
</tr>
<tr>
<td>6-55</td>
<td>Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0\frac{p_A}{H^2}, N_0 = \infty$, ISI is ignored)</td>
<td>189</td>
</tr>
<tr>
<td>7-1</td>
<td>Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0\frac{p_A}{H^2}, N_0 = 1$, ISI is ignored)</td>
<td>193</td>
</tr>
<tr>
<td>7-2</td>
<td>Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0\frac{p_A}{H^2}, N_0 = 10$, ISI is ignored)</td>
<td>193</td>
</tr>
<tr>
<td>7-3</td>
<td>Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0\frac{p_A}{H^2}, N_0 = \infty$, ISI is ignored)</td>
<td>193</td>
</tr>
</tbody>
</table>
List of Tables

7.1 Optimum $\frac{T}{T_i}$, $\alpha$ and $\frac{P_e}{N_0}$ for Mach-Zehnder and Fabry-Perot filters when 
   $i_{th} = 0 \frac{pA}{H^2T^3}$ (ISI is not ignored) .................................. 191

7.2 Optimum $\frac{T}{T_i}$, $\alpha$ and $\frac{P_e}{N_0}$ for Mach-Zehnder and Fabry-Perot filters when 
   $i_{th} = 1 \frac{pA}{H^2T^3}$ (ISI is not ignored) .................................. 191

7.3 Optimum $\frac{T}{T_i}$, $\alpha$ and $\frac{P_e}{N_0}$ for Mach-Zehnder and Fabry-Perot filters when 
   $i_{th} = 3 \frac{pA}{H^2T^3}$ (ISI is not ignored) .................................. 192

7.4 Optimum $\frac{T}{T_i}$, $\alpha$ and $\frac{P_e}{N_0}$ for Mach-Zehnder and Fabry-Perot filters when 
   $i_{th} = 10 \frac{pA}{H^2T^3}$ (ISI is not ignored) .................................. 192
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Chapter 1

Introduction

1.1 Background

The use of conventional repeaters to regenerate signals at points along the network is one of the major factors limiting the accessibility to the enormous bandwidth of optical fibers (on the order of 25,000 Ghz). This is caused by the speed limitations on electrical to optical and optical to electrical conversions that take place inside each repeater. The advent of optical amplifiers that use light to control light will eliminate the need for high-speed electronics to amplify signals, paving the way to a wideband all-optical fiber communication network that fully exploits the fiber's bandwidth and low attenuation.

An optical amplifier can be modeled as a linear optical field amplifier together with a source of white Gaussian noise over the bandwidth of interest [Yar 85]. This thesis is motivated by the potential use of optical filters in rejecting some of the incident amplifier noise power prior to detection. It analyzes the performance of two types of predetection optical filters used in direct detection receivers when the received signal consists of rectangular pulses of monochromatic light mixed with additive optical amplifier noise. The two modulations that will be considered are Amplitude Shift Keying (ASK) and wide deviation Frequency Shift Keying (FSK).
CHAPTER 1. INTRODUCTION

Direct detection receivers, often called power detecting receivers or noncoherent receivers, are used whenever the transmitted information occurs in the power variations of the received field. In the case of direct detection reception, the use of predetection optical filters matched to the transmitted rectangular pulses helped in fighting the effects of additive optical amplifier noise for ASK and FSK modulations [Hum 91]. A filter with impulse response $h(t)$ is said to be matched to a given signal $s(t)$ if $h(t)$ is proportional to $s(-t)$. Matched filters are used to optimally recover deterministic signals corrupted by additive white Gaussian noise; the output is sampled periodically and compared to a threshold based on which decisions are made [Van 68, WoJ 65].

In many situations, neighboring symbols overlap and interfere with one another at the proper sampling time resulting in a phenomenon given the descriptive name of intersymbol interference (ISI). The combined effects of noise and intersymbol interference may result in errors in the output message generated by the receiver. The use of a matched filter that reduces the effects of additive white Gaussian noise does not necessarily improve the system’s performance since it may very well introduce intersymbol interference at the receiver.

If the source of noise is white, the mean square value of the noise sample is reduced by narrowing the bandwidth of the receiver’s filter. Unfortunately, a smaller bandwidth will cause the filtered pulses to spread out, which would increase the ISI. Thus, there is a basic tradeoff between the amount of ISI and the intensity of the noise passed by the receiver’s filter.

The recent work in [Hum 91] focuses on the performance of one particular bandpass optical filter: the cascaded Mach-Zehnder filter. The analysis considers the special situation where the predetection Mach-Zehnder filter is matched to the transmitted rectangular pulses. It does not include the effects of ISI on the system’s performance and assumes the amplifier’s additive noise dominates the other source of uncertainty in the receiver: the thermal noise coming from the electronic post de-
CHAPTER 1. INTRODUCTION

tection circuitry. This thesis will examine the effects of both ISI and thermal noise on the performance of matched and unmatched Mach-Zehnder predetection optical filters. We will also study ways of reducing the amount of ISI (and run the risk of increasing the noise power at the sampler) through the use of another optical filter: the single-cavity Fabry-Perot interferometer. The performance analysis of Fabry-Perot interferometers will take into account the effects of ISI and thermal noise as well.

1.2 Thesis Overview

 Chapters 2 and 3 outline the structure of Mach-Zehnder and Fabry-Perot optical filters. In the time domain, we will focus on the response of these two filters to incident rectangular pulses and the resulting interference between neighboring filtered signals. In the frequency domain, we will derive the filters transfer functions and discuss their effectiveness (if any) in rejecting white optical noise with as little ISI created as possible.

 Chapter 4 outlines the receiver structure for ASK and FSK systems. One major problem with optical detection is that no optical energy can be collected by "sampling" the incoming pulse of light over an infinitesimal interval of time. It is therefore suggested to use a post detection filter that integrates the output current generated by the photodetector over a finite interval of time [Hum 91]. Chapter 4 also reviews the theory of detecting optical signals mixed with additive Gaussian fields and derives the corresponding electron count statistics. Several approximations and bounds for the bit error rate will be developed.

 In Chapters 5 and 6, we present the results of our performance analysis of Mach-Zehnder and Fabry-Perot predetection filters for both ASK and FSK modulations. Our analysis will be twofold: we first start by including the effects of ISI on the system’s performance and then repeat the same process while ignoring all interferences. A comparison between the two results will emphasize the impact of ISI on the
performance. The receiver will be subjected to low, moderate and strong electronic thermal noise.

Finally, Chapter 7 gives a summary of the work and some directions for further research.
Chapter 2

Mach-Zehnder optical filters

This chapter examines the structure and properties of Mach-Zehnder optical filters. It mainly focuses on (1) the response of such filters to incident rectangular pulses of monochromatic light, (2) their ability to reject additive white optical amplifier noise and (3) the interference between neighboring filtered pulses.

2.1 Structure of a single stage Mach-Zehnder filter

The Mach-Zehnder interferometer (MZI) shown in Figure 2-1 is composed of two 3dB optical couplers interconnected by two pieces of single mode fibers of different lengths.

![Diagram of a single-stage Mach-Zehnder filter]

Figure 2-1: Single-stage Mach-Zehnder filter
CHAPTER 2. MACH-ZEHNDER OPTICAL FILTERS

The two paths differ by a delay $\Delta T$. Such a single Mach-Zehnder filter involves the interference between two versions of the same optical signal separated in time by the differential delay $\Delta T$. The incident signal is first split evenly by the 3dB coupler at the input with half the optical power going in each of the two fibers, then merged together by the second coupler at the output after undergoing a delay of exactly $\Delta T$. This interferometer has little wavelength discrimination ability and will not be used as a predetection filter in our analysis. However, the cascading of a number of MZIs with properly adjusted differential delays can produce a narrow passband filter with a good noise rejection capability which is more suitable for use at the receiver to optimally recover rectangular pulses of monochromatic light.

2.2 Structure of a multistage Mach-Zehnder filter

2.2.1 Multistage Mach-Zehnder filter with 2 branches per stage

![Diagram of Mach-Zehnder filter with n stages (2 branches per stage)](image)

Figure 2-2: Mach-Zehnder filter with n stages (2 branches per stage)

Figure 2-2 shows a block diagram of one particular arrangement of Mach-Zehnder interferometers into a multistage structure that can be used as a predetection front end optical filter. Each stage introduces a differential delay that is $\frac{1}{2}$ the previous stage to its left. Cascaded filters with this particular delay relation of $\frac{T}{2^k}$ for stage
k have been proposed as a channel switching device to be used in frequency division multiple access optical networks. Several versions of this chain optical filter were recently implemented at the NTT Transmission Systems and Optoelectronics Laboratories (Ibaraki-ken, Japan) with silica-based optical waveguides deposited on a silicon substrate [InT 88, OdT 90, TaK 90].

Consider now an incident rectangular pulse \( s(t) \) of width \( T \), frequency \( \nu_0 \), and a total energy \( E \) as shown in Figure 2-3; the signal can be expressed as: \( s(t) = A \cos(2\pi \nu_0 t) \) for \( t \in [0, T] \), where the amplitude \( A \) is given by: \( A = \sqrt{\frac{2E}{T}} \).

![Figure 2-3: Incoming rectangular pulse \( s(t) \)](image)

We shall denote the filtered signal at the output by \( m(t) \). At the input of each stage, the incoming signal is split by a lossless 3dB coupler into two identical pulses each carrying half the incident energy. These two pulse are then recombined at the output after undergoing a differential delay of \( \frac{T}{2^n} \) with \( 1 \leq k \leq n \), \( n \) being the total number of stages. If we further assume that the smallest differential delay introduced by the cascaded Mach-Zehnder filter, \( \frac{T}{2^n} \), is a multiple of the optical period \( \frac{\lambda_0}{v_p} \):

\[
\frac{T}{2^n} = i \frac{\lambda_0}{v_p} \quad i = 1, 2, 3, \ldots \tag{2.1}
\]

where \( \lambda_0 \) is the optical wavelength and \( v_p = \nu_0 \lambda_0 \) is the phase velocity inside the fiber, then the pulses leaving each stage will add up in phase as illustrated in Figures 2-4 and 2-5.

Assuming no power is lost inside the waveguides or in any of the two couplers,
the amplitude of the recombined signal at \( t = T \) remains equal to \( A \) during the whole process.

![Diagram of output of the first stage](image)

**Figure 2-4: Output of the first stage**

The number of steps in the signal envelope grows by a factor of 2 each time the pulse travels through an additional stage: at the output of the \( k \)th stage, the total number of steps on either side of the envelope is equal to \( 2^k \). The width \( W_{base} \) of the pulse at the output of the \( k \)th stage, given by:

\[
W_{base} = T + \sum_{i=1}^{k} \frac{T}{2^i} = (2 - 2^{-k})T
\]  

(2.2)

converges geometrically to \( 2T \).

![Diagram of output of the second stage](image)

**Figure 2-5: Output of the second stage**
Figure 2-6: Output \( m(t) \) of a filter with an infinitely large number of stages

\[
\lim_{k \to \infty} W_{\text{base}} = 2T
\]  

(2.3)

As the number of stages \( n \) increases, the output signal envelope converges into a triangle of height \( A \) and a base of \( 2T \) as shown in Figure 2-6. The shaded area corresponds to the region of integration during post detection processing of the signal as will be explained later on in Chapter 4. A linear filter matched to an incoming rectangular pulse of width \( T \) produces a similar triangular pulse at the output.

It is possible to set the differential delays of the cascaded Mach-Zehnder filter of Figure 2-1 in such a way that the output signal envelope converges instead into a trapeze. Such an optical filter is shown in Figure 2-7 where the adjusting factor \( a \) is larger than 2.

Figure 2-7: Mach-Zehnder filter with \( n \) stages (2 branches per stage and \( a \geq 2 \))

Consider the same incident rectangular pulse shown in Figure 2-3. Each stage splits the incoming waveform into two identical pulses which are recombined at the
output after undergoing a differential delay of $\frac{T}{a^{2k-1}}$ with $1 \leq k \leq n$, $n$ being the total number of stages. The pulses leaving each stage will add up in phase as shown in Figures 2-8 and 2-9 only if the corresponding differential delay is a multiple of the optical period $\frac{\lambda}{v_p}$.

![Diagram](image)

Figure 2-8: Output of the first stage ($a \geq 2$)

By choosing the smallest differential delay introduced by the filter, $\frac{T}{a^{2n-1}}$, to be a multiple of $\frac{\lambda}{v_p}$:

$$\frac{T}{a^{2n-1}} = i \frac{\lambda}{v_p} \quad i = 1, 2, 3, \ldots$$  \hspace{1cm} (2.4)

we will ensure that the pulses leaving each stage are recombined in phase.

![Diagram](image)

Figure 2-9: Output of the second stage ($a \geq 2$)

The number of steps on either side of the signal envelope at the output of the kth stage is again $2^k$, however the pulse is narrower now and has a width $W_{base}$ equal to:
CHAPTER 2. MACH-ZEHNDER OPTICAL FILTERS

\[ W_{\text{base}} = T + \sum_{i=1}^{k} \frac{T}{a2^{i-1}} \]

\[ = (1 + \frac{2}{a}(1 - 2^{-k}))T \]

(2.5)

which converges exponentially to \((\frac{a+2}{a})T\) \((a \geq 2)\) as the number of stages increases:

\[ \lim_{k \to \infty} W_{\text{base}} = (\frac{a+2}{a})T \quad a \geq 2 \]

(2.6)

Moreover, the width \(W_{\text{top}}\) of the top flat region of the envelope, given by:

\[ W_{\text{top}} = T - \sum_{i=1}^{k} \frac{T}{a2^{i-1}} \]

\[ = (1 - \frac{2}{a}(1 - 2^{-k}))T \]

(2.7)

converges to \((\frac{a-2}{a})T\), \(a \geq 2\) (see Figure 2-10).

\[ \lim_{k \to \infty} W_{\text{top}} = (\frac{a-2}{a})T \quad a \geq 2 \]

(2.8)

Figure 2-10: Output \(m(t)\) of a filter with an infinitely large number of stages \((a \geq 2)\)
2.2.2 Multistage Mach-Zehnder filter with \( b \) branches per stage \((b \geq 2)\)

The same triangular output produced by a matched Mach-Zehnder filter with 2 branches per stage can be obtained using a cascaded optical filter with \( b \) branches per stage, where \( b \) is an integer larger or equal to 2. The configuration of such a filter is shown in Figure 2-11.

![Diagram of a multistage Mach-Zehnder filter](image)

Figure 2-11: Mach-Zehnder filter with \( n \) stages (\( b \) branches per stage)

The incident rectangular pulse \( s(t) \) shown in Figure 2-12 with width \( T \), frequency \( \nu_0 \), and a total energy \( E \), can be expressed as: \( s(t) = A \cos(2\pi \nu_0 t) \) for \( t \in [0, T] \) with \( A = \sqrt{\frac{2E}{T}} \). The optical signal arriving at the \( k \)th stage is split by a \( b \times b \) optical star into \( b \) identical pulses each carrying \( \frac{1}{b} \) of the incident energy and merge back together at the output after undergoing \( b - 1 \) differential delays of: \( \frac{T}{b^k}, \frac{2T}{b^k}, \frac{3T}{b^k}, \frac{4T}{b^k}, \ldots, \frac{(b-2)T}{b^k} \) and \( \frac{(b-1)T}{b^k} \). If all differential delays are multiples of the optical period \( \frac{\lambda}{\nu_0} \), the \( b \) pulses leaving each stage recombine in phase; this is satisfied when \( \frac{T}{b^k} \), the smallest
differential delay introduced by such a cascaded filter with \( n \) stages, is a multiple of the optical period \( \frac{\lambda_0}{v_p} \):

\[
\frac{T}{b^n} = i \frac{\lambda_0}{v_p} \quad i = 1, 2, 3, \ldots
\]  

(2.9)

![Diagram of incoming rectangular pulse](image)

Figure 2-12: Incoming rectangular pulse \( s(t) \)

As the signal travels along the cascaded filter the number of steps on either side of its envelope increases geometrically by a factor of \( b \) for each additional stage the signal goes through. At the output of the \( k \)th stage, the total number of steps on either side of the envelope is equal to \( b^k \) and the width \( W_{base} \) of the pulse increases by \( \frac{(b-1)T}{b^k} \) and is given by:

\[
W_{base} = T + \sum_{i=1}^{k} \frac{(b-1)T}{b^i} = (2 - b^{-k})T
\]

(2.10)

which converges exponentially to \( 2T \).

\[
\lim_{k \to \infty} W_{base} = 2T
\]

(2.11)

The envelope peak, located at \( t = T \) remains constant and equal to \( A \) during the whole process. As the number of stages \( n \) increases, the filtered signal envelope approaches a triangle of height \( A \) and a base of \( 2T \) as shown in Figure 2-13. The
CHAPTER 2. MACH-ZEHNDER OPTICAL FILTERS

shaded area corresponds to the region of integration during post detection filtering.

![Figure 2-13: Output $m(t)$ of a filter with an infinitely large number of stages](image)

Figure 2-13: Output $m(t)$ of a filter with an infinitely large number of stages

Figure 2-14 depicts a cascaded optical filter with 3 branches per stage. This 4 branches multistage Mach-Zehnder filter uses $3 \times 3$ optical stars to split and recombine the signal at the input and output of each stage. As long as $\frac{T}{3^n}$, the smallest differential delay introduced by the cascaded filter, is a multiple of the optical period $\frac{\lambda_0}{v_p}$:

$$\frac{T}{3^n} = i \frac{\lambda_0}{v_p} \quad i = 1, 2, 3, \ldots \quad (2.12)$$

the pulses leaving each stage add up in phase as shown in Figures 2-15 and 2-16.

![Figure 2-14: Mach-Zehnder filter with n stages (3 branches per stage)](image)
CHAPTER 2. MACH-ZEHNDER OPTICAL FILTERS

Figure 2-15: Output of the first stage

As the signal travels along the cascaded filter the number of steps on either side of its envelope increases geometrically by a factor of 3 for each new stage the pulse encounters in its path and is therefore equal to $3^k$ at the output of the $k$th stage. The width $W_{\text{base}}$ of the pulse at the $k$th stage output, given by:

$$W_{\text{base}} = T + \sum_{i=1}^{k} \frac{T}{3^i}$$
$$= (2 - 3^{-k})T \quad (2.13)$$

eventually converges to $2T$ and the envelope approaches the triangular waveform of Figure 2-13.

Figure 2-16: Output of the second stage

Although more complex and harder to build than the 2 branches multistage filter, this 3 branches Mach-Zehnder chain has the advantage of producing the desired output pulse shape with fewer stages: the filtered pulse envelope converges toward the
CHAPTER 2. MACH-ZEHNDER OPTICAL FILTERS

Figure 2-17: Mach-Zehnder filter with n stages (4 branches per stage)

desired triangular/trapezoidal configuration at a rate \( \frac{2}{3} \) times faster when a 3 stages filter is used. The rate of convergence can be increased even further by adding one more branch to each stage of the filter in Figure 2-14, raising the number of branches per stage to 4 as illustrated in Figure 2-17.

Figures 2-18 and 2-19 illustrate how the number of steps in the pulse envelope quadruples for each new stage it travels through; this is obviously true only when the signals are recombined in phase at each stage output, which in turn implies that:

Figure 2-18: Output of the first stage
$T = \frac{\lambda_0}{4^n v_p}$ \hspace{1cm} i = 1, 2, 3, \ldots \hspace{1cm} (2.14)

The filtered pulse width $W_{base}$ at the $k$th stage output is now:

$$W_{base} = T + \sum_{i=1}^{k} \frac{T}{4^i}$$

$$= (2 - 4^{-k})T \hspace{1cm} (2.15)$$

and clearly approaches the $2T$ limit twice as fast as the 2 branches Mach-Zehnder chain of Figure 2-2 and $\frac{4}{3}$ times faster than the 3 branches cascaded filter of Figure 2-14, unfortunately, such a complex filter is much harder to implement. The filtered pulse envelope eventually converges into the same triangular waveform generated by a linear filter matched to a rectangular input (Figure 2-13).

The incident rectangular signal can be shaped into a trapeze by adjusting the differential delays of the $b$ branches cascaded Mach-Zehnder filter of Figure 2-11. Consider the remodeled optical filter shown in Figure 2-20 where the adjusting factor $a$ is now larger than $b$.

Stage $k$ splits the incoming waveform into $b$ identical pulses each carrying $\frac{1}{b}$ of the incident energy and are recombined at the output after undergoing $b - 1$ differential delays of: $\frac{T}{ab^k-1}, \frac{2T}{ab^k-1}, \frac{3T}{ab^k-1}, \frac{4T}{ab^k-1}, \ldots, \frac{(b-2)T}{ab^k-1}$ and $\frac{(b-1)T}{ab^k-1}$, with $1 \leq k \leq n$, $n$ being the total number of stages. By choosing the smallest differential delay introduced by the
filter, $\frac{T}{a b^{n-1}}$, to be a multiple of $\frac{\lambda_0}{v_p}$:

$$\frac{T}{a b^{n-1}} = i \frac{\lambda_0}{v_p} \quad i = 1, 2, 3, \ldots \quad (2.16)$$

we will ensure that the $b$ pulses leaving each stage are recombined in phase. The width $W_{\text{base}}$ at the $k$ stage output becomes:

$$W_{\text{base}} = T + \sum_{i=1}^{k-1} \frac{T}{a b^i} = (1 + \frac{b}{a} (1 - b^{-k}))T \quad (2.17)$$

and approaches $\left(\frac{a+b}{a}\right)T \ (a \geq b)$ as the number of stages increases.

$$\lim_{k \to \infty} W_{\text{base}} = \left(\frac{a+b}{a}\right)T \quad a \geq b \quad (2.18)$$
CHAPTER 2. MACH-ZEHNDER OPTICAL FILTERS

The width $W_{\text{top}}$ of the top flat region of the envelope, given by:

\[ W_{\text{top}} = T - \sum_{i=1}^{k-1} \frac{T}{ab^i} \]
\[ = (1 - \frac{b}{a}(1 - b^{-k}))T \]

converges to $\left(\frac{a-b}{a}\right)T$ ($a \geq b$):

\[ \lim_{k \to \infty} W_{\text{top}} = \left(\frac{a-b}{a}\right)T \quad a \geq b \]

The output pulse envelope converges toward the trapeze shown in Figure 2-21 as the number of stages $n$ increases.

![Diagram](image)

Figure 2-21: Output $m(t)$ of a filter with an infinitely large number of stages ($a \geq b$)

If the adjusting factor $a$ is equal to the number of branches per stage, $b$, the incident rectangular pulse is shaped into a triangular pulse $m(t)$ of base $2T$ and height $A = \sqrt{\frac{2E}{T}}$ when the number of stages is moderately large. If $a > b$, the output $m(t)$ converges into a trapeze instead with a base $W_{\text{base}} = (1 + \frac{b}{a})T$, a flat top $W_{\text{top}} = (1 - \frac{b}{a})T$ and a height $A$. Since $a \geq b$, the ratio $\frac{b}{a}$ is always within the range $[0, 1]$. By decreasing the ratio $\frac{b}{a}$ toward 0, the base $W_{\text{base}}$ narrows and $W_{\text{top}}$ increases; in the extreme situation where $\frac{b}{a}$ approaches 0, $W_{\text{base}} = W_{\text{top}} = T$.

Figure 2-22 depicts a non-matched cascaded filter with 3 branches per stage. The factor $a$ is larger than 3.
CHAPTER 2. MACH-ZEHNDER OPTICAL FILTERS

Figure 2-22: Mach-Zehnder filter with n stages (3 branches per stage and \( a \geq 3 \))

If all differential delays are multiples of the optical period \( \frac{\lambda}{v_p} \), the merging three pulses recombine in phase as shown in Figures 2-23 and 2-24.

Figure 2-23: Output of the first stage

Once again, it is assumed here that:

\[
\frac{T}{a3^{i-1}} = \frac{\lambda_0}{v_p} \quad \text{for } i = 1, 2, 3, \ldots \quad (2.21)
\]

At the output of the \( k \)th stage, the pulse width \( W_{base} \) is:

\[
W_{base} = T + \sum_{i=1}^{k} \frac{T}{a3^{i-1}}
\]
Figure 2-24: Output of the second stage

\[ A = (1 + \frac{3}{a}(1 - 3^{-k}))T \]  \hspace{1cm} (2.22)

and converges to \((\frac{a+3}{a})T \ (a \geq 3)\) as shown in Figure 2-25.

The width \(W_{top}\) of the top flat region of the envelope, given by:

\[ W_{top} = T - \sum_{i=1}^{k} \frac{T}{a3^{i-1}} \]

\[ = (1 - \frac{3}{a}(1 - 3^{-k}))T \]  \hspace{1cm} (2.23)

converges to \((\frac{a-3}{a})T, \ a \geq 3\).

The height of the envelope remains constant and equal to \(A\) as the pulse travels along the filter while the number of steps in its envelope multiplies by a factor of 3 for each new stage it passes through.

Figure 2-25: Output \(m(t)\) of a filter with an infinitely large number of stages \((a \geq 3)\)
Figure 2-26 depicts a non-matched cascaded filter with 4 branches per stage. The factor \( a \) is now larger than 4.

![Diagram of Mach-Zehnder optical filters](image)

Figure 2-26: Mach-Zehnder filter with \( n \) stages (4 branches per stage \( a \geq 4 \))

Assuming the following condition is satisfied:

\[
\frac{T}{a4^{n-1}} = \frac{i\lambda_0}{v_p} \quad i = 1, 2, 3, \ldots
\]  

(2.24)

then the filtered waveforms at the outputs of the second and third stages are shown in Figures 2-27 and 2-28.

![Output of the first stage](image)

Figure 2-27: Output of the first stage
The width $W_{\text{base}}$ at the $k$ stage output becomes:

$$W_{\text{base}} = T + \sum_{i=1}^{k} \frac{T}{a4^{i-1}}$$

$$= (1 + \frac{4}{a}(1 - 4^{-k}))T$$

(2.25)

and approaches $(\frac{a+4}{a})T$ ($a \geq 4$) as the number of stages increases.

The width $W_{\text{top}}$ of the top flat region of the envelope, given by:

$$W_{\text{top}} = T - \sum_{i=1}^{k} \frac{T}{a4^{i-1}}$$

$$= (1 - \frac{4}{a}(1 - 4^{-k}))T$$

(2.26)
converges to \((\frac{a-d}{a})T\) \((a \geq 4)\).

### 2.2.3 ISI introduced by Mach-Zehnder filters

Figure 2-30 illustrates the interference between two consecutive "ON" pulses when \(a = b\) (triangular output). The shaded region corresponds to the range of integration during post detection processing of the signal. Note how each of the two neighboring pulses overlaps the signal of interest inside the shaded region: ISI is always present between triangular pulses regardless how short the interval of integration is. Furthermore, the amount of ISI energy increases rapidly as the range of integration is made wider.

![Diagram of a triangular signal with a shaded region indicating the region of integration.](image)

Figure 2-30: Triangular outputs (case where \(a = b\))

By contrast, the amount of ISI energy is greatly reduced when trapezoidal pulses are used as shown in Figure 2-31 (situation where \(a > b\)). Two neighboring pulses never overlap inside the flat region of the trapeze; therefore, ISI can be completely eliminated by confining the range of integration to within the flat region of the trapeze. As the range of integration is expanded beyond the flat region, the integrated ISI energy increases but remains significantly smaller than for triangular outputs.
CHAPTER 2. MACH-ZEHNDER OPTICAL FILTERS

2.3 Frequency response of a single stage Mach-Zehnder filter

The noise rejection properties of Mach-Zehnder filters become more apparent when visualizing the process in the frequency domain. Let us first derive the transfer function of the simplest possible Mach-Zehnder interferometer (MZI): the single stage, 2-branches filter of Figure 2-1. The magnitude of the transfer function, $\|H(\nu)\|$, $\nu$ being the optical frequency, is given by:

$$\|H(\nu)\| = \frac{1}{2} \|1 + e^{-j2\pi \Delta T \nu}\|$$

$$= |\cos(\pi \Delta T \nu)|$$  \hspace{1cm} (2.27)

Figure 2-32 shows a plot of the transfer function magnitude, $\|H(\nu)\|$, against the optical frequency $\nu$.

By inspecting equation 2.27, it can be easily seen that the transfer function is periodic. The period is commonly referred to as the free spectral range (FSR) of the
Figure 2.32: Frequency response of a single stage Mach-Zehnder filter ($\nu$ is expressed in units of $\frac{1}{\Delta T}$)

Mach-Zehnder filter, which in this case is equal to the inverse of the differential delay $\Delta T$:

$$FSR = \frac{1}{\Delta T}$$ (2.28)

Two quantities measured from the transfer function are of particular importance:

1. The spacing of the peaks (i.e. the free spectral range).
2. The full width at half maximum (FWHM) of the main lobes.

The filter's ability to combat additive optical white noise is improved by increasing the spacing between peaks and narrowing the main lobes (i.e. decreasing the FWHM). The single stage MZI is clearly not suitable for use as a predetection optical filter. Not only it is incapable of shaping the rectangular signal into the triangular
and trapezoidal outputs of Figures 2-30 and 2-31, but it also lacks the necessary wide 
FSR and narrow passband to effectively reject the additive noise power. The free 
spectral range can be increased by inserting more nulls between two neighboring main 
lobes of the transfer function; this is achieved by adding more stages to the filter re-
resulting in the complex 2-branches cascaded Mach-Zehnder filter structure of Figure 
2-7. Of course, the superiority of the 2-branches cascaded Mach-Zehnder filter over 
the single stage MZI lies also in its ability to shape the input pulse in the desired 
triangular and trapezoidal forms of Figures 2-30 and 2-31.

2.4 Frequency response of a multistage Mach-Zehnder 
filter

2.4.1 Multistage Mach-Zehnder filter with 2 branches per 
stage

Consider the 2 branches cascaded filter of Figure 2-7. The magnitude of the transfer 
function, $\|H(\nu)\|$, $\nu$ being the optical frequency, is given by:

\[
\|H(\nu)\| = \prod_{i=1}^{n} \left| \frac{1 + e^{-j2\pi \frac{T}{a^{2^i-1}}\nu}}{2} \right| \quad a \geq 2
\]

\[
= \prod_{i=1}^{n} \left| \frac{1 - e^{-j4\pi \frac{T}{a^{2^i-1}}\nu}}{1 - e^{-j2\pi \frac{T}{a^{2^i-1}}\nu}} \right| \quad a \geq 2
\]

\[
= \prod_{i=1}^{n} \left| \frac{\left(1 - \cos(4\pi \frac{T}{a^{2^i-1}}\nu)\right)^2 + \sin^2(4\pi \frac{T}{a^{2^i-1}}\nu)}{\left(1 - \cos(2\pi \frac{T}{a^{2^i-1}}\nu)\right)^2 + \sin^2(2\pi \frac{T}{a^{2^i-1}}\nu)} \right| \quad a \geq 2
\]

\[
= \prod_{i=1}^{n} \left| \frac{2 - 2\cos(4\pi \frac{T}{a^{2^i-1}}\nu)}{2 - 2\cos(2\pi \frac{T}{a^{2^i-1}}\nu)} \right| \quad a \geq 2
\]

\[
= \prod_{i=1}^{n} \left| \frac{\sin(\frac{1}{2}4\pi \frac{T}{a^{2^i-1}}\nu)}{\sin(\frac{1}{2}2\pi \frac{T}{a^{2^i-1}}\nu)} \right| \quad a \geq 2
\]
CHAPTER 2. MACH-ZEHNDER OPTICAL FILTERS

\[
\begin{align*}
&= \frac{1}{2^n} \left| \frac{\sin \left( \frac{1}{2} \frac{4\pi}{T_a} \nu \right)}{\sin \left( \frac{1}{2} \frac{2\pi}{T_a} \nu \right)} \sin \left( \frac{1}{2} \frac{4\pi}{a_2 T_a} \nu \right) \right| \ldots \\
&\ldots \sin \left( \frac{1}{2} \frac{4\pi}{a_{2^{n-1}} T_a} \nu \right) \sin \left( \frac{1}{2} \frac{4\pi}{a_{2^{n-2}} T_a} \nu \right) \sin \left( \frac{1}{2} \frac{2\pi}{a_{2^{n-1}} T_a} \nu \right) \\
&= \frac{1}{2^n} \left| \frac{\sin \left( 2\pi \frac{2}{a_2 \nu} \right)}{\sin \left( 2\pi \frac{2}{a_{2^n} \nu} \right)} \right| \quad a \geq 2
\end{align*}
\]

(2.29)

The transfer function as expressed in equation 2.29 is periodic; its free spectral range \(FSR\) is given by:

\[
FSR = \frac{a 2^{n-1}}{T} \quad a \geq 2
\]

(2.30)

The \(FSR\) of a matched Mach-Zehnder chain (situation where \(a = 2\)), given by:

\[
FSR = \frac{2^n}{T}
\]

(2.31)

increases geometrically with the number of stages \(n\) in the filter.

Figures 2-33, 2-34, 2-35, 2-36, 2-37 and 2-38 show plots of \(\|H(\nu)\|\) versus the optical frequency \(\nu\) for different values of \(n\) and \(a\).

The main lobes are located at multiples of \(\frac{a 2^{n-1}}{T}\). Between any two consecutive main lobes of the transfer function, there are \(2^n - 1\) nulls evenly spaced by intervals of \(\frac{\nu}{2^n T}\). The passband of the optical filter, equal in this case to the width of one main lobe \(\frac{\nu}{2^n T}\), is independent of the number of stages \(n\). The addition of new stages to the cascaded filter will definitively improve its noise rejection capability: the \(FSR\) explodes exponentially while the \(FWHM\) remains constant.
Figure 2-33: Frequency response when $n = 1$, $a = 2$, and $b = 2$ ($\nu$ is expressed in units of $\frac{1}{T}$)

Figure 2-34: Frequency response when $n = 2$, $a = 2$, and $b = 2$ ($\nu$ is expressed in units of $\frac{1}{T}$)

Figure 2-35: Frequency response when $n = 3$, $a = 2$, and $b = 2$ ($\nu$ is expressed in units of $\frac{1}{T}$)
Figure 2-36: Frequency response when $n = 1$, $a = 3$, and $b = 2$ ($\nu$ is expressed in units of $\frac{1}{T}$)

Figure 2-37: Frequency response when $n = 2$, $a = 3$, and $b = 2$ ($\nu$ is expressed in units of $\frac{1}{T}$)

Figure 2-38: Frequency response when $n = 3$, $a = 3$, and $b = 2$ ($\nu$ is expressed in units of $\frac{1}{T}$)
CHAPTER 2. MACH-ZEHNDER OPTICAL FILTERS

The transform $S(\nu)$ of the input rectangular pulse $s(t) = \sqrt{\frac{2E}{T}} \cos(2\pi \nu_0 t)$, $t \in [0, T]$ is the sum of two "sinc" functions centered at $\pm \nu_0$:

$$||S(\nu)|| = \sqrt{\frac{ET}{2}} \left| \frac{\sin(\pi T(\nu - \nu_0))}{\pi T(\nu - \nu_0)} \right| + \sqrt{\frac{ET}{2}} \left| \frac{\sin(\pi T(\nu + \nu_0))}{\pi T(\nu + \nu_0)} \right|$$  \hspace{1cm} (2.32)

It is therefore crucial to choose the optical carrier frequency $\nu_0$ to be a multiple of $\frac{a^{2n-1}}{T}$ in order to confine the signal energy within the passband of the optical filter:

$$\nu_0 = \frac{a^{2n-1}}{T} \quad a \geq 2, \quad i = 1, 2, 3, \ldots$$  \hspace{1cm} (2.33)

Equation 2.33 is actually identical to equation 2.4 since the frequency and the wavelength are related by:

$$\nu_0 \lambda_0 = v_p$$  \hspace{1cm} (2.34)

Most of the signal energy (92% of the total energy $E$ to be exact) is within the frequency bands $[\nu_0 - \frac{a}{2T}, \nu_0 + \frac{a}{2T}]$ and $[-\nu_0 - \frac{a}{2T}, -\nu_0 + \frac{a}{2T}]$. Note that since $a \geq 2$, most of the signal energy will be within the passband of the optical filter $[\pm \nu_0 - \frac{a}{2T}, \pm \nu_0 + \frac{a}{2T}]$ as well. If the number of stages $n$ is sufficiently large and equation 2.33 is satisfied, then for frequencies $\nu$ within the passband $[\pm \nu_0 - \frac{a}{2T}, \pm \nu_0 + \frac{a}{2T}]$, the denominator of transfer function as expressed in equation 2.29 can be approximated as follows:

$$\sin(2\pi \frac{T}{a^{2n}} \nu) = \sin(2\pi \frac{T}{a^{2n}} (\nu \pm \nu_0))$$

$$\approx 2\pi \frac{T}{a^{2n}} (\nu \pm \nu_0) \quad \text{for} \quad |\nu \pm \nu_0| \leq \frac{a}{2T} \ll \frac{a^{2n-1}}{T}$$  \hspace{1cm} (2.35)

Under these conditions, the transfer function magnitude $||H(\nu)||$ can be approximated by:
\[ \|H(\nu)\| \approx \left| \frac{\sin(2\pi T_a(\nu \pm \nu_0))}{2\pi T_a(\nu \pm \nu_0)} \right| \quad \text{for } |\nu \pm \nu_0| \leq \frac{a}{2T} \ll \frac{a^{2n-1}}{T} \quad (2.36) \]

In practice, the multistage Mach-Zehnder filter will be cascaded into another wideband optical filter of bandwidth \( \frac{a^{2n-1}}{T} \) centered at \( \pm \nu_0 \). The addition of such a filter is necessary to eliminate some of the additive noise power and to smooth the “staircase” feature of the filtered signal envelope. Consequently, as long as \( n \) is moderately large (i.e. the free spectral range \( \frac{a^{2n-1}}{T} \) is much larger than the main lobe width \( \frac{a}{T} \)) and equation 2.33 is satisfied, the transfer function magnitude \( \|H_r(\nu)\| \) for the two cascaded filters can be approximated by:

\[ \|H_r(\nu)\| \approx \left| \frac{\sin(2\pi T_a(\nu - \nu_0))}{2\pi T_a(\nu - \nu_0)} \right| + \left| \frac{\sin(2\pi T_a(\nu + \nu_0))}{2\pi T_a(\nu + \nu_0)} \right| \quad \text{for all } \nu \quad (2.37) \]

which is the frequency response of a filter matched to an optical rectangular pulse of frequency \( \nu_0 \), width \( \frac{2T}{a} \) and an amplitude \( \frac{a}{T} \). The magnitude squared of the transfer function of the two combined filters \( \|H_r(\nu)\|^2 \) is approximated by:

\[ \|H_r(\nu)\|^2 \approx \frac{\sin^2(2\pi T_a(\nu - \nu_0))}{(2\pi T_a(\nu - \nu_0))^2} + \frac{\sin^2(2\pi T_a(\nu + \nu_0))}{(2\pi T_a(\nu + \nu_0))^2} \quad \text{for all } \nu \quad (2.38) \]

Taking the Inverse Fourier Transform of both sides of equation 2.38, we get the following expression for the convolution of the equivalent filter impulse response \( h_r(\tau) \) with its time reversed conjugate replica \( h^*_r(-\tau) \):

\[ h_r(\tau) * h^*_r(-\tau) \approx 2 \frac{a}{2T} \left( 1 - \frac{|\tau|}{2T} \right) \cos(2\pi \nu_0 \tau), \quad \tau \in \left[ -\frac{2T}{a}, \frac{2T}{a} \right] \quad \text{with } a \geq 2 \quad (2.39) \]

It is important to remember that equation 2.39 was derived under the assumptions:
1. \( \frac{a_T}{2T} \ll \frac{a_{2n-1}}{T} \).

2. \( \nu_0 \) is a multiple of the FSR \( \frac{a_{2n-1}}{T} \).

Equation 2.39 will come in very handy when computing the autocorrelation function of the filtered optical amplifier noise later on.

### 2.4.2 Multistage Mach-Zehnder filter with \( b \) branches per stage \( (b \geq 2) \)

The magnitude of the transfer function of the 3 branches multistage Mach-Zehnder of Figure 2-22 can be computed in a similar fashion:

\[
\| H(\nu) \| = \prod_{i=1}^{n} \left| 1 + e^{-j2\pi \frac{T}{a_{3i-1}}} \nu + e^{-j4\pi \frac{T}{a_{3i-1}}} \nu \right| \quad a \geq 3
\]

\[
= \prod_{i=1}^{n} \left| \frac{1 - e^{-j6\pi \frac{T}{a_{3i-1}}} \nu}{1 - e^{-j2\pi \frac{T}{a_{3i-1}}} \nu} \right| \quad a \geq 3
\]

\[
= \prod_{i=1}^{n} \frac{(1 - \cos(6\pi \frac{T}{a_{3i-1}} \nu))^2 + \sin^2(6\pi \frac{T}{a_{3i-1}} \nu)}{(1 - \cos(2\pi \frac{T}{a_{3i-1}} \nu))^2 + \sin^2(2\pi \frac{T}{a_{3i-1}} \nu)} \quad a \geq 3
\]

\[
= \prod_{i=1}^{n} \frac{2 - 2 \cos(6\pi \frac{T}{a_{3i-1}} \nu)}{2 - 2 \cos(2\pi \frac{T}{a_{3i-1}} \nu)} \quad a \geq 3
\]

\[
= \prod_{i=1}^{n} \left| \frac{\sin\left(\frac{1}{2} 6\pi \frac{T}{a_{3i-1}} \nu\right)}{\sin\left(\frac{1}{2} 2\pi \frac{T}{a_{3i-1}} \nu\right)} \right| \quad a \geq 3
\]

\[
= \frac{1}{3^n} \left| \frac{\sin\left(\frac{1}{2} 6\pi \frac{T}{a} \nu\right) \sin\left(\frac{1}{2} 6\pi \frac{T}{a_3} \nu\right) \ldots}{\sin\left(\frac{1}{2} 2\pi \frac{T}{a} \nu\right) \sin\left(\frac{1}{2} 2\pi \frac{T}{a_3} \nu\right) \ldots} \right| \quad a \geq 3
\]

\[
= \frac{1}{3^n} \left| \frac{\sin(3\pi \frac{T}{a} \nu)}{\sin(3\pi \frac{T}{a_3} \nu)} \right| \quad a \geq 3
\]

(2.40)

The transfer function is periodic with a free spectral range FSR equal to:
CHAPTER 2. MACH-ZEHNDER OPTICAL FILTERS

\[ FSR = \frac{a^{3n-1}}{T} \quad a \geq 3 \quad (2.41) \]

If the filter is matched to the rectangular input (i.e. \( a = 3 \)), the \( FSR \) becomes:

\[ FSR = \frac{3^n}{T} \quad (2.42) \]

A comparison between equations 2.42 and 2.31 indicates that the free spectral range \((\frac{3}{2})^n\) times larger for a 3 branches matched multistage filter than a 2 branches filter with the same number of stages \( n \). Figures 2-39, 2-40, 2-41, 2-42, 2-43 and 2-44 show plots of \( \|H(\nu)\| \) versus the optical frequency \( \nu \) for different values of \( n \) and \( a \).

The main lobes, located at multiples of \( \frac{a^{3n-1}}{3T} \), are now separated by \( 3^n - 1 \) nulls evenly spaced by intervals of \( \frac{a}{3T} \). The width of one main lobe is \( 2\frac{a}{3T} \). Again, the optical frequency \( \nu_0 \) must be a multiple of \( \frac{a^{3n-1}}{3T} \) to insure that the signal energy is within the filter passband:

\[ \nu_0 = i \frac{a^{3n-1}}{T} \quad a \geq 3, \quad i = 1, 2, 3, \ldots \quad (2.43) \]

If the number of stages \( n \) is sufficiently large and equation 2.43 is satisfied, then for frequencies \( \nu \) within the passband \([\pm \nu_0 - \frac{a}{3T}, \pm \nu_0 + \frac{a}{3T}]\), the transfer function magnitude \( \|H(\nu)\| \) can be approximated by:

\[ \|H(\nu)\| \approx \left| \frac{\sin(3\pi \frac{a}{3} (\nu \pm \nu_0))}{3\pi a^{3n-1} \frac{T}{a} (\nu \pm \nu_0)} \right| \quad \text{for } |\nu \pm \nu_0| \leq \frac{a}{3T} \ll \frac{a^{3n-1}}{T} \quad (2.44) \]
Figure 2-39: Frequency response when $n = 1$, $a = 3$, and $b = 3$ ($\nu$ is expressed in units of $\frac{1}{\lambda}$)

Figure 2-40: Frequency response when $n = 2$, $a = 3$, and $b = 3$ ($\nu$ is expressed in units of $\frac{1}{\lambda}$)

Figure 2-41: Frequency response when $n = 3$, $a = 3$, and $b = 3$ ($\nu$ is expressed in units of $\frac{1}{\lambda}$)
Figure 2.42: Frequency response when \( n = 1, a = 4, \) and \( b = 3 \) (\( \nu \) is expressed in units of \( \frac{1}{P} \))

Figure 2.43: Frequency response when \( n = 2, a = 4, \) and \( b = 3 \) (\( \nu \) is expressed in units of \( \frac{1}{P} \))

Figure 2.44: Frequency response when \( n = 3, a = 4, \) and \( b = 3 \) (\( \nu \) is expressed in units of \( \frac{1}{P} \))
CHAPTER 2. MACH-ZEHNDER OPTICAL FILTERS

When cascading the 3 branches multistage Mach-Zehnder filter into a wideband optical filter of bandwidth $\frac{a^{3n-1}}{T}$ centered at $\pm \nu_0$, the transfer function of the resultant filter $\|H_r(\nu)\|$ can be approximated by:

$$\|H_r(\nu)\| \approx \left| \frac{\sin(3\pi \frac{T}{a}(\nu - \nu_0))}{3\pi \frac{T}{a}(\nu - \nu_0)} \right| + \left| \frac{\sin(3\pi \frac{T}{a}(\nu + \nu_0))}{3\pi \frac{T}{a}(\nu + \nu_0)} \right| \quad \text{for all } \nu \quad (2.45)$$

This is simply the frequency response of a filter matched to an optical rectangular pulse of frequency $\nu_0$, width $\frac{3T}{a}$ and an amplitude $2\frac{a}{3T}$. The magnitude squared $\|H_r(\nu)\|^2$ will therefore approach:

$$\|H_r(\nu)\|^2 \approx \frac{\sin^2(3\pi \frac{T}{a}(\nu - \nu_0))}{(3\pi \frac{T}{a}(\nu - \nu_0))^2} + \frac{\sin^2(3\pi \frac{T}{a}(\nu + \nu_0))}{(3\pi \frac{T}{a}(\nu + \nu_0))^2} \quad \text{for all } \nu \quad (2.46)$$

The autocorrelation function of the impulse response can be derived by rewriting the results from equation 2.46 in the time domain:

$$h_r(\tau) * h^*_r(-\tau) \approx 2\frac{a}{3T} \left( 1 - \frac{|\tau|}{\frac{3T}{a}} \right) \cos(2\pi \nu_0 \tau), \quad \tau \in \left[ -\frac{3T}{a}, \frac{3T}{a} \right] \quad \text{with } a \geq 3 \quad (2.47)$$

Equations 2.46 and 2.47 are derived under the assumptions:

1. $\frac{a}{3T} \ll \frac{a^{3n-1}}{T}$ (a large $n$ is needed).
2. $\nu_0$ is a multiple of the FSR $\frac{a^{3n-1}}{T}$.

The previous results can be generalized for the multistage Mach-Zehnder filter with $b$ branches per stage shown in Figure 2-20, where $b$ is an integer larger or equal to 2. The magnitude of the transfer function is given by:
\[
\|H(\nu)\| = \prod_{i=1}^{n} \frac{1}{b} \left| 1 + e^{-j2\pi \frac{T}{ab^{i-1}} \nu} + \cdots + e^{-j2(b-2)\pi \frac{T}{ab^{i-1}} \nu} + e^{-j2(b-1)\pi \frac{T}{ab^{i-1}} \nu} \right| \quad a \geq b
\]

\[
= \prod_{i=1}^{n} \frac{1}{b} \sqrt{1 - e^{-j2\pi \frac{T}{ab^{i-1}} \nu}} \quad a \geq b
\]

\[
= \prod_{i=1}^{n} \frac{2 - 2 \cos(2b\pi \frac{T}{ab^{i-1}} \nu)}{(1 - \cos(2\pi \frac{T}{ab^{i-1}} \nu))^2 + \sin^2(2\pi \frac{T}{ab^{i-1}} \nu)} \quad a \geq b
\]

\[
= \prod_{i=1}^{n} \frac{1}{b} \left| \frac{\sin\left(\frac{1}{2}2b\pi \frac{T}{ab^{i-1}} \nu \right)}{\sin\left(\frac{1}{2}2\pi \frac{T}{ab^{i-1}} \nu \right)} \right| \quad a \geq b
\]

\[
= \frac{1}{b^n} \left| \frac{\sin\left(\frac{1}{2}2b\pi \frac{T}{a} \nu \right)}{\sin\left(\frac{1}{2}2\pi \frac{T}{a} \nu \right)} \right| \cdots \frac{\sin\left(\frac{1}{2}2b\pi \frac{T}{ab^{n-1}} \nu \right)}{\sin\left(\frac{1}{2}2\pi \frac{T}{ab^{n-1}} \nu \right)} \sin\left(\frac{1}{2}2\pi \frac{T}{ab^{n-1}} \nu \right) \frac{\sin\left(\frac{1}{2}2b\pi \frac{T}{ab^{n-1}} \nu \right)}{\sin\left(\frac{1}{2}2\pi \frac{T}{ab^{n-1}} \nu \right)} \quad a \geq b
\]

\[
= \frac{1}{b^n} \left| \frac{\sin(b\pi \frac{T}{a} \nu)}{\sin(b\pi \frac{T}{ab^n} \nu)} \right| \quad a \geq b \quad (2.48)
\]

The free spectral range \(FSR\) of this periodic filter is given by:

\[
FSR = \frac{ab^{n-1}}{T} \quad a \geq b \quad (2.49)
\]

For a matched Mach-Zehnder chain (i.e. \(a = b\)), the \(FSR\) becomes:

\[
FSR = \frac{b^n}{T} \quad (2.50)
\]

and is \((\frac{a}{2})^n\) larger than the one for a 2 branches Mach-Zehnder chain with an equal number of stages \(n\).

Figures 2-45, 2-46, 2-47, 2-48, 2-49 and 2-50 show plots of \(\|H(\nu)\|\) versus the
optical frequency $\nu$ for different values of $n$ and $a$ when the number of branches per stage $b$ is 4.

The main lobes of $\|H(\nu)\|$, each having a width of $2\frac{a}{bT}$, are now located at multiples of $\frac{ab^{n-1}}{T}$ and alternate with sets of $b^n - 1$ nulls uniformly distributed at regular intervals of $\frac{a}{bT}$. In order for the signal energy to remain within the filter passband, the carrier frequency $\nu_0$ must be a multiple of $\frac{ab^{n-1}}{T}$:

$$
\nu_0 = i \frac{ab^{n-1}}{T} \quad a \geq b, \quad i = 1, 2, 3, \ldots \quad (2.51)
$$

This condition also guarantees that the pulses recombine in phase at the output of each of the $n$ stages. If, in addition, the number of stages $n$ is sufficiently large, then for frequencies $\nu$ within the passband $[\pm \nu_0 - \frac{a}{bT}, \pm \nu_0 + \frac{a}{bT}]$ (where more than 90% of the signal energy lies) the denominator of transfer function as expressed in equation 2.48 can be approximated as follows:

$$
\sin(b\pi \frac{T}{ab^n} \nu) = \sin(b\pi \frac{T}{ab^n} (\nu \pm \nu_0)) 
\approx b\pi \frac{T}{ab^n} (\nu \pm \nu_0) \quad \text{for} \quad |\nu \pm \nu_0| \leq \frac{a}{bT} \ll \frac{ab^{n-1}}{T} \quad (2.52)
$$

Therefore, under the following two conditions:

1. $\frac{a}{bT} \ll \frac{ab^{n-1}}{T}$.

2. $\nu_0$ is a multiple of the FSR $\frac{ab^{n-1}}{T}$.

the transfer function magnitude $\|H(\nu)\|$ as expressed in equation 2.48 approaches:

$$
\|H(\nu)\| \approx \left| \frac{\sin(b\pi \frac{T}{a} (\nu \pm \nu_0))}{b\pi \frac{T}{a} (\nu \pm \nu_0)} \right| \quad \text{for} \quad |\nu \pm \nu_0| \leq \frac{a}{bT} \ll \frac{ab^{n-1}}{T} \quad (2.53)
$$
Figure 2-45: Frequency response when $n = 1$, $a = 4$, and $b = 4$ ($\nu$ is expressed in units of $\frac{1}{T}$)

Figure 2-46: Frequency response when $n = 2$, $a = 4$, and $b = 4$ ($\nu$ is expressed in units of $\frac{1}{T}$)

Figure 2-47: Frequency response when $n = 3$, $a = 4$, and $b = 4$ ($\nu$ is expressed in units of $\frac{1}{T}$)
Figure 2-48: Frequency response when $n = 1$, $a = 5$, and $b = 4$ ($\nu$ is expressed in units of $\frac{1}{\Pi}$)

Figure 2-49: Frequency response when $n = 2$, $a = 5$, and $b = 4$ ($\nu$ is expressed in units of $\frac{1}{\Pi}$)

Figure 2-50: Frequency response when $n = 3$, $a = 5$, and $b = 4$ ($\nu$ is expressed in units of $\frac{1}{\Pi}$)
CHAPTER 2. MACH-ZEHNDER OPTICAL FILTERS

Note that in the case of the matched filter (i.e. \( a = b \)) the width of the main lobe (equal to \( 2\frac{a}{bT} \)), where most of the signal energy lies, reduces to:

\[
\frac{2}{T}
\] (2.54)

In contrast, the main lobe of a filter not perfectly matched to the input pulse (i.e. \( a > b \)) is wider by a factor of \( \frac{a}{b} \) which seems to indicate that a matched filter is more resistive to additive white optical noise. Unfortunately, a matched optical filter produces triangular outputs that introduce more ISI at the receiver than trapezoidal pulses generated by a non-matched Mach-Zehnder filter.

By cascading the Mach-Zehnder chain into a wideband optical filter of bandwidth \( \frac{a b^{n-1}}{T} \) centered at \( \pm \nu_0 \), the magnitude of the resulting transfer function \( \|H_r(\nu)\| \) can be approximated as:

\[
\|H_r(\nu)\| \approx \left| \frac{\sin(b\pi \frac{T}{a}(\nu - \nu_0))}{b\pi \frac{T}{a}(\nu - \nu_0)} \right| + \left| \frac{\sin(b\pi \frac{T}{a}(\nu + \nu_0))}{b\pi \frac{T}{a}(\nu + \nu_0)} \right| \quad \text{for all } \nu
\] (2.55)

which is the response of a filter matched to a rectangular pulse of frequency \( \nu_0 \), width \( \frac{bT}{a} \) and an amplitude \( 2\frac{a}{bT} \). The magnitude squared will therefore approach:

\[
\|H_r(\nu)\|^2 \approx \frac{\sin^2(b\pi \frac{T}{a}(\nu - \nu_0))}{(b\pi \frac{T}{a}(\nu - \nu_0))^2} + \frac{\sin^2(b\pi \frac{T}{a}(\nu + \nu_0))}{(b\pi \frac{T}{a}(\nu + \nu_0))^2} \quad \text{for all } \nu
\] (2.56)

which can be rewritten in the time domain as:

\[
h_r(\tau) \ast h^*_r(-\tau) \approx 2 \frac{a}{bT} \left( 1 - \frac{|\tau|}{bT} \right) \cos(2\pi \nu_0 \tau), \quad \tau \in \left[ -\frac{bT}{a}, \frac{bT}{a} \right] \quad \text{with } a \geq b
\] (2.57)
2.5 Concluding remarks about Mach-Zehnder chains

In summary, multistage Mach-Zehnder filters can be designed to shape incident rectangular pulses into triangular and trapezoidal signals. Under ideal conditions, the response of a \( b \) branches Mach-Zehnder chain cascaded into a wideband optical approaches that of a bandpass optical filter matched to a rectangular signal of width \( \frac{bT}{a} \) and a height \( 2 \frac{a}{b} \). When analyzing the performance of Mach-Zehnder filters in Chapter 5, we will always assume these conditions to be true (i.e. (1) a large number of stages is used to insure that \( \frac{a}{b} \ll \frac{a}{b}^{n-1} \) and (2) the signal's optical frequency \( \nu_0 \) is a multiple of the FSR \( \frac{a}{b}^{n-1} \)). Furthermore, we shall consider the input pulse \( s(t) \) to be centered at the origin: \( s(t) = A \cos(2\pi \nu_0 t), \, t\in[-\frac{T}{2}, \frac{T}{2}] \) and \( A = \sqrt{\frac{2E}{T}} \) as show in Figure 2-51.

![Figure 2-51: Incoming rectangular pulse \( s(t) \)](image)

To simplify our notations, we will define \( T' \) as the increase in the pulse duration after filtering:

\[
T' = \frac{bT}{a}
\]  \hspace{1cm} (2.58)

Note that since:

\[
a \geq b
\]  \hspace{1cm} (2.59)
CHAPTER 2. MACH-ZEHNDER OPTICAL FILTERS

then:

\[ 0 \leq T' \leq T \]

(2.60)

The width \( W_{base} \) of the filtered pulse \( m(t) \) as expressed in equation 2.18, can be rewritten in terms of \( T \) and \( T' \):

\[ W_{base} = T + T' \]

(2.61)

The width \( W_{top} \) of the flat portion of the trapezoidal output \( m(t) \) as expressed in equation 2.20 becomes:

\[ W_{base} = T - T' \]

(2.62)

Equations 2.60, 2.61 and 2.62 imply that:

\[ T \leq W_{base} \leq 2T \]

(2.63)

and

\[ 0 \leq W_{top} \leq T \]

(2.64)

In the extreme situation where the optical filter is exactly matched to the rectangular input of width \( T \) (i.e. \( a = b \)), then:

\[ T = T' \]

(2.65)

and the output signal \( m(t) \) converges into a triangular pulse of width \( 2T \) as illustrated in Figure 2-52. If the optical filter is matched to a rectangular signal of shorter width \( T' \) instead (i.e. \( a > b \)), the rectangular signal is shaped into the trapezoidal output of Figure 2-53.
CHAPTER 2. MACH-ZEHNDER OPTICAL FILTERS

Figure 2-52: Output $m(t)$ of a matched Mach-Zehnder filter (case where $T = T'$)

Figure 2-53: Output $m(t)$ of a Mach-Zehnder filter (case where $T > T'$)
CHAPTER 2. MACH-ZEHNDER OPTICAL FILTERS

The approximation to the transfer function as expressed in equation 2.55 can be rewritten as:

$$\|H_r(\nu)\| \approx \left| \frac{\sin(\pi T'(\nu - \nu_0))}{\pi T'(\nu - \nu_0)} \right| + \left| \frac{\sin(\pi T'(\nu + \nu_0))}{\pi T'(\nu + \nu_0)} \right| \text{ for all } \nu \quad (2.66)$$

where the width of the main lobe is now $\frac{1}{T'}$. The magnitude squared will therefore approach:

$$\|H_r(\nu)\|^2 \approx \frac{\sin^2(\pi T'(\nu - \nu_0))}{(\pi T'(\nu - \nu_0))^2} + \frac{\sin^2(\pi T'(\nu + \nu_0))}{(\pi T'(\nu + \nu_0))^2} \text{ for all } \nu \quad (2.67)$$

and the autocorrelation of the filter’s impulse response converges toward:

$$h_r(\tau) * h^*_r(-\tau) \approx \frac{1}{T'} \left( 1 - \frac{|\tau|}{T'} \right) \cos(2\pi \nu_0 \tau), \quad \tau \in [-T', T'] \quad (2.68)$$
Chapter 3

Fabry-Perot Interferometers

This chapter is entirely devoted to the study of a much simpler optical filter: the Fabry-Perot interferometer. We will again focus our attention on (1) the filter's response to incident rectangular signals, (2) the filter's frequency response and (3) the interference between consecutive pulses at the output.

3.1 Structure of Fabry-Perot interferometers

In some conventional passband communication systems that use wideband channels, the amount of intersymbol interference at the receiver is negligible. Such systems use a simple lowpass filter that consists mainly of a first order RC circuit instead of an optimal filter matched to the transmitted rectangular signal pulse. The effects of intersymbol interference can be reduced by decreasing the RC time constant of the receiver's filter. However, a smaller RC time constant results in a larger bandwidth that allows more noise to reach the decision slicer.

For an optical communication system where the transmitted signal consists of a rectangular pulse train of monochromatic light, a single-cavity Fabry-Perot interferometer with a carefully chosen cavity length $L$ can be used to produce the same output waveform as the one generated by a first order RC circuit fed with a sequence
of rectangular electrical pulses.

Tunable Fabry-Perot filters have been proposed for use as multiplexers in multi-access optical networks [FrL 88, KaI 88, KaI 89, MaJ 87, Mal 87, StS 87]. Figure 3-1 shows a single-cavity Fabry-Perot interferometer which consists of a resonant cavity of length \( L \) formed by two parallel mirrors with field reflectivity coefficients \( R_1 \) and \( R_2 \) and field transmittivity coefficients \( T_1 \) and \( T_2 \). Light coming from the input fiber on the left is reflected back and forth within the cavity. Upon each reflection, a fraction of the incident light is reflected back into the cavity and the remaining portion is transmitted through the mirror and into the adjacent fiber. The reflected field strength is \( R_i \), \( i \in \{1, 2\} \), times the incident field strength; likewise, the transmitted field strength is equal to the incident field multiplied by the transmittivity coefficient \( T_i \), \( i \in \{1, 2\} \). Moreover, the coefficients \( R_i \) and \( T_i \) satisfy the power conservation principle at the mirror's surface:

\[
R_i^2 + T_i^2 = 1 \quad i \in \{1, 2\}
\]  

(3.1)

![Figure 3-1: Fabry-Perot filter](image)

Let \( \beta \) be the power absorption loss suffered by light as it travels through the cavity from one mirror and reaches the opposite mirror.

Consider now an incident rectangular pulse with frequency \( \nu_0 \), width \( T \) and energy \( E \) as shown in Figure 3-2. The input pulse can be expressed as \( s(t) = A \cos(2\pi\nu_0 t) \) for \( 0 \leq t \leq T \) where \( A = \sqrt{\frac{2E}{T}} \).

Assume that the pulse width is much larger than the round trip propagation time
of light across the cavity $T_{\text{delay}} = \frac{2L}{v_p}$, where $v_p$ is the phase velocity of light traveling in the fiber:

$$T \gg T_{\text{delay}}$$  \hspace{1cm} (3.2)

with:

$$T_{\text{delay}} = \frac{2L}{v_p}$$  \hspace{1cm} (3.3)

When the incident light passes through the mirror on the left, a portion of strength $AT_1$ survives and the rest is lost as heat or is reflected to the left of the device and back into the fiber. The signal then reaches the mirror on the right with a field strength $A\sqrt{\beta}T_1$; a portion with strength $L_1 = A\sqrt{\beta}T_1T_2$ is passed to the output fiber while a portion with strength $A\sqrt{\beta}T_1R_2$ is reflected back into the cavity. When this reflected field arrives at the left mirror, a portion bounces back to the right where it emerges in the output fiber with a field strength of $L_2 = A\sqrt{\beta}T_1R_2R_1R_2$. If the round trip propagation time of light traveling across the cavity, $\frac{2L}{v_p}$, is a multiple of the optical period $\frac{\lambda_0}{v_p}$ ($\lambda_0$ being the incident light wavelength):

$$T_{\text{delay}} = i\frac{\lambda_0}{v_p} \quad i = 1, 2, 3, \cdots$$  \hspace{1cm} (3.4)

or equivalently:

![Figure 3-2: Incoming rectangular pulse $s(t)$](image-url)
then the output field with strength \( L_2 = A \sqrt{\beta T_1 T_2 R_1 R_2} \), produced by the second reflection of the signal against the left mirror, adds up in phase to the field of strength \( L_1 = A \sqrt{\beta T_1 T_2} \) produced by the first reflection. Thus, the output field strength increases by \( L_2 = A \sqrt{\beta T_1 T_2 R_1 R_2} \) following this second reflection (Figure 3-3).

![Figure 3-3: Output of a Fabry-Perot filter](image)

Using the same reasoning, we can deduce that \( L_k \), the \( k \)th increment in the output field strength, corresponding to the \( k \)th subsequent reflection of the signal by the right hand mirror at time \( t = (2k - 1) \frac{L}{v_p} \), is equal to \( A \sqrt{\beta \beta^{k-1} T_1 T_2 R_1^{k-1} R_2^{k-1}} \). Assuming \( T \geq (2k) \frac{L}{v_p} \), the output field strength becomes:

\[
\sum_{i=0}^{k-1} A T_1 T_2 \sqrt{\beta (R_1 R_2 \beta)^i} \quad \text{for} \quad t \in \left[ (2k - 1) \frac{L}{v_p}, (2k + 1) \frac{L}{v_p} \right]
\]

(3.6)

Since \( \beta \), \( |R_i| \) and \( |T_i| \), \( i \in \{1, 2\} \), are all smaller than 1, equation 3.6 reduces to:
\[
\frac{AT_1 T_2 \sqrt{\beta}}{1 - R_1 R_2 \beta} (1 - (\beta R_1 R_2)^k) \quad \text{for } t \in \left[ (2k - 1) \frac{L}{v_p}, (2k + 1) \frac{L}{v_p} \right]
\]

which converges to:

\[
\frac{AT_1 T_2 \sqrt{\beta}}{1 - R_1 R_2 \beta} \quad (3.8)
\]
as long as \( T' \gg \frac{2L}{v_p} \) (equation 3.2).

In the special situation where no power is dissipated inside the cavity (i.e. \( \beta = 1 \)) and both mirrors are identical (i.e. \( R_1 = R_2 = R \) and \( T_1 = T_2 = T \)), the output field converges toward \( \frac{AT_1^2}{1 - R^2} \) which is equal to the input field strength \( A = \sqrt{\frac{\tilde{E}}{T}} \).

As shown in Figure 3-3, the output field continues to approach the limit \( \frac{AT_1 T_2 \sqrt{\beta}}{1 - R_1 R_2 \beta} \) exponentially and in a “staircase” fashion, until the trailing edge of the pulse reaches the right hand mirror, at which point the output waveform starts to decay exponentially at regular intervals of \( \frac{2L}{v_p} \).

As long as equations 3.2 and 3.4 are both satisfied, the filtered pulse envelope will converge toward the convolution of the rectangular incident pulse envelope with the exponential impulse response \( h_r(t) \) of a linear lowpass RC filter as shown in Figure 3-4.

\[
h_r(t) = 2Ce^{-\frac{t}{T'}} \cos(2\pi v_0 t) \quad t \geq 0 \quad (3.9)
\]

The two constants \( C \) and \( T' \) will be evaluated in the following section in terms of \( R_1, R_2, T_1, T_2 \) and \( \beta \). The output signal \( m(t) \) can be then approximated as follows:

\[
m(t) \approx \begin{cases} 
\sqrt{\frac{2E}{T}} CT'(1 - e^{-\frac{t}{T'}}) \cos(2\pi v_0 t) & \text{if } 0 \leq t \leq T \\
\sqrt{\frac{2E}{T}} CT'(e^{\frac{T}{T'}} - 1)e^{-\frac{t}{T'}} \cos(2\pi v_0 t) & \text{if } t \geq T \\
0 & \text{otherwise}
\end{cases} \quad (3.10)
\]
3.2 ISI introduced by Fabry-Perot filters

Figure 3.4 illustrates the response of a Fabry-Perot filter operating under ideal conditions (equations 3.2 and 3.4 are both satisfied) to three consecutive “ON” pulses. Since the filtered pulses decay exponentially causing two neighbors to overlap inside the interval of integration (the shaded region in Figure 3-4), intersymbol interference cannot be avoided for this type of filters. Unlike the multistage Mach-Zehnder chains discussed in Chapter 2, a Fabry-Perot filter cannot be designed to completely eliminate the ISI energy leaking into the integration region. However, one can appreciate the simplicity in the design and implementation of the Fabry-Perot filter as opposed to the complexity of the more sophisticated Mach-Zehnder chain. Moreover, the ISI energy can be significantly reduced by shortening the filter’s time constant $T'$; unfortunately, as we will see shortly in the next section, a smaller $T'$ results in a wider bandwidth that passes more noise to the photodetector.
3.3 Frequency response of Fabry-Perot interferometers

Consider the Fabry-Perot filter of Figure 3-1. Its transfer function, \( H(\nu) \), \( \nu \) being the optical frequency, is given by:

\[
H(\nu) = \sum_{i=0}^{\infty} T_1 T_2 \sqrt{\beta} e^{-j2\pi \frac{L}{v_p} \nu} (R_1 R_2 \beta)^i e^{-j2\pi \frac{2iL}{v_p} \nu}
\]

\[
= \frac{T_1 T_2 \sqrt{\beta} e^{-j2\pi \frac{L}{v_p} \nu}}{1 - R_1 R_2 \beta e^{-j2\pi \frac{2L}{v_p} \nu}}
\]  
(3.11)

The magnitude squared of the transfer function, \( ||H(\nu)||^2 \), can be then easily derived:

\[
||H(\nu)||^2 = H(\nu)H^*(\nu)
\]

\[
= \frac{(T_1 T_2)^2 \beta}{(1 - R_1 R_2 \beta)^2} \frac{1}{1 + \frac{4R_1 R_2 \beta}{(1 - R_1 R_2 \beta)^2} \sin^2(2\pi \frac{L}{v_p} \nu)}
\]  
(3.12)

The transfer function as expressed in equation 3.12, also known as the Airy function, is periodic and has a free spectral range \( FSR \) equal to:

\[
FSR = \frac{v_p}{2L}
\]  
(3.13)

which is the inverse of the round trip delay \( T_{delay} \) of light traveling across the cavity.

The full width at half maximum length \( FWHM \) can be evaluated by solving for the frequencies for which \( ||H(\nu)||^2 \) is reduced to half its maximum value.
\[ \frac{1}{2} = \frac{1}{1 + \frac{4R_1 R_2 \beta}{(1 - R_1 R_2 \beta)^2} \sin^2(2\pi \frac{L}{v_p} \nu)} \]  

(3.14)

The expression for FWHM, as derived from the solution to equation 3.14, is:

\[ FWHM = 2 \frac{v_p}{2L \pi} \sin^{-1}\left(\frac{1 - R_1 R_2 \beta}{\sqrt{4R_1 R_2 \beta}}\right) \]  

(3.15)

Assuming that the round trip power loss inside the cavity is negligible (i.e. \( \beta \approx 1 \)) and that \( R_1 \) and \( R_2 \) are close to unity then:

\[ \sin^{-1}\left(\frac{1 - R_1 R_2 \beta}{\sqrt{4R_1 R_2 \beta}}\right) \approx \frac{1 - R_1 R_2 \beta}{\sqrt{4R_1 R_2 \beta}} \]  

(3.16)

and therefore:

\[ FWHM \approx \frac{v_p}{2L \pi} \frac{1}{\sqrt{R_1 R_2 \beta}} \]  

(3.17)

The ratio of FSR to FWHM is referred to as the finesse \( F \) of the Fabry-Perot filter and expresses the sharpness of the filter relative to its period.

\[ F = \frac{FSR}{FWHM} = \frac{\pi \sqrt{R_1 R_2 \beta}}{1 - R_1 R_2 \beta} \]  

(3.18)

Equation 3.12 can be expressed in terms of the finesse as follows:

\[ \|H(\nu)\|^2 = \frac{(T_1 T_2)^2 \beta}{(1 - R_1 R_2 \beta)^2} \frac{1}{1 + \left(\frac{2F}{\pi}\right)^2 \sin^2(2\pi \frac{L}{v_p} \nu)} \]  

(3.19)

Figure 3-5 shows plots of the transfer function for different choices of \( F \). Note that, unlike Mach-Zehnder filters, the transfer function of Fabry-Perot filters contains no nulls.

The height of the peaks, located at multiple values of the free spectral range, is given by:
\[ \|H(\nu)\|^2_{\text{max}} = \frac{(T_1 T_2)^2 \beta}{(1 - R_1 R_2 \beta)^2} \]

while the minimum of \(\|H(\nu)\|^2\) is equal to:

\[ \|H(\nu)\|^2_{\text{min}} = \frac{(T_1 T_2)^2 \beta}{(1 - R_1 R_2 \beta)^2} \frac{1}{1 + (\frac{2F}{\pi})^2} \]

Figure 3-5: Frequency response of a filter with finesse \(F\) (\(\nu\) is expressed in units of \(\frac{\nu_0}{2L}\))

The ratio of maximum to minimum of \(\|H(\nu)\|^2\) is therefore:

\[ \frac{\|H(\nu)\|^2_{\text{max}}}{\|H(\nu)\|^2_{\text{min}}} = 1 + \left(\frac{2F}{\pi}\right)^2 \]

and depends solely on the finesse \(F\). As the finesse increases, the peaks of the transfer function narrow and the attenuation in the stopband regions becomes more severe as illustrated in Figure 3-5.

As stated earlier on in Chapter 2, the transform \(S(\nu)\) of the input rectangular
pulse \( s(t) = \sqrt{\frac{2E}{T}} \cos(2\pi \nu_0 t) \), \( t \in [0, T] \) is the sum of two “sinc” functions centered at \( \pm \nu_0 \):

\[
\|S(\nu)\| = \sqrt{\frac{ET}{2} \left| \frac{\sin(\pi T(\nu - \nu_0))}{\pi T(\nu - \nu_0)} \right|} + \sqrt{\frac{ET}{2} \left| \frac{\sin(\pi T(\nu + \nu_0))}{\pi T(\nu + \nu_0)} \right|}
\]  

(3.23)

The input pulse energy must fall within the filter’s passband; this is achieved by choosing the optical frequency \( \nu_0 \) as a multiple of the free spectral range:

\[
\nu_0 = \frac{v_p}{2L} \quad i = 1, 2, 3, \ldots
\]  

(3.24)

Note that because:

\[
\nu_0 \lambda_0 = v_p
\]  

(3.25)

equations 3.4 and 3.24 are identical.

As mentioned in Chapter 2, more than 90% of the total signal energy \( E \) lies within the width \( \frac{2}{T} \) of the “sinc” function main lobe. Assuming the full width at half maximum \( FWHM = \frac{FSR}{F} = \frac{v_p}{2L\overline{F}} \) is larger than \( \frac{2}{T} \):

\[
FWHM \geq \frac{2}{T}
\]  

(3.26)

then most of the signal energy will be within the 3dB passband of the Fabry-Perot filter as well. If we further assume that a highly discriminating filter is chosen (one with a high finesse \( F \)):

\[
F \gg 1
\]  

(3.27)

then for all frequencies \( \nu \) within the 3dB passband \([\pm \nu_0 - \frac{v_p}{4L\overline{F}}, \pm \nu_0 + \frac{v_p}{4L\overline{F}}]\), the term \( \sin(2\pi \frac{L}{v_p} \nu) \) appearing in the the denominator of \( \|H(\nu)\|^2 \) as expressed in equation 3.19 can be approximated as follows:
\[
\sin\left(2\pi \frac{L}{v_p} \nu \right) = \sin\left(2\pi \frac{L}{v_p} (\nu \pm \nu_0) \right) \\
\approx 2\pi \frac{L}{v_p} (\nu \pm \nu_0) \text{ for } |\nu \pm \nu_0| \leq \frac{v_p}{4LF} \ll \frac{v_p}{2L} (3.28)
\]

Under these conditions, the transfer function magnitude squared \( ||H(\nu)||^2 \) can be approximated by:

\[
||H(\nu)||^2 \approx \frac{\left(T_1 T_2\right)^2 \beta}{(1 - R_1 R_2 \beta)^2} \frac{1}{1 + \frac{4R_1 R_2 \beta}{(1 - R_1 R_2 \beta)^2} \left(2\pi \frac{L}{v_p} (\nu \pm \nu_0) \right)^2} \text{ for } |\nu \pm \nu_0| \leq \frac{v_p}{4LF} \ll \frac{v_p}{2L} (3.29)
\]

When cascading the Fabry-Perot filter into a wideband optical filter of bandwidth \( \frac{v_p}{2L} \) centered at \( \pm \nu_0 \) that essentially attenuates the neighboring periodic peaks of \( ||H(\nu)||^2 \) (thus, smoothing the output pulse and eliminating some of the noise power), the transfer function magnitude squared \( ||H_r(\nu)||^2 \) of the resulting combined filter can be approximated by:

\[
||H_r(\nu)||^2 \approx \frac{\left(T_1 T_2\right)^2 \beta}{(1 - R_1 R_2 \beta)^2} \left( \frac{1}{1 + \frac{4R_1 R_2 \beta}{(1 - R_1 R_2 \beta)^2} \left(2\pi \frac{L}{v_p} (\nu + \nu_0) \right)^2} + \frac{1}{1 + \frac{4R_1 R_2 \beta}{(1 - R_1 R_2 \beta)^2} \left(2\pi \frac{L}{v_p} (\nu - \nu_0) \right)^2} \right) \text{ for all } \nu (3.30)
\]

The autocorrelation function of the impulse response \( h_r(\tau) * h^*_r(-\tau) \) can be then easily obtained from equation 3.30 by taking the Inverse Fourier Transform on both sides.

\[
h_r(\tau) * h^*_r(-\tau) \approx 2 \frac{\left(T_1 T_2\right)^2 \beta}{(1 - R_1 R_2 \beta)^2} \frac{v_p}{2L} \frac{1 - R_1 R_2 \beta}{2\sqrt{R_1 R_2 \beta}} e^{-\frac{v_p}{2L} \frac{1 - R_1 R_2 \beta}{\sqrt{R_1 R_2 \beta} |r|}} \cos(2\pi \nu_0 \tau)
\]
\[ H(\nu)H^*(\nu) \approx \frac{T_1 T_2 \sqrt{\beta}}{(1 - R_1 R_2 \beta) + \sqrt{R_1 R_2 \beta} \frac{2L}{v_p} j2\pi(\nu \pm \nu_0) \frac{2L}{4L\beta}} \frac{T_1 T_2 \sqrt{\beta}}{(1 - R_1 R_2 \beta) - \sqrt{R_1 R_2 \beta} \frac{2L}{v_p} j2\pi(\nu \pm \nu_0)} \text{ for } |\nu \pm \nu_0| \leq \frac{v_p}{4L\beta} \ll \frac{v_p}{2L} \] (3.32)

The transfer function \( H_r(\nu) \) of the Fabry-Perot interferometer cascaded into the wideband optical filter can then be approximated by:

\[ H_r(\nu) \approx \frac{T_1 T_2 \sqrt{\beta}}{(1 - R_1 R_2 \beta) + \sqrt{R_1 R_2 \beta} \frac{2L}{v_p} j2\pi(\nu \pm \nu_0)} \text{ for all } \nu \] (3.33)

Taking the Inverse Fourier Transform on both sides of equation 3.33 gives us the following expression for the impulse response \( h_r(t) \):

\[ h_r(t) = 2 \frac{T_1 T_2}{\sqrt{R_1 R_2}} \frac{v_p}{2L} e^{-\frac{t}{t^*}} \frac{1 - R_1 R_2 \beta t}{\sqrt{R_1 R_2} \beta} \cos(2\pi \nu_0 t) \quad t \geq 0 \] (3.34)

\[ = 2 \frac{T_1 T_2}{\sqrt{R_1 R_2}} \frac{v_p}{2L} e^{-\frac{t}{t^*}} \frac{1}{\sqrt{R_1 R_2} \beta} \cos(2\pi \nu_0 t) \quad t \geq 0 \] (3.35)
Of course, these approximations are valid only if the three assumptions made in equations 3.27, 3.26 and 3.24 hold. That is:

1. The full width at half maximum length is much smaller than the free spectral range (i.e. $\mathcal{F} \gg 1$).

2. The input pulse spectrum is within the filter's 3dB bandwidth.

3. The carrier frequency $\nu_0$ is a multiple of the free spectral range $\frac{v_p}{2L}$.

Under these three conditions, the filtered output pulse $m(t)$ is obtained by convolving the input rectangular pulse $s(t)$ with the exponential impulse response $h_r(t)$ derived in equation 3.35 resulting in the expression:

$$m(t) \approx \begin{cases} 
\sqrt{\frac{2E}{T}} \frac{T_1 T_2}{\sqrt{\mathcal{R}_1 \mathcal{R}_2}} \frac{\mathcal{F}}{\pi} (1 - e^{-\frac{\pi}{2} \frac{\mathcal{F}}{v_p} t}) \cos(2\pi \nu_0 t) & \text{if } 0 \leq t \leq T \\
\sqrt{\frac{2E}{T}} \frac{T_1 T_2}{\sqrt{\mathcal{R}_1 \mathcal{R}_2}} \frac{\mathcal{F}}{\pi} (e^{\frac{\pi}{2} \frac{\mathcal{F}}{v_p} T} - 1)e^{-\frac{\pi}{2} \frac{\mathcal{F}}{v_p} t} \cos(2\pi \nu_0 t) & \text{if } t \geq T \\
0 & \text{otherwise}
\end{cases} \quad (3.36)$$

This expression can be somehow simplified by introducing two new quantities $C$ and $T'$ defined as:

$$C = \frac{T_1 T_2}{\sqrt{\mathcal{R}_1 \mathcal{R}_2}} \frac{v_p}{2L} \quad (3.37)$$

and

$$T' = \frac{\mathcal{F}}{\pi \frac{v_p}{2L}} = \frac{1}{\pi FWHM} \quad (3.39)$$

The impulse response $h_r(t)$ and the output pulse $m(t)$ can be then rewritten in terms of $C$ and $T'$ to get back equations 3.9 and 3.10. Clearly, a Fabry-Perot
interferometer cascaded into a bandpass optical filter centered at the carrier frequency \( \nu_0 \), has an impulse response that approaches that of a linear bandpass RC filter with a time constant \( T' \) inversely proportional to the FWHM. It would be quite tempting to select a Fabry-Perot filter with a high finesse (i.e. its transfer function has a wide FSR and very narrow peaks) to effectively combat the amplifier additive noise power; unfortunately, a large finesse (or equivalently a narrow FWHM) results in a larger \( T' \) as suggested by equations 3.38 and 3.39: the corresponding filter will shape the rectangular input into a slowly decaying waveform thus worsening the interference between neighboring "ON" pulses.

### 3.4 Symmetric Fabry-Perot filters

When analyzing the performance of the Fabry-Perot filter in Chapter 6, we will assume that the power dissipated on a round trip journey across the resonant cavity is negligible and that both mirrors are identical:

\[
\beta \approx 1
\]  \hspace{1cm} \text{(3.40)}

\[
\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}
\]  \hspace{1cm} \text{(3.41)}

\[
\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}
\]  \hspace{1cm} \text{(3.42)}

For this particular situation, the expressions for \( C \) and \( T' \) become:

\[
C = \frac{T^2}{\mathcal{R}} \frac{\nu_p}{2L} = \frac{1 - \mathcal{R}^2}{\mathcal{R}} \frac{\nu_p}{2L}
\]  \hspace{1cm} \text{(3.43)}
$T' = \frac{\mathcal{R}}{(1 - \mathcal{R}^2) \frac{\nu_p}{2L}}$ (3.44)

And therefore:

$CT' = 1$ (3.45)

The Fabry-Perot filter to be analyzed will be operating under the ideal conditions specified in the previous section (i.e. (1) a large finesse is assumed such that $\frac{\nu_p}{2L \mathcal{F}} \ll \frac{\nu_p}{2L}$ and (2) the signal's optical frequency $\nu_0$ is a multiple of the FSR). We will also consider the input pulse $s(t)$ to be centered at $t = 0$: $s(t) = A \cos(2\pi \nu_0 t), t \in [-\frac{T}{2}, \frac{T}{2}]$ and $A = \sqrt{\frac{2E}{T}}$ as shown in Figure 3-6.

![Figure 3-6: Incoming rectangular pulse $s(t)$](image)

The corresponding output pulse $m(t)$, shown in Figure 3-7, approaches:

$$m(t) \approx \begin{cases} 
\sqrt{\frac{2E}{T}} (1 - e^{-\frac{\pi}{2\nu_0}} e^{-\frac{\pi}{2\nu_0}}) \cos(2\pi \nu_0 t) & \text{if } -\frac{T}{2} \leq t \leq \frac{T}{2} \\
\sqrt{\frac{2E}{T}} (e^{\frac{\pi}{2\nu_0}} - 1)e^{-\frac{\pi}{2\nu_0}} e^{-\frac{\pi}{2\nu_0}} \cos(2\pi \nu_0 t) & \text{if } t \geq \frac{T}{2} \\
0 & \text{otherwise}
\end{cases}$$ (3.46)

The autocorrelation function of the impulse response $h_r(\tau) \ast h^*_r(-\tau)$ as expressed in equation 3.31 can be rewritten as:
\[ h_r(\tau) \ast h^*_r(-\tau) \approx C^2 T' e^{-\frac{|\nu|}{2T}} \cos(2\pi \nu_0 \tau) \]
\[ = \frac{1}{T'} e^{-\frac{|\nu|}{2T}} \cos(2\pi \nu_0 \tau) \]

The result from equation 3.48 will be needed later on to evaluate the autocorrelation function of the filtered amplifier noise.
Chapter 4

Photodetection of Gaussian optical fields

This chapter examines the performance of direct detection receivers in recovering optical signals mixed with additive Gaussian noise as might be generated by an optical amplifier. The theory reviewed here will be applied in the next two chapters for the particular cases where Mach-Zehnder chains and Fabry-Perot interferometers are used as predetection optical filters. The receiver's structure for ASK and FSK systems will be outlined. We will determine the statistics of the photoelectron count and approximate the bit error rate with Chernoff bounds. Three major sources of uncertainty will be considered in our analysis: (1) the shot noise generated by the photodetectors, (2) the electronic thermal noise present in the post detection circuitry and (3) the interference between overlapping neighboring pulses (ISI).

4.1 Structure of receivers for ASK and FSK systems

The receiver to be used for ASK modulation is shown in Figure 4-1. It consists of a predetection optical filter cascaded with a wideband bandpass optical filter centered at
CHAPTER 4. PHOTODETECTION OF GAUSSIAN OPTICAL FIELDS

Figure 4-1: Receiver used for ASK systems

Figure 4-2: Receiver used for FSK systems
the signal's wavelength $\lambda_0$. The current $i(t)$ generated by the photodiode is integrated by a post detection filter over some interval of time $[T_{\text{initial}}, T_{\text{final}}]$ and compared to a certain threshold to determine whether a "0" or a "1" was sent. The normalizing factor $\frac{1}{q}$, where $q$ is the charge of one electron ($q = 1.6 \times 10^{-19} \text{C}$), is included here to convert the integrated current into an electron count $C_{\text{count}}$. The current $n_{\text{th}}(t)$ shown in Figure 4-1 refers to the additive electronic thermal noise and is assumed to be a white Gaussian stochastic process.

The receiver for FSK modulation, shown in Figure 4-2, requires two predetection optical filters, one for each wavelength ($\lambda_A$ and $\lambda_B$). The difference between the currents generated by both photodetectors is then integrated. The decisions made are based solely on the sign of the integrator's output.

4.2 Sources of Noise

Three sources of noise will be considered in this analysis: the optical amplifier noise, the electronic thermal noise, and the photodetector own shot noise.

4.2.1 The optical amplifier noise

An optical amplifier of power gain $G$ can be modeled as a linear optical field amplifier together with a source of white Gaussian noise over the bandwidth of interest. The noise has a two sided spectral density of $\frac{N_0}{2}$, with $N_0 = N_{sp} h \nu_0 (G - 1)$, where $h$ is Planck's constant ($h = 6.63 \times 10^{-34} \text{J}.\text{sec}$), $\nu_0$ is the optical frequency and $N_{sp}$ is a factor greater than 1 that takes into account imperfections in the amplifier [Yar 85].

It is convenient to express the optical noise spectral density $\frac{N_0}{2}$ in terms of the energy $h \nu_0$ of a single photon:

$$N_0 = N_{sp} (G - 1)$$  \hspace{1cm} (4.1)

The autocorrelation function $R(\tau)$ of the optical noise is then:
\[ R(\tau) = \frac{N_0}{2} \delta(\tau) \] (4.2)

where \( \delta(\tau) \) is the unit impulse function. The autocorrelation function \( R'(\tau) \) of the filtered optical noise \( n(t) \) is given by:

\[
R'(\tau) = h_r(\tau) \ast h^\ast_r(-\tau) \ast R(\tau)
\]

\[
= \frac{N_0}{2} h_r(\tau) \ast h^\ast_r(-\tau)
\] (4.4)

where \( h_r(t) \) is the impulse response of the predetection optical filter and the smoothing wideband filter cascaded together.

### 4.2.2 The electronic thermal noise

The additive thermal noise \( n_{th}(t) \) generated by the electronic circuitry during post detection filtering of the signal is assumed to be a 0 mean white Gaussian process with a two sided spectral density \( \frac{N_{th}}{2} \).

\[
E[n_{th}(t)] = 0 \quad \text{for all } t
\] (4.5)

\[
E[n_{th}(t + \tau)n_{th}^\ast(t)] = \frac{N_{th}}{2} \delta(\tau) \quad \text{for all } t, \tau
\] (4.6)

It is convenient to express the spectral density of thermal noise in terms of an equivalent current \( i_{th} \):

\[
\frac{N_{th}}{2} = i_{th}^2
\] (4.7)

The variable
\[ N = \frac{1}{q} \int_{T_{\text{initial}}}^{T_{\text{final}}} n_{th}(t)dt \]  

obtained by integrating the thermal noise \( n_{th}(t) \) over the range \([T_{\text{initial}}, T_{\text{final}}]\) is a 0 mean Gaussian random variable with a variance \( \sigma^2 = E[N^2] \) given by:

\[
\begin{align*}
\sigma^2 &= E[N^2] \\
&= \frac{1}{q^2} E[\int_{T_{\text{initial}}}^{T_{\text{final}}} \int_{T_{\text{initial}}}^{T_{\text{final}}} n_{th}(t)n_{th}(u)dtdu] \\
&= \frac{N_{th}}{2q^2} (T_{\text{final}} - T_{\text{initial}}) \\
&= \frac{i_{th}^2}{q^2} (T_{\text{final}} - T_{\text{initial}})
\end{align*}
\]  

\[ (4.9) \]

### 4.2.3 The shot noise

\[
\begin{align*}
\text{Incident Field: } m(t) + n(t) & \rightarrow i_1(t) \\
i_1(t) & \rightarrow i(t) \\
i(t) & \rightarrow n_{th}(t)
\end{align*}
\]

\[
\begin{align*}
i_d(t) & \rightarrow i(t) \\
i(t) & \rightarrow C_{\text{read}}
\end{align*}
\]

**Photodiode**

Figure 4-3: Schematic of the photodiode model
CHAPTER 4. PHOTODETECTION OF GAUSSIAN OPTICAL FIELDS

Figure 4.3 depicts a block diagram of a photodiode. The current \( i_l(t) \), often referred to as the light current, is generated only when incident light is present. Throughout this thesis, we will assume the source of light to be coherent and centered at a stable nominal frequency \( \nu_0 \) (or at any of two frequencies \( \nu_A \) and \( \nu_B \) in the case of FSK modulation). The field incident upon the diode's surface, assumed in this thesis to have a single spatial mode, will be modeled as follows:

\[
E_r(t) = m(t) + n(t)
\]

\[
= \sqrt{2} Re[m'(t)e^{j2\nu_0 t} + n'(t)e^{j2\nu_0 t}]
\]

(4.10)

Where \( m'(t) \) and \( n'(t) \) are the complex envelopes of the filtered amplified signal \( m(t) \) and the filtered optical noise \( n(t) \) respectively.

Conditioned on the incident light field \( E_r(t) \), the light current \( i_l(t) \) consists of electrons emitted by the diode according to a Poisson arrival process with rate \( \lambda_l(t) \) given by:

\[
\lambda_l(t) = \frac{\eta}{h\nu_0} \| m'(t) + n'(t) \|^2
\]

(4.12)

where \( \eta \), the quantum efficiency of the light sensitive surface, is a number between 0 and 1 that refers to the fraction of incident photons that are successful in creating hole-electron pairs.

The current \( i_l(t) \) is represented as a superposition of the effects from all released electrons up to time \( t \):

\[
i_l(t) = q \sum_{j=1}^{K(t)} \delta(t - \tau_j)
\]

(4.13)

where \( q\delta(t) \) is the current response due to a single electron, \( \tau_j \) is the time release of the \( j \)th electron, and \( K(t) \) is the total number of electrons released up to time \( t \).
CHAPTER 4. PHOTODETECTION OF GAUSSIAN OPTICAL FIELDS

The occurrence times \( \tau_j \) represent the arrival times of a conditional arrival process with rate \( \lambda_l(t) \).

Conditioned on the incident field \( E_r(t) \), the number of light current electrons generated in the interval \([T_{\text{initial}}, T_{\text{final}}]\) is a Poisson random variable with a mean \( \Lambda_l \) given by:

\[
\Lambda_l = \int_{T_{\text{initial}}}^{T_{\text{final}}} \lambda_l(t) \, dt \tag{4.14}
\]

\[
= \frac{\eta}{\hbar v_0} \int_{T_{\text{initial}}}^{T_{\text{final}}} ||m'(t) + n'(t)||^2 \, dt \tag{4.15}
\]

In effect, this means the density of the light current electrons is only conditionally Poisson, conditioned on a known value for \( \Lambda_l \) which is often referred to as the level of the Poisson probability. Therefore, the density function of the light emissions requires the averaging of a conditional Poisson count over the probability density of \( \Lambda_l \):

\[
P_k(k) = \int_0^\infty p_{\Lambda_l}(\Lambda_l) \left\{ \frac{\Lambda_l^k}{k!} e^{-\Lambda_l} \right\} d\Lambda_l \quad k = 0, 1, 2, \ldots \tag{4.16}
\]

Conditioned on the level \( \Lambda_l \), the mean and variance of the light electron count are both equal to \( \Lambda_l \).

\[
E[\text{light e- count}/\Lambda_l] = \Lambda_l \tag{4.17}
\]

\[
Var[\text{light e- count}/\Lambda_l] = \Lambda_l \tag{4.18}
\]

The current \( i_d(t) \) in Figure 4-3 arises when electrons are spontaneously emitted by the diode due to thermal effects. It is usually called dark current and consists of electrons released according to a Poisson arrival process with a rate \( \lambda_d \) that is constant with time. The current \( i_d(t) \) is modeled as:
where the occurrence times \( \{\tau'_{j}\} \) of the dark electrons represent the arrival times of a Poisson arrival process with rate \( \lambda_d \) and are independent of the family \( \{\tau_j\} \).

The number of dark current electrons generated in the interval \( [T_{\text{initial}}, T_{\text{final}}] \) is a Poisson random variable with mean \( \Lambda_d \) given by:

\[
\Lambda_d = \int_{T_{\text{initial}}}^{T_{\text{final}}} \lambda_d dt = \lambda_d (T_{\text{final}} - T_{\text{initial}})
\] (4.20)

and its density function is:

\[
P_k(k) = \frac{\Lambda_d^k e^{-\Lambda_d}}{k!} \quad k = 0, 1, 2, \ldots
\] (4.21)

The mean and variance of the dark electrons count are:

\[
E[\text{dark e- count}] = \Lambda_d
\] (4.22)

\[
\text{Var}[\text{dark e- count}] = \Lambda_d
\] (4.23)

Because of the “spiky” nature of both \( i_l(t) \) and \( i_d(t) \), the sum \( i(t) = i_l(t) + i_d(t) \) is commonly referred to in the literature as shot noise. To \( i(t) \), we then add thermal noise \( n_{th}(t) \). The sum is integrated over the range \( [T_{\text{initial}}, T_{\text{final}}] \) and the resulting random variable is normalized by the electron charge \( q \). The electron count \( C_{\text{count}} \) consists of a zero mean Gaussian random variable \( N \) with variance \( \sigma^2 \) and an added random variable equal to the number of light and dark electrons emitted by the diode in the interval \( [T_{\text{initial}}, T_{\text{final}}] \).

In this analysis, we will ignore dark emissions:

\[
i_d(t) = 0 \quad \text{for all } t
\] (4.24)
CHAPTER 4. PHOTODETECTION OF GAUSSIAN OPTICAL FIELDS

We will further assume that the quantum efficiency $\eta$ is equal to unity:

$$\eta = 1 \quad (4.25)$$

Therefore, conditioned on the incident field $E_\text{r}(t)$, the number of electrons generated by the diode in the interval $[T_{\text{initial}}, T_{\text{final}}]$ is a Poisson random variable with mean $\Lambda = \int_{T_{\text{initial}}}^{T_{\text{final}}} \lambda(t)\, dt$, the total optical energy received in the interval $[T_{\text{initial}}, T_{\text{final}}]$.

$$\Lambda = \int_{T_{\text{initial}}}^{T_{\text{final}}} \|m'(t) + n'(t)\|^2 \, dt \quad (4.26)$$

The density of the photoelectrons count is:

$$P_k(k) = \int_0^\infty p_\Lambda(\Lambda) \left\{ \frac{\Lambda^k}{k!} e^{-\Lambda} d\Lambda \right\} \quad k = 0, 1, 2, \ldots \quad (4.27)$$

The mean and variance of the count, conditioned on $\Lambda$, are:

$$E[\text{e- count}/\Lambda] = \Lambda \quad (4.28)$$

$$\text{Var}[\text{e- count}/\Lambda] = \Lambda \quad (4.29)$$

Note that in equation 4.26 the photon energy $h\nu_0$ has been absorbed through normalization into the quantum noise level $N_0$ and the amplified rectangular pulse energy $E$. Both $N_0$ and $E$ will be expressed in units of photon counts (i.e. units of $h\nu_0$) throughout this thesis.

4.3 Derivation of count statistics

In this section we derive the conditional Poisson count probabilities resulting from the photodetection of stochastic Gaussian fields ([GaK 76] is one good reference). We will determine the probability density of the electron count level $C_{\text{count}}$ and its transform
CHAPTER 4. PHOTODETECTION OF GAUSSIAN OPTICAL_FIELDS

by expanding the random incident field $E_r(t)$ into an orthonormal Karhunen-Loeve expansion \cite{Van 68} over the range of integration. The analysis will take into account the presence of thermal noise at the receiver as well as the effects of ISI.

4.3.1 Field mode decomposition

Consider the random field incident upon the photodiode to be composed of the sum of a deterministic signal

$$m(t) = \sqrt{2} \text{Re}[m'(t)e^{j2\pi \nu_0 t}]$$ (4.30)

and an additive zero mean Gaussian stochastic field

$$n(t) = \sqrt{2} \text{Re}[n'(t)e^{j2\pi \nu_0 t}]$$ (4.31)

where $m'(t)$ and $n'(t)$ are complex, narrow band envelopes about $\nu_0$. We assume that the combined field $E_r(t) = m(t) + n(t)$ has only one spatial mode incident upon the detector surface and that the noise complex envelope $n(t)$ has a known autocorrelation function $R'(\tau)$.

$$E_r(t) = \sqrt{2} \text{Re}[(m'(t) + n'(t))e^{j2\pi \nu_0 t}]$$ (4.32)

We will decompose the complex envelope of both signal and noise into a Karhunen-Loeve expansion over the interval $[T_{\text{initial}}, T_{\text{final}}]$, writing:

$$m(t) = \sqrt{2} \text{Re}\left[\sum_{i=0}^{\infty} m_i \phi_i(t)e^{j2\pi \nu_0 t}\right] \quad t\in[T_{\text{initial}}, T_{\text{final}}]$$ (4.33)

$$n(t) = \sqrt{2} \text{Re}\left[\sum_{i=0}^{\infty} n_i \phi_i(t)e^{j2\pi \nu_0 t}\right] \quad t\in[T_{\text{initial}}, T_{\text{final}}]$$ (4.34)

where the modes $m_i$ and $n_i$ are the projections of the envelopes $m'(t)$ and $n'(t)$ on the eigenfunction $\phi_i(t)$. 
CHAPTER 4. PHOTODETECTION OF GAUSSIAN OPTICAL FIELDS

\[ m_i = \int_{T_{initial}}^{T_{final}} m'(t) \phi_i^*(t) dt \quad i = 0, 1, 2, \cdots \]  
(4.35)

\[ n_i = \int_{T_{initial}}^{T_{final}} n'(t) \phi_i^*(t) dt \quad i = 0, 1, 2, \cdots \]  
(4.36)

The set of eigenfunctions \( \{\phi_i(t)\} \) forms an orthonormal basis over the interval \([T_{initial}, T_{final}]\).

\[ \int_{T_{initial}}^{T_{final}} \phi_i(t) \phi_j^*(t) dt = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \]  
(4.37)

Moreover each eigenfunction \( \phi_i(t) \) satisfies the integral equation:

\[ \int_{T_{initial}}^{T_{final}} R''(t - \tau) \phi_i(\tau) d\tau = \lambda_i \phi_i(t) \quad t \in [T_{initial}, T_{final}] \quad i = 0, 1, 2, \cdots \]  
(4.38)

for some positive constants \( \{\lambda_i\} \), where \( R''(\tau) \) is the envelope of the noise autocorrelation function \( R'(\tau) \). The constant \( \lambda_i \) is the eigenvalue associated with \( \phi_i(t) \).

This choice of orthonormal basis functions means that the complex random coefficients \( \{n_i\} \) in the series expansion of the noise field are independent Gaussian random variables with zero mean and independent real and imaginary parts each with variance \( \frac{\lambda_i}{2} \) [GaK 76].

\[ E[n_i] = 0 \quad i = 0, 1, 2, \cdots \]  
(4.39)

\[ E[Re\{n_i\}^2] = E[Im\{n_i\}^2] = \frac{\lambda_i}{2} \quad i = 0, 1, 2, \cdots \]  
(4.40)

Therefore the magnitude of the complex random variable \( n_i \) has a mean square value of \( \lambda_i \).
\[ E[||n_i||^2] = E[Re\{n_i\}^2] + E[Im\{n_i\}^2] = \lambda_i \quad i = 0, 1, 2, \ldots \quad (4.41) \]

### 4.3.2 Derivation of the received optical energy statistics

Let us ignore for the moment the electronic thermal noise \( n_{th}(t) \) corrupting the detected signal. The number of electron \( C_{\text{count}} \) generated in the interval \([T_{\text{initial}}, T_{\text{final}}]\) is then a conditional Poisson random variable, dependent on a known value of the incident stochastic field \( E_r(t) \). Its density is given by:

\[ P_{C_{\text{count}}}(C_{\text{count}}) = \int_0^\infty p_\Lambda(\Lambda) \frac{\Lambda^{C_{\text{count}}}}{C_{\text{count}}} e^{-\Lambda} d\Lambda \quad C_{\text{count}} = 0, 1, 2, \ldots \quad (4.42) \]

where \( p_\Lambda(\Lambda) \) is the probability density of \( \Lambda \), the total optical energy collected in the interval \([T_{\text{initial}}, T_{\text{final}}]\). The Poisson level \( \Lambda \) as expressed in equation 4.26 can be rewritten in terms of the modes \( \{m_i\} \) and \( \{n_i\} \).

\[
\Lambda = \int_{T_{\text{initial}}}^{T_{\text{final}}} ||m'(t) + n'(t)||^2 dt \\
= \int_{T_{\text{initial}}}^{T_{\text{final}}} \sum_{i=0}^{\infty} (m_i + n_i) \phi_i(t) \sum_{j=0}^{\infty} (m_j + n_j)^* \phi_j^*(t) dt \\
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (m_i + n_i)(m_j + n_j)^* \int_{T_{\text{initial}}}^{T_{\text{final}}} \phi_i(t) \phi_j^*(t) dt \quad (4.43)
\]

The orthonormality of the basis functions then yields:

\[
\Lambda = \sum_{i=0}^{\infty} ||m_i + n_i||^2 \quad (4.44)
\]

The mean of the energy count \( \Lambda \) is therefore given by:
\[ E[\Lambda] = \sum_{i=0}^{\infty} E[\|m_i + n_i\|^2] \] (4.45)

Since the modes \( \{n_i\} \) are all independent from each other, the variance of \( \Lambda \) is then equal to the sum of the variances of the individual random variables \( \{\|m_i + n_i\|^2\} \).

\[ Var[\Lambda] = \sum_{i=0}^{\infty} Var[\|m_i + n_i\|^2] \] (4.46)

For any nonnegative integer \( i \), we have:

\[ E[\|m_i + n_i\|^2] = E[(m_i + n_i)(m_i^* + n_i^*)] = \|m_i\|^2 + E[\|n_i\|^2] + m_i E[n_i^*] + m_i^* E[n_i] \] (4.47)

Using the results from equations 4.39 and 4.41, the mean of the random variable \( \|m_i + n_i\|^2 \) can be expressed as follows:

\[ E[\|m_i + n_i\|^2] = \lambda_i + \|m_i\|^2 \quad i = 0, 1, 2, \ldots \] (4.48)

which yields the following expression for the average total energy received:

\[ E[\Lambda] = \sum_{i=0}^{\infty} (\lambda_i + \|m_i\|^2) \] (4.49)

The mean of the total optical energy received in the interval \([T_{\text{initial}}, T_{\text{final}}]\) is therefore equal to the sum of the average total noise energy:

Average noise energy received in \([T_{\text{initial}}, T_{\text{final}}]\) = \[ E[\int_{T_{\text{initial}}}^{T_{\text{final}}} \|n'(t)\|^2 dt] \] (4.50)

= \[ \sum_{i=0}^{\infty} \lambda_i \] (4.51)
and the signal energy collected by the diode:

\[
\text{Total signal energy received in } [T_{\text{initial}}, T_{\text{final}}] = \int_{T_{\text{initial}}}^{T_{\text{final}}} \| m'(t) \|^2 \, dt \quad (4.52)
\]

\[
= \sum_{i=0}^{\infty} \| m_i \|^2 \quad (4.53)
\]

Using the results from equations 4.40 and 4.41 and the fact that the real and imaginary parts of the coefficient \( n_i \) are independent and identically distributed random variables, then for any nonnegative integer \( i \), the variance of the random number \( \| m_i + n_i \|^2 \) becomes:

\[
\text{Var}[\| m_i + n_i \|^2] = \text{Var}[(\text{Re}(m_i) + \text{Re}(n_i))^2] + \text{Var}[(\text{Im}(m_i) + \text{Im}(n_i))^2]
\]

\[
= 2\text{Var}[(\text{Re}(m_i) + \text{Re}(n_i))^2]
\]

\[
= 2E[((\text{Re}(m_i) + \text{Re}(n_i))^2 - (\text{Re}(m_i))^2 + \frac{\lambda_i}{2})^2]
\]

\[
= 2E[(\text{Re}(n_i)^2 + 2\text{Re}(m_i)\text{Re}(n_i) - \frac{\lambda_i}{2})^2]
\]

\( \text{Re}(n_i) \) is a zero mean Gaussian random variable with a variance of \( \frac{\lambda_i}{2} \). Thus the third and fourth moments of \( \text{Re}(n_i) \) are given by:

\[
E[\text{Re}(n_i)^3] = 0 \quad (4.54)
\]

\[
E[\text{Re}(n_i)^4] = \frac{3\lambda_i^2}{4} \quad (4.55)
\]

The variance of \( \| m_i + n_i \|^2 \) is therefore:

\[
\text{Var}[\| m_i + n_i \|^2] = \lambda_i^2 + 2\lambda_i \| m_i \|^2 \quad i = 0, 1, 2, \ldots \quad (4.56)
\]
CHAPTER 4. PHOTODETECTION OF GAUSSIAN OPTICAL FIELDS

The first term in the expression of the variance is referred to in the literature as the noise-noise beat term, and the second term is called the signal-noise beat. The variance of the energy count \(\Lambda\) given by:

\[
Var[\Lambda] = \sum_{i=0}^{\infty} \left( \lambda_i^2 + 2\lambda_i \|m_i\|^2 \right)
\]

is equal to sum of the total noise-noise beat variance:

\[
\text{Total noise-noise beat variance} = \sum_{i=0}^{\infty} \lambda_i^2
\]  \hspace{1cm} (4.58)

and the total signal-noise beat variance:

\[
\text{Total signal-noise beat variance} = \sum_{i=0}^{\infty} 2\lambda_i \|m_i\|^2
\]  \hspace{1cm} (4.59)

4.3.3 The electron count density function and its S-transform

We shall define the moment generating function \(\psi_x(s)\) of a random variable \(x\) as the S-transform of its density function \(p_x(x)\).

\[
\psi_x(s) = E[e^{sx}]
\]

\[
= \int_{-\infty}^{\infty} p_x(x)e^{sx}dx
\]  \hspace{1cm} (4.61)

For a discrete random variable \(k\), it is sometimes convenient to derive the Z-transform of its density which is defined as:

\[
G_k(z) = E[z^k]
\]

\[
= \sum_{i=-\infty}^{\infty} z^i Pr(k = i)
\]  \hspace{1cm} (4.63)
CHAPTER 4. PHOTODETECTION OF GAUSSIAN OPTICAL FIELDS

For any nonnegative integer \( i \), the two variables \( Re(m_i) + Re(n_i) \) and \( Im(m_i) + Im(n_i) \) are independent Gaussian random variables with means \( Re(m_i) \) and \( Im(m_i) \) and identical variances of \( \frac{1}{\lambda_i} \). The moment generating functions of \( Re(m_i) + Im(m_i) \) and \( Im(m_i) + Im(n_i) \) are:

\[
\psi_{Re(m_i)+Re(n_i)}(s) = E[e^{s(Re(m_i)+Re(n_i))}] \\
= e^{sRe(m_i) + \frac{s^2}{2}} \\
i = 0, 1, 2, \ldots \tag{4.64}
\]

and

\[
\psi_{Im(m_i)+Im(n_i)}(s) = E[e^{s(Im(m_i)+Im(n_i))}] \\
= e^{sIm(m_i) + \frac{s^2}{2}} \\
i = 0, 1, 2, \ldots \tag{4.65}
\]

The moment functions \( \psi_{(Re(m_i)+Re(n_i))^2}(s) \) and \( \psi_{(Im(m_i)+Im(n_i))^2}(s) \) of the squared Gaussian random variables can be obtained from equations 4.64 and 4.65 by completing the square in the exponent [GaK 76].

\[
\psi_{(Re(m_i)+Re(n_i))^2}(s) = E[e^{s(Re(m_i)+Re(n_i))^2}] \\
= \frac{1}{\sqrt{1 - s\lambda_i}} e^{-\frac{Re(m_i)^2}{1-2s\lambda_i}} \\
Re(s) < \frac{1}{\lambda_i}, \quad i = 0, 1, 2, \ldots \tag{4.66}
\]

\[
\psi_{(Im(m_i)+Im(n_i))^2}(s) = E[e^{s(Im(m_i)+Im(n_i))^2}] \\
= \frac{1}{\sqrt{1 - s\lambda_i}} e^{-\frac{Im(m_i)^2}{1-2s\lambda_i}} \\
Re(s) < \frac{1}{\lambda_i}, \quad i = 0, 1, 2, \ldots \tag{4.67}
\]

Since the variables \((Re(m_i) + Re(n_i))^2\) and \((Im(m_i) + Im(n_i))^2\) are independent,
the moment function for the variable

\[ ||m_i + n_i||^2 = (Re(m_i) + Re(n_i))^2 + (Im(m_i) + Im(n_i))^2 \]  \hspace{1cm} (4.68)

is given by:

\[
\psi_{||m_i+n_i||^2}(s) = E[e^{s||m_i+n_i||^2}] \\
= \psi_{(Re(m_i)+Re(n_i))^2}(s)\psi_{(Im(m_i)+Im(n_i))^2}(s) \\
= \frac{1}{1 - s\lambda_i} e^{\frac{||m_i||^2}{1 - s\lambda_i}} \quad Re(s) < \frac{1}{\lambda_i}, i = 0, 1, 2, \cdots \]  \hspace{1cm} (4.69)

The corresponding distribution function is a non-central Chi-square with 2 degrees of freedom (see [Pap 84]):

\[
p_{||m_i+n_i||^2}(y) = \begin{cases} 
\frac{1}{\lambda_i} e^{-\frac{||m_i||^2}{\lambda_i}} I_0\left(2\frac{||m_i||\sqrt{y}}{\lambda_i}\right) & \text{if } y \geq 0 \\
0 & \text{otherwise}
\end{cases} \]  \hspace{1cm} (4.70)

where \( I_0() \) is the 0-th order modified Bessel function of the first kind.

\[
I_0(x) = \int_0^{2\pi} e^{x\cos(\theta)} d\theta \\
= \sum_{i=0}^{\infty} \frac{1}{(i!)^2} \frac{x^{2i}}{2} \]  \hspace{1cm} (4.71)

All the moments of a random variable \( x \) can be obtained by differentiating its moment generating function \( \psi_x(s) \) [GaK 76, Pap 84]. In particular, the \( n \)th moment of variable \( x \) is given by:

\[
E[x^n] = \frac{\partial^n}{\partial s^n} \psi_x(s)_{s=0} \quad n = 1, 2, 3, \cdots \]  \hspace{1cm} (4.73)

Therefore, the mean and variance of \( ||m_i + n_i||^2 \) as expressed in equations 4.48 and
4.56 can be easily rederived by differentiating the characteristic function $\psi_{||m_i + n_i||^2}(s)$.

$$E[||m_i + n_i||^2] = \frac{\partial}{\partial s} \psi_{||m_i + n_i||^2}(s)|_{s=0}$$

$$= \lambda_i + ||m_i||^2 \quad i = 0, 1, 2, \ldots$$

(4.74)

$$Var[||m_i + n_i||^2] = E[||m_i + n_i||^4] - (E[||m_i + n_i||^2])^2$$

$$= \frac{\partial^2}{\partial s^2} \psi_{||m_i + n_i||^2}(s)|_{s=0} - (\lambda_i + ||m_i||^2)^2$$

$$= \lambda_i^2 + 2\lambda_i ||m_i||^2 \quad i = 0, 1, 2, \ldots$$

(4.75)

(4.76)

Since the modes $\{n_i\}$ are independent, then the random variables $\{||m_i + n_i||^2\}$ are independent, and the optical energy $\Lambda$ in equation 4.26 is the sum of independent random variables. Its moment generating function is therefore:

$$\psi_{\Lambda}(s) = \prod_{i=0}^{\infty} \psi_{||m_i + n_i||^2}(s)$$

$$= \prod_{i=0}^{\infty} \frac{1}{1 - s\lambda_i} e^{||m_i||^2 / \lambda_i} \quad Re(s) < \frac{1}{\lambda_{\max}}$$

(4.77)

where $\lambda_{\max}$ is the largest eigenvalue among the set $\{\lambda_i\}$.

Ignoring the thermal noise $n_{th}(t)$ for the moment, the probability density function of the number of electrons $C_{\text{count}}$ generated in the interval $[T_{\text{initial}}, T_{\text{final}}]$ can be obtained by inverse transforming its moment function $\psi_{C_{\text{count}}}(s)$ which is given by:

$$\psi_{C_{\text{count}}}(s) = E[e^{sC_{\text{count}}}]$$

$$= E_{\Lambda}[E[e^{sC_{\text{count}}}/\Lambda]]$$

(4.78)
where $E[e^{s C_{\text{count}}}/\Lambda]$ is the moment function of the electron count $C_{\text{count}}$ conditioned on a known value of the incident optical energy $\Lambda$. However, when conditioned on the optical energy $\Lambda$, $C_{\text{count}}$ is a Poisson random variable with mean $\Lambda$ and has a conditional distribution function $p_{C_{\text{count}}/\Lambda}(C_{\text{count}}/\Lambda)$ given by:

$$p_{C_{\text{count}}/\Lambda}(C_{\text{count}}/\Lambda) = \frac{\Lambda^{C_{\text{count}}}}{C_{\text{count}}!}e^{-\Lambda} \quad C_{\text{count}} = 0, 1, 2, \ldots \quad (4.79)$$

The corresponding moment function $E[e^{s C_{\text{count}}}/\Lambda]$ is the S-transform of a Poisson distribution with level $\Lambda$ and is given by:

$$E[e^{s C_{\text{count}}}/\Lambda] = e^{\Lambda(e^s - 1)} \quad (4.80)$$

The unconditional moment generating function $\psi_{C_{\text{count}}}(s)$ can then be easily computed using the results from equations 4.77 and 4.80.

$$\psi_{C_{\text{count}}}(s) = E_{\Lambda}[e^{\Lambda(e^s - 1)}]$$
$$= \psi_{\Lambda}(e^s - 1)$$
$$= \prod_{i=0}^{\infty} \frac{1}{1 + \lambda_i(1 - e^s)}e^{-\frac{\|m_i\|^2(1-e^s)}{1+\lambda_i(1-e^s)}} \quad \|e^s\| < \frac{1 + \lambda_{\text{max}}}{\lambda_{\text{max}}} \quad (4.81)$$

This moment generating function of $C_{\text{count}}$ appears as a product of individual moment functions. This means $\psi_{C_{\text{count}}}(s)$ can be interpreted as the moment function of an infinite sum of discrete random variables $\{C_i\}$, each with a moment function $\psi_{C_i}(s)$:

$$\psi_{C_i}(s) = \frac{1}{1 + \lambda_i(1 - e^s)}e^{-\frac{\|m_i\|^2(1-e^s)}{1+\lambda_i(1-e^s)}} \quad \|e^s\| < \frac{1 + \lambda_i}{\lambda_i}, \quad i = 0, 1, 2, \ldots \quad (4.82)$$

Inverting this transform yields the following Laguerre distribution (see [GaK 76, Hum 91]):
\[ p_{C_i}(C_i) = \frac{1}{1 + \lambda_i} \left( \frac{\lambda_i}{1 + \lambda_i} \right)^{C_i} e^{-\frac{\|m_i\|^2}{\lambda_i(1 + \lambda_i)}} L_{C_i}(-\frac{\|m_i\|^2}{\lambda_i(1 + \lambda_i)}) \quad C_i = 0, 1, 2, \ldots \] (4.83)

where \( L_k(x) \) is the Laguerre polynomial and is given by:

\[ L_k(x) = \sum_{i=0}^{k} \binom{k}{i} \frac{(-x)^i}{i!} \quad k = 0, 1, 2, \ldots \] (4.84)

In case \( m_i = 0 \), for some \( i \), implying no deterministic component exists in that particular mode, the moment generating function expressed in equation 4.82 reduces to:

\[ \psi_{C_i}(s) = \frac{1}{1 + \lambda_i(1 - e^s)} \quad \|e^s\| < \frac{1 + \lambda_i}{\lambda_i}, \quad i = 0, 1, 2, \ldots \] (4.85)

and the corresponding distribution is geometric with rate \( \frac{\lambda_i}{1 + \lambda_i} \).

\[ p_{C_i}(C_i) = \frac{(\lambda_i)^{C_i}}{(1 + \lambda_i)^{C_i+1}} \quad C_i = 0, 1, 2, \ldots \] (4.86)

This geometric distribution is also known as the Bosé-Einstein probability [GaK 76]. If all signal modes \( \{m_i\} \) are 0, the moment generating function for \( C_{\text{count}} \) becomes:

\[ \psi_{C_{\text{count}}}(s) = \prod_{i=0}^{\infty} \frac{1}{1 + \lambda_i(1 - e^s)} \quad \|e^s\| < \frac{1 + \lambda_{\text{max}}}{\lambda_{\text{max}}} \] (4.87)

The Z-transform of the distribution of \( C_{\text{count}} \) is found by replacing \( e^s \) by \( z \) in equation 4.87.

\[ G_{C_{\text{count}}}(z) = \psi_{C_{\text{count}}}(s)|_{e^s = z} = \prod_{i=0}^{\infty} \frac{1}{1 + \lambda_i(1 - z)} \quad \|z\| < \frac{1 + \lambda_{\text{max}}}{\lambda_{\text{max}}} \] (4.88)

The density of \( C_{\text{count}} \) is found by inverting the Z-transform in 4.88 and evaluating
all the residues, yielding the following distribution [Hum 91]:

\[
p(C_{count}) = \sum_{i=0}^{\infty} \frac{1}{1 + \lambda_i} \left( \frac{\lambda_i}{1 + \lambda_i} \right)^{C_{count}} \prod_{j=0, j \neq i}^{\infty} \frac{\lambda_i}{\lambda_i - \lambda_j} \quad C_{count} = 0, 1, 2, \ldots \tag{4.89}
\]

Similarly, a mode with no noise (i.e. \( n_i = 0 \) for some \( i \)) contributes an unconditional Poisson random variable with mean \( \|m_i\|^2 \).

Note that:

\[
\psi_{C_i}(s) = \psi_{\|m_i+n_i\|^2}(e^s - 1) \quad i = 0, 1, 2, \ldots \tag{4.90}
\]

However, since:

\[
\psi_{C_i}(s) = E[e^{sC_i}] = E_{\|m_i+n_i\|^2}[E[e^{sC_i}/\|m_i+n_i\|^2]] \tag{4.91}
\]

and

\[
\psi_{\|m_i+n_i\|^2}(e^s - 1) = E_{\|m_i+n_i\|^2}[e^{(e^s-1)\|m_i+n_i\|^2}] \tag{4.92}
\]

we can then write:

\[
E[e^{sC_i}/\|m_i+n_i\|^2] = e^{(e^s-1)\|m_i+n_i\|^2} \quad i = 0, 1, 2, \ldots \tag{4.93}
\]

Equation 4.93 suggests that the count \( C_i \) is a conditional Poisson random variable with an energy count of \( \|m_i+n_i\|^2 \). Conditioned on the \( i \)th mode \( \|m_i+n_i\|^2 \) of the incident optical energy, the mean and variance of the electron count \( C_i \) associated with it are:
\[ E[C_i/\|m_i + n_i\|^2] = \|m_i + n_i\|^2 \quad i = 0,1,2,\cdots \] (4.94)

\[ \text{Var}[C_i/\|m_i + n_i\|^2] = \|m_i + n_i\|^2 \quad i = 0,1,2,\cdots \] (4.95)

The unconditional mean of \( C_i \) is:

\[
E[C_i] = E_{\|m_i + n_i\|^2}[E[C_i/\|m_i + n_i\|^2]] \\
= E[\|m_i + n_i\|^2] \\
= \lambda_i + \|m_i\|^2 \quad i = 0,1,2,\cdots \] (4.96)

The unconditional second moment of \( C_i \) is:

\[
E[C_i^2] = E_{\|m_i + n_i\|^2}[E[C_i^2/\|m_i + n_i\|^2]] \\
= E[\|m_i + n_i\|^2 + \|m_i + n_i\|^4] \\
= \lambda_i + \|m_i\|^2 + \lambda_i^2 + 2\lambda_i\|m_i\|^2 + (\lambda_i + \|m_i\|^2)^2 \quad i = 0,1,2,\cdots \] (4.97)

The variance of \( C_i \) is therefore given by:

\[
\text{Var}[C_i] = E[C_i^2] - (E[C_i])^2 \\
= \lambda_i^2 + 2\lambda_i\|m_i\|^2 + \lambda_i + \|m_i\|^2 \quad i = 0,1,2,\cdots \] (4.98)

Note that the means of \( C_i \) and \( \|m_i + n_i\|^2 \) (equation 4.48) are equal, but the variance of \( C_i \) is larger than that of \( \|m_i + n_i\|^2 \) (equation 4.56) by a quantity equal to the mean. This increase in the variance is caused by the shot noise.

Thus, the number of electrons \( C_{\text{count}} \) generated in the interval \([T_{\text{initial}}, T_{\text{final}}]\) can
be interpreted as the sum

\[ C_{\text{count}} = \sum_{i=0}^{\infty} C_i \quad (4.99) \]

of independent counts \( C_i \), where each \( C_i \) has a Laguerre count probability given by equation 4.83. The distribution for the total count \( C_{\text{count}} \) is equal to the convolution of an infinite number of Laguerre densities. The mean and variance of \( C_{\text{count}} \) are:

\[ E[C_{\text{count}}] = \sum_{i=0}^{\infty} E[C_i] = \sum_{i=0}^{\infty} (\lambda_i + \|m_i\|^2) \quad (4.100) \]

\[ \text{Var}[C_{\text{count}}] = \sum_{i=0}^{\infty} \text{Var}[C_i] = \sum_{i=0}^{\infty} (\lambda_i^2 + 2\lambda_i\|m_i\|^2 + \lambda_i + \|m_i\|^2) \quad (4.101) \]

The mean of \( C_{\text{count}} \) is equal to that of the detected optical energy \( \Lambda \) (equation 4.49), but due to the presence of shot noise, the variance of \( C_{\text{count}} \) exceeds that of \( \Lambda \) (equation 4.57) by a quantity equal to the mean.

Equations 4.81, 4.100 and 4.101 were all derived under the assumption that no thermal noise is present at the receiver. By including electronic thermal noise into the analysis, the 0 mean Gaussian random variable \( N \) defined in equation 4.8 will be added to the total electron count at the integrator’s output. The moment generating function of \( N \) is given by:

\[ \psi_N(s) = e^{\frac{\sigma^2}{2}s^2} \quad (4.102) \]

where \( \sigma^2 \) is the variance of \( N \) and is given by equation 4.9. The moment function
for $C_{count}$ as expressed in equation 4.81 will include the additional term $e^{\frac{s^2}{2}r^2}$ to account for the thermal noise, assumed to be independent of the shot noise throughout this thesis.

$$\psi_{C_{count}}(s) = \prod_{i=0}^{\infty} \frac{1}{1 + \lambda_i(1 - e^s)} e^{-\frac{\|m_i\|^2(1-e^s)}{1+\lambda_i(1-e^s)}} e^{\frac{s^2r^2}{2}} \quad \|e^s\| < \frac{1 + \lambda_{\max}}{\lambda_{\max}} \quad (4.103)$$

The mean of $C_{count}$ remains the same but its variance is increased by $\sigma^2$.

$$E[C_{count}] = \sum_{i=0}^{\infty} E[C_i] = \sum_{i=0}^{\infty} (\lambda_i + \|m_i\|^2) \quad (4.104)$$

$$Var[C_{count}] = \sum_{i=0}^{\infty} Var[C_i] = \sum_{i=0}^{\infty} (\lambda_i^2 + 2\lambda_i\|m_i\|^2 + \lambda_i + \|m_i\|^2) + \sigma^2 \quad (4.105)$$

4.4 The bit error rate for ASK and FSK receivers

In this section, we shall obtain bounds to the error rate for the ASK and FSK receivers of Figures 4-1 and 4-2. Chernoff bounds will be used to approximate the error rate when thermal noise and intersymbol interference are included in the analysis. We cannot derive a closed formula for the exact error probability when thermal noise is included in our model because of the complexity of the electron count density function. The exact expression of the bit error rate can be obtained for the FSK receiver of Figure 4-2 by ignoring the thermal noise $n_{th}(t)$ and the interference between neighboring signal pulses [Hum 91].
4.4.1 The Chernoff bound

The Chernoff bound is a commonly used technique to study the tails of distributions [Pap 84, Pro 83, Van 68].

Consider a random variable $x$ with a probability density $p_x(x)$ and a moment generating function $\psi_x(s)$. Suppose we wish to approximate the tail of the distribution $Pr(x \geq \gamma)$ for some threshold $\gamma$. $Pr(x \geq \gamma)$ is given by:

$$Pr(x \geq \gamma) = \int_\gamma^\infty p_x(x) \, dx$$  \hspace{1cm} (4.106)

For any positive real number $s$, we have:

$$\frac{e^{sx}}{e^{s \gamma}} \geq 1 \quad \text{for } s \geq 0, \ x \geq \gamma$$  \hspace{1cm} (4.107)

We can therefore write:

$$Pr(x \geq \gamma) \leq \int_\gamma^\infty \frac{e^{sx}}{e^{s \gamma}} p_x(x) \, dx \leq \int_\gamma^\infty \frac{e^{sx}}{e^{s \gamma}} p_x(x) \, dx$$  \hspace{1cm} (4.108)

or simply:

$$Pr(x \geq \gamma) \leq \psi_x(s)e^{-s \gamma} \quad \text{for } s \geq 0$$  \hspace{1cm} (4.109)

The moment generating function can be used to upper bound the expression $Pr(x \leq \gamma)$ in a similar fashion.

For any negative real number $s$, we have:

$$\frac{e^{sx}}{e^{s \gamma}} \geq 1 \quad \text{for } s \leq 0, \ x \leq \gamma$$  \hspace{1cm} (4.110)

and therefore:

$$Pr(x \leq \gamma) \leq \int_{-\infty}^{\gamma} \frac{e^{sx}}{e^{s \gamma}} p_x(x) \, dx \leq \int_{-\infty}^{\gamma} \frac{e^{sx}}{e^{s \gamma}} p_x(x) \, dx$$  \hspace{1cm} (4.111)
CHAPTER 4. PHOTODETECTION OF GAUSSIAN OPTICAL FIELDS

We get back the same result expressed in equation 4.109 except for the constraint on $s$ which has to be negative:

$$Pr(x \leq \gamma) \leq \psi_e(s) e^{-s\gamma} \quad \text{for } s \leq 0$$ (4.112)

Equations 4.109 and 4.112 constitute Chernoff bounds when the right side is minimized (obtain the tightest bound) by choosing an optimal $s$ within the constraint region.

4.4.2 Bit error rate for ASK receivers

For the binary ASK receiver of Figure 4-1, the electron count $C_{\text{count}}$ at the integrator's output is compared to a certain threshold $\gamma$ based on which a decision is made about whether a "0" or a "1" was sent.

Define $p_m$, the miss probability, as the probability that the receiver fails to detect an incident pulse and decides "0". Define $p_f$, the false alarm probability, as the probability that a "1" is decided when no pulse is sent.

$$p_m = \int_{-\infty}^{\gamma} p_{C_{\text{count}}/"1" \text{ sent}(\theta)} d\theta$$ (4.113)

$$p_f = \int_{\gamma}^{\infty} p_{C_{\text{count}}/"0" \text{ sent}(\theta)} d\theta$$ (4.114)

The densities $p_{C_{\text{count}}/"1" \text{ sent}(\theta)}$ and $p_{C_{\text{count}}/"0" \text{ sent}(\theta)}$ are the distributions of the electron count $C_{\text{count}}$ conditioned on whether a pulse was sent or not.

Assuming that "0" and "1" are equally probable a priori, the error probability $p_e$ is then given by:

$$p_e = \frac{1}{2} (p_m + p_f)$$ (4.115)

We can bound $p_m$ and $p_f$ with Chernoff bounds using the expression for the
moment generating function for $C_{\text{count}}$ (equation 4.103) under both hypothesis (a "0" is sent or a "1" is sent). The signal modes \{m_i\} should incorporate the filtered pulse sent (if any) as well as the interference from any neighboring pulse. In the next two chapters, we shall use the Chernoff bounds for $p_m$ and $p_f$ to optimize over both the threshold and the required signal energy to achieve the desired performance of $p_e = 10^{-9}$.

### 4.4.3 Bit error rate for FSK receivers

For the binary FSK receiver of Figure 4-2, the decisions made are based on the number of electrons released by both diodes. The receiver decides signal A is sent if the electron count $C_A$ generated by diode A is the largest; if the opposite is true, then signal B is decided. The optimum threshold is therefore set at 0.

Assuming that signals A and B are equally probable, the error probability $p_e$ is then given by:

\[
p_e = \int_{-\infty}^{0} p_{C_A-C_B/\text{signal A sent}}(\theta) d\theta \\
= \int_{0}^{\infty} p_{C_A-C_B/\text{signal B sent}}(\theta) d\theta
\]

(4.116) (4.117)

where the densities $p_{C_A-C_B/\text{signal A sent}}(\theta)$ and $p_{C_A-C_B/\text{signal B sent}}(\theta)$ are the distributions of the difference between the random electron counts $C_A$ and $C_B$ conditioned on the signal received (signal A or B).

Chernoff bounds will be used to upperbound the error rate $p_e$. The electron counts $C_A$ and $C_B$ generated by two different diodes are assumed to be independent random variables. The moment generating function for the difference $C_A - C_B$ is then equal to the product $\psi_{C_A}(s)\psi_{C_B}(-s)$. 
\[ \psi_{C_A - C_B}(s) = E[e^{(C_A - C_B)s}] = E[e^{C_As}]E[e^{-C_Bs}] = \psi_{C_A}(s)\psi_{C_B}(-s) \] (4.118)

The conditional moment generating functions of variables \( C_A \) and \( C_B \) (equation 4.103), conditioned on both hypothesis (either signal A or signal B is sent), are needed for this analysis. The signal modes \( \{m_i\} \) will again incorporate the interference from neighboring pulses in addition to the signal of interest. Our task is somehow simplified for FSK receivers because the optimum threshold is already known (\( \gamma = 0 \)).

### 4.4.4 Exact bit error rate for FSK receivers

The exact expression for the error probability for binary FSK can be derived by ignoring both thermal noise and ISI [Hum 91].

Assume that signal A is sent. The receiver makes an error and decides signal B is sent if \( C_B > C_A \). If both counts are equal \( C_A = C_B \), an error is made with probability \( \frac{1}{2} \). Since both thermal noise and ISI are ignored, then, conditioned on the event signal A is sent, the random variable \( C_B \) has a density given by equation 4.89.

\[
p(C_B \text{/signal A sent}) = \sum_{i=0}^{\infty} \frac{1}{1 + \lambda_i} \left( \frac{\lambda_i}{1 + \lambda_i} \right)^{C_B} \prod_{j=0, j \neq i}^{\infty} \frac{\lambda_i}{\lambda_i - \lambda_j}
\]

\[ C_B = 0, 1, 2, \ldots \] (4.119)

The error probability conditioned on count \( C_A \) is:

\[
Pr(\text{error} / C_A, \text{signal A is sent}) = \sum_{i=1}^{\infty} \sum_{i=0}^{\infty} \frac{1}{1 + \lambda_i} \left( \frac{\lambda_i}{1 + \lambda_i} \right)^{C_A + i} \prod_{j=0, j \neq i}^{\infty} \frac{\lambda_i}{\lambda_i - \lambda_j}
\]
\[ C H A P T E R \ 4. \ \textit{PHOTODETECTION \ OF \ GAUSSIAN \ OPTICAL \ FIELDS} \ \ \ \ 109 \]

\[ + \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{1 + \lambda_i} \left( \frac{\lambda_i}{1 + \lambda_i} \right)^{C_A} \prod_{j=0, j \neq i}^{\infty} \frac{\lambda_i}{\lambda_i - \lambda_j} \]

\[ = \sum_{i=0}^{\infty} \frac{1}{2} + \lambda_i \left( \frac{\lambda_i}{1 + \lambda_i} \right)^{C_A} \prod_{j=0, j \neq i}^{\infty} \frac{\lambda_i}{\lambda_i - \lambda_j} \]  

(4.120)

The probability of error conditioned on signal A is obtained by averaging equation 4.120 over the probability density of count \( C_A \).

\[
Pr(\text{error/A sent}) = E_{C_A} \left[ \sum_{i=0}^{\infty} \frac{1}{2} + \lambda_i \left( \frac{\lambda_i}{1 + \lambda_i} \right)^{C_A} \prod_{j=0, j \neq i}^{\infty} \frac{\lambda_i}{\lambda_i - \lambda_j} \right] 
\]

\[ = \sum_{i=0}^{\infty} \frac{1}{2} + \lambda_i E\left[ \left( \frac{\lambda_i}{1 + \lambda_i} \right)^{C_A} \right] / A \text{ sent} \prod_{j=0, j \neq i}^{\infty} \frac{\lambda_i}{\lambda_i - \lambda_j} \]  

(4.121)

Note that:

\[ E\left[ \left( \frac{\lambda_i}{1 + \lambda_i} \right)^{C_A} \right] / A \text{ sent} = \psi_{C_A}(s)|_{s=(1/1+\lambda_i)} \]  

(4.122)

where \( \psi_{C_A}(s) \) is the moment generating function of \( C_A \) conditioned on signal A and is given by equation 4.81.

Therefore:

\[ E\left[ \left( \frac{\lambda_i}{1 + \lambda_i} \right)^{C_A} \right] / A \text{ sent} \]

\[ = \prod_{l=0}^{\infty} \left\{ \frac{1}{1 + \lambda_l(1 - \frac{\lambda_l}{1+\lambda_l})} \exp\left( -\frac{\|m_l\|^2(1 - \frac{\lambda_l}{1+\lambda_l})}{1 + \lambda_l(1 - \frac{\lambda_l}{1+\lambda_l})} \right) \right\} \]

\[ = \left\{ \prod_{l=0}^{\infty} \frac{1 + \lambda_l}{1 + \lambda_l} \right\} \exp\left( -\sum_{l=0}^{\infty} \frac{\|m_l\|^2}{1 + \lambda_l + \lambda_l} \right) \]  

(4.123)

Replacing \( E\left[ \left( \frac{\lambda_i}{1 + \lambda_i} \right)^{C_A} / A \text{ sent} \right] \) by its value in equation 4.121, we get the following expression for \( Pr(\text{error/signal A sent}) \):
\[ Pr(\text{error/signal A sent}) = \frac{1}{2} \sum_{i=0}^{\infty} \left( \prod_{j=0, j \neq i}^{\infty} \frac{\lambda_i}{\lambda_i - \lambda_j} \frac{1 + \lambda_i}{1 + \lambda_i + \lambda_j} \right) \exp \left( -\sum_{l=0}^{\infty} \frac{\|m_l\|^2}{1 + \lambda_i + \lambda_l} \right) \quad (4.124) \]

Since \( Pr(\text{error/signal A sent}) = Pr(\text{error/signal B sent}) \) and assuming that signals A and B are a priori equally likely to be transmitted, then the unconditional error probability \( p_e \) is:

\[ p_e = \frac{1}{2} Pr(\text{error/signal A sent}) + \frac{1}{2} Pr(\text{error/signal B sent}) \]

\[ = \frac{1}{2} \sum_{i=0}^{\infty} \left( \prod_{j=0, j \neq i}^{\infty} \frac{\lambda_i}{\lambda_i - \lambda_j} \frac{1 + \lambda_i}{1 + \lambda_i + \lambda_j} \right) \exp \left( -\sum_{l=0}^{\infty} \frac{\|m_l\|^2}{1 + \lambda_i + \lambda_l} \right) \quad (4.125) \]

The exact expression in 4.125 will be used in the next two chapters to study the accuracy of Chernoff bounds in approximating the bit error rate for FSK receivers. Equation 4.125 is valid only when the impacts of both ISI and thermal noise on the system's performance are negligible.
Chapter 5

Performance of Mach-Zehnder filters

In this chapter, we study the performance of ASK and FSK direct detection receivers in recovering rectangular pulses corrupted by additive Gaussian optical amplifier noise when Mach-Zehnder chains are used as predetection filters. This analysis takes into account the three major sources of uncertainty discussed in previous chapters: (1) the shot noise generated by the photodetectors, (2) the electronic thermal noise present in the post detection circuitry and (3) the interference between overlapping neighboring pulses (ISI). We first start by solving the integral equation and deriving the eigenfunctions, eigenvalues and modes in the Karhunen-Loeve expansions of the filtered signal and ISI. We then use Chernoff bounds to optimize on the signal-to-noise ratio $\frac{E}{N_0}$ that yields an error probability of $10^{-9}$.

5.1 Solving the integral equation

In this section, we shall solve the integral equation presented in the previous chapter (equation 4.38) for Mach-Zehnder predetection optical filters. We shall derive expressions for the eigenfunctions $\{\phi_i(t)\}$, the eigenvalues $\{\lambda_i\}$, the signal modes $\{m_i\}$ and
the ISI modes \( \{m_{ISI}\} \).

5.1.1 The filtered incident signal pulse \( m(t) \)

![Figure 5-1: Incoming rectangular pulse \( s(t) \)](image)

As we have seen in Chapter 2, the Mach-Zehnder filter's response \( m(t) \) to an incident amplified rectangular pulse \( s(t) \) with energy \( E \) shown in Figure 5-1 can be either a triangular or a trapezoidal waveform. The pulse amplitude in Figure 5-1 is \( A = \sqrt{\frac{3E}{T}} \).

Figures 5-2 through 5-9 depict the eight possible optical waveforms that can be incident upon the receiver's photodiode during the interval of integration. It is assumed that all eight possibilities are equally probable.

If, however, the range of integration is confined to within the flat region of the trapeze, ISI is no longer present and the number of possible waveforms reduces to two: the pulse of interest is either "ON" or "OFF". The error probability \( p_e \) is therefore given by:

\[
p_e = \frac{1}{2} (Pr(\text{error/signal is "ON"}) + Pr(\text{error/signal is "OFF"}))
\]  

(5.1)

When ISI is present at the receiver, it is reasonable to assume that the error probabilities conditioned on the waveforms in Figures 5-5 and 5-6 are the two most dominant terms in the bit error rate expression. Consequently, the bit error rate can
CHAPTER 5. PERFORMANCE OF MACH-ZEHNDER FILTERS

Figure 5-2: Filtered waveform corresponding to an “OFF” “OFF” “OFF” input

Figure 5-3: Filtered waveform corresponding to an “ON” “OFF” “OFF” input
Figure 5-4: Filtered waveform corresponding to an “OFF” “OFF” “ON” input

Figure 5-5: Filtered waveform corresponding to an “ON” “OFF” “ON” input
Figure 5-6: Filtered waveform corresponding to an "OFF" "ON" "OFF" input

Figure 5-7: Filtered waveform corresponding to an "ON" "ON" "OFF" input
Figure 5-8: Filtered waveform corresponding to an “OFF” “ON” “ON” input

Figure 5-9: Filtered waveform corresponding to an “ON” “ON” “ON” input
be bounded as follows:

\[
\begin{align*}
p_e &= \frac{1}{8}(Pr(\text{error/"OFF-OFF-OFF" sent}) + Pr(\text{error/"ON-OFF-OFF" sent}) \\
&\quad + Pr(\text{error/"OFF-OFF-ON" sent}) + Pr(\text{error/"ON-OFF-ON" sent}) \\
&\quad + Pr(\text{error/"OFF-ON-OFF" sent}) + Pr(\text{error/"ON-ON-OFF" sent}) \\
&\quad + Pr(\text{error/"OFF-ON-ON" sent}) + Pr(\text{error/"ON-ON-ON" sent})) \\
&\leq \frac{1}{2}(Pr(\text{error/"ON-OFF-ON" sent}) + Pr(\text{error/"OFF-ON-OFF" sent}))(5.2)
\end{align*}
\]

Throughout this chapter, we will consider only the two situations shown in Figures 5-5 and 5-6.

5.1.2 The filtered optical noise correlation function

The autocorrelation function \( R'(\tau) \) of the filtered optical noise \( n(t) \) is given by:

\[
R'(\tau) = h_r(\tau) \ast h^*_r(-\tau) \ast R(\tau) = \frac{N_0}{2} h_r(\tau) \ast h^*_r(-\tau) \quad (5.3)
\]

where \( h_r(t) \) is the impulse response of the predetection Mach-Zehnder optical filter and the smoothing wideband filter cascaded together.

The autocorrelation of the filter's impulse response, as derived in Chapter 2 (equation 2.68), converges toward:

\[
h_r(\tau) \ast h^*_r(-\tau) \approx 2 \frac{1}{T'}(1 - \frac{\tau}{T'}) \cos(2\pi \nu_0 \tau), \quad \tau \in [-T', T'] \quad (5.5)
\]

Therefore, \( R'(\tau) \) is approximated by:

\[ R'(\tau) = \frac{N_0}{T'} (1 - \frac{|\tau|}{T'}) \cos(2\pi v_0 \tau), \quad \tau \in [-T', T'] \] (5.6)

The autocorrelation function's envelope \( R''(\tau) \) is given by:

\[ R''(\tau) = \frac{N_0}{T'} (1 - \frac{|\tau|}{T'}), \quad \tau \in [-T', T'] \] (5.7)

### 5.1.3 Derivation of the eigenfunctions and eigenvalues

Now that the autocorrelation function envelope \( R''(\tau) \) is known, we can derive the eigenfunctions \( \{\phi_i(t)\} \) and the eigenvalues \( \{\lambda_i\} \) by solving the integral equation:

\[ \int_{T_{\text{initial}}}^{T_{\text{final}}} R''(t - \tau) \phi_i(\tau) d\tau = \lambda_i \phi_i(t) \quad t \in [T_{\text{initial}}, T_{\text{final}}] \quad i = 0, 1, 2, \ldots \] (5.8)

Solutions to this particular integral equation that involves a triangular correlation function were derived in the past [Kal 66]. Our solutions, as presented in this section, will differ somewhat.

It is convenient to express the bounds of integration \( T_{\text{initial}} \) and \( T_{\text{final}} \) in terms of the total width of the filtered pulse \( W_{\text{base}} = T + T' \).

\[ T_{\text{initial}} = -\alpha \frac{T + T'}{2} \] (5.9)

\[ T_{\text{final}} = \alpha \frac{T + T'}{2} \] (5.10)

where the dimensionless variable \( \alpha \) is the ratio of the integration range over the filtered pulse width:

\[ \alpha = \frac{T_{\text{final}} - T_{\text{initial}}}{T + T'} \] (5.11)
CHAPTER 5. PERFORMANCE OF MACH-ZEHNDER FILTERS

The ratio $\alpha$ is always smaller than 1.

$$\alpha \in [0, 1]$$  \hspace{1cm} (5.12)

We will see shortly, that the solutions to the integral equation can be easily obtained when:

$$-\frac{T'}{2} \leq T_{\text{initial}} \leq T_{\text{final}} \leq \frac{T'}{2}$$  \hspace{1cm} (5.13)

or simply:

$$0 \leq \alpha \leq \frac{1}{1 + \frac{T'}{T}}$$  \hspace{1cm} (5.14)

We will limit our performance analysis of the Mach-Zehnder filter to the special situation where equation 5.14 is satisfied.

Replacing $T_{\text{initial}}$ and $T_{\text{final}}$ by their expressions, the integral equation becomes:

$$\int_{\frac{\alpha T + T'}{2}}^{\frac{\alpha T + T'}{2}} R''(t - \tau) \phi_i(\tau) d\tau = \lambda_i \phi_i(t)$$

$$t \in \left[-\alpha \frac{T + T'}{2}, \alpha \frac{T + T'}{2}\right] \; i = 0, 1, 2, \cdots$$  \hspace{1cm} (5.15)

Note that:

$$R''(t - \tau) = \begin{cases} \frac{N_i}{T} (1 - \frac{|t - \tau|}{T'}) & \text{if } t - T' \leq \tau \leq t + T' \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (5.16)

If $\alpha$ is larger than $\frac{1}{1 + \frac{T'}{T}}$, then the integral equation in 5.15 will involve arguments of $R''(\cdot)$ outside the range $[t - T', t + T']$. To avoid that, we will only consider the situation when $\alpha \in [0, \frac{1}{1 + \frac{T'}{T}}]$.

If $0 \leq \alpha \leq \frac{1}{1 + \frac{T'}{T}}$, we then have:
\[ \frac{T + T'}{2} - T' \leq - \frac{T + T'}{2} \leq \frac{T + T'}{2} = - \frac{T + T'}{2} + T' \]  

(5.17)

Thus for every \( t \) in the interval \( [-\frac{T + T'}{2}, \frac{T + T'}{2}] \), we have:

\[ \frac{T + T'}{2} - T' \leq t \leq - \frac{T + T'}{2} + T' \]  

(5.18)

yielding the following inequalities:

\[ t - T' \leq - \frac{T + T'}{2} \leq t + T' \]  

(5.19)

Based on the results from equations 5.16 and 5.19, the integral equation becomes:

\[
\int_{-\frac{T + T'}{2}}^{t} (T' - t + \tau) \phi_i(\tau) d\tau + \int_{t}^{\frac{T + T'}{2}} (T' + t - \tau) \phi_i(\tau) d\tau = \frac{T'^2 \lambda_i}{N_0} \phi_i(t) \\
t \in [-\frac{T + T'}{2}, \frac{T + T'}{2}] \quad i = 0, 1, 2, \ldots 
\]  

(5.20)

Differentiating both sides of equation 5.20 on the interval \( [-\frac{T + T'}{2}, \frac{T + T'}{2}] \) and assuming all required derivatives exist, we get:

\[
- \int_{-\frac{T + T'}{2}}^{t} \phi_i(\tau) d\tau + \int_{t}^{\frac{T + T'}{2}} \phi_i(\tau) d\tau = \frac{T'^2 \lambda_i}{N_0} \phi_i(t) \\
t \in [-\frac{T + T'}{2}, \frac{T + T'}{2}] \quad i = 0, 1, 2, \ldots 
\]  

(5.21)

where \( \dot{\phi}_i(t) \) is the first order derivative of \( \phi_i(t) \) with respect to \( t \).

Differentiating once more with respect to \( t \), we get:

\[
\phi_i(t) = - \frac{T'^2 \lambda_i}{2N_0} \ddot{\phi}_i(t) \\
t \in [-\frac{T + T'}{2}, \frac{T + T'}{2}] \quad i = 0, 1, 2, \ldots 
\]  

(5.22)
where \(\ddot{\phi}_i(t)\) is the second order derivative of \(\phi_i(t)\) with respect to \(t\).

The boundary conditions for this differential equation are obtained by substituting for \(\phi_i(t)\) in the left hand side of equation 5.20 and evaluating both sides at \(t = -\alpha \frac{T + T'}{2}\) and \(t = \alpha \frac{T + T'}{2}\).

When \(t = -\alpha \frac{T + T'}{2}\), equation 5.20 becomes:

\[
\int_{-\alpha \frac{T + T'}{2}}^{\alpha \frac{T + T'}{2}} (T' - \alpha \frac{T + T'}{2} - \tau) \ddot{\phi}_i(\tau) d\tau = -2\phi_i(-\alpha \frac{T + T'}{2}) \quad i = 0, 1, 2, \ldots \quad (5.23)
\]

yielding the following boundary condition:

\[
(T' - \alpha(T + T')) \dot{\phi}_i(\alpha \frac{T + T'}{2}) - T' \dot{\phi}_i(-\alpha \frac{T + T'}{2}) + \phi_i(\alpha \frac{T + T'}{2}) + \phi_i(-\alpha \frac{T + T'}{2}) = 0 \quad i = 0, 1, 2, \ldots \quad (5.24)
\]

Evaluating equation 5.20 at \(t = \alpha \frac{T + T'}{2}\), we get:

\[
\int_{-\alpha \frac{T + T'}{2}}^{\alpha \frac{T + T'}{2}} (T' - \alpha \frac{T + T'}{2} + \tau) \ddot{\phi}_i(\tau) d\tau = -2\phi_i(\alpha \frac{T + T'}{2}) \quad i = 0, 1, 2, \ldots \quad (5.25)
\]

and the resulting boundary condition is:

\[
T' \phi_i(\alpha \frac{T + T'}{2}) - (T' - \alpha(T + T')) \dot{\phi}_i(-\alpha \frac{T + T'}{2}) + \phi_i(\alpha \frac{T + T'}{2}) + \phi_i(-\alpha \frac{T + T'}{2}) = 0 \quad i = 0, 1, 2, \ldots \quad (5.26)
\]

The two boundary conditions in 5.24 and 5.26 can be rearranged as follows:
\( \dot{\phi}_i(-\alpha \frac{T + T'}{2}) = -\dot{\phi}_i(\alpha \frac{T + T'}{2}) = \frac{\phi_i(\alpha \frac{T + T'}{2}) + \phi_i(-\alpha \frac{T + T'}{2})}{2(T' - \alpha \frac{T + T'}{2})} \quad i = 0, 1, 2, \ldots \) (5.27)

The set of eigenfunctions \( \{\phi_i(t)\} \) must also form an orthonormal basis over the interval \([-\alpha \frac{T + T'}{2}, \alpha \frac{T + T'}{2}]\).

\[
\int_{-\alpha \frac{T + T'}{2}}^{\alpha \frac{T + T'}{2}} \phi_i(t)\phi_j^*(t)dt = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}
\] (5.28)

The solutions \( \{\phi_i(t)\} \) to the differential equation in 5.22 can be divided into a set of even functions:

\[
\phi_i(t) = A_i \cos\left(\frac{\beta_i t}{\alpha \frac{T + T'}{2}}\right) \quad t \in [-\alpha \frac{T + T'}{2}, \alpha \frac{T + T'}{2}] \quad i = 0, 2, 4, 6, \ldots \) (5.29)

and a set of odd functions:

\[
\phi_i(t) = A_i \sin\left(\frac{\beta_i t}{\alpha \frac{T + T'}{2}}\right) \quad t \in [-\alpha \frac{T + T'}{2}, \alpha \frac{T + T'}{2}] \quad i = 1, 3, 5, 7, \ldots \) (5.30)

For both even and odd families of functions, the relationship between \( \beta_i \) and the eigenvalue \( \lambda_i \) is:

\[
\lambda_i = 2N_0\left(\frac{\alpha}{\beta_i}\right)^2 \quad i = 0, 1, 2, \ldots
\] (5.31)

The expression for \( \beta_i \) is found by applying the boundary conditions in equation 5.27 to the corresponding eigenfunction \( \phi_i(t) \).

The boundary conditions in 5.27, when applied to members of the odd set of eigenfunctions, imply that \( \dot{\phi}_i(\alpha \frac{T + T'}{2}) = 0 \). The odd-indexed \( \{\beta_i\} \) are therefore given by:


\[ \beta_i = \frac{i\pi}{2} \quad i = 1, 3, 5, 7, \ldots \quad (5.32) \]

and the corresponding eigenvalues are:

\[ \lambda_i = \frac{2N_0\alpha^2(1 + \frac{T_i}{T})^2}{\pi^2 i^2} \quad i = 1, 3, 5, 7, \ldots \quad (5.33) \]

For members of the even family, the boundary conditions in 5.27 yield the following transcendental equation in \( \beta_i \):

\[ \tan(\beta_i) = \frac{\frac{\alpha}{\beta_i} \left(1 + \frac{T}{T_i}\right)}{\left(1 - \frac{\alpha}{\beta_i} \left(1 + \frac{T}{T_i}\right)\right)\beta_i} \quad i = 0, 2, 4, 6, \ldots \quad (5.34) \]

The even-indexed eigenvalues are obtained from equation 5.31 by solving for the \( \{\beta_i\} \) coefficients first.

\[ \lambda_i = 2N_0\left(\frac{\alpha}{\beta_i} \left(1 + \frac{T}{T_i}\right)\right)^2 \quad i = 0, 2, 4, 6, \ldots \quad (5.35) \]

The normalizing factor \( A_i \), which ensures that the norm of eigenfunction \( \phi_i(t) \) is equal to unity (i.e. \( \int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_i(t)\phi_i^*(t)dt = 1, i = 0, 1, 2, \cdots \)), is given by:

\[ A_i = \begin{cases} 
\frac{1}{\sqrt{\frac{\alpha^{\frac{T}{2} + T'}(1 + \sin(\frac{T}{2\beta_i}))}{\beta_i}}} & \text{for } i = 0, 2, 4, \ldots \\
\frac{1}{\sqrt{\frac{\alpha^{\frac{T}{2} + T'}}{\beta_i}}} & \text{for } i = 1, 3, 5, \ldots 
\end{cases} \quad (5.36) \]

### 5.1.4 Derivation of the signal modes

Figure 5-10 depicts the envelope of the incident filtered pulse \( m'(t) \).

For any nonnegative integer \( i \), the coefficient \( m_i \) is the projection of \( m'(t) \) on the eigenfunction \( \phi_i(t) \).

\[ m_i = \int_{-\frac{T}{2}}^{\frac{T}{2}} m'(t)\phi_i^*(t)dt \quad i = 0, 1, 2, \ldots \quad (5.37) \]
Since \( m'(t) \) is an even function and all odd-indexed eigenfunctions are symmetric about the origin, then:

\[
\|m_i\|^2 = 0 \quad i = 1, 3, 5, \ldots
\]  \( (5.38) \)

The region of integration is confined inside the flat portion of \( m'(t) \) (i.e. no ISI exists) by having:

\[
\alpha \frac{T + T'}{2} \leq \min\left(\frac{T - T'}{2}, \frac{T'}{2}\right)
\]  \( (5.39) \)

or simply:

\[
\alpha \leq \min\left(\frac{T}{T'}, 1 + \frac{1}{\frac{T}{T'} + 1}\right)
\]  \( (5.40) \)

In this case, the expression for the even-indexed signal modes is:

\[
\|m_i\|^2 = \frac{2\alpha(1 + \frac{T'}{T})E \sin^2\beta_i}{1 + \frac{\sin(2\beta_i)}{2\beta_i}} \quad i = 0, 2, 4, \ldots \text{ when } \alpha \leq \min\left(\frac{T}{T'}, 1 + \frac{1}{\frac{T}{T'} + 1}\right)
\]  \( (5.41) \)
If, instead, the region of integration exceeds the flat portion of $m'(t)$ (i.e. ISI is present), the condition on $\alpha$ becomes:

$$\frac{T}{T'} - 1 \leq \alpha \leq \frac{1}{T'/T} + 1$$

(5.42)

and the expression for the even-indexed signal modes changes to:

$$\|m_i\|^2 = \left( \frac{T}{T'} - 1 \right) \frac{\sin(\beta_i)}{\beta_i} + \frac{\alpha(1 + \frac{T}{T'})}{\beta_i^2} \cos\left( \frac{T}{T'} - 1 \right) \beta_i \right) \frac{\frac{(1 + \frac{T}{T'})E}{\alpha(1 + \frac{T}{T'})}}{1 + \frac{\sin(2\beta_i)}{2\beta_i}}$$

$$i = 0, 2, 4, \ldots \text{ when } \frac{T}{T'} + 1 \leq \alpha \leq \frac{1}{T'/T}$$

(5.43)

### 5.1.5 Derivation of the ISI modes

Figure 5-11 depicts the envelope of the incident ISI waveform $m'_{ISI}(t)$.

![Figure 5-11: ISI envelope $m'_{ISI}(t)$](image)

We will denote by $m_{ISI_i}$ the projection of $m'_{ISI}(t)$ on the eigenfunction $\phi_i(t)$.

$$m_{ISI_i} = \int_{-\alpha \frac{T}{T'}}^{\alpha \frac{T}{T'}} m'_{ISI}(t) \phi_i^*(t) dt \quad i = 0, 1, 2, \ldots$$

(5.44)

Since $m'_{ISI}(t)$ is an even function, then the odd-indexed ISI modes are all equal.
to 0.

\[ \|m_{ISI_i}\|^2 = 0 \quad i = 1, 3, 5, \ldots \]  \hspace{1cm} (5.45)

Of course, the even-indexed ISI modes are all equal to 0 when the region of integration is within the interval \([-\frac{T-T'}{2}, \frac{T-T'}{2}]\).

\[ \|m_{ISI_i}\|^2 = 0 \quad i = 0, 2, 4, \ldots \quad \text{when} \quad \alpha \leq \min\left(\frac{T-1}{T'+1}, \frac{1}{T'+1}\right) \]  \hspace{1cm} (5.46)

If the region of integration exceeds the interval \([-\frac{T-T'}{2}, \frac{T-T'}{2}]\), the expression for the even-indexed ISI modes is given by:

\[
\|m_{ISI_i}\|^2 = \left\{ (3 - \frac{T}{T'}) \frac{\sin(\beta_i)}{\beta_i} - \frac{\alpha(1 + \frac{T}{T'})}{\beta_i^2} \cos\left(\frac{\frac{T}{T} - 1}{\beta_i}\right) \right\} \frac{\alpha}{2} \left(1 + \frac{T}{T'}\right) E \frac{1 + \frac{T}{T'}}{1 + \frac{\sin(2\beta_i)}{2\beta_i}}
\]

\[ i = 0, 2, 4, \ldots \quad \text{when} \quad \frac{T-1}{T'+1} \leq \alpha \leq \frac{1}{T'+1} \]  \hspace{1cm} (5.47)

5.2 The average noise energy

Recall from Chapter 4 that the average noise energy received in the interval \([-\alpha \frac{T+T'}{2}, \alpha \frac{T+T'}{2}]\) is given by:

\[
\text{Average noise energy received} = E[\int_{-\alpha \frac{T+T'}{2}}^{\alpha \frac{T+T'}{2}} |n'(t)|^2 dt]
\]

\[ = \sum_{i=0}^{\infty} \lambda_i \]  \hspace{1cm} (5.49)

However:

\[
E[\int_{-\alpha \frac{T+T'}{2}}^{\alpha \frac{T+T'}{2}} |n'(t)|^2 dt] = \int_{-\alpha \frac{T+T'}{2}}^{\alpha \frac{T+T'}{2}} R''(0) dt
\]
Figure 5-12: Average total noise energy $\sum_{i=0}^{\infty} \lambda_i$ (expressed in units of $N_0$)

Figure 5-13: Sum of the first 10 eigenvalues $\sum_{i=0}^{9} \lambda_i$ (expressed in units of $N_0$)
\[ = \alpha(1 + \frac{T}{T'})N_0 \]

The expression for the average noise energy is thus:

\[ \text{Average noise energy received} = \sum_{i=0}^{\infty} \lambda_i \]  
\[ = \alpha(1 + \frac{T}{T'})N_0 \] (5.50)

Figure 5-12 shows plots of the average noise energy versus \(\alpha\) for selected values of the ratio \(\frac{T}{T'}\). As suggested by equation 5.51, the average noise energy increases linearly with both \(\alpha\) and \(\frac{T}{T'}\). Recall from Chapter 2 that the larger \(\frac{T}{T'}\) is, the wider the Mach-Zehnder filter's bandwidth becomes (the bandwidth is inversely proportional to \(T'\)) resulting in more noise energy incident upon the photodetector. Also, as \(\alpha\) increases, the range of integration widens and more noise power is collected by the integrator.

Figure 5-13 shows plots of the sum of the first ten eigenvalues \(\sum_{i=0}^{9} \lambda_i\). A comparison between these plots and the ones in Figure 5-12 suggests that the first ten eigenvalues are clearly the dominant terms in the average noise energy expression.

### 5.3 The total signal energy

The total signal energy received in the interval \([-\alpha \frac{T+T'}{2}, \alpha \frac{T+T'}{2}]\) is given by:

\[ \text{Total signal energy received} = \int_{-\alpha \frac{T+T'}{2}}^{\alpha \frac{T+T'}{2}} \|m'(t)\|^2 dt \]  
\[ = \sum_{i=0}^{\infty} \|m_i\|^2 \] (5.53)

If the range of integration is within the flat region of the trapeze in Figure 5-10, the collected signal energy is:
Figure 5.14: Total signal energy received $\sum_{i=0}^\infty \|m_i\|^2$ (expressed in units of $E$)

Figure 5.15: Sum of the first 10 signal modes $\sum_{i=0}^9 \|m_i\|^2$ (expressed in units of $E$)
CHAPTER 5. PERFORMANCE OF MACH-ZEHNDER FILTERS

\[ \int_{-\alpha T' \frac{T}{2}}^{\alpha T' \frac{T}{2}} \| m'(t) \|^2 dt = \alpha \left( 1 + \frac{T'}{T} \right) E \quad \text{when} \quad \alpha \leq \min \left( \frac{p-1}{p+1}, \frac{1}{p+1} \right) \]  (5.54)

When the range of integration exceeds the flat region, the received signal energy becomes:

\[ \int_{-\alpha T' \frac{T}{2}}^{\alpha T' \frac{T}{2}} \| m'(t) \|^2 dt = \left\{ 1 - \frac{1}{3} \frac{T'}{T} - \frac{1}{12} \frac{T'}{T} \left( (1 + \frac{T}{T'}) (1 - \alpha)^3 \right) \right\} E \]

\[ \quad \text{when} \quad \frac{p-1}{p+1} \leq \alpha \leq \frac{1}{p+1} \]  (5.55)

The total energy carried by the incident filtered pulse \( m(t) \) is obtained by setting \( \alpha \) to 1 in equation 5.55.

\[ \text{Filtered pulse } m(t) \text{ total energy} = \int_{-\alpha T' \frac{T}{2}}^{\alpha T' \frac{T}{2}} \| m'(t) \|^2 dt = (1 - \frac{1}{3} \frac{T'}{T}) E \]  (5.56)

Figure 5-14 shows plots of the total signal energy collected \( \sum_{i=0}^{\infty} \| m_i \|^2 \) versus \( \alpha \) for selected values of the ratio \( \frac{T}{T'} \). Surprisingly, the signal energy is little affected by the filter's structure (i.e. \( \frac{T}{T'} \)); the impact of the filter's structure becomes evident as the range of integration is expanded. As expected, more signal energy is collected from a trapezoidal pulse (\( \frac{T}{T'} > 1 \)) than from a triangular one (\( \frac{T}{T'} = 1 \)) when \( \alpha \) is large.

Figure 5-15 shows plots of the sum of the first ten signal modes \( \sum_{i=0}^{9} \| m_i \|^2 \). The plots in Figures 5-14 and 5-15 are almost identical: the first ten signal modes clearly dominate in the signal energy expression.

5.4 The total ISI energy

The total ISI energy received in the interval \( [-\alpha \frac{T+T'}{2}, \alpha \frac{T+T'}{2}] \) is given by:
CHAPTER 5. PERFORMANCE OF MACH-ZEHNDER FILTERS

Figure 5-16: Total ISI energy received $\sum_{i=0}^{\infty} \|m_{ISI_i}\|^2$ (expressed in units of $E$)

Figure 5-17: Sum of the first 10 ISI modes $\sum_{i=0}^{9} \|m_{ISI_i}\|^2$ (expressed in units of $E$)
CHAPTER 5. PERFORMANCE OF MACH-ZEHNDER FILTERS

Total ISI energy received
\[
\text{Total ISI energy received} = \int_{-\frac{T}{2}}^{\frac{T}{2}} \|m'_{ISI}(t)\|^2 dt
\]  
(5.58)

\[
= \sum_{i=0}^{\infty} \|m_{ISI_i}\|^2
\]  
(5.59)

No ISI energy is collected when the range of integration is within the interval \([-\frac{T-T'}{2}, \frac{T-T'}{2}]\) (Figure 5-11).

\[
\int_{-\frac{T}{2}}^{\frac{T}{2}} \|m'_{ISI}(t)\|^2 dt = 0 \text{ when } \alpha \leq \min\left(\frac{\frac{T-T'}{T'}+1}{\frac{T}{T'}+1}, \frac{1}{\frac{T}{T'}+1}\right)
\]  
(5.60)

When the range of integration exceeds the interval \([-\frac{T-T'}{2}, \frac{T-T'}{2}]\), the received ISI energy becomes:

\[
\int_{-\frac{T}{2}}^{\frac{T}{2}} \|m'_{ISI}(t)\|^2 dt = \frac{1}{12} \frac{T'}{T} \left\{ \alpha \left(1 + \frac{T}{T'}\right) - \left(\frac{T}{T'} - 1\right)^3 \right\} E
\]  
when \(\frac{T-1}{\frac{T}{T'}+1} \leq \alpha \leq \frac{1}{\frac{T}{T'}+1}\)  
(5.61)

Figure 5-16 shows plots of the total ISI energy collected \(\sum_{i=0}^{\infty} \|m_{ISI_i}\|^2\) versus \(\alpha\) for selected values of the ratio \(\frac{T}{T'}\). As opposed to the received signal energy, the collected ISI energy is strongly affected by the filter’s structure (i.e. \(\frac{T}{T'}\)). As expected, the ISI energy decreases as \(\frac{T}{T'}\) is increased: a filter with a shorter \(T'\) shapes the rectangular input into a narrower signal resulting in a smaller interval over which neighboring pulses overlap.

Figure 5-17 shows plots of the sum of the first ten ISI modes \(\sum_{i=0}^{9} \|m_{ISI_i}\|^2\). The plots in Figures 5-16 and 5-17 are undistinguishable: the first ten ISI modes clearly dominate in the ISI energy expression.
5.5 The total noise-noise and signal-noise beat variances

Figures 5-18 depicts plots of the total noise-noise beat variance approximated as the sum the first 1000 noise-noise beat terms $\sum_{i=0}^{999} \lambda_i^2$ which are undistinguishable from plots of the sum of the first ten terms $\sum_{i=0}^{9} \lambda_i^2$ shown in Figure 5-19.

Plots of the total signal-noise beat variance (approximated as the sum of the first 1000 signal-noise beat terms $\sum_{i=0}^{999} 2\lambda_i \|m_i\|^2$) shown in Figure 5-20 are also undistinguishable from plots of the sum of the first ten terms $\sum_{i=0}^{9} 2\lambda_i \|m_i\|^2$ (Figure 5-21).

When analyzing the Mach-Zehnder filter’s performance in the remaining sections of this chapter, we will only consider the dominant terms in the Karhunen-Loeve expansion of the noise, signal and ISI. Our analysis will be therefore limited to the first ten eigenvalues and the first ten signal and ISI modes.
Figure 5-18: Total noise-noise beat variance $\sum_{i=0}^{\infty} \lambda_i^2$ (expressed in units of $N_0 \times N_0$)

Figure 5-19: Sum of the first 10 noise-noise beat terms $\sum_{i=0}^{9} \lambda_i^2$ (expressed in units of $N_0 \times N_0$)
Figure 5-20: Total signal-noise beat variance $\sum_{i=0}^{\infty} 2\lambda_i \|\mathbf{m}_i\|^2$ (expressed in units of $E \times N_0$)

Figure 5-21: Sum of the first 10 signal-noise beat terms $\sum_{i=0}^{9} 2\lambda_i \|\mathbf{m}_i\|^2$ (expressed in units of $E \times N_0$)
5.6 Performance of Mach-Zehnder filters in ASK receivers

In this section, we shall determine the optimum signal-to-noise ratio \( \frac{E}{N_0} \) necessary to achieve a bit error rate of \( 10^{-9} \) for ASK receivers that use Mach-Zehnder optical filters. Recall that \( E \) is the energy of the amplified rectangular pulse at the Mach-Zehnder filter input. Chernoff bounds will be used to approximate both the probability of a false alarm \( p_f \) and the probability of a miss \( p_m \). Our analysis will be twofold: the signal-to-noise ratio \( \frac{E}{N_0} \) is computed by first including then ignoring the effects of ISI. Such an analysis would give us a better understanding of how ISI affects the performance.

5.6.1 A performance analysis including ISI

Assuming that only the first ten dominant eigenvalues, signal modes and ISI modes are relevant for an accurate analysis, then using the results derived in Chapter 4, the Chernoff bounds to \( p_m \) and \( p_f \) are expressed as follows:

\[
 p_m = \Pr(C_{\text{count}} \leq \gamma / \text{"0-1-0" sent}) \\
\leq \left\{ \prod_{i=0}^{9} \frac{1}{1 + \lambda_i (1 - e^{s_1})} e^{-\frac{|m| s_1^2 (1 - e^{s_1})}{1 + \lambda_i (1 - e^{s_1})}} \right\} e^{\frac{s_1^2}{2} - s_1 \gamma} \quad \text{for } s_1 \leq 0 
\]

\[
 p_f = \Pr(C_{\text{count}} \geq \gamma / \text{"1-0-1" sent}) \\
\leq \left\{ \prod_{i=0}^{9} \frac{1}{1 + \lambda_i (1 - e^{s_2})} e^{-\frac{|m| s_2^2 (1 - e^{s_2})}{1 + \lambda_i (1 - e^{s_2})}} \right\} e^{\frac{s_2^2}{2} - s_2 \gamma} \\
\text{for } 0 \leq s_2 \leq \ln\left( \frac{1 + \lambda_{\text{max}}}{\lambda_{\text{max}}} \right)
\]
where $\gamma$ is the threshold and $\lambda_{\text{max}}$ is the largest of the first ten eigenvalues which in this case is $\lambda_0$.

The variance $\sigma^2$ of the thermal noise sample is given by:

$$
\sigma^2 = \frac{i_{\text{th}}^2}{q^2} \alpha (T + T')
$$

$$
= \frac{i_{\text{th}}^2}{q^2} \alpha T (1 + \frac{T'}{T})
$$

We will assume throughout this analysis that the rate of transmission is 1Gbit/sec.

$$
T = 10^{-9} \text{sec}
$$

![Graph](image)

Figure 5-22: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($N_0 = \infty$, ISI is not ignored)

For given values of $\alpha$, $\frac{T}{T'}$, and $N_0$ we will compute the signal-to-noise ratio $\frac{E}{N_0}$ that achieves an error rate $p_e = \frac{1}{2} (p_m + p_f)$ of $10^{-9}$ by optimizing over the threshold $\gamma$ and
the two variables $s_1$ and $s_2$. We will consider four different thermal noise levels:

$$i_{th} \in \{0, \frac{pA}{\sqrt{Hz}}, 1, \frac{pA}{\sqrt{Hz}}, 3, \frac{pA}{\sqrt{Hz}}, 10, \frac{pA}{\sqrt{Hz}}\}$$

(5.68)

Figure 5-22 shows plots of $\frac{E}{N_0}$ versus $\alpha$ when the optical noise is the most important source of uncertainty and completely dominates the electronic thermal noise.

The plots in Figure 5-22 suggest that the optimum Mach-Zehnder filter structure used in an ASK receiver exposed to a high-intensity optical noise field is the one matched to the incident rectangular pulses ($\frac{T}{\tau_p} = 1$). Recall that a matched Mach-Zehnder filter has a narrower bandwidth than an unmatched filter ($\frac{T}{\tau_p} > 1$), and is therefore more effective in combating optical noise. The range of integration must be very small ($\alpha \approx 0$) in order to reject as much optical noise power as possible and the corresponding signal-to-noise ratio $\frac{E}{N_0}$ is in the neighborhood of 90 (i.e. 90 amplified “signal photons” are needed for every additive “noisy photon” in order to achieve a bit error rate of $10^{-9}$). Notice how some of the curves ($\frac{T}{\tau_p} = 1.0, 1.1, \cdots, 1.5$) rise as $\alpha$ increases while others ($\frac{T}{\tau_p} = 1.6$ and 1.7) continue to decrease. This rise can be attributed to two factors: (1) the optical noise collected by the receiver and (2) the ISI energy which both increase with $\alpha$ (Figures 5-12 and 5-16). The rise in the signal-to-noise ratio curves is more severe for smaller values of $\frac{T}{\tau_p}$ and is therefore associated in a major way with ISI (recall that ISI increases as $\frac{T}{\tau_p}$ decreases; the opposite is true for the collected optical noise energy).

Figures 5-23 through 5-34 illustrate plots of $\frac{E}{N_0}$ versus $\alpha$ for different values of $N_0$ and $i_{th}$. Both $E$ and $N_0$ are expressed in units of photon count.
CHAPTER 5. PERFORMANCE OF MACH-ZEHNDER FILTERS

Figure 5-23: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 0 \frac{pA}{H z^2/\gamma}$, $N_0 = 1$, ISI is not ignored)

Figure 5-24: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 0 \frac{pA}{H z^2/\gamma}$, $N_0 = 10$, ISI is not ignored)

Figure 5-25: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 0 \frac{pA}{H z^2/\gamma}$, $N_0 = 50$, ISI is not ignored)
Figure 5-26: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 1 \frac{P_A}{H_z^2/T}$, $N_0 = 10$, ISI is not ignored)

Figure 5-27: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 1 \frac{P_A}{H_z^2/T}$, $N_0 = 50$, ISI is not ignored)

Figure 5-28: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 1 \frac{P_A}{H_z^2/T}$, $N_0 = 100$, ISI is not ignored)
Figure 5-29: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 3 \frac{P_A}{H_z^{1/3}}, N_0 = 10, \text{ISI is not ignored}$)

Figure 5-30: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 3 \frac{P_A}{H_z^{1/3}}, N_0 = 100, \text{ISI is not ignored}$)

Figure 5-31: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 3 \frac{P_A}{H_z^{1/3}}, N_0 = 1000, \text{ISI is not ignored}$)
Figure 5-32: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
($i_{th} = 10 \frac{E_A}{H s^{1/2}}$, $N_0 = 10$, ISI is not ignored)

Figure 5-33: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
($i_{th} = 10 \frac{E_A}{H s^{1/2}}$, $N_0 = 100$, ISI is not ignored)

Figure 5-34: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers
($i_{th} = 10 \frac{E_A}{H s^{1/2}}$, $N_0 = 1000$, ISI is not ignored)
5.6.2 A performance analysis ignoring ISI

Ignoring the effects of ISI on the receiver’s performance (i.e. the ISI modes \{m_{ISI}\} are all set to 0), the Chernoff bounds to \(p_m\) and \(p_f\) become:

\[
\begin{align*}
\frac{p_m}{\Pr(C_{\text{count}} \leq \gamma / \text{"0-1-0" sent})} & \leq \left\{ \prod_{i=0}^{9} \frac{1}{1 + \lambda_i (1 - e^{s_1})} e^{-\frac{\|m_i\|^2}{\lambda_i (1 - e^{s_1})}} \right\} e^{s_1^2 - s_1 \gamma} & \text{for } s_1 \leq 0 \\
\frac{p_f}{\Pr(C_{\text{count}} \geq \gamma / \text{"1-0-1" sent})} & \leq \left\{ \prod_{i=0}^{9} \frac{1}{1 + \lambda_i (1 - e^{s_2})} \right\} e^{s_2^2 - s_2 \gamma} \text{ for } 0 \leq s_2 \leq \ln\left( \frac{1 + \lambda_{\max}}{\lambda_{\max}} \right)
\end{align*}
\]

Figure 5-35: Plots of \(\frac{E}{N_0}\) that achieves a bit error rate of \(10^{-9}\) for ASK receivers \((N_0 = \infty, \text{ISI is ignored})\)

By optimizing over the threshold \(\gamma\) and the two variables \(s_1\) and \(s_2\), we compute
CHAPTER 5. PERFORMANCE OF MACH-ZEHNDER FILTERS

the signal-to-noise ratio necessary to achieve a bit error rate of $10^{-9}$.

Figure 5-35 shows plots of $\frac{E}{N_0}$ versus $\alpha$ when the optical noise completely dominates the electronic thermal noise.

A comparison between Figures 5-35 and 5-22 reveals once again that the collected ISI energy contributes in a major way to the rise of the optimum signal-to-noise ratio $\frac{E}{N_0}$ for large values of $\alpha$. The collected optical noise energy plays a marginal role in the degradation of the receiver's performance as the range of integration is expanded: most of the curves in Figure 5-35 decrease with $\alpha$ except the ones for $\frac{\tau}{T_i} = 1.0$ and $\frac{\tau}{T_i} = 1.1$ which rise by a mere 10%.

Figures 5-36 through 5-39 illustrate plots of $\frac{E}{N_0}$ versus $\alpha$ for different values of $N_0$ and $i_{th}$ when ISI is ignored.
Figure 5.36: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 0 \frac{p}{H^{2.75}}, N_0 = 1$, ISI is ignored)

Figure 5.37: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 0 \frac{p}{H^{2.75}}, N_0 = 10$, ISI is ignored)
Chapter 5. Performance of Mach-Zehnder Filters

Figure 5-38: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 10 \frac{A}{Hz^{1/2}}, N_0 = 10$, ISI is ignored)

Figure 5-39: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 10 \frac{A}{Hz^{1/2}}, N_0 = 1000$, ISI is ignored)
A comparison between Figures 5-36 and 5-23 and between Figures 5-37 and 5-24 illustrates the degradation in the receiver's performance due to heavy ISI introduced by a matched or a nearly matched Mach-Zehnder filter \((T_f = 1.0, 1.1, 1.2)\) when the range of integration is large. As \(T_f\) increases, the impact of ISI on the receiver's performance diminishes: the curves for \(T_f = 1.6\) and \(T_f = 1.7\) obtained by including then ignoring ISI are almost identical.

If thermal noise is the dominant source of uncertainty as it is the case in Figures 5-38 and 5-32, then both the ISI energy and the optical noise energy are no longer important factors affecting the receiver's performance: the plots in both figures are almost identical for any value of \(T_f\). The individual plots in each of the two figures are closely packed together as expected: a receiver that is completely swamped by thermal noise will perform poorly regardless how the predetection Mach-Zehnder filter is designed. In Figure 5-39, the optical noise is no longer dominated by thermal noise; the effects of the collected ISI energy on the receiver's performance become evident for large values of \(\alpha\) especially when a matched filter is used (refer to Figure 5-34).

5.7 Performance of Mach-Zehnder filters in FSK receivers

In this section, we determine the optimum signal-to-noise ratio \(\frac{E}{N_0}\) necessary to achieve a bit error rate of \(10^{-9}\) for FSK receivers that use Mach-Zehnder optical filters. The error probability \(p_e\) will be approximated with Chernoff bounds. The effects of ISI on the performance will be examined by first including then ignoring the received ISI energy in our analysis.

5.7.1 A performance analysis including ISI

The bit error rate \(p_e\) is given by:
CHAPTER 5. PERFORMANCE OF MACH-ZEHNDER FILTERS

\[ p_e = Pr(C_A \geq C_B/ \text{"A-B-A" sent}) \]  \hspace{1cm} (5.73)

\[ = Pr(C_B \geq C_A/ \text{"B-A-B" sent}) \]  \hspace{1cm} (5.74)

Considering only the first ten dominant eigenvalues, signal modes and ISI modes, the Chernoff bound to \( p_e \) is:

\[ p_e = Pr(C_A \geq C_B/ \text{"A-B-A" sent}) \]

\[ \leq \psi_{(C_A/ \text{"A-B-A" sent})} (s) \psi_{(C_B/ \text{"A-B-A" sent})} (-s) \text{ for } s \geq 0 \]  \hspace{1cm} (5.75)

\[ \leq \left\{ \prod_{i=0}^{9} \frac{1}{1 + \lambda_i (1 - e^s)} e^{-\|m_i\|_F^2 (1 - e^s)} \right\} \left\{ \prod_{j=0}^{9} \frac{1}{1 + \lambda_j (1 - e^{-s})} e^{-\|m_j\|_F^2 (1 - e^{-s})} \right\} e^{\frac{s^2}{2} \lambda_{\text{max}}} \text{ for } 0 \leq s \leq \ln \left( \frac{1 + \lambda_{\text{max}}}{\lambda_{\text{max}}} \right) \]  \hspace{1cm} (5.76)

where \( \lambda_{\text{max}} \) is the largest of the first ten eigenvalues which in this case is \( \lambda_0 \). Our task is somehow simplified because the optimum threshold is known and there is only one variable \( s \) to optimize over.

Figure 5-40 shows plots of \( \frac{E}{N_0} \) versus \( \alpha \) when the optical noise is the most important source of uncertainty and completely dominates the electronic thermal noise.

Clearly, when the noise optical power is extremely large, the optimum structure for Mach-Zehnder filters used in FSK receivers is the one matched to the incident rectangular pulses (\( \frac{E}{N_0} = 1 \)). The interval of integration must be as narrow as possible (\( \alpha \approx 0 \)) to prevent the high-intensity optical noise from reaching the slicer. The smallest signal-to-noise ratio that yields a bit error rate of \( 10^{-9} \) is in the neighborhood of 47 amplified “signal photons” for every additive “noisy photon”. The rise in the signal-to-noise ratio for large values of \( \alpha \) is attributed mostly to ISI because it is more severe for smaller values of \( \frac{E}{N_0} \) where ISI energy is the strongest and the received
Figure 5-40: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($N_0 = \infty$, ISI is not ignored)

optical noise energy is the weakest.

Figures 5-41 through 5-52 illustrate plots of $\frac{E}{N_0}$ versus $\alpha$ for different values of $N_0$ and $i_{th}$. Both $E$ and $N_0$ are expressed in units of photon count.
Figure 5-41: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0 \frac{P_A}{H_2^{1/3}}, N_0 = 1$, ISI is not ignored)

Figure 5-42: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0 \frac{P_A}{H_2^{1/3}}, N_0 = 10$, ISI is not ignored)

Figure 5-43: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0 \frac{P_A}{H_2^{1/3}}, N_0 = 50$, ISI is not ignored)
CHAPTER 5. PERFORMANCE OF MACH-ZEHNDER FILTERS

Figure 5-44: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 1 \frac{PA}{H_2^{1/3}}$, $N_0 = 10$, ISI is not ignored)

Figure 5-45: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 1 \frac{PA}{H_2^{1/3}}$, $N_0 = 50$, ISI is not ignored)

Figure 5-46: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 1 \frac{PA}{H_2^{1/3}}$, $N_0 = 100$, ISI is not ignored)
CHAPTER 5. PERFORMANCE OF MACH-ZEHNDER FILTERS

Figure 5-47: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers $(i_{th} = 3 \frac{p_A}{H_{21/3}}, N_0 = 10$, ISI is not ignored)

Figure 5-48: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers $(i_{th} = 3 \frac{p_A}{H_{21/3}}, N_0 = 100$, ISI is not ignored)

Figure 5-49: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers $(i_{th} = 3 \frac{p_A}{H_{21/3}}, N_0 = 1000$, ISI is not ignored)
CHAPTER 5. PERFORMANCE OF MACH-ZEHNDER FILTERS

Figure 5-50: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 10 \frac{p_A}{H x^{1/2}}, N_0 = 10$, ISI is not ignored)

Figure 5-51: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 10 \frac{p_A}{H x^{1/2}}, N_0 = 100$, ISI is not ignored)

Figure 5-52: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 10 \frac{p_A}{H x^{1/2}}, N_0 = 1000$, ISI is not ignored)
5.7.2 A performance analysis ignoring ISI

Ignoring the effects of ISI on the receiver's performance (i.e. the ISI modes \( m_{ISI} \) are all set to 0), the Chernoff bound to \( p_e \) becomes:

\[
p_e = Pr(C_A \geq C_B/ \text{"A-B-A" sent})
\leq \frac{1}{\prod_{i=0}^{9} 1 + \lambda_i(1 - e^s)} \psi_{(C_A/ \text{"A-B-A" sent})}(s) \psi_{(C_B/ \text{"A-B-A" sent})}(-s) \text{ for } s \geq 0
\]

(5.77)

\[
\leq \frac{1}{\prod_{j=0}^{9} 1 + \lambda_j(1 - e^{-s})} e^{-\frac{\|m_j\|^2(1 - e^{-s})}{1 + \lambda_j(1 - e^{-s})}} e^{\frac{s^2}{2}} \text{ for } 0 \leq s \leq \ln(\frac{1 + \lambda_{max}}{\lambda_{max}})
\]

(5.78)

The signal-to-noise ratio necessary to achieve a bit error rate of \( 10^{-9} \) is computed by first optimizing over \( s \).

Figure 5-53: Plots of \( \frac{E}{N_0} \) that achieves a bit error rate of \( 10^{-9} \) for FSK receivers \( (N_0 = \infty, \text{ISI is ignored}) \)
CHAPTER 5. PERFORMANCE OF MACH-ZEHNDER FILTERS

Figure 5-53 shows plots of $\frac{E}{N_0}$ versus $\alpha$ when the optical noise completely dominates the electronic thermal noise.

When Figure 5-53 is compared with 5-40, the effects of the ISI energy on the FSK receiver's performance become apparent. As expected, the impact of ISI on the receiver's performance is stronger for matched or nearly matched filters ($\frac{T}{T_i} = 1.0$ and $\frac{T}{T_i} = 1.1$) when the interval of integration is wide.

Figures 5-54 through 5-57 illustrate plots of $\frac{E}{N_0}$ versus $\alpha$ for different values of $N_0$ and $i_{th}$ when ISI is ignored.
Figure 5-54: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0 \frac{p_A}{H^2 f_3}, N_0 = 1$, ISI is ignored)

Figure 5-55: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0 \frac{p_A}{H^2 f_3}, N_0 = 10$, ISI is ignored)
Figure 5-56: Plots of $E/N_0$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 10 \frac{P_A}{H_2^{1/2}}$, $N_0 = 10$, ISI is ignored)

Figure 5-57: Plots of $E/N_0$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 10 \frac{P_A}{H_2^{1/2}}$, $N_0 = 1000$, ISI is ignored)
CHAPTER 5. PERFORMANCE OF MACH-ZEHNDER FILTERS

A comparison between Figures 5-54 and 5-41 and between Figures 5-55 and 5-42 illustrates the impact of ISI on the receiver's performance. The effects of ISI are mostly felt for smaller values of $\frac{\tau}{\lambda}$; as $\frac{\tau}{\lambda}$ increases, the collected ISI energy decreases until it becomes insignificant for $\frac{\tau}{\lambda} = 1.6$ and $\frac{\tau}{\lambda} = 1.7$ at which point the curves that ignores ISI are almost identical to those that don’t.

When thermal noise dominates all other sources of uncertainty (Figures 5-56 and 5-50), there is no longer one particular Mach-Zehnder filter structure that outperforms the rest; the receiver’s performance is practically independent of $\frac{\tau}{\lambda}$. Notice how the collected ISI energy is dwarfed by the thermal noise intensity.

5.8 Accuracy of Chernoff bounds

The use of Chernoff bounds to approximate the bit error rate will lead to errors in the evaluation of the signal-to-noise ratio $\frac{P_r}{N_0}$. These errors can be estimated in the case of FSK receivers by ignoring the effects of thermal noise and ISI. Recall from Chapter 4 that the exact expression for the error probability can be derived for FSK receivers by ignoring thermal noise and ISI (equation 4.125). Keeping only the first ten dominant eigenvalues and signal modes, the exact expression for the bit error rate $p_e$ is:

$$p_e = \frac{1}{2} \sum_{i=0}^{9} \left( \prod_{j=0}^{9} \frac{\lambda_i}{\lambda_i - \lambda_j} \frac{1 + \lambda_i}{1 + \lambda_i + \lambda_j} \right) \exp\left( -\sum_{i=0}^{9} \frac{\|m_i\|^2}{1 + \lambda_i + \lambda_i} \right)$$ (5.79)

Figures 5-58 through 5-60 illustrate plots of the signal-to-noise ratio for FSK receivers when both thermal noise and ISI are ignored. The plots shown in solid lines were obtained by approximating $p_e$ with Chernoff bounds, the ones in broken lines were obtained by using the exact expression for $p_e$ (equation 5.79). Chernoff bounds are quite reliable for analyzing the effects of optical noise, thermal noise and ISI on the receiver’s performance.
Figure 5-58: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0 \frac{pA}{H_z^{1/2}}$, $N_0 = 1$, ISI is ignored)

Figure 5-59: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0 \frac{pA}{H_z^{1/2}}$, $N_0 = 10$, ISI is ignored)

Figure 5-60: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0 \frac{pA}{H_z^{1/2}}$, $N_0 = \infty$, ISI is ignored)
Chapter 6

Performance of Fabry-Perot interferometers

In this chapter, we study the performance of ASK and FSK direct detection receivers in recovering rectangular pulses corrupted by additive optical amplifier noise when symmetric Fabry-Perot filters are used as predetection filters. This analysis takes into account (1) the shot noise generated by the photodetectors, (2) the electronic thermal noise present in the post detection circuitry and (3) the interference between overlapping neighboring pulses (ISI). We shall solve the integral equation and derive the eigenfunctions, eigenvalues and modes in the Karhunen-Loeve expansions of the filtered signal and ISI. We shall then use Chernoff bounds to optimize on the signal-to-noise ratio $\frac{E}{N_0}$ that yields an error probability of $10^{-9}$.

6.1 Solving the integral equation

In this section, we solve the integral equation presented in Chapter 4 (equation 4.38) for symmetric Fabry-Perot predetection optical filters. We shall also derive the eigenfunctions $\{\phi_i(t)\}$, eigenvalues $\{\lambda_i\}$, signal modes $\{m_i\}$ and ISI modes $\{m_{ISi}\}$. 

160
6.1.1 The filtered incident signal pulse $m(t)$

As we have seen in Chapter 3, a symmetric Fabry-Perot filter having a time constant $T'$ given by equation 3.44 and operating under ideal conditions shapes an incident amplified rectangular pulse $s(t)$ with energy $E$ shown in Figure 6-1 into an exponentially decaying waveform $m(t)$ shown in Figure 6-2. The pulse amplitude $A$ in Figure 6-1 is equal to $\sqrt{\frac{2E}{T'}}$. The region of integration ends at $t = \frac{T}{2}$ to exclude the ISI energy coming from the subsequent pulse (if any) which is more severe than both the signal energy collected after time $t = \frac{T}{2}$ and the interference from previous signals, especially when the ratio $\frac{T}{T'}$ is quite large (larger than 5).
CHAPTER 6. PERFORMANCE OF FABRY-PEROT INTERFEROMETERS 162

Figure 6-3: Filtered waveform envelope corresponding to an "OFF" "ON" input

Figure 6-4: Filtered waveform envelope corresponding to an "ON" "OFF" input
CHAPTER 6. PERFORMANCE OF FABRY-PEROT INTERFEROMETERS 163

The complex envelope $m'(t)$ of the filtered pulse is given by:

$$m'(t) \approx \begin{cases} \sqrt{\frac{E}{T}}(1 - e^{-\frac{t^2}{2T^2}} e^{-\frac{t^2}{2}}) & \text{if } -\frac{T}{2} \leq t \leq \frac{T}{2} \\ \sqrt{\frac{E}{T}}(e^{\frac{t^2}{2T^2}} - 1)e^{-\frac{t^2}{2}} e^{-\frac{t^2}{2}} & \text{if } t \geq \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$  \hfill (6.1)

Such an exponentially decaying signal will, in principle, interfere with all subsequent transmitted pulses. To simplify our analysis, we will ignore any interference between non-adjacent pulses. Throughout this chapter, we will consider only the two situations shown in Figures 6-3 and 6-4. In this case the bit error rate can be bounded as follows:

$$p_e \leq \frac{1}{2}(Pr(\text{error/"OFF-ON" sent}) + Pr(\text{error/"ON-OFF" sent}))$$  \hfill (6.2)

6.1.2 The filtered optical noise correlation function

The autocorrelation function $R'(\tau)$ of the filtered optical noise $n(t)$ is given by:

$$R'(\tau) = h_r(\tau) * h^*_r(-\tau) * R(\tau)$$  \hfill (6.3)

$$= \frac{N_0}{2} h_r(\tau) * h^*_r(-\tau)$$  \hfill (6.4)

where $h_r(t)$ is the impulse response of the predetection Fabry-Perot optical filter and the smoothing wideband filter cascaded together.

The autocorrelation of the filter's impulse response, as derived in Chapter 3 (equation 3.48), converges toward:

$$h_r(\tau) * h^*_r(-\tau) = \frac{1}{T'} e^{-\frac{|\tau|}{T'}} \cos(2\pi \nu_0 \tau)$$  \hfill (6.5)
Therefore, $R'(\tau)$ is approximated by:

$$R'(\tau) = \frac{N_0}{2T'} e^{-\frac{|\nu|}{T'}} \cos(2\pi \nu_0 \tau) \tag{6.6}$$

The autocorrelation function envelope $R''(\tau)$ is given by:

$$R''(\tau) = \frac{N_0}{2T''} e^{-\frac{|\nu|}{T''}} \tag{6.7}$$

### 6.1.3 Derivation of the eigenfunctions and eigenvalues

Now that the autocorrelation function envelope $R''(\tau)$ is known, we can derive the eigenfunctions $\{\phi_i(t)\}$ and the eigenvalues $\{\lambda_i\}$ by solving the integral equation:

$$\int_{T_{\text{initial}}}^{T_{\text{final}}} R''(t - \tau) \phi_i(\tau) d\tau = \lambda_i \phi_i(t) \quad t \in [T_{\text{initial}}, T_{\text{final}}] \quad i = 0, 1, 2, \ldots \tag{6.8}$$

Solutions to this particular integral involving an exponentially decaying correlation function were derived in the past for $T_{\text{initial}} = -T_{\text{final}}$ [Van 68, pages 187–189]. Our solutions, as presented in this section, will differ somewhat because the interval of integration is not symmetric.

We propose to integrate the filtered pulse $m(t)$ in the region of saturation in order to recover the signal energy with as little ISI as possible (see Figure 6-2). The bounds of integration $T_{\text{initial}}$ and $T_{\text{final}}$ are defined as follows:

$$T_{\text{initial}} = (1 - 4\alpha) \frac{T}{2} \tag{6.9}$$

$$T_{\text{final}} = \frac{T}{2} \tag{6.10}$$

where the dimensionless variable $\alpha$ is the ratio of the integration range over twice the incident pulse width:
\[ \alpha = \frac{T_{\text{final}} - T_{\text{initial}}}{2T} \] (6.11)

The normalizing factor of 2 is needed to confine \( \alpha \) within the range \([0, 0.5]\) (the same range used for Mach-Zehnder filters), making it possible to directly compare the performances of Fabry-Perot and Mach-Zehnder filters.

Replacing \( T_{\text{initial}} \), \( T_{\text{final}} \) and \( R''(\tau) \) by their expressions, the integral equation becomes:

\[
\int_{(1-4\alpha)\frac{T}{2}}^{\frac{T}{2}} \frac{N_0}{2T'} e^{-\frac{|u'-\tau'|}{\tau'}} \phi_i(\tau) d\tau' = \lambda_i \phi_i(t) \quad t \in [(1 - 4\alpha) \frac{T}{2}, \frac{T}{2}] \quad i = 0, 1, 2, \ldots \quad (6.12)
\]

The algebra becomes less tedious when the interval of integration is symmetric. We thus shift time by \((1 - 2\alpha)\frac{T}{2}\) and make the following change of variables:

\[
t' = t - (1 - 2\alpha) \frac{T}{2} \quad (6.13)
\]

\[
\tau' = \tau - (1 - 2\alpha) \frac{T}{2} \quad (6.14)
\]

The integral equation expressed in terms of \( t' \) and \( \tau' \) is:

\[
\int_{-\alpha T}^{\alpha T} \frac{N_0}{2T'} e^{-i\frac{|t'-\tau'|}{\tau'}} \phi_i(\tau') d\tau' = \lambda_i \phi_i(t') \quad t' \in [-\alpha T, \alpha T] \quad i = 0, 1, 2, \ldots \quad (6.15)
\]

The shifted eigenfunctions \( \{\phi_i(t')\} \) must also form an orthonormal basis over the interval \([-\alpha T, \alpha T]\).
\[ \int_{-\alpha T}^{\alpha T} \phi_i(t') \phi_j^*(t') dt' = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \quad (6.16) \]

The solutions to the symmetric integral equation given in 6.15 are well known [Van 68, pages 187–189] and there is no need to rederive them.

The eigenfunctions \( \{ \phi_i(t') \} \) can be divided into a set of even functions:

\[ \phi_i(t') = A_i \cos\left( \frac{\beta_i t'}{T} \right) \quad t' \in [-\alpha T, \alpha T] \quad i = 0, 2, 4, \ldots \quad (6.17) \]

and a set of odd functions:

\[ \phi_i(t') = A_i \sin\left( \frac{\beta_i t'}{T} \right) \quad t' \in [-\alpha T, \alpha T] \quad i = 1, 3, 5, \ldots \quad (6.18) \]

For both even and odd families of functions, the relationship between \( \beta_i \) and the eigenvalue \( \lambda_i \) is:

\[ \lambda_i = \frac{N_0}{1 + \left( \frac{T}{T'} \right)^2 \beta_i^2} \quad i = 0, 1, 2, \ldots \quad (6.19) \]

The even-indexed members of the set \( \{ \beta_i \} \) satisfy the following transcendental equation:

\[ \tan(\alpha \beta_i) = \frac{T}{\beta_i} \quad i = 0, 2, 4, 6, \ldots \quad (6.20) \]

The odd-indexed members of the set \( \{ \beta_i \} \) are solutions to the transcendental equation:

\[ \tan(\alpha \beta_i) = -\frac{\beta_i}{\frac{T}{2}} \quad i = 1, 3, 5, 7, \ldots \quad (6.21) \]

The normalizing factor \( A_i \), which ensures that the norm of eigenfunction \( \phi_i(t') \) is equal to unity (i.e. \( \int_{-\alpha T}^{\alpha T} \phi_i(t') \phi_i^*(t') dt' = \int_{(1-4\alpha)^{\frac{T}{2}}}^{\frac{T}{2}} \phi_i(t) \phi_i^*(t) dt = 1 \), \( i = 0, 1, 2, \ldots \)), is given by:
CHAPTER 6. PERFORMANCE OF FABRY-PEROT INTERFEROMETERS 167

\[ A_i = \begin{cases} 
\frac{1}{\sqrt{\alpha T(1 + \frac{\sin(2\beta_i \alpha)}{2\beta_i \alpha})}} & \text{for } i = 0, 2, 4, \ldots \\
\frac{1}{\sqrt{\alpha T(1 - \frac{\sin(2\beta_i \alpha)}{2\beta_i \alpha})}} & \text{for } i = 1, 3, 5, \ldots 
\end{cases} \quad (6.22) \]

Rewriting the eigenfunctions in terms of time \( t \), we get:

\[ \phi_i(t) = \frac{1}{\sqrt{\alpha T(1 + \frac{\sin(2\beta_i \alpha)}{2\beta_i \alpha})}} \cos\left(\frac{\beta_i(t - (1 - 2\alpha)\frac{T}{2})}{T}\right) \quad t \in \left[ (1 - 4\alpha)\frac{T}{2}, \frac{T}{2} \right] \quad i = 0, 2, 4, \ldots \quad (6.23) \]

\[ \phi_i(t) = \frac{1}{\sqrt{\alpha T(1 - \frac{\sin(2\beta_i \alpha)}{2\beta_i \alpha})}} \sin\left(\frac{\beta_i(t - (1 - 2\alpha)\frac{T}{2})}{T}\right) \quad t \in \left[ (1 - 4\alpha)\frac{T}{2}, \frac{T}{2} \right] \quad i = 1, 3, 5, \ldots \quad (6.24) \]

6.1.4 Derivation of the signal modes

Figure 6-5 depicts the envelope of the incident filtered pulse \( m'(t) \).

For any nonnegative integer \( i \), the coefficient \( m_i \) is the projection of \( m'(t) \) on the eigenfunction \( \phi_i(t) \).

\[ m_i = \int_{(1 - 4\alpha)\frac{T}{2}}^{\frac{T}{2}} m'(t)\phi_i^*(t)dt \quad i = 0, 1, 2, \ldots \quad (6.25) \]

The expression for the even-indexed signal modes is:

\[ \|m_i\|^2 = \frac{8E}{\beta_i^2\{2\alpha(1 + (\frac{T}{T'})^2\beta_i^2) + 2T'\}} \left\{1 - \cos^2(\alpha\beta_i)e^{-(1 - 2\alpha)\frac{T}{T'}}\right\}^2 \quad \text{for } i = 0, 2, 4, \ldots \quad (6.26) \]
Chapter 6. Performance of Fabry-Perot Interferometers 168

![Figure 6-5: Filtered signal pulse envelope $m'(t)$](image)

The expression for the odd-indexed signal modes is:

$$
\|m_i\|^2 = \frac{8E}{\beta_i^2\{2\alpha(1 + (T')^2/\beta_i^2) + 2T'/T\}}\sin^4(\alpha\beta_i)e^{-2(1-2\alpha)T'} \quad i = 1, 3, 5, \ldots \quad (6.27)
$$

6.1.5 Derivation of the ISI modes

Figure 6-6 depicts the envelope of the incident ISI waveform $m'_{ISI}(t)$.

We will denote by $m_{ISI_i}$ the projection of $m'_{ISI}(t)$ on the eigenfunction $\phi_i(t)$.

$$
m_{ISI_i} = \int_{(1-4\alpha)T/2}^{T/2} m'_{ISI}(t)\phi_i^*(t)dt \quad i = 0, 1, 2, \ldots \quad (6.28)
$$

The expression for the even-indexed ISI modes is given by:

$$
\|m_{ISI_i}\|^2 = \frac{8E}{\beta_i^2\{2\alpha(1 + (T')^2/\beta_i^2) + 2T'/T\}}(e^{T'/T} - 1)^2 e^{-4(1-\alpha)T'}\cos^4(\alpha\beta_i) \\
i = 0, 2, 4, \ldots \quad (6.29)
$$
 CHAPTER 6. PERFORMANCE OF FABRY-PEROT INTERFEROMETERS 169

![Graph showing the ISI envelope](image)

Figure 6-6: ISI envelope $m_{ISI}^i(t)$

The expression for the odd-indexed ISI modes is:

$$
||m_{ISI}^i||^2 = \frac{8E}{\beta_i^2 \{2\alpha(1 + (\frac{T_i}{T})^2 \beta_i^2) + 2 \frac{T_i}{T}\}} (e^{\frac{T_i}{T}} - 1)^2 e^{-4(1-\alpha) \frac{T_i}{T}} \sin^4(\alpha \beta_i)
$$

$$
i = 1, 3, 5, \ldots \quad (6.30)
$$

## 6.2 The average noise energy

The average noise energy received in the interval $[(1 - 4\alpha) \frac{T}{2}, \frac{T}{2}]$ is given by:

$$
\text{Average noise energy received} = E[\int_{(1-4\alpha)\frac{T}{2}}^{\frac{T}{2}} ||n'(t)||^2 dt] \quad (6.31)
$$

$$
= \sum_{i=0}^{\infty} \lambda_i \quad (6.32)
$$

However:
CHAPTER 6. PERFORMANCE OF FABRY-PEROT INTERFEROMETERS 170

Figure 6-7: Average total noise energy $\sum_{i=0}^{\infty} \lambda_i$ (expressed in units of $N_0$)

Figure 6-8: Sum of the first 20 eigenvalues $\sum_{i=0}^{19} \lambda_i$ (expressed in units of $N_0$)
\[ E \left[ \int_{\frac{T}{2}}^{\frac{T}{4}} ||n'(t)||^2 dt \right] = \int_{\frac{T}{4}}^{\frac{T}{2}} R''(0) dt \]
\[ = \alpha \frac{T}{T'} N_0 \]

The expression for the average noise energy is thus:

\[
\text{Average noise energy received} = \sum_{i=0}^{\infty} \lambda_i \quad (6.33)
\]
\[= \alpha \frac{T}{T'} N_0 \quad (6.34)\]

Figure 6-7 shows plots of the average noise energy versus \( \alpha \) for selected values of the ratio \( \frac{T}{T'} \). As suggested by equation 6.34, the average noise energy increases linearly with both \( \alpha \) and \( \frac{T}{T'} \). The larger \( \frac{T}{T'} \) is, the wider the Fabry-Perot filter’s bandwidth becomes (the filter’s FWHM is inversely proportional to \( T' \)) resulting in more noise energy incident upon the photodetector. Also, as \( \alpha \) increases, the range of integration widens and more noise power is collected by the integrator.

### 6.3 The total signal energy

The total signal energy received in the interval \([[(1 - 4\alpha)\frac{T}{2}, \frac{T}{2}] \) is given by:

\[
\text{Total signal energy received} = \int_{\frac{T}{4}}^{\frac{T}{2}} ||m'(t)||^2 dt \quad (6.35)
\]
\[= \sum_{i=0}^{\infty} ||m_i||^2 \quad (6.36)\]

The collected signal energy is:
Figure 6-9: Total signal energy received $\sum_{i=0}^{\infty} \| m_i \|^2$ (expressed in units of $E$)

Figure 6-10: Sum of the first 20 signal modes $\sum_{i=0}^{19} \| m_i \|^2$ (expressed in units of $E$)
\[
\int_{(1-4\alpha)\frac{T}{2}}^{\frac{T}{2}} \|m'(t)\|^2 dt = (2\alpha + \frac{1}{2} \frac{T'}{T} (e^{-2(1-2\alpha)\frac{T'}{T}} - e^{-2\frac{T'}{T}})
\]
\[
- 2 \frac{T'}{T} (e^{-(1-2\alpha)\frac{T'}{T}} - e^{-\frac{T'}{T}}) \} E \quad (6.37)
\]

Figure 6-9 shows plots of the total signal energy collected \(\sum_{i=0}^{\infty} \|m_i\|^2\) versus \(\alpha\) for selected values of the ratio \(\frac{T'}{T}\). Clearly, the signal energy is closely dependent on the filter's structure (i.e. \(\frac{T'}{T}\)). As expected, the signal energy collected by the detector increases when either the filter's time constant \(T'\) is decreased or the range of integration is expanded. For large values of \(\frac{T'}{T}\), the collected signal energy is practically the same (compare the plots for \(\frac{T'}{T} = 12, 15, 18\)); by contrast, the average noise energy increases by a factor of \(\frac{3}{2}\) as \(\frac{T'}{T}\) is changed from 12 to 18.

Figure 6-10 shows plots of the sum of the first twenty signal modes \(\sum_{i=0}^{19} \|m_i\|^2\). The plots in Figures 6-9 and 6-10 are almost identical: the first twenty signal modes clearly dominate in the signal energy expression.

6.4 The total ISI energy

The total ISI energy received in the interval \([(1 - 4\alpha)\frac{T}{2}, \frac{T}{2}]\) is given by:

\[
\text{Total ISI energy received} = \int_{(1-4\alpha)\frac{T}{2}}^{\frac{T}{2}} \|m'_{ISI}(t)\|^2 dt \quad (6.38)
\]
\[
= \sum_{i=0}^{\infty} \|m_{ISI_i}\|^2 \quad (6.39)
\]

As opposed to the more complex Mach-Zehnder chain, the Fabry-Perot can not be designed to completely avoid interference between neighboring pulses: ISI is always present regardless how narrow the range of integration is.

The received ISI energy is given by:
Figure 6-11: Total ISI energy received $\sum_{i=0}^{\infty} \| m_{ISI_i} \|^2$ (expressed in units of $E$)

Figure 6-12: Sum of the first 20 ISI modes $\sum_{i=0}^{19} \| m_{ISI_i} \|^2$ (expressed in units of $E$)
\[
\int_{(1-4\alpha)^{\frac{T}{T'}}}^{T} \|m'_{ISI}(t)\|^2 dt = \frac{1}{2} \frac{T'}{T} \left( e^{\frac{T'}{T}} - 1 \right)^2 e^{-2(2-\alpha)\frac{T'}{T}} (e^{2\alpha\frac{T'}{T}} - e^{-2\alpha\frac{T'}{T}}) E \tag{6.40}
\]

Figure 6-11 shows plots of the total ISI energy collected \(\sum_{i=0}^{\infty} \|m_{ISI,i}\|^2\) versus \(\alpha\) for selected values of the ratio \(\frac{T}{T'}\). As expected, the ISI energy collected decreases as \(\frac{T}{T'}\) is increased: the shorter the time constant \(T'\) is, the faster the filtered signal decays resulting in little interference with neighboring pulses. Note how severe the ISI energy is for smaller values of \(\frac{T}{T'}\): when \(\frac{T}{T'} = 2\) and \(\alpha = 0.5\) for instance, the collected ISI energy is nearly half the size of the received signal energy.

Figure 6-12 shows plots of the sum of the first twenty ISI modes \(\sum_{i=0}^{19} \|m_{ISI,i}\|^2\). The plots in Figures 6-11 and 6-12 are undistinguishable: the first twenty ISI modes clearly dominate in the ISI energy expression.

### 6.5 The total noise-noise and signal-noise beat variances

Figure 6-13 depicts plots of the total noise-noise beat variance approximated as the sum the first 1000 noise-noise beat terms \(\sum_{i=0}^{999} \lambda_i^2\) which are undistinguishable from plots of the sum of the first twenty terms \(\sum_{i=0}^{19} \lambda_i^2\) shown in Figure 6-14.

Plots of the total signal-noise beat variance (approximated as the sum of the first 1000 signal-noise beat terms \(\sum_{i=0}^{999} 2\lambda_i \|m_i\|^2\)) shown in Figure 6-15 are also undistinguishable from plots of the sum of the first twenty terms \(\sum_{i=0}^{19} 2\lambda_i \|m_i\|^2\) (Figure 6-16).

When analyzing the Fabry-Perot filter’s performance in the remaining sections of this chapter, we will only consider the dominant terms in the Karhunen-Loeve expansion of the noise, signal and ISI. Our analysis will be therefore limited to the first twenty eigenvalues and the first twenty signal and ISI modes.
CHAPTER 6. PERFORMANCE OF FABRY-PEROT INTERFEROMETERS 176

Figure 6-13: Total noise-noise beat variance $\sum_{i=0}^{\infty} \lambda_i^2$ (expressed in units of $N_0 \times N_0$)

Figure 6-14: Sum of the first 20 noise-noise beat terms $\sum_{i=0}^{19} \lambda_i^2$ (expressed in units of $N_0 \times N_0$)
CHAPTER 6. PERFORMANCE OF FABRY-PEROT INTERFEROMETERS 177

Figure 6-15: Total signal-noise beat variance $\sum_{i=0}^{\infty} 2\lambda_i \|m_i\|^2$ (expressed in units of $E \times N_0$)

Figure 6-16: Sum of the first 20 signal-noise beat terms $\sum_{i=0}^{19} 2\lambda_i \|m_i\|^2$ (expressed in units of $E \times N_0$)
6.6 Performance of Fabry-Perot filters in ASK receivers

In this section, we shall determine the optimum signal-to-noise ratio \( \frac{E}{N_0} \) necessary to achieve a bit error rate of \( 10^{-9} \) for ASK receivers that use symmetric Fabry-Perot optical filters. Chernoff bounds will be used to approximate both the probability of a false alarm \( p_f \) and the probability of a miss \( p_m \). The signal-to-noise ratio \( \frac{E}{N_0} \) is computed by first including then ignoring the effects of ISI. Such an analysis would give us a better understanding of how ISI affects the performance.

6.6.1 A performance analysis including ISI

Assuming that only the first twenty dominant eigenvalues, signal modes and ISI modes are relevant for an accurate analysis, then using the results derived in Chapter 4, the Chernoff bounds to \( p_m \) and \( p_f \) are expressed as follows:

\[
p_m = Pr(C_{\text{count}} \leq \gamma/ "0-1" \text{ sent}) \leq \left( \prod_{i=0}^{19} \frac{1}{1 + \lambda_i(1 - e^{*1})} e^{-\frac{\|m_i\|^2(1-e^{*1})}{1+\lambda_i(1-e^{*1})}} \right) e^{\frac{s_1^2}{2} - s_1 \gamma} \text{ for } s_1 \leq 0 \tag{6.41}
\]

\[
p_f = Pr(C_{\text{count}} \geq \gamma/ "1-0" \text{ sent}) \leq \left( \prod_{i=0}^{19} \frac{1}{1 + \lambda_i(1 - e^{*2})} e^{-\frac{\|m_i\|^2(1-e^{*2})}{1+\lambda_i(1-e^{*2})}} \right) e^{\frac{s_2^2}{2} - s_2 \gamma} \tag{6.42}
\]

\[
\text{for } 0 \leq s_2 \leq \ln \left( \frac{1 + \lambda_{\text{max}}}{\lambda_{\text{max}}} \right) \tag{6.44}
\]

where \( \gamma \) is the threshold and \( \lambda_{\text{max}} \) is the largest of the first twenty eigenvalues which in this case is \( \lambda_0 \).
The variance $\sigma^2$ of the thermal noise sample is given by:

$$
\sigma^2 = \frac{i_{th}^2}{q^2} 2\alpha T
$$

(6.45)

The rate of transmission is assumed to be 1Gbit/s throughout this analysis.

$$
T = 10^{-9}\text{sec}
$$

(6.46)

Figure 6-17: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($N_0 = \infty$, ISI is not ignored)

For given values of $\alpha$, $\frac{E}{H}$, and $N_0$ we will compute the signal-to-noise ratio $\frac{E}{N_0}$ that achieves an error rate $p_e = \frac{1}{2}(p_m + p_f)$ of $10^{-9}$ by optimizing over the threshold $\gamma$ and the two variables $s_1$ and $s_2$. We will consider four different thermal noise levels:

$$
i_{th} \in \{0 \frac{pA}{\sqrt{Hz}}, 1 \frac{pA}{\sqrt{Hz}}, 3 \frac{pA}{\sqrt{Hz}}, 10 \frac{pA}{\sqrt{Hz}}\}
$$

(6.47)
CHAPTER 6. PERFORMANCE OF FABRY-PEROT INTERFEROMETERS 180

Figure 6-17 shows plots of $\frac{E}{N_0}$ versus $\alpha$ when the optical noise is the most important source of uncertainty and completely dominates the electronic thermal noise.

The plots in Figure 6-17 suggest that the optimum Fabry-Perot filter structure used in an ASK receiver exposed to a high-intensity optical noise field is one with a short time constant $T'_n$ (the optimum $\frac{T}{T'_n}$ is somewhere around 10). As opposed to Mach-Zehnder filters, the range of integration must be quite large ($\alpha \approx 0.43$) and the corresponding signal-to-noise ratio $\frac{E}{N_0}$ is a bit larger, somewhere in the neighborhood of 104 (i.e. 104 amplified “signal photons” are needed for every additive “noisy photon” in order to achieve a bit error rate of $10^{-9}$). The small rise in the curves at large values of $\alpha$ is more severe for smaller values of $\frac{T}{T'_n}$, and is therefore mostly attributed to ISI rather than optical noise (the collected ISI energy decreases as $\frac{T}{T'_n}$ increases, the opposite is true for the collected optical noise energy).

Figures 6-18 through 6-29 illustrate plots of $\frac{E}{N_0}$ versus $\alpha$ for different values of $N_0$ and $i_{th}$. Both $E$ and $N_0$ are expressed in units of photon count.
Figure 6-18: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 0 \frac{pA}{Hz^{1/2}}, N_0 = 1$, ISI is not ignored)

Figure 6-19: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 0 \frac{pA}{Hz^{1/2}}, N_0 = 10$, ISI is not ignored)

Figure 6-20: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 0 \frac{pA}{Hz^{1/2}}, N_0 = 50$, ISI is not ignored)
CHAPTER 6. PERFORMANCE OF FABRY-PEROT INTERFEROMETERS 182

Figure 6-21: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 1 \frac{pA}{H_{2}^{1/3}}, N_0 = 10$, ISI is not ignored)

Figure 6-22: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 1 \frac{pA}{H_{2}^{1/3}}, N_0 = 50$, ISI is not ignored)

Figure 6-23: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 1 \frac{pA}{H_{2}^{1/3}}, N_0 = 100$, ISI is not ignored)
Figure 6-24: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 3 \frac{pA}{H^2T^3}$, $N_0 = 10$, ISI is not ignored)

Figure 6-25: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 3 \frac{pA}{H^2T^3}$, $N_0 = 100$, ISI is not ignored)

Figure 6-26: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 3 \frac{pA}{H^2T^3}$, $N_0 = 1000$, ISI is not ignored)
CHAPTER 6. PERFORMANCE OF FABRY-PEROT INTERFEROMETERS 184

Figure 6-27: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 10 \frac{pA}{Hz^{1/3}}$, $N_0 = 10$, ISI is not ignored)

Figure 6-28: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 10 \frac{pA}{Hz^{1/3}}$, $N_0 = 100$, ISI is not ignored)

Figure 6-29: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 10 \frac{pA}{Hz^{1/3}}$, $N_0 = 1000$, ISI is not ignored)
6.6.2 A performance analysis ignoring ISI

Ignoring the effects of ISI on the receiver's performance (i.e. the ISI modes \( \{m_{ISI}\} \) are all set to 0), the Chernoff bounds to \( p_m \) and \( p_f \) become:

\[
\begin{align*}
    p_m &= Pr(C_{count} \leq \gamma / \text{"0-1" sent}) \\
    &\leq \left\{ \prod_{i=0}^{19} \frac{1}{1 + \lambda_i(1 - e^{s_1})} e^{-\frac{\lambda_i s_1}{1 + \lambda_i(1 - e^{s_1})}} \right\} e^{\frac{e^{2s_1^2 - s_1\gamma}}{2}} \text{ for } s_1 \leq 0 \\
    &\leq \left\{ \prod_{i=0}^{19} \frac{1}{1 + \lambda_i(1 - e^{s_2})} \right\} e^{\frac{e^{2s_2^2 - s_2\gamma}}{2}} \text{ for } 0 \leq s_2 \leq \ln\left(\frac{1 + \lambda_{max}}{\lambda_{max}}\right)
\end{align*}
\]

**Figure 6-30:** Plots of \( \frac{E}{N_0} \) that achieves a bit error rate of \( 10^{-3} \) for ASK receivers \( (N_0 = \infty, \text{ISI is ignored}) \)

By optimizing over the threshold \( \gamma \) and the two variables \( s_1 \) and \( s_2 \), we compute
the signal-to-noise ratio necessary to achieve a bit error rate of $10^{-9}$.

Figure 6-30 shows plots of $\frac{E}{N_0}$ versus $\alpha$ when the optical noise completely dominates the electronic thermal noise.

A comparison between Figures 6-30 and 6-17 reveals once again that the collected ISI energy contributes in a major way to the small rise in the optimum signal-to-noise ratio $\frac{E}{N_0}$ for large values of $\alpha$. The collected optical noise energy plays a marginal role in the degradation of the receiver's performance as the range of integration is expanded: all of the curves in Figure 6-30 decrease with $\alpha$.

Figures 6-31 through 6-34 illustrate plots of $\frac{E}{N_0}$ versus $\alpha$ for different values of $N_0$ and $i_{th}$ when ISI is ignored.
Figure 6-31: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 0 \frac{p_d}{H_{2.175}}$, $N_0 = 1$, ISI is ignored)

Figure 6-32: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 0 \frac{p_d}{H_{2.175}}$, $N_0 = 10$, ISI is ignored)
Figure 6-33: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 10 \frac{PA}{H^2T^2}$, $N_0 = 10$, ISI is ignored)

Figure 6-34: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for ASK receivers ($i_{th} = 10 \frac{PA}{H^2T^2}$, $N_0 = 1000$, ISI is ignored)
A comparison between Figures 6-31 and 6-18 and between Figures 6-32 and 6-19 illustrates the degradation in the receiver's performance due to ISI. As expected the impact of ISI is mostly felt for smaller values of $\frac{S}{N}$.

When thermal noise is the dominant source of uncertainty as it is the case in Figures 6-33 and 6-27, the impact of ISI on the receiver's performance diminishes. The individual plots in each of the two figures are closely packed together: the performance is equally poor for any value of $\frac{S}{N}$. In Figure 6-34, the optical noise is no longer dominated by thermal noise; the effects of the collected ISI energy on the receiver's performance become evident for large values of $\alpha$.

6.7 Performance of Fabry-Perot filters in FSK receivers

In this section, we determine the optimum signal-to-noise ratio $\frac{S}{N_0}$ necessary to achieve a bit error rate of $10^{-9}$ for FSK receivers that use Fabry-Perot optical filters. The error probability $p_e$ will be approximated with Chernoff bounds. The effects of ISI on the performance will be examined by first including then ignoring the received ISI energy in our analysis.

6.7.1 A performance analysis including ISI

The bit error rate $p_e$ is given by:

$$p_e = Pr(C_A \geq C_B/ \text{“A-B” sent})$$

$$= Pr(C_B \geq C_A/ \text{“B-A” sent})$$

Considering only the first twenty dominant eigenvalues, signal modes and ISI modes, the Chernoff bound to $p_e$ is:
\[ p_e = Pr(C_A \geq C_B/ \text{"A-B" sent}) \]
\[ \leq \psi(C_A/ \text{"A-B" sent})^s \psi(C_B/ \text{"A-B" sent})(-s) \text{ for } s \geq 0 \]
\[ \leq \left\{ \prod_{i=0}^{19} \frac{1}{1 + \lambda_i(1 - e^s)} e^{-\frac{||m_{ij}||^2(1-e^{-s})}{1+\lambda_i(1-e^{-s})}} \right\} \]
\[ \left\{ \prod_{j=0}^{19} \frac{1}{1 + \lambda_j(1 - e^{-s})} e^{-\frac{||m_{ij}||^2(1-e^{-s})}{1+\lambda_j(1-e^{-s})}} \right\} e^{\frac{s^2}{2}} \text{ for } 0 \leq s \leq \ln\left(\frac{1 + \lambda_{\text{max}}}{\lambda_{\text{max}}}\right) \]

where \( \lambda_{\text{max}} \) is the largest of the first twenty eigenvalues which in this case is \( \lambda_0 \).
The optimum threshold is known and there is only one variable \( s \) to optimize over.

Figure 6-35: Plots of \( \frac{E}{N_0} \) that achieves a bit error rate of \( 10^{-9} \) for FSK receivers \( (N_0 = \infty, \text{ISI is not ignored}) \)

Figure 6-35 shows plots of \( \frac{E}{N_0} \) versus \( \alpha \) when the optical noise is the most important source of uncertainty and completely dominates the electronic thermal noise.

When the noise optical power is extremely large, the optimum structure for Fabry-Perot filters used in FSK receivers is the one for which \( \frac{T}{T'} \) is in the neighborhood of 10.
CHAPTER 6. PERFORMANCE OF FABRY-PEROT INTERFEROMETERS 191

The interval of integration must be quite large (\(\alpha \approx 0.43\)) as opposed to Mach-Zehnder filters. The smallest signal-to-noise ratio that yields a bit error rate of \(10^{-9}\) is in the neighborhood of 57 amplified "signal photons" for every additive "noisy photon". The rise in the signal-to-noise ratio for large values of \(\alpha\) is attributed mostly to ISI because it is more severe for smaller values of \(\frac{T}{T_i}\), where ISI energy is the strongest and the optical noise energy collected is the weakest.

Figures 6-36 through 6-47 illustrate plots of \(\frac{E}{N_0}\) versus \(\alpha\) for different values of \(N_0\) and \(i_{th}\). Both \(E\) and \(N_0\) are expressed in units of photon count.
Figure 6-36: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0 \frac{PA}{Hz^2/T}$, $N_0 = 1$, ISI is not ignored)

Figure 6-37: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0 \frac{PA}{Hz^2/T}$, $N_0 = 10$, ISI is not ignored)

Figure 6-38: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0 \frac{PA}{Hz^2/T}$, $N_0 = 50$, ISI is not ignored)
CHAPTER 6. PERFORMANCE OF FABRY-PEROT INTERFEROMETERS 193

Figure 6-39: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 1 \times \frac{pA}{Hz^{1/2}}$, $N_0 = 10$, ISI is not ignored)

Figure 6-40: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 1 \times \frac{pA}{Hz^{1/2}}$, $N_0 = 50$, ISI is not ignored)

Figure 6-41: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 1 \times \frac{pA}{Hz^{1/2}}$, $N_0 = 100$, ISI is not ignored)
CHAPTER 6. PERFORMANCE OF FABRY-PEROT INTERFEROMETERS

Figure 6-42: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 3 \frac{pA}{H^2/\gamma}, N_0 = 10$, ISI is not ignored)

Figure 6-43: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 3 \frac{pA}{H^2/\gamma}, N_0 = 100$, ISI is not ignored)

Figure 6-44: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 3 \frac{pA}{H^2/\gamma}, N_0 = 1000$, ISI is not ignored)
Figure 6-45: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 10 \frac{P_A}{H^2/3}$, $N_0 = 10$, ISI is not ignored)

Figure 6-46: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 10 \frac{P_A}{H^2/3}$, $N_0 = 100$, ISI is not ignored)

Figure 6-47: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 10 \frac{P_A}{H^2/3}$, $N_0 = 1000$, ISI is not ignored)
6.7.2 A performance analysis ignoring ISI

Ignoring the effects of ISI on the receiver's performance (i.e. the ISI modes \( m_{ISI} \) are all set to 0), the Chernoff bound to \( p_e \) becomes:

\[
p_e = Pr(C_A \geq C_B/ \text{"A-B" sent})
\leq \psi_{\{C_A/ \text{"A-B" sent}\}}(s) \psi_{\{C_B/ \text{"A-B" sent}\}}(-s) \text{ for } s \geq 0
\]

(6.56)

\[
\leq \left\{ \prod_{i=0}^{19} \frac{1}{1 + \lambda_i(1 - e^s)} \right\}
\cdot \left\{ \prod_{j=0}^{19} \frac{1}{1 + \lambda_j(1 - e^{-s})} e^{-\|m_{ls}\|_2^2(1-e^{-s})} \right\} e^{e^{-2}s^2} \text{ for } 0 \leq s \leq \ln\left( \frac{1 + \lambda_{max}}{\lambda_{max}} \right)
\]

(6.57)

The signal-to-noise ratio necessary to achieve a bit error rate of \( 10^{-9} \) is computed by first optimizing over \( s \).

Figure 6-48: Plots of \( \frac{E}{N_0} \) that achieves a bit error rate of \( 10^{-9} \) for FSK receivers \((N_0 = \infty, \text{ISI is ignored})\)
CHAPTER 6. PERFORMANCE OF FABRY-PEROT INTERFEROMETERS

Figure 6-48 shows plots of $\frac{E}{N_0}$ versus $\alpha$ when the optical noise completely dominates the electronic thermal noise.

The effects of the ISI energy on the FSK receiver's performance become apparent when comparing Figures 6-48 and 6-35. As expected, the impact of ISI on the receiver's performance is stronger for smaller values of $\frac{E}{I_t}$.

Figures 6-49 through 6-52 illustrate plots of $\frac{E}{N_0}$ versus $\alpha$ for different values of $N_0$ and $i_{th}$ when ISI is ignored.
Figure 6-49: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0 \frac{p_A}{Hz^{1/2}}, N_0 = 1$, ISI is ignored)

Figure 6-50: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0 \frac{p_A}{Hz^{1/2}}, N_0 = 10$, ISI is ignored)
Figure 6.51: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 10 \frac{PA}{H_s^{1/8}}$, $N_0 = 10$, ISI is ignored)

Figure 6.52: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 10 \frac{PA}{H_s^{1/8}}$, $N_0 = 1000$, ISI is ignored)
CHAPTER 6. PERFORMANCE OF FABRY-PEROT INTERFEROMETERS 200

A comparison between Figures 6-49 and 6-36 and between Figures 6-50 and 6-37 reveals the impact of ISI on the receiver's performance. The effects of ISI are mostly felt for smaller values of $\frac{T}{T_i}$.

When thermal noise dominates all other sources of uncertainty (Figures 6-51 and 6-45), there is no longer one particular Fabry-Perot filter structure that outperforms the rest; the receiver's performance is practically independent of $\frac{T}{T_i}$.

6.8 Accuracy of Chernoff bounds

In this section, we evaluate the error in determining the signal-to-noise ratio $\frac{E}{N_0}$ due to the use of Chernoff bounds in approximating the bit error rate. Recall from Chapter 4 that the exact expression for the error probability can be derived for FSK receivers by ignoring thermal noise and ISI (equation 4.125). Keeping only the first twenty dominant eigenvalues and signal modes, the exact expression for the bit error rate $p_e$ is:

$$p_e = \frac{1}{2} \sum_{i=0}^{19} \left( \prod_{j=0, j \neq i}^{19} \frac{\lambda_i}{\lambda_i - \lambda_j} \frac{1 + \lambda_i}{1 + \lambda_i + \lambda_j} \right) \exp \left( -\sum_{l=0}^{19} \frac{|m_l|^2}{1 + \lambda_l + \lambda_l} \right) (6.58)$$

Figures 6-53 through 6-55 illustrate plots of the signal-to-noise ratio for FSK receivers when both thermal noise and ISI are ignored. The plots shown in solid lines were obtained by approximating $p_e$ with Chernoff bounds, the ones in broken lines were obtained by using the exact expression for $p_e$ (equation 6.58). The error made in approximating $\frac{E}{N_0}$ using Chernoff bounds techniques is on the order of 10-20%. However, the Chernoff bound gives a reliable comparison of the two systems considered (receivers with Mach-Zehnder and those with Fabry-Perot filters).
Figure 6-53: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0 \frac{P_A}{Hz^{1/2}}, N_0 = 1$, ISI is ignored)

Figure 6-54: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0 \frac{P_A}{Hz^{1/2}}, N_0 = 10$, ISI is ignored)

Figure 6-55: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0 \frac{P_A}{Hz^{1/2}}, N_0 = \infty$, ISI is ignored)
Chapter 7

Conclusions

7.1 Summary

We studied the performance of Mach-Zehnder and Fabry-Perot filters predetection optical filters when used in ASK and FSK receivers. Some of our results are summarized in Tables 7.1 through 7.4. These tables list the optimum values of $\frac{T}{F}$, $\alpha$ and $\frac{E}{N_0}$ for various thermal and optical noise intensities. When the optical noise power is weak or is completely dominated by the thermal noise intensity, the Fabry-Perot filter outperforms the Mach-Zehnder chain. If, however, the optical noise is strong enough to dominate all other sources of uncertainty, a Mach-Zehnder filter is recommended instead. In spite of the superior performance of Mach-Zehnder filters for larger values of the optical noise power, the Fabry-Perot filter remains an attractive candidate due to its simpler structure: its implementation is by far easier and more cost-effective.

The results listed in these four tables can be somehow misleading because our analysis was restricted to values of $\frac{T}{F}$ between 1.0 and 1.7 for Mach-Zehnder filters and between 6 and 18 for Fabry-Perot interferometers. Consider the situation where $N_0 = 10$ and $i_{th} = 10 \frac{4}{H_0^2/3}$ (Table 7.4) for instance: the optimum values for $\alpha$ and $\frac{E}{N_0}$ may very well change from those listed if the analysis is extended for values of $\frac{T}{F}$ larger than 1.7 for Mach-Zehnder filters or 18 for Fabry-Perot interferometers.
### Table 7.1: Optimum $T/T'$, $\alpha$ and $E/No$ for Mach-Zehnder and Fabry-Perot filters when $i_{th} = 0$ $\frac{P_A}{H^2/\gamma}$ (ISI is not ignored)

<table>
<thead>
<tr>
<th>No (photons)</th>
<th>T/T'</th>
<th>$\alpha$</th>
<th>E/No ASK</th>
<th>E/No FSK</th>
<th>T/T'</th>
<th>$\alpha$</th>
<th>E/No ASK</th>
<th>E/No FSK</th>
</tr>
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<td>1</td>
<td>1.7</td>
<td>0.37</td>
<td>190</td>
<td>100</td>
<td>12</td>
<td>0.45</td>
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<td>91</td>
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<tr>
<td>10</td>
<td>1.2</td>
<td>0.18</td>
<td>115</td>
<td>60</td>
<td>10</td>
<td>0.43</td>
<td>112</td>
<td>60</td>
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<tr>
<td>50</td>
<td>1.1</td>
<td>0.10</td>
<td>100</td>
<td>52</td>
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<td>0.43</td>
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<tr>
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<td>0.01</td>
<td>90</td>
<td>47</td>
<td>10</td>
<td>0.43</td>
<td>104</td>
<td>57</td>
</tr>
</tbody>
</table>

### Table 7.2: Optimum $T/T'$, $\alpha$ and $E/No$ for Mach-Zehnder and Fabry-Perot filters when $i_{th} = 1$ $\frac{P_A}{H^2/\gamma}$ (ISI is not ignored)

<table>
<thead>
<tr>
<th>No (photons)</th>
<th>T/T'</th>
<th>$\alpha$</th>
<th>E/No ASK</th>
<th>E/No FSK</th>
<th>T/T'</th>
<th>$\alpha$</th>
<th>E/No ASK</th>
<th>E/No FSK</th>
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<tr>
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<td>90</td>
<td>47</td>
<td>10</td>
<td>0.43</td>
<td>104</td>
<td>57</td>
</tr>
</tbody>
</table>
### Table 7.3: Optimum $\frac{T}{T'}$, $\alpha$ and $\frac{E}{N_0}$ for Mach-Zehnder and Fabry-Perot filters when $i_{th} = 3 \frac{P_A}{H^{2.7}}$ (ISI is not ignored)

<table>
<thead>
<tr>
<th>No (photons)</th>
<th>$T/T'$</th>
<th>$\alpha$</th>
<th>$E/No$</th>
<th>$T/T'$</th>
<th>$\alpha$</th>
<th>$E/No$</th>
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</thead>
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<td>0.10</td>
<td>99</td>
<td>51</td>
<td>10</td>
<td>0.43</td>
</tr>
<tr>
<td>$\infty$</td>
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<td>0.01</td>
<td>90</td>
<td>47</td>
<td>10</td>
<td>0.43</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>No (photons)</th>
<th>$T/T'$</th>
<th>$\alpha$</th>
<th>$E/No$</th>
<th>$T/T'$</th>
<th>$\alpha$</th>
<th>$E/No$</th>
</tr>
</thead>
<tbody>
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<td>FSK</td>
<td>ASK</td>
<td>FSK</td>
</tr>
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<td>1790</td>
<td>18</td>
<td>0.46</td>
</tr>
<tr>
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<td>432</td>
<td>219</td>
<td>18</td>
<td>0.46</td>
</tr>
<tr>
<td>1000</td>
<td>1.4</td>
<td>0.27</td>
<td>119</td>
<td>62</td>
<td>12</td>
<td>0.45</td>
</tr>
<tr>
<td>$\infty$</td>
<td>1.0</td>
<td>0.01</td>
<td>90</td>
<td>47</td>
<td>10</td>
<td>0.43</td>
</tr>
</tbody>
</table>

Table 7.4: Optimum $\frac{T}{T'}$, $\alpha$ and $\frac{E}{N_0}$ for Mach-Zehnder and Fabry-Perot filters when $i_{th} = 10 \frac{P_A}{H^{2.7}}$ (ISI is not ignored)
Figure 7-1: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0 \frac{pA}{Hz^{1/2}}, N_0 = 1$, ISI is ignored)

Figure 7-2: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0 \frac{pA}{Hz^{1/2}}, N_0 = 10$, ISI is ignored)

Figure 7-3: Plots of $\frac{E}{N_0}$ that achieves a bit error rate of $10^{-9}$ for FSK receivers ($i_{th} = 0 \frac{pA}{Hz^{1/2}}, N_0 = \infty$, ISI is ignored)
A comparison between the results of the exact analysis and the Chernoff bound approximation for FSK receivers with both ISI and thermal noise ignored indicates the Chernoff bound technique is quite reliable in analyzing and comparing the performances of the two filters. Figures 7-1 through 7-3 compare the performance of Mach-Zehnder and Fabry-Perot filters when used in FSK receivers with both thermal noise and ISI ignored. The plots of the signal-to-noise ratio were obtained by using the exact expression for the error probability as derived in Chapter 4 (equation 4.125). The optimum $\frac{E}{N_0}$ and $\alpha$ are about the same for Mach-Zehnder and Fabry-Perot filters in all three figures. Notice how little $\frac{E}{N_0}$ changes with $\alpha$ for Mach-Zehnder filters in Figure 7-3. These plots suggest that both filters have identical performances; this conclusion (as well as the assumption that ISI may be ignored) is of course incorrect because it contradicts the results obtained from the more accurate analysis that includes ISI (Tables 7.1 through 7.4).

The optimum structure for Mach-Zehnder filters used in either ASK or FSK receivers exposed to high intensity optical noise is the one matched to the incoming rectangular pulses. The range of integration must be as small as possible. If the optical noise power is weak or is completely dominated by electronic thermal noise, a matched Mach-Zehnder filter is no longer the optimum choice but rather a filter with a wider bandwidth that introduces less ISI should be used. The range of integration should be increased to collect more of the signal energy.

The picture changes somehow when predetection Fabry-Perot filters are used instead. Surprisingly enough, the optimum range of integration is quite large ($\alpha \approx 0.43$) and seems to be independent of both the optical noise and the thermal noise intensities. Post detection filtering improves the receiver's performance tremendously. When the optical noise power is very strong, the optimum Fabry-Perot structure is one with a short time constant $T'$ ($\frac{E}{N_0} \approx 10$). If the optical noise is weak or is dominated by thermal noise, an even shorter time constant $T'$ is recommended ($\frac{E}{N_0} \approx 18$) to reduce the harmful interference between filtered pulses.
CHAPTER 7. CONCLUSIONS

7.2 Directions for further research

One potential area of research is to examine the performance of wide bandwidth filters particularly when $\frac{\Delta f}{f_0}$ exceeds 1.7 for Mach-Zehnder or 18 for Fabry-Perot filters. Unfortunately, the analysis of Mach-Zehnder filters can only be performed over the range (refer to equation 5.14):

$$0 \leq \alpha \leq \frac{1}{1 + \frac{\Delta f}{f_0}}$$

(7.1)

The performance analysis of Mach-Zehnder filters is therefore limited by the size of their bandwidth which is proportional to $\frac{\Delta f}{f_0}$. As the Mach-Zehnder bandwidth increases, the range of $\alpha$ over which the performance analysis can be performed shrinks. Solving the integral equation over the interval:

$$\frac{1}{1 + \frac{\Delta f}{f_0}} \leq \alpha \leq 0.5$$

(7.2)

might give us a better insight on the Mach-Zehnder performance when both the bandwidth and the range of integration are large.

Consider the plots in Figure 5-22 for instance. If the curves for $\frac{\Delta f}{f_0} = 1.6$ and 1.7 were to be extended all the way up to $\alpha = 0.5$, they may very well drop below the optimum signal-to-noise ratio of 90 obtained for $\frac{\Delta f}{f_0} = 1$. Of course, this would require solving the integral equation over the interval $\frac{1}{1 + \frac{\Delta f}{f_0}} \leq \alpha \leq 0.5$.

As we mentioned in the previous section, Chernoff bounds are very useful in approximating and comparing the performance of both filters. More accurate results can be obtained by finding the S-transform $\varphi_{C_{count}}(s)$ of the cumulative distribution of $C_{count}$

$$F_{C_{count}}(\gamma) = Pr(C_{count} \leq \gamma)$$

(7.3)
CHAPTER 7. CONCLUSIONS

\[ = \int_{-\infty}^{\gamma} p_{C_{\text{count}}}(\theta) d\theta \]  

(7.4)

which is given by:

\[ \varphi_{C_{\text{count}}}(s) = \frac{\psi_{C_{\text{count}}}(s)}{s} \]  

(7.5)

where \( \psi_{C_{\text{count}}}(s) \) is the moment generating function of \( C_{\text{count}} \) as expressed in equation 4.103.

The cumulative distribution \( F_{C_{\text{count}}}(\gamma) \) can be evaluated at a particular threshold \( \gamma = \gamma_0 \) by inverse transforming \( \varphi_{C_{\text{count}}}(s) \) through integration in the complex s-plane.

\[ F_{C_{\text{count}}}(\gamma_0) = \int_{-\infty}^{\infty} \varphi_{C_{\text{count}}}(Re(s) + jIm(s)) e^{-(Re(s) + jIm(s))\gamma_0} dIm(s) \]

(7.6)

\[ = \int_{-\infty}^{\infty} \frac{\psi_{C_{\text{count}}}(Re(s) + jIm(s))}{Re(s) + jIm(s)} e^{-(Re(s) + jIm(s))\gamma_0} dIm(s) \]

(7.7)

The integration is carried out along a vertical axis intercepting the real axis \( Im(s) = 0 \) at some point \( Re(s) \) between 0 and \( \ln(\frac{1+\lambda_{\text{max}}}{\lambda_{\text{max}}} \) where \( \lambda_{\text{max}} \) is the largest eigenvalue. It is necessary to choose the value for \( Re(s) \) wisely in order to simplify the integration (the integral is evaluated through numerical procedures); a good choice for \( Re(s) \) would be the optimum value that minimizes the Chernoff bound:

\[ \psi_{C_{\text{count}}}(Re(s)) e^{-Re(s)\gamma_0} \]

(7.8)
Bibliography


