An Integral Turbulent Boundary-Layer Method
and the Residuary Resistance of Ships

by

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Ingénieur de l’Armement

Submitted to the Department of Ocean Engineering
in partial fulfillment of the requirements for the degree of
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Abstract

Few results have been derived describing the influence of turbulence and viscosity on the Wave-resistance acting on a ship. An integral turbulent boundary-layer method has been used here to provide estimations of the integral parameters which describe the development of a boundary-layer past a body, such as the momentum thickness, the shape parameter or the entrainment coefficient. The evolution of these parameters is described by a differential system, and some numerical integrations of this differential scheme were performed for a flat-plate, a two-dimensional Karman-Trefftz strut and for an axisymmetric ellipsoid. These informations were used afterwards to compute the Wave-resistance as well as all relevant forces acting on three different bodies: a vertical strut and two hulls. These computations were performed by combining the viscous information provided by the turbulent method with the results of the computer code SWAN which evaluates the potential wake flow past a body advancing at constant speed in calm water.

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Chapter 1

An integral turbulent boundary-layer method

1.1 Introduction

This method was derived in 1973 by Green, Weeks and Brooman in an R.A.E. publication (cf ref.[5]). Its purpose is to predict the development of turbulent boundary-layers and wakes past two-dimensional and axisymmetric bodies. It involves three differential equations: the momentum integral equation, the entrainment equation and an equation for the streamwise rate of change of the entrainment coefficient. This third equation was derived by Bradshaw, Ferris and Atwell in 1967 (cf ref.[2]) from the turbulent kinetic energy equation. The forward integration of these three simultaneous differential equations provides a prediction of three independent parameters: momentum thickness $\theta$, shape parameter $H$ and entrainment coefficient $C_E$ which describe completely the boundary-layer and constitute meaningful pieces of information for engineering purpose.

1.2 Main parameters

As opposed to conventionnal turbulent methods which evaluate the mean velocity field at any point, integral methods such as this one deal only with a few parameters
which are computed as functions of the streamwise curvilinear abscisse and are various integrals of the mean velocity along the transverse coordinate. Such a method is much more economic in terms of computing time and space requirements, and it provides useful informations which prove to be sufficient for many purposes (forces evaluation ...).

The main parameters are:

- displacement thickness

\[ \delta^* = \int_0^\infty \left( 1 - \frac{\rho U}{\rho_e U_e} \right) dy \]

- momentum thickness

\[ \theta = \int_0^\infty \frac{\rho U}{\rho_e U_e} \left( 1 - \frac{U}{U_e} \right) dy \]

- shape parameter

\[ H = \frac{\delta^*}{\theta} \]

- mass-flow shape parameter

\[ H_1 = \frac{1}{\theta} \int_0^{\delta^*} \frac{\rho U}{\rho_e U_e} dy = \frac{(\delta - \delta^*)}{\theta} \]

- skin-friction coefficient

\[ C_f = \frac{\tau_w}{\frac{1}{2}\rho_e U_e} \]

\( \tau_w \) is the shear stress on the body itself.

- entrainment coefficient

\[ C_E = \frac{1}{\tau \rho_e U_e} \frac{d}{dx} (r \int_0^\delta \rho U dy) \]

In all these equations, \( x \) is the curvilinear abscisse, \( y \) the transverse coordinate, \( r \) the body radius (set to unity in the 2-D flows).
Figure 1-1: Coordinate System

$U$ is the mean turbulent velocity, $U_e$ the mean turbulent velocity at the edge of the layer, and the boundary-layer thickness is defined by $\delta = y$ where $U/U_e \cong 0.995$.

In incompressible flow, the entrainment coefficient can be related to the distribution of normal velocity at the edge of the boundary-layer, known as the "breathing effect" (the boundary-layer tends to push the flow away from the body, it "breathes"): \[ v(\delta) = U_e C_E \]

This relation is obtained by integrating the continuity equation along the transverse coordinate.

Of these coefficients, three are independent and can be chosen as unknowns of a system of three simultaneous differential equations.

1.3 The differential equation system

The boundary-layer can be described by the prediction of only three independent parameters: momentum thickness $\theta$, shape parameter $H$ and entrainment coefficient $C_E$. The evolution of these parameters along the streamwise curvilinear absicisse is governed by three differential equations.
The first equation is the momentum integral equation, usually known as the Karman relation, which is derived from the Navier-Stokes equation by integration along the transverse coordinate:

$$\frac{d}{dx}(r\theta) = \frac{rC_f}{2} - (H + 2)\frac{\theta}{U_e} \frac{dU_e}{dx}$$  \hspace{1cm} (1.1)

The second equation is the entrainment equation which comes directly from the definition of the entrainment coefficient

$$C_E = \frac{1}{rU_e} \frac{d}{dx}(rU_e H_1 \theta)$$

which, combined with equation (1.1) gives:

$$\theta \frac{dH}{dx} = \frac{dH}{dH_1} \left\{ C_E - H_1(C_f/2 - (H + 1)\frac{\theta}{U_e} \frac{dU_e}{dx}) \right\}$$  \hspace{1cm} (1.2)

The third one is a rate equation for the entrainment coefficient, known as the "lag equation". It comes from the turbulent kinetic equation which for 2-D flows is:

$$\frac{1}{\rho} \left( \frac{\partial q^2}{\partial x} + V \frac{\partial q^2}{\partial y} \right) - \frac{\partial}{\partial y} \left( \frac{\partial U}{\partial y} \right) - \frac{\partial}{\partial y} \left( \rho \frac{\partial (\rho v)}{\partial y} \right) + \frac{1}{2} \frac{\partial q^2}{\partial y} + \frac{1}{2} \frac{\partial q^2}{\partial y} + \frac{\partial}{\partial x} \left( \frac{\partial \varepsilon}{\partial x} \right) = 0$$  \hspace{1cm} (1.3)

where

$$q^2 = u^2 + v^2$$  

$$\tau = -\rho \bar{u} \bar{v}$$  

$$\varepsilon \approx \nu \left( \frac{\partial u_i}{\partial x_j} \right)^2$$

From that, Bradshaw and al. define the following parameters:

$$a_1 = \frac{\tau}{\rho q^2}$$  

$$L = (\tau/\rho)^{3/2} / \varepsilon$$
\[ G = \left( \frac{\rho \bar{u}}{\rho} + \frac{1}{2} \frac{\rho^2 v^2}{\rho} \right) / \left( \frac{\tau_{\text{max}}}{\rho} \right)^{1/2} \]

They postulate then that \( a_1 \) is a constant and that \( L \) and \( G / (\tau_{\text{max}} / \rho U_e^2)^{1/2} \) are functions of \( (y/\delta) \) only:

\[ \frac{G}{(\tau_{\text{max}} / \rho U_e^2)^{1/2}} = \zeta = f\left( \frac{y}{\delta} \right) \]

Equation (1.3) reduces then to an ordinary differential equation for maximum shear stress:

\[ \frac{U}{2a_1} \frac{d}{dx} \left( \frac{\tau}{\rho} \right)_{\text{max}} - \left( \frac{\tau}{\rho} \right)_{\text{max}} \frac{\partial U}{\partial y} + \frac{1}{\delta U_e} \left( \frac{\tau}{\rho} \right)_{\text{max}}^2 \lambda' + \frac{(\tau / \rho)_{\text{max}}^{3/2}}{L} = 0 \quad (1.4) \]

Writing \( \tau_{\text{max}} / \rho U_e^2 = C_{\text{rm}} \) and using the "equilibrium flows" for which \( \frac{dH}{dx} \) and \( \frac{dC_{\text{rm}}}{dx} \) are both zero, this becomes:

\[ \frac{\delta}{C_{\text{rm}}} \frac{dC_{\text{rm}}}{dx} = 2a_1 \frac{U_e}{U} \frac{\delta}{L} \left( C_{\text{rm,eq}}^{1/2} - C_{\text{rm}}^{1/2} \right) + \left( \frac{2 \delta}{U_e} \frac{dU_e}{dx} \right)_{\text{eq}} - \frac{2 \delta}{U_e} \frac{dU_e}{dx} \quad (1.5) \]

where the subscript \( \text{EQ} \) means equilibrium flows quantities.

The reason for using "equilibrium flows" is the fact that \( G \) is known empirically for this particular flows, and this allows to determine \( C_{\text{EQ}} \) and \( (\frac{\delta}{U_e} \frac{dU_e}{dx})_{\text{EQ}} \) as functions of the unknown parameters \( H, H_1 \) and \( C_f \). Using then an analytic approximation derived from experimental results which relates \( C_{\text{rm}} \) to \( C_{E} \),

\[ C_{\tau} = 0.024C_{E} + 1.2C_{E}^2 + 0.32C_{f0} \]

the "lag-equation" is obtained:

\[ \theta \frac{dC_{E}}{dx} = F \left( \frac{2.8}{H + H_1} \right) \left\{ \left( (C_{\tau})^{1/2}_{\text{EQ}} - \lambda C_{\tau}^{1/2} \right)_{\text{EQ}} + \left( \frac{\theta}{U_e} \frac{dU_e}{dx} \right)_{\text{EQ}} - \frac{\theta}{U_e} \frac{dU_e}{dx} \right\} \quad (1.6) \]

The various dependent variables and functions in equations (1.1),(1.2) and (1.6) are evaluated as follows:

- For \( U_e \) and \( \frac{dU_e}{dx} \): this two terms are respectively the mean turbulent velocity and
its z-derivative both evaluated at the edge of the boundary-layer. Before trying to solve the system, one must therefore be able to obtain an estimate of this two terms at any point, which can be done using as a first approximation the potential flow solution of the flow past the body considered.

- For $C_f$:

Local Reynolds number $R_\theta$

$$R_\theta = \frac{\theta_e U_e \theta}{\mu_e} \quad (1.7)$$

Skin-friction coefficient for the flat-plate $C_{f0}$

$$C_{f0} = \frac{0.01013}{\log_{10}(R_\theta) - 1.02} - 0.00075 \quad (1.8)$$

Shape-parameter for the flat-plate $H_0$

$$1 - \frac{1}{H_0} = 6.55(C_{f0}/2)^{1/2} \quad (1.9)$$

Skin-friction coefficient $C_f$

$$C_f = C_{f0}(0.9(\frac{H}{H_0} - 0.4)^{-1} - 0.5) \quad (1.10)$$

- For $H_1$ and $\frac{dH}{dH_1}$:

$$H_1 = 3.15 + \frac{1.72}{H - 1} - 0.01(H - 1)^2 \quad (1.11)$$

$$\frac{dH}{dH_1} = \frac{(H - 1)^2}{1.72 + 0.02(H - 1)^3} \quad (1.12)$$

- For $C_r$ and $F$:

$$C_r = 0.024C_E + 1.2C_E^2 + 0.32C_{f0} \quad (1.13)$$

$$F = \frac{0.02C_E + C_E^2 + 0.8C_{f0}/3}{0.01 + C_E} \quad (1.14)$$
• For secondary influences (coefficient $\lambda$):

- longitudinal streamline curvature:

$$\lambda_1 = 1 + \beta Ri$$  \hspace{1cm} (1.15)

with

$$Ri = \frac{2}{3} \frac{\theta}{R} (H + H_1)(\frac{H_1}{H} + 0.3)$$

where $R$ is the radius of longitudinal curvature, and $\beta = 7$ for $R \geq 0$.

- Flow convergence or divergence:

$$\lambda_2 = 1 - \frac{7}{3} \left( \frac{H_1}{H} + 0.3 \right) (H + H_1) \frac{\theta}{r} \frac{dr}{d\ell}$$  \hspace{1cm} (1.16)

Finally,

$$\lambda = \lambda_1 \lambda_2$$

and since these influences are justified only when they are small to moderate, some arbitrary limits have been imposed:

$$0.4 \leq \lambda \leq 2.5$$

• For equilibrium quantities:

$$\left( \frac{\theta}{U_e} \frac{dU_e}{dx} \right)_{EQ_0} = \frac{1.25}{H} \left\{ \frac{C_f}{2} - \frac{(H - 1)^2}{(6.432H)^2} \right\}$$  \hspace{1cm} (1.17)

$$(C_E)_{EQ_0} = H_1 \left\{ \frac{C_f}{2} - (H + 1) \left( \frac{\theta}{U_e} \frac{dU_e}{dx} \right)_{EQ_0} \right\}$$  \hspace{1cm} (1.18)

$$(C_r)_{EQ_0} = 0.024 (C_E)_{EQ_0} + 1.2 (C_E)_{EQ_0}^2 + 0.32 C_{f0}$$  \hspace{1cm} (1.19)

$$C = (C_r)_{EQ_0} \lambda^{-2} - 0.32 C_{f0}$$  \hspace{1cm} (1.20)
\[(C_E)_{EQ} = (C/1.2 + 0.0001)^{1/2} - 0.01 \quad (1.21)\]
\[\left(\frac{\theta \frac{dU_e}{dx}}{U_e} \right)_{EQ} = \left(\frac{C_f}{2} - \frac{(C_E)_{EQ}}{H_1}\right)(H + 1) \quad (1.22)\]

The differential equation system made of the three equations (1.1), (1.2) and (1.6) is non-linear with respect to its unknowns \(\theta\), \(H\) and \(C_E\). Therefore, it can only be solved by numerical integration using a finite-differences scheme such as a Runge-Kutta scheme. All the informations contained in equations (1.7) to (1.22) can be arranged as subroutines of a computer program so that each quantity might be evaluated at each step of the forward numerical integration.

### 1.4 Wake flows

The entrainment method derived for the flow along a 2-D body was extended by Green and al. to the treatment of the 2-D wake past the trailing edge of this body. The corresponding method is obtained by by-passing the equations (1.7) to (1.10) of the original model and setting \(C_f = C_{f0} = 0\) and \(\lambda = 0.5\) in all the other equations.

The new differential system is then:

\[\frac{d\psi}{dx} = -(H + 2)\frac{\theta}{U_e} \frac{dU_e}{dx} \quad (1.23)\]

\[\theta \frac{dH}{dx} = \frac{dH}{dH_1} (C_E - H_1 (H + 1) \frac{\theta}{U_e} \frac{dU_e}{dx}) \quad (1.24)\]

\[\theta \frac{dC_E}{dx} = F \left(\frac{2.8}{H + H_1}\right) \{((C_r)_{EQ0} - \lambda C_r^{1/2}) + \left(\frac{\theta}{U_e} \frac{dU_e}{dx}\right)_{EQ} - \frac{\theta}{U_e} \frac{dU_e}{dx}\} \quad (1.25)\]

This new system can be solved the same way as the first one, using as starting values the predicted values obtained by running the original scheme unto the trailing edge of the body.
1.5 An assessment of the method: comparison with the 1/7-power method for the flat-plate

1.5.1 The 1/7-power approximation

This method consists on assuming that the velocity distribution within the boundary-layer

\[ \frac{U}{U_e} = f\left(\frac{y}{\delta}\right) \]

is in fact a power-relation:

\[ \frac{U}{U_e} = (\frac{y}{\delta})^{1/7} \]

This approximation has less empirical support than the logarithmic profile but it is useful as a basis for qualitative comparisons. By integrating the Karman relation, one can compute the \( x \)-dependence of the main integral parameters:

\[ \delta = 0.373xR_x^{-1/5} \]

\[ \theta = 0.0363xR_x^{-1/5} \]

\[ C_f = 0.0592R_x^{-1/6} \]

\[ \delta^* = 0.0467xR_x^{-1/5} \]

where \( R_x \) is the local Reynolds number:

\[ R_x = \frac{\rho_e U_e x}{\mu_e} \]

1.5.2 The Green and al. method for the flat-plate

In the case of the flat-plate, the differential equation system made of equations (1.1), (1.2) and (1.6) becomes:

\[ \frac{d\theta}{dx} = \frac{C_{f0}}{2} \]  

(1.26)
\[
\theta \frac{dH}{dz} = \frac{dH}{dh_1}(C_E - H_1 \frac{C_{f_0}}{2}) \\
\theta \frac{dC_E}{dz} = F\left(\frac{2.8}{H + H_1}\right)\left((C_r)^{1/2} - C_r^{1/2}\right) + \left(\frac{\theta}{U_e} \frac{dU_e}{dz}\right)_{EQ}
\]

(1.27)

(1.28)

In these equations, r has been set to unity (and therefore \(\frac{d\phi}{dz} = 0\)), the potential flow solution is \(U_e = U_\infty\) everywhere in the fluid and the pressure gradient \(\frac{dU_e}{dz}\) is also equal to zero, and the extraneous influence coefficient \(\lambda\) is equal to 1 since there is no longitudinal curvature nor divergence of the flow.

These three equations were displayed into a computer code using a Runge-Kutta finite-differences scheme with adaptative stepsize, and a numerical integration was conducted.

1.5.3 Results

Comparisons were made for a Reynolds number of

\[
R = \frac{\rho_e U_e}{\mu_e} = 10^7
\]

The initial values for the finite-differences code were taken from the 1/7-power approximation: for \(x = 0.001m\)

\[
\theta = 5.864 \times 10^{-6}m
\]

\[
H = 1.285
\]

\[
C_E = 5.27 \times 10^{-2}
\]

As functions of \(x\), the 1/7-power approximation gives the following expressions for the main integral parameters:

\[
C_f = 2.3568 \times 10^{-3}x^{-0.2}
\]

\[
\delta^* = 1.891 \times 10^{-3}x^{0.8}
\]
\[ \theta = 1.473 \times 10^{-3} x^{0.8} \]

\[ \delta = 1.5128 \times 10^{-3} x^{0.8} \]

\[ C_E = \frac{d}{dx} (\delta - \delta^*) = 1.3237 \times 10^{-2} x^{-0.2} \]

This curves were plotted together with the results of the numerical integration, and the following figures were obtained.

Figure 1-2: Skin-friction coefficient
Figure 1-3: Displacement thickness

Figure 1-4: Momentum thickness
Figure 1-5: Entrainment coefficient

All the curves show a good accordance between the two methods, both in amplitude and in behaviour of these parameters along the $z$-direction. Because of the shortcomings of the $1/7$-power approximation, one cannot expect a perfect accordance, but this results constitute a good qualitative assessment of this integral turbulent Boundary-layer method.
Chapter 2

Two-dimensional flow:
Karman-Trefftz strut

2.1 Introduction

A first application of the integral boundary-layer method derived by Green and al. consists on two-dimensional flows. The body considered in our case is a particular strut, whose profile is made of two symmetrical circular arcs, and is in fact the image of a circle by a Karman-Trefftz conformal mapping. Such a profile was chosen because the corresponding potential flow solution is known analytically everywhere in the fluid, and this allows to perform the numerical integration of the differential without the cost of a panel method used to evaluate the potential flow.

These computations were performed in order to provide reliable turbulent information (mainly the thickness of the layer and the breathing effect) which was used afterwards to improve the evaluation of the wave-pattern past a vertical strut piercing an undisturbed free-surface and also to ameliorate the estimation of the residual drag on this body.
2.2 Geometry of the body

The body considered is a vertical symmetric strut whose transverse-cut profile is made of two circular arcs. Using complex variables in the transverse plane, this profile is the image of a circle of radius $R$ by the Karman-Trefftz transform:

$$Y(Z) = \frac{\lambda R\{Z + R)^\lambda + (Z - R)^\lambda\}}{(Z + R)^\lambda - (Z - R)^\lambda}$$

where $\lambda = 2 - \tau/\pi$ and $\tau$ is the angle shown on the figure:

![Diagram showing the Karman-Trefftz transform](image)

Figure 2-1: Karman-Trefftz transform

For the numerical integration, the strut chosen has a length $A = 1.0$ and a beam $B = 0.1$. The coefficients $\lambda$ and $R$ appearing in the Karman-Trefftz transform can be expressed as follows in terms of these parameters:

$$\lambda = \frac{4}{\pi} \tan^{-1}\left(\frac{A}{B}\right) \quad R = \frac{A}{2\lambda}$$
The equation of the superior circular arc is:

\[ X^3 + (Y - b)^2 = \varphi^2 \]

where

\[ b = -\frac{\lambda R}{\tan(\tau/2)} \]

is the ordinate of the center of the circle and

\[ \varphi = \frac{\lambda R}{\sin(\tau/2)} \]

is the corresponding radius.

![Figure 2-2: Geometric profile](image)

The curvilinear abscisse along the body \( s \), which is the variable of integration for
the differential scheme, can be expressed in terms of \( x \):

\[
s = \frac{\varrho}{\sqrt{\varrho^2 - x^2}} \left\{ \sin^{-1} \left( \frac{x}{\varrho} \right) + \sin^{-1} \left( \frac{\lambda R}{\varrho} \right) \right\}
\]

and its \( x \)-derivative is:

\[
\frac{ds}{dx} = \frac{\varrho}{\sqrt{\varrho^2 - x^2}}
\]

### 2.3 The differential system

Setting \( r \) to unity and therefore \( \frac{dr}{dz} \) to zero in the equations (1.1), (1.2) and (1.6), the following system is obtained for the evolution of the three integral parameters along the body (\( s \) has been substituted to \( x \) to avoid confusions):

\[
\frac{d\theta}{ds} = \frac{C_f}{2} - (H + 2) \frac{\theta}{U_e} \frac{dU_e}{ds}
\]

(2.1)

\[
\theta \frac{dH}{ds} = \frac{dH}{dH_1} \left\{ C_E - H_1 \left( \frac{C_f}{2} - (H + 1) \frac{\theta}{U_e} \frac{dU_e}{ds} \right) \right\}
\]

(2.2)

\[
\theta \frac{dC_E}{ds} = F(\frac{2.8}{H + H_1}) \left\{ \left( (C_r)^{1/2} - \lambda C_r^{1/2} \right) + \left( \frac{\theta}{U_e} \frac{dU_e}{ds} \right) - \frac{\theta}{U_e} \frac{dU_e}{ds} \right\}
\]

(2.3)

And in the wake:

\[
\frac{d\theta}{ds} = -(H + 2) \frac{\theta}{U_e} \frac{dU_e}{ds}
\]

(2.4)

\[
\theta \frac{dH}{ds} = \frac{dH}{dH_1} \left\{ C_E + H_1 (H + 1) \frac{\theta}{U_e} \frac{dU_e}{ds} \right\}
\]

(2.5)

\[
\theta \frac{dC_E}{ds} = F(\frac{2.8}{H + H_1}) \left\{ \left( (C_r)^{1/2} - \lambda C_r^{1/2} \right) + \left( \frac{\theta}{U_e} \frac{dU_e}{ds} \right) - \frac{\theta}{U_e} \frac{dU_e}{ds} \right\}
\]

(2.6)

But, since \( s \) is a function of \( x \) only the \( s \)-derivatives can be expressed as

\[
\frac{d}{dx} = \frac{ds}{dx} \frac{d}{ds}
\]

where

\[
\frac{ds}{dx} = \frac{\varrho}{\sqrt{\varrho^2 - x^2}} \quad \text{on the body and}
\]

\[
\frac{ds}{dx} = 1.0 \quad \text{in the wake}
\]
and the numerical integration can therefore be conducted with respect to \( z \) by simply multiplying the right-hand-side of the equations by \( \frac{dz}{dx} \).

The numerical integration of this differential system requires also the knowledge of the term

\[
\frac{1}{U_e} \frac{dU_e}{ds}
\]

which should be theoretically evaluated at the edge of the layer (unknown at the beginning). A good estimate of this term can be obtained using the potential flow solution of the flow past this body. Because of the particular shape of the strut considered, this quantity is in fact known analytically everywhere in the fluid, and can therefore be evaluated wherever it is required.

### 2.4 The potential flow solution

![Z-plane](image)

**Figure 2-3: The Z-plane**

In the Z-plane, where the profile is circular of radius \( R \), the complex potential \( F(Z) \) corresponding to an uniform flow \( (u = U, v = 0 \) at infinity) past this body has the well-known expression:

\[
F(Z) = U(Z + \frac{R^2}{Z})
\]
In the $Y$-plane, by virtue of the properties of conformal mapping, this expression remains the same:

$$F(Z(Y)) = U(Z(Y) + \frac{R^2}{Z(Y)})$$

but $Z$ must then be expressed in terms of $Y$ using the inverse Karman-Trefftz transform:

$$Z(Y) = R\frac{1 + C(Y)}{1 - C(Y)}$$

for which

$$C(Y) = \left(\frac{Y - \lambda R}{Y + \lambda R}\right)^{1/\lambda}$$

The term we need to evaluate is

$$U_e = \sqrt{\Phi_\chi^2 + \Phi_\psi^2}$$

and its $s$-derivative

$$\frac{dU_e}{ds} = \frac{\partial U_e}{\partial x} \frac{dx}{ds} + \frac{\partial U_e}{\partial y} \frac{dy}{ds}$$

for which

$$\frac{\partial U_e}{\partial x} = \frac{\Phi_\chi \Phi_{\chi\chi} + \Phi_\psi \Phi_{\chi\psi}}{U_e}$$
\[
\frac{\partial U_e}{\partial y} = \frac{\Phi_X \Phi_{XY} + \Phi_Y \Phi_{YY}}{U_e}
\]

We need therefore to determine \(\Phi_X, \Phi_Y\) and their derivatives \(\Phi_{XX}, \Phi_{XY}\) and \(\Phi_{YY}\) which can be expressed in terms of the derivatives of the complex potential \(F(Y)\) as follows:

\[
\Phi_X = RE\left(\frac{dF}{dY}\right)
\]

\[
\Phi_Y = -IM\left(\frac{dF}{dY}\right)
\]

\[
\Phi_{XX} = RE\left(\frac{d^2F}{dY^2}\right) = -\Phi_{YY}
\]

\[
\Phi_{XY} = -IM\left(\frac{d^2F}{dY^2}\right)
\]

Finally, expressing \(\frac{dF}{dY}\) and \(\frac{d^2F}{dY^2}\) in terms of \(Y\), we have all the information we need to perform the numerical integration:

\[
\frac{dF}{dY} = \frac{16R^2C^2U}{(Y^2 - \lambda^2 R^2)(1 - C^2)^2}
\]

\[
\frac{d^2F}{dY^2} = \frac{32R^2C^2U}{(Y^2 - \lambda^2 R^2)^2(1 - C^2)^2}\left\{ -Y + \frac{2R(1 + C^2)}{1 - C^2} \right\}
\]

where

\[
C = \left(\frac{Y - \lambda R}{Y + \lambda R}\right)^{1/\lambda}
\]

### 2.5 The iteration process

#### 2.5.1 General description

The whole point of this process is to evaluate correctly the term \(\frac{1}{U_e} \frac{\partial U_e}{\partial y}\) using potential flow solutions as approximations and thus converge to a stable solution. \(U_e\) is the actual turbulent mean velocity evaluated at the edge of the boundary-layer, but since the thickness of the layer is initially unknown, this term cannot be estimated exactly for the first integration. However, an approximation of \(U_e\) can be obtained from the potential flow solution described above, by evaluating \(U\) and its \(s\)-derivative along
the profile itself on the body, and along the \( z \)-axis in the wake.

After this first integration, two improvements can be made using the intermediate results: first, from the evaluation of the displacement thickness, the edge of the boundary-layer is now defined by a curve along which the potential flow term can be evaluated during the next iterations. Furthermore, some additionnal corrections can be made due to the knowledge of "the breathing effect": the distribution of normal velocity is known at the edge of the layer, and to obtain a better approximation of the actual velocity at the edge of the layer, one may superpose to the initial potential flow solution some perturbations which will account for the breathing effect.

An improved estimate of the velocity at the edge of the layer was finally obtained by superposing three potential flows:

- the initial potential corresponding to the flow past a circular cylinder mapped into the \( Y \)-plane.

- a first perturbation potential which accounts for the distribution of normal velocity along the body.

- a second perturbation potential which accounts for the distribution of normal velocity in the wake.

The corresponding velocity \( U_e \) and its \( s \)-derivative where then evaluated along the edge of the layer and some additionnal iterations were performed using for each of them the outputs of the previous one as inputs.

2.5.2 Breathing effect on the body

From the first integration, the entrainment coefficient \( C_E \) and therefore the distribution of normal velocity at the edge of the layer

\[
V_n = \frac{\partial \phi}{\partial n} = U_e C_E
\]

is known for a given set of points.
To find the exact velocity potential satisfying both Laplace’s equation and this boundary condition is a difficult task, but an approximate solution can be obtained by enforcing this normal condition along the body itself and not along the edge of the layer. The new normal condition is then

\[ \frac{\partial \phi}{\partial n} = V_n \]

for a given set of points along the strut in the \( \text{Y-plane} \). Using the properties of conformal mapping, this can be mapped into the \( \text{Z-plane} \), and it corresponds to a distribution of normal velocity along the circle:

\[ \frac{\partial \phi}{\partial n'} = \frac{V_n}{\left| \frac{df}{dY} \right|} \]

where \( \frac{df}{dY} \) is the \( \text{Y-derivative of the inverse Karman-Trefftz transform} \):

\[ \frac{df}{dY} = \frac{4R^2 C}{(Y^2 - \lambda R \lambda^2)(1 - C^2)^2} \quad \text{with} \quad C = \left( \frac{Y - \lambda R}{Y + \lambda R} \right)^{1/\lambda} \]

In the \( \text{Z-plane} \), we can now look for a velocity potential satisfying Laplace’s equation, which using the principle of separations of variables will have the form:

\[ \phi(r, \theta) = B_0 \log(r) + \sum_{n=1}^{\infty} B_n \frac{\cos(n\theta)}{r^n} \]

where \( r \) and \( \theta \) are the polar coordinates in the \( \text{Z-plane} \) \( (Z = r e^{i\theta}) \). Differentiating with respect to \( r \), the normal velocity is obtained:

\[ \frac{\partial \phi}{\partial r}(r, \theta) = \frac{B_0}{r} - \sum_{n=1}^{\infty} n B_n \frac{\cos(n\theta)}{r^{n+1}} \quad \text{(2.7)} \]

On the circle, the normal velocity is a function of \( \theta \) only, and since it is periodic of period \( 2\pi \), it can be expanded into a Fourier series:

\[ \frac{\partial \phi}{\partial r}(R, \theta) = \sum_{n=1}^{\infty} A_n \cos(n\theta) \quad \text{(2.8)} \]
Since \( \frac{\partial \psi}{\partial r}(R, \theta) \) is known numerically for a given set of points as:

\[
\frac{\partial \phi}{\partial r}(R, \theta) = \frac{V_n}{|\frac{df}{d\theta}|} = v(\theta)
\]

its Fourier coefficients can be computed by evaluating numerically the following integrals:

\[
A_0 = \frac{1}{\pi} \int_0^\pi v(\theta)d\theta
\]

\[
A_n = \frac{2}{\pi} \int_0^\pi v(\theta)\cos(n\theta)d\theta \quad \text{for} \quad n \geq 1
\]

Once these coefficients known, the \( B_n \) coefficients can be obtained by equating equation 2.8 and 2.7 where \( r \) has ben set to \( R \) and then by identifying the coefficients of the series:

\[
B_0 = A_0 R
\]

\[
B_n = -A_n \frac{R^{n+1}}{n} \quad \text{for} \quad n \geq 1
\]

And the velocity potential can thus be reconstructed as:

\[
\phi(r, \theta) = A_0 R \log(r) - \sum_{n=1}^{\infty} \frac{A_n \cos(n\theta)R^{n+1}}{n \cdot r^n}
\]

and its derivatives:

\[
\frac{\partial \phi}{\partial r}(r, \theta) = \frac{A_0 R}{r} + \sum_{n=1}^{\infty} A_n \cos(n\theta) \frac{R^{n+1}}{r^{n+1}}
\]

\[
\frac{\partial \phi}{\partial \theta}(r, \theta) = \sum_{n=1}^{\infty} A_n \sin(n\theta) \frac{R^{n+1}}{r^n}
\]

\[
\frac{\partial^2 \phi}{\partial \theta^2}(r, \theta) = \sum_{n=1}^{\infty} n A_n \cos(n\theta) \frac{R^{n+1}}{r^{n+1}}
\]

\[
\frac{\partial^2 \phi}{\partial \theta \partial r}(r, \theta) = -\sum_{n=1}^{\infty} n A_n \sin(n\theta) \frac{R^{n+1}}{r^{n+1}}
\]

These expressions must then be mapped back into the original \( Y \)-plane, so that
the real velocity field along the body can evaluated as:

\[
\frac{\partial \phi}{\partial s} = -\frac{1}{R \theta} \left| \frac{df}{dY} \right| \tag{2.9}
\]

\[
\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial r} \left| \frac{df}{dY} \right| \tag{2.10}
\]

and its \( s \)-derivatives:

\[
\frac{\partial^2 \phi}{\partial s^2} = \frac{1}{R^2 \theta^2} \left| \frac{df}{dY} \right|^2 + \frac{1}{R^2 \theta} \frac{df}{dY} \frac{\partial}{\partial \theta} \left( \frac{df}{dY} \right) \tag{2.11}
\]

\[
\frac{\partial^2 \phi}{\partial s \partial n} = -\frac{1}{R \theta \theta r} \left| \frac{df}{dY} \right|^2 - \frac{1}{R \theta r} \left| \frac{df}{dY} \right|^2 \frac{\partial}{\partial \theta} \left( \frac{df}{dY} \right) \tag{2.12}
\]

where

\[
\frac{\partial}{\partial \theta} \left( \frac{df}{dY} \right) = -\frac{IM(f(Y)f''(Y)f'(Y)/f'(Y))}{|f'(Y)|}
\]

Using the expressions of equations 2.9, 2.10, 2.11 and 2.12, this potential flow solution can be superposed to the original stream flow and the term \( \frac{V_s}{U_0} \frac{dU_s}{ds} \) can be corrected.

2.5.3 Breathing effect in the wake

The evaluation of the entrainment coefficient during the first integration allows to determine the distribution of normal velocity at the edge of the layer in the wake. Once again, evaluating the exact corresponding potential flow solution is very difficult, but a good estimate can be obtained by enforcing this normal condition on the z-axis itself.

Since the flow is symmetric with respect to the z-axis, this corresponds in fact to a discontinuity of the normal velocity across the z-axis, and it can be therefore represented by a linear arrangement of sources along the axis from the trailing-edge \( z = 0.5 \) to the end of the computational domain \( z = L \).
The velocity potential induced by such an arrangement can be written as:

\[
\phi(x, y) = \frac{1}{2\pi} \int_{0.5}^{L} \sigma(\xi) \log(\sqrt{(x - \xi)^2 + y^2}) d\xi
\]

where the source strength at the point \( \xi \) is:

\[
\sigma(\xi) = 2V_n(\xi) = 2U_n(\xi)C_E(\xi)
\]

The components of the velocity field can be obtained by differentiation:

\[
\phi_x(x, y) = \frac{1}{2\pi} \int_{0.5}^{L} \sigma(\xi) \frac{x - \xi}{(x - \xi)^2 + y^2} d\xi
\]

\[
\phi_y(x, y) = \frac{1}{2\pi} \int_{0.5}^{L} \sigma(\xi) \frac{y}{(x - \xi)^2 + y^2} d\xi
\]

The numerical integration produces in fact discrete information: the source strength is therefore known only for a given set of points \( x = \xi_i, i = 1 \) to \( n \) and must be assumed constant over the segments \([\xi_i, \xi_{i+1}]\). The velocity field can then be evaluated as discrete sums of integrals over the segments:
\[ \phi_e(x,y) = \frac{1}{2\pi} \sum_{i=1}^{n} \sigma(\xi_i) \int_{\xi_i}^{\xi_{i+1}} \frac{x-\xi}{(x-\xi)^2 + y^2} \, d\xi \]

\[ \phi_v(x,y) = \frac{1}{2\pi} \sum_{i=1}^{n} \sigma(\xi_i) \int_{\xi_i}^{\xi_{i+1}} \frac{y}{(x-\xi)^2 + y^2} \, d\xi \]

which after evaluation of the integrals gives:

\[ \phi_e(x,y) = \frac{1}{4\pi} \sum_{i=1}^{n} \sigma(\xi_i) \log \left[ \frac{(x-\xi_i)^2 + y^2}{(x-\xi_{i+1})^2 + y^2} \right] \]

\[ \phi_v(x,y) = \frac{1}{4\pi} \sum_{i=1}^{n} \sigma(\xi_i) \left[ \tan^{-1} \left( \frac{\xi_{i+1} - x}{y} \right) - \tan^{-1} \left( \frac{\xi_i - x}{y} \right) \right] \]

These two expressions were used to evaluate the influence of the sources along the body itself. Unfortunately, they cannot be used directly in the wake itself since \( \phi_e(x,y) \) is singular for each \( x = \xi_i \) and \( \phi_v(x,y) \) is singular for \( y = 0 \).

In order to get nevertheless a broad estimate of the corrections to the velocity field in the wake, the following calculations were conducted:

- \( \phi_e \) was estimated at the mid-points \( x_i = (\xi_i + \xi_{i+1})/2 \) using the previous expression of \( \phi_e(x,y) \) which is not singular at these points. These datas were then interpolated using cubic-splines.

- \( \phi_v \) was known at the points \( x = \xi_i \) and this set of data was also interpolated in order to get a continuous set of information.

### 2.6 Results

Four successive numerical integrations were conducted both along the body itself and downstream in the wake. The process proved to converge rapidly since the results obtained for each of the three independent parameters \( \theta \), \( H \) and \( C_E \) were identical after the third and after the fourth integration.

The initial values of \( \theta \), \( H \) and \( C_E \) were borrowed from the 1/7-power approximation for the flat-plate. Except for the behaviour just after the leading edge, the differential scheme proved to be relatively unsensitive to the choice of this starting
values: the coupling between the three equations enforces the behaviour of the three parameters more than the choice of their initial values.

The passage from the body to the wake was made by interrupting the integration just before the trailing edge \((X_f = 0.495)\) and using the final values of \(\theta\), \(H\) and \(C_E\) as initial values to start the integration in the wake just after the trailing edge \((X_i = 0.505)\). The results obtained in the wake proved to be sufficiently insensitive to the position of \(X_f\) and \(X_i\) to justify this operation.

Interpretation of the results:

- momentum thickness (cf figure 3-2): \(\theta\) grows rapidly along the body, reaches a maximum at the trailing edge and tends to a constant in the far wake. The pike at the trailing edge is due to the term \(\frac{1}{U_e} \frac{dU_e}{ds}\) which gets large and changes sign at this point, its magnitude is exaggerated at the first integration since the potential flow solution is singular at the trailing edge and \(\frac{1}{U_e} \frac{dU_e}{ds}\) becomes therefore infinite. The discontinuity in slope at the trailing edge is due both to the change of sign of \(\frac{1}{U_e} \frac{dU_e}{ds}\) and to the fact that \(C_f\) is set to zero. Far downstream, \(\frac{d\theta}{ds}\) tends to zero since the flow becomes uniform, and \(\theta\) tends to a constant limit in accordance with the asymptotic behaviour observed empirically.

- shape-parameter (cf figure 3-3): \(H\) is slightly constant along the body then reaches a maximum at the trailing edge and tends to 1.0 in the far wake. As for the momentum thickness, the pike at the trailing edge is due to the term \(\frac{1}{U_e} \frac{dU_e}{ds}\), and its magnitude is smoothed by the successive iterations. The limit \(H = 1.0\) in the far wake has also been observed empirically.

- breathing effect (cf figure 3-4): the normal velocity \(U_eC_E\) remains almost constant along the first half of the body, then grows across the trailing edge to reach a maximum at the beginning of the wake and finally decreases slowly to zero in the far wake. The term \(\frac{1}{U_e} \frac{dU_e}{ds}\) is not in this case predominant and therefore there is no discontinuity in slope across the trailing edge. The limit \(C_E = 0\) is also in accordance with the experiments.
Figure 2-6: Momentum thickness
Figure 2-7: Shape-parameter
Figure 2-8: Breathing effect
Chapter 3

Axisymmetric flow: Ellipsoid of revolution

3.1 Introduction

The ultimate application of the method derived by Green & al. consists on the evaluation of flows past axisymmetric bodies. In order to apply this method, we have considered one of the bodies which was the most studied in hydrodynamics: an ellipsoid of revolution.

As for the Karman-Trefftz strut, such a choice was justified by the knowledge of the analytic solution for the potential flow past the selected body, which allows to avoid extra-computational cost due to the evaluation of the potential flow using a panel method. However, some difficulties were encountered due to the infinite slope of the body at the trailing edge.

3.2 The differential system

The boundary-layer development is as usual specified by the three independent parameters $\theta$, $H$ and $C_E$ and is predicted by the numerical integration of the three simultaneous ordinary differential equations which can be written along the body as:
\[
\frac{d\theta}{ds} = \frac{C_f}{2} - (H + 2) \frac{\theta}{U_e} \frac{dU_e}{ds} - \frac{\theta r}{r ds} 
\] (3.1)

\[
\frac{dH}{ds} = \frac{dH_1}{dH_1} \{C_E - H_1(C_f/2 - (H + 1) \frac{\theta}{U_e} \frac{dU_e}{ds}) \}
\] (3.2)

\[
\frac{dC_E}{ds} = F \left(\frac{2.8}{H + H_1}\right) \left\{\left(\left(\frac{C_f}{U_e^2}\right)_{EQ} - \lambda C_{r/2}^1\right) + \left(\frac{\theta}{U_e} \frac{dU_e}{ds}\right)_{EQ} - \frac{\theta}{U_e} \frac{dU_e}{ds}\right\}
\] (3.3)

and in the wake as:

\[
\frac{d\theta}{ds} = -(H + 2) \frac{\theta}{U_e} \frac{dU_e}{ds}
\]

\[
\frac{dH}{ds} = \frac{dH_1}{dH_1} \{C_E - H_1(H + 1) \frac{\theta}{U_e} \frac{dU_e}{ds}\}
\]

\[
\frac{dC_E}{ds} = F \left(\frac{2.8}{H + H_1}\right) \left\{\left(\left(\frac{C_f}{U_e^2}\right)_{EQ} - \lambda C_{r/2}^1\right) + \left(\frac{\theta}{U_e} \frac{dU_e}{ds}\right)_{EQ} - \frac{\theta}{U_e} \frac{dU_e}{ds}\right\}
\]

If we compare this to the system obtained in the case of 2D-flows, there is an additionnal term in equation 3.1 which accounts for the variations of the body radius \( r \). The correction factor \( \lambda \) in equation 3.3 is also modified and accounts now both for the variations of longitudinal curvature and for the flow convergence and divergence (cf chapter 1, equations 1.15 and 1.16).

As for 2D-flows, the numerical integration requires the knowledge of the term \( \frac{1}{U_e} \frac{dU_e}{ds} \), an estimate of which can be obtained using potential flow theory.

### 3.3 The potential flow solution

#### 3.3.1 Geometry of the body and ellipsoidal coordinates

The body considered is axisymmetric around the \( X \)-axis, and any meridian section of it is an ellipse. If we consider the \( X \) – \( Y \) plane, the equation of the ellipse is:

\[
\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1
\]

where \( a \) and \( b \) are the length and beam of the ellipsoid.
Figure 3.1: Ellipsoid of revolution

A new coordinates system \((\zeta, \mu)\) associated to the meridian elliptic section can be defined as:

\[
x = c\mu\zeta
\]
\[
y = c\sqrt{1 - \mu^2}\sqrt{\zeta^2 - 1}
\]

where \(c\) is the abscissa of the foci of the meridian section \((c = \sqrt{a^2 - b^2})\).

The surfaces \(\zeta = \text{const}, \mu = \text{const}\) are respectively confocal ellipses and hyperboles, the common foci being the points \((\pm c, 0)\). The section of the ellipsoid is defined in this new system by the curve \(\zeta = \zeta_0 = 1/e\) for which \(e\) is the excentricity of the meridian section \((e = c/a)\). The ellipsoidal coordinates can be obtained for any point \((x, y)\) by inverting the system:

\[
\zeta = \frac{\sqrt{(x^2 + y^2 + c^2) + \sqrt{(x^2 + y^2 + c^2)^2 - 4c^2x^2}}}{c\sqrt{2}}
\]
\[
\mu = \frac{x}{c\zeta} = \frac{x\sqrt{2}}{\sqrt{(x^2 + y^2 + c^2) + \sqrt{(x^2 + y^2 + c^2)^2 - 4c^2x^2}}}
\]
3.3.2 The potential flow solution

Using the ellipsoidal coordinates in a referential moving with the body, the velocity potential can be written as the sum of two potentials:

$$\Phi_1 = Ux = Uc\mu\zeta$$

which corresponds to the changement of referential from an absolute frame and:

$$\Phi_2 = \sum_n A_n P_n(\mu) Q_n(\zeta)$$

which is an expansion in ellipsoidal harmonics using Legendre’s polynomials, and is the expression an arbitrary solution, vanishing at the infinite, of Laplace’s equation \( \nabla^2 \Phi = 0 \).

In this particular case \( \Phi_2 \) has a very simple expression:

$$\Phi_2 = A_1 P_1(\mu) Q_1(\zeta) = A_1 \mu \left\{ \frac{1}{2} \zeta \log \left[ \frac{\zeta + 1}{\zeta - 1} \right] - 1 \right\}$$

where

$$A_1 = \frac{Ua}{\frac{1}{1-e^2} - \frac{1}{2e} \log \left[ \frac{1+e}{1-e} \right]}$$

Setting \( U = 1.0 \) we obtain the final expression of the total potential:

$$\Phi = \Phi_1 + \Phi_2 = c\zeta \mu + A_1 \mu \left\{ \frac{1}{2} \zeta \log \left[ \frac{\zeta + 1}{\zeta - 1} \right] - 1 \right\}$$

In order to compute the term \( \frac{1}{Uc} \frac{dU}{dx} \), the first and second partial derivatives of \( \Phi \) with respect to \( x \) and \( y \) had to be evaluated using a symbolic calculation software:

$$\Phi_x(\zeta, \mu) = 1 + \frac{A_1}{c} \left\{ \frac{\zeta}{\mu^2 - \zeta^2} + \frac{1}{2} \log \left[ \frac{\zeta + 1}{\zeta - 1} \right] \right\}$$

$$\Phi_y(\zeta, \mu) = \frac{A_1}{c} \frac{\mu}{\mu^2 - \zeta^2} \sqrt{\frac{1 - \mu^2}{\zeta^2 - 1}}$$

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\[ \Phi_{XX}(\zeta, \mu) = -\Phi_{YY}(\zeta, \mu) = \frac{A_1}{c^2} \frac{\mu}{(\mu^2 - \zeta^2)} (\mu^4 + (1 - 5\zeta^2)\mu^2 + 3\zeta^2) \]

\[ \Phi_{XY}(\zeta, \mu) = \Phi_{YX}(\zeta, \mu) = \frac{A_1}{c^2} \frac{\zeta}{(\mu^2 - \zeta^2)^2} \sqrt{\frac{1 - \mu^2}{\zeta^2 - 1}} (\mu^4 + (3 - 5\zeta^2)\mu^2 + \zeta^2) \]

From this, \( U_e = \sqrt{\Phi_{XX}^2 + \Phi_{YY}^2} \) and \( \frac{dU_e}{da} = \nabla U_e \cdot \vec{r} \) can be evaluated since we know:

\[ \vec{r} = \left( \frac{a\sqrt{a^2 - \zeta^2}}{\sqrt{a^2 - \zeta^2}}, \frac{-b\zeta}{\sqrt{a^2 - \zeta^2}} \right) \quad \text{and} \quad \nabla U_e = \left( \begin{array}{c} \Phi_{XX} \Phi_{YY} + \Phi_{YY} \Phi_{XX} \\ \Phi_{XX} \Phi_{YY} + \Phi_{YY} \Phi_{XX} \\ \end{array} \right) \]

### 3.4 The iteration process

#### 3.4.1 General description

As for 2D-flows, the purpose of accomplishing successive numerical integrations is to evaluate the term \( \frac{1}{U_e} \frac{dU_e}{da} \) as accurately as possible and thus converge to a stable solution.

During the first integration, \( \frac{1}{U_e} \frac{dU_e}{da} \) is computed from the potential flow solution described in section 3.3.2. For the first half of the body, \( U_e \) and its \( s \)-derivative were evaluated along the body itself (curve \( \zeta = \zeta_0 \)). To get rid of the singular behaviour due to the infinite slope of the ellipsoid at its trailing edge, a streamlined body was considered, whose edge is no more the curve \( \zeta = \zeta_0 \) but a streamline close to it, along which \( U_e \) and its \( s \)-derivative were evaluated both for the second half of the body and in the wake.

After this first integration, the intermediate results were used to improve the evaluation of \( \frac{1}{U_e} \frac{dU_e}{da} \), and the velocity at the edge of the layer was obtained from the superposition of three potentials:

- the initial potential described in section 3.3.2 corresponding to the flow past an ellipsoid of revolution.

- a first perturbation potential which accounts for the breathing effect on the body.
a second perturbation potential which accounts for the breathing effect in the wake.

Adding these three potentials, the velocity $U_e$ was computed at the edge of a displacement body which is the body considered for the first integration enlarged by the displacement thickness, and $\frac{dU_s}{ds}$ was then obtained numerically from a spline-interpolation subroutine.

3.4.2 Breathing effect on the body

From the first integration, the distribution of normal velocity $\frac{\partial \Phi}{\partial n} = U_e C_E$ is known at the edge of the layer. To find directly the corresponding potential is not possible, but an approximation can be done by enforcing this distribution on the meridian elliptic section itself. $\frac{\partial \Phi}{\partial n}$ is then known for a given set of points $(\zeta_0, \mu_i)_{i=1,n}$ along the ellipse.

An arbitrary solution of Laplace’s equation vanishing at the infinite can be expressed as:

$$\Phi(\zeta, \mu) = \sum_{n=0}^{\infty} A_n P_n(\mu) Q_n(\zeta)$$

where the $P_n$-functions are the Legendre’s polynomials and the $Q_n$-functions can be defined as:

$$Q_n(\zeta) = P_n(\zeta) \int_\zeta^\infty \frac{d\zeta}{[P_n(\zeta)]^2(\zeta^2 - 1)}$$

and their numerical values can be obtained from appropriate subroutines. The $\zeta$-derivative of $\Phi$ is then:

$$\Phi'_{\zeta}(\zeta, \mu) = \sum_{n=0}^{\infty} A_n P_n(\mu) Q'_n(\zeta)$$

and along the meridian section this becomes:

$$\Phi_{\zeta}(\zeta_0, \mu) = \sum_{n=0}^{\infty} B_n P_n(\mu)$$  \(\text{(3.8)}\)

Since $\frac{\partial \Phi}{\partial n}(\zeta_0, \mu)$ is known for a given set of points $\mu_i$, and can be written as:

$$\frac{\partial \Phi}{\partial n}(\zeta_0, \mu) = \frac{1}{c} \sqrt{\frac{\zeta_0^2 - 1}{\zeta_0^2 - \mu^2}} \Phi_{\zeta}(\zeta_0, \mu)$$
\[ \Phi_\zeta(\zeta_0, \mu) \] is therefore also known for the same set of points. Multiplying equation 3.8 by \( P_k(\mu) \) and integrating with respect to \( \mu \) from \(-1\) to \(1\) we obtain:

\[
\int_{-1}^{1} \Phi_\zeta(\zeta_0, \mu) P_k(\mu) d\mu = \sum_{n=0}^{\infty} B_n \int_{-1}^{1} P_n(\mu) P_k(\mu) d\mu
\]

From the orthogonality property of the Legendre's polynomials, this becomes:

\[
\int_{-1}^{1} \Phi_\zeta(\zeta_0, \mu) P_k(\mu) d\mu = B_k \frac{2}{2k+1}
\]

Since \( \Phi_\zeta(\zeta_0, \mu) \) is known for a discrete set of points, the leften-side integrals can be estimated numerically, and the \( B_n \)-coefficients are therefore also known. From that, the \( \zeta \)- and \( \mu \)-derivatives of the potential can be derived as:

\[
\Phi_\zeta(\zeta, \mu) = \sum_{n=0}^{\infty} B_n P_n(\mu) \frac{Q'_n(\zeta)}{Q'_n(\zeta_0)}
\]

\[
\Phi_\mu(\zeta, \mu) = \sum_{n=0}^{\infty} B_n P'_n(\mu) \frac{Q_n(\zeta)}{Q'_n(\zeta_0)}
\]

where the \( B_n \)-coefficients are now given. The velocity field can then be evaluated using the curvilinear referential as:

\[
\frac{\partial \Phi}{\partial n} = \frac{1}{c} \sqrt{\frac{\zeta^2 - 1}{\zeta^2 - \mu^2}} \Phi_\zeta
\]

\[
\frac{\partial \Phi}{\partial t} = \frac{1}{c} \sqrt{\frac{1 - \mu^2}{\zeta^2 - \mu^2}} \Phi_\mu
\]

### 3.4.3 Breathing effect in the wake

From the first integration, the distribution of normal velocity at the edge of the boundary-layer is given by \( \frac{\partial \Phi}{\partial n} = U_c C_E \). The expression of the corresponding velocity potential can be obtained as the result of a linear distribution of sources along the \( x \)-axis in the wake:

\[
\Phi(x, y) = -\frac{1}{4\pi} \int_{0.5}^{\hat{z}} \frac{\sigma(\xi)}{\sqrt{(x - \xi)^2 + y^2}} d\xi
\]
\( \sigma(\xi) \) is in fact known only for a discrete set of points \( \xi_i \) as

\[
\sigma(\xi_i) = 4\pi \delta(\xi_i) \frac{\partial \Phi}{\partial n}(\xi_i) = 4\pi \delta(\xi_i) U_e(\xi_i) C_E(\xi_i)
\]

where \( \delta \) is the Boundary-layer thickness and \( C_E \) the entrainment coefficient at the point \( \xi_i \) known from the first integration.

\( \Phi \) has to be rewritten as a discrete sum:

\[
\Phi(x, y) = -\frac{1}{4\pi} \sum_{i=1}^{n} \sigma(\xi_i) \int_{\xi_i}^{\xi_{i+1}} \frac{1}{\sqrt{(x-\xi)^2 + y^2}} d\xi
\]

which becomes after integration:

\[
\Phi(x, y) = -\frac{1}{4\pi} \sum_{i=1}^{n} \sigma(\xi_i) [\text{arg sinh} \left( \frac{\xi_{i+1} - x}{y} \right) - \text{arg sinh} \left( \frac{\xi_i - x}{y} \right)]
\]

and the velocity field is then obtained by differentiation:

\[
\Phi_x(x, y) = \frac{1}{4\pi} \sum_{i=1}^{n} \sigma(\xi_i) \left[ \frac{1}{\sqrt{(x-\xi_i+1)^2 + y^2}} - \frac{1}{\sqrt{(x-\xi_i)^2 + y^2}} \right]
\]

\[
\Phi_y(x, y) = -\frac{1}{4\pi} \sum_{i=1}^{n} \sigma(\xi_i) \left[ \frac{1}{\sqrt{(x-\xi_i-1)^2 + y^2}} - \frac{1}{\sqrt{(x-\xi_i)^2 + y^2}} \right]
\]

And it can be expressed in the curvilinear system using the following transformation:

\[
\frac{\partial \Phi}{\partial t} = \frac{1}{c} \sqrt{\frac{1 - \mu^2}{\zeta^2 - \mu^2}} \left[ \Phi_x \frac{\partial x}{\partial \mu} + \Phi_y \frac{\partial y}{\partial \mu} \right]
\]

\[
\frac{\partial \Phi}{\partial n} = \frac{1}{c} \sqrt{\frac{\zeta^2 - 1}{\zeta^2 - \mu^2}} \left[ \Phi_x \frac{\partial x}{\partial \zeta} + \Phi_y \frac{\partial y}{\partial \zeta} \right]
\]

so that it can be superposed to the velocity fields obtained in sections 3.3.2 and 3.4.2
3.5 Results

As for the strut, four successive numerical integrations were conducted both along the body itself and downstream in the wake. The process proved to converge rapidly since the results obtained for each of the three independent parameters $\theta$, $H$ and $C_E$ were identical after the third and after the fourth integration.

The initial values of $\theta$, $H$ and $C_E$ were borrowed from the 1/7-power approximation for the flat-plate. Except for the behaviour just after the leading edge, the differential scheme proved to be relatively unsensitive to the choice of this starting values: the coupling between the three equations enforces the behaviour of the three parameters more than the choice of their initial values. Interpretation of the results:

- momentum thickness (cf figure 3-2): $\theta$ grows rapidly along the body, reaches a maximum at the trailing edge and tends to a constant in the far wake. The pike at the trailing edge is due to the term $\frac{1}{U_e} \frac{dU}{ds}$ which gets large and changes sign at this point, its magnitude is exaggerated at the first integration since the potential flow solution is singular at the trailing edge and $\frac{1}{U_e} \frac{c_{1f} \lambda}{ds}$ becomes therefore infinite. The discontinuity in slope at the trailing edge is due both to the change of sign of $\frac{1}{U_e} \frac{dU}{ds}$ and to the fact that $C_f$ is set to zero. Far downstream, $\frac{d\theta}{ds}$ tends to zero since the flow becomes uniform, and $\theta$ tends to a constant limit in accordance with the asymptotic behaviour observed empirically.

- shape-parameter (cf figure 3-3): $H$ is slightly constant along the body then reaches a maximum at the trailing edge and tends to 1.0 in the far wake. As for the momentum thickness, the pike at the trailing edge is due to the term $\frac{1}{U_e} \frac{dU}{ds}$, and its magnitude is smoothed by the successive iterations. The limit $H = 1.0$ in the far wake has also been observed empirically.

- breathing effect (cf figure 3-4): the normal velocity $U_e C_E$ decreases along the body due to the parameter $\lambda$ which accounts for extraneous influences. The upper limit of $\lambda$ had to be set to a lower value (1.5 instead of 2.5) than the one prescribed by Green & al., so that these influences remain moderate, otherwise
the entrainment coefficient $C_E$ would decrease and become negative before the trailing edge, which makes no sense. The global behaviour is still similar to the one observed for the strut, with a maximum reached at the beginning of the wake, and whose value does not depend on the upper limit of $\lambda$.

![Diagram showing momentum thickness](image)

**Figure 3-2: Momentum thickness**
Figure 3-3: Shape-parameter

Figure 3-4: Breathing effect
Chapter 4

Drag force on a ship: Turbulent Boundary-Layer & Residuary Resistance

4.1 Introduction

The estimation of the total drag force on a ship advancing at constant speed in calm water is one of the most important problems of Naval Hydrodynamics. According to Froude's hypothesis, this force can be decomposed as follows:

\[ C_T(R, F) = C_F(R) + C_W(F) \]

where \( C_F \) accounts for the viscous effects (shear-stress) and depends mainly on the Reynolds number \( R \), and \( C_W \) accounts for the wave-effects (pressure) and depends mainly on the Froude number \( F \).

This hypothesis needs in fact to be refined, since it assumes that there is no drag force resulting of the integration of the pressure obtained by solving the double-body flow (D’Alembert paradox). In reality, it exists a small force, due
to the fact that the pressure does not recover at the trailing edge in the actual flow, and this can be evaluated by solving the double-body flow past a displaced body which is the original one enlarged by the displacement thickness of the boundary-layer. Once the pressure evaluated, it may be translated back to the original body, and the form drag $C_P(R)$ can be obtained by integrating this pressure distribution over the wetted-surface of the hull.

The equation obtained by Froude can then be rewritten as:

$$CT(R, F) = CP(R) + C_P(R) + CW(F)$$

The three corresponding components of the drag force where then evaluated for three different bodies: a vertical strut and two ship hulls. The horizontal sections of these three hulls where taken as Karman-Trefftz profiles so that the results derived in Chapter 2 could be used.

The evaluation of these three forces was performed by combining two computer codes: the code derived in Chapter 2 from the Green & al. boundary-layer method which allows to evaluate the skin-friction as well as all relevant parameters concerning the boundary-layer, and the code SWAN developed by P.D. Sclavounos & D. Nakos\(^1\) in 1988 which solves numerically the wave-flow around ships advancing at a constant speed and whose output includes wave fields, pressure distribution on the hull, as well as all relevant forces acting on the hull. From that, the influence of the viscous effects on the wave pattern and on the wave-drag (dependence of $CW$ on the Reynolds number $R$) was estimated.

All these computations were performed for a range of Froude numbers varying from 0.25 to 0.4, and a corresponding range for the Reynolds number from $10^8$ to $1.6 \times 10^8$ which is beyond the transition to turbulence. These ranges were chosen from the common length and speed range of a realistic ship.

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\(^1\)See references 8 & 10 for details
4.2 The code SWAN

4.2.1 General description

The evaluation of the wake-flow past a ship advancing at constant speed in calm water is one of the most difficult problem of Naval Hydrodynamics. Two main theories have been derived since the 19th century: the Neumann-Kelvin approach which expresses the velocity potential as an integral of a particular Green function, and the Double-body solution developed by Dawson in 1977 which writes the potential as the sum of a double-body potential and a perturbation potential accounting for the disturbance of the free-surface.

The code SWAN solves the steady and time-harmonic wave-flows around ships advancing at a constant speed. The flow is supposed to be symmetric with respect to the z-axis. The code solves either using the Neumann-Kelvin approach or the Double-body solution and provides a wide range of outputs: wave fields, velocity fields, pressure distribution, forces...

Figure 4-1: The moving frame of reference
The discretization of the free-surface and the ship hull is based on a bi-quadratic spline-approximation of the unknown velocity potential. The grid is characterized by two main parameters: NBX, the number of panels in the z-direction, and AR, the aspect ratio which is the ratio of the width of the panels to their length. The convergence of the results may be assumed only if they become independent of these two parameters.

4.2.2 Evaluation of the Wave-Resistance

The Wave-Resistance can first be evaluated by integrating the pressure distribution over the wetted-surface of the hull, the corresponding expression being:

\[ R_w = -\rho \int \int_{S_0} \left[ \nabla \Phi \nabla \phi + \frac{1}{2} \nabla \phi \nabla \phi \right] n_z \, dS - \rho \int \int_{\Delta S} \left[ \nabla \Phi \nabla \phi + g z \right] n_z \, dS \quad (4.1) \]

*\( S_0 \): portion of the hull below \( z = 0 \)
*\( \Delta S \): difference between \( S_0 \) and the actual wetted surface

In Neumann-Kelvin theory:

\[ \Phi = -U x \]

\( \phi \): perturbation potential, solution of Laplace's equation satisfying the linearized free-surface condition

In Double-Body theory:

\[ \Phi \): Double-Body potential

\( \phi \): perturbation potential accounting for the disturbance of the free-surface

In both cases the assumption \( \nabla \phi \ll \nabla \Phi \) has been made.

The second expression is obtained from momentum conservation: considering a fluid region bounded by the hull, the free-surface and a vertical control surface,
some momentum is supplied to this control volume by the forward motion of the ship and escapes downstream in the form of the free waves.

If we consider a cut of the wave field perpendicular to the ship track at some distance \( x = x_0 < 0 \), the Fourier transform of the wave-elevation \( \zeta(x, y) \) and its \( x \)-derivative can be expressed as follows:

\[
C(x, v) = \int_{-\infty}^{+\infty} dy \, \zeta(x, y) e^{iuv} = 2 \int_{0}^{+\infty} \hat{\zeta}(x, y) \cos(vy)
\]

\[
C_x(x, v) = \int_{-\infty}^{+\infty} dy \, \frac{\partial \zeta}{\partial x}(x, y) e^{iuv} = \frac{\partial C}{\partial x}(x, v)
\]

According to the transverse-cut method derived by Eggers, Sharma & Ward (1967)\(^2\), the complex free-wave spectrum may be written as:

\[
H(u) = \frac{1}{8\pi} \frac{2u^2 - 1}{u^2} \left[ C(x, v) + i \frac{C_x(x, v)}{u} \right] e^{iu^2}
\]

where the non-dimensional wave-numbers \( u, v \) satisfy the dispersion relation

\[
u^2 = \sqrt{u^2 + v^2}
\]

The Wave-Resistance can then be derived from the free wave-spectrum using the formula:

\[
R_w = \frac{1}{8\pi} \int_{0}^{+\infty} |H(u)|^2 \frac{\sqrt{1 + 4u^2}}{1 + \sqrt{1 + 4u^2}} du \quad (4.2)
\]

The code SWAN solves first the potential flow, from that the pressure distribution along the hull and the wave-elevation in the wake-field are evaluated, and the Wave-Resistance can therefore be computed by quadrature using the two different expressions. The Wave-Resistance is then usually normalized using the wetted-surface of the hull \( S_w \):

\[
C_w = \frac{R_w}{\frac{1}{2} \rho U^2 S_w}
\]

\(^2\text{See reference [4] for details}\)
4.3 Geometry of the bodies

4.3.1 Karman-Trefftz strut

The original body considered is a vertical strut of length 1.0, beam 0.1 and draft 0.5. Its horizontal cut is a Karman-Trefftz profile identical to the 2D-profile for which boundary-layer computations were performed in Chapter 2.

![Diagram of Karman-Trefftz strut]

Figure 4.2: The original Karman-Trefftz strut

The solution of the potential flow past this body shows that the streamlines along the hull are almost horizontal. The pressure gradient used to perform the numerical integration of the boundary-layer scheme can therefore be computed as in Chapter 2, using the analytical expression obtained from conformal mapping. The convergent results obtained in Chapter 2 can then be used to construct a displaced body, made of the original body enlarged by the displacement thickness of the boundary-layer.
4.3.2 The ship hulls

For both of the hulls, the horizontal cuts are Karman-Trefftz profiles as defined in Chapter 2. Vertical sections are parabolic of the form:

\[ Y = Y_{\text{max}}(1 - \frac{Z^2}{D^2}) \]

where \( D \) is the draft of the body, and where \( Y_{\text{max}} \) is a function of \( X \) derived from:

\[ X^2 + (Y_{\text{max}} - b)^2 = \varrho^2 \]

The characteristics of the hull-1 were chosen as: length 1.0, beam 0.1 and draft 0.2, so that the stream-lines are still almost horizontal, and that the effect of three-dimensionnality can be compared to the 2-D flow past the vertical strut of same beam to length ratio. The characteristics of the hull-2 were chosen as: length 1.0, beam 0.15 and draft 0.2, so that it approaches the dimensions of a realistic ship hull.
As for the strut, two corresponding displaced bodies were contructed using the results of the boundary-layer code. The convergent results for the different parameters describing the layer were however obtained differently: in order to account for the three-dimensionnality of the hulls, the pressure gradient was no more obtained analytically from the 2-D potential flow solution, but by running SWAN to compute the double-body flow around these hulls. The boundary-layer code was then run for different profiles corresponding to the horizontal cuts of the body (which match the discretization) and a first estimate of the boundary-layer thickness was obtained. SWAN was then rerun to get a new estimate of the pressure gradient along this enlarged body, and this process was repeated three times to obtain the convergent results.

Figure 4.4: The hull-2
4.4 Pressure drag

The pressure drag or form drag is due to the fact that in real flows, the pressure does not recover its original value at the trailing edge.

![Diagram of fluid distribution](image)

**Figure 4-5: Pressure distribution along the meridian section of an axisymmetric body**

The integration of such a distribution produces a small force known as the form drag $C_P(R)$ which depends mainly on the Reynolds number, since it is an effect of the viscous flow. For the strut, the corresponding pressure was obtained by solving the double-body flow past the enlarged body on which a new normal condition was enforced taking into account the breathing effect. For the two hulls, each horizontal section was treated as a 2-D strut of the proper Beam to Length ratio and the pressure was therefore obtained for each row of panels along the hull.

The total force is then obtained by numerical integration as:

$$R_P = \int \int_{S_w} p \, n_z \, dS$$
where $S_w$ is the wetted-surface of the body. The force was then normalized as

$$C_p(R) = \frac{R_p}{\frac{1}{2} \rho U^2 S_w}$$

and the following results were derived:

![Graph showing pressure drag as a function of Reynolds number.

**Figure 4-6:** Pressure drag as a function of the Reynolds number

### 4.5 Frictional drag

The skin-friction along horizontal sections of the three bodies was evaluated by running the boundary-layer code with the proper geometries and pressure gradients. It can be computed as a function of the converged results obtained for the momentum thickness $\theta$ and shape-parameter $H$ using the following expressions:

Local Reynolds number $R_{\theta}$

$$R_{\theta} = \frac{\theta U_e}{\mu_e}$$
Skin-friction coefficient for the flat-plate $C_{f0}$

$$C_{f0} = \frac{0.01013}{\log_{10}(R_e) - 1.02} - 0.00075$$

Shape-parameter for the flat-plate $H_0$

$$1 - \frac{1}{H_0} = 6.55(C_{f0}/2)^{1/2}$$

Skin-friction coefficient $C_f$

$$C_f = C_{f0}(0.9\left(\frac{H}{H_0} - 0.4\right)^{-1} - 0.5)$$

The Skin-friction coefficient $C_f$ must then be integrated over the all wetted-surface as:

$$C_F(R) = \frac{1}{S_w} \int \int_{S_w} C_f \alpha \, dS$$

where $\alpha$ is the direction cosine between the tangential vector and $e_z$, the unitary vector in the z-direction.

Plotted against the Reynolds number, the frictional drag obtained for the three bodies is shown on figure 4-7.

These results were then compared to the flat-plate frictional drag obtained from the following empirical expression known as ITTC 57:

$$C_{f0} = \frac{0.075}{(\log R - 2.0)^2}$$

which was integrated over a flat-plate of the same wetted-area than the body-considered. As expected, the ratio between the real frictional drag and the flat-plate frictional drag proved to be independent of the Reynolds number and function of the geometry of the hull only, according to the expression:

$$C_F = (1 + k)C_{F0}$$
and the following results were obtained for the form coefficient $k$:

Strut: $k = 0.0167$

Hull-1: $k = 0.003$

Hull-2: $k = 0.018$

![Frictional Drag as a function of the Reynolds number](image)

Figure 4-7: Frictional drag as a function of the Reynolds number

### 4.6 Wave-resistance

As shown in section 4.2.2, the wave-resistance can be evaluated using two different methods, either by pressure integration or by momentum considerations. Nevertheless, the first method is the most reliable to compute $C_W(F)$ since the second expression may not account for all the transfer of momentum.

For the three different bodies, the same method was used: the code SWAN was run around the displaced body with a new normal condition accounting for the breathing effect. To assess the convergence of the results, different discretizations were considered.
4.6.1 Perturbation potential and boundary conditions

The steady-flow potential of the problem can be decomposed as:

$$\Phi(x, y, z) = \Psi(x, y, z) + \phi(x, y, z)$$

where $\Psi$ accounts for the "double-body" flow, and the perturbation "wave" potential $\phi$ verifies the system:

Laplace's equation $\Delta \phi = 0$

Linearized free-surface condition

$$\nabla \Psi \cdot \nabla (\nabla \Psi \cdot \nabla \phi) + \frac{1}{2} \nabla (\nabla \Psi \cdot \nabla \Psi) \cdot \nabla \phi + g \phi_z - \Psi_{zz} \nabla \Psi \cdot \nabla \phi$$

$$= -\frac{1}{2} \nabla (\nabla \Psi \cdot \nabla \Psi) \cdot \nabla \Psi - \frac{1}{2} (U^2 - \nabla \Psi \cdot \nabla \Psi) \Psi_{zz}$$

Ship hull boundary condition

$$\frac{\partial \phi}{\partial n} = 0$$

However, instead of considering the usual potential flow condition along the body, we have considered a new normal condition:

$$\frac{\partial \phi}{\partial n} = -U_e C_E$$

where $C_E$ is the entrainment coefficient and $n$ is the normal vector pointing inside the body.
4.6.2 Discretization and computational domain

The original bodies were discretized as follows:

- 25, 50 or 75 panels in the $x$-direction.
- 5 panels in the $z$-direction for the strut.
- 7 panels in the $z$-direction for the two hulls.

The displaced body was discretized as follows:

- 50, 75 or 150 panels in the $x$-direction.
- 5 panels in the $z$-direction for the strut.
- 7 panels in the $z$-direction for the two hulls.

The computational domain was the following for all the runs:

- the upstream limit was 0.2 from the bow of the original body.
- the downstream limit was 1.0 from the stern of the original body.
- the lateral limit was 0.5 from the $x$-axis.

In the case of the displaced body, the size of the domain was identical, but the downstream limit was the stern itself, since a tail of length 1.0 had been added to the original body, accounting for the development of the boundary-layer in the wake.
Figure 4-8: Original body: Strut

Figure 4-9: Displaced body: Strut
4.6.3 Convergence of the pressure integration

The Wave-Resistance was derived from the following expression:

\[ R_w = -\theta \int \int_{S_0} [\nabla \Phi \nabla \phi + \frac{1}{2} \nabla \phi \nabla \phi] n_x \, dS - \theta \int \int_{\Delta S} [\nabla \Phi \nabla \phi + g x] n_x \, dS \]

where the pressure has been evaluated using Bernouilli’s equation.

To assess the numerical convergence of this integral, several discretizations have been considered. The two parameters governing the grid fitting the hull and the free-surface are the number of panels in the z-direction NBX, and the aspect ratio AR (which characterizes the size of the panels in the y-direction on the free-surface: the larger AR gets, the finer the grid becomes). The values chosen for NBX were 50, 75 and 150, and for AR, 1.0, 2.0 and 4.0. For each of the three bodies, 8 runs were conducted involving the different possible combinations of the parameters AR and NBX (the combination AR=4.0, NBX=150 was not run because it exceeded the capacity of the computer).

As shown on figures 4-10 and 4-11, the pressure integration proved to be independent of the aspect ratio for a fixed NBX, and proved to converge with the number of panels NBX for a fixed aspect ratio.

The results obtained for the combination AR=2.0, NBX=150 were therefore assumed to be convergent, and the results shown in figure 4-12 were obtained for the three bodies, where \( C_W \) is plotted as a function of the Froude number \( F \)
Figure 4-10: Pressure integration versus NBX: fixed aspect ratio

Figure 4-11: Pressure integration versus AR: fixed NBX
4.6.4 Wave-cut analysis

As described in section 4.2.2, the Wave-resistance can also be evaluated by spectrum analysis of the wave-elevation in the wake behind the ship. The corresponding formula is the following:

\[ R_w = \frac{1}{8\pi} \int_0^{+\infty} |H(u)|^2 \frac{\sqrt{1 + 4u^2}}{1 + \sqrt{1 + 4u^2}} du \]

where \( H(u) \) is the complex free-wave spectrum whose expression is:

\[ H(u) = \frac{1}{8\pi} \frac{2u^2 - 1}{u^2} \left[ C(x, v) + i \frac{C_x(x, v)}{u} \right] e^{iu\omega} \]

The code SWAN computes both the wave-resistance from pressure integration and by spectrum analysis. It was run twice for each of the hulls, once around the original body and a second time for the enlarged body with a new boundary condition. The convergence of the Wave-cut analysis can be assumed when it proves to be independent of the wave-cut considered. An example of convergence is shown in the following figure where the wave-resistance has been plotted.
against the wave-cut in the wake. The evaluation is not accurate just after the stern nor at the end of the computational domain, but a reliable estimation can be obtained in between where the curve is almost flat.

![Wave-resistance Cw graph](image)

**Figure 4-13: Convergence of the Wave-cut analysis**

After running SWAN for the three original bodies, the results show a significant difference between the two evaluations, the spectrum evaluation being always lower than the pressure integration as if there was a term missing. However, once the body has been enlarged by the boundary-layer thickness and once a new boundary condition has been enforced, the agreement is much better. In the case of the strut, the two curves are very similar, and the relative error between the two is always less than 10%. The results for the two hulls are not as good, even so they show a significant improvement in the accordance of the curves: this may be due to the fact that the breathing is not determined as accurately as for the strut since the flow is no more two-dimensional. Furthermore, this improvement seems to be due mainly to the enforcement of a new boundary condition accounting for the breathing effect, because the modification of the geometry showed to have little influence on the evaluation of this forces.
Figure 4-14: Wave-resistance: Original strut

Figure 4-15: Wave-resistance: Enlarged strut with breathing
Figure 4-16: Wave-resistance: Original hull-1

Figure 4-17: Wave-resistance: Enlarged hull-1 with breathing
Figure 4-18: Wave-resistance: Original hull-2

Figure 4-19: Wave-resistance: Enlarged hull-2 with breathing
4.6.5 Breathing and wave pattern

Since the enforcement of a new boundary condition has shown to bring some significant modifications in the evaluation of the wave-resistance, it is also interesting to study the differences in the wave patterns: as shown in the following figures the waves seem to be damped by the viscous effects. The modification of the wave-elevation is always lower than 15% in all cases, with the largest difference obtained for the highest Froude number considered.

Figure 4-20: Wave-elevation on the first line of panels: Hull-1, F=0.4
Figure 4-21: Wave-elevation on the first line of panels: Strut, F=0.4

4.6.6 Wave-resistance and Reynolds number

Running SWAN for the original body and evaluate the wave-resistance is like setting arbitrarily the Reynolds number to infinity and neglecting the viscous effects. By comparing the results obtained for the original bodies and the one for the enlarged bodies with breathing, one can see the influence of the viscosity
on the Wave-resistance: for the three bodies, the results show that the Wave-resistance is reduced by the viscous effects as one could expect it. The viscosity tends to damp the waves and therefore reduce their resistance. This may not be perfectly true for the hull-2, but these results cannot be trusted as the one obtained for the strut because of the three-dimensionnality of the flow.

Figure 4-22: Strut: Wave-resistance and Reynolds number
Figure 4-23: Hull-1: Wave-resistance and Reynolds number

Figure 4-24: Hull-2: Wave-resistance and Reynolds number
4.7 Total Drag force

Once the three components of the drag have been computed, the total drag force exerted on the three hulls can be evaluated by simply adding the three components:

\[ C_T(R, F) = C_P(R) + C_F(R) + C_W(F) \]

where \( C_P \) is the pressure drag obtained in section 4.4, \( C_F \) is the frictional drag derived in section 4.5, and \( C_W \) is the Wave-resistance obtained by pressure integration in section 4.6.4.

The following figures show the relative importance of the three components for the three different bodies.

Figure 4-25: Strut: Total Drag
Figure 4-26: Hull-1: Total Drag

Figure 4-27: Hull-2: Total Drag
Conclusions

The influence of viscosity on the evaluation of the Wave-resistance of ships has been established for three different bodies: a vertical strut and two hulls. The results show that viscous effects tend to reduce the wave-resistance as compared to potential flow, and they seem to damp the wave-pattern. The enforcement of a new boundary-condition around a displaced body also brings a better agreement between the two possible estimations of the Wave-resistance: pressure integration and Wave-cut analysis. All this is especially true for the strut where the flow is really two-dimensionnal and for which the results of the boundary-layer have been derived more carefully. A three-dimensional boundary-layer method should probably be used to get better results for the hulls, since the estimation of the breathing effect relies deeply on evaluation of the pressure gradient. This will be the subject of a future study.
Bibliography


