A NONCONSERVATIVE SOLUTION TO THE GENERAL MIXED 
$H_2/H_\infty$ OPTIMIZATION PROBLEM

by

Darrell Brett Ridgely, Capt., USAF

B.S.A.E., University of Maryland, 1980
M.S.A.E., Air Force Institute of Technology, 1981
M.S.(Guidance & Control), Air Force Institute of Technology, 1983

Submitted to the Department of AERONAUTICS AND ASTRONAUTICS
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 1991

© Darrell Brett Ridgely, 1991

The author hereby grants to MIT permission to reproduce and to distribute copies of this thesis in whole or in part.

Signature of Author

Department of Aeronautics and Astronautics
September 1991

Certified by

Lena Valavani
Thesis Supervisor

Accepted by

Harold Wachman, Chairman
Department Committee on Graduate Students
A NONCONSERVATIVE SOLUTION TO THE GENERAL MIXED H₂/H∞ OPTIMIZATION PROBLEM

by

DARRELL BRETT RIDGELY

Submitted to the Department of Aeronautics and Astronautics on September 12, 1991 in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the Massachusetts Institute of Technology

ABSTRACT

A synthesis method is developed to design an output feedback controller to minimize the two-norm of one transfer function while ensuring the infinity-norm of another is held below a chosen level. This is known as the general mixed H₂/H∞ optimization problem. The solution minimizes the actual two-norm, rather than an upper bound to it, and therefore is not conservative. Seven coupled nonlinear matrix equations are derived which represent the necessary conditions such a solution must satisfy. By analyzing the case where the desired controller has the same order as the plant, it is found that the solution lies on the boundary of the infinity-norm constraint whenever the two objectives are competing. In this case, the mixed controller requires the neutrally stabilizing solution to a Riccati equation and solution of a Lyapunov equation which has no unique solution. A suboptimal approach is developed to solve the problem, which avoids problems associated with these equations. A numerical solution technique is developed to solve the problem, and two examples are solved and examined in detail.

Thesis Supervisor: Lena Valavani
Title: Associate Professor of Aeronautics and Astronautics

Thesis Supervisor: Wallace E. VanderVelde
Title: Professor of Aeronautics and Astronautics

Thesis Supervisor: Munther Dahleh
Title: Associate Professor of Electrical Engineering

Thesis Supervisor: Gunter Stein
Title: Adjunct Professor of Electrical Engineering
ACKNOWLEDGEMENTS

So many people have been involved with this work over the past few years, it is inevitable I will miss thanking some of them. I'd like to start by saying, "Thank you", to anyone I leave out below that I shouldn't have.

I'd like to thank my advisor, Prof. Lena Valavani, for all the support she's given me, and all her encouragement when I needed it. I'd also like to sincerely thank my committee, Profs. Gunter Stein, Munther Dahleh, and Wallace VanderVelde, for all their technical help and for supporting me through the trials of completing this long distance.

Next, I'd like to thank all of my old office mates at MIT, in particular Dragan Obradovic, Tony (T.O.) Rodriguez, Petros Voulgaris, Jeff Shamma, and Petros Kappasouris. They made my life at MIT a lot more enjoyable. I'd also like to thank Prof. Michael Athans, who was responsible for my coming to MIT in the first place. A sincere thanks also goes out to all of my friends, colleagues, and students at AFIT and the Flight Dynamics Laboratory.

Two people deserve special recognition. First, Major Curt Mracek, whose help in coding the numerics was invaluable to me. His advice and assistance were instrumental, and his help will not soon be forgotten. Next, I'd like to say thanks to one of the best friends anyone could ask for, Dr. Siva Banda. Even though both of our lives have gone through many changes and upheavals, his friendship was the one thing that always remained constant.

Of course, a heartfelt thanks to my mother and father, without whom this work would not have been possible. They both provided me with much needed support, each in their own unique ways. Yes, Dad, it's finally finished. I'd also like to say thanks to my wife's family, who always had an encouraging word to say just when I needed it.

Saving the best for last, I'd like to thank the two people who mean the most to me in this world. First, to my son Teddy, thanks for all the understanding you've shown me when I couldn't do things with you because I had to work on this. I'm really looking forward to making up for the lost time with you. Now I can leave going to school to you. Finally, the things I'd like to say in appreciation to my wife, Amy, really can't be put onto paper. Even though you entered my life right in the middle of all this,
I feel as though you've been there for me every step of the way. There's no way I can ever give back to you all the long days and nights I didn't get to be with you, but I want you to know that without your support, encouragement, understanding, and especially your love, I never could have finished this. This thesis is as much yours as mine, and since I can't put your name on it, the least I can do is dedicate it to you.

Dedicated to my wife, Amy, who has shown me what love truly is.
# TABLE OF CONTENTS

**ABSTRACT**  
2

**ACKNOWLEDGEMENTS**  
3

**LIST OF FIGURES**  
8

**LIST OF TABLES**  
12

**NOTATION**  
13

1. **INTRODUCTION**  
18

1.1 **Overview**  
18

1.2 **Review of Related Work**  
24

1.2.1 Bernstein & Haddad  
24

1.2.2 Zhou, Doyle, Glover & Bodenheimer  
25

1.2.3 Yeh, Banda & Chang  
25

1.2.4 Mustafa & Glover  
25

1.2.5 Khargonekar & Rotea  
26

1.2.6 Boyd  
26

1.2.7 Rotea & Khargonekar  
27

1.3 **Research Objectives and Contributions**  
29

1.4 **Thesis Outline**  
30

2. **MATHEMATICAL PRELIMINARIES**  
31

2.1 **State Space and Transfer Functions**  
31

2.2 **Stability Theory**  
33

2.3 **Spaces, Norms and Operators**  
36

2.3.1 The Two- and Infinity-Norms  
36

2.3.2 Computation of the Two- and Infinity-Norms  
37

2.3.3 Entropy  
39
2.4 Lyapunov Equations

2.5 Riccati Equations
  2.5.1 \( H_2 \) (LQG) Type
  2.5.2 \( H_\infty \) Type
  2.5.3 Examples
    2.5.3.1 Example One
    2.5.3.2 Example Two
    2.5.3.3 Example Three
    2.5.3.4 Example Four

3. \( H_2 \) AND \( H_\infty \) OPTIMIZATION
  3.1 \( H_2 \) Optimization
  3.2 \( H_\infty \) Optimization
  3.3 Connections between \( H_2 \) and \( H_\infty \) Optimization

4. MIXED \( H_2/H_\infty \) OPTIMIZATION
  4.1 General Derivation of the Mixed \( H_2/H_\infty \) Problem
  4.2 Fixed Order
    4.2.1 Full Order
    4.2.2 Increased Order
    4.2.3 Reduced Order

5. NUMERICAL SOLUTION OF THE MIXED \( H_2/H_\infty \)
   OPTIMIZATION PROBLEM
  5.1 Suboptimal Problem Derivation
  5.2 DFP Numerical Algorithm
6. SISO MIXED OPTIMIZATION EXAMPLE 133

6.1 Problem Set-Up 133

6.2 Mixed Results 153
6.2.1 \( \gamma = 2.2 \) Results 153
6.2.2 Other \( \gamma \) Results 158

6.3 Extensions 170
6.3.1 Order 170
6.3.2 Mixing Norms of the Same Transfer Function 171

7. MIMO MIXED OPTIMIZATION EXAMPLE 173

7.1 Problem Set-Up 173

7.2 Mixed Results 188

8. CONCLUSIONS AND RECOMMENDATIONS 193

8.1 Summary 193

8.2 Directions for Future Work 196

REFERENCES 198

APPENDIX A - Differentiation 206

APPENDIX B - Proof of Lemma 4.2.1 208

APPENDIX C - Computer Code 220
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-1</td>
<td>General nominal block diagram</td>
<td>18</td>
</tr>
<tr>
<td>1-2</td>
<td>Block diagram with uncertainty</td>
<td>20</td>
</tr>
<tr>
<td>1-3</td>
<td>General robust block diagram</td>
<td>21</td>
</tr>
<tr>
<td>1-4</td>
<td>Uncertainty block diagram</td>
<td>22</td>
</tr>
<tr>
<td>2-1</td>
<td>General nominal block diagram</td>
<td>33</td>
</tr>
<tr>
<td>2-2</td>
<td>Internal stability diagram</td>
<td>35</td>
</tr>
<tr>
<td>2-3</td>
<td>$H_\infty$ locus for Example 1</td>
<td>58</td>
</tr>
<tr>
<td>2-4</td>
<td>$H_2$ locus for Example 1</td>
<td>62</td>
</tr>
<tr>
<td>2-5</td>
<td>$H_\infty$ locus for Example 2</td>
<td>64</td>
</tr>
<tr>
<td>2-6</td>
<td>$H_2$ locus for Example 2</td>
<td>65</td>
</tr>
<tr>
<td>2-7</td>
<td>$H_\infty$ locus for Example 3</td>
<td>67</td>
</tr>
<tr>
<td>2-8</td>
<td>$H_\infty$ locus for Example 4</td>
<td>69</td>
</tr>
<tr>
<td>3-1</td>
<td>General $H_2$ feedback problem</td>
<td>70</td>
</tr>
<tr>
<td>3-2</td>
<td>$H_2$ feedback problem with scalings</td>
<td>72</td>
</tr>
<tr>
<td>3-3</td>
<td>$H_2$ suboptimal compensator diagram</td>
<td>74</td>
</tr>
<tr>
<td>3-4</td>
<td>$H_2$ optimal diagram with scalings to compensator</td>
<td>76</td>
</tr>
<tr>
<td>3-5</td>
<td>General $H_\infty$ feedback problem</td>
<td>77</td>
</tr>
<tr>
<td>3-6</td>
<td>$H_\infty$ feedback problem with scalings</td>
<td>79</td>
</tr>
<tr>
<td>3-7</td>
<td>Feedback problem after moving scalings to</td>
<td>83</td>
</tr>
<tr>
<td></td>
<td>compensator</td>
<td></td>
</tr>
<tr>
<td>3-8</td>
<td>General optimization problem</td>
<td>85</td>
</tr>
<tr>
<td>3-9</td>
<td>Closure of the $P-J$ loop through $H_\infty$</td>
<td>86</td>
</tr>
<tr>
<td></td>
<td>optimization</td>
<td></td>
</tr>
<tr>
<td>4-1</td>
<td>General mixed optimization problem</td>
<td>89</td>
</tr>
<tr>
<td>4-2</td>
<td>Local minima case with $\gamma &gt; \gamma_2$</td>
<td>106</td>
</tr>
<tr>
<td>4-3</td>
<td>Local minima case with $\gamma &lt; \gamma_2$</td>
<td>106</td>
</tr>
<tr>
<td>4-4</td>
<td>No local minima case with $\gamma \geq \gamma_2$</td>
<td>110</td>
</tr>
<tr>
<td>4-5</td>
<td>No local minima case with $\gamma &lt; \gamma_2$</td>
<td>110</td>
</tr>
<tr>
<td>5-1</td>
<td>$\text{tr}(X^T C)$ versus $\gamma$ for Example 2</td>
<td>122</td>
</tr>
<tr>
<td>5-2</td>
<td>$T_{zw}$ versus $T_{ed}$ plot for Theorem 5.1.1</td>
<td>125</td>
</tr>
<tr>
<td>5-3</td>
<td>Typical $T_{zw}$ versus $T_{ed}$ plot</td>
<td>126</td>
</tr>
<tr>
<td>5-4</td>
<td>Flow diagram for DPP</td>
<td>128</td>
</tr>
<tr>
<td>6-1</td>
<td>Mixed optimization block diagram</td>
<td>133</td>
</tr>
<tr>
<td>6-2</td>
<td>Magnitude plot of the SISO plant</td>
<td>134</td>
</tr>
</tbody>
</table>
Figure 6-3. Singular value plot of $T_{ed}$ for $K_{\infty 2,1426}$
Figure 6-4. Singular value plot of $K_{\infty 2,1426}$
Figure 6-5. Singular value plot of $T_{ZW}$ for $K_{\infty 2,1426}$
Figure 6-6. Singular value plot of $K_{2_{opt}}$
Figure 6-7. Singular value plots of $T_{ZW}$ for $K_{\infty 2,1426}$ and $K_{2_{opt}}$
Figure 6-8. Singular value plot of $T_{ed}$ for $K_{2_{opt}}$
Figure 6-9. Mixed optimization block diagram with $Q(s)$
Figure 6-10. Singular value plots of $T_{ed}$ for $H_2$ optimization on $P_{ed}$ and $P_{ZW}$
Figure 6-11. Singular value plots of $T_{ZW}$ for $H_2$ optimization on $P_{ed}$ and $P_{ZW}$
Figure 6-12. $G_{ed}$ at various $\gamma$ levels and for $H_2$ optimization on $P_{ed}$
Figure 6-13. $G_{ZW}$ at various $\gamma$ levels and for $H_2$ optimization on $P_{ed}$ & $P_{ZW}$
Figure 6-14. Infinity-norm vs. two-norm for $Q(s) = 0$
Figure 6-15. $G_{er}$ or $G_{vd}$ at various $\gamma$ levels
Figure 6-16. $G_{vr}$ at various $\gamma$ levels
Figure 6-17. $G_{zr}$ at various $\gamma$ levels
Figure 6-18. $G_{zw}$ at various $\gamma$ levels
Figure 6-19. Closure of the P-J loop through $H_{\infty}$ optimization
Figure 6-20. Singular value plot of $Q(s)$ for $\gamma = 4.5364$
Figure 6-21. Singular value plots of $T_{ed}$ for $K_{\infty 4.5364}$ and $K_{mix}$
Figure 6-22. Singular value plots of $T_{zw}$ for $K_{\infty 4.5364}$ and $K_{mix}$
Figure 6-23. Singular value plots of $Q(s)$ for various values of $\gamma \geq \gamma_2$
Figure 6-24. $T_{ed}$ comparison plot for $\gamma = 2.2$
Figure 6-25. $T_{ZW}$ comparison plot for $\gamma = 2.2$
Figure 6-26. Equivalence of K and P-J compensators
Figure 6-27. Block diagram used to compute $Q(s)$
Figure 6-28. $Q(s)$ plot for $\gamma = 2.2$
Figure 6-29. $T_{ed}$ comparison plot for $\gamma = 2.25$
Figure 6-30. $T_{ed}$ comparison plot for $\gamma = 2.35$
Figure 6-31. $T_{ed}$ comparison plot for $\gamma = 2.5$
Figure 6-32. $T_{ed}$ comparison plot for $\gamma = 2.75$
Figure 6-33. $T_{ed}$ comparison plot for $\gamma = 3.0$
Figure 6-34. $T_{ed}$ comparison plot for $\gamma = 3.25$
Figure 6-35. $T_{ed}$ comparison plot for $\gamma = 3.5$
Figure 6-36. $T_{ed}$ comparison plot for $\gamma = 4.0$
Figure 6-37. $T_{ZW}$ comparison plot for $\gamma = 2.25$
Figure 6-38. $T_{zw}$ comparison plot for $\gamma = 2.35$

Figure 6-39. $T_{zw}$ comparison plot for $\gamma = 2.5$

Figure 6-40. $T_{zw}$ comparison plot for $\gamma = 2.75$

Figure 6-41. $T_{zw}$ comparison plot for $\gamma = 3.0$

Figure 6-42. $T_{zw}$ comparison plot for $\gamma = 3.25$

Figure 6-43. $T_{zw}$ comparison plot for $\gamma = 3.5$

Figure 6-44. $T_{zw}$ comparison plot for $\gamma = 4.0$

Figure 6-45. Mixed $T_{ed}$ plot for $\gamma = 2.1426, 2.25, 2.5, 3, 3.5, 4,$ and $4.5364$

Figure 6-46. Mixed $T_{zw}$ plot for $\gamma = 2.1426, 2.25, 2.5, 3, 3.5, 4,$ and $4.5364$

Figure 6-47. $Q(s)$ plot for $\gamma = 2.25, 2.5, 3, 3.5, 4,$ and $4.5364$

Figure 6-48. Comparison of $Q(s) = 0$ and mixed compensators

Figure 6-49. Infinity-norm vs. two-norm for $Q(s) = 0$ for $w = d$ and $z = e$

Figure 7-1. Mixed optimization block diagram

Figure 7-2. Singular value plot of the MIMO plant

Figure 7-3. Singular value plot of $T_{ed}$ for $K_{\infty 2.3012}$

Figure 7-4. Singular value plot of $K_{\infty 2.3012}$

Figure 7-5. Singular value plot of $T_{zw}$ for $K_{\infty 2.3012}$

Figure 7-6. Singular value plot of $K_{2_{opt}}$

Figure 7-7. Singular value plots of $T_{zw}$ for $K_{\infty 2.3012}$ and $K_{2_{opt}}$

Figure 7-8. Singular value plot of $T_{ed}$ for $K_{2_{opt}}$

Figure 7-9. Mixed optimization block diagram with $Q(s)$

Figure 7-10. Singular value plots of $T_{ed}$ for $H_2$ optimization on $P_{ed}$ and $P_{zw}$

Figure 7-11. Singular value plots of $T_{zw}$ for $H_2$ optimization on $P_{ed}$ and $P_{zw}$

Figure 7-12. $G_{ed}$ at various $\gamma$ levels and for $H_2$ optimization on $P_{ed}$

Figure 7-13. $\bar{G}_{zw}$ at various $\gamma$ levels and for $H_2$ opt on $P_{ed}$ & $P_{zw}$

Figure 7-14. Blow-up of Figure 7-13

Figure 7-15. $G_{zw}$ at various $\gamma$ levels and for $H_2$ opt on $P_{ed}$ and $P_{zw}$

Figure 7-16. Infinity-norm vs. two-norm for $Q(s) = 0$

Figure 7-17. $G_{er}$ or $G_{vd}$ at various $\gamma$ levels

Figure 7-18. $G_{yr}$ at various $\gamma$ levels

Figure 7-19. $G_{zt}$ at various $\gamma$ levels

Figure 7-20. $G_{vw}$ at various $\gamma$ levels
Figure 7-21. Mixed $T_{ed}$ plot for various $\gamma$  189
Figure 7-22. Mixed $T_{zw}$ plot for various $\gamma$  190
Figure 7-23. Expanded view of Figure 7-22  190
Figure 7-24. Comparison of $Q(s) = 0$ and mixed compensators  192
LIST OF TABLES

Table 6-1. SISO Example Results, Q = 0 143
Table 6-2. SISO Example Full Results 169
Table 7-1. MIMO Example Results, Q = 0 184
Table 7-2. MIMO Example Full Results 191
NOTATION

\( \mathbb{R} \) field of real numbers

\( \mathbb{R}_+ \) positive real numbers

\( \mathbb{R}^n \) space of ordered \( n \)-tuples of real numbers

\( \mathbb{R}^{n \times m} \) set of \( n \times m \) matrices with elements in \( \mathbb{R} \)

\( \mathbb{C} \), \( \mathbb{C}^n \), \( \mathbb{C}^{n \times m} \) complex analog of \( \mathbb{R} \), \( \mathbb{R}^n \), \( \mathbb{R}^{n \times m} \)

\( x^T \), \( A^T \) vector/matrix transpose

\( A^{-1} \) inverse of \( A \)

\( A^+ \) Moore-Penrose pseudoinverse

\( A^\dagger \) complex conjugate transpose of \( A \)

\( A > 0 \ ( \geq 0 \) A is positive (negative) definite

\( A < 0 \ ( \leq 0 \) A is positive (negative) semidefinite

\( \sqrt{A} \) matrix square root of \( A \)

\( \lambda_i(A) \) eigenvalues of \( A \)

\( \sigma(A) \) spectrum (set of all eigenvalues) of \( A \)

\( \sigma_i(A) \) singular values of \( A \)

\( \sigma(A) \) maximum singular value of \( A \)

\( \rho(A) \) spectral radius of \( A \equiv \max_i \left| \lambda_i(A) \right| \)
\[ \det(A) \quad \text{determinant of } A \]

\[ \text{rank}(A) \quad \text{rank of } A \]

\[ \text{tr}(A) = \sum_{i=1}^{n} a_{ii} \quad \text{trace of } A \]

\[ \text{vec}(A) \quad \text{vector operator on } A \]

\[ \text{Im}(A) \equiv \{ y \in \mathbb{F}^m \mid y = Ax \text{ for some } x \in \mathbb{F}^n \} \quad \text{image of } A \]

\[ \text{Ker}(A) \equiv \{ x \in \mathbb{F}^n \mid Ax = 0 \} \quad \text{kernel of } A \]

\[ \frac{\partial A}{\partial B} \quad \text{partial of } A \text{ with respect to } B \]

\[ \text{Re}(x) \quad \text{real component of } x \]

\[ \text{Im}(x) \quad \text{imaginary component of } x \]

\[ \bar{x} \quad \text{complex conjugate of } x \]

\[ \mathcal{RL}_2 \quad \text{space of all real-rational, strictly proper transfer matrices (or vector signals) with no poles on the imaginary axis} \]

\[ \mathcal{RH}_2 \quad \text{space of all real-rational, strictly proper, stable transfer matrices (or vector signals)} \]

\[ \mathcal{RH}_2^\perp \quad \text{space of all real-rational, strictly proper, antistable transfer matrices (or vector signals)} \]

\[ \mathcal{RL}_\infty \quad \text{space of all real-rational, proper, transfer matrices with no poles on the imaginary axis} \]

\[ \mathcal{RH}_\infty \quad \text{space of all real-rational, proper, stable transfer matrices} \]

\[ \| \cdot \|_2 \quad \text{vector or matrix norm on } L_2 \]
\[ \| \cdot \|_\infty \quad \text{matrix norm on } L_\infty \]
\[ \| \cdot \| \quad \text{operator norm} \]
\[ I[ G(s), \gamma] \quad \text{entropy (at infinity) of } G(s) \text{ at } \gamma \]
\[ < \cdot, \cdot > \quad \text{inner product} \]
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]
\[
\text{transfer function matrix notation } \equiv C(sI - A)^{-1}B + D
\]
\[ G^*(s) \quad \text{complex conjugate transpose of } G(s) = G^T(-s) \]
\[ \text{Ric}(M) \quad \text{Riccati operator on Hamiltonian matrix } M \]
\[ \text{Ric}(M) \quad \text{extended Riccati operator on Hamiltonian matrix } M \]
\[ \mathcal{X}_-(M) \quad \text{stable spectral subspace of Hamiltonian matrix } M \]
\[ \mathcal{X}_+(M) \quad \text{antistable spectral subspace of Hamiltonian matrix } M \]
\[ \sigma(M|\nu) \quad \text{spectrum of } M \text{ restricted to subspace } \nu \]
\[ X_+ \quad \text{maximal solution of a Riccati equation} \]
\[ X_- \quad \text{minimal solution of a Riccati equation} \]
\[ \text{dom}(\cdot) \quad \text{domain of an operator} \]
\[ \text{span}(\cdot) \quad \text{span of a vector space} \]
\[ \text{inf} \quad \text{infimum} \]
\[ \lim \quad \text{limit} \]
\[ \ln \quad \text{natural logarithm} \]
\[ F_1(P, K) \quad \text{lower linear fractional transformation of } P \text{ with } K \]
a ≡ b  a identically equal to b, a defined as b

(A | B)  set of all A such that B
ARE  Algebraic Riccati Equation
DFP  Davidon-Fletcher-Powell
LFT  Linear Fractional Transformation
LQG  Linear Quadratic Gaussian
LQG/LTR  Linear Quadratic Gaussian with Loop Transfer Recovery
MIMO  Multiple-Input, Multiple-Output
SISO  Single-Input, Single-Output
\cap  intersected with
\subset  subset of
\epsilon  element of
\not\epsilon  not an element of
\exists  there exists
\forall  for all
\blacksquare  end of proof
s  Laplace variable
\omega  frequency variable
\delta_{ij}  dirac delta function ≡ 0 if i ≠ j, 1 if i = j
γ  value of the infinity norm
\[ \gamma_0 \quad \inf_{K \text{ admissible}} \| T_{ed} \|_{\infty} \]
\[ \gamma_2 \quad \| T_{ed} \|_{\infty} \text{ when } K(s) = K_{2\text{opt}} \]
\[ \gamma^* \quad \| T_{ed} \|_{\infty} \text{ when } K(s) = K_{\text{mix}} \]
\[ \alpha \quad \text{value of the two-norm} \]
\[ \alpha_0 \quad \inf_{K \text{ admissible}} \| T_{zw} \|_{2} \]
\[ \alpha^* \quad \| T_{zw} \|_{2} \text{ when } K(s) = K_{\text{mix}} \]
\[ \mu \quad \text{real number } \in [0, 1] \]
\[ n \quad \text{number of plant states} \]
\[ n_c \quad \text{number of controller states} \]
\[ J \quad \text{performance index} \]
\[ J_\mu \quad \text{performance index at a given } \mu \]
\[ L \quad \text{Lagrangian} \]
\[ L_\mu \quad \text{Lagrangian at a given } \mu \]
\[ K_{\text{mix}} \quad \text{a } K(s) \text{ that solves the mixed } H_2/H_\infty \text{ problem at some } \gamma \]
\[ K_{2\text{opt}} \quad \text{the unique } K(s) \text{ that makes } \| T_{zw} \|_{2} = \alpha_0 \]

**Definition:** Let \( A : \mathbb{C}^n \rightarrow \mathbb{C}^n \) be a linear transformation. A subspace \( S \subset \mathbb{C}^n \) is *invariant* for \( A \), or *\( A \)-invariant*, if \( Ax \in S \) for every vector \( x \in S \). In other words, \( S \) is \( A \)-invariant means that the image of \( S \) under \( A \) is contained in \( S \), or \( AS \subset S \).
CHAPTER 1
INTRODUCTION

1.1 Overview

Design of feedback controllers for linear, time-invariant systems, whether single-input single-output (SISO) or multiple-input multiple-output (MIMO), is a non-trivial task when uncertainties are included in the problem. As characterizations of uncertainty and performance specifications become more demanding, the design problem becomes more difficult. It has long been argued that uncertainty is the primary reason for using feedback control ([Bod45], [Hor63]). Over the past decade, a great deal of work has been done to directly incorporate uncertainty into "modern" control design techniques, but the problem still lacks a complete solution.

Control design can be broken down into two categories, each of which has two objectives. The first category is nominal control. Here, the designer would only consider a design on the nominal model of the plant. The two objectives are stability and performance - thus, we have the concepts of nominal stability and nominal performance. Given a plant model $G(s)$, the nominal stability requirement implies that the closed-loop system; i.e., the system with the designed feedback controller $K(s)$ in place, is asymptotically stable. A general block diagram is shown in Figure 1-1.

![Figure 1-1. General nominal block diagram](image-url)
For nominal stability, let \( z = y, w = 0, \) and \( P(s) = G(s) \). Thus, the nominal stability requirement would be that all of the poles of \( P(s)[I - P(s)K(s)]^{-1} \) be in the open left-half s-plane. A broader requirement which will be necessary later, but equivalent here, would be for the closed-loop system to be internally stable, which requires stability throughout the loop.

For nominal performance, let \( z \) be in general different from \( y \), and \( w \) not necessarily zero. Thus \( P(s) \) is not in general equal to \( G(s) \), but is still assumed to be completely known. Here, \( w \) would typically be "well known", such as step commands or white Gaussian noise of given intensity. A measure of performance is defined, and a \( K(s) \) is sought to provide nominal stability and to provide an adequate level of the performance measure. If the performance measure is mathematically quantified and a \( K(s) \) is sought to either minimize or maximize it, this is called optimal control.

The second category of feedback design is robust control. Here, the plant is recognized to be just a model of the true physical system, \( P \). Typically, \( P \) is parameterized as a set of \( P_1(s) \), or as \( P = P(s) + \Delta P \), where \( \Delta P \) is an unknown perturbation of the nominal. Robust stability is then the requirement that \( K(s) \) stabilize \( P \), not just \( P(s) \). Another way to model this is to include an uncertainty \( \Delta(s) \) in the system, as shown in Figure 1-2. Robust stability requires the \( K(s) \) in Figure 1-2 to internally stabilize the system for all possible \( \Delta(s) \) in a given set.

The last feedback design objective is that of robust performance. In general, this is the major problem most of the control community has been interested in for the past 10 years. Globally stated, robust performance requires closed-loop stability and acceptable performance for the nominal plant, as well as stability and guaranteed performance in the face of plant uncertainties. If uncertainties are modelled as in Figure 1-2, then the system must be internally stable and some performance measure must be satisfied for all \( \Delta(s) \) in an allowable set. This is, in general, an extremely difficult problem.

Much of the control community has acknowledged that \( \mu \)-synthesis ([Dai90], [Doy82], [Doy85], [Pac88]) is currently the best technique for addressing the robust performance problem. \( \mu \) is the structured singular value, which in general allows for nonconservative evaluation of the effects of uncertainties injected into the feedback loop at multiple locations. While the theory for analysis of \( \mu \) is now well developed, its
Figure 1-2. Block diagram with uncertainty

computation is still difficult, and more importantly, general synthesis techniques based on it are incomplete and highly numerical in nature. It is, therefore, important to consider related problems which may be solvable and lead to insights into the robust performance problem.

One such problem is based on a generalization of the block diagram in Figure 1-2. To allow a more general and realistic setting, allow two sets of exogenous inputs, d and w, and two sets of controlled outputs, e and z, where e and d are not necessarily equal to z and w, respectively. Thus, consider the block diagram in Figure 1-3, where P(s) and K(s) are linear time-invariant systems.

Let $T_{ed}$ and $T_{zw}$ denote the closed-loop transfer matrices from d to e and w to z, respectively. A typical measure of performance is the quadratic (LQG) performance index. This assumes the exogenous input w to be a zero-mean white Gaussian noise of fixed (known) intensity, and K(s) is designed to internally stabilize the system and minimize the energy of the output, $\|z\|_2$. This is equivalent to minimizing $\|T_{zw}\|_2$. K(s) is known as the $H_2$ optimal (LQG) controller. The major drawback to doing this alone is
that there is no guaranteed robustness ([Doy78]); gain and phase margins of LQG systems may be arbitrarily small.

In an effort to directly include robustness into LQG, the methodology known as LQG/LTR was developed ([DS79], [RB86], [SA87]). In LQG/LTR, the loop transfer function is made to asymptotically recover the full state loop, thereby achieving the excellent margins of the regulator or filter alone, in the limit. The recovery is done at the expense of LQG performance; thus, an implicit trade-off is made between LQG performance and robustness. This trade-off is made from an open-loop point-of-view, with closed-loop stability guaranteed at all points in the recovery process.

The LQG performance index is not the only performance measure that can be defined. Assume the exogenous inputs $d$ are modelled as deterministic with unknown but bounded energy. Then minimizing the energy of the output, $\| e \|_2$, to any $d$ such that $\| d \|_2 \leq 1$ is equivalent to minimizing $\| T_{ed} \|_\infty$. $K(s)$ is then known as the $H_\infty$ optimal controller. If $d$ is a bounded energy disturbance, then minimizing $\| T_{ed} \|_\infty$ guarantees a certain level of disturbance rejection.

A potentially more important application of the transfer matrix infinity-norm is robustness to plant variations. This is based on the Small Gain Theorem, given by

**Theorem 1.1.1 (Small Gain Theorem):** Let $T_{ed} \in \mathcal{RH}_\infty$. Assume $\Delta \in \mathcal{RH}_\infty$ is connected from $e$ to $d$ as shown in Figure 1-4. Then the closed-loop
system is internally stable if

\[ \| T_{ed}(s) \Delta(s) \|_\infty \leq \| T_{ed}(s) \|_\infty \| \Delta(s) \|_\infty < 1 \]

**Proof**: See [Zam66].

Thus, the smaller \( \| T_{ed} \|_\infty \) is made, the more uncertainty the system can tolerate and still be guaranteed to be stable. In particular, if \( K(s) \) is designed such that \( \| T_{ed} \|_\infty \leq \gamma \), the system is guaranteed to be robustly stable for all \( \Delta(s) \) such that \( \| \Delta \|_\infty < \gamma^{-1} \).

It may seem that minimizing \( \| T_{ed} \|_\infty \) is exactly what we want to do from a stability robustness point-of-view. \( H_\infty \) optimality may not be what we wish to achieve, however. Not only is the optimal \( H_\infty \) norm difficult to compute exactly due to the iterative nature of the problem, \( H_\infty \) optimal controllers tend to have undesirable characteristics compared to slightly suboptimal ones [ZDGB90]. In particular, consider a system whose optimal closed-loop infinity-norm is exactly one. Now consider two compensators which stabilize the closed-loop system and make its infinity-norm equal to 1.0001 and 1.02, respectively. From a robustness point-of-view, the two systems would be literally indistinguishable. However, the one with the slightly higher infinity-norm would typically have a much lower bandwidth, and the closed-loop two-norm would be much smaller. Therefore, we will consider \( \gamma \) as a design variable which specifies a desired level of stability robustness. Allowing increased \( \gamma \) releases degrees of freedom, which may be used to optimize some other measure of performance, which in this thesis will be \( \| T_{ZW} \|_2 \). A given controller \( K(s) \) is called *admissible* for \( P(s) \) if \( K(s) \) is real-rational, proper, and the
minimal realization of \( K(s) \) internally stabilizes \( P(s) \). The problem addressed in this thesis is therefore defined as follows:

The Mixed \( H_2/H_\infty \) Optimization Problem: For the plant \( P(s) \) in Figure 1-3, find an admissible \( K(s) \) that achieves

\[
\inf \left( \| T_{zw} \|_2 \mid K(s) \text{ admissible and } \| T_{ed} \|_\infty \leq \gamma \right)
\]

This is a problem of optimal nominal performance with robust stability. By choosing \( \gamma \), the designer has control over a direct trade-off between stability robustness and quadratic performance. Consider the case where \( e = z \) and \( d = w \), so that only one transfer function is involved. The mixed \( H_2/H_\infty \) problem is then effectively equivalent to LQG/LTR, except that the design is done from a closed-loop perspective and the trade-off between LQG performance and robustness is now explicit. Allowing \( e \neq z \) and \( d \neq w \) gives the designer freedom to specify performance and robustness requirements at different points in the feedback system, which LQG/LTR cannot do.

A great deal of work has been done on the mixed \( H_2/H_\infty \) problem, under various simplifying assumptions and even for "modified" versions of the problem. The work contained here generalizes the previous work to the case of two potentially independent sets of exogenous inputs and controlled outputs, uses general output feedback, and no overbounds to the two-norm are used. It is worth noting that in [KR91b], the problem addressed in this work is called the constrained optimal control problem (COCP). This author feels that the term COCP should be used for the general problem

\[
\inf \left( \| T_{zw} \|_{\alpha} \mid K(s) \text{ admissible and } \| T_{ed} \|_\beta \leq \gamma \right)
\]

for which mixed \( H_2/H_\infty \) optimization is a specific case.

Next, related work will be reviewed in order to highlight the contributions of this work and justify the comments just made.
1.2 Review of Related Work

1.2.1 Bernstein & Haddad

Perhaps the earliest work done in the mixed $H_2/H_\infty$ optimization area was done by Bernstein and Haddad ([BH89]). They considered the case of only one exogenous input; i.e., $d = w$. Also, the two controlled outputs cannot be completely independent of each other (the control penalty matrices must be scalar multiples of each other). Finally, rather than minimize $\| T_{zw} \|_2$ directly, they minimized an upper bound to it. Denote the upper bound by $I(T_{zw})$. Then their problem is to find an admissible $K(s)$ of a fixed order such that $K(s)$ achieves

$$\inf \left( I(T_{zw}) \mid \| T_{ew} \|_\infty < \gamma \right) \tag{1.1}$$

The fixed order requirement comes about due to their use of Lagrange multiplier methods to solve the problem. That is, they derive necessary conditions for optimality of controllers of prespecified order which satisfy (1.1). While this is lacking in its ability to give sufficient conditions for optimality and to identify the optimal controller order, it is very useful in that optimal reduced-order controllers are a direct by-product. From a practical point-of-view, a reduced-order controller will most likely be desired. Therefore, the major limitations of this technique are considered to be:

1) only one exogenous input is allowed, and the controlled outputs cannot be completely independent

2) the solution only minimizes an upper bound to the two-norm of $T_{zw}$, and it is not known how tight the upper bound is, in general

3) only necessary conditions are obtained

4) an analytical solution is not obtained - the solution is a set of coupled nonlinear matrix (Riccati-like) equations. However, these are solvable by homotopy methods ([MB85], [RC90], [Ric87]) when the $H_\infty$ control penalty is set to zero
1.2.2 Zhou, Doyle, Glover & Bodenheimer

In a paper by Zhou, Doyle, Glover and Bodenheimer ([ZDGB90]), which was a follow-on to the work in [DZB89], a related problem to that of Bernstein and Haddad was considered. In their work, it was assumed that $e = z$, but $d$ and $w$ may be different. Here, $d$ and $w$ can be completely independent. The signal $w$ was assumed to be a white noise input while $d$ was a deterministic signal of unknown but bounded energy. This gives an insightful physical interpretation to the problem, since in real systems some signals will be well modelled as white noises while others (commands) do not fit into this class very well and are better modelled as bounded energy signals. Rather than set up the problem as a Lagrange multiplier problem, they define a new performance index such that when $d \equiv 0$, the induced norm is $\| T_{zw} \|_2$ and when $w \equiv 0$, the induced norm is $\| T_{zd} \|_\infty$. For both nonzero, a mixed problem is obtained. Necessary and sufficient conditions are derived for the existence of a mixed controller, and the parameterization of the controller requires the solution of coupled matrix equations very similar in form to those of Bernstein and Haddad, but with an additional coupling equation since $d$ and $w$ are allowed to be independent. The reason for the similarity was shown in the following work.

1.2.3 Yeh, Banda & Chang

In a paper by Yeh, Banda, and Chang ([YBC90]), it was shown that for the order of the compensator equal to the order of the plant, the two problems in the preceding sections are the duals of each other. That is, even though they are derived from different algebraic frameworks, in the full order case, the two optimality criteria are duals. Thus, Bernstein and Haddad's necessary conditions are also sufficient, and the problems are equivalent. This also yields a solution (in the full order case) to Bernstein and Haddad's problem when $e$ and $z$ are independent. However, in this case, the homotopy approach to a solution does not apply.

1.2.4 Mustafa & Glover

In a paper by Mustafa and Glover ([MG88]), expanded upon in [MG90], another related problem was addressed. First, they assumed that both $e = z$ and $d = w$, so that $T_{ed} = T_{zw}$. Under this assumption, the mixed $H_2/H_\infty$ optimization problem becomes that of finding the controller in the family of all suboptimal $H_\infty$ controllers which has the smallest resulting closed-loop two-norm. As previously stated, this is effectively equivalent.
to the objective of LQG/LTR. Instead of solving this problem, they defined and solved the minimum entropy/H∞ control problem, where the entropy of a transfer function will be defined in more detail in Chapter 2 of this work. For now, it suffices to say that the entropy is an upper bound to the H₂ (LQG) cost, and minimum entropy/H∞ control contains the two extremes of H₂ and H∞ optimal control. The key developments Mustafa and Glover showed in relation to this thesis are that the upper bound in Bernstein and Haddad's work is precisely the entropy, so that specializing their work to the Tₐₑ = Tᵢₙ case yields the minimum entropy/H∞ controller. Furthermore, this controller is the "central" H∞ optimal controller ([DGKF89]), which will be described in detail in Chapter 3. This controller requires the solution of two uncoupled Riccati equations, and thus possesses an analytical solution. However, even for the Tₐₑ = Tᵢₙ case, this is not necessarily the mixed H₂/H∞ controller, since the entropy is an upper bound to the two-norm. The bound becomes tight when γ is large, but could be very "loose" when γ is small.

1.2.5 Khargonekar & Rotea

In the work of Khargonekar and Rotea ([KR91b]), the full order problem of Bernstein and Haddad was rigorously transformed into a convex optimization problem. This approach provides an effective alternative to solving the coupled nonlinear equations of Bernstein and Haddad. Software for solving convex optimization problems is readily available, and this formulation is very powerful in that local minima are nonexistent due to convexity. It was shown that in the case of full state feedback, Bernstein and Haddad's problem has an arbitrarily close to optimal static gain solution, which may be found through reduction to a convex programming problem over a bounded set of real matrices. The full order output feedback case was then shown to reduce to an H∞ estimator problem together with a state feedback problem as described above. It was also shown that there always exists an output feedback controller of order no greater than that of the plant, provided a solution exists at all. This final result is not surprising, since [ZDGB90] showed this result and it is the dual problem of Bernstein and Haddad.

1.2.6 Boyd

The software program QDES, which is described in [BB91], [BBB+89], and [BBN89], is a very powerful convex optimization control design package. It allows minimization of a convex objective function subject to convex constraints. The mixed H₂/H∞ optimization problem falls into this class of
problems. However, there are a number of drawbacks to using this approach. First, as this is a purely numerical solution technique, general analytical-type results are extremely difficult to obtain. Secondly, the current version of the program requires digitization of an analog plant, with a digital controller being produced. Since $H_\infty$ optimization tends to produce high bandwidth controllers, digitization effects become severe. Also, since the two-norm is involved and finite-length impulse responses are used, non-strictly proper compensators can result which produce closed-loop systems with infinite two-norms. Another problem is that the current version of the program cannot compute singular values, so for a MIMO system the $H_\infty$ constraint cannot be enforced. This capability would need to be added, and a modification to allow continuous systems would also be desired. Finally, the singular value constraint would have to be exact, not a magnitude approximation as exists in the current program. This is due to the optimal being achieved on the boundary of the constraint, as the work in this thesis will show.

1.2.7 Rotea & Khargonekar

The work of Rotea and Khargonekar ([RK91]) constitutes the first work done towards solving a special case of the true (nonconservative) mixed $H_2/H_\infty$ optimization problem. That is, no upper bound to the two-norm is used. They also allow for two sets of exogenous inputs and controlled outputs, as in the general problem shown in Figure 1-3. However, they do restrict to the case of full state availability, so that $y = x$. They define two problems:

**Problem A:** Mixed $H_2/H_\infty$ optimization. Find a $K(s)$ which achieves

$$\inf \{ \| T_{zw} \|_2 \mid K(s) \text{ admissible and } \| T_{ed} \|_\infty < \gamma \}$$

This is the problem addressed in this thesis, when the assumption of full state availability is removed.

**Problem B:** Simultaneous $H_2/H_\infty$ optimization. Assuming it is nonempty, from the family of all $K(s)$ which achieve

$$\inf \{ \| T_{zw} \|_2 \mid K(s) \text{ admissible } \}$$

find one that also satisfies $\| T_{ed} \|_\infty < \gamma$. 

27
Note that if a $K(s)$ exists which solves Problem B, then that $K(s)$ must also satisfy Problem A, since a solution to Problem B achieves the global minimum two-norm of $T_{ZW}$ as well as meeting the infinity-norm bound on $T_{ed}$. Thus, Problem B provides sufficient conditions for the existence of a solution to Problem A, under the $y = x$ assumption. One solution to

$$\inf \left( \| T_{ZW} \|_2 \mid K(s) \text{ admissible} \right)$$

is the well-known (LQR) static feedback gains found by solving a Riccati equation. Assuming this as a solution would fix $T_{ed}$, and thus Problem B would have a solution for any $\gamma$ greater than the resulting $\| T_{ed} \|_\infty$, say $\gamma_2$. Rotea and Khargonekar show that under certain conditions, there is also a family of dynamic compensators which achieve the same minimum two-norm. The additional freedom provided by these dynamics can often be used to reduce the $\gamma$ for which Problem B has a solution to less than $\gamma_2$.

Given this family of dynamic compensators, necessary and sufficient conditions for the existence of solutions to Problem B are given, which require the solutions to three uncoupled Riccati equations. One is an $H_2$-type Riccati equation (to find the LQR gains), while the other two are $H_\infty$-type (to meet the infinity-norm bound). Riccati equations "types" will be explained in more detail in Chapter 2. Thus, the solution to Problem B uses a new characterization of all solutions to the $H_2$ state feedback problem, which consists of the "usual" LQR static solution as well as a family of dynamic compensators.

In general, the sufficient conditions for the existence of a controller which solves Problem A provided by the existence of a solution to Problem B are too restrictive. That is, a solution to Problem A is not, in general, a solution to Problem B, since Problem A is not required to achieve the unconstrained minimal two-norm. However, when $\text{Im}(B_1) \cap \text{Im}(B_2) = 0$ (independence of the two exogenous inputs), a solution to Problem A exists iff a solution to Problem B exists, and they are the same. Even though solutions are not unique, it was shown that there are cases when all solutions must be dynamic, even though the full state is available. This leads the authors to conjecture that in the output feedback case, controllers of higher order than the plant may be required. This conjecture, as well as the solution to Problem B in the output feedback case, will be addressed later in this thesis.

28
1.3 Research Objectives and Contributions

The purpose of this work is to develop a synthesis method to minimize the two-norm of one transfer function while ensuring the infinity-norm of another is held below a chosen level. Of course, internal stability of the closed-loop system is also required. Many researchers have been actively involved in this area, as detailed in Section 1.2. These methods have not been able to solve the general problem – that of an output feedback mixed controller which minimizes the actual two-norm of one transfer function while bounding the infinity-norm of a potentially completely independent second transfer function.

The first contribution of this thesis is to provide a better understanding of the nature and existence of solutions to $H_\infty$-type Riccati equations. Of particular interest in this work are neutrally stabilizing solutions to such equations, which to this author's knowledge no computer software currently available can find directly. An integrally related topic is that of solutions to Lyapunov equations with neutrally stable $A$ matrices and no constant terms. Both of these topics are examined and methods for finding such solutions are investigated.

The second contribution is that of analytically developing the necessary conditions which a mixed $H_2/H_\infty$ controller must satisfy. These consist of seven coupled nonlinear matrix equations. From these equations, several powerful results are obtained. First, regions where the two objectives compete and where they do not for the case of a controller of equal order to the plant are defined. In the region where they do not compete (where the infinity-norm constraint is inactive), the solution is the unique $H_2$ optimal controller. In the region where they compete, it is shown that the solution always lies on the boundary of the $H_\infty$ constraint, thus requiring the neutrally stabilizing solution to a Riccati equation.

Finally, since no analytical solution to the set of seven equations is known, a suboptimal approach is developed which avoids the neutral stability problems. This leads to a numerical solution algorithm for the necessary conditions through Davidson-Fletcher-Powell optimization. Two academic examples, one SISO and one MIMO, are developed to analyze the nature of the mixed controller and the potential benefits of using such a controller.
1.4 Thesis Outline

This thesis is divided into eight chapters, including the present introductory chapter. Chapter 2 highlights some of the essential mathematics necessary to develop and analyze the problem, as well as to collect numerous theorems on Lyapunov and Riccati equations. Also included are some new insights on solutions to Riccati equations, illustrated through several examples. Chapter 3 then summarizes the basic state space $H_2$ and $H_\infty$ optimization results appearing in the literature, as well as defining assumptions and nomenclature for the remainder of the thesis.

Chapter 4 formally defines the mixed $H_2/H_\infty$ optimization problem and formulates it as a Lagrange multiplier problem. From this, the necessary conditions a mixed controller must satisfy are derived. These equations are analyzed in detail for the case of a controller of equal order to that of the plant, with the major result being that the solution is either the global $H_2$ optimal controller or one for which the infinity-norm constraint is an equality constraint. Chapter 5 then proceeds to develop a suboptimal approach to solving the mixed problem which avoids numerical difficulties at mixed optimality, as well as a numerical algorithm for implementing it.

Chapter 6 performs mixed $H_2/H_\infty$ optimization on a single-input, single-output example. The results show very clearly exactly what the mixed controller does, and the potential benefit in using this controller. The mixed controller is computed over a wide range of infinity-norm bound choices, which a designer would need to do to determine the best mix of $H_2$ and $H_\infty$ performance. Chapter 7 does the same for a multiple-input, multiple-output example. The results are similar in nature but even more dramatic in the sense of showing the benefit of designing a mixed controller.

Finally, Chapter 8 concludes the work by summarizing the results and giving recommendations for future research in this area.
CHAPTER 2
MATHEMATICAL PRELIMINARIES

2.1 State Space and Transfer Functions

Throughout this section, consider the system

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &=Cx + Du
\end{align*}
\] (2.1)

where \(x(t) \in \mathbb{R}^n\), \(u(t) \in \mathbb{R}^m\), and \(y(t) \in \mathbb{R}^p\) are the state, input, and output vectors. \(A, B, C,\) and \(D\) are real matrices of the appropriate dimensions.

The stability of (2.1) is determined by the eigenvalues of \(A\). Define the following terms:

i) (2.1) is **stable** if \(\text{Re} \lambda_i(A) < 0 \ \forall \ i\)

ii) (2.1) is **neutrally stable** if \(\text{Re} \lambda_i(A) \leq 0 \ \forall \ i\) and \(\text{Re} \lambda_i(A) = 0\) for some \(i\)

iii) (2.1) is **unstable** if \(\text{Re} \lambda_i(A) > 0\) for some \(i\) and \(\text{Re} \lambda_j(A) < 0\) for some \(j\)

iv) (2.1) is **neutrally antistable** if \(\text{Re} \lambda_i(A) \geq 0 \ \forall \ i\) and \(\text{Re} \lambda_i(A) = 0\) for some \(i\)

v) (2.1) is **antistable** if \(\text{Re} \lambda_i(A) > 0 \ \forall \ i\)

Furthermore, we will apply these terms and definitions to the \(A\) matrix itself, as well. That is, if \(\text{Re} \lambda_i(A) < 0 \ \forall \ i\), we will also say that \(A\) is stable.

Next, we introduce transfer functions and their properties. Taking the Laplace transform of (2.1) yields

\[
y(s) = G(s)u(s)
\]

where
\[ G(s) = C(sI - A)^{-1}B + D \]

As a shorthand notation, let

\[
\begin{bmatrix}
A & B \\
\hline
C & D
\end{bmatrix} \equiv C(sI - A)^{-1}B + D
\]

Conversely, starting with a real rational \( G(s) \) which is proper, i.e., analytic at \( s = \infty \), there exists a quadruple \((A, B, C, D)\) such that

\[ G(s) = \begin{bmatrix}
A & B \\
\hline
C & D
\end{bmatrix} \]

\((A, B, C, D)\) is called a realization of \( G(s) \), and is minimal if \( A \) has the minimum possible dimension. A realization is minimal iff \((A, B)\) is controllable and \((C, A)\) is observable.

The following are some useful formulae for transposes and conjugates of transfer functions and their properties:

\[ G^T(s) = \begin{bmatrix}
A^T & C^T \\
\hline
B^T & D^T
\end{bmatrix} \quad G(-s) = \begin{bmatrix}
-A & B \\
\hline
-C & D
\end{bmatrix} \]

\[ G^*(s) = G^T(-s) = \begin{bmatrix}
-A^T & -C^T \\
\hline
B^T & D^T
\end{bmatrix} \]

\[ \text{poles}[G(s)] = \text{poles}[G^T(s)] = -\text{poles}[G(-s)] = -\text{poles}[G^*(s)] \]

For \( G \in \mathbb{C}^{n \times m} \), let \( p = \min(n, m) \). Then for \( i = 1, 2, \ldots, p \)

\[ \sigma_i[G(s)] = \sigma_i[G^T(s)] = \sigma_i[G(-s)] = \sigma_i[G^*(s)] \]
2.2 Stability Theory

In this section, we will examine the concept of stability of a closed-loop feedback system, both from a transfer function and a state space viewpoint. A general feedback diagram considered throughout this thesis has the form shown in Figure 2-1.

![General nominal block diagram](image)

Figure 2-1. General nominal block diagram

Let $P(s)$ and $K(s)$ be known proper transfer function matrices. The function-of-s notation will often be dropped. Then Figure 2-1 represents the equations

$$
\begin{bmatrix}
  z \\
  y
\end{bmatrix} = P
\begin{bmatrix}
  w \\
  u
\end{bmatrix} \quad \quad u = Ky
$$

where $P$ can be partitioned as

$$
P = \begin{bmatrix}
P_{zw} & P_{zu} \\
P_{yw} & P_{yu}
\end{bmatrix}
$$

The system is well-posed iff $[I - K(\infty)P_{yu}(\infty)]$ is invertible, which is assumed throughout the remainder of this thesis. Well-posedness is equivalent to all the signals in the system being physically realizable. Typically, the above inverse will trivially exist since we assume $P_{yu}(s)$ and/or $K(s)$ is strictly proper, i.e., $P_{yu}(\infty) = 0$ and/or $K(\infty) = 0$.
We can also write Figure 2-1 in state space form, as

\[
P = \begin{bmatrix}
A & B_w & B_u \\
C_z & D_{zw} & D_{zu} \\
C_y & D_{yw} & D_{yu}
\end{bmatrix}
\tag{2.2}
\]

\[
K = \begin{bmatrix}
A_c & B_c \\
C_c & D_c
\end{bmatrix}
\tag{2.3}
\]

Notice that well-posedness is then equivalent to \([I - D_c D_{yu}]\) being invertible. The open-loop system is stable if \(A\) is stable. The closed-loop system is said to be \textit{internally stable} if its "A" matrix is stable. The following theorems give some basic conditions for internal stability.

\textbf{Theorem 2.2.1}: Assume the realization of \(P\) in (2.2) is minimal. Then there exists a proper \(K\) achieving internal stability in Figure 2-1 iff \((A, B_u)\) is stabilizable and \((C_y, A)\) is detectable.

\textbf{Proof}: See [Fran87], Chapter 4.

We will define a controller \(K\) which achieves internal stability to be \textit{admissible}, or say that \(K\) is a \textit{stabilizing controller}. If such a \(K\) exists, \(P\) is said to be \textit{stabilizable}.

\textbf{Theorem 2.2.2}: \(K\) is a stabilizing controller for \(P\) iff \(K\) is a stabilizing controller for \(P_{yu}\).

\textbf{Proof}: See [Fran87], Theorem 4.2.

Theorem 2.2.2 says that we only need to consider \(P_{yu}\) in analyzing internal stability of Figure 2-1. Consider Figure 2-2, for which internal stability implies that for all bounded inputs \((v_1, v_2)\), the outputs \((e_1, e_2)\) are bounded. Figure 2-1 is internally stable iff Figure 2-2 is, and

\textbf{Theorem 2.2.3}: The system in Figure 2-2 is internally stable iff \([I - P_{yu}K]\) is invertible and all four transfer functions in
\[
\begin{bmatrix}
  I & -K \\
-\mathcal{P}_{yu} & I
\end{bmatrix}^{-1} = \begin{bmatrix}
  I + K[I - \mathcal{P}_{yu}K]^{-1}\mathcal{P}_{yu} & K[I - \mathcal{P}_{yu}K]^{-1} \\
[I - \mathcal{P}_{yu}K]^{-1}\mathcal{P}_{yu} & [I - \mathcal{P}_{yu}K]^{-1}
\end{bmatrix}
\]

which transfers \((v_1, v_2)\) to \((e_1, e_2)\), are proper and stable.

![Diagram](image)

Figure 2-2. Internal stability diagram

**Proof**: See [Fran87], Chapter 4.

To conclude this section, consider once again Figure 2-1, with \(P\) and \(K\) minimal. The closed-loop transfer function, denoted by \(T_{zw}\), is a linear fractional transformation (LFT) of \(P\) and \(K\). It is given by

\[
T_{zw} = \hat{F}_1(P, K) = P_{zw} + P_{zu}K[I - \mathcal{P}_{yu}K]^{-1}\mathcal{P}_{yw}
\]

where \(\hat{F}_1(P, K)\) denotes the lower LFT of \(P\) with \(K\) (i.e., \(K\) "below" \(P\)). Given the state space descriptions of \(P\) and \(K\) in (2.2) and (2.3), \(T_{zw}\) is given in state space by

\[
T_{zw} = \begin{bmatrix}
\tilde{A} & \tilde{B} \\
\tilde{C} & \tilde{D}
\end{bmatrix}
\]

where

\[
\tilde{A} = \begin{bmatrix}
A + BuZ_1Dc_y & BuZ_1c_c \\
B_cZ_2c_y & A_c + B_cZ_2D_{yu}c_c
\end{bmatrix}
\]
\[ \tilde{B} = \begin{bmatrix} B_w + B_u Z_1 D_c D_Y w \\ B_c Z_2 D_Y w \end{bmatrix} \]

\[ \tilde{C} = \begin{bmatrix} C_z + D_{zu} D_c Z_2 C_Y \\ D_{zu} Z_1 C_c \end{bmatrix} \]

\[ \tilde{D} = \begin{bmatrix} D_{zw} + D_{zu} D_c Z_2 D_Y w \end{bmatrix} \]

and

\[ Z_1 = [I - D_c D_Y u]^{-1} \]

\[ Z_2 = [I - D_Y u D_c]^{-1} \]

2.3 Norms and Entropy

2.3.1 The Two- and Infinity-Norms

This section is not meant to be a complete treatment of normed spaces or operators, but rather only to define and highlight the specific portions used in this work. For a more complete treatment, see [Fran87], [Lue69], or [NS82].

In this thesis, two major norms will be considered. These norms will be motivated by the input/output spaces that generate them. Assume the input of a system, \( w \), to be zero-mean white Gaussian noise of unit intensity (or a unit impulse at a fixed frequency). Suppose we wish to find the (expected) value of the energy of the output, \( z \), which is given by the vector two-norm. It is relatively straightforward to show that

\[ \| z \|_2 = \| T_{zw} \|_2 \quad \text{where} \quad z = T_{zw} w \]

and where the two-norm of \( T_{zw} \) is defined as

\[ \| T_{zw} \|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}[T_{zw}(j\omega)^* T_{zw}(j\omega)] \, d\omega \]

The other major norm is the infinity-norm, which is an induced operator norm. Thus, a brief discussion of operators and induced norms is appropriate.
Let \( \mathbf{x} \) and \( \mathbf{y} \) be linear vector spaces. An operator \( T \) from \( \mathbf{x} \) to \( \mathbf{y} \) is a mapping of a subset \( S \) of \( \mathbf{x} \) into \( \mathbf{y} \), or

\[
T : S \subset \mathbf{x} \longrightarrow \mathbf{y}
\]

The operator \( T \) associates with every element \( x \in S \) a unique element \( y = Tx \in \mathbf{y} \). \( S \) is called the domain of \( T \), denoted \( \text{dom}(T) \). The image of \( T \) is defined by

\[
\text{Im}(T) = \{ y \in \mathbf{y} \mid y = Tx, x \in \mathbf{x} \} \subset \mathbf{y}
\]

Now suppose we choose the "input" and "output" spaces \( \mathbf{x} \) and \( \mathbf{y} \) to be \( L_2 \). Then, for \( z = T_{zw} w \),

\[
\| T_{zw} \| = \| T_{zw} \|_\infty = \sup_{w \neq 0} \frac{\| T_{zw} w \|_2}{\| w \|_2} = \sup_{\| w \|_2 \leq 1} \| z \|_2
\]

\[
= \sup_{\omega} \sigma(T_{zw})
\]

Thus, if we assume a deterministic input of unknown but bounded energy, and we wish to find the maximum energy of the resulting output, it is equal to the maximum "gain" of the transfer function. To minimize the energy of the output in the face of the worst possible bounded energy input, just minimize \( \| T_{zw} \|_\infty \). Also, the submultiplicative property of operator norms says that if \( F, G \in \mathcal{R}L_\infty \), then

\[
\| FG \|_\infty \leq \| F \|_\infty \| G \|_\infty
\]

which does not hold for the two-norm. This is very important from a robustness viewpoint, as detailed in Chapter 1 of this thesis.

### 2.3.2 Computation of the Two- and Infinity-Norms

Calculation of the two- and infinity-norms will now be discussed. First, consider calculation of the two-norm of a transfer function, as given by
\[ \| G(j\omega) \|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}[G^*(j\omega)G(j\omega)] \, d\omega \] (2.4)

The form of (2.4) is not amenable to easy computation. However, it is a fairly standard result that for

\[ G(s) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \in \mathcal{RH}_2 \]

the two-norm is given by

\[ \| G(s) \|_2^2 = \text{tr}[L_c C^T C] = \text{tr}[L_0 BB^T] \] (2.5)

where \( L_c \) and \( L_0 \) are the controllability and observability gramians of \( G(s) \), respectively. The gramians are the positive semidefinite solutions to the Lyapunov equations

\[ 0 = AL_c + L_c A^T + BB^T \]
\[ 0 = L_0 A + A^T L_0 + C^T C \]

Lyapunov equations will be described in more detail in Section 2.4. If \( G(s) \in \mathcal{RL}_2 \), write

\[ G(s) = G_1(s) + G_2(s) \]

where \( G_1 \in \mathcal{RH}_2 \) and \( G_2 \in \mathcal{RH}^\perp_2 \). Since \( \langle G_1, G_2 \rangle = 0 \) (\( \langle \cdot, \cdot \rangle \) is the inner product on \( \mathcal{RL}_2 \)),

\[ \| G(s) \|_2^2 = \| G_1(s) \|_2^2 + \| G_2(s) \|_2^2 \]

The two-norm of \( G_1 \) is computable from (2.5), and

\[ \| G_2(s) \|_2^2 = \| G_2^s(s) \|_2^2 \]

where \( G_2^s(s) \in \mathcal{RH}_2 \) and thus is also computable from (2.5).
Next, consider the infinity-norm, given by

\[ \| G(j\omega) \|_\infty = \sup_{\omega} |G(j\omega)| \] (2.6)

The most obvious way to compute the infinity-norm would be to compute \( |G(j\omega)| \) over a large number of frequency points and then select the largest value. Refinements of the frequency grid would most likely be desired if there was a "spike" in the \( \sigma \)-plot. This approach is easy to do on a computer, but is basically ad-hoc and suffers from producing a lower bound to the true norm with no upper bound available. A more precise and algorithmically useful approach is provided by checking eigenvalues of an associated Hamiltonian matrix, which will be discussed in more detail in Section 2.5.

### 2.3.3 Entropy

The final topic in this section is that of the entropy of a system. Let \( G \in \mathfrak{R}_{L_\infty} \) and let \( \| G \|_\infty < \gamma \). The entropy (at infinity) of \( G(s) \) is defined by

\[
I[G(s), \gamma] = \lim_{s_0 \to \infty} \left\{ -\frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \ln |\det[I - \gamma^2 G^*(j\omega)G(j\omega)]| \left[ \frac{2}{s_0 + \omega^2} \right] d\omega \right\}
\]

The entropy satisfies:

i) \( I[G(s), \gamma] \geq 0 \)

ii) \( I[G(s), \gamma] = 0 \) iff \( G(s) = 0 \)

iii) \( I[G(s), \gamma] < \infty \) iff \( G(\infty) = 0 \)

If \( G(\infty) = 0 \), \( I[G(s), \gamma] \) is a monotonically decreasing function of \( \gamma \). Some of the important connections between entropy and the two- and infinity-norms are given in the following theorems.
**Theorem 2.3.1** : Let $G \in \mathcal{RH}_2$ and let $\gamma$ be such that $\|G\|_\infty < \gamma$. Then $I[G(s), \gamma] \geq \|G(s)\|_2^2$ and $I[G(s), \infty] = \|G(s)\|_2^2$.

**Proof** : See [MG90], Theorem 2.4.4.

Theorem 2.3.1 shows that the entropy is an overbound to the two-norm, and is equal to it as $\gamma \to \infty$. The next theorem relates computation of the entropy to solution of an *algebraic Riccati equation* (ARE).

**Theorem 2.3.2** : Let $G \in \mathcal{RH}_2$ with $\|G\|_\infty < \gamma$ and $G(s) = \begin{bmatrix} A & B \\ \hline C & 0 \end{bmatrix}$.

Then

$$I[G(s), \gamma] = \text{tr}\{ QC^T C \}$$

where $Q = Q^T > 0$ is the stabilizing solution to the ARE

$$0 = AQ + QA^T + \gamma^2 QC^T C Q + BB^T$$

**Proof** : See [MG90], Lemma 5.3.2.

ARE's will be discussed in more detail in Section 2.5.

### 2.4 Lyapunov Equations

First, a theorem on uniqueness of solutions to a Lyapunov equation.

**Theorem 2.4.1** : If $Z \geq 0$ and $A$ is stable, the Lyapunov equation

$$A^T X + XA + Z = 0$$

has a unique solution, and $X \geq 0$.

**Proof** : See [Won85], Lemma 12.1.
Next, two theorems are given which relate existence of solutions to the stability of the A matrix.

**Theorem 2.4.2** : Suppose $X \geq 0$, $Z \geq 0$, $(\sqrt{Z}, A)$ is detectable and

$$A^T X + XA + Z = 0$$

Then A is stable. If $(\sqrt{Z}, A)$ is observable, then $X > 0$.

**Proof** : See [Won85], Lemma 12.2.

**Theorem 2.4.3** - The Lyapunov equation

$$A^T X + XA + C^T C = 0$$

has a unique solution iff A is stable. If A is stable, then $X = 0$ is the only solution to

$$A^T X + XA = 0$$

**Proof** : See [SZ70], Theorem 2.1.

The final part of Theorem 2.4.3 will be useful in Chapter 4, as well as its extension to the case where A has one or more eigenvalues on the imaginary axis. This will be explored next.

**Definition** : Let $A$ be any square matrix. Let $\lambda$ be an eigenvalue of $A$ and $k$ be the dimension of the Jordan block corresponding to $\lambda$. Then there exist $k$ linearly independent vectors $(x_1, x_2, \ldots, x_k)$ that satisfy

$$Ax_1 = \lambda x_1$$

$$Ax_j = \lambda x_j + x_{j-1} \quad 2 \leq j \leq k$$

The set $(x_1, x_2, \ldots, x_k)$ is a characteristic chain of $A$ related to $\lambda$. Any member of a chain is an eigenvector.
Definition: Let \( p \) denote the number of characteristic chains of the square matrix \( A \) associated with zero eigenvalues, and \( q \) be the number associated with \( j\omega \)-axis eigenvalues (but not at the origin). The letters \( \beta \) and \( \zeta \) stand for integers such that \( 0 \leq \beta \leq p \) and \( 0 \leq \zeta \leq \frac{q}{2} \).

**Theorem 2.4.4:** Let \( A \) be any real square matrix whose eigenvalues have a zero real part. Let \( X \) be a nonnegative real matrix such that

\[
A^TX +XA = 0
\]

Then \( \text{rank}(X) = \beta + 2\zeta \).

**Proof:** See [SZ70], Lemma 4.1.

**Theorem 2.4.5:** Let \( A \) be any nonzero real square matrix whose eigenvalues have a zero real part. Then, given \( \beta + 2\zeta \neq 0 \), the equation

\[
A^TX +XA = 0
\]

has infinitely many real nonnegative solutions \( X \) whose rank is \( \beta + 2\zeta \).

**Proof:** See [SZ70], Lemma 4.2.

The two theorems above do not change at all if \( A \) has some of its eigenvalues in the open left-half plane as well. This is clear through the proofs in [SZ70] as well as in the example shown below.

**Example.** Let \( A \) be a real matrix with at least one eigenvalue on the imaginary axis, but all in the closed left-half plane. For ease of development, let the eigenvalues of \( A \) be distinct. As a general case, assume the eigenvalues of \( A \) are given by

\[
\lambda_i(A) = \{ \pm j\omega_1, 0, -\lambda_1, -\lambda_2 \}
\]

where \( \omega_1, \lambda_1 \) and \( \lambda_2 \) are real, positive, and \( \lambda_1 \neq \lambda_2 \). Then \( A \) can be put into the block real form
\[
\Lambda = \begin{bmatrix}
0 & \omega_1 & 0 & 0 & 0 \\
-\omega_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_1 & 0 \\
0 & 0 & 0 & 0 & \lambda_2 \\
\end{bmatrix}
\]

through the transformation matrix

\[
T = [ \text{Re}(v_1) \quad \text{Im}(v_1) \quad v_3 \quad v_4 \quad v_5 ]
\]

where \( \text{Im}(v_i) \) is the imaginary part of \( v_i \), \( v_i \) is the eigenvector associated with \( \lambda_i \), and the eigenvalues are ordered as in the \( \Lambda \) matrix above. In other words,

\[
\Lambda = T^{-1} \Lambda T \quad \text{or} \quad \Lambda = T \Lambda T^{-1}
\]

Suppose we wish to find \( X = X^T \geq 0 \) such that

\[
AX + XA^T = 0
\]

One solution is obviously \( X = 0 \). This is not the only solution, as stated by the above theorems. Using the transformation on \( A \) turns the Lyapunov equation into

\[
T \Lambda T^{-1} X + X T^{-T} \Lambda T^{-T} = 0
\]

Premultiply by \( T^{-1} \) and postmultiply by \( T^{-T} \) to get

\[
\Lambda(T^{-1}XT^{-T}) + (T^{-1}XT^{-T})\Lambda^T = 0
\]

or

\[
\Lambda Y + Y\Lambda^T = 0
\]

where \( Y = T^{-1}XT^{-T} \). Solving this Lyapunov equation yields

\[
Y = \begin{bmatrix}
y_{11} & 0 & 0 & 0 & 0 \\
0 & y_{11} & 0 & 0 & 0 \\
0 & 0 & y_{33} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
where $y_{11}$ and $y_{33}$ are arbitrary. Restricting $Y$ to be real and positive semidefinite requires $y_{11}$ and $y_{33}$ to be nonnegative real numbers. Thus, we have the possibilities

\begin{align*}
a) & \quad y_{11} = y_{33} = 0 \quad \Rightarrow \quad \text{rank}(Y) = 0 \quad Y \equiv 0 \\
b) & \quad y_{11} > 0, \ y_{33} = 0 \quad \Rightarrow \quad \text{rank}(Y) = 2 \quad \infty \quad \# \text{ of solutions} \\
c) & \quad y_{11} = 0, \ y_{33} > 0 \quad \Rightarrow \quad \text{rank}(Y) = 1 \quad \infty \quad \# \text{ of solutions} \\
d) & \quad y_{11} > 0, \ y_{33} > 0 \quad \Rightarrow \quad \text{rank}(Y) = 3 \quad \infty \quad \# \text{ of solutions}
\end{align*}

This matches the results of the previous theorem, since

\[
\beta = 0 \text{ or } 1 \quad \Rightarrow \quad \text{rank}(Y) = 0, 1, 2, \text{ or } 3 \\
\zeta = 0 \text{ or } 1
\]

and $\text{rank}(Y) = \text{rank}(X)$. For completeness, the corresponding $X$ is given by

\[
X = TYT^T \\
= y_{11}[\text{Re}(v_1)(\text{Re}(v_1))^T + \text{Im}(v_1)(\text{Im}(v_1))^T] + y_{33}[v_3(v_3)^T]
\]

2.5 Riccati Equations

The material presented in this section is a collection and expansion of results on Riccati equations from a large number of sources. Some of the major sources not referenced in the remainder of this section are [Bro70], [Frag89], [Mol73b], [Mus91], [RV88], and [Wim84].

2.5.1 $H_2$ (LQG) Type

The general algebraic Riccati equation of concern for $H_2$ (LQG) problems is

\[
A^TX + XA - XBp^{-1}B^TX + C^TC = 0 \quad \text{(ARE21)}
\]
where \( P = P^T > 0 \).

First, a theorem which characterizes solutions when \((A, B, C)\) is minimal.

**Theorem 2.5.1**: Assume \((A, B)\) controllable and \((C, A)\) observable. Then there exists a real symmetric solution \( X = X^T \) to \((\text{ARE}_2)\) with the property that \( \text{Re} \lambda_i(A - BP^{-1}B^T X) < 0 \ (> 0) \ \forall \ i \). Moreover, it is unique and such that \( X > 0 \ (< 0) \). In fact, it is the only solution in the set of all positive (negative) semidefinite matrices.

**Proof**: See [Wil71], Lemma 4.

Relaxing controllability and observability produces the following two results.

**Theorem 2.5.2**: Assume \((A, B)\) is stabilizable. If \((\text{ARE}_2)\) has a real symmetric solution, then it has a maximal solution \( X_+ \) such that \( X_+ \geq X \) for all \( X \) satisfying \((\text{ARE}_2)\). The matrix \( X_+ \) is such that
\[
\text{Re} \lambda_i(A - BP^{-1}B^TX_+) \leq 0 \ \forall \ i.
\]

**Proof**: See [GLR86], Theorem 2.1.

**Theorem 2.5.3**: Assume \((A, B)\) is stabilizable. Then a real symmetric solution to \((\text{ARE}_2)\) exists, and its maximal solution is positive semidefinite. If \((C, A)\) is detectable, then \( \text{Re} \lambda_i(A - BP^{-1}B^TX_+) < 0 \ \forall \ i \), and if \((C, A)\) is observable, then \( X_+ > 0 \).

**Proof**: See [GLR86], Theorem 2.2.

All of the above results hold true for the Riccati equation
\[
AX + XA^T - XCP^{-1}CX + BB^T = 0 \quad \text{(ARE}_2)\]

by simply making the substitutions
\[
A = A^T, \quad B = C^T, \quad C = B^T
\]
Notice that the resulting expressions \((A^T, C^T)\) stabilizable (controllable) and \((B^T, A^T)\) detectable (observable) are equivalent to \((C, A)\) detectable (observable) and \((A, B)\) stabilizable (controllable), respectively. When examining the matrices that are stabilized by this Riccati equation, it is more convenient and more common to rewrite the resulting expressions using the identity \(\sigma(Z^T) = \sigma(Z)\).

### 2.5.2 \(H_\infty\) Type

The general algebraic Riccati equation of concern for \(H_\infty\) problems is

\[
A^TX + XA + XB^{-1}B^TX + C^TC = 0 \tag{ARE_{\infty}}
\]

Define

\[
\Phi(s) = P - G^*(s)G(s) \quad \text{where} \quad G(s) = C(sI - A)^{-1}B
\]

Any Riccati equation can be related to an associated Hamiltonian matrix. The Hamiltonian matrix associated with \(\text{(ARE}_{\infty}\)) is given by

\[
M_{\infty} = \begin{bmatrix}
A & B^{-1}B^T \\
-C^TC & -A^T
\end{bmatrix}
\]

The following theorem establishes relationships between the eigenvalues of \(M_{\infty}\), the function \(\Phi(s)\), and \(\text{(ARE}_{\infty}\)).

**Theorem 2.5.4**: \(M_{\infty}\) is such that

\[
\det \left[ sI - M_{\infty} \right] = (-1)^n \det \left[ sI - A \right] \det \left[ -sI - A^T \right] \det \left[ \Phi(s) \right]
\]

and if \(X\) is a solution to \(\text{(ARE}_{\infty}\)), then

\[
\det \left[ sI - M_{\infty} \right] = \det \left[ sI - (A + B^{-1}B^TX) \right] \det \left[ sI + (A + B^{-1}B^TX) \right]
\]

**Proof**: See [Wil71] and use of a similarity transform. \(\blacksquare\)
The next theorem establishes existence and uniqueness of solutions to (ARE\(_{\infty 1}\)) when (A, B) is assumed to be controllable.

**Theorem 2.5.5**: Assume (A, B) controllable. Then the following are equivalent:

i) there exists a solution \(X\) of (ARE\(_{\infty 1}\)) such that \(X = X^*\)

ii) there exists a solution \(X_-\) of (ARE\(_{\infty 1}\)) such that
\[
\text{Re} \lambda_i (A + BP^{-1}B^T X_-) \leq 0 \quad \forall i
\]

iii) there exists a solution \(X_+\) of (ARE\(_{\infty 1}\)) such that
\[
\text{Re} \lambda_i (A + BP^{-1}B^T X_+) \geq 0 \quad \forall i
\]

iv) the partial multiplicities (see below) of the imaginary axis eigenvalues of the Hamiltonian matrix
\[
M_{\infty 1} = \begin{bmatrix} A & BP^{-1}B^T \\ -C^T C & -A^T \end{bmatrix}
\]

(if any) are all even

v) \(\Phi(j\omega) \geq 0\) \(\forall \omega \in [0, \infty]\)

Moreover, if i) - v) hold, then the following are true:

vi) the solution \(X_-\) of (ARE\(_{\infty 1}\)) with the properties of ii) is unique

vii) the solution \(X_+\) of (ARE\(_{\infty 1}\)) with the properties of iii) is unique

viii) \(X_- = X_+\) iff all the eigenvalues of \(M_{\infty 1}\) are on the imaginary axis

**Proof**: See [LR80], Theorem 1, 3 & Corollary 5, [Not90], Theorem 3.11, and [Mol73a], Theorem 3.
Write the characteristic polynomial of a \( n \times n \) matrix \( A \) as

\[
\det(\lambda I - A) = (\lambda - \lambda_1)^{p_1}(\lambda - \lambda_2)^{p_2} \cdots (\lambda - \lambda_m)^{p_m}
\]

\( p_i \) is called the \textit{partial multiplicity} of \( \lambda_i \). It should be obvious that

\[
\sum_{i=1}^{m} p_i = n \quad \text{and if} \quad m = n, \quad \text{the eigenvalues are distinct. If} \quad J \quad \text{is the} \quad \textit{Jordan form} \quad \text{of} \quad A, \quad \text{and} \quad J_i \quad \text{is the} \quad \textit{Jordan block} \quad \text{corresponding to} \quad \lambda_i, \quad \text{the partial multiplicity of} \quad \lambda_i \quad \text{is the dimension of} \quad J_i.
\]

The next theorem allows for relaxation of the controllability requirement on \((A, B)\) and establishes ordering of solutions.

\textbf{Theorem 2.5.6:} Define

\[
Q(X) = A^T X + XA + XBP^{-1}B^T X + C^T C
\]

Assume there exists an \( X = X^* \) such that \( Q(X) \leq 0 \). Then

\begin{enumerate}
  \item[i)] if \((A, B)\) is stabilizable, there exists a unique minimal solution \( X_- \) to \((ARE_{\infty 1})\). Further, \( X_- \leq X \) for all \( X \) such that \( Q(X) \leq 0 \) and \( \Re \lambda_i(A + BP^{-1}B^T X_-) \leq 0 \ \forall \ i \)
  \item[ii)] if \((-A, B)\) is stabilizable, there exists a unique maximal solution \( X_+ \) to \((ARE_{\infty 1})\). Further, \( X_+ \geq X \) for all \( X \) such that \( Q(X) \leq 0 \) and \( \Re \lambda_i(A + BP^{-1}B^T X_+) \geq 0 \ \forall \ i \)
  \item[iii)] if \((A, B)\) is controllable, both \( X_+ \) and \( X_- \) exist. Further,
    \[
    X_+ > X_- \iff \Re \lambda_i(A + BP^{-1}B^T X_-) < 0 \ \forall \ i \quad \text{iff} \quad \Re \lambda_i(A + BP^{-1}B^T X_+) > 0 \ \forall \ i
    \]
  \item[iV)] if \( Q(X) < 0 \), then i) and ii) can be strengthened to \( X_- < X, \)
    \[
    \Re \lambda_i(A + BP^{-1}B^T X_-) < 0 \ \forall \ i \quad \text{and} \quad X_+ > X, \Re \lambda_i(A + BP^{-1}B^T X_+) > 0 \ \forall \ i
    \]
\end{enumerate}

\textbf{Proof:} See [Not90], Corollary 3.4. \( \blacksquare \)
Next, two theorems are given which relate positive semidefinite solutions of \((\text{ARE}_{\infty 1})\) to stability of the A matrix.

**Theorem 2.5.7**: Assume \((A, B)\) stabilizable and \(\text{Re} \lambda_i(A) \leq 0\) \(\forall i\). If \(X = X^*\) satisfies \((\text{ARE}_{\infty 1})\), then \(X \succeq 0\).

**Proof**: See [Wim85], Lemma.

**Theorem 2.5.8**: Assume \((C, A)\) is detectable. Then there exists an \(X \succeq 0\) satisfying \((\text{ARE}_{\infty 1})\) only if \(A\) is stable.

**Proof**: Immediate from considering \((\text{ARE}_{\infty 1})\) to be a Lyapunov equation, and using [Won85], Theorem 3.6.

The next group of theorems deals with characterizing all solutions to a Riccati equation and how to compute them. Those that do not specify what \(P\) is do not require \(P\) to be positive definite, and thus apply to \(H_2\)-type Riccati equations as well.

**Theorem 2.5.9**: Let \(\mathbf{v} \in \mathbb{C}^{2n}\) be an n-dimensional invariant subspace of \(M_{\infty 1}\), and let \(X_1, X_2 \in \mathbb{C}^{n \times m}\) be such that

\[
\mathbf{v} = \text{span} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}
\]

If \(X_1\) is invertible, then \(X \equiv X_2X_1^{-1}\) is a solution to \((\text{ARE}_{\infty 1})\) and \(\sigma(A + BP^{-1}B^T X)\) equals \(\sigma(M_{\infty 1}\vert \mathbf{v})\), where \(\sigma(Z)\) is the spectrum of \(Z\). Further, the solution \(X\) is independent of a specific choice of bases of \(\mathbf{v}\).

**Proof**: See [Not90], Theorem 3.1.

The converse of Theorem 2.5.9 also holds, as seen in:

**Theorem 2.5.10**: If \(X \in \mathbb{C}^{n \times n}\) is a solution to \((\text{ARE}_{\infty 1})\), then there exist
$X_1, X_2 \in \mathbb{C}^{n \times n}$ with $X_1$ invertible such that $X \equiv X_2 X_1^{-1}$ and the columns of

$$\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}$$

form a basis of an $n$-dimensional invariant subspace of $M_{\infty 1}$.

**Proof:** See [Not90], Theorem 3.2. □

The eigenvectors and generalized eigenvectors of $M_{\infty 1}$ can be used to form an invariant subspace of $M_{\infty 1}$. If $\lambda_i$ has multiplicity $k$, the generalized eigenvectors are given by

$$( M_{\infty 1} - \lambda_i I )v_i = 0$$

$$( M_{\infty 1} - \lambda_i I )v_{i+1} = v_i$$

$$\vdots$$

$$( M_{\infty 1} - \lambda_i I )v_{i+k-1} = v_{i+k-2}$$

and span($v_i$) is an invariant subspace of $M_{\infty 1}$. It is easy to show that all of the lower ranking generalized eigenvectors must be used if a generalized eigenvalue is used to form the invariant subspace.

From the set of all solutions, we can characterize those that are Hermitian by the following theorem.

**Theorem 2.5.11:** Suppose $\nu$ is an $n$-dimensional $M_{\infty 1}$-invariant subspace, and $X_1, X_2 \in \mathbb{C}^{n \times n}$ are such that

$$\nu = \text{span} \begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}$$

Then for all $i, j = 1, 2, \ldots, n$, $\lambda_i, \lambda_j \in \sigma(M_{\infty 1}|\nu)$, $\lambda_i + \overline{\lambda_j} \neq 0$ implies $X_2^* X_1$ is Hermitian. Furthermore, if $X_1$ is nonsingular, then $X = X_2 X_1^{-1}$ is Hermitian.

**Proof:** See [Not90], Theorem 3.3. □
The following theorem gives necessary and sufficient conditions for a solution to be real.

**Theorem 2.5.12**: Suppose \( \mathcal{V} \) is an \( n \)-dimensional \( M_{\infty 1} \)-invariant subspace, and let \( X_1, X_2 \in \mathbb{C}^{n \times n} \) be such that the columns of \( \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \) form a basis of \( \mathcal{V} \) with \( X_1 \) assumed nonsingular. Then \( X = X_2 X_1^{-1} \) is real iff \( \mathcal{V} \) is conjugate symmetric, that is, \( v \in \mathcal{V} \) implies \( \overline{v} \in \mathcal{V} \).

**Proof**: See [Not90], Theorem 3.4.

Next, we will characterize stabilizing solutions and the Riccati operator. Assume that a Hamiltonian matrix \( M \) has no eigenvalues on the imaginary axis. Thus, it must have \( n \) eigenvalues in \( \text{Re} \, s < 0 \) and \( n \) in \( \text{Re} \, s > 0 \).

Consider the two \( n \)-dimensional subspaces \( \mathcal{X}_-(M) \) and \( \mathcal{X}_+(M) \). \( \mathcal{X}_-(M) \) is the invariant subspace corresponding to the eigenvalues in \( \text{Re} \, s < 0 \) and \( \mathcal{X}_+(M) \) is the invariant subspace corresponding to those in \( \text{Re} \, s > 0 \).

Finding a basis for \( \mathcal{X}_-(M) \), stacking up the basis vectors to form a matrix, and partitioning yields

\[
\mathcal{X}_-(M) = \text{Im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} 
\]

where \( X_1, X_2 \in \mathbb{R}^{n \times n} \). \( X_1 \) is nonsingular iff the two subspaces

\[
\mathcal{X}_-(M), \quad \text{Im} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (2.7)
\]

are complementary. In this case, we can set \( X = X_2 X_1^{-1} \). Then \( X \) is uniquely determined by \( M \), so that \( M \rightarrow X \) is a function, which will be denoted \( \text{Ric} \). Thus, \( X = \text{Ric}(M) \).

The domain of the function \( \text{Ric} \), denoted \( \text{dom}(\text{Ric}) \), will be chosen to consist of Hamiltonian matrices with two properties. The first is that \( M \) has no eigenvalues on the imaginary axis, and the second is that the two
subspaces in (2.7) are complementary.

**Theorem 2.5.13**: Suppose \( M_{\infty 1} \in \text{dom}(\text{Ric}) \) and \( X = \text{Ric}(M_{\infty 1}) \). Then

i) \( X \) is symmetric

ii) \( X \) satisfies \((\text{ARE}_{\infty 1})\)

iii) \( \text{Re} \, \lambda_i(A + BP^{-1}BTX) < 0 \quad \forall \, i \)

\( X \) is therefore referred to as the **stabilizing solution** to \((\text{ARE}_{\infty 1})\). If it exists, it is unique.

**Proof**: See [Not90], Theorems 3.5 and 3.6.

Finally, the existence of a stabilizing solution is linked to the infinity-norm of the corresponding \( G(s) \).

**Theorem 2.5.14**: Assume that \( A \) is stable, \( G(s) = C(sI - A)^{-1}B \), and that \( P = \gamma^2 I \). Then the following are equivalent:

i) \( \| G(s) \|_{\infty} < \gamma \)

ii) \( M_{\infty 1} \) has no imaginary axis eigenvalues

iii) \( M_{\infty 1} \in \text{dom}(\text{Ric}) \)

iv) \( M_{\infty 1} \in \text{dom}(\text{Ric}) \) and \( X = \text{Ric}(M_{\infty 1}) \geq 0 \quad (\geq 0 \text{ if } (C, A) \text{ is observable}) \)

**Proof**: See [GD89], Lemma 2.4 (I) with modifications to allow general \( \gamma \).
\[
M \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} T_x \quad \text{Re} \lambda_i(T_x) \leq 0 \quad \forall \ i
\]

Assume that this is chosen (it is not unique) such that the subspaces in (2.7) are complementary. Set \( X = X_2 X_1^{-1} \) if \( X \) is symmetric. Define the function \( \text{Ric} \), whose domain is denoted as \( \text{dom}(\text{Ric}) \) and consists of Hamiltonian matrices with the property that an \( \mathcal{X}_-(M) \) exists satisfying complementarity of the subspaces in (2.7) and with the resulting \( X = X_2 X_1^{-1} \) being symmetric. This may not always be a function, as \( X \) may not be uniquely determined. Whenever it is needed here, it will be a well-defined function, so that the uniqueness of \( X = \text{Ric}(M) \) will be required.

**Theorem 2.5.15**: Suppose \( M_{\infty 1} \in \text{dom}(\text{Ric}) \) and \( X = \text{Ric}(M_{\infty 1}) \) is unique. Then

i) \( X \) is symmetric

ii) \( X \) satisfies \( (\text{ARE}_{\infty 1}) \)

iii) \( \text{Re} \lambda_i(A + BP^{-1}B^T X) \leq 0 \quad \forall \ i \)

When iii) has at least one \( i \) such that equality holds, \( X \) is referred to as the *neutrally stabilizing solution* to \( (\text{ARE}_{\infty 1}) \).

**Proof**: Modification of [Not90], Theorem 3.5 using discussion above.

The relationship between neutrally stabilizing solutions and the infinity-norm of \( G(s) \) is given by:

**Theorem 2.5.16**: Assume that \( A \) is stable, \( G(s) = C(sI - A)^{-1}B \), and that \( P = \gamma^2 I \). Then the following are equivalent:

i) \( \| G(s) \|_\infty \leq \gamma \)

ii) the partial multiplicities of the imaginary axis eigenvalues of \( M_{\infty 1} \) (if any) are all even

53
iii) $M_{\infty 1} \in \text{dom}(\text{Ric})$

iv) $M_{\infty 1} \in \text{dom}(\text{Ric})$ and $X = \text{Ric}(M_{\infty 1}) \geq 0$ is unique ($> 0$ if $(C, A)$ is observable)

**Proof**: See [GD89], Lemma 2.4 (II) with modifications to remove D term and to allow general $\gamma$.

Theorem 2.5.16 provides an alternate means of computing the infinity-norm of a stable transfer function, which produces both a lower and an upper bound to any desired degree of accuracy. That is, choose a $\gamma$ and compute the eigenvalues of $M_{\infty 1}$. If there are $\omega$-axis eigenvalues with odd partial multiplicities, the guess of $\gamma$ was too low. If there are no $\omega$-axis eigenvalues, $\gamma$ can be reduced. Choose a new $\gamma$ and iterate to any desired degree of accuracy. This procedure can also be used for transfer functions with a non-zero D matrix - see [Dai90] for the details.

All of the above theorems hold true for the Riccati equation

$$AX + XA^T + XC^TP^{-1}CX + BB^T = 0$$

(ARE$_{\infty 2}$)

by simply making the substitutions

$$A = A^T \quad B = C^T \quad C = B^T$$

Notice that the resulting expressions $(A^T, C^T)$ stabilizable (controllable) and $(B^T, A^T)$ detectable (observable) are equivalent to $(C, A)$ detectable (observable) and $(A, B)$ stabilizable (controllable), respectively. Also note that the matrix $G(s)$ gets replaced with $G^T(s)$, so that $\det[P - G^*(s)G(s)]$ should be replaced with $\det[P - G(s)G^*(s)]$. Finally, $\lambda_i(A + BP^{-1}B^TX)$ is replaced with $\lambda_i(A + XC^TP^{-1}C)$. In particular, consider one final theorem:

**Theorem 2.5.17**: Suppose A is stable. If $\exists$ an $X = X^T \geq 0$ satisfying

$$0 = AX +XA^T + \gamma^2XC^TCX + BB^T$$

(ARE$_{\infty 2}$)

then

$$\|C(sI - A)^{-1}B\|_\infty \leq \gamma$$
Proof: Assume there exists \( X = X^T \geq 0 \) satisfying \((\text{ARE}_{\infty}^2)\). Rewrite \((\text{ARE}_{\infty}^2)\) as

\[
sX - sX - AX - XA^T = \gamma^2 XC^T CX + BB^T
\]

or

\[
(sI - A)X + X(-sI - A^T) = \gamma^2 XC^T CX + BB^T
\]

Premultiply by \( C(sI - A)^{-1} \) and postmultiply by \((-sI - A^T)^{-1}C^T\) to get

\[
CX(-sI - A^T)^{-1}C^T + C(sI - A)^{-1}XC^T
\]

\[
= \gamma^2 C(sI - A)^{-1}XC^T CX(-sI - A^T)^{-1}C^T + C(sI - A)^{-1}BB^T(-sI - A^T)^{-1}C^T
\]

\[\text{(2.8)}\]

Let

\[
L(s) = C(sI - A)^{-1}XC^T
\]

so that

\[
L^*(s) = CX(-sI - A^T)^{-1}C^T
\]

Then \((2.8)\) becomes

\[
L^*(s) + L(s) - \gamma^2 L(s)L^*(s) = G(s)G^*(s)
\]

\[\text{(2.9)}\]

Note that

\[
[\gamma I - \gamma^1 L] [\gamma I - \gamma^1 L]^* = \gamma^2 I - L^*(s) - L(s) - \gamma^2 L(s)L^*(s)
\]

Writing \((2.9)\) as

\[
\gamma^2 I - L^*(s) - L(s) + \gamma^2 L(s)L^*(s) = \gamma^2 I - G(s)G^*(s)
\]

we get
\[
[ \gamma I - \gamma^{-1} L \| \gamma I - \gamma^{-1} L ]^* = \gamma^2 I - G(s)G^*(s)
\]

The quantity on the left is Hermitian on the \(j\omega\)-axis, and thus is \(\geq 0\). Therefore

\[
\gamma^2 I - G(s)G^*(s) \geq 0
\]
or

\[
\| G(s) \|_{\infty} \leq \gamma
\]

### 2.5.3 Examples

Since in the latter development we will only be interested in positive semidefinite solutions to \(H_{\infty}\)-type Riccati equations, we will restrict all but the last example to stable \(A\) matrices (see Theorem 2.5.8). For simplicity, all of the examples will be for a strictly proper, two-state SISO system. Assuming a minimal canonical realization, the state space description can be written in the general canonical form

\[
A = \begin{bmatrix}
-a_1 & -a_2 \\
1 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
1 \\
0
\end{bmatrix}, \quad C = \begin{bmatrix}
c_1 & c_2
\end{bmatrix}, \quad D = \{0\}
\]

The ARE of primary interest later in this work is \((\text{ARE}_{\infty 2})\), with \(P = \gamma^2 I\). Therefore, we have

\[
AX + XA^T + \gamma^2 XC^T CX + BB^T = 0 \quad \text{(ARE}_{\infty 2})
\]

and

\[
M_{\infty 2} = \begin{bmatrix}
A^T & \gamma^2 C^T C \\
-BB^T & -A
\end{bmatrix} = \begin{bmatrix}
-a_1 & 1 & \alpha c_1^2 & \alpha c_1 c_2 \\
-a_2 & 0 & \alpha c_1 c_2 & \alpha c_2^2 \\
-1 & 0 & a_1 & a_2 \\
0 & 0 & -1 & 0
\end{bmatrix}
\]

56
where $\alpha = \gamma^2$. The eigenvalues of $M_{\infty 2}$ are found by setting the determinant of $[\lambda I - M_{\infty 2}] = 0$, which produces the equation

$$
\lambda^4 + (2a_2 - a_1^2 + \alpha c_1^2)\lambda^2 + (a_2^2 - \alpha c_2^2) = 0
$$

It is clear that there will be four roots to this equation, symmetric with respect to both the real and imaginary axes. Four specific examples will now be examined.

2.5.3.1 Example One

Let

$$
A = \begin{bmatrix}
-6 & -8 \\
1 & 0
\end{bmatrix} \quad B = \begin{bmatrix}
1 \\
0
\end{bmatrix} \quad C = \begin{bmatrix}
0 & 8
\end{bmatrix}
$$

so that

$$
G_1(s) = C(sI - A)^{-1}B = \frac{8}{(s + 2)(s + 4)}
$$

It is easy to verify that

$$
\|G_1(s)\|_\infty = 1.0
$$

For $(A R E_{\infty 2})$ we have

$$
M_{\infty 2} = \begin{bmatrix}
-6 & 1 & 0 & 0 \\
-8 & 0 & 0 & 64\alpha \\
-1 & 0 & 6 & 8 \\
0 & 0 & -1 & 0
\end{bmatrix}
$$

whose eigenvalues are

$$
\lambda_{1,2} = \pm \sqrt{s_1} \quad \lambda_{3,4} = \pm \sqrt{s_2}
$$

where

$$
s_{1,2} = 10 \pm 2\sqrt{25 - 16(1 - \alpha)}
$$

57
For $\alpha < 1$ ($\gamma > 1 = \| G_1(s) \|_\infty$), this results in four real roots, two positive and two negative. For $\alpha = 1$ ($\gamma = 1 = \| G_1(s) \|_\infty$), the roots are \( \{0, 0, \sqrt{20}, -\sqrt{20}\} \). For $\alpha > 1$ ($\gamma < 1 = \| G_1(s) \|_\infty$), there will be two roots on the imaginary axis and two on the real axis which are opposite in sign to each other. Plotting the roots as $\alpha$ varies yields Figure 2-3.

Note that for $\gamma < 1 = \| G_1(s) \|_\infty$, the imaginary axis eigenvalues have unity partial multiplicities, and thus by Theorem 2.5.5, no Hermitian solutions to \( (ARE_{\infty2}) \) exist.

\[ \begin{align*} M_{\infty2}(1.2) &= \begin{bmatrix} -6 & 1 & 0 & 0 \\ -8 & 0 & 0 & 44.44 \\ -1 & 0 & 6 & 8 \\ 0 & 0 & -1 & 0 \end{bmatrix} \end{align*} \]

Figure 2-3. $H_{\infty}$ locus for Example 1

Choose $\gamma = 1.2$ so that
The eigenvalues and eigenvectors of $M_{\infty 2}(1.2)$ are

$$
\lambda_1 = -4.3554 \quad \lambda_2 = -1.0153 \quad \lambda_3 = 4.3554 \quad \lambda_4 = 1.0153
$$

\[
\begin{align*}
\mathbf{v}_1 &= \begin{bmatrix} -0.5190 \\ -0.8536 \\ -0.0426 \\ -0.0098 \end{bmatrix} & \mathbf{v}_2 &= \begin{bmatrix} 0.1967 \\ 0.9803 \\ 0.0132 \\ 0.0130 \end{bmatrix} & \mathbf{v}_3 &= \begin{bmatrix} 0.0855 \\ 0.8856 \\ -0.4450 \\ 0.1022 \end{bmatrix} & \mathbf{v}_4 &= \begin{bmatrix} 0.1408 \\ 0.9877 \\ -0.0486 \\ 0.0479 \end{bmatrix}
\end{align*}
\]

Since all four eigenvectors are real, Theorem 2.5.12 says that all solutions to $(\text{ARE}_{\infty 2})$ will be real. There are six different $M_{\infty 2}$-invariant subspaces that can be formed from these eigenvectors, corresponding to different combinations of two of them. Label them as

\[
\mathbf{x}_1(M_{\infty 2}) = \text{Im} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \quad \mathbf{x}_2(M_{\infty 2}) = \text{Im} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_3 \end{bmatrix} \\
\mathbf{x}_3(M_{\infty 2}) = \text{Im} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_4 \end{bmatrix} \quad \mathbf{x}_4(M_{\infty 2}) = \text{Im} \begin{bmatrix} \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \\
\mathbf{x}_5(M_{\infty 2}) = \text{Im} \begin{bmatrix} \mathbf{v}_2 & \mathbf{v}_4 \end{bmatrix} \quad \mathbf{x}_6(M_{\infty 2}) = \text{Im} \begin{bmatrix} \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix}
\]

Notice that $\mathbf{x}_1(M_{\infty 2}) \equiv \mathbf{x}_-(M_{\infty 2})$ and $\mathbf{x}_6(M_{\infty 2}) \equiv \mathbf{x}_4(M_{\infty 2})$. Now, from Theorem 2.5.9, we can get the six solutions to $(\text{ARE}_{\infty 2})$ by the formula

$$
X_i = X_{2i}(X_{1i})^{-1}
$$

where

$$
\mathbf{x}_i(M_{\infty 2}) = \text{Im} \begin{bmatrix} X_{1i} \\ X_{2i} \end{bmatrix}
$$

Doing this produces

\[
\begin{align*}
X_1 &= \begin{bmatrix} 0.0893 & -0.0045 \\ -0.0045 & 0.0142 \end{bmatrix} & X_2 &= \begin{bmatrix} 1.0800 & -0.6068 \\ -0.2032 & 0.1350 \end{bmatrix} \\
X_3 &= \begin{bmatrix} 0.2129 & -0.0796 \\ -0.0796 & 0.0598 \end{bmatrix} & X_4 &= \begin{bmatrix} 4.9591 & -0.9814 \\ -0.9814 & 0.2102 \end{bmatrix}
\end{align*}
\]

59
\[
X_5 = \begin{bmatrix}
1.0800 & -0.2032 \\
-0.6068 & 0.1350
\end{bmatrix} \quad X_6 = \begin{bmatrix}
9.8586 & -1.4545 \\
-1.4545 & 0.2558
\end{bmatrix}
\]

\(X_2\) and \(X_5\) can be discarded immediately if symmetric solutions are required, and this is expected from Theorem 2.5.11 since for these two cases \(\lambda_i + \bar{\lambda}_j = 0\). The remaining four solutions are all positive definite, and \(X_1 \equiv X_- < X_i\) as well as \(X_6 \equiv X_+ > X_i\), as expected from Theorem 2.5.6. \(X_1\) and \(X_6\) are the unique stabilizing and antistabilizing solution, respectively, with

\[
\lambda_i \left[ A + \frac{1}{\gamma^2} X_1 C^T C \right] = (-4.3554, 1.0153)
\]

and

\[
\lambda_i \left[ A + \frac{1}{\gamma^2} X_6 C^T C \right] = (4.3554, 1.0153)
\]

The other two solutions yield one root in the right-half plane one in the left-half plane.

Now choose \(\gamma_0 = 1.0 = \|G_1(s)\|_{\infty}\) so that

\[
M_{\infty 2}(1.0) = \begin{bmatrix}
-6 & 1 & 0 & 0 \\
-8 & 0 & 0 & 64 \\
-1 & 0 & 6 & 8 \\
0 & 0 & -1 & 0
\end{bmatrix}
\]

The eigenvalues and eigenvectors of \(M_{\infty 2}(1.0)\) are

\[
\lambda_1 = -4.4721 \quad \lambda_2 = 0.0 \quad \lambda_3 = -4.4721 \quad \lambda_4 = 0.0
\]

\[
v_1 = \begin{bmatrix}
-0.5471 \\
-0.8358 \\
-0.0446 \\
-0.0100
\end{bmatrix} \quad v_2 = \begin{bmatrix}
0.1644 \\
0.9862 \\
0.0000 \\
0.0205
\end{bmatrix} \quad v_3 = \begin{bmatrix}
0.0891 \\
0.9326 \\
-0.3412 \\
0.0763
\end{bmatrix} \quad v_4 = \begin{bmatrix}
0.1644 \\
0.9862 \\
0.0000 \\
0.0205
\end{bmatrix}
\]

Using the same ordering and notation as for the \(\gamma = 1.2\) case yields the following six solutions to (ARE\(\infty 2\))
\[ X_1 = \begin{bmatrix} 0.1094 & -0.0182 \\ -0.0182 & 0.0239 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0.7500 & -0.4375 \\ -0.1250 & 0.0938 \end{bmatrix} \]

\[ X_3 = X_1 \quad X_4 = X_6 \]

\[ X_5 \text{ does not exist} \quad X_6 = \begin{bmatrix} 5.1406 & -0.8568 \\ -0.8568 & 0.1636 \end{bmatrix} \]

\( X_2 \) can be discarded immediately if symmetric solutions are required, and this is expected from Theorem 2.5.11 since for this case \( \lambda_i + \lambda_j = 0 \). \( X_5 \) does not exist since the two eigenvalues are identical and \( X_1 \) is not invertible. Note that there are only two distinct symmetric solutions, which are positive definite, and \( X_1 \equiv X_- \prec X_6 \equiv X_+ \). \( X_1 \) and \( X_6 \) are the neutrally stabilizing and neutrally antistabilizing solution, respectively, with

\[ \lambda_i [ A + \frac{1}{\gamma^2} X_1C^T C ] = ( -4.4721, 0.0 ) \]

and

\[ \lambda_i [ A + \frac{1}{\gamma^2} X_6C^T C ] = ( 4.4721, 0.0 ) \]

Now assume that we had been looking for the solution to an \( H_2 \) (LQG) type Riccati equation for this example. Thus, the ARE of interest is

\[ AX + XA^T - XCTP^{-1}CX + BB^T = 0 \quad (\text{ARE}22) \]

and the associated Hamiltonian is

\[ M_{22} = \begin{bmatrix} A^T & -\gamma^2 C^T C & -6 & 1 & 0 & 0 \\ -8 & 0 & 0 & -64\alpha \\ -1 & 0 & 6 & 8 \\ 0 & 0 & -1 & 0 \end{bmatrix} \]

where \( \alpha = \gamma^2 \). The eigenvalues of \( M_{22} \) are
\[ \lambda_{1,2} = \pm \sqrt{s_1} \quad \lambda_{3,4} = \pm \sqrt{s_2} \]

where

\[ s_{1,2} = 10 \pm 2 \sqrt{25 - 16(1 + \alpha)} \]

Plotting the roots as \( \alpha \) varies yields Figure 2-4. The four roots are never on the imaginary axis, and thus a solution to \((\text{ARE}_{22})\) exists for any nonnegative \( \alpha \), as guaranteed by Theorem 2.5.1. Theorem 2.5.1 also guarantees that for any value of \( \alpha \), there will be exactly one stabilizing, \( > 0 \) solution and one antistabilizing, \( < 0 \) solution. The breakaway point on Figure 2-4 corresponds to \( \alpha = 0.5625 \). For \( \alpha > 0.5625 \), computing the six solutions as in the \((\text{ARE}_{\infty 2})\) case yields only two real solutions, one of which is positive definite and stabilizing and the other being negative definite and antistabilizing. This is due to the restriction of conjugate roots. For \( \alpha < 0.5625 \), there are four distinct real solutions: one is positive definite and stabilizing, one is negative definite and antistabilizing, and the other two are indefinite and result in one stable and one antistable root. For \( \alpha = 0.5625 \), the two indefinite solutions become the same.

![Figure 2-4. \( H_2 \) locus for Example 1](image)

62
2.5.3.2 Example Two

Let

\[
A = \begin{bmatrix}
-6 & -8 \\
1 & 0
\end{bmatrix} \quad B = \begin{bmatrix}
1 \\
0
\end{bmatrix} \quad C = \begin{bmatrix}
2 & 12
\end{bmatrix}
\]

so that

\[
G_2(s) = C(s I - A)^{-1}B = \frac{2(s + 6)}{(s + 2)(s + 4)}
\]

Again, it is easy to verify that

\[
\|G_2(s)\|_\infty = 1.5
\]

For this example

\[
M_{\infty 2} = \begin{bmatrix}
-6 & 1 & 4\alpha & 24\alpha \\
-8 & 0 & 24\alpha & 144\alpha \\
-1 & 0 & 6 & 8 \\
0 & 0 & -1 & 0
\end{bmatrix}
\]

so that the eigenvalues of \(M_{\infty 2}\) are given by

\[
\lambda_{1,2,3,4} = \pm \sqrt{2(5 - \alpha) \pm 2 \sqrt{(5 - \alpha)^2 - 4(4 - 9\alpha)}}
\]

There seems to be two critical values of \(\alpha\) in the above equation. For \(\alpha = 4/9\), two eigenvalues are at the origin. Note that this value of \(\alpha\) corresponds to \(\gamma_0 = 1.5\), the infinity-norm of \(G_2(s)\). For \(\alpha < 4/9\), all four roots are real, and for \(\alpha\) greater than this, two roots are always on the imaginary axis. For \(\alpha = 5\), all four roots lie on a circle centered at the origin. This value does not correspond to a change in the character of the eigenvalues. The locations of the eigenvalues as \(\alpha\) varies is shown in Figure 2-5. The corresponding (ARE\(_{\infty 2}\)) has real symmetric solutions only for \(\alpha \leq 4/9\), which corresponds to \(\gamma \geq \|G_2(s)\|_\infty\). All solutions are positive definite, the minimal one is the (neutrally) stabilizing solution and the maximal is the (neutrally) antistabilizing solution.
For $\gamma = \gamma_0 = \| G_2(s) \|_\infty$, the neutrally stabilizing solution is

$$X_\gamma = \begin{bmatrix} 0.1169 & -0.0250 \\ -0.0250 & 0.0322 \end{bmatrix}$$

which has

$$\lambda_1[A + \frac{1}{\gamma_0^2} X_\gamma^{T}C] = (-4.2687, 0.0)$$

For $\gamma$ smaller than $\gamma_0$, no Hermitian solution exists, and for $\gamma$ greater than $\gamma_0$, there always exists a minimal, positive definite, stabilizing solution as well as a maximal, positive definite, antistabilizing solution.

Now consider solving (ARE22) for the same system. The eigenvalues of $M_{22}$ are given by
\[
\lambda_{1,2,3,4} = \pm \sqrt{2(5 + \alpha) \pm 2 \sqrt{\alpha^2 - 26\alpha + 9}}
\]

Setting the inner radical equal to zero yields the values

\[
\alpha_{1,2} = 0.3509, 25.6491
\]

which corresponds to the breakaway and break-in points, located at

\[
\lambda_{1,2,3,4} = \pm 3.2714, \pm 7.8293
\]

Figure 2-6 shows the locations of the eigenvalues as \(\alpha\) varies.

![Figure 2-6. H_2 locus for Example 2](image)

Solutions to (ARE_{22}) exist for all nonnegative values of \(\alpha\). The unique positive definite solution is stabilizing and the unique negative definite solution is antistabilizing. As this type of result always occurs for stable minimal systems, (ARE_{22}) will not be examined in the next example.
2.5.3.3 Example Three

Let

\[ A = \begin{bmatrix} -4 & -8 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 10 \end{bmatrix} \]

so that

\[ G_3(s) = C(sI - A)^{-1}B = \frac{10}{(s^2 + 4s + 8)} \]

and

\[ \| G_3(s) \|_\infty = 1.25 \]

For \((\text{ARE}_\infty)\)

\[ M_\infty = \begin{bmatrix} -4 & 1 & 0 & 0 \\ -8 & 0 & 0 & 100\alpha \\ -1 & 0 & 4 & 8 \\ 0 & 0 & -1 & 0 \end{bmatrix} \]

The eigenvalues of \(M_\infty\) are given by

\[ \lambda^4 + 4(16 - 25\alpha) = 0 \]

For \(\alpha = 16/25\), all four eigenvalues are at the origin. Note that this value of \(\alpha\) corresponds to \(\gamma_0 = 1.25\), the infinity-norm of \(G_3(s)\). For \(\alpha < 16/25\), all four roots are complex with a nonzero real part, and for \(\alpha > 16/25\), two roots are always on the imaginary axis. The locations of the roots as \(\alpha\) varies is shown in Figure 2-7. The corresponding \((\text{ARE}_\infty)\) has Hermitian solutions only for \(\alpha \leq 16/25\), which corresponds to \(\gamma \geq \| G_3(s) \|_\infty\). All solutions are positive definite; the minimal one is the (neutrally) stabilizing solution and the maximal is the (neutrally) antistabilizing solution. For \(\gamma = \| G_3(s) \|_\infty\), the only Hermitian solution to \((\text{ARE}_\infty)\) is given by

\[ X_- = X_+ = \begin{bmatrix} 0.5000 & -0.1250 \\ -0.1250 & 0.0625 \end{bmatrix} \]
which has

\[ \lambda_i \left[ A + \frac{1}{\gamma^2} X_C C^T \right] = \left( 0.0, 0.0 \right) \]

This was expected from the last part of Theorem 2.5.5.

2.5.3.4 Example Four

For the final example, let

\[
A = \begin{bmatrix}
-3 & 4 \\
1 & 0
\end{bmatrix} \quad B = \begin{bmatrix}
1 \\
0
\end{bmatrix} \quad C = \begin{bmatrix}
-10 & 4
\end{bmatrix}
\]

so that

\[
G_4(s) = \frac{-10(s - 0.4)}{(s + 4)(s - 1)}
\]
and from a highly refined magnitude plot

\[ \| G_4(s) \|_\infty = 2.04228974 \]

This example is being shown to illustrate what happens for unstable A matrices, and also to show that the eigenvalues of \( M_\infty \) as \( \alpha \) increases do not always first reach the imaginary axis at the origin, as they have in the previous examples. For \( (\text{ARE}_\infty) \) we have

\[
M_\infty = \begin{bmatrix}
-3 & 1 & 100\alpha & -40\alpha \\
4 & 0 & -40\alpha & 16\alpha \\
-1 & 0 & 3 & -4 \\
0 & 0 & -1 & 0
\end{bmatrix}
\]

whose eigenvalues are the roots of

\[ \lambda^4 + (-17 + 100\alpha)\lambda^2 + 16(1-\alpha) = 0 \]

A simple analysis reveals that there are roots on the imaginary axis for all \( \alpha \geq 0.23975368 \), which corresponds to \( \gamma \leq \| G_4(s) \|_\infty \). The roots as \( \alpha \) varies are shown in Figure 2-8. Using the eigenvector technique from Example 1 reveals that all the real solutions to \( (\text{ARE}_\infty) \) are indefinite for \( \gamma \geq \| G_4(s) \|_\infty \), as expected from Theorem 2.5.8. Since the infinity-norm sees an antistable pole at \( s = \beta \) to be the same as a stable one at \( s = -\beta \), using a \( G(s) \) of

\[ G(s) = \frac{-10(s - 0.4)}{(s + 4)(s + 1)} \]

produces the same infinity-norm and the same \( \alpha \)-plot as in Figure 2-8. However, in this case, all real solutions are positive definite, as predicted by Theorem 2.5.7.

As a means of wrap-up, note that from Theorem 2.5.4,

\[
\det \{ sI - M_\infty \} = (-1)^n \det \{ sI - A \} \det \{ -sI - A^T \} \det \{ \gamma^2 I - G(-s)G^T(s) \} \\
= (-1)^{n+1} \det \{ sI - A \} \det \{ sI - (-A) \} \det \{ \gamma^2 I - G(s)G^*(s) \} \\
= \det \{ sI - (A + \gamma^2 XC^TC) \} \det \{ sI + (A + \gamma^2 XC^TC) \}
\]

68
where $X$ is a solution to (ARE$_{\infty 2}$). Therefore, in the SISO case, the eigenvalues of $M_{\infty 2}$ as $\gamma$ varies can be found from a root-locus of $G(s)G^*(s)$, where the gain parameter $\alpha = \gamma^2$. Formal construction rules may be found in [CLBY90a]. This locus will always include the entire imaginary axis. The infinity-norm of $G(s)$ is given by $\|G(j\omega_0)\|$, where $\omega_0$ is the frequency where the locus first touches the imaginary axis. The roots of $M_{22}$ are given by the locus formed by using "negative gain", which never has branches on the imaginary axis. The locus roots are the eigenvalues of

$$\pm (A + \frac{1}{\gamma^2} XC^T C) \quad \text{or} \quad \pm (A - \frac{1}{\gamma^2} XC^T C)$$

for (ARE$_{\infty 2}$) or (ARE$_{22}$), respectively. The results are also true for the MIMO case, using a multivariable root-locus.

Figure 2-8. $H_\infty$ locus for Example 4
CHAPTER 3
H₂ AND H∞ OPTIMIZATION

3.1 H₂ Optimization

H₂ optimization can be thought of as a generalization and formalization of
the standard LQG problem, which appears in countless references, a few of
which are [AM90], [DGKF89], [KS72], and [Mac89]. Consider the feedback
problem shown in Figure 3-1.

![Diagram of feedback problem](image)

Figure 3-1. General H₂ feedback problem

The plant P can be partitioned as

\[
P = \begin{bmatrix}
P_{zw} & P_{zu} \\
P_{yw} & P_{yu}
\end{bmatrix}
\]

or, in other words,

\[
z = P_{zw}w + P_{zu}u
\]

\[
y = P_{yw}w + P_{yu}u
\]

The exogenous input w is assumed to be a zero-mean white Gaussian noise
of unit intensity. It is desired to minimize the energy or two-norm of the
controlled output z; that is, find an admissible (internally stabilizing) K(s)
such that \( \| z \|_2 \) is minimized. By the development given in Sections 2.2 and 2.3, this is given by

\[
\inf_{K \text{ admissible}} \| z \|_2 = \inf_{K \text{ admissible}} \| T_{zw} \|_2
\]

\[
= \inf_{K \text{ admissible}} \| P_{zw} + P_{zu}K[I - P_{yu}K]^{-1}P_{yw} \|_2
\]

where

\[
\| T_{zw} \|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}[T_{zw}^*(j\omega)T_{zw}(j\omega)] \, d\omega
\]

In state space, the plant \( P \) is described by the equations

\[
\dot{x} = Ax + B_u w + B_u u \quad (3.2)
\]

\[
z = C_z x + D_{zw} w + D_{zu} u \quad (3.3)
\]

\[
y = C_y x + D_{yw} w + D_{yu} u \quad (3.4)
\]

Assume the following are true:

i) \( D_{zw} = 0 \)

ii) \( D_{yu} = 0 \)

iii) \((A, B_u)\) stabilizable & \((C_y, A)\) detectable

iv) \( D_{zu}^T D_{zu} \) full rank \quad \( D_{yw} D_{yw}^T \) full rank

v) \[
\begin{bmatrix}
A-j\omega I & B_u \\
C_z & D_{zu}
\end{bmatrix}
\] has full column rank for all \( \omega \)

vi) \[
\begin{bmatrix}
A-j\omega I & B_w \\
C_y & D_{yw}
\end{bmatrix}
\] has full row rank for all \( \omega \)
Condition i) is required or the $H_2$ problem is not well defined, as the closed-loop transfer function will then have a non-zero $D$ term for any choice of compensator, thus making the closed-loop two-norm infinite. Condition ii) makes the development easier, but can be completely removed ([SLC89]). Condition iii) is necessary for the existence of stabilizing solutions. Condition iv) ensures that the penalty on control usage and the sensor noise intensity are nonsingular - relaxation leads to singular control problems. Conditions v) and vi) guarantee the existence of stabilizing solutions to the two Riccati equations which appear in the solution to the problem.

To make the final form of the equations simpler, strengthen condition iv) to read

$$ iv) \quad D_{zu}^T D_{zu} = I \quad D_{yw}^T D_{yw} = I $$

Condition iv) can always be satisfied (given iv) is true) by doing a scaling on $u$ and $y$, which we will denote as $S_u$ and $S_y$, respectively. The scaled problem may be viewed in Figure 3-2.

![Figure 3-2. $H_2$ feedback problem with scalings](image)

The scaled equations are

$$ u = S_u^{-1} \tilde{u} $$

(3.5)
\[ \tilde{y} = S_y y \]  \hspace{1cm} (3.6)

where \( S_u \) and \( S_y \) are such that

\[ S_u^T S_u = D_{zu}^T D_{zu} \]  \hspace{1cm} (3.7)

\[ S_y^{-1} (S_y^{-1})^T = D_{yw} D_{yw}^T \]  \hspace{1cm} (3.8)

If \( A \in \mathbb{R}^{nxn} \) is symmetric positive definite, then there exists an upper triangular \( S \in \mathbb{R}^{nxn} \) with positive diagonal entries such that \( A = S^T S \). This is known as a Cholesky decomposition, which may be used to find \( S_u \) and \( S_y \). The "new" plant state and output equations become

\[ \dot{x} = Ax + B_w w + \tilde{B}_u \tilde{u} \]  \hspace{1cm} (3.9)

\[ z = C_z x + \tilde{D}_{zu} \tilde{u} \]  \hspace{1cm} (3.10)

\[ \tilde{y} = \tilde{C}_y x + \tilde{D}_{yw} w \]  \hspace{1cm} (3.11)

where

\[ \tilde{B}_u = B_u S_u^{-1} \] \hspace{1cm} \( \tilde{C}_y = S_y C_y \) \hspace{1cm} (3.12)

\[ \tilde{D}_{zu} = D_{zu} S_u^{-1} \] \hspace{1cm} \( \tilde{D}_{yw} = S_y D_{yw} \) \hspace{1cm} (3.13)

The compensator that minimizes (3.1) is unique. Denote the minimum value of \( \| T_{zw} \|_2 \) in (3.1) as \( \alpha_0 \). If suboptimal compensators are considered, the family of all stabilizing compensators such that

\[ \| T_{zw} \|_2 \leq \alpha \quad \alpha \geq \alpha_0 \]

is a family of LFT's of the optimal compensator, denoted as \( J_{uy}(s) \), with a constrained freedom parameter \( Q(s) \). This is shown in Figure 3-3. The parameterization is given by

\[ \hat{K}(s) = F_j [ J(s), Q(s) ] \]  \hspace{1cm} (3.14)
Figure 3-3. H$_2$ suboptimal compensator diagram

where

\[
J(s) = \begin{bmatrix} J_{uy} & J_{ur} \\ J_{vy} & J_{vr} \end{bmatrix} = \begin{bmatrix} A_J & K_f & K_{fl} \\ -K_c & 0 & 1 \\ K_{cl} & 1 & 0 \end{bmatrix}
\]  \hspace{1cm} (3.15)

\[
A_J = A - K_f \bar{C}_y - \bar{B}_u K_c  \hspace{1cm} (3.16)
\]

\[
K_c = \bar{B}_u^T X_2 + \bar{D}_{zu} C_z  \hspace{1cm} (3.17)
\]

\[
K_f = Y_2 \bar{C}_y + B_w \bar{D}_{yw}^T  \hspace{1cm} (3.18)
\]

\[
K_{cl} = - \bar{C}_y  \hspace{1cm} (3.19)
\]

\[
K_{fl} = \bar{B}_u  \hspace{1cm} (3.20)
\]

Here, $X_2$ and $Y_2$ are the real, unique, symmetric positive semidefinite solutions of the algebraic Riccati equations.
\[
( A - \tilde{B}_u \tilde{D}_{zu}^T C_z )^T X_2 + X_2( A - \tilde{B}_u \tilde{D}_{zu}^T C_z ) - X_2 \tilde{B}_u \tilde{B}_u^T X_2 + \tilde{C}_z^T \tilde{C}_z = 0 \quad (3.21)
\]

where
\[
\tilde{C}_z = ( I - \tilde{D}_{zu} \tilde{D}_{zu}^T ) C_z \quad (3.22)
\]

and
\[
( A - B_w \tilde{D}_{yw}^T \tilde{C}_y ) Y_2 + Y_2( A - B_w \tilde{D}_{yw}^T \tilde{C}_y )^T - Y_2 \tilde{C}_y^T \tilde{C}_y Y_2 + \tilde{F}_w \tilde{B}_w^T = 0 \quad (3.23)
\]

where
\[
\tilde{F}_w = B_w ( I - \tilde{D}_{yw}^T \tilde{D}_{yw} ) \quad (3.24)
\]

Finally,
\[
Q \in \mathcal{RH}_2 \quad \| Q \|_2^2 \leq \alpha^2 - \alpha_0^2 \quad (3.25)
\]

The \( Q \) term in the parameterization of all \( H_2 \) suboptimal compensators is identically equal to zero if the actual optimal is desired. That is, \( Q \) is nonzero only if suboptimal compensators are acceptable. As means of a summary, assume the optimal \( H_2 \) compensator is desired, and the plant satisfies the conditions i) - iv). Note that the "plant" here has all weighting functions already absorbed into it. First, do the scaling indicated in (3.5) - (3.8) to put the plant into the form of (3.9) - (3.13). Next, solve (3.21) - (3.24) for the unique positive semidefinite \( X_2 \) and \( Y_2 \). The scaled \( H_2 \) optimal compensator is then given by

\[
\tilde{K}_{2,\text{opt}}(s) = \begin{bmatrix}
A_J & K_f \\
-K_c & 0
\end{bmatrix}
\]

where \( A_J, K_c, \text{ and } K_f \) are given by (3.16) - (3.18). To get the resulting unscaled compensator, the scalings must be "given back to" the compensator, as shown in Figure 3-4. The resulting \( K_{2,\text{opt}}(s) \) is given by

\[
K_{2,\text{opt}}(s) = \begin{bmatrix}
A_J & \tilde{K}_f \\
-K_c & 0
\end{bmatrix}
\]
Figure 3-4. $H_2$ optimal diagram with scalings to compensator

where

$$\tilde{K}_f = K_f S_y$$
$$\tilde{K}_c = S_u^{-1} K_c.$$

3.2 $H_\infty$ Optimization

$H_\infty$ optimization is a relatively new subject area which began with the seminal paper by Zames ([Zam81]). The problem was then posed and solved in an operator-theoretic framework, in such work as [FD87], [Fran87], [Glo84], and [SVVL87]. Finally, a state space solution was derived, first for state feedback ([KPR88], [Pet87], and [ZK88]), and then for output feedback ([DGKF89]). Further expansions are contained in [GD88], [GD89], [GLDK91], [KPZ90], [Hvo90], [LC91], and [SLC89].

In general, the $H_\infty$ optimal compensator is not unique. Therefore, a $(J, Q)$-type parameterization will be the starting point for this section. To avoid confusion with the $H_2$ case, call the exogenous input $d$ and the controlled output $e$. This convention will be followed throughout the remainder of the thesis. Consider the feedback problem shown in Figure 3-5. The plant $P$ can be partitioned as

$$P = \begin{bmatrix} P_{ed} & P_{eu} \\ P_{yd} & P_{yu} \end{bmatrix}$$
or, in other words,

\[ e = P_{ed}d + P_{eu}u \]

\[ y = P_{yd}d + P_{yu}u \]

The exogenous input \( d \) is assumed to be a deterministic signal of unknown but bounded energy. The bound can be normalized so that

\[ \|d\|_2 \leq 1 \]  \hspace{1cm} (3.26)

It is desired to minimize the energy of the controlled output \( e \); that is, find an admissible (internally stabilizing) \( K(s) \) such that \( \|e\|_2 \) is minimized, for the worst possible input in the class described by (3.26). By the development given in Section 2.3, this is given by

\[
\inf_{K \text{ admissible}} \sup_{\|d\|_2 \leq 1} \|e\|_2 = \inf_{K \text{ admissible}} \|Ted\|_\infty \\
= \inf_{K \text{ admissible}} \|P_{ed} + P_{eu}K[I - P_{yu}K]^{-1}P_{yd}\|_\infty \]  \hspace{1cm} (3.27)
where

$$\| T_{ed} \|_{\infty} = \sup_{\omega} \sigma [ T_{ed} ]$$

In state space, the plant $P$ is described by the equations

$$\dot{x} = Ax + B_d d + B_u u$$  \hspace{1cm} (3.28)

$$e = C_e x + D_{ed} d + D_{eu} u$$  \hspace{1cm} (3.29)

$$y = C_y x + D_y d + D_{yu} u$$  \hspace{1cm} (3.30)

Assume the following are true:

i) $D_{ed} = 0$

ii) $D_{yu} = 0$

iii) $(A, B_u)$ stabilizable & $(C_y, A)$ detectable

iv) $D_{eu}^T D_{eu}$ full rank $D_y D_{yd}^T$ full rank

v) $\begin{bmatrix} A-j\omega I & B_u \\ C_e & D_{eu} \end{bmatrix}$ has full column rank for all $\omega$

vi) $\begin{bmatrix} A-j\omega I & B_d \\ C_y & D_{yd} \end{bmatrix}$ has full row rank for all $\omega$

Condition i) is not required for the $H_{\infty}$ problem, since the closed-loop transfer function having a non-zero $D$ term does not make the closed-loop infinity-norm infinite. The development is much simpler if i) holds, but it can be completely removed ([SLC89]). Condition ii) also makes the development easier, but can be completely removed ([SLC89]). Condition iii) is necessary for the existence of stabilizing solutions. Condition iv) ensures that the penalty on control usage and the sensor noise intensity are nonsingular - relaxation leads to singular control problems. Conditions v) and vi) along with iii) guarantee that the two Hamiltonian matrices in

78
the corresponding $H_2$ problem belong to $\text{dom} (\text{Ric})$. They are necessary for the solution method given below to be applicable ([GD89]).

To make the final form of the equations simpler, strengthen condition iv) to read:

$$
\text{iv) } D_{eu}^T D_{eu} = I \quad D_{yd}^T D_{yd} = I
$$

Condition iv) can always be satisfied (given iv) is true) by doing a scaling on $u$ and $y$, which we will denote as $S_u$ and $S_y$, respectively. The scaled problem may be viewed in Figure 3-6. The resulting scaled equations are:

$$
\begin{align*}
    u &= S_u^{-1} \tilde{u} \\
    \tilde{y} &= S_y y
\end{align*}
$$

where $S_u$ and $S_y$ are such that (can be found by a Cholesky decomposition)

$$
S_u^T S_u = D_{eu}^T D_{eu}
$$

Figure 3-6. $H_\infty$ feedback problem with scalings
\[ S_y^{-1} (S_y^{-1})^T = D_y d_y d_y^T \] (3.34)

The "new" plant state and output equations become

\[ \dot{x} = Ax + B_d d + B_u \tilde{u} \] (3.35)

\[ e = C_e x + \tilde{D}_{eu} \tilde{u} \] (3.36)

\[ \tilde{y} = \tilde{C}_y x + \tilde{D}_{yd} d \] (3.37)

where

\[ \tilde{B}_u = B_u S_u^{-1} \quad \tilde{C}_y = S_y C_y \] (3.38)

\[ \tilde{D}_{eu} = D_{eu} S_u^{-1} \quad \tilde{D}_{yd} = S_y D_{yd} \] (3.39)

Define

\[ \gamma_0 = \inf_{K \text{ admissible}} \| T_{ed} \|_\infty \]

The compensator that achieves this minimum value is not unique, in general. Furthermore, just as the infinity-norm of a given transfer function must be found iteratively (through a frequency sweep of the maximum singular value or the eigenvalues of an associated Hamiltonian matrix), the value of \( \gamma_0 \) must be found iteratively. The parameterization of all \( \mathcal{H}_\infty \) optimal compensators is very complicated; first, consider parameterizing all \( \mathcal{H}_\infty \) suboptimal compensators. That is, find the family of all admissible compensators such that

\[ \| T_{ed} \|_\infty < \gamma \] (3.40)

Obviously, this set should be empty if \( \gamma \leq \gamma_0 \). While this excludes characterization of \( \mathcal{H}_\infty \) optimal controllers, \( \gamma \) can theoretically be chosen arbitrarily close to \( \gamma_0 \) and thus be suboptimal to an arbitrarily small degree. The family of all admissible compensators such that (3.40) is satisfied is given by
\[ \tilde{K}(s) = F_l[J(s), Q(s)] \quad (3.41) \]

\[ J(s) = \begin{bmatrix} J_{uy} & J_{ur} \\ J_{vy} & J_{vr} \end{bmatrix} = \begin{bmatrix} A_J & K_f & K_{fl} \\ -K_c & 0 & 1 \\ K_{cl} & 0 & 0 \end{bmatrix} \quad (3.42) \]

where

\[ A_J = A - K_f \tilde{C}_y - \tilde{B}_uK_c + \gamma^{-2}Y_\infty C_e^T (C_e - \tilde{D}_{eu}K_c) \quad (3.43) \]

\[ K_c = (\tilde{B}_uX_\infty + \tilde{D}_{eu}C_e) (I - \gamma^{-2}Y_\infty X_\infty)^{-1} \quad (3.44) \]

\[ K_f = Y_\infty \tilde{C}_y^T + B_d\tilde{D}_{yd}^T \quad (3.45) \]

\[ K_{cl} = -(\gamma^{-2}\tilde{D}_{yd}B_d^TX_\infty + \tilde{C}_y)(I - \gamma^{-2}Y_\infty X_\infty)^{-1} \quad (3.46) \]

\[ K_{fl} = \gamma^{-2}Y_\infty C_e^T \tilde{D}_{eu} + \tilde{B}_u \quad (3.47) \]

Here, \( X_\infty \) and \( Y_\infty \) are solutions of the algebraic Riccati equations

\[ (A - \tilde{B}_u\tilde{D}_{eu}^T C_e)^T X_\infty + X_\infty (A - \tilde{B}_u\tilde{D}_{eu}^T C_e) \]

\[ + X_\infty (\gamma^{-2}B_dB_d^T - \tilde{B}_u\tilde{B}_u^T)X_\infty + \tilde{C}_e^T \tilde{C}_e = 0 \quad (3.48) \]

where

\[ \tilde{C}_e = (I - \tilde{D}_{eu}\tilde{D}_{eu}^T)C_e \quad (3.49) \]

and

\[ (A - B_d\tilde{D}_{yd}^T \tilde{C}_y)Y_\infty + Y_\infty (A - B_d\tilde{D}_{yd}^T \tilde{C}_y)^T \]

\[ + Y_\infty (\gamma^{-2}C_e^T C_e - \tilde{C}_y^T \tilde{C}_y)Y_\infty + \tilde{B}_d\tilde{B}_d^T = 0 \quad (3.50) \]
where

\[ \hat{B}_d = B_d (I - \hat{D}_{Yd}^T \hat{D}_{Yd}) \]  

(3.51)

Finally,

\[ Q \in \mathcal{R}H_\infty \quad \|Q\|_{\infty} < \gamma \]  

(3.52)

For this parameterization to be valid, three conditions on the solutions \(X_\infty\) and \(Y_\infty\) in relation to \(\gamma\) must be met. Consider the Hamiltonian matrices \(H_x\) and \(H_Y\) associated with (3.48) and (3.50) given by

\[ H_x = \begin{bmatrix}
A - \hat{B}_u \hat{D}_{eu}^T \hat{C}_e & \gamma^2 B_d B_d^T - \hat{B}_u \hat{B}_u^T \\
- \hat{C}_e^T \hat{C}_e & -(A - \hat{B}_u \hat{D}_{eu}^T \hat{C}_e)^T
\end{bmatrix} \]

\[ H_Y = \begin{bmatrix}
(A - B_d \hat{D}_{Yd}^T \hat{C}_y)^T & \gamma^2 \hat{C}_e^T \hat{C}_e - \hat{C}_y^T \hat{C}_y \\
\hat{B}_d \hat{B}_d^T & -(A - B_d \hat{D}_{Yd}^T \hat{C}_y)
\end{bmatrix} \]

Then there exists an admissible controller \(K(s)\) such that (3.40) holds iff

\[ i) \quad H_x \in \text{dom}(\text{Ric}) \text{ with } X_\infty = \text{Ric}(H_x) \geq 0 \]  

(3.53)

\[ ii) \quad H_Y \in \text{dom}(\text{Ric}) \text{ with } Y_\infty = \text{Ric}(H_Y) \geq 0 \]  

(3.54)

\[ iii) \quad \rho(Y_\infty X_\infty) < \gamma^2 \]  

(3.55)

If any of the above three conditions are violated, the chosen value of \(\gamma\) must be increased - the minimum \(\gamma\) which satisfies the above three is the \(H_\infty\) suboptimal solution (the minimum can be approached but not achieved). This search is referred to as \(\gamma\) iteration. Once \(\gamma\) is found to any desired degree of accuracy, the scaled compensator may be found from (3.41) - (3.47). To produce the unscaled compensator, move the scalings back into the compensator, as shown in Figure 3-7.
The unscaled compensator $K(s)$ is then given by

$$K(s) = F[J(s), Q(s)]$$  \hspace{1cm} (3.56)

where

$$J(s) = \begin{bmatrix} A_J & \tilde{K}_f & K_{fl} \\ -\tilde{K}_c & 0 & S_u^{-1} \\ K_{cl} & S_y & 0 \end{bmatrix}$$  \hspace{1cm} (3.57)

where

$$\tilde{K}_f = K_f S_y$$  \hspace{1cm} (3.58)

$$\tilde{K}_c = S_u^{-1} K_c$$  \hspace{1cm} (3.59)

Now consider parameterizing all $H_\infty$ optimal compensators; that is, find the family of all admissible compensators such that
\| T_{ed} \|_\infty \leq \gamma

The optimal is achieved at the infimum of \( \gamma \) such that (3.53) - (3.55) hold, with this \( \gamma \) denoted by \( \gamma_0 \). Typically, the optimum occurs at \( \rho(Y_\infty X_\infty) = \gamma_0^2 \). In this case, \( (I - \gamma_0^2 Y_\infty X_\infty)^{-1} \) does not exist, but this difficulty can be removed by using a descriptor system form of the problem ([GLDKS91]). The resulting optimal compensator will have infinite bandwidth in this case - that is, \( K(s) \) will have a nonzero \( D \) term in its state space realization.

It is also possible that the optimum occurs when one of the Riccati equations fails to have a positive semidefinite solution but \( \rho(Y_\infty X_\infty) < \gamma^2 \). It is possible to characterize solutions in this case as well, and in this case, it is possible to have optimal compensators which are strictly proper. For more on \( H_\infty \) optimal compensators, see [CLBY90b], [CLBY91], [GD89], [GLDKS91], [IG90], and [SLC89].

### 3.3 Connections between \( H_2 \) and \( H_\infty \) Optimization

Now the connections between \( H_2 \) and \( H_\infty \) optimization can easily be seen. Let \( Q = 0 \) and let \( \gamma \) be very large in (3.41) - (3.51). Then (3.41) - (3.51) become the \( H_2 \) equations given by (3.14) - (3.24). In other words, as \( \gamma \) becomes large, the \( (Q = 0) - H_\infty \) compensator becomes the optimal \( H_2 \) compensator. The \( (Q = 0) - H_\infty \) compensator will be referred to as the central \( H_\infty \) controller. The connection here is actually much deeper. Recall the definition of entropy from Section 2.3.3. Mustafa and Glover([MG88], [MG90]) have shown that the controller that solves the minimum entropy/\( H_\infty \) control problem, given by:

**Minimum entropy/\( H_\infty \) control problem** - Let \( P \) satisfy conditions i) - vi) in Section 3.2, and let \( \gamma > \gamma_0 \). Minimize the closed-loop entropy \( I(T_{ed}, \gamma) \) over all admissible \( K(s) \) such that \( \| T_{ed} \|_\infty < \gamma \).

is precisely the central \( H_\infty \) controller of the parameterization given in Section 3.2.
Now consider adding another exogenous input and controlled output to the system in Figure 3-5, as shown in Figure 3-8.

![Figure 3-8. General optimization problem](image)

Assume that the state space representation of \( P(s) \) in Figure 3-8 is given by

\[
\begin{bmatrix}
    A & B_d & B_w & B_u \\
    C_e & 0 & D_{ew} & D_{eu} \\
    C_z & D_{zd} & 0 & D_{zu} \\
    C_y & D_{yd} & D_{yw} & 0
\end{bmatrix}
\]

(3.60)

From (3.56) - (3.59), the family of all admissible \( K(s) \) such that (3.40) is satisfied is given by

\[
\begin{align*}
\dot{x}_J &= A_J x_J + \tilde{K}_f y + K_{fl} r \\
 u &= -\tilde{K}_c x_J + S_u^{-1} r \\
v &= K_{cl} x_J + S_y y
\end{align*}
\]

(3.61a)  (3.61b)  (3.61c)
and $Q(s)$ is such that (3.52) is satisfied. Combining (3.60) with equations (3.61) yields

$$\dot{x}_n = A_n x_n + B_D d + B_W w + B_R r$$  \hfill (3.62a)

$$e = C_E x_n + D_{ew} w + D_{eu} S_u^{-1} r$$  \hfill (3.62b)

$$z = C_Z x_n + D_{zd} d + D_{zu} S_u^{-1} r$$  \hfill (3.62c)

$$v = C_V x_n + S_y D_{yd} d + S_y D_{yw} w$$  \hfill (3.62d)

where

$$A_n = \begin{bmatrix} A & -B_u \tilde{K}_c \\ \tilde{K}_f C_y & A_J \end{bmatrix}$$

$$B_D = \begin{bmatrix} B_d \\ \tilde{K}_f D_{yd} \end{bmatrix} \quad B_W = \begin{bmatrix} B_w \\ \tilde{K}_f D_{yw} \end{bmatrix} \quad B_R = \begin{bmatrix} B_u S_u^{-1} \\ K_{fl} \end{bmatrix}$$

$$C_E = [C_e \quad -D_{eu} \tilde{K}_c] \quad C_Z = [C_Z \quad -D_{zu} \tilde{K}_c] \quad C_V = [S_y C_y \quad K_{cl}]$$

This is just the closure of the $(P, J)$ loop. The corresponding block diagram is shown in Figure 3-9, where $G(s)$ is given by (3.62) or

![Block diagram](image)

**Figure 3-9.** Closure of the P-J loop through $H_\infty$ optimization
\[ G(s) = \begin{bmatrix} A_n & B_D & B_w & B_R \\ C_E & 0 & D_{ew} & D_{eu}S_u^{-1} \\ C_Z & D_{zd} & 0 & D_{zu}S_u^{-1} \\ C_V & S_YD_{yd} & S_YD_{yw} & 0 \end{bmatrix} \]  \tag{3.63}

In transfer function form, \( T_{ed} \) is given by

\[ e = T_{ed} d = (G_{ed} + G_{er}(I - QG_{vr})^{-1}QG_{vd})d \]  \tag{3.64}

where

\[ G_{ed} = C_E(sI - A_n)^{-1}B_D \]  \tag{3.65}

\[ G_{er} = C_E(sI - A_n)^{-1}B_R + D_{eu}S_u^{-1} \]  \tag{3.66}

\[ G_{vd} = C_V(sI - A_n)^{-1}B_D + S_YD_{yd} \]  \tag{3.67}

\[ G_{vr} = C_V(sI - A_n)^{-1}B_R \]  \tag{3.68}

and \( \|T_{ed}\|_\infty < \gamma \) is satisfied iff \( Q \) is chosen such that (3.52) holds.

Similarly, \( T_{zw} \) is given in transfer function form by

\[ z = T_{zw} w = (G_{zw} + G_{zr}(I - QG_{vr})^{-1}QG_{vw})w \]  \tag{3.69}

where

\[ G_{zw} = C_Z(sI - A_n)^{-1}B_W \]  \tag{3.70}

\[ G_{zr} = C_Z(sI - A_n)^{-1}B_R + D_{zu}S_u^{-1} \]  \tag{3.71}

\[ G_{vw} = C_V(sI - A_n)^{-1}B_W + S_YD_{yw} \]  \tag{3.72}

and \( G_{vr} \) is given by (3.68).
Let $Q(s) = C_c(sI - A_c)^{-1}B_c$ so that

$$\dot{x}_c = A_c x_c + B_c v$$

$$r = C_c x_c$$

Then the closed-loop transfer functions in state space form are

$$T_{ed} = C_e (sI - \mathcal{A})^{-1}B_d$$

$$T_{zw} = C_z (sI - \mathcal{A})^{-1}B_w$$

where

$$\mathcal{A} = \begin{bmatrix} A_n & B_R C_c \\ B_C C_V & A_c \end{bmatrix}$$

$$B_d = \begin{bmatrix} B_W \\ B_C S_y D_y d \end{bmatrix}$$

$$B_w = \begin{bmatrix} B_W \\ B_C S_y D_y w \end{bmatrix}$$

$$C_e = [ C_E, D_e u S_u^{-1} C_c ]$$

$$C_z = [ C_z, D_z u S_u^{-1} C_c ]$$

The problem this thesis considers is called the mixed $H_2/H_\infty$ problem, which will be defined and developed in the next chapter. Basically, the problem is to find an admissible $K(s)$ that minimizes $\| T_{zw} \|_2$ subject to $\| T_{ed} \|_\infty \leq \gamma$. The problem will be solved numerically by asymptotically approaching the infinity-norm bound, so that $\| T_{ed} \|_\infty$ will always be strictly less than $\gamma$. Thus, the suboptimal mixed compensator must be in the family of $H_\infty$ suboptimal compensators, and must therefore correspond to some $Q(s) \in \mathcal{RH}_\infty$ with $\| Q \|_\infty < \gamma$. 

88
CHAPTER 4
MIXED $H_2/H_\infty$ OPTIMIZATION

4.1 General Derivation of the Mixed $H_2/H_\infty$ Problem

Consider the block diagram shown in Figure 4-1, which has two sets of exogenous inputs and controlled outputs. Note that this formulation is completely general in that no relationship is assumed to hold between $e$ and $z$ or $d$ and $w$ — that is, either pair or both pairs can be equal, dependent, or completely independent.

![Block Diagram](image)

Figure 4-1. General mixed optimization problem

The plant $P$ can be partitioned as

$$
P = \begin{bmatrix}
P_{ed} & P_{ew} & P_{eu} \\
P_{zd} & P_{zw} & P_{zu} \\
P_{yd} & P_{yw} & P_{yu}
\end{bmatrix}
$$

or the input-output equations

$$
e = P_{ed}d + P_{ew}w + P_{eu}u$$

$$z = P_{zd}d + P_{zw}w + P_{zu}u$$

$$y = P_{yd}d + P_{yw}w + P_{yu}u$$
A state space realization of $P$ is given by

$$
P = \begin{bmatrix}
    A & B_d & B_w & B_u \\
    C_e & D_{ed} & D_{ew} & D_{eu} \\
    C_z & D_{zd} & D_{zw} & D_{zu} \\
    C_y & D_{yd} & D_{yw} & D_{yu}
\end{bmatrix}
$$

The exogenous input $d$ is assumed to be a deterministic signal of unknown but bounded energy. The bound can be normalized so that

$$
\| d \|_2 \leq 1
$$

It is desired to minimize the energy of the controlled output $e$ for the worst possible input in the class described by (4.1); that is, find an admissible (internally stabilizing) $K(s)$ such that $\| e \|_2$ is minimized. By the development given in Section 3.2, this is equivalent to finding the family of $K(s)$ that achieves

$$
\inf_{K \text{ admissible}} \| T_{ed} \|_\infty
$$

or equivalently, finding the set of all admissible $K(s)$ such that

$$
\| T_{ed} \|_\infty \leq \gamma
$$

and lowering $\gamma$ until the set becomes empty. From Chapter 1, this guarantees a given level of robustness, and the smaller $\gamma$ is made, the more robust the system is.

At the same time, the exogenous input $w$ is assumed to be a zero-mean white Gaussian noise of unit intensity. It is also desired to minimize the energy of the controlled output $z$ given such a $w$ as an input; that is, find an admissible $K(s)$ such that $\| z \|_2$ is minimized in the face of $w$. By the development given in Section 3.1, this is equivalent to finding the family of $K(s)$ that achieves

$$
\inf_{K \text{ admissible}} \| T_{zw} \|_2
$$

and this "family" is a single unique $K(s)$. Since this is unique, there is no freedom left to affect $T_{ed}$, and it seems unlikely that (4.2) would be
satisfied for an arbitrary choice of $\gamma$. Therefore, a trade-off between the
two objectives (4.2) and (4.3) is indicated.

First, assume that the plant $P$ satisfies the following assumptions:

i) $D_{ed} = 0$
ii) $D_{zw} = 0$
iii) $D_{yu} = 0$

Condition i) is not required for the existence of admissible $K(s)$ such that
(4.2) is finite, but the development is much simpler with it enforced.
Condition ii) is required or the $H_2$ problem is not well defined, as the
closed-loop transfer function will then have a non-zero D term for any
choice of compensator, thus making the closed-loop two-norm infinite.
Condition iii) again makes the development easier, but could be completely
removed.

The plant $P$ is now

$$
P = \begin{bmatrix}
A & B_d & B_w & B_u \\
C_e & 0 & D_{ew} & D_{eu} \\
C_z & D_{zd} & 0 & D_{zu} \\
C_y & D_{yd} & D_{yw} & 0
\end{bmatrix}
$$

(4.4)

The mixed $H_2/H_\infty$ problem can now be stated as follows:

**The Mixed $H_2/H_\infty$ Problem** - Consider the feedback system of Figure
4-1, where the plant $P$ is given by (4.4). Find an admissible controller $K(s)$
that achieves

$$
\inf_{K \text{ admissible}} \|T_{zw}\|_2
$$

subject to the constraint

$$
\|T_{ed}\|_\infty \leq \gamma
$$
Now make the following additional assumptions on the plant:

iv) \((A, B_u)\) stabilizable & \((C_y, A)\) detectable

v) \(D_{eu}^T D_{eu}\) full rank \(D_{yd} D_{yd}^T\) full rank

vi) \(D_{zu}^T D_{zu}\) full rank \(D_{yw} D_{yw}^T\) full rank

vii) \[
\begin{bmatrix}
A - j\omega I & B_u \\
C_e & D_{eu}
\end{bmatrix}
\] has full column rank for all \(\omega\)

viii) \[
\begin{bmatrix}
A - j\omega I & B_d \\
C_y & D_{yd}
\end{bmatrix}
\] has full row rank for all \(\omega\)

ix) \[
\begin{bmatrix}
A - j\omega I & B_u \\
C_z & D_{zu}
\end{bmatrix}
\] has full column rank for all \(\omega\)

x) \[
\begin{bmatrix}
A - j\omega I & B_w \\
C_y & D_{yw}
\end{bmatrix}
\] has full row rank for all \(\omega\)

Condition iv) is necessary for the existence of stabilizing controllers. Conditions v) through x) ensure that each of the individual problems (that is, the \(H_2\) or \(H_\infty\) problem considered separately) has nonsingular admissible solutions. In the mixed problem, it may be possible to remove some of these; for example, enforcing the full rank conditions in vi) guarantees nonsingular controls for the \(H_2\) part of the problem, so one or both of the \(H_\infty\) counterparts in v) may be unnecessary since the same controller is being used. This relaxation will not be considered in this work, however.

The compensator in Figure 4-1 is given by

\[ u = K(s) y \]

or, in state space form

\[
\dot{x}_c = A_c x_c + B_c y \quad (4.5a)
\]

\[ u = C_c x_c + D_c y \quad (4.5b) \]
Combining (4.4) and (4.5) produces the closed-loop state space equations

\[
\dot{x} = (A + B_u D_c C_Y)x + B_u C_c x_c \\
+ (B_d + B_u D_c D_Y d) d + (B_w + B_u D_c D_Y w) w \tag{4.6a}
\]

\[
\dot{x}_c = B_c C_Y x + A_c x_c + B_c D_Y d + B_c D_Y w \tag{4.6b}
\]

\[
e = (C_e + D_{eu} D_c C_Y)x + D_{eu} C_c x_c \\
+ D_{eu} D_c D_Y d + (D_{ew} + D_{eu} D_c D_Y w) w \tag{4.6c}
\]

\[
z = (C_z + D_{zu} D_c C_Y)x + D_{zu} C_c x_c \\
+ (D_{zd} + D_{zu} D_c D_Y d) d + D_{zu} D_c D_Y w \tag{4.6d}
\]

As may be seen from (4.6d), the transfer function \( T_{zw} \) would have the D term \( D_{zu} D_c D_Y w \). In order for \( T_{zw} \) to be finite, it cannot have a D term, so

\[
D_{zu} D_c D_Y w = 0
\]

Given condition vi) on \( D_{zu} \) and \( D_Y w \), the pseudoinverses \( D_{zu}^+ \) and \( D_Y^+ \) must exist, so the above implies that \( D_c = 0 \).

Therefore, without loss of generality, assume the compensator \( K(s) \) is strictly proper, so \( D_c = 0 \). Equations (4.6) can then be written in the form

\[
\dot{x} = A x + B_d d + B_w w \tag{4.7a}
\]

\[
e = C_e x + D_{ew} w \tag{4.7b}
\]

\[
z = C_z x + D_{zd} d \tag{4.7c}
\]

where
\[ \mathbf{A} = \begin{bmatrix} A & B_u C_c \\ B_c C_y & A_c \end{bmatrix} \quad (4.8) \]

\[ \mathbf{B}_{d} = \begin{bmatrix} B_d \\ B_c D_{yd} \end{bmatrix} \quad (4.9) \]

\[ \mathbf{B}_{w} = \begin{bmatrix} B_w \\ B_c D_{yw} \end{bmatrix} \quad (4.10) \]

\[ \mathbf{C}_{e} = \begin{bmatrix} C_e \\ D_{eu} C_c \end{bmatrix} \quad (4.11) \]

\[ \mathbf{C}_{z} = \begin{bmatrix} C_z \\ D_{zu} C_c \end{bmatrix} \quad (4.12) \]

so that

\[ T_{ed} = \mathbf{C}_{e} (s \mathbf{I} - \mathbf{A})^{-1} \mathbf{B}_{d} \quad (4.13) \]

\[ T_{zw} = \mathbf{C}_{z} (s \mathbf{I} - \mathbf{A})^{-1} \mathbf{B}_{w} \quad (4.14) \]

The mixed H$_2$/H$_\infty$ problem can now be restated as:

**The Mixed H$_2$/H$_\infty$ Problem** - Determine $K(s) = C_c(s \mathbf{I} - A_c)^{-1} B_c$ such that

1) the closed-loop system is internally stable; i.e., $\mathbf{A}$ is stable

2) for a given $\gamma > 0$, the transfer function $T_{ed}$ as given in (4.13) satisfies the $H_\infty$ constraint

\[ \| T_{ed} \|_\infty \leq \gamma \]

3) the performance index

\[ J = \| T_{zw} \|_2^2 \]

is minimized, where $T_{zw}$ is given by (4.14)
We now state one of the main theorems of this section.

**Theorem 4.1.1**: Let \((A_C, B_C, C_C)\) be given and assume there exists a \(Q_\infty = Q_\infty^T \succeq 0\) satisfying

\[
A Q_\infty + Q_\infty A^T + \gamma^2 Q_\infty C e^T e Q_\infty + B_d B_d^T = 0
\]  \(4.15\)

Then the following are equivalent:

i) \((A, B_d)\) is stabilizable

ii) \(A\) is stable

Moreover, if i) - ii) hold, then the following are true:

iii) the transfer function \(T_{ed}\) satisfies

\[\|T_{ed}\|_\infty \leq \gamma\]

iv) the transfer function \(T_{zw}\) is given by

\[\|T_{zw}\|_2^2 = \text{tr}[C_z Q_2 C_z^T] = \text{tr}[Q_2 C_z C_z]\]

where \(Q_2 = Q_2^T \succeq 0\) is the solution to the Lyapunov equation

\[A Q_2 + Q_2 A^T + B_w B_w^T = 0\]

v) all real symmetric solutions to \((4.15)\) are positive semidefinite

vi) there exists a unique minimal solution to \((4.15)\) in the class of real symmetric solutions

vii) \(Q_\infty\) is the minimal solution to \((4.15)\) iff

\[\text{Re}[\lambda_i(A + \gamma^2 Q_\infty C e^T e)] \leq 0 \quad \forall i\]
viii) \( \| T_{ed} \|_\infty < \gamma \) iff \( \mathcal{A} + \gamma^2 Q_\infty C_e^T C_e \) is stable, where \( Q_\infty \) is the minimal solution to (4.15)

ix) if \( Q_\infty \) is the minimal solution to (4.15) and \( \| T_{ed} \|_\infty < \gamma \), then

\[
I(T_{ed}, \gamma) = \text{tr}[Q_\infty C_e^T C_e]
\]

Proof: i) \( \rightarrow \) ii) From the dual of Theorem 2.5.8, the assumed existence of a \( Q_\infty = Q_\infty^T \geq 0 \) satisfying (4.15) and \( (\mathcal{A}, B_d) \) stabilizable implies \( \mathcal{A} \) is stable. ii) \( \rightarrow \) i) \( \mathcal{A} \) stable trivially implies \( (\mathcal{A}, B_d) \) is stabilizable. With \( \mathcal{A} \) stable, Part iii) is immediate from Theorem 2.5.17. Part iv) comes directly from the discussion of two-norms in Section 2.3.2. Part v) follows from the dual of Theorem 2.5.7, since \( \mathcal{A} \) stable implies \( (C_e, \mathcal{A}) \) is detectable. Parts vi) and vii) follow from the dual of Theorem 2.5.6. Part viii) follows from the duals of Theorem 2.5.13, 2.5.14, and from part vii). Finally, Part ix) follows from Theorem 2.3.2.

Using Theorem 4.1.1, the mixed \( H_2/H_\infty \) problem can again be restated as:

**The Mixed \( H_2/H_\infty \) Problem** - Determine \((A_c, B_c, C_c)\) that minimizes

\[
J(A_c, B_c, C_c) = \text{tr}[Q_2 C_{z_2}^T]
\quad (4.16)
\]

where \( Q_2 \) is the real, symmetric, positive semidefinite solution to

\[
\mathcal{A} Q_2 + Q_2 \mathcal{A}^T + B_w B_w^T = 0
\quad (4.17)
\]

and such that

\[
\mathcal{A} Q_\infty + Q_\infty \mathcal{A}^T + \gamma^2 Q_\infty C_e C_e Q_\infty + B_d B_d^T = 0
\quad (4.18)
\]

has a real, symmetric, positive semidefinite solution.
The problem now has the form of minimizing an objective function (4.16), subject to the equality constraints (4.17) and (4.18). Therefore, it can be cast as a Lagrange multiplier problem as follows:

**The Mixed $H_2/H_\infty$ Lagrange Multiplier Problem** - Assume $Q_2 = Q_2^T \geq 0$ and $Q_\infty = Q_\infty^T \geq 0$. Minimize the Lagrangian

$$L = \text{tr}[Q_2 C_2^T C_2] + \text{tr}(AQ_2 + Q_2 A^T + B_w B_w^T)\chi)$$

$$+ \text{tr}(AQ_\infty + Q_\infty A^T + \gamma^2 Q_\infty C_e^T C_e Q_\infty + B_d B_d^T)\Upsilon) \quad (4.19)$$

Here, $\chi$ and $\Upsilon$ are the Lagrange multiplier matrices.

Let

$$B = B_u \quad C = C_y$$

$$Q_2 = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} \quad \chi = \begin{bmatrix} X_1 & X_{12} \\ X_{21} & X_2 \end{bmatrix}$$

$$Q_\infty = \begin{bmatrix} Q_a & Q_{ab} \\ Q_{ab}^T & Q_b \end{bmatrix} \quad \Upsilon = \begin{bmatrix} Y_1 & Y_{12} \\ Y_{21} & Y_2 \end{bmatrix}$$

From (4.9) - (4.12),

$$B_d B_d^T = \begin{bmatrix} B_d \\ B_c D_y b \end{bmatrix} \begin{bmatrix} B_d^T & D_y b^T \end{bmatrix} = \begin{bmatrix} V_a & V_{ab} B_c^T \\ B_c V_{ab}^T & B_c V_b b^T \end{bmatrix}$$

97
\[
\mathbf{B}_w \mathbf{B}_w^T = \begin{bmatrix}
\mathbf{B}_w^T \\
\mathbf{B}_c \mathbf{D}_{yw}
\end{bmatrix} \begin{bmatrix}
\mathbf{B}_w^T \\
\mathbf{D}_{yw} \mathbf{B}_c^T
\end{bmatrix} = \begin{bmatrix}
\mathbf{V}_1 \\
\mathbf{V}_{12} \mathbf{B}_c^T
\end{bmatrix}
\]

\[
\mathbf{C}_e \mathbf{C}_e = \begin{bmatrix}
\mathbf{C}_e^T \\
\mathbf{C}_c \mathbf{D}_{eu}
\end{bmatrix} \begin{bmatrix}
\mathbf{C}_e \\
\mathbf{D}_{eu} \mathbf{C}_c
\end{bmatrix} = \begin{bmatrix}
\mathbf{R}_a \\
\mathbf{C}_c \mathbf{R}_{ab}
\end{bmatrix}
\]

\[
\mathbf{C}_z \mathbf{C}_z = \begin{bmatrix}
\mathbf{C}_z^T \\
\mathbf{C}_c \mathbf{D}_{zu}
\end{bmatrix} \begin{bmatrix}
\mathbf{C}_z \\
\mathbf{D}_{zu} \mathbf{C}_c
\end{bmatrix} = \begin{bmatrix}
\mathbf{R}_1 \\
\mathbf{C}_c \mathbf{R}_{12}
\end{bmatrix}
\]

Using the formulas for differentiation of a scalar with respect to a matrix, notice that

\[
\frac{\partial \mathbf{L}}{\partial \mathbf{Q}_2} = 0 = \mathbf{A}^T \mathbf{X} + \mathbf{X} \mathbf{A} + \mathbf{C}_z \mathbf{C}_z
\]

and

\[
\frac{\partial \mathbf{L}}{\partial \mathbf{Q}_\infty} = 0 = \mathbf{Y} \left( \mathbf{A} + \mathbf{C}_e^T \mathbf{C}_e \right) + \left( \mathbf{A} + \mathbf{C}_e^T \mathbf{C}_e \right)^T \mathbf{Y}
\]

so \( \mathbf{X} \) and \( \mathbf{Y} \) are symmetric. Thus, \( X_{21} = X_{12}^T, Y_{21} = Y_{12}^T \) and \( X_1, X_2, Y_1 \), and \( Y_2 \) are symmetric. After substituting all the above into the Lagrangian (4.19), and using (4.8) for \( \mathbf{A} \), (4.19) becomes

98
\[ \mathbf{L} = trl \ Q_1 R_1 + Q_{12} C_{c}^T R_{12}^T + Q_{12}^T R_{12} C_{c} + Q_{2} C_{c}^T R_{2} C_{c} + A Q_{1} X_{1} + Q_{1} A_{X_1}^T X_{1} \\
+ B C_{c} Q_{12}^T X_{1} + Q_{12} C_{c} B_{X_1}^T X_{1} + V_{1} X_{1} + A Q_{12} X_{1}^T + Q_{12} A_{c X_1}^T X_{12} + B C_{c} Q_{2} X_{12}^T \\
+ Q_{1} C_{c} B_{c} X_{12}^T + V_{12} B_{X_1}^T X_{12} + B C_{c} Q_{12} X_{12} + A c Q_{12} X_{12}^T + Q_{12} A_{c}^T X_{12} \\
+ Q_{2} C_{c} B_{c} X_{12} + B c V_{12} X_{12} + A c Q_{2} X_{2} + Q_{2} A_{c X_2}^T X_{2} + B C_{c} Q_{12} X_{2}^T \\
+ Q_{12} C_{c} B_{c} X_{2} + B c V_{2} B_{c}^T X_{2} + A Q_{a Y_1} + Q_{a A_{Y_1}}^T Y_{1} + B C_{c} Q_{ab Y_1}^T \\
+ Q_{ab} C_{c} B_{Y_1}^T Y_{1} + V_{a Y_1} + A Q_{ab Y_1}^T Y_{1} + Q_{ab A_{c Y_1}}^T X_{12} + B C_{c} Q_{ab Y_1}^T \\
+ Q_{a} C_{c} B_{c}^T Y_{12} + V_{a b} C_{c}^T Y_{12} + B C_{c} Q_{a} Y_{12} + A c Q_{a b} Y_{12} + Q_{a b A_{c Y_1}}^T Y_{12} \\
+ Q_{b} C_{c} B_{c}^T Y_{12} + B c V_{a b} Y_{12} + A c Q_{b} Y_{2} + Q_{b A_{c Y_2}}^T Y_{12} + B C_{c} Q_{ab Y_2} \\
+ Q_{a b} C_{c} B_{c}^T Y_{2} + B c V_{2} B_{c}^T Y_{2} + \gamma^2 \left\{ Q_{a} R_{a} Q_{a Y_1} + Q_{ab} C_{c} R_{ab} Q_{a Y_1} \\
+ Q_{a R_{a c Y_1}}^T C_{c} Q_{a b} Y_{1} + Q_{a b} C_{c} R_{b c} Q_{a b} Y_{1} + Q_{a R_{a Y_1}}^T Q_{a b} Q_{a b Y_1} + Q_{a b} C_{c} R_{ab} Q_{a b Y_1} \\
+ Q_{a R_{a c Y_1}}^T C_{c} Q_{b Y_1} + Q_{a b} C_{c} R_{b c} Q_{b} Q_{a b Y_1} + Q_{a R_{a Y_1}}^T Q_{a b} Q_{a b Y_1} + Q_{a b} C_{c} R_{ab} Q_{a b Y_1} \\
+ Q_{a b} R_{a b c Y_1}^T Q_{a b Y_1} + Q_{a b} C_{c} R_{b c} Q_{a b} Q_{a b Y_1} + Q_{a R_{a b Y_1}}^T Q_{a b} Q_{a b Y_1} + Q_{a b} C_{c} R_{ab} Q_{a b Y_1} \\
+ Q_{a b} R_{a b c Q_{b} Y_2} + Q_{b} C_{c} R_{b c} Q_{b} Q_{a b Y_1} \right\} \tag{4.20} \]

Now take the partial derivatives of (4.20) with respect to the unknown matrices to produce the first-order necessary conditions for a minimum. These are

\[ \frac{\partial \mathbf{L}}{\partial A_{c}} = X_{12}^T Q_{12} + X_{2} Q_{2} + Y_{12}^T Q_{ab} + Y_{2} Q_{b} = 0 \tag{4.21} \]
\[
\frac{\partial L}{\partial B_c} = X_{12}^TQ_1C^T + X_{2Q_12}^T + X_{12}^TV_{12} + X_2B_cV_2 + Y_{12}^TQ_aC^T + Y_{2Q_2}^T + Y_{12}^TV_{ab} + Y_2B_cV_b = 0
\]  
\hspace{1cm} (4.22)

\[
\frac{\partial L}{\partial C_c} = B^TX_{1Q_12} + B^TX_{12Q_2} + R_{1Q_12}^T + R_{2Q_2} + B^TY_{1Q_ab} + B^TY_{12Q_b}
+ \gamma^2 \left\{ R_{ab}^TQ_aY_{1Q_ab} + R_{ab}^TQ_aY_{12Q_b} + R_{ab}^TQ_abY_{1Q_ab}
+ R_{ab}^TQ_abY_{12Q_b} + R_{ab}^TQ_abY_{1Q_ab} + R_{ab}^TQ_abY_{12Q_b}
+ R_{ab}^TQ_abY_{12Q_b} + R_{ab}^TQ_abY_{12Q_b} \right\} = 0
\]  
\hspace{1cm} (4.23)

\[
\frac{\partial L}{\partial \mathbf{X}} = \mathbf{A}Q_2 + Q_2\mathbf{A}^T + B_wB_w^T = 0
\]  
\hspace{1cm} (4.24)

\[
\frac{\partial L}{\partial Q_2} = \mathbf{A}^T\mathbf{X} + \mathbf{X}A + \mathbf{C}_2^T\mathbf{C}_2 = 0
\]  
\hspace{1cm} (4.25)

\[
\frac{\partial L}{\partial \mathbf{y}} = \mathbf{A}Q_\infty + Q_\infty\mathbf{A}^T + \gamma^2Q_\infty\mathbf{C}_e^T\mathbf{C}_eQ_\infty + B_dB_d^T = 0
\]  
\hspace{1cm} (4.26)

\[
\frac{\partial L}{\partial Q_\infty} = [\mathbf{A} + \gamma^2Q_\infty\mathbf{C}_e^T\mathbf{C}_e]^T\mathbf{y} + \mathbf{y}[\mathbf{A} + \gamma^2Q_\infty\mathbf{C}_e^T\mathbf{C}_e] = 0
\]  
\hspace{1cm} (4.27)

Consider the final equation, (4.27). From Theorem 2.4.3, if 
\((\mathbf{A} + \gamma^2Q_\infty\mathbf{C}_e^T\mathbf{C}_e)\) is stable, \(\mathbf{y} = 0\) is the only solution. Note that if

\((\mathbf{A} + \gamma^2Q_\infty\mathbf{C}_e^T\mathbf{C}_e)\) is stable, \(Q_\infty\) is the stabilizing solution to (4.26), so
\[\|T_{\text{ed}}\|_\infty < \gamma\] by Theorem 4.1.1. More importantly, we have the Lagrange multiplier associated with the \(H_\infty\) Riccati equation equal to zero.
Therefore, this is an "off boundary" solution - the $H_\infty$ constraint is not active. This is seen by noting that the Lagrangian (4.19) becomes

$$L = \text{tr}(Q_2 C_z^T C_z) + \text{tr}( [A Q_2 + Q_2 A^T + B_w B_w^T ] X )$$  \hspace{1cm} (4.28)

which seems to indicate that the above problem, which is just $H_2$ optimization, solves the mixed $H_2/H_\infty$ problem, and thus the $H_\infty$ constraint is trivially satisfied. To better understand what is happening, consider a very simple illustrative example.

**Example.** Consider minimizing the scalar objective function

$$f(x) = (x - 2)^2 \quad \quad x \in \mathbb{R}$$

subject to

$$x \leq \beta \quad \beta \text{ a known real scalar}$$  \hspace{1cm} (4.29)

The standard solution technique for a problem of this nature is to transform the inequality constraint (4.29) into an equality constraint by introducing a *slack variable*, given by $\alpha^2$. The slack variable is a measure of how much "slack" exists between the value of $x$ and its boundary value $\beta$. Therefore, the problem is transformed into

$$\min (f(x) = (x - 2)^2)$$  \hspace{1cm} (4.30)

subject to

$$x - \beta + \alpha^2 = 0$$  \hspace{1cm} (4.31)

Note that when $\alpha = 0$, the inequality constraint (4.29) becomes an equality, and for any nonzero $\alpha$, the inequality in (4.29) is strict.

Form the Lagrangian

$$L = (x - 2)^2 + \lambda (x - \beta + \alpha^2)$$

101
where $\lambda$ is a Lagrange multiplier. The first-order necessary conditions for a minimum are

\[
\frac{\partial L}{\partial x} = 0 = 2(x - 2) + \lambda \\
\frac{\partial L}{\partial \lambda} = 0 = x - \beta + \alpha^2 \\
\frac{\partial L}{\partial \beta} = 0 = 2\alpha \lambda
\]  

(4.32)  

(4.33)  

(4.34)

From (4.34), there are two possible solutions: $\alpha = 0$ or $\lambda = 0$. Examining each yields:

<table>
<thead>
<tr>
<th>$\alpha = 0$</th>
<th>&quot;on boundary&quot; - slack = 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = \beta$</td>
<td>$f(x) = (\beta - 2)^2$</td>
</tr>
<tr>
<td>$\lambda = 2(2 - \beta)$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\lambda = 0$</th>
<th>&quot;off boundary&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 2$</td>
<td>$f(x) = 0$</td>
</tr>
</tbody>
</table>

$\alpha^2 = \beta - 2 \Rightarrow \alpha$ is imaginary for $\beta < 2$

use "on boundary" solution if $\beta < 2$

Therefore, we know the solution is "on the boundary" if $\beta < 2$, and the slack is nonzero. The constraint is active. What if $\beta > 2$? Since the value of $f(x)$ is always positive for the "on boundary" solution when $\beta > 2$, and always equals zero for the "off boundary" solution, the minimal solution is "off the boundary" if $\beta > 2$. The Lagrange multiplier is always zero here, and the constraint is inactive. The only remaining case is $\beta = 2$. In this case, both solutions are exactly the same, and thus the solution is both "on and off the boundary", in some sense. The Lagrange multiplier is zero in this case, but the slack is zero. The overall result is that
\[
\min \{ f(x) = (x - 2)^2 \} = \begin{cases} 
0 & \text{if } \beta \geq 2 \\
(\beta - 2)^2 & \text{if } \beta \leq 2 
\end{cases}
\]

This example will be referred to when describing the behavior of the mixed $H_2/H_\infty$ problem solution.

Make the following definitions, which will be used throughout the rest of this thesis:

\begin{align*}
\gamma_0 & = \inf_{K \text{ admissible}} \| T_{ed} \|_\infty \\
\alpha_0 & = \inf_{K \text{ admissible}} \| T_{zw} \|_2 \\
K_{2\text{opt}} & = \text{the unique } K(s) \text{ that makes } \| T_{zw} \|_2 = \alpha_0 \\
\gamma_2 & = \| T_{ed} \|_\infty \text{ when } K(s) = K_{2\text{opt}} \\
K_{\text{mix}} & = \text{a } K(s) \text{ that solves the mixed } H_2/H_\infty \text{ problem at some } \gamma \\
\gamma^* & = \| T_{ed} \|_\infty \text{ when } K(s) = K_{\text{mix}} \\
\alpha^* & = \| T_{zw} \|_2 \text{ when } K(s) = K_{\text{mix}}
\end{align*}

To proceed further, we need to fix the order of the compensator. This is done in the next section.

### 4.2 Fixed Order

The order of the compensator will now be assumed to be fixed. Define the dimension of the state vector of $P(s)$ to be $n$, and the order of the state vector of $K(s)$ to be $n_c$. The problem will be broken down into the three cases: i) *full order*, where $n_c = n$, ii) *reduced order*, where $n_c < n$, and iii) *increased order*, where $n_c > n$. 

103
4.2.1 Full Order

Consider the full order case where \( n_c = n \), so that the compensator has the same order as the plant. It is obvious that if \( \gamma \) is chosen such that \( \gamma \geq \gamma_2 \), we can use \( K_{2_{\text{opt}}} \) (which has order \( n \)) and still satisfy the \( H_\infty \) bound. In this case the two-norm of \( T_{ZW} \) would be \( \alpha_0 \), the minimum achievable two-norm using any compensator (of any order), so it must be the solution to the problem in this case. This is what (4.21) - (4.27) are saying for \( \gamma > \gamma_2 \) - the solution is "off boundary" and \( \mathbf{y} = 0 \) is the correct choice. The problem then reduces to that of \( H_2 \) optimization, with the Riccati equation trivially satisfied, and thus it may be ignored for all practical purposes. For \( \gamma = \gamma_2 \), the solution is still \( \mathbf{y} = 0 \), but now the solution is on the boundary. This is precisely like the special case of \( \beta = 2 \) in the example of the previous section. Again, \( K_{\text{mix}} = K_{2_{\text{opt}}} \) for \( \gamma = \gamma_2 \). In summary, we have

**Theorem 4.2.1**: Assume \( n_c \geq n \), and \( \gamma \) is selected such that \( \gamma \geq \gamma_2 \). Then

i) \( K_{\text{mix}} = K_{2_{\text{opt}}} \)

ii) \( \alpha^* = \alpha_0 \)

iii) \( \gamma^* = \gamma_2 \)

**Proof**: For \( n_c = n \), the \( H_2 \) optimal compensator is unique, and given by the results in Section 3.1. Using this compensator, defined as \( K_{2_{\text{opt}}} \) in (4.37), as \( K(s) \) in Figure 4-1 yields \( \| T_{ZW} \|_2 = \alpha_0 \) and \( \| T_{ed} \|_\infty = \gamma_2 \), by (4.36) and (4.38), respectively. Since \( \alpha_0 \) is the absolute minimum achievable two-norm of \( T_{ZW} \), and for \( \gamma \geq \gamma_2 \) the constraint \( \| T_{ed} \|_\infty \leq \gamma \) is obviously satisfied, we must have part i). Parts ii) and iii) follow immediately from the definitions in (4.40) and (4.41).

For \( n_c > n \), \( K_{2_{\text{opt}}} \) makes \( \| T_{ZW} \|_2 = \alpha_0 \), which is the minimum achievable value of \( \| T_{ZW} \|_2 \) using any compensator of any order. Thus, it must be the solution, and ii) and iii) follow as above.

\[ \square \]
What if $\gamma$ is chosen such that $\gamma < \gamma_2$? By the definition of $K_{2\text{opt}}$, it cannot be the solution for $\gamma < \gamma_2$, since $K_{2\text{opt}}$ is unique and would violate the $H_\infty$ bound. We can restrict our choice of $\gamma$ further by the next theorem.

**Theorem 4.2.2**: For $n_c$ of any order and $\gamma < \gamma_0$, there is no solution to the mixed $H_2/H_\infty$ problem.

**Proof**: This is immediate from the fact that no compensator (of any order) satisfies the $H_\infty$ bound in this case.

Therefore, we can now restrict our attention to solving the full order mixed $H_2/H_\infty$ problem for $\gamma_0 \leq \gamma < \gamma_2$. The case $\gamma_0 = \gamma$ will be briefly considered at the end of this section, and we now only consider $\gamma_0 < \gamma < \gamma_2$. Relating the mixed $H_2/H_\infty$ problem to the example shown in the last section, the "boundary" is where the inequality constraint on $\|T_{ed}\|_\infty$ is an equality - that is, the boundary is $\|T_{ed}\|_\infty = \gamma$. Therefore, an "off boundary" solution is one where $\|T_{ed}\|_\infty < \gamma$, thus $\gamma = 0$ and the solution we seek to (4.26) is the unique stabilizing solution. An "on boundary" solution would be one where $\|T_{ed}\|_\infty = \gamma$, and thus $\gamma \neq 0$.

Consider off boundary solutions when $\gamma$ is chosen such that $\gamma_0 < \gamma < \gamma_2$. Since $\gamma = 0$, the necessary conditions are just (4.21) - (4.25), which correspond to the Lagrangian (4.19) with the last term equal to zero. This is just $H_2$ optimization. Can (4.21) - (4.25) have solutions other than $K_{2\text{opt}}$? In other words, could there be off boundary local minima? If so, the situation would conceptually look like Figures 4-2 and 4-3. For $\gamma > \gamma_2$, the solution would always be off boundary and $\alpha^*$ would equal $\alpha_0$, as shown in Figure 4-2 and proven in Theorem 4.2.1. For $\gamma < \gamma_2$, however, the solution could be on or off boundary. This is clearly seen in Figure 4-3, where, for the chosen $\gamma$, the solution is off boundary. Increasing $\gamma$ a bit would put the solution on the boundary. This type of behavior would make the problem very difficult to solve. Fortunately, local minima cannot occur in the full order case, which produces the key result stated next.
Figure 4-2. Local minima case with $\gamma > \gamma_2$

Figure 4-3. Local minima case with $\gamma < \gamma_2$
**Theorem 4.2.3:** Assume $n_c = n$, and $\gamma$ is selected such that $\gamma_0 < \gamma < \gamma_2$. Then the solution to the mixed $H_2/H_\infty$ problem lies on the boundary of the $H_\infty$ constraint. In other words, $K_{mix}$ is such that $\gamma^* = \gamma$.

**Proof:** Assume the solution is off boundary, so that $\gamma = 0$. The gradient equations are then just (4.21) - (4.25). Substituting $\gamma = 0$ into (4.21) - (4.23) and expanding (4.24) and (4.25) into their subblock equations yields the nine necessary conditions:

\begin{align*}
X_{12}^T Q_{12} + X_2 Q_2 &= 0 \\
X_{12}^T Q_{12}^T C^T + X_{2}^T Q_{12}^T C^T + X_{12}^T V_{12} + X_2 B_c V_2 &= 0 \\
B^T X_{12} Q_{12} + B^T X_{12} Q_2 + R_{12}^T Q_{12} + R_{2C} Q_2 &= 0 \\
A Q_1 + Q_{1A}^T + B C_c Q_{12} + Q_{12}^T C_c B_c^T + V_1 &= 0 \\
A Q_{12} + Q_{12}^T A_c^T + B C_c Q_2 + Q_{1C}^T B_c^T + V_{12} B_c^T &= 0 \\
A_c Q_2 + Q_{2A}^T + B_c Q_{12} + Q_{12}^T C_b^T B_c^T + B_c V_2 B_c^T &= 0 \\
A^T X_1 + X_1 A + C_b^T B_c X_{12} + X_{12} B_c C + R_1 &= 0 \\
A^T X_{12} + X_{12} A_c + C_b^T B_c X_2 + X_1 B_c C + R_{12C} &= 0 \\
A_c^T X_2 + X_2 A_c + C_c^T B^T X_{12} + X_{12} B_c C + C_c R_{2C} &= 0
\end{align*}
First, assume \((A_c, B_c, C_c)\) is minimal. Consider the following lemma, which follows the reasoning of [BJ76] and [BJ78].

**Lemma 4.2.1:** Assume \((A_c, B_c, C_c)\) is minimal and such that \(n_c = n\). Consider minimizing the Lagrangian

\[
L = \text{tr} \left( Q_2 C_2 \tilde{C}_2 \right) + \text{tr} \left( A Q_2 + Q_2 \tilde{A}^T + B_w B_w^T \right) X
\]  

(4.51)

where \(Q_2 = Q_2^T \geq 0\). The set of all admissible minimizing solutions is given by state space transformations of the unique compensator

\[
K_{2_{\text{opt}}}(s) = \tilde{C}_c(sI - \tilde{A}_c)^{-1}\tilde{B}_c
\]  

(4.52)

where

\[
\tilde{A}_c = A + B \tilde{C}_c - \tilde{B}_c C
\]  

(4.53)

\[
\tilde{B}_c = [ Q_0 C^T + V_{12} ] V_2^{-1}
\]  

(4.54)

\[
\tilde{C}_c = -R_2^{-1} [ B^T X_0 + R_{12}^T ]
\]  

(4.55)

and \(Q_0\) and \(X_0\) are the unique, positive semidefinite stabilizing solutions to the Riccati equations

\[
0 = \begin{bmatrix} A - V_{12} V_2^{-1} C \end{bmatrix} Q_0 + Q_0 \begin{bmatrix} A - V_{12} V_2^{-1} C \end{bmatrix}^T - Q_0 C^T V_2^{-1} C Q_0
\]

\[
+ V_1 - V_{12} V_2^{-1} V_{12}^T
\]  

(4.56)

\[
0 = \begin{bmatrix} A - B R_2^{-1} R_{12}^T \end{bmatrix} X_0 + X_0 \begin{bmatrix} A - B R_2^{-1} R_{12}^T \end{bmatrix} - X_0 B R_2^{-1} B^T X_0
\]

\[
+ R_1 - R_{12} R_2^{-1} R_{12}^T
\]  

(4.57)

**Proof of Lemma 4.2.1:** See Appendix B. 

108
If \((A_c, B_c, C_c)\) is nonminimal, decompose the compensator into its controllable and uncontrollable subspaces. Then, using a technique similar to the proof of Lemma 4.2.1, it is straightforward to show that the set of all admissible minimizing solutions to (4.51) is nothing more than state space transformations of a unique compensator plus arbitrary pole-zero cancellations.

Thus, the set of all compensators that satisfy the necessary conditions and stabilize the closed-loop system are just state space transformations of the same unique compensator, which is \(K_{2opt}\). This proves that there are no local minima or saddle points that are admissible compensators.

We have shown that the only admissible solution to the necessary conditions with \(y = 0\) is \(K_{2opt}\). For \(K(s) = K_{2opt}\), \(\|T_{ed}\|_\infty = \gamma_2\), which is outside the set of allowable \(\gamma\) in the statement of the theorem. Therefore, we have a contradiction, and \(y \neq 0\). Thus, the solution must lie on the boundary, so \(\gamma^* = \gamma\).

Now that we know that there cannot be local minima, the situation must look like Figure 4-4 for the case \(\gamma \geq \gamma_2\). For \(\gamma_0 < \gamma < \gamma_2\), the constraint is active, and the solution must therefore lie on the boundary. This is illustrated in Figure 4-5. This shows that for \(\gamma_0 < \gamma < \gamma_2\), the two objectives of two- and infinity-norm minimization are competing; that is, the two-norm is minimized by allowing the infinity-norm to be as large as allowed. In this region, \(y = 0\) is not the correct solution. Looking at (4.27), we have a Lyapunov equation with no constant term. Theorem 2.4.3 says that if \((A + \gamma^2Q_{\infty}C_e^Tc_e)\) is stable, \(y = 0\) is the only solution. Therefore, \((A + \gamma^2Q_{\infty}C_e^Tc_e)\) cannot be stable. The minimal solution to (4.26) places the eigenvalues of \((A + \gamma^2Q_{\infty}C_e^Tc_e)\) in the closed left-half plane, so we are not looking for the stabilizing solution to (4.26). Rather, we need the neutrally stabilizing solution, as shown in the next theorem.
Figure 4-4. No local minima case with $\gamma \geq \gamma_2$

Figure 4-5. No local minima case with $\gamma < \gamma_2$
**Theorem 4.2.4**: Assume $\mathcal{A}$ is stable. If there exists a $Q_\infty \geq 0$ satisfying

$$
\mathcal{A}Q_\infty + Q_\infty \mathcal{A}^T + \gamma^2 Q_\infty \mathcal{C}_e^T \mathcal{C}_e Q_\infty + B_d^T B_d = 0
$$

then the following are equivalent:

i) $\| T_{ed} \|_\infty = \gamma$

ii) $(\mathcal{A} + \gamma^2 Q_\infty \mathcal{C}_e^T \mathcal{C}_e)$ is neutrally stable

Furthermore, in this case, $Q_\infty$ is unique.

**Proof**: Assume $\mathcal{A}$ is stable and $\exists Q_\infty \geq 0$ satisfying (4.26).

i) $\rightarrow$ ii). Assume $\| T_{ed} \|_\infty = \gamma$. Since $\| T_{ed} \|_\infty \neq \gamma$, from the dual of Theorem 2.5.14, the Hamiltonian matrix

$$
M_{\infty 2} = \begin{bmatrix}
\mathcal{A}^T & \gamma^2 \mathcal{C}_e^T \mathcal{C}_e \\
-B_d^T B_d & -\mathcal{A}
\end{bmatrix}
$$

must have at least one imaginary axis eigenvalue. By Theorem 2.5.16, it must actually have at least two, since it cannot have imaginary axis eigenvalues with odd partial multiplicities. By Theorem 2.5.4, the eigenvalues of $M_{\infty 2}$ are the eigenvalues of $(\mathcal{A} + \gamma^2 Q_\infty \mathcal{C}_e^T \mathcal{C}_e)$ and $-(\mathcal{A} + \gamma^2 Q_\infty \mathcal{C}_e^T \mathcal{C}_e)$, so each of these must contain at least one imaginary axis eigenvalue. Thus, we have part ii).

ii) $\rightarrow$ i). Assume $(\mathcal{A} + \gamma^2 Q_\infty \mathcal{C}_e^T \mathcal{C}_e)$ is neutrally stable. From the dual of Theorem 2.5.14, $\| T_{ed} \|_\infty \geq \gamma$. But from Theorem 2.5.17, $\| T_{ed} \|_\infty \leq \gamma$. Therefore, $\| T_{ed} \|_\infty = \gamma$.

Uniqueness of $Q_\infty$ is immediate from the dual of Theorem 2.5.6.
Theorem 4.2.4 shows that the solution we seek is a neutrally stabilizing solution to the Riccati equation. The associated Lyapunov equation (4.27) then has an infinite number of solutions by Theorem 2.4.5 - one of these is $\gamma = 0$. This is the correct solution if $\gamma$ is chosen such that $\gamma \geq \gamma_2$. For lower values of $\gamma$ the solution is not $\gamma = 0$, and it is unclear whether all solutions to (4.27) lead to the same compensator or whether they characterize nonuniqueness.

The only value of $\gamma$ not yet considered is the $\gamma = \gamma_0$ case. As mentioned in Section 3.2, there are two types of $H_\infty$ optimality:

1) $\rho(X_\infty Y_\infty) = \gamma_0^2$ : in this case, $D_c$ is not equal to zero, and thus the two-norm of $T_{zw}$ is infinite by the development in (4.6). Therefore, the mixed $H_2/H_\infty$ problem is ill-posed at $\gamma = \gamma_0$.

2) Breakdown of a Riccati equation solution while $\rho(X_\infty Y_\infty) < \gamma_0^2$ : in this case, it is possible for $D_c$ to be equal to zero at $\gamma = \gamma_0$. Thus, it may be possible to find a mixed $H_2/H_\infty$ compensator which yields a finite two-norm.

It would be possible to check the above conditions at $H_\infty$ optimal upon problem definition. If condition 1) holds, we don't need to worry about $\gamma = \gamma_0$. If condition 2) holds, the problem is well defined at $H_\infty$ optimal and all of the development above still holds.

4.2.2 Increased Order

In this case consider choosing a compensator order higher than that of the plant; that is, $n_c > n$. Theorem 4.2.2 states that for $\gamma < \gamma_0$, no solution to the mixed $H_2/H_\infty$ problem exists. On the other hand, Theorem 4.2.1 shows that for $\gamma \geq \gamma_2$, the optimal solution is the $n$th order compensator $K_{2_{opt}}$ and $\alpha^* = \alpha_0$. Therefore, the region of competing objectives in the increased order case is again $\gamma_0 < \gamma < \gamma_2$. Recall Problem B from [RK91], as discussed in Section 1.2.7:
Problem B: Simultaneous $H_2/H_\infty$ optimization. Assuming it is nonempty, from the family of all $K(s)$ which achieve

$$\inf \{ \| T_{zw} \|_2 \mid K(s) \text{ admissible} \} \quad (4.58)$$

find one that also satisfies $\| T_{ed} \|_\infty \leq \gamma$.

Here, an equality sign has been added to the infinity-norm condition, which does not affect the problem, except possibly at $\gamma = \gamma_0$. Now, under the assumption of output feedback, the family of all admissible compensators that satisfy (4.58) is given by the development in Section 3.1. In particular, the family of all admissible $K(s)$ such that

$$\| T_{zw} \|_2 \leq \alpha \quad \alpha \geq \alpha_0$$

is given by a lower LFT of a transfer function matrix $J(s)$, given by (3.15)-(3.24), together with a $Q(s)$ such that

$$Q \in \mathcal{RH}_2 \quad \| Q \|_2^2 \leq \alpha^2 - \alpha_0^2$$

Note that for $\alpha = \alpha_0$, the family becomes a single compensator, which is $K_{2, \text{opt}}$. Thus, Problem B has no solution for $\gamma < \gamma_2$ in the output feedback case - the additional dynamics in the compensator do not allow $\alpha_0$ to be achieved for $\gamma < \gamma_2$. Thus, we have:

Theorem 4.2.5: For any $\gamma$ in the range $\gamma_0 \leq \gamma < \gamma_2$ and for $n_c > n$, the value of $\alpha^*$ may be lower than the corresponding value of $\alpha^*$ with $n_c = n$, but it is greater than $\alpha_0$.

Proof: Since Problem B has no solution for $\gamma < \gamma_2$, no compensator of any order can achieve $\alpha_0$ in this region. For $\alpha > \alpha_0$, the family of all $H_2$ suboptimal compensators has an infinite number of members, so further reduction of $\alpha^*$ from the full order case may be possible.

The possible reduction in $\alpha^*$ over the full order case by using a higher order compensator mentioned in Theorem 4.2.5 will indeed be shown to be possible in an example in Chapter 6. Equations (4.21)-(4.27) are still the necessary conditions for $n_c > n$, and the numerical approach to solving these equations developed in the next chapter applies for increased order.
as well. Since the unconstrained $H_2$ optimization problem in this case has the unique solution $K_{2_{opt}}$, a reasonable conjecture would be:

**Conjecture 4.2.1**: In the case $n_c > n$ with $\gamma_0 < \gamma < \gamma_2$, the solution to the mixed $H_2/H_{\infty}$ problem lies on the boundary of the infinity-norm constraint.

This still requires a formal proof.

### 4.2.3 Reduced Order

If we choose $n_c < n$, then several people including [HyB84], [HyB85], [HaB90b] and [Mer90] have shown that there are indeed possible local minima in the solution of the $H_2$ problem. The unconstrained (by an infinity-norm bound) $H_2$ equations are then known as the optimal projection equations. The solution to these equations consists of coupled Riccati equations, which must be solved by homotopy methods ([MB85], [RC90], [Ric87]). Since there are possible local minima that arise in the solution to these equations, Theorem 4.2.3 does not hold, and the solution may or may not lie on the boundary of the infinity-norm constraint. Furthermore, in this case $\gamma_0$, $\gamma_2$, and $\alpha_0$ will be different (higher) than in the full or increased order case, and if $n_c$ is chosen too small, no stabilizing compensators may exist at all. If $n_c$ is chosen larger than this value, but still less than $n$, $\gamma_0$ and $\alpha_0$ are much more difficult to compute than in the full order case, requiring the solution of coupled Riccati equations. Since this case would require substantial additional development, it is not considered further in this thesis.

Now that we have the necessary conditions for a solution to the mixed problem, can we solve them for $K_{mix}$? This will be discussed in the next chapter.
CHAPTER 5
NUMERICAL SOLUTION OF THE MIXED H₂/H∞ OPTIMIZATION PROBLEM

5.1 Suboptimal Problem Derivation

A summary of the results thus far will be given, in order to motivate the solution technique detailed in this chapter. The problem to be solved is:

The Mixed H₂/H∞ Problem - Consider the feedback system of Figure 4-1, where the plant P is given by (4.4). Assume conditions i) - x) in Section 4.1 hold. Find an admissible controller K(s) that achieves

\[ \inf_{\text{K admissible}} \| T_{zw} \|_2 \]  

subject to the constraint

\[ \| T_{ed} \|_\infty \leq \gamma \]  

Using Theorem 4.1.1, this was shown to be equivalent to:

The Mixed H₂/H∞ Problem - Determine \((A_c, B_c, C_c)\) that minimizes

\[ J(A_c, B_c, C_c) = \text{tr}[Q_z C_z C_z^T] \]  

where \(Q_z\) is the real, symmetric, positive semidefinite solution to

\[ A Q_z + Q_z A^T + B_w B_w^T = 0 \]  

and such that

\[ A Q_\infty + Q_\infty A^T + \gamma^2 Q_\infty C_c C_c^T Q_\infty + B_d B_d^T = 0 \]  

has a real, symmetric, positive semidefinite solution.
The problem was then formulated as a Lagrange multiplier problem and necessary conditions for a minimizing $K(s)$ were given. By fixing the order of the compensator to be equal to that of the plant (which we will do for the remainder of this chapter), the analysis of these equations led to the result that no solution exists for $\gamma < \gamma_0$, and for $\gamma \geq \gamma_2$, the solution is the optimal two-norm compensator for $T_{zw}$. For $\gamma_0 < \gamma < \gamma_2$, Theorem 4.2.3 showed that the solution is found by solving the seven nonlinear matrix equations given by (4.21) - (4.27), where the solution must be such that $\| T_{ed} \|_\infty = \gamma$, and thus $y \neq 0$.

The set of necessary conditions given by (4.21) - (4.27) comprises a set of highly coupled nonlinear matrix equations, for which an exact analytical solution may not even exist. This is a potential research topic in itself. Rather, a numerical solution will be sought. Even this appears to be very difficult; consider (4.26) and (4.27), repeated here for convenience.

$$A_{Q_{\infty}} + Q_{\infty}A_T^T + \gamma^2Q_{\infty}C_eC_e^TQ_{\infty} + B_dB_d^T = 0 \quad (5.6)$$

$$[A + \gamma^2Q_{\infty}C_eC_e^T]^TY + Y[A + \gamma^2Q_{\infty}C_eC_e^T] = 0 \quad (5.7)$$

From Theorem 4.2.4, we are looking for the neutrally stabilizing solution (which is unique) to (5.6); i.e., the matrix

$$A_Y = A + \gamma^2Q_{\infty}C_eC_e^T$$

must be neutrally stable. Neutrally stabilizing solutions to Riccati equations are on the boundary of non-existence of solutions, as highlighted in Section 2.5. Existing Riccati equation solvers, such as those in PRO-MATLAB™, cannot find this solution. Furthermore, (5.7) has an infinite number of nonzero solutions - PRO-MATLAB™ will return this statement but will not give any of the solutions. Therefore, a suboptimal approach, reminiscent of infinity-norm computations, may be more amenable to computation.

Consider adding a small symmetric, positive semidefinite term to (5.7), such as
\[ [ \mathcal{A} + \gamma^2 Q_\infty C_e C_e^T ]^T y + y^T [ \mathcal{A} + \gamma^2 Q_\infty C_e C_e^T ] + \mu Z \]
\[ = \mathcal{A}_y y + y^T \mathcal{A}_y + \mu Z = 0 \]  
(5.8)

where \( Z = Z^T \geq 0, (\sqrt{Z}, \mathcal{A}_y) \) is detectable, and \( \mu \in \mathbb{R}_+ \). For \( \mu \neq 0 \), Theorem 2.4.2 says that \( \mathcal{A}_y \) is stable, and thus we desire a stabilizing solution to (5.6). From Theorem 4.1.1, this also implies \( \| \text{Teq} \|_\infty < \gamma \). As \( \mu \to 0 \), (5.8) becomes (5.7) and the solution to (5.6) becomes the neutrally stabilizing solution. In order to produce the necessary condition given by (5.8), modify the performance index in (5.3) to read

\[ J_\mu(A_C, B_C, C_C) = \text{tr}[Q_2 C_Z C_Z^T] + \mu \text{tr}[Q_\infty Z] \]

It is easy to verify that this produces the original necessary conditions, but with (5.8) rather than (5.7). In order to make the performance index look like a multiobjective optimization problem ([HaB90a], [KR91a], [VG81]), further modify the performance index to be

\[ J_\mu(A_C, B_C, C_C) = (1 - \mu) \text{tr}[Q_2 C_Z C_Z^T] + \mu \text{tr}[Q_\infty Z] \]  
(5.9)

where \( \mu \in [0, 1] \). We now have the problem:

**The General Suboptimal Mixed H_2/H_\infty Lagrange Multiplier Problem** - Assume \( Q_2 = Q_2^T \geq 0, Q_\infty = Q_\infty^T \geq 0, Z = Z^T \geq 0 \), and \( (\sqrt{Z}, \mathcal{A}_y) \) is detectable. Minimize the Lagrangian

\[ L_\mu = (1 - \mu) \text{tr}[Q_2 C_Z C_Z^T] + \mu \text{tr}[Q_\infty Z] \]
\[ + \text{tr}( [ A Q_2 + Q_2 A^T + B_w B_w^T ] X ) \]
\[ + \text{tr}( [ A Q_{\infty} + Q_{\infty} A^T + \gamma^2 Q_{\infty} C_e C_e^T Q_{\infty} + B_d B_d^T ] Y ) \]  
(5.10)

When \( \mu = 0 \), we have the optimal mixed H_2/H_\infty problem. When \( \mu = 1 \), the first term vanishes, and the third term becomes a superfluous constraint.
as it only imposes stability of $\mathcal{A}$, which is already imposed by existence of $Q_\infty$ from the fourth term. Notice that if $Z$ is chosen to be

$$Z = C_e^T C_e$$

(5.11)

which meets the requirements on the choice of $Z$, then for $\mu = 1$ we have

$$L_1 = \text{tr}[Q_\infty C_e^T C_e] + \text{tr}([A Q_\infty + Q_\infty A^T + \gamma^2 Q_\infty C_e^T C_e Q_\infty + B_d B_d^T] y)$$

(5.12)

This was shown in [MG88] and [MG90] to be equivalent to the minimum entropy/H$_\infty$ control problem; that is

$$\inf_{K \text{ admissible}} \ I[T_{ed}, \gamma]$$

subject to

$$\|T_{ed}\|_\infty < \gamma$$

Note that the infinity-norm constraint is a strict inequality here, but that causes no problems since with $\mu \neq 0$, equality is prohibited. The solution to this problem is just the H$_\infty$ central [ $Q(s) = 0$ ] controller from Section 3.2. This will provide an easily computable initial guess for the numerical optimization routine. For this reason, the choice of $Z$ in (5.11) will be made - it is emphasized that this is not necessary, but rather allows the connection to the minimum entropy problem, which has an analytical solution. Therefore, we have:

**The Suboptimal Mixed H$_2$/H$_\infty$ Problem** - Determine $(A_c, B_c, C_c)$ that minimizes

$$J_\mu = (1 - \mu) \|T_{zw}\|_2^2 + \mu \text{tr}[Q_\infty C_e^T C_e]$$

(5.13)

or

118
\[ J_\mu(A_c, B_c, C_c) = (1 - \mu) \text{tr}[Q_2C_2^T C_2] + \mu \text{tr}[Q_\infty C_e^T C_e] \]

where \( \mu \in [0, 1] \), \( Q_2 \) is the real, symmetric, positive semidefinite solution to

\[ \mathcal{A}Q_2 + Q_2\mathcal{A}^T + B_wB_w^T = 0 \]

and such that

\[ \mathcal{A}Q_\infty + Q_\infty\mathcal{A}^T + \gamma^2 Q_\infty C_e^T C_e Q_\infty + B_dB_d^T = 0 \]

has a real, symmetric, positive semidefinite solution.

Note that for \( \mu \neq 0 \), (5.13) is a weighted mix of minimum two-norm of \( T_{zw} \) and minimum entropy of \( T_{ed} \), since Theorem 2.3.2 says that \( \text{tr}[Q_\infty C_e^T C_e] \) equals \( I[T_{ed}, \gamma] \) so long as \( \| T_{ed} \|_\infty < \gamma \), which is assured for \( \mu \neq 0 \).

Defining the Lagrangian in (5.10) with \( Z = C_e^T C_e \) and computing the first partials produces the necessary conditions for a minimum given by

\[ X_{12}^T Q_{12} + X_2Q_2 + Y_{12}^T Q_{ab} + Y_2Q_b = 0 \quad (5.14) \]

\[ X_{12}^T Q_{1}C^T + X_2Q_{12}^T C^T + X_{12}^T V_{12} + X_2B_cV_2 + Y_{12}^T Q_{ab} C^T + Y_2Q_{ab} C^T \]

\[ + Y_{12}^T V_{ab} + Y_2B_cV_b = 0 \quad (5.15) \]
\[ B^T X_1 Q_{12} + B^T X_{12} Q_2 + (1 - \mu) R_{12}^T Q_{12} + (1 - \mu) R_2 C_2 Q_2 + B^T Y_1 Q_{ab} \]
\[ + B^T Y_{12} Q_b + \mu R_{ab}^T Q_{ab} + \mu R_b C_b Q_b + \gamma^2 \left( R_{ab}^T Q_a Y_1 Q_{ab} \right) \]
\[ + R_{ab}^T Q_a Y_{12} Q_b + R_{ab}^T Q_{ab} Y_{12} Q_{ab} + R_{ab}^T Q_{ab} Y_2 Q_b + R_b C_b Q_{ab}^T Y_{12} Q_{ab} \]
\[ + R_b C_b Q_b Y_{12} Q_{ab} + R_b C_b Q_{ab} Y_{12} Q_b + R_b C_b Q_b Y_2 Q_b \} = 0 \quad (5.16) \]

\[ A Q_2 + Q_2 A^T + B_{\mu} B_{\mu}^T = 0 \quad (5.17) \]

\[ A^T X + X A + (1 - \mu) C_2 C_2^T = 0 \quad (5.18) \]

\[ A Q_\infty + Q_\infty A^T + \gamma^2 Q_\infty C_e C_e^T Q_\infty + B_d B_d^T = 0 \quad (5.19) \]

\[ \left[ A + \gamma^2 Q_\infty C_e C_e^T \right]^T Y + Y \left[ A + \gamma^2 Q_\infty C_e C_e^T \right] + \mu C_e C_e^T = 0 \quad (5.20) \]

Notice that (5.20) has a nonzero constant term for \( \mu \neq 0 \). \( \mu \) acts as a factor to control how close to the boundary of the infinity-norm constraint the solution is - as long as \( \mu \neq 0 \). \( \| T_{ed} \|_\infty \) cannot equal \( \gamma \), since a stabilizing solution to (5.19) is always sought. For \( \mu = 0 \), the equations above become the mixed \( H_2/H_\infty \) equations presented in Chapter 4.

The only thing that remains is to show that the suboptimal mixed problem converges to the optimal mixed problem as \( \mu \to 0 \). First, consider the behavior of the entropy of a transfer function as \( \gamma \) approaches the infinity-norm of the transfer function (this behavior has been misinterpreted in the literature).
**Theorem 5.1.1**: Let \( G(s) = C(sI - A)^{-1}B \), with \( A \) stable. Define \( \| G(s) \|_\infty \equiv \gamma_0 \) and let \( \gamma \geq \gamma_0 \). Define \( \epsilon = \gamma - \gamma_0 \geq 0 \). Then as \( \epsilon \to 0 \), the value of the entropy (which is singular at \( \epsilon = 0 \)) converges to

\[
\text{tr}[X_0C^T C]
\]

where \( X_0 \) is the neutrally stabilizing solution to the ARE

\[
AX_0 + X_0A^T + \gamma_0^{-2}X_0C^T CX_0 + BB^T = 0
\]

(5.22)

**Proof**: First, Theorem 2.3.2 says that for any \( \epsilon \neq 0 \), the entropy of \( G(s) \) at \( \epsilon \) is given by

\[
\text{tr}[XC^T C]
\]

where \( X \) is the stabilizing solution to the ARE

\[
AX + XA^T + (\epsilon + \gamma_0)^{-2}XC^T CX + BB^T = 0
\]

(5.24)

From Proposition 2.3.2 of [MG90], the entropy is a continuous, monotonically increasing function of decreasing \( \epsilon \). Finally, from Theorem 1 of [Wim85], \( X_0 \geq X \), where \( X_0 \) is the neutrally stabilizing solution to (5.22) and \( X \) is the stabilizing solution to (5.24) for any \( \epsilon > 0 \). Thus, (5.23) is continuous and monotonically increasing as \( \epsilon \to 0 \), and is bounded by (5.21), which is the value at \( \epsilon = 0 \). Therefore, (5.23), the entropy, converges to (5.21) as \( \epsilon \to 0 \). Note that from the definition of the entropy in Section 2.3.3

\[
I[G(s), \gamma] = \lim_{s_0 \to \infty} \left\{ -\frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \ln|\det[1 - \gamma^2G^*(j\omega)G(j\omega)]| \left| \frac{s_0^2}{s_0^2 + \omega^2} \right| \, d\omega \right\}
\]

which may be written as
\[ I[G(s), \gamma] \equiv \lim_{s_0 \to \infty} \left\{ -\frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \sum_i \ln|1 - \gamma^2 \sigma_i^2[G(j\omega)]| \left[ \frac{s_0^2}{s_0^2 + \omega^2} \right] \text{d}\omega \right\} \]

it is clear that the integral is singular for \( \gamma = \gamma_0 \), since \( \sigma[G(j\omega)] = \gamma_0 \) at some \( \omega \).

In order to see this result graphically, consider Example 2 from Section 2.5.3.2. For this example, \( \|G(s)\|_\infty = 1.5 \). Computing the stabilizing solution to (5.24) over a wide range of \( \gamma \), evaluating \( \text{tr}[X^T C] \), and plotting this value versus \( \gamma \) yields the plot shown in Figure 5-1. For \( \gamma > 1.5 \), this is a plot of the entropy versus \( \gamma \). Using the neutrally stabilizing solution to (5.22) given in Section 2.5.3.2 yields \( \text{tr}[X_0^T C] = 3.9044 \), which is the value the entropy is asymptotically approaching as \( \gamma \to 1.5 \) from above.

![Figure 5-1. \( \text{tr}[X^T C] \) versus \( \gamma \) for Example 2](attachment:image.png)
Now that we know that for a strictly proper, stable transfer function the entropy is bounded and converges to its bound as $\gamma \rightarrow \|G(s)\|_\infty$ from above, we can show the following key theorem.

**Theorem 5.1.2**: For $\gamma > \gamma_0$, $J_\mu$ given by (5.13) converges to the optimal mixed $H_2/H_\infty$ problem as $\mu \rightarrow 0$.

**Proof**: $\gamma > \gamma_0$ implies there exists a strictly proper suboptimal $H_\infty$ compensator, so the $H_\infty$ bound can be satisfied with $T_{ed}$ stable and $K(s)$ strictly proper. The latter is necessary for $\|T_{zw}\|_2$ to be finite. Thus, we can assume $T_{ed}$ is stable and strictly proper at all values of $\mu$. For $\mu \neq 0$, the second term in (5.13) is $\mu \|T_{ed}, \gamma\|$ by the fact that $\|T_{ed}\|_\infty < \gamma$ and Theorem 2.3.2. From Theorem 5.1.1, the value of the closed-loop entropy is bounded, and converges to its bound as $\mu \rightarrow 0$. Since $\text{tr}\{Q_{\infty}C_e^T C_e\}$ remains bounded as $\mu \rightarrow 0$, $\mu \text{tr}\{Q_{\infty}C_e^T C_e\} \rightarrow 0$ as $\mu \rightarrow 0$. Thus, $J_\mu \rightarrow J_0$. $J_0$ is the optimal mixed $H_2/H_\infty$ performance index.

The existence of a solution to the mixed problem for $\gamma > \gamma_0$ is immediate, since there exists a central $H_\infty$ compensator, which is strictly proper and thus is an admissible solution. The case of $\gamma = \gamma_0$ is excluded due to potential ill-posedness of the problem.

For $\mu = 1$, the solution is the central $H_\infty$ compensator. The next theorem proves that for any other value of $\mu \in [0,1)$, the resulting compensator has a value of $\|T_{zw}\|_2^2$ less than or equal to the value at $\mu = 1$.

**Theorem 5.1.3**: For $\gamma > \gamma_0$ and $\mu \in [0,1)$, the compensator which minimizes the performance index

$$J_\mu = (1 - \mu) \|T_{zw}\|_2^2 + \mu \text{tr}\{Q_{\infty}C_e^T C_e\}$$

has a value of $\|T_{zw}\|_2^2$ less than or equal to that at $\mu = 1$. 

123
Proof: Consider Figure 5-2, which is a plot of \( \| T_{zw} \|_2^2 \) versus \( \| T_{ed} \|_\infty \) for the central H\(_\infty\) compensator (\( \mu = 1 \)). Assume Point A exists such that it is the optimal solution to the performance index above at some \( \mu \) and some \( \gamma \). Let Point B be the point on the \( \mu = 1 \) curve which has the same \( \| T_{ed} \|_\infty \) value. Then by construction

\[
\| T_{zw} \|_2^2 \leq \| T_{zw} \|_2^2
\]

We also have

\[
J_{\mu A} < J_{\mu B}
\]

since Point A was assumed to be the optimal (minimizing) solution. Then

\[
J_{\mu A} = (1 - \mu) \| T_{zw} \|_2^2 + \mu \text{tr}[Q_{\infty}C_e^TC_e]_A
\]

\[
< (1 - \mu) \| T_{zw} \|_2^2 + \mu \text{tr}[Q_{\infty}C_e^TC_e]_B = J_{\mu B}
\]

or

\[
(1 - \mu) [\| T_{zw} \|_2^2 - \| T_{zw} \|_2^2] < \mu [\text{tr}[Q_{\infty}C_e^TC_e]_B - \text{tr}[Q_{\infty}C_e^TC_e]_A]
\]

(5.25)

The term on the left-hand side of (5.25) is \( > 0 \), so we have

\[
\mu [\text{tr}[Q_{\infty}C_e^TC_e]_B - \text{tr}[Q_{\infty}C_e^TC_e]_A] > 0
\]

(5.26)

However, Point B has the minimum value of \( \text{tr}[Q_{\infty}C_e^TC_e] = I[T_{ed}, \gamma] \) possible for the given \( \| T_{ed} \|_\infty \) value, and thus the left-hand side of (5.26) is \( < 0 \). This is a contradiction, so Point A cannot exist. Thus, the \( \mu = 1 \) curve is an upper bound for minimizing solutions to the performance index for any value of \( \mu \).
Figure 5-2. $T_{zw}$ versus $T_{ed}$ plot for Theorem 5.1.3

A similar result holds at the other extreme. That is,

**Theorem 5.1.4**: For $\gamma > \gamma_0$ and $\mu \in (0,1]$, the compensator which minimizes the performance index

$$J_{\mu} = (1 - \mu) \| T_{zw} \|_2^2 + \mu \text{tr} \{ Q_{\infty} C_e^T C_e \}$$

has a value of $\| T_{zw} \|_2^2$ greater than or equal to that at $\mu = 0$.

**Proof**: Immediate from realizing that an optimal compensator for $\mu = 0$ minimizes the value of $\| T_{zw} \|_2^2$.

The above two theorems indicate that as $\mu$ is varied from one to zero, the solution always lies on or in between the bounds shown in the Figure 5-3; that is, it is never in the shaded region.
Figure 5-3. Typical $T_{zw}$ versus $T_{ed}$ plot

The $\mu = 0$ curve in Figure 5-3 is shown as monotonically decreasing. This is actually the case, as proven in the following theorem.

**Theorem 5.1.5**: A plot of $\alpha^*$ versus $\gamma^*$ for $\gamma > \gamma_0$ is monotonically decreasing with $\gamma$. This plot will be referred to as the *mixed plot*.

**Proof**: To show this, consider the two problems:

(A) \[ \inf_{K \text{ admissible}} \| T_{zw} \|_2 \quad \text{subject to} \quad \| T_{ed} \|_\infty \leq \gamma \]

and

(B) \[ \inf_{K \text{ admissible}} \| T_{zw} \|_2 \quad \text{subject to} \quad \| T_{ed} \|_\infty \leq \gamma + \varepsilon \]
where \( \varepsilon \) is a positive real number. Assume there exists a solution to Problem A, say \( K_{\text{mix}}(s) \). Then \( K_{\text{mix}}(s) \) is an admissible compensator for Problem B. That is, it may not be optimal, but it definitely meets the constraint. Thus, the answer to Problem B has a two-norm less than or equal to that of Problem A, for any \( \varepsilon \). Therefore, the mixed plot is monotonically decreasing.

For \( \mu \neq 1 \), the analytical solution to (5.14) - (5.20) is unknown, and there may indeed be no such solution. However, a numerical solution for \( \mu \neq 0 \) is possible, as described next.

### 5.2 DFP Numerical Algorithm

The basic approach used to solve the problem is the Davidon-Fletcher-Powell (DFP) algorithm ([Fox71]). This will be discussed first, both in general and specifically as it applies to the mixed \( H_2/H_\infty \) problem.

DFP is a powerful optimization algorithm that encompasses the features of a variable metric method, a "quasi-Newton" approach, and a conjugate direction method. It is a quadratically convergent first-order method. The advantage of being a first-order method is that the second partials from Sections 4.1 and 5.1 do not need to be calculated. These are fourth-order tensors which are very cumbersome. The original approach used in solving the problem was a gradient method which required these - they were successfully coded up, but required massive amounts of storage space and severely hampered the speed associated with solving the problem. For this reason, the DFP approach was chosen.

The basic problem to be solved is to minimize a scalar function \( J_\mu \) which is a function of several matrix unknowns and subject to matrix equality constraints. The basic flow diagram for DFP is seen in Figure 5-4. Each block as it pertains to the mixed problem will be described.

As developed in Section 5.1, the suboptimal mixed \( H_2/H_\infty \) problem is to minimize the performance index

\[
J_\mu = (1 - \mu) \text{tr} \left[ Q_2 C_Z^T C_Z \right] + \mu \text{tr} \left[ Q_\infty C_e C_e^T \right] \quad \mu \in (0, 1)
\]
subject to

\[ A_2 Q_2 + Q_2 A^T + B_w B^T_w = 0 \]

and

\[ A_\infty Q_\infty + Q_\infty A^T + \gamma^2 Q_\infty C^T e C_\infty + B_d B^T_d = 0 \]
Referring to Figure 5-4, the function to be minimized is $F$, so here

$$F \equiv J_\mu$$

$X$ represents all of the unknowns to be found, stretched out as a column vector. $\nabla F(X)$ represents the gradient of $F$ stretched out as a column vector, which in this case would be given by (5.14) - (5.20). In general, $X$ would therefore consist of $A_C$, $B_C$, $C_C$, $Q_2$, and $Q_\infty$, as well as the Lagrange multiplier matrices $\lambda$ and $\gamma$. However, once a guess of $A_C$, $B_C$, and $C_C$ is made, the remaining unknowns can be solved for exactly (within numerical limits) by solving (5.17) - (5.20), which would then be Lyapunov and Riccati equations with known coefficients. Therefore, the vector of unknowns $X$ will be taken to be $A_C$, $B_C$, and $C_C$ stretched out into a column vector, or

$$X = \begin{bmatrix} \text{vec}(A_C) \\ \text{vec}(B_C) \\ \text{vec}(C_C) \end{bmatrix}$$

where $\text{vec}(\cdots)$ is the column vector operator. $\nabla F(X)$ then becomes (5.14) - (5.16), also stretched out into a column vector, or

$$\nabla F(X) = \begin{bmatrix} \text{vec}\left(\frac{\partial L}{\partial A_C}\right) \\ \text{vec}\left(\frac{\partial L}{\partial B_C}\right) \\ \text{vec}\left(\frac{\partial L}{\partial C_C}\right) \end{bmatrix}$$

$S$ is the direction matrix that the current guess of the solution should be moved in order to reduce the value of $F$. In a second-order method, this matrix should be

$$S = -J^{-1}\nabla F(X) \quad (5.27)$$

where $J^{-1}$ is the inverse of the Hessian or second derivative matrix. In DFP, this is replaced by $H$, a symmetric positive definite matrix which serves as an approximation to $J^{-1}$. Thus, (5.27) becomes
$$S = -H \nabla F(X)$$

Finally, $\kappa$ is the length of a move in the $S$ direction. The $\kappa$ which makes the value of $F$ smallest, that is, minimizes $F(X + \kappa S)$, is denoted by $\kappa^*$. 

First, an initial guess needs to be made for the unknowns, $X$. This involves choosing the compensator matrices. The compensator chosen must be acceptable in that it is stabilizing and satisfies the $\gamma$-bound on the infinity-norm of $T_{ed}$. If not, equations (5.17) - (5.20) will not have solutions. Also needed is an initial guess for $S$. The initial guess for $H$ is chosen to be the identity matrix, so that the initial $S$ is just the negative of the gradient.

Next, a one-dimensional search for a $\kappa$ that minimizes the function $F(X + \kappa S)$ is performed; the result is $\kappa^*$. There are numerous ways to do this, and the one used here will be briefly outlined.

i) choose three values of $\kappa$: $\kappa_1 = 0$, $\kappa_2 = \beta$, and $\kappa_2 = 2\beta$, where $\beta$ is some preset constant

ii) calculate $F(X + \kappa S)$ at the three $\kappa$ values, and identify the smallest value; call it $F_{\text{min}}$

iii) a) if $F_{\text{min}}$ is associated with $\kappa_1$, reset $\kappa_1 = \kappa_1$, $\kappa_2 = \kappa_2/2$, and $\kappa_3 = \kappa_2$ and go to ii)

b) if $F_{\text{min}}$ is associated with $\kappa_3$, reset $\kappa_1 = \kappa_1$, $\kappa_2 = \kappa_3$, and $\kappa_3 = 2\kappa_3$ and go to ii)

c) if $F_{\text{min}}$ is associated with $\kappa_2$, compute the value of $F$ at $\kappa_4 = \kappa_1 + (\kappa_2 - \kappa_1)/2$ and $\kappa_5 = \kappa_2 + (\kappa_3 - \kappa_2)/2$; identify which three consecutive $\kappa$'s contain the minimum, define these as $\kappa_1$, $\kappa_2$, and $\kappa_3$, and go to ii)

iv) continue until $(\kappa_2 - \kappa_1)/\kappa_2$ is below some preset value; when it is, define $\kappa^*$ as the $\kappa$ that produces $F_{\text{min}}$

Now $X$ is replaced with $(X + \kappa^* S)$ and convergence is checked. Numerically, convergence is declared when

130
\[ \frac{\nabla F^T H \nabla F}{|F(X)|} < \epsilon \]

where \( \epsilon \) is a user-specified small positive number. If the algorithm has not converged, then an update on \( S \) is made as follows. Call the previous step of the iteration \( q \) and the new one \( q+1 \). Thus, we have \( X_q \) from the previous guess as well as the new \( X_{q+1} \) just computed above. Let

\[ H_{q+1} = H_q + M_q + N_q \]

where

\[ Y_q = \nabla F(X_{q+1}) - \nabla F(X_q) \]

\[ M_q = \kappa^*_q \frac{S_q S_q^T}{S_q Y_q} \]

\[ N_q = -\frac{(H_q Y_q)(H_q Y_q)^T}{Y_q^T H_q Y_q} \]

Then the new \( S \) is given by

\[ S_{q+1} = -H_q \nabla F(X_{q+1}) \]

The algorithm then returns to the \( \kappa^* \) computation step using the new \( X \) and \( S \).

The algorithm will remain stable as long as \( H \) remains positive definite, which theoretically must be true. However, due to numerical inaccuracies and round-off error, \( H \) may become indefinite. If this happens, \(-\nabla F^T H \nabla F\) will eventually become positive and the method would not move in a descending direction. The aforementioned quantity is therefore checked after each update and if positive, the \( H \) matrix is reset to identity.

The DFP algorithm was coded into PRO-MATLAB\textsuperscript{TM} through a script file and a number of M-files. These are shown in Appendix C. Recall that the initial guess must satisfy the infinity-norm bound, or the \( H_\infty \) Riccati
equation (5.19) will not have a solution. Also, when searching for $\mathbf{K}^*$ some of the guesses for the $\kappa_i$ along the way will put the solution in an unacceptable region, which means some of the equations will blow up, have no stabilizing solution, and/or not be positive semidefinite. This problem was circumvented by setting $F ( = J_\mu )$ to a very large value at these points so that they will not be considered to be a minimum.

Each and every new guess here will be acceptable, which actually causes a minor problem. The infinity-norm bound is seen as a hard constraint which may not be violated. The overall problem formulated this way is not convex, and problems arise when the algorithm hits a constraint and the minimum lies in the direction of "through the boundary". As the boundary here is very hard to characterize algebraically (and causes numerical instability), it is not clear how to get the algorithm to run along the boundary, the common solution technique in a problem such as this.

The process was sped up considerably by coding this approach into Fortran and using Riccati and Lyapunov equation solver subroutines. This allowed for faster convergence of the algorithm than in PRO-MATLAB™. The code for this is not shown, due to its size as well as straightforwardness of the conversion.

The basic approach for solving a mixed $H_2/H_\infty$ problem using this algorithm is to choose a $\gamma$ near $\gamma_0$, and compute the central $H_\infty$ controller for this $\gamma$ value. This is an acceptable compensator, and is the solution for $\mu = 1$. Then $\mu$ is chosen to be slightly less than one, and the algorithm is started. The converged result then becomes the initial guess for a smaller $\mu$. The value of $\mu$ is decreased until the two-norm of $T_{ZW}$ stops changing. This produces $K_{mix}$ for the chosen value of $\gamma$. Then $\gamma$ is increased slightly. Since the previous $K_{mix}$ is an acceptable compensator for a larger $\gamma$, this $K_{mix}$ is used as an initial guess with a relatively small value of $\gamma$. Again, $\mu$ is decreased until the value of $\| T_{ZW} \|_2$ stops changing. The entire process is repeated until $\gamma = \gamma_2$, where $K_{mix} = K_{2_{opt}}$ for all $\gamma \geq \gamma_2$.

In the next two chapters, a SISO and a MIMO example will be solved. The results will be examined in detail to highlight the analytical results and to describe the nature of the solution.
CHAPTER 6
SISO MIXED OPTIMIZATION EXAMPLE

6.1 Problem Set-Up

In this simple example to illustrate the mixed $H_2/H_\infty$ optimization results, all signals are assumed to be scalars. Thus, $d, w, u, e, z,$ and $y$ in Figure 6-1 are all scalar signals.

![Block Diagram](image)

Figure 6-1. Mixed optimization block diagram

The plant $P(s)$ has the state space representation

$$P(s) = \begin{bmatrix}
A & B_d & B_w & B_u \\
C_e & D_{ed} & D_{ew} & D_{eu} \\
C_z & D_{zd} & D_{zw} & D_{zu} \\
C_y & D_{yd} & D_{yw} & D_{yu}
\end{bmatrix}$$

where the individual matrices are given by

$$A = \begin{bmatrix}
-0.3908 & -0.4565 & 1.2657 \\
1.4453 & -1.0491 & -1.2077 \\
-0.1288 & 0.6744 & 1.0324
\end{bmatrix}$$

(6.2a)
\[
B_d = \begin{bmatrix}
0.0488 \\
0.3608 \\
0.3564
\end{bmatrix}, \quad B_w = \begin{bmatrix}
1.4077 \\
0.9723 \\
-1.6050
\end{bmatrix}, \quad B_u = \begin{bmatrix}
-0.4275 \\
-0.4470 \\
-0.9172
\end{bmatrix} \quad (6.2b)
\]

\[
C_e = \begin{bmatrix}
0.9420 & 0.0144 & 0.1187
\end{bmatrix}
\]

\[
C_z = \begin{bmatrix}
-0.0450 & 0.3606 & 1.8972
\end{bmatrix} \quad (6.2c)
\]

\[
C_y = \begin{bmatrix}
-1.5567 & -1.9432 & -0.0914
\end{bmatrix}
\]

\[
D_{ed} = \begin{bmatrix}
0
\end{bmatrix}, \quad D_{ew} = \begin{bmatrix}
0
\end{bmatrix}, \quad D_{eu} = \begin{bmatrix}
1.3575
\end{bmatrix}
\]

\[
D_{zd} = \begin{bmatrix}
0
\end{bmatrix}, \quad D_{zw} = \begin{bmatrix}
0
\end{bmatrix}, \quad D_{zu} = \begin{bmatrix}
0.5781
\end{bmatrix} \quad (6.2d)
\]

\[
D_{yd} = \begin{bmatrix}
0.5185
\end{bmatrix}, \quad D_{yw} = \begin{bmatrix}
0.3899
\end{bmatrix}, \quad D_{yu} = \begin{bmatrix}
0
\end{bmatrix}
\]

The actual "unweighted" plant, given by \( P_{yu}(s) = C_e(sI - A)^{-1}B_d + D_{yu} \), is unstable, minimum phase with a singular value plot as shown in Figure 6-2.

![Magnitude plot of the SISO plant](figure)
The objective of mixed $H_2/H_\infty$ optimization is to design a $K(s)$ that minimizes the two-norm of $T_{ZW}$ while keeping the infinity-norm of $T_{ed}$ below a chosen value. To fully evaluate what the mixed controller does, the chosen value of the infinity-norm of $T_{ed}$, $\gamma$, will be ranged over the allowable set of choices, which is $\gamma = (\gamma_0, \infty]$. The choice of $\gamma = \gamma_0$ cannot actually be made, but the result in this case will become apparent. The plant $P(s)$ in (6.1) and (6.2) meets all of the assumptions given in Section 3.1 for $H_2$ optimization on $P_{ZW}$ as well as those in Section 3.2 for $H_\infty$ optimization on $P_{ed}$. First, the "limiting" cases will be evaluated.

Consider doing $H_\infty$ optimization on $P_{ed}$, as described in Section 3.2. Doing this yields a nearly optimal infinity-norm of roughly 2.1426. Since the choice of the freedom parameter $Q(s) = 0$ (the central $H_\infty$ compensator) corresponds to the minimum entropy controller, and entropy overbounds the two-norm, the choice of $Q(s) = 0$ is made. The resulting singular value plot of $T_{ed}$ is shown in Figure 6-3. Note that this plot is virtually all-pass with a gain of 2.1426. The resulting 3-state $K_{\infty,2.1426}$ singular value plot is shown in Figure 6-4. The controller does roll off at a frequency beyond that shown, but still has nearly infinite bandwidth. The true optimal $K_{\infty,\text{opt}}$ would have infinite bandwidth. Closing the loop in Figure 6-1 with $K_{\infty,2.1426}$ produces the $T_{ZW}$ plot shown in Figure 6-5. This actually does roll off at very high frequency, and the two-norm is very large with a value of 94.323. If $K_{\infty,\text{opt}}$ was used (for SISO it is unique), the $T_{ZW}$ plot would not roll off and the two-norm would be infinite. Therefore, at $\gamma = \gamma_0$ the mixed problem here is actually ill-posed since no finite solution exists. Thus, we eliminate $\gamma = \gamma_0$ from consideration in the mixed optimization results. For convenience, we will refer to the value 2.1426 as $\gamma_0$ in what follows.

Now consider the other extreme; that is, doing $H_2$ optimization on $P_{ZW}$, which will produce $\alpha_0$. Carrying this out as described in Section 3.1 produces the 3-state compensator $K_{2,\text{opt}}$, whose singular value plot is shown in Figure 6-6. Figure 6-7 shows a singular value plot of $T_{ZW}$ for both $K_{\infty,2.1426}$ and $K_{2,\text{opt}}$. From this figure it is clear that $K_{\infty,2.1426}$ produces a much higher two-norm, since $T_{ZW}$ has a large spike and does not roll off until very high frequency. For the $K_{2,\text{opt}}$ case, $\|T_{ZW}\|_2 \leq \alpha_0 = 9.9263$. Since $K_{2,\text{opt}}$ is unique, no other $K(s)$ produces a two-norm this small. Closing the loop in Figure 6-1 with $K_{2,\text{opt}}$ produces the $T_{ed}$ plot shown in Figure 6-8.
Figure 6-3. Singular value plot of $T_{ed}$ for $K_{\infty 2.1426}$

Figure 6-4. Singular value plot of $K_{\infty 2.1426}$
Figure 6-5. Singular value plot of $T_{ZW}$ for $K_{\infty 2.1426}$

Figure 6-6. Singular value plot of $K_{2_{opt}}$
Figure 6-7. Singular value plots of $T_{ZW}$ for $K_{\infty 2.1426}$ and $K_{2_{opt}}$

Figure 6-8. Singular value plot of $T_{ed}$ for $K_{2_{opt}}$
which has \( \| T_{ed} \|_{\infty} = 4.5364 \), obviously higher than the optimal infinity-norm.

Doing the analysis above defines the achievable limits of the problem; that is, it defines the minimum achievable infinity-norm of \( T_{ed} \) and the minimum achievable two-norm of \( T_{zw} \). It is now instructive to consider the equations derived in Section 3.3; that is, consider the system given in Figure 6-1 to be represented by Figure 6-9, so that

\[
e = T_{ed}d = (G_{ed} + G_{er}[I - QG_{vr}]^{-1}QG_{vd})d \quad (6.3)
\]

\[
z = T_{zw}w = (G_{zw} + G_{zr}[I - QG_{vr}]^{-1}QG_{vw})w \quad (6.4)
\]

where

\[
G_{ed} = C_{\mathcal{E}}(sI - A_{n})^{-1}B_{D} \quad (6.5)
\]

\[
G_{er} = C_{\mathcal{E}}(sI - A_{n})^{-1}B_{R} + D_{eu}S_{u}^{-1} \quad (6.6)
\]

\[
G_{vd} = C_{\mathcal{V}}(sI - A_{n})^{-1}B_{D} + S_{y}D_{yd} \quad (6.7)
\]

\[
G_{vr} = C_{\mathcal{V}}(sI - A_{n})^{-1}B_{R} \quad (6.8)
\]

\[
G_{zw} = C_{\mathcal{Z}}(sI - A_{n})^{-1}B_{w} \quad (6.9)
\]

\[
G_{zr} = C_{\mathcal{Z}}(sI - A_{n})^{-1}B_{R} + D_{zu}S_{u}^{-1} \quad (6.10)
\]

\[
G_{vw} = C_{\mathcal{V}}(sI - A_{n})^{-1}B_{w} + S_{y}D_{yw} \quad (6.11)
\]

\[
Q \in RH_{\infty} \quad \| Q \|_{\infty} < \gamma \quad (6.12)
\]

and the matrices in (6.5) - (6.11) are given in Section 3.3. Once a value of \( \gamma \) is chosen, doing \( H_{\infty} \) optimization on \( P_{ed} \) makes all of the transfer functions in (6.5) - (6.11) known. Since the optimal SISO \( H_{\infty} \) compensator is unique, the next logical step would be to back away from \( H_{\infty} \)-optimality. If \( Q(s) \) is chosen to be zero, then \( T_{ed} = G_{ed} \) and \( T_{zw} = G_{zw} \) and the other 5 transfer functions are unnecessary. Recall that as we let the chosen infinity-norm level \( \gamma \) get larger, the resulting compensator for \( Q(s) = 0 \) becomes \( H_{2} \)

139
optimal, but that is the optimal $H_2$ compensator for $P_{ed}$, not for $P_{zw}$. Thus, while the choice of $Q(s) = 0$ minimizes an overbound to the two-norm of $T_{ed}$, it may not be a good choice for minimizing the two-norm of $T_{zw}$.

Figure 6-10 shows singular value plots of $T_{ed}$ when the optimal two-norm compensator for $P_{ed}$ and for $P_{zw}$ are used in Figure 6-1. The former results in an infinity-norm of 4.1629, which is slightly lower than that for the latter. However, these compensators produce the singular value plots of $T_{zw}$ shown in Figure 6-11. The two-norm optimal $P_{ed}$ compensator results in a $\|T_{zw}\|_2 = 14.553$, which is considerably higher than $\alpha_o$. Recall that these results are equivalent to choosing $\gamma$ to be very large with $Q(s) = 0$ in $H_\infty$ optimization on $P_{ed}$.

Next, various values of $\gamma$ were chosen. Figure 6-12 shows a plot of $G_{ed}$ for the following values of $\gamma$: 2.1426, 2.145, 2.2, 2.25, 2.35, 2.5, 2.75, 3, 3.25, 3.5, 4, 4.5364, 6, 10, and $\infty$ ($H_2$ on $P_{ed}$). Figure 6-13 shows a plot of $G_{zw}$ for the same values of $\gamma$, as well as for $H_2$ optimization on $P_{zw}$. The results are summarized in Table 6-1, with several other $\gamma$ values added. Notice that as we choose a larger and larger $\gamma$, the resulting infinity-norm of $G_{ed}$ gets further from the requested, finally leveling off at a value around 4.16. This is approaching the value of the $H_2$ optimal compensator.
Figure 6-10. Singular value plots of $T_{ed}$ for $H_2$ optimization on $P_{ed}$ and $P_{zw}$

Figure 6-11. Singular value plots of $T_{zw}$ for $H_2$ optimization on $P_{ed}$ and $P_{zw}$
Figure 6-12. $G_{ed}$ at various $\gamma$ levels and for H$_2$ optimization on Ped

Figure 6-13. $G_{zw}$ at various $\gamma$ levels and for H$_2$ optimization on Ped & Pzw
<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$| T_{\text{ed}} |_\infty$, $Q=0$</th>
<th>$| T_{ZW} |_2$, $Q=0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1426</td>
<td>2.1426</td>
<td>94.323</td>
</tr>
<tr>
<td>2.145</td>
<td>2.145</td>
<td>20.387</td>
</tr>
<tr>
<td>2.15</td>
<td>2.15</td>
<td>16.099</td>
</tr>
<tr>
<td>2.175</td>
<td>2.1747</td>
<td>14.229</td>
</tr>
<tr>
<td>2.2</td>
<td>2.1991</td>
<td>14.055</td>
</tr>
<tr>
<td>2.225</td>
<td>2.2232</td>
<td>14.042</td>
</tr>
<tr>
<td>2.25</td>
<td>2.247</td>
<td>14.076</td>
</tr>
<tr>
<td>2.3</td>
<td>2.2936</td>
<td>14.185</td>
</tr>
<tr>
<td>2.35</td>
<td>2.3389</td>
<td>14.305</td>
</tr>
<tr>
<td>2.5</td>
<td>2.4675</td>
<td>14.622</td>
</tr>
<tr>
<td>2.75</td>
<td>2.6589</td>
<td>14.953</td>
</tr>
<tr>
<td>3.0</td>
<td>2.8243</td>
<td>15.107</td>
</tr>
<tr>
<td>3.25</td>
<td>2.9673</td>
<td>15.161</td>
</tr>
<tr>
<td>3.5</td>
<td>3.0909</td>
<td>15.164</td>
</tr>
<tr>
<td>4.0</td>
<td>3.2913</td>
<td>15.107</td>
</tr>
<tr>
<td>4.5364</td>
<td>3.4532</td>
<td>15.025</td>
</tr>
<tr>
<td>4.6</td>
<td>3.4695</td>
<td>15.016</td>
</tr>
<tr>
<td>6.0</td>
<td>3.7263</td>
<td>14.851</td>
</tr>
<tr>
<td>10.0</td>
<td>3.9948</td>
<td>14.666</td>
</tr>
<tr>
<td>50.0</td>
<td>4.1559</td>
<td>14.557</td>
</tr>
<tr>
<td>100.0</td>
<td>4.1611</td>
<td>14.554</td>
</tr>
</tbody>
</table>
for \( P_{ed} \), which is 4.1629, as expected. Also notice that the two-norm of \( G_{zw} \) is approaching the value of 14.553 given above. Most importantly, notice that as \( \gamma \) gets larger, the resulting two-norm of \( G_{zw} \) does not approach that of \( T_{zw} \) for \( K_{2opt} \).

Finally, the results in Table 6-1 are presented graphically in Figure 6-14, which shows a plot of the two-norm of \( T_{zw} \) versus the infinity-norm of \( T_{ed} \), when we choose \( Q(s) = 0 \). Notice that the plot is not monotonically decreasing! This is not as surprising as it may seem - while it has been proven that the two- vs. infinity-norm is monotonically decreasing with \( \gamma \) using the mixed compensator, here we only used the central \( H_{\infty} \) compensator for \( T_{ed} \). This compensator design is actually unaware of the existence of \( z \) or \( w \), and is obviously a poor choice for a mixed controller. Strangely enough, for the choice \( Q(s) = 0 \), there is a "best" \( \gamma \) to select if one is willing to sacrifice some \( H_{\infty} \)-optimality to gain some \( H_{2} \)-optimality. This is not always true - the MIMO example in the next section does not exhibit this behavior.

![Figure 6-14. Infinity-norm vs. two-norm for Q(s) = 0](image_url)
A summary is now in order, to set the stage for the main results of this example. The minimum achievable infinity-norm of $T_{ed}$ is 2.1426, which results in a very large (actually infinite) two-norm of $T_{ZW}$. Conversely, the minimum achievable two-norm of $T_{ZW}$ is 9.9263, which results in an infinity-norm of $T_{ed}$ equal to 4.5364. This is the reason for 4.5364 appearing in Table 6-1 - it would be foolish to choose a $\gamma$ greater than this, since the optimal $H_2$ compensator can be used and still satisfy the infinity-norm requirement. Using the central $H_\infty$-controller for any value of $\gamma$ between 2.1426 and 4.5364 results in the fixed value of the two-norm of $T_{ZW}$ shown in Table 6-1.

Note that for $Q(s) = 0$ and $\gamma > \gamma_0$, the achieved value of $\| T_{ed} \|_\infty$ is always less than the chosen $\gamma$. From Theorem 4.2.3, for $n_c = n$ (as it is in the $Q(s) = 0$ case) and for $\gamma < \gamma_2$, we know that the mixed solution should lie on the boundary of the infinity-norm constraint. For this example, $\gamma_2 = 4.5364$, and for $\gamma \geq \gamma_2$, $K_{mix} = K_{2opt}$. Thus, the central $H_\infty$-controller is not using its "full power" to minimize the two-norm, which is to be expected since the two-norm is not included in the problem. Therefore, the mixed solution must be one for which $Q(s) \neq 0$. The cost of doing this will be an increase in the value of $\| T_{ed} \|_\infty$ over that of the central solution, but with a reduction in $\| T_{ZW} \|_2$. Order of the compensator is eliminated as an increased cost by only searching for compensators of order equal to the plant.

The potential to reduce the two-norm of $T_{ZW}$ by backing away from $H_\infty$-optimality of $T_{ed}$ and choosing a nonzero $Q(s)$ is best seen by examining the plots of the transfer functions given in (6.5) - (6.11) for various choices of $\gamma$. No matter what method is used to perform the mixed optimization, (6.3) and (6.4) still characterize all possible $T_{ed}$'s and $T_{ZW}$'s such that $\| T_{ed} \|_\infty < \gamma$. It is critical to realize that using a dynamic $Q(s)$ does not necessarily increase the order of $K(s)$ - pole/zero cancellations may occur when closing the P-J loop. Figures 6-15 and 6-16 show plots of the 3 transfer functions given by (6.6) - (6.8) for the values of $\gamma$: 2.1426, 2.145, 2.2, 2.25, 2.35, 2.5, 2.75, 3, 3.25, 3.5, 4, 4.5364, 6, and 10. The transfer function in (6.5) has already been presented as Figure 6-12. Figures 6-12, 6-15, and 6-16 influence $T_{ed}$. Figure 6-12 represents the central $H_\infty$-compensator, so that $T_{ed} = G_{ed}$ if $Q(s) = 0$. Thus, the entire effect of $Q(s)$ is determined by Figures 6-15 and 6-16, as seen in (6.3). Figure 6-16 is the transfer function inside the inverse along with $Q(s)$. For
Figure 6-15. $G_e$ or $G_v$ at various $\gamma$ levels

Figure 6-16. $G_{vr}$ at various $\gamma$ levels
$T_{ed} \in RH_\infty$, $[I - QG_{Vr}]^{-1}$ must be an element of $RH_\infty$, since all the other terms in (6.3) are elements of $RH_\infty$ and $Q(s)$ is arbitrary (can't count on unstable pole/zero cancellations). Thus, we must have the result:

**Lemma 6.1.1**: Given that any $Q(s) \in RH_\infty$ with $\|Q(s)\|_\infty < \gamma$ yields $[I - QG_{Vr}]^{-1} \in RH_\infty$, it must be true that

$$\|G_{Vr}\|_\infty \leq \frac{1}{\gamma} \tag{6.13}$$

**Proof**: Assume $\|G_{Vr}\|_\infty > 1/\gamma$, and since $Q(s)$ is arbitrary, let it be such that its infinity-norm is calculated at the same frequency as that of $G_{Vr}$, with a value of the reciprocal of $\|G_{Vr}\|_\infty$. Then $[I - QG_{Vr}]$ is singular on the $j\omega$-axis, and thus $[I - QG_{Vr}]^{-1} \notin RH_\infty$, a contradiction.

Therefore, (6.13) dictates the behavior of $G_{Vr}$. Notice that Figure 6-16 does indeed start with a value close to the inverse of the optimal infinity norm, and then decreases as $\gamma$ is increased. The bottom line is that when $Q(s)$ is chosen to be as large as allowed over the frequencies where $G_{Vr}$ is large, the product is close to unity and thus the inverse is potentially large. This would have the greatest effect on $T_{ed}$, considering the $[I - QG_{Vr}]^{-1}Q$ term only. Unfortunately, Figure 6-15 (note that $G_r$ and $G_{Vd}$ are identical) shows that when $\gamma$ is chosen very close to optimal, the transfer functions which pre- and post-multiply this term are very close to zero over the bandwidth where $G_{Vr}$ is large. The net result is that when $\gamma$ is chosen very close to $\gamma_0$, no $Q(s)$ will have very much of an effect on $T_{ed}$. This is another way to show that the SISO $H_\infty$-optimal compensator is unique.

It is more important for the purposes of this work to examine the transfer functions in (6.4), as they affect $T_{ZW}$. Figure 6-13 shows a large spike in the plot of $G_{ZW}$ for all values of $\gamma$. By looking at the $H_2$-optimal plot of $T_{ZW}$ also shown on this figure, it is clear that the spike is not desirable and needs to be removed through proper selection of $Q(s)$, if possible. The spike in $G_{ZW}$ occurs at roughly the same frequency as the dip in $P_{yu}$ as seen in Figure 6-2 -- the dip is the cause of the spike. This occurs because the optimization performed thus far is only concerned with $T_{ed}$. The mathematics are not even aware of the presence of the input $w$. 

147
or output $z$. Thus, the infinity-norm optimization tries to notch out this dip, causing the large spike in $G_{zw}$. This spike is the main contributor to the increased two-norm.

The transfer function inside the inverse along with $Q(s)$ in (6.4) is again $G_{vr}$, so all of the comments from the preceding paragraphs regarding $G_{vr}$ and its implications on the choice of $Q(s)$ apply here as well. All that remains is to examine the plots of the "outer" transfer functions, shown in Figures 6-17 and 6-18. Notice that once again when $\gamma$ is chosen close to $\gamma_o$, these functions are both nearly zero and little can be done with $Q(s)$. As $\gamma$ is increased from optimal, however, these functions get larger, especially around the frequency of the troublesome spike. This indicates that some significant reduction in the two-norm of $T_{zw}$ may be possible by proper selection of $Q(s)$, as will be shown to be true.

We will now examine the limiting case of $\gamma = \gamma_2 = 4.5364$. Choosing this $\gamma$ and performing $H_\infty$ optimization on $P_{ed}$ produces a 3-state $J(s)$ as shown in Figure 6-9. Then closing the P-J loop produces the system shown in Figure 6-19.

![Figure 6-17. $G_{zt}$ at various $\gamma$ levels](image-url)
Figure 6-18. $G_{yw}$ at various $\gamma$ levels

Figure 6-19. Closure of the P-J loop through $H_\infty$ optimization

$G(s)$ is given by

$$G(s) = \begin{bmatrix} G_{ed} & G_{ew} & G_{er} \\ G_{zd} & G_{zw} & G_{zr} \\ G_{vd} & G_{vw} & G_{vr} \end{bmatrix}$$  \hspace{1cm} (6.14)
The infinity norm of $T_{ed}$ will be less than 4.5364 if $Q(s)$ is chosen such that

$$Q(s) \in RH_\infty \quad \|Q(s)\|_\infty < \gamma$$  \hspace{1cm} (6.15)

The above condition is also sufficient to guarantee internal stability of the system. If we ignore the signals $e$ and $d$ and the requirement in (6.15), we can do $H_2$ optimization on the block $2 \times 2$ subsystem

$$G_2 = \begin{bmatrix} G_{ew} & G_{er} \\ G_{zw} & G_{zr} \end{bmatrix}$$

In general, this will guarantee internal stability of the system given by (6.14), but may not yield $\|T_{ed}\|_\infty < 4.5364$, depending upon the resulting $Q(s)$. Doing this produced the 6th order, stable $Q(s)$ shown in Figure 6-20. This transfer function has an infinity norm of 4.5363, so $Q(s)$ does indeed satisfy (6.15).

![Figure 6-20. Singular value plot of $Q(s)$ for $\gamma = 4.5364$](image)

150
$K_{mix}$ can now be found by

$$K_{mix} = F_I[J(s), Q(s)]$$  \hspace{1cm} (6.16)

While it seems that $K_{mix}$ should be ninth order (three states from $J(s)$ and six from $Q(s)$), there are six pole/zero cancellations that occur when computing (6.16), resulting in $K_{mix}$ being third order. Upon examination, we see that $K_{mix} = K_{2_{opt}}$. Figure 6-21 shows the singular value plots of $T_{ed}$ for the central $H_\infty$ compensator (this will be denoted $K_{\infty 4.5364}$) and for $K_{mix}$. As expected, the mixed $T_{ed}$ plot is identical to the $H_2$ optimal plot in Figure 6-8. Figure 6-22 shows the singular value plots of $T_{zw}$ for $K_{\infty 4.5364}$ and $K_{mix}$. Again, the mixed plot is identical to the $H_2$ optimal $T_{zw}$ plot shown in Figure 6-7. As proven in Theorem 4.2.1, $\alpha^* = \alpha_0$.

As also shown in Theorem 4.2.1, if $\gamma$ is chosen to be greater than $4.5364 = \gamma_2$, we still have $K_{mix} = K_{2_{opt}}$ and $\alpha^* = \alpha_0$. The compensator will no longer produce a $T_{ed}$ plot on the boundary of the infinity-norm constraint. A $Q(s)$ which satisfies (6.15) will always be able to be found, as shown in Figure 6-23 for the values of $\gamma$: 4.5364, 4.6, 4.8, 5, 5.5, 6, 8, 10, and 100.

![Graph showing singular value plots](image)

Figure 6-21. Singular value plots of $T_{ed}$ for $K_{\infty 4.5364}$ and $K_{mix}$
Figure 6-22. Singular value plots of $T_{zw}$ for $K_{\infty4,5364}$ and $K_{mix}$.

Figure 6-23. Singular value plots of $Q(s)$ for various values of $\gamma \geq \gamma_2$. 

152
We have now defined the limits of this problem. For \( \gamma < 2.1426 \), no compensator of any order exists which satisfies the mixed optimization problem. For \( \gamma = 2.1426 \), \( K_{mix} = K_{\infty2.1426} \), the \( T_{ed} - H_\infty \) optimal controller, which has a very large (theoretically infinite) two-norm that cannot be reduced. For \( \gamma \geq 4.5364 \), \( K_{mix} = K_{2_{opt}} \), the \( T_{zw} - H_2 \) optimal controller. The true problem is to determine what can be done when \( 2.1416 < \gamma < 4.5364 \).

### 6.2 Mixed Results

In what follows, we will use the terminology:

- "\( K_{\infty****} \)" to mean the central ( \( Q(s) = 0 \) ) \( H_\infty \) compensator for \( T_{ed} \) at the value of \( \gamma = **** \)
- "\( K_{2_{opt}} \)" to mean the optimal \( H_2 \) compensator for \( T_{zw} \)
- "\( K_{mix****} \)" to mean the compensator which minimizes \( \| T_{zw} \|_2 \) subject to \( \| T_{ed} \|_\infty < \gamma = **** \)

The numerical solution technique used for the following results was the approach detailed in Section 5.2. A total of nine different \( \gamma \) levels were chosen between 2.1426 and 4.5364 to get a picture of what the mixed optimization was doing. For each value of \( \gamma \), a combined plot of \( T_{ed} \) using \( K_{\infty****}, K_{mix****}, \) and \( K_{2_{opt}} \) is shown. A similar plot is also shown for \( T_{zw} \). From these, the critical data for each \( \gamma \) is obtained. Recall that using this approach we are solving for the (sub)optimal mixed compensator \( K_{mix****} \) of order equal to the plant, and thus the same order as the other two compensators.

#### 6.2.1 \( \gamma = 2.2 \) Results

Figure 6-24 shows the \( T_{ed} \) comparison plot. Notice how the mixed \( T_{ed} \) plot tries to match the \( H_2 \) plot where it dips below 2.2. Through the Bode integral relations, pushing down on a transfer function in one frequency range must cause a rise in another range. This accounts for the higher bandwidth of the mixed plot.
Figure 6-24. \( T_{ed} \) comparison plot for \( \gamma = 2.2 \)

The desire for the mixed plot to match the \( H_2 \) plot makes sense, since \( K_{2_{\text{opt}}} \) minimizes the two-norm. The infinity-norm of the mixed plot is \( \gamma^* = 2.19999 \), increased from 2.1991 using \( K_{\infty 2.2} \). This shows that \( K_{\text{mix2.2}} \) is using the full freedom available to it and that the two objectives in the performance index are truly competing. This will be the case for every chosen value of gamma up to 4.5364. Beyond 4.5364, they no longer compete.

Figure 6-25 shows the \( T_{ZW} \) comparison plot. As we are very close to \( H_{\infty} \) optimal at this \( \gamma \) level, there is little freedom to reduce the two-norm. The main contributor to the increased two-norm is the large spike - notice that \( K_{\text{mix2.2}} \) uses the freedom it has to reduce the spike. From the two-norm of the mixed plot we get \( \alpha^* = 13.868 \). This is indeed a reduction in the two-norm from \( K_{\infty 2.2} \), for which \( \| T_{ZW} \|_2 = 14.055 \).

Finally, one could ask whether or not there exists some \( Q(s) \) that such that when it is fed back around the \( J(s) \) for this \( \gamma \), the \( K_{\text{mix2.2}} \) found here is
obtained. The answer must be yes, since $K_{\text{mix}}_{2,2}$ is $H_\infty$ admissible and the $Q$ parameterization is complete. We therefore wish to determine $Q(s)$ such that the equivalence shown in Figure 6-26 is established. In Figure 6-26, both $K_{\text{mix}}(s)$ and $J(s)$ are completely known. $J(s)$ is given in Section 3.2 as

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} A_J & K_f S_Y & K_f l \\ -S_u^{-1} K_c & 0 & S_u^{-1} \\ K_{cl} & S_Y & 0 \end{bmatrix}$$

In terms of transfer functions, we desire

$$K_{\text{mix}} = J_{11} + J_{12} Q(I - J_{22} Q)^{-1} J_{21}$$

or
\[ J_{12}^{-1} [ K_{\text{mix}} - J_{11} ] J_{21}^{-1} = Q[I - J_{22}Q]^{-1} \]  \hspace{1cm} (6.17)

\( J_{12} \) and \( J_{21} \) are always square and their inverses always exist, since their D terms are \( S_u^{-1} \) and \( S_y \), respectively, which are invertible. Define

\[ X = J_{12}^{-1} [ K_{\text{mix}} - J_{11} ] J_{21}^{-1} \]  \hspace{1cm} (6.18)

Note that \( X \) is completely known. Then Equation (6.17) becomes

\[ X = Q[I - J_{22}Q]^{-1} \]

or

\[ Q = [I + XJ_{22}]^{-1}X \]  \hspace{1cm} (6.19)

where the right-hand side of Equation (6.19) consists of known quantities. Equation (6.19) now gives us a formula to compute the \( Q(s) \) that would
produce the $K_{\text{mix}}(s)$ found by the DFP (or any other) algorithm. From a computational viewpoint, this is done by computing $X$ using (6.18) and then recognizing that $Q(s)$ in (6.19) is the closed-loop transfer function of the block diagram shown in Figure 6-27. This was done for this example, with the resulting sixth order $Q(s)$ shown in Figure 6-28. The $Q(s)$ here is indeed stable and has its infinity-norm less than $\gamma (\|Q\|_{\infty} = 2.19999)$. This shows that $Q(s) = 0$ is not the correct choice for mixed $H_2/H_\infty$ optimization. This procedure for computing $Q(s)$ was used for all of the results that follow. The resulting $Q(s)$ plots will be shown near the end of this section.

![Block diagram used to compute Q(s)](image)

**Figure 6-27.** Block diagram used to compute $Q(s)$

![Magnitude plot](image)

**Figure 6-28.** $Q(s)$ plot for $\gamma = 2.2$
6.2.2 Other $\gamma$ Results

Figures 6-29 through 6-36 show the $T_{ed}$ comparison plots for the following $\gamma$ values, respectively: $\gamma = 2.25, 2.35, 2.5, 2.75, 3, 3.25, 3.5$, and $4$. It is easy to see from these figures that the mixed $T_{ed}$ plot is trying to match the $H_2-T_{ed}$ plot in each case, with better and better "recovery" with increasing $\gamma$. We have already seen in Figure 6-21 that for $\gamma = 4.5364$ (or higher) we get perfect recovery. Figures 6-37 through 6-44 show the $T_{zw}$ comparison plots for the same values of $\gamma$. Notice that as $\gamma$ is increased, the magnitude of the spike is reduced by $K_{mix}$. Finally, once the spike is nearly removed, the low frequency portion of the mixed plot is increased in order to reduce the high frequency portion even further (recall that the two-norm is frequency dependent, and high frequency "decades" have a much greater contribution to the norm). As seen back in Figure 6-22, we get perfect recovery of the $T_{zw}-H_2$ plot by the mixed plot for $\gamma = 4.5364$ (or higher).

![Graph](image)

**Figure 6-29.** $T_{ed}$ comparison plot for $\gamma = 2.25$
Figure 6-30. $T_{ed}$ comparison plot for $\gamma = 2.35$

Figure 6-31. $T_{ed}$ comparison plot for $\gamma = 2.5$
Figure 6-32. $T_{ed}$ comparison plot for $\gamma = 2.75$

Figure 6-33. $T_{ed}$ comparison plot for $\gamma = 3.0$
Figure 6-34. $T_{ed}$ comparison plot for $\gamma = 3.25$

Figure 6-35. $T_{ed}$ comparison plot for $\gamma = 3.5$
Figure 6-36. $T_{ed}$ comparison plot for $\gamma = 4.0$

Figure 6-37. $T_{zw}$ comparison plot for $\gamma = 2.25$
Figure 6-38. $T_{zw}$ comparison plot for $\gamma = 2.35$

Figure 6-39. $T_{zw}$ comparison plot for $\gamma = 2.5$
Figure 6-40. $T_{ZW}$ comparison plot for $\gamma = 2.75$

Figure 6-41. $T_{ZW}$ comparison plot for $\gamma = 3.0$
Figure 6-42. $T_{zw}$ comparison plot for $\gamma = 3.25$

Figure 6-43. $T_{zw}$ comparison plot for $\gamma = 3.5$
A summary of the above results is seen most clearly in Figures 6-45 and 6-46. Figure 6-45 shows the mixed $T_{ed}$ plot for a representative sampling of $\gamma$ values. Here, a definite recovery of the $H_2$ plot is seen as $\gamma$ is increased. Figure 6-46 shows a similar plot for $T_{zw}$. Again, a definite recovery is seen. Also included is Figure 6-47, which shows the $Q(s)$ transfer functions that produce the mixed compensators when wrapped around the corresponding $H_\infty$ compensators. These also show the recovery tendency; that is, as $\gamma$ is increased, they tend to recover the $(\gamma = 4.5364)$-$Q(s)$ function. Both Figures 6-45 and 6-47 also show the competitive nature of the mixed problem, in that the full amount of infinity-norm allowed to be used by either $T_{ed}$ or $Q(s)$ is used.
Figure 6-45. Mixed $T_{ed}$ plot for $\gamma = 2.1426, 2.25, 2.5, 3, 3.5, 4$, and 4.5364

Figure 6-46. Mixed $T_{zw}$ plot for $\gamma = 2.1426, 2.25, 2.5, 3, 3.5, 4$, and 4.5364
Figure 6-47. Q(s) plot for $\gamma = 2.25, 2.5, 3, 3.5, 4, \text{ and } 4.5364$

The true bottom line of the mixed compensator design is seen in Table 6-2 and Figure 6-48. Table 6-2 shows a summary of $\| T_{ed} \|_\infty$ and $\| T_{zw} \|_2$ for both the central $H_\infty$ compensator and the mixed compensator at each design value of $\gamma$. The value of $\| T_{ed} \|_\infty$ for the mixed controller in each case is literally the chosen value of $\gamma$ (typically within 0.00001), and is shown rounded off to $\gamma$. The value of $\| T_{zw} \|_2$ for the mixed case is always lower than the Q(s) = 0 case, except very close to $\gamma_0$, where no real improvement is possible. The results are best visualized in Figure 6-48, which is a plot of $\| T_{zw} \|_2$ versus $\| T_{ed} \|_\infty$ for both Q(s) = 0 and the $K_{mix}$ controllers. This figure dramatically shows the benefit in using the mixed compensator. The two-norm of $T_{zw}$ for $K_{mix}$ is always less than or equal to that of the central $H_\infty$ compensator for any value of $\gamma$. For a value of $\gamma$ not very far from optimal, the improvement can be dramatic. It is vital to remember that the compensator designed here is the same order as the plant, so no price is paid by using $K_{mix}$ in that respect. Perhaps just as interesting is considering the reverse of the normal objective here - design an $H_2$ compensator that reduces the infinity norm of $T_{ed}$, if possible.
<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$| T_{ed} |_\infty, Q = 0$</th>
<th>$| T_{zw} |_2, Q = 0$</th>
<th>$| T_{ed} |_\infty, \text{ mix}$</th>
<th>$| T_{zw} |_2, \text{ mix}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1426</td>
<td>2.1426</td>
<td>94.323</td>
<td>2.1426</td>
<td>-94.323</td>
</tr>
<tr>
<td>2.145</td>
<td>2.145</td>
<td>20.387</td>
<td>2.145</td>
<td>-20.387</td>
</tr>
<tr>
<td>2.15</td>
<td>2.15</td>
<td>16.099</td>
<td>2.15</td>
<td>16.091</td>
</tr>
<tr>
<td>2.2</td>
<td>2.1991</td>
<td>14.055</td>
<td>2.2</td>
<td>13.868</td>
</tr>
<tr>
<td>2.25</td>
<td>2.247</td>
<td>14.076</td>
<td>2.25</td>
<td>13.295</td>
</tr>
<tr>
<td>2.35</td>
<td>2.3389</td>
<td>14.305</td>
<td>2.35</td>
<td>12.501</td>
</tr>
<tr>
<td>2.5</td>
<td>2.4675</td>
<td>14.622</td>
<td>2.5</td>
<td>11.616</td>
</tr>
<tr>
<td>2.75</td>
<td>2.6589</td>
<td>14.953</td>
<td>2.75</td>
<td>10.870</td>
</tr>
<tr>
<td>3.0</td>
<td>2.8243</td>
<td>15.107</td>
<td>3.0</td>
<td>10.460</td>
</tr>
<tr>
<td>3.25</td>
<td>2.9673</td>
<td>15.161</td>
<td>3.25</td>
<td>10.225</td>
</tr>
<tr>
<td>3.5</td>
<td>3.0909</td>
<td>15.164</td>
<td>3.5</td>
<td>10.086</td>
</tr>
<tr>
<td>4.0</td>
<td>3.2913</td>
<td>15.107</td>
<td>4.0</td>
<td>9.9539</td>
</tr>
<tr>
<td>4.5364</td>
<td>3.4532</td>
<td>15.025</td>
<td>4.5364</td>
<td>9.9263</td>
</tr>
<tr>
<td>10.0</td>
<td>3.9948</td>
<td>14.666</td>
<td>4.5364</td>
<td>9.9263</td>
</tr>
<tr>
<td>50.0</td>
<td>4.1559</td>
<td>14.557</td>
<td>4.5364</td>
<td>9.9263</td>
</tr>
<tr>
<td>100.0</td>
<td>4.1611</td>
<td>14.554</td>
<td>4.5364</td>
<td>9.9263</td>
</tr>
</tbody>
</table>
Figure 6-48. Comparison of $Q(s) = 0$ and mixed compensators

Figure 6-48 shows that by giving up only a small amount of $H_2$ optimality a significant reduction in the infinity-norm is possible.

6.3 Extensions

Two extensions to the above results will now be briefly addressed.

6.3.1 Order

The question of optimal order of the mixed controller has not been directly addressed thus far. While it is nearly impossible to obtain results of this nature using Lagrange multiplier methods, some important results are obtainable. Consider allowing the order of the compensator to be higher than that of the plant; that is, choose $n_c$ to be greater than $n$. The equations developed in Chapter 4 do not change in this case, but it is no
longer known if the solution always lies on the boundary of the \( H_\infty \) constraint on \( T_{ed} \). However, since the equations do not change, the numerical procedure in Chapter 5 still works and a mixed compensator of any chosen order may be found (actually, even for \( n_c < n \)). This was done for \( n_c = 9 \) and \( \gamma = 2.5 \). The results were \( \gamma^* = 2.5 \) and \( \alpha^* = 11.507 \). While this is by no means sufficient to prove the solution always lies on the boundary (see Conjecture 4.2.1), it is sufficient to prove that the optimal order of the mixed compensator is not, in general, the order of the plant but rather higher order, as implied by Theorem 4.2.5. The ninth order \( K_{mix_{2.5}} \) produced a \( T_{ed} \) which did a slightly better job of approximating the dip in the \( H_2-T_{ed} \) plot and a \( T_{zw} \) which reduced the spike in that plot a little more than the third order case. As proven in Theorem 4.2.5, while \( \alpha^* \) was reduced, it cannot be reduced to the value of \( \alpha_0 \).

### 6.3.2 Mixing norms of the same transfer function

Consider letting \( w = d \) and \( z = e \) so that the mixed \( H_2/H_\infty \) optimization problem becomes

\[
\inf_{K \text{ admissible}} \| T_{ed} \|_2
\]

subject to

\[
\| T_{ed} \|_\infty \leq \gamma
\]

For this example, choosing \( Q(s) = 0 \) and varying \( \gamma \) produces the plot shown in Figure 6-49. This plot is indeed monotonically decreasing, and corresponds to the minimum entropy solution. Since entropy overbounds the two-norm, and the bound is tight as \( \gamma \to \infty \), this curve may be a very good approximation to the mixed solution and is the solution for \( \gamma \geq 4.1611 \). While it may be a good approximation for \( \gamma_0 < \gamma < \gamma_2 \), it is not the true mixed solution. Theorem 4.2.3 still holds in this case, so the mixed solution must lie on the boundary of the infinity-norm constraint, which the minimum entropy solution does not. Therefore, \( Q(s) = 0 \) is not the mixed solution.
Figure 6-49. Infinity-norm vs. two-norm for $Q(s) = 0$ for $w = d$ and $z = e$
CHAPTER 7
MIMO MIXED OPTIMIZATION EXAMPLE

7.1 Problem Set-Up

In this multivariable example, all signals are two-dimensional vectors. Thus, d, w, u, e, z, and y in Figure 7-1 are all 2x1.

![Block Diagram]

Figure 7-1. Mixed optimization block diagram

The plant P(s) has the state space representation

\[
P(s) = \begin{bmatrix}
A & B_d & B_w & B_u \\
C_e & D_{ed} & D_{ew} & D_{eu} \\
C_z & D_{zd} & D_{zw} & D_{zu} \\
C_y & D_{yd} & D_{yw} & D_{yu}
\end{bmatrix}
\]

(7.1)

where the individual matrices are given by

\[
A = \begin{bmatrix}
-5 & 2 & 14 & 20 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
B_d = \begin{bmatrix}
0.03 & 0.008 \\
0.05 & 0.38 \\
0.53 & 0.07 \\
0.67 & 0.42
\end{bmatrix}
\]

\[
B_w = \begin{bmatrix}
0.22 & 0.93 \\
0.05 & 0.38 \\
0.68 & 0.52 \\
0.68 & 0.83
\end{bmatrix}
\]

\[
B_u = \begin{bmatrix}
0.07 & 0.44 \\
0.63 & 0.77 \\
0.88 & 0.48 \\
0.27 & 0.24
\end{bmatrix}
\]
\[
C_e = \begin{bmatrix}
0.55 & 0.33 & 1.8 & 0.12 \\
0.72 & 0.97 & 1.82 & 1.81
\end{bmatrix}
\]

\[
C_z = \begin{bmatrix}
0.07 & 0.38 & 0.91 & 0.46 \\
0.50 & 0.28 & 0.53 & 0.94
\end{bmatrix}
\]

\[
C_y = \begin{bmatrix}
0.05 & 0.77 & 0.13 & 0.69 \\
0.76 & 0.83 & 0.02 & 0.87
\end{bmatrix}
\]

\[
D_{ed} = D_{ew} = D_{zd} = D_{zw} = D_{yu} = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

\[
D_{zu} = D_{yd} = D_{yw} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

\[
D_{eu} = \begin{bmatrix}
0.1 & 0 \\
0 & 0.1
\end{bmatrix}
\]

The actual "unweighted" plant, given by \( P_{yu}(s) = C_e(sI - A)^{-1}B_d + D_{yu} \), is unstable, nonminimum phase with a singular value plot as shown in Figure 7-2.

![Figure 7-2. Singular value plot of the MIMO plant](image-url)
The system $P(s)$ above has the proper form for both $H_2$ and $H_\infty$ optimization. Consider doing state space $H_\infty$ optimization on $P_{ed}$, as described in Section 3.2. Doing this yields an "optimal" infinity-norm of roughly 2.3012, with the singular value plot of $G_{ed}$ shown in Figure 7-3. The minimum singular value for this transfer function is zero, which indicates that in this example the optimal compensator will produce zero outputs for an input in a certain direction. This is unusual, but obviously is possible, as this example shows. Now choose the freedom parameter $Q(s)$ to be zero, so that $T_{ed} = G_{ed}$. The resulting four-state controller $K_{\infty 2.3012}$ is shown in Figure 7-4. Closing the loop in Figure 7-1 with $K_{\infty 2.3012}$ produces the $T_{zw}$ plot shown in Figure 7-5. This actually does roll off at very high frequency, and the two-norm is extremely large with a value of 1887.3. If $K_{\infty \text{opt}}$ was used, the $T_{zw}$ plot would not roll off and the two-norm would be infinite. Therefore, at $\gamma = \gamma_0$ the mixed problem here is actually ill-posed since no finite solution exists. Thus, we eliminate $\gamma = \gamma_0$ from consideration in the mixed optimization results. For convenience, we will refer to the value 2.3012 as $\gamma_0$ in what follows.

Now consider the other extreme; that is, doing $H_2$ optimization on $P_{zw}$, which will produce $\alpha_0$. Carrying this out as described in Section 3.1 produces the four-state compensator $K_{2 \text{opt}}$, whose singular value plot is shown in Figure 7-6. Figure 7-7 shows a singular value plot of $T_{zw}$ for both $K_{\infty 2.3012}$ and $K_{2 \text{opt}}$. From this figure it is clear that $K_{\infty 2.3012}$ produces a much higher two-norm. For the $K_{2 \text{opt}}$ case, $\| T_{zw} \|_2 = \alpha_0 = 0.7975$. Since $K_{2 \text{opt}}$ is unique, no other $K(s)$ produces a two-norm this small. Closing the loop in Figure 7-1 with $K_{2 \text{opt}}$ produces the $T_{ed}$ plot shown in Figure 7-8, which has $\| T_{ed} \|_\infty = 40.548$, obviously higher than the optimal infinity-norm.

Doing the analysis above defines the achievable limits of the problem; that is, it defines the minimum achievable infinity-norm of $T_{ed}$ and the minimum achievable two-norm of $T_{zw}$. It is again instructive to consider the equations derived in Section 3.3; that is, consider the system given in Figure 7-1 to be represented by Figure 7-9, so that

\[
e = T_{ed}d = ( G_{ed} + G_{er}[I - QG_{vr}]^{-1}QG_{vd} )d \tag{7.2}
\]

\[
z = T_{zw}w = ( G_{zw} + G_{zr}[I - QG_{vr}]^{-1}QG_{vw} )w \tag{7.3}
\]
Figure 7-3. Singular value plot of $T_e d$ for $K_{\infty 2.3012}$

Figure 7-4. Singular value plot of $K_{\infty 2.3012}$
Figure 7-5. Singular value plot of $T_{ZW}$ for $K_{\infty}$2.3012

Figure 7-6. Singular value plot of $K_{2,opt}$
Figure 7-7. Singular value plots of $T_{ZW}$ for $K_{\infty_{2.3012}}$ and $K_{2_{\text{opt}}}$

Figure 7-8. Singular value plot of $T_{ed}$ for $K_{2_{\text{opt}}}$
where the transfer functions in (7.2) and (7.3) were defined in the previous chapter by (6.5) - (6.12). Once a value of $\gamma$ is chosen, doing $H_\infty$ optimization on $P_{ed}$ makes all of the transfer functions in (7.2) and (7.3) known. To follow along with the SISO example and to fully show what the mixed compensator can do, we now back away from $H_\infty$-optimality and examine the results for $Q(s) = 0$. If $Q(s)$ is chosen to be zero, then $T_{ed} = G_{ed}$ and $T_{zw} = G_{zw}$ and the other 5 transfer functions in (7.2) and (7.3) are unnecessary. Recall that as we let the chosen infinity-norm level $\gamma$ get larger, the resulting compensator for $Q(s) = 0$ becomes $H_2$ optimal, but that is the optimal $H_2$ compensator for $P_{ed}$, not for $P_{zw}$. Figure 7-10 shows singular value plots of $T_{ed}$ when the optimal two-norm compensator for $P_{ed}$ and for $P_{zw}$ are used in Figure 7-1. The former results in an infinity-norm of 4.6022, which is considerably lower than that for the latter. These compensators produce the singular value plots of $T_{zw}$ shown in Figure 7-11. The two-norm optimal $P_{ed}$ controller results in $\| T_{zw} \|_2 = 6.0664$. While this is considerably lower than the value of the two-norm near $H_\infty$-optimal, it still is higher than the $H_2$ optimal value of 0.7975. Recall that these results are equivalent to choosing $\gamma$ to be very large with $Q(s) = 0$ in $H_\infty$ optimization on $P_{ed}$. 
Figure 7-10. Singular value plots of $T_{ed}$ for $H_2$ optimization on $P_{ed}$ and $P_{zw}$

Figure 7-11. Singular value plots of $T_{zw}$ for $H_2$ optimization on $P_{ed}$ and $P_{zw}$
Next, various values of $\gamma$ were chosen. Figure 7-12 shows a plot of the maximum singular value of $G_{ed}$ for the following values of $\gamma$: 2.3012, 2.31, 2.4, 2.5, 2.6, 2.75, 3, 4, 5, 7.5, 10, 40, 548 and $\infty$ ($H_2$ on $P_{ed}$). The minimum singular values are all zero. Figure 7-13 shows a plot of $\sigma(G_{zw})$ for the same values of $\gamma$, as well as for $H_2$ optimization on $P_{zw}$. Figure 7-14 shows the same plot on a different scale so that more detail is visible. Finally, Figure 7-15 shows a plot of the minimum singular value of $G_{zw}$ for the various $\gamma$ values. The results are summarized in Table 7-1, with several other $\gamma$ values added. Notice that as we choose a larger and larger $\gamma$, the resulting infinity-norm of $G_{ed}$ gets further from the requested, approaching the value of the $H_2$ optimal compensator for $P_{ed}$ as expected. Also notice that the two-norm of $G_{zw}$ is approaching the value of 6.0664 given above. Most importantly, notice that as $\gamma$ gets larger, the resulting two-norm of $G_{zw}$ does not approach that of $T_{zw}$ for $K_{2opt}$. Finally, the results in Table 7-1 are presented graphically in Figure 7-16, which shows a plot of two-norm of $T_{zw}$ versus infinity norm of $T_{ed}$, when we choose $Q(s) = 0$. Notice that here the plot is monotonically decreasing, but never reaches the absolute minimum achievable two-norm.

![Figure 7-12. $G_{ed}$ at various $\gamma$ levels and for $H_2$ optimization on $P_{ed}$](image)
Figure 7-13. $\bar{\sigma}[G_{zw}]$ at various $\gamma$ levels and for $H_2$ optimal on $P_{ed}$ & $P_{zw}$

Figure 7-14. Blow-up of Figure 7-13
Figure 7-15. $|G_{zw}|$ at various $\gamma$ levels and for $H_2$ opt on $P_{ed}$ and $P_{zw}$

Figure 7-16. Infinity-norm vs. two-norm for $Q(s) = 0$
### Table 7-1. MIMO Example Results, Q = 0

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$|T_{ed}|_\infty, Q = 0$</th>
<th>$|T_{zw}|_2, Q = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.3012</td>
<td>2.3012</td>
<td>1887.3</td>
</tr>
<tr>
<td>2.31</td>
<td>2.31</td>
<td>188.32</td>
</tr>
<tr>
<td>2.32</td>
<td>2.32</td>
<td>119.13</td>
</tr>
<tr>
<td>2.35</td>
<td>2.3495</td>
<td>62.307</td>
</tr>
<tr>
<td>2.4</td>
<td>2.3979</td>
<td>37.008</td>
</tr>
<tr>
<td>2.5</td>
<td>2.4914</td>
<td>22.155</td>
</tr>
<tr>
<td>2.6</td>
<td>2.5807</td>
<td>16.799</td>
</tr>
<tr>
<td>2.75</td>
<td>2.7069</td>
<td>13.11</td>
</tr>
<tr>
<td>3.0</td>
<td>2.8975</td>
<td>10.415</td>
</tr>
<tr>
<td>3.5</td>
<td>3.2133</td>
<td>8.3741</td>
</tr>
<tr>
<td>4.0</td>
<td>3.4578</td>
<td>7.5491</td>
</tr>
<tr>
<td>4.5</td>
<td>3.6482</td>
<td>7.1158</td>
</tr>
<tr>
<td>5.0</td>
<td>3.7978</td>
<td>6.8949</td>
</tr>
<tr>
<td>6.0</td>
<td>4.0121</td>
<td>6.5647</td>
</tr>
<tr>
<td>7.0</td>
<td>4.1534</td>
<td>6.4128</td>
</tr>
<tr>
<td>8.0</td>
<td>4.2505</td>
<td>6.3224</td>
</tr>
<tr>
<td>9.0</td>
<td>4.3198</td>
<td>6.2638</td>
</tr>
<tr>
<td>10.0</td>
<td>4.3708</td>
<td>6.2236</td>
</tr>
<tr>
<td>20.0</td>
<td>4.5421</td>
<td>6.1036</td>
</tr>
<tr>
<td>30.0</td>
<td>4.5753</td>
<td>6.0828</td>
</tr>
<tr>
<td>40.548</td>
<td>4.5874</td>
<td>6.0753</td>
</tr>
<tr>
<td>50.0</td>
<td>4.5925</td>
<td>6.0723</td>
</tr>
<tr>
<td>100.0</td>
<td>4.5997</td>
<td>6.0678</td>
</tr>
</tbody>
</table>
To summarize, the minimum achievable infinity-norm of $T_{ed}$ is $\gamma_0 = 2.3012$, which results in a very large (actually infinite) two-norm of $T_{zw}$. Conversely, the minimum achievable two-norm of $T_{zw}$ is $\alpha_0 = 0.7975$, which results in an infinity-norm of $T_{ed}$ equal to $\gamma_2 = 40.548$. Using the central $H_\infty$-controller for any value of $\gamma$ between 2.3102 and 40.548 results in the fixed value of the two-norm of $T_{zw}$ shown in Table 7-1. Note that for $Q(s) = 0$ and $\gamma > \gamma_0$, the achieved value of $\|T_{ed}\|_\infty$ is always less than the chosen $\gamma$. From Theorem 4.2.3, for $n_c = n$ (as it is in the $Q(s) = 0$ case) and for $\gamma < \gamma_2$, we know that the mixed solution should lie on the boundary of the infinity-norm constraint. For $\gamma \geq \gamma_2 = 40.548$, $K_{mix} = K_{2opt}$.

The potential to reduce the two-norm of $T_{zw}$ by backing away from $H_\infty$-optimality of $T_{ed}$ and choosing a nonzero $Q(s)$ is best seen by examining the plots of the transfer functions given in (7.2) and (7.3) for various choices of $\gamma$. Figures 7-17 and 7-18 show plots of the transfer functions $G_{er}$, $G_{vd}$, and $G_{vr}$ for the values of $\gamma$: 2.3012, 2.31, 2.4, 2.5, 2.6, 2.75, 3, 4, 5, 7.5, 10, and 40.548. The transfer function $G_{ed}$ has already been presented as Figure 7-12. Figures 7-12, 7-17, and 7-18 influence $T_{ed}$. The entire effect of $Q(s)$ is determined by Figures 7-17 and 7-18, as seen in (7.2). Figure 7-18 is the transfer function inside the inverse along with $Q(s)$. The minimum singular value of this transfer function is equal to zero. As shown in the SISO example, this transfer function must satisfy $\|G_{vr}\|_\infty \leq 1 / \gamma$. Notice that the maximum singular value in Figure 7-18 does indeed start with a value close to the inverse of the optimal infinity-norm, and then decreases as $\gamma$ is increased. When $Q(s)$ is chosen to be as large as allowed over the frequencies where $G_{vr}$ is large, the product is close to unity and thus the inverse is potentially large. This would have the greatest effect on $T_{ed}$, considering the $[I - QG_{vr}]^{-1}Q$ term only. Figure 7-17 shows something that is potentially very different from the SISO case. While the minimum singular value has properties very similar to that of the SISO case (being very small when $\gamma$ is chosen close to $\gamma_0$), the maximum singular value is always equal to one. This implies that it may be possible to "do something" with $Q(s)$ even close to or at optimal. This is an indication of the fact that the $H_\infty$-optimal compensator is not unique for MIMO systems.

Next, look at the transfer functions that affect $T_{zw}$, which are $G_{zw}$, $G_{vr}$, $G_{zd}$, and $G_{ww}$. The first two have already been shown in Figures 7-13 to 7-15 and 7-18, respectively. Figure 7-13 or 7-14 shows large
Figure 7-17. $G_{er}$ or $G_{vd}$ at various $\gamma$ levels

Figure 7-18. $G_{vr}$ at various $\gamma$ levels
magnitudes of $G_{zw}$ at relatively high frequencies, especially near $\gamma_0$. By looking at the $H_2$-optimal plot of $T_{zw}$ also shown on these figures, it is clear that it is not desirable to have the magnitude so large at high frequency, or even as large at low frequency. This needs to be reduced through proper selection of $Q(s)$, if possible. From (7.3), the transfer function inside the inverse along with $Q(s)$ is again $G_{vr}$, so all of the comments from the preceding paragraph regarding $G_{vr}$ and its implications on the choice of $Q(s)$ apply here as well. All that remains is to examine the plots of the "outer" transfer functions $G_{zf}$ and $G_{vw}$ shown in Figures 7-19 and 7-20, respectively. As in Figure 7-17, these transfer functions have their minimum singular values near zero when gamma is near $\gamma_0$, and increasing with $\gamma$. The maximum singular values, however, are not small near or away from $\gamma_0$, which indicates that some significant reduction in the two-norm of $T_{zw}$ may be possible by proper selection of $Q(s)$, even close to $\gamma_0$. Near $\gamma_0$ this does not seem to be true, but significant reduction is possible away from $\gamma_0$.

![Figure 7-19. $G_{zf}$ at various $\gamma$ levels](image-url)
Figure 7-20. $G_{vw}$ at various $\gamma$ levels

We have now defined the limits of this problem. For $\gamma < 2.3102$, no compensator of any order exists which satisfies the mixed optimization problem. For $\gamma = 2.3102$, $K_{mix} = K_{\infty 2.1426}$, the $T_{ed-H_\infty}$ optimal controller, which has a very large (theoretically infinite) two-norm that cannot be reduced. For $\gamma \geq 40.548$, $K_{mix} = K_{2_{opt}}$, the $T_{zw-H_2}$ optimal controller. The true problem is to determine what can be done when $2.3102 < \gamma < 40.548$.

7.2 Mixed Results

In what follows, we will use the terminology defined at the beginning of Section 6.2. The same numerical technique as used in the SISO example was used here. A total of ten different $\gamma$ levels were chosen between 2.3102 and 40.548 to get a picture of what the mixed optimization was doing. Recall that using this approach we are looking for the optimal mixed compensator of order equal to the plant, and thus the same order as the central $H_\infty$ and the $H_2$ optimal controllers.
A summary of the mixed results are now shown. Figure 7-21 shows the mixed $T_{ed}$ plot for the following $\gamma$ values: 2.3102, 2.32, 2.35, 2.4, 2.5, 2.6, 2.75, 3, 4, 5, 10, and 40.548. Only the maximum singular values are shown. A definite recovery of the $H_2$ plot is seen as $\gamma$ is increased. The minimum singular values of the $T_{ed}$ plots also exhibit the same type of recovery, but are not involved in the infinity-norm computation. Figure 7-22 shows a similar plot for the maximum singular values of $T_{ZW}$, with Figure 7-23 showing a more detailed view. Again, a definite recovery is seen. The minimum singular values behave similarly. Also computed were the $Q(s)$ transfer functions that produce the mixed compensators when wrapped around the corresponding $H_\infty$ compensators. These also showed the recovery tendency; that is, as $\gamma$ was increased, they recovered the $(\gamma = 40.548)$-$Q(s)$ function. Both Figures 7-21 and the $Q(s)$ functions show the competitive nature of the mixed problem, in that the full amount of infinity-norm allowed to be used by either $T_{ed}$ or $Q(s)$ is used.

![Graph showing mixed $T_{ed}$ plot for various $\gamma$.](image)

Figure 7-21. Mixed $T_{ed}$ plot for various $\gamma$
Figure 7-22. Mixed $T_{ZW}$ plot for various $\gamma$

Figure 7-23. Expanded view of Figure 7-22
The true bottom line of the mixed compensator design is seen in Table 7-2 and Figure 7-24. Table 7-2 shows a summary of $\| T_{ed} \|_\infty$ and $\| T_{zw} \|_2$ for both the central $H_\infty$ compensator and the mixed compensator at each design value of $\gamma$. The value of $\| T_{ed} \|_\infty$ for the mixed controller in each case is literally the chosen value of $\gamma$ (typically within 0.00001), and is shown rounded off to $\gamma$. The value of $\| T_{zw} \|_2$ for the mixed case is always lower than the $Q(s) = 0$ case, except very close to $\gamma_0$, where no real improvement is possible. The results are best visualized in Figure 7-24, which is a plot of $\| T_{zw} \|_2$ versus $\| T_{ed} \|_\infty$ for both $Q(s) = 0$ and the $K_{mix}$ controllers. This figure again shows the benefit in using $K_{mix}$. The two-norm of $T_{zw}$ for $K_{mix}$ is always less than or equal to that of the central $H_\infty$

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$| T_{ed} |_\infty, \ Q = 0$</th>
<th>$| T_{zw} |_2, \ Q = 0$</th>
<th>$| T_{ed} |_\infty, \ \text{mix}$</th>
<th>$| T_{zw} |_2, \ \text{mix}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.3012</td>
<td>2.3012</td>
<td>1887.3</td>
<td>2.302</td>
<td>$\sim 1887.3$</td>
</tr>
<tr>
<td>2.32</td>
<td>2.3199</td>
<td>119.129</td>
<td>2.32</td>
<td>115.713</td>
</tr>
<tr>
<td>2.35</td>
<td>2.3495</td>
<td>62.307</td>
<td>2.35</td>
<td>47.619</td>
</tr>
<tr>
<td>2.4</td>
<td>2.3979</td>
<td>37.008</td>
<td>2.4</td>
<td>23.664</td>
</tr>
<tr>
<td>2.5</td>
<td>2.4914</td>
<td>22.155</td>
<td>2.5</td>
<td>10.713</td>
</tr>
<tr>
<td>2.6</td>
<td>2.5807</td>
<td>16.799</td>
<td>2.6</td>
<td>6.9406</td>
</tr>
<tr>
<td>2.75</td>
<td>2.7069</td>
<td>13.11</td>
<td>2.75</td>
<td>4.6072</td>
</tr>
<tr>
<td>3.0</td>
<td>2.8975</td>
<td>10.415</td>
<td>3.0</td>
<td>3.1622</td>
</tr>
<tr>
<td>4.0</td>
<td>3.4578</td>
<td>7.5491</td>
<td>4.0</td>
<td>1.9115</td>
</tr>
<tr>
<td>5.0</td>
<td>3.7978</td>
<td>6.8949</td>
<td>5.0</td>
<td>1.4582</td>
</tr>
<tr>
<td>10.0</td>
<td>4.3708</td>
<td>6.2236</td>
<td>10.0</td>
<td>0.9800</td>
</tr>
<tr>
<td>40.548</td>
<td>4.5874</td>
<td>6.0753</td>
<td>40.548</td>
<td>0.7975</td>
</tr>
<tr>
<td>50.0</td>
<td>4.5925</td>
<td>6.0723</td>
<td>40.548</td>
<td>0.7975</td>
</tr>
<tr>
<td>100.0</td>
<td>4.5997</td>
<td>6.0678</td>
<td>40.548</td>
<td>0.7975</td>
</tr>
</tbody>
</table>

Table 7-2. MIMO Example Full Results
compensator for any value of \( \gamma \). For a value of \( \gamma \) not very far from optimal, the improvement can be dramatic.

It is vital to remember that the compensator designed here is restricted to be the same order as the plant, so no price is paid by using the mixed compensator in that respect. Perhaps the most interesting result here is considering the reverse of the normal objective - design an \( H_2 \) compensator that reduces the infinity norm of \( T_{ed} \), if possible. Figure 7-24 shows that by giving up only a small amount of \( H_2 \)-optimality of \( T_{zw} \), a significant reduction in the infinity-norm of \( T_{ed} \) is possible. In particular, by choosing the \( \gamma = 4 \) case, the infinity-norm of \( T_{ed} \) can be reduced by a factor of ten for only roughly a doubling in the two-norm of \( T_{zw} \). This would produce a compensator which trades off a small amount of \( H_2 \) performance in order to make the closed-loop system ten times as robust as the \( H_2 \) optimal!
CHAPTER 8
CONCLUSIONS AND RECOMMENDATIONS

8.1 Summary

This thesis has developed a synthesis method to design an output feedback controller to minimize the two-norm of one transfer function while ensuring the infinity-norm of another is held below a chosen level. This is known as the general mixed $H_2/H_\infty$ optimization problem, general in that the two transfer functions mentioned above can be completely independent. The solution is the first one presented to the general problem - it also has the added feature that the actual two-norm is minimized, rather than an upper bound to it.

Notationally, the mixed $H_2/H_\infty$ optimization problem is:

**The Mixed $H_2/H_\infty$ Problem** - Find an admissible controller $K(s)$ that achieves

$$\inf_{K \text{ admissible}} \|T_{zw}\|_2$$

subject to the constraint

$$\|T_{ed}\|_\infty \leq \gamma$$

The problem was first transformed into a Lagrange multiplier problem, as follows:

**The Mixed $H_2/H_\infty$ Lagrange Multiplier Problem** - Minimize the Lagrangian

$$L = \text{tr}(Q_2^{T}C_z^{T}C_z) + \text{tr}(\{AQ_2 + Q_2A^T + B_wB_w^T\}X)$$

$$+ \text{tr}(\{AQ_\infty + Q_\inftyA^T + \gamma^2Q_{\infty}C_eC_eQ_{\infty} + B_uB_d^T\}Y)$$

(8.1)

where $Q_2 = Q_2^T \geq 0$ and $Q_\infty = Q_\infty^T \geq 0$
From this, the seven nonlinear matrix necessary conditions were derived, as given in (4.21) - (4.27).

While no analytical solution to these seven equations was found (it is not clear that one even exists), analysis of the equations led to several strong theoretical results, particularly in the case of the controller being of equal order to the plant. Define

\[ \gamma_0 \equiv \inf_{K \text{ admissible}} \| T_{ed} \|_\infty \]

\[ \alpha_0 \equiv \inf_{K \text{ admissible}} \| T_{zw} \|_2 \]

\[ K_{2_{opt}} = \text{the unique } K(s) \text{ that makes } \| T_{zw} \|_2 = \alpha_0 \]

\[ \gamma_2 = \| T_{ed} \|_\infty \text{ when } K(s) = K_{2_{opt}} \]

\[ K_{mix} = \text{a } K(s) \text{ that solves the mixed } H_2/H_\infty \text{ problem at some } \gamma \]

\[ \gamma^* = \| T_{ed} \|_\infty \text{ when } K(s) = K_{mix} \]

\[ \alpha^* = \| T_{zw} \|_2 \text{ when } K(s) = K_{mix} \]

In Theorem 4.2.1, it was shown that for \( \gamma \geq \gamma_2 \) and \( n_c \geq n \), the solution is just the \( H_2 \) optimal controller and thus \( \alpha^* = \alpha_0 \). Theorem 4.2.2 showed that no solution exists for \( \gamma < \gamma_0 \) for any order controller. Finally, in Theorem 4.2.3 it was shown that for \( \gamma_0 < \gamma < \gamma_2 \) and \( n_c = n \), the mixed solution lies on the boundary of the infinity-norm constraint on \( T_{ed} \), so that \( \gamma^* = \gamma \). This led to the result in Theorem 4.2.4; that is, the optimal mixed controller is found through the neutrally stabilizing solution to the Riccati equation

\[ A Q_\infty + Q_\infty A^T + \gamma^2 Q_\infty C^T e C_\infty + B_d B_d^T = 0 \]  \hspace{1cm} (8.2)

Section 2.5 presented numerous theorems on existence of such solutions and methods to compute them. Several examples were given, along with a root-locus view of the problem. Finally, Theorem 4.2.5 showed that it may be possible to achieve further reduction in \( \| T_{zw} \|_2 \) by choosing \( n_c > n \), but \( \alpha_0 \) cannot be achieved for any \( \gamma < \gamma_2 \).
The requirement for a neutrally stable solution to (8.2) creates a problem in solving the last of the necessary conditions, given by

\[ [\mathbf{A} + \gamma^2 Q_{\infty} C_e^T C_e]^T y + y [\mathbf{A} + \gamma^2 Q_{\infty} C_e^T C_e] = 0 \]  \hspace{1cm} (8.3)

Theorems 2.4.4 and 2.4.5 showed that there are infinitely many nonzero solutions to (8.3), of varying ranks. Because of this and the nature of the solution to (8.2), existing computer software is of little help in directly solving the necessary conditions. Therefore, a small constant term was added to (8.3) to push the solution slightly off the boundary. The Lagrangian in (8.1) then becomes

\[ L_\mu = (1 - \mu) \text{tr} [Q_{\infty} C_e^T C_e] + \mu \text{tr} [Q_{\infty} Z] \]
\[ + \text{tr} \{ [\mathbf{A} Q_{\infty} + Q_{\infty} A^T + B_w B_w^T] \mathbf{X} \} \]
\[ + \text{tr} \{ [\mathbf{A} Q_{\infty} + Q_{\infty} A^T + \gamma^2 Q_{\infty} C_e C_e^T Q_{\infty} + B_d B_d^T] y \} \]  \hspace{1cm} (8.4)

and as \( \mu \to 0 \), Theorem 5.1.2 proved that the above converges to the mixed \( H_2/H_\infty \) problem. By choosing

\[ Z = C_e^T C_e \]

in (8.4) with \( \mu \neq 0 \), the problem becomes a convex combination of minimum entropy of \( T_{ed} \) at \( \gamma \) and minimum two-norm of \( T_{zw} \). When \( \mu = 1 \), we have the minimum entropy/\( H_\infty \) problem, for which the solution is the central \( H_\infty \) controller. This provides an acceptable starting point for a numerical solution of the necessary conditions using the Davidon-Fletcher-Powell algorithm. The value of \( \mu \) is then lowered until the solution stops changing, at which point we are arbitrarily close to the mixed \( H_2/H_\infty \) solution.

Two academic examples, one SISO and one MIMO, are developed to analyze the nature of the mixed controller and the potential benefits of using such a controller. It was found that the mixed controller does its best to approximate the \( H_2 \) optimal controller for \( T_{zw} \) in light of the bound on \( \| T_{ed} \|_\infty \). By finding the solution over a wide range of infinity-norm
bound choices, a designer can determine the best mix of $H_2$ and $H_\infty$ performance. It was found in both examples that a significant reduction in either norm was possible by allowing an increase in the other. This trade-off will always occur in the full order case, as the objectives compete and Theorem 5.1.5 shows that the mixed plot is always monotonically decreasing. In many cases, a great deal of improvement in one norm is possible for only a small penalty in the other. This method should allow for the design of compensators which incorporate the desire for good quadratic performance as well as requiring a reasonable level of robustness.

8.2 Directions for Future Work

This work has raised as many or more questions than it has answered. First, to make this method truly practical, especially for large systems, either an analytical solution to the seven necessary conditions needs to be derived (if one exists) or a better form for numerical optimization needs to be developed. For instance, in the work of Bernstein and Haddad ([BH89]), they were able to reduce their problem to a set of coupled Riccati equations, for which homotopic continuation methods ([MB85], [RC90], [Ric87]) are often applicable. These allow for rather fast solutions with fewer numerical difficulties than arise here. As the solution here gets very close to the boundary, the numerics tend to become unstable, since we are looking for neutrally stabilizing solutions to a Riccati equation and for solutions to a Lyapunov equation which does not have a unique solution.

In conjunction with the last comment just made, the question of uniqueness of the solution to the mixed problem needs to be addressed. While the neutrally stabilizing solution of a Riccati equation is unique, a Lyapunov equation with a neutrally stable $A$ matrix and no constant term has an infinite number of nonzero solutions. Since this type of Lyapunov equation appears as one of the necessary conditions, it is not clear whether or not any solution of this equation is acceptable or if only one is the correct choice. The nonuniqueness may characterize nonuniqueness of the overall solution. This topic needs to be addressed, as well as determining whether the mixed problem itself has a unique solution or not. The issue of the existence of local minima using the numerical algorithm presented here also needs to be studied.
Third, the question of optimal order has not been fully addressed. While Lagrange multiplier methods are extremely powerful when solutions of a fixed order are desired, they do not lend themselves to proving what the optimal order of a solution is. As shown in Chapter 6 and Theorem 4.2.5, the optimal order for the mixed $H_2/H_\infty$ optimization problem is, in general, of order higher than the plant, and actually depends on the bound placed on $\| T_{ed} \|_\infty$. Even though the optimal order for a given $\gamma < \gamma_2$ may be infinite, this will not produce $\alpha^* = \alpha_0$, since that is achieved only by the unique $H_2$ optimal compensator which has $\gamma = \gamma_2$. Conjecture 4.2.1, which is about increased order solutions being on the infinity-norm boundary, needs to be proven. Also, the reduced order case needs to be analyzed. Both areas here are currently under investigation.

Fourth, this method needs to be performed on systems for which $z = e$ and $w = d$, to compare with the minimum entropy solution. $Q(s) = 0$ is not the mixed $H_2/H_\infty$ solution, and it needs to be determined how loose an upper bound the entropy is to the two-norm. Directly along these lines, the choice of either $z = e$ or $w = d$, but not both, needs to be examined. This corresponds to the problem of Bernstein and Haddad ([BH89]) or Doyle, et al ([ZDGB90]), where an upper bound to the actual two-norm is minimized. Using the method in this thesis, the tightness of that upper bound can be evaluated. This work is also currently in progress.

Finally, some physically motivated realistic examples need to be developed, to show the true benefit in using this method. This would proceed along the lines of defining $w$ to be white noises, $d$ to be command type signals, $z$ to be control power and other relevant outputs, $e$ to be tracking errors, and so on, as well as $e$ and $d$ being chosen from a robustness point of view. It is hoped that many such examples will be formulated to apply this new methodology to.
REFERENCES


202


APPENDIX A
DIFFERENTIATION

The main results of this thesis involve application of Lagrange multiplier techniques to minimize a constrained optimization problem. This will require the evaluation of partial derivatives of a scalar Lagrangian with respect to certain matrices. Formulas for taking the partials are most easily derived in tensor notation. We have

\[
\frac{\partial \text{tr}(AX)}{\partial X} = \frac{\partial}{\partial X_{kl}} A_{ij} X_{ji} = A_{ij} \frac{\partial X_{ji}}{\partial X_{kl}} = A_{ij} \delta_{lj} \delta_{ii} = A_{ik} = A^T
\]

Using this technique, the following formulae can be easily derived:

\[
\frac{\partial \text{tr}(AX^T)}{\partial X} = A
\]

\[
\frac{\partial \text{tr}(XA)}{\partial X} = A^T
\]

\[
\frac{\partial \text{tr}(X^TA)}{\partial X} = A
\]

\[
\frac{\partial \text{tr}(AXB)}{\partial X} = A^T B^T
\]

\[
\frac{\partial \text{tr}(AX^TB)}{\partial X} = BA
\]

\[
\frac{\partial \text{tr}(AXBXC)}{\partial X} = A^T C^T X^T B^T + B^T X^T A^T C^T
\]

206
\[
\frac{\partial \text{tr}(AXBX^T C)}{\partial X} = A^T C X B^T + C A X B
\]

For more detailed information on computing matrix derivatives, see [AS65] or [Gee76].
APPENDIX B
PROOF OF LEMMA 4.2.1

Lemma 4.2.1: Assume \((A_c, B_c, C_c)\) is minimal and such that \(n_c = n\). Consider minimizing the Lagrangian

\[
L = \text{tr}(Q_2C_z^Tz_z^2) + \text{tr}(A_Q Z_2 + Q_2A^T + B_wB_w^T X) \tag{B.1}
\]

which produces the nine necessary conditions given by

\[
X_{12}^TQ_{12} + X_2Q_2 = 0 \tag{4.42}
\]

\[
X_{12}^TQ_1C^T + X_2Q_{12}^TC^T + X_{12}^TV_12 + X_2B_cV_2 = 0 \tag{4.43}
\]

\[
B^TX_{12}Q_2 + B^TX_{12}Q_2 + R_{12}^TQ_{12} + R_{2C}Q_2 = 0 \tag{4.44}
\]

\[
AQ_1 + Q_1A^T + BC_cQ_{12}^T + Q_{12}C_c^TB^T + V_1 = 0 \tag{4.45}
\]

\[
AQ_{12} + Q_{12}A_c^T + BC_cQ_2 + Q_1C^TB_c^T + V_{12}B_c^T = 0 \tag{4.46}
\]

\[
A_cQ_2 + Q_2A_c^T + B_cQ_{12} + Q_{12}C^TB_c^T + B_cV_2B_c^T = 0 \tag{4.47}
\]

\[
A^TX_1 + X_1A + C^TB_c^TX_{12} + X_{12}B_cC + R_1 = 0 \tag{4.48}
\]

\[
A^TX_{12} + X_{12}A_c + C^TB_c^TX_2 + X_1B_cC + R_{12}C_c = 0 \tag{4.49}
\]

\[
A_c^TX_2 + X_2A_c + C_cB^TX_{12} + X_{12}B_cC + C_cR_{2c}C_c = 0 \tag{4.50}
\]
The set of all admissible minimizing solutions to (4.42)-(4.50) is given by state space transformations of the unique compensator

\[ K_{2_{\text{opt}}} (s) = \tilde{C}_c (sI - \tilde{A}_c)^{-1} \tilde{B}_c \]  

(B.2)

where

\[ \tilde{A}_c = A + B\tilde{C}_c - \tilde{B}_cC \]  

(B.3)

\[ \tilde{B}_c = [Q_o C^T + V_{12} V_2^{-1}] \]  

(B.4)

\[ \tilde{C}_c = -R_2^{-1} [B^T X_o + R_{12}^T] \]  

(B.5)

and \(Q_o\) and \(X_o\) are the unique, positive semidefinite stabilizing solutions to the Riccati equations

\[ 0 = [A - V_{12} V_2^{-1} C] Q_o + Q_o [A - V_{12} V_2^{-1} C]^T + Q_o C^T V_2^{-1} C Q_o + V_1 - V_{12} V_2^{-1} V_{12}^T \]  

(B.6)

\[ 0 = [A - BR_2^{-1} R_{12}^T] X_o + X_o [A - BR_2^{-1} R_{12}^T] - X_o BR_2^{-1} B^T X_o + R_1 - R_{12} R_2^{-1} R_{12} \]  

(B.7)

That is, the set of all solutions is given by

\[ A_c = S^{-1} \tilde{A}_c S \]  

(B.8)

\[ B_c = S^{-1} \tilde{B}_c \]  

(B.9)

\[ C_c = \tilde{C}_c S \]  

(B.10)

where \(S\) is any \(n \times n\) nonsingular matrix.
Proof: Begin by looking at (4.47). Since $Q_2 \geq 0$, from [KJ72] or [Alb69] we can write

$$Q_2 \geq 0 \quad \text{(B.11)}$$

and

$$Q_{12} = Q_{12}Q_2^+Q_2$$

where $Q_2^+$ is the Moore-Penrose pseudoinverse of $Q_2$. The Moore-Penrose pseudoinverse of a matrix $A$ satisfies the four conditions:

i) $AA^+A = A$

ii) $A^+AA^+ = A$

iii) $(AA^+)^* = AA^+$

iv) $(A^+A)^* = A^+A$

$A^+$ is unique, and can be found through a singular value decomposition. Thus, (4.47) can be rewritten as

$$(A_c + B_cCQ_{12}Q_2^+)Q_2 + Q_2(A_c + B_cCQ_{12}Q_2^+)^T + B_cV_2B_c^T = 0 \quad \text{(B.12)}$$

Since $(A_c, B_c, C_c)$ was assumed to be minimal, $(A_c, B_c)$ is controllable, and by [Won85, Lemma 2.1], so is $(A_c + B_cCQ_{12}Q_2^+, B_c)$. This implies that $(A_c + B_cCQ_{12}Q_2^+, B_cD_{YW})$ is also controllable, due to $D_{YW}$ having full row rank. Thus, by the dual of Theorem 2.4.2, (B.11) and (B.12) imply $Q_2 > 0$, and thus $Q_2^{-1}$ exists. A similar argument applied to (4.50) implies that $X_2^{-1}$ exists. Therefore, (4.43) and (4.44) can be written as

$$B_c = -[X_2^{-1}X_{12}^TQ_1C^T + Q_{12}^TC^T + X_2^{-1}X_{12}^TV_{12}IV_2^{-1}] \quad \text{(B.13)}$$
\[ C_c = -R_2^{-1} \left[ B^T X_1 Q_{12} Q_2^{-1} + B^T X_{12} + R_{12}^T Q_{12} Q_2^{-1} \right] \] (B.14)

Rewrite (4.42) as
\[ X_2 Q_2 = -X_{12}^T Q_{12} \] (B.15)

In the full order case, all four matrices in (B.15) are square. Since \( X_2 \) and \( Q_2 \) are full rank, so is their product. Thus, \( X_{12}^T Q_{12} \) has full rank, which implies both \( X_{12} \) and \( Q_{12} \) have full rank.

Without loss of generality, let
\[ Q_{12} = SQ_2 \] (B.16)

where \( S \) is an \( n \times n \) nonsingular matrix. This is always possible, since for any given square, nonsingular \( Q_2 \) and \( Q_{12} \), a square nonsingular \( S \) exists and is given by
\[ S = Q_{12} Q_2^{-1} \]

Substituting (B.16) into (4.42) yields
\[ X_{12}^T SQ_2 + X_2 Q_2 = [X_{12}^T S + X_2] Q_2 = 0 \]

or
\[ X_2 = -X_{12}^T S \] (B.17)

Using (B.16) and (B.17), equations (B.13) and (B.14) become
\[ B_c = [S^{-1} Q_1 C^T - Q_2 S^T C^T + S^{-1} V_{12} V_2^{-1}] \\ = S^{-1} \left[ (Q_1 - SQ_2 S^T) C^T + V_{12} V_2^{-1} \right] \] (B.18)
\[
C_c = -R_2^{-1} \left[ B^T X_1 S - B^T S^{-T} X_2 + R_{12}^T S \right]
- R_2^{-1} \left[ B^T (X_1 - S^{-T} X_2 S^{-1}) + R_{12}^T S \right]
\] (B.19)

Define
\[
Q_o = Q_1 - SQ_2 S^T
\] (B.20)

and
\[
X_o = X_1 - S^{-T} X_2 S^{-1}
\] (B.21)

Notice that both \(Q_o\) and \(X_o\) are symmetric. Then (B.18) and (B.19) become
\[
B_c = S^{-1} \left[ Q_o C^T + V_{12} \right] V_2^{-1}
\] (B.22)

\[
C_c = -R_2^{-1} \left[ B^T X_o + R_{12}^T \right] S
\] (B.23)

Substituting (B.16) into (4.46) and (4.47) yields
\[
A S Q_2 + S Q_2 A_c^T + B C_c Q_2 + Q_1 C^T B_c^T + V_{12} B_c^T = 0
\] (B.24)

\[
S A_c Q_2 + S Q_2 A_c^T + S B_c C S Q_2 + S Q_2 S^T C B_c^T + S B_c V_2 B_c^T = 0
\] (B.25)

where the latter has been premultiplied by \(S\). Now, (B.25) subtracted from (B.24) is
\[
0 = A S Q_2 + B C_c Q_2 + Q_1 C^T B_c^T + V_{12} B_c^T - S A_c Q_2 - S B_c C S Q_2
- S Q_2 S^T C B_c^T - S B_c V_2 B_c^T
\]
\[
= - \left[ S A_c - A S - B C_c + S B_c C S \right] Q_2
+ \left[ (Q_1 - S Q_2 S^T) C^T + V_{12} - S B_c V_2 \right] B_c^T
\]

212
\[
= - [SA_c - AS - BC_c + SB_c CS]Q_2 + [Q_o C^T + V_{12} - SB_c V_2]B_c^T
\]

\[
= SA_c - AS - BC_c + SB_c CS
\]

where the last equality follows from (B.22). Thus,

\[
A_c = S^{-1}AS + S^{-1}BC_c - B_c CS
\]  (B.26)

We now have the controller matrices in terms of \( S, Q_o, \) and \( X_o \). Putting (B.16) into (4.45) yields

\[
0 = AQ_1 + Q_1 A^T + BC_c Q_2 S^T + SQ_2 C_c B_c^T + V_1
\]  (B.27)

Postmultiplying (B.24) by \( S^T \) and subtracting from (B.27) yields

\[
0 = AQ_1 + Q_1 A^T + BC_c Q_2 S^T + SQ_2 C_c B_c^T + V_1 - ASQ_2 S^T - SQ_2 A_c S^T
\]

\[- BC_c Q_2 S^T - Q_1 C_c T B_c S^T - V_{12} B_c S^T \]

\[
= A(Q_1 - SQ_2 S^T) + Q_1 A^T + BC_c Q_2 S^T + SQ_2 C_c B_c^T + V_1 - SQ_2 A_c S^T
\]

\[- BC_c Q_2 S^T - Q_1 C_c T B_c S^T - V_{12} B_c S^T \]

\[
= AQ_o + Q_1 A^T + V_1 - SQ_2 S^T A^T + SQ_2 S^T C_c B_c S^T
\]

\[- Q_1 C_c T B_c S^T - V_{12} B_c S^T \]

\[
= AQ_o + (Q_1 - SQ_2 S^T) A^T - (Q_1 - SQ_2 S^T) C_c T B_c S^T + V_1 - V_{12} B_c S^T
\]

\[
= AQ_o + Q_o A^T + V_1 - [Q_o C^T + V_{12}] B_c S^T
\]
\[
= A Q_o + Q_o A^T + V_1 - \{ Q_o C^T + V_{12} \} V_2^{-1} \{ C Q_o + V_1^T \}
\]

\[
= \{ A - V_{12} V_2^{-1} C \} Q_o + Q_o \{ A - V_{12} V_2^{-1} C \}^T - Q_o C^T V_2^{-1} C Q_o + V_1 - V_{12} V_2^{-1} V_{12}^T \quad (B.28)
\]

Note that (B.28) is solvable for \( Q_o \). Putting (B.20) and (B.23) into (B.27) yields

\[
0 = A (Q_o + S Q_2 S^T) + (Q_o + S Q_2 S^T) A^T - B R_2^{-1} \{ B^T X_o + R_{12}^T \} S Q_2 S^T
\]

\[
- S Q_2 S^T \{ X_o B + R_{12} \} R_2^{-1} B^T + V_1
\]

\[
= A Q_o + Q_o A^T + V_1 + A S Q_2 S^T + S Q_2 S^T A^T
\]

\[
- B R_2^{-1} \{ B^T X_o + R_{12}^T \} S Q_2 S^T - S Q_2 S^T \{ B^T X_o + R_{12}^T \} R_2^{-1} B^T
\]

\[
= [A - B R_2^{-1} (B^T X_o + R_{12}^T)] S Q_2 S^T + S Q_2 S^T [A - B R_2^{-1} (B^T X_o + R_{12}^T)]^T
\]

\[
+ A Q_o + Q_o A^T + V_1
\]

Using (B.28) in the last expression above yields

\[
0 = [A - B R_2^{-1} (B^T X_o + R_{12}^T)] S Q_2 S^T + S Q_2 S^T [A - B R_2^{-1} (B^T X_o + R_{12}^T)]^T
\]

\[
+ \{ Q_o C^T + V_{12} \} V_2^{-1} \{ Q_o C^T + V_{12} \}^T
\]

\[
= [A - B R_2^{-1} (B^T X_o + R_{12}^T)] Q + Q [A - B R_2^{-1} (B^T X_o + R_{12}^T)]^T
\]

\[
+ \{ Q_o C^T + V_{12} \} V_2^{-1} \{ Q_o C^T + V_{12} \}^T \quad (B.29)
\]

where \( Q \equiv S Q_2 S^T \). This is a Lyapunov equation in terms of \( Q \).
Now, put (B.17) into (4.48) - (4.50) to get

\[ 0 = A^T X_1 + X_1 A - C^T B_c^T X_2 S^{-1} - S^{-T} X_2 B_c C + R_1 \]  \hspace{1cm} (B.30) \]

\[ 0 = -A^T S^{-T} X_2 - S^{-T} X_2 A_c + C^T B_c^T X_2 + X_1 B_c C + R_{12} C_c \]  \hspace{1cm} (B.31) \]

\[ 0 = C_c^T X_2 + X_2 A_c - C_c^T B^T S^{-T} X_2 - X_2 S^{-1} B_c C_c + C_c^T R_2 C_c \]  \hspace{1cm} (B.32) \]

Premultiplying (B.32) by \( S^{-T} \), adding to (B.31), and using (B.23) yields the following equation for \( A_c \)

\[ A_c = S^{-1} AS + S^{-1} B_c C - B_c CS \]

which is the same as (B.26). Thus, the set of equations is under-determined. This will be addressed shortly. Next, add (B.30) to (B.31) postmultiplied by \( S^{-1} \) and use (B.21), (B.23), and (B.26) to get

\[ 0 = A^T X_1 + X_1 A - C^T B_c^T X_2 S^{-1} - S^{-T} X_2 B_c C + R_1 - A^T S^{-T} X_2 S^{-1} \]

\[-S^{-T} X_2 A_c S^{-1} + C^T B_c^T X_2 S^{-1} + X_1 B_c C_s S^{-1} + R_{12} C_c S^{-1} \]

\[ = A^T \{ X_1 - S^{-T} X_2 S^{-1} \} + X_1 A - S^{-T} X_2 B_c C + R_1 \]

\[-S^{-T} X_2 \{ S^{-1} A S + S^{-1} B_c C - B_c CS \} S^{-1} + X_1 B_c C_s S^{-1} + R_{12} C_c S^{-1} \]

\[ = A^T X_0 + X_0 A + R_1 + \{ X_0 B + R_{12} \} C_c S^{-1} \]

\[ = A^T X_0 + X_0 A + R_1 - \{ B^T X_0 + R_{12}^T \} R_2^{-1} \{ B^T X_0 + R_{12}^T \} \]

\[ = [ A - BR_2^{-1} R_{12}^T ]^T X_0 + X_0 \{ A - BR_2^{-1} R_{12}^T \} - X_0 BR_2^{-1} B^T X_0 \]

\[ + R_1 - R_{12} R_2^{-1} R_{12} \]  \hspace{1cm} (B.33) \]
This is a Riccati equation solvable for \( X_0 \). Putting (B.21) into (B.30), and using \( (B.22) \) and \( (B.33) \) yields

\[
0 = A^T [X_0 + S^{-T}X_2S^{-1}] + [X_0 + S^{-T}X_2S^{-1}]A - C^T B_c^T X_2 S^{-1} - S^{-T} X_2 B_c C + R_1
\]

\[
= A^T X_0 + A^T S^{-T} X_2 S^{-1} + X_0 A + S^{-T} X_2 S^{-1} A
- C^T V_2^{-1} [CQ_0 + V_1^T S^{-T} X_2 S^{-1} - S^{-T} X_2 S^{-1} [Q_0 C^T + V_{12}] V_2^{-1} C + R_1
\]

\[
= [A - (Q_0 C^T + V_{12}) V_2^{-1} C]^T S^{-T} X_2 S^{-1}
+ S^{-T} X_2 S^{-1} [A - (Q_0 C^T + V_{12}) V_2^{-1} C] + A^T X_0 + X_0 A + R_1
\]

\[
= [A - (Q_0 C^T + V_{12}) V_2^{-1} C]^T H + H [A - (Q_0 C^T + V_{12}) V_2^{-1} C]
+ [B^T X_0 + R_{12}^T R_2^{-1} [B^T X_0 + R_{12}^T]
\]

\[
(B.34)
\]

which is a Lyapunov equation in terms of \( H = S^{-T} X_2 S^{-1} \).

A summary of the pertinent equations (those required to generate a solution) is:

\[
Q_{12} = SQ_2 \quad (B.35)
\]

\[
X_2 = -X_{12} S \quad (B.36)
\]

\[
Q_0 = Q_1 - Q \quad (B.37)
\]

\[
X_0 = X_1 - H \quad (B.38)
\]

\[
Q = SQ_2 S^T \quad (B.39)
\]

\[
H = S^{-T} X_2 S^{-1} \quad (B.40)
\]
\[ A_c = S^{-1}AS + S^{-1}BC_c - B_cCS \quad (B.41) \]

\[ B_c = S^{-1}[Q_0C^T + V_{12}V_2^{-1}] \quad (B.42) \]

\[ C_c = -R_2^{-1}[B^TX_o + R_{12}^T]S \quad (B.43) \]

\[ 0 = [A - V_{12}V_2^{-1}C]Q_o + Q_o[A - V_{12}V_2^{-1}C]^T - Q_oC^TV_2^{-1}CQ_o \]
\[ + V_1 - V_{12}V_2^{-1}V_{12} \quad (B.44) \]

\[ 0 = [A - BR_2^{-1}(B^TX_o + R_{12}^T)]Q + Q[A - BR_2^{-1}(B^TX_o + R_{12}^T)]^T \]
\[ + [Q_oC^T + V_{12}V_2^{-1}[Q_oC^T + V_{12}]^T \quad (B.45) \]

\[ 0 = [A - BR_2^{-1}R_{12}^T]^TX_o + X_o[A - BR_2^{-1}R_{12}^T] - X_oBR_2^{-1}B^TX_o \]
\[ + R_1 - R_{12}R_2^{-1}R_{12} \quad (B.46) \]

\[ 0 = [A - (Q_oC^T + V_{12})V_2^{-1}C]^TH + H[A - (Q_oC^T + V_{12})V_2^{-1}C] \]
\[ + [B^TX_o + R_{12}^T]R_2^{-1}[B^TX_o + R_{12}^T] \quad (B.47) \]

This is a set of 13 matrix equations in 14 unknowns, so there may exist more than one solution. In fact, there are infinitely many solutions to these equations. However, some of these are not of interest, as they are not admissible. To see this, consider only equations (B.44) - (B.47). These are independent of S. The Riccati equations (B.44) and (B.46) in general have many solutions. We now show that only the unique stabilizing solutions to (B.44) and (B.46) lead to a closed-loop stable system. Using (B.42) and (B.43), we can rewrite (B.45) and (B.47) as

\[ 0 = [A + BC_cS^{-1}]Q + Q[A + BC_cS^{-1}]^T + SB_cV_2B_c^TS^T \quad (B.48) \]
Notice that $Q$ and $H$ are positive definite, from (B.39) and (B.40) and the fact that $Q_2$ and $X_2$ are positive definite and $S$ is square with full rank. Also notice from the classic $H_2$ separation principle that the eigenvalues of the "A" matrices of (B.48) and (B.49) are exactly the closed-loop poles. Thus, to stabilize the closed-loop system we want the unique positive definite solutions of (B.48) and (B.49) [which are the same as (B.45) and (B.47)] and the "A" matrices of both equations are stable. Now looking at (B.44) and (B.46), it is clear that we require the unique stabilizing solutions to these two Riccati equations, since the matrices that are guaranteed to be stabilized are exactly the "A" matrices of (B.45) and (B.47). Thus, $Q$, $H$, $Q_0$, and $X_0$ exist and are unique. From (B.37) and (B.38), $Q_1$ and $X_1$ exist and are unique. Choose $S$ to be any $n \times n$ nonsingular matrix, and all of the remaining variables may be uniquely solved for from the remaining equations. Ranging $S$ over all $n \times n$ nonsingular matrices defines all admissible solutions to the set of equations. The key is to notice that $S$ is just a similarity transformation of the compensator matrices. This is seen by looking at the set of matrices $(SA_cS^{-1}, SB_c, C_cS^{-1})$ from equations (B.41) - (B.43). These are unique, and are just state space transformations of each other for different $S$ matrices. Thus, the set of all compensators that satisfy the necessary conditions and stabilize the closed-loop system are just state space transformations of the same unique compensator, which is $K_{2\text{opt}}$. 


APPENDIX C
COMPUTER CODE

This appendix contains the PRO-MATLAB™ script and m-files to solve the mixed $H_2/H_\infty$ problem using Davidon-Fletcher-Powell optimization. A Fortran version has been found to run much faster - conversion of this routine to Fortran is all that is required. Note that this will require Riccati and Lyapunov equation solver subroutines, which are available.

The code begins on the following page.
Main script file for doing H2/H-infinity optimization using the
Davidon-Fletcher-Powell algorithm. The mat-file named "user_chosen_name" must contain the following variables:
The plant matrices: a, bd, bw, bu, ce, cz, cy, deu, dzu, dyd, dyw.
An admissible initial guess of the controller matrices: ac, bc, cc
The number of states, inputs, and outputs of the plant: ns, nu, ny
The order of the controller: nc
The desired value of gamma: gam
The desired value of mu: mu
The tolerance of the convergence check: tol
The maximum number of iterations to be run: niter
The string name of the mat-file containing this data: source
(that is, a variable source whose value is "user_chosen_name")
It cannot contain the variable xmat!
The program is started by typing source = 'user_chosen_name'
and then typing dfp

Load data

eval(['load ',source])
format short e

Put initial guess of ac, bc, cc into column vector named xmat

for i = 1:nc,
    for j = 1:nc,
        xmat((i-1)*nc + j,1) = ac(i,j);
    end;
end;

nadd = nc*nc;
for i = 1:nc,
    for j = 1:ny,
        xmat((i-1)*ny + j + nadd,1) = bc(i,j);
    end;
end;
nadd = nadd + nc*ny;
for i = 1:nu,
    for j = 1:nc,
        xmat((i-1)*nc + j + nadd,1) = cc(i,j);
    end;
end;

221
% Call m-file to evaluate the derivatives w.r.t. to ac, bc, and cc at the%
% initial guess - return result as column vector delf%

delf = evdelf(xmat,source);
%
% Initialize h = identity, set initial s and counters to zero%
numb = nc*nc + nc*ny + nc*nu;
h = eye(numb,numb);
s = -h*delf;
iflag = 0;
icount = 0;
%
% Begin main computations%

while (iflag<1)
  % Call m-file to find step size (kappa)* to minimize function for current
  % xmat and s
  kapstar = findkap(xmat,s,source)
  % Save old delf, compute new xmat based on (kappa)*
  delfold = delf;
  xmat = xmat + kapstar*s;
  % Call m-file to compute & print new two-norm
  nm2mixcur = evalu2(xmat,source)
  % Evaluate derivatives at new guess
  delf = evdelf(xmat,source)
  % Compute new s matrix based on DFP algorithm
  ymat = delf - delfold;
  mmat = kapstar*(s'*s)/(s'*ymat);
  nmat = -(h'*ymat)*(h'*ymat)/(ymat'*h'*ymat);
  h = h + mmat + nmat;
  s = -h*delf;
\%
\% If new \textbf{h} is not positive definite, reset \textbf{h} to identity
\%
if (s'\textbf{delf}>0),
  'had to update \textbf{h}'
  \textbf{h} = \textbf{eye(numb, numb)};
  s = -\textbf{h}'\textbf{delf};
end;
\%
\% Update counter and store current \textbf{xmat} in \textbf{temp}
\%
  icount = icount + 1
  save temp \textbf{xmat}
\%
\% Check if iteration limit is exceeded
\%
if (icount>niter),
  'did not converge in allowed iterations'
  iflag = 1;
end;
\%
\% Check if converged
\%
  \textbf{chkstop} = \textbf{delf}'\textbf{h}'\textbf{delf}
  if (\textbf{chkstop}<\textbf{tol}),
    iflag = 1;
  end;
end;
\%
\% Take final results from \textbf{xmat} and put into new \textbf{ac}, \textbf{bc}, \textbf{cc}
\%
for \textbf{i}=1:nc,
  for \textbf{j}=1:nc,
    \textbf{ac}(i,j)=\textbf{xmat}((i-1)*nc+j,1);
  end;
end;
\textbf{nadd}=nc*nc;
for \textbf{i}=1:nc,
  for \textbf{j}=1:ny,
    \textbf{bc}(i,j)=\textbf{xmat}((i-1)*ny+j+nadd,1);
  end;
end;
\textbf{nadd}=\textbf{nadd}+nc*ny;
for i=1:nu,
    for j=1:nc,
        cc(i,j)=xmat((i-1)*nc+j+nadd,1);
    end;
end;
%
% Clear extra variables
%
clear chkstop; clear xmat; clear nadd; clear kapstar; clear delf; clear delfold;
clear iflag; clear icount; clear numb; clear nadd; clear nm2mixcur;
clear nmat; clear mmat; clear h; clear ymat; clear i; clear j; clear s;
%
% Save final results back into source
%
eval(['save ',source])
%
% End of program
function X = are2(F,G,H)
% ARE2  X = are2(F, G, H) returns the stabilizing solution (if it
% exists) to the continuous-time Riccati equation:
%  
% F*X + X*F - X*G*X + H = 0
%  
% assuming G and H are symmetric. This is a minor modification of the
% MATLAB Control Toolbox routine which returns X = 1e+10*eye if no
% stabilizing solution exists rather than an error message.
%
% ARE m-file Copyright (c) 1987-88 by The MathWorks, Inc.
%
% check for correct input problem
%  
[nr,nc] = size(F); n = nr;
if (nr ~= nc), error('Nonsquare F matrix'), end;
[nr,nc] = size(G);
if (nr~=n | nc~=n), error('Incorrectly dimensioned G matrix'), end;
[nr,nc] = size(H);
if (nr~=n | nc~=n), error('Incorrectly dimensioned H matrix'), end;
%
% Find Schur decomposition of Hamiltonian
%  
[q,t] = schur([F -G; -H -F']*(1.0+eps*eps*sqrt(-1)));
tol = 10.0*eps*max(abs(diag(t)));  % ad hoc tolerance
ns = 0;
%
% Prepare an array called index to send message to ordering routine
% giving location of eigenvalues with respect to the imaginary axis.
% -1  denotes open left-half-plane
% 1   denotes open right-half-plane
% 0   denotes within tol of imaginary axis
%  
for i = 1:2*n,
   if (real(t(i,i)) < -tol),
       index = [ index -1 ];
       ns = ns + 1;
   elseif (real(t(i,i)) > tol),
       index = [ index 1 ];
   else,
       index = [ index 0 ];
   end;
end;

225
Compute solution to ARE - if there are eigenvalues of the Hamiltonian on the imaginary axis, return \( X = 1e+10 \) rather than error.

\[
\text{if}(\text{ns} \neq \text{n}), \\
X = 1e+10 \cdot \text{eye}(\text{n}, \text{n});
\]
\[
\text{else,} \\
[q, t] = \text{schord}(q, t, \text{index}); \\
X = \text{real}(q(n+1:n+n, 1:n)/q(1:n, 1:n));
\]
\[
\text{end};
\]

% End of file

%
function [x] = evalu2(xmat,source)
%
% [x] = evalu2(xmat,source)
%
% M-file for two-norm evaluation in H2/H-infinity optimization using the
% Davidon-Fletcher-Powell algorithm.
%
% Load data
%
eval(['load ',source])
%
% Take xmat and form ac, bc, cc
%
for i = 1:nc,
    for j = 1:nc,
        ac(i,j) = xmat((i-1)*nc + j,1);
    end;
end;

nadd = nc*nc;
for i = 1:nc,
    for j = 1:ny,
        bc(i,j) = xmat((i-1)*ny + j + nadd,1);
    end;
end;

nadd = nadd + nc*ny;
for i = 1:nu,
    for j = 1:nc,
        cc(i,j) = xmat((i-1)*c + j + nadd,1);
    end;
end;
%
% Form state space matrices of the closed-loop system from w to z and find
% two-norm
%
b = bu; c = cy;
ati1 = [a b*cc;bc*c ac]; bwt1 = [bw;bc*dyw]; czti1 = [cz du*cc];
q2ti1 = lyap(ati1,bwt1*bwt1');
x = sqrt(trace(q2ti1*cti1*cti1));
%
% End of file
%
function [x] = evaluf(xmat,source)
  %
  % [x] = evaluf(xmat,source)
  %
  % M-file for function evaluation in H2/H-infinity optimization using the
  % Davidon-Fletcher-Powell algorithm. The function x is the Lagrangian.
  
  % Load data
  %
  eval(['load ',source])
  %
  % Take xmat and form ac, bc, cc
  %
  for i = 1:nc,
    for j = 1:nc,
      ac(i,j) = xmat((i-1)*nc + j,1);
    end;
  end;
  nadd = nc*nc;
  for i = 1:nc,
    for j = 1:ny,
      bc(i,j) = xmat((i-1)*ny + j + nadd,1);
    end;
  end;
  nadd = nadd + nc*ny;
  for i = 1:nu,
    for j = 1:nc,
      cc(i,j) = xmat((i-1)*nc + j + nadd,1);
    end;
  end;

  % Form state space matrices of the closed-loop system
  %
  b = bu;
  c = cy;
  atil = [a b*cc;bc*c ac];
  bdtil = [bd;bc*dyd];
  bwtil = [bw;bc*dyw];
  cetil = [ce deu*cc];
  cztil = [cz dzu*cc];
% Solve for gramians, H-infinity Riccati solution, form stabilized matrix, % and compute eigenvalues of these solutions
%
g = 1/(gam^2);
q2til = lyap(atil,bwtil*bwtil');
xtil = lyap(atil',(1-mu)*cztil'*cztil);
qinf = are2(atil',-g*ctil'*ctil,bdti*bdti');
ast = atil + g*qinf*ctil'*ctil;
ei = eig(ast);
eq2 = eig(q2til);
eqi = eig(qinf);
ex2 = eig(xtil);
%
% Compute value of the Lagrangian - reset to 1e+10 if any of the solutions % above were unacceptable
%
x = (1-mu)*trace(q2til*ctil'*ctil) + mu*trace(qinf*ctil'*ctil);
if (max(real(ei))>=1e-8),
    x = 1e+10;
end;
if (min(real(eqi))<=1e-10),
    x = 1e+10;
end;
if (min(real(eq2))<=1e-10),
    x = 1e+10;
end;
if (min(real(ex2))<=1e-10),
    x = 1e+10;
end;
%
% End of file
%
229
function [x] = evdelf(xmat,source)
%
% [x] = evdelf(xmat,source)
%
% M-file for derivative evaluation in H2/H-infinity optimization using the
% Davidon-Fletcher-Powell algorithm. The vector of derivatives is
% returned.
%
% Load data
%
eval(['load ',source])
%
% Take xmat and form ac, bc, cc
%
for i = 1:nc,
    for j = 1:nc,
        ac(i,j) = xmat((i-1)*nc + j,1);
    end;
end;

nadd = nc*nc;
for i = 1:nc,
    for j = 1:ny,
        bc(i,j) = xmat((i-1)*ny + j + nadd,1);
    end;
end;

nadd = nadd + nc*ny;
for i = 1:nu,
    for j = 1:nc,
        cc(i,j) = xmat((i-1)*nc + j + nadd,1);
    end;
end;

% Form state space matrices of the closed-loop system
%
b = bu;
c = cy;

atil = [a b*cc;bc*c ac];
bdt1l = [bd;bc*dyd];
bwtil = [bw;bc*dyw];
cetil = [ce deu*cc];

czt1l = [cz dzu*cc];
% Solve for gramians, H-infinity Riccati solution, form stabilized matrix, 
% and compute Y Lagrange multiplier 
%
g = 1/(gam^2);
q2til=lyap(ati1,bwtil*bwtil');
xtil=lyap(ati1,(1-mu)*czt1l'*czt1l);
qinft1l=are(ati1,-g*cet1l'*cet1l,bdtil'*bdtil');
ast=ati1+g*qinft1l*cet1l'*cet1l;
ytil=lyap(ast1,mu*cet1l'*cet1l);
%
% Split above solutions into block components and form V's and R's
%
[q1,q12,q21,q2]=split(q2til,dims(1));
[xl,x12,x21,x2]=split(xtil,dims(1));
[qa,qab,qba,gb]=split(qinft1l,dims(1));
[y1,y12,y21,y2]=split(ytil,dims(1));
v1=bd*dv';
v2=dyd';
v1=bd*bw';
v2=dyw';
va=ce'*ce;
rab=ce'*deu;
rb=deu'*ce;
r2=duz';
%
% Compute gradients w.r.t. ac, bc, and cc
%
eqn1 = x21*q12 + x12'*q21' + x2*q2 + x2'*q2' + ... 
y21*qab + y12'*qba' + y2*qb + y2'*qb';
eqn2 = x21*q1*c' + x12'*q1' + x21*v12 + x12'*v12 + x2'*q12'*c' + ... 
x2'*q12';
eqn2 = eqn2 + x2*bc*v2 + x2*cc*v2 + y21*qa*c' + y12'*qa'*c' + y12'*vab;
eqn2 = eqn2 + y21*vab + y2*qab*c' + y2*xba*c' + y2*bc*vb + y2*bc*vb;
mp = rab*qa*y1*qab + rab*qa'*y1'*qba' + rb*cc*qba*y1*qab;
mp = mp + rb*cc*qba*y1'*qba' + rab*qa*y12*qb + rab*qa'*y21'*qb;
mp = mp + rb*cc*qba*y21*qab + rb*cc*qba'y12*qab + rb*cc*qba'y12*qab;
mp = mp + rb*cc*qba'y21*qab + rb*cc*qba'y21*qab + rb*cc*qba'y12*qab;
tmp = tmp + rb*cc*qb'*y2'*qb';
eqn3 = (1-mu)*r12'*q12 + (1-mu)*r12'*q21' + (1-mu)*r2*cc*q2 + ...
(1-mu)*r2*cc*q2';
eqn3 = eqn3 + mu*rab'*qab + mu*rab'*qa' + mu*rb*cc*qb + mu*rb*cc*qb';
eqn3 = eqn3 + b'*x1'*q21' + b'*x1'*q12 + b'*x2'*q2' + b'*x12'*q2;
eqn3 = eqn3 + b'*y1'*qba' + b'*y1'*qab + b'*y21'*qb' + b'*y12'*qb + g*tmp;

% Stack above results into column vector, which is the final result
%
for i = 1:nc,
    for j = 1:nc,
        x((i-1)*nc + j,1) = eqn1(i,j);
    end;
end;
nadd = nc*nc;
for i = 1:nc,
    for j = 1:ny,
        x((i-1)*ny + j + nadd,1) = eqn2(i,j);
    end;
end;
nadd = nadd + nc*ny;
for i = 1:nu,
    for j = 1:nc,
        x((i-1)*nc + j + nadd,1) = eqn3(i,j);
    end;
end;

% End of file

232
function x = findkap(xmat,s,source)
%
% [x] = evdelf(xmat,source)
%
% M-file for finding (kappa)* in H2/H-infinty optimization using the
% Davidon-Fletcher-Powell algorithm.
%
% Store original xmat
%
xmatold = xmat;
%
% Set the search tolerance and initial guesses for kappa (these are ad-hoc)
%
tolser = 1e-3;
kappa1 = 0;
kappa2 = 0.0001;
kappa3 = 0.0002;
%
% Begin search - terminate if normalized difference is less than tolser
%
while ((kappa2 - kappa1)/kappa2 > tolser),
%
% Display current guesses, evaluate function (Lagrangian) at them, and
% display resulting function values
%
    cur_step = [kappa1,kappa2,kappa3]
    xmat = xmatold + kappa1*s;
    x1 = evaluf(xmat,source);
    xmat = xmatold + kappa2*s;
    x2 = evaluf(xmat,source);
    xmat = xmatold + kappa3*s;
    x3 = evaluf(xmat,source);
    cur_val = [x1,x2,x3]
%
% Check if function at first kappa is less than second; if so, expand search
% to between first and second values and restart
%
    if (x1 < x2),
        kappa3 = kappa2;
        kappa2 = (kappa2 + kappa1)/2;
    else
        kappa1 = kappa2;
        kappa2 = kappa3;
        kappa3 = (kappa3 + kappa1)/2;
    end
    if (x1 < x2),
        kappa3 = kappa2;
        kappa2 = (kappa2 + kappa1)/2;
    end
end

233
% Check if function at second kappa is less than third and less than or
% equal to the first; if so, compute and check midpoints, called kappa4 and
% kappa5
%
else,
  if (x2 < x3),
    kappa4 = (kappa2 - kappa1) / 2 + kappa1;
    xmat = xmatold + kappa4 * s;
    x4 = evalf(xmat, source);
    kappa5 = (kappa3 - kappa2) / 2 + kappa2;
    xmat = xmatold + kappa5 * s;
    x5 = evalf(xmat, source);
%
% Check if min lies between kappa1 and kappa2; if so, redefine search area
% to be from kappa1 to kappa2 and restart
%
  if (x4 < x2),
    kappa3 = kappa2;
    kappa2 = kappa4;
%
% Check if min lies between kappa2 and kappa3; if so, redefine search area
% to be from kappa2 to kappa3 and restart
%
  else,
    if (x5 < x2),
      kappa1 = kappa2;
      kappa2 = kappa5;
%
% If above two fail, redefine search area to be from kappa4 to kappa5 and
% restart
%
  else,
    kappa1 = kappa4;
    kappa3 = kappa5;
  end;
end;
%
% If kappa3 produced lowest of three original function values, double
% length of search area and restart
%
else,
  kappa2 = kappa3
  kappa3 = 2 * kappa3
end;
end;
end;
%
% Once difference in kappa's is small, find best value
%
xmat = xmatold + kappa1*s;
x1 = evaluf(xmat,source);
xmat = xmatold + kappa2*s;
x2 = evaluf(xmat,source);
xmat = xmatold + kappa3*s;
x3 = evaluf(xmat,source);
if (x1<x2),
  if (x1<x3),
    x = kappa1;
  end;
end;
if (x2<=x1),
  if (x2<x3),
    x = kappa2;
  end;
end;
if (x3<=x1),
  if (x3<=x2),
    x = kappa3;
  end;
end;
xmat = xmatold;
%
% End of file
%