Operations in Complex-Oriented Cohomology Theories Related to Subgroups of Formal Groups

by

Matthew Ando

Submitted to the Department of Mathematics in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Mathematics at the Massachusetts Institute of Technology June 1992

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ABSTRACT

Using the character theory of Hopkins-Kuhn-Ravenel and the total power operation
in complex cobordism of tom Dieck, we develop a theory of power operations in
Landweber-exact cohomology theories. We give a description of the total power
operation in terms of the theory of subgoups of formal group laws developed by
Lubin.

We apply this machinery in two cases. For the cohomology theory $E_h$, we obtain a
formal-group theoretic condition on the orientation which is an obstruction to the
compatibility of $H_\infty$ structures in $MU$ and $E_h$. We show that there is a unique choice
of orientation for which this obstruction vanishes, allowing us to build a large family
of unstable cohomology operations based on $E_h$. We show that the the multiplicative
formal group law of $K$-theory satisfies our condition.

In elliptic cohomology, our machinery is naturally related to quotients of elliptic curves
by finite subgroups. We adapt our machinery to elliptic cohomology, and produce
both the Adams operation and versions of the Hecke operators of Baker as power
operations.

Thesis Supervisor: Haynes Miller
Title: Professor of Mathematics
Acknowledgments

At a time like this, it is natural to experience gratitude in two forms, mathematical and personal. In my case the overlap is substantial, which may or may not be a good thing, but at least makes for a shorter Acknowledgments section.

I am very grateful to my advisor, Haynes Miller, for his generosity and his patience. The idea for this project came out of a course he taught my first year here. Most of the ideas he's suggested to me have yet to be studied seriously, and if I can remember just a few, I'll be in good shape for several more years.

By now, Mike Hopkins has been my teacher for five years, poor guy. My debt to him runs throughout this paper, both in citations and in the mathematics itself. Theorem 4.0.2, especially, came into being in consultation with Mike, and I never would have started on this project without [HKR91].

If the credit for the inception of this thesis goes to Haynes's course and [HKR91], then at least some of the credit for its completion goes to Dan Kan, who told me to start writing in August, which was barely in time but at least three months before I might otherwise have started. I benefitted from his counsel on several crucial occasions, and I never had to take time out of my day otherwise to do so.

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Looking at my work over the last few years, it's clear that I made a lot of progress this year. Everyone who knows me well knows how much Amy helped me. Only Amy and I know how difficult I was about it.

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1 Introduction

Recent work in topology [DHS88, Hop87] and in mathematical physics [Wit88, BT89] has focused attention on a collection of complex-oriented cohomology theories that includes $K$-theory, elliptic cohomology [Lan88, LRS88, Seg88], and various $E_h$. With the exception of $K$-theory, these theories are most easily defined using an algebraic technique due to Landweber [Lan76], and one of the fundamental questions in the subject is how to render geometric or analytic descriptions of these theories.

The algebraic description begins with a "genus", that is, a ring homomorphism

$$MU^* \xrightarrow{t_E} E^*,$$  \hspace{1cm} (1.0.1)

where $MU^*$ is the cobordism ring of stably almost-complex manifolds. Landweber's theorem [Lan76] is a sufficient criterion on a genus for the resulting functor

$$X \mapsto E^* \otimes_{MU^*} X$$  \hspace{1cm} (1.0.2)

to be a cohomology theory on finite complexes. Because of the Mayer-Vietoris axiom, this amounts to showing that tensoring with $E^*$ via the genus $t_E$ is exact for $MU^*$ modules of the form $MU^* X$, where $X$ is a finite complex. For this reason, a cohomology theory determined by (1.0.2) is called "exact."

Knowing only the description (1.0.2) is a little like having a Greek-Greek dictionary when one speaks only English: given such a theory $E$ and a space $X$, one can attempt to compute its cohomology groups, but then would have very little idea what has been accomplished when the task is finished.

The paradigm for this problem is $K$-theory. The genus associated to $K$-theory is the Todd genus

$$MU^* \xrightarrow{\text{Td}} K^*.$$

A theorem of Conner and Floyd tells us that

**Theorem 1.0.3 ([CF66, Lan76])** The map

$$K^* \otimes_{MU^*} MU^* X \xrightarrow{1_X} K^* X$$  \hspace{1cm} (1.0.4)

is an isomorphism when $X$ is a finite complex.

The left-hand-side of (1.0.4) is the description of $K$-theory provided by exactness, and the proof of (1.0.4) using Landweber’s theorem makes no use of vector bundles. For the theories $Ell$ and $E_h$, it’s as if we knew of the existence of $K$-theory, but not about its relationship to vector bundles.
One thorny problem in understanding the geometry of a theory like $E_h$ is the choice of the genus itself. In stable homotopy, this problem is usually studied in terms of the associated group law: by Quillen's theorem, specifying a genus

$$MU^* \to E_h^*$$

is equivalent to specifying a formal group law $F$ over $E^*$. More precisely, there is a formal group law $F^{MU}$ over $MU^*$, and $MU^*$ together with the formal group law $F^{MU}$ represents functor

$$R \xrightarrow{FGL} \{ \text{Formal group laws over } R \}$$

by the correspondence

$$\text{Hom}_{\text{ring}}[MU^*, R] \to FGL(R)$$

$$f \mapsto f_* F^{MU}.$$

The ring $E_h^*$ is a complete, local ring. The usual genus

$$MU^* \xrightarrow{t_1} E_h^*$$

over $E_h$ results in a formal group law $F_h$ whose reduction modulo the maximal ideal we call $\varphi$. It turns out that any formal group law $F$ over $E_h^*$ which reduces modulo the maximal ideal to $\varphi$ detects the same information in stable homotopy. The formal group law $F_h$ has the feature that it is "$p$-typical", and is an obvious choice to make when $E_h$ is constructed from $BP$. However, geometric considerations, when they are available, lead to other genera.

For example, $E_1$ is $p$-adic $K$-theory. With respect to the genus $t_1$ of [Rav86] (see section 5.1), the $p$-series of the resulting formal group law $F_1$ satisfies

$$[p]_{F_1}(x) = px + u^{p-1}x^p \quad (\text{mod degree } p+1).$$

From the point of view of geometry, however, the most natural formal group law is the multiplicative formal group law $G_m$, which is the group law classified by the Todd genus. It satisfies

$$[p]_{G_m}(x) = px + u \left( \frac{p}{2} \right) x^2 + \ldots + u^{p-1}x^p.$$

If we were to have any hope, starting with just the description (1.0.2), of discovering the relationship between $E_1$ and vector bundles, we would surely have needed to start our search with the Todd genus and the multiplicative formal group law!

The main result of this paper is the description of a new canonical genus for the cohomology theories $E_h$. In the case that $h = 1$, we recover the multiplicative group law over $p$-adic $K$-theory.
Let $F$ be any formal group law over $E_h$ which reduces modulo the maximal ideal to $\varphi$. Over any complete, local $E_h^\ast$-algebra $R$, the elements of the maximal ideal of $R$, together with the addition law specified by the formal group law, form a group, which we denote $F(R^\ast)$.

Let $D^\ast$ be the smallest extension of $E_h^\ast$ which contains the roots of the $p^k$-series for all $k > 0$. Then ([Hop91]; see 2.3) $D^\ast$ is a Galois extension of $E^\ast$ with Galois group $\text{Aut}_{\text{gp}}[F(D^\ast)_{\text{tors}}]$. Let $f_p(x) \in D^\ast[[x]]$ be the power series

$$f_p(x) = \prod_{v \in F(D^\ast) \mid \frac{v}{p}(v) = 0} (v + x).$$

By Galois invariance, the power series $f_p$ in fact has coefficients in $E_h^\ast$. It is the power series considered by Lubin in his study of finite subgroups of formal groups, in the case that the subgroup is the $p$-torsion points of $F(D^\ast)$. Associated to it is a formal group law $F/p$ which is related to $F$ by the homomorphism of formal group laws

$$F \xrightarrow{f_p} F/p.$$ 

$f_p$ and $F/p$ are constructed so that

$$pF(D^\ast) = \text{Ker}[F(D^\ast) \xrightarrow{f_p} F/p(D^\ast)].$$

On the other hand, the $p$-series $[p]F(x)$ an endomorphism of $F$, and certainly

$$pF(D^\ast) = \text{Ker}[F(D^\ast) \xrightarrow{[p]} F(D^\ast)].$$

Let $u$ denote the periodicity element of $E_h$; we use a version of $E_h$ in which this element has degree $-2$ (5.1). Our genus is given by

**Theorem A** (see section 5.4) There is a unique genus

$$MU^\ast \xrightarrow{\text{flip}} E_h^\ast$$

such that the resulting formal group law is $\ast$-isomorphic to $F_h$ and satisfies

$$[p](x) = u^{p-1} f_p(x).$$

(1.0.5)

The reason why we hope that the genus described in Theorem A is related to geometry is its relationship to “power operations”, which we now describe.

Once one has a good geometric understanding of a cohomology theory, one expects to find rich additional structure in these cohomology theories coming from the geometry. In particular, it is natural to examine symmetries of the geometry in search of cohomology operations. This thesis represents an attempt to turn this idea on its
head, and to study cohomology operations in exact cohomology theories as a means of learning something about their conjectural geometry.

"Power operations" is the name of a program for constructing cohomology operations in a ring theory $E$. The reason for singling this method out for investigation is that on the one hand, one can describe in purely algebraic terms what it is one would like to do, while on the other, in every cohomology theory where power operations have been successfully constructed, the construction relies heavily on a good description of the cohomology theory. Some examples are mod-$p$ cohomology, in which one obtains the Steenrod operations $\{\text{SE}62\}$, $K$-theory, where one obtains the exterior powers and Adams operations $\{\text{Ati}66\}$, and complex cobordism $\{\text{tDi}68\}$, where Quillen used them to prove his theorem about $MU^*$ $\{\text{Qui}71\}$.

The basic problem in constructing power operations is as follows. Let $E$ be a cohomology theory with products, and let $S_n$ denote the symmetric group on $n$ letters. For any space $X$, $S_n$ acts on $X^n$. One wants to construct a "total power operation" $P_n^E$ which factors the $n^{th}$ power map

\[ E^n X \xrightarrow{\times^n} E^n X^n \]

through a suitable equivariant cohomology theory $E_{S_n}$. If no better candidate is available, one simply takes $E_{S_n}^*(X^n)$ to be $E^*(D_n X)$, where $D_n X$ is the Borel construction

\[ D_n X = ES_n \times X^n. \]

In other words, the idea is to find a natural transformation $P_n^E$ filling in the diagram

\[ E^n X \xrightarrow{\times^n} E^n X^n \]

\[ \downarrow P_n^E \]

\[ \downarrow \quad i^* \quad \]

\[ E_{S_n}^n(X^n), \]

where $i^*$ is the forgetful map

\[ E_{S_n}^n(X^n) \xrightarrow{i^*} E^*(X^n). \]

In fact it is often enough to construct, for $\pi$ an abelian $p$-group of order $n$, a "total power operation based on $\pi$"

\[ E^n X \xrightarrow{P_{\pi}^E} E^n D_{\pi} X, \]

where $D_{\pi} X$ is the Borel construction

\[ D_{\pi} X = E_{\pi} \times X_{\pi}, \]
and

\[ X^\pi = \text{Maps}[\pi, X] \]

denotes the left \( \pi \)-space obtained by using the right action of \( \pi \) on itself.

In any case, with \( P_n^E \) in hand, one proceeds to produce natural transformations

\[ E_{S_n}^*(X^n) \overset{\theta}{\to} E^*X; \]

then the composite

\[ E^*X \overset{P_n^E}{\to} E_{S_n}^*(X^n) \overset{\theta}{\to} E^*X \]

is an operation on \( E \).

For example, Atiyah [Ati66] constructed a total power operation

\[ KX \overset{P_n^K}{\to} K_{S_n}(X^n). \]

If

\[ X \xrightarrow{\Delta} X^n \]

is the diagonal map, then \( \Delta^*P_n^K \) lands in \( K_{S_n}X \), which is isomorphic to \( RS_n \otimes KX : \)

\[ KX \overset{P_n^K}{\to} K_{S_n}(X^n) \overset{\Delta^*}{\to} K_{S_n}(X) \cong RS_n \otimes KX. \]

Evaluation of characters on the class of an \( n \)-cycle in \( S_n \) yields a ring homomorphism

\[ RS_n \overset{\text{eval}}{\to} \mathbb{Z}, \]

and the composite

\[ KX \overset{P_n^K}{\to} K_{S_n}(X^n) \overset{\Delta^*}{\to} RS_n \otimes KX \overset{\text{eval}}{\to} KX \]

turns out to be the (integral!) Adams operation \( \Psi^n \) on \( KX \).

It is a hard problem in general to construct the total power operation \( P_n \) in an arbitrary cohomology theory (see e.g. [BMMS86]), primarily because it has never been clear what \( P_n \) should mean in any generality. One approach is to try to use tom Dieck’s [tDi68, Qui71] construction of a total power operation

\[ MU^{2*}X \overset{P_n^{MU}}{\to} MU^{2n*}(D_nX) \]

for complex cobordism. Then for a cohomology theory \( E \) with a complex orientation

\[ MU \overset{\iota_E}{\to} E, \]

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(e.g. any exact theory), one might ask whether a putative total power operation $P^n$ be compatible in the sense that the diagram

$$
\begin{array}{ccc}
MU^{2n}X & \xrightarrow{P_{MU}} & MU^{2n}(D_nX) \\
\downarrow^{t_E} & & \downarrow^{t_E} \\
E^{2n} & \xrightarrow{P^n} & E^{2n}(D_nX)
\end{array}
$$

commutes. If moreover $E$ is exact, then it is determined by $MU$ via $t_E$ (1.0.2), so $P^n$ is determined by the diagram (1.0.8) if it exists.

This optimism is supported by the example of $K$-theory: tom Dieck’s operation $P^n_{MU}$ and Atiyah’s operation $P^n_K$ are compatible under the Atiyah-Bott-Shapiro orientation: if

$$MU^{2n}X \xrightarrow{t_{ABS}} KX$$

is the Atiyah-Bott-Shapiro orientation, then

$$
\begin{array}{ccc}
MU^{2n}X & \xrightarrow{P^n_{MU}} & MU^{2n}(D_nX) \\
\downarrow^{t_{ABS}} & & \downarrow^{t_{ABS}} \\
KX & \xrightarrow{P^n_K} & K(D_nX)
\end{array}
$$

commutes [tDi68].

In practice, though, it is a tricky matter to carry this program out, for example because $P^n_{MU}$ isn’t additive (2.1.5), and so one doesn’t expect it to behave well under tensor products like (1.0.2).

### 1.1 Splitting the total power operation via character theory

Our idea is to appeal at this point to the character theory of Hopkins-Kuhn-Ravenel [HKR91]. To focus the discussion, suppose that $E^*$ is a Noetherian local domain, complete with respect to its maximal ideal $m$. Suppose that the residue characteristic of $E^*$ is $p$, and that the formal group law of $E^*$ has height $h$. Once again, let $D^*$ be the ring (see section 2.3) obtained from $E^*$ by adjoining the roots of the $p^k$-series for all $k$. For every subgroup

$$H \subset F(D^*)$$

of order $n$, we use [HKR91] in section 3.1 to construct a character map

$$E^*(D_nX) \xrightarrow{X_H} D^* \otimes E^* X.$$  

(1.1.1)
The $\chi_H$ are the building blocks of a "total character map" which detects $E^*(D_nX)$ rationally (for more on the total character map, see section 3.3). It turns out that $D^*$ is faithfully flat over $E^*$ ([Hop91]; see section 2.3), so

$$D_*X = D_* \otimes_{E_*} E_*X$$

is a homology theory.

We shall analyze the operation

$$MU^{2n}(D_nX) \xrightarrow{\Phi_n} E^{2n}(D_nX)$$

in terms of the composites

$$Q^H : MU^{2n}(D_nX) \xrightarrow{t \circ \Phi_n^{PMU}} E^{2n}(D_nX) \xrightarrow{\chi_H} D^{2n}X$$

as $H$ runs over the subgroups of $F(D^*)_{\text{tor}}$. The first indication that these operations are dramatically simpler to study than $P_n^{PMU}$ is

**Theorem B** The operation

$$MU^{2n}(D_nX) \xrightarrow{Q^H} D^{2n}X$$

is additive. In fact, it is a homomorphism of graded rings

$$MU^{2n}(D_nX) \xrightarrow{Q^H} \Phi_nD^{2n}X,$$

where for a graded object $M^*$, $\Phi_n M^*$ is the graded object which is $M^{nk}$ in degree $k$.

### 1.2 The operation $Q^H$ and Lubin's quotient formal group law.

The real power of Theorem B as a tool for understanding power operations lies in the fact that the operations $Q^H$ have an elegant description in terms of the formal group law. Lubin ([Lub67]; see 2.2) associates to the subgroup $H$ of $F$ a quotient formal group law $F/H$ and a formal homomorphism

$$F \xrightarrow{\iota_H} F/H,$$

both defined over $D^*$. They are constructed so that

$$H = \text{Ker}[F(D^*) \xrightarrow{\iota_H} F/H(D^*)].$$

Our description of $Q^H$ is
Theorem C  On coefficients, $Q^H$ is determined via Quillen's theorem by the equation
\[ Q_* F^{MU} = F/H. \]
Moreover, if $L$ is a complex line bundle, then
\[ \text{where } e_{MU}L \text{ and } e_EL \text{ are respectively the } MU \text{ and } E \text{ Euler classes of } L. \]

1.3  Factoring $Q^H$ through the orientation $t_E$.

In relation to our original goal of producing operations in $E$-cohomology, the operations $Q^H$ have two overriding unsatisfactory features: they have the wrong source, $MU$, and the wrong target, $D$. Making the target $E$ instead of $D$ can be arranged: as has already been mentioned, the ring $D^*$ is a Galois extension of $E^*$; the Galois group is $\text{Aut}[F(D^*)_{\text{tors}}]$. If $H \subset F(D^*)_{\text{tors}}$ is a finite subgroup and $g \in \text{Aut}[F(D^*)_{\text{tors}}]$, then it is not hard to show that
\[ gQ^H = Q^gH. \]
If $\rho$ is a polynomial over $E^*$ on the set of subgroups $H \subset F(D^*)_{\text{tors}}$ of order $n$, denote by $Q^\rho$ the resulting operation
\[ MU^{2*}X \xrightarrow{Q^\rho} E^{2*}X. \]

Theorem D  If a polynomial $\rho$ is invariant under $\text{Aut}[F(D^*)_{\text{tors}}]$, then the operation $Q^\rho$ factors through the natural map
\[ E^*X \to D^*X \]
to produce an operation
\[ MU^{2*}X \xrightarrow{Q^\rho} E^{2*}X, \]
where the degrees indicated by the asterisk on either side may not coincide.

For example, if $H = p^k F(D^*)$ is the full subgroup of points of order $p^k$, then the operation $Q^H = Q^{p^k}$ lands in $E^*:
\[ MU^{2*}X \xrightarrow{Q^{p^k}} E^{2p^k*}X. \]

To make an operation with $E$ as source, we can take advantage of the fact that $Q^H$, unlike the total operation $t_E \circ p^{MU}$, is $MU$-linear. When $E$ is exact (1.0.2), it suffices by Theorem B to find a ring homomorphism
\[ E^* \xrightarrow{\beta^H} D^{n*} \]
such that
\[ \beta^H_* F = F/H. \] (1.3.1)

Note that the existence of a homomorphism \( \beta^H \) satisfying (1.3.1) follows from the existence of a total power operation \( P^n_F \) for \( F \) which is compatible with \( P^n_{MU} \). For suppose one can find \( P^n_E \) and \( t_E \) such that the diagram
\[
\begin{array}{ccc}
MU^{2*}X & \xrightarrow{P^n_MU} & MU^{2n*}D_nX \\
\downarrow t_E & & \downarrow t_E \\
E^{2*}X & \xrightarrow{P^n_E} & E^{2n*}D_nX & \xrightarrow{X^H} & D^{2n*}X
\end{array}
\]
(1.3.2)

commutes. In case \( X \) is a point space, the left vertical arrow is the genus (1.0.1), which satisfies
\[ t_{E*} F^{MU} = F. \] (1.3.3)
The clockwise composite is \( Q^H \), which on coefficients classifies \( F/H \), according to Theorem B. If \( E^* \) is concentrated in even dimensions (as it is in for \( E_h \) and \( Ell \)), then the bottom row is a ring homomorphism
\[ E^* \xrightarrow{\beta^H} D^{n*} \]
which by (1.3.2), (1.3.3), and Theorem B must satisfy (1.3.1).

1.4 A new orientation on \( E_h \).

We can now explain Theorem A; for details, see section 5. One studies homomorphisms out of \( E_h^* \) in terms of the functor it represents: recall that \( \varphi \) is the reduction of the group law \( F_h \) modulo the maximal ideal of \( E_h \). Then according to [LT66], the ring \( E_h^* \) represents the functor of complete local \( \mathbb{Z}_p \)-algebras
\[ R \mapsto \left\{ \text{*-isomorphism classes of lifts of } \varphi \right\}. \]
The quotient formal group laws \( F/H \) are lifts of \( \varphi \), so there is a unique homomorphism
\[ E_h^* \xrightarrow{\beta^H} D^{n*} \]
such that \( \beta^H_* F \) is *-isomorphic to \( F/H \). According to (1.3.1), we need strict equality in order for the operation \( Q^H \) to factor through \( E_h \). When \( H = pF(D^*) \) is the full subgroup of points of order \( p \), the fact that the \( p \)-series is an endomorphism of the
formal group law implies that \( \beta^p \overset{\text{def}}{=} \beta^H \) is (up to a multiple of the periodicity element) the identity. The grading determines that this multiple is \( u^{p^h-1} \). Since the \( p \)-series is the unique endomorphism of a universal formal group law over \( E_h^* \) with kernel \( pF(D^*) \), we conclude that

\[
\beta^p F = F/p
\]

if and only if

\[
[p]F(x) = u^{p^h-1} f_p(x),
\]

which is the condition of Theorem A. This orientation is the only one that has the possibility of making the diagram (1.3.2) commute, if \( P_n \) exists.

By combining Theorems A and B, we obtain

**Theorem E** Let \( t_{PO} \) denote the orientation given by Theorem A. Then there is a unique operation

\[
E_h^{2*} X \xrightarrow{\psi^p} E_h^{2p^h*} X
\]

such that

\[
\mu_{2*} X \xrightarrow{P_{\mu^H}} \mu_{2p^h*}(D_{p^h} X) \xrightarrow{t_{PO}} \mu_{2p^h*}(X) \xrightarrow{\chi_{p^h \text{ot PO}}} E_h^{2*} X \xrightarrow{\psi^p} E_h^{2p^h*} X
\]

commutes. On coefficients, \( \psi^p \) is given by

\[
\psi^p(m) = u^{-\frac{|m|}{2}(p^h-1)} m.
\]

If \( L \) is a complex line bundle, then

\[
\psi^p(eL) = f_p(eL) = u^{1-p^h}[p^k](eL).
\]

Because of its relationship to the \( p \)-series, the operation \( \psi^p \) should be thought of as an unstable Adams operation in \( E_h \). Thus the orientation provided by Theorem A is the unique orientation in which the Adams operation is obtained as a power operation. In theories such as \( E_h \) there is a well-known construction of a stable Adams operation

\[
E_h \xrightarrow{\psi^p} E_h \left[ \frac{1}{p} \right]
\]

which is however not integral: one has to invert \( p \) in the range. It turns out that \( \psi^p \) and our \( \psi^p \) are related by

\[
\psi^p(z) = (\frac{u^{p^h-1}}{p})^{\frac{|z|}{2}} \psi(z),
\]
so we recover the well-known fact (5.4.10) that the unstable operation

\[ E^2 X \xrightarrow{p^* \psi^p} E^2 X \]

(1.4.1)

is integral on 2r-dimensional classes.

When the height \( h \) is 1, the only subgroups of the formal group law are the subgroups of the form

\[ p^* F. \]

This is the case that corresponds to \( K \)-theory: the character map

\[ E^*_1 D_p X \xrightarrow{x_p} E^*_1 X \]

corresponds to evaluation on the class of a \( p \)-cycle, and our construction is exactly analogous to Atiyah’s description of the unstable integral Adams operation in \( K \)-theory; see section 5.6.

When the height is greater than 1, there are other subgroups. In that case, the diagram (1.3.2) provides for each subgroup \( H \) a condition on \( t_{PO} \) for the orientation to be compatible with \( P_{TMU} \) and any (conjectural) total power operation \( P^E \). However, already the condition (1.0.5) was enough to determine the orientation. In fact, in section 5.5 we prove

**Theorem F** *The formal group law over \( E_h \) obtained in Theorem A satisfies (1.3.1) for every finite subgroup \( H \).*

With Theorem F, we can use the operations \( Q^H \) to produce an operation

\[ E^* X \xrightarrow{\Psi^H} D^{n*} X \]

for every subgroup \( H \) of \( F(D^*)_{tor} \). We assemble these operations into a description of a “total power operation” (5.3.5) in \( E_h \), at least in terms of the character map of Hopkins–Kuhn–Ravenel (5.3.5). Let \( \Lambda_\infty \) denote an abelian group isomorphic to \( \mathbb{Z}_p^h \). For \( \pi \) an abelian \( p \)-group, the description of \( E^* D_\pi X \) by Hopkins–Kuhn–Ravenel is a character map

\[ E^*_h D_\pi X \xrightarrow{\Lambda_\infty \xrightarrow{\alpha} \pi} D^*(X^{\pi/(\alpha)}). \]

The operations \( \Psi^H \) enable us to construct an operation

\[ E^* X \xrightarrow{p^* \psi^p} \prod_{\Lambda_\infty \xrightarrow{\alpha} \pi} D^{2n*}(X^{\pi/(\alpha)}), \]
where \( n \) is the order of \( \pi \), such that

\[
\begin{align*}
MU^{2*}X & \xrightarrow{p_{MU}} MU^{2n*}(D_\pi X) \\
t_{PO} \downarrow & \quad \downarrow t_{PO} \\
E^2_X & \xrightarrow{\prod_{\lambda}} E^{2n*}_\lambda(D_\pi X) \\
\prod_{\lambda} & \quad \bigotimes D^{2n*}(X^{*/(\alpha)})
\end{align*}
\]

commutes. This is as close as we have been able to come to constructing a total power operation in \( E_\lambda \); it is recorded as Theorem 5.3.5.

Another indication of interesting applications of the operations \( \Psi^H \) is provided by

1.5 Elliptic cohomology and Hecke operators.

In the last chapter, we begin a study of the application of our theory to elliptic cohomology. The study of subgroups of formal groups is a formal analogue of the study of subgroups of elliptic curves, which are the source of "Hecke operators" [Ser70]. For details, see section 6. Let \( Z_0(2) \) denote the space of pairs

\[ (\Xi, \eta) \]

consisting of a lattice \( \Xi \subset \mathbb{C} \) and a point of order two \( \eta \in 2\mathbb{C}/\Xi. \) An element \( f \) of \( E\ell^{2\pi} \) is a function

\[ Z_0(2) \xrightarrow{f} \mathbb{C} \]

which satisfies (among other properties: see section 6)

\[ f(t(\Xi, \eta)) = t^* f(\Xi, \eta). \]

Fix an odd prime \( p. \) A subgroup \( H \) of order \( p \) of \( F^{E\ell} \) is a rule with assigns to each pair \((\Xi, \eta)\) a subgroup of \( \mathbb{C}/\Xi \) of order \( p \), for which we also write \( H \) by abuse of notation. Equivalently, it assigns to each lattice \( \Xi \) a super-lattice \( \Xi_H \supset \Xi \) such that

\[ [\Xi_H : \Xi] = p. \]

The \( p \)-th Hecke operator \( T_p \) applied to \( f \) is given by

\[ T_p f(\Xi, \eta) = \frac{1}{p} \sum_{H \subset F^{E\ell}, \#H = p} f(\Xi_H, \eta). \]
It turns out that if \( f \in \text{Ell}^{-2r} \) then also \( T_pf \in \text{Ell}^{-2r} \), and Andrew Baker has shown [Bak90] that \( T_p \) can be extended to a stable operation on elliptic cohomology

\[
\text{Ell}^* X \xrightarrow{T_p} \text{Ell}^1 X;
\]

trying to understand his operations was one of the starting points of this investigation. Actually, Baker uses a slightly different version of elliptic cohomology, but his construction and ours work in either context.

It turns out that \( \text{Ell} \) is more “geometric” than the \( E_h \) in that the exponential of the Euler formal group law is an elliptic function, and so is determined (up to constant multiple) by its divisor. Because of this, the Euler formal group is related to its quotient formal group law \( F^{\text{Ell}}/H \) in a simple manner, and one can write down a homomorphism

\[
\text{Ell}^* \xrightarrow{\beta^H} D^p^* \]

such that

\[
\beta^H F^{\text{Ell}} = F^{\text{Ell}}/H.
\]

Therefore one obtains an operation

\[
\text{Ell}^* X \xrightarrow{\Psi^H} D^p^* X \tag{1.5.1}
\]

for each subgroup \( H \) of order \( p \), where \( D^* \) is the range of the character map for elliptic cohomology (see [Hop89] and section 6.1). By Theorem D, the sum

\[
R_p = \sum_{H \in \text{Ell}^* \#H=p} \Psi^H
\]

is an operation

\[
\text{Ell}^* X \xrightarrow{R_p} \text{Ell}^p^* X
\]

which is we use to recover Baker’s Hecke operation in \( \text{Ell} \) (6.5.2). Actually, it seems likely that one can do better than this, and obtain integrality results for the Hecke operators analogous to the integrality of the unstable Adams operations (1.4.1). The main issue is to find a better version of the ring \( D^* \) for elliptic cohomology.

As Mike Hopkins has explained ([Hop89], see 6.2), for groups of exponent \( p^k \) this ring is roughly ring of meromorphic modular forms for the congruence subgroup \( \Gamma(2p^k) \). The trick is to show, for example, that this ring is flat over \( \text{Ell}^* \). Brylinski [Bry90] has studied such a ring for elliptic cohomology which does not invert \( p \), but which only contains modular forms of even weight. Our constructions rely heavily on certain modular forms of weight \(-1\) (6.2.6), and at the present time we can only see our way to showing that \( D^* \) is flat over \( \text{Ell}^* \) if we admit \( 1/p \). However, we believe that it will be possible to show that a ring without \( 1/p \) is flat. Our constructions lead us to conjecture that

**Conjecture G** The operation \( p^{r+1}T_p \) is integral on \( 2r \)-dimensional classes.
1.6 Organization

The rest of this paper is organized as follows. In section 2 we collect facts about tom Dieck’s total power operation in $MU$ (2.1) and about Lubin’s theory of subgroups of formal groups (2.2) which we have used in our work. The most important facts are the computation (Proposition 2.1.9) of

$$P_{\pi}^{MU}(eL)$$

where $eL$ is the Euler class of a line bundle $L$, and Lubin’s theorem (2.2.2) on the existence of the quotient formal group law. In section 2.3 we describe the ring $D^*$ which is the smallest ring over which all of the subgroups of the formal group law occur. The essential facts about $D^*$ were provided by Mike Hopkins [Hop91].

In section 3 we use the character theory of Hopkins–Kuhn–Ravenel to study the operations $Q^H$. In 3.1 we construct the operation $Q^H$ and in 3.2 we give proofs of Theorems B and C. Beyond the character theory, the essential ingredient is the similarity between the expression for $P_{\pi}^{MU}(eL)$ (2.1.9) and Lubin’s homomorphism $f_H(x)$ (2.2.1). The $Q^H$ can be used to give a description of the image of

$$MU^*X \xrightarrow{P_{\pi}^{MU}} MU^nD_{\pi}X \xrightarrow{i_E} E^nD_{\pi}X$$

under the character map of Hopkins–Kuhn–Ravenel. We work this out in 3.3 as Theorem 3.3.3.

In section 3.4, we study the action of $\text{Gal}(D^*/E^*)$ on $Q^H$ to obtain Theorem D.

In section 4 we explain how to use the operations $Q^H$ to produce operations out of exact cohomology theories. The key point is to find the homomorphism $\beta^H$ satisfying (1.3.1). The result (Theorem 4.0.1) is almost trivial, but it does not appear to have been used before to produce unstable operations. The operation $Q^H$ is naturally associated to a stable operation $\nu Q^H$ (4.0.6). The comparison of these two operations (4.0.7) is the basis of integrality statements like (1.4.1) and Conjecture G.

In section 5 we apply these results to $E_h$. After describing briefly the “completed” theory $E_h$, we use the statement of Theorem F to obtain the operations $\Psi^H$. With these we can write down (5.3.5) a total operation out of $E_h^*X$ which lands in the character-theoretic description of $E_h^*(D_{\pi}X)$. I believe that the construction of a total operation

$$E_h^*X \xrightarrow{P_E} E_h^*(D_{\pi}X)$$

is within reach of the methods in this paper. Theorem 5.3.5 represents the best statement we can make at this time.

In section 5.4 we study the particular operation $\Psi^P$, completing the proof of Theorem E. Finally, section 5.5 is devoted to the proof of Theorems A and F. As an
illustration of Theorem A, we show in 5.6 that the multiplicative formal group law satisfies (1.0.5).

In section 6 we turn to elliptic cohomology. In section 6.1, we set up the character map and the operation \( Q^H \) for elliptic cohomology, following [Hop89] but with some adjustments for improvements in the character theory and to make room for conjectural integrality statements. The main result is the comparison of Lubin's quotient formal group law \( F^{Ell}/H \) with the Euler formal group law for the quotient elliptic curve (6.3.8), and the consequent existence of the homomorphisms \( \beta^H \) for elliptic cohomology (6.3.16). These combine to prove Theorem 6.3.15, which is the existence of the operation \( \Psi^H \) of equation (1.5.1). In section 6.4, we use apply this theorem to the full subgroup

\[ pF^{Ell} \]

\[ \rightarrow E_{\infty} \]

to obtain an Adams operation in \( E_{\infty} \) (6.4.2). Finally, in 6.5, we turn to the operation \( R_p \) and Hecke operations. We provide sufficient information to prove Conjecture G as soon as a sufficiently good ring \( D^* \) becomes available.
2 Prerequisites

2.1 Power operations in MU

We shall frequently write the Borel construction

\[ D_\pi X, \]

when \( \pi \) is an abelian group, \textit{without} reference to an ambient symmetric group. In this case, we shall mean by that notation

\[ D_\pi X \overset{\text{def}}{=} E_\pi \times X^\pi, \]

where

\[ X^\pi = \text{Maps}[\pi, X], \]

and the action of \( \pi \) on \( X^\pi \) is the left action coming from right multiplication on \( \pi \).

tom Dieck and Quillen constructed a total power operation

\[ MU^{2n}X \xrightarrow{P_{n}^{MU}} MU^{2n^*}(D_nX) \]

for complex cobordism [tDi68, Qui71], where \( S_n \) is the symmetric group on \( n \) letters.

Briefly, the construction is as follows. If a map

\[ M^k \xrightarrow{f} X^l \]

of manifolds is complex-oriented, then there is a Gysin homomorphism

\[ MU^* M \xrightarrow{f_*} MU^{*+l-k} X. \]

In particular,

\[ f_1 \in MU^{l-k} X. \]

Thom’s theorem [Tho54] shows that every class in \( MU^* X \) can be obtained in this way.

Now suppose that

\[ M \xrightarrow{f} X \]

is a complex-oriented map of even dimension \( 2d \). Then

\[ D_\pi M \xrightarrow{D_{\pi}f} D_\pi X \]

inherits a complex orientation, and by definition

\[ P_{\pi}^{MU}(f_1) = (D_{\pi}f)_*1. \quad (2.1.1) \]

From this construction, one can quickly check the following properties of \( P_n^{MU} \) and \( P_{\pi}^{MU} \). When there is no possibility of confusion, we abbreviate \( P_{\pi}^{MU} \) as \( P_{\pi} \).
Lemma 2.1.2 [tDi68] $P_\pi$ is a “total power operation” in the sense that if

$$X^n \xrightarrow{i} D_\pi X$$

denotes the inclusion of the fiber, then

$$i^* P_\pi z = z^{x_n}.$$  

Moreover, the operation $P_\pi$ is natural with respect to pull-backs: for a map $X \xrightarrow{f} Y$, the diagram

$$
\begin{array}{ccc}
MU^{2*}Y & \xrightarrow{P_\pi^M U} & MU^{2*n}(D_\pi Y) \\
\downarrow{f^*} & & \downarrow{(D_\pi f)^*} \\
MU^{2*}X & \xrightarrow{P_\pi^M U} & MU^{2*n}(D_\pi X)
\end{array}
$$

(2.1.3)

commutes.

Lemma 2.1.4 (compare [tDi68]) $P_\pi$ is multiplicative: for $x \in MU^{2k}X$ and $y \in MU^{2l}X$,

$$P_\pi(xy) = P_\pi(x)P_\pi(y) \in MU^{2n(k+l)}D_\pi X.$$  

One of the features of power operations that makes subject difficult is that they are not additive. However, the failure of $P_\pi$ to be additive can be expressed as a sum of terms which are transfers. A critical feature of our operations $Q^H$ is that they are additive. The demonstration (3.2.1) depends on

Lemma 2.1.5

$$P_n(x + y) = \sum_{j=0}^{n} Tr_{j,n}^{MU} d^*(P_j x \times P_{n-j} y),$$

where

$$Tr_{j,n}^{MU} : MU^*(ES_n \times_{S_j \times S_{n-j}} X^n) \rightarrow MU^*(D_n X)$$

(2.1.6)

is the MU-transfer associated to the fibration

$$S_n/(S_j \times S_{n-j}) \rightarrow ES_n \times_{S_j \times S_{n-j}} X^n \rightarrow D_n X,$$

(2.1.7)

and $d$ is the map

$$ES_n \times_{S_j \times S_{n-j}} X^n \xrightarrow{d} D_j X \times D_{n-j} X.$$

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$P^MU_{\pi}$ on Euler classes

Now let \( \downarrow \) be a complex vector bundle of rank \( r \), and let

\[ eV \in MU^{2r}X \]

be its \( MU \) Euler class. Then \( \pi \) acts on the product \( \downarrow \), and the resulting Borel construction \( \downarrow \) is a complex vector bundle over \( D_\pi X \) with rank \( nr \). We have

**Proposition 2.1.8 ([tDi68])**

\[
P_\pi(eV) = e \left( \begin{array}{c} D_\pi V \\ D_\pi X \end{array} \right).
\]

Proof: Let \( \zeta : X \to V \) be the zero-section. Then the Thom class of \( \downarrow \) is the image of 1 under the push-forward

\[ MU^*X \xrightarrow{\zeta_*} MU^{*+2r}X^V. \]

By definition, the Euler class \( eV \) is the pull-back of the Thom class by \( \zeta \); in other words,

\[ eV = \zeta^*\zeta_*1. \]

We have

\[
P_\pi(eV) = (D_\pi \zeta)^*P_\pi(\zeta_*1)
= (D_\pi \zeta)^*(D_\pi \zeta)_*1
= e \left( \begin{array}{c} D_\pi V \\ D_\pi X \end{array} \right),
\]

where the first equality is the naturality of \( P_\pi \) with respect to pull-backs (2.1.3), the second is the definition of \( P_\pi \) (2.1.1), and the last is the definition of the Euler class. \( \square \)
$P_{\pi}^{MU}(eL)$ when $L$ is a line bundle

Let $\downarrow$ be a complex line bundle and $eL \in MU^2X$ its Euler class. Let $\Delta$ denote the "diagonal map"

$$B\pi \times X \xrightarrow{\Delta} D\pi X.$$ 

One of the keystones of this paper is the formal similarity between the expression for $\Delta^*P_x(eL)$, which we now give, and Lubin's homomorphism $f_H(x)$ from a formal group law to its quotient by a finite subgroup (2.2.1).

When $A$ is an abelian topological group, we denote by $A^*$ its continuous complex dual

$$A^* = \text{Hom}[A, \mathbb{C}^\times].$$

Proposition 2.1.9 (compare [Qui71], p. 42)

$$\Delta^*P_x(eL) = \prod_{u \in \pi^*} \left[ e \left( \begin{array}{c} E\pi \times \mathbb{C} \\ \downarrow \\ B\pi \end{array} \right) + eL \right] \in MU^{2n}(B\pi \times X),$$

where $x + y$ denotes the formal sum with respect to the formal group law of $MU$, and we omit symbols for the pull-backs under the projections $B\pi \times X \rightarrow X$ and $B\pi \times X \rightarrow B\pi$.

Proof: By (2.1.8),

$$\Delta^*P_x = e \left( \begin{array}{c} \Delta^*(E\pi \times L^\pi) \\ \downarrow \\ B\pi \times X \end{array} \right)$$

$$= e \left( \begin{array}{c} \text{Reg}_\pi \otimes L \\ \downarrow \\ B\pi \times X \end{array} \right)$$

$$= e \left[ \bigoplus_{u \in \pi^*} \left( \begin{array}{c} E\pi \times \mathbb{C} \\ \downarrow \\ BA \end{array} \right) \otimes L \right]$$

$$= \prod_{u \in \pi^*} \left[ e \left( \begin{array}{c} E\pi \times \mathbb{C} \\ \downarrow \\ BA \end{array} \right) \right] + eL \right] \cdot \Box$$

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2.2 Subgroups of formal groups

To understand the notion of a subgroup of a formal group law, it is helpful to consider the situation in which a formal group law yields an actual group (see also section 6 for the example of subgroups of the group law of an elliptic curve): let $R$ be a Noetherian ring and $m$ a maximal ideal of $R$; suppose moreover that $R$ is complete in the $m$-adic topology. Let $F(x, y) \in R[[x, y]]$ be a formal group law. If $a, b \in m$ are elements of this maximal ideal then the series $F(a, b)$ converges to an element of $m$. The elements of $m$ with the addition specified by $F$ form a group, denoted $F(m)$.

Now suppose $H \subset F(m)$ is a finite subgroup. Lubin’s main result is to show that the quotient $F(m) \to F(m)/H$ is realizeable as a homomorphism of formal groups: that is, there is a formal group law $F/H$ over $R$, a homomorphism $F \to F/H$, and a commutative diagram

$$
\begin{array}{ccc}
H & \longrightarrow & F(m) \\
\| & & \| \\
H & \longrightarrow & F(m)
\end{array}
\quad \xrightarrow{f_H} \quad
\begin{array}{ccc}
F(m) & \longrightarrow & F(m)/H \\
\| & & \| \\
F/H(m) & \longrightarrow & F/H(m).
\end{array}
$$

In other words, every subgroup $H$ of $F(m)$ is “formal” in the sense that it occurs as the kernel some formal homomorphism $(F \to F/H)(m)$.

The homomorphism $f_H$ is constructed in terms of its kernel $H$ and by the formal group law $F$ : one takes $f_H(x)$ to be

$$f_H(x) = \prod_{h \in H} (h + x). \tag{2.2.1}$$

Since $0 \in R$ is the identity of $F(m)$, equation (2.2.1) guarantees that

$$H = \{ h \in F(m) \mid f_H(h) = 0 \}.$$

Also, the power series $f_H(x)$ has $f_H(0) = 0$ and leading term

$$f'_H(0) = a_H = \prod_{0 \neq h \in H} h$$

given by the product of the non-zero elements of $H$.

Lubin’s theorem is that $f_H(x)$ is in fact a homomorphism of formal group laws.

**Theorem 2.2.2 (Lubin [Lub67])** Let $R$ be a Noetherian domain which is complete in the topology induced by an ideal $I$. Let $F$ be a formal group law over $R$, and suppose that $H$ is a finite subgroup of $F(I)$. Let $f_H(x)$ be defined by

$$f_H(x) = \prod_{h \in H} (h + x).$$
Then there is a unique formal group law \( F/H \) defined over \( R \) such that
\[
f_H F(x, y) = F/H(f_H(x), f_H(y)),
\]
so \( H = \text{Ker}(F(I) \overset{f_H}{\rightarrow} F/H(I)) \).

Lubin also shows that when \( F \) is not the additive group, \( F/H \) is universal, because homomorphisms of formal groups are formally surjective:

**Theorem 2.2.3 ([Lub67])** Let \( R \) be a complete Noetherian local ring with maximal ideal \( m \). Suppose that \( F \) is a formal group law over \( R \) whose reduction to the residue field has finite height. Suppose that there are homomorphisms of formal group laws
\[
F \overset{f_i}{\rightarrow} F_i, \ i = 1, 2,
\]
such that
\[
\text{Ker}[f_1(m)] \subset \text{Ker}[f_2(m)].
\]
Then there is a unique homomorphism of formal group laws
\[
F_1 \overset{g}{\rightarrow} F_2
\]
such that
\[
f_2 = g \circ f_1. \quad \square
\]

If \( H \subset G \subset F(m) \), then the three finite subgroups
\[
H \subset F(m), \quad G \subset F(m), \quad \text{and} \quad G/H \subset F/H(m)
\]
produce three formal homomorphisms
\[
F \overset{f_H}{\rightarrow} F/H, \quad F \overset{f_G}{\rightarrow} F/G, \quad \text{and} \quad F/H \overset{f_{G/H}}{\rightarrow} (F/H)/(G/H).
\]

An important property of Lubin's isogeny is

**Proposition 2.2.4**

\[
f_G = f_{G/H} \circ f_H,
\]
and so
\[
(F/H)/(G/H) = F/G.
\]
Proof: Let \( S \subset G \) be a set of coset representatives of \( G/H \). Then

\[
f_G(x) = \prod_{g \in G} (g + x)
= \prod_{g \in S} \prod_{h \in H} (h + g + x)
= \prod_{g \in S} f_H(g + x)
= \prod_{g \in S} (f_H(g) + f_H(x))
= \prod_{\sigma \in G/H} (\sigma + f_H(x))
= f_{G/H}(f_H(x)). \]

Note the similarity between the expression (2.1.9) for the total power operation \( P_{x}^{\pi} \) with respect to an abelian group \( \pi \) on the Euler class of a line bundle and the expression (2.2.1) for the homomorphism \( f_H(x) \) defined by a subgroup of a formal group law. This simple observation is the key to Theorem C; see section 3.2.

Formal group laws over graded rings

Our method directly produces unstable integral operations. The formal groups that occur in complex-oriented cohomology theories are graded, and keeping careful track of the gradings will enable us to prove integrality results about associated stable operations (4.0.7) including stable Adams operations (Theorem 5.4.10) and Hecke operations (Conjecture G)). In this section we cast the theory of subgroups in a graded setting.

Suppose then that \( R^* \) is a graded Noetherian ring which is complete with respect to a maximal ideal \( m \). Let \( F(x, y) \in R^*[[x, y]] \) be a formal group law which is of degree \( 2n \) not as a power series but as an element of the graded ring \( R^*[[x, y]] \), where \( x \) and \( y \) are both taken to have degree \( 2n \). In this case we say \( F \) is “homogeneous of degree \( 2n \)”. Notice that the coefficients of \( F \) will be concentrated in \( R^{2n*} \).

One makes this definition so that if \( a \) and \( b \) are elements of \( m \) of degree \( 2n \), then \( F(a, b) \) will also be an element of \( m \) of degree \( 2n \). The elements of \( m \) of degree \( 2n \) with addition defined by \( F \) is a subgroup of \( F(m) \) which will be denoted \( F(m, 2n) \). If \( F \) is a homogeneous formal group law of degree \( 2n \) and \( f(x) = x + o(x^2) \in R^*[[x]] \) is a power series which is of degree \( 2n \) when the degree of \( x \) is \( 2n \), then the power series

\[
G(x, y) = f F(f^{-1}(x), f^{-1}(y))
\]

is also a homogeneous formal group law of degree \( 2n \), and \( f(x) \) defines a homomorphism

\[
F(m, 2n) \xrightarrow{f(x)} G(m, 2n).
\]
More generally, we have

**Lemma 2.2.5** Let $F \xrightarrow{f(x)} G$ be a homomorphism of formal group laws over $R^*$. Suppose that $F$ is homogeneous of degree $2n$, and that $f(x)$ satisfies

1. $f(x)$ has degree $2k$ when $x$ is taken to be of degree $2n$, and
2. the element $a = f'(0)$ of $R^{2(k-n)}$ is not a zero-divisor.

Then $G$ is also homogeneous and has degree $2k$. Moreover, if $R$ is complete with respect to the topology defined by an ideal $m$, then $f(x)$ restricts to a homomorphism

$$F(m, 2n) \xrightarrow{f(x)} G(m, 2k).$$

Proof: Since $f(x) = ax + o(x^2), |a| = 2(k - n), \overline{1}_a R^*$ one can study the power series $f^{-1}(x)$ which has degree $2n$ when $x$ is taken to have degree $2k$. Over $\overline{1}_a R^*$, $G$ is given by

$$G(x, y) = fF(f^{-1}(x), f^{-1}(y))$$

which is homogeneous of degree $2k$. Since $a$ is not a zero-divisor, the localization $R^* \to \overline{1}_a R^*$ is injective, and $G$ has degree $2k$ as a formal group law over $R^*$. This situation is summarized, and the proof of the second part of the lemma given, by the diagram

$$
\begin{array}{ccc}
m^{2n} \times m^{2n} & \xrightarrow{F} & m^{2n} \\
\downarrow f \times f & & \downarrow f \\
m^{2k} \times m^{2k} & \xrightarrow{G} & m^{2k}.
\end{array}
$$

**Corollary 2.2.6** In the situation of Lemma 2.2.5, the coefficients of $G(x, y)$ are concentrated in $R^{2k*}$.

Now we apply (2.2.5) to Lubin's quotient formal group law $F/H$. Let $F$ be a homogeneous formal group law of degree 2 over a graded domain $R^*$ which is complete with respect to the ideal $m$. Let $H$ be a subgroup of $F(m, 2)$ of order $n$. Then by (2.2.1), $f_H(x)$ has degree $2n$ when $x$ is taken to be of degree 2, and $a_H = f_H'(0)$ has degree $2(n - 2)$.

**Corollary 2.2.7** $F/H$ is homogeneous of degree $2n$. In particular, the coefficients of $F/H$ are concentrated in $R^{2n*}$.
The formal group laws of complex-oriented cohomology theories are homogeneous of degree 2. If $H$ is a subgroup of order $n$, then the quotient formal group law $F/H$ is related to a formal group law of degree 2 which we call $\nu F/H$, the normalized quotient. It is defined in terms of the normalized homomorphism $\nu f_H(x)$,

$$\nu f_H(x) = \frac{f_H(x)}{a_H} \in R[\frac{1}{a_H}]^*[[x]].$$

(2.2.8)

The normalized formal group law $\nu F/H$ is defined so that

$$F \xrightarrow{\nu f_H} \nu F/H;$$

(2.2.9)

it is defined over $R[\frac{1}{a_H}]$ since $\nu f_H(x) = x + o(x^2)$ is an invertible power series over $R[\frac{1}{a_H}]$. By Corollary 2.2.7, $\nu F/H$ is homogeneous of degree 2.

The integrality results (4.0.7) and (5.4.10) are based on the comparison of $\nu F/H$ and $F/H$. By construction, they are related by

$$F \xrightarrow{\nu f_H} \nu F/H \xrightarrow{\mu_{H1}} F/H.$$  

Let $\mu^H : R[\frac{1}{a_H}]^* \rightarrow R[\frac{1}{a_H}]^{n*}$ and $\delta^H : R^{2n*} \rightarrow R[\frac{1}{a_H}]^{2*}$ be defined by

$$\mu^H(m) = a_H^{-\frac{|m|}{2n}} m$$

(2.2.10)

and

$$\delta^H(m) = a_H^{-\frac{|m|}{2n}} m.$$  

(2.2.11)

Recall (2.2.7) that the coefficients of $F/H(x, y)$ are concentrated in $R^{2n*}$.

**Proposition 2.2.12** $F/H$ and $\nu F/H$ are related by

$$F/H = \mu^H \nu F/H$$

and

$$\nu F/H = \delta^H F/H.$$  

**Proof:** This is a consequence of the more general

**Lemma 2.2.13** Let $R^*$ be a graded ring in which $a$ is a unit such that

$$|a| = 2(n - k).$$

Suppose that $F$ is a formal group law which is homogeneous of degree $2k$. Then $F$ and $F^{ax}$ are related by

$$F^{ax} = \mu_a F$$

and

$$F = \delta_a F^{ax},$$

where $R^{2k*} \xrightarrow{\mu^a} R^{2n*}$ and $R^{2n*} \xrightarrow{\delta^a} R^{2k*}$ are the homomorphisms

$$\mu^a(m) = a^{\frac{|m|}{2k}} m$$

and

$$\delta^a(m) = a^{-\frac{|m|}{2n}} m.$$
Proof: We prove one case, that \( F = \delta^a F'^a \). Let \( F'^a \) be given by
\[
F'^a(x, y) = x + y + \sum_{i,j \geq 1} \gamma_{i,j} x^i y^j \in R[[x, y]].
\]
Note that by Lemma 2.2.5, we have
\[
\gamma_{i,j} \in R^{2n(1-i-j)}.
\]
It follows that
\[
\delta^a F'^a(x, y) = x + y + \sum_{i,j \geq 1} \gamma_{i,j} a^{i+j-1} x^i y^j.
\]
On the other hand,
\[
F(x, y) = a^{-1} F'^a(ax, ay)
= x + y + a^{-1} \sum_{i,j \geq 1} \gamma_{i,j} (ax)^i (ay)^j
= x + y + \sum_{i,j \geq 1} \gamma_{i,j} a^{i+j-1} x^i y^j. \qed
\]

2.3 \( D^* \) : a universal ring for formal subgroups

In this section we assume that \( E^* \) is a complete local Noetherian domain with maximal ideal \( m \) and residue characteristic \( p \), and we suppose that \( F \) is a homogeneous formal group law over \( E^* \) of degree 2, whose mod-\( m \) reduction has height \( h \). Thus this section covers the case of \( E_h \) which we study in detail in section 5. The analogous constructions for elliptic cohomology are the subject of section 6.1.

If \( F \) is a formal group law over \( E^* \) and \( A \) is a finite abelian group, a monomorphism
\[
A \hookrightarrow F(m)
\]
is called a "level-\( A \) structure" on \( F \). According to (2.2.2), a level-\( A \) structure on \( F \) over a complete Noetherian domain determines quotient formal group laws
\[
F/B \in E^*[x, y]
\]
for every subgroup \( B \) of \( A \).

Since an element of \( F(E^*) \) of order \( p^k \) is exactly a root of the \( p^k \)-series, which has Weierstrass degree \( p^{kh} \), the subgroups \( A \) which occur can be studied inside the algebraic closure of (the fraction field of) \( E^* \). Over a complete local ring \( E^* \) with maximal ideal \( m \) we write \( F(E^*) \) for \( F(m) \).

**Theorem 2.3.1** ([LT65]; see also [HKR91]) Let \( \mathcal{O}^* \) be the ring of integers in the algebraic closure of the fraction field of \( E^* \). Then
\[
\begin{align*}
p^k F(\mathcal{O}^*, 2) &\cong (\mathbb{Z}/p^k \mathbb{Z})^h \\
F(\mathcal{O}^*, 2)_{\text{tors}} &\cong (\mathbb{Q}_p/\mathbb{Z}_p)^h.
\end{align*}
\]
Let $\Lambda_\infty$ be an abelian group isomorphic to $\mathbb{Z}_p^h$, so

$$\Lambda_\infty^* \cong (\mathbb{Q}_p/\mathbb{Z}_p)^h,$$

although no explicit isomorphism has been chosen. Let $\Lambda_k^* = p^k\Lambda_\infty^*$ be the subgroup of elements of order $p^k$; $\Lambda_k^*$ is precisely the dual of

$$\Lambda_k = \Lambda_\infty/p^k\Lambda_\infty.$$

We can rephrase Theorem 2.3.1 as saying that there exist compatible isomorphisms

$$\Lambda_k^* \xrightarrow{\cong} p^*F(\mathcal{O}^*, 2). \quad (2.3.2)$$

We describe next an observation of Hopkins-Kuhn-Ravenel to the effect that in the case of a ring $E^* = E^*(pt)$ which comes from a cohomology theory with a complex orientation, the orientation and the group $\Lambda_k$ conspire to provide an extension $D_k^*$ of $E^*$ which comes equipped with a canonical level-$\Lambda_k^*$ structure

$$\Lambda_k^* \xrightarrow{\text{univ}} p^*F(D_k^*, 2).$$

Moreover, the ring $D_k$ is as small as it can be.

A complex orientation on $E$ determines a map

$$A^* \ni u \mapsto e \left( \begin{array}{c} EA \times C \\ u \downarrow \\ BA \end{array} \right) \in E^2BA \quad (2.3.3)$$

which is an element of

$$\text{Hom}_{gps}[A^*, F(E^*BA, 2)].$$

The role of $E^*BA$ in describing subgroups of the formal group law is moderated by the following observation.

**Lemma 2.3.4 ([HKR91])** The natural transformation

$$\text{Hom}_{E^*\text{-alg}}[E^*BA, R^*] \to \text{Hom}_{gps}[A^*, F(R^*, 2)]$$

given by

$$f \mapsto f \circ \phi$$

is an equivalence of functors of pairs $(A, R)$, where $A$ is a finite abelian group, and $R^*$ is a complete, local, graded $E^*$-algebra.
Lemma 2.3.4 says that $E^*BA$ represents the functor

$$R^* \mapsto \text{Hom}_{\text{gps}}[A^*, F(R^*, 2)].$$

However, $E^*BA$ is not a domain, so Lubin's theorem doesn't apply. Moreover, not every map

$$E^*BA \rightarrow R^*$$

represents a monomorphism. Note, however, that when $R^*$ is a domain, then a homomorphism

$$A^* \xrightarrow{\phi} F(R^*)$$

is a monomorphism precisely when $\phi(u)$ is not a zero-divisor, for $u \neq 0$.

Now take the case of $\Lambda_k$, and let $S \subset E^*B\Lambda_k$ be the multiplicative set generated by the set

$$\{\phi(u) \in E^2B\Lambda_k \mid 0 \neq u \in \Lambda^*_k\}$$

of Euler classes of non-trivial line bundles over $B\Lambda_k$. The ring $D^*_k$ is defined by

$$D^*_k = \text{Im}[E^*B\Lambda_k \rightarrow S^{-1}E^*B\Lambda_k].$$

$D^*_k$ comes with a homomorphism

$$\phi_{\text{univ}} : \Lambda^*_k \xrightarrow{\phi_{\text{univ}}} F(E^*B\Lambda_k, 2) \rightarrow F(D^*_k, 2) \quad (2.3.5)$$

which is represented by the localization map

$$E^*B\Lambda_k \xrightarrow{\hat{\phi}} D^*_k.$$

The basic results about the ring $D^*_k$ are

**Theorem 2.3.6** ([Dri73, KM85, Hop91]) *The ring $D^*_k$ is a domain and is faithfully flat over $E^*$. It represents the functor (of complete local $E^*$-algebras)*

$$D_k(R^*) = \{\text{level-$\Lambda^*_k$ structures on } F(R^*, 2)\}. \quad \square$$

**Proposition 2.3.7** *The homomorphism

$$\Lambda^*_k \xrightarrow{\phi_{\text{univ}}} \rho_*F(D^*_k)$$

is an isomorphism.*

Proof: $\phi_{\text{univ}}$ is the universal monomorphism which gives Proposition 2.3.6. According to Theorem 2.3.1, the group $\rho_*F(D^*_k)$ can be no larger than $\Lambda^*_k$. \quad \square
We proceed to construct a ring $D^* = D^*_\infty$ with a full level $\Lambda^*_\infty$-structure as a limit of the $D^*_k$. The inclusions

$$\Lambda^*_{k-1} \rightarrow \Lambda^*_k$$

yield forgetful transformations

$$D_k \rightarrow D_{k-1}.$$

These are represented (2.3.6) by maps

$$D^*_{k-1} \rightarrow D^*_k.$$

The ring $D^*$ is the colimit

$$D^* = \text{colim}(... \rightarrow D^*_{k-1} \rightarrow D^*_k \rightarrow ...).$$

It comes with an isomorphism

$$\Lambda^*_\infty \xrightarrow{\phi_{\text{uni}}} F(D^*, 2)_{\text{tors}}.$$

Since the $D^*_p$ are free over $E^*$, the ring $D^*$ is flat over $E^*$, and so

**Proposition 2.3.10** The functor $D^*(-)$

$$X \mapsto D^* \otimes_{E^*} E^* X$$

is a cohomology theory for finite spaces.

**Proposition 2.3.11** If $H$ is a finite subgroup of $F(D^*, 2)_{\text{tors}}$ whose exponent divides $p^k$, then Lubin’s isogeny $f_H$ and the quotient formal group law $F/H$ are defined over $D^*_k$:

$$f_H(x) \in D^*_k[[x]] \subset D^*[[x]], \text{ and}$$

$$F/H(x, y) \in D^*_k[[x, y]] \subset D^*[[x, y]].$$

Proof: This follows from Lubin’s theorem (2.2.2) and the fact (2.3.6) that $D^*_k$ is a Noetherian domain. □
3 Character theory

The main results of this section are Theorems B and C of the introduction. They are proved in section 3.2, after we define the operation $Q^H$ using character theory in section 3.1.

3.1 The cohomology operation $Q^H$ defined by a subgroup $H$

Generalized characters

The rings $D^*$ and $D_k^*$ are the natural range of the character map of Hopkins-Kuhn-Ravenel. Suppose that $G$ is a finite group and that $k$ is so large that $p^k$ kills the $p$-torsion in $G$. Let

$$\Lambda_\infty \to \Lambda_k \to G$$

be a homomorphism: if one chooses an isomorphism $\mathbb{Z}_p^h \cong \Lambda_\infty$ then $\alpha$ determines an $h$-tuple of commuting elements of $G$ of $p$-power order. The character map corresponding to $\alpha$ is the ring homomorphism

$$E^*BG \xrightarrow{\alpha} D^*$$

given by the composite

$$E^*BG \xrightarrow{\alpha} E^*B\Lambda_k \xrightarrow{\gamma} D^*_k \to D^*.$$  

Remark: Hopkins-Kuhn-Ravenel use the ring $L^* = \frac{1}{p}D^*$ as the range of their character map, because only after inverting $p$ does the “total character map”

$$L^* \otimes_{E^*} E^*BG \xrightarrow{\alpha} \prod_{\text{conj. classes}} L^*$$

become an isomorphism. However, for our purposes it will be essential not to invert $p$, because we must retain the maximal ideal of $D^*$ in order to realize the quotient formal group laws $F/H$ as deformations (See (5.2.1)).

The character map defined by a formal subgroup $H$

Now let $H \subset F(D^*,2)_{\text{tors}}$ be a finite subgroup of order $n$ and exponent $p^k$. Specifying $H$ is equivalent to specifying the subgroup

$$\phi_{\text{univ}} H \subset \Lambda_k^r \subset \Lambda_\infty^r.$$
Let $\text{Ann}[H] \subset \Lambda_k \subset \Lambda_\infty$ be the kernel of the evaluation map
\[ \Lambda_\infty \xrightarrow{\text{eval}} \text{Hom}_{\mathcal{C}^*,\mathcal{C}}[\phi^{-1} H, \mathcal{C}], \]
and let $G$ be the cokernel
\[ \text{Ann}[H] \hookrightarrow \Lambda_\infty \xrightarrow{\alpha_H} G. \]

**Lemma 3.1.1** There is a natural isomorphism
\[ G^* \cong H. \]

Let $\chi_H$ be the composite
\[ E^*DGX \xrightarrow{\Delta^*} E^*(BG \times X) \xrightarrow{\cong} E^*BG \otimes_{E^*} E^*X \xrightarrow{\chi_H \otimes 1} D^* \otimes_{E^*} E^*X. \]  

The operation $Q^H$ is the natural transformation
\[ MU^2_\ast X \xrightarrow{Q^H} D^{2n}X \]
given by
\[ MU^2_\ast X \xrightarrow{P_G} MU^{2n}_\ast DGX \xrightarrow{t_E} E^{2n}_\ast DGX \xrightarrow{\chi_H} D^{2n}X. \]  

We are ready to prove

**3.2 Theorems B and C.**

**Theorem 3.2.1** ("Theorem B") $Q^H$ is additive. Moreover, it is a graded ring homomorphism
\[ MU^2_\ast X \xrightarrow{Q^H} \Phi_n D^{2\ast}X, \]
where $\Phi_n R^\ast$ is the graded ring which is $R^n$ in degree $k$. In particular, $Q^H$ is a ring homomorphism
\[ MU^* \xrightarrow{Q^H} \Phi_n D^\ast. \]
Theorem 3.2.4 ("Theorem C") Let \( L \to X \) be a complex line bundle, and let \( e_{MU} L \) and \( e_{E} L \) denote its Euler classes in \( MU \) and \( E \) cohomology. Then \( Q^H \) on Euler classes is given by

\[
Q^H(e_{MU} L) = f_H(e_{E} L) \in D^{2n}X.
\] (3.2.5)

Its effect on coefficients is determined by the equation

\[
Q^* F^{MU} = F/H.
\] (3.2.6)

The proof of the additivity of \( Q^H \) in Theorem 3.2.1 imitates Atiyah’s proof [Ati66] of the additivity of the Adams operations in \( K \)-theory. The basic ingredient is the formula of Hopkins–Kuhn–Ravenel for induced characters:

**Theorem 3.2.7 ([HKR91])** Let \( G \) be a finite group and let \( S \leq G \) be a subgroup. For \( u \in E^* BS \) and \( \lambda_{\infty} \to G \) one has the formula

\[
\chi_f(Tr^E u) = \sum_{g \in G/G/S} \chi_{g^{-1} f g}(u),
\]

where \( Tr^E : E^* BS \to E^* BG \) is the transfer map in \( E \)-cohomology associated to the fibration \( G/S \to BS \to BG \).

Proof of Theorem 3.2.1: In (3.1.3), all the maps are ring homomorphisms except \( P_G \). By Lemma 2.1.4, \( P_G \) is multiplicative. Suppose we order the elements of \( G \), which determines an isomorphism

\[
\text{Aut}_{sets} G \to S_n
\]

and so a monomorphism

\[
G \to S_n.
\]

Then one obtains a character map

\[
E^* D_n X \to D^* X
\]

such that the diagram

\[
\begin{array}{ccc}
MU^2 X & \xrightarrow{P_n} & MU^{2n} D_n X \\
\downarrow P_G & & \downarrow t_E \\
MU^{2n} D_G X & \xrightarrow{\omega^*} & E^{2n} D_n X \\
\downarrow t_E & & \downarrow \chi \omega \\
E^{2n} D_G X & \xrightarrow{\chi_H} & D^{2n} X \\
\end{array}
\]
commutes. By (2.1.5), $P_n$ is given on sums by
\[
P_n(x + y) = \sum_{j=0}^{n} T^{MU}_{j,n}(P_j x \times P_{n-j} y),
\]
so
\[
Q^H(x + y) = \sum_{j=0}^{n} \chi_{\omega} T^{E}_{j,n} t_E P_j x P_{n-j} y,
\]
since $t_E$ commutes with the transfer. By the formula for induced characters (3.2.7),
\[
\chi_{\omega} T^{E}_{j,n} u = \sum_{g \in S_j \times S_{n-j}, t \in (S_j \times S_{n-j})^*/\mu} \chi_{g^{-1} \omega} u,
\]
Since the image of $G$ in $S_n$ is transitive, the sum is empty unless $j = 0$ or $j = n$. So
\[
Q^H(x + y) = \chi_{H} t_E P_G x + \chi_{H} t_E P_G y = Q^H(x) + Q^H(y).
\]
Thus $Q^H$ is additive. □

Proof of Theorem 3.2.4: First we prove (3.2.5); then (3.2.6) will follow from Lemma 3.2.8. In the diagram
\[
\begin{array}{cccc}
MU^2 X & \xrightarrow{P_G} & MU^2 D_G X & \xrightarrow{t_E} & E^{2n} D_G X \\
& \downarrow \Delta^* & & \downarrow \Delta^* \\
MU^2 (BG \times X) & \xrightarrow{t_E} & E^{2n} (BG \times X) & & \\
& \downarrow \cong & & & \\
& (E^{*} BG \otimes E^{*} E^{*} X)^{2n} & & & \\
& \downarrow \chi_{\omega_H} & & & \\
& D^{2n} X, & & & \\
\end{array}
\]
the right outer arrows give $Q^H$. Going around the left outer arrows, we get
\[
Q^H(e_{MU L}) = \hat{\phi} t_E \Delta^* P_G e_{MU L}
\]
\[
= \hat{\phi} t_E \prod_{u \in G^{*H}} \left[ e_{MU} \left( E^{u} \times \frac{C}{u} \uparrow \downarrow \right) \right] + e_{MU L}
\]
\[
= \hat{\phi} \prod_{h \in H} \left[ e_{E} \left( E^{u} \times \frac{C}{u} \uparrow \downarrow \right) \right] + e_{E L}
\]
\[
= \prod_{h \in H} \left[ h + e_{E L} \right]
\]
\[
= f_{H}(e_{E L}).
\]
where the second equality is Lemma 2.1.9.

Now recall (2.3.6) that $D^*$ is a domain. The effect of $Q^H$ on coefficients (3.2.6) follows from (3.2.5), Theorem 3.2.1, and

**Lemma 3.2.8** Let $C$ be a complex-oriented cohomology theory. Suppose that

$$MU^{2*}(-) \overset{Q}{\to} \Phi_n C^{2*}(-)$$

is a natural transformation of ring-valued functors. Suppose that the effect of $Q$ on Euler classes is given by

$$Q(e_{MU}L) = f(e_CL),$$

where $f(x)$ is a homomorphism of formal group laws

$$F^C \to F'$$

such that $f'(0)$ is not a zero-divisor in $\Phi_n C^*$. Then $Q$ is determined on $MU^*$ as the homomorphism classifying $F'$:

$$Q_*F^{MU} = F'.$$

Proof: Let $L_l$ and $L_r$ be the two tautological bundles over $CP^\infty \times CP^\infty$. By (3.2.9),

$$Q(e_{MU}L_l + e_{MU}L_r) = Q(e_{MU}L_l \oplus L_r) = \theta(e_CL_l + e_CL_r).$$

On the other hand, we know that $Q$ is a ring homomorphism

$$MU^*(CP^\infty \times CP^\infty) \overset{Q}{\to} C^*(CP^\infty \times CP^\infty),$$

so

$$Q(e_{MU}L_l + e_{MU}L_r) = \theta(e_CL_l + e_CL_r).$$

Comparing (3.2.10) and (3.2.11), we see that

$$\theta F^C(x, y) = Q_*F^{MU}(\theta(x), \theta(y)),$$

so $Q_*F^{MU} = F'$. □
3.3 The character-theoretic description of the total power operation

Hopkins–Kuhn–Ravenel give a description of the cohomology of the full Borel construction $D_\pi X$ in terms of a total character map

$$E^*(E_\pi \times X^\pi) \xrightarrow{\chi} \prod_{\Lambda_\infty \overset{a}{\to} \pi} D^*(X^{\pi/(a)})$$

which we now describe. For a map $\Lambda_\infty \overset{\alpha}{\to} \pi$, let $G = \text{(Im} \alpha)$ be the subgroup generated by the image of $\alpha$. The isomorphism of $G$-sets

$$\pi \cong G \times (\pi/G)$$

gives an isomorphism

$$\text{Fixed}_G[\text{Maps}[\pi, X]] \cong \text{Maps}[\pi/G, X]$$

between the fixed point set of $X^\pi$ by the action of $G$ and $X^{\pi/G}$. The component of $\chi$ corresponding to a map $\alpha$ is

$$E^*(E_\pi \times X^\pi) \to E^*(BG \times X^{\pi/G}) \xrightarrow{\chi_\alpha} D^*(X^{\pi/G}). \quad (3.3.1)$$

Let $T^E_\pi$ be the natural transformation

$$MU^*X \xrightarrow{P^*} MU^{n*}D_\pi X \xrightarrow{\Lambda_\infty \overset{a}{\to} \pi} E^{n*}D_\pi X \xrightarrow{\chi} \prod_{\Lambda_\infty \overset{a}{\to} \pi} D^{n*}X^{\pi/(a)} \quad (3.3.2)$$

which is the image of the total power operation of $MU$ under the character map of Hopkins–Kuhn–Ravenel.

**Theorem 3.3.3** For a map $\Lambda_\infty \overset{\alpha}{\to} \pi$, let $G = \text{(Im} \alpha)$ be the subgroup generated by the image of $\alpha$ and let $H = \text{(Im} \alpha^*)$ be the subgroup of $\Lambda_\infty^*$ generated by the image of the map

$$\pi^* \overset{\alpha^*}{\to} \Lambda_\infty^*.$$ 

Suppose that $|\pi| = n$, $|G| = r$, and $[\pi : G] = n/r = k$. Then the component of $T^E_\pi$ corresponding to $\alpha$ is given by

$$\text{pr}_\alpha T^E_\pi(z) = Q^H(z^{x_k}) \in D^{2n^*}(X^k).$$

**Proof:** Comparison of (3.1.2) and (3.3.1) shows that if $\alpha$ is surjective, then

$$\text{pr}_\alpha T^E_\pi = Q^H.$$

For general $\alpha$, the result follows from the case of surjective $\alpha$, the definition of $Q^H$ (3.1.3), and
Lemma 3.3.4 In the situation of the theorem, the diagram

\[
\begin{array}{ccc}
MU^{2*} X & \xrightarrow{P_*} & MU^{2n*} D_\pi X \\
\downarrow_{x \mapsto z^{xk}} & & \downarrow \\
MU^{2k*} X^* / G & \xrightarrow{P_G} & MU^{2n*} D_G(\pi / G)
\end{array}
\]

commutes.

Proof: Represent a class \( z \in MU^{2*} X \) as \( f_1 \) where \( f \) is a complex-oriented map

\[
Z \xrightarrow{f} X.
\]

Then \( P_* z \) is represented by the map

\[
D_z Z \xrightarrow{D_z f} D_z X.
\]

The isomorphism of \( G \)-sets

\[
\pi \cong G \times \pi / G
\]

provides the vertical isomorphisms in the commutative diagram

\[
\begin{array}{ccc}
EG \times Z^* & \xrightarrow{EG \times f^*} & EG \times X^* \\
\cong & & \cong \\
D_G(Z^*/G) & \xrightarrow{D_G(f^*/G)} & D_G(\pi / G).
\end{array}
\]

The bottom row of the diagram represents \( P_G(z^{xk}) \) by definition. The top row represents the composite

\[
MU^{*} X \xrightarrow{P_*} MU^{n*} D_\pi X \to MU^{n*} D_G X
\]

because it fits into the pull-back diagram

\[
\begin{array}{ccc}
EG \times Z^* & \longrightarrow & D_z Z \\
\downarrow & & \downarrow_{D_z f} \\
EG \times X^* & \longrightarrow & D_z X.
\end{array}
\]

Remarks: 1) For example, the component of \( T^E_\pi \) corresponding to the trivial map \( \Lambda_\infty \xrightarrow{0} \pi \) is the \( n \)th external power, which we knew already since \( P_* \) is a total power
operation and evaluation at 0 corresponds to the pull-back by the inclusion of the fiber

\[ X^* \to D_* X. \]

2) Also, a computation similar to (2.1.9) shows that for

\[ \Lambda_\infty \to H \hookrightarrow G, \]

we have

\[ \chi_\alpha (Bi)^* P_G eL = (f_H (eL))^{[G/H]}, \]

which is a special case of Theorem 3.3.3 in view of the multiplicativity of \( Q^H \).

### 3.4 Galois theory and the proof of Theorem D

The group \( \text{Aut}[\Lambda^*_k] \) acts on \( D^*_k \), because it acts on the functor \( D_k \). The fact we shall need is

**Proposition 3.4.1 ([Hop91])** The ring of \( \text{Aut}[\Lambda^*_k] \)-invariants in \( D^*_k \) is exactly \( E^* \).

The group \( \text{Aut}[\Lambda^*_k] \) also acts on the set

\[ \mathcal{S}_G_k = \left\{ \text{subgroups of } \Lambda^*_k \right\}. \]

We can use Proposition 3.4.1 and the action on \( \mathcal{S}_G_k \) to produce operations whose target is \( E \) rather than \( D \). In what follows, we shall not distinguish notationally between a subgroup

\[ H \subset \Lambda^*_k \]

and its image

\[ \phi(H) \subset p^* F(D^*_k, 2) \]

under the isomorphism (2.3.7).

**Proposition 3.4.2** For \( H \in \mathcal{S}_G_k \) and \( a \in \text{Aut}[\Lambda^*_k] \), we have

\[ Q^{aH} = aQ^H, \]

where on the right the action of \( a \) refers to the action of \( \text{Aut}[\Lambda^*_k] \) on the ring \( D^*_k \) induced by the action on the functor it represents (2.3.6, 3.4.1), together with the definition (2.3.10) of \( D^*_k X \) as

\[ D^*_k X = D^*_k \otimes_{E^*} E^* X. \]
Proof: It suffices to check this when $X$ is a point space, since $\chi_H$ factors through

$$E^*(BG \times X) \cong E^*BG \otimes_{E^*} E^*X$$

and the action takes place entirely on the left of the tensor product.

$\text{Aut}[\Lambda_k^*]$ also acts on $\Lambda_k$, by adjointness. The commutative square

$$
\begin{array}{ccc}
B\Lambda_k & \xrightarrow{B_a} & B\Lambda_k \\
\downarrow{\chi_{aH}} & & \downarrow{\chi_H} \\
BG & \xrightarrow{B_a} & BaG
\end{array}
$$

shows that

$$E^*BaG \xrightarrow{a^*} E^*BG$$

commutes. The proof is complete upon observing that the diagram

$$
\begin{array}{ccc}
MU^{2*}X & \xrightarrow{P_{aG}} & MU^{2*}X \\
\downarrow{P_G} & & \downarrow{P_G} \\
MU^{2n*}D_{aG}X & \xrightarrow{a^*} & MU^{2n*}DX
\end{array}
$$

commutes. □

Let $SG_k(n)$ denote the subset of $SG_k$ consisting of subgroups of order $n$, and let $\mathbb{Z}[SG_k(n)]$ be the polynomial ring on the set $SG_k(n)$. The action of $\text{Aut}[\Lambda_k^*]$ on $SG_k$ extends to an action of $\text{Aut}[\Lambda_k^*]$ on $\mathbb{Z}[SG_k(n)]$. We define the graded set $Op^*$ by

$$Op^n = \mathbb{Z}[SG_k(n)]^{|\text{Aut}[\Lambda_k^*]}|.$$

Note that $Op^n$ doesn’t depend on $k$ as long as $n|p^k$. An element $\rho$ of $\mathbb{Z}[SG_k(n)]$ can be represented by

$$\rho = \sum_{i \in I} a_i \prod_{H \in \alpha_i} H$$

where $I$ is a finite set, $a_i \in \mathbb{Z}$, and the $\alpha_i$ are lists of elements of $SG_k(n)$, with possible repetitions. For $\rho \in Op^n$, let $Q^\rho$ be the operation

$$MU^{2*}X \xrightarrow{Q^\rho} D_k^{2*}X$$

given by

$$Q^\rho = \sum_{i \in I} a_i \prod_{H \in \alpha_i} Q^H.$$

In this situation, Proposition 3.4.2 imples
Theorem 3.4.4 (Theorem D) For an element $\rho \in Op$, that is, a polynomial in $\mathbb{Z}[SG_k(n)]$ which is invariant under $\text{Aut}[\Lambda_k^*]$, the operation $Q^\rho$ factors through $E$; that is, $Q^\rho$ defines an operation

$$MU^{2*}X \xrightarrow{Q^\rho} E^{2*}X.$$ 

Corollary 3.4.5 ("Adams operations") The full subgroup $\nu^*F(D^*,2)$ of points of order $p^k$ defines an operation

$$MU^{2*}X \xrightarrow{Q^{p^k}} E^{2p^k}X$$

which has all the properties described in Theorems 3.2.1 and 3.2.4.

Corollary 3.4.6 ("Hecke operations") The operation

$$R_n = \sum_{H \subset \Lambda_k^*} Q^H$$

$|H|=n$

is an additive operation

$$MU^{2*}X \xrightarrow{R_n} E^{2n}X$$
4  Cohomology operations out of exact cohomology theories

In this section we describe a recipe (Theorem 4.0.1) which we shall use repeatedly to produce cohomology operations in exact complex-oriented cohomology theories.

Suppose that $E$ is an exact theory; recall (1.0.2) that this means that

$$E^*X = E^* \otimes_{MU^*} MU^*X,$$

where the tensor product is with respect to a genus

$$MU^* \to E^*.$$

The resulting natural transformation

$$MU^*X \xrightarrow{t_E} E^*X$$

will be denoted $t_E$.

Let $C$ be a cohomology theory with a ring structure, and let $Q$ be a natural transformation of ring-valued functors

$$MU^*X \xrightarrow{Q} \Phi_nC^*X.$$

Let $F^Q$ be the formal group law over $\Phi_nC^*$ classified by $Q(pt)$.

Such $C$ and $Q$ are given, for example, by $D$ and $Q^H$ as in Theorem 3.2.1. Other important examples are produced by Lemma 4.0.2.

Our main result is an immediate consequence Quillen's theorem and the properties of the tensor product.

**Theorem 4.0.1** The operation $Q$ factors through $t_E$ to an operation

$$E^*X \xrightarrow{\beta} \Phi_nC^*X$$

if and only if there is a ring homomorphism

$$\beta : E^* \to \Phi_nC^*$$

such that

$$\beta_*F^E = F^Q.$$

The same result obtains if one restricts to even-dimensional classes, provided that the ring $E^*$ is concentrated in even degrees. □
The operations

\[ MU^{2*}X \xrightarrow{QH} \Phi_x D^{2*}X \]

of section 3.1 can be used as input to Theorem 4.0.1, and section 5 we shall produce the homomorphisms \( \beta^H \) required by the theorem. Another method of producing operations from Theorem 4.0.1 is provided by strict isomorphisms of the formal group law. In this case, it produces a stable operation.

**Lemma 4.0.2** Let \( C \) be a complex-oriented cohomology theory, and let

\[ e_C L \in C^{2 \mathbb{CP}^\infty} \]

be the Euler class of the tautological bundle. For each power series

\[ \theta(e_C L) = e_C L + o(e_C L)^2 \in C^{2 \mathbb{CP}^\infty} \]

there is a unique transformation of ring theories

\[ MU^{*}X \xrightarrow{t_\theta} C^{*}X \]

such that

\[ t_\theta(e_{MU} L) = \theta(e_C L) \]

and

\[ t_{\theta*} F^MU = (F^C)^\theta. \]

Proof: See, for example, Adams [Ada74]. □

**Corollary 4.0.3** (e.g. Miller [Mil89]) Let \( E \) and \( C \) be complex-oriented cohomology theories, and suppose that \( E \) is exact. Suppose that \( \beta : E^{*} \rightarrow C^{*} \) is a ring homomorphism, and that there is a strict isomorphism of formal group laws

\[ F^Q \xrightarrow{\theta} \beta_* F^E. \]  (4.0.4)

Then \( \beta \) extends to a stable natural transformation of functors of finite complexes (to rings)

\[ E^{*}X \xrightarrow{\lambda} C^{*}X \]

whose effect on Euler classes is given by

\[ \lambda(e_E L) = \theta(e_C L). \]  (4.0.5)

Proof: Use Lemma 4.0.2 to produce a natural transformation of ring theories

\[ MU^{*}X \xrightarrow{t_\theta} C^{*}X \]
whose effect on Euler classes is

\[ t_\theta(e_{MU}L) = \theta(e_CL). \]

By Lemma 3.2.8, the effect of \( t_\theta \) on coefficients is given by

\[ F^C \xrightarrow{t_\theta} t_\theta F^MU. \]

Now apply Theorem 4.0.1 to \( t_\theta \) and \( \beta \). \( \square \)

Our main application of (4.0.2) is to the normalized quotient (2.2.8)

\[ \nu f_H(x) = \frac{f_H(x)}{a_H} \in \frac{1}{a_H} D^*[x]. \]

By applying Lemma 4.0.2 to \( \nu f_H(x) \) we obtain

**Proposition 4.0.6** There is a stable natural transformation

\[ MU^*X \rightarrow \frac{1}{a_H} D^*X \]

of functors of finite complexes to rings. On coefficients, it classifies the normalized quotient

\[ \nu Q^H F^MU = \nu F/H. \]

The effect of \( \nu Q^H \) on the Euler class of a line bundle is

\[ \nu Q^H(eL) = \nu f_H(eL). \]

Proof: By construction (2.2.8), \( f_H(x) = x + o(x^2) \) is homogeneous of degree 2 when the degree of \( x \) is 2. Also by definition

\[ \nu F/H = F^{\nu f_H}. \]

Lemma 4.0.2 provides us with the operation \( \nu Q^H \). \( \square \)

The relationship between the operations \( \nu Q^H \) and \( Q^H \) will allow us to prove integrality statements about stable operations.

**Proposition 4.0.7** Let \( H \subset \Lambda^*_\infty \) be a subgroup of order \( n \). On even-dimensional classes, the diagram

\[
\begin{array}{ccc}
MU^{2*}X & \rightarrow & MU^{2*}X \\
\xrightarrow{Q^H} & \downarrow \xrightarrow{\nu Q^H} & \\
D^{2n*}X & \rightarrow & \xrightarrow{\delta H} D^{2n*}X \\
\end{array}
\]

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commutes, where $\delta^H$ is the homomorphism of (2.2.10)

$$\delta^H(m) = \frac{\text{im}}{a_H} m.$$  

Proof: By the definition of $f_H$ and $\nu f_H$, and by Theorems 3.2.4 and 4.0.6, the diagram commutes on Euler classes of line bundles. By the definition of $F/H$ and $\nu F/H$ and Theorems 3.2.1 and 4.0.6, it commutes on coefficients. Moreover both the clockwise and counterclockwise composites are maps of rings. Since $\nu Q^H$ is stable, the result follows since the facts listed so far imply that the two operations determine the same element of 

$$\text{Hom}_{\pi_* MU}[MU_* MU, \frac{1}{a_H} D^*].$$

For odd-dimensional classes, it will be useful to know that

**Proposition 4.0.8** The operation $a_H^r \nu Q^H$ is integral on $(2r-1)$-dimensional classes:

$$MU^{2r-1} X \xrightarrow{a_H^r \nu Q^H} D^{2nr-1} X \subset \frac{1}{a_H} D^* X.$$  

Proof: Let $\sigma$ denote the suspension isomorphism, and suppose $m \in MU^{2r-1} X$. By Proposition 4.0.7, we have

$$a_H^r \nu Q^H(\sigma m) = Q^H(m) \in D^{2nr} X.$$  

Since $\nu Q^H$ is a stable operation,

$$a_H^r \nu Q^H(\sigma m) = a_H^r \sigma \nu Q^H(m).$$

Finally, we have

$$D^{2nr-1} X \xrightarrow{\frac{1}{a_H}} D^{2nr-1} X$$

$$\sigma$$

$$D^{2nr} X \xrightarrow{\frac{1}{a_H}} D^{2nr} X. \square$$
5 Power operations in $E_h$

In this section we focus on the cohomology theory $E_h$, and study the problem of constructing a ring homomorphism

$$E_h^* \xrightarrow{\beta^H} \Phi_n D^*$$

such that

$$\beta^H_* F = F/H,$$  \hspace{1cm} (5.0.1)

as required by Theorem 4.0.1. Our main result (Theorem 5.3.1) is that there is an (essentially unique) genus

$$MU^* \xrightarrow{\text{top.}} E_h^*$$

such that the resulting formal group law satisfies this condition for all finite subgroups $H$. This is the genus announced in Theorem A.

As a result, one obtains an operation

$$E^{2n}X \xrightarrow{\Phi^H} D^{2n}X$$

for every $H$. In section 5.4 we study the case of the group $H = \pi^* F(D^*, 2)$, which yields the unstable Adams operation $\Psi^p$ (Theorem E). We begin by describing

5.1 The cohomology theory $E_h$

Let $E_h^*$ be the complete, Noetherian, local domain

$$E_h^* = \mathbb{Z}[u_1, \ldots, u_{h-1}][u, u^{-1}], |u_i| = 0, |u| = -2.$$

Let $t_h$ be the genus

$$MU^* \xrightarrow{\text{top.}} E_h^*$$

which is given on the Araki generators [Rav86] by

$$t_h(u_i) = \begin{cases} u_i u^{p^{i-1}} & i \leq h - 1, \\ u^{p^{h-1}} & i = h, \text{ and} \\ 0 & i > h. \end{cases}$$  \hspace{1cm} (5.1.1)

It is an immediate consequence of Landweber's exact functor theorem that

**Proposition 5.1.2** The functor

$$E_h^* X = E_h^* \otimes_{MU^*} MU^* X$$

is a cohomology theory.
The important feature of $E_h$ is that the coefficient ring $E_h^*$ and the formal group law $F_h = t_h F^{MU}$ represent a functor. Denote by

$$K_h^* = F_p[u, u^{-1}]$$

the graded residue field of $E_h^*$, and by

$$\varphi(x, y) \in K_h^*[x, y]$$

the formal group law over $K_h^*$ obtained by reducing the coefficients of $F_h$ modulo the maximal ideal $E_h^*$. Note that since the formal group law $F_h$ satisfies [Rav86]

$$[p]_{F_h}(x) = px + u_1 u^{p-1} x^p + \ldots + u_{h-1} u^{p(h-1)-1} x^{p(h-1)} + u^{p^h-1} x^{p^h}, \quad (5.1.3)$$

the formal group law $\varphi$ satisfies

$$[p]_{\varphi}(x) = u^{p^h-1} x^{p^h},$$

and so has height $h$.

Let $R^*$ be a complete, local, graded $\mathbb{Z}[u, u^{-1}]$-algebra, with residue field $\mathcal{K}^*$. When $f$ is a power series over a local ring, we denote by $\overline{f}$ the power series over the residue field obtained by reducing the coefficients of $f$ modulo the maximal ideal.

**Definition 5.1.4** A formal group law $G$ over $R^*$ which is homogeneous of degree 2 is a deformation of $\varphi$ to $R$ if

$$G(x, y) = \varphi(x, y) \in \mathcal{K}^*[x, y].$$

Two deformations $G_1$ and $G_2$ are $\ast$-isomorphic if there is an isomorphism of formal groups

$$G_1 \overset{f}{\rightarrow} G_2$$

such that

$$\overline{f}(x) = x.$$

The fundamental result about $E_h^*$ is

**Theorem 5.1.5 ([LT66])** Let $F$ be any formal group law $\ast$-isomorphic to $F_h$. The ring $E_h^*$ and the formal group law $F$ represent the functor

$$R^* \longrightarrow \left\{ \ast - \text{isomorphism classes of deformations of } \varphi \text{ to } R^* \right\}$$

In fact, if $G$ is a deformation of $\varphi$ to $R$, then there is a unique homomorphism

$$E_h^* \overset{\varphi}{\rightarrow} R$$
such that for any choice of $F$, $G$ is $\ast$-isomorphic to $\beta_\ast F$. Moreover, the $\ast$-isomorphism

$$G \xrightarrow{\imath} \beta_\ast F$$

is uniquely determined by $G$ and $F$.

Let $D_k^\ast$ be the ring of section 2.3 for the pair $(E_n^\ast, F_h)$.

**Corollary 5.1.6** The ring $D_k^\ast$ represents the functor

$$R \mapsto \left\{ \begin{array}{c} \ast - \text{isomorphism classes of deformations } G \text{ of } \varphi \text{ to } R^\ast, \\ \text{with a level-} \Lambda_k^\ast \text{ structure} \\ \Lambda_k^\ast \xrightarrow{\varphi} G(R, 2)_{\text{tors}} \end{array} \right\}.$$

**Proof:** This follows from Theorem 5.1.5 and Theorem 2.3.6.

### 5.2 Lubin's quotients $F/\mathbb{H}$ are deformations

Because the theories $E_n^\ast$ are periodic, constructing a homomorphism (for a subgroup $H \subset \Lambda_\infty^\ast$ of order $n$)

$$E_n^\ast \xrightarrow{\beta^H} \Phi_n D^\ast$$

such that

$$\beta_\ast^H F = F/H$$

is equivalent to constructing a homomorphism

$$E_n^\ast \xrightarrow{\gamma^H} D^\ast$$

such that

$$\gamma_\ast^H F = (F/H)^{n-1} z :$$

by Lemma 2.2.13, $\beta^H$ and $\gamma^H$ are related by

$$\beta^H = \mu^n \circ \gamma^H,$$

where

$$E_n^\ast \xrightarrow{\mu^n} E_n^\ast$$

is the homomorphism

$$\mu^n(m) = u^{\frac{(1-n)\text{max}}{2}} m.$$

**For this chapter only**, we take Lubin's quotient $f_H(x)$ to be homogeneous of degree 2. In other words, the homomorphism which is elsewhere written

$$u^{n-1} f_H(x).$$
Similarly, we take the quotient formal group law
\[ F/H \]
to be homogeneous of degree 2, so one gets a homomorphism
\[ H \to F(D^*, 2) \xrightarrow{\text{inh}} F/H(D^*, 2). \]
A $\ast$-isomorphism
\[ F \xrightarrow{\delta} F_h \]
yields an isomorphism
\[ F(R^*, 2) \xrightarrow{\delta} F_h(R^*, 2), \]
so a level structure
\[ A^* \xrightarrow{\phi} F(R^*, 2) \]
is equivalent to a level structure
\[ A^* \xrightarrow{\delta \phi} F_h(R^*, 2). \]

The utility of Theorem 5.1.5 in our work stems from

**Lemma 5.2.1** The formal group law $F/H$ is a deformation of $\varphi$ to $D^\ast$. Moreover, the correspondence
\[ F \leftrightarrow F/H \]
preserves $\ast$-isomorphism classes.

Proof: Recall that $F/H$ is defined by the homomorphism
\[ F \xrightarrow{J_h} F/H \]
where
\[ J_h(x) = u^{n-1}x^n. \]
$F/H$ is a deformation since $u^{n-1}x^n$ is an endomorphism of $\varphi$, so the diagram
\[ \begin{array}{ccc}
F & \xrightarrow{J_h(x)} & F/H \\
\downarrow & & \downarrow \\
\varphi & \xrightarrow{u^{n-1}x^n} & \varphi
\end{array} \]
commutes, where the vertical arrows represent reduction modulo the maximal ideal. To prove the part about preservation of $\ast$-isomorphism classes, suppose that
\[ F \xrightarrow{\delta} F' \]
is a $\star$-isomorphism. Suppose that

$$F' \xrightarrow{f_H} F'/H$$

is the homomorphism based on the group law $F'$. Since

$$F(D^*, 2) \xrightarrow{g} F'(D^*, 2) \xrightarrow{f_H} F'/H(D^*, 2)$$

and

$$F(D^*, 2) \xrightarrow{f_H} F/H(D^*, 2)$$

have the same kernel, Theorem 2.2.3 implies that there is a unique isomorphism

$$F/H \xrightarrow{g'} F'/H$$

such that

$$
\begin{array}{ccc}
F & \xrightarrow{g} & F' \\
\downarrow f_H & & \downarrow f_H' \\
F/H & \xrightarrow{g'} & F'/H \\
\end{array}
$$

commutes. $g'$ is a $\star$-isomorphism since

$$f_H(x) = f_H(x) = u^{-1} x^n. \quad \square$$

Let

$$E_h^* \xrightarrow{\gamma^H} D^*$$

be the homomorphism determined by the $\star$-isomorphism class corresponding to $H$, and let

$$F/H \xrightarrow{g_H} \gamma^H_{\ast} F_h$$

be the $\star$-isomorphism determined by $H$ and a group law $F$ which is $\star$-isomorphic to $F_h$. Then

**Lemma 5.2.3** $\gamma^H$ is the unique homomorphism

$$E^*_h \to D^*$$

such that $\gamma^H_{\ast} F$ is $\star$-isomorphic to $F/H$ as deformations of $\varphi$. \quad \square

We really want strict equality in equation (5.2.2).
5.3 Theorem F and power operations in $E_h$

The main result of this chapter is

**Theorem 5.3.1 ("Theorem F")** There is a unique formal group law $F_{po}$ over $E_H$ which is $\ast$-isomorphic to $F_h$ and which satisfies

$$\gamma^H_{*} F_{po} = F_{po}/H$$

(5.3.2)

for all finite subgroups $H$.

We give a proof in section 5.5. For now, we examine the consequences of Theorem 5.3.1 for cohomology operations. Let

$$MU^\ast X \xrightarrow{t_{po}} E_h^\ast X$$

denote the orientation specified by Theorem 5.3.1. As remarked in the introduction, equation (5.3.2) is guaranteed to be satisfied by a formal group law $F$ if there is a total power operation

$$E^2_n X \xrightarrow{P_{\pi}} E^2_n D_{\pi} X$$

which is compatible with the orientation

$$MU^\ast X \xrightarrow{t} E_h^\ast X$$

in the sense that

$$MU^2_n X \xrightarrow{P_{MU}} MU^2_n D_{\pi} X$$

(5.3.3)

$$E^2_n X \xrightarrow{P_{\pi}} E^2_n D_{\pi} X$$

commutes. On the other hand, using the orientation provided by Theorem 5.3.1, the operation $Q^H$, and Theorem 4.0.1, we have

**Corollary 5.3.4** If $H \subset \Lambda^\ast_\infty$ is a finite subgroup of order $n$, then there is a unique operation

$$E^2_n X \xrightarrow{\psi_H} D^2_n X$$

such that

$$MU^2_n X \xrightarrow{t_{po}} E^2_n X$$

$$E^2_n X \xrightarrow{\psi_H} D^2_n X$$

commutes.

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Theorem 5.3.5 The operation

\[ MU^{2*}X \xrightarrow{T^\mathbb{F}_h} \prod_{\Lambda_\infty} D^{2n*}X^{\pi/\langle \alpha \rangle}, \]

which is the image of the total power operation of \( MU \) under the character map of Hopkins-Kuhn-Ravenel (3.3.2), factors through the orientation

\[ MU^*X \xrightarrow{ip^G} E^*_hX \]
to an operation

\[ E^{2*}X \xrightarrow{\mathcal{P}_h} \prod_{\Lambda_\infty} D^{2n*}X^{\pi/\langle \alpha \rangle}. \]

The projection to the component corresponding to a homomorphism

\[ \Lambda_\infty \xrightarrow{\alpha} \pi \]
is

\[ pr_\alpha \mathcal{P}_h(z) = \Psi^H(z^{x^k}) \in D^{n*}(X^{\pi/\langle \alpha \rangle}), \]
where \( k = [\pi : \langle \alpha \rangle] \).

Proof: This is just Corollary 5.3.4 combined with Theorem 3.3.3. \( \square \)

Corollary 5.3.4 is of particular interest in the case \( H = \Lambda_\infty^* \), for then one obtains

5.4 The unstable Adams operation \( \Psi^p_\pi \) as a power operation.

Denote by \( f_p, F/p, \) etc., the quotient constructions corresponding to the subgroup \( \Lambda_1^* \cong \mathbb{F}(D^*, 2) \). By (3.4.5), we have

\[ f_p(x) \in E^*_h[[x]], \]
\[ F_h/p(x, y) \in E^*_h[[x, y]], \] and
\[ MU^{2*}X \xrightarrow{Q^p} E^{2\mathbb{F}^h*}X. \]

Proposition 5.4.1 For any formal group law \( F \ast \)-isomorphic to \( F_h \), there is a unique \( \ast \)-isomorphism

\[ F/p \xrightarrow{g^p} F. \]

Proof: First of all, note that \([p]_F(x)\) is an endomorphism of \( F \). Because of the co-equalizer diagram

\[ pF(D^*, 2) \twoheadrightarrow F(D^*, 2) \xrightarrow{[p]_F(x)} F(D^*, 2), \]
it follows from Theorem 2.2.3 that there is a unique isomorphism of formal group laws

\[ F/p \xrightarrow{\varphi} F \]

such that

\[ [p]_F(x) = g^p(f_p(x)). \tag{5.4.2} \]

The fact that

\[ f_p(x) = u^{p^h-1}x^{p^h} = [p]_F(x), \]

shows that

\[ g^p(x) = x, \]

so \( g^p \) is a \( * \)-isomorphism. \( \square \)

**Corollary 5.4.3** The homomorphism \( \gamma^p \) determined by the deformation \( F/p \) is the identity, and \( \beta^p \) is the homomorphism

\[ \beta^p = \mu^{p^h}. \]

**Corollary 5.4.4** The condition

\[ \gamma^p_* F = F/p, \]

which is a special case of equation (5.3.2), is equivalent to the condition

\[ f_p(x) = [p]_F(x). \tag{5.4.5} \]

Proof: The proof of (5.4.3) is just Proposition 5.4.1. (5.4.4) follows from equation (5.4.2), since with the formal group law of Theorem 5.3.1, we have

\[ g^H(x) = x \]

for all \( H \). \( \square \)

**Remark:** Equation (5.4.5) turns out to determine the formal group law \( F_{PO} \) and figures in the proof of Theorem 5.3.1. See section 5.5.

Theorem 5.3.4 applied to \( Q^p \), together with Corollaries 5.4.3 and 5.4.4, are the case \( k = 1 \) of

**Theorem 5.4.6 (“Theorem E”)** With the orientation

\[ MU_* X \xrightarrow{f_{PO}} E_\lambda_* X \]

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described in Theorem 5.3.1, the operation

\[ E^*_h X \xrightarrow{\psi^k} E^*_{h^2} X \]

obtained in Corollary 5.3.4 satisfies

\[ \Psi^k(eL) = u^{1-p^h} [p^k]_F(eL) \]  

(5.4.7)

when \( eL \in E^*_h X \) is the Euler class of a complex line bundle. On coefficients, \( \Psi^k \) is given by

\[ \Psi^k = \mu^{p^h}. \]

The cases \( k > 1 \) are verified in the same manner. □

Remarks: Because of (5.4.7), we call \( \Psi^k \) the “\( p^h \)th unstable Adams operation” in \( E_h \). Although an Adams operation could have been defined with any orientation on \( E_h \), our orientation is the unique one in which the Adams operation is obtained as a power operation, in the spirit of [Ati66]. More familiar in the context of theories like \( E_h \) is the stable Adams operation

\[ E^*_h X \xrightarrow{\psi^p} \frac{1}{p} E^*_h X. \]

This operation is obtained, using the method of Corollary 4.0.3, from the power series

\[ \frac{[p]_F(eL)}{p} \in \frac{1}{p} E^*_h \mathbb{CP}^\infty \]  

(5.4.8)

and the homomorphism

\[ E^*_h \xrightarrow{1} \frac{1}{p} E^*_h \]

\[ m \mapsto p^{\frac{|m|}{2}} m. \]  

(5.4.9)

With the orientation \( t_{PO} \), it coincides with the operation produced from the stable operation \( \nu Q^p \) (4.0.6) using the homomorphism (5.4.9). Then our comparison theorems for \( \nu Q^p \) and \( Q^p \) (4.0.7, 4.0.8) prove that on even-dimensional classes, we have

\[ \psi^p(x) = \left( \frac{u^{p^h} - 1}{p} \right) \frac{|x|}{2} \Psi^p(x), \]

and moreover that
Proposition 5.4.10 The unstable operation $p^i \psi^p$ is integral on $2i$ and $(2i - 1)$-dimensional classes:

\[
E_{2i}^X \xrightarrow{p^i \psi^p} E_{2i}^X \quad \text{and} \quad E_{2i-1}^X \xrightarrow{p^i \psi^p} E_{2i-1}^X. \]

Remark: This result is well-known to the experts; see e.g. [Wil82]. The attractive feature of our method is that the proofs imitate the “geometric” proofs available in $K$-theory.

The rest of this section is devoted to a

5.5 Proof of Theorem F.

Our proof is divided into two steps, of which the most important is Theorem 5.5.1. Since the formal group laws $F/H$ are deformations of $\varphi$ (5.2.1), it shows there is a unique formal group law $F_H$ which is $\ast$-isomorphic to $F/H$ and which satisfies equation (5.4.5). The case $H = 0$ yields $F_{PO}$. Since $\gamma^H$ is a ring homomorphism, it preserves this formula for the $p$-series, and so must send $F_{PO}$ to $F_H$. The second step (5.5.16) is to show that the $p$-series of $F_{PO}/H$ also satisfies equation (5.4.5). By uniqueness, it will follow that

\[ F_{PO}/H = F_H = \gamma^H_{\ast} F_{PO}. \]

Theorem 5.5.1 ("Theorem A") Let $E^\ast$ be a complete, graded, local domain which is a $\mathbb{Z}[u, u^{-1}]$-algebra, and let $F$ be a deformation of $\varphi$ to $E^\ast$. Let $E_1^\ast \supset E^\ast$ be an extension over which there is a level-$\Lambda_1^\ast$ structure

\[ \Lambda_1^\ast \xrightarrow{\beta} pF(E_1^\ast, 2). \]

There is exactly one representative of the $\ast$-isomorphism class of $F$ over $E^\ast$ which satisfies (5.4.5).

Proof: Existence: It suffices to prove the statement in the universal case: suppose we can construct a formal group law $F_{PO}$ over $E_1^\ast$ satisfying (5.4.5). If

\[ E_1^\ast \xrightarrow{\beta} E^\ast \]

is the homomorphism such that there is a $\ast$-isomorphism

\[ F \xrightarrow{\varphi} \beta_{\ast} F_{PO}, \]

then the level-$\Lambda_1^\ast$ structure

\[ \Lambda_1^\ast \xrightarrow{\beta} pF(E_1^\ast, 2) \]
determines an extension of $\beta$ to a homomorphism
\[ D_1^* \xrightarrow{\beta} E_1^* \]
such that
\[ \Lambda_1^* \xrightarrow{\phi_{\text{univ}}} \Lambda_1^* \]
\[ pF_{\text{PO}}(D_1^*, 2) \xrightarrow{\beta} p\beta_*F_{\text{PO}}(E_1^*, 2). \]
commutes. Then
\[ (p[\beta_*F_{\text{PO}}(x)] = \beta_* \left[ \prod_{c \in \Lambda_1^*} (x + \phi_{\text{univ}}(c)) \right] \]
\[ = \prod_{c \in \Lambda_1^*} (x + \beta_*\phi_{\text{PO}}(\phi(c))), \quad (5.5.2) \]
which is (5.4.5) for $\beta_*F_{\text{PO}}$.

We turn to the universal case. For the purposes of the exposition, it is extremely convenient to take advantage of the distinction between formal groups and formal group laws: let $L$ denote the Lubin-Tate formal group on $E_h$ for lifts of the formal group law $\varphi$. For any choice of coordinate $y$ on $L$, denote by $F^y$ the resulting formal group law. Then the homomorphism
\[ \Lambda_1^* \xrightarrow{\phi} F^y(D_1^*) \]
factors as
\[ \Lambda_1^* \xrightarrow{\phi} L(D_1^*) \xrightarrow{\varphi} F^y(D_1^*). \]
Under a change of coordinate
\[ F^y \xrightarrow{\delta} F^x, \quad \text{x}-\text{iso} \]
the level-$\Lambda_1^*$ structure changes by
\[ \Lambda_1^* \xrightarrow{\phi} L(D_1^*) \xrightarrow{\varphi} F^y(D_1^*) \]
\[ \Downarrow \quad \Downarrow \quad \Downarrow \delta \]
\[ \Lambda_1^* \xrightarrow{\phi} L(D_1^*) \xrightarrow{\varphi} F^x(D_1^*) \]
In this language, Lubin's homomorphism $f_p$ becomes
\[ f_p^y(x) = \prod_{c \in \Lambda_1^*} (x + y(\phi(c))). \]

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The proof is inductive, on powers of the maximal ideal $I$ of $D^*_1$. Let $y$ be any coordinate on $\mathcal{L}$ which gives a group law $F^y$ *-isomorphic to $F_h$. Let $g^y(t) \in E_h[[t]]$ denote the unique *-isomorphism

$$F^y/p \xrightarrow{g^y} F^y$$

such that

$$[p]_{F^y}(t) = g^y(f^y_p(t)),$$  \hspace{1cm} (5.5.3)

whose existence and uniqueness are the matter of Proposition 5.4.1. Write

$$g^y(t) = t + a(t).$$

Since $g^y$ is a *-isomorphism, we get automatically the case $n = 2$ of the hypothesis that

$$a(t) = \sum_{j \geq 1} a_j t^j, \text{ with } a_j \in I^{n-1}. $$

Let $\delta(t)$ be the power series

$$\delta(t) = t - a(t).$$

Since $g^y$ is defined over $E_h^*$, the coordinate

$$x = \delta(y)$$

on $\mathcal{L}$ yields a formal group law $F^x$ over $E_h^*$. We shall show that the *-isomorphism $g^x(t)$ such that

$$[p]_{F^x}(t) = g^x(f^x_p(t))$$  \hspace{1cm} (5.5.4)

satisfies

$$g^x(t) \equiv t \pmod{I^n}. $$ \hspace{1cm} (5.5.5)

By construction, $\delta$ is a homomorphism of formal group laws

$$F^y \xrightarrow{\delta} F^x.$$

Thus the diagram

$$\begin{array}{ccc}
F^y & \xrightarrow{\delta} & F^x \\
\downarrow{[p]^y} & & \downarrow{[p]^x} \\
F^y & \xrightarrow{\delta} & F^x \\
\end{array}$$

commutes; in other words, we have

$$\delta([p]^y(y)) = [p]^x \delta(y).$$ \hspace{1cm} (5.5.6)
Substituting (5.5.3) and (5.5.4) into this equation yields
\[ \delta[g^\gamma(f_p^\gamma(y))] = g^\gamma[f_p^\gamma(\delta(y))] \] (5.5.7)

Notice that
\[ f_p^\gamma(\delta(y)) = u^{\phi_b} \prod_{c \in A^*} \left( \delta(y) + x(\phi(c)) \right) \]
\[ = u^{\phi_b} \prod_{c \in A^*} \delta(\delta^{-1}(\delta(y)) + \delta^{-1}(x(\phi(c)))) \]
\[ = u^{\phi_b} \prod_{c \in A^*} \delta(y + y(\phi(c))) \]
so (5.5.7) becomes
\[ \delta[g^\gamma(f_p^\gamma(y))] = g^\gamma[u^{\phi_b} \prod_{c \in A^*} \delta(y + y(\phi(c)))]. \] (5.5.8)

We can evaluate the left side modulo \( I^n \) very easily:
\[ \delta(g^\gamma(f_p^\gamma(y))) = \delta(f_p^\gamma(y) + a(f_p^\gamma(y))) \]
\[ \equiv \delta(f_p^\gamma(y) + a(u^{\phi_b} y^{\phi_b})) \]
\[ \equiv f_p^\gamma(y) + a(u^{\phi_b} y^{\phi_b}) - a(f_p^\gamma(y) + a(u^{\phi_b} y^{\phi_b})) \]
\[ \equiv f_p^\gamma(y) + a(u^{\phi_b} y^{\phi_b}) - a(f_p^\gamma(y)) \]
\[ \equiv f_p^\gamma(y), \] (5.5.9)

where the equivalences are all modulo \( I^n \). The first equivalence uses the fact that \( f_p^\gamma(y) \equiv u^{\phi_b} y^{\phi_b} \) (mod \( I \)) plus the fact that \( a(t) \) is a power series with constant term zero and coefficients in \( I^{n-1} \) (with \( n \geq 2! \)). The third equivalence follows from this description of \( a(t) \), and the last equivalence is like the first.

All we know \( \text{a priori} \) about \( g^\gamma \) is that it is a power series of the form
\[ g^\gamma(t) = t + b(t) \]
where
\[ b(t) = \sum_{j \geq 1} b_j t^j, \quad b_j \in I, \] (5.5.10)
so we start by evaluating right-hand-side of (5.5.8) modulo $I^2$. We can compute

$\prod_{c \in \Lambda_1^*} \delta(y + y(\phi(c)))$ modulo $I^n$:

$$\prod_{c \in \Lambda_1^*} \delta(y + y(\phi(c))) = \prod_{c \in \Lambda_1^*} [(y + y(\phi(c)) - a(y + y(\phi(c)))]$$

$$\equiv \prod_{c \in \Lambda_1^*}(y + y(\phi(c))) - \sum_c \left[ \prod_{d \neq c} (y + y(\phi(d))) \right] a(y + y(\phi(c)))$$

$$\equiv \prod_{c \in \Lambda_1^*}(y + y(\phi(c))) - \sum_c y^{p^h-1} a(y)$$

$$\equiv \prod_{c \in \Lambda_1^*}(y + y(\phi(c))) - p^h y^{p^h-1} a(y)$$

$$\equiv \prod_{c \in \Lambda_1^*}(y + y(\phi(c)))$$

(Recall that $p \in I$ is contained in the maximal ideal.) Continuing (5.5.11) modulo $I$ shows that

$$\prod_{c \in \Lambda_1^*} \delta(y + y(\phi(c))) \equiv y^{p^h} \pmod{I}.$$ 

Equipped with these observations, we compute that if

$$b(t) \equiv 0 \pmod{I^{i-1}}, \quad j \geq 2,$$

then for $j \leq n$,

$$g^x(u^{p^h-1} \prod_{c \in \Lambda_1^*} \delta(y + y(\phi(c)) = u^{p^h-1} \prod_{c \in \Lambda_1^*} \delta(y + y(\phi(c)))$$

$$+ b(u^{p^h-1} \prod_{c \in \Lambda_1^*} \delta(y + y(\phi(c)))$$

$$\equiv f_p^y(y) + b(u^{p^h-1}y^{p^h}) \pmod{I^j}$$

Comparing (5.5.9) and (5.5.12), it follows in view of (5.5.10) that

$$b(t) \equiv 0 \pmod{I^j},$$

and inductively that

$$b(t) \equiv 0 \pmod{I^n}.$$

**Uniqueness:** Suppose that $F^y$ satisfies (5.4.5) and that

$$F^y \xrightarrow{\delta} F^x$$

is a $*$-isomorphism

$$\delta(t) = t + \sum_{j \geq 1} a_j t^j.$$
Let \( m \) be the maximal ideal of \( E^* \), and let \( n \) be as large as possible so that for all \( j \). Note that \( n \geq 1 \), and that if \( n = \infty \) then \( F = F' \), so we may suppose that \( n \) is finite. Then, working modulo \( m^{n+1} \), we have

\[
[p]^x(x) = \delta(\delta^{-1}(x))
\]

\[
= \delta(\delta^{-1}(x) + y(\phi(c)))
\]

\[
= \delta(\sum_{c \in A^*_1} \delta^{-1}(x + \delta(y(\phi(c)))))
\]

\[
= \prod_{c \in A^*_1} \delta^{-1}(x + \delta(y(\phi(c))))
\]

\[
= \prod_{c \in A^*_1} \delta^{-1}(x + \delta(y(\phi(c)))) + a(u^{p^{n-1}}x^{p^n})
\]

\[
= f_p^x(x) - \prod_{c \in A^*_1} \delta^{-1}(x + \delta(y(\phi(c)))) + a(u^{p^{n-1}}x^{p^n})
\]

\[
= f_p^x(x) - u^{p^{n-1}} \prod_{c \in A^*_1} \delta^{-1}(x + \delta(y(\phi(c)))) + a(u^{p^{n-1}}x^{p^n})
\]

\[
= f_p^x(x) + a(u^{p^{n-1}}x^{p^n})
\]

so \( F^x \) fails to satisfy (5.4.5).

Denote by \( F_{PO} \) the formal group law over \( E^*_k \) *-isomorphic to \( F_h \) which is constructed in Theorem 5.5.1. For a finite subgroup \( H \subset A^*_\infty \) of exponent \( p^k \) and order \( n \) and for \( l \geq k \) we get level-\( \Lambda^*_l/H \) structures on the formal group law \( F_{PO}/H \) via

\[
\begin{align*}
H & \xrightarrow{\phi} H \\
\Lambda^*_l & \xrightarrow{\phi} F_{PO}(D^*_l, 2) \\
\Lambda^*_l/H & \xrightarrow{\phi/H} F_{PO}/H(D^*_l, 2).
\end{align*}
\]

Corollary 5.5.14 The formal group law \( \gamma^H_{*} F_{PO} \) is the unique formal group law in the *-isomorphism class determined by \( H \) over \( D^*_k \) such that over \( D^*_{k+1} \),

\[
[p]_{\gamma^H_{*} F_{PO}}(x) = \prod_{c \in A^*_k/H \atop \phi(c) = 0} \delta^H_{F_{PO}}(x + \delta^H_{F_{PO}}(c)),
\]

where \( \delta \) is the *-isomorphism

\[
F_{PO}/H \xrightarrow{\delta} \gamma^H_{*} F_{PO}
\]

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Proof: Let $\Lambda'_1$ denote the group

$$\Lambda'_1 = \rho(\Lambda_{k+1}/H).$$

Any isomorphism

$$\Lambda'_1 \cong \Lambda'_1$$

determines a level-$\Lambda_1$ structure

$$\Lambda'_1 \cong \Lambda'_1 \xrightarrow{\partial H} F_{PO}/H(D^*_k,2).$$

and so an extension of $\gamma^H$ to a homomorphism

$$D^*_1 \xrightarrow{\gamma^H} D^*_k.$$

We are now in the situation in the beginning of the proof of Theorem 5.3.1; compare (5.5.2). □

The proof of Theorem 5.3.1 is completed by

**Proposition 5.5.16** For any finite subgroup $H \subset \Lambda^*_\infty$, the quotient formal group law $F_{PO}/H$ satisfies (5.5.15).

Proof: The quotient $F_{PO}/H$ is determined by the homomorphism $f_H$. The $p$-series $[p]_{F_{PO}/H}(x)$ is determined by the functional equation

$$F_{PO} \xrightarrow{I_H} F_{PO}/H$$

$$[p]_{F_{PO}} \downarrow \quad \quad \quad [p]_{F_{PO}} \downarrow (5.5.17)$$

$$F_{PO} \xrightarrow{I_H} F_{PO}/H.$$

Let $Z(x)$ be the product

$$Z(x) = u^{n-1} \prod_{c \in \Lambda^*_k/H} (x + f_{PO}/H(c)).$$

We are going to show that $Z(x)$ satisfies (5.5.17). Let $p^{-1}H$ denote the subgroup

$$p^{-1}H = \{c \mid pc \in H\} \subset \Lambda^*_k.$$
for the group law \( F_{\mathcal{P}O}/H \). The composition rule (2.2.4) applied to this case is
\[
f_{p^{-1}H/H}(f_{H}(x)) = f_{p^{-1}H}.
\]
On the other hand, by construction
\[
[p]_{F_{\mathcal{P}O}}(x) = f_{p}(x),
\]
so
\[
f_{H}([p]_{F_{\mathcal{P}O}}(x)) = f_{H}(f_{p}(x)) = f_{p^{-1}H}(x).
\]

5.6 Example: K-theory

The group law of K-theory is the multiplicative group law
\[
x \circ_{K} y = x + y - vxy,
\]
where \( v \in K^{-2} \) is the Bott element. This group law arises from the fact that the Euler class of a line bundle \( L \)
is
\[
e_{L} = v^{-1}(1 - L).
\]
The roots of the \( p \)-series of \( F^{K} \) are
\[
v^{-1}(1 - \zeta^{j}), \quad 0 \leq j < p,
\]
where \( \zeta = e^{2\pi i/p} \).

Suppose \( p = 2 \). Then
\[
[2](x) = 2x - vx^{2}.
\]
On the other hand,
\[
f_{2}(x) = v^{-1}(1 - L)[v^{-1}(1 - L) + 2v^{-1}]
\]
\[
= v^{-2}(1 - L)[1 - L + 2 - 2 + 2L]
\]
\[
= v^{-2}(1 - L^{2})
\]
\[
= v^{-1}[2](x).
\]
Similarly, for \( p \) odd, we have
\[
[p](x) = v^{-1}(1 - L^{p})
\]
while

\[ f_p(x) = \prod_{j=0}^{p-1} [v^{-1}(1 - L) + v^{-1}(1 - \zeta^j)] \]

\[ = v^{-p} \prod_{j=0}^{p-1} (1 - \zeta^j L) \]

\[ = v^{-p} \zeta^p(p-1)/2 \prod_{j=0}^{p-1} (\zeta^{-j} - L) \]

\[ = v^{-p}(1 - L^p) \]

\[ = v^{1-p}[p](x). \]

Thus the multiplicative group law is the unique formal group law over \((p\text{-adic}) K\text{-theory}\) which is \(*\)-isomorphic to \(F_1\) and which satisfies (5.4.5). In the case of \(E_1\), Theorem A picks out the multiplicative formal group law.

The construction of the Adams operation \(\Psi^p\) given as Theorem E in \(p\text{-adic} K\text{-theory}\) is in the same spirit as Atiyah's construction \([Ati66]\). In the case of the multiplicative group law, the only finite subgroups of the formal group law are the groups \(nF^K\), since the height of the group law is one. For greater heights, however, there are other interesting subgroups of order \(n\). As an example, we turn now to the study of elliptic cohomology.
6 Elliptic cohomology

Mike Hopkins [Hop89] has described the application of the character theory of Hopkins-Kuhn-Ravenel to the elliptic cohomology theory \( \text{Ell} \) of Landweber-Ravenel-Stong [LRS88]. In this section we add this information to the ideas developed in sections 3 and 4 to produce two sorts of operations in elliptic cohomology. In section 6.4 we produce an "Adams operation"

\[
\text{Ell}^2 X \xrightarrow{\Psi^n} \frac{1}{p} \text{Ell}^{2p^2} X
\]

whose effect on the Euler class of a line bundle \( \downarrow \) is given by

\[
\Psi^p(eL) = e^{ \frac{1-p^2}{p} } [p]_{\text{Ell}}(eL),
\]

where \( e \) is a unit in \( \text{Ell}^* \); see below. The technique is the same as that of Theorem 5.4.6.

In the case of \( K \)-theory, this is essentially the only kind of power operation there is, because the only subgroup of \( G \) of order \( n \) is the group \( nG \). In the case of the formal group law \( F_{\text{Ell}} \) of elliptic cohomology, however, we have

\[
nF_{\text{Ell}} \cong (\mathbb{Z}/n\mathbb{Z})^2,
\]

so there are interesting subgroups of order \( n \). These non-trivial subgroups are the source of \textit{Hecke operators}. As operations on Elliptic cohomology, they have been constructed by Andrew Baker [Bak90]. In section 6.5 we show how to realize these operations as power operations. To prepare for the discussion in sections 6.4 and 6.5, we adapt in sections 6.1 and 6.2 the machinery developed in section 3 to the case of elliptic cohomology.

In our work on \( E_h \), the operations \( Q^H \) were only a small part of the battle, and the really interesting problem was to produce the homomorphism

\[
E_h^* \xrightarrow{\beta^H} D^{**}
\]

such that

\[
\beta^H F_{PO} = F_{PO}/H.
\]

Part of the reason why we think that the orientation \( F_{PO} \) which we discovered in the process is "analytic" is that in the case of \( K \)-theory, we recover the multiplicative group. Another reason is that in the case of \( \text{Ell} \), the exponential of the formal group law is an elliptic function, so it is determined up to a constant multiple by its divisor. This simple fact identifies the quotient formal group law \( F_{\text{Ell}}/H \), and enables us to produce the homomorphisms \( \beta^H \) (section 6.3).
6.1 A review of elliptic cohomology

We use the version of elliptic cohomology whose coefficient ring is the ring of modular forms over \( \mathbb{Z}[\frac{1}{2}] \) for curves with \( \Gamma_0(2) \) structure, with possible poles at the cusps; thus

\[
\text{Ell}^* \cong \mathbb{Z}[\frac{1}{2}][\delta, \varepsilon, \Delta^{\pm 1}] / (\Delta = \varepsilon(\delta^2 - \varepsilon)),
\]

where \( \delta \in \text{Ell}^{-4} \) and \( \varepsilon \in \text{Ell}^{-8} \) are modular forms of weight 2 and 4 respectively.

Let \( Z_0 \) denote the space of pairs

\[(\Xi, \eta)\]

where \( \Xi \) is a lattice in \( \mathbb{C} \), and \( \eta \) is a non-trivial point of order two of \( \mathbb{C}/\Xi \). \( Z_0 \) comes with an action of \( \mathbb{C}^\times \) by homotheties:

\[t(\Xi, \eta) = (t\Xi, t\eta), \ t \in \mathbb{C}^\times.\]

A modular function \( f \) of weight \( r \) is a function

\[Z_0(2) \xymatrix{	o \mathbb{C}}\]

such that for \( t \in \mathbb{C}^\times \),

\[f(t(\Xi, \eta)) = t^{-r}f(\Xi, \eta). \quad (6.1.1)\]

Let \( \mathcal{h} \) denote the complex upper half-plane; then the quotient \( Z_0/\mathbb{C}^\times \) is the modular curve

\[Y_0 = \Gamma_0(2) \backslash \mathcal{h}.\]

For \( \tau \in \mathcal{h} \) let \([\tau]\) denote its orbit modulo \( \Gamma_0(2) \). The projection \( Z_0 \to Y_0 \) comes with a section

\[\tau \xymatrix{\longrightarrow \ar@{-}[r] & (4\pi i \tau \mathbb{Z} + 4\pi i \mathbb{Z}, 2\pi i)}. \quad (6.1.2)\]

Via the section \( \sigma \) and the projection \( \mathcal{h} \xymatrix{	o \ar@{..>}[r] & Y_0} \), a modular function \( f \) pulls back to a function \( \tilde{f} \) on \( \mathcal{h} \) such that

\[\tilde{f}(\alpha \tau) = (c\tau + d)^r f(\tau),\]

where

\[\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2).\]

A meromorphic modular form \( f \in \text{Ell}^{-2r} \) is a modular function of weight \( r \) on \( Z_0 \) such that \( \tilde{f} \) is an analytic function on \( \mathcal{h} \) whose \( q \)-expansions at the two inequivalent cusps of \( \Gamma_0(2) \) have coefficients in \( \mathbb{Z}[\frac{1}{2}] \).
There is a formal group law $F_{Ell}$ defined over $Ell^*$ called the Euler formal group law [Igu59], given by

$$F_{Ell}(x_1, x_2) = \frac{x_1 \sqrt{R(x_2)} + x_2 \sqrt{R(x_1)}}{1 - \varepsilon x_1^2 x_2^2}, \quad (6.1.3)$$

where

$$R(x) = 1 - 26x^2 + \varepsilon x^4.$$  

This determines a genus

$$MU^* \xrightarrow{\text{Igu}*} Ell^*.$$  

Once again the exact functor theorem applies and shows

**Theorem 6.1.4 ([LRS88, Lan88])** The functor

$$X \mapsto Ell^* X \overset{\text{def}}{=} Ell^* \otimes_{MU^*} MU^* X$$

is a cohomology theory on finite complexes. □

**Remark:** Franke [Fra92] has recently shown that the extension of $Ell^*(-)$ to infinite complexes is unique.

The exponential of $F_{Ell}$ is a power series

$$s(z) \in \mathbb{Q} \otimes Ell^*[z]$$

which has a characterization as an elliptic function.

**Lemma 6.1.5 ([CC88, Zag88])** For pair $(\Xi, \eta) \in \mathbb{Z}_0$, there is a unique meromorphic function

$$s(-, \Xi, \eta) : \mathbb{C} \to \mathbb{C}$$

which satisfies

i. $s(z + \xi, \Xi, \eta) = s(z, \Xi, \eta)$ for $\xi \in \Xi$,

ii. $s(z + \eta, \Xi, \eta) = -s(z, \Xi, \eta)$, and

iii. $s(z, \Xi, \eta) = z + o(z^2)$.

The function $s(-, \Xi, \eta)$ and its derivative satisfy the functional equation

$$s'^2 = 1 - 2\delta(\Xi, \eta)s^2 + \varepsilon(\Xi, \eta)s^4, \quad (6.1.6)$$

and so uniformize the *Jacobi quartic*

$$C : y^2 = 1 - 2\delta(\Xi, \eta)x^2 + \varepsilon(\Xi, \eta)x^4. \quad (6.1.7)$$
via the map

$$\mathbb{C}/\Xi \xrightarrow{(s(-\Xi, \eta), s'(-\Xi, \eta), 1)} C \subset \mathbb{P}^2.$$  \hfill (6.1.8)

As written, the Jacobi quartic has a singularity at $\infty$, but this singularity is minor. The compactification of $C$ in $\mathbb{P}^3$ via the map

$$C \xrightarrow{(1, x, x^2, y)} \mathbb{P}^3$$

is a smooth curve $C_0$ such that

$$C_0 \cap \mathcal{O} \neq \emptyset \cong C.$$

So by choosing the origin to be $(0, 1, 1)$, the Jacobi quartic becomes an elliptic curve; as such, it is a group. Because of (6.1.5, iii), the function $s$ determines a parameter near the origin, with respect to which the group law of the curve (6.1.7) becomes the Euler formal group law specialized at $(\Xi, \eta)$:

$$F_{\Pi l}(x_1, x_2) = \frac{x_1 y_1 + x_2 y_2}{1 - \varepsilon(\Xi, \eta)x_1^2 x_2^2}. \hfill (6.1.9)$$

As promised, [Zag88] $s$ is the exponential for the Euler formal group law:

$$s(z + w, \Xi, \eta) = s(z, \Xi, \eta) + s(w, \Xi, \eta),$$  \hfill (6.1.10)

so the uniformization (6.1.8) is a homomorphism of groups.

We shall use two additional facts about $F_{\Pi l}$. First, equation (6.1.6) and the fact that $s$ is characterized by the three properties listed above imply, in view of the weights of $\delta$ and $\varepsilon$, that $s$ satisfies the homogeneity property

$$s(tz, t\Xi, t\eta) = ts(z, \Xi, \eta).$$  \hfill (6.1.11)

The second fact about the group law $F_{\Pi l}$ is that for $N$ odd, the series $[N]_{F_{\Pi l}}(x)$ has a “Weierstrass factorization” already over $\Pi l^*$:

**Theorem 6.1.12 ([Igu59])** There are polynomials

$$p_N(x), g_N(x) \in \Pi l^*[x]$$

with

$$p_N(x) = N x + \ldots \pm \varepsilon^{N^2-1} x^{N^2} \text{ and } g_N(0) = 1$$

such that

$$[N]_{F_{\Pi l}}(x) = \frac{p_N(x)}{g_N(x)}.$$
The operations $Q^H$ in the case of elliptic cohomology

The ring “$D^*$” for elliptic cohomology

Now let $p$ be an odd prime. As in section 2.3, we let $\Lambda_\infty$ be an abelian group isomorphic to $\mathbb{Z}_p^2$, so that its dual is

$$\Lambda_\infty^* \cong (\mathbb{Q}_p/\mathbb{Z}_p)^2.$$  

We denote by

$$\Lambda^*_k = p^* \Lambda^*_\infty$$

the $p^k$-torsion subgroup of $\Lambda^*_\infty$. For the purposes of this chapter, it will be convenient to choose explicit compatible isomorphisms

$$\Lambda^*_k \cong (\mathbb{Z}/p^k)^2;$$

then we refer to the chosen generators of $\Lambda^*_k$ as $e_k$ and $f_k$.

In this version of elliptic cohomology, also we have to keep track of the point $\eta$ of order 2, so we let $\Lambda'_k$ be the family of groups

$$\Lambda'_k = (\mathbb{Z}/2p^k)^2$$

with generators $e_{2k}$ and $f_{2k}$. The point of order 2, $\eta$, will be the point $p^k e_{2k}$. Thus there are compatible inclusions

$$
\begin{array}{ccc}
\Lambda^*_k & \longrightarrow & \Lambda'_k \\
\downarrow & & \downarrow \\
\Lambda^*_{k+1} & \longrightarrow & \Lambda'_{k+1}
\end{array}
$$

under which

$$e_k \mapsto 2e_{2k} \text{ and } \quad f_k \mapsto 2f_{2k}.$$  

Let $Z(2p^k)$ be the space of pairs

$$(\Xi, \alpha)$$

consisting of a lattice $\Xi \subset \mathbb{C}$ and an isomorphism

$$\Lambda'_k \xrightarrow{\alpha} 2p^*(\mathbb{C}/\Xi)$$

such that

$$w_{2p^k}(\alpha(e_{2k}), \alpha(f_{2k})) = \zeta_{2p^k} \overset{\text{def}}{=} e^{\pi i/p^k}, \quad (6.2.1)$$

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where $w_N$ is the *Weil pairing* for points of order $N$ of $\mathbb{C}/\Xi$ given by ([KM85], p. 90)

\[
w_N(v_1/N, v_2/N) = \exp(\frac{2\pi i}{N} \cdot \text{Im}(\overline{v_1}v_2)),
\]

where $v_1, v_2 \in \Xi$ and $A(\Xi)$ is the area of a fundamental parallelogram for $\Xi$.

$\mathbb{C}^\times$ acts on $Z(2p^k)$ by homotheties; that is,

\[
t(\Xi, \alpha) = (t\Xi, t\alpha),
\]

and the quotient $Z(2p^k)/\mathbb{C}^\times$ is the modular curve

\[
Y(2p^k) = \Gamma(2p^k) \backslash \mathbb{H}.
\]

The projection $Z(2p^k) \to Y(2p^k)$ comes with a section

\[
Y(2p^k) \xrightarrow{\sigma} Z(2p^k)
\]

\[
[r] \mapsto (4p^k\pi i \tau \mathbb{Z} + 4\pi i \mathbb{Z}, \alpha(bf + ce) = 2\pi i b + \frac{2\pi i c}{p^k}).
\]

The group $SL_2\mathbb{Z}/2p^k \cong SL_2\mathbb{Z}/p^k \times SL_2\mathbb{Z}/2$ acts on $Z(2p^k)$. Let $G_k \subset SL_2\mathbb{Z}/2 \subset SL_2\mathbb{Z}/2p^k$ be the subgroup (isomorphic to $\mathbb{Z}/2$) such that

\[
g(\eta) = \eta.
\]

Let $Z_0(p^k)$ be the orbit space of the action of $G_k$ on $Z(2p^k)$. It is the space of triples

\[(\Xi, \alpha, \eta)\]

where $\Xi$ is a lattice in $\mathbb{C}$, $\eta$ is a non-trivial point of order two in $\mathbb{C}/\Xi$, and $\alpha$ is an isomorphism

\[
\Lambda^* \xrightarrow{\alpha} \mathbb{C}/\Xi
\]

such that

\[
w_{p^k}(\alpha(e_k), \alpha(f_k)) = \zeta_{p^k}.
\]

Once again, there is a map

\[
Y(2p^k) \xrightarrow{\sigma} Z_0(p^k)
\]

\[
[r] \mapsto (4p^k\pi i \tau \mathbb{Z} + 4\pi i \mathbb{Z}, \alpha(bf_k + ce_k) = 4\pi i b + \frac{4\pi i c}{p^k}, 2\pi i)
\]

(6.2.2)

A "meromorphic modular form of level $2p^k$ and weight $r$" is a function $f$ on $Z(2p^k)$ which behaves as (6.1.1) with respect to homotheties and whose pull-back via the section $\sigma$ is analytic on $\mathbb{H}$ and has $q$-expansions at every cusp which are finite-tailed laurent series with coefficients in $\mathbb{Z}[\frac{1}{2p^k}, \zeta_{2p^k}]$. The ring of meromorphic modular forms of level $2p^k$ will be denoted $Ell^*_{\Gamma(2p^k)}$. 

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The ring which is the range of the character map for elliptic cohomology is the ring of modular functions on \( \mathbb{Z}_0(p^k) \) which pull back via \( \sigma \) to analytic functions on \( \mathfrak{h} \) whose \( q \)-expansions at the cusps of \( Y(2p^k) \) have coefficients in the ring \( \mathbb{Z}[[\frac{1}{2p^k}, \zeta_{2p^k}]] \). It is exactly the subring \( (Ell^*_{\Gamma_0(2p^k)})^{G_k} \) of \( Ell^*_{\Gamma_0(2p^k)} \) invariant under the action of \( G_k \). We denote this ring by \( Ell^*_{p^k} \).

The map "forget \( \alpha \)" gives a projection \( \mathbb{Z}_0(p^k) \rightarrow \mathbb{Z}_0 \) and exhibits \( Ell^*_{p^k} \) as an \( Ell^* \)-algebra. Brylinski has shown

**Proposition 6.2.3** ([Bry90]; see also [KM85]) \( Ell^*_{p^k} \) is a faithfully flat \( Ell^* \)-algebra.

Proof: The proof of [Bry90] is based on the study of the moduli schemes for level structures on elliptic curves [KM85]. His proof can be imitated in our situation, with one important modification. Brylinski studies modular forms of even weight, and obtains a ring \( Ell^*_{\Gamma(N)} \) of modular forms whose \( q \)-expansions have coefficients in \( \mathbb{Z}[[\frac{1}{2}, \zeta_N]] \). We shall need modular forms of odd weight, namely the functions \( \phi(a) \) (6.2.5). For a moduli space \( M \) with a universal curve \( E \rightarrow M \), one defines the invertible sheaf

\[
\omega = \pi_* \Omega_{E/M}.
\]

Let \( T \) be the circle bundle of \( \omega \). It is a principal \( G_m \)-bundle over \( M \), and one can view modular functions as functions on the space \( T \), graded by the \( G_m \)-action. In order study \( q \)-expansions of modular forms in this setting, one must be able to extend the sheaf \( \omega \) to a sheaf on the compactified moduli space \( \overline{M} \), as the cusps are precisely the points one adds in compactifying. Actually, only the square \( \omega^{\otimes 2} \) extends in general ([KM85], section 10.13), which enables Brylinski to study modular forms of even weight. Fortunately, there is a natural extension of \( \omega \) in case the moduli problem is "elliptic curves with level \( 2p^k \)-structure over \( \mathbb{Z}[[\frac{1}{2p^k}, \zeta_{2p^k}]] \)," ([KM85], (10.13.9.1), [Kat73]) and this moduli space can be used in Brylinksi's argument, now for the ring of modular forms with possibly odd weight. Unfortunately, we have had to invert \( p \). See the remarks at the end of this chapter. \( \square \)

We shall need the additional facts that

**Proposition 6.2.4** (compare 3.4.1) \( Ell^*_{p^k} \) is a domain (since it is a subring of a ring of functions on \( Y(2p^k) \)). The group

\[
\text{Aut}[\mathbb{Z}_0(p^k)/\mathbb{Z}_0] \overset{def}{=} SL_2 \mathbb{Z}/p^k \times \text{Gal}(\mathbb{Z}[[\zeta_{2p^k}]!/\mathbb{Z})
\]

acts on \( Ell^*_{p^k} \), and the ring of invariants is \( \frac{1}{p} Ell^* \).

Proof: By definition, an element of \( Ell^*_{p^k} \) which is invariant under \( SL_2 \mathbb{Z}/p^k \) is a modular form for the group \( \Gamma_0(2) \) whose \( q \)-expansions have coefficients in \( \mathbb{Z}[[\frac{1}{2p^k}, \zeta_{2p^k}]] \).
Denote this ring by

\[ M(\Gamma_0(2), \mathbb{Z}[\frac{1}{2p}, \zeta_{2p^k}]). \]

Then [Kat73]

\[ M(\Gamma_0(2), \mathbb{Z}[\frac{1}{2p}, \zeta_{2p^k}]) = \mathbb{Z}[\frac{1}{2p}, \zeta_{2p^k}] \otimes \text{Ell}^*. \]

(Actually, Katz shows this for the ring of cusp forms for the congruence subgroup \( \Gamma(2) \). His argument applies also to the group \( \Gamma_0(2) \). Then [Bak90] we can use multiples of \( \Delta \), which is a cusp form, to convert a modular form \( f \) to a cusp form \( \Delta^d f \). Since \( \Delta^{-1} \in \text{Ell}^{16} \) is already defined over \( \mathbb{Z}[\frac{1}{2}] \), the result follows for the full ring of modular forms). [证明]

The character map for elliptic cohomology

We need an assignment

\[ \Lambda^*_k \rightarrow \text{Ell}_{p^k}^* \]

that will play the role of the map \( \phi_{\text{univ}} \) constructed in (2.3.5). The key idea is to use the exponential map for elliptic cohomology to find roots of the \( p^k \)-series, in the same way that roots of the \( p^k \)-series in K-theory are

\[ 1 - e^{\frac{2\pi i j}{p^k}}, 0 \leq j < p^k. \]

We define the map \( \phi \) to be

\[ \phi(a) = \{ (\Xi, \alpha, \eta) \mapsto s(\alpha(a), \Xi, \eta). \} \tag{6.2.5} \]

This definition only makes \( \phi(a) \) a complex-valued function on \( \mathbb{Z}_0(p^k) \). Notice, however, that by the homogeneity property of \( s \) (6.1.11), \( \phi \) satisfies

\[ \phi(a)(i\Xi, t\alpha, t\eta) = t \phi(a)(\Xi, \alpha, \eta), \]

so \( \phi(a) \) has the right behavior with respect to homotheties to be an element of \( \text{Ell}_{p^k}^2 \).

**Proposition 6.2.6** \( \phi(a) \) is an element of \( \text{Ell}_{p^k}^2 \).

Proof: It remains to show that the \( q \)-expansions of \( \phi(a) \) have coefficients in \( \mathbb{Z}[\frac{1}{2p}, \zeta_{2p^k}] \) at every cusp. To simplify the notation, let \( N = p^k \). The cusps of \( Y(2N) \) correspond to choices of level structure on the "Tate curve" \( \text{Tate}(q^{2N}) \) [Kat73]. Via the map \( \sigma \) (6.2.2), we are led to consider the lattice

\[ \Xi = 4\pi i N \tau \mathbb{Z} + 4\pi i \mathbb{Z} \]
and the function
\[ \tilde{s}(z, q) = s(z, \Xi, 2\pi i), \]
where \( q = e^{2\pi i \tau} \). According to [CC88, Zag88], \( \tilde{s}(z, q) \) is given by
\[ \tilde{s}(z, q) = (e^\frac{i}{2} - e^{-\frac{i}{2}}) \prod_{m \geq 1} \left\{ \frac{(1 - q^{Nm} e^z)(1 - q^{Nm} e^{-z})}{(1 - q^{Nm})^2} \right\}^{(-1)^m}. \]

Then
\[ \phi(bf_k + cc_k a)(\Xi, 2\pi i, \alpha) = s(4\pi i b + \frac{4\pi ic}{N}, \Xi, 2\pi i) \]
\[ = (\zeta^c q^b - \zeta^{-c} q^{-b}) \prod_{m \geq 1} \left\{ \frac{(1 - q^{mN+2b}\zeta^2c)(1 - q^{mN-2b}\zeta^{-2}c)}{(1 - q^{mN})^2} \right\}^{(-1)^m}, \]
where \( \zeta = e^{\frac{2\pi i}{p}} \) is a \( p^k \)-th root of unity. For each point \( a \in (\mathbb{Z}/p^k)^2 \), then, the expression (6.2.7) has coefficients of \( q \) in \( \mathbb{Z}[\frac{1}{p}, \zeta_N] \). Fixing this level \( p^k \) structure and letting \( a \) vary over \( (\mathbb{Z}/p^k)^2 \) is equivalent to fixing an \( a \) and varying over all level \( p^k \) structures. There are other cusps which correspond to varying the point of order 2. However, it in fact suffices to check each \( \phi(a) \) at one cusp, because of the \( q \)-expansion principle ([Kat73]): by choosing a value of the Weil pairing (6.2.1), we have restricted ourselves to one component of the moduli space Katz calls \( M_{2p^k} \). \( \square \)

The significance of the modular forms \( \phi(a) \) is that they are the roots of the \( p^k \) series of \( F^{Ell} \):
\[ [p^k]_{F^{Ell}}(\phi(a))(\Xi, \alpha, \eta) = [p^k]_{F^{Ell}} s(\alpha(a), \Xi, \eta) \]
\[ = s(p^k \alpha(a), \Xi, \eta) = 0. \]

**Proposition 6.2.8** Let \( N = p^k \). The \( p^2k \) modular forms \( \phi(a) \) are the roots of \( f_N \):
\[ f_N(x) = \pm e^{\frac{N^2-1}{4}} \prod_{a \in \Lambda^*_k} (x - \phi(a)). \]

Let \( S = \text{Im}(\phi) \) be the image of the map (of sets) \( \phi \). Notice that for any two elements \( \phi(a) \) and \( \phi(b) \), their formal sum is
\[ \phi(a + b)(\Xi, \alpha, \eta) = s(\alpha(a + b), \Xi, \eta) \]
\[ = s(\alpha(a) + \alpha(b), \Xi, \eta) \]
\[ = s(\alpha(a), \Xi, \eta) + _{F^{Ell}} s(\alpha(b), \Xi, \eta) \]
\[ = (\phi(a) + _{F^{Ell}} \phi(b))(\Xi, \alpha, \eta). \]

In particular, \( _{F^{Ell}} \phi(a) +_ {F^{Ell}} \phi(b) = \phi(a + b) \) is an element of \( S \). Let \( F^{Ell}(S) \) be the group made of the set \( S \) together with the formal sum \( F^{Ell} \). We have shown
Proposition 6.2.9 $\phi$ is an isomorphism of groups

$$\Lambda^*_k \xrightarrow{\phi} F^{E_{\ell}}(S).$$

In order to construct the character map, we have to complete $E_{\ell p^*_k}$ with respect to the ideal $I$ generated by the set $S$. Let $\overline{E_{\ell p^*_k}}$ denote the completion of $E_{\ell p^*_k}$ with respect to $I$, and $\overline{I}$ the extension of $I$ to $\overline{E_{\ell p^*_k}}$.

Proposition 6.2.10 $\phi$ induces an isomorphism

$$\Lambda^*_k \xrightarrow{\hat{\phi}} p^*_k F^{E_{\ell}}(\overline{I}, 2).$$

Moreover, $\phi$ is represented (in the sense of (2.3.4)) by a homomorphism

$$E_{\ell} B\Lambda^*_k \xrightarrow{\hat{\phi}} \overline{E_{\ell p^*_k}}.$$ 

Proof: $E_{\ell p^*_k}$ is a domain, so by Krull's theorem

$$E_{\ell p^*_k} \rightarrow \overline{E_{\ell p^*_k}}$$

is injective. The first part then follows from (6.2.9) since by Igusa's theorem (6.1.12), $[p^k]_{F^{E_{\ell}}}(x)$ has $p^{2k}$ roots. The definition of the map $\hat{\phi}$ is (compare 2.3.3)

$$e \left( \begin{array}{c}
E\Lambda^*_k \times C \\
\downarrow \\
B\Lambda^*_k
\end{array} \right) \mapsto \phi(a)$$

(6.2.11)

for $a \in \Lambda^*_k$. It defines a map since power series in the $\phi(a)$ converge in $\overline{E_{\ell p^*_k}}$. 

Theorems B and C for $E_{\ell}$.

We can use the maps $\phi$ and $\hat{\phi}$ to imitate the constructions in section 3.1, and so to obtain versions of Theorems B and C. Let $H \subset \Lambda^*_k$ be a subgroup of order $n$.

Theorem 6.2.12 There is a natural transformation

$$MU^2_* X \xrightarrow{Q^H} \Phi_n E_{\ell p^*_k} X$$

of functors of finite spaces to graded rings, whose effect on the Euler classes of a line bundle $L$ is

$$Q^H(eL) = f_H(eL) = \prod_{h \in H} (\phi(h) + eL).$$

Its effect on coefficients is determined by the equation

$$Q^*_* F^{MU} = F^{E_{\ell}/H}.$$
6.3 The Euler form of the quotient formal group law.

In the case of elliptic cohomology, the theory of elliptic functions enables us to show that the quotient formal group law $F^{Ell}/H$ takes a particularly nice form. A finite subgroup of odd order

$$H \subset \mathbb{C}/\Xi$$

determines a quotient map

$$H \to \mathbb{C}/\Xi \to \mathbb{C}/\Xi_H,$$

where $\Xi_H$ denotes the lattice in $\mathbb{C}$ obtained from the union of $\Xi$ and the preimage of $H$ under $\mathbb{C} \to \mathbb{C}/\Xi$, and we have denoted again by $\eta$ the image of the point $\eta \in \mathbb{C}/\Xi$ under the projection. This quotient uniformizes the Jacobi quartic

$$C_H : y^2 = 1 - 2\delta(\Xi_H, \eta)x^2 + \epsilon(\Xi_H, \eta)x^4,$$  \hspace{1cm} (6.3.1)

which is the quotient of $C$ by the subgroup $H$, via the map

$$\mathbb{C}/\Xi_H \to \mathbb{C}/\Xi_{\mathbb{C}} \	o \mathbb{C}/\Xi = \mathbb{C}/\Xi_H \subset \mathbb{P}^2.$$  \hspace{1cm} (6.3.2)

With respect to the parametrization (6.3.2), the group law of the curve becomes the Euler formal group law for the quotient curve:

$$F_H^{Ell}(x_1, x_2) = \frac{x_1\sqrt{R_H(x_2)} + x_2\sqrt{R_H(x_1)}}{1 - \epsilon(\Xi_H, \eta)x_1^2x_2^2},$$  \hspace{1cm} (6.3.3)

where

$$R_H(x) = 1 - 2\delta(\Xi_H, \eta)x^2 + \epsilon(\Xi_H, \eta)x^4.$$  

Now suppose the $H \subset \Lambda_r^*$. Then by using the level structure component of $Z_0(p^k)$, we can associate a quotient curve to each triple $(\Xi, \alpha, \eta)$.

**Proposition 6.3.4** For $H \subset \Lambda_r^*$ and a modular form $f \in Ell^{-2r}$, the modular function $\gamma^H f$ given by

$$\gamma^H f(\Xi, \alpha, \eta) = f(\Xi_H, \eta)$$  \hspace{1cm} (6.3.5)

is an element of $Ell_{p^k}^{-2r}$.

**Proof:** The only thing to show is that $\gamma^H f$ has $q$-expansions in $\mathbb{Z}[\frac{1}{p^k}, \zeta_{2p^k}][(q)]$. By decomposing the subgroup $H$ via short exact sequences, it suffices to prove this when the order of $H$ is $p$. Suppose that

$$f(\Xi, 2\pi i) = \sum_{n \geq -\infty} a_n q^n \in \mathbb{Z}[\frac{1}{2}][(q)],$$

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where \( q = e^{2\pi i r} \) and once again \( \Xi \) is the lattice
\[
\Xi = 4\pi i r \mathbb{Z} + 4\pi i \mathbb{Z}.
\]

The \( q \)-expansions we need to check are obtained by evaluating \( f \) on the quotients by subgroups of order \( p \) of \( \mathbb{C}/\Xi(p) \), where \( \Xi(p) \) is the lattice
\[
\Xi(p) = 4\pi ip r \mathbb{Z} + 4\pi i \mathbb{Z},
\]
so
\[
f(\Xi(p), 2\pi i) = \sum_{n > -\infty} a_n q^n.
\]

For any level-\( p \) structure on \( \mathbb{C}/\Xi(p) \), the quotients by subgroups of order \( p \) correspond to the lattices
\[
\Xi_j = 4\pi i (\tau + \frac{j}{p}) \mathbb{Z} + 4\pi i \mathbb{Z}, \quad 0 \leq j < p, \text{ and}
\]
\[
\Xi_p = 4\pi i p r \mathbb{Z} + 4\pi i \mathbb{Z}.
\]

We have
\[
f(\Xi_j, 2\pi i) = \sum_{n > -\infty} a_n q^n \zeta ^{2nj}, \quad 0 \leq j < p, \quad \text{and} \quad (6.3.6)
\]
\[
f(\Xi_p, 2\pi i) = f\left(\frac{1}{p} (4\pi i p^2 r \mathbb{Z} + 4\pi i \mathbb{Z}, 2\pi i)\right) = p^r \sum_{n > -\infty} a_n q^{2n},
\]
where \( \zeta = e^{2\pi i / p} \). \( \square \)

**Corollary 6.3.7** The formal group law \( F^{Ell}_H \) is defined over \( Ell'_p \). It is classified by the homomorphism
\[
Ell^* \xrightarrow{\gamma} Ell'_p^*.
\]

By the miracle of complex analysis, the group law \( F^{Ell}_H \) is very simply related to Lubin's quotient \( F^{Ell}/H \). Let
\[
a_H = f'_H(0) = \prod_{0 \neq h \in H} \phi(h) \in Ell'^*_{p^2}.
\]

**Theorem 6.3.8** \( F^{Ell}/H \) is related to the formal group law \( F^{Ell}_H \) by
\[
F^{Ell}_H \xrightarrow{\alpha_H} F^{Ell}/H.
\]

In other words, \( F^{Ell}_H \) is exactly the "normalized" quotient formal group law \( \nu F^{Ell}/H \) of (2.2.9).
Proof: Since the ring $\hat{\mathbb{E}l}_{p^k}$ is torsion-free (6.2.4), formal group laws over $\hat{\mathbb{E}l}_{p^k}$ are determined by their exponentials over $\mathbb{Q} \otimes \hat{\mathbb{E}l}_{p^k}$. Let $t(z)$ denote the power series

$$t(z) = f_H(s(z)) \in \mathbb{Q} \otimes \hat{\mathbb{E}l}_{p^k}[z],$$

and let $\mathbb{G}_a$ denote the additive group. Then $t$, $s$, and $f_H$ fit into a diagram of formal group laws:

$$\begin{array}{ccc}
\mathbb{G}_a & \xrightarrow{s} & F_{\mathbb{E}l}
\\
\downarrow & & \downarrow f_H
\\
\mathbb{G}_a & \xrightarrow{t} & F_{\mathbb{E}l/H}
\end{array}$$

The function $t(z)$ is an element of $\mathbb{Q} \otimes \hat{\mathbb{E}l}_{p^k}[z]$, so we evaluate it on the triple $(\Xi, \alpha, \eta) \in Z_0(p^k)$.

$$t(z)(\Xi, \alpha, \eta) = \prod_{h \in H} (\phi(h) + F_{\mathbb{E}l} s(z))(\Xi, \alpha, \eta)$$

$$= \prod_{h \in H} (s(\alpha(h), \Xi, \eta) + F_{\mathbb{E}l} s(z, \Xi, \eta))$$

$$= \prod_{h \in H} s(z + \alpha(h), \Xi, \eta). \tag{6.3.9}$$

Now we use the key fact that as a function on $\mathbb{C}/\Xi$, $s(z, \Xi, \eta)$ is characterized by Lemma 6.1.5. Think of $z$ as a complex parameter. Then by (6.3.9), $t(z)(\Xi, \alpha, \eta)$ is a complex-valued function

$$\mathbb{C} \xrightarrow{t(-)(\Xi, \alpha, \eta)} \mathbb{C}$$

which satisfies

$$t(z + \xi) = t(z), \quad \xi \in \Xi, \tag{6.3.10}$$

$$t(z + \eta) = \prod_{h \in H} s(z + \eta, \Xi, \eta) \tag{6.3.11}$$

$$= (-1)^n \prod_{h \in H} s(z, \Xi, \eta)$$

$$= -t(z), \text{ and}$$

$$t(z + \alpha(h)) = t(z), \quad h \in H. \tag{6.3.12}$$

Together, equations (6.3.10-6.3.12) imply that for each triple $(\Xi, \alpha, \eta)$, $t(-, \Xi, \alpha, \eta)$ is a constant multiple of the function $s(-, \Xi_H, \eta) = \gamma s(-, \Xi, \eta)$. We can compute this multiple using the last part of Lemma 6.1.5:

$$t(z, \Xi, \alpha, \eta) = c(\Xi, \alpha, \eta)s(z, \Xi_H, \eta)$$

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where
\[
c(\Xi, \alpha, \eta) = \lim_{z \to 0} \frac{t(z, \Xi, \alpha, \eta)}{s(z, \Xi_H, \eta)} \\
= \lim_{z \to 0} \frac{t'(z, \Xi, \alpha, \eta)}{s'(z, \Xi_H, \eta)} \\
= \prod_{0 \neq h \in H} s(\alpha(h), \Xi, \eta) \\
= a_H(\Xi, \alpha, \eta).
\]

It follows that
\[
F^{Ell/H} = G_a = (F^{Ell}_H)^{a_H x},
\]
since \( t \) fits into the diagram of formal group laws over \( Q \otimes \hat{Ell}^*_p \)
\[
\begin{array}{ccc}
G_a & \overset{\gamma_H}{\longrightarrow} & F^{Ell}_H \\
\downarrow{\scriptstyle s} & & \downarrow{\scriptstyle a_H} \\
F^{Ell} & \overset{f_H}{\longrightarrow} & F^{Ell/H}.
\end{array}
\]

**Proposition 6.3.14** The stable operation

\[
MU^*X \overset{\nu^H}{\longrightarrow} \frac{1}{a_H} \hat{Ell}^*_p X
\]
factors through the map

\[
\frac{1}{a_H} Ell^*_p X \to \frac{1}{a_H} \hat{Ell}^*_p kX.
\]

Moreover, \( \nu^H \) factors through the orientation

\[
MU^*X \overset{i_{Ell}}{\longrightarrow} Ell^*X
\]
to produce a stable ring operation

\[
Ell^* \overset{\nu^H}{\longrightarrow} \frac{1}{a_H} Ell^*_p X
\]
for \( X \) a finite complex.

Proof: By (6.3.8), the homomorphism of formal group laws

\[
F \to F^{Ell}_H
\]
is given by the normalized quotient

\[
\nu f_H(x) = \frac{1}{a_H} f_H(x) \in \frac{1}{a_H} Ell^*_p[x],
\]

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and so one gets the operation $\nu Q^H$ by (4.0.2). By (6.3.7), the ring homomorphism which classifies $F_{E}^{2n}\ell^*$ is

$$\gamma^H \circ t_{E\ell^*}.$$  

Since it factors through $t_{E\ell^*}$, we get $\nu \gamma^H$ by (4.0.3).  

Factoring $Q^H$ through the orientation $t_{E\ell}$

Combining (6.3.8) and (6.3.7) with (6.2.12), we obtain

**Theorem 6.3.15** Let $H \subset \Lambda^*_k$ be a subgroup of order $n$. The operation

$$MU^{2*}X \xrightarrow{Q^H} \tilde{El}^{2*}_{p^k}X$$

factors through the orientation

$$MU^*X \xrightarrow{t_{E\ell}} El^*X$$

to produce a natural transformation of functors of finite complexes to rings

$$El^{2*}X \xrightarrow{\gamma^H} \tilde{El}^{2*}_{p^k}X.$$

In particular, if $L$ is a complex line bundle, then

$$\Psi^H(eL) = f_{H}(eL) \in El^{2*}_{p^k}X.$$

On coefficients, $\Psi^H$ is given by

$$\Psi^H(m) = \beta^H(m) = (a_H)^{\frac{m}{2}} \gamma^H(m);$$

more succinctly, we have the equation

$$\beta^H F^{E\ell} = F^{E\ell/H}.$$ 

6.4 An example: the Adams operation.

When the machinery of 6.2 is applied to the full group $\Lambda^*_k$, one obtains an Adams operation for $E\ell$. Let $N = p^k$, and write $\Psi^N$ for $\Psi^{\Lambda^*_k}$. The main task is to use (6.3.8) to identify the formal group law $F^{E\ell}/N$. 

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Lemma 6.4.1 For the full subgroup $\Lambda^*_k$ of $N$ torsion points, the constant $a_N$ of (6.3.8) is given by

$$a_N = \prod_{0 \neq h \in N \setminus \{0\}} \phi(h) = \pm N \varepsilon^{\frac{1-N^2}{4}}$$

Proof: The $\phi(h)$ are exactly the non-zero roots of the $N$-series for $F^{E_{\ell}}$ by (6.2.9). The result follows from (6.2.8). $\square$

Theorem 6.4.2 The operation

$$E_{\ell}^{2*}X \xrightarrow{\Psi^N} E_{\ell}^{2N^2*}X$$

factors through an operation

$$E_{\ell}^{2*}X \xrightarrow{\Psi^N} \frac{1}{p} E_{\ell}^{2N^2*}X.$$

The effect of $\Psi^N$ on the Euler class of a line bundle $L \to X$ is given by

$$\Psi^N(eL) = f_N(eL) = \pm \varepsilon^{\frac{1-N^2}{4}} [N]_{F_{\ell}E_{\ell}}(eL).$$

(6.4.3)

On coefficients, $\Psi^N$ acts by

$$\Psi^N(f) = (\pm \varepsilon^{\frac{1-N^2}{4}})^{\ell/2} f.$$

The associated stable operation

$$E_{\ell}^*(-) \xrightarrow{\nu \Psi^N} \frac{1}{p} E_{\ell}^*(-)$$

is the usual stable Adams operation whose effect on Euler classes is given by

$$\Psi^N(eL) = \frac{1}{N} [N]_{F_{\ell}E_{\ell}}(eL).$$

(6.4.4)

Proof: According to (6.3.14), the effect of $\nu \Psi^N$ on coefficients is given by

$$E_{\ell}^* \xrightarrow{\gamma^N} \frac{1}{p} E_{\ell}^*$$

(6.4.5)

$$f \mapsto \{(\Xi, \eta) \mapsto f\left(\frac{1}{N}(\Xi, \eta)\right) = N^{-\ell/2} f(\Xi, \eta)\}.$$
Moreover, the formal group law $F_N^\text{Ell}$ has exponential

\[
\exp_{F_N}(z) = s(z, \frac{1}{N} \Xi, \frac{1}{N} \eta) = \frac{1}{N} s(Nz, \Xi, \eta) = \frac{1}{N}[N]_{\text{Ell}} s(z, \Xi, \eta).
\]

which proves (6.4.4). Since $\nu \Psi^N$ on coefficients and on Euler classes lands in $\frac{1}{p}\text{Ell}$, it is in fact an operation

\[
\text{Ell}^* X \xrightarrow{\nu \Psi^H} \frac{1}{p}\text{Ell}^* X.
\]

According to (4.0.7), $\nu Q^N$ and $Q^N$ are related by

\[
\nu Q^N(x) = a_N^{-\frac{|x|^2}{2}} Q^N(x).
\]

Since $a_N$ is a unit in $\text{Ell}^*\times$, it follows that

\[
Q^N(x) = a_N^{\frac{|x|^2}{2}} \nu Q^N(x) = (\pm Ne^{-\frac{1-\zeta^2}{4}})^{\frac{|x|^2}{2}} \nu Q^N(x) \in \frac{1}{p}\text{Ell}^{2|\zeta|^2} X.
\]

The proof is complete once we observe that by (6.3.8), the homomorphism

\[
\text{Ell}^* \xrightarrow{\beta^N} \frac{1}{p}\text{Ell}^{2*}
\]

such that

\[
\beta^N F^\text{Ell} = F^\text{Ell}/N
\]

is

\[
\beta^N(f) = a_N^{-\frac{|x|^2}{2}} \gamma^N f = (\pm e^{-\frac{1-\zeta^2}{4}})^{\frac{|x|^2}{2}} f \quad \Box
\]

### 6.5 Hecke operators as power operations

Associated to a finite subgroup $H \subset \Lambda^*_\Lambda$ and the Jacobi quartic $C$ we have seen that there is a quotient curve $C_H$. The Hecke operator $T_n$ on a modular form $f$ is obtained by evaluating $f$ on the quotient curve $C_H$ as $H$ runs over all subgroups of a given order. Specifically, the $p$-th Hecke operator $T_p$ is given by

\[
T_p f = \frac{1}{p} \sum_{\substack{H \subset \Lambda^*_\Lambda \text{ s.t. } H \neq p \# \text{ of subgroups}}} \gamma^H f,
\]  

(6.5.1)

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where $\gamma^H$ is the map defined in (6.3.5). Indeed, the computations (6.3.6) in the proof of Lemma 6.3.4 are exactly the famous computation of the effect of $T_p$ on the $q$-expansion of a modular form [Ser70].

Let $T_p$ be the operation

$$T_p = \frac{1}{p} \sum_{\#H=p} \nu \Psi^H.$$

**Theorem 6.5.2 (compare [Bak90])** The operation

$$T_p = \frac{1}{p} \sum_{\#H=p} \nu \Psi^H$$

is an additive operation

$$\text{Ell}^* X \rightarrow \frac{1}{p} \text{Ell}^* X$$

Its effect on coefficients is given by (6.5.1).

Proof: The only thing to check is the claim about the range: *a priori*, the operation lands in

$$\frac{1}{A} \text{Ell}^* p X,$$

where

$$A = \prod_{\#H=p} a_H.$$

At the end of this section we shall prove

**Proposition 6.5.3** Each of the constants $a_H$ is a unit in $\text{Ell}^* p$.

Granting this, the operation $T_p$ lands in $\text{Ell}^* p X$. Visibly, it is invariant under $SL_2 \mathbb{Z}/p$, so it lands in $\mathbb{Z}[\frac{1}{2p}, \zeta_{2p}] \otimes \text{Ell}^*(-)$ by (3.4.6). It is invariant under $\text{Aut}[Z_0(p^k)/Z_0]$ because it is on coefficients and on Euler classes: on coefficients by the formulas (6.3.6), and on Euler classes because by equation (6.3.13), we have

$$f_H(s(z)) = a_H \gamma^H s(z) \in \mathbb{Q} \otimes \text{Ell}^*_p[z]. \Box$$

This operation is entirely analogous in its effect to Baker's [Bak90] Hecke operation $T_p$ for his slightly different version of elliptic cohomology. What is new here is the realization of the operation as a sum of power operations $\nu \Psi^H$. It shows that the Hecke operator is a very natural operation in elliptic cohomology in much the way that the Adams operation is a natural operation in $K$-theory.
Our description of the situation in elliptic cohomology is far from ideal. A better approach would have used as the range of the character map a ring in which $p$ is not a unit: for example, the ring of meromorphic modular forms for $Z_0(p^k)$ whose $q$-expansions have coefficients lying in the ring $Z[\frac{1}{2}][\zeta_{2p^k}]$. Then, as in the case of $K$-theory and the $E_n$, one could use the unstable operation $Ψ^H$ to prove integrality theorems about their stable counterparts. For example, we conjecture

**Conjecture ("Conjecture G")** The operation $p^{r+1}T_p$ is integral on $2r$-dimensional classes.

This conjecture would follow immediately from the computation of the constants $a_H$ in the proof of Proposition 6.5.3, below, once a better version of $Ell^*_p$ is available. Additionally, we would get the integrality theorem about the Adams operation $Ψ^N$ analogous to (5.4.10).

All the constructions in this chapter would apply with this more stringent integrality condition: most crucially, the computations in (6.2.6) show that the roots of the $p$-series $ϕ(a)$ have $q$-expansions with coefficients in $Z[\frac{1}{2}][\zeta_{2p^k}]$. What is missing is a proof that the ring of such modular forms is flat over $Ell^*$ and so defines a cohomology theory. In fact, Brylinski shows that such a ring is flat, but only for modular forms with even weight. I believe that the right ring for this discussion exists, is within reach, and is even possibly interesting in its own right; however, I don’t know enough to say more about it, let alone use it, at this time.

Finally, I owe the reader a

### 6.6 Proof of Proposition 6.5.3.

This is simply a matter of computation. Let $Ξ(p)$ be the lattice

$$Ξ(p) = 4πi prZ + 4πiZ.$$

Then the subgroups of $C/Ξ(p)$ of order $p$ correspond the superlattices

$$Ξ_j = 4πi(\tau + \frac{j}{p})Z + 4πiZ, \ 0 \leq j < p,$$

$$Ξ_p = 4πi prZ + \frac{4πi}{p}Z.$$

Varying the level structure interchanges these lattices and so the constants $a_H$, so it is enough to fix the level structure, and check that $a_H(Ξ(p), α, 2πi)$ is a unit in $Z[\frac{1}{2p}, \zeta_{2p^k}][(q)]$ for various $H$. Let the level structure be given by

$$α(bf + ce) = 4πitb + \frac{4πic}{p}.$$
Let $H$ be the subgroup of $\Lambda_1^*$ generated by $f + je$, where $0 \leq j < p - 1$. Then we claim that

$$a_H(\Xi(p), \alpha, 2\pi i) = (-1)^{(1-p)/2}(\zeta q^{-1})(p^2-1)/4 \prod_{n \geq 1} \left[ \frac{1 - q^n \zeta^{nj}}{1 - q^{pn}} \right]^{2(-1)^n}, \quad (6.6.1)$$

where $\zeta = e^{2\pi i/p}$. This is even a unit in $\mathbb{Z}[[\zeta]]((q))$. We do this in a series of steps. First of all, by (6.2.7),

$$a_H(\Xi(p), \alpha, 2\pi i) = \prod_{r=1}^{p-1} s(4\pi r + \frac{4\pi jr}{p}, \Xi(p), 2\pi i)$$

$$= \prod_{r=1}^{p-1} \left( \zeta^{2jr} q^{2r} - 1 \right) \prod_{m \geq 1} \left\{ \frac{(1 - q^{mp+2r} \zeta^{2jr})(1 - q^{mp-2r} \zeta^{-2jr})}{(1 - q^{mp})^2} \right\}^{(-1)^m}. \quad (6.6.2)$$

We move the product over $r$ inside, and collect terms. The terms of the form $(1 - q^{mp})^2$ contribute a factor

$$\prod_{s \geq 1} \left[ \frac{(1 - q^{2sp-1})}{(1 - q^{2s})} \right]^{2(p-1)} \quad (6.6.3)$$

to the infinite product. We treat the other part of the infinite product in two steps, the numerator and denominator. The numerator corresponds to $m$ even in (6.6.2). It produces a term of the form

$$\prod_{s \geq 1} (1 - q^{2sp+2s \zeta^{2j}}) \cdot \ldots \cdot (1 - q^{2sp+2s(p-1)} \zeta^{2j(p-1)}) \cdot (1 - q^{2sp-2s \zeta^{-2j}}) \cdot \ldots \cdot (1 - q^{2sp-2s(p-1)} \zeta^{-2j(p-1)}).$$

Collecting terms in $q^{2l}$, we find for each $l$ prime to $p$ and also greater than $p$ a term of the form

$$(1 - q^{2l \zeta^{2jl}})^2,$$

where one factor comes from the $(1 - q^{2sp+2r \zeta^{2r}})$ side, and the other from the $(1 - q^{2sp-2r \zeta^{-2r}})$ side.

When $l$ is less than $p$, we get since $s \geq 1$ only one term of the form

$$1 - q^{2l \zeta^{2jl}}.$$

Here, however, the numerator of the leading factor

$$\prod_{r=1}^{p-1} \left( \zeta^{2jr} q^{2r} - 1 \right) \quad \zeta^{jr} q^r$$

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comes to the rescue, and contributes the missing terms
\[ (\zeta^{2j}q^2 - 1)(\zeta^{4j}q^4 - 1) \ldots \]
up to a factor of \((-1)^{p-1} = 1!\)

The denominator \((m \text{ odd in (6.6.2)})\) is a product of the form
\[
\prod_{s \geq 1} (1 - q^{(2s-1)p+2\zeta^{2j}}) \ldots (1 - q^{(2s-1)p+2(p-1)\zeta^{2j}(p-1)})
\]
\[
(1 - q^{(2s-1)p-2\zeta^{2j}}) \ldots (1 - q^{(2s-1)p-2(p-1)\zeta^{2j}(p-1)}) \text{.} \quad (6.6.4)
\]

Proceeding analogously, we find, upon collecting terms of the form \((1 - q^{2l-1}\zeta^{j(2l-1)})\),
that for \(2l - 1 \text{ prime to and greater than } p\), we get a term of the form
\[
(1 - q^{2l-1}\zeta^{j(2l-1)})^2,
\]
where one factor comes from the \((1 - q^{(2s-1)p+2\zeta^{2r}})\) and the other from the \((1 - q^{(2s-1)p-2\zeta^{2r}})\) side. Now however there are no terms of the form \((1 - q^{2l-1}\zeta^{j(2l-1)})\) at all for \(2l - 1 < p\). However, there are messy terms arising on the \((1 - q^{(2s-1)p-2\zeta^{2r}})\) side when \(s = 1\): the factor is
\[
(1 - q^{p-2\zeta^{2j}})(1 - q^{p-4\zeta^{-4}}) \ldots (1 - q^{-p+2\zeta^{-2(p-1)j}}) \text{.} \quad (6.6.5)
\]
It includes negative exponents of \(q\), which would be very bad in the denominator. Once again, the denominator of the leading factor saves the day, this time by contributing a factor of
\[ q^{1+2+\ldots+(p-1)}. \]
Multiplying (6.6.5) by this factor yields
\[
q^{2+4+\ldots+(p-1)}(q^{p-2 - \zeta^{2(p-1)j}})(q^{p-4 - \zeta^{-2(p-2)j}})
\]
\[
\ldots (q - \zeta^{(p+1)j})(1 - q\zeta^{(p-1)j}) \ldots (1 - q^{(p-2)\zeta^{-2j}})
\]
\[
= q^{(p^2-1)/4(-1)(p-1)/2\zeta^{2(p-1)j}i/4(q\zeta)^2 \ldots (1 - q^{p-2}\zeta^{-2j})^2}
\]
which are exactly the missing terms.

Assembling the numerator, denominator, and (6.6.3) gives
\[
a_H(\Xi(p), \alpha, 2\pi i) = (-1)^{(1-p)/2}(\zeta q^{-1}(p^2-1)/4 \prod_{s \geq 1} \left[ \frac{1 - q^{2s-1}}{1 - q^{2s}} \right]) \prod_{s \geq 1} (1 - q^s\zeta^{2s})^{2(-1)^s}
\]
\[
= (-1)^{(1-p)/2}(\zeta q^{-1}(p^2-1)/4 \prod_{n \geq 1} \left[ \frac{1 - q^n\zeta^{2n}}{1 - q^{np}} \right])^{2(-1)^n}
\]
which is (6.6.1).
The remaining subgroup of $\Lambda^*_1$ is the one generated by $e$. In that case, we have

$$a_H(\Xi p, \alpha, 2\pi i) = \prod_{r=1}^{p-1} \frac{(\zeta^{2r} - 1)}{\zeta^r} \prod_{m \geq 1} \left\{ \frac{(1 - q^{mp}\zeta^{2r})(1 - q^{mp}\zeta^{-2r})}{(1 - q^{mp})^2} \right\}^{(-1)^m}$$

Since $p$ is odd, we get

$$\zeta^{1+2+\ldots+(p-1)} = \zeta^{p(p-1)/2} = 1$$

and

$$\prod_{r=1}^{p-1} (\zeta^{2r} - 1) = p.$$

Also,

$$\prod_{r=1}^{p-1} (1 - q^{mp}\zeta^{\pm 2r}) = 1 + q^{mp} + \ldots + q^{mp(p-1)} = \frac{1 - q^{mp^2}}{1 - q^{mp}}.$$

Putting these together, we find

$$a_H(\Xi p, \alpha, 2\pi i) = \prod_{m \geq 1} \left[ \frac{1 - q^{mp^2}}{(1 - q^{mp})^p} \right]^{2(-1)^m}$$

which is a unit in $\mathbb{Z}[\frac{1}{2p}, \zeta_{2p^*}][(q)]$. □
References


