ACOUSTIC TRANSMISSION THROUGH FLUID-FILLED PIPES
IN BOREHOLES

by

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Abstract

A study of propagation of sound through a fluid-filled pipe in a borehole is undertaken in order to understand Mud Pulse Telemetry. MPT is a means of data transmission in the mud inside a drill pipe, through propagating pressure pulses which are created at the bottom of the borehole, during the process of drilling.

A low frequency model which idealizes the drill pipe, mud inside the pipe, mud outside the pipe and the surrounding formation as a multi-layered, cylindrical waveguide was considered. The solution to the equations of motion along with the concept of Transfer matrices was used in calculating the Transfer function between a pressure/velocity input to one of the three media, at an input location and the pressure/velocity response in any of the three media, at an output location. The capability to model changes in geometry/physical properties of the pipe, borehole or the formation was included in the Transfer function calculation. Further modelling of the top and bottom boundary conditions and the source of MPT inputs was also discussed.

The time domain behaviour of the system was then studied by calculating the Impulse response from the Transfer function. The Impulse response revealed salient geometric/physical details of the system like locations of strong reflections. Finally, the Impulse response was convolved with typical MPT inputs like rectangular waveforms and surface pressure transients were predicted. This is useful in deciphering observed surface pressure transients and is a step towards making MPT a more efficient and robust data transmission system.

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To

Ilayaraja,

for the song

and you,

for being the tune
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Chapter 1

INTRODUCTION

... kallura pārkum pārvai ullura pāyumé, thullāmal thullum ullam sallābamé
villodu ambu rendu kollāmal kolluthé, penpāvai kango enru poi solluthé ...¹

The recovery of petrochemical and geothermal resources from beneath the earth’s surface requires the drilling of deep wells. The efficiency of the drilling process is limited by the amount of information that is available at the surface about events occurring downhole. Inclement conditions between the bottom of the hole and the top impede the measurement and transmission of downhole data. One has to contend with dissipation, distortion and dispersion when considering any kind of data transmission in this environment. In spite of this bleak scenario there do exist commercial data transmission systems which function, albeit, at rather low transmission rates.

¹from Rettai Vāl Kuruvi
Further discussion of this problem would be eased by getting to know the various parts of the drilling rig and its attendant auxiliary machinery. The schematic of a typical land-based rig is as shown in Fig 1.1. The terms that will be occurring frequently are:

**Formation** It is the extended rock mass in which the hole has been drilled.

**Bit** It is the drilling tool that is at the bottom of the drilling assembly.

**Drill pipe** It is used to transfer torque from the rotary table to the bottom hole assembly and to support the weight of the drill string.

**Drill collar** It is heavier and of a larger cross section than a drill pipe and serves to provide sufficient load on the bit to penetrate the rock. It also transmits torque to the bit.

**Drill string** It consists of kelly, drill pipe, drill collars and a variety of special tools.

**Bottom Hole Assembly (BHA)** It is the section of the drill string from the bit to the beginning of the drill pipe.

Typical information that can be transmitted to the surface currently includes position and direction of the bit, pressure and temperature of the drilling mud, rock properties and ambient gamma-ray radiation. Commercial data transmission

\(^2\)Like any machining process this too requires a fluid that acts as coolant/lubricant and flushes away the cuttings from the actual machining site.
Figure 1.1: Schematic of a typical land-based rig.
techniques accomplish this by encoding the measured data in the form of pressure pulses propagating in the drilling mud. Specifically, the pulses are created close to the bit, at the bottom, in the mud inside the BHA\(^3\). The process is referred to as mud pulse telemetry (MPT). Other processes that have been tried include electromagnetic radiation through the rock formation, electrical transmission through insulated conductors and acoustic wave propagation through the metal drill string. These processes are plagued to varying degrees by high attenuation, ambient noise and incompatibility with standard drilling procedures.

As mentioned earlier MPT works only at low transmission rates, typically around one bit per second. Of all the data that can be acquired at the bottom, we can transmit only those that vary at far less than 1 Hz. This precludes any real time drill string dynamics data. This data is of great interest since bending vibration and whirling of the BHA are a major source of connection fatigue failures and heavy surface abrasion. Thus there is a need to increase transmission rate. Further there is also a need to understand borehole conditions which may lead to poor MPT performance.

In this thesis we shall address the latter issue and possibly lay the background for future investigations of the former. When a signal travels from a source point to a receiver point, it gets modified by the structure through which it travels. Thus the received signal is distorted by information about the structure, which it picks up as

\(^3\)Alternatively shall also be referred to as inner fluid. The mud that is in the annulus between the pipe and the hole shall be referred to as annulus/outer fluid.
it travels through the structure. Hence the 'path' effects have to be subtracted from the received signal to arrive at the source signal. The transfer function between a pressure input to the inner fluid, at the bottom of the hole and the pressure response at the surface, in the inner fluid, is derived for the purpose of surface signal prediction and source waveform recovery.

The second chapter gives the background theory in the form of coupled wave equations of an infinite pipe system. The eigen-value problem is solved and the three natural frequencies and modeshapes of the pipe-borehole system are calculated. Mode superposition is used to describe the initial decomposition of any introduced disturbance into the three modes of the structure. Further, additional mode conversions due to presence of geometric/physical discontinuities in the structure, are then explained.

In the third chapter, the excitation is introduced as a boundary condition and the Transfer matrix which describes the acoustics of a finite, uniform, pipe system is derived. It relates kinematic and dynamic variables at one boundary to those at the other boundary of a pipe system. The Transfer matrix, along with appropriate boundary conditions, is then used in estimating the natural frequencies and modeshapes and in getting the forced response of the pipe system. The forced response leads to the Transfer function of the pipe system. The modelling of the top and bottom boundaries as well as the source is then discussed. Results of a Transfer function calculation for an example problem are presented.
In the fourth chapter, Fourier transform techniques are used to obtain the Impulse response of the pipe system from the Transfer function. The idea of group delay is used to interpret the Impulse response results. Convolving the Impulse response with input waveforms gives the transient behaviour of the pipe system to the specified input. An example problem is then worked out.

The fifth chapter concludes the study and looks at future possibilities.
Chapter 2

BACKGROUND

... anbé un nyabagam vazhum ennodu, onralla āyiram, jenmam unnodu

ilanchittu, unaí vittu, ini engum pogathu,

iru ullam, pudhu vellam, anai pottal thangāthu ...

This thesis is based on the solution to the coupled equations of motion of a pipe system, as derived by H. Y. Lee [5]. The assumptions that were made in deriving the coupled equations are given below.

† The pressures in the inner and outer fluid are coupled to the axial motion of the pipe through radial deformation and poisson effects in the pipe.

1from Annanagar Mudal theru
The change in sound propagation speed in the mud, due to depth was neglected, since the speed in water increases by only 0.34\% per 1000 ft. of depth, due to increase of static pressure. Therefore the mud and the pipe were assumed to be homogeneous.

Viscosity of mud was neglected initially in deriving the equations of motion but its effects were included later.

Steady mud velocity being slow compared to the speed of sound in mud and static pressure rise in the mud due to depth being small compared to its Bulk modulus were reasons for neglecting their effects.

A low frequency approximation was made, which implies that wavelengths are long compared to borehole radius.

The pipe system was assumed to be axially symmetric.

2.1 The Homogeneous Equations of Motion

The assumptions detailed in the previous section are employed in deriving the coupled equations of motion for an infinitely long pipe system. A sketch of the derivation is given here. For more details one may refer to H.Y.Lee's thesis [5].
2.1. THE HOMOGENEOUS EQUATIONS OF MOTION

2.1.1 The Pipe Equation

The assumption about axial wavelengths being long compared to pipe diameter or low frequencies is as follows:

\[ \omega \ll \frac{2\pi c_{\phi}}{b}, \quad (2.1) \]
\[ \omega \ll \frac{2\pi c_{\psi}}{b}, \quad (2.2) \]
\[ \omega \ll \frac{2\pi c_{x}}{b}, \quad (2.3) \]

where

\[ \omega \] : circular frequency
\[ b \] : outer radius of pipe
\[ c_{\phi} \] : propagation speed of dilational wave in pipe
\[ c_{\psi} \] : propagation speed of shear wave in pipe
\[ c_{x} \] : wave propagation speed in the axial direction in the pipe

The equations of motion in cylindrical co-ordinates \((r, \theta, z)\) for an axisymmetric pipe are [8]:

\[ \rho_p \frac{\partial^2 u_r}{\partial t^2} = \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{\partial \tau_{zr}}{\partial x}, \quad (2.4) \]
\[ \rho_p \frac{\partial^2 u_z}{\partial t^2} = \frac{\partial \sigma_z}{\partial x} + \frac{\partial \tau_{zr}}{\partial r} + \frac{\tau_{zr}}{r}. \quad (2.5) \]
2.1. THE HOMOGENEOUS EQUATIONS OF MOTION

If we neglect the viscosity of the mud, there are no shear stresses on the inner and outer surfaces of the pipe. Further, \( \tau_{zx} \) can also be neglected inside the pipe based on the low-frequency assumption. The above equations then are:

\[
\rho_p \frac{\partial^2 u_r}{\partial t^2} = \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r}, \quad (2.6)
\]

\[
\rho_p \frac{\partial^2 u_x}{\partial t^2} = \frac{\partial \sigma_x}{\partial x}. \quad (2.7)
\]

We can write the above two equations in terms of axial and radial displacements alone by introducing stress-strain and strain-displacement relations. If we assume a harmonic solution in time \( (\sim e^{i\omega t}) \) we can see that the radial inertia term in eqn. 2.6 is negligible from our assumption that the radius is small compared to the axial wavelength. Thus we arrive at the following quasi-static equation of motion in the radial direction by neglecting the fluctuation of \( u_x \) (axial displacement of pipe) in the radial direction,

\[
r^2 \frac{\partial^2 u_r}{\partial r^2} + r \frac{\partial u_r}{\partial r} - u_r \approx 0. \quad (2.8)
\]

Imposing the boundary conditions

\[
\sigma_r \big|_{(r=a)} = -p_i, \quad (2.9)
\]

\[
\sigma_r \big|_{(r=b)} = -p_o, \quad (2.10)
\]
we can get a static solution for the radial displacement \( u_r \). With that and the stress-strain and the strain-displacement relations we arrive at the dynamic equation relating axial displacements and mud pressures:

\[
\rho_p \frac{\partial^2 u_x}{\partial t^2} = E \frac{\partial^2 u_x}{\partial x^2} + \frac{2\nu}{b^2-a^2} \left( a^2 \frac{\partial p_i}{\partial x} - b^2 \frac{\partial p_o}{\partial x} \right). \tag{2.11}
\]

Assuming one-dimensional pressure fields in the mud layers, the mud pressures are given in terms of axial displacements as

\[- \frac{\partial p}{\partial x} = \rho_m \frac{\partial^2 u}{\partial t^2} . \tag{2.12}\]

Substituting into the axial equation of motion above, yields the equation governing the pipe as:

\[
\rho_p \frac{\partial^2 u_p}{\partial t^2} = E \frac{\partial^2 u_p}{\partial x^2} + \frac{2\nu}{b^2-a^2} \rho_m \left( -a^2 \frac{\partial^2 u_i}{\partial t^2} + b^2 \frac{\partial^2 u_o}{\partial t^2} \right), \tag{2.13}
\]

where

\[
\begin{align*}
\rho_p, E, \nu & : \text{Density, Youngs modulus and Poisson's ratio of pipe} \\
p_i, p_o & : \text{Dynamic pressure inside and outside the pipe} \\
u_p & : \text{axial displacement in pipe} \\
u_i, u_o & : \text{axial displacement in inner and outer mud} \\
a, b & : \text{inner and outer radius of pipe}
\end{align*}
\]
2.1.2 The Inner and Outer Fluid Equations

The low frequency assumption for the fluid translates as [8, 3]:

\[
\omega \ll 2\pi \frac{c_o}{r}, \quad (2.14)
\]

\[
\omega \ll 2\pi \frac{c_x}{r}, \quad (2.15)
\]

where

\[
\omega \quad : \text{circular frequency}
\]

\[
r \quad : \text{radius}
\]

\[
c_o \quad : \text{wave propagation speed in mud}
\]

\[
c_x \quad : \text{wave propagation speed in the axial direction in the mud}
\]

With the low frequency assumption all field variables can be approximated as functions of the axial coordinate \((x)\) and time \((t)\) alone. The continuity and momentum equations in the axial directions then become:

\[
\frac{\partial}{\partial t} (\rho_t A_t) = -\frac{\partial}{\partial x} (\rho_t A_t v_t), \quad (2.16)
\]

\[
\frac{\partial}{\partial t} (\rho_t A_t v_t) = -\frac{\partial}{\partial x} \left( \rho_t A_t v_t^2 \right) - \frac{\partial}{\partial x} (p_t A_t) + \rho_m g A_t. \quad (2.17)
\]
2.1. THE HOMOGENEOUS EQUATIONS OF MOTION

Equation 2.17 can be simplified using equation 2.16 as follows:

\[
\rho_t A_t \left( \frac{\partial}{\partial t} + v_t \frac{\partial}{\partial x} \right) v_t = - \frac{\partial}{\partial x} (p_t A_t) + \rho_m g A_t ,
\]

where

- \( \rho_t = \rho_m + \rho \) : density
- \( A_t = A_o + A \) : cross-sectional area
- \( v_t = v_o + v \) : axial velocity
- \( p_t = p_o + \rho_m g x + p \) : pressure
- \( g \) : acceleration due to gravity

The quantities \( \rho_m, A_o, v_o, p_o + \rho_m g x \), are constant values corresponding to steady flow and \( \rho, A, v, p \), are fluctuations (as a function of \( x \) and \( t \)) about those constant values.

If products of fluctuations are neglected and if we use the relation \( \rho = (\rho_m/B)p \), the linearized continuity and momentum equations are obtained as:

\[
\rho_m A_o \frac{\partial v}{\partial x} = - \frac{\rho_m A_o}{B} \left( \frac{\partial}{\partial t} + v_o \frac{\partial}{\partial x} \right) p - \rho_m \left( \frac{\partial}{\partial t} + v_o \frac{\partial}{\partial x} \right) A \quad (2.19)
\]

\[
\rho_m A_o \left( \frac{\partial}{\partial t} + v_o \frac{\partial}{\partial x} \right) v = -A_o \frac{\partial p}{\partial x} - (p_o + \rho_m g x) \frac{\partial A}{\partial x} . \quad (2.20)
\]

If we assume harmonic solutions \( \sim e^{ik_x x - \omega t} \) for \( v, p \) and \( A \), it can be shown that \( v_o \frac{\partial}{\partial x} \) can be neglected compared to \( \frac{\partial}{\partial t} \) when the steady mud velocity is slow.
2.1. THE HOMOGENEOUS EQUATIONS OF MOTION

compared to the wave propagation speed in the axial direction \( c_x = \frac{w}{k_x} \). If we differentiate equation 2.19 with respect to \( t \) and equation 2.20 with respect to \( x \) and combine them, we obtain:

\[
\frac{\partial^2 p}{\partial t^2} + \frac{B}{A_o} \frac{\partial^2 A}{\partial t^2} - c_o^2 \frac{p_x}{A_o} \frac{\partial^2 A}{\partial x^2} = c_o^2 \frac{\partial^2 p}{\partial x^2},
\]

(2.21)

where

\[
\begin{align*}
p_x & : p_o + \rho_m g x \\
c_o & : \sqrt{\frac{B}{\rho_m}} \\
B & : \text{Bulk modulus}
\end{align*}
\]

Harmonic solutions show that the third term is small compared to the second term in the left hand side of equation 2.21 when the static pressure is very low compared to the bulk modulus.

If we now differentiate equation 2.21 with respect to \( x \) and substitute equation 2.20 in the resulting equation and integrate twice with respect to \( t \) we get:

\[
\rho_m \frac{\partial^2 u}{\partial t^2} = B \frac{\partial^2 u}{\partial x^2} + B \frac{\partial A'}{\partial x},
\]

(2.22)

where

\[
\begin{align*}
A' & : \text{area strain, } A/A_o \\
A & : \text{cross-sectional area change}
\end{align*}
\]
\[ A_o \quad : \text{initial cross-sectional area} \]

Equation 2.22 is the equation of motion for any fluid volume that is, in general, confined inside the pipe or in the annulus between two cylinders. The expression for the derivative of the appropriate area strain is to be substituted in the above equation to arrive at the equations of motion of the inner and outer fluid. From the solution of the radial displacement \( u_r \), we get,

\[
\begin{align*}
\frac{\partial u_{ra}}{\partial x} &= -\frac{\nu a \rho_p \partial^2 u_x}{E \partial t^2} + \frac{1}{K_a} \frac{\partial p_i}{\partial x} - \frac{1}{K_{ab}} \frac{\partial p_o}{\partial x}, \\
\frac{\partial u_{rb}}{\partial x} &= -\frac{\nu b \rho_p \partial^2 u_x}{E \partial t^2} + \frac{1}{K_{ba}} \frac{\partial p_i}{\partial x} - \frac{1}{K_b} \frac{\partial p_o}{\partial x}, \\
u_{rc} &= \frac{p_o}{K_c},
\end{align*}
\]

where

\[ \rho_m, B \quad : \text{Density and Bulk modulus of mud} \]

\[ u_{ra}, u_{rb}, u_{rc} \quad : \text{radial displacement at } r = a, r = b, r = c \]

\[ K_a \quad : \text{Static inner mud pressure } (p_i) \text{ required to obtain unit radial displacement at inner surface of pipe } (r = a), \text{ in the absence of outer mud pressure} \]

\[ = \frac{E}{a} \frac{b^2 - a^2}{(1-\nu)a^2 - (1+\nu)b^2} \]
2.1. THE HOMOGENEOUS EQUATIONS OF MOTION

\( K_{ab} \) : Static outer mud pressure \( (p_o) \) required to obtain unit radial displacement at inner surface of pipe \( (r = a) \), in the absence of inner mud pressure

\[
K_{ab} = \frac{E}{a} \frac{b^2 - a^2}{2b^3}
\]

\( K_b \) : Static outer mud pressure \( (p_o) \) required to obtain unit radial displacement at outer surface of pipe \( (r = b) \), in the absence of inner mud pressure

\[
K_b = \frac{E}{b} \frac{b^2 - a^2}{(1+\nu)a^2 + (1-\nu)b^2}
\]

\( K_{ba} \) : Static inner mud pressure \( (p_i) \) required to obtain unit radial displacement at outer surface of pipe \( (r = b) \), in the absence of outer mud pressure

\[
K_{ba} = \frac{E}{b} \frac{b^2 - a^2}{2a^3}
\]

\( K_c \) : Static outer mud pressure \( (p_o) \) required to obtain unit radial displacement of the formation \( (r = c) \)
The equivalent spring constant distributed around the borehole wall

\[
K_c = \frac{2G}{c}
\]

For the inner fluid (subscript 'i'):

\[
A_i = 2\pi a u_{ra}, \quad (2.26)
\]

\[
A'_{i} = \frac{2u_{ra}}{a}, \quad (2.27)
\]
2.1. THE HOMOGENEOUS EQUATIONS OF MOTION

where

\[ a \quad : \text{inner radius of pipe} \]
\[ u_{ra} \quad : \text{radial displacement at } r = a \]
\[ A_i \quad : \text{cross-sectional area change of inner fluid} \]
\[ A_i' \quad : \text{area strain of inner fluid} \]

Taking the derivative, with respect to \( x \), of \( A_i' \), and substituting in it for \( \frac{\partial u_{ra}}{\partial x} \), we get the expression for \( \frac{\partial A_i'}{\partial x} \). This, when substituted in equation 2.22, yields the equation governing the inner fluid:

\[
\rho_m \left( 1 + \frac{B}{K_a'} \right) \frac{\partial^2 u_i}{\partial t^2} + \rho_m \frac{B}{K_{ab}'} \frac{\partial^2 u_o}{\partial t^2} + \rho_p \frac{2B\nu}{E} \frac{\partial^2 u_p}{\partial t^2} = B \frac{\partial^2 u_i}{\partial x^2} ,
\]

where

\[ K_a' : \text{Static inner mud pressure } (p_i) \text{ required to obtain unit} \]
\[ \text{area strain of inner mud layer, in the absence of outer mud} \]
\[ = \frac{a}{2} K_a \]
\[ K_{ab}' : \text{Static outer mud pressure } (p_o) \text{ required to obtain unit} \]
\[ \text{area strain of inner mud layer, in the absence of inner mud} \]
\[ = \frac{a}{2} K_{ab} \]
2.1. THE HOMOGENEOUS EQUATIONS OF MOTION

For the outer fluid (subscript 'o'):

\[ A_o = 2\pi(c u_{rc} - b u_{rb}), \quad (2.29) \]
\[ A'_o = \frac{2(c u_{rc} - b u_{rb})}{c^2 - b^2}, \quad (2.30) \]

where

- \( b \): outer radius of pipe
- \( c \): borehole radius
- \( u_{rb}, u_{rc} \): radial displacement at \( r = b, r = c \)
- \( A_o \): cross-sectional area change of outer fluid
- \( A'_o \): area strain of outer fluid

In a similar manner we get the equation governing the outer fluid as:

\[-\rho_m \frac{B}{K'_{bo}} \frac{\partial^2 u_i}{\partial t^2} + \rho_m \left( 1 + \frac{B}{K'_b} + \frac{B}{K'_c} \right) \frac{\partial^2 u_o}{\partial t^2} - \rho_p \frac{2B\nu}{E} \frac{b^2}{c^2 - b^2} \frac{\partial^2 u_p}{\partial x^2} = B \frac{\partial^2 u_o}{\partial x^2}, \quad (2.31)\]

where

- \( K'_b \): Static outer mud pressure \((p_o)\) required to obtain unit area strain of outer mud layer due to \(u_{rb}\) only, in the absence of inner mud pressure

\[ = \frac{c^2-b^2}{2b} K_b \]
2.1. THE HOMOGENEOUS EQUATIONS OF MOTION

\[ K'_{ba}: \text{Static inner mud pressure } (p_i) \text{ required to obtain unit} \]
area strain of outer mud layer, in the absence of outer mud pressure
\[ = \frac{c^2 - b^2}{2b} K'_{ba} \]

\[ K'_{c}: \text{Static outer mud pressure } (p_o) \text{ required to obtain unit} \]
area strain of outer mud layer due to \( u_{rc} \) only, in the absence of inner mud pressure
\[ = G \frac{c^2 - b^2}{c^3} \]

\[ G: \text{Shear modulus of the formation} \]

2.1.3 The Coupled Equations

The three coupled equations of motion derived in sub-section 2.1.1 and sub-section 2.1.2, are assembled in matrix form as follows:

\[
\frac{\partial^2}{\partial t^2}[M] \begin{bmatrix} u_p \\ u_i \\ u_o \end{bmatrix} - \frac{\partial^2}{\partial x^2}[K] \begin{bmatrix} u_p \\ u_i \\ u_o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.32}
\]

where
\[ [M] = \begin{bmatrix} \rho_p & \frac{2\nu a^2}{b^3-a^3} \rho_m & -\frac{2\nu b^2}{b^3-a^3} \rho_m \\ \frac{2B\nu}{E} \rho_p & (1 + \frac{B}{K_a}) \rho_m & -\frac{B}{K_{ab}} \rho_m \\ -\frac{B^2}{c^2-b^2} \frac{2B\nu}{E} \rho_p & -\frac{B}{K_{ba}} \rho_m & (1 + \frac{B}{K_b} + \frac{B}{K_c}) \rho_m \end{bmatrix}, \quad (2.33) \]

and

\[ [K] = \begin{bmatrix} E & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{bmatrix}. \quad (2.34) \]

### 2.2 An Eigen-Solution

If we assume a harmonic solution \( \sim e^{(-k_x z + \omega t)} \) for the axial displacements \( u_p, u_i, \) and \( u_o \), the equations of motion reduce to an eigen-value problem where the eigen-values are the axial wave propagation speeds \( (c_x = w/k_x) \) i.e.

\[ \begin{vmatrix} c_x^2 \end{vmatrix} - [M] = [K] \begin{bmatrix} u_p \\ u_i \\ u_o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (2.35) \]

A non-trivial solution is possible only if the characteristic determinant vanishes.

\[ \begin{vmatrix} c_x^2 \end{vmatrix} - [M] = [K] = 0. \quad (2.36) \]

The solution consists of three eigen-values \( (c^2_{x_n}, n = 1, 2, 3) \) and three eigen-
vectors \((\psi_n, n = 1, 2, 3)\). Each of the eigen-values represent an axial propagation speed, of the corresponding eigen-vector, which is a pattern of axial displacements in the three layered cylindrical waveguide that is our pipe system. Based on the relations between the various field variables, we can, if the axial displacement modes are known, get the velocity, stress/pressure and radial displacement modes uniquely. The relations of interest are:

\[
\begin{align*}
\begin{pmatrix}
v_p \\
v_i \\
v_o
\end{pmatrix}_n & = \omega \begin{pmatrix}
u_p \\
u_i \\
u_o
\end{pmatrix}_n, \\
\begin{pmatrix}
\sigma_p \\
p_i \\
p_o
\end{pmatrix}_n & = c_{zn} \begin{pmatrix}
-\rho_p & 0 & 0 \\
0 & \rho_m & 0 \\
0 & 0 & \rho_m
\end{pmatrix} \begin{pmatrix}
v_p \\
v_i \\
v_o
\end{pmatrix}_n.
\end{align*}
\] (2.37) (2.38)

Thus, associated with each axial propagation speed we have a unique axial displacement pattern or a radial displacement pattern or an axial stress/pressure pattern. There exists a linear transformation that relates modes of any one field variable to another.
2.3 Modal Analysis

There exist three pressure modeshapes \( (\phi_n, n = 1, 2, 3) \), each of which, propagates at a different axial speed. An arbitrary pressure disturbance \( \{P\}e^{i\omega t} \) introduced into a uniform semi-infinite pipe system at \( z = 0 \), gets decomposed into a linear combination of these three pressure modeshapes.

The stress/pressure distribution at any position and at any instant is:

\[
\begin{bmatrix}
\sigma_p \\
p_i \\
p_o
\end{bmatrix} = D_1 \begin{bmatrix} \phi_1 \end{bmatrix} e^{-i k_{x1} z} + D_2 \begin{bmatrix} \phi_2 \end{bmatrix} e^{-i k_{x2} z} + D_3 \begin{bmatrix} \phi_3 \end{bmatrix} e^{-i k_{x3} z},
\]

(2.39)

where

\[ k_{xn} = \omega/c_{xn}, n = 1, 2, 3 \]

\( D_n \): amplitude of the \( n^{th} \) mode (modal amplitude)

The boundary condition at \( z = 0 \) is:

\[
\begin{bmatrix}
\sigma_p \\
p_i \\
p_o
\end{bmatrix} = D_1 \begin{bmatrix} \phi_1 \end{bmatrix} + D_2 \begin{bmatrix} \phi_2 \end{bmatrix} + D_3 \begin{bmatrix} \phi_3 \end{bmatrix}.
\]

(2.40)
Thus

\[
\begin{pmatrix}
D_1 \\
D_2 \\
D_3
\end{pmatrix} = \left[
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix}
\right]^{-1} \begin{pmatrix}
P
\end{pmatrix}.
\tag{2.41}
\]

Since each mode travels at a different axial speed, the introduced pulse gets decomposed and distorted as it travels through the structure.

In the case of a finite, uniform pipe system, each of the three travelling modes can create standing wave patterns. Imposing boundary conditions then allows standing wave solutions to be calculated. The behaviour of a single mode in a finite, uniform pipe system is akin to that of a plane wave in an uniform organ pipe — both have resonances at an infinite set of discrete frequencies. The finite, uniform pipe system, in general (when all the three modes are present), then can be considered to behave like a complex organ pipe with a triple infinite set of resonances.

### 2.3.1 Mode Conversions

In a realistic pipe system the physical and geometric properties are unlikely to be homogeneous over its entire length. It is likely to have changes in drill pipe geometry, changes in borehole geometry, transition from drill pipe to drill collars and changes in the physical properties of the formation.

We have up to now considered the forcing to be harmonic (~ \(e^{i\omega t}\)) and have
hence been concerned with the steady state behaviour of our system. But now, we briefly consider transient 'pressure pulse' behaviour, since it clarifies the events at an interface (a plane where there is a change in geometric/physical property).

The axial propagation speeds and pressure (or axial displacement or radial displacement) modeshapes that were derived in section 2.2 were for a uniform pipe system with no change in geometry or physical properties. Another pipe system with any or all of the changes mentioned above would have a different set of eigenvalues and eigen-vectors. Consider the case of two semi-infinite pipe sections (I and II) of different geometry joined together at $x = 0$ as shown in figure 2.1. A pressure impulse introduced far to the left in Section I, decomposes into three modal pulses (corresponding to the three modes of that section) and travel independently to the interface between the two sections. The three modal pulses arrive at different times because their axial propagation speeds are not the same. At the interface there are compatibility conditions to be satisfied between the left and right sections. They are axial force and axial displacement compatibilities between the pipe media (force balance and continuity of displacement) and pressure and axial velocity compatibilities between the fluid media (continuity of pressure and mass conservation) in the left and right sections.
Figure 2.1: An interface between two pipe systems of different geometries.
2.3. MODAL ANALYSIS

Kinematic compatibility:

\[
\begin{align*}
\begin{cases}
    u_p \\
    A_i v_i \\
    A_o v_o
\end{cases}
    =
    \begin{cases}
    u_p \\
    A_i v_i \\
    A_o v_o
\end{cases}
\end{align*}
\]

\hspace{1cm} (2.42)

Dynamic compatibility:

\[
\begin{align*}
\begin{cases}
    \sigma_p A_p \\
    p_i \\
    p_o
\end{cases}
    =
    \begin{cases}
    \sigma_p A_p \\
    p_i \\
    p_o
\end{cases}
\end{align*}
\]

\hspace{1cm} (2.43)

A single modal pulse that arrives at the interface, from the left section, will produce a reflected pulse and a transmitted pulse. The reflected pulse goes back into the left section and the transmitted pulse goes into the right section. But these pulses can only travel as the modeshapes of the corresponding section in which they exist. Thus a single modal pulse arriving at an interface results in three reflected and three transmitted modal pulses. This is referred to as Mode Conversion. Note that the three reflected modes are different from the three transmitted modes because they are the eigen-vectors of different pipe systems. This has been the story of one modal component (three such components exist) of an impulse arriving at an interface. Cumulatively, the impulse disturbance that was introduced initially results in nine reflected pulses (three of each mode) in the left section and nine
transmitted pulses (three of each mode) in the right section. An observer in Section I would record three incident arrivals and nine reflected arrivals at different times and an observer in Section II would record nine transmitted arrivals. Thus there is considerable potential for distortion of a pressure pulse when it encounters an interface.

2.4 Example 2.1

Consider two uniform, semi-infinite pipe systems (Section I and Section II) joined together. We will calculate the propagation speeds, modeshapes and the mode conversions in travelling from Section I to Section II. We shall do the above for two cases.

Case (a) : There is a change in geometric properties

Case (b) : There is change in physical properties

in going from Section I to Section II.

2.4.1 Case(a) – Change in Geometry

Consider two different sections (I and II) and an interface as shown in fig 2.1. The only change in going from Section I to II is a change in the borehole diameter.
### 2.4. EXAMPLE 2.1

<table>
<thead>
<tr>
<th>Input Data for Example 2.1 – Case (a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner dia. of pipe</td>
</tr>
<tr>
<td>Outer dia. of pipe</td>
</tr>
<tr>
<td>Density of pipe</td>
</tr>
<tr>
<td>Youngs Modulus of pipe</td>
</tr>
<tr>
<td>Poissons ratio of pipe</td>
</tr>
<tr>
<td>Density of mud</td>
</tr>
<tr>
<td>Bulk Modulus of mud</td>
</tr>
<tr>
<td>Wall stiffness</td>
</tr>
<tr>
<td>Attenuation constants a₁, a₂, a₃²</td>
</tr>
<tr>
<td>Borehole dia. Sec I</td>
</tr>
<tr>
<td>Sec II</td>
</tr>
</tbody>
</table>

The propagation speeds\(^3\) and pressure modeshapes of Section I and II are as below:

\(^3\)The terms *axial propagation speed* and *propagation speed* will be used synonymously.
The modeshapes of Section I and II are as shown in figure 2.2 and figure 2.3 respectively. The two fluid media are said to be strongly *coupled* if the fluid pressures in the two fluid modes are comparable. In such a case, the two modes travel at comparable speeds and have modeshapes that look similar. Coupling is dependent on the relative magnitudes of the pipe radial stiffness and the formation radial stiffness. Hard formations engender strong coupling. The consequence of strong
coupling is that, a pressure disturbance introduced into the inner fluid medium would communicate to the outer fluid medium and travel as a combination of the two fluid modes. The fluid media in both the above sections are strongly coupled as can be seen from their modeshapes and propagation speeds.

The consequences of a single mode arriving at the interface, from the far left, is now detailed. Based on kinematic and dynamic compatibility between the two sections, at the interface, we have three reflected and three transmitted modes. If it is required that the sum of the incident and reflected waves equal the transmitted wave and if we express the result in modal terms,

\[
I + R = T, \tag{2.44}
\]

\[
\phi_i' + c_1\phi_1' + c_2\phi_2' + c_3\phi_3' = d_1\phi_1'' + d_2\phi_2'' + d_3\phi_3'', \tag{2.45}
\]

where

| \( \phi \) | \( = \{u_p, A_i v_i, A_o v_o, A_p \sigma_p, p_i, p_o\}^T \) |
| \( I \) | \( = \phi_i' \), \( i = 1,2 \text{ or } 3 \) |
| \( R \) | \( = c_1\phi_1' + c_2\phi_2' + c_3\phi_3' \) |
| \( T \) | \( = d_1\phi_1'' + d_2\phi_2'' + d_3\phi_3'' \) |
Figure 2.2: The modeshapes of Section I, Example 2.1 - Case (a) and Case (b).
Figure 2.3: The modeshapes of Section II, Example 2.1 - Case (a).
2.4. EXAMPLE 2.1

: Transmitted modes (three)

\[ c_i \] modal amplitudes in Section I, \( i = 1, 2, 3 \)

\[ d_i \] modal amplitudes in Section II, \( i = 1, 2, 3 \)

subscript: node number, \( i = 1, 2 \text{ or } 3 \)
superscript: number, I or II

<table>
<thead>
<tr>
<th>Incident Mode</th>
<th>reflected components</th>
<th>Transmitted components</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( c_1 )</td>
<td>( c_2 )</td>
</tr>
<tr>
<td>( \phi_1^I )</td>
<td>-0.31389</td>
<td>-0.32610</td>
</tr>
<tr>
<td>( \phi_2^I )</td>
<td>-0.04140</td>
<td>-0.04320</td>
</tr>
<tr>
<td>( \phi_3^I )</td>
<td>0.00027</td>
<td>0.00029</td>
</tr>
</tbody>
</table>

A pattern emerges from the above numbers. Mode 2 and Mode 3 of Section I and Section II are almost the same (Mode 1 is the one that is different when we go from one section to another). Hence when Mode 2 or Mode 3 are incident on the interface, they suffer little reflection and are transmitted through, almost completely, to Section II.

The change that occurs in going from Section I to Section II, is in the outer fluid and this results in Mode 1 of the two sections being considerably different. Hence Mode 1 suffers considerable reflection (\( c_1 \) comparable to \( d_1 \)) when it arrives at the interface.
2.4. EXAMPLE 2.1

2.4.2 Case (b) – Change in Physical Property

We now consider the two sections with no change in geometry but with a change in wall stiffness in going from Section I to Section II.

The data is the same as Case (a), except for the wall stiffness and the borehole diameter. The details are as shown in Figure 2.4.

<table>
<thead>
<tr>
<th>Input Data for Example 2.1 – Case (b)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Borehole dia.</td>
<td>12.347 in. 31.36 cm</td>
</tr>
<tr>
<td>Wall stiffness Sec I</td>
<td>$9.089 \times 10^8 \text{ (lb/ft}^2\text{)/ft}$ $4.3518 \times 10^{10} \text{ Pa/m}$</td>
</tr>
<tr>
<td>Sec II</td>
<td>$0.114 \times 10^8 \text{ (lb/ft}^2\text{)/ft}$ $5.458 \times 10^8 \text{ Pa/m}$</td>
</tr>
</tbody>
</table>

The propagation speeds and Modeshapes of the two sections are as shown below:

<table>
<thead>
<tr>
<th>Section I</th>
<th>Mode 1</th>
<th>Mode 2</th>
<th>Mode 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\phi_1$</td>
<td>$\phi_2$</td>
<td>$\phi_3$</td>
</tr>
<tr>
<td>$\sigma_p$</td>
<td>0.2156</td>
<td>-0.0871</td>
<td>1.0000</td>
</tr>
<tr>
<td>Mode shapes</td>
<td>$p_i$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.9607</td>
<td>1.0000</td>
<td>0.0058</td>
</tr>
<tr>
<td></td>
<td>$p_o$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.0000</td>
<td>0.1325</td>
<td>-0.0009</td>
</tr>
<tr>
<td>Speeds $ft/sec$</td>
<td>4451</td>
<td>4820</td>
<td>16943</td>
</tr>
<tr>
<td>$m/sec$</td>
<td>1356</td>
<td>1469</td>
<td>5164</td>
</tr>
</tbody>
</table>
Figure 2.4: An interface between two pipe systems of different physical properties.
The modes shapes of Section II are as shown in Figure 2.5. As can be seen from the above numbers, the two fluid media are strongly coupled in Section I but weakly coupled in Section II. This is so because the radial stiffness of the formation has gone down by a factor of $\sim 100$ in going from Section I to Section II (formation radial stiffness is now low compared to the pipe radial stiffness). In Section II, a pressure disturbance introduced into one of the fluid media does not communicate with the other fluid media and travels almost entirely in one media as a single mode.

The mode conversion calculation when carried out for this case yields:
Figure 2.5: The modeshapes of Section II, Example 2.1 - Case (b).
2.4. EXAMPLE 2.1

<table>
<thead>
<tr>
<th>Incident Mode</th>
<th>reflected components</th>
<th>Transmitted components</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$c_1$</td>
<td>$c_2$</td>
</tr>
<tr>
<td>$\phi_1^l$</td>
<td>-0.53741</td>
<td>-0.56788</td>
</tr>
<tr>
<td>$\phi_2^l$</td>
<td>-0.07226</td>
<td>-0.07647</td>
</tr>
<tr>
<td>$\phi_3^l$</td>
<td>0.00057</td>
<td>0.00061</td>
</tr>
</tbody>
</table>

We see that Mode 2 is transmitted better ($d_2$ is larger now, than in Case (a)) into Section II. This is because Section II is weakly coupled. Any pressure introduced into the inner fluid, of Section II, has to travel almost entirely as Mode II, since, the other two modes have small inner fluid pressures. Further Mode 2 of Section I does not have a large outer mud pressure and hence we do not need a large Mode 1 of Section II to balance it.

Mode I undergoes a stronger reflection now, as compared to Case (a), as the difference in modesheapes between the two sections is much starker. Mode 1 of Section I has almost equal pressures in the inner and outer fluid and hence it requires comparable amounts of Mode 1 and Mode 2 of Section II to balance it, at the interface. Hence $d_1$ and $d_2$ are comparable as they were in Case (a) but for different reasons. In Case (a), Mode 1 of Section II has a very large inner mud pressure. The amount of outer mud pressure ($d_1$) required in Section II, for compatibility at the interface, resulted in very large pressures in the inner mud. Hence the amount $d_2$ of
Mode 2 necessary to offset the inner mud pressure produced by Mode 1 is substantial (note: the inner mud pressure in Mode 1 and Mode 2 are of opposite sign). This is why $d_2$ is positive in Case (a) – it cancels the inner mud pressure produced by Mode 1, and it is negative in Case (b) – it adds to the inner mud pressure produced by Mode 1.
Chapter 3

THE TRANSFER FUNCTION

... anrum inrum enrum, undan kayil thanjam
pavayalla parvai pesum oviyam
katri vilangum muchilum, kanni pesum pechilum
nejamanathu, undhan thanjamananathu ...

We get the displacement response at any position \(0 \leq x \leq l\) due to applied axial forces at a boundary from the homogeneous equations of motion derived in chapter 2. It is then used in building up a transfer matrix for a uniform, finite pipe system. A Transfer matrix relates kinematic and dynamic variables on one boundary of the structure (say, the left, in the case of a pipe system) to those at another boundary (the right). With such a relation between the field variables and suitable boundary conditions, we can get a Transfer Function for a single uniform,

1 from Agni Natchathiram
finite pipe system — response at one boundary, to a forcing at the other. A Transfer function, in general, relates inputs and outputs at any two locations in the structure. But since the Transfer matrix of a single uniform, finite pipe system relates only the field variables at the boundaries, the Transfer function that we derive from the Transfer matrix by imposing boundary conditions, would be only capable of relating inputs and outputs which are at the boundaries. A real borehole, which would have changes in properties along its length, would be modelled by many uniform, finite pipe systems (whose physical/geometric properties are constant within an individual pipe system, but vary from one to another) linked together in a chain-like fashion. The individual Transfer matrices of the pipe systems above would be used in getting a Transfer function which would be able to relate field variables at any two locations in the structure, as we shall detail in Section 3.3.

A knowledge of the Transfer Function between the pressure response in the inner mud, at the surface (i.e. top), and a pressure input to the inner mud, at the bottom of the borehole would be valuable in understanding and improving current MWD techniques. Further, a Transfer Function between secondary sources like mud pumps, the bit etc. and the pressure response in the inner mud, at the surface, would help us understand these noise sources and their impact on information transmission and reception.

---

2 The forcing could be an imposed axial displacement or force in any of the three media and the response likewise could be the resulting axial displacement or force in any of the three media.

3 Measurement While Drilling
3.1 A Mode Superposition Solution

If we are interested in response, to forces applied only at the boundaries, we can use the homogeneous equations of motion and introduce the forcing as a boundary condition.

In seeking the solution (axial displacements response to applied forces) for the forced problem, we may use Mode superposition. This was detailed in section 2.3 under Modal analysis.

A homogeneous, finite pipe system of length \( l \) is considered. The problem formulation is:

\[
[M] \frac{\partial^2}{\partial t^2} \{u\} - [K] \frac{\partial^2}{\partial x^2} \{u\} = 0 ,
\]  

(3.1)

with the following boundary conditions:

\[
\{f\} = 0 \quad \text{at } x = 0 ,
\]  

(3.2)

\[
\{f\} = \{F\} e^{i\omega t} \quad \text{at } x = l ,
\]  

(3.3)

where

\[
\{f\} \quad : \text{time varying vector of axial forces} - \{f_p \ f_i \ f_o\}^T
\]

\[
\{F\} \quad : \text{time independent vector of applied axial forces} - \{F_p \ F_i \ F_o\}^T
\]

The displacement pattern can be expressed as a linear combination of the three
modes in the form of standing waves between the two boundaries i.e.

\[
\{u(x)\}e^{i\omega t} = \left[ D_1\{\psi_1\}e^{ik_p x} + D_2\{\psi_1\}e^{-ik_p x}\right]e^{i\omega t} + \\
\left[ D_3\{\psi_2\}e^{ik_1 x} + D_4\{\psi_2\}e^{-ik_1 x}\right]e^{i\omega t} + \\
\left[ D_5\{\psi_3\}e^{ik_2 x} + D_6\{\psi_3\}e^{-ik_2 x}\right]e^{i\omega t}.
\] (3.4)

where

\[
\{u(x)\} : \text{vector of axial displacements} - \{u_p(x) \ u_i(x) \ u_o(x)\}^T
\]

\[
\{\psi_n\} : \text{Displacement modeshape} \ (n = 1, 2, 3)
\]

\[
k_p , k_1 , k_2 : \text{wavenumber in pipe, inner and outer fluid respectively}^4
\]

\[
D_n : \text{Modal amplitudes} \ (n = 1, 2, \ldots 6)
\]

\(D_2, D_4, \text{ and } D_6\) are waves travelling in the \(+z\) direction and \(D_1, D_3 \text{ and } D_5\) are waves travelling in the \(-z\) direction.

In section 2.2 we had touched upon the duality between axial displacements and stress/pressure. The relation between them is:

\[
\frac{\partial}{\partial x} \begin{bmatrix}
\sigma_p \\
p_i \\
p_o
\end{bmatrix}
= -\omega^2 
\begin{bmatrix}
\rho_p & 0 & 0 \\
0 & -\rho_m & 0 \\
0 & 0 & -\rho_m
\end{bmatrix}
\begin{bmatrix}
u_p \\
u_i \\
u_o
\end{bmatrix}.
\] (3.5)
Thus the stress/pressure distribution assumes the following form:

\[
\{ p(x) \} = \omega^2 \{ \rho \} \times \left\{ \frac{D_1 \{ \psi_1 \}}{i k_p} e^{i k_p x} - \frac{D_2 \{ \psi_1 \}}{i k_p} e^{-i k_p x} \right\} + \left\{ \frac{D_3 \{ \psi_2 \}}{i k_1} e^{i k_1 x} - \frac{D_4 \{ \psi_2 \}}{i k_1} e^{-i k_1 x} \right\} + \left\{ \frac{D_5 \{ \psi_3 \}}{i k_2} e^{i k_2 x} - \frac{D_6 \{ \psi_3 \}}{i k_2} e^{-i k_2 x} \right\},
\]

where

\[
\{ p(x) \} : \text{vector of stress/pressures } \{ \sigma_p \quad p_1 \quad p_3 \}^T
\]

\[
\{ \rho \} : \text{diagonal matrix of densities } \{-\rho_p \quad \rho_m \quad \rho_m\} \text{ is the principal diagonal}
\]

The first boundary condition, equation 3.2, equation 3.6 and the fact that the modes are orthogonal imply that

\[
D_1 = D_2 , \quad (3.7)
\]
\[
D_3 = D_4 , \quad (3.8)
\]
\[
D_5 = D_6 . \quad (3.9)
\]

The amplitude of the \( +x \) direction travelling component of a mode has the same amplitude as the \( -x \) direction travelling component.

The displacement and stress/pressure vectors now become

\[
\{ u(x) \} = 2 \left( D_1 \{ \psi_1 \} \cos(k_p x) + D_2 \{ \psi_2 \} \cos(k_1 x) + D_6 \{ \psi_3 \} \cos(k_2 x) \right) (3.10)
\]
3.1. A MODE SUPERPOSITION SOLUTION

\[ \{p(x)\} = 2 \left( D_1 \{\phi_1\} \sin(k_p x) + D_2 \{\phi_2\} \sin(k_1 x) + D_3 \{\phi_3\} \sin(k_2 x) \right) \]  \hspace{1cm} (3.12)

\[ \{\Phi(x)\} \{D\}, \]  \hspace{1cm} (3.13)

where

\[ \{\phi_n\} = \omega c_i \rho \{\psi_n\}, \quad (n = 1, 2, 3) \]

\[ c_i = \omega / k_i, \text{ axial wave propagation speed, } (i = p, 1, 2) \]

\[ [\Psi(x)] = \begin{bmatrix} \{\psi_1\} \cos(k_p x) & \{\psi_2\} \cos(k_1 x) & \{\psi_3\} \cos(k_2 x) \end{bmatrix} \]

\[ [\Phi(x)] = \begin{bmatrix} \{\phi_1\} \sin(k_p x) & \{\phi_2\} \sin(k_1 x) & \{\phi_3\} \sin(k_2 x) \end{bmatrix} \]

\[ \{D\} = \begin{bmatrix} 2D_1 & 2D_2 & 2D_3 \end{bmatrix}^T \]

\[ [\rho] : \text{ diagonal matrix of densities - } \{-\rho_p, \rho_f, \rho_f\} \text{ is the principal diagonal} \]

\[ \rho_p, \rho_f : \text{ density of pipe, fluid} \]

The second boundary condition, equation 3.3, gives the following result.

\[ [A]\{p(x = l)\} = \{F\}, \]  \hspace{1cm} (3.14)

where

\[ [A] : \text{ diagonal matrix of areas - } \{A_p, A_i, A_o\} \text{ is the principal diagonal} \]
From the above equation and equation 3.13, we can get

\[ \{D\} = [\Phi(x)]_{x=1}^{-1}[A]^{-1}\{F\} . \]  

(3.15)

Once \(\{D\}\) is determined, the axial displacement response to the applied forces, is obtained from equation 3.11. This response is used to get the Transfer matrix of a uniform, finite pipe system.

### 3.2 The Transfer matrix

A transfer matrix relates kinematic and dynamic variables at one boundary of a structure to those at another [7]. It is usually used for structures that have a 'chain-like' topology, i.e. cables, beams, rods, shafts etc. A state vector at a point \(i\) in an elastic system \(z_i\), is a column vector of displacements (or rotations) and internal forces (or moments) at the position \(i\). Thus a Transfer matrix of our uniform, finite pipe system would relate the state vector \(z_R\) to \(z_L\) as shown in figure 3.1. The variables of interest, in our case are, axial displacement in the three media and axial force in the three media\(^5\). The Transfer matrix then fits into the following relation:

\[ z_R = [T] z_L . \]  

(3.16)

\(^5\)Our state vectors \(z = \{u_p, u_i, u_o, f_p, f_i, f_o\}^T\), are \(6 \times 1\) vectors and our Transfer matrices are \(6 \times 6\) matrices.
Figure 3.1: The schematic of the Transfer matrix for a pipe system.
3.2. THE TRANSFER MATRIX

The Transfer matrix is a function of \( \omega \), the frequency. The transfer matrices are called so because they 'transfer' or map the state vector at position \( i \) (\( z_i \)) to a state vector at position \( i + 1 \) (\( z_{i+1} \)). Equation 3.16 in expanded form is:

\[
\begin{align*}
    u_R &= T_{11}u_L + T_{12}f_L , \\
    f_R &= T_{21}u_L + T_{22}f_L ,
\end{align*}
\]  

(3.17)  

(3.18)

where

\( T_{ij} \) : a sub-matrix of \( T \) – a \( 3 \times 3 \) matrix

\( u \) : a column vector of displacements – \( \{ u_p, u_i, u_o \} \)

\( f \) : a column vector of forces – \( \{ f_p, f_i, f_o \} \)

\( R, L \) : subscripts indicating right and left boundaries

Consider now a Mobility matrix \( \{ B \} \). It relates kinematic variables at both boundaries to dynamic variables at both boundaries. This is just equation 3.16 rearranged, but using the Mobility matrix as an intermediate step in deriving the Transfer matrix allows us to see reciprocity and symmetry come into play in our structure.

\[
\begin{bmatrix}
    u_L \\
    u_R
\end{bmatrix} =
\begin{bmatrix}
    B_{LL} & B_{LR} \\
    B_{RL} & B_{RR}
\end{bmatrix}
\begin{bmatrix}
    f_L \\
    f_R
\end{bmatrix},
\]

(3.19)

where

3.2. THE TRANSFER MATRIX

\( B_{LR} \) : Displacement at the 'L'eft boundary due
to unit force at the 'R'ight boundary — a 3 \( \times \) 3 matrix

In section 3.1 we derived the response \( \{u(x)\} \) to force \( \{F\} \) applied at \( x = l \).

Specifically, it was:

\[
\{u(x)\} = [\Psi(x)] [\Phi(x)]_{x=l}^{-1} [A]^{-1} \{F\} .
\]

(3.20)

Specifying the position \( x \), would give us the response at that position. Hence

\[
u_L = [\Psi(x)]_{x=0} [\Phi(x)]_{x=l}^{-1} [A]^{-1} \{F\} ,
\]

(3.21)

\[
u_R = [\Psi(x)]_{x=l} [\Phi(x)]_{x=l}^{-1} [A]^{-1} \{F\} .
\]

(3.22)

The above two equations identify \( B_{LR} \) and \( B_{RR} \). i.e.

\[
B_{LR} = [\Psi(x)]_{x=0} [\Phi(x)]_{x=l}^{-1} [A]^{-1} ,
\]

(3.23)

\[
B_{RR} = [\Psi(x)]_{x=l} [\Phi(x)]_{x=l}^{-1} [A]^{-1} .
\]

(3.24)

Further,

\[
B_{RL} = -B_{LR} , \text{because of reciprocity}
\]

(3.25)

\[
B_{LL} = -B_{RR} , \text{because of symmetry}
\]

(3.26)
3.3. MODELLING A COMPLEX PIPE SYSTEM

Notice that a negative sign has sneaked into the above two relations because of the sign convention regarding the direction of \( f_L \) (refer to figure 3.1). Thus the Mobility matrix is now fully determined. We can now juggle equation 3.19 to get it to resemble equations 3.17 and 3.18. As a result we get the following relations between the sub-matrices of the Mobility and Transfer matrices.

\[
T_{11} = B_{RR}B_{LR}^{-1}, \quad (3.27)
\]

\[
T_{12} = -B_{LR} + B_{RR}B_{LR}^{-1}B_{RR}, \quad (3.28)
\]

\[
T_{21} = B_{LR}^{-1}, \quad (3.29)
\]

\[
T_{22} = -B_{LR}^{-1}B_{RR}. \quad (3.30)
\]

With equations 3.23 and 3.24 and the expressions above the Transfer matrix is now fully determined. The fruits of these labours are borne in the next section.

3.3 Modelling a Complex Pipe System

In the previous section, we developed a mathematical model for the acoustics of a finite, uniform pipe system. Now we endeavour to do the same for a finite, complex pipe system, the complexity being inhomogeneity in geometric/physical properties, along its length.

The Transfer matrix of a uniform pipe system was derived in section 3.2. In a similar manner, we can derive transfer matrices of common elastic elements like...
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### 3.3 Modelling a Complex Pipe System

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The Transfer matrix of a uniform pipe system was derived in section 3.2. In a similar manner, we can derive transfer matrices of common elastic elements like
3.3. MODELLING A COMPLEX PIPE SYSTEM

uniform bars and lumped elements like a mass or a spring.

... is based the idea of breaking up a complicated system into component parts with simple elastic and dynamic properties that can be readily expressed in matrix form. These component matrices are considered as building blocks that, when fitted together according to a set of predetermined rules, provide the static and dynamic properties of the entire system. The matrix formulation of these rules is superbly adapted for consumption by digital computers, and furthermore the concise matrix notation brings to light basic properties of linear elastic systems formerly obscured in a mass of algebraic baroque.

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Thus, with transfer matrices of common elastic elements on hand, we can model complicated linear elastic structures by linking the elements together in a chain-like fashion with appropriate compatibility requirements imposed between adjacent elements.

Consider a non-uniform, finite pipe system as shown in the figure 3.2. A mathematical model of it can be constructed with just a knowledge of transfer matrices of uniform, finite pipe systems.

We first break down the non-uniform pipe system into a number of sub-systems of uniform pipe systems linked together. In any sub-system/element $i$, the relation
Figure 3.2: The Transfer matrix relation for a non-uniform, finite pipe system.
between the state vectors at its boundaries are:

\[ z_i^R = [T_i] z_i^L. \]  \hspace{1cm} (3.31)

Between any two adjacent elements the compatibility requirement is:

\[ z_{i+1}^L = [C_i] z_i^R, \]  \hspace{1cm} (3.32)

where \([C_i]\) is a compatibility matrix, whose elements ensure the kinematic and dynamic compatibility that was detailed in section 2.3.1. For our pipe system \([C_i]\) is a diagonal matrix given as:

\[
[C_i] = \begin{bmatrix}
1 & & & \\
\frac{a_i^2}{a_{i+1}^2} & 1 & & \\
\frac{c_i^2-b_i^2}{c_{i+1}^2-b_{i+1}^2} & \frac{c_i^2-b_i^2}{c_{i+1}^2-b_{i+1}^2} & 1 & \\
\frac{a_{i+1}^2}{a_i^2} & & \frac{c_{i+1}^2-b_{i+1}^2}{c_i^2-b_i^2} & \\
\end{bmatrix}, \hspace{1cm} (3.33)
\]

where

\[ a_i, b_i, c_i \] : pipe inner radius, pipe outer radius and borehole radius of the \(i^{th}\) element

Now the algebra of the synthesis of a non-uniform pipe system is as shown in
3.3. MODELLING A COMPLEX PIPE SYSTEM

Proceeding from left to right within an element and from the first element to the \( n \)th element, we multiply the transfer and compatibility matrices, based on equations 3.31 and 3.32, as follows:

\[
\mathbf{z}_n^R = [T_n][C_{n-1}][T_{n-1}] \cdots [C_i][T_i] \cdots [T_2][C_1][T_1] \mathbf{z}_1^L ,
\]

\[= [\Pi] \mathbf{z}_1^L, \tag{3.35}\]

\[\Rightarrow u_n^R = \Pi_{11}u_1^L + \Pi_{12}f_1^L, \tag{3.36}\]

\[f_n^R = \Pi_{21}u_1^L + \Pi_{22}f_1^L, \tag{3.37}\]

where

\[\Pi_{ij} : \text{a sub-matrix of } \Pi\]

Thus the state vectors at the boundaries of the structure are related through the simple relation above.

3.3.1 Natural Frequencies and Modeshapes

If the product of matrices is carried out in equation 3.34, it reduces to a set of six equations in twelve variables. If any six of the state variables are specified, then the other six can be related to one another. In other words, if we specify six of the state variables by imposing boundary conditions on the above structure, we can relate the remaining unknown displacements and forces. Two results fall out of the above equation. When we specify that the solution be non-trivial, the
natural frequencies of the system are determined. And, through the relation between
unknown displacements and forces we can get the normal modes of the system.

For example, if we had our pipe system fixed at the left boundary and free at
the right boundary, then,

\[ u_1^L = 0 \quad \text{fixed boundary} \quad (3.38) \]
\[ f_n^R = 0 \quad \text{free boundary} \quad (3.39) \]

Then, from equations 3.36 and 3.37 we have:

\[ u_n^R = \Pi_{12} f_1^L \quad (3.40) \]
\[ 0 = \Pi_{22} f_1^L \quad (3.41) \]

From equation 3.41, if \( f_1^L \) is non-trivial i.e. \( \neq 0 \) then:

\[ \text{det} |\Pi_{22}| = 0 \quad (3.42) \]

Note that since \( \Pi_{22} \) is a sub-matrix of \([\Pi]\), which is a product of transfer matrices,
and since any Transfer matrix is a function of the frequency \( \omega \), \( \Pi_{22} \) is a function of \( \omega \).
The above determinant is evaluated for various values of \( \omega \), and the zero-crossings
of the resulting curve of the determinant versus the frequency gives us an infinite set
of natural frequencies associated with this structure (with fixed-free boundaries).
3.3. MODELLING A COMPLEX PIPE SYSTEM

Equation 3.40 tells a different story. It relates the six unknown variables through three equations. Hence we can write any three of the variables in terms of the remaining.

\[ f_1^L = \Pi_{12}^{-1} u_n^R, \quad (3.43) \]

\[ \Rightarrow z_1^L = \begin{cases} 
0 \\
\Pi_{12}^{-1} u_n^R
\end{cases}, \quad (3.44) \]

and \[ z_n^R = \begin{cases} 
u_n^R \\
0
\end{cases}. \quad (3.45) \]

The first three elements of \( z_1^L \) are zero because of the fixed (displacement \( = 0 \)) boundary condition there and the last three of \( z_n^R \) are zero because of the free (force \( = 0 \)) boundary condition there. Now by multiplying the appropriate Transfer and Compatibility matrices by either of the boundary state vectors \( z_1^L \) or \( z_n^R \) in the usual fashion, we can step from state vector to state vector. Thus we can determine all the state vectors of the system in terms of \( u_n^R \) only. However \( u_n^R \) remains undetermined. Thus

\[ z_1^R = [T_1] z_1^L, \quad (3.46) \]

\[ \vdots \quad (3.47) \]

\[ z_{n-1}^R = [T_{n-1}] \cdots [C_i][T_i] \cdots [T_2][C_1][T_1] z_1^L. \quad (3.48) \]
3.3. MODELLING A COMPLEX PIPE SYSTEM

Setting these three displacements of $u^R_n$ to unity or an arbitrary constant fixes the relative magnitudes of all state vectors. So we now have the variation/pattern of all six field variables of a state vector, along the length of the system. In other words we have the displacement and force modeshapes. Note that in carrying out actual products of Transfer matrices, the frequency $\omega$ has to be specified since $[T_i] = [T_i(\omega)]$. So setting $\omega$ to equal the natural frequency of the system, in the above calculations, gives us the normal modes of the system. Note further that in a real estimation of the modeshapes, we also have to be concerned about the number of state vectors we have, to track the modeshape along the length of the system. Analogous to sampling a signal in the time domain we are interested in sampling the modeshape in the space domain. The higher modes require correspondingly larger number of sample points/state vector locations, than the lower modes. Hence we might have to introduce state vectors at locations where they are not otherwise mandatory, i.e. within uniform sections, for purposes of tracking the modeshape along the length of the structure.

3.3.2 Forced Response

We have seen how free vibration problems involving non-uniform pipe systems can be handled using the transfer matrix technique in section 3.3.1. Now we venture to attack the forced problem.

Consider the pipe shown in figure 3.3 with an external, harmonic forcing applied
as shown in the boundary between element \( i \) and \( i + 1 \). Then

\[
\mathbf{z}_{i+1}^L = [C_i] \left[ \{ \mathcal{F} \} + [T_i] \cdots [T_2][C_1][T_1] \mathbf{z}_1^L \right].
\] (3.49)

Thus the state vector \( \mathbf{z}_{i+1}^L \) gets modified by the presence of a forcing \( \{ \mathcal{F} \} \) term. Note that the forcing \( \{ \mathcal{F} \} \) used in the above equation and to be used henceforth is a vector of applied displacement and forces\(^6\) as opposed to the forcing that was meant in section 3.1, which was a vector of applied forces only. Thus forcing as defined now can be either imposed displacements or forces. When we assemble the whole structure composed of \( n \) elements, we get

\[
\mathbf{z}_n^R = [T_n][C_{n-1}][T_{n-1}] \cdots [C_i] \left[ \{ \mathcal{F} \} + [T_i] \cdots [T_2][C_1][T_1] \mathbf{z}_1^L \right],
\] (3.50)

\[
\Rightarrow \quad \mathbf{z}_n^R = [T_n][C_{n-1}][T_{n-1}] \cdots [T_2][C_1][T_1] \mathbf{z}_1^L
\]

\[
+ \quad [T_n][C_{n-1}][T_{n-1}] \cdots [T_{i+1}][C_i] \{ \mathcal{F} \},
\] (3.51)

\[
\mathbf{z}_n^R = [\Pi] \mathbf{z}_1^L + [\Gamma] \{ \mathcal{F} \}.
\] (3.52)

Again we have six equations in twelve variables and imposing boundary conditions (specifying six of the displacements and forces) would determine the remaining six variables. Thus all components of the state vectors \( \mathbf{z}_n^R \) and \( \mathbf{z}_1^L \) would be fully known. These are boundary quantities. If our interest is in getting the response at some

\(^6\{ \mathcal{F} \} = \{ u_p \ u_i \ u_o \ f_p \ f_i \ f_o \}^T\)
Figure 3.3: The Transfer matrix relation for a non-uniform, finite pipe with an external force.
interior location in the structure it is a simple matter to track back to that location to determine the state vector at that point. If that location was (say, the left boundary of element \( 'k' \)) then we can get \( \mathbf{z}_k^L \) by moving in from either boundary. Specifically,

\[
\mathbf{z}_k^L = [T_k]^{-1}[C_k]^{-1}[T_{k+1}]^{-1} \cdots [T_{n-1}]^{-1}[C_{n-1}]^{-1}[T_n]^{-1} \mathbf{z}_n^R, \quad (3.53)
\]

or

\[
= [C_{k-1}][T_{k-1}] \cdots [C_i] \left[ \{ \mathcal{F} \} + [T_i] \cdots [T_2][C_1][T_1] \right] \mathbf{z}_i^L. \quad (3.54)
\]

In an actual calculation, the second equation might be preferable as it does not involve matrix inversions.

We now have a means of getting the response at any interior or boundary point in a structure in response to an imposed harmonic displacement or force. The only conditions being that the excitation and response points have to be on a boundary between two elements, i.e. there should be a state vector defined at that location. If the location of interest, for either excitation or response occurs within an element, we can split the element at the location of interest into two uniform elements and thus define a state vector there.

The Transfer Function \( H(\omega) \) between an excitation and a response, at specified locations, is equal to, the response at that location, due to a unit harmonic input
3.3. MODELLING A COMPLEX PIPE SYSTEM

at the excitation location. Or

\[ H(\omega) = \frac{Y(\omega)}{X(\omega)} , \]  

(3.55)

where

\[ Y(\omega) \quad : \quad \text{Fourier Transform of response} \]
\[ X(\omega) \quad : \quad \text{Fourier Transform of excitation} \]

Since in calculating the response, we get the state vector at the response location, we get six Transfer Functions. That is, we get six response quantities at the response location due to single, unit, harmonic input at the excitation location.

To illustrate:

If the Transfer Function between a force input to the pipe at the right boundary of element ‘i’ and displacement response in the outer fluid at the right boundary of element ‘k’ is desired:

Then \( \{ F \} \) is taken to be \( \{ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \}^T \) and is substituted in equation 3.52. Imposing the appropriate boundary conditions we can get \( z_k^R \).

Thus we have obtained six response quantities at the output location out of which the third element in \( z_k^R \) is the desired Transfer Function.
3.4 Modelling the Boundaries

We have till now glossed over the specifics of what conditions we impose at the boundary to realistically model an oil well. This is a tricky question since the boundary conditions which prevail in an actual well are not well understood. Further, of the two boundaries, the conditions existing downhole, at the bit, are the most obscure. In this section we shall discuss some models for the top and bottom boundaries.

Surface Boundary Conditions

In general, the inner fluid extends above the ground inside the kelly and/or the drill pipe (typically steel) and connects at the swivel to a flexible hose. The fluid path continues into the stand pipe (steel) which then communicates through a network of pipes to mud pumps and then to mud reservoirs. The length of surface piping is of the order of hundreds of ft. Between the stand pipe and the mud pumps it is common to have have pulsation dampeners\(^7\) fitted on the pipe. They serve to reduce the pulsations introduced in the fluid by mud pumps and pressure surges caused by closing/opening of valves, as these have the potential to damage surface equipment and tend to interfere with Mud Pulse Telemetry.

The pulsation dampener is a pressure vessel attached to the piping through a tee joint. Commercial dampeners have gas under pressure, occupying a part of

---

\(^7\)Also termed as Accumulators or Surge Absorbers.
the volume of the vessel. The fluid that enters the vessel then expends its energy in compressing the pressurized gas. Some also have an orifice plate at the throat of the dampener to dissipate additional energy through viscous losses. Since the pulsations downstream of these dampers are muted, they are mounted upstream of the pressure sensors of the MPT system and downstream of the mud pumps. The modelling of the pulsation dampeners is not addressed in this thesis and the mud pumps are modelled as fixed boundaries.

The outer fluid on the other hand has a less exciting conclusion. It is led off through pipes to the mud reservoirs. So, a mathematically tractable and realistic boundary condition for the outer fluid would be pressure release at the surface.

The drill pipe rises above the surface to the swivel which is connected to the pulley blocks which are suspended by a set of cables. The swivel and blocks act as masses while the cables function as a spring. A mass spring system would model the boundary constraint for the pipe.

Bottom Boundary Conditions

The BHA terminates at the bottom of the hole with the bit, a drilling tool. Interaction of the bit with the formation into which it is drilling is a strong source of drill string vibration. As an approximation we can consider the bit to be in contact with the formation at all times (i.e. bit bounce on the formation is not considered) and the formation itself to be modelled as an equivalent spring-damper system.
3.4. MODELLING THE BOUNDARIES

The inner fluid flows out of the bit, through small orifices in it. These jets serve in dislodging the cuttings and in flushing the cutting site. The orifices dissipate energy and their effect could be included, at very low frequencies, by terminating the inner fluid in a dashpot.

The outer fluid, in soft formation, has a kinematic boundary condition based on continuity of fluid flow from inside the pipe to the annulus, at the bottom. The formation is soft, relative to the BHA but is hard, relative to the fluid. In hard formation, its boundary can be approximated by an M-S-D, similar to the one at the end of the pipe.

In our numerical dramatizations of the acoustics of the rig, we shall model the boundaries as follows:

As a first step we will use simple pressure release surfaces (free) and fixed surfaces in each of the three media, to model the boundaries.

When drilling in soft formation, the resistance of the rock that the bit sees is small. In that situation the pipe is terminated at the bottom by almost a free boundary condition. The inner fluid, always being in contact with the pipe undergoes the same displacement as the pipe (we assume that the orifice areas are negligible compared to cross-sectional area of the BHA and hence the inner fluid sees a closed end). The outer fluid undergoes a displacement at the boundary, given by the motion of the bit, cross-sectional area of the BHA and the cross-sectional
area of the annulus. Specifically,

\[ u_i = u_p, \quad (3.56) \]
\[ u_o = -\frac{A_h u_p}{A_a}, \quad (3.57) \]

where

- \( A_b \) : cross-sectional area of the BHA
- \( A_a \) : cross-sectional area of the annulus

For the purpose of the examples to follow we will assume \( A_b \approx A_h \), which means that the outer fluid displacement at the bottom boundary is equal and opposite to that of the pipe.

The above model could be extended to the case when the formation being drilled into is not soft. In this case the drilling action of the bit can be modelled by imposing a displacement to the bottom of the BHA. Then

\[ u_p = x_t, \quad (3.58) \]
\[ u_i = u_p, \quad (3.59) \]
\[ u_o = -\frac{A_h u_p}{A_h}, \quad (3.60) \]

where

- \( x_t \) : amplitude of the imposed bit displacement
Another possible boundary model is Mass-Spring-Dashpot systems at attached to each of the three media, at either boundary and the parameters that govern them reflect the type of termination that exist at the boundary. The details are as in figure 3.4.

In our simulations we shall use the first two models - simple, fixed and pressure release boundaries and the soft formation model.

3.5 Modelling the Source

The source that is of interest in MPT is the device that encodes the measured data as pressure pulses in the inner mud. There are secondary sources present, like the excitation at the bit, mud pumps etc. which can be analysed separately. The total solution can be obtained by superposition.

If our interest is confined to low frequencies (as defined in subsections 2.1.1 and 2.1.2), we can develop lumped parameter models for most acoustic 'elements' like ducts, cavities etc. These acoustic models can be used in building up acoustic 'circuits' which can model the low frequency behaviour of aggregates of such elements, as in [1, 6].

Acoustic sources can be classified to be of two types– constant volume velocity\(^8\) generators and constant pressure generators. Further the circuits themselves can be based on the mobility analogy or the impedance analogy, i.e. based on whether

---

\(^8\)Volume velocity – amount of fluid displaced per second = cross-sectional area × linear velocity.
Figure 3.4: Mass-spring-dashpot boundary condition.
3.5. MODELLING THE SOURCE

Figure 3.5: Constant volume velocity and constant pressure sources.

the pressure is the flow variable or volume velocity is the flow variable. Since pressure can be usually measured without disturbing the acoustical set-up it is common practice to consider it to be analogous to voltage. Hence we favour the impedance analogy – pressure being the drop variable and volume velocity being the flow variable. Thus with the impedance analogy the two types of sources are as shown in the figure 3.5.

A constant volume velocity source maintains a constant volume velocity in the acoustical circuit in which it is present (similar to a current source in an electrical circuit). An example of such a source is a rigid piston reciprocating harmonically in a cylinder. Ideally, it displaces a constant volume of fluid irrespective of the ambient pressure. Another example is a valve, either capable of restricting the flow or introducing new fluid, in a pipeline in which fluid is being pumped at a uniform rate. The opening and closing of the valve can be modelled as startup or shutdown
3.5. MODELLING THE SOURCE

behaviour of a constant volume velocity source. These kinds of sources are high impedance sources.

A constant pressure source maintains a constant pressure drop across its terminals (similar to a voltage source in an electrical circuit). An approximation of such a source is a ideal fan or propeller as the pressure drop across this is, ideally, independent of the compressibility of the fluid in which it operates. These kinds of sources are low impedance sources.

One of the sources that is used in MPT systems is a valve, in the inner fluid, which is closed for a specified duration and then opened. The measurements are coded in terms of valve closings of two different durations. This procedure creates a pressure drop downstream and a pressure rise upstream of the valve. It is these pressure pulses that travel up the mud column and are received and decoded at the surface. To model this, we first get the Transfer function between pressure response in one of the fluid media at the surface and a volume velocity input to the inner fluid, near the bottom of the BHA. Taking the Fourier transform of that Transfer function yields the Impulse response of the system to a volume velocity input. This when convolved with typical input waveforms (rectangular waveforms of durations equal to the valve closing durations, with finite rise time) gives the resulting predicted transients at the surface. A positive impulse of volume velocity at a location would produce a pressure rise upstream and downstream of the source.

In the present formulation the Transfer function has as inputs, either axial force
or axial displacement. We need a Transfer function with a *velocity* input to model sources like valves. This is a simple extension once the corresponding Transfer function with a displacement input has been evaluated.

\[
H_{XP}(\omega) = \frac{P(\omega)}{X(\omega)},
\]

(3.61)

where

\[
H_{XP}(\omega) \quad : \quad \text{Transfer function between a displacement input and a pressure output}
\]

\[
P(\omega) \quad : \quad \text{Fourier transform of the pressure, in the specified medium, at the response location}
\]

\[
X(\omega) \quad : \quad \text{Fourier transform of the displacement, in the specified medium, at the input location}
\]

Now extending this further

\[
H_{VP}(\omega) = \frac{P(\omega)}{V(\omega)},
\]

\[
= \frac{P(\omega)}{i\omega X(\omega)},
\]

\[
= \frac{H_{XP}(\omega)}{i\omega},
\]

(3.62)

and

\[
H_{UP}(\omega) = \frac{P(\omega)}{U(\omega)},
\]

\[
= \frac{P(\omega)}{AV(\omega)},
\]

\[
= \frac{H_{XP}(\omega)}{i\omega A},
\]

(3.63)
where

\[ H_{VP}(\omega) \] : Transfer function between a velocity input and a pressure output

\[ V(\omega) \] : Fourier transform of the velocity, in the specified media at the response location

\[ H_{UP}(\omega) \] : Transfer function between a volume velocity input and a pressure output

\[ U(\omega) \] : Fourier transform of the volume velocity, in the specified media at the response location

\[ A \] : Cross-sectional area

### 3.6 Modelling the Damping

Losses in a medium may be divided into three basic types, viscous losses, heat conduction losses and molecular exchanges of energy. The viscous losses result from relative motion occurring between the various portions of the medium during the compressions and rarefactions that accompany a transmitted wave. Since the basic wave equations have been derived assuming that the pressure changes are adiabatic, the medium has a change in temperature accompanying pressure changes. Thus there is a tendency for heat to be conducted from regions of condensation to regions of rarefaction. In this process of heat transfer there is a tendency towards pressure equalization. The dissipation of acoustic energy that is associated with changes in
the molecular structure of the medium results from the finite time that is required for these changes to take place.

The propagation of plane waves in an absorbing medium can be modelled by making the wavenumber complex. This results in an exponential rate of decay of the amplitude of the wave with distance which depends on the complex part of the wavenumber $iak$.

$$k = k(1 - ia)$$ \hspace{1cm} (3.64)

One source of attenuation in fluids contained within pipes is associated with the viscous resistance offered to fluid motion at the walls of the pipe. It can be shown that the decay rate due to this mechanism is given by [4]:

$$\alpha = \frac{1}{rc_n} \sqrt{\frac{\eta \omega}{2\rho}}$$ \hspace{1cm} (3.65)

$$= ak$$ \hspace{1cm} (3.66)

where

$\eta$ : coefficient of shear viscosity
$r$ : radius of pipe
$c_n$ : axial propagation speed
$\rho$ : fluid density

The above attenuation model for typical values of mud viscosity ($2.1 \times 10^{-4} \text{ lb sec/ft}^2$),
mud density \((2.027 \text{ slugs/ft}^3)\), mud propagation speeds \((4500 \text{ ft/sec})\) and drill pipe radius \((2.5 \text{ inch})\) and \(\alpha = 1 \text{ Hz}\) gives an \(\alpha\) of 0.0866. This means that the amplitude attenuates by a factor of 2 in travelling a distance of 36016 \(\text{ ft}\). This is a rather low value of attenuation.

In a realistic situation there are other mechanisms apart from viscous losses at pipe walls that tend to raise the absorption levels in the fluid. Such mechanisms include turbulence, the presence of inhomogeneities such as suspended particles in the fluid, the presence of suspended bubbles and losses due to flow past a porous formation. The presence of inhomogeneities causes scattering, i.e., removal of a small amount of energy from a directed beam by each particle and subsequent reradiation in all directions. The drilling fluid contains inhomogeneities in the form of cuttings from the drilling site and sometimes, gas bubbles which leak in from the formation.

Incorporating all the above forms of attenuation is likely to change the frequency dependence of \(\alpha\). As a simple model which gives reasonable attenuation we use the following definition of the wavenumber.

\[ k = k(1 - ia) \]  \hspace{1cm} (3.67)

where

\[ a : \text{a constant depending on all sources of attenuation} \]

The attenuation constant \(a\) is usually frequency dependent. However care must be
exercised in specifying a frequency dependence which does not lead to an Impulse response which violates causality such as when one assumes 'a' to be a constant over all frequencies. It may be assumed to be a constant over bands of frequency. A constant 'a' is analogous to having a constant damping ratio or a damping constant that increases linearly with frequency.

Since we have a wavenumber for each of the modes we can stipulate the attenuation characteristic of each mode, separately.

\[ k_p = k_p(1 - ia_p) , \quad (3.68) \]
\[ k_1 = k_1(1 - ia_1) , \quad (3.69) \]
\[ k_2 = k_2(1 - ia_2) , \quad (3.70) \]

In all our examples we assume constant 'a' over narrow bands of frequency (\( \leq 20 \) Hz).

### 3.7 Examples

We shall calculate the Transfer function for pipe systems of two kinds— one being uniform and the other being composite. The calculation for a uniform, single section, simple boundary, pipe system will familiarize us with the information latent in the Transfer function. This will be analysed for the case of a hard and a soft formation. The second example would consider a uniform pipe system made up of two sections.
3.7. EXAMPLES

The top section is in soft formation and the bottom is in hard formation.

3.7.1 Example 3.1 - Hard Formation

The structure is as shown in figure 3.6 and the data is given in the following table.

<table>
<thead>
<tr>
<th>Input Data for Example 3.1 - Hard formation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner dia. of pipe</td>
</tr>
<tr>
<td>Outer dia. of pipe</td>
</tr>
<tr>
<td>Borehole dia.</td>
</tr>
<tr>
<td>Length</td>
</tr>
<tr>
<td>Density of pipe</td>
</tr>
<tr>
<td>Youngs Modulus of pipe</td>
</tr>
<tr>
<td>Poissons ratio of pipe</td>
</tr>
<tr>
<td>Density of mud</td>
</tr>
<tr>
<td>Bulk Modulus of mud</td>
</tr>
<tr>
<td>Wall stiffness</td>
</tr>
<tr>
<td>Attenuation constants a₁</td>
</tr>
<tr>
<td>a₂, a₃</td>
</tr>
</tbody>
</table>

**Excitation** : harmonic volume velocity (1 inch³/sec) applied to the inner fluid, 50 ft from the bottom boundary
3.7. EXAMPLES

Response : pressure in the pipe, inner and outer fluid, 50 ft from the top boundary

Boundary condition : fixed at both boundaries

We now pick a frequency range \((f_{\text{min}} - f_{\text{max}} = 0 - 20.37 \text{Hz})\) and a step size \((\Delta f = 0.0159 \text{Hz})\) and evaluate the state vector, at the response location, for all the above specified frequencies as we have shown in section 3.3.2 with displacement as our input. From this, Transfer functions for a volume velocity input can be calculated as shown in the previous section. The Transfer functions between pressure response in the pipe, inner fluid and outer fluid, 50 ft from the top boundary and a volume velocity input to the inner fluid, 50 ft from the bottom boundary, are as shown in figures 3.7, 3.8 and 3.9 respectively.

The speeds and modeshapes are:
Figure 3.6: Schematic of example for Transfer function calculation - Example 3.1
3.7. EXAMPLES

<table>
<thead>
<tr>
<th>Hard formation</th>
<th>Mode 1</th>
<th>Mode 2</th>
<th>Mode 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_p$</td>
<td>0.2156</td>
<td>-0.0871</td>
<td>1.0000</td>
</tr>
<tr>
<td>$p_i$</td>
<td>-0.9607</td>
<td>1.0000</td>
<td>0.0058</td>
</tr>
<tr>
<td>$p_o$</td>
<td>1.0000</td>
<td>0.1325</td>
<td>-0.0009</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Speeds</th>
<th>ft/sec</th>
<th>m/sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4451$</td>
<td>$4820$</td>
<td>$16943$</td>
</tr>
<tr>
<td>$1356$</td>
<td>$1469$</td>
<td>$5164$</td>
</tr>
</tbody>
</table>

Discussion

The stress/pressure in any medium is the sum of the contributions from all three pressure modes, as we have seen earlier in equations 3.12 and 3.13. Specifically they can be written as

\[
\begin{aligned}
\begin{bmatrix}
\sigma_p(x) \\
\end{bmatrix} e^{i\omega t} &= 2D_1 \begin{bmatrix}
\sigma_p \\
p_i \\
p_o \\
\end{bmatrix} + 2D_3 \sin(k_p x) e^{i(\omega t - \alpha_1)} + 2D_5 \sin(k_1 x) e^{i(\omega t - \alpha_2)} + 2D_6 \sin(k_3 x) e^{i(\omega t - \alpha_3)}, \\
\begin{bmatrix}
\sigma_p \\
p_i \\
p_o \\
\end{bmatrix} &= \begin{bmatrix}
\sigma_p \\
p_i \\
p_o \\
\end{bmatrix},
\end{aligned}
\]

(3.71)

where
Figure 3.7: Transfer function between a volume velocity input (1 in³/sec), to the inner fluid, 50 ft from the bottom boundary and stress response in pipe, 50 ft, from the top boundary. Example 3.1 Hard formation.
Figure 3.8: Transfer function between a volume velocity input (1 inch$^3$/sec), to the inner fluid. 50 ft from the bottom boundary and pressure response in inner fluid, 50 ft from the top boundary. Example 3.1 Hard formation.
Figure 3.9: Transfer function between a volume velocity input (1 inch$^3$/sec), to the inner fluid, 50 ft from the bottom boundary and pressure response in outer fluid, 50 ft from the top boundary - Example 3.1 Hard formation.
\[ \sigma_p(x) \] : cumulative stress in pipe due to the three modal contributions

\( \{ \sigma_p, p_i, p_o \}^T_i \) : \( i^{th} \) pressure modeshape

\( \alpha_i \) : phase angle of the \( i^{th} \) mode

In general, the stress in the pipe at the response location, in the frequency domain, would reveal peaks at the resonant frequencies of each of the three modes. But since in most cases, the stress in the pipe created due to the two fluid modes is small (\( 2D_1\sigma_{p,1} \) and \( 2D_3\sigma_{p,2} \) are small in magnitude compared to \( 2D_5\sigma_{p,3} \)), the peaks are mainly at those frequencies at which the pipe mode is resonating.

Similarly, each fluid medium should reveal peaks at the resonant frequencies of each of the three modes. However the pressure in the two fluid media due to the pipe mode is, in general, small (\( 2D_5p_{i,3} \) is small) as this mode is dominated by stress in the pipe and travels at a speed almost equalling the sound propagation speed in the material of the pipe. So, in the fluids we are likely to see peaks corresponding to resonances of the two fluid modes only. Further, the coupling between the inner and outer fluids decides whether each fluid medium would reveal peaks at one or both fluid modes (coupling between the two fluid media is described in section 2.4.1). A fluid medium will reveal peaks at resonances of both fluid modes if there is strong coupling and at only one of the fluid modes when there is weak coupling. Thus when there is weak coupling, response in each of the fluid media will be dominated by one of the fluid modes. The hard formation example under consideration is a
3.7. EXAMPLES

case of strong coupling and resonances due to both fluid modes are evident in the two fluid media.

Damping also plays a significant role. If the damping of a particular mode is small and the system is not too long, the individual resonances appear as peaks in the Transfer function. In effect standing wave behaviour dominates the response. However as the damping or the length of the system increases, it is possible that a wave created at one boundary, in the form of one mode, may damp out before it reaches the other boundary. In this case, travelling wave behaviour dominates the response.

The two fluid media Transfer functions display resonant peaks due to standing waves at low frequencies ($\leq 5\, Hz$ for inner fluid and $\leq 2.5\, Hz$ for the outer fluid) and smooth travelling wave behaviour at higher frequencies. Damping controls the frequency range of travelling versus standing wave behaviour. The magnitude of the Transfer function for stress in the pipe displays resonant peaks over the full frequency range of interest since the structural damping in the pipe was assumed to be low.

The phase portion of the Transfer function provide additional information. Beyond a few hertz the damping in the system smoothens out the phase transitions due to poles (resonances) and zeros (anti resonances) resulting in a smooth phase curve. The concept of group delay is now useful [2]. It is a measure of how fast the energy in each frequency component of a signal travels in the structure. It does in
general depend upon the frequency and is defined as the negative of the slope of the phase versus the frequency curve.

\[ \tau_\theta(\omega) = -\frac{d\theta}{d\omega}, \]  

(3.72)

where

\[ \tau_\theta \quad : \text{Group delay} \]
\[ \theta \quad : \text{phase angle} \]

If the group delay is a constant over a range of frequency, it then indicates that energy at those frequency components travels at the same speed – the medium is non-dispersive. The equations governing this example of a pipe in fluid indicate that over significant ranges of frequency the media are non-dispersive, but is sensitive to damping. Further, the separation between the input and output locations along with the group delay provides a measure of group speed, the propagation speed of energy.

\[ c_g = \frac{\Delta l}{\tau_\theta}, \]  

(3.73)

where

\[ c_g \quad : \text{Group speed} \]
\[ \Delta l \quad : \text{source-receiver separation} \]
From the phase curve of figure 3.7, we can estimate the group delay in the pipe medium, as a mean slope over a frequency range of 0 to 20 Hz,

\[ \tau_g \approx 0.29 \text{ sec}, \]

\[ \Delta l = 4900 \text{ ft}, \]

\[ \Rightarrow c_g \approx 16377 \text{ ft/sec}. \]

The speed of the pipe mode is 16943 ft/sec. Since the group speed is nearly equal to the pipe mode propagation speed, we can say that the most of the observed energy in the pipe, at the output location, traveled from the input, in the pipe mode.

The phase curves of the two fluid media are characterized by one group delay below 15 Hz and a different one above 15 Hz. For the inner fluid,

\[ \tau_{g,1} \approx 1.02, \]

\[ \Rightarrow c_{g,1} \approx 4783.4 \text{ ft/sec}. \]

\[ \tau_{g,2} \approx 0.29, \]

\[ \Rightarrow c_{g,2} \approx 16650 \text{ ft/sec}, \]

where

sub-script 1 : quantities below 15 Hz

sub-script 2 : quantities above 15 Hz
\[ \Delta l = 4900 \text{ ft} \]

This indicates that below 15 Hz energy travels at a speed comparable to the fluid mode propagation speed for the second mode and above 15 Hz it travels at a speed almost equalling the pipe mode propagation speed. The reason for this is apparent when we look at the magnitude of the Transfer function for the two fluid media, figures 3.8 and 3.9. Both magnitudes attenuate to very low levels, because of damping, by 15 Hz. In this case the input was to the inner fluid and the second mode had the greatest participation. The second mode contribution has attenuated to the level of the pipe mode contribution at 12 – 15Hz at the response location. Thus, we see that, beyond 15 Hz the two fluid modes are attenuated almost completely and the burden of energy transmission as observed in the two fluid media, falls on the pipe mode. Hence the group speed changes, beyond 15 Hz, to equal the pipe mode propagation speed.

Furthermore, the bumps in the Transfer functions at about 15Hz, we believe, are interference patterns developed by superposition of the second mode and the pipe mode which at that frequency are of approximately equal amplitude, but different wavelength. The bumps nearer to 20 Hz are the pipe mode resonances.

For the outer fluid,

\[ \tau_{g,1} \approx 1.02 , \]

\[ \Rightarrow c_{g,1} \approx 4780.5 \text{ ft/sec} . \]
3.7. EXAMPLES

\[ \tau_{g,2} \approx 0.30 , \]

\[ \Rightarrow c_{g,2} \approx 16317 \text{ ft/sec} . \]

These numbers bear out the reasoning above.

To summarize:

<table>
<thead>
<tr>
<th>Group delay calc. - Hard formation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau_g )</td>
</tr>
<tr>
<td>sec</td>
</tr>
<tr>
<td>Pipe</td>
</tr>
<tr>
<td>Inner fluid</td>
</tr>
<tr>
<td>0.29</td>
</tr>
<tr>
<td>Outer fluid</td>
</tr>
<tr>
<td>0.30</td>
</tr>
</tbody>
</table>

The group delay gives us the speed of energy propagation as a function of frequency i.e. it gives us the dispersion characteristics of the structure. This is indicative of the distortion the input waveform is likely to undergo as it travels through the structure.

3.7.2 Example 3.1 - Soft Formation

We now consider the above problem, with a soft formation. The schematic of the structure is as shown in figure 3.6.
### Input Data for Example 3.1 – Soft formation

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner dia. of pipe</td>
<td>4.19</td>
<td>10.64</td>
</tr>
<tr>
<td>in.</td>
<td></td>
<td>cm</td>
</tr>
<tr>
<td>Outer dia. of pipe</td>
<td>5.00</td>
<td>12.7</td>
</tr>
<tr>
<td>in.</td>
<td></td>
<td>cm</td>
</tr>
<tr>
<td>Borehole dia.</td>
<td>12.347</td>
<td>31.36</td>
</tr>
<tr>
<td>in.</td>
<td></td>
<td>cm</td>
</tr>
<tr>
<td>Length</td>
<td>5000</td>
<td>1524</td>
</tr>
<tr>
<td>ft.</td>
<td></td>
<td>m</td>
</tr>
<tr>
<td>Density of pipe</td>
<td>15.18</td>
<td>7823.46</td>
</tr>
<tr>
<td>slugs/ft³</td>
<td></td>
<td>kg/m³</td>
</tr>
<tr>
<td>Youngs Modulus of pipe</td>
<td>4.32×10⁹</td>
<td>2.0684×10¹¹</td>
</tr>
<tr>
<td>lb/ft²</td>
<td></td>
<td>Pa</td>
</tr>
<tr>
<td>Poissons ratio of pipe</td>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>Density of mud</td>
<td>2.027</td>
<td>1044.6752</td>
</tr>
<tr>
<td>slugs/ft³</td>
<td></td>
<td>kg/m³</td>
</tr>
<tr>
<td>Bulk Modulus of mud</td>
<td>5.30×10⁷</td>
<td>2.5376×10⁹</td>
</tr>
<tr>
<td>lb/ft²</td>
<td></td>
<td>Pa</td>
</tr>
<tr>
<td>Wall stiffness</td>
<td>0.162×10⁸</td>
<td>(lb/ft²)/ft</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.5448×10⁹</td>
</tr>
<tr>
<td>Attenuation constants a₁,a₂,a₃</td>
<td>0.01, 0.08, 0.08</td>
<td></td>
</tr>
</tbody>
</table>

The data is the same as before except for the wall stiffness of the formation which is down by a factor of about hundred.

**Excitation** : harmonic volume velocity (1 inch³/sec) applied to the inner fluid, 50 ft from the bottom boundary

**Response** : pressure in the pipe, inner and outer fluid, 50 ft from the top boundary

**Boundary condition** : fixed at both boundaries
3.7. EXAMPLES

The speeds and modeshapes are:

<table>
<thead>
<tr>
<th>Soft formation</th>
<th>Mode 1</th>
<th>Mode 2</th>
<th>Mode 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_p$</td>
<td>0.0099</td>
<td>-0.1050</td>
<td>1.0000</td>
</tr>
<tr>
<td>Mode shapes</td>
<td>$p_i$</td>
<td>-0.0109</td>
<td>1.0000</td>
</tr>
<tr>
<td></td>
<td>$p_o$</td>
<td>1.0000</td>
<td>0.0015</td>
</tr>
<tr>
<td>Speeds</td>
<td>$ft/sec$</td>
<td>1269</td>
<td>4775</td>
</tr>
<tr>
<td></td>
<td>$m/sec$</td>
<td>387</td>
<td>1455</td>
</tr>
</tbody>
</table>

Discussion

As can be seen above the two fluid modes are uncoupled and the fluid propagation speeds are very different. So a pressure disturbance introduced into one of the fluid media propagates as a single dominant mode without communicating to the other fluid medium.

The Transfer functions between pressure response in the pipe, inner fluid and outer fluid, at one boundary and a volume velocity input to the inner fluid, at the other boundary, are as shown in figures 3.10, 3.11 and 3.12 respectively.

The two fluid medium Transfer functions have the same general features as the those of the hard formation example. The transition to the bumps that we observed before in the magnitude plots, now take place at lower frequencies. The pipe medium
Figure 3.10: Transfer function between a volume velocity input (1 inch³/sec), to the inner fluid, 50ft from the bottom boundary and stress response in pipe, 50ft from the top boundary - Example 3.1 Soft formation.
Figure 3.11: Transfer function between a volume velocity input (1 inch$^3$/sec), to the inner fluid, 50ft from the bottom boundary and pressure response in inner fluid, 50ft from the top boundary - Example 3.1 Soft formation.
Figure 3.12: Transfer function between a volume velocity input (1 inch$^3$/sec), to the inner fluid, 50ft from the bottom boundary and pressure response in outer fluid, 50ft from the top boundary - Example 3.1 Soft formation.
Transfer function remains largely unchanged implying that energy transfer in the pipe is almost independent of the properties of the surrounding formation.

Further, the Group delay calculations as in the previous case are as follows:

<table>
<thead>
<tr>
<th></th>
<th>( \tau_g )</th>
<th>( \Delta l )</th>
<th>( \Rightarrow c_g )</th>
<th>Frequency range Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>sec</td>
<td>ft</td>
<td>m</td>
<td>ft/sec</td>
</tr>
<tr>
<td>Pipe</td>
<td>0.28</td>
<td>4900</td>
<td>1493</td>
<td>16904</td>
</tr>
<tr>
<td>Inner fluid</td>
<td>1.03</td>
<td>4900</td>
<td>1493</td>
<td>4725</td>
</tr>
<tr>
<td></td>
<td>0.29</td>
<td>4900</td>
<td>1493</td>
<td>16450</td>
</tr>
<tr>
<td>Outer fluid</td>
<td>1.03</td>
<td>4900</td>
<td>1493</td>
<td>4725</td>
</tr>
<tr>
<td></td>
<td>0.29</td>
<td>4900</td>
<td>1493</td>
<td>16415</td>
</tr>
</tbody>
</table>

In this case (soft formation) we excite the inner fluid and calculate the response in the three media. This excites all the three modes since all have an inner mud pressure (however small). We see that energy propagation in both of the fluid media is accomplished primarily by the second fluid mode at low frequencies and by the pipe mode at higher frequencies. This is similar to the case of the hard formation. The difference is in the frequency at which the transition from fluid to pipe mode dominance takes place in the outer fluid. It occurs at 9 Hz in the outer fluid and at 12.5 Hz in the inner fluid. Since the fluid media are weakly coupled, a forcing applied to the inner fluid excites Mode 2 predominantly. Thus the response that is measured in the outer fluid is mainly due to Mode 2. Since Mode 2 has small
outer fluid pressure, the pressure in the outer fluid is small. Hence it attenuates to a smaller magnitude than the inner fluid and thus the transition to the pipe mode takes places earlier. Thus if the fluid modes are strongly coupled (hard formation) the transitions in the two fluid media occur at about the same frequency. In the case of weak coupling (soft formation) the transitions in the two fluid media take place at different frequencies.

### 3.7.3 Example 3.2 – Composite formation

In Example 3.1, we considered two cases of a 5000 ft borehole - one with a hard formation and a soft formation. We now consider a borehole with the first 6021 ft being of a soft formation and the bottom 4050 ft being a hard formation.

The schematic of the borehole is shown in figure ???. The data is as follows:

<table>
<thead>
<tr>
<th>Input Data for Example 3.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner dia. of pipe</td>
</tr>
<tr>
<td>Outer dia. of pipe</td>
</tr>
<tr>
<td>Density of pipe</td>
</tr>
<tr>
<td>Youngs Modulus of pipe</td>
</tr>
<tr>
<td>Poissons ratio of pipe</td>
</tr>
<tr>
<td>Density of mud</td>
</tr>
<tr>
<td>Bulk Modulus of mud</td>
</tr>
</tbody>
</table>
Figure 3.13: Schematic of example for Transfer function calculation - MSD boundaries.
<table>
<thead>
<tr>
<th></th>
<th>Length (ft)</th>
<th>Hole dia. (in)</th>
<th>Hole dia. (cm)</th>
<th>Wall stiffness (lb/ft³) × 10⁸</th>
<th>Wall stiffness (Pa/m) × 10⁸</th>
<th>Attenuation const. (a₁, a₂, a₃)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sec. 1</td>
<td>6021</td>
<td>12.347</td>
<td>31.36</td>
<td>0.162</td>
<td>25.44</td>
<td>0.01 0.08 0.08</td>
</tr>
<tr>
<td>Sec. 2</td>
<td>4050</td>
<td>1234.4</td>
<td>12.347</td>
<td>9.089</td>
<td>1427.8</td>
<td>0.01 0.05 0.05</td>
</tr>
</tbody>
</table>

**Excitation**: harmonic volume velocity (1 inch³/sec) applied to the inner fluid, 50 ft (15.24 m) from the bottom boundary.

**Response**: pressure in the pipe, inner and outer fluid, 50 ft (15.24 m) from the top boundary.

**Boundary condition**: fixed at both boundaries.

The Transfer functions between a volume velocity input (1 inch³/sec) to the inner fluid, 50 ft from the bottom boundary and pressure response in the pipe, inner fluid and the outer fluid, 50 ft from the top boundary are as shown in figures 3.14, 3.15 and 3.16 respectively.

The inner fluid Transfer function displays resonant peaks at low frequencies and travelling wave behaviour at intermediate frequencies. Beyond that the appearance of resonant peaks at the spacing of the pipe medium Transfer function can be seen, clearly indicating pipe mode dominance in the inner fluid at these frequencies. By
Figure 3.14: Transfer function between a volume velocity input (1 inch$^3$/sec) to the inner fluid, 50 ft from the top boundary and pressure response in the pipe, 50 ft from the bottom boundary - Example 3.2 Composite formation.
Figure 3.15: Transfer function between a volume velocity input (1 inch\(^3\)/sec) to the inner fluid, 50 ft from the top boundary and pressure response in the inner fluid, 50 ft from the bottom boundary - Example 3.2 Composite formation.
Figure 3.16: Transfer function between a volume velocity input (1 inch³/sec) to the inner fluid, 50 ft from the top boundary and pressure response in the outer fluid, 50 ft from the bottom boundary. Example 3.2 Composite formation.
now we recognize that this is because the two fluid modes have attenuated to levels lower than the inner fluid pressure in the pipe mode. Hence we see the low magnitude pressures in the inner fluid created by the pipe mode at high frequencies.

The outer fluid on the other hand displays considerable number of resonances and zeros in the low frequency region before it gets into the pipe mode behaviour. This is unlike the outer fluid Transfer function of either of the two previous cases.

The pipe medium displays resonant peaks with no indication of the fact that we have a transition from a soft to a hard formation.

The outer fluid phase displays three group delays in different frequency ranges.

\[ \tau_{g1} = 5.67 , \]
\[ \tau_{g2} = 2.31 , \]
\[ \tau_{g2} = 0.59 . \]

(3.74)

It is possible that \( \tau_{g1} \) corresponds to wave travelling as Mode 2 (4820 ft/sec) in the hard formation (4000 ft) and then as Mode 1 (1269 ft/sec) in the soft formation (5971 ft). This is a travel time of 5.53 sec. \( \tau_{g2} \) corresponds to a wave travelling as Mode 2 (4451 ft/sec) in the hard formation and as Mode 2 (4775 ft/sec) in the soft formation. This is travel time of 2.15 sec. \( \tau_{g3} \) corresponds to a wave travelling as Mode 3 (16943 ft/sec) in the soft formation and as Mode 3 (16932 ft/sec) in the
hard formation. The travel time is 0.59 sec.

The fact that different modal combinations dominate in different frequency bands is a consequence of the initial relative amplitudes of each mode and different attenuation rates, resulting in sometimes very complicated interpretations.

The inner fluid phase curve displays two group delays corresponding to propagation in the second mode, in both hard and soft formations, at low frequencies and propagation in the pipe mode, in both hard and soft formations, at high frequencies.

The pipe medium phase has a single group delay corresponding propagation in the pipe mode, in both hard and soft formations at all frequencies.
Chapter 4

SURFACE SIGNAL PREDICTION

... édo édo ānanda thāgam, unnāl thāné undānathu

kāl pona pāthaigal nān pona bothu, kai serthu needhané mei sertha mādhū ...¹

In Chapter 3 we derived the Transfer Function of a finite pipe system. In this Chapter we resort to Fourier transform techniques to move from the Transfer Function to the Impulse response. Once the Impulse response is known transient responses to specific inputs can be calculated. In particular, if we start off with the Transfer function between the pressure response in the inner mud, at the surface, and a pressure input to the inner mud, at the bottom of the borehole, we can get the corresponding Impulse response. If we convolve it with common MPT inputs like rectangular waves we can predict the pressure transients that might be observed at the surface. This would help in understanding/interpreting observed surface signals.

¹from Pudhu Pudhu Arthangal
4.1 Impulse Response of a Finite Pipe System

The Transfer Function $H(\omega)$ and the Impulse response $h(t)$ are related as follows:

\[
H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt , \\
h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega t} d\omega ,
\]

where

$H(\omega)$ : Transfer function

$h(t)$ : Impulse response

We can get the transient response to specific inputs, from the Impulse response as

\[
x(t) = \int_{-\infty}^{\infty} G(\tau) h(t - \tau) d\tau ,
\]

where

$x(t)$ : transient response

$G(t)$ : input waveform

4.2 Example 4.1

The first pipe system that is analysed is as shown in figure 4.1. This example is taken from a field experiment conducted by Philippe Bres of Elf Aquitaine as de-
4.2. **EXAMPLE 4.1**

scribed in H. Y. Lee's thesis [5]. The data is as follows:

<table>
<thead>
<tr>
<th>Input Data for Example 4.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner dia. of pipe</td>
</tr>
<tr>
<td>Outer dia. of pipe</td>
</tr>
<tr>
<td>Density of pipe</td>
</tr>
<tr>
<td>Youngs Modulus of pipe</td>
</tr>
<tr>
<td>Poissons ratio of pipe</td>
</tr>
<tr>
<td>Density of mud</td>
</tr>
<tr>
<td>Bulk Modulus of mud</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Length</th>
<th>Hole dia.</th>
<th>Wall stiffness</th>
<th>Attenuation const.</th>
</tr>
</thead>
<tbody>
<tr>
<td>ft</td>
<td>m</td>
<td>in.</td>
<td>cm</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sec. 1</td>
<td>6021</td>
<td>1835</td>
<td>12.347</td>
</tr>
<tr>
<td>Sec. 2</td>
<td>50</td>
<td>15.24</td>
<td>17.5</td>
</tr>
<tr>
<td>Sec. 3</td>
<td>4000</td>
<td>1219.02</td>
<td>12.347</td>
</tr>
</tbody>
</table>

**Excitation**: harmonic volume velocity (1 inch$^3$/sec) applied to the inner fluid, at the top boundary

**Response**: pressure in the outer fluid, 50 ft away from the excitation
Boundary condition: applied excitation at the top boundary

: pipe is free at the bottom boundary,

\( u_i = u_p \) at the bottom boundary and

\( u_o = -u_p \) at the bottom boundary

The Transfer function between a volume velocity input to the inner fluid at the input location and the stress/pressure response in the pipe, inner fluid and outer fluid at the response location are as in figures 4.2, 4.3 and 4.4. The Impulse response is calculated from the Transfer function and it is as shown in the figure 4.5.

The Impulse response, figure 4.5, displays the direct arrival of the input signal and subsequent echoes, a result of reflection at boundaries and discontinuities. The significant arrivals correspond to strong reflections occurring early in the life of the introduced pulse. Weak reflections that occur early do not show up at the response location because of the small magnitudes of the reflected waves and reflections occurring late do not show up because of damping. The Impulse response thus indicates locations of strong reflections.

From figure 4.5 we can see a first arrival almost at the time origin. The second arrival is at 2.32 sec and the third arrival is at 2.65 sec. The fourth and fifth arrivals are at 3.88 and 4.27 sec respectively. The next strong arrival can be seen at 5.28 sec. Based on the geometry of our structure and upon the propagation speeds of a
Figure 4.1: Schematic of example for Impulse response calculation - Example 4.1.
Figure 4.2: Transfer function between a volume velocity input (1 inch$^3$/sec) to the inner fluid, at the top boundary and pressure response in the pipe, at the response location, 50 ft away. Example 4.1.
Figure 4.3: Transfer function between a volume velocity input \(1 \text{ inch}^3/\text{sec}\), to the inner fluid, at the top boundary and pressure response in the inner fluid, at the response location, 50 ft away. Example 4.1.
Figure 4.4: Transfer function between a volume velocity input (1 inch$^3$/sec), to the inner fluid, at the top boundary and pressure response in the outer fluid, at the response location, 50 ft away - Example 4.1.
Figure 4.5: The transient response of pressure \((\text{lb/}ft^2)\) in the outer fluid, at the response location, due to an impulse of volume velocity \((\text{in}^3/\text{sec})\) applied to the inner fluid at the boundary, 50 ft away - Example 4.1.
disturbance through our structure we can arrive at the history of each arrival.

We introduce a nomenclature for the various interfaces as follows. The boundary between Section 1 and Section 2 is Interface I (hard formation-washout) and the boundary between Section 2 and Section 3 is Interface II (washout-soft formation). The first arrival is the direct arrival, from the input location to the response location (distance travelled, $\Delta l = 50 \text{ ft}$). The next arrival is a reflection, of the introduced pulse, from the Interface I (hard formation-washout) - $\Delta l = 11992 \text{ ft}$. The following arrival is a reflection from the far end of Section 3\(^2\), the bottom boundary - $\Delta l = 20092 \text{ ft}$. Subsequent arrivals would be additional reflections of the above arrivals at the various interfaces and boundaries as time goes by and can be traced in a similar manner.

Strong reflections occur at interfaces where there is a large change in acoustical impedance. The indicators of acoustical impedance, in our structure, are the three modal propagation speeds and the stress/pressure modeshapes. When two sections differ in acoustical impedance by a large amount, the three propagation speeds and modeshapes are different in the two sections. Section 1 is a region which is strongly coupled\(^3\) and Section 2 is weakly coupled. The two fluid modes propagation speeds are comparable in Section 1 and they are vastly different in Section 2. This is because there is a reduction in the wall stiffness of the formation by a factor of

\(^2\)The modeshapes and propagation speeds of Section 2 and Section 3 are almost the same. This indicates that there is almost complete transmission through interface II (washout-soft formation). There is only a weak reflection at this interface which can be neglected.

\(^3\)Example 2.1 and 2.2 in Chapter 2 detail strong and weak coupling.
hundred in going from Section 1 to Section 2. It was shown in Example 2.2 in Chapter 2 that a reduction in wall stiffness by a factor of ten produced strong reflections. Thus Interface I (hard formation-washout) can be identified as a strong reflector. Whereas Interface II (washout-soft formation) is a very weak reflector with almost complete transmission since the propagation speeds and modeshapes in Section 2 and Section 3 are almost identical. So we shall neglect any reflections off this interface. That part of the initial pulse that has transmitted through Interface I (hard formation-washout) and reflected off the bottom boundary suffers a strong reflection when it arrives at Interface I again. Thus Interface I (hard formation-washout) traps a certain amount of the input signal in the region between it and the bottom boundary.

The introduced impulse gets decomposed into three transients corresponding to the three modes of the structure. The pipe mode has negligible fluid pressure and hence is not usually observable in the impulse response of the fluid media. The impulse to the inner fluid would excite comparable amounts of both fluid modes since Section 1 is strongly coupled. Further, at time $t = 0$, when the impulse is applied to the inner fluid, the outer fluid is in a quiescent state. Hence the two fluid modes have opposite signs so as to equal the applied disturbance in the inner fluid and cancel each other in the outer fluid. Thus, the introduced impulse is replaced by two transients, corresponding to the two fluid modes, travelling at their respective propagation speeds.
Thus, in figure 4.5, we see two arrivals, of opposite sign, at 2.32 and 2.65 sec because of reflection of the two fluid modes of Section 1 (hard formation), at Interface I (hard formation-washout).

\[ \Delta l = 11992 \text{ft} \]

propagation speed – Mode 1 = 4451 ft/sec

propagation speed – Mode 2 = 4820 ft/sec

⇒ arrival time – Mode 1 = 11992/4451

= 2.693 sec

⇒ arrival time – Mode 2 = 11992/4820

= 2.487 sec

An inherent difficulty in interpreting arrivals from the impulse response is the problem of identifying the ‘arrival time’ i.e. what do we consider as the arrival time of a pulse that has travelled through a dispersive structure – is it the time of arrival of the leading edge of the pulse or the time of arrival of the center of the pulse, ...? This accounts for the disagreement between arrival times picked off the impulse response and arrival times predicted by calculations.

The next two arrivals, in figure 4.5 correspond to reflection off the bottom boundary. Both fluid modes of Section 2 are excited as the modes of Section 1 reflect off Interface I (hard formation-washout). The fourth arrival that we see in the Impulse
response at 3.94 sec is a disturbance that travels as Mode 2 (4820 ft/sec) in Section 1 and as the Mode 2 (∼4775 ft/sec) in Section 2 and Section 3 and reflects off the bottom boundary and comes back as Mode 2 (∼4775 ft/sec) in Section 2 and Section 3 and as Mode 2 (4820 ft/sec) in Section 1.

\[ \Delta l = 11992 \text{ ft in Sec. 1} + 8100 \text{ ft in Sec. 2} \]

arrival time = \frac{6021}{4820} + \frac{8100}{4775} + \frac{5971}{4820}

= 4.18 \text{sec}

The fifth arrival is due to a disturbance that travels as Mode 1 (4451 ft/sec) in Section 1 and as Mode 2 (∼4775 ft/sec) in Section 2 and Section 3 and reflects off the bottom boundary and comes back as Mode 2 (∼4775 ft/sec) in Section 2 and Section 3 and as Mode 2 (4820 ft/sec) in Section 1.

\[ \Delta l = 11992 \text{ ft in Sec. 1} + 8100 \text{ ft in Sec. 2} \]

arrival time = \frac{6021}{4451} + \frac{8100}{4775} + \frac{5971}{4820}

= 4.28 \text{sec}

The other arrivals in the Impulse response can be analysed in a similar manner.
The Transfer function phase gives us insight into the frequency content of the Impulse response. The phase curve of interest is in figure 4.4. The phase curve at frequencies $\leq 5 \ Hz$ has higher slopes than at higher frequencies i.e. low frequencies have higher group delays. Higher group delays correspond to longer travel distances or slower group speeds.

Further, in the low frequency regime, the slope is not a constant but is a function of frequency i.e. different frequencies have travelled different distances. This indicates that each low frequency component of the input has survived a different number of reflections. The higher frequencies have an almost constant, low slope, with no perturbations. This indicates that all the higher frequencies have travelled almost the same distance from the source to the receiver.

The phase curve of the outer fluid has two, distinct, piecewise, linear regions, one below $5 \ Hz$ and one above $5 \ Hz$. The group delays of those sections of the phase curves are $\tau_{g,1}$ and $\tau_{g,2}$ and are as given below.

$$\tau_{g,1} \approx 2.96 \ sec , \ 1.33 \leq f(Hz) \leq 3.81$$

$$\tau_{g,2} \approx 0.003 \ sec . \ 5.0 \leq f(Hz)$$

The phase curve of the inner fluid is characterized by an almost constant, group delay indicating that the pressure at the response location is dominated by the
direct arrivals.

\[ \tau_g \approx 0.01 \text{sec}. \]

The phase curve of the pipe, has two group delays, one below 7 Hz and one above 7 Hz:

\[ \tau_{g1} \approx 1.22 \text{sec}, \quad 0 \leq f(\text{Hz}) \leq 7.5 \]

\[ \tau_{g2} \approx 0.003 \text{sec}, \quad 7.5 \leq f(\text{Hz}) \]

The group speed is a function of the three modal propagation speeds. With the possible values of the group speed and the group delay we can arrive at possible values of source-receiver separation. Based on the geometry of the structure we get a set of values for the source-receiver separation (taking into account reflections at boundaries and discontinuities). From the intersection of these two sets of values we can get the actual distance that the frequency component with the specified group delay travelled. This helps us to identify locations of strong reflections and the origin of the various arrivals.

With a knowledge of the medium that we excite and the nature of coupling between the two fluid media, we can narrow down the possible values of the group speed to either two or one. When we excite the pipe medium the main mode that responds is the pipe mode. The propagation speed of the disturbance (group speed)
is then the pipe propagation speed. When we have strong coupling and excite one of the fluid media both fluid modes respond. The disturbance would decompose into the the two fluid modes and they would travel at their respective propagation speeds. When we have weak coupling and excite one of the fluid media only one of the fluid modes respond. Thus when we excite one of the fluid media there is one possible propagation speed, in the case of weak coupling, and two possible propagation speeds in the case of strong coupling.

The propagation speeds and modeshapes for the three sections are as follows:

<table>
<thead>
<tr>
<th>Section 1</th>
<th>Mode 1</th>
<th>Mode 2</th>
<th>Mode 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\phi_1$</td>
<td>$\phi_2$</td>
<td>$\phi_3$</td>
</tr>
<tr>
<td>$\sigma_p$</td>
<td>0.2156</td>
<td>-0.0871</td>
<td>1.0000</td>
</tr>
<tr>
<td>Mode shapes</td>
<td>$p_i$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.9607</td>
<td>1.0000</td>
<td>0.0058</td>
</tr>
<tr>
<td></td>
<td>$p_o$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.0000</td>
<td>0.1325</td>
<td>-0.0009</td>
</tr>
<tr>
<td>Speeds</td>
<td>$ft/sec$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4451</td>
<td>4820</td>
<td>16943</td>
</tr>
<tr>
<td></td>
<td>$m/sec$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1356</td>
<td>1469</td>
<td>5164</td>
</tr>
</tbody>
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4.2. EXAMPLE 4.1

<table>
<thead>
<tr>
<th>Section 2</th>
<th>Mode 1</th>
<th>Mode 2</th>
<th>Mode 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\phi_1$</td>
<td>$\phi_2$</td>
<td>$\phi_3$</td>
</tr>
<tr>
<td>Mode shapes</td>
<td>$\sigma_p$</td>
<td>0.0108</td>
<td>-0.1051</td>
</tr>
<tr>
<td></td>
<td>$p_i$</td>
<td>-0.0120</td>
<td>1.0000</td>
</tr>
<tr>
<td></td>
<td>$p_o$</td>
<td>1.0000</td>
<td>0.0008</td>
</tr>
<tr>
<td>Speeds</td>
<td>$ft/sec$</td>
<td>1325</td>
<td>4774</td>
</tr>
<tr>
<td></td>
<td>$m/sec$</td>
<td>404</td>
<td>1455</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Section 3</th>
<th>Mode 1</th>
<th>Mode 2</th>
<th>Mode 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\phi_1$</td>
<td>$\phi_2$</td>
<td>$\phi_3$</td>
</tr>
<tr>
<td>Mode shapes</td>
<td>$\sigma_p$</td>
<td>0.0099</td>
<td>-0.1050</td>
</tr>
<tr>
<td></td>
<td>$p_i$</td>
<td>-0.0109</td>
<td>1.0000</td>
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<tr>
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<td>$p_o$</td>
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<td>0.0015</td>
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<tr>
<td>Speeds</td>
<td>$ft/sec$</td>
<td>1269</td>
<td>4775</td>
</tr>
<tr>
<td></td>
<td>$m/sec$</td>
<td>387</td>
<td>1455</td>
</tr>
</tbody>
</table>

The possible distances that the input can travel, from source to the receiver, are 50 $ft$ [direct arrival], 11992 $ft$ [reflection from Interface I (hard formation-washout)], 20092 $ft$ [reflection from bottom boundary] and so on. Thus we get that
The inference of the history of the various pulses is summarized in the above table. The group delays have been calculated as a mean slope over a frequency range of the phase plot and hence are not exact. This accounts for the difference between the inferred value of $\Delta l$ and the actual distances.

We can conclude from figure 4.4, that in the outer fluid medium, the frequency components above 5 $Hz$ reach the response location only as direct arrivals. The damping in the system attenuates them before they can reach the response location after one or more reflections. Further they propagate in the pipe mode and not in one of the fluid modes. This is because the fluid modes get damped out by 5 $Hz$ and the pipe mode is the only means of transferring energy in the outer fluid medium. Below 5 $Hz$ we can see the two group delays. These correspond to reflections from Interface
I (hard formation-washout). The additional perturbations that are present below 5 Hz account for arrivals with further reflections. This accounts for the presence of the arrivals after two, three and four reflections at Interface I (hard formation-washout) in the transient pressure response of the outer fluid. The various arrivals in the transient response, smear the input impulse, in time, because they are rich in low frequencies and low on high frequencies. The range of frequency over which a group delay is constant is indicative of the strength of the arrival corresponding to that group delay. Hence we can conclude that the strongest arrivals are those that reflect off interface I (apart from the direct arrival).

In the inner fluid, from figure 4.3, we observe a single mean group delay corresponding to a direct arrival. Since the response location is close to the source it is dominated by the direct arrivals. Yet frequencies below 2 Hz indicate that they have high group delays corresponding to reflections off Interface I (hard formation-washout) and the bottom boundary.

In the pipe, from figure 4.2, we see two group delays, one corresponding to the direct arrival and another corresponding to reflection off the bottom boundary. There is no change in geometry or physical property of the pipe in passing past Interface I (hard formation-washout). The pipe propagation speeds in Section 1 and Section 2 and 3 are almost the same. Hence there is almost complete transmission in passing through Interface II. Thus the only reflection is from the bottom for the pipe mode.
A similar problem was done in H.Y. Lee's thesis [5], where the various arrivals were calculated by ray tracing. It involves keeping track of the fate of all the involved modes to arrive at the Impulse response. The method dealt with in this thesis, calculating the Transfer function and inverse fourier transforming it to get the Impulse response, is a relatively easier exercise as the interactions of the various modes and the boundaries are incorporated in the Transfer function calculation, by insuring kinematic and dynamic compatibility at all interfaces and by considering the response as a linear combination of the three modes.

4.3 Example 4.2

In this example we consider a pipe system that is similar to the one that was considered in Example 4.1. However the input is near the bottom and the response is measured near the surface. This is a simulation of a MPT system. The schematic of the pipe system is shown in figure 4.6.
Figure 4.6: Schematic of example for Impulse response calculation - Example 4.2.
### Input Data for Example 4.2

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner dia. of pipe</td>
<td>4.19</td>
<td>in.</td>
</tr>
<tr>
<td>Outer dia. of pipe</td>
<td>5.00</td>
<td>in.</td>
</tr>
<tr>
<td>Density of pipe</td>
<td>15.18</td>
<td>slugs/ft³</td>
</tr>
<tr>
<td>Young's Modulus of pipe</td>
<td>4.32×10⁹</td>
<td>lb/ft²</td>
</tr>
<tr>
<td>Poissons ratio of pipe</td>
<td>0.2561</td>
<td></td>
</tr>
<tr>
<td>Density of mud</td>
<td>2.027</td>
<td>slugs/ft³</td>
</tr>
<tr>
<td>Bulk Modulus of mud</td>
<td>5.30×10⁷</td>
<td>lb/ft²</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Length</th>
<th>Hole dia.</th>
<th>Wall stiffness</th>
<th>Attenuation const.</th>
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<tr>
<td></td>
<td>ft</td>
<td>m</td>
<td>in.</td>
<td>cm</td>
</tr>
<tr>
<td>Sec. 1</td>
<td>6021</td>
<td>1835</td>
<td>12.347</td>
<td>31.36</td>
</tr>
<tr>
<td>Sec. 2</td>
<td>4000</td>
<td>1219.02</td>
<td>12.347</td>
<td>31.36</td>
</tr>
</tbody>
</table>

**Excitation**: harmonic volume velocity (1 inch³/sec) applied to the inner fluid, 50 ft from the bottom boundary

**Response**: pressure in the inner fluid, 50 ft from the top boundary

**Boundary condition**: pipe is fixed at the top boundary, inner fluid is fixed at the top boundary and
outer fluid has a pressure release at the top
boundary
: pipe is free at the bottom boundary
$u_i = u_p$ at the bottom boundary and
$u_o = -u_p$ at the bottom boundary

The calculated Transfer function for the pipe, inner and outer fluid are in figures 4.7, 4.8 and 4.9.

The Impulse response of the inner fluid is shown in figure 4.10. It shows the introduced impulse arriving at the response location at regular time intervals, with the interarrival times equalling the round trip travel time. The pulses get attenuated as they travel through the structure.

The impulse was applied to the inner fluid in the soft formation section. This excites the second mode predominantly. When this mode arrives at Interface I (hard formation-washout), it excites both the fluid modes of the hard formation section. So the impulse response should display two arrivals corresponding to the two fluid modes at the response location. The speeds of the two modes are 4451 and 4820 $ft/sec$ respectively. In travelling 5971 $ft$ from Interface I (hard formation-washout) to the response location, they arrive at a time separation of 0.102 $sec$. Hence we cannot distinguish the arrivals as dual arrivals. This time separation increases as the arrivals suffer additional reflections (it is 0.306 $sec$ the second time it arrives at
the response location) and this causes the succeeding arrivals to spread out in time.

We can observe this behaviour in the impulse response as broader peaks at later times.

We follow the same argument that we developed in the previous example and arrive at the following results, based on the Transfer function phase. The distance between source and receiver i.e., \( \Delta l \) is 9921 \( ft \).

<table>
<thead>
<tr>
<th>Medium</th>
<th>( \tau_g )</th>
<th>( c_g )</th>
<th>( \Rightarrow \Delta l )</th>
<th>Notes</th>
<th>Frequency range Hz</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>sec ( ft/sec ) ( m/sec ) ( ft ) ( m )</td>
<td></td>
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<td>Outer fluid</td>
<td>2.13</td>
<td>4451 (1356) in Sec. 1</td>
<td>direct arr.</td>
<td>0 - 9</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>4775 (1455) in Sec. 2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.58</td>
<td>16938</td>
<td>5162</td>
<td>9804</td>
<td>2988</td>
</tr>
<tr>
<td>Inner fluid</td>
<td>2.08</td>
<td>4820 (1469) in Sec. 1</td>
<td>direct arr.</td>
<td>0 - 10</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>4451 (1356) in Sec. 2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.58</td>
<td>16938</td>
<td>5162</td>
<td>9804</td>
<td>2988</td>
</tr>
<tr>
<td>Pipe</td>
<td>0.58</td>
<td>16938</td>
<td>5162</td>
<td>9866</td>
<td>3007</td>
</tr>
</tbody>
</table>

The outer fluid has two group delays, one corresponding to a fluid mode propagation below 9 Hz and one corresponding to a pipe mode propagation above 9 Hz. Damping attenuates the fluid modes beyond 9 Hz and the energy transfer in the outer fluid takes place due to the pipe mode.
Figure 4.7: Transfer function between a volume velocity input (1 inch$^3$/sec), to the inner fluid, 50 ft from the bottom boundary and stress response in the pipe, at the response location, 50 ft from the top boundary - Example 4.2.
Figure 4.8: Transfer function between a volume velocity input ($1 \text{ inch}^3/\text{sec}$), to the inner fluid, 50 ft from the bottom boundary and pressure response in the inner fluid, at the response location, 30 ft from the top boundary. Example 4.2.
Figure 4.9: Transfer function between a volume velocity input (1 in$^3$/sec) to the inner fluid, 50 ft from the bottom boundary and pressure response in the outer fluid, at the response location, 50 ft from the top boundary - Example 4.2.
Impulse response of pipe system: input to inner fluid

Figure 4.10: The transient response of pressure (lb/ft²) in the inner fluid, 50 ft from the top boundary, due to an impulse of volume velocity (inch³/sec) applied to the inner fluid at the input location, 50 ft from the bottom boundary - Example 4.2.
The inner fluid exhibits a similar behaviour. The pipe has a group delay corresponding to a direct arrival.

If the receiver were placed in the outer fluid instead of the inner fluid, the observed transient response to a unit impulse of volume velocity to the inner fluid is shown in figure 4.11. This is considerably different from the transient response of the inner fluid to the same input. But it has some similarity with the impulse response of the outer fluid that was calculated in Example 4.1 in that we see two arrivals that follow closely, with opposite signs. The direct arrival has them arriving with a time separation of 0.102 sec at the response location, which accounts for the observed shape. In the second arrival the two modes get separated in time and we can observe them separately. The fact that the introduced impulse had to communicate through the radial stiffness of the pipe and poisson effects of the pipe for the outer fluid to experience the introduced pressure disturbance accounts for the comparatively low magnitudes of pressure.

4.4 Predicted Transient

We now convolve the Impulse response the we have calculated for a given pipe system, with a rectangular waveform to obtain the pressure transient that would be observed at the surface. Consider the pipe system of Example 4.2. The transient pressure response of the inner fluid, at the response location to an impulse of vol-
Figure 4.11: The transient pressure response (lb/ft²) of the outer fluid, 50 ft from the top boundary, due to an unit impulse of volume velocity (inch³/sec) applied to the inner fluid at the input location, 50 ft from the bottom boundary - Example 4.2.
ume velocity to the inner fluid at the input location is known from Example 4.2, figure 4.10. The pressure transient when this is convolved with a unit volume velocity input of 2 and 4 sec durations, is as shown in figure 4.12. We see that the arrivals that we had observed in the Impulse response are now of approximately two second durations in Case (a). This very soon gets smeared out in time and begins to look like a sine wave. The period of the observed sinusoid is dictated by the geometry of the borehole. In our case the time separation between succeeding arrivals in the Impulse response is $\sim 4.2$ sec. It is this time separation that is manifested as the period of the sinusoid in the transient response.

In Case (b), we see the effect of the longer signal is to cause the succeeding arrivals of the input waveform to superimpose and create a step-like response.

For comparison, we get the transient response in the outer fluid 50 ft from the top boundary, to an impulse of volume velocity to the inner fluid, 50 ft from the bottom boundary and convolve it with the same inputs as above. The Impulse response is in figure 4.11 and the predicted response is in figure 4.13.

The transient response of the outer fluid shows the arrival of the leading edge and the trailing edge of the input signal. The outer fluid pressure exhibits a non-zero pressure only at the leading edge and the trailing edge of the input waveform. This is because of the two fluid modes of opposing sign arriving at the response location$^4$. The two fluid modes are so because they need to sum up to the input excitation,

$^4$This was detailed in the discussion on the Impulse response of the outer fluid in Example 4.1.
Figure 4.12: Predicted transient pressure in the inner fluid, 50 ft from the top boundary, due to a long unit volume input (inch^3/sec) to the inner fluid, 50 ft from the bottom boundary. Example 4.2, (Case (a) T = 2 sec, Case (b) T = 4 sec).
in the inner fluid and cancel each other in the outer fluid, which was initially in a quiescent state. Thus, in the Transient response we have two waveforms, delayed in time, and of opposing sign. They cancel each other out in the regions where they overlap and create non-zero pressure at the beginning and the end of the input waveform duration, where they are staggered with respect to one another because of their differences in propagation speeds.

We also see that echoes of the input signal are unlikely to be observed in the outer fluid, under the given conditions of damping.
Figure 4.13: Predicted transient pressure (lb/ft²) in the outer fluid, 50 ft from the top boundary, due to a 'T' sec long unit volume velocity input (inch³/sec) to the inner fluid, 50 ft from the bottom boundary - Example 4.2. [Case (a) T = 2 sec, Case (b) T = 4 sec]
Chapter 5

CONCLUSIONS

Devanin koil, mudiya neram, nān yenna kétpén deivamē
inru en jeevan theyuthê, en manam eno sāyuthê ...¹

In this thesis we have used the mathematical model developed by H. Y. Lee for a three-layered cylindrical waveguide, in understanding acoustic transmission in boreholes.

The Transfer matrix method for analyzing structures was used to get the Transfer function of the pipe system. The modelling of the source and boundaries was conducted.

A constant volume velocity source model was incorporated in calculations to arrive at the Transfer function between the pressure response in one of the fluid media and a volume velocity input in any one of the fluid media; two cases of a

¹from Aruvadai Naal

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uniform pipe system with different wall stiffnesses were considered. The Transfer function was calculated and the concept of group delay was employed to interpret the Transfer function phase.

The above analysis revealed the dispersion characteristics of the structure (introduced due to damping) and indicated the nature of distortion that an input might suffer. Further, it was used to predict the dominant modes that would be responsible for energy transfer in each of the three media.

The Transfer function was used in getting the Impulse response of the pipe system. The group delay was used in identifying strong arrivals at the response location. It was found that the group delay was a more conclusive indicator of strong arrivals than the Impulse response. A typical borehole was considered and the Impulse response was calculated. It was convolved with square wave inputs of 2 and 4 sec durations and the resulting transient pressure at the response location in the two fluid media was predicted. While the inner fluid response was due to the arrival of two fluid modes of the same sign the outer fluid response was due to the arrival of two fluid modes of opposing sign. As a result one could observe the arrival of the complete input signal in the inner fluid but only see the leading and trailing edges of it in the outer fluid.

The damping in the structure was introduced as a complex wavenumber, producing an exponential decay with distance. The attenuation constant, which is the complex part of the wavenumber, was assumed to be a constant over our frequency
range of interest (≤ 20 Hz). This damping model could be improved by choosing a frequency dependant attenuation constant.

We now have the capability to successfully model acoustic transmission through a borehole.
Appendix A

Transfer Matrices

The Transfer matrix for a Mass-spring-dashpot system:

\[
\begin{pmatrix}
  u_R \\
  F_R
\end{pmatrix} =
\begin{bmatrix}
  1 & \frac{1}{K + \omega C} \\
  -\omega^2 M & 1 - \frac{\omega^2 M}{K + \omega C}
\end{bmatrix}
\begin{pmatrix}
  u_L \\
  F_L
\end{pmatrix}
\]

\(M, K, C\) : mass, spring constant, damping coefficient

The Transfer matrix for a uniform bar:

\[
\begin{pmatrix}
  u_R \\
  F_R
\end{pmatrix} =
\begin{bmatrix}
  \cos kl & \frac{\sin kl}{E Ak} \\
  -E Ak \sin kl & \cos kl
\end{bmatrix}
\begin{pmatrix}
  u_L \\
  F_L
\end{pmatrix}
\]

\[k^2 = \frac{\rho \omega^2}{E} (1 - 2\zeta)\]

\[\zeta = \frac{c}{2 \rho \omega A}\]

\(\rho, E\) : density, Young's Modulus

\(A, l\) : cross-sectional area, length
Appendix B

English to Metric Units

<table>
<thead>
<tr>
<th>Quantity</th>
<th>FPS equivalent</th>
<th>MKS equivalent</th>
<th>Multiply by</th>
</tr>
</thead>
<tbody>
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<td>0.3048</td>
</tr>
<tr>
<td></td>
<td>in.</td>
<td>cm</td>
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</tr>
<tr>
<td>Area</td>
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<tr>
<td></td>
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<td></td>
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<td>m³</td>
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</tr>
<tr>
<td>Quantity</td>
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<td>MKS equivalent</td>
<td>Multiply by</td>
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<td>----------------</td>
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Bibliography


