FEEDBACK CONTROL OF A
CONVECTIVE INSTABILITY

by

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ABSTRACT

Continuum instabilities may be divided into non-convective instabilities, which grow in time, and convective instabilities which grow in space. In this work the techniques of electronic feedback control are used to stabilize a particular convective instability.

The possibility of continuum feedback control is shown with an ideal feedback system in which the disturbance at every point in the system is measured and a force proportional to this disturbance, or to its derivatives in space and time is applied to the same point in the system.

In most cases the ideal continuum feedback must be implemented by a number of finite stations, in which the disturbance at some point within the station is sampled, and a feedback force proportional to this disturbance is applied throughout the section. Although disturbances can affect points both upstream and downstream as a result of the sampled feedback, the theory of characteristics shows that the boundary conditions remain those associated with a supercritical convective system. The response of the single station indicates that if the feedback gain exceeds a certain critical value, the system becomes non-convectively unstable.

To study the convective stability of a collection of sampled feedback stations a new convective stability criterion has been formulated. This criterion indicates that, while the sampled feedback may induce convective instabilities at certain frequencies, it is possible to stabilize the system studied under certain operating conditions for all frequencies.

Various experiments carried out on a water jet confirmed the existence of the feedback induced non-convective instability and show that it is possible to stabilize a convectively unstable system by using sampled feedback. In these experiments the electric pressure was calibrated using the growth rate of the instability on an uncontrolled jet,
while the feedback gain was calibrated using the known response of the jet to excitation.

Thesis Supervisor: James R. Melcher
Title: Assistant Professor of Electrical Engineering
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Chapter 1: Introduction

A. Background

Continuum Instabilities

The instabilities of a continuum have held the interest of scientists for the last century, but only recently have they recognized the two broad classes into which instabilities may be divided. As Twiss first pointed out, the standard classical techniques will predict the presence of an instability but they do not distinguish between the two forms in which the instability can appear. The first and most common is the non-convective or absolute instability which grows in time at any point in space such as the inverted glass of water (Rayleigh-Taylor instability). Another system which might be regarded as a non-convective instability although not really a continuum, is a feedback control system in which the gain of the feedback loop has been made too high. In both of these systems, the disturbance at any point grows without bound, either monotonically, as in the inverted water glass, or as an oscillation of increasing amplitude, as in the feedback system. The other type of instability, which is found in distributed systems, is the convective instability or amplifying wave. A common example is the jet of water from a faucet breaking up into droplets under the influence of surface tension. Because the growing disturbance is swept downstream, the amplitude at any point on the jet is bounded in time, and the disturbance grows in space. This growth in
space coupled with a bounded time response at any point is characteristic of the convective instability.

The distinction between convective and non-convective instabilities becomes important only when the finite length of any physical system cannot be ignored. If the water jet were infinitely long, for example, and a uniform disturbance were initially set up over its whole length, the disturbance seen by a stationary observer would appear to be growing in time at his observation point, since the disturbance at later times, which is convected from original points farther away, has grown more when it reaches the observation point. This distinction must always be made before boundary conditions can be applied to the system. A non-convective system, such as the inverted water glass, requires spacelike boundary conditions (the rim of the glass) while a convective system requires timelike boundary conditions (at the faucet nozzle).

Previous Stability Criteria

There is a close connection between the boundary conditions which may be imposed on a system and the type of instability it supports. In a non-convective system, such as a vibrating string, the conditions must be applied at boundaries separated in space and they determine the solution at all intermediate points. In a convective system such as supersonic flow, on the other hand, the boundary conditions must be applied upstream, and they determine the solution at all downstream points. This relation has been
exploited by Melcher, who used the allowed boundary conditions, as determined by the theory of characteristics, to deduce the nature of the instability.

Sturrock has suggested that it is also possible to distinguish between these two forms of instability by studying the properties of free waves in the system. In this approach, expanded and perfected by Fainberg, Kurilko, and Shapiro, Polovin, and Briggs, a small pulse is applied to the system, and the resulting disturbance followed in space and time to determine whether the system is stable. This method has been used quite successfully with homogeneous systems, but can not be easily applied to systems in which the disturbance is driven in space and time or whose properties change in the direction of propagation. Since the feedback control system discussed in this paper is characterized by a driving force which changes abruptly in passing between spatial regions, the question of stability will be more involved than in previous work. It will in fact be shown that our system can support either convective or non-convective instabilities, or both.

**Continuum Feedback Control**

Previous work in the control of continuum systems has been performed in the field of chemical processes where the systems to be controlled are basically diffusive. In the majority of this work, the continuum systems have been treated as a collection of lumped circuits. More recently, Murray has used the continuum nature of these systems to
consider the detection and control of the spatial eigenmodes of chemical systems. A similar technique has been used by Wiberg\textsuperscript{11} in the control of xenon oscillations in nuclear reactors.

The feedback control of a system which supports waves has received less attention, although Bateman\textsuperscript{12} in 1953 discussed the problems and possibilities of control of an elastic fluid. Recently Melcher\textsuperscript{13} showed that a non-convective instability can be controlled by sampling the disturbance at regular spatial intervals and feeding back a force proportional to this sampled disturbance to the surface of the system in the neighborhood of the sampled point.

In this paper we will consider the control of a particular convective instability.

**Goals for the Control of an Instability**

The object of a scheme to control an unstable system will be somewhat different from the object of the usual control system. Normally, we would like to force the output of the system, such as the purity of a gasoline, to be as close as possible to some ideal output. With a system which is naturally unstable, however, success is achieved when the output of the system remains bounded, regardless of its form. Thus, we need not require that the unwanted disturbances disappear, but only that they do not grow.

In this work we will consider the use of feedback to control the convective instability of a liquid jet in a normal electric field. Since this particular system is well
modelled by an equation whose form is general enough to
describe a wide variety of convective instabilities, the
results obtained should indicate the capabilities of
feedback control for many similar systems, such as charged
particle beams and traveling wave amplifiers.

To implement the continuum feedback control we must
usually break the system into many segments of finite size.
The restrictions placed on the operation of the control
system by this segmentation will be studied to determine the
practical possibilities of the continuum feedback control of
convective instabilities.
8. Outline

Chapter II describes the system to be controlled: the kink mode of a liquid jet in a normal electric field. The possibility of controlling the convective instability of this system with an idealized continuum feedback system is demonstrated in Chapter III.

In Chapters IV and V the technique of implementing this ideal continuum control by breaking the system into small sections of finite length is investigated. In Chapter IV the theory of characteristics is used to determine the allowed boundary conditions on the response, and also to investigate the stability of the system by following the development of a small input pulse. In Chapter V the boundary conditions obtained in the previous section are used to obtain an analytic solution which shows the possibility of non-convective instability due to the time lag in the feedback loop.

In Chapter VI a convective stability criterion suited to the continuum feedback control system is developed. This criterion is applied to the jet to determine the range of feedback gain, electric pressure, and surface tension which results in convectively stable operation.

Chapter VII describes experiments undertaken to study both the non-convective and convective instabilities predicted by the theory of Chapters V and VI.
Chapter II Description of the System

The Kink Mode of a Liquid Jet

The kink mode of a liquid jet traveling through a normal electric field has several advantages for an experiment on the control of a convective instability. First, because the instability is due entirely to the presence of the electric field, the growth rate can be controlled externally by adjusting the source of the field. This would not be possible with the usual droplike mode of the liquid jet because the instability, while it is affected by the electric field, is fundamentally due to the capillary velocity of the jet, a parameter which is determined primarily by the geometry of the jet and the surface tension of the liquid.

Second, if a sampled feedback system is used to control an instability, it seems reasonable that if adequate control is to be achieved the wavelength of the disturbance should be much longer than the sampling length. Thus we must rely on some physical mechanism in the system to control disturbances of very short wavelengths. In many systems, such as plasmas and electron beams, the effects which might stabilize short disturbances are extremely difficult to model satisfactorily and the behavior of the system is therefore poorly described at high frequencies. The liquid jet does not suffer this drawback, since the relevant high frequency effect is the surface tension, which enters the equation of motion of a thin jet quite simply.
Finally, it is a relatively simple matter to produce and study a liquid jet in the laboratory, since the elevated temperatures and complex diagnostic techniques of plasma and electron beam technologies are not required. Most measurements on the instability of the jet can be performed optically or photoelectrically, since the disturbances are on the order of centimeters in length.

In this chapter we will sketch the derivation of the equations of motion of a planar and a circular jet for the mode of interest, the kink mode. These equations, which differ only in the form of the constant coefficients, are general enough to describe many other convectively unstable systems, such as electron beams. We will then solve the generalized equations to show first the possibility of convective instability, and then the behavior of the jet when the electric field is driven by a time varying voltage. This voltage, which excites a disturbance on the jet, will be the basis of the feedback system.

The Planar Jet

The system under consideration, sketched in figure 2.1, is a thin liquid jet of thickness Δ, infinite in width, and moving to the right with a velocity \( v_0 \) between two parallel metal plates, which are separated from the jet by a distance d. The liquid has a mass density \( \rho \), surface tension \( T \), and infinite conductivity. In all of the following work, primed variables, such as \( x' \) and \( t' \), will refer to physically measureable quantities while unprimed
2.1. The planar jet flows between two parallel electrodes whose voltage is a function of both space and time.
quantities, such as \( x \) and \( t \), will refer to dimensionless quantities.

An electric field is set up between the grounded jet and the metal plates by voltage sources whose output is the sum of a constant bias voltage and a controlled voltage, which may vary in space and time. That is

\[
\phi' = \phi_o' + \phi_1'(x', t')
\]  \hspace{1cm} (2.1)

on the upper and lower surfaces, respectively.

We shall assume that the steady motion of the jet is perturbed slightly and that the wavelength of these perturbations is large compared to the thickness of the jet and to the separation between the jet and the metal electrodes. Under these conditions, the only important component of the electric field is in the \( y \)-direction,

\[
E'_y = \left[ \frac{\phi_o' + \phi_1'(x', t')}{d + u} \right]
\]  \hspace{1cm} (2.2)

in the regions above and below the jet, respectively, where \( u \) is the amplitude of the perturbation.

Consider a differential length of the system. The forces acting on this length will be due to the surface tension of the jet and to the electric field between the jet and the electrodes. The total force per unit area on the jet due to surface tension will be proportional to the total length of exposed surface edges, the surface tension coefficient \( T \), and the curvature of the jet.
\[ E'_e = 2T \frac{\partial^2 u}{\partial x'^2} \quad (2.3) \]

The factor of 2 occurs because the jet has two surfaces. The net electrical force per unit area on the jet, found by integrating the Maxwell stress tensor over the surface is

\[ E'_e = \varepsilon_0 (\varphi' + \varphi'_L)^2 / (d - u')^2 \quad (2.4) \]

Linearized, this becomes

\[ E'_e = (2 \varepsilon_0 \varphi'^2 / d^2) \left( u'/d + \varphi'_L / \varphi'_0 \right) \quad (2.5) \]

The electric field increases if the surface of the jet moves toward the electrode or if the applied voltage is increased. The increase of force as the jet moves accounts for the instability of the system.

Now apply Newton's Law to the differential length to obtain the equation of motion of the jet

\[
\rho \Delta \left( \frac{\partial}{\partial t} + V_0 \frac{\partial}{\partial x'} \right)^2 u' = 2T \frac{\partial^3 u'}{\partial x'^3} + \varepsilon_0 \varphi'^2 \frac{\partial u'}{\partial x'} + \varepsilon_0 \varphi' \frac{\partial^2 \varphi'}{\partial x'^2} (x', t')
\]

\[ (2.6) \]

In this equation, the acceleration of a particle of the jet, represented by the term on the left, is equal to the restoring force of the surface tension, the destabilizing force of the electric pressure, and the controlled force. Although this equation was derived for a particular, and somewhat unphysical situation, its form is general enough to describe a great variety of systems which exhibit the
possibility of convective instability.

**The Circular Jet**

So far we have been concerned exclusively with the two dimensional planar jet. While a planar jet lends itself easily to theoretical analysis, it is extremely difficult to realize in the laboratory. A cylindrical jet, however, is easy to produce, and its theoretical analysis, while not so straightforward as that of the planar jet, is entirely possible. Just as we were concerned only with the kink mode in the discussion of the planar jet, we shall only consider those motions of the circular jet in which the cross-section of the jet remains essentially unaltered, so that the jet moves uniformly in the direction transverse to the downstream direction.

The physical arrangement of the system is shown in figure 2.2. In the region downstream of the exciter, the outer electrode is at a constant potential and the jet is grounded. The outer electrode is split into two semicircles, one at potential \( \phi_0' + \phi_1' \) and the other at potential \( \phi_0' - \phi_1' \). This difference in the two potentials leads to a force per unit length on the jet which can be calculated using the Maxwell stress tensor. This procedure gives

\[
F_{el} = 8\varepsilon_0 \phi_0' \phi_1' \left( \frac{d}{dR} \left( \ln \frac{d}{R} \right) \left( 1 - \frac{R^2}{d^2} \right) \right)^{-1}
\]

(2.7)

The force per unit length due to surface tension may be calculated by considering a short length of the circular jet
2.2. The circular jet (in cross section) flows between two concentric semicircular electrodes.
displaced as a whole from its equilibrium position. The total equilibrium tension on the jet is given by the product of the surface tension and the length of the line on which it acts, and by a process similar to that used for a string under tension,

$$F'_e = 2 \pi RT \frac{\partial^2 u}{\partial x'}$$  \hspace{1cm} (2.8)

There will also be a force on the jet due to its displacement from equilibrium in the steady electric field. This force has been calculated elsewhere in connection with electrical capacitors and is given by

$$F'_{e2}/u' = 2 \pi \varepsilon_0 \varepsilon_r \left( \frac{d^2 - R^2}{d} \right)^{-1} \left( \ln (d/R) \right)^{-2}$$  \hspace{1cm} (2.9)

With these forces, the equation of motion of the jet is

$$\pi R^2 \rho \left( \frac{\partial}{\partial t'} + v_0 \frac{\partial}{\partial x'} \right)^2 u' = 2 \pi RT \frac{\partial^2 u'}{\partial^2 x'} + \frac{2 \pi \varepsilon_0 \varepsilon_r}{(d^2 - R^2)(\ln d/R)^2}$$ + \frac{8 \varepsilon_0 \varepsilon_r}{d(\ln d/R)(1 - R^2/d^2)}$$  \hspace{1cm} (2.10)

This equation differs from that of the planar jet in the constant coefficients as a result of the different transverse geometry, but the equation describing disturbances propagating along the jet is identical in form to the equation for a planar jet. Similarities like this also exist with the ion and electron beams used in microwave amplifiers and thermonuclear devices. Thus, although the work described here will be couched in terms of a liquid jet, the concepts and techniques, with necessary
modification, should be applicable in a variety of areas.

**Solution of the General Equation**

The general equation of this form may be normalized as

\[
\left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right)^2 u = \alpha^2 \frac{\partial^2 u}{\partial x^2} + Nu + P\phi
\]

where

\[
\phi = \phi_0' \rho_0' \\
x = x' / L \\
t = t' V_o / L \\
u = u' / L \\
\alpha^2 = 2T (\rho_0 \Delta V_o^2)^{-1} (\text{surface tension}) \\
N = \frac{\epsilon_0}{\epsilon'_0} \rho_0^2 (\rho \Delta V_o^2 \Delta V_o^3)^{-1} L^3 (\text{electric pressure}) \\
P = \frac{\epsilon_0}{\epsilon'_0} \rho_0^2 (\rho \Delta V_o^2 \Delta V_o^3)^{-1} L^3 (\text{controlled force})
\]

for the planar jet. For the other systems described by an equation of this form, of course, the parameters \(\alpha, N,\) and \(P,\) representing the surface tension, electric pressure, and controlled force will be different.

If the perturbation has a sinusoidal time dependence,

\[
u(x,t) = \text{Re} \left[ \hat{u}(x) e^{i\omega t} \right]
\]

where \(\omega = \omega' L / V_o\)

the equation of motion reduces to

\[
\left( 1 - \alpha^2 \right) \frac{d^2 \hat{u}}{dx^2} + 2i\omega \frac{d\hat{u}}{dx} - (N + \omega^2) \hat{u} = P\phi
\]

This equation has a solution of the form
\[ \hat{U}(x) = D e^{-ik_x x} + Be^{-ik_{-x} x} + C \]  

(2.14)

where \( k \) satisfies the dispersion relation

\[ k_{\perp} L = \frac{\omega \pm i \sqrt{N(1-\alpha^2) - (\alpha \omega)^2}}{1-\alpha^2} = k_\perp L + k_{\parallel} L \]  

(2.15)

**Excitation**

If the controlled force is driven sinusoidally, a sinusoidal disturbance will be set up on the jet as the electric field alternately pulls and pushes successive particles of the jet. The displacement of the jet as a result of the sinusoidal drive is given by

\[ \frac{\hat{U}(x)}{\varnothing} = \rho \left[ \cosh k_{\perp} x + (i k_{\perp}/k_{\parallel}) \sinh k_{\perp} x \right] \frac{e^{-i k_{\perp} x} - 1}{N + \omega^2} \]  

(2.16)

It is thus possible to excite a disturbance on the jet of arbitrary frequency with a controlled voltage source. This exciter effect can also be used to reduce an unwanted disturbance on the jet, if suitable methods are found for detecting the offending disturbance.

**A Simpler Approach**

The description of the exciter is complicated by the interaction between the surface and the electric field perturbations. This restriction may be relaxed if the forces due to the perturbation of the shape of the surface are much smaller than the force set up by the change in the
applied electric field. Under these conditions the force on
the surface is independent of the surface perturbations, and
the equation expressing conservation of momentum for a
differential length of the jet may be written as
\[
\int_0^\infty \rho \Delta u/\Delta t \, d(\rho \Delta u/\Delta t') = \int_{-L/V_o}^0 \frac{2\varepsilon_0 \rho_0 q'}{d^2} \, dt'
\]  
(2.17)

For a sinusoidal signal voltage, the exit velocity is
\[
(du/dt)/\rho = B \sin \omega t / \omega \sin \omega t / 2 \, e^{i\omega t / 2}
\]  
which is equivalent to the velocity obtained by taking the
limit of the derivative of equation 2.16 as \( \alpha \to 0 \) and
\( N \to 0 \).

Two features of this response deserve mention. The
first is the existence of minima in the response at the
frequencies \( \omega = 2n\pi \). At these frequencies, the jet
passes through the entire length of the exciter in a time
equal to an integral multiple of one cycle of the input
signal. Since the force varies sinusoidally and the net
velocity is approximately equal to the integral of the
electric force over the time the jet spends in the exciter,
the exit velocity is very small.

Also, the response falls off at higher frequencies
because the integral of the force is very small after one
period of the exciting voltage and the signal up to this
time is "wasted". Thus, at higher frequencies the exciter,
which is longer than a wavelength, is effectively shortened,
and the magnitude of the response drops accordingly.
2.3. At low frequencies, the growing waves amplify the exciter response ($N = 0.695$). At higher frequencies the response exhibits a series of minima near the frequencies $f = L/V_0$. 
The velocity of the jet at the exit of the exciter as predicted by the approximate and the exact approach is plotted in figure 2.3. At very low frequencies, growing waves exist inside the exciter which amplify the response to \((\sinh k_L L)/k_L L\) times its value when the waves are neglected. For higher frequencies, the waves no longer grow, and their effect is to shift the frequencies at which the maxima and minima of the response occur. In most physically realizable excitors, however, these effects are slight.

**The Downstream Response of the Jet**

The continuity of the displacement and slope at the exit of the exciter determine the disturbance downstream of the exciter, which satisfies equation 2.16 with \(N=0\). By writing the downstream displacement in the form

\[
\hat{u}(x) = \left[ x \left( \partial \hat{u}_0 / \partial x + i k_i \hat{u}_0 \right) \right] (k_i x)^{-1} \sinh k_i x + \hat{u}_0 \cosh k_i x \]  
\[e^{-ik_i x} \quad (2.19)\]

where \(\hat{u}_0\) is the exit displacement and \(\partial \hat{u}_0 / \partial x\) is the exit slope, we can see that it consists of two waves, one excited by the exit displacement and one by the exit velocity.

If \(k_i\) is imaginary, a stationary pattern of waves is set up on the jet. This pattern may be considered as the superposition of two forward traveling waves, the slow wave with a propagation velocity slightly less than the jet velocity, and the fast wave with a propagation velocity slightly greater than the jet velocity. These two waves
interact to give a wave pattern which is stationary in space, in much the same way as two waves with slightly different frequencies interact to give the phenomenon of "beating" in time. If $k$ is real, the fast and slow waves become decaying and growing waves. At some distance from the exciter the decaying wave becomes negligible, and the only disturbance on the jet is an exponentially growing wave.

**Experimental Results**

Two types of measurements were made to test the theory of the exciter. The first was a measurement of the frequencies at which the minima in the response curve occur. To make these extremes evident, especially at the higher frequencies where the response of the exciter is small, an electric field large enough to cause growing waves was applied to the jet downstream of the exciter. The signal frequency was then adjusted until the disturbance reached a minimum and the frequency measured with an electronic counter. The results of this measurement are plotted in figure 2.4. The scatter around the theoretically predicted position of the minima may be partly attributed to the zero slope of the response curve at these points. From these data, we can conclude that the exciter does have the predicted minima in its response curve.

The response of the exciter as a function of frequency was also measured. Photographs of the jet were taken with the aid of a strobotachometer flashing at twice the
2.4. The measured nulls in the exciter response are near the predicted values of $f = \frac{L}{V_0}$. 
frequency of the signal voltage. Measurements of the amplitude of the wave versus the distance along the jet were made from these photographs and the results plotted. These plots, one of which is shown in figure 2.5, indicate that the displacement response is sinusoidal in space, with the wavelength of the sinusoid decreasing as the frequency of the signal increases, as predicted by equation 2.13. From these plots the displacement response of the jet at any particular position may be found.

The values of the displacement were read from the graphs at a distance of 25 cm. from the exciter, and these displacement values were plotted versus frequency to find the frequency response of the exciter. One of these plots, for the conditions

\[ L = 1.4 \times 10^{-2} \text{ m}, \]
\[ R = 1.59 \times 10^{-3} \text{ m}, \]
\[ V_0 = 3.26 \text{ m/sec}, \]
\[ \rho = 10^3 \text{ kg/m} \]

is shown in figure 2.6. Again, there is good agreement between the predictions of the theory and the experimental results.
2.5. As the frequency of the excited wave increases, the beat length of the standing waves decreases.
2.6. The measured frequency response of a circular jet exciter agrees with the prediction of the theory.
Chapter III. Ideal Continuum Feedback

In our model, the feedback will be represented by the controlled voltage source, which can in principle be made proportional to the displacement of the jet at any point or to its derivatives or integrals with respect to space or time. Not all of these combinations will lead to improved stability, and it is our task to ascertain which form of feedback will enable us to stabilize the convective instability most efficiently.

The simplest case is ideal continuum feedback, in which the controlled voltage at each point is proportional to some function of the displacement at this same point. Conceptually, this is just an outgrowth of the effect of a normal electric field on the jet. Usually as the jet moves closer to the electrode, the electric field, which is inversely proportional to the jet-electrode separation, increases. This increase leads to a more negative electric pressure at the surface of the jet, which pulls the jet toward the electrode and tends to decrease the separation still more. If this is the dominant effect in the system, the jet will be unstable.

Here the feedback system described above also leads to a change in the electric pressure as a result of the movement of the jet, but this change is controlled by the nature of the feedback system, which can be adjusted at will. For example, if the feedback were arranged to decrease the electrode voltage as the jet moved toward the
electrode, the electric field, and hence the electric pressure would tend to decrease. This effect can counterbalance the usual destabilizing effect of the electric field or even overpower it, to stabilize the jet.

With ideal continuum feedback, the feedback term in the equation of motion may be written as

$$ P \delta (x) = M(\omega, \partial / \partial x) \hat{U}(x) $$

(3.1)

and the equation of motion, 2.13, takes the form

$$ (1 - \alpha^2) \frac{d^2 \hat{U}}{dx^2} + 2i \omega \frac{d \hat{U}}{dx} - (N+M(\omega, \partial / \partial x) + \omega^2) \hat{U} = 0 $$

(3.2)

Since the equation is homogeneous, the dispersion relation, an equation in $\omega$ and $k$

$$ (1 - \alpha^2) (kL)^2 - 2i \omega (kL) + (N+M(\omega, kL) + \omega^2) = 0 $$

(3.3)

must be satisfied.

The dispersion relation is commonly used in the study of instabilities to determine the behavior of the system. With non-convective instabilities, the boundary conditions are used to fix the wavelength, $2\pi/k$, and the imaginary part of the resulting $\omega$ corresponds to either growth (instability) or decay (stability). For a convective instability, the procedure is similar, but the boundary conditions in this case fix the frequency, $\omega$, and the dispersion relation is solved for $k$. A complex value of $k$ corresponds either to a wave growing in space (amplifying,
convectively unstable) or to a wave decaying in space (evanescent). Since convective stability is the foremost concern here, we will usually fix the frequency and solve for the wavenumber, \( k \). To illustrate the procedure of using feedback control to stabilize a convective instability, we will briefly consider several different types of feedback. The first, and simplest, is a feedback force directly proportional to the disturbance

\[
M(\omega, k) = \text{constant} = M
\]

In this case, the dispersion relation may be solved for \( k \)

\[
k_{\perp} = \frac{\omega + i \sqrt{(N+M)(1-\alpha^2) - (\alpha^\omega)^2}}{1-\alpha^2}
\]  \hspace{1cm} (3.4)

In the absence of feedback (\( M=0 \)), the quantity under the square root is positive for low enough frequencies, so that the propagation constant has an imaginary part. This signifies a disturbance growing in the +x direction which, in this case, is a convective instability.

The simplest way to stabilize this convective instability is to make the feedback term a constant larger in magnitude and opposite in sign to the electric field term,

\[
M = \text{constant} \leftarrow -N
\]

so that the quantity under the radical sign will always be negative, and the disturbances will always be sinusoidal in nature. In effect, we have used the feedback mechanism to
synthesize a physical effect similar to the electric pressure in every respect, except for its direction of action, thus forcing the electric field to contain the jet, rather than destabilize it.

It is also possible to duplicate other physical effects on the jet by means of the continuum feedback if the feedback is allowed to be more general in form. For example, if \( M \sim k^2 \), the feedback term will be identical in form to the surface tension term, and the effect on the jet will be to alter the observed surface tension of the jet. This will decrease the range of instability of the jet, and shorten the wavelength of the stable disturbance.

Although the synthetic electric field has prevented the disturbance on the jet from growing, it still does not decay away in space. This could be achieved by using the feedback to simulate the effect of a viscous force in the jet, which is proportional to

\[
\frac{\partial^2}{\partial x^2} \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) u(x,t)
\]

A feedback gain of the form

\[ M \sim I(kL)^2(\omega - kL) \]

should therefore cause the disturbance to damp out in space. The rate of damping can be controlled by varying the gain of the amplifier.

It is of course possible to simulate any number of physical effects simultaneously by making the feedback
proportional to the sum of the feedbacks necessary to simulate any one effect individually. For example, a very viscous jet with no electric field could be simulated by making the feedback equal to the sum of a constant, to eliminate the destabilizing effects of a normal electric field, plus a viscous type feedback. It is also possible to make a stable disturbance unstable by means of feedback, for example, by effectively removing the surface tension force or simulating a negative viscosity. Another possibility is using a frequency filter in the feedback path to eliminate growing waves over a range of frequencies only, or conversely, to make only some frequencies unstable.

In this work, however, we will limit ourselves to a feedback independent of either the frequency or the wavelength of the disturbance, and investigate in some detail the possibility of synthesizing a positional feedback system by dividing the system into small segments. For each segment the disturbance will be measured at some point within the segment, and a feedback force proportional to the disturbance at this point will be applied over the whole length of the segment. The mathematical description and non-convective stability of this spacewise sampled system is described in the next two chapters, while the possibility of implementing convective stability is investigated in Chapter VI.
Chapter IV  Sampled Feedback: The Response of the Single Station

So far we have not discussed how the continuum control of the jet may be accomplished. One possibility is finding some natural mechanism which will give a force of the desired form on the jet. Since the forces which occur naturally in the system will not usually have the form required, we can try to control the instability by breaking the system into many small sections, measuring the disturbance at some point within each section, and applying an external force controlled by the measured disturbance. This space-sampled feedback will use electric forces to control the instability studied here by breaking the electrodes into many small sections, each one of which will have the same voltage over its entire length. The voltage will be made a function of the displacement of the jet at one particular point within the section, called the sampling point.

Under these conditions it is apparent that if the characteristic length of the disturbance is much larger than the region of space under consideration our system might be a good approximation to the continuum feedback system discussed in the last section. If the characteristic length of the disturbance is much shorter than the sampled region, however, we can expect the control to be of little help, and perhaps even responsible for instability.

In order to understand the dynamics of the system we will consider just one station (sampling and feedback) in
some detail. With sampled feedback, the equation of motion becomes

\[ \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right)^2 u = \alpha \frac{\partial^2 u}{\partial x^2} + Nu + Mu(x=a,t) \]  

(4.1)

where the quantity \( a \) is the position of the sampling point within the section.
A. Boundary Conditions

Before we can solve the equation of motion, we must decide which type of boundary condition to impose. There are basically two types of boundary conditions which can be applied to a flowing one-dimensional system described by a second differential equation of the form of equation 2.1. The most common type, which occurs when disturbances can propagate both upstream and downstream in the system, fixes conditions at two different points in space which then determine the solution at all intermediate points. A well-known example is the acoustic disturbance in an organ pipe stopped at both ends, in which the flow velocity vanishes. If the system is flowing so fast that the waves are both swept downstream, however, as in supersonic flow, this type of boundary condition is no longer valid, and the solution is determined by conditions on the upstream side of the flow only.

Thus there are two possible regimes for the flowing acoustic wave system, and the behavior of the flow as well as the techniques of solution are markedly different in these two regimes. As is well known from fluid mechanics, the behavior of the system is determined by the velocity of the flow relative to its fixed boundaries.\(^{15}\) If this relative velocity is less than the speed of sound in the fluid, the flow is termed subsonic, and the disturbance at any point in the flow is influenced by the disturbances at all other points in the fluid, both upstream and downstream
of the point in question. Physically the explanation of this effect is simple. The disturbances are acoustic and travel at the acoustic wave velocity. In the subsonic case this velocity is greater than the flow velocity, so the disturbance will eventually propagate to all upstream points. In the other flow regime, called supersonic flow, the velocity of the flow relative to the boundaries is greater than the acoustic velocity. Under this condition, the disturbances propagating upstream at the acoustic velocity can not overcome the flow velocity in the downstream direction, and as a result, a disturbance can never be felt upstream of the point at which it originates. The flow in this case is determined only downstream of the boundary conditions.

**Limitations of the Wave Speed Concept**

Although these concepts have been widely applied to simple wave systems, such as supersonic flow, they are not immediately useful with more complicated wave systems, in which these concepts are not as clearly defined. The wave equation which represents the simplest form of wave motion is

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \]  \hspace{1cm} (4.2)

In the last few years, a great deal of work on wave systems which are described by more complicated equations has been performed. With an equation of the form
\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + 2 \nu \frac{\partial u}{\partial t} \quad (4.3) \]

for example, which may describe a situation as simple as a viscously damped wave, it is not immediately apparent what the velocity which separates the subcritical and supercritical regimes of the flow should be. For a disturbance sinusoidal in space and time of the form

\[ u(x,t) = \text{Re} \left[ u e^{i(\omega t - kx)} \right] \quad (4.4) \]

There are at least two velocities which could be used, the phase velocity

\[ v_p = \frac{\omega}{k} = \frac{i\gamma}{k} \pm \sqrt{1 - \frac{\omega^2}{k^2}}^{-1/2} \quad (4.5) \]

and the group velocity

\[ v_g = \frac{\partial \omega}{\partial k} = \pm \left( 1 - \frac{\gamma^2}{k^2} \right)^{-1/2} \quad (4.6) \]

As a first objection to both of these velocities we might point out that both may be complex, and have no physical significance. If we take only the real part of these velocities, we note that both are functions of the frequency (or wavelength) of the disturbance. If either of these were taken as the critical velocity, the disturbances at one frequency would propagate both upstream and downstream, requiring resonance boundary conditions, while disturbances at some other frequency would all be swept downstream. Physically, we do not expect this type of behavior with a slightly viscous fluid. There are examples of wave systems which behave in this manner, however,
although they can not be described by a simple wave equation. A fishing line dragged through still water sets up a characteristic wave pattern in which the short wavelength disturbances are visible upstream of the string, while longer wavelength disturbances can only be seen on the downstream side of the string. In this case, the sub- or supercritical nature of the flow apparently differs for different surface disturbances.

A further difficulty with this criterion is that it is only applicable to wave-like disturbances. In the equation of motion of the controlled jet there is a term representing the feedback force which is a function of time alone, since it is proportional to the disturbance at one particular point, the sampling point. Because the feedback force is applied over the whole length of the section instantaneously, the disturbance in effect can propagate upstream at an infinite velocity. This adds yet another possibility to the selection of wave speeds that may be defined in the system, and complicates the problem even more. Obviously, the rather vague criterion of wave speed is only useful when the system is so simple that all possible definitions of wave speed are identical. For complicated systems, the criterion must be made more definite.

Application of the Theory of Characteristics

We can extricate ourselves from this dilemma by turning to the theory of characteristics. This approach enables us
to apply a simple principle of causality directly to the solution of the equation of motion to determine the appropriate boundary conditions. This principle states that the disturbance at any point in space can only be influenced by disturbances which occurred at earlier times. For ease of solution we can rewrite the equation of motion of the sampled feedback system, equation 4.1, as two first order equations of the form

\[
\frac{\partial u}{\partial t} + (1 + \alpha) \frac{\partial u}{\partial x} = v \tag{4.7a}
\]

\[
\frac{\partial v}{\partial t} + (1 - \alpha) \frac{\partial v}{\partial x} = N u(x,t) + M u(x=a,t) \tag{4.7b}
\]

This set of equations is valid over the entire x-t plane. The left hand terms of the equations represent derivatives along an arbitrary path in the x-t plane. For each of the equations, however, there is a special path along which the terms on the right will be an exact differential. For the first equation, if the differentiation is carried out along the C + characteristic line given by the first characteristic equation

\[
x = (1 + \alpha) t + \text{constant} \tag{4.8}
\]

the expression

\[
\frac{\partial u}{\partial t} + (1 + \alpha) \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} + \frac{dx}{dt} \frac{\partial u}{\partial x}
\]

can be rewritten as du/dt. Along this special line, the equation 4.7a is simplified to a perfect differential
(called the second characteristic equation).

\[
    u - u_0 = \int_{C^+} v(x,t) \, dt
\]  
(4.9)

The line integral is taken along one of the \( C^+ \) family of characteristic lines. The second equation can be treated in exactly the same manner, to yield

\[
    v - v_0 = \int_{C^-} \left[ Mu(x=a,t) + Nu(x,t) \right] \, dt
\]  
(4.10)

which is only valid on the \( C^- \) family of characteristic lines,

\[
    x = (1 - \alpha) t + \text{constant}
\]  
(4.11)

Although these new equations are valid only along very special lines in the \( x-t \) plane, they can now be used to find the disturbance at any point in the plane. Suppose we would like to find the disturbance at the point \( A \), and that at \( t=0 \) the disturbance is specified all along the \( x \)-axis, as shown in figure 4.1. We can cover the whole plane with the two families of characteristic lines given by equations 4.8 and 4.11. The principle of causality states that only disturbances at positive time, \( t>0 \), are determined by conditions at \( t=0 \). The point at \( D \), for instance, is not affected by the conditions at \( t=0 \). To demonstrate the application of the method, we will show how to find the response at point \( A \), by integrating the two equations, 4.8 and 4.10, along their respective characteristic lines. To avoid unnecessary complication we will at first neglect the
4.1. The characteristics through the point A which extend backward in time intersect the boundary conditions at $t = 0$ at the points B and C. The point at D, at previous time, is not affected by these conditions.
feedback term (M=0). Since the integral which gives the change in \( u \) and \( v \) is dependent upon the variation of these quantities along the characteristic lines, we must find \( u \) and \( v \) at all points along the two characteristic lines from the initial conditions at \( t=0 \) to the point A. To find \( u \) and \( v \) along these two lines, however, we must repeat the process of integrating equations 4.8 and 4.10 along all the characteristic lines through the intermediate points, such as \( A' \) and \( A'' \) in figure 4.2. As this process is repeated for all points on the line, it becomes apparent that the disturbance at point A will depend on the disturbance at all points within the triangle ABC, and only on these points. This region is known as the domain of dependence of point A.

In practice we can determine the disturbance at a point A by working out from the line \( t=0 \) in small steps, as indicated in figure 4.3.\(^\text{18} \) To go from the line \( t=0 \), where the disturbance is known, to the first step, we approximate the integral equations by

\[
\Delta u = v \Delta t \quad (4.12a)
\]

\[
\Delta v = \left[ u(x=a,t) + Nu(x,t) \right] \Delta t \quad (4.12b)
\]

Although this process only gives the disturbance at discrete points along a line parallel to the \( x \)-axis, the disturbance at intermediate points along this line can be calculated by interpolation. This step by step process can be repeated to find the disturbance at all points.
4.2. To find the disturbance at A, the disturbance at all points in the domain of dependence (triangle ABC) must be found.
4.3. The disturbance at A can be found by a carrying out the whole solution at small increments in time, starting from the initial conditions.
**Determination of Allowed Boundary Conditions**

This method not only yields a numerical solution, but also reveals where boundary conditions may be applied, not only for this method of solution, but for any method, if that is desired. So far we have been discussing the case in which the characteristics slope both upstream and downstream ($\alpha = 1$). Since one characteristic passes through upstream points and the other through downstream points, the disturbance at $A$ depends on points both upstream and downstream, and the boundary conditions must therefore be applied on either side of the point in question. Suitable boundary conditions in this case might require that $u=0$ at $x=0$ and $x=L$, a typical resonance boundary condition.

**Determination of Allowed Boundary Conditions**

If $\alpha = 1$, however, as is the case if the jet is moving faster than its capillary velocity, the characteristics all have positive slope, as shown in figure 4.4, and the domain of dependence of the point $A$ lies entirely upstream of this point.

Consider a point $A$ so close to the line $x=0$ that both characteristics through the point intersect the line $x=0$ for positive time. As we let the point $A$ approach closer to the $x=0$ axis, we find that the integrated terms in the second characteristic equations, 4.9 and 4.10, vanish. Thus both $u$ and

$$y = \frac{\partial u}{\partial t} + (1 + \alpha) \frac{\partial u}{\partial x}$$
4.4. When the jet is supercritical, only the disturbances at upstream points can affect any point in space, because all points previous in time along either characteristic also lie upstream.
must be continuous at the entrance of the section if $\alpha < 1$, and can therefore be specified externally by the manner of exciting the disturbance at the entrance of the section. Since the quantity $\alpha$ is the ratio squared of the surface tension velocity of the jet to the convective velocity, the condition $\alpha < 1$ states that the surface tension velocity is the critical velocity which separates the convective and non-convective flow regimes, even in the presence of a destabilizing electric field. The critical velocity really depends on the slope of the characteristic lines, and these are only functions of the coefficient of the second derivative in the wave equation. Thus in the example of the viscous wave, the viscous term was of the first order and had no effect on the critical velocity or the allowed boundary conditions.

**Effect of Sampled Feedback on Boundary Conditions**

The addition of a forcing term over the whole length of the section does not add any complications to the above argument if the force is externally controlled, since it merely contributes an extra integral term to the second characteristic equations 4.9, 10. In our case, however, the extra term is a function of the disturbance at some point within the section, which may be downstream of the point of interest. To the normal domain of dependence of the point $A$ must be added the entire domain of dependence of the line $x=a$, as shown in figure 4.5. Since the disturbance at $A$ is now affected by disturbances both upstream and downstream,
4.5. With sampled feedback, the domain of dependence must be expanded to include the domain of dependence of the sampling point.
we must investigate the possibility that the boundary conditions will have to be changed or that the basic nature of the flow will be altered.

We can find the boundary condition required at the entrance by considering the disturbance at a point A very near the entrance line $x=0$. As we let this point approach the $x=0$ axis, the integrated terms again vanish in the second characteristic equations, 4.9 and 4.10, which then state that both $u$ and

$$v = \frac{\partial u}{\partial t} + (1+\alpha) \frac{\partial u}{\partial x}$$

are continuous at the line $x=0$ in figure 4.4, regardless of feedback. Thus it is apparent that the sampled if our theory is to be consistent, the sampled feedback can have no effect on the entrance boundary conditions. In the next section, the actual dynamics due to some initial condition of the jet will be followed. Since we can arrive at a solution consistent with the above boundary conditions and causality, this provides a complete justification of the validity of these conditions.

From another viewpoint, because we have shown that no disturbance can propagate upstream on the jet if $\alpha < 1$, and that the disturbance which is carried upstream by the feedback system has a negligible effect just inside the entrance, the slope and displacement of the jet must be continuous at the entrance.
Summary

The allowed boundary conditions in a system which contains a flowing fluid are usually determined by the ratio of the flow velocity to the wave velocity. When a space-sampled feedback is applied to a convectively unstable jet, this criterion becomes ambiguous since there are several wave velocities which may be used. The theory of characteristics can resolve this difficulty by showing that the critical velocity of the system is the velocity of capillary waves, independent of the electric pressure or sampled feedback.
B. Numerical Solution of the Characteristic Equations

The feedback from one point within the section to another, however, can change the nature of the disturbance within the system even though the external boundary conditions are not affected. We shall first discuss the behavior of the system when a special pulse is applied at the entrance. The pulse, which is of the form

$$u(x=0, t) = te^{-t} \quad (4.13a)$$

$$\frac{\partial u(x=0, t)}{\partial x} = 0 \quad (4.13b)$$

excites a similar disturbance which is swept down the jet. When the feedback is absent this pulse passes through the section under consideration, and the jet returns to its undisturbed equilibrium. When the feedback is included, however, the feedback apparatus will apply a force to the whole length of the jet within the section as the disturbance passes the sampling point. This force tends to increase the disturbance if the feedback is positive or to oppose it if the feedback is negative. Since the force is applied upstream of the sampling point as well as downstream, it excites an additional disturbance which will again pass the sampling point, and again give rise to a force over the whole section. We shall examine the effect of this feedback with the aid of a few examples, using a numerical solution of the characteristic equations which follows the procedure outlined above. In all of these
examples, the sampling point is at the center of the section 
(a=.5).

Positive Feedback

The first example, figure 4.6, shows the effect of positive feedback (M=1). As described above, when the pulse passes the pickup point, the feedback mechanism creates a disturbance over the whole length of the section. The first picture of the jet, at the time when the original pulse is just being swept out of the section (t=1), shows the pulse trailing a wake as a result of the feedback loop. The overshoot at x=.8 is caused by the interaction of the fast and slow waves on the jet, and is present even in the absence of feedback. In the next two pictures, the pulse leaves the section, the trailing wake decays away monotonically with increasing time, and the jet returns to its original undisturbed state.

If the positive feedback is made very large, however, (M=100), the behavior of the jet is radically altered, as shown in figure 4.7. In the first picture the original pulse, which is being swept out of the section, is not even noticeable against the much larger disturbance created by the feedback mechanism. As the time goes on this disturbance does not decay away, as it did before, but grows monotonically in time at every point within the section. From the discussion of Chapter 1, it appears that making the feedback gain too high will result in a non-convective instability of the jet, and it is interesting to note that
4.6. The response of the jet with small positive sampled feedback ($M = 1$) to a short pulse. At $t = 1.0$ the main part of the pulse is swept out of the section, and the response then decays monotonically to zero, indicating non-convective stability.
4.7. The response of the jet to the same input when a large feedback gain exists ($M = 100$). The response grows monotonically, indicating non-convective instability.
although a non-convective instability is present, the boundary conditions at the entrance remain those associated with a convective instability, and in fact, the convective instability can still be present if $N > 0$. In this system we have the unusual possibility of either convective or non-convective instability with the same boundary conditions, which as we have seen depend only on the capillary velocity of the jet.

**Negative Feedback**

A similar behavior can be seen if negative feedback is used. With a small value of negative feedback ($M = -1$), the pulse again trails a wake which oscillates as it decays away, as shown in figure 4.8. This tendency to oscillation is characteristic of most negative feedback systems with inherent time lags. As the negative feedback gain is made higher ($M = -100$), the oscillating wake no longer decays, but grows in time as shown in figure 4.9. In this case the disturbance at any point in the section is an oscillation whose magnitude is continuously increasing, indicating an oscillatory non-convective instability, or overstability.

**Summary**

The results of the numerical solution of the equation of motion have shown that in our attempt to stabilize the convective instability of the jet by using a sampled feedback system, we have introduced the possibility of making the situation even worse by setting up a
4.8. The response of the jet with small negative feedback ($M=-1$) to the same input. The disturbance at any point is a decaying sinusoidal, indicating non-convective stability.
4.9. The response to the same output, with a large negative feedback gain ($M = -100$). The convective growth of the disturbance can be visualized by following a particle of the jet, labeled X.
non-convective instability. The instability can not exist on the jet in the absence of feedback and represents a new mode of behavior for the system. The sampled feedback system can exhibit either convective or non-convective instabilities, a most unusual mode of behavior. If $N$ is finite, the usual convective instability will appear on the jet, while if $M$ is large, the system will exhibit non-convective instability. Since the steady state tacitly assumed in discussion of convective instability can never be reached in the presence of non-convective instability we must ensure non-convective stability before we can hope to control a convective instability.

Physically, the non-convective instability occurs because the feedback system allows a disturbance at a particular point, the sampling point, to influence points both upstream and downstream, a situation usually associated with non-convective systems. This mixture of the two types of behavior might be called, for want of a better name, a mixed convective-non-convective system.
Chapter V  Sampled Feedback: Non-Convective Stability

Although the method of characteristics furnishes a solution for any arbitrary boundary conditions, its application to the problem of convective and non-convective stability is quite unwieldy, requiring a numerical solution of the characteristic equations for each section, and many repetitions of this process if more stations are considered. A much simpler approach is to find an analytic solution to the equation of motion of the sampled feedback system, equation 4.1, which describes the displacement at any point in terms of the displacement and slope at the entrance to the section.

Since we have seen in Chapter IV that the disturbance within any section is determined externally by the condition at the entrance of the section, the analytical solution is straightforward. We can find the disturbance in one section, use the exit conditions of this section as the entrance condition of the next section, and continue one step at a time to solve the entire problem. This "splicing" technique is not as simple for a system of \( n \) sections which requires boundary conditions at both ends of the section, since it leads to a set of \( 2n \) simultaneous equations.\(^{13}\)

The assumption of a steady state sinusoidal response of the form

\[
u(x,t) = \text{Re} \left[ \hat{u}(x) e^{i\omega t} \right] \quad (5.1)\]
reduces the equation of motion to
\[
(1 - \alpha^2) \frac{d^2 \hat{u}}{dx^2} + 2i \omega \frac{d \hat{u}}{dx} - (N + \omega^2) \hat{u} = M \hat{U}(x = a) \tag{5.2}
\]

which has the general solution
\[
\hat{u} = De^{-ik_+x} + Be^{-ik_-x} + C \tag{5.3}
\]

where
\[
k_{\pm L} = \frac{\omega \pm i \sqrt{N(1 - \alpha^2) - (\alpha \omega)^2}}{1 - \alpha^2} \tag{5.4}
\]

As before, the conditions on this solution require that the displacement and slope at the entrance to the section, \(x = 0\), be equal to the externally imposed displacement, \(u_0\), and slope, \(\partial u_0 / \partial x\).

\[
\hat{u}_0 = D + B + C \tag{5.5}
\]

\[
\frac{\partial \hat{u}_0}{\partial x} = -i(k_+ L)D - i(k_- L)B \tag{5.6}
\]

In the analytic solution, however, there are three arbitrary constants to be determined, and only two conditions. The third independent condition is the satisfaction of the equation of motion at the sampling point, \(x = a\).

\[
Me^{-ik_+x} D + Me^{-ik_-x} B - (M + N + \omega^2) C = 0 \tag{5.7}
\]

Since this third requirement is not directly influenced by the external conditions, it did not appear explicitly in the characteristics solution, where the feedback was represented.
by integrated terms in the second characteristic equations, 4.9, 10. The solution of these three equations yields the response in terms of the entrance conditions.

\[
\hat{u}(x) = A_{11} \hat{u}_o + A_{12} \frac{\partial}{\partial x} \hat{u}_o \tag{5.8}
\]

\[
\frac{\partial}{\partial x} \hat{u}_o(x) = A_{21} \hat{u}_o + A_{22} \frac{\partial}{\partial x} \hat{u}_o \tag{5.9}
\]

where

\[
KA_{11} = \left[1 + \frac{M}{N+\omega^2}\right] c(x) e(x) - \frac{M}{N+\omega^2} c(a) e(a) + i k_L \left[\left(1 + \frac{M}{N+\omega^2}\right) s(x) e(a)\right. - \frac{M}{N+\omega^2} s(a) e(x)\left.\right] \tag{5.10}
\]

\[
KA_{12} = \left[1 + \frac{M}{N+\omega^2}\right] s(x) e(x) - \frac{M}{N+\omega^2} \left(s(a) e(a) + s(x-a) e(x+a)\right) \tag{5.11}
\]

\[
KA_{21} = \left[1 + \frac{M}{N+\omega^2}\right] \left[(k_L)^2 + (k_{1L})^2\right] s(x) e(x) \tag{5.12}
\]

\[
KA_{22} = \left[1 + \frac{M}{N+\omega^2}\right] c(x) e(x) - \frac{M}{N+\omega^2} c(x-a) e(x+a) - i k_L \left[\left(1 + \frac{M}{N+\omega^2}\right) s(x) e(x) - \frac{M}{N+\omega^2} s(x-a) e(x+a)\right] \tag{5.13}
\]

where

\[
K = 1 - \frac{M}{N+\omega^2} \left[c(a) + i k_L a s(a)\right] e(a) - 1 \tag{5.14}
\]
and
\[ e(z) = e^{-ik_x z} \]
\[ c(z) = \cosh k_x z \]
\[ s(z) = (\sinh k_x z)/k_x L \]

**Stability of the Transient Solution**

A solution of this form presupposes that a sinusoidal steady state may be reached, which will only be possible if the transient part of the response decays away in time, leaving only the forced solution. In general, the transient solution consists of terms of the form \( e^{i\omega_n t} \) where \( \omega_n \) is the natural frequency given by the poles of the response, which occur at the zeros of the denominator

\[
K = N + \omega^2 + M \left[ \left( c(a) + ik_x a s(a) \right) e(a) \right] - 1 = 0
\]

(5.15)

If one or more of these terms grow in time, corresponding to a negative imaginary part for \( \omega_n \), the system can never attain a sinusoidal steady state, since the disturbance becomes unbounded throughout the section. This unstable transient response corresponds to the non-convective instability that we detected in the last section using a numerical solution of the characteristic equations.

The transient response can be represented in terms of the spatial eigenvalues of the feedback controlled jet.
These eigenvalues take the form

\[ \hat{\omega}(x) = \left[ c(x) + ik_xL \sin(x) \right] e(x) - 1 \]  \hspace{1cm} (5.16)

**The Nature of the Non-Convective Instability**

From physical grounds, the jet will be stable when the feedback is absent. As the feedback gain, \( M \), is increased, we would expect the roots of the stability equation to migrate toward the real axis and eventually to cross it. As soon as one root of the equation crosses the real axis the solution becomes non-convectively unstable. At this critical point the natural frequency has only a real component, \( \omega_n = \omega_c \), and the eigenvalue equation may be separated into its imaginary and real parts

\[ \tan \frac{k_x a}{k_x a} = \tanh \frac{k_x a}{k_x a} \]  \hspace{1cm} (5.17)

\[ -M = \frac{N + \omega^2}{\cosh k_x a \cos k_x a + k_x a \sinh k_x a \sin k_x a - 1} \]  \hspace{1cm} (5.18)

The first equation, 5.17, determines the critical eigenfrequency, \( \omega_c \), which is independent of the feedback gain. This frequency, when substituted into the second equation, 5.18, gives the maximum allowed gain for stable operation, \( M_c \).

The non-convective instability is primarily a result of the phase shift introduced as the disturbance at the sampling point excites an upstream disturbance which requires a finite amount of time to return to the sampling
point. The secondary effects of surface tension and electric pressure, however, modify the critical values of gain and frequency. Because the convection is the dominant physical mechanism, an analysis which neglects surface tension and electric pressure will give a good qualitative picture of the non-convective instability. Under these conditions, \( (N=0, \alpha = 0) \) the critical values of frequency and gain are

\[
\omega_c = 4.49 \\
M_c = -3.61
\]

The real and imaginary parts of the marginally unstable disturbance, calculated from the equation 5.16, are shown in figure 5.1. Both the real and imaginary parts have physical significance, which follows from the definition of the disturbance as

\[
u(x, t) = \text{Re} \left[ \hat{u}(x) e^{i\omega t} \right]
\]

If \( \hat{u} = \hat{u}_r + i\hat{u}_i \), the disturbance may be written

\[
\hat{u}_r(x) \cos \omega t - \hat{u}_i(x) \sin \omega t
\]

showing that the real and imaginary parts represent the disturbance at quarter-period intervals, while for intermediate times, the disturbance is a mixture of the two. At the critical frequency the imaginary part vanishes at the pickup point, \( x=a \), and the real part is the negative reciprocal of the feedback gain. In other words, the feedback loop has a gain of \(-1\) at marginal instability.
5.1. The real and imaginary parts of the marginally unstable eigen-function with no surface wave effects ($\alpha = 0$, $N = 0$).
At the critical point the gain of the feedback loop is $-1$. 
The Effect of Electric Pressure

When an electric field is applied to the jet, the disturbance will grow convectively. The effect of this convective growth on the non-convective instability can be determined by setting $\alpha=0$, and allowing $N$ to approach infinity, corresponding to a large convective growth per section. The frequency equation approaches

$$\tan \frac{\omega_c}{\omega_c} = \tanh \frac{\sqrt{N}}{\sqrt{N}}$$  \hspace{1cm} (5.19)

Since $\tanh \frac{\sqrt{N}}{\sqrt{N}} \to 0$ as $N \to \infty$

the lowest eigenfrequency shifts down to $\pi$. Substitution of this value of $\omega_c$ into equation 5.18 yields for the critical gain

$$M_c \to N e^{-\sqrt{N}}$$

Thus, very little gain can be applied before the system becomes non-convectively unstable. A convective instability in these circumstances would be very difficult to control, but it should be noted that decreasing $L/V_o$ will reduce $N$ (the growth per section) without changing the growth rate of the convective instability. In other words, if the length of interest is split into many short sections, the electric field will not significantly affect the non-convective instability.

The critical eigenfunction for large convective growth ($N=10$), shown in figure 5.2, grows much faster than the
5.2. A large electric pressure \((N = 10)\) causes the real part of the marginally unstable disturbance to grow convectively within the section, reducing the gain necessary for instability.
eigenfunction for N=0. It is this growth, a result of the growing wave inside the section, that limits the allowed feedback gain.

The Effect of Surface Tension

The surface tension of the fluid can stabilize very short disturbances by providing a restoring force. The effect of this restoring force on the non-convective instability can be determined by letting N=0, and allowing \( \alpha \) to vary from 0 to 1. (The value of \( \alpha \) can not exceed unity if the boundary conditions derived above are to remain valid, as required by the results of the characteristics theory.)

With N=0, the eigenvalues are determined by equation 5.17, which becomes

\[
(1-\alpha) \cos \frac{\omega_c}{1-\alpha} - (1+\alpha) \sin \frac{\omega_c}{1-\alpha} = 0
\]

(5.20)

In the limit \( \alpha \to 1 \), the eigenfrequencies are

\[
\omega_c = 2n\pi \quad n = 1, 2, 3, \ldots
\]

indicating that an increase in surface tension (or capillary wave velocity) increases the critical frequency of the system. Substitution of this value into the equation for critical gain yields

\[
M_C = 2\pi^2
\]

in the limit \( \alpha \to 1 \), N=0. This indicates that as the surface tension increases, more feedback can be applied before the
system becomes unstable. This is to be expected, since surface tension has a stabilizing effect on the system.

The critical eigenfunction (figure 5.3) shows the critical disturbance as the sum of two sinusoidal parts with the two wavenumbers

\[ kL = \omega / (1 + \alpha) \]

The short wavelength component is very small and is therefore dominated by the long wavelength component.

The solution of the critical equations for intermediate values of electric pressure with \( \alpha = .1 \), shown in figures 5.4-5, indicates that as the electric pressure is increased the critical frequency decreases monotonically as does the maximum allowed value of the feedback gain.

**Summary**

Thus we have seen that the time lag caused by the motion of the jet will give rise to a non-convective instability if the feedback gain is made too high. Increasing the electric pressure will make the system more unstable and lower the frequency of the oscillation, while increasing the surface tension will make the system more stable and increase the frequency, however.
5.3. A large surface tension ($\alpha = .9$) reduces the growth of the marginally unstable disturbance, thus making the system more stable. The undulations in the real part of the eigenvalue are due to the interaction of the fast and slow waves.
5.4. As the electric pressure increases, the gain necessary for instability decreases, approaching the $Ne^{-\frac{1}{2}N^2}$ as $N$ becomes very large.
5.5. As the electric pressure increases, the frequency of the marginally unstable disturbance decreases approaching $\pi$ at very large $N$. 
The Nyquist Criterion

The Nyquist criterion can also be used to study the non-convective instability of the system. This approach yields not only the stability of the system but also an indication of the margin of stability, since it shows the amount of gain or phase shift tolerable before the system becomes unstable. With the aid of the Nyquist criterion, it is also possible to apply conventional control system techniques to the non-convective instability.

To use the Nyquist criterion we trace out the path of the function derived from $K$, the denominator of the response

$$G(i\omega) = \frac{M}{N+i\omega^2} \left[ \frac{\cosh kia + ikra \sinh kia}{kia} \right] e^{-ikrX} - 1$$

in the complex plane as the frequency varies from 0 to $\infty$. The number of unstable zeros of $G$ is given by the algebraic sum of the encirclements of the -1 point by the $G$ contour and the number of poles of $G$ in the left half of the complex plane. There will be one pole of $G$ in this region due to the term $N+\omega^2$ in the denominator of $G$, but this is a result of the normalization of the response and is canceled by a similar pole in the numerator of the transfer response. The Nyquist plot yields the same conditions on $\omega^2$ and $M$ for critical operation as the previous equations 5.17-18.

An Example of Nyquist Techniques

We will now show by an example how the Nyquist plot can be used in the improvement of a control system. A typical Nyquist plot ($N=.001, \alpha=.1$) shown in figure 5.6, spirals
5.6. The Nyquist Plot shows how the addition of a low pass filter can help to stabilize the system.
into the origin, crossing the negative real axis an infinite number of times in the process. Although each of these crossings represents a possible instability, the first crossing which is the most negative, will place the limit on the amount of feedback gain allowable for stable operation. If we can force the first crossing point closer to the origin by an appropriate filter, the allowed gain for stable operation will increase, and the control of the convective instability will be improved. In the example chosen, the cutoff frequency for the convective instability is well below the critical frequency of the feedback loop, so a low pass filter which passes signals at the frequencies of the convective instability, but attenuates signals near the critical frequency should help to stabilize the system. In figure 5.6, a low pass filter of the form

\[ F(i\omega) = (1+2.5i\omega) \]

has been added to the feedback loop, increasing the allowed gain from \( M=-3.61 \) to \( M=-15.5 \). The filter also introduces a phase lag into the feedback system, in addition to the inherent phase lag, which reduces the critical frequency from \( \omega_c = 4.45 \) to \( \omega_c = 2.54 \). Thus the Nyquist plot can provide a quick insight into the effects of various control schemes in the non-convective instability of the system.
Chapter VI  Sampled Feedback: Convective Stability

In the traditional dispersion relation approach to convective stability, the disturbance in the linear system is described by an exponential of the form

\[ \text{Re} \left[ e^{i(\omega t-kx)} \right] \]

When this function is substituted into the equation of motion, each term of the equation contains a function of the form \( e^{i(\omega t-kx)} \) which may be cancelled out, leaving an algebraic equation in \( \omega \) and \( k \) to be satisfied if the assumed solution is valid. The boundary conditions on a convective system allow us to specify the frequency in this equation, called the dispersion relation, and then solve the equation for the allowed values of \( k \). The imaginary part of \( k \), when substituted into the assumed solution, tells us whether the disturbance grows or decays in space.

In the sampled feedback system this technique is not possible, since the feedback term in the equation of motion is independent of space. If a solution of the form \( \text{Re} \ e^{i(\omega t-kx)} \) is substituted into this equation the term \( e^{i(\omega t-kx)} \) cannot be cancelled from the equation, indicating that the assumed spatial dependence of the solution is not valid. This difficulty is eliminated by adding a particular solution which is a function of time, to the homogeneous solution, which satisfies the equation with no feedback (\( M=0 \)). This was the procedure followed in Chapter V.
When this solution of motion is substituted into the equation we find that the same dispersion relation must be satisfied by the frequency, $\omega$, and $k$, but this dispersion relation characterizes only the homogeneous part of the solution. For this reason the dispersion relation can not be used to determine the stability of the system with feedback. If, in the absence of feedback, the dispersion relation indicates convective instability, it will give the same indication when the feedback is applied. The feedback system must then furnish a particular solution which will counteract the growth of the homogeneous solution within the section.

It might be thought that the convective stability could be determined from the ratio of the slope and displacement at the exit to the corresponding quantities at the entrance of the section. A simple example, however, will show that this criterion is ambiguous. In general, both fast and slow waves will be excited on the jet, and these two waves will "beat" against each other to give the standing wave pattern, described in Chapter II, in the absence of both electric pressure and feedback. Depending on where the section is placed with respect to the standing wave pattern, the slope or displacement ratio would indicate either growth or decay, despite the fact that the wave is neutrally stable.

This simple criterion, therefore, is not immediately useful, but it can be the basis of a practical criterion in which we work through the system one station at a time, to
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This simple criterion, therefore, is not immediately useful, but it can be the basis of a practical criterion in which we work through the system one station at a time, to
determine whether the disturbance grows at large distances.

**Concept of Finite Station Convective Stability**

When we are dealing with a small number of stations, it is possible to determine the convective stability in the absence of feedback from the displacement at the exit if the slope and displacement at the entrance are known. Consider the behavior of the jet when no feedback is applied. At the entrance of the section two boundary conditions must be satisfied, as we have seen in Chapter IV. If the disturbance has zero displacement and unit slope at the entrance, it will take the form \((\sinh k_i x/k_i x) e^{ik_i x}\) inside the section. At the point of neutral stability, \(k_i=0\), the disturbance at the exit will have a magnitude of unity. If the magnitude is greater than unity, the jet is convectively unstable, and conversely, if the magnitude is less than unity, the jet is convectively stable.

With the addition of feedback, this criterion is no longer strictly true, since the disturbance inside the section is no longer described by a simple hyperbolic sine function, but by the complicated expression of equations 5,8,9. As a straightforward and simple approach, however, we can use as a stability criterion the condition that the displacement at the exit be unity when a unit slope is applied at the entrance.

A similar procedure can be used if the disturbance at the entrance has a unit displacement and zero slope. Under these conditions the disturbance inside the section takes
the form \( \cosh k_x x e^{-ik_x x} \) in the absence of feedback. With marginal convective stability, the disturbance at the exit will again have the magnitude unity, and we can again take as a stability criterion the condition that the magnitude of the displacement at the exit be unity with feedback applied.

**The Effect of Feedback**

The disturbance at the entrance will in general have Fourier components of all frequencies, so we must require that the jet be convectively stable for all frequencies, a condition which is analogous to requiring that a non-convective system be stable for all wavelengths.

We can gain some physical insight into the operation of the control system at different frequencies if we start with an uncontrolled jet (\( M=0 \)), add a small amount of feedback (we shall see later that \( M \) is small enough), and calculate the resulting change in the displacement at the exit of the system from equations 5.8,9. This relation between \( \partial |u| / \partial M \) and \( M \) at \( M=0 \) is shown in figure 6.1. This curve, calculated by taking the derivative of the response numerically, shows that at very low frequencies the feedback system should act to reduce the convective growth by decreasing the exit displacement magnitude. As the frequency increases, however, the effect of the feedback becomes less helpful, and finally, at \( \omega = 6.5 \), \( \partial |u| / \partial M \) changes direction and exerts the opposite of the desired effect. This increase in exit displacement, apparently a destabilizing effect, is caused by the feedback loop, which at these frequencies is
6.1. At low frequencies the feedback reduces the displacement at the exit of a single station. At higher frequencies the feedback alternately increases and decreases the exit displacement.
operating with 180° phase shift, and is the same effect which led to the non-convective instability discussed in Chapter V.

As the frequency is raised still further the effect of the feedback alternates from stabilizing to destabilizing, but with continually decreasing amplitude. This is also apparent from the Nyquist plot of figure 5.6, which shows a continuously decreasing response as the frequency is increased. The \( \partial |u|/\partial M \) curve is practically independent of \( N \) for \( N \ll 1 \), since it is determined primarily by the exciter response of the section.

From this curve of \( \partial |u|/\partial M \) versus we may find some conditions which must be satisfied if the feedback system is to provide effective control of the convective instability. The first requirement is that the system be stable in the absence of feedback throughout the frequency range where the function \( \partial |u|/\partial M \) is positive. In this range, application of the feedback will always cause the growth of the disturbance, and if the disturbance is initially unstable, the instability becomes even greater. In the liquid jet, where stability at high frequencies is furnished by the surface tension, the growth in the absence of feedback is described by the imaginary part of the growth constant given by equation 2.15. At the point of marginal instability, \( k_{m}^{2} = 0 \), or

\[
N = (\alpha_{m})^2 / (1 - \alpha^2)
\]
As a necessary condition that the jet can be controlled, we must require that

\[ N < \alpha_0^2 / (1 - \alpha^2) \]  \hspace{1cm} (6.1)

where \( \alpha_0 \) is the lowest frequency at which \( \partial |u| / \partial M \) is positive.

At very low frequencies, \( k_L = N \), which is greater than zero, and the uncontrolled jet will be unstable. The feedback system must then cause enough decrease in the exit displacement to counteract the unstable growth due to the electric pressure if the jet is to be stable.

The Location of Stable Operating Regions

These stability requirements can be placed on a firmer footing by an exact calculation using the response equations 5,8,9. If values for \( N, \alpha \), and \( g \) are chosen, the value of feedback gain, \( M \), needed to bring the magnitude of the exit displacement to unity can be calculated by trial and error for every frequency. A plot of these critical values of \( M \) versus frequency for a typical stable situation is shown in figure 6.2. From \( \omega = 0 \) to the cutoff frequency of the jet, the gain must be greater than the critical value indicated to overcome the usual convective instability of the jet. From the cutoff frequency to the frequency at which \( \partial |u| / \partial M \) becomes positive, the system is stable for all negative values of gain. Above this range, \( \partial |u| / \partial M \) becomes negative, thus making the jet less stable.
6.2. For stability at all frequencies, the gain must be in the range between the least value needed to stabilize the usual convective instability and largest value which does not induce instability near $\omega = \omega_c$. 
If the gain is greater than the critical value indicated in the figure, the system becomes convectively unstable in the frequency range near $\omega = \omega_c$. As the frequency increases to infinity, these regions of stability and instability alternate, but the gain necessary for instability becomes greater as the frequency rises. Therefore, for convective stability at all frequencies we need only require that the system be stable at frequencies near $\omega = 0$ and $\omega = \omega_c$, since these represent the two most unstable frequency ranges of the system.

This figure shows that there is a band of feedback gain in which disturbances at all frequencies can be controlled. If this diagram is recalculated for all values of $N$, we can map out a stable region of operation in the $M$-$N$ plane for particular values of $\alpha$ and $\alpha$. The stable region for the conditions $a = .5$, $\alpha = .05$, which corresponds closely to the values obtained in the experiments of Chapter VII, is shown in figure 6.3. The lower limit represents the gain necessary to counteract the low frequency growth, while the upper limit represents the gain necessary to make the high frequency disturbance unstable. Operation at any point within the stable area will give apparent stability for one station.

The Response for Large Gain

In the work so far, the assumption that the effect of the feedback gain can be described by a straight line with slope $\alpha \frac{\Delta M}{\Delta}$ has led to predictions which are in accord
6.3. As $N$ increases, the stable range of $M$ becomes smaller. Stable operation is not possible above $N = .11$ for $\alpha = .05$. 
with the results of exact calculations of the stable regions of operation, in which the threshold gain was on the order of \( M=1 \). To determine the behavior of the system at higher values of gain, we can calculate the relation between the exit displacement magnitude, \(|u|\), and gain, \( M \), for frequencies in the two critical ranges near \( \omega = 0 \) and \( \omega = \omega_c \) (figure 6.4).

At very low frequencies the magnitude decreases as the feedback gain is increased until the gain reaches a value in the neighborhood of \( M=9 \). Above this point the magnitude of the displacement increases. Physically, as the gain is increased the exit displacement continues to decrease until it becomes very small. As the gain is increased still more, the displacement at the exit becomes negative. The effect of this phenomenon on the convective stability can be clearly seen in the limiting case in which \( N \) and \( \alpha \) are negligibly small. Under these conditions, the displacement is proportional to

\[
  u \sim x - \gamma x^2
\]

where \( \gamma \) is a constant proportional to the gain of the feedback loop. The slope of the jet at the exit when \( N=\alpha=0 \) is given by

\[
  \frac{\partial u}{\partial x} = 1 - 2\gamma x
\]

At the point where the exit displacement vanishes, \( \gamma = 1 \), and the exit slope is \(-1\). If a second station is added, the
6.4. As $M$ is increased at low frequencies, the exit displacement decreases until $M = 9$ then increases, indicating convective overstability. At high frequencies the response increases, reaching a pole of response at $M = 13.4$ which corresponds to the non-convective instability.
slope at its exit will again reverse, being equal to +1. This process will be repeated indefinitely as more stations are added, the the disturbance will have a constant amplitude along the jet. If the gain is raised above the value corresponding to $\gamma=1$, however, the exit slope will be greater than the entrance slope, and as more stations are added, the magnitude of the slope, and hence of the displacement will become greater. This then is an additional type of convective instability possible on the jet, which might be called a convective overstability, since it continually overcorrects for the detected disturbance.

Although the possibility of the convective overstability can be seen from this discussion, the exact value of $M$ for instability can not be determined easily from the finite station considerations. For this reason, further discussion of convective overstability will be deferred until the infinite station stability criterion is introduced.

At frequencies in the range $\omega=\omega_c$, increasing the feedback increases the exit displacement monotonically, reaching infinity at the gain $N_c$. This brings out the relation between the convective instability induced near a pole of the response, and the non-convective instability which corresponds to the pole itself. For this reason, the instability near $\omega=\omega_c$ will be called a resonant convective instability.
Other Effects

As the surface tension increases the inherent stability of the jet also increases, and the allowed gain for resonant convective instability increases. Figure 6.5 shows the stable regions of operation for $\alpha = 0.05$ and $\alpha = 0.1$. The gain necessary to overcome the usual convective instability is little affected by surface tension if the cutoff frequency of the instability is lower than the frequency at which $\Im \phi$ changes sign.

As more stations are added, the apparent stability diagrams can be constructed in a similar manner, merely by repeating the operation of finding the displacement at the exit in terms of the entrance conditions, and requiring that the output be that associated with neutral stability. The stability curves for 1, 2, and 4 stations are shown in figure 6.6. As more stations are added, the changes in the stability diagram becomes smaller, suggesting that stability is independent of the number of stations if there are a large number of stations.

Infinite Station Convective Stability

If we knew the response at every point on the jet, and not just within a single section, it would be a simple matter to determine the convective stability of the system. We merely take the limit of the response to an arbitrary input as the distance approaches infinity. If this response is bounded, the system is convectively stable; if it grows without bound the system is convectively unstable. This
6.5. As α increases, the stable range in the M-N plane also increases.
6.6. As more stations are added, the stable region in the M-N plane appears to approach the limit given by the infinite station stability criterion.
approach is conceptually simple, but requires an inordinate amount of calculation, since after the response at the exit of the first station is calculated in terms of the entrance disturbance, the procedure must be repeated for the second and all succeeding stations, until the nature of the response becomes clear.

To lighten the computational load, we can modify this procedure. The displacement and slope of the jet at the exit of the first control section can be expressed in terms of the displacement and slope at the entrance as

$$
\begin{bmatrix}
  u_1 \\
  \frac{\partial u \partial x}
\end{bmatrix}
= \begin{bmatrix}
  A_{11} & A_{12} \\
  A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
  u_0 \\
  \frac{\partial u \partial x}_0
\end{bmatrix}
$$

(6.4)
in general. Similarly, the displacement and slope at the exit of the second stage can be expressed as

$$\begin{bmatrix} u_2 \\ \partial u_2 / \partial x \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ \partial u_1 / \partial x \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix}^2 \begin{bmatrix} u_0 \\ \partial u_0 / \partial x \end{bmatrix}$$ (6.5)

and continuing in this fashion, the response at the exit of the nth stage can be expressed as

$$\begin{bmatrix} u_n \\ \partial u_n / \partial x \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix}^n \begin{bmatrix} u_0 \\ \partial u_0 / \partial x \end{bmatrix}$$ (6.6)

We shall call the $A_{ij}$ the response matrix, and it must satisfy the relations

$$A_{21} = \partial A_{11} / \partial x \quad A_{22} = \partial A_{12} / \partial x$$

by the definition of $\partial u / \partial x$.

Now we must determine the response as $n$ approaches infinity. For the equation above this is a somewhat difficult task, but if we first transform the equation into the canonical form

$$\begin{bmatrix} u_n^* \\ u_n^{**} \end{bmatrix} = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}^n \begin{bmatrix} u_0^* \\ u_0^{**} \end{bmatrix}$$ (6.7)

our task is lightened. In these equations $u^*$ and $u^{**}$ are the eigenvectors and $q_1$ and $q_2$ are the eigenvalues of the response matrix, given by the equation
\[
\begin{bmatrix}
A_{11} - q & A_{12} \\
A_{21} & A_{22} - q
\end{bmatrix} = 0
\]  

(6.8)

Now we can determine the limiting form of the response easily, since, for a diagonal matrix

\[
\begin{bmatrix}
q_1 & 0 \\
0 & q_2
\end{bmatrix}^n = \begin{bmatrix}
q_1^n & 0 \\
0 & q_2^n
\end{bmatrix}
\]  

(6.9)

Thus, if the magnitude of either eigenvalue is greater than unity, the limiting response, which is proportional to the eigenvalue raised to a very large power, will be unbounded. If the magnitudes of both eigenvalues are less than unity, the response will approach zero as the number of stations is increased, and the disturbance will die away. Finally, if the magnitude of the eigenvalue is exactly unity, the magnitude of the response will remain constant. The magnitude of the largest eigenvalue can then be considered as the index of performance of the control system, and the smaller this magnitude, the more quickly will disturbances be damped out. The condition for stability of the jet is

\[ |q_1| < 1 \text{ and } |q_2| < 1. \]

Before we evaluate the stable regions of operation of the infinite number of stations, we can gain some insight into the physical meaning of this criterion by applying it to an uncontrolled jet excited by a constant disturbance. The response matrix under these conditions is
\[
\begin{bmatrix}
\cosh k_{1}L & \sinh k_{1}L/k_{1}L \\
k_{1}L \sinh k_{1}L & \cosh k_{1}L
\end{bmatrix}
\]

Diagonalized, this matrix may be written
\[
\begin{bmatrix}
e^{k_{1}L} & 0 \\
0 & e^{-k_{1}L}
\end{bmatrix}
\]

This represents the growing and decaying waves normally found on the jet. For a real value of \(k_{1}L\), one of the eigenvalues will be greater than unity, indicating convective instability. Thus this new criterion is identical to the usual convective stability criterion when the feedback may be neglected.

**The Infinite Station Stable Regions**

Now that a criterion for the convective stability of an infinite station system has been formulated, we can find the stable region of operation corresponding to the finite station stability regions just discussed. To delimit this stable range, we choose values of \(\alpha\), \(N\) and \(M\), and evaluate the eigenvalues of the response matrix for all frequencies. If none of the eigenvalues has a magnitude greater than unity, the chosen operating point lies in the stable region. By repeating this procedure at different operating point, we can map out a stable operating region in the \(M-N\) plane for each value of \(\alpha\).

The stable region of operation determined in this manner for \(\alpha=.05\) is almost identical to that for 4 stations, shown in figure 5.6.
The one or two station response does not always give this close an indication of the convective stability of a very long system, however. One example of this failure is the convective overstability mentioned before, in which the exit displacement of a single station is close to zero at the point of marginal instability. Although the increase in exit magnitude due to this effect many not show up for many stations, it is indicated immediately when the eigenvalues of the response matrix are calculated.

The threshold gain for convective overstability is shown in figure 6.7 for $\alpha = .1$. Since this instability is due primarily to the response of the jet to controlled excitation, and not to any wave effects, it is relatively independent of $\alpha$, in contrast to the resonant instability in which the threshold gain is roughly given by

$$M \sim \alpha^2$$

for $\alpha < .1$. As a result, at high values of $\alpha$, the allowed gain for stable operation may be limited by the convective overstability and not by the resonant instability. This effect can be seen in figure 6.8, which shows the stable operating region for $\alpha = .1$ and $\alpha = .5$. At $\alpha = .1$ the upper limit on $M$ is imposed by the resonant instability, while at $\alpha = .5$, the limit is imposed by the convective overstability for low values of $N$. 
6.7. The complete M-N stability diagram for $\alpha = .1$, showing the threshold gain for non-convective instability, convective overstability, and $\omega = \omega_c$ resonant convective instability, and the gain needed to stabilize the usual convective instability. The completely stable region of operation is in the lower left hand corner.
6.8. As $\alpha$ becomes large the convective overstability sets the upper limit in gain for stable operation.
Entrance and Exit Sampling
In all of the work so far, we have been discussing only
the special cases of sampling at the center of each station.
It is possible to sample at any point within the section,
and two special cases, sampling at the entrance (a=0) and at
the exit (a=1) were considered. In both of these cases, the
stability for the two special cases \( \omega =0 \) and \( N=0 \) was studied
to determine whether stability was possible in the simplest
of cases, before the complicating wave effects are brought
into the discussion.

For entrance sampling (a=0), evaluation of the
eigenvalues of the response matrix shows that the special
case \( N=0, \omega =0 \) in which there is usually no convective
growth, is always destabilized by the application of
feedback. This effect is certainly not apparent from a one
or two station stability diagram, which is similar to the
diagram for center sampling. Some physical justification
can be given by considering a disturbance entering at
constant amplitude. The sampling at the entrance causes a
force proportional to the entrance displacement which tends
to reduce the disturbance. Since this disturbance is
smaller at all points inside the section than at the
entrance, however, the force is slightly greater than
intended. When the disturbance becomes negative, the
opposite effect occurs. The magnitude at the entrance is
smaller than at any point within the section, and the force
is therefore slightly less than intended. Thus the force is
slightly greater when the magnitude is decreasing and
slightly less when increasing. These two effects, alternating as the slope of the disturbance alternates, tend to increases the amplitude of the disturbance, leading to convective overstability. Thus the entrance sampling is unstable for any amount of feedback, even in the absence of electric pressure.

A difficulty also arises when exit sampling is used. Here it is found that the resonant instability at \( \omega = \omega_c \) occurs for all values of \( M \) when \( N=0 \). The physical explanation of this effect is not clear, but it appears that the feedback loop causes a continuous amplification of disturbances at the critical frequency. With center sampling, the same amplification occurs for the first half of the section, but in the second half the system is apparently able to undo this effect.

All of these results assume that only positional feedback is available. The possibility of stable entrance or exit sampling can not be ruled out is more sophisticated detection schemes are available. In this work, however, we will consider only positional feedback, and will therefore be restricted to center sampling.

**Design Techniques**

In the discussion of infinite station stability, several types of instability which have very little influence over a small number of stations have occurred. Since the stability diagram in the \( M-N \) plane for one and infinite station systems are quite similar, it might be
thought that there is no advantage to using many stations. It should be remembered, however, that the stability diagrams are normalized on the basis of a single station. Thus if a disturbance must be controlled over a given length, the largest value of N which can be controlled will be four times as great if the given length is divided into two sections.

For the experimental conditions ($\alpha = 0.05$), we found that N, which is roughly equivalent to the fractional growth per section with $M = 0$, must be less than 0.1 if the system is to be stabilized at all frequencies. In a practical case, however, we might not bother to stabilize a convective instability unless the growth over the length of interest were by a factor of 100, not by 10%. Since growth by a factor of 100 corresponds roughly to $N = 4.5$, based on the total length, the system would have to be divided into $\sqrt[4.5]{1} = 7$ sections to control this growth. Thus if the convective growth is large, it is not possible to use only one or two stations to control the instability, and the effects associated with a large number of stations must be considered in the design of the system.

The importance of the inherent high frequency stability, such as that furnished by the surface tension, can be seen by doubling the surface tension in the above example. Under these new conditions ($\alpha = 0.1$), the controlled growth per section must be less than 0.4, and only 4 stations, instead of the previous 7, are needed to control
the instability. If the high frequency stability were completely absent, of course, it would be impossible to control the disturbance, due to the destabilizing effect of the feedback loop.

Summary

The application of sampled feedback to the system can lead to new types of convective instability. One, resonant instability, is caused by operation near a pole of the response associated with non-convective instability. The other, called convective overstability, is caused by overcorrecting for the disturbance at longer wavelengths. In spite of these new difficulties, it is possible to define a stable region of operation with the aid of an extended convective stability criterion.
Chapter VII   Sampled Feedback: Experiments

A. Introduction

In the system sketched in figure 7.1, the jet leaves the reservoir described in Appendix A, falls through the exciter described in Chapter II, and then through the control region which consists of four different sampling stations and electrodes. These four stations may be connected together to reduce the number of stations or to lengthen the stations. The exciter sets up a disturbance of known frequency and spatial distribution on the downstream jet, and it is this disturbance which we shall attempt to control.

In each of the control sections a 300 W. projection lamp casts a shadow of a short length of the jet onto a phototube. As the jet moves, the edge of its shadow sweeps back and forth across the phototube, whose output is therefore proportional to the transverse displacement of the jet. This signal is amplified to a maximum of 2000 volts peak to peak, added to a variable bias supplied by a floating high voltage power supply, and applied to the electrodes of the section.
7.1. In the experiment, a phototube detects transverse motion of the jet. This signal is amplified and fed back to electrodes along the jet to control the disturbance.
Measurement of Electric Pressure (N)

The electric pressure, N, is a critical parameter in the experiments performed here, since it represents the convective growth in the system. Unfortunately, this parameter cannot be easily calculated, since it depends on the complicated transverse geometry of the system. The electric pressure can be calibrated experimentally in terms of the applied voltage, however, by using the results of Chapter II.

As shown in Chapter II, a disturbance excited on the jet by a sinusoidally varying electric field will take the form

\[ \hat{U}(x) = \frac{\sinh k_x x e^{-i k_x x}}{k_x L} \]  \hspace{1cm} (7.1)

where

\[ k_x L = \frac{\omega}{1 - \alpha^2} \]  \hspace{1cm} (7.2a)

\[ k_x L = \sqrt{N(1 - \alpha^2) - (\alpha \omega)^2} / (1 - \alpha^2) \]  \hspace{1cm} (7.2b)

and x=0 at the exciter. If N=0, this response will be sinusoidal, with a node at the exciter. As N is increased, the wavelength of the sinusoidal response becomes longer, and the amplitude of the response becomes larger, as shown in Figure 7.2, because the increased electric pressure counteracts the surface tension. As the electric pressure is increased to
7.2. As the electric pressure increases, the exit displacement for a constant input slope increases. When this magnitude exceeds unity, the system is convectively unstable.
\[ N = \frac{(\alpha \omega)^2}{(1 - \alpha^2)} \]

the disturbance becomes a straight line in space, and as \( N \) is increased still more, the disturbance changes to a hyperbolic sine, indicating convective growth. Since \( \alpha \) is the surface tension coefficient, and \( \alpha \) can be easily measured on the laboratory jet, we can relate the amplitude or wavelength of the disturbance to the electric pressure, \( N \), as the applied voltage is increased.

It might be noted here that an increase in the amplitude of the disturbance on the jet when an electric field is applied does not necessarily indicate a convective instability. This increase may be caused by the decrease in the effective surface tension at the frequency of the excited signal, or from another viewpoint, by the increase in the beat length, and merely indicates that the jet is "floppier", but not unstable.

As an example of this technique, we will describe the construction of a calibration curve used in the measurements of \( \omega_c \) and \( M_c \) versus \( N \). With a constant amplitude sinusoid applied to the exciter, the steady voltage on the control sections was increased to the breakdown voltage of air and back to zero in small steps. The magnitude of the signal at the exit of the section for each step was plotted against the voltage as shown in figure 7.3.

In Chapter II, it was shown that for a planar or circular jet the constant \( N \), representing the electric pressure, is proportional to \( \rho_0^2 \). Using the Maxwell stress
7.3. As the bias voltage, $\phi_0^2$, is raised, the exit displacement increases. By measuring the change in $\phi_0^2$ required to produce neutral convective stability, we can calibrate $N$ in terms of $\phi_0^2$. 
tensor, it can be shown that in a linear system $N$ is always proportional to $\varrho_0^4$, so that it is only necessary to determine $N$ at two values of $\varrho$ to calibrate the system. In this experiment we choose two special values of $\varrho_0 = 0$ (which corresponds to $N = 0$) and the $\varrho$ at which $N = (\alpha_0^2 / (1 - \alpha^2))$, where $\omega$ is the driving frequency.

At this last value of $N$, $k_4 = 0$, and the response is neutrally stable. Since the response at $N = 0$ is

$$\frac{\sin (\alpha_0^2 / (1 - \alpha^2))}{\alpha_0^2 / (1 - \alpha^2)}$$

we can form the ratio of these two displacements as

$$\frac{u(k_4 = 0)}{u(N = 0)} = \frac{\alpha_0^2 / (1 - \alpha^2)}{\sin (\alpha_0^2 / (1 - \alpha^2))}$$

By reading from the graph of figure 7.3 the value of $\varrho_0$ at which the new value of $u$ is reached, the second point needed for the calibration of $N$ can be found. From the curve shown in figure 7.3, taken at $f = 20$ cps, the calibration equation is

$$N = 0.0197 \varrho_0^2$$

while from a second curve, not shown, which was taken at $f = 35$ cps, the calibration equation is

$$N = 0.0174 \varrho_0^2$$

The average of these two equations was used as the calibration equation for the measurements.
Calibration of the Feedback Gain (M)

The calculation of the feedback gain is subject to greater difficulties than calculation of the electric pressure, since the gain depends on such things as the amount of light falling on the phototube, as well as the electrode geometry. The gain can be determined experimentally, however, by using the theory of the exciter.

As we have seen in Chapter III, the condition for marginal convective instability is that the gain of the feedback loop should be exactly -1. The prediction can be checked quite easily in the laboratory. With the jet unexcited, the feedback gain is increased until the system just breaks into oscillation. The frequency of oscillation is the critical frequency, \( \omega_c \), and the gain at this point is the critical gain, \( N_c \). To measure the open loop gain at this critical point, we break the loop at the point marked X in figure 7.1 and apply a signal at the critical frequency. The resulting signal on the other side of the break should have the same magnitude and the opposite phase of the applied signal, if the loop gain is -1. This proves to be the case, within 3-5\%, when the experiment is carried out.

The gain of the feedback loop may be divided into two parts, the response of the detector and amplifier which applies a voltage proportional to the sampled disturbance to the electrode, and the response of the jet at the sampling point to a voltage applied to the electrodes. The total open loop gain of the system is thus the product of the
individual section gains.

\[ G = \left( M_c \right) \left[ \frac{e(L) + ik_L s(L)}{N + \frac{e(L)}{L^2}} \right] \]

The response of the jet to excitation is determined by the exciter theory of Chapter II, which has been checked independently by direct measurement. Since the total loop gain at the critical point is known to be -1, the effective value of \( M_c \) is determined. Thus to calibrate the gain of the feedback loops, we need only increase the gain of each station until non-convective instability sets in. At this point, \( M_c \) is given by the reciprocal of the jet-electrode gain. A operational amplifier controlled by a decade resistor will then furnish any desired fraction of the gain to the loop.

In Appendix B, it is shown that the gravitational acceleration affects the value of the critical frequency and feedback gain, but that the fractional change in these quantities as the electric pressure or surface tension changes is not strongly affected. Thus it is possible by scaling the values of gain and frequency to neglect the effect of gravity. The experimental calibration of the system achieves the same simplification since the calibrated values of feedback gain and electric pressure include the effect of gravity.
B. Non-Convective Measurements

Since only one station is needed in the non-convective experiment, all the electrodes are connected together with the sampling point at the exit since the non-convective instability is most evident with exit sampling. If the feedback gain is now increased, the system will break into an oscillation at the critical eigenfrequency.

The first measurement investigates the effects of increasing the electric pressure on the critical eigenfrequency and gain. The gain of the feedback loop is increased until the system breaks into oscillation. The frequency of this oscillation, which is about 10 cycles per second, is measured with an electronic counter, normalized to the length of the section and the velocity of the jet at the entrance, and plotted against the current value of \( N \), which is determined from the calibration described above. The results of this measurement, along with the prediction of the theory described in Chapter V, modified by the acceleration of gravity as described in Appendix B, are shown in figure 7.4.

The relative gain of the feedback amplifier at the critical point was read from the setting of the decade resistor in the operational amplifier, multiplied by the bias voltage on the electrodes (\( M \) is proportional to the product of the loop gain and the bias voltage, \( V_o \)), and plotted versus \( N \) in figure 7.5. Since the absolute magnitude of the loop gain is not known, the experimental
7.4. As $N$ increases, the eigenfrequency for neutral non-convective stability decreases, as expected.
7.5. As the electric pressure ($N$) increases, the measured threshold gain for non-convective instability decreases as expected. (Since the exact magnitude of $M$ cannot be measured, the results have been scaled.)
results have been scaled to give a good fit with the theoretical curve. With this scaling, the measured variation of the critical gain with electric pressure gives good agreement with the predicted variation.
C. Convective Stability Experiments

Four experimental results with center sampling are presented here to validate the theory and justify the experimental procedure. The first quantity measured was $|u|$, the magnitude of the displacement at the exit, versus the feedback gain, $M$, for several frequencies at a constant electric pressure, $N$. The results of this measurement indicate that the effect of feedback on the exit displacement is well modeled by the theory of Chapter VI. The $|u|$ versus $M$ curve is practically a straight line for $M < 1$. Under these conditions the gain needed for stability can be calculated directly from a measurement of the slope of the $|u|$ versus $M$ curve, $\partial |u| / \partial M$, obviating the need for the exact value of $|u|$ for all $M$.

The second measurement is that of $\partial |u| / \partial M$ versus for several different values of $N$. The results of this measurement confirm the theoretical prediction that $\partial |u| / \partial M$ is independent of $N$ when $N < 1$, and therefore eliminates the need for a separate measurement of $\partial |u| / \partial M$ for each value of $N$.

The third set of results shows the stability region for both one and two station systems. These regions were calculated from the exciter response of the jet and the measured curve of $\partial |u| / \partial M$ versus $\omega$.

Finally, the one and two station stability regions are plotted with the theoretical limit of infinite stations to show that the experimental stability approaches the
theoretical limit as the number of stations increases. The details of these experiments are given in the next few pages.

Method

The experimental study of the convective stability can of course only be carried out on a finite number of stations, so the theory of the infinite station convective stability can only be checked as an apparent limit of a finite system. However, since the infinite station stability theory is merely an algebraic extension of the single station theory, we would expect that if this theory predicts the performance of a one or two station controller satisfactorily it should be equally valid as more stations are added, particularly if the stability regions quickly approach the theoretical limit as the number of stations increases.

If a procedure similar to that used in the theoretical discussion of Chapter VI is used, only the variation of the magnitude of the displacement magnitude as a function of M must be measured to determine the convective stability of a one or two station system. Once the effect of M on the exit displacement is determined, the gain needed to give neutral stability can be easily calculated from the known response of the jet to excitation.

An exciter, similar to that described in Chapter II, at the entrance of the controlled region of the jet sets up a disturbance which enters the control system with a
negligible displacement but finite slope. With an electric field applied, this disturbance grows in space, taking the form \( \frac{\sinh k_iL}{k_iL} \) as described in Chapter II. So far the behavior of the system is known to be well described by the model used here.

In experimental studies of instability, the usual procedure is to slowly decrease the stability of the system, in this case by slowly increasing the electric field, until finally a single frequency in the noise spectrum becomes unstable. With a convective instability, an additional approach is available. By exciting a disturbance on the jet at a single frequency, we can study the behavior of a known signal, rather than noise, since we can easily excite a signal so large that even if there are very unstable frequencies in the noise, the excited disturbance will still dominate over the noise disturbance, at least for the length of the apparatus.

The analogous approach in a non-convective system would be to excite a disturbance at a single wavelength. It is, of course, possible to eliminate disturbances whose wavelength is longer than some desired value by suitably restricting the dimensions of the system, but it is not possible to eliminate very short wavelengths in this manner in a non-convective system.

Strictly speaking, the convective stability of the system should be determined for all frequencies, but the theoretical work of Chapter VI indicates that the disturbance
in one of two frequency ranges is always the most unstable. At very low frequencies, the usual convective instability grows most rapidly if the feedback gain is too low, and the convective overstability is most pronounced if the feedback gain is too high. At higher frequencies in the vicinity of the lowest critical frequency of the feedback system, there is a tendency to amplify normally stable disturbances if the feedback gain is made too high. Thus in theory, it is only necessary to determine the stability in these two frequency ranges to specify the stability of the system at all frequencies experimentally. Before this concept can be used, however, it must be checked.

**Measurement of Exit Displacement versus Gain**

A first step in determining the stable operating region is to determine the amplitude of the exit disturbance for all gain at several frequencies. The theory of Chapter VI indicates that there are two frequency ranges in which the behavior of the controlled system is vitally important. At very low frequencies, where the uncontrolled system is unstable, the feedback system must be able to reduce the exit disturbance in order to counteract the unstable growth. At higher frequencies, in the neighborhood of the critical frequency, we expect the feedback to increase the exit displacement, but not so much that the normally stable disturbance becomes larger than unity.

The results of a measurement of $\mu$ versus $M$ for $N=.38$, $a=.5$ (center sampling), and $\alpha=.050$ is shown in figure 7.6.
7.6. As predicted, the feedback increases the exit displacement at high frequencies. At low frequencies, the displacement first decreases, stabilizing the jet, then increases, as the convective overstability sets in. For convective stability, both curves must be below \( u = 1 \).
As expected, at the low frequencies the displacement decreases as the feedback gain is increased until a minimum is reached in the neighborhood of $M=10.6$. Above this value of gain the amplitude again rises, indicating a convective overstability, since the exit slope has a larger magnitude than the entrance slope.

At the higher frequency, the amplitude increases with increasing gain, as expected. When the gain is increased to the point $M=13.3$, non-convective instability sets in, and the exit displacement is very little affected by the entrance conditions. The limits on $M$ for stable operation could be read from this graph by noting the values of $M$ at which the amplitude first becomes less than unity for low frequencies, and the value of $M$ at which the displacement first becomes greater than unity at the higher frequencies. This procedure is complicated, however, by the necessity of measuring the curve of $|u|$ versus $M$ for many frequencies at each value of $N$ and then repeating this measurement for many values of $N$.

**The Slope of the Output-Gain Curve versus Frequency**

We can take advantage of the characteristics of the controller to bring the number of measurements needed to determine the stable region of operation. From the first experimental results, figure 7.6, it is apparent that for $M<1$, the change in $|u|$ with increasing $M$ is almost linear at both high and low frequencies. Thus we need only measure the change in $|u|$ for an arbitrary change in $M$ to determine
the slope of the \(|u|\) versus \(M\) curve under any particular operating condition \((N, \alpha\) and \(a)\). The measured values of \(\frac{\partial|u|}{\partial M}\) can then be used to calculate the gain necessary to reduce the disturbance to the marginally stable point.

In the discussion of \(\frac{\partial|u|}{\partial M}\) in Chapter VI it was mentioned that this quantity is essentially independent of \(N\) for \(N < 1\), because \(\frac{\partial|u|}{\partial M}\) is principally determined by the behavior of the single section operated as a controlled exciter. The results of the measurement of \(\frac{\partial|u|}{\partial M}\) versus for several different values of \(N\), shown in figure 7.7, bears out this expectation, since the difference between points taken at different values of \(N\) is slight. The curve shows a good agreement between the theoretical and experimental values of \(\frac{\partial|u|}{\partial M}\) except for the range below the dimensionless frequency \(\omega = 3\), which corresponds to a physical frequency of 9 cps. Below this frequency the low frequency gain of the electronic amplifier starts to fall off, and the feedback control therefore becomes less effective. (This effect can be avoided by using an amplifier which passed all frequencies down to zero, instead of the RC coupled amplifiers used in this experiment. For practical control of this instability a zero-frequency amplifier is a necessity).

The Stability Region in the Gain-Electric Pressure Plane

Now that we have shown experimentally that \(\frac{\partial|u|}{\partial M}\) is essentially independent of \(M\) and \(N\) in the region in which this particular system operates, we need only measure the
7.7. The curve of $\frac{\partial|u|}{\partial M}$ versus $\omega$, which indicates the effect of feedback on the exit displacement, is practically independent of $N$, as expected. (The low frequency cutoff of the electronic amplifiers reduces the effective gain of the feedback loop below $f = 9$ cps $\omega = 3$).
relation at one arbitrary value of \( N \) to be able to calculate the gain necessary to stabilize the system at any value of \( N \), for the given frequency. The only difference between the experimentally determined stability region and the theoretical stability region of Chapter VI is in the determination of the relation between \( |u| \) and \( M \). In Chapter VI this relation was derived from the response matrix, equations 5.8, 9, while in the experimental work, it is determined by measurement of the change in \( |u| \) caused by a change in \( M \). Except for this difference, the calculation of the stable region of operation is similar to the corresponding theoretical calculation of Chapter VI.

To find the experimental stability region, we assume some value of \( N \), and calculate the magnitude of the response at the exit of the section from equation 7.2 for the range of frequencies around \( \omega = 0 \) and \( \omega = \omega_c \). Using this response we can calculate the fractional change in the exit displacement which corresponds to neutral stability in the absence of feedback (\( M = 0 \)).

\[
\frac{\Delta |u|}{|u|} \bigg|_{\text{needed}} = 1 - \frac{|u|}{1} \quad (7.3)
\]

Then from the measured value of \( \frac{\partial |u|}{\partial M} \) at each frequency, we can calculate the gain needed for apparent neutral stability as

\[
\Delta M \bigg|_{\text{needed}} = \frac{\Delta |u|}{\frac{\partial |u|}{\partial M}} \quad (7.4)
\]
Since the same experimental curve of $\delta w / \delta M$ versus is used for all values of $N$, the resulting stability curve in the $M-N$ plane shows no scatter as $N$ changes, but is displaced from the theoretical lines by an almost constant distance. The neutral stability curve for one and two station systems with slope input determined in this manner are shown in figure 7.8 and 7.9 along with the theoretical prediction.

We expect from Chapter VI that as the number of stations increases, the stability region will approach the stability region for an infinite number of stations. Figure 7.10, which shows the experiment for one and two station response with the theoretical limit for an infinite number of stations indicates that this is indeed the case.
7.8. The one station stability region, determined from a measurement of $\frac{\partial |u|}{\partial M}$ versus $\omega$, along with the theoretically expected stability region.
7.9. The two station stability region, determined from a measurement of \( \frac{\partial |\mu|}{\partial M} \) versus \( \omega \), along with the theoretically expected stability region.
7.10. As the number of stations increases, the measured stability region approaches that of the infinite station limit.
Appendix A  Formation of the Jet

Requirements

The success of the experiments described here depends to a large extent on the quality of the liquid jet produced. A corrugated jet swaying from side to side will yield erratic results which can not be reproduced, and may short circuit the high voltage electrodes.

There are several requirements which the jet must meet to be acceptable. The surface should not be wrinkled or bumpy, since any small protrusions of the surface will grow in the electric field and induce corona and arcing from the jet to electrode. The low current corona is particularly annoying, since the high voltage system can continue to operate, although the electrode voltage is seriously reduced. As a result, the effective values of M and N can fluctuate erratically in the course of a single measurement, making the whole result meaningless.

As an additional requirement the jet should not be noticeably affected by its naturally unstable axisymmetric mode, in which the jet breaks up into droplets. This breakup will obscure the growth of the kink mode in the electric field, and in extreme cases break the continuity of the jet, which then ceases to be an equipotential surface. To avoid droplike breakup the disturbances with wavelengths in the unstable range of the droplet mode must be extremely small as the jet leaves the exciter.
Finally the jet should not sway from side to side, since the electric field, and hence $N$ and $M$ depend critically of the separation between the jet and the electrode. Since this distance is usually on the order of millimeters, even a very slight swaying will noticeably affect the calibration of the feedback loop.

**Disturbing Influences and Preventive Measures**

An important source of both high frequency noise and low frequency swaying is the vibration of the fluid source (50 gallon barrel) and supporting structure. To reduce disturbances from this source, the entire structure, which is rigidly braced, is supported on a concrete floor by shock absorbers. The barrel is also isolated from the rest of the structure by shock absorbers to provide a steady frame for the formation of the jet.

Within the barrel itself, there are several sources of noise. As the water enters the barrel, it can cause splashing or aeration of the water already present, both of which will disturb the jet noticeably. If the water enters the barrel obliquely, a slow steady swirling is set up in the reservoir and jet.

Because the natural frequency of gravity waves in the reservoir is very low, and the dissipation in these waves is also low, any sloshing set up in the barrel may take many minutes to die away. This sloshing causes the jet to sway and change its velocity as the hydrostatic pressure varies.
If the surface of the reservoir is open to the air, an effect similar to the bathtub vortex appears. As a result of this vortex, the jet changes from solid and circular to hollow and bell-shaped. The hollow center of the jet is a continuation of the hollow center of the vortex in the barrel. Although this phenomenon is interesting in itself, it must be prevented in this experiment.

Finally, any suspended particles in the reservoir can cause premature breakup of the jet at irregular intervals. This breakup appears to be caused by the gross disturbance in the flow as the particle passes through the nozzle at high speed.

All of these sources of disturbance inside the barrel can be effectively counteracted by a system of filtering and settling spaces described below. The water enters the barrel vertically, as shown in figure A.1.1, flows through a wire mesh, and is then deflected radially to the edge of the barrel by a wooden floating baffle, where it flows into the water already in the tank. This procedure reduces the speed of the water as it enters, and also eliminates the swirling.

The water then passes through a settling region, and then through a baffle containing numerous 1/2" holes. After passing through this coarse baffle, the water passes through another settling space, and then through a fine honeycomb mesh of the type used to reduce turbulence in wind tunnels. This mesh is 6" long, with roughened hexagonal holes about 1/8" in diameter. The water finally flows into
A.1.1. The interior of the barrel contains a series of baffles and settling spaces to quiet the flow of the jet.
a third settling region, and then through the nozzle.

In the nozzle itself there are two possible sources of noise. If the Reynolds number of the flow in the nozzle exceeds the critical Reynolds number, the flow at the exit may be turbulent and the jet will not be smoothly formed. The Reynolds number of the jet depends in part on the length of the nozzle, and shortening the nozzle will improve the flow pattern. Due to the large diameter of the jet (1 cm.) it was necessary to use a polished knife edge orifice in the bottom of the barrel. This polished edge also eliminates the second source of noise in the nozzle: the irregularities in the surface which forms the jet.

The jet produced by this method had no disturbance visible to the naked eye as it flowed into a second barrel 6 feet below the nozzle.
Appendix B: The Effect of Acceleration on the Non-Convective Instability

In the experimental work the velocity and radius of the jet are not constant since the jet is falling in the gravitational field. As the jet falls its velocity will increase and, as a result of conservation of mass, its radius will decrease. The changing radius will in turn affect the mass per unit length of the jet, as well as the capillary velocity and the electric field.

The equation of motion of the system, derived in Chapter II on a quasi-one-dimensional basis, must be modified to include the effect of the changing electric field, jet radius, and velocity. If we assume that the change in velocity due to gravitational acceleration is small over a length on the order of the transverse dimensions of the system, the equation of motion can be rederived from the force balance on a very short length of the jet. For the long wave limit of the planar jet, the equation would be

\[ \rho \Delta(x) \left( \frac{\partial}{\partial t} + V_0(x) \frac{\partial}{\partial x} \right)^2 u' = 2T \frac{\partial^2 u'}{\partial x^2} + \frac{\epsilon_p \rho \sigma}{d(x)} \frac{\partial^2 u'}{\partial d(x)^2} \]

(A.2.1)

In this equation \( V_0(x) \), \( \Delta(x) \), and \( d(x) \) now vary along the jet as a result of the acceleration and contraction of the jet as if falls. The form of the velocity is well known,

\[ V_0(x) = V_0(x=0) \left( 1 + \beta x \right)^{1/2} \]

(A.2.2)
where \( \beta = 2gL/V_0 \) is a measure of the change of velocity inside the section due to gravity. The exact transverse geometry factor (\( \Delta(x) \) and \( d(x) \) for the planar jet) however, will depend on the configuration of the jet and electrodes. The general equation of motion of this system, normalized to conditions at the entrance, is

\[
\begin{pmatrix}
1 - \alpha^2(x) \\
2
\end{pmatrix} \frac{d^2 \hat{u}}{dx^2} + 2i \omega(x) \frac{d \hat{u}}{dx} - \left( N(x) + \omega^2(x) \right) \hat{u} = -M(x) \hat{u}(x=a)
\]

(A.2.3)

with the assumption

\[
u = \text{Re} \left[ \hat{u}(x) e^{i \omega t} \right]
\]

(A.2.4)

The function

\[
\omega(x) = \left( \omega' L/V_0(x=0) \right) \left( V_0(x=0)/V_0(x) \right)
\]

(A.2.4)

changes only because the velocity changes, while the functions \( \Delta(x) \), \( M(x) \), and \( N(x) \) are also affected by the change in transverse geometry.

If \( \hat{u} = \hat{u}_r + i \hat{u}_i \), and \( \omega = \omega_c \) is real, as it will be at the point of neutral stability, this equation may be separated into its real and imaginary parts

\[
\begin{pmatrix}
1 - \alpha^2(x) \\
2
\end{pmatrix} \frac{d^2 \hat{u}_r}{dx^2} - 2 \omega_c(x) \frac{d \hat{u}_i}{dx} - \left( N(x) + \omega_c^2(x) \right) \hat{u}_r =
\]

\[+M(x) \hat{u}_r(x=a)
\]

(A.2.5)
\[-140-\]

\[
\left( \frac{d^2}{dx^2} - \alpha^2(x) \right) \hat{u}(x) + 2 \omega_c(x) \frac{d \hat{u}}{dx} - \left[ N(x) + \omega_c^2(x) \right] \hat{u}(x) = 0
\]

\[M_c(x) \hat{u}(x) = 0 \quad (A,2.6)\]

Since \( u_r \) and \( u_i \) are orthogonal in time, both must satisfy the same boundary conditions at \( x=0 \)

\[\hat{u}_r(0) = \hat{u}_i(0) = 0 \quad (A,2.7a)\]

\[\frac{d \hat{u}_r}{dx}(0) = \frac{d \hat{u}_i}{dx}(0) = 0 \quad (A,2.7b)\]

This set of coupled differential equations must now be solved to find the frequency \( \omega_c \) and feedback gain \( M_c \) to give marginal stability. At the point of marginal stability, the solution must satisfy the two conditions at \( x=0 \), equations A.2.7a and A.2.7b, and the further condition that the loop gain is -1. Thus, as we have seen in Chapter V, the imaginary part of the solution will vanish at the sampling point, and the reciprocal of the real part will be the allowed gain, \( M_c \). To carry out the solution, we let \( M=1 \), substitute a value of \( \omega_c \) into the equations, and solve them numerically. If the imaginary part of the solution, \( u_i \), does not vanish at the sampling point, the process is repeated until it does. The current value of \( \omega_c \) will then be the correct one, and the reciprocal of \( u_r \) at the sampling point will be the allowed gain.

Before this solution can be carried out, the exact form the variable coefficients must be known. The form of
\( \omega(x) \) can be determined immediately by the definition, equation A.2.3. The variation of \( N(x) \) and \( \alpha(x) \) for the experimental setup is more difficult to determine since both of these terms depend on the electric field, which is a complicated function of the transverse geometry. However, since the electric pressure and surface tension have little effect on the non-convective instability, as we have seen in Chapter V, these terms may be neglected, because their variation along the jet should have even less effect. The variation in the feedback force, \( M(x) \), can not be neglected, however, since this term is vital to the instability.

Since the feedback term can not be easily calculated, we will assume different variations to determine its effect on the non-convective stability of the system. To begin, we will assume that \( M \) is unaffected by the change in radius, \( (M(x) = \text{constant} = M) \). Under these conditions, only the acceleration of the jet affects the critical frequency and gain. The real part of the eigenfunction on an accelerated jet with \( \beta = 3 \), shown in figure A.2.1 with that for an unaccelerated jet, has a smaller magnitude, since the jet passes through the section more quickly, and is thus influenced less by the feedback force than the unaccelerated jet. The figure also shows that the eigenfunction passes through zero closer to the entrance, because the jet spends more time in the first half of the section than in the last half.
A.2.1. With gravitational acceleration, ($\beta = 3$) the critical eigenfrequency changes sign sooner and is smaller in magnitude than when the acceleration is absent ($\beta = 0$).
The shift in the eigenfrequency, shown in figure A.2.2 normalized to the entrance velocity, increases as the effect of the acceleration, $\beta$, becomes larger. This shift of eigenfrequency can be approximated as due to the change in the transit time of a particle of the jet. For the unaccelerated jet, the normalized eigenfrequency is

$$\omega_c = \omega'_c L/V_0 = 4.49$$ (A.2.8)

Here $L/V_0$ represents the transit time. For an accelerated jet, the transit time is

$$T = (L/V_0(x=0)) \sqrt{1 + \beta x}$$ (A.2.9)

The value of $\omega_c = \omega'_c L/V_0(x=0)$ calculated using this transit time approximation, shown in figure A.2.2, gives good agreement with the exact solution for $\beta<.5$. Since $\beta<.5$ for all the experimental work, this approximation has been used throughout.

If the effective feedback increases along the jet, most of the force will be applied close to the pickup and the frequency of instability, because the effective length of the section is shorter, will increase. Conversely if the effective feedback is highest near the entrance, the section is apparently lengthened and the critical frequency will be lower. In either case, figure A.2.2 indicates that the change in transit time is still the dominant effect on the accelerated jet.
A.2.2. As the acceleration ($\beta$) increases, the critical eigen-frequency increases.
Experiments

The effect of gravitational acceleration on the eigenfrequency is shown in figure A.2.3, determined as described in Chapter VII. In this experiment, $N$ and $\alpha$ are both much less than unity, so that only the change in the transit time of the jet can shift the critical eigenfrequency. The gravitation parameter, $\beta$, is controlled by varying the jet velocity at the entrance to the section. The results indicate that for $\beta = .5$ the transit time approximation is valid. At higher values of $\beta$, the measured eigenfrequency is greater than that predicted by the transit time approximation, suggesting that the feedback force is more effective near the exit.
A.2.3. The experimental points, which are higher than the elapsed time approximation, indicate that the feedback force increases due to the acceleration. For $\beta < .5$ the elapsed time approximation is valid.
Appendix C  Biography

Joseph Michael Crowley was born in Philadelphia, Pennsylvania on September 9, 1940. After graduating from LaSalle College High School in 1958, he entered M.I.T. on a General Motors Scholarship. He received a B.S. degree in Electrical Engineering in 1962, and continued in M.I.T. on an U.S. Atomic Energy Commision Special Fellowship, where he received an M.S. degree in Electrical Engineering in 1963.

He spent several summers as an electrical engineer at the Naval Air Material Center in Philadelphia and taught classes at M.I.T. while studying under the A.E.C. Fellowship.

He is a member of Eta Kappa Nu, Tau Beta Pi, Sigma Xi, and the American Physical Society, and the author of the paper


He is married to the former Barbara Ann Sauerwald of Philadelphia and has one child, Joseph William.
Appendix D  References


16. Reference 2, p. 468
