ACTIVE ELECTROMECHANICAL CONTROL

OF FOURTH-ORDER CONTINUA

by

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ABSTRACT

The stabilization of two continuum systems is analyzed. These systems are related by the fact that they are defined by fourth-order (in space) differential equations.

The first system consists of a current carrying wire, surrounded by an axial magnetic field. Control forces are generated by application of a magnetic field transverse to the wire. This field interacts with the current to produce restoring forces on the wire. The stability is analyzed for the case where the feedback signal is a continuum and also for the case where the effects of spatial sampling are considered. The limits, due to the effects of spatial sampling, on the regime of stability were found.

The second system is a layer of fluid, levitated by the pressure of a gas. The stability is analyzed by use of the dispersion relation. Two types of ideal feedback are applied to the film. The first applies an electrical traction to the top surface. The second applies a traction to the lower surface in proportion to the fluid thickness. This second method is successful and a possible implementation of the control system with a television camera and cathode ray tube is suggested.

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# TABLE OF CONTENTS

Chapter

I. Introduction 1

II. Stabilization of Two Dimensional Motion

2.1.0 Introduction 2
2.1.1 System Dynamics 4
2.1.2 Stability Requirements 11

2.2 Ideal Feedback

2.2.1 Position Feedback 12
2.2.2 Cross Coupling Feedback 16
2.2.3 Comparison of Ideal Feedback Techniques 19

2.3 Finite Sampling Interval Feedback

2.3.1 One Station Average Displacement Feedback 20
2.3.2 One Station Midpoint Displacement Feedback 24
2.3.3 Two Station Midpoint Displacement Feedback 29
2.3.4 Summary and Discussion 36

III. Stabilization of a Levitated Fluid

3.0 Introduction 39
3.1 Fluid Bulk Equation 41
3.2 Electrical Surface Traction 43
3.3 Fluid Pressure Boundary Conditions 47
3.4 Film Dynamics No Electrical Traction 48

3.4.1 Long Wave Limit 50

3.5 Film Dynamics with Top Surface Feedback 51
3.6 Film Dynamics with Lower Surface Feedback 54

3.7 Experimental Results 57
3.8 Suggested Feedback Realization 59
3.9 Summary 61
I. Introduction

This thesis concerns the application of feedback control to two forth-order systems. These two systems have no similarity except the order of their differential equations. Because of this lack of coupling, the thesis is divided into two non-related sections. Each section has its own introduction and summary and is entirely self contained.

The first section, Chapter II, concerns the stabilization of a continuum, which can exhibit motion in two dimensions. This problem is of interest because of the similarity of its instabilities to those which are produced by a plasma in a thermonuclear pinch experiment. This system is illustrated in Figure 1.

The second section, Chapter III, is concerned with the stabilization of instabilities which occur when a layer of liquid is levitated by the pressure of a gas. This problem is of interest because of its application to the manufacture of thin glass. Figure 2 is an illustration of the example which is analyzed.
CHAPTER II

STABILIZATION OF TWO DIMENSIONAL MOTION

2.1.0 Introduction

The system to be studied is illustrated in Figure 1. It consists of a wire which is stretched along the z axis and can be displaced in either the x or y directions. If there are no other forces acting on the spring, the deflections along the x axis will be independent of the y deflections and each would obey the equations of a wire with a one dimensional deflection. To provide coupling between the modes, a current I is carried in the wire and a z directed B field is applied. The interaction of the current and magnetic field will cause deflections on the wire to rotate. This will couple the deflections of the x and y directions and can cause instabilities.

This system is of interest because of the similarity of its instability and the instability exhibited by a plasma in thermonuclear pinch experiments. The mechanics of the two instabilities are probably very different in nature. This is a result of the plasma being considered a perfect conductor, high $R_m$, and the spring being considered to be a zero conductor, low $R_m$. Despite this difference, the instabilities are similar in appearance. It is hoped that stabilization of the wire may provide some insight for the stabilization of the plasma.
To control the wire, control coils would be placed in a manner such that they could create magnetic fields in the x and y directions. By interaction with the z directed current in the wire, tractions in the x and y directions could be created. By connecting these coils in a suitable feedback loop, it would be possible to control some of the instabilities of the wire.

The first analysis is done with the approximation that the control coils are very small compared to the length of the disturbances on the wire. With this assumption, the control coils may be considered to be a continuum and the magnetic field generated by these coils may be independently specified for each point in space. This type of feedback is referred to as ideal feedback. Two different types of generation of the feedback signal are analyzed and compared. One method creates a signal proportional to the displacement. This is referred to as displacement feedback. The second method tries to cancel the coupling between the x and y displacements of the spring which causes the instability. This is referred to as cross-coupling feedback.

The stability of the wire is also analyzed when the size of the control coils is significant compared to the disturbances of the wire. In this case, the control field produced will be a constant value over a section of the wire. The wire is analyzed with one coil, which produces a uniform field over the length of the wire, and with two coils, each of which produces a uniform field over one half of the wire.
Two different feedback signals are coupled to the wire when the control field is uniform over the length of the wire. One signal represents the average displacement and the other signal represents the displacement of the midpoint. When two control sections are placed on the wire, the feedback signal used is proportional to the displacement at the midpoint of each section.

The results of the two ideal feedback systems and the three finite interval feedback systems are compared. These results are used to determine how well the finite feedback approximates the ideal feedback and how much improvement is made by adding additional feedback stations.

2.1.1 System Dynamics

The system to be described is shown in Figure 1. A wire is fastened at both ends along the z axis and it is free to deflect in the x and y directions. This deflection can be described as

\[ \xi(x,t) = u(z,t) \bar{i}_x + v(z,t) \bar{i}_y \]  

(1)
A current $I$ flows in the $z$ direction in the wire and a magnetic field $B_0$ is applied to the wire by an external source. The $x$ and $y$ displacements will each obey the equations which describe one-dimensional deflection of a wire. This is a consequence of the fact that there is no mechanical force which couples the two deflections. These equations are

$$R \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial z^2} + T_x \tag{2}$$

$$R \frac{\partial^2 v}{\partial t^2} = T \frac{\partial^2 v}{\partial z^2} + T_y \tag{3}$$

where $R$ is the mass per unity length, $T$ is the tension and $T_x$ and $T_y$ are the force per unit length acting in the $x$ and $y$ directions respectively.

The coupling which occurs between the two deflections of the wire must be the result of interaction with the magnetic field. Any displacement of the wire will cause a radial current to flow. This current will interact with the $B$ field to produce a force that tries to rotate the wire. To compute this force on the wire the following assumptions are made. The currents which are produced by the motions of the wire in the magnetic field are considered to be negligible. The current in the wire will then have the same direction as a tangent
to the wire. To linear terms this current is

$$I = \vec{i}_x I \frac{\partial u}{\partial z} + \vec{i}_y I \frac{\partial v}{\partial z} + I \vec{i}_z$$

(4)

The applied magnetic field $B_0$ is much larger than the magnetic field produced by the current $I$ flowing in the wire. The magnetic reynolds number of the system is also small so that deflections of the wire won't affect the magnetic field. The force per unit length on the wire is

$$T = I \times B_0 \vec{i}_z$$

(5)

which with Equation 4 is

$$T = B_0 I \frac{\partial v}{\partial z} \vec{i}_x - B_0 I \frac{\partial u}{\partial z} \vec{i}_y$$

(6)

With these forces, equations 2 and 3 become

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{R} \frac{\partial^2 u}{\partial z^2} + \frac{B_0 I}{R} \frac{\partial v}{\partial z}$$

(7)
\[
\frac{\partial^2 v}{\partial t^2} = \frac{T}{R} \frac{\partial^2 v}{\partial z^2} - \frac{B_o I}{R} \frac{\partial u}{\partial z}
\]  
\hspace{3cm} (8)

The solutions of equations 7 and 8 will be waves that propagate on the wire. They are assumed to be of the form

\[
u(z, t) = \text{Re} \ \hat{u} \ \exp j(\omega t - kz)
\]  
\hspace{3cm} (9)

\[
v(z, t) = \text{Re} \ \hat{v} \ \exp j(\omega t - kz)
\]  
\hspace{3cm} (10)

The equations of motion are satisfied by these solutions if

\[
\hat{u}(\omega - \frac{T}{R} k^2) - \hat{v}(jk \frac{B}{R} I) = 0
\]  
\hspace{3cm} (11)

\[
\hat{v}(jk \frac{B}{R} I) + \hat{u}(\omega^2 - \frac{T}{R} k^2) = 0
\]  
\hspace{3cm} (12)

For finite solutions to exist for \(\hat{u}\) and \(\hat{v}\), the determinant of equations 11 and 12 must be zero. This gives the dispersion relation for the waves on the wire

\[
(\omega^2 - \frac{T}{R} k^2) = \pm \frac{k B I}{R}
\]  
\hspace{3cm} (13)
From this dispersion equation, the wave numbers which exist on the wire for any given $\omega$ can be found.

Using the $+$ sign in Equation 13 gives

$$k = k_2, -k_1$$  \hspace{1cm} (14)

and using the $-$ sign gives

$$k = k_1, -k_2$$  \hspace{1cm} (15)

where $k_1$ and $k_2$ are defined by

$$k_1 = \frac{B_0 I}{2T} + \left( \frac{B_0 I}{2T} \right)^2 + \frac{R_0 \omega}{2T} \right)^{1/2}$$ \hspace{1cm} (16)

$$k_2 = -\frac{B_0 I}{2T} + \left( \frac{B_0 I}{2T} \right)^2 + \frac{R_0 \omega}{2T} \right)^{1/2}$$ \hspace{1cm} (17)

It is important that $k_1$ is always greater than $k_2$ and they are related by

$$k_1 = k_2 + \frac{B_0 I}{T}$$ \hspace{1cm} (18)
The dispersion relation, Equation 13, along with Equation 11 or Equation 12, shows that

\[ \hat{u} = \pm j\hat{v} \]  

(19)

where the + sign is used for wave numbers given by Equation 14 and the minus sign is used for wave numbers given by Equation 15. It follows that for a wave whose x displacement is given by

\[ u = \text{Re} \left[ u_+^a e^{j(\omega t-k_x x)} + u_+^b e^{j(\omega t-k_y y)} + u_-^a \exp j(\omega t+k_1 z) + u_-^b \exp j(\omega t+k_2 z) \right] \]  

(20)

the y displacement will be

\[ v = \text{Re} j \left[ + u_+^b \exp j(\omega t-k_1 z) - u_+^b \exp j(\omega t-k_2 z) ight. \\
- \left. u_-^a \exp j(\omega t-k_1 z) + u_-^b \exp j(\omega t+k_2 z) \right] \]  

(21)

These two equations can be rearranged to give

\[ u = \text{Re} e^{j\omega t} \left[ A \sin k_1 z + B \cos k_1 z + C \sin k_2 z + D \cos k_2 z \right] \]  

(22a)
\[ v = \text{Re} \ e^{j\omega t} \left[ -A \cos k_1 z + B \sin k_1 z + C \cos k_2 z - D \sin k_2 z \right] \]  

(22b)

From equations 22a. and 22b., along with the boundary conditions that \( \hat{\nu} \) and \( \hat{\gamma} \) are zero at \( z = 0 \) and \( z = \ell \), the eigenvalues of \( k_1 \) and \( k_2 \) can be found. The boundary condition at \( z = 0 \) determines that

\[ B = -D \]  

(23a)

\[ A = C \]  

(23b)

The remaining two conditions at \( z = \ell \) require that

\[ A(\sin k_1 \ell + \sin k_2 \ell) + B(\cos k_1 \ell - \cos k_2 \ell) = 0 \]  

(24)

\[ A(\cos k_1 \ell - \cos k_2 \ell) - B(\sin k_1 \ell + \sin k_2 \ell) = 0 \]  

(25)

For finite values of \( A \) and \( B \) to exist, the determinant of the coefficients must be zero. This gives the conditions that

\[ \cos \ell(k_1 + k_2) = 1 \]  

(26)
or

\[ k_1 + k_2 = \frac{2n\pi}{\ell} \quad n = 0,1,2 \quad (27) \]

Addition of equations 16 and 17 yields the following expression for the eigenfrequencies.

\[ \omega_n^2 = \frac{T}{4R} \left( (k_1 + k_2)^2 - \left( \frac{B I}{T} \right)^2 \right) \quad (28) \]

which with the eigenvalues for \( k_1 + k_2 \) given by Equation 27 becomes

\[ \omega_n^2 = \frac{T}{4R} \left( \left( \frac{2n\pi}{\ell} \right)^2 - \left( \frac{B I}{T} \right)^2 \right) \quad n = 1,2,3,\ldots \quad (29) \]

The value for \( n = 0 \) has been deleted because it corresponds to a trivial displacement.

### 2.1.2 Stability Requirements

The equations for the eigenfrequencies, Equation 20, shows that application of the magnetic field and current will make the frequency decrease in magnitude. When the magnetic field and current become large enough, \( \omega \) becomes imaginary
and the wire will exhibit a static instability. From Equation 13 it can be seen that the wire will be stable for

\[
\frac{B I}{O T} < k_1
\]

As long as the wave number \( k_1 \) is greater than \( \frac{B I}{O T} \) the system will be stable. By use of Equation 29, this stability criterion may be restated as

\[
\frac{B I}{O T} < \frac{2\pi}{\ell}
\]

where the worst case of \( n = 1 \) has been used.

2.2 Ideal Feedback

2.2.1 Position Feedback

One method of stabilizing the spring displacement, would be to apply a restoring force which is proportional to the displacement of the spring. This type of force would be similar to that of a spring which produces a force proportional to its displacement. The applied force would be of the form

\[
T_x(z,t) = G u(z,t)
\]

(32)
\[ T_y(z,t) = G \nu(z,t) \]  \hspace{1cm} (33)

With the applied feedback forces of equations 36 and 37, the equations of motion are changed from the form of equations 7 and 8 to

\[ \frac{\partial^2 u}{\partial t^2} = \frac{T}{R} \frac{\partial^2 u}{\partial z^2} + \frac{B \Omega}{R} \frac{\partial \nu}{\partial z} - \frac{G}{R} u \]  \hspace{1cm} (34)

\[ \frac{\partial^2 \nu}{\partial t^2} = \frac{T}{R} \frac{\partial^2 \nu}{\partial z^2} - \frac{B \Omega}{R} \frac{\partial u}{\partial z} - \frac{G}{R} \nu \]  \hspace{1cm} (35)

These equations of motion will be satisfied by the complex solutions given by equations 9 and 10 if

\[ \hat{u}(\omega^2 - \frac{T}{R} k^2 - \frac{G}{R}) - \hat{\nu}(jk \frac{B \Omega}{R}) = 0 \]  \hspace{1cm} (36)

\[ \hat{u}(jk \frac{B \Omega}{R}) + \hat{\nu}(\omega^2 - \frac{T}{R} k^2 - \frac{G}{R}) \]  \hspace{1cm} (37)

Again non-trivial solutions exist if the determinant of the coefficients is zero. This yields the dispersion relation of

\[ \omega^2 = k^2 \left[ \frac{T}{R} \pm \frac{B \Omega}{\kappa R} \right] + G/R \]  \hspace{1cm} (38)
Similar to the case of no feedback, two discrete wavenumbers are found. They are

\[ k_1 = \frac{B_0 I}{2T} + \left[ \left( \frac{B_0 I}{2T} \right)^2 + \frac{R}{T} (\omega^2 - \frac{G}{R}) \right]^{1/2} \]  

(39)

\[ k_2 = -\frac{B_0 I}{2T} + \left[ \left( \frac{B_0 I}{2T} \right)^2 + \frac{R}{T} (\omega^2 - \frac{G}{R}) \right]^{1/2} \]  

(40)

These wavenumbers are still related by the relation given in Equation 18.

The system is stable for positive values of \( \omega^2 \). This condition is met when

\[ \frac{G}{R} + \frac{k_1^2 T}{R} > \frac{k_1 B_0 I}{R} \]  

(41)
By defining the normalized variables

\[ N = \frac{B I}{4T} \ell \]  \hspace{1cm} (42a)

\[ Q = k_1 \ell \]  \hspace{1cm} (42b)

\[ M = \frac{G}{T} \ell^2 \]  \hspace{1cm} (42c)

the stability criterion can be charged to

\[ M > Q \left[ 4N - Q \right] \]  \hspace{1cm} (43)

Equation 27 allows this to be changed to

\[ M > 4N^2 - \pi^2 \]  \hspace{1cm} (44)

where the worst case value of \( n = 1 \) was used.

A plot of the region of stability for ideal position feedback is given in Figure 3. For a given magnetic force \( N \), the system will be stable if the feedback gains \( M \) is in the region defined by Equation 44. With no feedback, the stability criterion reduces to that of Equation 31, which was derived for the case of no feedback.
2.2.2 Cross-Coupling Feedback

Another scheme which could stabilize the wire would be to apply a force in the x direction which is proportional to the spatial derivative of \( v(z,t) \), the y displacement. In a similar manner, a force in the y direction would be applied which was proportional to the spatial derivative of \( u(z,t) \). The feedback forces would be of the form

\[
T_x(z,t) = -G \frac{\partial v}{\partial z}(z,t) \\
T_y(z,t) = G \frac{\partial u}{\partial z}(z,t)
\]

With the applied feedback forces, the equations of motion of the system become

\[
\frac{\partial^2 u}{\partial t^2} = \frac{T}{R} \frac{\partial^2 u}{\partial z^2} + \frac{\partial v}{\partial z} \left( \frac{B}{R} \frac{I}{R} - \frac{G}{R} \right) \tag{47}
\]

\[
\frac{\partial^2 v}{\partial t^2} - \frac{T}{R} \frac{\partial^2 v}{\partial z^2} - \frac{\partial u}{\partial z} \left( \frac{B}{R} \frac{I}{R} - \frac{G}{R} \right) \tag{48}
\]

These equations will be satisfied by the complex solutions given by equations 9 and 10 if

\[
\hat{\omega}^2 \left( \frac{T}{R} k^2 \right) - \hat{\nu}(jk) \left( \frac{B}{R} \frac{I}{R} - \frac{G}{R} \right) = 0 \tag{49}
\]
\[ \hat{\omega} j k \left( \frac{B I}{R} - \frac{G}{R} \right) + \hat{\nabla}(\omega^2 - \frac{T}{R} k^2) = 0 \]  

(50)

Equating the determinant of the coefficient matrix to zero gives the following dispersion relation.

\[ \omega^2 = k^2 \frac{T}{R} + k \left( \frac{B I}{R} - \frac{G}{R} \right) \]  

(51)

Two distinct wave numbers are found from Equation 51, they are

\[ k_1 = \frac{1}{2} \left( \frac{B I}{O T} - \frac{G}{T} \right) + \left( \frac{1}{4} \left( \frac{B I}{O T} - \frac{G}{T} \right)^2 + \frac{R T}{2} \omega^2 \right)^{1/2} \]  

(52a)

\[ k_2 = -1/2 \left( \frac{B I}{O T} - \frac{G}{T} \right) + \left( \frac{1}{4} \left( \frac{B I}{O T} - \frac{G}{T} \right)^2 + \frac{R T}{2} \omega^2 \right)^{1/2} \]  

(52b)

These wave numbers are related by

\[ k_1 = k_2 + \left( \frac{B I}{O T} - \frac{G}{T} \right) \]  

(53)

For the wire to be stable, \( \omega^2 \) must be positive. This condition occurs when

\[ \frac{k_1 T}{R} > \left| \frac{B I}{O R} - \frac{G}{R} \right| \]

(54)
With the values of $k_1$ found from equations 53 and 27, and with the normalized quantities defined by equation 42a., and 42b., the stability criterion can be converted to

$$\frac{M}{2} > 2N - \pi \quad (55a)$$

$$\frac{M}{2} < \pi + 6N \quad (55b)$$

where $M$ is the normalized gain $G\ell/T$. The worst case of $n = 1$ was used in Equation 27.

The region of stability in the MN plane is shown in Figure 4. For a given $N$, the wire will be stable if $M$ is in the required band. If the gain is too large or too small, the system is unstable with $\omega$ purely imaginary. The feedback force and the magnetic interaction force each act to rotate a disturbance of the wire but in opposite directions. To achieve stability, the two forces should be approximately equal so that the net effect will be zero. When either the magnetic interaction or the feedback dominates, instability is produced.
2.2.3 Comparison of Ideal Feedback Techniques

From the information given in Figures 3 and 4, the following comparisons between the two types of ideal feedback can be made. For a given value of $N$ the gain required for the cross-coupling feedback is less than that needed for the displacement feedback. This is because the gain for the proportional feedback increases with the square of $N$ while for the cross-coupling feedback, the value of $M$ necessary for stability increases linearly with $N$.

While it does require a higher gain, the displacement feedback has several other advantages. Increasing the value of the gain will not induce instability as it does for cross-coupling feedback. If the system were to be stabilized for a large value of $N$, the gain $M$ could be made large enough to control the system and then the magnetic pressure $N$ could be applied. With cross-coupling feedback, both $M$ and $N$ would have to be adjusted together to keep the system in the stable band shown in Figure 4.

The displacement feedback is also advantageous because it would be simpler to measure the displacement of the wire than it would be to measure the spatial derivative.
2.3 Finite Sampling Interval Feedback

2.3.1 One Station, Average Displacement Feedback

Due to practical considerations, the feedback force cannot be specified for each point but is determined for a finite interval of the spring. The situation analyzed here is one where the average displacement of the spring is measured, and a force proportional to this average is applied over the length of the wire. The motivation for this technique is that by measuring the capacitance between the wire and a flat plate, an indication of the average displacement may be made.

This feedback would produce restoring forces of the form

\[ \hat{T}_x(z) = \frac{G}{\ell} \int_{-\ell/2}^{+\ell/2} \hat{u}(z) \, dz \]  
(56a)

\[ \hat{T}_y(z) = \frac{G}{\ell} \int_{-\ell/2}^{\ell/2} \hat{v}(z) \, dz \]  
(56b)

These feedback forces will convert the equations of motion to

\[ \frac{\partial^2 u}{\partial t^2} = \frac{T}{R} \frac{\partial^2 u}{\partial z^2} + \frac{B}{R} \frac{I}{R} \frac{\partial v}{\partial z} - \frac{G}{\ell} \int_{-\ell/2}^{\ell/2} u(z,t) \, dz \]  
(57)
\[
\frac{\partial^2 v}{\partial t^2} = \frac{T}{R} \frac{\partial^2 v}{\partial z^2} - \frac{B}{R} \frac{\partial u}{\partial z} - \frac{C}{\ell R} \int_{-\ell/2}^{\ell/2} v(z) \, dz
\]  
(58)

Solutions of these equations will be

\[
\hat{u}(z) = A \sin k_1 z + B \cos k_1 z + C \sin k_2 z + D \cos k_2 z + E
\]  
(59)

\[
\hat{v}(z) = -A \cos k_1 z + B \sin k_1 z + C \cos k_2 z - D \sin k_2 z + F
\]  
(60)

where \(k_1\) and \(k_2\) are the same as for the case with no feedback. They are defined by equations 16 and 17.

To simplify the computation, the solutions for \(\hat{u}(z)\) and \(\hat{v}(z)\) are factored into their even and odd parts. This generates two pairs of solutions. One pair is the even part of \(\hat{u}(z)\) and the odd part of \(\hat{v}(z)\). The first pair of factored solutions is

\[
\hat{u}(z) = B \cos k_1 z + D \cos k_2 z + E
\]  
(61)

\[
\hat{v}(z) = B \sin k_1 z - D \sin k_2 z
\]  
(62)
The second pair of solutions generated by factoring is the same as equation 61 except that they are rotated 90° around the z axis. The second set provides no more information about stability than does the first set so the second set will no longer be considered. Using the equation of motion and the boundary conditions that \( \hat{u}(z) \) and \( \hat{v}(z) \) are zero at the ends of the spring, \( z = \pm \ell/2 \), three independent equations of motion are found.

\[-B \frac{2G}{k_1R} \sin \left( \frac{k_1 \ell}{2} \right) - D \left( \frac{-2G}{\ell k_2 R} \right) \sin \left( \frac{k_2 \ell}{2} \right) + \left( \omega_0^2 - \frac{G}{R} \right) E = 0\]  
\[(63)\]

\[+B \cos k_1 \ell/2 + D \cos k_2 \ell/2 + E = 0\]  
\[(64)\]

\[+B \sin k_1 \ell/2 - D \sin k_2 \ell/2 = 0\]  
\[(65)\]

For finite values of the constants to exist, the determinant of the coefficients must be zero. This gives the dispersion equation for the system

\[\left( \omega_0^2 - \frac{G}{R} \right) \sin \left( \frac{k_1 + k_2}{2} \right) \ell + \frac{2G}{R} \left( \frac{1}{k_1} + \frac{1}{k_2} \right) \sin k_1 \ell/2 \sin k_2 \ell/2 = 0\]  
\[(66)\]
By using the normalized quantities defined in Equation 42, along with the definitions of \(k_1, k_2,\) and \(\omega\) given in Equations 16, 18, and 27, the dispersion relation can be converted to

\[
(Q^2 - 4QN - M)\sin(Q - 2N) + 2M \sin(Q/2) \frac{\sin(Q/2 - 2N)(2Q - 4N)}{Q^2 - 4QN}
\]

(67)

The condition for stability is the same as that stated in Equation 30, which in dimensionless form is

\[Q > 4N\]

(68)

This means that all of the possible wavenumbers \(Q\), for which finite displacements can exist, must be greater than \(4N\) for stability.

A root locus plot of the roots \(Q\) of Equation 67 is given in Figure 5. With no gain, the roots are located at

\[Q = 2N + n\pi \quad n = 1, 2, \ldots\]

(69)

There are also roots at \(0, 2N\) and \(4N\) for which no displacement can exist and which are not affected by the feedback.

As the gain \(M\) is increased, the roots defined in Equation 69 will migrate to higher values. As \(M\) approaches infinity, the lowest root approaches a value less than \(2N + 2\pi\). For values of gain such that this root is larger than \(4N\), the system is stable.
Figure 6 is a plot of the stable region for all values of M and N. Asymptotically, the largest value of N for stability approaches $3/4 \pi$ as M goes to infinity.

2.3.2 One Station, Midpoint Displacement Feedback

Another technique of creating a feedback signal would be to sense the position of the spring at only the midpoint. This would replace the average displacement used in the previous example. With one sampling station placed at the middle of the spring, a restoring force density of the following form could be applied to the system.

\[
T_x = -G u(0) \quad (70)
\]

\[
T_y = -G v(0) \quad (71)
\]

These feedback forces will convert the equations of motion to

\[
\frac{\partial^2 u}{\partial t^2} = \frac{T}{R} \frac{\partial^2 u}{\partial z^2} + \frac{B I}{R} \frac{\partial v}{\partial z} - \frac{G}{R} u(0, t) \quad (72)
\]
\[ \frac{\partial^2 v}{\partial t^2} = \frac{T}{R} \frac{\partial^2 v}{\partial z^2} - \frac{B}{R} \frac{\partial u}{\partial z} - \frac{G}{R} v(0,t) \] (73)

The solutions of equations 72 and 73 are similar to those found in Section 2.3.1.

\[ \hat{u}(z) = B \cos k_1 z + D \cos k_2 z + E \] (74)

\[ \hat{v}(z) = B \sin k_1 z - D \sin k_2 z \] (75)

where \( k_1 \) and \( k_2 \) are defined by equations 16 and 17. There is also another pair of solutions similar to these but rotated 90° on the \( z \) axis.

The equations of motion and the boundary conditions \((\hat{u}(z) \text{ and } \hat{v}(z) \text{ are zero at } z = \pm \ell/2)\) give three independent equations

\[ - \frac{G}{R} B \quad - \frac{G}{R} D \quad + (\omega^2 - \frac{G}{R}) E = 0 \] (76a)

\[ + B \cos k_1 \ell/2 \quad + D \cos k_2 \ell/2 \quad + E = 0 \] (76b)
\[
B \sin k_1 \ell/2 - D \sin k_2 \ell/2 = 0
\]

(76c)

For finite values of \( B, D, \) and \( E \) to exist, the determinant of the coefficient matrix must be zero. This yields the dispersion relation

\[
\left( \frac{\omega^2 - G/R}{G/R} \right) \sin \frac{(k_1 + k_2) \ell}{2} + \sin k_1 \ell/2 + \sin k_2 \ell/2 = 0
\]

(77)

By use of equations 16 and 18, which define \( \omega \) and \( k_2 \) in terms of \( k_1 \), and the normalized variables defined in Equation 42, the dispersion equation can be rewritten as

\[
\sin \left( \frac{\omega}{2} - N \right) = 0
\]

(78)

\[
(Q^2 - 4QN - M) \left[ \cos \left( \frac{\omega}{2} - N \right) \right] + \left[ \cos (N) \right] \quad M = 0
\]

(79)

where Equation 77 has been factored into equations 78 and 79.

The eigenvalues for the wavenumber \( Q \) given by Equation 78 are

\[
Q = 2N + 2n\pi \quad \quad n = 0, 1, 2, \ldots
\]

(80)
and they are unaffected by the feedback of the system. For these values of \( Q \), the displacement at midpoint, \( z = 0 \), is zero and the pickoff transducer does not detect any motion when these modes are excited.

The eigenvalues of Equation 79 must be found by numerical methods. A root locus plot of the roots is given in Figure 7. The roots are at

\[
Q = 2N + n\pi \quad n \text{ odd}
\]

(81)

when the gain \( M \) is zero and move as shown in the figure as \( M \) is increased. For \( N \) less than \( \pi/2 \), the movement is described by Figure 7a. Figure 7b. shows the movement for \( N \) greater than \( \pi/2 \).

For the system to be stable, the stability criterion developed in Section 2.1.2 must be satisfied. This criterion is that the roots corresponding to non-zero displacements must be greater than \( \frac{B}{I} \). In dimensionalized form this is

\[
Q > 4N
\]

(82)
For $N$ greater than $\pi/2$, the root which is at $2N + \pi$, when the gain is zero, will cause instability. Increasing the gain will move this root to a higher value and stabilize the system. For values of $N$ greater than $\pi$, the root at $2N + 2\pi$ will cause instability. This root isn't affected by the feedback so the wire can't be stabilized for values of $N$ greater than $\pi$.

In Figure 7a, as the gain is increased, the roots which start at $2N + \pi$ and $2N + 3\pi$ move toward each other. When they meet, they break away from the real axis and become complex. This produces complex values of $\omega$ which lead to growing oscillations of amplitude. This type of motion is referred to as overstability. A further increase in gain moves the two roots back to the real axis and the system becomes stable again.

For values of $N$ greater than $\pi/2$ the behavior is somewhat different. The direction that the roots move is reversed except for the first root which still increases with increased gain. Now the roots which start at $2N + 3\pi$ and $2N + 5\pi$ move together and cause overstability.

For $N = \pi/2$, none of the roots except the lowest one move as the gain is increased. This is because all of the modes have zero displacement at the midpoint, the location of the feedback transducer. The feedback transducer won't be affected by the displacement of these modes and the feedback network can't couple to them.
A plot of the regions of stability, overstability, and static instability are plotted in Figure 8.

2.3.3 Two Station, Mid-point Displacement Feedback

In an effort to provide better control of the wire, the control structure was divided into two sections. A sampling station is placed in the middle of each section and is coupled to a control system which applies forces on the section. There is no coupling between the feedback network of each section.

The technique used for finding the motion on the wire is the splicing technique\(^3\). Solutions are assumed for each section and these solutions are spliced together at the sampling interface. For splicing the solutions, the requirements are made that \(\xi(z,t)\) and \(\frac{\partial \xi}{\partial z}(z,t)\) are continuous at the interface.

The control structure is divided at the point \(z = 0\). The section of negative \(z\) is designated as Section 1 and the Section of positive \(z\) is designated as Section 2. The sampling station for Section 1 is at \(x = -\ell/4\) and the sampling station for Section 2 is at \(z = +\ell/4\).
This feedback system will apply the following restoring forces to the system

\[ T_{x1} + \frac{G}{R} u_1(-\ell/4) \]  

(83a)

\[ T_{y1} = -\frac{G}{R} v_1(-\ell/4) \]  

(83b)

\[ T_{x2} = -\frac{G}{R} u_2(\ell/4) \]  

(84a)

\[ T_{y2} = -\frac{G}{R} v_2(\ell/4) \]  

(84b)

where the subscripts 1 and 2 refer to the section where the force is applied.

The equations of motion of the system for either section will be

\[ \frac{\partial^2 u_i}{\partial t^2} = \frac{T}{R} \frac{\partial^2 u_i}{\partial z^2} + \frac{B}{R} \frac{\partial v_i}{\partial z} + T_{xi} \]  

(85)

\[ \frac{\partial^2 v_i}{\partial t^2} = \frac{T}{R} \frac{\partial^2 v_i}{\partial z^2} - \frac{B}{R} \frac{\partial u_i}{\partial z} + T_{xi} \]  

(86)

where \( u_i, v_i \) are solutions valid in the \( i \)th section. The solutions for equations 85 and 86 will be
\begin{equation}
\hat{u}_i = A_i \sin k_1 x + B_i \cos k_1 x + C_i \sin k_2 x + D_i \cos k_2 x + E_i
\end{equation}
\begin{equation}
\hat{v}_i = -A_i \cos k_1 x + B_i \sin k_1 x + C_i \cos k_2 x - D_i \sin k_2 x + F_i
\end{equation}

The procedure is similar to that used for the one station feedback. The solutions may be factored into their even and odd parts. This creates two pairs of solutions. One pair is even in \( \hat{u} \) and odd in \( \hat{v} \) while the other has \( \hat{v} \) even and \( \hat{u} \) odd. Each of these pairs of solutions is sufficient to determine the stability of the system. The solution for which \( \hat{u} \) is even, \( \hat{u}(-z) = \hat{u}(+z) \), and for which \( \hat{v} \) is odd, \( \hat{v}(-z) = -\hat{v}(+z) \), establish the following relations for the constants of equations 86 and 87.

\begin{align}
A_1 &= -A_2 \\
C_1 &= -C_2 \\
B_1 &= B_2 \\
D_1 &= D_2
\end{align}
\[ E_1 = E_2 \] (89e)

\[ F_1 = -F_2 \] (89f)

For this system there are eight boundary equations. At the ends of the wire, \( \hat{u}_1(-\ell/2), \hat{u}_2(\ell/2), \hat{v}_1(-\ell/2), \hat{v}_2(\ell/2) \) are zero. At the interface of the two sections \( \hat{u}_1(0) = \hat{u}_2(0), \hat{v}_2(0) = \hat{v}_1(0), \frac{\partial \hat{u}_1}{\partial z}(0) = \frac{\partial \hat{u}_2}{\partial z}(0) \), and \( \frac{\partial}{\partial z} \hat{v}_2(0) = \frac{\partial}{\partial z} \hat{v}_1(0) \).

With the four equations of motion, given by equations 85 and 86, and the six relations of equation 89, the following six independent equations are found.

\[ -\frac{G}{R} A \sin k_1 \ell/2 \quad -\frac{G}{R} B \cos k_1 \ell/2 \quad -\frac{G}{R} C \sin k_2 \ell/2 \quad -\frac{G}{R} D \cos k_2 \ell/2 \]

\[ + (\omega^2 - \frac{G}{R}) E = 0 \] (90a)

\[ A \sin k_1 \ell + B \cos k_1 \ell + C \sin k_2 \ell + D \cos k_2 \ell + E = 0 \] (90b)

\[ -\frac{G}{R} A \cos k_1 \ell/2 \quad +\frac{G}{R} B \sin k_1 \ell/2 \quad +\frac{G}{R} C \cos k_2 \ell/2 \quad -\frac{G}{R} D \sin k_2 \ell/2 \]

\[ + (\omega^2 - \frac{G}{R}) F = 0 \] (90c)
\[ A \cos k_1 \ell - B \sin k_1 \ell - C \cos k_2 \ell + D \sin k_2 \ell + F = 0 \]  
(90d)
\[ A - C + F = 0 \]  
(90e)
\[ k_1 A + k_2 C = 0 \]  
(90f)

For finite values of the constants to exist, the determinant of the matrix of coefficients must be zero. This establishes the dispersion relation for the system. As in Section 2.3.2, the dispersion relation can be factored into two equations. With the normalized quantities defined by Equations 42. These two equations are

\[ \sin (Q/4 - N/2) = 0 \]  
(91)
\[ (Q-2N) \left[ (Q^2-4QN-M)^2 \right] \begin{array} {l} 2 \cos (Q/4-N/2) \cos (Q/2-N) + M^2 \cos (Q/4-N/2) \\ + (Q^2-4QN-M) \left[ \cos (Q/4-N/2) \left[ \cos (Q/4-N) + \cos (Q/4-N) \right] \\ + \cos (Q/2-N) \cos (N/2) \right] \end{array} \]  
\[ + (Q/2) \left[ (Q^2-4QN-M)M \cos (Q/2-N/2) + M^2 \cos (Q/4+N/2) \right] \]  
\[ + (Q/2-2N) \left[ (Q^2-4QN-M)M \cos (Q/2-\frac{3N}{2}) + M^2 \cos (Q/4-\frac{3N}{2}) \right] = 0 \]  
(92)
The eigenvalues of Equation 91 are

$$Q = 2N + 4n\pi \quad n = 0, 1, 2...$$  \hspace{1cm} (93)

Comparison of these values to those of Equation 81 shows that the two station feedback is able to couple to more of the modes of the system than the one station feedback.

The roots of Equation 93 must be found by graphical methods. As found for the one station case, the system is unstable with no feedback for $N$ greater than $\pi/2$. Increasing the gain moves the roots which start at $2N + \pi$ and $2N + 2\pi$ to higher values so the stability requirement, Equation 82, is satisfied and the system is stable. As the gain $M$ goes to infinity, the root which began at $2N + \pi$ reaches a limit of $2N + 2.5\pi$. The largest value of $N$ for which the system could be stable is then $1.25\pi$.

The behavior of the roots of this system are much more complicated than for the one station system. The direction that the roots move is illustrated in Figure 9. For $N$ less than $\frac{4\pi}{5}$, the three roots below the first fixed root move the higher values and the three roots above the first fixed root decrease in value. For values of $N$ greater than $\frac{4\pi}{5}$, the root which begins at $2N + 5\pi$ changes direction and increases as $M$ increases. The root which begins at $2N + 6\pi$ begins to
increase for increasing gain when N is greater than π. The root at 2N + 7π increases for increasing gain for N greater than \( \frac{6\pi}{5} \). For values of N greater than \( \frac{6\pi}{5} \), the six lowest moveable roots will increase with increasing gain.

The roots which first break away from the real axis to cause overstability also change as N is increased. For N less than \( \frac{4\pi}{5} \), the roots which started at 2N + 2π and 2N + 3π meet on the real axis and then become complex. For N greater than \( \frac{4\pi}{5} \) and less than π, the roots which began at 2N + 5π and 2N + 6π meet on the real axis and then become complex.

A plot of the values of M and N for which this system is overstable, statically unstable, and stable is shown in Figure 10.
2.3.4 Summary and Discussions

From Figures 6, 8, and 10, the following comparisons of the several types of finite interval feedback can be made.

In regard to overstability, the average displacement feedback was superior because it wouldn't induce this type of instability. The two station feedback was generally better than the one station case except for values of $N$ slightly greater than $\pi/2$. In this region the one station feedback does not become overstable for high values of gain.

From the point of view of stability, the two station feedback case has the most improvement. The highest value of $N$ for which the wire will be stable is approximately $1.1 \pi$. This compares to $0.9\pi$ for the case of one station and $0.75\pi$ for the average feedback case.
In Figures 8 and 10, a comparison has been made between the stabilization possible with one and two station feedback and that possible with the ideal position feedback. The results show that the one and two station feedback are very close to the ideal feedback for small values of $N$. The two station feedback is a better approximation to the ideal case for only a small range of $N$. While it can not be definitely concluded without analyzing the situation with more than two sampling stations, it appears that the addition of each new sampling station does not provide a significant increase in the maximum $N$ for which the system is stable.

Some explanation for this poor rate of return for increased sampling stations may be given by comparing the stability criterion of Equation (82) to that given by Melcher$^{(3)}$. In the problem analyzed by Melcher, the stability criterion is for the square of the lowest root to be greater than the perturbing force $N$. The stability criterion for this problem requires that the root be less than $4N$. It is probable that the maximum value of $N$ for this problem will only increase linearly with the number of feedback stations compared to the problem discussed by Melcher, where the maximum value of $N$ increased as the square of the number of stations.

If it is assumed that the stability criteria derived for this system were applicable to the thermonuclear pinch experiment, then a large number of feedback stations would be needed. This is a consequence of the effective tension in a plasma column is caused by the sheath effect$^{(2)}$ and is very small.
When the tension is small, the system is unstable for very small values of current and magnetic field. To obtain thermonuclear temperatures, the plasma pinch would require a large increase in magnetic field and current above the values which lead to instability. Due to the low improvement in stability caused by the addition of sampling stations, it would require a large number of stations to stabilize the system sufficiently. For this reason, a true continuum feedback amplifier will have to be developed for the plasma to be stabilized in the thermonuclear temperature range.
CHAPTER III

STABILIZATION OF A LEVITATED FLUID

3.0 Introduction

The second system to be analyzed is a system for levitating a fluid film against the force of gravity. The problem is illustrated in Figure 2. A layer of conducting fluid of thickness $2d$, is supported against the force of gravity by the pressure of a gas held beneath the fluid. The top surface of the fluid is a free surface and the pressure above it is zero. The density of the fluid is $\rho$ and the density of the supporting gas is considered to be zero. The feedback forces, which will be applied are created by electrodes placed a distance $\ell$ above and below the film. These electrodes will be able to exert electrical tractions on the fluid surface which may stabilize the film.

The system will be analyzed for stability first with no electrical forces applied. To increase the stability of the film, feedback control will be applied to both the top and bottom surfaces. The first method of control will be to apply a traction to the top surface, proportional to the deflection of the top surface. This method doesn't improve the stability of the film so a second method is attempted. A traction is applied to the lower surface of the fluid which is proportional to the thickness of the fluid. This type of control is successful in creating a stable film.
This problem is of interest because of the applications this system would have in the manufacture of thin glass sheets. If a film of molten glass could be held flat until it hardened, an inexpensive way of making thin sheets of glass will have been found.
3.1 Fluid Bulk Equations

In the bulk of the fluid, the only applied force will be gravity. The linearized equation of motion is then

\[ \rho \frac{\partial v}{\partial t} = -\nabla P - \rho g \]  \hspace{1cm} (1)

Because the fluid is incompressible, the velocity satisfies the condition that

\[ \nabla \cdot v = 0 \]  \hspace{1cm} (2)

By taking the curl of Equation 1 twice and using Equation 2, the following relation for the velocity is obtained

\[ \nabla^2 v = 0 \]  \hspace{1cm} (3)

For the given system, the proper solution for \( v_y(t,x,y) \) will be of the form

\[ v_y(t,x,y) = \text{Re} \left[ \hat{v}_y(y) \exp j(\omega t - kx) \right] \]  \hspace{1cm} (4)

where, from Equation 3, \( \hat{v}_y(y) \) obeys the relation

\[ \left[ \frac{\partial^2}{\partial y^2} - k^2 \right] \hat{v}_y(y) = 0 \]  \hspace{1cm} (5)
The solutions for $\hat{V}_x$ can be found from $\hat{V}_y$ by use of Equation 2.

From Equation 5, the solution for $\hat{V}_y$ is of the form

$$\hat{V}_y(y) = A \sinh ky + B \cosh ky$$  \hspace{1cm} (6)

The constants of Equation 6 are to be determined by the boundary conditions at the surfaces of the film. At the surfaces, the velocity must be equal to the rate of change of the surface displacement. That is the velocities must obey the conditions

$$\hat{V}_y(+d) = \frac{\partial \hat{\xi}_1}{\partial t}$$ \hspace{1cm} (7)

$$\hat{V}_y(-d) = \frac{\partial \hat{\xi}_2}{\partial t}$$ \hspace{1cm} (8)

where $\hat{\xi}_1$ and $\hat{\xi}_2$ are the displacements from equilibrium of the top and bottom surfaces respectively.

By substituting Equations 4 and 6 into Equations 7 and 8, the constants A and B may be found in terms of $\hat{\xi}_1$ and $\hat{\xi}_2$. They are

$$A = \frac{j\omega (\hat{\xi}_1 - \hat{\xi}_2)}{\sinh kd}$$ \hspace{1cm} (9)

$$B = j\omega \frac{\hat{\xi}_1 + \hat{\xi}_2}{\cosh kd}$$ \hspace{1cm} (10)
The solution for $\hat{v}_y$ in the bulk of the fluid then becomes

$$\hat{v}_y = j\omega \frac{\hat{v}_1 - \hat{v}_2}{\sinh kd} \sinh ky + \frac{j\omega (\hat{v}_1 + \hat{v}_2)}{\cosh kd} \cosh ky$$  \hspace{1cm} (11)$$

With equations 11 and 1, the pressure in the fluid is found to be

$$P(y) = \left( \frac{\rho \omega^2}{k} \right) \left[ \frac{(\hat{v}_1 - \hat{v}_2)}{\sinh kd} \cosh ky + \frac{(\hat{v}_1 + \hat{v}_2)}{\cosh kd} \sinh ky \right]$$  \hspace{1cm} (12)$$

$$-\rho gy + C$$

where $C$ is found from boundary conditions imposed on the surface of the film.

### 3.2 Electrical Surface Traction

The pressure distribution throughout the fluid will be determined by the electrical traction applied at the surfaces. The traction applied to the upper fluid surface of Figure 1, due to the voltage on the top electrode will be computed. The coordinates will be shifted so that the fluid equilibrium is at $y = -\xi$ and the plate is at $y = 0$. The voltage on the
top electrode will be

$$V(x,t) = V_o + \text{Re} \hat{V} e^{j(\omega t - kx)} \quad (13)$$

In the air space between the fluid and the electrode, the electric field obeys the following conditions.

$$\nabla \times E = 0 \quad (14)$$

$$\nabla \cdot E = 0 \quad (15)$$

The curl of Equation 14 plus Equation 15 gives

$$\nabla^2 E = 0 \quad (16)$$

Solutions for $E_x$ will be of the form

$$E_x(t,x,y) = \text{Re} \left[ \hat{E}_x(y) \exp j(\omega t - kx) \right] \quad (17)$$

where $\hat{E}_x(y)$ satisfies the differential equation

$$\left[ \frac{\partial^2}{\partial y^2} + k^2 \right] \hat{E}_x(y) = 0 \quad (18)$$
Solutions of Equation 18 for $\hat{E}_x(y)$ are of the form

$$\hat{E}_x(y) = A \cosh ky + B \sinh ky$$  \hspace{1cm} (19)

The $y$ directed component of the electric field can be found from Equation 12 or Equation 13.

The boundary conditions defined by the system are that the tangential component of the field at the fluid surface must be zero.

$$n_x E = 0$$  \hspace{1cm} (20)

and that at the electrode, the tangential $E$ field is fixed by the applied voltage

$$\hat{E}_x = -\frac{\partial \hat{V}}{\partial x} = jk \hat{V}$$  \hspace{1cm} (21)

At the surface of the fluid, the linearized normal vector is

$$n = \frac{i}{y} + \frac{\delta \xi}{\partial x} \frac{i}{x}$$  \hspace{1cm} (22)

With Equations 20, 21, and 22, the constants for Equation 19 can be found. The solution for $\hat{E}_x(y)$ is

$$\hat{E}_x = jk \hat{V} \cosh ky + \left\{ \frac{jk \hat{V} \cosh kl}{\sinh kl} - \frac{jk \xi (\frac{V}{d})}{\sinh kl} \right\} \sinh ky$$  \hspace{1cm} (23)
The y directed electric field is then found to be

\[ E_y = k \mathbf{V} \sinh ky - \left\{ \frac{k \mathbf{V} \cosh k \ell}{\sinh k \ell} - \frac{k \mathbf{V} \frac{o}{d}}{\sinh k \ell} \right\} \cosh ky \]

(24)

This was found from equations 15 and 23.

The traction which is exerted on the fluid surface is found by the use of the Maxwell stress tensor. The general expression for this traction is

\[ \tau_\alpha = \left[ T^1_{\alpha \beta} - T^2_{\alpha \beta} \right] n_\beta \]

(25)

where \( \tau_\alpha \) is the traction in the \( \alpha \) direction, \( T^1_{\alpha \beta} \) and \( T^2_{\alpha \beta} \) are the Maxwell stress tensors above and below the surface respectively, and \( n_\beta \) is the component of the normal vector in the \( \beta \) direction.

To linearized terms this gives the electrical surface traction as

\[ \tau_x = \frac{1}{2} \varepsilon_o \left[ E_o^2 + 2E_o k \mathbf{V} \frac{\cosh k \ell}{\sinh k \ell} - \frac{2E_o k \mathbf{V}}{\sinh k \ell} \right] \]

(26)

This first term is the d.c. term, the second term is the perturbation caused by irregularity of the fluid surface, and the last term is the perturbation caused by the feedback voltage on the electrode.
3.3 Fluid Pressure Boundary Conditions

At a fluid interface, there is a discontinuity in pressure due to the surface tension. This pressure discontinuity is given by Lamb. \(^{(1)}\)

\[
p - p' = T \left( \frac{1}{R_1} + \frac{1}{R_2} \right)
\]

(27)

where \(p - p'\) is the change in pressure across the surface, \(T\) is the surface tension, and \(R_1\) and \(R_2\) are the radii of curvature with centers in the unprimed side of the interface. For the two dimensional case being considered, this reduces to

\[
\Delta p + T \frac{\partial^2 \xi}{\partial x^2} = 0
\]

(28)

where \(\Delta p\) is the decrease in pressure when the interface is traversed in the \(+y\) direction.

Equating the pressure in the fluid at the top and bottom surfaces to the pressure of the gas, the electrical surface traction, and the discontinuity caused by surface tension gives

\[
\hat{p}_1 = k^2 T \xi_1 - \tau_e^a
\]

(29)

\[
\hat{p}_2 = -k^2 T \xi_2 - \tau_e^b + p_o
\]

(30)
where \( \hat{p}_1 \) and \( \hat{p}_2 \) are the complex pressures at the top and bottom surfaces of the fluid respectively, \( P_o \) is the pressure of the supporting gas, \( \tau_e^a \) is the electrical traction applied to the top surface and \( \tau_e^b \) is the traction applied to the lower surface.

3.4 Film Dynamics, No Electrical Traction

With no potentials applied to the electrodes, equations 29 and 30 reduce to

\[
\hat{p}_1 = k^2 \hat{T} \xi_1 \tag{31}
\]

\[
\hat{p}_2 = -k^2 \hat{T} \xi_2 + P_o \tag{32}
\]

Evaluation of Equation 12 at \( y = \pm d \) to find \( \hat{p}_1 \) and \( \hat{p}_2 \) changes equations 31 and 32 to

\[
\left( \frac{\rho \omega^2}{k} \right) \left[ \frac{\xi_1 - \xi_2}{\sinh k_1 d} \cosh kd + \frac{(\xi_1 + \xi_2)}{\cosh kd} \sinh kd - \rho g \xi_1 \right] = k^2 \hat{T} \xi_1 \tag{33}
\]

\[
\left( \frac{\rho \omega^2}{k} \right) \left[ \frac{\xi_1 - \xi_2}{\sinh kd} \cosh kd - \frac{(\xi_1 + \xi_2)}{\cosh kd} \sinh kd - \rho g \xi_2 \right] = -k^2 \hat{T} \xi_2 \tag{34}
\]
for the first order terms and gives

\[ p_o = 2\rho gd \]  \hspace{1cm} (35)

\[ C = \rho gd \]  \hspace{1cm} (36)

for the zero order terms.

Reorganizing equations 33 and 34 in matrix form gives

\[
\hat{\xi}_1 \left( \frac{\rho \omega^2}{k} \left( \coth 2kd - k^2 T - \rho g \right) \right) + \hat{\xi}_2 \left( \frac{\rho \omega^2}{k} \left( \frac{-1}{\sinh 2kd} \right) \right) = 0
\]

(37a)

\[
\hat{\xi}_1 \left( \frac{\rho \omega^2}{k} \left( \frac{1}{\sinh 2kd} \right) \right) + \hat{\xi}_2 \left( \frac{\rho \omega^2}{k} \left( -\coth 2kd - \rho g + k^2 T \right) \right) = 0
\]

(37b)

For finite values of \( \xi_1 \) and \( \xi_2 \) to exist, the determinant of the coefficients must be zero, this gives the dispersion relation for the system

\[
\left( \frac{\rho \omega^2}{k} \right)^2 - \left( \frac{\rho \omega^2}{k} \right) \left( \coth 2kd \right) \left( 2k^2 T \right) + \left( k^2 T \right)^2 - \left( \rho g \right)^2 = 0
\]

(38)
This film will be stable for $\omega^2$ greater than zero. Equation 38 shows that this condition will occur as long as

$$k^2T > \rho g$$  \hspace{1cm} (39)

This is the condition for stability. It is of interest that the thickness of the fluid has no effect on the stability of the fluid film.

3.4.1 Long Wave Limit

If the wave length of the disturbances are long compared to the thickness of the fluid, the following two dispersion relations can be made from Equation 38.

$$\rho \omega^2 = k^4Td - (\rho g)^2 \frac{d}{T}$$  \hspace{1cm} (40)

$$d\rho \omega^2 = k^2T + (\rho g)^2 \frac{d^2}{T}$$  \hspace{1cm} (41)

The first of these relations, Equation 40, represents the odd mode on the film, where the disturbances form as thick spots on the film. This is the mode which goes unstable for values of $\rho g$ larger than $k^2T$. The restoring force term, $k^4Td$, and the term which gives the instability, $(\rho g)^2 \frac{d}{T}$, are both multiplied by the thickness $d$. Since the thickness affects both of
these terms equally, the thickness doesn't have any effect on the stability of the fluid film.

The second dispersion relation, Equation 41, represents the even mode, where the film acts like a membrane. As the thickness decrease, \( \omega \) goes to infinity because the effective mass per unit area \( \rho d \), goes to zero. Also the term due to gravity becomes negligible, compared to the restoring force \( k^2T \). This is a consequence of the fact that the membrane force term, \( k^2T \), depends only on the surface tension and isn't affected by changes in the thickness of the film.

3.5 Film Dynamics With Top Surface Feedback

To improve the stability of the film, an electrical traction could be applied to the top surface of the fluid. The perturbation voltage \( \hat{V} \) would be placed in a feedback loop and made proportional to the displacement of the top surface \( \xi_1 \). That is,

\[
\hat{V} = G\hat{\xi}_1
\]  

(42)

The motivation is that when the film starts to sag, an increase in traction applied to the top surface would pull it back to equilibrium.

With an electrical traction on the top surface, the boundary conditions of the top and bottom surfaces are
\[ \hat{p}_1 = k^2 T \hat{\xi}_1 - 1/2 \varepsilon_0 \left[ E_o^2 + 2E_o^2 k \hat{\xi} \coth k \ell - \frac{2E_o kG \hat{\xi}}{\sinh k \ell} \right] \]

(43)

\[ \hat{p}_2 = -k^2 T \hat{\xi}_1 + p_o \]  

(44)

where \( E_o \) is the zero order electric field produced by the electrode. Equation 12, evaluated at \( y = \pm d \) to determine \( p_1 \) and \( p_2 \) gives for the zero order pressure terms

\[ p_o = 2\rho gd - \frac{1}{2} \varepsilon_0 E_o^2 \]  

(45a)

\[ C = \rho gd - 1/2 \varepsilon_0 E_o^2 \]  

(45b)

The first order terms for \( \hat{p}_1 \) and \( \hat{p}_2 \) are rearranged in matrix form in terms of \( \hat{\xi}_1 \) and \( \hat{\xi}_2 \) to give

\[ \hat{\xi}_1 \left[ \left( \frac{\rho \omega^2}{k} \right) \left( \coth 2kd \right) - k^2 T - \rho g + \tau_e - \tau_f \right] + \hat{\xi}_2 \left[ \left( \frac{\rho \omega^2}{k} \right) \left( \frac{-1}{\sinh 2kd} \right) \right] = 0 \]

(46)

\[ \hat{\xi}_1 \left[ \left( \frac{\rho \omega^2}{k} \right) \left( \frac{1}{\sinh 2kd} \right) \right] + \hat{\xi}_2 \left[ \left( \frac{\rho \omega^2}{k} \right) \left( -\coth 2kd \right) - \rho g + k^2 T \right] = 0 \]

(47)
where
\[
\tau_e = \varepsilon_0 E^2 k \coth k \ell
\]  
(48a)

and
\[
\tau_f = \frac{\varepsilon_0 E k G}{\sinh k \ell}
\]  
(48b)

The dispersion relation formed by setting the determinant of the coefficient matrix equal to zero is
\[
\begin{align*}
\left( \frac{\rho \omega^2}{k} \right)^2 + \left( \frac{\rho \omega^2}{k} \right)(\coth 2kd)(-2k^2 T + \tau_e - \tau_f) \\
+ (k^2 T)^2 - (\rho g)^2 + (\tau_e - \tau_f)(-k^2 T + \rho g) &= 0
\end{align*}
\]  
(49)

For stability, the sum of the last three terms of Equation 49 must be positive to make \( \omega^2 > 0 \). These three terms will be positive if
\[
k^2 T > \rho g
\]  
(50)

This is the same requirement given in Equation 39 for stability with no feedback. With this type of feedback, no improvement in stability is achieved. This can be explained by finding \( \xi_1 \).
in terms of $\xi_2$ with Equation 46. As the point of impending instability is approached, $\omega$ approaches 0 and by Equation 46, $\xi_1$ will go to zero. The feedback force is proportional to the displacement $\xi_1$, and it will also approach zero. Because the feedback force approaches zero as the film approaches the point of impending instability, this type of control cannot stabilize an unstable film.

The failure of this type of feedback indicates that for the film to be stabilized, part of the feedback signal must be generated by the displacement of the lower surface.

3.6 Film Stability With Lower Surface Feedback

The second attempt to stabilize the film applied a traction to the bottom surface of the film. Due to the results of Section 3.5, the feedback signal was made proportional to the difference between the top and bottom surface displacements. With this feedback, the perturbation voltage on the lower electrode would be

$$\hat{V} = G(\hat{\xi}_2 - \hat{\xi}_1)$$  \hspace{1cm} (51)

This type of feedback would be easy to produce by sensing the thickness of the film by optical techniques.

With the electrical traction applied to the lower surface, the boundary conditions for the pressures at the film surfaces are
\[ \hat{p}_1 = k^2 T \xi_1 \]

\[ \hat{p}_2 = - k^2 T \xi_2 + p_o - \frac{1}{2} \varepsilon \left[ E_o^2 k \xi_2 \coth k \ell \left( \frac{2E_o kG(\xi_1 - \xi_2)}{\sinh k \ell} + E_o^2 \right) \right] \]

Evaluation of Equation 12 at \( y = \pm d \) to determine \( \hat{p}_1 \) and \( \hat{p}_2 \) gives for the zero-order terms

\[ p_o = 2 \rho gd + \frac{1}{2} \varepsilon E_o^2 \]  

(54a)

\[ C = \rho gd \]  

(54b)

The first order terms are rearranged in matrix form to give

\[ \xi_1 \left[ \left( \frac{\rho \omega^2}{k} \right) (\coth 2kd) - k^2 T - \rho g \right] + \xi_2 \left[ \left( \frac{\rho \omega^2}{k} \right) \left( \frac{-1}{\sinh 2kd} \right) \right] = 0 \]

(55)

\[ \xi_1 \left[ \left( \frac{\rho \omega^2}{k} \right) \left( \frac{1}{\sinh 2kd} \right) + \tau_f \right] + \xi_2 \left[ \left( \frac{\rho \omega^2}{k} \right)(-\coth 2kd) \right. \]

\[ - \rho g + k^2 T + \tau_e - \tau_f \right] = 0 \]

(56)
where $\tau_e = \varepsilon_0 E_0^2 k \coth k \ell$

and $\tau_f = \frac{\varepsilon_0 E_0 k G}{\sinh k \ell}$

The dispersion relation resulting from the determinant of these two equations is

\[
\left(\frac{\rho \omega}{k}\right)^2 + \left(\frac{\rho \omega}{k}\right) \left[ (\coth 2kd) (-2k^2 T - \tau_e + \tau_f) - \frac{\tau_f}{\sinh 2kd} \right] + (k^2 T)^2 - (\rho g)^2 + (\tau_e - \tau_f) k^2 T + \rho g = 0
\]

(57)

The film will be stable as long as the sum of the last three terms is positive. That is,

\[
(k^2 T)^2 - (\rho g)^2 + (\tau_e - \tau_f) \left[ k^2 T + \rho g \right] > 0
\]

(58)

This can be rearranged to

\[
\tau_f > \rho g - k^2 T + \tau_e
\]

(59)

as the condition for stability.
3.7 Experimental Results

The results of Section 3.4.1 were that the fluid film would be unstable for all wavelengths less than a given value and that this value was independent of the film's thickness.

To test the results of section 3.1.4, the following experiment was performed. A soap film was made in a metal ring which was sealed to one end of a plexi-glass cylinder. The other end of the cylinder was sealed. An opening was made in the wall of the cylinder through which the pressure for supporting the fluid was adjusted. A drawing of the apparatus is shown in Figure 11.

The soap film was made approximately six inches in diameter. This diameter was picked because it was several times larger than the largest stable wavelength predicted by Equation 39. The purpose of the experiment was to see if a fluid film was made very thin, the stability condition of Equation 39 would still apply.

It was found that with the proper pressure in the chamber, the fluid was unstable except when the fluid was very thin. If a small amount of fluid was added to the stable film, an instability would form. This instability would result in a droplet on the lower surface of the film.

This experiment seemed to verify that Equation 39 was correct when there was enough fluid in the film to form instabilities. When the fluid is extremely thin Equation 39 is
invalid because extremely small surface deflections, nevertheless involve nonlinear effects, which presumably tend to stabilize the interface.

If a feedback system would be built to control the film, as Section 3.6 predicts that it could, it would be necessary to measure the thickness of the film at many points. To make this measurement electrically on a film which covered a large area would be difficult. A possible way of making the measurement would be to use an optical scanning device such as a television camera.

For the television camera to work, the instabilities must be made to appear as dark spots, which the camera can detect. To see if optical detection was possible, methyl violet, a purple dye, was dissolved in the soap solution. The photograph in Figure 12 shows the appearance of an instability with this soap solution. The instabilities appear as dark areas which could easily be detected by a television camera. It is, therefore, possible to use a television camera as a pickoff transducer for a feedback system which could stabilize the film.
3.8 Suggested Feedback System

The results of the experiment show that detection of fluid perturbations could be accomplished by a television camera. This suggests that the feedback may also be accomplished by another television component, the cathode ray tube.

The face of the cathode ray tube is at a high positive potential with respect to the cathode that emits the electron beam. At the point where the electron beam strikes the face, the voltage is decreased.\(^{(6)}\) If the electron beam is modulated, the voltage at the point the beam strikes will also be modulated. By sweeping the electron beam over the face of the cathode ray tube, it would be possible to create a potential with a desired spatial variation.

By using the face of a cathode ray tube as the electrode, and a television camera as the transducer, it would be possible to create a feedback voltage proportional to the thickness of the film. Proper adjustment of the gain could make it possible to meet the conditions for stability given by Equation 59.

There are several problems which must be solved before this system can be built. The first is the effect of temporal sampling of the film thickness. A television camera does not monitor each point continuously, but scans an area. The thickness at a point would be measured only once each time the area is scanned. At the present time, the effect that
temporal sampling will have in a continuum feedback system is not known.

The television camera will also introduce spatial sampling. When the camera scans an area, it doesn't cover each point but scans along parallel lines. These lines have a finite distance between them. The effect of this spatial sampling was not considered in Section 3.6.

If a certain spatial frequency is to be detected, it must be sampled at least twice every period. Therefore, if the television camera is to be effective as a pickoff transducer, it must have a sampling interval which is less than one half of the smallest wavelength of interest.

From Equation 59, the highest spatial frequency which leads to instability can be found. The television camera must be made so that the spacing between the lines is such that the unstable frequencies are sampled at least twice. With the large number of scans a television camera can make, it doesn't appear that spatial sampling will cause any difficulties.
3.9 Summary

The stability requirement for a layer of fluid, supported by the pressure of a gas, was derived. It was found that the stability was independent of the thickness. An experiment was conducted and it was found that the theory is correct, except in the limit of an extremely thin film. For these thin films, the stability is probably caused by non-linear effects which are not included in the theory.

Two possible systems for stabilizing the film were analyzed. The first used an active feedback loop to apply a traction to the top surface. This traction was proportional to the deflection of the top surface. It was found that this type of feedback could not improve the stability of the film. The technique was not successful in suppressing instability because the top surface remained fixed at the point of impending instability. The feedback force is proportional to the top surface displacement and it also becomes small at the point of impending instability.

The failure of the top surface feedback indicated that a successful feedback system must generate at least part of the feedback signal from the displacement of the lower surface.

Using the information from the first feedback system, a second approach to the problem was made. A feedback signal proportional to the difference between the top and bottom surface displacements was made. This signal created a force on
the lower surface of the film. It was found that this feedback method could stabilize the film.

An experiment was conducted to see if the thickness of the film could be easily detected. It was found that by placing dye in the soap solution, the thickness could be determined optically by the darkness of the film.

The theoretical and experimental results of this section indicate that it may be possible to build a feedback system that would stabilize the film. Since the thickness of the film can be detected optically, a television camera could be used to generate the feedback signal. To apply the electrical traction to the lower fluid surface, a cathode ray tube may be used. Proper modulation and scanning of the beam of this tube can create the desired feedback voltage on the face of the tube.
BIBLIOGRAPHY


7. Melcher, J. R.,"Continuum Feedback Control of Instabilities of an Infinite Fluid Interface," (To be Published).
$N > \frac{\pi}{2}$

**Figure 7b**
FIGURE 12

INSTABILITIES OF A LEVITATED SOAP FILM