FUNCTIONAL ANALYSIS OF SYSTEMS CHARACTERIZED

BY NONLINEAR DIFFERENTIAL EQUATIONS

by

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Submitted to the Department of Electrical Engineering on
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for the degree of Doctor of Philosophy.

ABSTRACT

An analysis, by functional calculus, of a class of nonlinear
systems is presented in this thesis. The class of nonlinear sys-
tems which are analyzed are all those analytic systems which are
characterized by nonlinear differential equations. Applications
of this analysis are shown for several actual nonlinear physical
systems which are analytic.

The precise definition of an analytic system is given in the
body of this thesis. Loosely speaking, an analytic system is any
system with the following three properties: (1) It is determinis-
tic (for a given input signal, the system can have one and only
one corresponding output signal). (2) It is time-invariant.
(3) It is "smooth" (the system cannot introduce any abrupt or
switch-like changes into its output. All such changes in the
output must be due to the input rather than the system).

Given a nonlinear differential equation, the conditions are
shown under which it characterizes an analytic system. Given an
analytic system characterized by a nonlinear differential equation,
it is shown how that system can be analyzed by an application of
functional calculus. Specifically, an inspection technique is
developed whereby a Volterra functional power series is obtained
for that system's input-output transfer relationship. Applications
are given for: (1) The demonstration of a pendulum's nonlinear
resonance phenomenon. (2) The computation of a shunt-wound motor's
response to white noise excitation. (3) The computation of a
varector frequency doubler's transient response. (4) The determina-
tion of the stability of a magnetic suspension device which is
currently being used in space vehicles.

An experimental confirmation of this latter stability deter-
mination is appended.

(190 pages)

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CHAPTER 1

INTRODUCTION

Preview

In this thesis we present an analysis, by functional calculus, of a class of nonlinear systems. The class of nonlinear systems that we shall analyze are all those analytic systems which are characterized by nonlinear differential equations. We shall also show applications of this analysis to several actual nonlinear physical systems which are analytic.

The precise definition of analytic systems shall be given later. For the time being, we shall say that analytic systems are all those systems which have the following three properties: (1) Analytic systems are deterministic (that is, for a given input they may have one and only one corresponding output). (2) Analytic systems are time-invariant (that is, the system's inputs and outputs co-translate in time). (3) Analytic systems are "smooth" (loosely speaking, by smooth we mean that the system cannot introduce an abrupt or switch-like change into the system's output. If such a change is present in the system's output, then it must be due to a similar switch-like change in the system's input (or one of the derivatives of the system's input)).

Given an analytic system characterized by a nonlinear differential equation, then we shall show how that system can be analyzed by an application of functional calculus. On the other hand, given a nonlinear differential equation, we shall show the conditions
under which it characterizes an analytic system. Specifically, we shall show a technique whereby a Volterra functional power series solution for that system's input-output transfer relationship can be obtained from its characterizing equation by inspection. We shall then show some applications of that functional solution of the system's input-output relationship. These applications will include: (1) The demonstration of a pendulum's nonlinear resonance phenomenon, (2) The computation of a shunt-wound motor's response to white noise, (3) The computation of a varactor frequency doubler's transient response, and (4) The determination of the stability of a magnetic suspension device which is currently being used in our nation's space vehicles.

History

The analysis of nonlinear systems by functional calculus is not new; Wiener introduced such analysis in 1942. The Volterra series solution of nonlinear differential equations is not new; Volterra did so in the nineteenth century. Barrett gave a solution technique for a certain kind of nonlinear differential equation in 1957. In 1963, Liou gave a procedure which will often (but not always) yield a solution for another type of differential equation. Van Trees, in 1964, solved the particular differential equation which characterized a phase-locked loop. The technique he used there parallels many of the essential features of the inspection technique that we shall present here.
An Illustration

Since many of the laws of physics are most readily stated by a differential equation, it is quite natural to specify a system's behavior by a differential equation. The system is then said to be characterized by that differential equation. However, for some applications, a functional description (the Volterra series solution of the characterizing differential equation) is more useful. (For example, if we want an explicit expression for a system's output or if we wish to calculate the properties of a system's output when its input is stochastic, then it will be seen that the functional description is more desirable than the differential equation description.) As an illustration of a method (but not the inspection technique which we shall present later) whereby a functional characterization of a system may be obtained from its differential equation characterization, we now present two examples—a linear low-pass filter and an associated nonlinear low-pass filter.

A Linear Low-pass Filter. Consider the low-pass filter shown in Figure I-1. If we are primarily interested in the relation of this filter's external variables (the voltages $x$ and $y$) rather than in its internal variables (such as the current $i$) then we can express that relation by a differential equation

$$0 = \frac{dy}{dt} + y - x \quad (1.1)$$

and the boundary condition of initial rest

$$y(t) = 0 \text{ until } x(t) \neq 0 \quad (1.2)$$
Figure I-1 A Linear Low-pass Filter
Another way in which we can express that relation is by the convolution of $x$ with $h$, the filter's impulse response. That is

$$y(t) = h(t)(*)x(t)$$

$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

(1.3)

Eq. 1.3 shows that the filter's output, $y$, is a functional of its input, $x$.

A Nonlinear Low-pass Filter. Consider the low-pass filter, squarer, low-pass filter system shown in Figure I-2. We can express the relation of its input voltage $x$ to its output voltage $z$ by an integro-differential equation

$$0 = \frac{dz}{dt} + z - (\int_{-\infty}^{\infty} u_{-1}(\tau)\exp(-\tau)x(t-\tau)d\tau)^2$$

(1.4)

(where $u_{-1}$ is the unit step function), and the boundary condition of initial rest

$$z(t) = 0 \text{ until } x(t) \neq 0$$

(1.5)

We can also express that relation in a form analogous to Eq. 1.3. That is

$$z(t) = h_0 + \int_{-\infty}^{\infty} h_1(\tau_1)x(t-\tau_1)d\tau_1$$

$$+ \int_{-\infty}^{\infty} h_2(\tau_1, \tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1d\tau_2 + \ldots$$

(1.6)
Figure 1-2. A Nonlinear Low-pass Filter

Circuit
Eq. 1.6 is called a Volterra Functional Power Series and the constant $h_0$ together with the functions $h_1, h_2, h_3, \ldots$ are called its Volterra kernels. Wiener introduced the use of such a Volterra series to express the input-output transfer relation of a nonlinear system in 1942.$^{39}$

Not all nonlinear systems have input-output transfer relations that can be expressed by means of a Volterra series (Eq. 1.6). We shall deal with the question of which nonlinear systems can and which nonlinear systems cannot be expressed by Eq. 1.6 later in this thesis. For the time being, we shall merely state that the system shown in Figure I-2 is expressible by Eq. 1.6 and show how we can solve for its Volterra kernels.

**An Illustration.** A major topic of this thesis will be the development of a technique whereby the Volterra series description of a nonlinear system can be found from its differential equation or integro-differential equation description by inspection. In order to illustrate this technique we shall solve for the Volterra series description of the system shown in Figure I-2 (but not by inspection.) In order to make this illustration clearer, we shall first use this technique to solve for the convolution integral description of the linear system shown in Figure I-1. In this case, the technique that we shall use will be unorthodox and more tedious than the usual well known methods but it will help to illustrate the use of the same technique on the nonlinear system.

Consider Eq. 1.3 where $h$ is unknown. By differentiating we
obtain the result that

\[
\frac{dy}{dt} = h^{(1)}(t)(*) x(t)
\]

\[
= \int_{-\infty}^{\infty} h^{(1)}(\tau) x(t-\tau) \, d\tau
\]  \hspace{1cm} (1.7)

where \( h^{(1)} \) is the derivative of \( h \). Any input \( x \) is related to itself, through the unit impulse, \( u_0 \), in the form

\[
x(t) = u_0(t)(*) x(t)
\]

\[
= \int_{-\infty}^{\infty} u_0(\tau) x(t-\tau) \, d\tau
\]  \hspace{1cm} (1.8)

When Eqs. 1.3, 1.7 and 1.8 are substituted into Eq. 1.1 then the result is that

\[
0 = \left[ h^{(1)}(t) + h(t) - u_0(t) \right] (*) x(t)
\]

\[
= \int_{-\infty}^{\infty} \left[ h^{(1)}(\tau) + h(\tau) - u_0(\tau) \right] x(t-\tau) \, d\tau
\]  \hspace{1cm} (1.9)

Eq. 1.9 must be true for any \( x \). This implies that the entire kernel of Eq. 1.9 is zero. That is

\[
0 = h^{(1)}(\tau) + h(\tau) - u_0(\tau) \quad , \quad \text{all}\ \tau \quad . \hspace{1cm} (1.10)
\]

If we take the bilateral Laplace transform of Eq. 1.10 we get that

\[
0 = (s+1)H(s) - 1 \hspace{1cm} (1.11)
\]

where
\[ H(s) = \int_{-\infty}^{\infty} h(\tau) \exp(-s \tau) \, d\tau \]  

(1.12)

Solving Eq. 1.11 for \( H \) we get that

\[ H(s) = \frac{1}{s+1} \]  

(1.13)

By taking the inverse transform of Eq. 1.13 and by taking into account the boundary condition, Eq. 1.2, we find that the system's impulse response is

\[ h(t) = u_-(t) \exp(-t) \]  

(1.14)

We have thus found the desired function \( h \) so that the system shown in Figure I-1 can be expressed by means of Eq. 1.3, the convolution integral. We shall now show an analogous method by which we will find the Volterra kernels whereby the system shown in Figure I-2 can be expressed by means of Eq. 1.6.

Consider Eq. 1.6. By differentiating, we can obtain the result that

\[
\frac{dz}{dt} = \int_{-\infty}^{\infty} h_1^{(1)}(\tau_1) x(t-\tau_1) \, d\tau_1 + \iint_{-\infty}^{\infty} h_2^{(1,0)}(\tau_1, \tau_2) x(t-\tau_1) \, d\tau_1 \, d\tau_2 + \\
+ \int_{-\infty}^{\infty} h_2^{(0,1)}(\tau_1, \tau_2) x(t-\tau_1) x(t-\tau_2) \, d\tau_1 \, d\tau_2 + \\
+ \iiint_{-\infty}^{\infty} h_3^{(1,0,0)}(\tau_1, \tau_2, \tau_3) + h_3^{(0,1,0)}(\tau_1, \tau_2, \tau_3) + \\
+ h_3^{(0,0,1)}(\tau_1, \tau_2, \tau_3) x(t-\tau_1) x(t-\tau_2) x(t-\tau_3) \, d\tau_1 \, d\tau_2 \, d\tau_3 + \ldots
\]  

(1.15)
where $h_{p}^{(1,j,k,...)}$ is the $i$th partial derivative with respect to the first argument, $j$th partial derivative with respect to the second argument, and so on, of the function $h_{p}$.

When Eqs. 1.6 and 1.15 are substituted into Eq. 1.4 the result is

$$0 = h_0 + \int_{-\infty}^{\infty} \left[ h_1(t_1) + h_1^{(1)}(t_1) \right] x(t-t_1) \, dt_1$$

$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ h_2(t_1,t_2) + h_2^{(1,0)}(t_1,t_2) + h_2^{(0,1)}(t_1,t_2) \right] x(t-t_1)x(t-t_2) \, dt_1 \, dt_2$$

$$- u_{-1}(t_1)u_{-1}(t_2) \exp(-t_1-t_2) \int x(t-t_1)x(t-t_2) \, dt_1 \, dt_2$$

$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ h_3(t_1,t_2,t_3) + h_3^{(1,0,0)}(t_1,t_2,t_3) + \ldots \right] \cdot$$

$$x(t-t_1)x(t-t_2)x(t-t_3) \, dt_1 \, dt_2 \, dt_3 + \ldots \quad (1.16)$$

Eq. 1.16 must hold for any $x$. One way in which this can be true is if each kernel in Eq. 1.16 is zero. That is

$$0 = h_0 \quad (1.17)$$

$$0 = h_1(t_1) + h_1^{(1)}(t_1) \quad (1.18)$$

$$0 = h_2(t_1,t_2) + h_2^{(1,0)}(t_1,t_2) + h_2^{(0,1)}(t_1,t_2)$$

$$- u_{-1}(t_1)u_{-1}(t_2) \exp(-t_1-t_2) \quad (1.19)$$
\[ 0 = h_k^{(1,0,0,\ldots,0)} + h_k^{(0,1,0,\ldots,0)} + h_k^{(0,0,1,\ldots,0)} + \ldots + h_k^{(0,0,0,\ldots,1)} \quad ; \quad k = 3,4,5,\ldots \quad (1.20) \]

Eqs. 1.18 - 1.20 are analogous to Eq. 1.10. They are multilinear, rather than nonlinear and can be solved with the aid of the multivariate bilateral Laplace transform.² It is

\[ H_k(s_1,\ldots,s_k) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_k(t_1,\ldots,t_k) \exp(-s_1 t_1 - \ldots - s_k t_k) \, dt_1 \ldots dt_k \quad (1.21) \]

The transformation of Eqs. 1.18 - 1.20 yields that

\[ 0 = (s_1 + 1) H_1(s_1) \quad (1.22) \]

\[ 0 = (s_1 + s_2 + 1) H_2(s_1,s_2) - \frac{1}{(s_1+1)(s_2+1)} \quad (1.23) \]

\[ 0 = (s_1 + s_2 + \ldots + s_k + 1) H_k(s_1,s_2,\ldots,s_k) \quad ; \quad k = 3,4,5,\ldots \quad (1.24) \]

By solving these equations for the kernel transforms, taking the inverse transform and taking into account the boundary condition, Eq. 1.5, we find that all the kernels except \( h_2 \) are zero. The transform of \( h_2 \) is

\[ H_2(s_1,s_2) = \frac{1}{(s_1+1)(s_2+1)(s_1+s_2+1)} \quad (1.25) \]

and the second-order kernel is
\begin{equation}
h_2(t_1, t_2) = \frac{u_1(t_1) u_1(t_2)}{\exp\left(-\bar{m}(t_1, t_2)\right) - \exp(-t_1 - t_2)}
\end{equation}

where \(\bar{m}\) is the max function

\begin{equation}
\bar{m}(t_1, t_2) = \begin{cases}
  t_1, & t_1 \geq t_2; \\
  t_2, & t_1 \leq t_2
\end{cases}
\end{equation}

It can now be verified, by direct substitution into Eq. 1.4, that the input-output relation of the system shown in Figure I-2 is

\begin{equation}
z(t) = \int_{-\infty}^{\infty} h_2(t_1, t_2) x(t - t_1) x(t - t_2) \, dt_1 \, dt_2
\end{equation}

Eqs. 1.4 and 1.28 both represent the input-output transfer relation of the nonlinear system. Depending upon the application, either representation has some advantages over the other. In the case of stochastic inputs, Eq. 1.28 is more tractable than Eq. 1.4.

One would normally write Eq. 1.11 by inspection of Eq. 1.1. We shall show a technique whereby Eqs. 1.17, 1.22 to 1.24 could have been written by inspection of Eq. 1.4.
CHAPTER II
FUNCTIONALS AND SYSTEMS

In this thesis, we shall develop an inspection technique for finding Volterra series solutions to those nonlinear differential equations which characterize analytic systems. In this chapter, we shall summarize those results of system theory and functional calculus which we shall have need to use.

Functionals

Functionals are somewhat like functions. Functions assign points to points but functionals assign points to functions (that is, functionals assign points to the way in which points are assigned to points). Volterra, the father of functionals, puts it that the "definition of a functional recalls especially the ordinary general definition of a function given by Dirichlet."* The formal definitions of functions and functionals presuppose set theory. An informal definition of a functional is:

DEFINITION OF A FUNCTIONAL. If F assigns a point \( F[x] \) to a function \( x \), then \( F \) is a functional, \( x \) is the argument of \( F \) and \( F[x] \) is the value of \( F \) for \( x \).

Figure II-1 illustrates the definition of a functional. Each function is a bundle of lines going from its domain to its range.

* The italics are Volterra's.
Figure II-1 Illustration of a Functional
A string is tied around each of these bundles of lines (function) and goes from it to some point in a set of points $\mathcal{Q}_F$. The bundle of all these strings is a functional $F$.

**Systems**

A system associates pairs of signals (a signal is a function of time). An informal definition of a deterministic system is:

**DEFINITION OF A DETERMINISTIC SYSTEM.** If $S$ assigns a signal $y$ to a signal $x$, then $S$ is a deterministic system, $x$ is the input to $S$ and $y$ is the system's output for $x$. The set of inputs to $S$ is called the system's input ensemble. The set of outputs of $S$ is called the system's output ensemble.

Not all systems are deterministic systems. For example, the communication channels that are studied in Information Theory, which for input $x_i$ can have output $y_j$ with probability $p(y_j|x_i)$, are not deterministic systems. Such systems might be called probabilistic systems. Also, systems with hysteresis may or may not be deterministic systems, depending upon whether or not they are always started from the same state.

The input-output relations of deterministic systems can be expressed by a functional. For example, if a system is linear, deterministic, time-invariant, realizable and stable, then the value of its output $y$ at a time $t$ is the convolution of its input $x$ with its impulse response $h$. That is
\[ y(t) = \int_{0}^{\infty} h(\tau) x(t-\tau) d\tau = \int_{-\infty}^{t} x(\tau) h(t-\tau) d\tau \quad (2.1) \]

Eq. 2.1 shows that at any given time \( t_0 \) the system's output is the value of some functional \( L_0 \) for \( x \). That is

\[ y(t_0) = L_0[x] \quad (2.2) \]

On the other hand, Eq. 2.1 also shows that for any given input signal \( x_0 \) the system's output is the value of some function \( L_{x_0} \) at \( t \). That is

\[ y(t) = L_{x_0}(t) \quad (2.3) \]

Eqs. 2.2 and 2.3 together show that there is an \( L \) which is both a function and a functional such that the value of the system's output for \( x \) at \( t \) is

\[ y(t) = L[x, (t)] \quad (2.4) \]

Eq. 2.1 is more specific than Eq. 2.4 as to the way in which the system's input influences its output. For example, Eq. 2.1 shows that only the interval \(( -\infty, t)\) of the domain of \( x \) pertains to \( y(t) \). Whenever we wish to explicitly exhibit the domain of a function which influences the value of a functional, we shall do so by the notation \(^{10,33}\)

\[ y = F[x(\tau)]^{b}_{\tau=a} \quad (2.5) \]

Eq. 2.5 is read as "\( y \) is the value of the functional \( F \) for the
function \( x \) from the interval \( (a, b) \) of its domain."

With the added notational explicitness of Eq. 2.5, for the linear system we can now write that

\[
y(t) = \int_{-\infty}^{t} x(\tau) h(t-\tau) d\tau = L \left[ x(\tau), (t) \right]_{\tau=-\infty}^{t} \tag{2.6}
\]

The input-output relations of nonlinear systems can also be expressed by functionals. Consider any deterministic system \( S \) with input signal ensemble \( X = \{ x_n \} \) and output signal ensemble \( Y = \{ y_m \} \), as is shown in Figure II-2. Since \( S \) is deterministic, then for any given input signal \( x_k \in X \) the system's output can be one and only one signal \( y_k \in Y \). Thus the value of any deterministic system's output signal \( y \) at a time \( t \) is, by definition, a functional of that system's input signal \( x \). That is

\[
y(t) = S \left[ x(\tau), (t) \right]_{\tau=a(t)}^{b(t)} \tag{2.7}
\]

where the interval \( (a(t), b(t)) \) is any common interval of the domains of all the \( x_k \in X \) such that each \( x_k \) that assigns a distinct \( y_k(t) \) is distinct (in other words, the interval is long enough so that we can tell which input signal it is).

If system \( S \) is time-invariant as well as deterministic, then there are input-output pairs of signals, \( (x, y) \) and \( (x_T, y_T) \), where \( x, x_T \in X \) and \( y, y_T \in Y \), such that for any \( T \)

\[
x(\xi) = x_T(\xi - T) \tag{2.8}
\]

\[
y(\xi) = y_T(\xi - T) \tag{2.9}
\]
Figure II-2 A Deterministic System
Eq. 2.7 shows that the value of the system's output at a time \( t_0 \) for the input \( x_T \) is

\[
y_T(t_0) = S\left[ x_T(\tau), (t_0) \right]_{\tau=a(t_0)}^{b(t_0)}
\] (2.10)

When we substitute Eqs. 2.8 and 2.9 in Eq. 2.10 and choose \( T = t - t_0 \), then the result is that *

\[
y(t) = S\left[ x(t+\tau), (t_0) \right]_{\tau=a}^{b}
\] (2.11)

where \( a = a(t_0) - t_0 \), and \( b = b(t_0) - t_0 \).

That is, if we know a time-invariant, deterministic system's functional at some instant of time, then we know that system's output at any time for any input.

Figure II-3 illustrates Eq. 2.11. Based upon Figure II-3, it is clear why (1) the interval \((a,b)\) is called the system's memory, (2) if \( b \geq 0 \) then the system is unrealizable, and (3) if \( b \leq 0 \) then the system is realizable. (4) If \( a = b = 0 \), then the value of the system's output at time \( t \) depends exclusively upon the value of the system's input at time \( t \). That is, the value of the system's output is a function (rather than a functional) of its input. Such a system is called a no-memory system.

* Eq. 2.6 is not a contradiction of Eq. 2.11. Eq. 2.6 could have been written as

\[
y(t) = \int_{0}^{\infty} h(\tau) x(t-\tau) \, d\tau = L\left[ x(t+\tau), (0) \right]_{\tau=-\infty}^{0}
\]
Figure II-3 A Time-Invariant Deterministic System

\[ S \left[ x(t+\tau), \left(t_0 \right) \right]_{\tau=a}^{b} = S \left[ x(\xi), \left(t_0 \right) \right]_{\xi=t+a}^{t+b} = y(t) \]
Two results about finite memory, time-invariant, deterministic systems that can be shown from Eq. 2.11 are: (1) Input signals which are constant in time yield output signals which are also constant in time. (2) Input signals which are periodic in time T yield output signals which are also periodic in time T. Thus oscillators, counters, "divide-by" circuits, etc., are not systems which are deterministic, time-invariant, and finite memoried (the latter systems because their outputs can contain frequency components which are subharmonics of their input signals).

Calculus of Functionals

Volterra, and others, have developed a calculus of functionals. Continuity for functionals has been defined. The first functional derivative,

\[ F^{(1)} \left[ x(\tau), (\tau_1) \right]_a^b \]

and the n-th functional derivative,

\[ F^{(n)} \left[ x(\tau), (\tau_1), \ldots, (\tau_n) \right]_a^b \]

of the functional

\[ F \left[ x(\tau) \right]_a^b \]

have been defined. Volterra has shown a theorem for analytic
functionals which is an analog of Taylor's Theorem for analytic functions. Volterra's Theorem is

\[
F \left[ x_0(\tau) + x(\tau) \right]^{b}_{\tau=a} = F \left[ x_0(\tau) \right]^{b}_{\tau=a} \\
+ \int_a^b F^{(1)} \left[ x_0(\tau), (\tau_1) \right]^{b}_{\tau, \tau_1=a} x(\tau_1) \, d\tau_1 + \ldots \\
+ \frac{1}{n!} \int_a^b \ldots \int F^{(n)} \left[ x_0(\tau), (\tau_1), \ldots, (\tau_n) \right] x(\tau_1) \ldots \ldots x(\tau_n) \, d\tau_1 \ldots d\tau_n + \ldots
\]  

(2.12)

For our applications, we shall be interested in Eq. 2.12 when the function \(x_0\) is zero. That is

\[
F \left[ x(\tau) \right]^{b}_{\tau=a} = F \left[ (0) \right] + \int_a^b F^{(1)} \left[ (0), (\tau_1) \right] x(\tau_1) \, d\tau_1 + \ldots \\
+ \frac{1}{n!} \int_a^b \ldots \int F^{(n)} \left[ (0), (\tau_1), \ldots, (\tau_n) \right] x(\tau_1) \ldots \ldots x(\tau_n) \, d\tau_1 \ldots d\tau_n + \ldots
\]  

(2.13)
DEFINITION OF AN ANALYTIC SYSTEM. If the functional of a time-
invariant deterministic system S (see Eq. 2.11) is analytic about
zero input at some time \( t_0 \), then S is an analytic system.

Eqs. 2.11 and 2.13 can be used to show that the value of
the output \( y \) of an analytic system S for input \( x \) is

\[
y(t) = S \left[ (0), (t_0) \right] + \int_a^b S^{(1)} \left[ (0), (t_0), (\tau_1) \right] x(t+\tau_1) \, d\tau_1 + \ldots
\]

\[
+ \frac{1}{n!} \int_a^b \cdots \int_a^b S^{(n)} \left[ (0), (t_0), (\tau_1), \ldots, (\tau_n) \right] x(t+\tau_1) \ldots x(t+\tau_n) \, d\tau_1 \ldots d\tau_n + \ldots
\]

(2.14)

For our purposes, much of the notation in Eq. 2.14 is
unnecessary. For example, \( S \left[ (0), (t_0) \right] \) is merely a constant and
\( S^{(n)} \left[ (0), (t_0), (\tau_1), \ldots, (\tau_n) \right] \) is merely a symmetric function
on \( n \) variables. For notational ease and in order to conform with
the works of others, we shall write in place of Eq. 2.14 that the
value of the output \( y \) of an analytic system H for input \( x \) is

\[
y(t) = H \left[ x \right]
\]

\[
= h_0 + \int_{-\infty}^\infty h_1(\tau_1) x(t-\tau_1) \, d\tau_1 + \ldots
\]

\[
+ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) x(t-\tau_1) \ldots x(t-\tau_n) \, d\tau_1 \ldots d\tau_n + \ldots
\]

(2.15)
where

\[ h_0 = S \int_{a}^{b} (0), (t_0), \tau \]  

(2.16)

and

\[ \text{Sym} \left\{ \frac{1}{n!} S^{(n)} \int_{a}^{b} (0), (t_0), (-\tau_1), \ldots, (-\tau_n) \right\} = \prod_{k=1}^{n} \frac{u_{-1}(\tau_k - \alpha)}{\tau - \tau_k} + \frac{u_{-1}(\tau_k + \beta)}{\tau - \tau_k} \]

(2.17)

where

\[ \text{Sym} \left( \tau_1, \ldots, \tau_n \right) f(\tau_1, \ldots, \tau_n) \]

indicates the operation of symmetrization of the function \( f \) in its \( n \) arguments. This is accomplished by summing the values of \( f \) at all \( n! \) possible permutations of its \( n \) arguments and dividing the sum by \( n! \). This symmetrization operator must be introduced because \( S^{(n)} \) is symmetric in its \( n \) arguments (this is a property of functional derivatives) but \( h_n \) need not be symmetric in its \( n \) arguments. Eq. 2.17 is a necessary and sufficient condition such that like order integrals in Eqs. 2.14 and 2.15 have the same values for all signals \( x \). (The symmetrization operator corresponds to permuting the dummy variables of integration.)

We shall refer to Eq. 2.15 as a Volterra series and shall call the constant \( h_0 \) and the functions \( h_n \) the kernels of the
Volterra series. We shall also have need to refer to separate terms within the Volterra series. For this purpose, when we say "the k-th order term in the Volterra series", we shall mean

\[ y_0(t) = H_0[x] = h_0 \quad , \text{when } k = 0 , \] (2.18)

and when \( k \neq 0 \),

\[ y_k(t) = H_k[x] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_k(\tau_1, \ldots, \tau_k)x(t-\tau_1)\ldots x(t-\tau_k) \]
\[ d\tau_1 \ldots d\tau_k \] (2.19)

Combinations of Analytic Systems

The combinations of analytic systems have been studied extensively. In this section, we shall summarize those results which we shall use. These results are more easily stated in the frequency domain than in the time domain, hence we shall define the multivariate bilateral Laplace transform of a function of k variables. It is

\[ H_k(s_1, \ldots, s_k) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_k(\tau_1, \ldots, \tau_k) \exp(-s_1 \tau_1 - \cdots - s_k \tau_k) \]
\[ d\tau_1 \ldots d\tau_k \] (2.20)

where \( s_k = \sigma_k + j \omega_k \)

* Of course \( h_0 \) is not a kernel. However, it would be awkward to forever have to refer to "\( h_0 \) and the kernels."
Consider three analytic systems, F, G, and H, whose outputs are p, q, and r. That is, by Eqs. 2.15, 2.18, and 2.19, when the input to these systems is x, their outputs are

\[ p(t) = F[x] = \sum_{k=0}^{\infty} F_k[x] \]  

(2.21)

\[ q(t) = G[x] = \sum_{k=0}^{\infty} G_k[x] \]  

(2.22)

\[ r(t) = H[x] = \sum_{k=0}^{\infty} H_k[x] \]  

(2.23)

**ADDITIVE COMBINATION.** \( F + G = H \) (See Figure II-4).

If the outputs of two analytic systems, F and G, are added, then their sum, \( r = p+q \), is the output of an analytic system H and

\[ H_k(s_1, \ldots, s_k) = F_k(s_1, \ldots, s_k) + G_k(s_1, \ldots, s_k) \]  

(2.24)

**MULTIPLICATIVE COMBINATION.** \( F \cdot G = H \) (See Figure II-5).

If the outputs of two analytic systems, F and G, are multiplied, then their product, \( r = pq \), is the output of an analytic system H and

\[ H_n(s_1, \ldots, s_n) = \sum_{m=0}^{n} F_m(s_1, \ldots, s_m) G_{n-m}(s_{m+1}, \ldots, s_n) \]  

(2.25)
Figure II-4 The Additive Combination of Analytic Systems
Figure II-5 The Multiplicative Combination of Analytic Systems
CASCADE COMBINATION. \( G(*)F = H \) \ (See Figure II-6). 

If the input to an analytic system \( F \) is \( x \), and its output \( p (p = F[x]) \) is the input to an analytic system \( G \), then its output \( q (q = G[p] = G[F[x]]) \) is the output of an analytic system \( H \) and \( 6 \)

**Case 1:** If \( G \) is linear, then

\[
H_n(s_1, \ldots, s_n) = F_n(s_1, \ldots, s_n) G_1(s_1 + \ldots + s_n)
\] \hspace{1cm} (2.26)

**Case 2:** If \( F \) is linear, then

\[
H_n(s_1, \ldots, s_n) = F_1(s_1) F_2(s_2) \ldots F_l(s_l) G_n(s_1, \ldots, s_n)
\] \hspace{1cm} (2.27)

**Case 3:** If neither \( F \) nor \( G \) is linear, then see Brilliant for the combination formula. \( 6 \)

FEEDBACK COMBINATION. \ (See Figure II-7). 

The feedback combination of analytic systems is an analytic system (except when the feedback makes the system unstable). See Brilliant, George, and Zames. \( 7, 17, 43 \)

George's Association Technique 

The sole remaining result of functional calculus which we will use but have not yet introduced is George's frequency association technique. \( 19 \) It is a technique for evaluating, by inspection, the transforms of signals which are the outputs of analytic systems.
Figure II-6 The Cascade Combination of Analytic Systems
Figure II-7 The Feedback Combination of Analytic Systems
The $k$-th order term of the system's output (see Eq. 2.19) has a \textbf{multilinear correspondent}. It is

$$y(k)(t_1, \ldots, t_k) = \int_{-\infty}^{\infty} \cdots \int \ h_k(\tau_1, \ldots, \tau_k) \ x(t_1 - \tau_1) \cdots \ x(t_k - \tau_k) \ d\tau_1 \cdots d\tau_k$$

(2.28)

The transform of the multilinear correspondent is a product.

$$Y(k)(s_1, \ldots, s_k) = H_k(s_1, \ldots, s_k) \ X(s_1) \cdots X(s_k)$$

(2.29)

An example of the application of George's frequency association technique is:

If

$$Y(2)(s_1, s_2) = A(s_1 + s_2) \ \frac{B}{s_1 + b} \ \frac{C}{s_2 + c}$$

(2.30)

then

$$Y_2(s) = A(s) \ \frac{BC}{s + b + c}$$

(2.31)

If

$$Y(2)(s_1, s_2) = A(s_1 + s_2) \ \frac{B}{(s_1 + b)^n} \ \frac{C}{(s_2 + c)^m}$$

(2.32)

then

$$Y_2(s) = A(s) \ \frac{BC \ (n+m-2)!}{(n-1)! \ (m-1)! \ (s + b + c)^{n+m-1}}$$

(2.33)
In our later applications of George's frequency association technique it will become clear that the evaluation of the transforms of the output signals of analytic systems is straightforward so long as the transforms of the multi-linear correspondents have partial fraction expansions.

Multi-input Systems

The extension of these results to multi-input systems is straightforward. The value of the output $y$ of an analytic system whose simultaneous input signals are $u, v, w, \ldots$ is

$$y(t) = H[u, v, w, \ldots]$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \cdots H_{i,j,k,\ldots}[u, v, w, \ldots]$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \int \int \cdots \int_{\infty}^{\infty} h_{i,j,k,\ldots}(\tau_1, \tau_2, \ldots)$$

$$u(t-\tau_1) \cdots u(t-\tau_1) v(t-\tau_{i+1}) \cdots v(t-\tau_{i+j}) w(t-\tau_{i+j+1}) \cdots$$

$$w(t-\tau_{i+j+k}) \cdots d\tau_1 d\tau_2 \cdots$$

(2.34)

where it must be understood (as a convention) that a zero subscript obviates the corresponding integrations.
CHAPTER III

THE VOLterra SERIES SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

Based upon the results summarized in Chapter II, in this chapter we shall show (1) which differential equations characterize analytic systems, and (2) an inspection technique for finding the Volterra series solutions to those differential (or for that matter, integral or integro-differential equations) which characterize analytic systems. This technique's rational is: (1) If a differential equation characterizes an analytic system, then there is a Volterra series which satisfies that differential equation. (2) Because the system combination theorems show that the addition of, the multiplication of or an analytic operation upon (the cascade) a Volterra series yields a Volterra series, then its substitution into the differential equation yields a Volterra series. (3) If two Volterra series are equal, then the symmetrization of their like order kernels must be equal.

The justification of this last statement follows: Consider two Volterra series which are equal for all inputs. That is

$$ F[x] = G[x] , \text{ all } x \quad (3.1) $$

Eq. 2.19 shows that

$$ H_k[\lambda x] = \lambda^k H_k[x] \quad (3.2) $$
for any constant \( \lambda \). Then, by Eqs. 3.1, 3.2, and 2.15, we see that

\[
\sum_{n=0}^{\infty} \lambda^n F_n(x) = \sum_{m=0}^{\infty} \lambda^m G_m(x)
\]  

(3.3)

for any constant \( \lambda \).

Eq. 3.3 equates two power series in \( \lambda \). For two power series to be equal, their like order coefficients must be equal. That is

\[
F_k(x) = G_k(x), \text{ all } x
\]  

(3.4)

But Eq. 2.17 shows that the kernels of both of these k-order functionals, when symmetrized, equal the same k-order functional derivative. Thus

\[
\text{Sym} \left\{ F_k(s_1, \ldots, s_k) \right\} = \text{Sym} \left\{ G_k(s_1, \ldots, s_k) \right\}
\]  

(3.5)

The application of this technique will be illustrated by examples.

Example One

Consider a system whose input is \( x \) and whose output \( y \) obeys the differential equation

\[
\frac{dy}{dx} = 1 + \frac{dx}{dt}
\]  

(3.6)

and the boundary condition of initial rest. That is

\[
y(t) = 0 \text{ until } x(t) \neq 0
\]  

(3.7)
Because the system characterized by Eqs. 3.6 - 3.7 is an analytic system (we shall postpone until later the question of how we know that it is an analytic system), then there is some Volterra series $H$ such that

$$y(t) = H[x] = \sum_{k=0}^{\infty} H_k[x] , \text{ any } x$$

(3.8)

The term $\frac{dy}{dx}$ in Eq. 3.6 is not the result of an analytic operation upon $y$ (Eq. 3.8) so Eq. 3.6 is not yet in the form whereby we can exploit the system combination theorems. However, since

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

(3.9)

then Eq. 3.6 can be rewritten as

$$\frac{dy}{dt} = \left[1 + \frac{dx}{dt}\right] \frac{dx}{dt}$$

(3.10)

All the terms in Eq. 10 are the results of analytic operations upon either $x$ or $y$. Specifically, $\frac{dy}{dt}$ is the result of a cascade of system $H$ with a linear system whose 1-order (its sole term) kernel's transform is $s$. (See Figure III-1.) The other two terms in Eq. 3.10 are trivial Volterra series in $x$. (See Figures III-2 and III-3.) Thus, by the system combination theorems of Chapter II and Eq. 3.5, we may write from Eq. 3.10 by inspection that

$$s_1 H_1(s_1) = s_1$$

(3.11)
Figure III-2 The Synthesis of $\frac{dx}{dt}$
Figure III-3 The Synthesis of the Square of $\frac{dx}{dt}$
\[
\text{Sym}_{(s_1, s_2)} \left\{ (s_1 + s_2) H_2(s_1, s_2) \right\} = s_1 s_2 \quad (3.12)
\]

\[
\text{Sym}_{(s_1, \ldots, s_k)} \left\{ (s_1 + \ldots + s_k) H_k(s_1, \ldots, s_k) \right\} = 0 ; \quad k = 3, 4, \ldots \quad (3.13)
\]

By considering the boundary condition (Eq. 3.7) when \( x = 0 \), we can add that

\[
H_0(x) = 0 \quad (3.14)
\]

For solutions to Eqs. 3.11 - 3.13, let us choose the remaining kernels to be

\[
H_1(s_1) = 1 \quad (3.15)
\]

\[
H_2(s_1, s_2) = \frac{s_1 s_2}{s_1 + s_2} \quad (3.16)
\]

\[
H_k(s_1, \ldots, s_k) = 0 ; \quad k = 3, 4, \ldots \quad (3.17)
\]

These solutions are amenable to synthesis by another application of the system combination theorems. (See Figure III-4.) The time domain expression for \( y \) can be written by inspection of this synthesis (Figure III-4). It is

---

* Which "choice" of kernels we chose that solve Eqs. 3.11 - 3.13 is irrelevant. Any kernels which satisfy Eqs. 3.11 - 3.13 will, when substituted into Eq. 3.8, yield the same value for \( y \). The only difference in the answers will be the way in which the integration over the dummy variables takes place.
Figure III-4 The Synthesis of H
\[ y(t) = u_0(t)(\ast)x(t) + u_{-1}(t)(\ast)(u_{+1}(t)(\ast)x(t))^2 \]

\[ = x(t) + \int_{-\infty}^{t} \left( \frac{dx}{dt} \right)^2 dt \]

(3.18)

The observation that Eq. 3.18 is indeed the solution to Eqs. 3.6 and 3.7 is trivial. However, Example 1 was not chosen to show the solution of a difficult differential equation, rather, it was chosen to be an easy illustration of the solution technique. An example using a less trivial differential equation follows.

**Example Two**

Consider a system whose input is \( x \) and whose output is \( y \) which obeys the differential equation

\[ x \left[ 1 + \frac{d}{dt} \right]^3 y - \frac{dx}{dt} \left[ 1 + \frac{d}{dt} \right]^2 y = 2 x^3 \]

(3.19)

and the boundary condition of initial rest. That is

\[ y(t) = 0 \text{ until } x(t) \neq 0 \]

(3.20)

Because the system characterized by Eqs. 3.19 - 3.20 is an analytic system (we shall postpone until later the question of how we know that it is an analytic system), then there is some Volterra series \( F \) such that

\[ y(t) = F[x] = \sum_{k=0}^{\infty} F_k[x], \text{ all } x \]

(3.21)
Eq. 3.19 is already written in a form such that all the terms are the results of analytic operations upon either \(x\) or \(y\). Hence, by inspection, we write

\[
1 \left(1 + 0\right)^3 F_0 - s_1 \left(1 + 0\right)^2 F_0 = 0
\]  \hspace{1cm} (3.22)

\[
\text{Sym}_{(s_1, s_2)} \left\{ 1 \left(1 + s_2\right)^3 F_1(s_2) - s_1 \left(1 + s_2\right)^2 F_1(s_2) \right\} = 0
\]  \hspace{1cm} (3.23)

\[
\text{Sym}_{(s_1, s_2, s_3)} \left\{ 1 \left(1+s_2+s_3\right)^3 F_2(s_2, s_3) - s_1 \left(1+s_2+s_3\right)^2 F_2(s_2, s_3) \right\} = 2
\]  \hspace{1cm} (3.24)

\[
\text{Sym}_{(s_1, \ldots, s_{k+1})} \left\{ 1 \left(1+s_2+\ldots+s_{k+1}\right)^3 F_k(s_2, \ldots, s_{k+1}) - s_1 \left(1+s_2+\ldots+s_{k+1}\right)^2 F_k(s_2, \ldots, s_{k+1}) \right\} = 0
\]  \hspace{1cm} (3.25)

For Eqs. 3.22, 3.23, and 3.25 we shall choose the solutions

\[
0 = F_0 = F_1 = F_3 = F_4 = \ldots
\]  \hspace{1cm} (3.26)

But for Eq. 3.24 we must choose an \(F_2\) such that

\[
\text{Sym}_{(s_1, s_2, s_3)} \left\{ \left(1+s_2+s_3\right)^2 \left(1-s_1+s_2+s_3\right) F_2(s_2, s_3) \right\} = 2
\]  \hspace{1cm} (3.27)

An \(F_2\) which satisfies Eq. 3.27 can be found quickly if we restrict our choice of \(F_2\)'s to those which are symmetrical. That is

\[
F_2(s_a, s_b) = F_2(s_b, s_a) \quad \text{all } s_a, s_b
\]  \hspace{1cm} (3.28)
When we evaluate Eq. 3.27 at \((s_1, s_2, s_3) = (0, 0, 0)\), then the result is that

\[
F_2(0, 0) = 2 \tag{3.29}
\]

When we evaluate Eq. 3.27 at \((s_1, s_2, s_3) = (s, 0, 0)\), then the result is that

\[
\frac{1}{3} \left\{ [1]^2 [1-s] F_2(0, 0) + [1+s]^2 [1+s] F_2(s, 0) + [1+s]^2 [1+s] F_2(0, s) \right\} = 2 \tag{3.30}
\]

When Eqs. 3.29 and 3.29 are substituted into Eq. 3.30, then the result is that

\[
F_2(0, s) = F_2(s, 0) = \frac{s + 2}{(s + 1)^3} \tag{3.31}
\]

When we evaluate Eq. 3.27 at \((s_1, s_2, s_3) = (s_1, s_2, 0)\), then the result is that

\[
\frac{1}{6} \left\{ (1+s_2)^2 (1-s_1+s_2) F_2(s_2, 0) + (1+s_1)^2 (1-s_1+s_2) F_2(0, s_2) + (1+s_2)^2 (1-s_2+s_1) F_2(0, s_1) + (1+s_1+s_2)^2 (1+s_1+s_2) F_2(s_1, s_2) + (1+s_1+s_2)^2 (1+s_1+s_2) F_2(s_2, s_1) \right\} = 2 \tag{3.32}
\]

When Eqs. 3.28 and 3.31 are substituted into Eq. 3.32, then the
result is that

\[ F_2(s_1, s_2) = \frac{s_1 + s_2 + 2}{(s_1 + 1)(s_2 + 1)(s_1 + s_2 + 1)^2} \]  
(3.33)

The inverse transform of Eq. 3.33 is

\[ f_2(\tau_1, \tau_2) = u_1(\tau_1) u_1(\tau_2) \, m(\tau_1, \tau_2) \, \exp(-\tilde{m}(\tau_1, \tau_2)) \]  
(3.34)

where \( m \) and \( \tilde{m} \) are the minimum and maximum functions. They are

\[ m(\tau_1, \tau_2) = \begin{cases} \tau_1, & \tau_1 \leq \tau_2 \\ \tau_2, & \tau_1 > \tau_2 \end{cases} \]  
(3.35)

\[ \tilde{m}(\tau_1, \tau_2) = \begin{cases} \tau_2, & \tau_1 \leq \tau_2 \\ \tau_1, & \tau_1 > \tau_2 \end{cases} \]  
(3.36)

When the expression for \( f_2 \) (the only nonzero kernel) is substituted into Eq. 3.21, then the result (the solution to Eqs. 3.19 - 3.20) is

\[ y(t) = \iint_{-\infty}^{\infty} f_2(\tau_1, \tau_2) \, x(t-\tau_1) \, x(t-\tau_2) \, d\tau_1 \, d\tau_2 \]

\[ = 2 \iint_{0}^{\infty} \tau_2 \, \exp(-\tau_2) \, x(t-\tau_1) \, x(t-\tau_2) \, d\tau_1 \, d\tau_2 \]  
(3.37)

* It should be clear that the method that we have just used to get Eq. 3.33 from Eq. 3.27 is general. That is, whenever a k-order kernel transform is in a constraint equation with more than k frequencies, then that kernel's transform can be found in k+1 steps; namely by evaluating that equation at \((0,0,0,\ldots), (s_1,0,0,\ldots), (s_1, s_2,0,\ldots), \ldots, \) and \((s_1, s_2,\ldots, s_k,0,\ldots).\)
A frequency domain synthesis of this system (from Eq. 3.33) is shown by Figure III-5. Upon examination of Figure III-5, two qualitative observations about the system's behavior may be made by inspection:

(i) If \( x \) is a low frequency signal (\( |s| \ll 1 \)), then \( y \approx 2x^2 \).

(ii) If \( x \) is a high frequency signal (\( |s| \gg 1 \)) of narrow band-width (small compared to \( 1 \)), then \( y \) is approximately twice the low frequency component of the square of the integral of \( x \).

Both of these observations are simple by-products of the Volterra series solution of the system characterized by Eqs. 3.19 - 3.20. These equations completely describe this system but neither observation (i) nor (ii) were, a priori, obvious from them. Thus we see that the Volterra series description of a system may well provide more insight about the behavior of that system than its differential equation description.

In the two examples considered thus far, the extraction of like order terms in the kernel transforms from the system's characteristic equation gave equations which were independent. This is not usually the case. The usual result is a set of simultaneous equations in the kernels. In this event, solution for the kernel transforms involves the formulation of a recursive formula for the kernel transforms. An example follows.
Figure III-5 The Synthesis of F
Example Three

Consider a system whose input is x and whose output is y which obeys the differential equation

$$y + \frac{dy}{dt} - [2y + 1]x = y^2 [x - 1]$$  \hspace{1cm} (3.38)

and the boundary condition of initial rest. That is

$$y(t) = 0 \text{ until } x(t) \neq 0$$  \hspace{1cm} (3.39)

Because the system characterized by Eqs. 3.38 - 3.39 is analytic, for a certain class of input signals, (we shall postpone until later the questions of which class of input signals and how we know that it is an analytic system for them), then there is some Volterra series G such that

$$y(t) = G[x] = \sum_{k=0}^{\infty} G_k[x], \text{ all } x \text{ in a class}$$  \hspace{1cm} (3.40)

Eq. 3.38 is already in a form such that all the terms are the results of analytic operations upon either x or y. Hence, by inspection, we may write that

$$G_0 + 0 - 0 = 0 - G_0^2$$  \hspace{1cm} (3.41)

$$G_1(s_1) + s_1G(s_1) - (2G_0 + 1)l = G_0^2 - 2G_0G_1(s_1)$$  \hspace{1cm} (3.42)
\[ \text{Sym} \left\{ G_k(s_1, \ldots, s_k) + (s_1 + \ldots + s_k)G_k(s_1, \ldots, s_k) \right\} - 2G_{k-1}(s_1, \ldots, s_{k-1}) = \text{Sym} \left\{ \sum_{n=0}^{k-1} G_n(s_1, \ldots, s_n) \right\} \]

\[ \cdots, s_n)G_{k-n-1}(s_{n+1}, \ldots, s_{k-1}) - \sum_{m=0}^{k} G_m(s_1, \ldots, s_m) \cdot \]

\[ G_{k-m}(s_{m+1}, \ldots, s_k) \right\} ; \quad k = 2, 3, \ldots \] (3.43)

Eq. 3.41 can be rewritten as

\[ (G_0 + 1) G_0 = 0 \] (3.44)

Eq. 3.42 can be solved for \( G_1 \). The result is

\[ G_1(s_1) = \frac{(G_0 + 1)^2}{s_1 + 2G_0 + 1} \] (3.45)

The highest order term in Eq. 3.43 can be isolated and solved for. The result is

\[ \text{Sym} \left\{ G_k(s_1, \ldots, s_k) \right\} = \text{Sym} \left\{ G_{k-1}(s_1, \ldots, s_{k-1}) + \sum_{n=1}^{k-1} G_n(s_1, \ldots, s_n) \right\} \] \[ \begin{aligned} \cdots, s_{k-1} \right\} (G_0 + 2) + \sum_{n=1}^{k-1} G_n(s_1, \ldots, s_n) \right\} \]

\[ s_1 + \ldots + s_k + 2G_0 + 1 \]

\[ \cdots, s_{k-1} - G_{k-n}(s_{n+1}, \ldots, s_k) \right\} ; \quad k = 2, 3, 4, \ldots \] (3.46)
Eq. 3.46 is a recursive formula for \( G_k \). That is, given \( G_0 \) and \( G_1 \), it constrains \( G_2 \); given \( G_0, G_1 \) and \( G_2 \), it constrains \( G_3 \); and so on.

Eq. 3.44 allows us two choices of values for \( G_0 \), 0 and \(-1\). These two distinct values of \( G_0 \) correspond to the two distinct states possessed by that system which obeys Eq. 3.38. They are: the \(-1\) State \( ' \) (Denote by a single prime(')), and the \(0\) State \( " \) (Denote by a double prime('')).

\(-1\) State \( ' \). If, for solution to Eq. 3.44, we choose

\[
G_0' = -1
\]  \( (3.47) \)

then the solution to Eq. 3.45 is

\[
G_1'(s_1) = 0
\]  \( (3.48) \)

and, for Eq. 3.46, we may choose the solutions

\[
G_k'(s_1, \ldots, s_k) = 0 \quad ; \quad k = 2, 3, \ldots
\]  \( (3.49) \)

The substitution of the kernels whose transforms are given by Eqs. 3.47 - 3.49 into Eq. 3.40 yields the Volterra series \( G' \) for the system's \(-1\) state. It is

\[
y(t) = G'[x] = -1
\]  \( (3.50) \)

That is, in its \(-1\) state, this system's output is \( y = -1 \), regardless of the input \( x \).

For input signals \( x \) such that
\[
\lim_{t \to \infty} [x(t)] \neq 0
\]

(3.51)

The output signal \( y(t) = -1 \), all \( t \), is indeed a solution to the system's characteristic equations, Eq. 3.38 - 3.39. However, for inputs which do not begin until time \( T \), that is, if

\[
x(t) = u(-1)(t-T) x(t), \text{ all } t
\]

(3.52)

then the boundary condition of initial rest, Eq. 3.39, requires that

\[
y(t) = u(-1)(t-T) y(t), \text{ all } t
\]

(3.53)

In order to satisfy Eq. 3.53, we must choose \( G_0 = 0 \). Thus, for input signals which do not begin until time \( T \), the system must be in its 0 state for at least the time interval \( (-\infty, T) \).

"0 State". If, for solution to Eq. 3.44, we choose

\[
G_0^n = 0
\]

(3.54)

then the solution to Eq. 3.45 is

\[
G_1^n(s) = \frac{1}{s_1 + 1}
\]

(3.55)

and, for Eq. 3.46, we may choose the solutions

\[
G^n_k(s_1, \ldots, s_k) = \frac{1}{(s_1 + 1) \cdots (s_k + 1)} \quad ; \quad k = 2, 3, \ldots
\]

(3.56)
The substitution of the kernels whose transforms are given by
Eqs. 3.54 - 3.56 into Eq. 3.40 yields the Volterra series \( G^n \) for
the system's 0 state. A frequency domain synthesis of this state
of the system is shown by Figure III-6. A closed form of that
synthesis is shown by Figures III-7 and III-8. Figure III-7 uses
feedback, Figure III-8 uses one linear but memoried system and one
nonlinear no-memory system. Inspection of any of these syntheses
yields that

\[
y(t) = G^n[x] = \frac{\int_0^\infty \exp(-\tau) x(t-\tau) d\tau}{\int_0^\infty \exp(-\sigma) x(t-\sigma) d\sigma}
\]  

(3.57)

Direct substitution will verify that Eq. 3.57 does indeed solve
the system's characteristic equations, Eqs. 3.38 - 3.39.

While inputs that do not begin until time \( T \), as in Eq. 3.52,
require that the system begin in its 0 state, this does not mean
that the system has to stay there. For example, consider an input
\( x_c \) which begins at time \( T \) but is such that later, at time \( T_s > T \),
it has caused an output \( y_c \)

\[
y_c = G^n[x_c]
\]  

(3.58)

such that

\[
y_c(T_s) = -1
\]  

(3.59)

and
Figure III-6 A Parallel Synthesis of $G$"
Figure III-7 A Feedback Synthesis of $G$
\[ z(t) = g(t)(x) x(t) \]

\[ g(t) = u_1(t) \exp(-t) \]

\[ f(z) = \frac{z}{1-z} = y \]
\[
\frac{d}{dt} y_c(t) \bigg|_{t = T_s} = 0
\] (3.60)

For input \( x_c \), the system could switch states at time \( t = T_s \). Having switched states, the system could then stay in its -1 state forevermore, or, at some still later time, it could decide to switch back to its 0 state—\textbf{all without violating its characteristic equations} (Eqs. 3.38 - 3.39). For inputs like \( x_c \), the system has the opportunity to exercise free will. Therefore, for this class of input signals, the system is nondeterministic and is not analytic.

Having raised the question of analyticity again, let us now answer it.

Equations Which Characterize Analytic Systems

In the three examples considered, we stated that the equations considered characterized analytic systems. Any differential equation (or for that matter, any integral or integro-differential equation) which characterizes an analytic system has a Volterra series solution. But not every differential equation characterizes an analytic system. For example, the differential equation

\[
(\frac{dy}{dt})^2 = x^2
\] (3.61)

characterizes a nondeterministic system (at any instant of time \( t \), the system may choose either \( \frac{dy}{dt} = x \) or \( \frac{dy}{dt} = -x \)) and, therefore, it characterizes a system which is not analytic.
On the other hand, the differential equation

\[ xy + \frac{dy}{dt} + 1 = 0 \]  \hspace{1cm} (3.62)

can, with an appropriate boundary condition, characterize a
deterministic system, but not one which is also time-invariant
(consider \( x(t) = u_{-1}(t-T) \) for various \( T \)) and, therefore, it
characterizes a system which is not analytic.

Otherwise still, the differential equation

\[ \frac{dy}{dt} = |x| \]  \hspace{1cm} (3.63)

can, with an appropriate boundary condition, characterize a
time-invariant deterministic system, but not one which is analytic.
(It can be shown that if the input to an analytic system is infinitely
differentiable, then that system's output must also be infinitely
differentiable. The input \( x(t) = \cos(t) \) is infinitely differentiable
but the third derivative of its corresponding output does not exist.)

In Chapter II, an analytic system was defined to be a time-
invariant deterministic system whose functional was analytic about
zero input at some time. Therefore, an equation characterizes an
analytic system if and only if (1) solution pairs to the equation
exist (so that the equation characterizes a system), (2) for a
given input signal the equation's output solutions are unique (so
that the equation characterizes a deterministic system), (3) the
equation's input-output solution pairs must co-translate in time
(so that the equation characterizes a time-invariant system), and
(4) certain limits of the equation's output solutions for small perturbations of the input signal about zero are well behaved (all the terms in Eq. 2.13 must exist and the series must converge absolutely in order for that functional to be analytic).

Given an equation, there is an assortment of ways in which to show that it has properties (1) to (4), if it does, and thereby prove that it characterizes an analytic system. There are several general tests for existence (property 1) but there are no general tests for uniqueness (property 2). In general, a uniqueness proof must be constructed for each given equation. Once uniqueness has been shown, then the proof of time-invariance is usually trivial (property 3). Showing that the functional is analytic about zero at some time (property 4) by a "brute-force" application of Volterra's definition is tedious and unnecessary. Instead, once existence, uniqueness, and time-invariance have been shown, then it is far easier to use our inspection technique to find a Volterra series solution to the equation (if there is one) and prove that it converges absolutely (perhaps for a certain class of inputs). If this can be done, then, due to uniqueness, this is sufficient to prove analyticity.

* The author's experience has been that if our inspection technique is used, on a trial basis, to get a Volterra series solution to the equation and if that solution implies a feedback synthesis for the system, then that feedback synthesis often suggests an approach to a uniqueness proof.
Multi-input Systems

The extension of our inspection technique to multi-input systems is straightforward. The only significant change is that when two Volterra series are equal (see Eq. 2.34) then the symmetrization in parts of their like order terms are equal.

That is, if

\[ F[u,v,w,\ldots] = G[u,v,w,\ldots] \quad \text{all } u,v,w,\ldots \]  \hspace{1cm} (3.64)

then

\[ \text{Sym} \left( s_1, \ldots, s_1 \right) (s_{i+1}, \ldots, s_{i+j}) (s_{i+j+1}, \ldots, s_{i+j+k}) \ldots \left\{ F_1, j, k, \ldots (s_1, s_2, \ldots) \right\} - G_1, j, k, \ldots (s_1, s_2, \ldots) \right\} = 0 \]  \hspace{1cm} (3.65)
CHAPTER IV
THE SIMPLE PENDULUM

"Molecules, pendulums, violin strings, structures, etc., all have oscillatory motion similar to that of a mass attached to a spring ... The spring-type force, $F = -kx$, ... increases linearly with $x$, the displacement from equilibrium position. An oscillator with a force of this type is called a linear or a harmonic oscillator, and the corresponding oscillation is called harmonic motion. If the force depends on $x$ in any other way, the oscillator is called nonlinear. We find that although many oscillators are nonlinear, most are linear or approximately so at sufficiently small amplitudes of oscillations." 22

In this chapter we shall present an application of the inspection technique developed in Chapter III. We shall consider an actual nonlinear physical system--a simple pendulum. We choose this as our first example because (1) the reader is intuitively familiar with a pendulum's physical behavior, (2) the reader is familiar with the standard mathematical solution of a pendulum's behavior for "small amplitudes of oscillations", (3) the Volterra series solution of a pendulum's behavior is easily obtained by our inspection technique, and (4) the Volterra series solution shows a new aspect of a pendulum's behavior--nonlinear resonance. We shall see that this nonlinear resonance phenomenon cannot be explained by the standard "small amplitudes" solution but that it is nonetheless clearly part of a pendulum's physical behavior.

Consider the damped simple pendulum, oscillating in a plane, which is shown in Figure IV-1. It is characterized by the well known differential equation
Figure IV-l A Damped Simple Pendulum
\[ 2 \frac{d^2 \theta}{dt^2} = \tau - a \frac{d \theta}{dt} - mgL \sin \theta \] (4.1)

and the boundary condition of initial rest

\[ \theta(t) = 0 \text{ until } \tau(t) \neq 0 \] (4.2)

where \( \tau \) is the input torque to the pendulum and \( \theta \) is its output angle.*

For a certain class of input torques, this pendulum is an analytic system. Thus there exists some functional \( H \) such that

\[ \theta(t) = H[\tau] = \sum_{m=0}^{\infty} H_m[\tau] \] (4.3)

Observe that if the input \( \tau \) produces the output \( \theta \), then Eq. 4.1 shows that the input \(-\tau\) produces the output \(-\theta\). The pendulum's output angle \( \theta \) is thus an odd functional of its input torque \( \tau \). That is

\[ -\theta(t) = H[-\tau] = \sum_{m=0}^{\infty} (-)^m H_m[\tau] \] (4.4)

where we have used the result given by Eq. 3.2.

Eqs. 4.3 and 4.4 are simultaneously true. Therefore all the

---

* A pendulum problem is not out of place in an Electrical Engineering thesis. Minorsky shows that, with appropriate identification of variables, Eq. 4.1 also describes the motion of a synchronous machine.
even-order terms must vanish. That is

\[ 0 = H_0 = H_2 [t] = \ldots = H_{2k} [t] = \ldots \] (4.5)

In order to find the odd-order kernels by our inspection technique, we shall substitute a Taylor series for \( \sin \theta \) in Eq. 4.1. The result is

\[ \mathcal{T} = mL^2 \frac{d^2 \theta}{dt^2} + a \frac{d \theta}{dt} + mgL \sum_{k=0}^{\infty} (-)^k \frac{\theta^{2k+1}}{(2k+1)!} \] (4.6)

By inspection, the first-order terms in Eq. 4.6 gives us that

\[ 1 = mL^2 a_1 H_1(s_1) + as_1 H_1(s_1) + mgLH_1(s_1) \] (4.7)

In order to write our solutions in a compact form, we shall define a dimensionless linear filter whose system function is \( G \). \( G(s) \) is

\[ G(s) = \frac{mgL}{ml^2 s^2 + as + mgL} \]

\[ = \frac{\omega_0^2}{(s-s_0)(s-s_0^*)} \] (4.8)

Where we have used the standard notation for the filter's natural frequencies--namely

\[ s_0 = -\alpha + j \omega_0 \] (4.9)

\[ \alpha = \frac{s}{2ml^2} \] (4.10)
\[ \omega_d = \sqrt{\omega_0^2 - \alpha^2} \quad (4.11) \]

\[ \omega_0 = \sqrt{\frac{k}{L}} \quad (4.12) \]

The s-plane plot of G is shown in Figure IV-2.

The solution to Eq. 4.7 can now be written as

\[ H_1(s_1) = \frac{1}{mgL} G(s_1) \quad (4.13) \]

By inspection, the third-order terms in Eq. 4.6 gives us that

\[ 0 = \sum_{(s_1, s_2, s_3)} \left\{ \left[ a_0L^2(s_1 + s_2 + s_3)^2 + a(s_1 + s_2 + s_3) + mgL \right] H_3(s_1, s_2, s_3) \right. \]

\[ \left. - \frac{mgL}{3!} H_1(s_1)H_1(s_2)H_1(s_3) \right\} \quad (4.14) \]

The symmetrical solution to Eq. 4.14 is *

\[ H_3(s_1, s_2, s_3) = \frac{1}{3! (mgL)^3} G(s_1)G(s_2)G(s_3)G(s_1 + s_2 + s_3) \quad (4.15) \]

By inspection, the fifth-order terms in Eq. 4.6 gives us that

\[ \frac{\alpha}{3! (mgL)^3} G(s_1)G(s_2)G(s_3)G(s_1 + s_2 + s_3) \]

\[ \omega_0 = \sqrt{\frac{k}{L}} \quad (4.12) \]

* As ever, we could have chosen any kernel which satisfied Eq. 4.14. It makes no difference in the final answer for \( \theta \). Throughout this chapter, however, we will always choose a symmetrical kernel because it will make some of our later frequency domain arguments easier to write.
Figure IV-2 The s-plane Plot of $G(s)$
0 = (s_1, \ldots, s_5)^{\text{Sym}} \left\{ \left[ mL^2 s^2 + as + mgL \right] H_5(s_1, \ldots, s_5) \right. \\
\left. - \frac{mgL}{3!} \left[ H_1(s_1) H_1(s_2) H_3(s_3, s_4, s_5) \right. \right. \\
\left. + H_1(s_1) H_3(s_2, s_3, s_4) H_1(s_5) \right. \right. \\
\left. + H_3(s_1, s_2, s_3) H_1(s_4) H_1(s_5) \right. \right. \\
\left. + \frac{mgL}{5!} H_1(s_1) \ldots H_1(s_5) \right\} \right\} \quad (4.16)

The symmetrical solution to Eq. 4.16 is

\[ H_5(s_1, \ldots, s_5) = \frac{1}{5! (mgL)^5} G(s_1) \ldots G(s_5) G(s_1+\ldots+s_5) \]

\[ (s_1, \ldots, s_5)^{\text{Sym}} \left\{ 10 G(s_1+s_2+s_3) - 1 \right\} \quad (4.17) \]

The higher order odd kernel's transforms can be found similarly. The key to writing the \((2k+1)\)-order terms is in finding all the possible combinations of an odd number of positive odd integers that add to \(2k+1\). *

A parallel synthesis of our pendulum system (from Eqs. 4.5, 4.13, 4.15, and 4.17) is shown in Figure IV-3. Note that every term in the system's functional (except for the first) is the result of both a pre and a post filtering by \(G\). This suggests the feedback synthesis of the system which is shown in Figure IV-4.

* Zames' trees can aid us here. 44
Figure IV-3 A Parallel Synthesis of H
Figure IV-4 A Feedback Synthesis of H

de\(Z = f(\theta) = \theta - \sin(\theta)\)
This feedback synthesis can be verified by first rewriting Eq. 4.1 as

\[ L \cdot \frac{d^2 \theta}{dt^2} + \frac{a}{mgL} \cdot \frac{d\theta}{dt} + \theta = \frac{T}{mgL} + \theta - \sin \theta \]  \hspace{1cm} (4.18)

and then taking into account the boundary condition (Eq. 4.2), inverting Eq. 4.8 to get \( g \), and convolving \( g(t) \) with Eq. 4.18. The final result is

\[ \theta(t) = g(t)(n) \left[ \frac{T(t)}{mgL} + \theta(t) - \sin(\theta(t)) \right] \]  \hspace{1cm} (4.19)

Eq. 4.19 verifies the feedback synthesis in Figure IV-4 and, therefore, indirectly verifies the synthesis in Figure IV-3 and our solutions for the kernels which led to it.

The feedback synthesis, Figure IV-4, uses one linear memoried system, one linear no-memory system, and one nonlinear no-memory system \( f \) for feedback. The transfer curve, over the interval \(( -\pi, \pi )\), of \( f \) is graphed in Figure IV-5. This graph shows that the nonlinear system has a "dead-band" for small \( \theta \). (For example, if \( |\theta| < 45^\circ \), then \( |z| < 0.08 \).) Thus we have arrived (albeit roundabout) at the standard mathematical solution to a pendulum's "small amplitudes of oscillations" behavior: If the pendulum's output angle \( \theta \) is such that we can neglect \( \theta - \sin \theta \) with respect to \( \frac{T}{mgL} \) as the input to filter \( g \), then \( \theta \) is approximately equal to the convolution of the impulse response of \( g \) with \( \frac{T}{mgL} \) and the pendulum is therefore a "linear" or a "harmonic" oscillator. That is
Figure IV-5 A Graph of the Transfer Curve of $f$
\[ \theta(t) \sim g(t)(s) \frac{\tau(t)}{mgL} = \frac{1}{mgL} \int_0^\infty g(\sigma) \tau(t-\sigma) \, d\sigma \quad (4.20) \]

However, it is not true to say that Eq. 4.20 follows if "\( \theta \) is small enough" or if "\( \theta \) is small compared to \( \frac{\tau}{mgL} \)." Indeed, we shall show that there are cases where \( \theta \) can be as small as we might wish and yet it is still not valid to neglect \( \theta - \sin \theta \) with respect to \( \frac{\tau}{mgL} \). This, we shall see, is because when filter \( g \) is highly resonant, then the critical consideration will not be "what are the relative magnitudes of \( \theta \) and \( \tau \)" but rather "what are the frequency components of \( \theta \) and \( \tau \).

As an illustration of the critical dependence of a pendulum's behavior upon frequency, consider the single frequency input torque signal

\[ \tau(t) = T_0 \exp(st) \quad (4.21) \]

Eqs. 2.19, 4.3, 4.5, 4.13, 4.15, and 4.17 then give us that

\[ \theta(t) = H \left[ T_0 \exp(st) \right] \]

\[ = \sum_{k=0}^{\infty} T_0^{2k+1} H_{2k+1}(s, \ldots, s) \exp\left(\left[2k+1\right] st\right) \]

\[ = \frac{T_0}{mgL} G(s) \exp(st) + \frac{T_0^3}{3!(mgL)^3} G^3(s) G(3s) \exp(3st) \]

\[ + \frac{T_0^5}{5!(mgL)^5} G^5(s) G(5s) \left[ \log(3s) - 1 \right] \exp(5st) \]

\[ + \ldots \quad (4.22) \]
Note that the \((2k+1)\)-order term in Eq. 4.22 is multiplied by \(G(\left[\frac{2k+1}{s}\right])\). (It comes from the post filtering by \(g\).) Since the frequencies \(s_0\) and \(\frac{s_0}{2k+1}\) are the natural frequencies (poles) of \(G(s)\) (see Eq. 4.8), then Eq. 4.22 shows that \(\theta\) fails to exist at the frequencies \(\frac{s_0}{2k+1}\) and \(\frac{s_0^*}{2k+1}\) \((k=0,1,2,\ldots)\).

We shall call these frequencies the **nonlinear natural frequencies** of the pendulum. They are shown in Figure IV-6.

The Nonlinear Resonances of a Pendulum

To illustrate the nonlinear resonances of a pendulum, in the vicinity of its nonlinear natural frequencies, consider the sinusoidal input torque signal

\[
\tau(t) = \text{Re}\left[ T_0 \exp(j \omega t) \right]
\]

(4.23)

Eqs. 2.19, 4.3, 4.5, 4.13, 4.15, and 4.17 then give us that

* Cf. Eq. 4.22. Eq. 4.24 came out in this simple form because we made the kernels symmetric.
Figure IV-6 An s-plane Plot of a Pendulum's Natural Frequencies
\[ \theta(t) = H \left[ \Re T_0 \exp(j \omega t) \right] \]

\[
= \Re \left[ \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} \frac{(2k+1)!}{(k+m+1)! (k-m)!} \frac{T_0^{k+m+1} T_0^*}{2^k} \right] \\
= \Re \left[ \frac{T_0}{mgL} G(j \omega) + \frac{T_0^2 T_0^*}{8(mgL)^3} G^3(j \omega) G(-j \omega) \right] \\
+ \frac{T_0^3 (T_0^*)^2}{192(mgL)^5} G^3(j \omega) G^2(-j \omega) \left[ G(3j \omega) + 6G(j \omega) \right] \\
+ 3G(-j \omega) - 1 \right] + \ldots \exp(j \omega t) \\
+ \left( \frac{T_0^3}{24(mgL)^3} G^3(j \omega) G(3j \omega) \right) \\
+ \frac{T_0^4 T_0^*}{384(mgL)^5} G^4(j \omega) G(-j \omega) G(3j \omega) \left[ 6G(j \omega) \right] \\
+ 4G(3j \omega) - 1 \right] + \ldots \exp(3j \omega t) \\
+ \left( \frac{T_0^5}{1920(mgL)^5} G^5(j \omega) G(5j \omega) \left[ 10G(3j \omega) - 1 \right] + \ldots \right) \exp(5j \omega t) \\
+ \ldots \right] \\
(4.24)\]

Note the presence of terms like \( G([2k+1]j \omega) \) in Eq. 4.24. Since filter \( g \) is resonant, for sinusoidal frequencies in the vicinity
of \( \omega_0 \) (see Figure IV-2), then Eq. 4.24 shows that the pendulum is resonant for sinusoidal frequencies in the vicinity of \( \frac{\omega_0}{2k+1} \) (\( k=0,1,2,\ldots \)). To be specific, let us consider sinusoidal inputs at the first odd sub-harmonic of \( \omega_0 \) (\( k=1 \)). That is

\[
\omega = \frac{\omega_0}{3}
\]  

(4.25)

Let us assume that the pendulum is very lightly damped, that is

\[
a \ll \omega_0
\]

(4.26)

Then Eq. 4.11 gives us that

\[
\omega_0 \approx \omega_0
\]

(4.27)

and Eq. 4.8 gives us that

\[
G \left( \frac{3\omega_0}{3} \right) \approx \frac{2}{8}
\]

(4.28)

\[
G \left( \omega_0 \right) \approx -\frac{3\omega_0}{2a}
\]

(4.29)

\[
G \left( \frac{5\omega_0}{3} \right) \approx -\frac{9}{16}
\]

(4.30)

If we define two real constants, A and B and a phase angle \( \phi \) such that

\[
\frac{\mathcal{H}_0}{8mgL} = A \exp(j\phi)
\]

(4.31)
\[ \frac{\omega_0}{48a} = B \]  

(4.32)

Then Eqs. 4.24 - 4.32 give us that

\[
\theta(t) \sim \Re \left[ (A + \frac{9}{64} A^3 - \frac{1}{8} A^5B + \ldots) \exp(j \left[ \frac{\omega_0 t}{3} + s \right]) 
+ (-jA^3B - 6A^5B^2 + \ldots) \exp(j \left[ \omega_0 t + 3s \right]) 
+ \left( \frac{19}{128} A^5B + \ldots \right) \exp(j \left[ \frac{5\omega_0 t}{3} + 5s \right]) 
+ \ldots \right] 
\]

(4.33)

Eq. 4.33 shows us that if we choose B large enough (make the damping a small enough compared to \( \omega_0 \)), then the major component of the output will not be at the same frequency as the input. An examination of Eq. 4.20 shows that this aspect of a pendulum's behavior cannot be predicted or explained on the basis of the standard "small amplitudes of oscillations" solution for a pendulum.

Yet it is not true that this phenomenon does not take place at "small amplitudes of oscillations". Eq. 4.33 also shows that if we choose A small enough (make \( T_0 \) small enough compared to \( mgL \)), then this phenomenon can take place at arbitrarily small amplitudes of oscillation.
Physical Explanation of a Pendulum's Nonlinear Resonances

The pendulum's nonlinear resonance phenomena, while unexplainable by the standard mathematical solution of its behavior for "small amplitudes of oscillations", is nonetheless, easily explained physically. Consider Figure IV-4. It shows that if an odd subharmonic of G's resonant frequency is present in \( \tau \), then a component at that frequency must also be present in \( \theta \). Since the feedback network's transfer curve (see Figure IV-5) is an odd function of \( \theta \), then the feedback signal \( z \) must contain a component at the filter's resonant frequency. If filter G is resonant enough, then this component can easily induce a significant sinusoidal component in \( \theta \) at G's resonant frequency— even if the input \( \tau \) contains no component at that frequency.
CHAPTER V

COMMUTATOR MACHINES

In this chapter we shall present another application to an actual nonlinear physical system of the inspection technique developed in Chapter III. We shall investigate one of the oldest devices known to Electrical Engineering, the commutator machine.

Figure V-1 is a schematic of a single axis commutator machine which is driving a load that is both inertial and frictional. The parameters of this system are:

- $v_f$, the field voltage
- $v_a$, the armature voltage
- $i_f$, the field current
- $i_a$, the armature current
- $\omega$, the rotation speed of the machine
- $R_f$, the field resistance
- $R_a$, the series resistance of the armature and the brushes
- $L_f$, the field's self-inductance
- $L_a$, the armature's self-inductance
- $G$, the machine's speed coefficient
- $A$, the load's coefficient of torque drag per rotation speed
Figure V-1 Schematic of a Single Axis Commutator Machine Driving a Load
\[ J, \text{ the load's moment of inertia} \]

We shall assume that \( R_f, R_a, L_f, L_a, G, A, \) and \( J \) are constants (that is, coil saturation and variable loading will not be considered). With this restriction, the system's equations of motion are

\[ (R_f + L_f \frac{d}{dt}) \ i_f = v_f \]  \hspace{1cm} (5.1)

\[ G\omega_f + (R_a + L_a \frac{d}{dt}) \ i_a = v_a \]  \hspace{1cm} (5.2)

\[ (A + J \frac{d}{dt}) \ \omega = G \ i_a \ i_f \]  \hspace{1cm} (5.3)

with the boundary condition of initial rest

\[ i_f(t), i_a(t), \text{ and } \omega(t) = 0 \text{ until } v_f(t), v_a(t) \neq 0 \]  \hspace{1cm} (5.4)

When excited by a single source, the two configurations of the device are as a shunt-wound motor (Figure V-2) and as a series-wound motor (Figure V-3).

**Shunt-wound**

When wired as a shunt-wound commutator machine, the systems have two additional constraints. They are

\[ v_f = v_a = v \]  \hspace{1cm} (5.5)

\[ i_f + i_a = 1 \]  \hspace{1cm} (5.6)
Figure V-2 Wiring Diagram of a Shunt-wound Motor
Figure V-3 Wiring Diagram of a Series-wound Motor
The author has shown elsewhere that Eqs. 5.1 - 5.6 characterize an analytic system for a certain class of inputs. Therefore, there are three functionals, F, I, and W, such that

\[ i_f(t) = F[v] \quad (5.7) \]

\[ i_a(t) = I[v] \quad (5.8) \]

\[ \omega(t) = W[v] \quad (5.9) \]

Observe from the characterizing equations (Eqs. 5.1 - 5.6) that if input v yields outputs \( i_f \), \( i_a \), and \( \omega \), then input \(-v\) yields outputs \(-i_f\), \(-i_a\), and \(+\omega\). Thus F and I are odd functionals and W is an even functional. Observe also that if input v yields output \( i_f \), then input \( \lambda v \) yields output \( \lambda i_f \), where \( \lambda \) is any constant. Thus F is a linear functional of v. In addition, observe that the boundary condition, Eq. 5.4 gives us that \( W_0 \) is zero (we already knew that \( F_0 \) and \( I_0 \) were zero because F and I are odd). These observations allow us to rewrite Eqs. 5.7 - 5.9 as

\[ i_f(t) = F_1[v] \quad (5.10) \]

\[ i_a(t) = \sum_{m=0}^{\infty} I_{2m+1}[v] \quad (5.11) \]

\[ \omega(t) = \sum_{m=1}^{\infty} W_{2m}[v] \quad (5.12) \]
By inspection of Eqs. 5.1 and 5.5, we get that

\[(R_f + s_1 L_f) F_1(s_1) = 1 \quad (5.13)\]

By inspection of Eqs. 5.2 and 5.5 for first-order terms, we get that

\[0 + (R_a + s_1 L_a) I_1(s_1) = 1 \quad (5.14)\]

For \((2k+1)\)-order terms \((k=1,2,3,...)\), we get that

\[\text{Sym} \left\{ \begin{array}{l}
GW_{2k}(s_1,\ldots,s_{2k}) F_1(s_{2k+1}) \\
+ (R_a + s_1 L_a) I_{2k+1}(s_1,\ldots,s_{2k+1}) \end{array} \right\} = 0 \quad (5.15)\]

s = s_1 + \ldots + s_{2k+1}

By inspection for \((2k)\)-order terms \((k=1,2,3,...)\), Eqs. 5.3 and 5.5 give us that

\[\text{Sym} \left\{ \begin{array}{l}
(A+sJ) W_{2k}(s_1,\ldots,s_{2k}) \\
s = s_1 + \ldots + s_{2k} \end{array} \right\} = \text{Sym} \left\{ \begin{array}{l}
GW_{2k}(s_1,\ldots,s_{2k}) F_1(s_{2k}) \\
GI_{2k-1}(s_1,\ldots,s_{2k-1}) F_1(s_{2k}) \end{array} \right\} \quad (5.16)\]

For solutions to Eqs. 5.13 - 5.16 we shall choose

\[F_1(s_1) = \frac{1}{R_f + s_1 L_f} \quad (5.17)\]

\[I_1(s_1) = \frac{1}{R_a + s_1 L_a} \quad (5.18)\]
\[ I_{2k+1}(s_1, \ldots, s_{2k+1}) = \frac{-G \ W_{2k}(s_1, \ldots, s_{2k})}{(R_f + s_{2k+1} L_f)(R_a + [s_1 + \ldots + s_{2k+1}] L_a)} \quad ; \quad k=1,2,3,\ldots \]  

(5.19)

\[ W_{2k}(s_1, \ldots, s_{2k}) = \frac{G \ I_{2k-1}(s_1, \ldots, s_{2k-1})}{(R_f + s_{2k} L_f)(A + [s_1 + \ldots + s_{2k}] J)} \quad ; \quad k=1,2,3,\ldots \]  

(5.20)

Syntheses of these solutions are shown in Figures V-4, V-5 and V-6. Taken together, these syntheses imply the feedback syntheses shown in Figure V-7. This feedback synthesis is directly verified by Eqs. 5.1 - 5.3, and 5.5. Thus Eqs. 5.17 - 5.20 are also verified.

Eqs. 5.19 and 5.20 make up a two step recursive formula for \( I_{2k+1} \) and \( W_{2k} \). They result in closed forms which are

\[ I_{2k+1}(s_1, \ldots, s_{2k+1}) = \frac{(-)^k G^{2k}}{(R_a + s_1 L_a)} \]

\[ \prod_{m=1}^{k} \ \frac{1}{(R_f + s_{2m} L_f)(R_f + s_{2m+1} L_f)(A + [s_1 + \ldots + s_{2m}] J)(R_a + [s_1 + \ldots + s_{2m+1}] L_a)} \quad ; \quad k=1,2,3,\ldots \]  

(5.21)

\[ W_2(s_1, s_2) = \frac{G}{(R_a + s_1 L_a)(R_f + s_2 L_f)(A + [s_1 + s_2] J)} \]  

(5.22)

* Cf. Eq. 3.46.
Figure V-4 A Synthesis of $i_f$
Figure V-5 A Synthesis of $i_{e,1}$
Figure V-6 A Synthesis of \( \omega_{2k} \) and \( i_{e,2k+1} \)
Figure V-7 A Feedback Synthesis of $I_f$, $I_a$, and $\omega$
\[ W_{2k}(s_1, \ldots, s_{2k}) = \frac{(-1)^k \Gamma^{2k-1}}{(R_a + s_1 L_a)(R_f + s_2 L_f)(A + [s_1 + \ldots + s_{2k}] J)} \]
\[
\prod_{n=1}^{k-1} \frac{1}{(R_f + s_2 L_f)(R_f + s_{2n+1} L_f)(A + [s_1 + \ldots + s_{2n}] J)(R_a + [s_1 + \ldots + s_{2n+1}] L_a)}
\]
\[ ; \quad k = 2, 3, 4, \ldots \quad (5.23) \]

Two Cases of Inputs

For a quick check on our results, let us consider the trivial case where \( v(t) = V_0 \), a constant. Eqs. 5.10 - 5.12 then give us that

\[ i_f(t) = F[V_0] = V_0 \phi_1(0) \quad (5.24) \]

\[ i_a(t) = I[V_0] = \sum_{n=0}^{\infty} \phi_{2n+1}(0, \ldots, 0) \quad (5.25) \]

\[ \omega(t) = W[V_0] = \sum_{m=1}^{\infty} \phi_{2m}(0, \ldots, 0) \quad (5.26) \]

Eqs. 5.17, 5.18 and 5.21 - 5.23 give us that

\[ \phi_1(0) = \frac{1}{R_f} \quad (5.27) \]
\[ I_{2k+1}(0,...,0) = \frac{1}{R_a} \left( \frac{-G^2}{A R a_f R_e^2} \right)^k ; \, k = 0,1,2,... \quad (5.28) \]

\[ W_{2k}(0,...,0) = \frac{-R_f}{G} \left( \frac{-G^2}{A R a_f R_e^2} \right)^k ; \, k = 1,2,3,... \quad (5.29) \]

If \[ |V_0| < \frac{R_f}{G} \sqrt{A R_a} \], then the substitution of Eqs. 5.27 - 5.29 in Eqs. 5.24 - 5.26 gives the result that

\[ I_f(t) = \frac{V_0}{R_f} \quad (5.30) \]

\[ I_a(t) = \frac{V_0}{R_a} \frac{G V_0^2}{1 + \frac{G V_0^2}{A R a_f R_e^2}} \quad (5.31) \]

\[ \omega(t) = \frac{G V_0^2}{A R a_f R_e^2} \frac{G V_0^2}{1 + \frac{G V_0^2}{A R a_f R_e^2}} \quad (5.32) \]

The substitution of Eqs. 5.30 - 5.32 into Eqs. 5.1 - 5.3 will show that they are indeed the correct d-c solutions for the shunt-wound motor.

Let us next consider the non-trivial case where \( v \) is \( g \), stationary white Gaussian noise of power density \( P \). The statistical
expectation (ensemble average) of the product of an odd number of g's is zero. \[ 20, 30 \]

\[
E \left[ \prod_{k=1}^{2n+1} g(t-T_k) \right] = 0 \tag{5.33}
\]

The statistical expectation of an even number of g's is the sum of the products of their expectations taken in pairs. \[ 20, 30 \]

\[
E \left[ \prod_{k=1}^{2m} g(t-T_k) \right] = \sum_{\text{pairs } j} \prod_{i, j} E \left[ g(t-T_i) g(t-T_j) \right] \tag{5.34}
\]

For example, \[ 20 \]

\[
E \left[ g(t-T_1) g(t-T_2) \right] = F_{u_0}(T_1 - T_2) \tag{5.35}
\]

\[
E \left[ g(t-T_1) g(t-T_2) g(t-T_3) g(t-T_4) \right] = F^2 \left[ u_0(T_1-T_2) u_0(T_3-T_4) \right] + u_0(T_1-T_3) u_0(T_2-T_4) + u_0(T_1-T_4) u_0(T_2-T_3) \tag{5.36}
\]

Since F and I are odd, then the substitution of Eq. 5.33 into the expectations of the expressions given by Eqs. 5.10 - 5.11 shows that when \( \nu = g \), then the statistical expectations of \( i_F \) and \( i_g \) are zero. That is

\[
E \left[ i_F(t) \right] = E \left[ F \left[ g(t) \right] \right] = E \left[ F_1 \left[ g(t) \right] \right] = 0 \tag{5.37}
\]
\[
E[I_a(t)] = E[I[g]] = \sum_{n=0}^{\infty} E[I_{2n+1}[g]] = 0 \tag{5.38}
\]

But \(W\) is an even functional of \(v\), hence its statistical expectation need not vanish. It is

\[
E[\omega(t)] = E[W[g]] = \sum_{m=1}^{\infty} E[W_{2m}[g]] \tag{5.39}
\]

For example, the first two terms in Eq. 5.39 are

\[
E[\omega_2(t)] = E[W_2[g]] = \int_{-\infty}^{\infty} v_2(\tau_1, \tau_2) P_0(\tau_1 - \tau_2) d\tau_1 d\tau_2 \\
= P \int_{-\infty}^{\infty} v_2(\tau_1, \tau_1) d\tau_1 \tag{5.40}
\]
\[ E[\omega_4(t)] = E[u_4(s)] \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_4(\tau_1, \tau_2, \tau_3, \tau_4) P^2 \left[ u_0(\tau_1 - \tau_2) u_0(\tau_3 - \tau_4) \right. \]

\[ + u_0(\tau_1 - \tau_3) u_0(\tau_2 - \tau_4) + u_0(\tau_1 - \tau_4) u_0(\tau_2 - \tau_3) \]

\[ + d\tau_1 d\tau_2 d\tau_3 d\tau_4 \]

\[ = P^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v_4(\tau_1) d\tau_1 d\tau_2 \]

\[ + v_4(\tau_1, \tau_2) + v_4(\tau_1, \tau_2, \tau_3) \]

\[ + v_4(\tau_1, \tau_2, \tau_3, \tau_4) \]

\[ d\tau_1 d\tau_2 \]

\[ (5.41) \]

These terms can be evaluated, in a few steps, with the aid of George's association theorem.\(^{19,20}\) Consider the particular input \( \tilde{v} \)

\[ \tilde{v}(t) = \sqrt{P} u_0(t) \]

\[ (5.42) \]

which yields the particular output \( \tilde{\omega} \). Then

\[ \tilde{\omega}_2(t) = \hat{w}_2(\tilde{v}) \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_2(\tau_1, \tau_2) P u_0(t - \tau_1) u_0(t - \tau_2) d\tau_1 d\tau_2 \]

\[ = P u_2(t, t) \]

\[ (5.43) \]

Eqs. 5.40 and 5.43 show that the expectation of \( \omega_2 \) is equal to
the bilateral Laplace transform of $\tilde{\omega}_2$ at $s = 0$. That is

$$\tilde{\mathcal{L}}_2(0) = \int_{-\infty}^{\infty} \omega_2(t) \exp(-st) \, dt \bigg|_{s=0}$$

$$= P \int_{-\infty}^{\infty} \omega_2(t,t) \, dt$$

$$= \mathbb{E} [\omega_2(t)] \quad (5.44)$$

The value of $\tilde{\mathcal{L}}_2(s)$ is easily found by considering the multilinear correspondent of $\tilde{\omega}_2$ (Eq. 2.28). It is

$$\omega_2(t_1,t_2) = \int_{-\infty}^{\infty} v_2(\tau_1, \tau_2) \rho u_0(t_1 - \tau_1) u_0(t_2 - \tau_2) \, d\tau_1 d\tau_2$$

$$= P \omega_2(t_1,t_2) \quad (5.45)$$

whose bilateral Laplace transform, by Eq. 5.22, is

$$\tilde{\mathcal{L}}_2(s_1,s_2) = P \tilde{W}_2(s_1,s_2)$$

$$= \frac{PG}{L_a L_f J (s_1 + \frac{R}{L_a})(s_2 + \frac{R_f}{L_f})(s_1 + s_2 + \frac{A}{J})} \quad (5.46)$$

The application of George's association theorem to Eq. 5.46 shows by inspection that

$$\tilde{\mathcal{L}}_2(s) = \frac{PG}{L_a L_f J (s + \frac{R}{L_a} + \frac{R_f}{L_f})(s + \frac{A}{J})} \quad (5.47)$$
When Eq. 5.47 is evaluated at $s = 0$ and that value is substituted into Eq. 5.44, then the result is that

$$E[\omega_2(t)] = \frac{P_G}{A(L_f R_a + L_a R_f)} \quad (5.48)$$

The higher order terms in Eq. 5.39 are found similarly. Thus we can find the average rotation speed of a shunt-wound motor when it is excited by stationary white Gaussian noise.

The two cases of input signals just considered serve to illustrate some of the areas of usefulness for the Volterra series characterization of nonlinear systems. Namely, in addition to the straight-forwardness of the computation of the output of an analytic system for a constant input signal or a sinusoidal input signal (eg. the pendulum output Chapter IV pp. 84-86), the statistical moments of the system's output for stochastic inputs can also be calculated—even in those cases where there are no known methods for executing such computations from the system's characterizing differential equation.

This is not to say that the Volterra series is not without disadvantages. One of its most serious disadvantages is that in being an infinite series, we have to decide how many terms we must compute in order to get an adequate approximation for its value. This question need not be without answer. For example, in the case of the shunt-wound motor, the author has shown elsewhere that if the input voltage is everywhere bounded then the truncation error introduced by approximating $\omega$ by its first $N$ non-zero terms is
also everywhere bounded. That is, if

$$\left| v(t) \right| \leq V \leq V_B = \frac{R_f \sqrt{A R_s}}{G}; \text{ for all } t \quad (5.49)$$

then

$$\left| \omega(t) - \sum_{m=1}^{N} \omega_{2m}[v] \right| \leq \frac{R_f}{G} \cdot \frac{(\frac{V}{V_B})^{2N+2}}{1 - (\frac{V}{V_B})^2}; \text{ all } t \quad (5.50)$$

Series Motor

When wired as a series-wound commutator machine (see Figure V-3, and Eqs. 5.1 - 5.3), then the single axis commutator machine's characteristic differential equations are

$$(R + L \frac{d}{dt})i + G\omega i = v \quad (5.51)$$

$$(A + J \frac{d}{dt})\omega = G i^2 \quad (5.52)$$

where

$$v = v_f + v_a \quad (5.53)$$

$$i = i_f + i_a \quad (5.54)$$

$$R = R_f + R_a \quad (5.55)$$

$$L = L_f + L_a \quad (5.56)$$
Eqs. 5.51 and 5.52 show that \( i \) is an odd functional and \( \omega \) is an even functional of \( v \). The boundary condition, Eq. 5.4, shows that the zero-order term of \( \omega \) is zero. Thus

\[
i(t) = H[v] = \sum_{n=0}^{\infty} H_{2n+1} [v] \quad (5.57)
\]

\[
\omega(t) = Q[v] = \sum_{m=1}^{\infty} Q_{2m} [v] \quad (5.58)
\]

The first-order terms of Eq. 5.51 are

\[(R + s_{1} L)H_{1}(s_{1}) + C = 1 \quad (5.59)\]

The (2k)-order terms of Eq. 5.52 are

\[
\begin{align*}
\text{Sym} (s_{1}, \ldots, s_{2k}) \left\{ \left( A + s J \right) Q_{2k}(s_{1}, \ldots, s_{2k}) \right\} \\
\end{align*}
\]

\[
= \text{Sym} (s_{1}, \ldots, s_{2k}) \left\{ \sum_{n=0}^{k-1} H_{2n+1}(s_{1}, \ldots, s_{2n+1}) H_{2(k-n)-1}(s_{2n+2}, \ldots, s_{2k}) \right\} \quad (5.60)
\]

The (2k+1)-order terms of Eq. 5.51 are
\[ \text{Sym} \left( s_1, \ldots, s_{2k+1} \right) \left\{ (R + sL) H_{2k+1} \left( s_1, \ldots, s_{2k+1} \right) \right\} = 0 \]

\[ + G \sum_{m=1}^{k} Q_{2m} \left( s_1, \ldots, s_{2m} \right) H_2(k-m)+1 \left( s_{2m+1}, \ldots, s_{2k+1} \right) \left\{ s_1 + \ldots + s_{2k+1} \right\} = 0 \]

; \ k = 1, 2, 3, \ldots \tag{5.61} \]

Eq. 5.59 gives us that \( H_1 \) is

\[ H_1(s_1) = \frac{1}{R + s_1L} \tag{5.62} \]

Eqs. 5.60 and 5.61 give us two step recursive formulas for all the \( Q \)'s and all the remaining \( H \)'s. We shall choose the solutions that

\[ Q_{2k} \left( s_1, \ldots, s_{2k} \right) \]

\[ = G \sum_{n=0}^{k-1} H_{2n+1} \left( s_1, \ldots, s_{2n+1} \right) H_2(k-n)+1 \left( s_{2n+2}, \ldots, s_{2k} \right) \]

\[ + \left[ s_1 + \ldots + s_{2k} \right] J \]

; \ k = 1, 2, 3, \ldots \tag{5.63} \]

and

\[ H_{2k+1} \left( s_1, \ldots, s_{2k+1} \right) \]

\[ = -G \sum_{m=1}^{k} Q_{2m} \left( s_1, \ldots, s_{2m} \right) H_2(k-m)+1 \left( s_{2m+1}, \ldots, s_{2k+1} \right) \]

\[ + \left[ s_1 + \ldots + s_{2k+1} \right] L \]

; \ k = 1, 2, 3, \ldots \tag{5.64} \]
CHAPTER VI

THE VARACTOR

When a semi-conductor diode is operated back-biased, then it behaves like a nonlinear capacitor and displays variable reactance—hence the name varactor. In this chapter we shall present an application of the inspection technique, developed in Chapter III, to a varactor circuit.

Model of a Varactor

The abrupt junction semi-conductor diode, shown in Figure VI-1, stores a charge +q in the N-type region of its depletion layer and a charge -q in the P-type region of its depletion layer. The voltage across the depletion layer is $kq^2$, where $k$ is a positive constant. The diode has a bulk resistance $R$ and a Fermi contact potential $\phi$ (due to the two leads attached to it). We shall assume that both $R$ and $\phi$ are positive constants. The diode's voltage $v_D$ can then be modeled by the usual equation for an abrupt junction varactor. It is

$$v_D = R \frac{dq}{dt} + kq^2 - \phi$$  \hspace{1cm} (6.1)

Eq. 6.1 is a model, rather than a characterization, because it fails to match all of the varactor's nuances. Some of its defects are: (1) Negative values of $q$ are physically unobtainable. (2) When $v_D$ is a positive constant, then Eq. 6.1 predicts that
Figure VI-1 An Abrupt-junction Semi-conductor Diode
(3) In an actual diode, the bulk resistance $R$ changes slightly with the width of the depletion layer (which depends upon $q$). Nonetheless, we shall use Eq. 6.1 as our characteristic differential equation for an abrupt junction varactor but we must bear in mind that our analysis will be subject to its defects.

One Varactor Imbedded in a Linear Network

Consider a system which consists of one abrupt junction varactor imbedded in a linear network. If we form the Thévenin equivalent of the linear network, then Figure VI-2 shows a circuit model for that system. Here $Z$ is the Thévenin equivalent output impedance of the linear network, as seen by the varactor. We have divided the Thévenin equivalent open-circuit voltage into two parts:

$E_0$, the varactor's bias voltage (a constant) and $e$, a variable. By inspection of Figure VI-2 and Eq. 6.1, the circuits characteristic equations are

$$E_0 + e - R \frac{dq}{dt} - kq^2 + \phi - v = 0$$  \hspace{1cm} (6.2)

and

$$V(s) = Z(s)q(s)$$  \hspace{1cm} (6.3)

We shall assume that $Z$ has no series capacitor. That is

* In a practical circuit this is true. If there was a series capacitor then it would keep the bias $E_0$ from the varactor.
Figure VI-2 The Thévenin Equivalent of One Varactor Imbedded in a Linear Network
\[
\lim_{s \to 0} \left[ sZ(s) \right] = 0 \tag{6.4}
\]

Then there are two d-c solutions to Eq. 6.2. They are

\[
q_0 = \pm \sqrt{\frac{E_0 + s}{k}} \tag{6.5}
\]

Depending on which of these two values we choose as the zero-order term of a functional solution for \( q \), we get two different functional power series (Cf. Eq. 3.44). We might eliminate the negative value on physical grounds (the charge stored in the N-type region of the depletion layer must be positive) but there is a more interesting reason for not choosing the negative value—it is unstable. To show this, consider Eq. 6.2 when both \( e \) and \( v \) are zero. We may then rewrite it as

\[
\frac{dq}{dt} = \frac{E_0 + s}{R} - \frac{k}{R} q^2
\]

\[
= \frac{k}{R} (q_0^2 - q^2) \tag{6.6}
\]

The phase-plane graph of Eq. 6.6 is shown in Figure VI-3. It shows that the positive d-c solution is stable but that the negative d-c solution is not. The significant fact is that had we chosen the negative value as the zero-order term of a functional for \( q \), then the Volterra series that we get by our inspection technique has natural frequencies in the right half-plane. This example serves to remind us that, given a differential equation,
Figure VI-3 The Phase-plane Graph of Eq. 6.6
we cannot rashly assume that it characterizes an analytic system.

Having established which d-c solution is tenable, we see that the boundary condition of initial rest for this varactor system is

\[ v(t) = 0, q(t) = \sqrt{\frac{E_0 + \phi}{k}} \quad \text{until } e(t) \neq 0 \quad (6.7) \]

It can be shown that, for a class of inputs about zero, Eqs. 6.2, 6.3 and 6.7 characterize an analytic system. Hence

\[ q(t) = G[e] \quad (6.8) \]

\[ v(t) = H[e] \quad (6.9) \]

From the boundary condition, Eq. 6.7 the zero-order terms of G and H are

\[ G_0 = \sqrt{\frac{E_0 + \phi}{k}} \quad (6.10) \]

\[ H_0 = 0 \quad (6.11) \]

The first-order terms of Eq. 6.2 are

\[ 1 - s_1 R G_1(s_1) - 2k G_0 G_1(s_1) - H_1(s_1) = 0 \quad (6.12) \]

The n-order terms of Eq. 6.2 are
\[ \text{Sym} \left\{ \left[ -s_1 + \ldots + s_n \right] R_n(s_1, \ldots, s_n) - k \sum_{m=0}^{n} G_m(s_1, \ldots, s_m) \right\} \]
\[ G_{n-m}(s_{m+1}, \ldots, s_n) - H_n(s_1, \ldots, s_n) \}
= 0 \]

; \ n = 2, 3, 4, \ldots \hspace{1cm} (6.13)

The m-order terms of Eq. 6.3 are

\[ \text{Sym} \left\{ H_m(s_1, \ldots, s_m) \right\} \]
\[ = \text{Sym} \left\{ \left[ s_1 + \ldots + s_m \right] Z(s_1 + \ldots + s_m) G_m(s_1, \ldots, s_m) \right\} \]

; \ m = 1, 2, 3, \ldots \hspace{1cm} (6.14)

For solution to Eq. 6.14 we shall choose

\[ H_m(s_1, \ldots, s_m) = s Z(s) \int G_m(s_1, \ldots, s_m) \quad ; \ m = 1, 2, 3, \ldots \hspace{1cm} (6.15) \]

\[ s = s_1 + \ldots + s_m \]

When Eq. 6.15 is evaluated for m=1 and the resultant value for \( H_1(s_1) \) is substituted into Eq. 6.12, then the result is that

\[ G_1(s_1) = \frac{1}{2kG_0 + s_1 \left[ R + Z(s_1) \right]} \hspace{1cm} (6.16) \]

When the value for \( H_m \) given by Eq. 6.15 is substituted into Eq. 6.13 and the highest-order term of G is isolated, then it is
seen that we may choose the solutions

\[
G_n(s_1, \ldots, s_n) = \frac{-k \sum_{m=1}^{n-1} G_m(s_1, \ldots, s_m) G_{n-m}(s_{m+1}, \ldots, s_n)}{2kG_0 + s[R + Z(s)]} \quad s = s_1 + \ldots + s_n
\]

; \quad n = 2, 3, 4, \ldots \quad (6.17)

Parallel syntheses of G and H are shown in Figures VI-4 and VI-5. Together they imply the feedback synthesis of both G and H shown in Figure VI-6. Figure VI-6 can be directly verified by Eqs. 6.2, 6.3, and 6.7 which therefore verifies Eqs. 6.10, 6.11 and 6.15 - 6.17.

Transient Response of a Varactor Frequency Doubler

As an example of the application of our solution, we shall compute a frequency doubler circuit's transient response. For simplicity, we shall let Z be an inductor, that is

\[
Z(s) = sL \quad (6.18)
\]

and we shall restrict our attention to the second-order term in the output voltage.* The multilinear correspondent to the second-

---

* There is no zero-order term (see Eq. 6.11). There is no doubler effect from the first-order term (Cf. Eq. 4.20). If we consider small amplitude input signals and choose our frequencies judiciously, then the doubler effect from terms higher than the second-order term can be made small compared to it.
Figure VI-4 A Parallel Synthesis of G
Figure VI-5 A Parallel Synthesis of H
\[ g_1(s) = \frac{\sqrt{E_0 + \phi}}{R} \]

\[ w = f(g) = -k_q^2 + 2k_q g_0 q + k g_0^2 \]
order term is
\[ v_2(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_2(\tau_1, \tau_2) e(t_1 - \tau_1) e(t_2 - \tau_2) d\tau_1 d\tau_2 \]  
(6.19)

Its multivariate bilateral Laplace transform is
\[ V_2(s_1, s_2) = \mathcal{L}_2(s_1, s_2) \mathcal{E}(s_1) \mathcal{E}(s_2) \]  
(6.20)

The value of \( H_2 \) is found from Eqs. 6.10 and 6.15-6.18. They give us that

\[ H_2(s_1, s_2) = \frac{-k(s_1+s_2)Z(s_1+s_2)}{(2kG_0+s_1[R+Z(s_1)])(2kG_0+s_2[R+Z(s_2)])(2kG_0+[s_1+s_2][R+Z(s_1+s_2)])} \]

\[ = \frac{-(s_1+s_2)^2kL}{(s_1+s_2)^2L+s_1R+2kG_0)(s_1^2L+s_1R+2kG_0)(s_2^2L+s_2R+2kG_0)} \]

\[ = \frac{+F(s_1+s_2)}{(s_1-s_0)(s_1-s_0^*)(s_2-s_0)(s_2-s_0^*)} \]  
(6.21)

where

\[ F(s) = \frac{-ks^2}{L^2(s-s_0)(s-s_0^*)} \]  
(6.22)

and

* Cf. Eqs. 4.9 - 4.12.
\[ s_0 = -a + j\omega_D \]  
(6.23)

\[ a = \frac{R}{2L} \]  
(6.24)

\[ \omega_D = \sqrt{\omega_0^2 - a^2} \]  
(6.25)

\[ \omega_0 = \sqrt{\frac{2kG_0}{L}} \]

\[ = \sqrt{\frac{2}{L}} \left[ k(E_0 + \phi) \right]^{\frac{1}{4}} \]  
(6.26)

In preparation for application of George's Association Technique, we shall rewrite Eq. 6.21 in a partial fraction form. That is

\[ H_2(s_1, s_2) = F(s_1 + s_2) \left[ \frac{A}{s_1 - s_0} + \frac{A^*}{s_1 - s_0^*} \right] \left[ \frac{A}{s_2 - s_0} + \frac{A^*}{s_2 - s_0^*} \right] \]  
(6.27)

where

\[ A = \frac{1}{s_0 - s_0^*} = \frac{-1}{2\omega_D} \]  
(6.28)

We shall let the input voltage signal \( e \) be a step of sinusoid. That is

\[ e(t) = Re \left[ E_1 \exp(s_1 t) \right] u_{-1}(t) \]  
(6.29)

The bilateral Laplace transform of \( e \) is therefore
\[ E(s) = \frac{B}{s-s_1} + \frac{B^*}{s-s_1^*} \] (6.30)

where
\[ B = \frac{1}{2} E_1 \] (6.31)

When Eqs. 6.27 and 6.30 are substituted into Eq. 6.20, we find that the transform of the multilinear correspondent to \( v_2 \) is

\[ V(2)(s_1, s_2) = \mathcal{F}(s_1 + s_2) \left[ \frac{A}{s_1 - s_0} + \frac{A^*}{s_1 - s_0^*} \right] \left[ \frac{B}{s_1 - s_1^*} + \frac{B^*}{s_1 - s_1} \right]. \]

\[ = \mathcal{F}(s_1 + s_2) \left[ \frac{C}{s_1 - s_0} + \frac{C^*}{s_1 - s_0^*} + \frac{D}{s_1 - s_1^*} + \frac{D^*}{s_1 - s_1} \right] \]

\[ \left[ \frac{C}{s_2 - s_0} + \frac{C^*}{s_2 - s_0^*} + \frac{D}{s_2 - s_1^*} + \frac{D^*}{s_2 - s_1} \right] \] (6.32)

where
\[ C = \mathcal{A} \left[ \frac{B}{s_0 - s_1} + \frac{B^*}{s_0 - s_1^*} \right] \] (6.33)

and
\[ D = \mathcal{B} \left[ \frac{A}{s_1 - s_0} + \frac{A^*}{s_1 - s_0^*} \right] \] (6.34)
Using George's Frequency Association Technique, the transform of \( v_2 \) is found from Eq. 6.32 by inspection to be

\[
F(s) = \frac{G}{s} + \frac{(C^*)^2}{s-2s_0} + \frac{D^2}{s-2s_1} + \frac{(D^*)^2}{s-2s_1} + \frac{2GD^*}{s-s_0-s_1} + \frac{2DD^*}{s-s_0-s_1} + \frac{2GD}{s-s_0-s_1} + \frac{2G}{s-s_0-s_1}
\]

(6.35)

Before evaluating \( v_2(t) \) from Eq. 6.35, let us take a moment out to study \( v_2 \) in the vicinity of \( t = 0 \). Observe from Eq. 6.22 that

\[
\lim_{s \to \infty} \left[ F(s) \right] = -\frac{k}{L^2}
\]

(6.36)

Therefore, from Eq. 6.35 we find that

\[
\lim_{s \to \infty} \left[ V_2(s) \right] = 0
\]

(6.37)

That is, \( v_2(t) \) has no singularities at \( t = 0 \). We also find that

\[
v_2(0^+) = \lim_{s \to \infty} \left[ sV_2(s) \right]
\]

\[
= -\frac{k}{L^2} \left[ C^2 + (C^*)^2 + D^2 + (D^*)^2 + 2CC^* + 2DD^* + 2GD + 2CD + 2C^*D + 2GD^* + 2G^*D \right]
\]

\[
= -\frac{k}{L^2} \left( C + C^* + D + D^* \right)^2 = -\frac{4k}{L^2} \left( \text{Re} [C+D] \right)^2
\]

(6.38)
From Eqs. 6.33 and 6.34 we find that

\[
C+D = \frac{AB}{s_0-s_1} + \frac{AB}{s_1-s_0} + \frac{AB^*}{s_0-s_1} + \frac{A^*B}{s_1-s_0}
\]

\[
= 2j\text{Im} \left[ \frac{AB^*}{s_0-s_1} \right]
\]

Therefore

\[
\text{Re} \left[ C+D \right] = 0
\]

(6.39)

and by Eq. 6.38 we have that

\[
v_2(0^+) = 0
\]

(6.40)

That is, \(v_2(t)\) is continuous at \(t = 0\). Similarly, it can be shown that the derivative of \(v_2\) at \(t = 0\) is zero. Thus we see that the transient response is a smooth build-up to the steady state.

The steady state terms in Eq. 6.35 are due to the poles at \(s = 2s_1\) and \(s = 2s_1^*\). This shows that the circuit does actually double input frequencies. The residue of \(V_2\) at \(s = 2s_1\), by Eq. 6.35, is

\[
\text{Res}(2s_1) = \lim_{s \to 2s_1} \left[ (s-2s_1)V_2(s) \right]
\]

\[
= F(2s_1)D^2
\]

\[
= \frac{-4\text{Im}B^2}{L^2(2s_1-s_0)(2s_1-s_0^*)} \left[ \frac{A}{s_1-s_0} + \frac{A^*}{s_1-s_0^*} \right]^2
\]

(6.42)
where we have used Eqs. 6.22 and 6.34 to evaluate $F$ and $D$.

Eqs. 6.42 shows that if we wish to maximize the amplitude of the doubler's steady state output, then we should choose $s_1$ in the vicinity of $s_0$. This may be somewhat of a surprise. The natural frequency of the linear incremental model of the doubler's circuit is $s_0$. By "physical insight", we, a priori, might have guessed that the doubler's output would have been maximized when the input's frequency was at half the resonant frequency of the linear incremental model. Our reasoning would be: "If $s_1$ is near $\frac{1}{2} s_0$, then when $s_1$ gets doubled it resonates the incremental circuit and, therefore, gives the largest possible output." Eq. 6.42 shows that such reasoning is false. The residue of the doubled term does have a relative maximum near $s_1 = \frac{j\omega_D}{2}$ (when $2s_1 - s_0 = a$) but its absolute maximum is near $s_1 = j\omega_D$. A posteriori, we can explain this result physically: When the input frequency is near the linear incremental circuit's resonant frequency, then the voltage source's input current is maximized. When the maximum current gets "doubled", it produces the maximum double frequency output. Thus we choose

$$s_1 = j\omega_D$$  \hspace{1cm} (6.43)

The distinct frequencies present in Eq. 6.35 are then

$$s_{0,1} = s_0 + s_1$$  

$$= -a$$  \hspace{1cm} (6.44)

* Frequencies $s_{0,3}$ and $s_{0,3}^*$ came from $F(s)$ in Eq. 6.35. Eq. 6.35 looks as if there is also a frequency at $s=s_1+s_1^*=0$ but Eq. 6.38 shows that it has a residue of zero.
\[ s_{0,2} = s_0 + s_0^* \]
\[ = -2a \quad (6.45) \]
\[ s_{0,3} = s_0 \]
\[ = -a + j\omega_D \quad (\text{and } s_{0,3}^*) \quad (6.46) \]
\[ s_{0,4} = 2s_0 \]
\[ = -2a + 2j\omega_D \quad (\text{and } s_{0,4}^*) \quad (6.47) \]
\[ s_{0,5} = s_0 + s_1 \]
\[ = -a + 2j\omega_D \quad (\text{and } s_{0,5}^*) \quad (6.48) \]
\[ s_{0,6} = 2s_1 \]
\[ = 2j\omega_D \quad (\text{and } s_{0,6}^*) \quad (6.49) \]

An s-plane plot of these frequencies is shown in Figure VI-7. It shows that \( v_2(t) \) consists of two dying exponentials, \( (s_{0,1} \text{ and } s_{0,2}) \), one exponentially damped sinusoid at half the frequency of the steady state \( (s_{0,3}) \), two exponentially damped sinusoids at the same frequency as the steady state \( (s_{0,4} \text{ and } s_{0,5}) \) and the steady state \( (s_{0,6}) \). The value of \( v_2(t) \) is found by rewriting Eq. 6.35 as

\[
v_2(s) = \frac{\text{Res}(s_{0,1})}{s-s_{0,1}} + \frac{\text{Res}(s_{0,2})}{s-s_{0,2}} + \sum_{n=3}^{6} \left[ \frac{\text{Res}(s_{0,n})}{s-s_{0,n}} + \frac{\text{Res}(s_{0,n}^*)}{s-s_{0,n}^*} \right] \quad (6.50)
\]
Figure VI-7 An $s$-plane Plot of the Frequencies in $v_2$
Then it is seen that

\[ v_2(t) = \left\{ \text{Res}(s_{0,1}) \exp(s_{0,1}t) + \text{Res}(s_{0,2}) \exp(s_{0,2}t) \right\} + 2 \sum_{n=3}^{6} \text{Re} \left[ \text{Res}(s_{0,n}) \exp(s_{0,n}t) \right] \{ u_{-1}(t) \} \]

(6.51)

For the sake of illustration, we shall compute the double frequency terms in Eq. 6.51 in the case of small damping. That is

\[ a \ll \omega_D \]

(6.52)

So that Eq. 6.25 shows that

\[ \omega_D \approx \omega_0 \]

(6.53)

then

\[ \text{Res}(s_{0,4}) \approx \frac{kE_1^2}{3(\omega_0 R)^2} \]

(6.54)

\[ \text{Res}(s_{0,5}) \approx \frac{-2kE_1^2}{3(\omega_0 R)^2} \]

(6.55)

\[ \text{Res}(s_{0,6}) \approx \frac{kE_1^2}{3(\omega_0 R)^2} \]

(6.56)

When Eqs. 6.54 - 6.56 are substituted into Eq. 6.51 then the result is that the doubled frequency terms are
\[ v_2(t)_{2\omega_0} \approx \frac{2k}{3(\omega_0 R)^2} \left[ 1 - 2\exp(-at) + \exp(-2at) \right] \text{Re} \left[ E_1^2 \exp(2\omega_0 t) \right] u_{-1}(t) \]

(6.57)

Figure VI-8 shows a graph of the envelope on the transient build-up to the steady state of the doubled frequency terms that we have just computed.
Figure VI-8 A Graph of the Envelope on the Double Frequency Components in $\nu_2$
CHAPTER VII

AN ANALYSIS OF A MAGNETIC SUSPENSION DEVICE

The inertial guidance instruments of some of our nation's space vehicles are magnetically suspended by a device developed at the Instrumentation Laboratory, M.I.T.\textsuperscript{9,12,21,23} A single axis version of this magnetic suspension device is shown in Figure VII-1. The device's principle of operation is as follows: If the suspended block is moved towards the right, then it detunes the circuit on the right but it tunes the circuit on the left. This causes the current flow on the right to decrease and the current flow on the left to increase. This imbalance in currents produces a net magnetic force to the left which restores the block to center.

In this chapter we shall present an application of our inspection technique, developed in Chapter III, to this single axis version of the magnetic suspension device. Specifically, we shall use the functional characterization of this magnetic suspension device, obtained by our inspection technique, in order to (1) determine the device's static suspension stability, (2) determine the device's dynamic suspension stability, and (3) show that the suspension will have an oscillatory instability whenever the device has insufficient mechanical damping. (The results of an experimental verification of these predicted oscillatory instabilities are given in appendix.)

In order to use our inspection technique to determine this magnetic suspension device's functional characterization, we shall first derive this device's characterizing differential equations
Figure VII-1 A Single Axis Magnetic Suspension Device
by the well known state function, state variables, Hamilton's principle approach.

The Differential Equations Which Characterize The Device

In a state function, state variables, Hamilton's principle derivation of the equations which characterize the device shown in Figure VII-1, we must first extract the device's lossless magnetic energy storage structure, which is shown in Figure VII-2. Figure VII-2 shows that the differential increment \( dw \) in the structure's stored magnetic energy \( w \), due to differential increments in the variable pairs at the structure's one mechanical and four electrical ports, is

\[
\begin{align*}
\frac{dw}{dt} &= -f_r \, dx + \sum_{k=1}^{4} e_k \, i_k \, dt \\
&= -f_r \, dx + \sum_{k=1}^{4} i_k \, d\lambda_k \\
&= -f_r \, dx + \sum_{k=1}^{4} i_k \, d\lambda_k \\
&= -f_r \, dx + \sum_{k=1}^{4} i_k \, d\lambda_k \quad (7.1)
\end{align*}
\]

where \( f_r \) is the device's restoring force. The differential form of Eq. 7.1 demonstrates that \( w \) is a state function of five state variables (the four flux-linkages \( \lambda_k \) and the block displacement \( x \)) and that their port pairs are

\[
i_k = \frac{\partial}{\partial \lambda_k} w(\lambda_1, \lambda_2, \lambda_3, \lambda_4, x) \quad ; \quad k = 1, 2, 3, 4. \quad (7.2)
\]
Figure VII-2 The Suspension Device's Lossless Magnetic Energy Storage Structure
\[-f_r = \frac{\partial}{\partial x} w(\lambda_1, \lambda_2, \lambda_3, \lambda_4, x) \quad (7.3)\]

Although the structure's stored energy is a function of four flux-linkages, if we neglect leakage, then there are only two independent flux-linkages in the entire structure. We shall choose the structure's mean flux-linkage \( \lambda \)

\[\lambda = \frac{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}{4} \quad (7.4)\]

and its mean-difference flux-linkage \( \delta \)

\[\delta = \frac{-\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4}{4} \quad (7.5)\]

to be the two independent flux-linkages. The structure's other flux-linkages are then

\[\lambda_k = (-)^k \delta + \lambda \quad ; \ k = 1, 2, 3, 4. \quad (7.6)\]

If we neglect fringing in the gaps, then the structure's stored magnetic energy \( w \) is

\[w = \frac{2g(\lambda^2 + \delta^2) - 4x \lambda \delta}{\mu A N^2} \quad (7.7)\]

When the expression for \( w \), given by Eq. 7.7, is substituted into Eq. 7.3, then the result is that
\[ f_r = \frac{4 \lambda \delta}{\mu A N^2} \quad (7.8) \]

The partials of Eq. 7.7 with respect to \( \lambda \) and \( \delta \) yield, with the aid of Eqs. 7.2 and 7.2, two additional equations. They are

\[ 4 \frac{g\lambda - x\delta}{\mu A N^2} = \frac{\partial v}{\partial \lambda} = \sum_{k=1}^{4} \frac{\partial w}{\partial \lambda_k} \cdot \frac{\partial \lambda_k}{\partial \lambda} = \sum_{k=1}^{4} 1_k \quad (7.9) \]

\[ 4 \frac{g\delta - x\lambda}{\mu A N^2} = \frac{\partial v}{\partial \delta} = \sum_{k=1}^{4} \frac{\partial w}{\partial \lambda_k} \cdot \frac{\partial \lambda_k}{\partial \delta} = \sum_{k=1}^{4} (-)^k 1_k \quad (7.10) \]

Figure VII-3 shows the connections external to the lossless magnetic energy storage structure, shown in Figure VII-2, which completes the formation of the device shown in Figure VII-1 except that the capacitors have been generalized to any linear impedance \( Z \). These external connections impose seven independent constraints. They are

\[ v_1(t) = z(t)*i_1(t) \quad (7.11) \]

\[ v_2(t) = z(t)*i_2(t) \quad (7.12) \]

\[ i_1 = i_3 \quad (7.13) \]

\[ i_2 = i_4 \quad (7.14) \]
\[ v = v_1 + i_1 R + e_1 + i_3 R + e_3 \quad (7.15) \]

\[ v = v_2 + i_2 R + e_2 + i_4 R + e_4 \quad (7.16) \]

\[ f + f_r - K \frac{dx}{dt} = M \frac{d^2x}{dt^2} \quad (7.17) \]

There are seven external constraints (Eqs. 7.11 - 7.17), seven internal constraints (Eq. 7.6 four times, Eqs. 7.8 - 7.10), and four definitions \( e_k \frac{dt}{dt} = d\lambda_k \) (Eq. 7.1)) for a total of eighteen equations in twenty-one variables. When we eliminate the fifteen internal variables \( i_k, e_k, \lambda_k, v_1, v_2, \) and \( f_r, \) then we are left with three equations in the remaining six variables \( t, \lambda, \delta, v, x, \) and \( f. \) These three equations characterize the device.*

They are

\[ \mu AN^2 v = \left[ z(t) + 2Ru_0(t) \right] (g+x)(\lambda - \delta) + 2\mu AN^2 \frac{d}{dt} (\lambda - \delta) \quad (7.18) \]

\[ \mu AN^2 v = \left[ z(t) + 2Ru_0(t) \right] (g-x)(\lambda + \delta) + 2\mu AN^2 \frac{d}{dt} (\lambda + \delta) \quad (7.19) \]

\[ f = \frac{d}{dt} (M \frac{dt}{dt} + K) x - \frac{4\lambda \delta}{\mu A N^2} \quad (7.20) \]

---

* Together with the boundary condition of initial rest.
Functional Solutions of the System's Characteristic Equations

These three equations, * Eqs. 7.18 - 7.20, in six variables, t, λ, δ, v, x, and f, characterize a system with two input signals (two functions of t = three variables) and three output signals. As is shown in Figure VII-4, we will choose v and x, rather than v and f, as the input signals to this multi-input system. We will do so because the functional characterization which we shall then obtain for the system is simpler and is analytic for the stability demonstrations which we wish to show.

If the signals v, x, λ, and δ are solutions of Eqs. 7.18, 7.19, ** then for any constant c, the signals cv, x, cλ, and cδ, are also solutions. Therefore λ and δ are linear functionals of v. If the signals v, x, λ, δ, and f are solutions of Eqs. 7.18 - 7.20, then the signals v, -x, λ, -δ, and -f are also solutions. Therefore λ is an even functional of x, δ and f are odd functionals of x. Thus, by Eq. 2.34, we may write that

\[ \lambda = L[v, x] = \sum_{n=0}^{\infty} L_{1,2n}[v, x] \]  

(7.21)

\[ \delta = D[v, x] = \sum_{m=0}^{\infty} D_{1,2m+1}[v, x] \]  

(7.22)

* Together with the boundary condition of initial rest.

** We have assumed the boundary condition of initial rest. Namely f, λ, and δ are zero until either v or x are not zero.
Figure VII-4 A System of Five Signals Characterized by Three Equations
\[ f = F[v, x] = F_{0,1}[v, x] + \sum_{p=0}^{\infty} F_{2,2p+1}[v, x] \quad (7.23) \]

The \((1,0)\)-order terms of either Eq. 7.18 or Eq. 7.19 are

\[ \mu AN^2 - 1 = \left[ gZ(s_1) + 2Rg - 1 + 2 \mu AN^2 s_1 \right] L_{1,0}(s_1) \quad (7.24) \]

The \((1,n)\)-order terms \((n \neq 0)\) of either Eq. 7.18 or Eq. 7.19 are (see Eq. 3.65)

\[ 0 = \text{Sym} \left\{ \left[ gZ(s_1^{+} + \ldots + s_{2n+1}) + 2Rg + 2\mu AN^2(s_1^{+} + \ldots + s_{2n+1}) \right] L_{1,2n}(s_1^{+}, \ldots, s_{2n+1}) - \left[ Z(s_1^{+} + \ldots + s_{2n+1}) \right] + 2R \right\} D_{1,2n-1}(s_1^{+}, \ldots, s_{2n}) \quad ; \; n = 1, 2, 3, \ldots \quad (7.25) \]

The \((1,2m+1)\)-order terms of either Eq. 7.18 or Eq. 7.19 are

\[ 0 = \text{Sym} \left\{ \left[ gZ(s_1^{+} + \ldots + s_{2m+2}) + 2Rg + 2\mu AN^2(s_1^{+} + \ldots + s_{2m+2}) \right] D_{1,2m+1}(s_1^{+}, \ldots, s_{2m+2}) - \left[ Z(s_1^{+} + \ldots + s_{2m+2}) \right] + 2R \right\} L_{1,2m}(s_1^{+}, \ldots, s_{2m+1}) \quad ; \; m = 0, 1, 2, \ldots \quad (7.26) \]

Let us define two dimensionless linear filters, \(G\) and \(H\), whose system functions are
\[ G(s) = \frac{2R}{Z(s) + 2R + sL} \]  \hspace{1cm} (7.27)

\[ H(s) = \frac{Z(s) + 2R}{Z(s) + 2R + sL} \]  \hspace{1cm} (7.28)

where \[ L = \frac{2\mu AN^2}{g} \]  \hspace{1cm} (7.29)

The syntheses of \( G \) and \( H \) are shown in Figure VII-5. We can now write the solution to Eq. 7.24 as

\[ L_{1,0}(s_1) = \frac{L}{4R} G(s_1) \]  \hspace{1cm} (7.30)

For solutions to Eqs. 7.25 - 7.26 we shall choose

\[ L_{1,2n}(s_1, \ldots, s_{2n+1}) = \frac{L}{4Rg^{2n}} G(s_1) \prod_{k=2}^{2n+1} H(s_1 + \ldots + s_k) \; ; \; n = 1, 2, 3, \ldots \]  \hspace{1cm} (7.31)

\[ D_{1,2m+1}(s_1, \ldots, s_{2m+2}) = \frac{L}{4Rg^{2m+1}} G(s_1) \prod_{k=2}^{2m+2} H(s_1 + \ldots + s_k) \; ; \; m = 0, 1, 2, \ldots \]  \hspace{1cm} (7.32)

The products in Eqs. 7.31 and 7.32 imply the feedback synthesis of \( \lambda \) and \( \delta \) which is shown in Figure VII-6. It can be directly verified by Eqs. 7.18 - 7.19.

The kernels of the functional for \( f \) are found from Eq. 7.20. Its \( (0,1) \) -order term is

\[ F_{0,1}(s_1) = s_1(s_1 M + K) \]  \hspace{1cm} (7.33)
Figure VII-5 The Syntheses of Two Filters
Figure VII-6 The Feedback Synthesis of $\lambda$ and $S$
The \((2,2p+1)\)-order terms of Eq. 7.20 gives us that

\[
\text{Sym} \left\{ \frac{F_{2,2p+1}(s_1, \ldots, s_{2p+3})}{s_1,s_2} \right\} = \\
\text{Sym} \left\{ -\frac{g}{gL} \sum_{k=0}^{p} L_{1,2p-2k}(s_1,s_2,k+4, \ldots, s_{2p+3}) \right\} D_{1,2k+1}(s_2,s_3, \ldots, s_{2k+4})
\]

\[
; p = 0,1,2, \ldots 
\]

(7.34)

We shall now use these solutions to study the system's stability.

**D-C Stability**

Let us first consider d-c block displacement signals (that is, \(x(t) = X_0, \text{ a constant}\)). For this case, Eq. 7.17 gives us that

\[
f = F[v,X_0] = -f_r
\]

(7.35)

That is, in order to hold the block displaced from center by \(X_0\), we have to apply a force \(f = F[v,X_0]\) to balance the device's restoring force \(f_r\). If this device has achieved stable d-c suspension, then \(f_r\) must, on the average, act so as to restore the block to center \((x = 0)\). Thus a necessary condition for stable d-c suspension is that the time average of the applied force,

\[
\langle F[v,X_0] \rangle
\]

must be such that
\[
\langle F[v, x_0] \rangle = \begin{cases} 
> 0, & \text{if } x_0 > 0 \\
= 0, & \text{if } x_0 = 0 \\
< 0, & \text{if } x_0 < 0 
\end{cases} \tag{7.36}
\]

In the vicinity of the center, the slope of the curve of the time average force vs. \(x_0\)

\[
S = \left[ \frac{\partial}{\partial x_0} \langle F[v, x_0] \rangle \right]_{x_0 = 0} \tag{7.37}
\]

is called the stiffness coefficient of the device (by analogy between the device's restorative action and that of a spring).\(^{13}\)

For d-c stability, Eq. 7.36 shows that the device's stiffness coefficient must be positive. That is

\[
d-c \text{ stability } \Rightarrow S > 0 \tag{7.38}
\]

Eq. 7.23 shows us that the time average of the force, that must be applied to the block in order to make \(x = x_0\), a constant, is

\[
\langle f \rangle = \langle F[v, x_0] \rangle \\
= x_0 F_{0,1}[v, 1] + \sum_{p=0}^{\infty} x_0^{2p+1} \langle F_{2,2p+1}[v, 1] \rangle \\
= x_0 F_{0,1}(0) + \sum_{p=0}^{\infty} x_0^{2p+1} \int_{-\infty}^{\infty} F_{2,2p+1}(\tau_1, \tau_2, 0, \ldots, 0) \cdot \varphi_v(\tau_1 - \tau_2) d\tau_1 d\tau_2 \tag{7.39}
\]
where by $P_{2,2p+1}(\tau_1, \tau_2, 0, ..., 0)$ we mean the inverse transform of $P_{2,2p+1}(s_1, s_2, 0, ..., 0)$ and $\phi_v$ is the autocorrelation function of $v$.

When Eq. 7.33 is evaluated at $s_1 = 0$, then it shows us that the $f_{0,1}$ term of Eq. 7.39 is zero. When the inverse Fourier transform of the spectral function of $v$ is substituted for the autocorrelation function of $v$ in Eq. 7.39, then the result is that

$$\langle f \rangle = \langle F_v(x_0) \rangle$$

$$= \sum_{p=0}^{\infty} x_0^{2p+1} \int_{-\infty}^{\infty} P_{2,2p+1}(-j\omega, j\omega, 0, ..., 0) \phi_v(j\omega) \frac{d\omega}{2\pi} \quad (7.40)$$

where $P_{2,2p+1}$ has regained its usual meaning. When Eqs. 7.30 - 7.32 and 7.34 are evaluated at the appropriate frequencies, then the result is that

$$P_{2,2p+1}(-j\omega, j\omega, 0, ..., 0) = -\frac{L}{2\pi^2 \sigma^{2p+2}} |G(j\omega)|^2 \cdot$$

$$\sum_{k=0}^{P} \left[ H(j\omega) \right]^{2k+1} \left[ H(-j\omega) \right]^{2p-2k} \quad (7.41)$$

When the expression for $P_{2,2p+1}$ given by Eq. 7.41 is substituted into Eq. 7.40 and appropriate reorderings of summation are made, then the result is that
\[ \langle f \rangle = \langle F[v, x_0] \rangle \]

\[ = -\frac{L}{4\pi R^2 g} \int_{-\infty}^{\infty} \Xi_v(j\omega) |G(j\omega)|^2 \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \left[ \frac{x_0}{g} H(j\omega) \right]^{2k+1} \left[ \frac{x_0}{g} H(-j\omega) \right]^{2n} \, d\omega \]

\[ = -\frac{x_0 L}{2\pi R^2 g^2} \int_{0}^{\infty} \left\{ \frac{\Xi_v(j\omega) |G(j\omega)|^2 \text{Re}[H(j\omega)]}{1 - \left( \frac{x_0}{g} \right)^2 [H(j\omega)]^2} \right\} \, d\omega \]  \hspace{1cm} (7.42)

The stiffness coefficient of the device, Eq. 7.37, is thus

\[ S = -\frac{L}{2\pi R^2 g^2} \int_{0}^{\infty} \Xi_v(j\omega) |G(j\omega)|^2 \text{Re}[H(j\omega)] \, d\omega \]  \hspace{1cm} (7.43)

By inspection of either Eq. 7.42 or 7.43, it is evident that if the device is to achieve stable d-c suspension (see Eq. 7.38) then the real part of the system function of the filter \( H \) must be negative over some frequency range. With reference to Figure VII-5, it is then clear that the impedance \( Z \) must contain at least one capacitor. In order to maximize the stiffness of the device, the power of the input \( v \) should be concentrated at that frequency \( \omega \) where \( |G(j\omega)|^2 \text{Re}[H(j\omega)] \) is a minimum. Since \( G \) and \( H \) have the same poles, then this frequency is higher than the frequency which maximizes \( |G(j\omega)|^2 \) but lower than the frequency which minimizes \( \text{Re}[H(j\omega)] \). This forces us to "trade-off" between \( G \) and \( H \) in order to maximize \( S \).

We shall next study the device's dynamic stability.
A-C Stability

Let us next consider a-c block displacement signals (that is, x is a sum of sinusoids). For this case, when we apply the force \( f = F[v, x] \) then we are doing mechanical work. The rate at which we are doing work is our mechanical input power \( p \)

\[
p = f \frac{dx}{dt} \tag{7.44}
\]

If this device has achieved stable a-c suspension, then \( p \) must, on the average, be positive. That is, we must have to work against the device's efforts to keep the block from moving.

Eqs. 7.23 and 7.44 show that \( p \) is a functional of \( v \) and \( x \).

That is

\[
p = R[v, x] = R_{0,2}[v, x] + \sum_{q=1}^{\infty} R_{2,2q}[v, x] \tag{7.45}
\]

where we shall choose

\[
R_{0,2}(s_1, s_2) = F_{0,1}(s_1) s_2 \tag{7.46}
\]

and

\[
R_{2,2q}(s_1, \ldots, s_{2q+2}) = F_{2,2q-1}(s_1, \ldots, s_{2q+1}) s_{2q+2} \tag{7.47}
\]

The \((0,2)\)-order term of \( p \) always has a non-negative average.
\[
\langle p_{0,2} \rangle = \langle R_{0,2} [v, x] \rangle \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r_{0,2}(\tau_1, \tau_2) \varphi_x(\tau_1 - \tau_2) \, d\tau_1 \, d\tau_2 \\
= \frac{K}{\pi} \int_0^\infty \omega^2 \Phi_x(j\omega) \, d\omega 
\]

That is, so long as there is some viscous mechanical damping \((K \neq 0)\), then it aids the a-c suspension stability.

For some applications of this device, the presence of viscous mechanical damping might seem undesirable (e.g. when the suspended block is the rotor of a gyro). In order to achieve stable a-c suspension when \(K = 0\), the terms of \(p\) other than \(p_{0,2}\) would have to have positive averages.

If \(v\) and \(x\) are independent, then the time average of \(p_{2,2}\) is

\[
\langle p_{2,2} \rangle = \langle R_{2,2} [v, x] \rangle \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r_{2,2}(\tau_1, \tau_2, \tau_3, \tau_4) \varphi_v(\tau_1 - \tau_2) \varphi_x(\tau_3 - \tau_4) \, d\tau_1 \ldots d\tau_4 \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{2,2}(\omega, \omega, -j\omega, j\omega) \Phi_v(j\omega) \Phi_x(j\omega) \frac{d\omega}{2\pi} \frac{d\omega}{2\pi} \\
= \frac{-jL}{8\pi^2 R^2 g^2} \int_{-\infty}^{\infty} \left[ \Phi_x(j\omega) \Phi_v(j\omega) |G(j\omega)|^2 \right] \frac{H(j[\omega - \omega]) - H(j[\omega + \omega])}{2\pi} \, d\omega \\
= \frac{L}{4\pi^2 R^2 g^2} \int_0^{\infty} \left[ \Phi_x(j\omega) \Phi_v(j\omega) |G(j\omega)|^2 \right] \frac{\Im[H(j[\omega - \omega]) - H(j[\omega + \omega])]}{2\pi} \, d\omega 
\]

\((7.49)\)
Consider a particular input \( x \) whose spectral function is:

\[
\Phi_x(j\omega) = \frac{2 x_1^2 \omega_0}{\omega^2}
\]  \hspace{1cm} (7.50)

where \( x_1 \) and \( \omega_0 \) are any two positive constants with the dimensions of displacement and frequency respectively. When Eq. 7.50 is substituted into Eq. 7.49 and the Hilbert transform is used to

\[
\text{Re}[H(j\omega)] = \int_0^\infty \frac{\text{Im}[H(j(\omega-\delta)] - H(j(\omega+\delta)]}{\pi \delta} \, d\delta
\]  \hspace{1cm} (7.51)

relate the imaginary part of filter \( H \)'s system function to its real part, then the result is that

\[
\langle p_{2,2} \rangle = -\omega_0 x_1^2 S
\]  \hspace{1cm} (7.52)

where \( S \) is the device's stiffness coefficient (see Eq. 7.43).

Eq. 7.52 is highly significant. If the device is d-c stable, then \( S \) is positive. If \( S \) is positive, then Eq. 7.52 shows that there is at least one \( x \) (Eq. 7.50) such that \( \langle p_{2,2} \rangle \) is negative. If \( K = 0 \), then there is some \( x_1 > 0 \) which is small enough such that \( \langle p \rangle \) is negative. If \( \langle p \rangle \) is negative for some \( x \), then the device is a-c unstable (because the power to sustain the block's movement is flowing out of the device rather than into it). Thus Eq. 7.52 shows that: \textit{If viscous mechanical damping is not present in the device, then its magnetic suspension is unstable.}

\* Integrated white Gaussian noise (a random walk) has such a spectral function. Strictly speaking, this function does not exist but this mathematical flaw can be bypassed by deletion of the singular point at \( \omega = 0 \).
Since the device has now been shown to be either a-c or d-c unstable whenever viscous mechanical damping is absent, then let us next determine just how much damping is needed in order for the device to be both a-c and d-c stable.

Under normal operating conditions, the input voltage \( v \) is a simple sinusoid.

\[
v = \text{Re} \left[ V_1 \exp(j \omega_0 t) \right] \tag{7.53}
\]

Then the voltage spectrum consists of a pair of impulses

\[
\tilde{v}_v(j \omega) = \frac{\pi |V_1|^2}{2} \left[ u_0(\omega - \omega_0) + u_0(\omega + \omega_0) \right] \tag{7.54}
\]

and the evaluation of Eq. 7.42 yields that the time average force necessary to displace the device's block by \( x_0 \), a constant, is

\[
\left< F[v,x_0] \right> = -\frac{L |V_1|^2 x_0 |G(j \omega_0)|^2 \text{Re}[H(j \omega_0)]}{4R^2 g^2 \left[ 1 - \left( \frac{x_0 H(j \omega_0)}{g} \right)^2 \right]^2} \tag{7.55}
\]

The substitution of Eq. 7.54 into Eq. 7.43 gives that the stiffness coefficient is then

\[
S = -\frac{L |V_1|^2}{4R^2 g^2} |G(j \omega_0)|^2 \text{Re}[H(j \omega_0)] \tag{7.56}
\]

The necessary condition for d-c stability is then simply that

\[
\text{Re}[H(j \omega_0)] < 0 \tag{7.57}
\]
If the impedance $Z$ is simply a capacitor $C$, as in Figure VII-1, then Eq. 7.28 yields that

$$\text{Re}[H(j\omega)] = \frac{1 - \omega^2 (LC - 4R^2C^2)}{(LC \omega^2 - 1)^2 + (2RC \omega)^2}$$  \hspace{1cm} (7.58)

Thus the device achieves d-c stability whenever both

$$C < \frac{L}{4R^2}$$  \hspace{1cm} (7.59)

and

$$\omega_e^2 > \frac{1}{LC - 4R^2C^2}$$  \hspace{1cm} (7.60)

(Incidentally, the conditions given by Eqs. 7.59 - 7.60 are strong enough such that the denominator of Eq. 7.55 (the time average force) can never go to zero.)

After Eq. 7.27 is solved for $|G(j\omega_e)|^2$, then the stiffness coefficient can be found from Eqs. 7.56 and 7.58. It is

$$S = \frac{L|V_1|^2 \left[ \omega_e^2 (LC - 4R^2C^2) - 1 \right]}{16 \omega_e^2 g^2 R^4 C^2 \left[ \left( \frac{LC \omega_e^2 - 1}{2\omega_e RC} \right)^2 + 1 \right]}$$  \hspace{1cm} (7.61)

Curves for Eqs. 7.55 and 7.61 are plotted in Frazier and Kingsley.\(^*\)

\(^*\) Frazier and Kingsley's notation is

$$Q_o = \frac{\omega_e L}{2R}, \quad Q = \frac{\omega_e^2 LC - 1}{2\omega_e RC}, \quad S = -\frac{F_o K_n \omega_o}{g} = \frac{4 F_o f(q)}{g}$$
In order to determine how much viscous mechanical damping is necessary to achieve both a-c and d-c stability when the input voltage is a simple sinusoid (see Eqs. 7.53-7.54) we shall consider displacement signals which are also simple sinusoids.

\[ x = \text{Re}\left[ X_1 \exp(j \omega_m t) \right] \]  

(7.62)

\[ \Phi_x(j\omega) = \frac{\pi}{2} \frac{|X_1|^2}{2} \left[ u_0(\omega - \omega_m) + u_0(\omega + \omega_m) \right] \]  

(7.63)

Eq. 7.48 evaluates to

\[ \langle p_{0,2} \rangle = \frac{k \omega_m^2 |X_1|^2}{2} \]  

(7.64)

If \( \omega_e^2 \neq \omega_m^2 \), then Eq. 7.49 evaluates to

\[ \langle p_{2,2} \rangle = \frac{L|X_1V_1|^2}{16R^2 g^2} \omega_m |G(j\omega_e)|^2 \text{Im}\left[ H(j[\omega_e - \omega_m]) - H(j[\omega_e + \omega_m]) \right] \]  

(7.65)

If the device is to be a-c stable, then Eqs. 7.64 - 7.65 show that as a necessary condition, for all \( \omega_m \), \( k \) must be such that

\[ k > \frac{L|X_1|^2 |G(j\omega_e)|^2}{8R^2 g^2} \text{Im}\left[ \frac{H(j[\omega_e + \omega_m]) - H(j[\omega_e - \omega_m])}{\omega_m} \right] \]  

(7.66)

The Hilbert transform, Eq. 7.51, may be used to demonstrate that the expression on the right of Eq. 7.66 is indeed positive so long as the device is d-c stable (see Eq. 7.57).

Thus: Eq. 7.66 is the lower bound upon the viscous mechanical
damping which must be present in the device in order for the magnetic suspension to be stable.

If the damping is less than the bound given by Eq. 7.66, then the suspension is unstable.

Free Oscillations

Whenever \( K \) fails Eq. 7.66, then the device can exhibit self-sustained free oscillations. If these oscillations are periodic

\[
x = \text{Re} \left[ \sum_{n=0}^{\infty} X_n \exp(jn\omega_m t) \right]
\]

(7.67)

and the input voltage \( v \) is a simple sinusoid (Eq. 7.53), then Eq. 7.23 shows that \( \omega_e \) and \( \omega_m \) must be commensurate

\[
2\omega_e = k\omega_m ; \; k \text{ an integer}
\]

(7.68)

in order for these oscillations to be force free (if the frequencies do not satisfy Eq. 7.68, then the frequency components present in Eq. 7.23 could not balance so as to sum to zero force for all time). The power to drive these oscillations is being provided by \( p_{2,2} \) (and the higher order terms) which involves \( \text{Im} H(j\omega) \). When the impedance \( Z \) is simply a capacitor \( C \), as in Figure VII-1, then Eq. 7.28 yields that

\[
\text{Im} \left[ H(j\omega) \right] = \frac{-2\omega^3 RLC^2}{(LC\omega^2 - 1)^2 + (2RC \omega)^2}
\]

(7.69)

If the first harmonic of \( x \) dominates, then Eqs. 7.65 and 7.69 show
that $p_{2,2}$ will contribute power to the oscillations most strongly whenever

$$\omega_e - \omega_m = \omega_\delta \approx \frac{1}{\sqrt{IC}} = \omega_0$$  \hspace{1cm} (7.70)

Eqs. 7.68 and 7.70 then show that

$$k = \frac{2 \omega_e}{\omega_e - \omega_\delta} \approx \frac{2 \omega_e}{\omega_e - \omega_0}$$  \hspace{1cm} (7.71)

$$\omega_e = \frac{\omega_\delta k}{k - 2} \approx \frac{\omega_0 k}{k - 2}$$  \hspace{1cm} (7.72)

$$\omega_m = \frac{2 \omega_\delta}{k - 2} \approx \frac{2 \omega_0}{k - 2}$$  \hspace{1cm} (7.73)

If the third harmonic of $x$ dominates, then instead of Eq. 7.70, we must write

$$\omega_e - 3\omega_m = \omega_\delta \approx \frac{1}{\sqrt{IC}} = \omega_0$$  \hspace{1cm} (7.74)

and then

$$k = \frac{6 \omega_e}{\omega_e - \omega_\delta} \approx \frac{6 \omega_e}{\omega_e - \omega_0}$$  \hspace{1cm} (7.75)

$$\omega_e = \frac{\omega_\delta k}{k - 6} \approx \frac{\omega_0 k}{k - 6}$$  \hspace{1cm} (7.76)
\[ \omega_m = \frac{2 \omega_0}{k - 6} \approx \frac{2 \omega_0}{k - 6} \] (7.77)

Eqs. 7.68 - 7.77 predict certain properties of the magnetic suspension device's self-sustained periodic oscillatory instabilities. These predictions have been verified experimentally. The results of these experimental confirmations of Eqs. 7.68 - 7.77 are presented in Appendix A. The device's self-sustained oscillations can also be explained physically.

Physical Explanation of the Free Oscillations

The magnetic suspension device's self-sustained oscillations can be explained physically by the phase shift of filter \( H \). The frequency response of filter \( H \) is

\[ H(j\omega) = \text{Re}[H(j\omega)] + j \text{Im}[H(j\omega)] \]

\[ = |H(j\omega)| \exp(j\theta(\omega)) \] (7.78)

where \( \text{Re}[H(j\omega)] \) and \( \text{Im}[H(j\omega)] \) are given by Eqs. 7.58 and 7.69 and \( |H(j\omega)| \) and \( \theta(\omega) \) are graphed in Figure VII-7.

By Eq. 7.60, the condition for d-c stability is

\[ \omega_e > \frac{1}{\sqrt{LC - 4R^2C^2}} \] (7.79)

which, by Figure VII-7, is equivalent to

\[ -\pi < \theta(\omega_e) < -\frac{\pi}{2} \] (7.80)
Figure VII-7 The Frequency Response of $H(j\omega)$
Note from Figure VII-7 that \( \omega_e \) must be past the resonant peak of \( |H(j\omega)| \).

When \( \omega_e \) satisfies Eqs. 7.79 or 7.80 and \( x \) is either a simple sinusoid (Eq. 7.62) or a periodic function (Eq. 7.67) whose first harmonic dominates, then the line spectrums of \( v, x, \lambda, \delta \), and \( u \), given by Figure VII-8, follows from Figure VII-6. That is, the input \( v \) (Eq. 7.53) gives \( \lambda \) a dominant component at \( \omega_e \).

\[
\lambda \approx \text{Re} \left[ \Lambda \exp(j \omega_e t) \right] \tag{7.81}
\]

The product \( x \lambda \), therefore, has dominant components at \( \omega_e + \omega_m \) and \( \omega_e - \omega_m \). The component of \( x \lambda \) at \( \omega_e + \omega_m \) is further past the resonant peak of \( |H(j\omega)| \) than \( \omega_e \) was but the component at \( \omega_e - \omega_m = \omega_\delta \) can be near the peak of \( |H(j\omega)| \). (That is, \( \omega_\delta \approx \omega_0 \), see Eq. 7.70). Thus when \( x \lambda \) passes through \( \frac{1}{g} H \) to form \( \delta \) (see Figure VII-6), then the dominant component of \( \delta \) is at \( \omega_e - \omega_m = \omega_\delta \).

\[
\delta \approx \text{Re} \left[ \frac{\Lambda X_{1,n}}{2g} H(j\omega_\delta) \exp(j\omega_\delta t) \right] \tag{7.82}
\]

The product \( x \delta \), therefore, has dominant components at \( \omega_e \) and \( \omega_e - 2\omega_m \). These two components are nearly centered about the peak of \( |H(j\omega)| \). Thus when \( x \delta \) passes through \( \frac{1}{g} H \) to form \( u \), the feedback signal, then \( u \) contributes one component which reinforces the dominant component of \( \lambda \) at \( \omega_e \) and one subsidiary component of

---

* In those cases where \( x \) is periodic and its \( k \)-th harmonic dominates, then \( \omega_e - k\omega_m \) is near the peak of \( |H(j\omega)| \). (See Eq. 7.74.)
Figure VII-8 The Line Spectrum of Free Oscillations
\[ \lambda at \omega = 2\omega_m. \]

By Eqs. 7.8, 7.81, and 7.82, the device's restoring force \( f_r \) is

\[ f_r \approx \text{Re} \left[ \frac{|\Lambda|^2}{A N^2 g} \left( H^*(j\omega_0) \exp(j\omega_m t) \right) \right. \]

\[ + \frac{\Lambda^2 x_1}{A N^2 g} H(j\omega_0) \exp(j[2\omega - \omega_m]t) \right] \]

(7.83)

Eq. 7.83 shows that the component of the device's restoring force at \( \omega_m \) has a phase shift, relative to \( x \), of \( \theta(\omega_m) \). With reference to Figure VII-7 and Eq. 7.80, the range of this phase shift is

\[ 0 < \theta(\omega_0) < \theta(\omega_m) < \pi \]

(7.84)

which is shown in Figure VII-9.

If we rewrite Eq. 7.17 as

\[ f + f_r = M \frac{d^2x}{dt^2} + K \frac{dx}{dt} \]

(7.85)

then we see that the phase shift, relative to \( x \), due to the block's mass and the viscous mechanical damping \( K \), is given by the polynomial \(-\omega_m^2 + j\omega_m K\) and has a range from \( \frac{\pi}{2} \) to \( \pi \), as is also shown in Figure VII-9. Figure VII-9 shows that these two ranges overlap. Thus it is possible for the \( \omega_m \) component of \( f_r \) to balance \( M \frac{d^2x}{dt^2} + K \frac{dx}{dt} \) in Eq. 7.85 when
Figure VII-9 The Phase Angles of $-\theta(\omega)$ and $j\omega(\omega K + K)$
\[-\pi < \theta(\omega_e) < \theta(\omega_0) \leq \frac{-\pi}{2}\]  \hfill (7.86)

That is, when $\omega_m$ is such that

\[0 < \omega_m < \omega_e - \frac{1}{\sqrt{LC - 4R^2C^2}}\]  \hfill (7.87)

then the device's restoring force can balance the block's mechanical forces even though $f = 0$. (See Eq. 7.85) This is the physical basis of the device's free oscillations when the damping is insufficient. The bound upon the damping which will inhibit these oscillations is given by Eq. 7.66. The physical basis for that bound (which is not due to the phase angle argument that we have just completed) is in the fact that Figure VII-9 tells only half the story. That is, even when there is a phase angle balance between the device's restoring force and the block's mechanical forces, this does not prove that $f$ can be zero (force free oscillations) unless their amplitudes also balance. If $K$ is large enough to satisfy Eq. 7.66 for all $\omega_m$ in the range of Eq. 7.87, then this amplitude balance is prevented and the device's oscillatory instabilities are overcome.
CHAPTER VIII

CONCLUSION

If a system's behavior can be characterized by a differential, integral, or integro-differential equation, and if solutions to that equation exist, are unique (for a given input), time-invariant and "smooth" enough so that certain limits exist), then that system is an analytic system and its output can be expressed as a Volterra series of its input. If the system's characteristic equation (or equations) is (are) in the appropriate form, then that system's Volterra series can be found, in a few steps, by our inspection technique (developed here in Chapter III.) If the characteristic equation is not in an appropriate form, it can usually be rewritten in an appropriate form by multiplying through to eliminate those terms which represent division (division is not the result of an operation by an analytic system) by one of the system's variables, (e.g. Eq. 3.6 rewritten as Eq. 3.10).

An analytic system's Volterra series is a functional characterization of that system's behavior. This functional characterization of the system is useful because (1) it can lead to interesting syntheses of that system, (2) it is an explicit statement of that system's output for an arbitrary input (or perhaps for an arbitrary input from within a class of inputs), and (3) from it we can compute that system's output for particular inputs. Some examples of such computations which we have seen were for (1) a d-c input signal (Eqs. 5.24 - 5.32)*, (2) an a-c input signal (Eqs. 4.23 - 4.33),

* The computation of the output for d-c input signals should always be made. It is simple and it serves as a useful check on the writing of the program of the computations.
(3) an exponential input signal (4.21 - 4.22), (4) an exponential-step input signal (Eqs. 6.19 - 6.57), (5) an unknown input signal whose autocorrelation was known (Eqs. 7.39 - 7.42), and (6) a stochastic input signal. The example of the computation of a system's output (or rather the statistical expectation of that system's output) when the input signal is stochastic, from the system's functional characterization, is of special interest because such computations may not be possible from the system's differential equation characterization.

The functional characterization of an analytic system by a Volterra series is not without disadvantages. The Volterra series is usually an infinite series. It need not always be an infinite series (e.g. Eq. 3.37) and even when it is an infinite series we may be able to find a closed form for its evaluation (e.g. Eq. 7.42) but usually we are forced to approximate it by a finite number of terms. When we approximate a Volterra series by a finite number of its terms, then we should compute a bound upon the truncation error that we have introduced (e.g. Eq. 5.50) but finding such a bound can be very difficult.

The extension of the work presented in this thesis to non-analytic systems is questionable. We suppose that, faced with a nonlinear differential equation which did not characterize an analytic system (e.g. Eqs. 3.61 - 3.63), then we could use the inspection technique to get "an answer" but the meaning, if any, of such "an answer" is doubtful. This is one of the disadvantages of our inspection technique—it is too easy. That is, given a
nonlinear differential equation we may be tempted to evade the hard
work of testing it to see if it characterizes an analytic system
and, rashly assuming that it does, proceed to use our inspection
technique to find "an answer". If the differential equation did
not characterize an analytic system, then our "answer" can be
grossly erroneous and misleading.

Additional research is needed on the problem of testing a
differential equation in order to determine if it characterizes
an analytic system. The known tests are too hard. The three
major known existence theorems can be quite difficult to apply.
All too often, a uniqueness proof has to be invented for each
particular differential equation. The proof of time-invariance,
one existence and uniqueness have been shown, is usually trivial.
The proof that the system's functional is analytic at some instant
of time through Volterra's definition (that is, by showing from
limits of small perturbations of the input about zero that all of
Volterra's functional derivatives exist at zero), is far too
tedious to use. As has been said (p. 70), once existence, unique-
ness and time-invariancy have been established, then it is easier
to use our inspection technique to find a Volterra series solution,
if there is one. If we get an answer, and if we can show that it
converges absolutely, then, due to uniqueness, we have shown
analyticity. A simpler series of tests than these would be
extremely useful.
APPENDIX A

SELF-SUSTAINED OSCILLATIONS OF A MAGNETIC SUSPENSION DEVICE EXPERIMENTALLY VERIFIED

In this appendix we shall present the experimental verification of the predictions in Chapter VII as to the self-sustained oscillations of a magnetic suspension device.

Figure A-1 is a sketch of the suspension device upon which the measurements presented in this section were made. The device consisted of a steel rotor (the suspended block) on a shaft set into a pivot bearing for the vertical suspension. Two orthogonal single-axis suspension devices provided the horizontal suspension of the device. In order to measure the device's static parameters, the current \( i \) (see Figure A-2 and Figure VII-3) was monitored with paper shims inserted in the gaps (\( g=0.006" \)) to fix the block displacement at zero. When \( x = 0 \), then it can be shown from Eqs. 7.9, 7.18, 7.22, and 7.29 that

\[
2V(s) = \left[ Z(s) + 2R + sL \right] I(s)
\]  

(A.1)

From the tabulation of the relative phase angles between \( v \) and \( i \), given in Table A-1, we can determine \( \omega_0 \), \( L \), and \( R \). They are

---

* Prof. R. H. Frazier and Mr. P. J. Gilinson, Jr., of the Instrumentation Laboratory, M.I.T., provided the suspension device and the facilities at which the author, with the assistance of Mr. J. Scoppettuolo, conducted these experiments during the early part of June, 1965.
Figure A-2 A Sketch of the Suspension Device
Figure A-2 An Electrical Schematic of the Suspension Device
<table>
<thead>
<tr>
<th>Relative Phase Angle of (v) and (i)</th>
<th>Frequency of (v), According To Oscillator Dial</th>
<th>Frequency of (v), According To Oscilloscope</th>
<th>Frequency of (v), According To E-put Meter</th>
</tr>
</thead>
<tbody>
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</tr>
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</tr>
<tr>
<td>67°</td>
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<td>1.1</td>
<td>1.0971</td>
</tr>
</tbody>
</table>

All frequencies in kilo-cycles-per-second, \(Z(s) = \frac{1}{sC}\), \(C = 0.1 \mu F\).

Table A-1.
\[ \omega_0 = \frac{1}{\sqrt{L C}} = 6.1 \times 10^3 \text{ radians/second} \quad (A.2) \]

\[ L = 0.273 \text{ H} \quad (A.3) \]

\[ R = 103 \Omega \quad (A.4) \]

The paper shims were then removed and \( \psi \) was made large enough so that self-sustained oscillations along the x-axis ensued (the y-axis \( \psi \) was set such that there was no oscillation along the y-axis). Three signals were then monitored: \( v \), the input voltage to the x-axis suspension device; \( v_S \), the stator voltage (which shows when the rotor is touching the stator (see Figure A-1)); and \( v_B \), the bridge voltage (shown in Figures A-2 and VII-3). It is useful to monitor the bridge voltage, \( v_B \), because it provides information about the mean-difference flux-linkage \( \delta \). That is, it can be shown from Eqs. 7.10 - 7.14, and 7.18 - 7.19 that

\[ v_B(s) = -\frac{4s Z(s)}{2R + Z(s)} \Delta(s) \quad (A.5) \]

Eq. A.5 demonstrates that the bridge voltage contains the same frequencies as \( \delta \).

Oscilloscope photographs of the three signals, \( v \), \( v_S \), and \( v_B \) recorded nine separate observations of the magnetic suspension device's self-sustained oscillations. Contrapositives of the photographs of four of these observations are presented as exhibits here.
Interpretation of Exhibits

If the reader will take a sheet of paper and mark an interval on its edge that is as long as the black line under the text for \( v(t) \) in Exhibit A, then he will find that that interval is as long as (i) 5 periods of \( v \), (ii) 4 periods of \( v_s \), (iii) 3 cycles of \( v_B \), and (iv) 1 fundamental period of \( v_B \) (the interval between the two highest peaks). In this experiment, the rotor was hitting the stator twice each period of the mechanical oscillation (that is, it hit once per side). Exhibit A thus demonstrates that

\[
\frac{3.0 \text{ msec.}}{2\pi} = \frac{5}{\omega_e} = \frac{2}{\omega_m} = \frac{3}{\omega_0} = \frac{1}{\omega_{00}} \tag{A.6}
\]

which verifies Eq. 7.68 and shows that \( k = 5 \) in this observation of the device's self-sustained oscillations. The verifications of Eqs. 7.70 - 7.73 by Exhibit A are tabulated on line 1 of Table A-2.

In Exhibit B, the reader will find that the length of the black line under the text for \( v(t) \) equals the length of (i) 9 periods of \( v \), (ii) 4 periods of \( v_s \), and (iii) 1 fundamental period of \( v_B \). The rotor was hitting the stator once per side (twice per mechanical period). Exhibit B thus demonstrates that

\[
\frac{6.5 \text{ msec.}}{2\pi} = \frac{9}{\omega_e} = \frac{2}{\omega_m} = \frac{1}{\omega_0} \tag{A.7}
\]

which verifies Eq. 7.68 and shows that \( k = 9 \) in this observation. The signal \( v_B \) is too rich in harmonics to observe \( \omega_0 \) directly but
\[ v_B(t), \left( 10 \frac{\text{V}}{\text{cm.}} \right) \times \left( 1 \frac{\text{msec.}}{\text{cm}} \right) \]

The y-axis bridge voltage signal.

\[ v(t), \left( 20 \frac{\text{V}}{\text{cm.}} \right) \times \left( 1 \frac{\text{msec.}}{\text{cm}} \right) \]

\[ v_s(t), \left( 20 \frac{\text{V}}{\text{cm.}} \right) \times \left( 1 \frac{\text{msec.}}{\text{cm}} \right) \]

\[ v_B(t), \left( 10 \frac{\text{V}}{\text{cm.}} \right) \times \left( 1 \frac{\text{msec.}}{\text{cm}} \right) \]

\[ v_s(t), \left( 20 \frac{\text{V}}{\text{cm.}} \right) \times \left( 1 \frac{\text{msec.}}{\text{cm}} \right) \]
\[ v_B(t), \ (10 \ \frac{V}{cm.}) \times (1 \ \frac{msec.}{cm.}) \]

The y-axis bridge voltage signal.

\[ v(t), \ (20 \ \frac{V}{cm.}) \times (1 \ \frac{msec.}{cm.}) \]

\[ v_S(t), \ (20 \ \frac{V}{cm.}) \times (1 \ \frac{msec.}{cm.}) \]

\[ v_B(t), \ (10 \ \frac{V}{cm.}) \times (1 \ \frac{msec.}{cm.}) \]

\[ v_S(t), \ (20 \ \frac{V}{cm.}) \times (1 \ \frac{msec.}{cm.}) \]
\[ v_B(t), \ (10 \ \frac{V}{\text{cm.}}) \times (1 \ \frac{\text{msec.}}{\text{cm.}}). \]

The y-axis bridge voltage signal,
\[ (10 \ \frac{V}{\text{cm.}}) \times (1 \ \frac{\text{msec.}}{\text{cm.}}). \]

\[ v(t), \ (20 \ \frac{V}{\text{cm.}}) \times (1 \ \frac{\text{msec.}}{\text{cm.}}). \]

The y-axis input voltage signal,
\[ (20 \ \frac{V}{\text{cm.}}) \times (1 \ \frac{\text{msec.}}{\text{cm.}}). \]

\[ v_B(t), \ (10 \ \frac{V}{\text{cm.}}) \times (1 \ \frac{\text{msec.}}{\text{cm.}}). \]

\[ v_S(t), \ (20 \ \frac{V}{\text{cm.}}) \times (1 \ \frac{\text{msec.}}{\text{cm.}}). \]
\[ v_B(t), \left( 10 \frac{V}{cm.} \right) \times \left( 1 \frac{msec.}{cm.} \right). \]

The y-axis bridge voltage signal,
\[ v(t), \left( 20 \frac{V}{cm.} \right) \times \left( 1 \frac{msec.}{cm.} \right). \]

The y-axis input voltage signal,
\[ v_B(t), \left( 10 \frac{V}{cm.} \right) \times \left( 1 \frac{msec.}{cm.} \right). \]

\[ v_g(t), \left( 20 \frac{V}{cm.} \right) \times \left( 1 \frac{msec.}{cm.} \right). \]
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<td>1.7</td>
<td>9</td>
<td>6.7</td>
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<tr>
<td>8.3</td>
<td>7.5</td>
<td>1.5</td>
<td>1.4</td>
<td>11</td>
<td>7.6</td>
</tr>
</tbody>
</table>
Eq. 7.70 can be verified indirectly by Eq. A.7. That is

\[ \omega_\delta = \omega_e - \omega_m \]

\[ = \frac{18\pi}{6.5 \text{ msec.}} - \frac{4\pi}{6.5 \text{ msec.}} \]

\[ = 6.8 \times 10^3 \text{ radians/sec.} \quad (A.8) \]

which verifies Eq. 7.70 (via Eq. A.2). The verifications of Eqs. 7.70 - 7.73 by Exhibit B are tabulated on line 5 of Table A-2.

In Exhibit C, the reader will find that the length of the black line under the text for \( v(t) \) equals the length of (i) 11 periods of \( v \), (ii) 3 intervals of \( v_S \), and (iii) 1 fundamental period of \( v_B \). The rotor was hitting the stator erratically (as is seen by the lack of symmetry in the waveform) and a longer sequence of pictures (not included here) shows the occasional presence of one more contact per this length. Thus the 3 intervals of \( v_S \) shown in Exhibit C really represent 2 mechanical periods. Exhibit C thus demonstrates that

\[ \frac{8.3 \text{ msec.}}{2\pi} = \frac{11}{\omega_e} = \frac{2}{\omega_m} = \frac{1}{\omega_{60}} \quad (A.9) \]

which verifies Eq. 7.68 and shows that \( k = 11 \) in this observation.

As before, Eq. 7.70 is verified indirectly by

\[ \omega_\delta = \omega_e - \omega_m \]

\[ = \frac{22\pi}{8.3 \text{ msec.}} - \frac{4\pi}{8.3 \text{ msec.}} \]

\[ = 6.8 \times 10^3 \text{ radians/sec.} \quad (A.10) \]
\[ v_B(t), \ (10 \frac{V}{cm.}) \times (1 \frac{msec.}{cm.}). \]

The y-axis bridge voltage signal,
\[ v(t), \ (20 \frac{V}{cm.}) \times (1 \frac{msec.}{cm.}). \]

The y-axis input voltage signal,
\[ v_S(t), \ (20 \frac{V}{cm.}) \times (1 \frac{msec.}{cm.}). \]

\[ v_B(t), \ (5 \frac{V}{cm.}) \times (1 \frac{msec.}{cm.}). \]

\[ v_S(t), \ (20 \frac{V}{cm.}) \times (1 \frac{msec.}{cm.}). \]
$v_B(t), \ (10 \ \frac{V}{cm.}) x (1 \ \frac{msec.}{cm.})$.

The y-axis bridge voltage signal,
$\ (10 \ \frac{V}{cm.}) x (1 \ \frac{msec.}{cm.})$.

$v(t), \ (20 \ \frac{V}{cm.}) x (1 \ \frac{msec.}{cm.})$.

The y-axis input voltage signal,
$\ (20 \ \frac{V}{cm.}) x (1 \ \frac{msec.}{cm.})$.

$v_S(t), \ (20 \ \frac{V}{cm.}) x (1 \ \frac{msec.}{cm.})$.

$v_B(t), \ (5 \ \frac{V}{cm.}) x (1 \ \frac{msec.}{cm.})$.

$v_S(t), \ (20 \ \frac{V}{cm.}) x (1 \ \frac{msec.}{cm.})$. 

---

Exhibit C. A Self-sustained 11.2 Oscillation
The verifications of Eqs. 7.70 - 7.73 by Exhibit C are tabulated on line 7 of Table A-2.

In Exhibit D, the reader will find that the length of the black line under the text for \( v(t) \) equals the length of (i) 19 periods of \( v \), (ii) 8 periods of \( v_S \) (N.B. the change of time calibration), and (iii) 1 fundamental period of \( v_B \). The rotor was hitting the stator 4 times each mechanical period (the third harmonic of \( x \) was dominant). Exhibit D thus demonstrates that

\[
\frac{12.5 \text{ msec.}}{2\pi} = \frac{19}{\omega_e} = \frac{2}{\omega_m} = \frac{1}{\omega_{\delta 0}} \tag{A.11}
\]

which verifies Eq. 7.68 and shows that \( k = 19 \) in this observation. Unlike the preceding three exhibits, in this case the third harmonic of \( x \) dominated. Thus we can verify Eq. 7.74 indirectly via Eq. A.11. That is

\[
\omega_{\delta} = \omega_e - 3\omega_m
\]

\[
= \frac{38\pi}{12.5 \text{ msec.}} - \frac{12\pi}{12.5 \text{ msec.}}
\]

\[
= 6.5 \times 10^3 \text{ radians sec.} \tag{A.12}
\]

The verification of Eqs. 7.74 - 7.77 are tabulated on line 1 of Table A-3.

It will be noted in Tables A-2 and A-3 that the crudest predictions are those of \( k \). This is because the denominator of the formula for \( k \) is of the same order of magnitude as the approximation \( \omega_{\delta} \approx \omega_0 \).
\[ v(t), \ (20 \ \frac{V}{\text{cm.}}) \times (2 \ \frac{\text{msec.}}{\text{cm.}}). \]

\[ v_B(t), \ (10 \ \frac{V}{\text{cm.}}) \times (2 \ \frac{\text{msec.}}{\text{cm.}}). \]

\[ v_S(t), \ (20 \ \frac{V}{\text{cm.}}) \times (0.5 \ \frac{\text{msec.}}{\text{cm.}}). \]

\[ v(t), \ (20 \ \frac{V}{\text{cm.}}) \times (0.5 \ \frac{\text{msec.}}{\text{cm.}}). \]

\[ v_B(t), \ (10 \ \frac{V}{\text{cm.}}) \times (0.5 \ \frac{\text{msec.}}{\text{cm.}}). \]

The y-axis bridge voltage signal, \((10 \ \frac{V}{\text{cm.}})(0.5 \ \frac{\text{ms}}{\text{cm}}).\)
v(t), (20 \frac{V}{cm.}) \times (2 \frac{msec.}{cm} ).

v_B(t), (10 \frac{V}{cm.}) \times (2 \frac{msec.}{cm} ).

v_S(t), (20 \frac{V}{cm.}) \times (0.5 \frac{msec.}{cm} ).

v(t), (20 \frac{V}{cm.}) \times (0.5 \frac{msec.}{cm} ).

v_B(t), (10 \frac{V}{cm.}) \times (0.5 \frac{msec.}{cm} ).

The y-axis bridge voltage signal, (10 \frac{V}{cm.}) \times (0.5 \frac{msec.}{cm} ).
### TABLE A-3

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<th>( \omega_e \approx \frac{k \omega_0}{k-6} )</th>
<th>( \omega_m \approx \frac{2 \omega_0}{k-6} )</th>
<th>( k \approx \frac{2 \omega_e}{\omega_m} = \frac{6 \omega_e}{\omega_e - \omega_0} )</th>
<th>( \omega_0 = \omega_e - 3 \omega_m \approx \omega_0 )</th>
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<td>(pred.)</td>
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<td>Lower freq.</td>
</tr>
</tbody>
</table>

(All frequencies in kilo-radians-per-second)
BIBLIOGRAPHY


   Equation 31 is the definition of the multivariate bilateral Laplace transform.

3. Ibid., pp. 30-44.
   In these four sections, Brilliant studies the combinations of analytic systems.

4. Ibid., p. 31.
   Equation 44 is our Eq. 2.24.

5. Ibid., p. 32.
   Equation 49 is our Eq. 2.25.

6. Ibid., p. 33.
   Equation 54 is the general cascade combination formula.

7. Ibid., pp. 37-44.
   In these two sections, Brilliant studies the feedback combinations of analytic systems.

   In Chapter IV, Davis covers the three most powerful known existence theorems: The Calculus of Limits, The Method of Successive Approximations, and The Cauchy-Lipschitz Method.

   This section was written by P. J. Gilinson, Jr.


13. Ibid., p. 270.
Equation 12-93, with appropriate identification of the variables, is our Eq. 7.61.


The definitions of function and functional are given here.


17. Ibid., pp. 23-27.

18. Ibid., p. 33.
Here George introduces the multilinear correspondent.

19. Ibid., pp. 36-38.
Here George presents his frequency association technique.

20. Ibid., pp. 60-69.
Here George shows the application of his frequency association technique to systems with white noise input.


This chapter, entitled "Electromechanical Components" was written by P. J. Gilinson, Jr.


27. Ibid., p. 207.
The truncation error bound.


33. Ibid., p. 4.
Volterra's definition of and notation for a functional.

34. Ibid., p. 10.
The definition of a continuous functional.

35. Ibid., pp. 23-25.
The derivatives of functionals.

Volterra's Theorem and the definition of an analytical functional.


38. Ibid., p. 280, Eq. 4-57.


Here Wiener defines symmetrization and explains that it has no effect upon the value of the Volterra series. He writes his Volterra series in terms of Stieltjes' integral, we write ours in terms of Riemann's integral.

41. Ibid., p. 89.

I interpret Wiener's comments in the last part of the penultimate paragraph and in the first part of the last paragraph as being about this result.


43. Ibid., pp. 7-9.

Zames' treatment of feedback.

44. Ibid., pp. 73-74.

Zames' trees.
BIографICAL NOTE

The author was born on September 10, 1936 in New York City where he graduated from Newtown High School. He entered M.I.T. in 1954 where, as an undergraduate, he held offices in student government, athletics, and acted in dramatic productions. He has since received the degrees of S.B.E.E., S.M.E.E., and E.E. from M.I.T. and is a member of Sigma Xi, Tau Beta Pi, Eta Kappa Nu, and Hex-alpha. He has held the appointment of Instructor of Electrical Engineering at M.I.T. and has taught courses in Circuit Theory, Electronics, Energy Radiation and Transmission, Linear System Theory, and Information Theory. His professional specialties cover Linear and Nonlinear System Theory, Statistical Communication Theory and Information Theory, and Operations Research. His non-academic experiences include the development of the electrical power control system on the Air Force's B-52 bombers, the Cape Kennedy telemetry timing decoders, a cryogenic laboratory facility, the arming and fusing of the Hydrogen device warheads on the Atlas I.C.B.M., and the planning of the Air Force's response to a BMDS alert of an enemy I.C.B.M. attack. Since July 1, 1964 he has held the appointment of Assistant Professor of Engineering at the University of California, Los Angeles.
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