Numerical Evaluation of Nonlinear Energy Transfer to Short Gravity Waves in the Presence of Long Waves

by

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Submitted to the Department of Ocean Engineering  
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Abstract

Conventional perturbation theory has been the standard tool in calculating the energy transfer rates due to resonant nonlinear interactions among wave components. This subject has been studied extensively from the standpoint of mathematical perturbation theory, and it is safe to say that what can be taught by the perturbation theory is well understood. A fundamental question that remains however is: “How well does the perturbation theory explain the energy transfer that actually takes place in the ocean?” The accuracy of the perturbation calculation for energy input rate to a short wave in the presence of long, large waves is questionable as the long wave amplitude is often large in comparison to the short wave length.

A direct numerical method that avoids the perturbation theory approximation is developed to compute the energy transfer. This has been accomplished by solving an integral equation based on Green’s theorem in three–dimensional space. Although the numerical computation itself is fully nonlinear, our best results for initial conditions have been to base them on high–order perturbation theory. Calculated energy transfer rates for an interacting quartet have been compared with those from the classical perturbation theory and from the spectral method. The long–term evolution of nonlinear wave interactions is examined for moderately steep intermediate–length waves, and it is concluded that the perturbation theory compares reasonably well with the direct numerical method. In the presence of a long wave, first–order amplitudes of the quartet are altered by the high–order cross–interactions of the long wave with the members of the quartet. Based on these corrected amplitudes, perturbation theory provides spectral energy transfer rates in reasonable agreement with the direct calculation.

Thesis Supervisor: Jerome H. Milgram
Title: Professor of Ocean Engineering
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Chapter 1

Introduction

Microwave remote sensing of the sea surface is used for large field scientific ocean measurements, ocean forecasting and ship monitoring by detection of their wakes. At incidence angles more than a few degrees from nadir, much of the microwave backscattering by the sea surface is due to Bragg scattering from short water waves in resonance with the microwaves. The water wavelengths responsible for Bragg scattering with typical instruments are of the order of a few centimeters for X-band radars, or tens of centimeters for L-band. Variable ocean currents, long gravity waves, internal waves, ship wakes and wind have little direct influence on microwave back scattering. Short gravity and capillary waves are the dominant scatterers of microwave radiation from the ocean surface under most of the oceanic and radar-viewing conditions. For example, radar images of long waves are due to modulation of the short Bragg scattering waves by the long ones. As these short sea waves are also influential on several other oceanographic phenomena (e.g., wind stress), it is crucial to understand the short wave dynamics and the ways in which such components are modified. Once we have a profound understanding of the physical mechanisms that transfer energy to and from these waves, we will be most of the way towards predicting the behavior of the short wave energy.

The imaging of the sea surface has motivated a great deal of interest in the short-long wave interaction. Phillips (1981a) and Longuet-Higgins (1987) have studied the
propagation and modulation of linear short waves on a finite-amplitude long wave. In a recent study, Zhang and Melville (1990) have extended these calculations for weakly nonlinear short gravity waves using a multiple-scale perturbation expansion on the surface of the long wave. In this study, we shall focus on energy transfer to short waves by nonlinear interactions among longer waves. A numerical technique is devised to study the initial growth rate of a short gravity wave due to nonlinear interactions in a resonant quartet in the presence of a long, large wave.

1.1 Background and motivation

Growth, decay and equilibrium energy levels of short sea waves is one of the least physically understood areas of near-surface oceanography. The short wave energy is governed by a balance between wave energy input and wave energy dissipation. The balance includes scattering from other waves and currents, and energy propagation effects when the energy is spatially varying. Energy input comes from the wind and from energy transfer via nonlinear wave-wave interactions.

The evolution of ocean wave spectrum can be described by the spectral energy balance equation. In the absence of currents, it takes the following form for deep water waves (Phillips, 1985):

\[ \frac{\partial F(k, x, t)}{\partial t} + c_g \cdot \nabla F(k, x, t) = S_\omega(k) + S_n(k) - S_d(k) \tag{1.1} \]

where \( F \) is the directional power density spectrum of the sea surface elevation, \( t \) is time, \( k \) is the wavenumber vector and \( c_g \) is the group velocity for each spectral component. The first term on the left-hand side is the local rate of change of spectral energy and the second term is the propagation of energy away from a local region (convection by the wave group velocity for spatially non-uniform conditions). For typical oceanic conditions, we can presume that the spatial length scales of spectral inhomogeneities are large in comparison to the wavelengths of interest. The right-hand side contains the energy source terms for the spectral component at wavenumber
k. $S_w$ and $S_n$ represent the rates of energy input to waves from, respectively, the wind and nonlinear wave–wave interactions. $S_d$ is the rate of energy dissipation for each spectral component. The calculation of the distribution of spectral energy and the understanding of the physics require knowledge of the individual source terms which are often coupled. Both the equilibrium short wave energy levels and their rates of change in non–equilibrium conditions depend on the source terms in the spectral energy balance equation. Earlier studies of the source terms include: wind energy input to long waves, Snyder et al. (1981); wind energy input, nonlinear wave–wave interactions and white–cap dissipation in the low frequency region of the spectrum generated in a laboratory wave tank, Wu et al. (1979); wind energy and nonlinear energy transfer input to short waves with frequencies near the spectral peak in a laboratory wind–wave tank, Plant (1980); nonlinear energy transfer in the main energy containing part of a wave spectrum, Hasselmann (1962), Fox (1976), Longuet–Higgins (1976), Webb (1978), Dungey and Hui (1979) and Masuda (1980). Komen et al. (1984) studied the energy balance equation numerically in a fully developed sea up to frequencies that are 2.5 times that of the peak. Waves responsible for radar backscattering, however, have frequencies several times as large as do waves at the spectral peak. Non of these studies was directed to short waves whose frequencies are much higher than that of the spectral peak.

The dissipation of short waves contained in a full spectrum poses a difficult problem. In realistic environments, turbulence, surface films, wave breaking and molecular viscosity lead to decay of these short waves. Viscous decay rate of ocean waves is proportional to $|k|^2$ and is not too crucial for Bragg waves (L–band) whose typical lengths are about 30 centimeters. For example, e–fold decay time for these waves due to molecular viscosity is over 15 minutes. Attenuation of these short waves, however, can significantly increase in the presence of surface films (Peltzer et al., 1990). Although microscale breaking might be of importance as a dissipative process, there are neither direct measurements nor reliable calculations concerning the spectral rate of energy dissipation by wave breaking. The effect of white capping on
the spectral energy balance of waves has been studied by Hasselmann (1974). The effect of turbulence on the attenuation of short gravity waves is recently investigated by Ölmez and Milgram (1991). Laboratory experiments suggest that downward convection of energy by the turbulent velocities might be the principal mechanism for the wave attenuation in the presence of turbulence. Turbulence could be generated artificially by the passage of a ship, wave breaking, air flow over the wavy surface, intense currents, et cetera. We neglected the scattering in the spectral energy balance equation. Turbulence generated scattering would re-distribute the energy among the wave components rather than diminishing it (Phillips, 1958).

As we pointed out, $S_w$ and $S_n$ have been extensively studied for longer waves. For these long waves, $S_d$ has been estimated as the sum of $S_w$ and $S_n$ in equilibrium conditions where the left–hand side of the above equation is zero (Komen et al., 1984 and Phillips, 1985). For the wind–induced growth rate, the most widely used ‘prescription’ is due to Plant (1982). He parameterized the rate of energy input from the wind on the basis of recently published data taken in wind–wave tanks and on the ocean. Under the guidance of very general theoretical considerations, he suggested that the growth rate $\beta$ due to wind energy input take the following form:

$$\beta(k, \theta) = \frac{(0.04 \pm 0.02)u_*^2|k|\cos \theta}{c_p}$$ (1.2)

where $u_*$ is the air friction velocity of the wind, $c_p$ is the phase velocity of the wave component, and $\theta$ is the angle between wind and the wave propagation direction. The available data used by Plant are based on the wave frequencies that are close to that of the spectral peak. The validity of his formulation for frequencies much higher than the spectral peak is uncertain and is an area of active research.

The state–of–the–art method in estimating the energy transfer due to nonlinear wave–wave interactions is a computational implementation of Hasselmann (1962) who investigated the nonlinear spectral energy transfer within a continuous spectrum of ocean waves. The underlying idea in Hasselmann’s theory is to express the wave
field as a regular perturbation expansion in wave slope about the mean free–surface level and to calculate the unsteady energy transfer resulting from resonant nonlinear interactions of spectral components. Perturbation expansion has been carried out to fifth–order in wave amplitude to include the leading–order contributions for the spectral energy transfer. In a wave spectrum consisting of many spectral components, the interaction sets are not closed since a specific wave component might exist in a number of resonant sets (quartets). The problem then is a statistical one. In his theory, Hasselmann treated the underlying statistical process as Gaussian and assumed a Gaussian sea–state which eventually led to the fifth–order theory. The lowest–order interactions for gravity waves involve quartets of first–order waves. For gravity waves, Hasselmann found that the leading–order effect which yields a non–zero mean transfer is a quartet interaction among four waves expressed as a complicated quadruple integral in wavenumber space. Hasselmann’s theory depends critically on two properties:

1. First–order spectral components are statistically independent.

2. Transfer rates are determined entirely by resonance.

Based on dimensional analyses of Kitaigorodskii (1983) and Phillips (1985), Alpers and Hühnerfuss (1989) concluded that the energy input from the wind and from non-linear interactions are both of importance in the equilibrium range of the spectrum and have comparable magnitudes. We have applied the Hasselmann theory to wide bandwidth wave spectra that cover the short gravity waves. Wind energy input is computed using Plant (1982). Computations are based on a JONSWAP spectrum with the following parameters: Phillips’ constant 0.01, peak frequency 0.3 Hz, peak enhancement factor 3.3, left and right spectral widths 0.07 Hz. A ‘cosine–square’ spreading factor is chosen for angular distribution of the spectrum. Air friction velocity which is needed for wind energy input is computed from the relation between Charnock’s drag coefficient and the wind velocity at 10 m as given in Komen (1984). Our preliminary computations for a comparison of energy input rates computed from
wind stress and from nonlinear interactions indicate that the latter is roughly one-fifth of the former for the frequency range corresponding to L-band scattering (see Figure 1-1). If Hasselmann’s nonlinear energy transfer theory is accurate for transfer from long waves to short waves which are outside the main energy containing part of the spectrum, but which are responsible for the Bragg scattering that is crucial to microwave remote sensing of the sea, the nonlinear energy transfer to short waves in a typical sea spectrum is nearly of the same order of magnitude as a rough estimate of the wind energy to short waves based on extrapolating the wind energy input to longer waves. One should bear in mind, however, that the reliability of energy input rates computed for these source terms is questionable in the high frequency end of the spectrum. For the wind–induced growth rate, there is indeed experimental evidence that the short waves grow less rapidly than what Plant’s formula predicts.

![Figure 1-1: Computed energy transfer to short waves from the wind and nonlinear interactions](image)

There are no direct measurements of the spectral energy transfer associated with the nonlinear interactions for the range of frequencies of interest. Wu et al. (1979)
studied the energy balance equation to establish the role of the nonlinear wave-wave interactions for wind-generated waves in a tank. This experimental set-up was designed for a laboratory-scale verification of Hasselmann's theory. Based on these experimental results, Hasselmann's theory seems to be valid in the main energy containing region of the wave spectrum. As the experiments covered frequencies that are typically 2.5 times that of the spectral peak, the energy balance in the high frequency end of the spectrum was not addressed.

In another experimental study, Plant (1980) measured the energy transfer rate to short waves with and without airflow and compared the nonlinear energy transfer rate with the predictions of the perturbation theory of Hasselmann. He generated a spectrum in the tank with a peak value in the high frequency range and evaluated the energy transfer rate due to nonlinear interactions. He found that in the case of waves generated without airflow, perturbation theory predicted energy transfer rates smaller than the measured values. Plant conjectured that one possible explanation for this could be that the first-order spectral components are not completely uncorrelated. If some correlation existed among the first-order spectral components, nonlinear energy transfer could appear at a lower order in the energy expansion leading to a stronger interaction and thus would be larger than presently predicted.

Aside from the assumption of uncorrelated spectral components, a more important question for us is whether Hasselmann's perturbation theory prediction is physically correct for the short Bragg waves whose wavelengths are much shorter than the amplitude of long waves. Applications of the Hasselmann theory have been limited to the energy containing range in the wave spectrum which approximately extends up to frequencies 2.5 times that of the peak for typical wave spectra. If the theory is reliable for only up to these frequencies, prediction of the nonlinear energy transfer to short gravity waves would be erroneous. This potential flaw in Hasselmann's analysis was pointed out by Holliday (1977). According to Brueckner and West (1988), Holliday's arguments have not found universal acceptance, in part because their applicability to interacting surface waves widely separated in scale was unclear at that time. A
simple example will make this point clear. In the perturbation theory, the velocity potential is expanded in a Taylor series about the equilibrium level \( z = 0 \) so that if \( \zeta(x, t) \) is the deviation from that reference surface, the velocity potential can then be approximated by (West, 1981):

\[
\phi(x, z = \zeta, t) = \sum_{k} \phi(k, t) e^{ikx} \sum_{n=0}^{\infty} \frac{(k\zeta)^n}{n!}.
\] (1.3)

When truncated at a finite order, \( e^{k\zeta} \) is approximated by the first few terms of the series expansion:

\[
e^{k\zeta} \simeq 1 + k\zeta + \frac{1}{2} k^2 \zeta^2 + \frac{1}{6} k^3 \zeta^3 + ... \] (1.4)

which is useful if \( k\zeta \) is not too large (e.g., \( k\zeta < 1 \)). When there are two or more wave components widely separated in scale (as in a spectrum of ocean waves), an expansion about the equilibrium level (reference plane) for a wave whose wavelength is shorter than the displacement from that reference plane would fail to converge rapidly enough to produce meaningful lower-order corrections. Holliday (1977) rightly attributes this to the fact that the short wave rides on the surface of the long wave and is displaced quite far from the reference plane. As can be seen from this example, a Taylor series expansion about the equilibrium level displays different convergence characteristics for different wavelengths involved. How much of this range can be justifiably covered in an expansion of the form above is crucial in assessing the usefulness of the theory in the presence of disparate length scales.

### 1.2 Objective and approach

An underlying assumption of the perturbation theory is that all wave amplitudes are small in comparison to all wavelengths. Although there are a number of physical situations that follow this premise, this assumption is clearly violated in typical sea wave spectra which contain widely separated length scales. A fundamental question is whether the ‘small parameter expansion’, currently used in nonlinear wave–wave
interaction problems based on conventional perturbation techniques, is reliable when the parameter is not small. A questionable situation is one where waves of intermediate lengths interact to provide energy to short waves in the presence of much longer, larger waves which may invalidate the perturbation theory. There is little doubt that the perturbation theory is accurate in a narrow-banded spectrum. However, the presence of a wave that is widely separated in scale will displace the short waves quite far from the mean level about which the perturbation expansion is made. We will try to address whether such a perturbation expansion can be valid when the long-wave amplitude is comparable to the short-wave length.

Our approach in answering the validity of the current theory is to compare the results of the perturbation theory with those of a direct numerical method which avoids the perturbation expansion altogether. The approach for the direct numerical method can be explained rather simply. The idea is to consider the free-surface flow having a velocity potential in a bounded region and to determine the time evolution of the free-surface shape and of the potential on the boundaries of the region by time-stepping the boundary conditions. Initial conditions provide the velocity potential and the free-surface shape at the initial time. The procedure is aimed at determining these quantities for the succeeding time steps. Once this is done, the free-surface quantities can be determined at the next time step, and so forth. The free-surface elevation after a time step is easily determined from knowledge of its time rate of change in the kinematic free-surface boundary condition. Similarly, the potential on the free-surface after a time step is easily determined from the dynamic free-surface boundary condition. These boundary conditions are fully nonlinear, time-dependent partial differential equations which contain the gradients of the velocity potential and the surface wave height. The horizontal gradients of the potential and the wave elevation can be computed from knowledge of the free-surface shape and the potential on the free-surface. The 'difficult part' is the calculation of the normal gradient of the velocity potential (normal velocity) on the free-surface. Determination of this quantity is numerically intensive and requires considerable computational resources.
Various numerical methods exist for determining the normal gradient of the potential. A direct approach is to apply Green's theorem using the Rankine source Green function and to solve an integral equation at each time step. The principal advantage of this approach is that it involves no wave parameter expansions, so they cannot be a source of error. We will also compare the results from the direct numerical method with those of a three-dimensional spectral method which utilizes the presumptions of perturbation theory, but which includes perturbations of very high order.

For comparing theoretical predictions of the perturbation theory with the numerical method, we will not use the complete Hasselmann theory which involves the stochastic interaction of all possible groups of four waves, but rather we will consider individual groups of four waves and look at the problem of deterministic interactions. Such a quartet forms the fundamental hydrodynamic resonant nonlinear interaction that is the basis of the Hasselmann theory, but avoids the complexities associated with the stochastic analysis. Therefore, the deterministic quartet interaction is found to be suitable for comparing the perturbation theory to the direct numerical method.
Chapter 2

A review on nonlinear wave–wave interactions

As in many other physical systems, one (perhaps the most) striking feature of the water waves is nonlinearity which leads to the following properties:

1. Existence of bound higher harmonics.

2. Energy transfer among wave components.

Higher–order corrections to linear wave field were first done by Stokes in the mid–19th century. The effects of these higher–order terms appeared in the wave profile as steepened crests and flattened troughs. This effect on wave profile immediately appears at the second–order and continues at higher orders. Another well–known effect is the modification to the phase velocity of a single wave train triggered at the third–order. These higher–order effects were investigated by Stokes (1847). An intriguing outcome of the nonlinearity is the energy exchange among wave components via resonant nonlinear interactions. For gravity waves, this occurs at the third–order in wave slope when only certain (resonance) conditions are satisfied as we shall see later, thus suggesting that nonlinear interactions are not only weak but also selective. Their consequences, however, are important. For example, Hasselmann et al. (1973) inferred from the JONSWAP measurements that the nonlinear resonant wave–wave
interactions account for a major part of the wave growth on the forward face of the wave spectrum, and the overshoot phenomenon.

This effect of nonlinearity was first discovered by Phillips (1960). There was a great deal of skepticism about the whole phenomenon when Phillips first presented his research in a conference (Phillips, 1961). Later however, these effects of nonlinearity on gravity waves were experimentally verified by Longuet-Higgins and Smith (1966) and McGoldrick et al. (1966). The results of these experiments are summarized by Phillips (1967). Recently, Tomita (1989) carried out more accurate experimental studies verifying and extending the results of these earlier experiments. For a history of developments on the subject, see Phillips (1981b).

We mentioned that nonlinear energy transfer occurs when certain conditions are satisfied. We will draw an analogy to the case in water waves by studying a simple mechanical system. This example is given in Phillips (1974), and re-produced here to point out the basic mechanism behind the resonant nonlinear wave–wave interactions. Let us examine a linear undamped oscillator which is subjected to an infinitesimal forcing function. Then, the governing equation for this system can be described by the following:

$$F_{tt} + \omega^2 F = \varepsilon e^{i\Omega t}, \quad (2.1)$$

which has a solution of

$$F = \frac{\varepsilon e^{i\Omega t}}{\omega^2 - \Omega^2} \quad \text{if} \quad \omega^2 \neq \Omega^2$$

$$F = -\frac{i\varepsilon e^{i\Omega t}}{2\Omega} \quad \text{if} \quad \omega^2 = \Omega^2 \quad (2.2)$$

where \(\omega\) is the natural frequency of the system, and \(\Omega\) is the frequency of the forcing function. The response of the system from an initial state of rest is small (of order \(\varepsilon\)) unless \(\Omega\) is nearly equal to \(\omega\). If the two are equal, the amplitude of the oscillation will grow linearly with time and become arbitrarily large. This phenomenon is called 'resonance'. The only way that the response can become large, indeed, is through the resonant excitation.
A very similar effect is involved in the interaction of water waves. For gravity waves, three wave trains with wavenumbers $k_1, k_2,$ and $k_3$ can act as a small amplitude forcing function and provide an excitation at the wavenumber $k_1 \pm k_2 \mp k_3$ and frequency $\omega_1 \pm \omega_2 \mp \omega_3$. The response of the system will be small (of order $\epsilon$) unless the wavenumber and frequency at which the forcing is applied correspond to the wavenumber and frequency of a natural wave mode. If they do, this new free wave will grow in amplitude with energy transfer from the primary waves as a result of the resonant excitation. Otherwise it will be a bound component and remain small forever. For gravity waves, the resonance phenomenon is a third-order effect and requires a quartet of waves. The conditions under these interactions take place can be written as:

$$\omega_1 \mp \omega_2 \mp \omega_3 \mp \omega_4 = 0,$$
$$k_1 \mp k_2 \mp k_3 \mp k_4 = 0,$$

(2.3)

provided that the linear dispersion relation holds for each wave component. As a side note, we should perhaps mention that, among capillary waves, nonlinear energy transfer requires a triad of waves and takes place at the second–order leading to a stronger interaction (McGoldrick, 1965 and 1970). A review of the early papers published on the classical theory and on Hasselmann's theory can be found in Kinsman (1984). West (1981) and Craik (1985) provide a very through review on nonlinear interactions with a good account of the underlying principles and assumptions. Many other aspects of the nonlinear interactions including the statistical mechanics of sea waves have been surveyed by Phillips (1977). A detailed discussion of the basic aspects of nonlinearity in water waves can be found in several other references such as: Lamb (1945), Stoker (1957), Whitham (1974), Lighthill (1978), Drazin and Reid (1981), and Mei (1983).
2.1 The classical theory

Of all possible wavenumber configurations of the vectors $k_1, k_2, k_3,$ and $k_4$, one is particularly convenient for theoretical study. As illustrated in Figure 2-1 below, two of the primary wave trains coincide (say $k_1$) and are perpendicular to another ($k_2$) so that there are only three distinct wavenumbers in the interacting resonant quartet in which the resonant tertiary wave is given by $k_{ter} = 2k_1 - k_2$. Phillips (1960) and later

![Figure 2-1: Wave configuration for the special triad](image)

Longuet-Higgins (1962) studied this special case and derived a theoretical prediction for the amplitude of the resonant tertiary wave. This special case was also used as a benchmark in a series of experiments by Longuet-Higgins and Smith (1966) and McGoldrick et al. (1966) to confirm the theoretical predictions of the tertiary wave growth. Two groups mentioned above generated these primary wave components ($k_1$ and $k_2$) in the tank and observed and quantified the tertiary wave component, whose amplitude is initially zero, building up solely as a result of nonlinear energy transfer. In their theoretical studies, both Phillips and Longuet-Higgins have used essentially the same method, a perturbation analysis, in which the variables of interest in the problem were expressed as a power series in some small parameter $\epsilon$ being proportional to the wave slope. The formalism of the perturbation is such the lower-order solutions are used to drive a linear equation at the next higher order. The right-hand sides of these equations contain the known quantities from solutions to lower-order components. By collecting the coefficients of the various powers of $\epsilon$, this analysis yields, at the third-order ($\epsilon^3$), a tertiary wave component whose amplitude
grows linearly with time. The amplitude of the tertiary wave is given by:

$$|a_{ter}(t)| = Kt$$  \hspace{1cm} (2.4)$$

where $K$ is the interaction (coupling) coefficient involving the wavenumbers, wave frequencies and the first–order amplitudes of the two primary wave components. We will skip the mathematical formulation for the time being as this method (the classical theory) will form the basis of the third–order perturbation analysis in the next chapter. Typical development time of the nonlinear interactions, or the time for the amplitude of the tertiary to become comparable with those of the primaries, is of order

$$t_{int} \sim \frac{T}{a_1 k_1 a_2 k_2},$$  \hspace{1cm} (2.5)$$

that is, of order of the wave period of the tertiary component divided by the product of the maximum slopes of the primary waves. In this analysis, amplitudes of the primary wave components are treated as constants throughout the interaction, and the resonant tertiary has been found to grow linearly with time. In the language of nonlinear mechanics, this tertiary component is a secular term. When such terms appear in the perturbation analysis, some care should be exercised in interpreting the results physically. In an initial value problem, the classical theory must be correct. In other words, the initial growth rates are correct, but this growth cannot occur for a long time since it would require a continuous supply of energy from the primary waves. Therefore when such terms appear, the physical interpretation of the mathematical analysis should be restricted to some very brief initial time. This successive approximation scheme is valid only when,

$$a^2 k^2 \omega t \ll 1$$  \hspace{1cm} (2.6)$$

i.e., for times small in comparison with the interaction time. In a conservative system, the amplitude of the tertiary wave grows at the expense of the primary wave
components which is not indicated by the theory above. It gives no indication of the long time effect of the interaction on for example, the amplitudes of the other modes or the level to which the new mode will grow.

Benney (1962) was the first to show the energy sharing mechanism in the interacting quartet by analyzing the full problem of energy exchanges among the set of four resonant waves whose amplitudes are of comparable magnitude and to obtain a closed set of equations to specify the development in time of all four wave components. The problem Benney worked out was basically to determine the slow rate of change of the Fourier or wave amplitudes due to nonlinearity. Benney assumed that the wave amplitudes in the interacting quartet are slowly varying functions of time rather than strict constants. Benney's approach differs from the classical theory in this respect. As pointed out by Bretherton (1964) though, a weak point in this approach is that the precise meanings of slowly varying and rapidly oscillating amplitude functions are not well defined except that the latter should be small when averaged over a time period that is small compared to typical interaction times.

Consider a set of four interacting gravity wave trains in which the free-surface height is represented by

\[ \zeta = \sum_{m=1}^{4} [a_m(t)e^{i(k_m \cdot x - \omega_m t)} + c.c.] \tag{2.7} \]

where \(\omega\) and \(k\) satisfy the linear dispersion relation. Imposing the assumption that first-order amplitudes \(a_i\) are slowly varying functions of time, we can write:

\[ \left| \frac{da_i}{dt} \right| \ll \omega_i |a_i|. \tag{2.8} \]

By substituting an expected solution for the surface elevation and the velocity potential into the governing equations, we would like to examine the amplitude variation in equation 2.8. From here on, the techniques of the classical perturbation theory follow. All of the rapidly oscillating terms are dropped by averaging over a time

\(^1c.c. \text{ denotes the complex conjugate.}\)}
that is short compared to the interaction time. Then, we are left with the following evolution equations of Benney (1962):

\[
\begin{align*}
    i\frac{da_1}{dt} &= a_1 \sum_{m=1}^{4} g_{1m} a_m^* a_m^* + iha_2^* a_3 a_4 \\
    i\frac{da_2}{dt} &= a_2 \sum_{m=1}^{4} g_{2m} a_m^* a_m^* + iha_1^* a_3 a_4 \\
    i\frac{da_3}{dt} &= a_3 \sum_{m=1}^{4} g_{3m} a_m^* a_m^* + iha_1^* a_2 a_4^* \\
    i\frac{da_4}{dt} &= a_4 \sum_{m=1}^{4} g_{4m} a_m^* a_m^* + iha_1^* a_2 a_3^*.
\end{align*}
\]

(2.9)

The coefficient of \( a_i \) in the first term is always real and it makes \( a_i \) vary sinusoidally with time without changing its magnitude. The terms involving \( g_{ij} \) are associated with changes in the phase velocities that result from self- and mutual-interactions. The mutual-interaction between pairs of waves was noted by Longuet-Higgins and Phillips (1962). \( g_{ii} \) term describes the interaction of a wave with itself to third-order which produces the well-known Stokes correction to the phase velocity. What describes the energy exchange among the wave components is the remaining term involving the coupling coefficient \( h \) (a real number) which is a complicated function of the wavenumber configurations.

Nonlinear interactions are conservative, i.e., the total energy is conserved. Therefore, growth in amplitude of one of the wave components occurs at the expense of the others. The set of equations given above shows how the growth of the new mode is quenched by the nonlinear interactions and how this modifies the amplitudes of the existing modes and this change in turn modifies the growth of the new mode, forming a feedback system. There is no restriction on time for which these results are valid as there is for the secular results of perturbation theory. Among others, Bretherton (1964), McGoldrick (1965), Zakharov (1968), Inoue (1975) and Case and Chiu (1977) used similar techniques to those of Benney.
2.2 Zakharov's theory for surface gravity waves

Zakharov (1968) investigated the instability of nonlinear wave trains to perturbations and described these modulations for weakly nonlinear waves in a manner similar to the nonlinear Schrödinger equation. His derivation did not make any assumptions on the narrow–bandedness and differs from the nonlinear Schrödinger equation in that respect. This improved description of the nonlinear wave dynamics has motivated quite a few researchers on the subject. Crawford et al. (1980) gave a full re–derivation of the Zakharov integral equation with some minor misprints corrected. A few of the recent applications of the method include Crawford et al. (1981), Stiassnie and Shemer (1984 and 1987). Yuen and Lake (1982) describes the method in detail and provides a broad review on several related applications.

In treating the free–surface boundary conditions, Zakharov introduced a velocity potential at the free–surface instead of referring to quantities at the mean level $z = 0$ as is usually done. We will give a derivation of the free–surface boundary conditions used by Zakharov (so–called Zakharov equations) in Chapter 4. In deriving Zakharov’s integral equation, the basic assumptions are the weak nonlinearity and slow modulations as in the nonlinear Schrödinger equation. With a normalized Fourier representation of the free–surface in the governing equations, we decompose the wave field into slowly varying, and small, rapidly varying parts. We will side–step the details as these can be found in the above references and only write down the Zakharov’s integral equation. Collecting terms up to third–order in the slowly varying part, we get (Crawford et al. 1981):

$$i \frac{\partial B(k,t)}{\partial t} = \int \int \int_{-\infty}^{\infty} T(k, k_1, k_2, k_3) \delta(k + k_1 - k_2 - k_3) e^{i(\omega - \omega_1 - \omega_2 - \omega_3)} B^*(k_1, t) B(k_2, t) B(k_3, t) dk_1 dk_2 dk_3$$

(2.10)

where $B(k,t)$ can be interpreted as kind of a generalized amplitude spectrum. This
relates to the free–surface by the following expression:

\[ \zeta(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{k}{2\omega} \right)^{1/2} \left[ B(k, t)e^{i(kx - \omega t)} + B^*(k, t)e^{-i(kx - \omega t)} \right] dk \]  

(2.11)

where \( \omega \) is the wave frequency which is related to \( k \) through the linear dispersion relation. \( T(k, k_1, k_2, k_3) \) is a real scalar coupling coefficient composed of wave frequencies and wavenumbers and given in Appendix A. \( B(k, t) \) can be written as the superposition of discrete modes:

\[ B(k, t) = \sum_n B_n(t) \delta(k - k_n). \]  

(2.12)

Considering the special triad in the previous section, substitution of equation 2.12 into 2.10 and evaluation of the delta functions would yield the following discrete evolution equation for each mode in the system:

\[ i \frac{dB_1}{dt} = \left[ T_{1111}|B_1|^2 + 2T_{1212}|B_2|^2 + 2T_{1313}|B_3|^2 \right] B_1 + 2T_{1123}B_1^*B_2B_3e^{i\Delta\omega t}, \]

\[ i \frac{dB_2}{dt} = \left[ 2T_{2121}|B_1|^2 + T_{2222}|B_2|^2 + 2T_{2323}|B_3|^2 \right] B_2 + T_{2311}B_3^*B_2^2e^{-i\Delta\omega t}, \]

\[ i \frac{dB_3}{dt} = \left[ 2T_{3131}|B_1|^2 + 2T_{3232}|B_2|^2 + T_{3333}|B_3|^2 \right] B_3 + T_{3211}B_2^*B_3^2e^{-i\Delta\omega t}, \]  

(2.13)

where \( \Delta\omega = 2\omega_1 - \omega_2 - \omega_3 \). In the above formulation, subscript 3 refers to the tertiary wave. These are the evolution equations for the special case in which one of the primary waves is counted twice to form the interacting quartet. These are very similar to Benney’s evolution equations for discrete modes. Spectral amplitude \( B_n \) is related to the physical amplitude \( a_n \) as:

\[ a_n = \frac{1}{\pi} \left( \frac{k_n}{2\omega_n} \right)^{1/2} |B_n|. \]  

(2.14)

Corrections to the linear frequencies due to self- and mutual-interactions between
modes are given by:

$$\tilde{\omega}_i = \omega_i + T_{i111}|B_1|^2 + 2 \sum_{j \neq i} T_{ijij}|B_j|^2. \quad (2.15)$$

The first term in the right-hand side is the linear frequency. The second term is the Stokes correction to the linear wave frequency due to self-interactions at the third-order. The final term is another correction due to mutual-interactions between wave components (Longuet-Higgins and Phillips, 1962).

The system of three nonlinear complex ordinary differential equations in equation 2.13, along with the initial conditions, defines the evolution problem of the discrete system. Having separated the real and imaginary parts, these equations can be solved numerically. Following the procedure given in McGoldrick (1972) for capillary waves, these equations lend themselves to a few analytic solutions in terms of Jacobian elliptic functions. Tomita (1989) worked out the numerical and analytical solutions to the system of equations given above. Shemer and Stiassnie (1985) analyzed the long-time evolution of a Stokes wave with two initially small perturbations superposed on the Stokes wave and derived the analytical solution to the amplitudes of each component in time. To point out the link between the classical theory and the Zakharov formulation, we will follow a simple, approximate analytical solution given by Tomita (1989). In his formulation of this approximate solution, Tomita makes the assumption that the amplitude of the resonant tertiary wave is much smaller than those of the primary waves, and the primary wave amplitudes remain constant. Under the guidance of these assumptions, we can neglect the terms that contain $B_3$ and treat the quantities in brackets of equation 2.13 as constants. Using Tomita’s notation, we express the terms in the brackets as:

$$\theta_1 = T_{1111}|B_1|^2 + 2T_{1212}|B_2|^2 + 2T_{1313}|B_3|^2,$$

$$\theta_2 = 2T_{2112}|B_1|^2 + T_{2222}|B_2|^2 + 2T_{2323}|B_3|^2,$$

$$\theta_3 = 2T_{3131}|B_1|^2 + 2T_{3232}|B_2|^2 + T_{3333}|B_3|^2. \quad (2.16)$$

Writing $B_n(t) = b_n(t)exp[i\chi_n(t)]$ given that $b_n$ and $\chi_n$ are real quantities, the last row

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in equation 2.13 yields the following expressions for the tertiary wave component:

\[
\frac{db_3}{dt} = T_{3211} b_1^2(0) b_2(0) \cos[(2\theta_1 - \theta_2)t + \chi_3],
\]
\[
(2.17)
\]

\[
\frac{d\chi_3}{dt} = -\theta_3 - T_{3211} b_1^2(0) b_2(0) b_3^{-1} \sin[(2\theta_1 - \theta_2)t + \chi_3].
\]
\[
(2.18)
\]

If initially \( b_3(0) = 0 \) and \( \chi_3 = 0 \), then the time variation of the tertiary wave amplitude, using equation 2.14, can be written as:

\[
a_3(t) = \frac{K}{|\theta_1 - \frac{1}{2}\theta_2 - \frac{1}{2}\theta_3|} |\sin(\theta_1 - \frac{1}{2}\theta_2 - \frac{1}{2}\theta_3)t|.
\]
\[
(2.19)
\]

in which the coupling (interaction) coefficient \( K \) is given by:

\[
K = 2\pi^2 \omega_1^2 T_{3211} (2r - 1)^{1/2} a_1^2 a_2
\]
\[
(2.20)
\]

where \( r = \omega_1 / \omega_2 \). In the early stages of evolution, i.e., for small \( t \), the solution in equation 2.19 reduces to the one given by the classical theory (equation 2.4):

\[
a_3(t) = Kt.
\]
\[
(2.21)
\]
Chapter 3

Third-order perturbation problem for a triad of waves

A direct numerical implementation of the resonant wave-wave interactions requires that the initial velocity potential and the surface wave profile be specified. In doing this, a perturbation expansion is utilized to express the free-surface quantities initially. The simulation can then be performed in time domain for subsequent time steps.

We will derive the velocity potential and the surface wave elevation for an interacting triad of waves in which one of the primary waves is counted twice so as to form a quartet, but not necessarily normal to the other primary wave. This covers the special case mentioned in the previous chapter and will be our benchmark for the direct numerical method when small wave amplitudes are used. From a theoretical standpoint, working with only two distinct wavenumbers to create the resonant tertiary wave is relatively simpler in algebra and less susceptible to algebraic errors. We will then repeat the derivation for the special triad in the presence of a long wave. The motivation behind this task is to provide the direct numerical method with proper initial conditions.

We will follow the analysis of Longuet-Higgins (1962) which is what we called earlier: The classical theory. The derivation employs a perturbation expansion for
the free–surface quantities about the mean level \( z = 0 \). The present analysis includes all the bound components resulting from both self– and mutual–interactions up to and including third–order.

### 3.1 Initial conditions for the special triad

For the idealized problem of an irrotational flow and an incompressible, inviscid fluid, the velocity field can be described as the gradient of a scalar potential. By the conservation of mass, this potential can be shown to satisfy the Laplace equation in the fluid domain:

\[
\nabla^2 \phi(x, y, z, t) = 0.
\]

We will seek solutions to deep water gravity waves governed by the Laplace equation subjected to following kinematic and dynamic boundary conditions satisfied on the free–surface \( z = \zeta \):

\[
g \zeta + \frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 = 0,
\]

\[
\frac{\partial \zeta}{\partial t} - \frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial x} \frac{\partial \zeta}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \zeta}{\partial y} = 0.
\]

Taking the substantial derivative \( D/Dt \) of equation 3.2 and subtracting \( g \) times equation 3.3 give the free–surface boundary condition in terms of the velocity potential:

\[
\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} + \frac{\partial}{\partial t} (\nabla \phi)^2 + \frac{1}{2} \nabla \phi \cdot \nabla (\nabla \phi)^2 = 0
\]

which is to be satisfied on the free–surface \( z = \zeta \).

We can express the variables of interest as a power series in some small, independent parameter being proportional to wave slope. The surface wave height and the velocity potential can be written as:

\[
\zeta(x, y, t) = (\alpha \zeta_{10} + \beta \zeta_{01}) + (\alpha^2 \zeta_{20} + \alpha \beta \zeta_{11} + \beta^2 \zeta_{02})
\]

\[
+(\alpha^3 \zeta_{30} + \alpha^2 \beta \zeta_{21} + \alpha \beta^2 \zeta_{12} + \beta^3 \zeta_{03}) + ...
\]
\[
\phi(x, y, z, t) = (\alpha \phi_{10} + \beta \phi_{01}) + (\alpha^2 \phi_{20} + \alpha \beta \phi_{11} + \beta^2 \phi_{02}) \\
+ (\alpha^3 \phi_{30} + \alpha^2 \beta \phi_{21} + \alpha \beta^2 \phi_{12} + \beta^3 \phi_{03}) + \ldots \tag{3.6}
\]

from which the orders are apparent. \(\alpha\) and \(\beta\) are small, independent parameters proportional to surface slopes of the two primary waves. \(\zeta_{10}\) and \(\zeta_{01}\) represent the first-order approximations to two intersecting wave trains. The interaction terms are identified as \(\zeta_{11}, \zeta_{21}\) and \(\zeta_{12}\) which appear at the second- and third-orders.

The next step is to expand the kinematic and combined free-surface boundary conditions in a Taylor series about the equilibrium level \((z = 0)\). The expansion of the left-hand sides of 3.2 and 3.3 yields the following equations, respectively:

\[
g \zeta = -\left[ \frac{\partial \phi}{\partial t} + \zeta \frac{\partial^2 \phi}{\partial z \partial t} + \frac{1}{2} \frac{\partial^3 \phi}{\partial z^2 \partial t} + \ldots \right] \\
- \left[ \frac{1}{2} \nabla \cdot \nabla \phi + \zeta \frac{\partial}{\partial z} \left( \frac{1}{2} \nabla \cdot \nabla \phi \right) + \ldots \right] \tag{3.7}
\]

and

\[
- \left( \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right) = \left[ \zeta \frac{\partial}{\partial z} \left( \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} \right) + \frac{1}{2} \zeta^2 \frac{\partial^3}{\partial z^2} \left( \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} + \ldots \right) \right] \\
+ \left[ \frac{\partial}{\partial t} (\nabla \phi)^2 + \zeta \frac{\partial^2}{\partial z \partial t} (\nabla \phi)^2 + \ldots \right] + \left[ \frac{1}{2} \nabla \cdot \nabla (\nabla \phi)^2 + \ldots \right] \tag{3.8}
\]

which are correct up to and including third-order and satisfied on the quiescent free-surface \(z = 0\).

Now we can sort out the equations according to the order in small parameters and solve for the variables of interest at successive orders. Having substituted the formal expressions for the wave height and the velocity potential, equations 3.5 and 3.6, into equation 3.8, the coefficients of like powers of \(\alpha^i \beta^j\) are equated. Starting with the first-order, we solve a linear partial differential equation at each successive order for
the velocity potential which also satisfies the Laplace equation.

At the first-order, the terms in $\alpha$ from equation 3.8 give:

$$\frac{\partial^2 \phi_{10}}{\partial t^2} + g \frac{\partial \phi_{10}}{\partial z} = 0$$  (3.9)

which is to be satisfied at $z = 0$. To the first-order of approximation, we have
the well-known linear solution for deep water waves which also satisfies the Laplace
equation:

$$\phi_{10} = \frac{a_1 g}{\omega_1} e^{k_1 z} \sin \psi_1$$  (3.10)

where we write $\psi_m = k_m(x \mathbf{i} + y \mathbf{j}) - \omega_m t$ for convenience. The corresponding wave
elevation from equation 3.7 is:

$$\zeta_{10} = -\frac{1}{g} \left. \frac{\partial \phi_{10}}{\partial t} \right|_{z=0}$$

$$= a_1 \cos \psi_1.$$  (3.11)

We have similar first-order solutions for the terms in $\beta$:

$$\phi_{01} = \frac{a_2 g}{\omega_2} e^{k_2 z} \sin \psi_2,$$

$$\zeta_{01} = a_2 \cos \psi_2.$$  (3.12)

Having found the first-order approximations, we can now proceed with the second-order quantities. The terms in $\alpha^2$ provide the following equation which is to be satisfied at $z = 0$:

$$-\left( \frac{\partial^2 \phi_{20}}{\partial t^2} + g \frac{\partial \phi_{20}}{\partial z} \right) = \zeta_{10} \frac{\partial}{\partial z} \left( \frac{\partial^2 \phi_{10}}{\partial t^2} + g \frac{\partial \phi_{10}}{\partial z} \right) + \frac{\partial}{\partial t} (\nabla \phi_{10})^2$$  (3.13)

in which $(\nabla \phi_{10})^2 = \omega_1^2 a_1^2 e^{2k_1 z}$ is independent of the horizontal coordinates and time.

The first term in the right-hand side is identically zero from equation 3.9. The second
term is independent of time, so its partial derivative with respect to time is zero. The
above equation along with the Laplace equation is satisfied identically by $\phi_{20} = \phi_{20}(t)$. 

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Therefore, the second-order correction to the velocity potential is zero, and the first-order potential is a solution of the second-order boundary value problem as well. This result confirms the well-known fact that the second-order velocities vanish for irrotational deep water waves. Note that the absence of the second-order terms in the potential holds for deep water waves only (see Wehausen and Laitone, 1960, for nonlinear corrections to the velocity potential in finite water depth). However, the surface wave elevation does not vanish and can be found from equation 3.7:

\[
g \zeta_{20} = -\frac{\partial \phi_{20}}{\partial t} - \zeta_{10} \frac{\partial^2 \phi_{10}}{\partial z \partial t} - \frac{1}{2}(\nabla \phi_{10})^2 \quad \text{on } z = 0 \quad (3.14)
\]

which yields the classical second-order Stokes correction to the surface elevation of a single wave component:

\[
\zeta_{20} = \frac{1}{2} a_1^2 k_1 \cos 2\psi_1. \quad (3.15)
\]

A similar solution is found for the second primary wave:

\[
\zeta_{02} = \frac{1}{2} a_2^2 k_2 \cos 2\psi_2. \quad (3.16)
\]

The equation to be solved for the terms in \( \alpha \beta \) is readily found from equation 3.8:

\[
- \left( \frac{\partial^2 \phi_{11}}{\partial t^2} + g \frac{\partial \phi_{11}}{\partial z} \right) = \zeta_{10} \frac{\partial}{\partial z} \left( \frac{\partial^2 \phi_{01}}{\partial t^2} + g \frac{\partial \phi_{01}}{\partial z} \right) + \zeta_{01} \frac{\partial}{\partial t} \left( \frac{\partial^2 \phi_{10}}{\partial t^2} + g \frac{\partial \phi_{10}}{\partial z} \right) + \frac{\partial}{\partial t} \left( 2 \nabla \phi_{10} \cdot \nabla \phi_{01} \right) \quad (3.17)
\]

which is to be satisfied at \( z = 0 \). From equation 3.9 and its counterpart for \( \beta \), the first two terms vanish. For two waves propagating in arbitrary directions, scalar product of the velocities yields the following expression:

\[
\nabla \phi_{10} \cdot \nabla \phi_{01} = a_1 a_2 \omega_1 \omega_2 e^{(k_1 + k_2)z} \left[ \cos \psi_1 \cos \psi_2 \cos \theta + \sin \psi_1 \sin \psi_2 \right]
\]

\[
= \cos^2 \frac{\theta}{2} \cos(\psi_1 - \psi_2) - \sin^2 \frac{\theta}{2} \cos(\psi_1 + \psi_2) \quad (3.18)
\]
where \( \theta \) is the angle between the vector wavenumbers \( k_1 \) and \( k_2 \). Now the equation becomes:

\[
- \left( \frac{\partial^2 \phi_{11}}{\partial t^2} + g \frac{\partial \phi_{11}}{\partial z} \right) = 2a_1a_2\omega_1\omega_2 \left[ (\omega_1 - \omega_2) \cos^2 \frac{\theta}{2} \sin (\psi_1 - \psi_2) \\
- (\omega_1 + \omega_2) \sin^2 \frac{\theta}{2} \sin (\psi_1 + \psi_2) \right]
\]

(3.19)

which is satisfied at \( z = 0 \). This equation along with the Laplace equation is satisfied by the following solution:

\[
\phi_{11} = \Phi_{(1,-1)}^{(1,1)} e^{i|k_1 - k_2|z} \sin (\psi_1 - \psi_2) - \Phi_{(1,1)}^{(1,1)} e^{i|k_1 + k_2|z} \sin (\psi_1 + \psi_2)
\]

(3.20)

where

\[
\Phi_{(1,-1)}^{(1,1)} = \frac{2a_1a_2\omega_1\omega_2(\omega_1 - \omega_2) \cos^2 \frac{\theta}{2}}{(\omega_1 - \omega_2)^2 - g|k_1 - k_2|},
\]

(3.21)

and

\[
\Phi_{(1,1)}^{(1,1)} = \frac{2a_1a_2\omega_1\omega_2(\omega_1 + \omega_2) \sin^2 \frac{\theta}{2}}{(\omega_1 + \omega_2)^2 - g|k_1 + k_2|}.
\]

(3.22)

In our notation, the subscripts define the phase of the component associated with this particular coefficient. For example, an index of \((1, -1)\) represents the phase \((k_1 - k_2)\). The summation of the superscripts indicates the order of approximation. The superscript \((1, 1)\) in the above coefficients indicates that these are second-order quantities. The corresponding wave height can be found using equation 3.7 at \( z = 0 \):

\[
g\zeta_{11} = -\frac{\partial \phi_{11}}{\partial t} - \zeta_{10} \frac{\partial^2 \phi_{01}}{\partial z \partial t} - \zeta_{01} \frac{\partial^2 \phi_{10}}{\partial z \partial t} - \nabla \phi_{10} \cdot \nabla \phi_{01}.
\]

(3.23)

Substituting the quantities in the right-hand side, \( \zeta_{11} \) can be obtained as follows:

\[
\zeta_{11} = \left[ \frac{\Phi_{(1,-1)}^{(1,1)}}{g} (\omega_1 - \omega_2) + \frac{a_1 a_2}{2g} (\omega_1^2 + \omega_2^2 - \omega_1 \omega_2 \cos^2 \frac{\theta}{2}) \right] \cos (\psi_1 - \psi_2)
+ \left[ -\frac{\Phi_{(1,1)}^{(1,1)}}{g} (\omega_1 + \omega_2) + \frac{a_1 a_2}{2g} (\omega_1^2 + \omega_2^2 + \omega_1 \omega_2 \sin^2 \frac{\theta}{2}) \right] \cos (\psi_1 + \psi_2).
\]

(3.24)
We will now examine the third-order terms. Taylor expansion of the combined free-surface boundary condition yields the following equation for the terms in \( \alpha^3 \):

\[
- \left( \frac{\partial^2 \phi_{30}}{\partial t^2} + g \frac{\partial \phi_{30}}{\partial z} \right) = \xi_{10} \frac{\partial}{\partial z} \left( \frac{\partial^2 \phi_{20}}{\partial t^2} + g \frac{\partial \phi_{20}}{\partial z} \right) + \xi_{20} \frac{\partial}{\partial z} \left( \frac{\partial^2 \phi_{10}}{\partial t^2} + g \frac{\partial \phi_{10}}{\partial z} \right) \\
+ \frac{1}{2} \xi_{10} \frac{\partial^2}{\partial z^2} \left( \frac{\partial^2 \phi_{10}}{\partial t^2} + g \frac{\partial \phi_{10}}{\partial z} \right) + \xi_{10} \frac{\partial^2}{\partial z \partial t} (\nabla \phi_{10})^2 \\
+ \frac{1}{2} \nabla \phi_{10} \cdot \nabla (\nabla \phi_{10})^2 \tag{3.25}
\]

which is to be satisfied at \( z = 0 \). From the first- and second-order quantities we found earlier, the first three terms in the right-hand side are zero. We found that \( \nabla (\phi_{10})^2 \) was independent of time. This leaves us only with the last term in the right-hand side. Thus the equation to be solved reduces to:

\[
- \left( \frac{\partial^2 \phi_{30}}{\partial t^2} + g \frac{\partial \phi_{30}}{\partial z} \right) = a_1^2 \omega_1^2 k_1 \sin \psi_1. \tag{3.26}
\]

Let us assume a solution of the form \( \phi_{30} = \Phi_{(1,0)}^{(3,0)} e^{k_1 z} \sin \psi_1 \) which also satisfies the Laplace equation. From equation 3.26, one would get the following solution:

\[
\Phi_{(1,0)}^{(3,0)} = \frac{a_1^2 \omega_1^2 k_1}{\omega_1^2 - g k_1}. \tag{3.27}
\]

Observe that this particular solution remains bounded for all time if \( \omega_1^2 \neq g k_1 \). A secularity arises, however, if the wavenumber and the wave frequency of the component correspond to those of a free (natural) wave mode, that is, if the wavenumber and the wave frequency obey the linear dispersion relation for deep water gravity waves. Then this solution becomes incorrect (recall the linear undamped oscillator we mentioned in Chapter 2). The correct solution has an amplitude which grows in time (see Bender and Orszag, 1978, pp.544–545):

\[
\phi_{30} = -\frac{a_1^2 \omega_1^2 k_1 t}{2 \omega_1} e^{k_1 z} \cos \psi_1. \tag{3.28}
\]

Unlike the resonance phenomenon which causes a genuine energy transfer (as we shall
see later for the \( \alpha^2\beta \) term), this case has a rather special interpretation. This is an example of a developing tertiary component in which the perturbation frequency, or the perturbation wavenumber for that matter, is equal to that of a free gravity wave component. In other words, tertiary wave component that grows under resonant interaction is the same as the primary wavenumber. However, it is clear from equation 3.28 that the phase of this growing tertiary is different from that of the primary by \( \pi/2 \). Phillips (1960) explains this transfer of potential energy to the same wavenumber with a different phase as an increase in the phase speed of the primary wave. This, indeed, is the well-known effect in the phase velocity of a Stokes wave at the third-order due to self-interactions. The wave height for this component is given by:

\[
\frac{1}{g} \frac{\partial \phi_{30}}{\partial t} \bigg|_{t=0} = \frac{1}{2} a_1(a_1 k_1)^2 \omega_1 t \sin \psi_1 . \tag{3.29}
\]

Knowing that our perturbation solution is valid for a short time period, say:

\[
t \ll \frac{1}{\frac{1}{2}(a_1 k_1)^2 \omega_1} , \tag{3.30}
\]

for this time interval we can write:

\[
\frac{1}{2} a_1(a_1 k_1)^2 \omega_1 t \sin \psi_1 \approx a_1 \sin \left( \frac{1}{2} a_1^2 k_1^2 \omega_1 t \right) \sin \psi_1 . \tag{3.31}
\]

When we combine this with the first-order solution to the wave height for this component (Phillips, 1960), we get:

\[
\eta_1 = a_1 \cos \psi_1 + a_1 \sin \left[ \frac{1}{2} a_1^2 k_1^2 \omega_1 t \right] \sin \psi_1 , \tag{3.32}
\]

which in turn yields the following expression:

\[
\eta_1 \approx a_1 \cos [k_1 x - (1 + \frac{1}{2} a_1^2 k_1^2) \omega_1 t] . \tag{3.33}
\]

As can be seen from the above equation, this correction at the third-order only
changes the phase velocity of the wave component. The increase in the phase velocity is in agreement with the classical Stokes correction at the third-order as given in Lamb (1932):

\[
\tilde{c}_1 = \frac{\omega_1}{k_1} (1 + \frac{1}{2} a_1^2 k_1^2) + O(a_1^3 k_1^3) \tag{3.34}
\]

where \(\tilde{c}_1\) denotes the corrected phase velocity. The phase velocity, as is well-known, depends on the wavelength in a dispersive system. In the equation above, one can see that the phase velocity, though weakly, depends also on the wave amplitude. A wave of high steepness travels faster than a wave of the same wavenumber with less steepness. This phenomenon is known as the amplitude dispersion.

In terms of our perturbation solution, we can show that the first-order solution for the velocity potential indeed satisfies the third-order boundary conditions provided that the dispersion relation is corrected for a second-order effect of the form:

\[
\tilde{\omega}_1^2 = g k_1 (1 + a_1^2 k_1^2) + O(a_1^3 k_1^3) \tag{3.35}
\]

where \(\omega_1 = \sqrt{g k_1}\) is the linear frequency. Thus the corrected wave frequency is given by:

\[
\tilde{\omega}_1 = \omega_1 (1 + \frac{1}{2} a_1^2 k_1^2). \tag{3.36}
\]

The solution to the third-order problem (equation 3.28) is a secular term which always appears whenever the inhomogeneous term (right-hand side in equation 3.26) is itself a solution to the associated homogeneous differential equation. An alternate solution technique (Wehausen and Laitone, 1960) is to expand the wave frequency in a perturbation series of the following form:

\[
\tilde{\omega} = \omega + \epsilon \omega_1 + \epsilon^2 \omega_2 + \ldots \tag{3.37}
\]

At the second-order, a non-secular solution to equation 3.13 implies that \(\omega_1 = 0\). In the same manner, a non-secular solution at the third-order requires that the coefficient of the inhomogeneous term be zero, which in turn yields \(\omega_2 = \frac{1}{2} a_1^2 k_1^2\).
Similar expressions exist for the terms in $\beta^3$. The corrected frequency for the second wave component is:

$$\tilde{\omega}_2 = \omega_2 (1 + \frac{1}{2} a_2^2 k_2^2).$$ (3.38)

Now let us find the surface wave height at the third–order. Using 3.7, we get the following equation at $z = 0$:

$$g\zeta_{30} = -\frac{\partial \phi_{30}}{\partial t} - \zeta_{20} \frac{\partial^2 \phi_{10}}{\partial z \partial t} - \frac{1}{2} \zeta_{10} \frac{\partial^3 \phi_{10}}{\partial z^2 \partial t} - \frac{1}{2} \zeta_{10} \frac{\partial}{\partial z} (\nabla \phi_{10})^2$$ (3.39)

Equation 3.39 provides the third–order Stokes correction to the surface wave height of a single wave component (Fenton, 1985):

$$\zeta_{30} = \frac{3}{8} a_3^2 k_1^2 \cos 3\psi_1 - \frac{3}{8} a_2^3 k_1^2 \cos \psi_1.$$ (3.40)

Results at this order are completely identical for the terms in $\beta^3$. The third–order surface elevation for the second wave component is given by:

$$\zeta_{33} = \frac{3}{8} a_3^2 k_2^2 \cos 3\psi_2 - \frac{3}{8} a_2^3 k_2^2 \cos \psi_2.$$ (3.41)

For the terms in $\alpha^2 \beta$, we get the following equation from 3.8:

$$- \left( \frac{\partial^2 \phi_{21}}{\partial t^2} + g \frac{\partial \phi_{21}}{\partial z} \right) = \zeta_{20} \frac{\partial}{\partial z} \left( \frac{\partial^2 \phi_{01}}{\partial t^2} + g \frac{\partial \phi_{01}}{\partial z} \right) + \zeta_{01} \frac{\partial}{\partial z} \left( \frac{\partial^2 \phi_{20}}{\partial t^2} + g \frac{\partial \phi_{20}}{\partial z} \right)$$

$$+ \zeta_{11} \frac{\partial}{\partial z} \left( \frac{\partial^2 \phi_{10}}{\partial t^2} + g \frac{\partial \phi_{10}}{\partial z} \right) + \zeta_{10} \frac{\partial}{\partial z} \left( \frac{\partial^2 \phi_{11}}{\partial t^2} + g \frac{\partial \phi_{11}}{\partial z} \right)$$

$$+ \frac{1}{2} \zeta_{10} \frac{\partial^2}{\partial z \partial t} \left( \frac{\partial^2 \phi_{01}}{\partial t^2} + g \frac{\partial \phi_{01}}{\partial z} \right)$$

$$+ \frac{1}{2} \zeta_{10} \frac{\partial^2}{\partial z \partial t} \left( \frac{\partial^2 \phi_{10}}{\partial t^2} + g \frac{\partial \phi_{10}}{\partial z} \right)$$

$$+ \zeta_{10} \frac{\partial^2}{\partial z \partial t} \left( \frac{\partial^2 \phi_{11}}{\partial t^2} + g \frac{\partial \phi_{11}}{\partial z} \right)$$

$$+ \nabla \phi_{10} \cdot \nabla (\nabla \phi_{10} \cdot \nabla \phi_{01}) + \frac{1}{2} \nabla \phi_{01} \cdot \nabla (\nabla \phi_{10})^2$$ (3.42)

which is to be satisfied at $z = 0$. From the first– and second–order combined free–
surface boundary conditions, the first three terms are zero. We showed earlier that
the second-order velocity $\nabla \phi_{01}$ vanishes for irrotational deep water, and $(\nabla \phi_{10})^2$
is independent of time. This leaves us with the following non-zero terms for the
right-hand side when evaluated at $z = 0$:

$$
\zeta_0 \frac{\partial}{\partial z} \left( \frac{\partial^2 \phi_{11}}{\partial t^2} + g \frac{\partial \phi_{11}}{\partial z} \right) =
-2a_1a_2^2\omega_1\omega_2 \left[ (\omega_2 - \omega_1)(k_1 - k_2) \cos^2 \frac{\theta}{2} \sin(\psi_2 - \psi_1) \cos \psi_2 \right.
- (\omega_1 + \omega_2)(k_1 + k_2) \sin^2 \frac{\theta}{2} \sin(\psi_1 + \psi_2) \cos \psi_2 \left. \right],
$$

(3.43)

$$
\frac{\partial}{\partial t} \left( 2 \nabla \phi_{11} \cdot \nabla \phi_{01} \right) =
-2\Phi_{(1,1)}^{(1,1)} a_2 \omega_2 |k_1 - k_2| \left[ \omega_1 \sin^2 \frac{\theta}{2} \sin \psi_1 - (2\omega_2 - \omega_1) \cos^2 \frac{\theta}{2} \sin(2\psi_2 - \psi_1) \right]
-2\Phi_{(1,1)}^{(1,1)} a_2 \omega_2 |k_1 + k_2| \left[ \omega_1 \sin^2 \frac{\theta}{2} \sin \psi_1 - (2\omega_2 + \omega_1) \cos^2 \frac{\theta}{2} \sin(2\psi_2 + \psi_1) \right],
$$

(3.44)

$$
\zeta_0 \frac{\partial^2}{\partial z \partial t} \left( 2 \nabla \phi_{01} \cdot \nabla \phi_{10} \right) =
2a_1a_2^2\omega_1\omega_2 (k_1 + k_2) \left[ (\omega_2 - \omega_1) \cos^2 \frac{\theta}{2} \sin(\psi_2 - \psi_1) \cos \psi_2 \right.
- (\omega_1 + \omega_2) \sin^2 \frac{\theta}{2} \sin(\psi_1 + \psi_2) \cos \psi_2 \left. \right],
$$

(3.45)

$$
\nabla \phi_{10} \cdot \nabla (\nabla \phi_{10} \cdot \nabla \phi_{01}) =
a_1^2a_2\omega_1^2\omega_2 \left[ (k_1 + 2k_2 \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2}) \sin \psi_2 \right.
+k_2 \cos^4 \frac{\theta}{2} \sin(2\psi_1 - \psi_2) - k_2 \sin^4 \frac{\theta}{2} \sin(2\psi_1 + \psi_2) \left. \right],
$$

(3.46)

and

$$
\frac{1}{2} \nabla \phi_{01} \cdot \nabla (\nabla \phi_{10})^2 = a_1^2a_2\omega_1^2\omega_2k_1 \sin \psi_2.
$$

(3.47)

The angles $\delta_1$, $\delta_2$ and $\theta$ are illustrated in Figure 3.1. If we sort out all the terms with
respect to phases, the third-order equation for $\alpha^2\beta$ reduces to the following form:

$$
- \left( \frac{\partial^2 \phi_{21}}{\partial t^2} + g \frac{\partial \phi_{21}}{\partial z} \right) = \Phi_{(2,-1)}^{(2,1)} \sin(2\psi_1 - \psi_2) + \Phi_{(2,1)}^{(2,1)} \sin(2\psi_1 + \psi_2) + \Phi_{(0,1)}^{(2,1)} \sin \psi_2
$$

(3.48)
Figure 3.1: Definition of angles $\delta_1$ and $\delta_2$

which is to be satisfied at $z = 0$. The solution to $\phi_{21}$ will be the superposition of the three solutions that correspond to the terms in the right-hand side. In our present calculations, the first term in the right-hand side of equation 3.48 plays a central role. Starting with this term, we have:

$$- \left( \frac{\partial^2 \phi_{21}}{\partial t^2} + g \frac{\partial \phi_{21}}{\partial z} \right) = \Phi_{(2, -1)}^{(2, 1)} \sin(2\psi_1 - \psi_2) \quad \text{on} \quad z = 0. \quad (3.49)$$

A solution that satisfies this equation together with Laplace's equation is:

$$\phi_{21} = -\frac{\Phi_{(2, -1)}^{(2, 1)}}{(2\omega_1 - \omega_2)^2 - g|2k_1 - k_2|^2} e^{2|k_1 - k_2|z} \sin(2\psi_1 - \psi_2) \quad (3.50)$$

where

$$\Phi_{(2, -1)}^{(2, 1)} = a_1^2 a_2 \omega_1 \omega_2 (\omega_1 - \omega_2) \cos^2 \frac{\theta}{2} \left\{ -|k_1 + k_2| + (k_1 + k_2) \right. \left. + \frac{\omega_1 k_2 \cos^2 \frac{\theta}{2}}{\omega_1 - \omega_2} - \frac{4\omega_1 (2\omega_1 - \omega_2)|k_1 - k_2| \cos^2 \frac{\delta_1}{2}}{(\omega_1 - \omega_2)^2 - g|k_1 - k_2|} \right\}. \quad (3.51)$$

As we have seen in the solution of the velocity potential to the third-order boundary value problem in $\alpha^3$, this solution remains bounded for all time only if $(2\omega_1 - \omega_2)^2 \neq g|2k_1 - k_2|$. Again a secularity arises when the perturbation frequency and pertur-
bation wavenumber correspond to those of a free wave mode, i.e., when the linear dispersion relation holds for this frequency and wavenumber pair (resonance phenomenon). We started with a wave configuration that satisfies the resonance conditions for the so-called 'special case'. For this special case, this component is indeed a resonant tertiary wave which results from a genuine energy transfer from the primary waves. Then the correct solution for this resonant term has an amplitude which grows linearly with time:

$$\phi_{21} = -\frac{\Phi_{(2,1)}^{(2,1)} t}{2(\omega_1 - \omega_2)} e^{2|k_1 - k_2|z} \cos(2\psi_1 - \psi_2).$$

(3.52)

This is an example of a developing tertiary wave component under the influence of nonlinear resonant type interactions. The surface wave height of this resonant tertiary wave is given by:

$$\left. -\frac{1}{g} \frac{\partial \phi_{21}}{\partial t} \right|_{z=0} = \frac{\Phi_{(2,1)}^{(2,1)}}{2g} t \sin(2\psi_1 - \psi_2).$$

(3.53)

The amplitude of the growing tertiary wave can be written as:

$$a_{ter}(t) = K t$$

$$= \frac{\Phi_{(2,1)}^{(2,1)}}{2g} t$$

(3.54)

where $K$ is the coupling coefficient given by equation 2.20 in Chapter 2. We confirmed that this equation agrees with the coupling coefficient of the classical perturbation theory found above.

Now let us continue with the second term in the right-hand side of equation 3.48. We will solve the following equation:

$$- \left( \frac{\partial^2 \phi_{21}}{\partial t^2} + g \frac{\partial \phi_{21}}{\partial z} \right) = \Phi_{(2,1)}^{(2,1)} \sin(2\psi_1 + \psi_2) \quad \text{on } z = 0.$$

(3.55)
A solution that will also satisfy the Laplace equation is:

\[
\phi_{21} = \frac{\Phi_{(2,1)}^{(2,1)}}{(2\omega_1 + \omega_2)^2 - g|2k_1 + k_2|} e^{i|k_1 + k_2|z} \sin(2\psi_1 + \psi_2) \tag{3.56}
\]

where

\[
\Phi_{(2,1)}^{(2,1)} = a_1^2 a_2 \omega_1 \omega_2 (\omega_1 + \omega_2) \sin^2 \frac{\theta}{2} \left\{ |k_1 + k_2| - (k_1 + k_2) \right. \\
- \frac{\omega_1 k_2 \sin^2 \frac{\theta}{2}}{\omega_1 + \omega_2} + \frac{4\omega_1 (2\omega_1 + \omega_2)|k_1 + k_2| \cos^2 \frac{\theta}{2}}{(\omega_1 + \omega_2)^2 - g|k_1 + k_2|} \left\}. \tag{3.57}
\]

Note that this solution remains bounded for all times as the wave frequency and wavenumber for this case do not belong to those of a free wave mode.

The last term in equation 3.48 gives rise to a secular term and is quite similar to the third–order boundary value problem we solved for the velocity potential of a single wave component. Thus, it can be interpreted in the same context. The equation at hand is:

\[
- \left( \frac{\partial^2 \phi_{21}}{\partial t^2} + g \frac{\partial \phi_{21}}{\partial z} \right) = \Phi_{(0,1)}^{(2,1)} \sin \psi_2 \quad \text{on} \quad z = 0. \tag{3.58}
\]

This has a solution of the form:

\[
\phi_{21} = -\frac{\Phi_{(0,1)}^{(2,1)} t}{2\omega_2} \cos \psi_2 \tag{3.59}
\]

where

\[
\Phi_{(0,1)}^{(2,1)} = a_1^2 a_2 \omega_1 \omega_2 \left\{ (\omega_1 - \omega_2)|k_1 - k_2| \cos^2 \frac{\theta}{2} \left[ 1 + \frac{4\omega_1 \omega_2 \sin^2 \frac{\theta}{2}}{(\omega_1 - \omega_2)^2 - g|k_1 - k_2|} \right] \\
+ (\omega_1 + \omega_2)|k_1 + k_2| \sin^2 \frac{\theta}{2} \left[ 1 - \frac{4\omega_1 \omega_2 \sin^2 \frac{\theta}{2}}{(\omega_1 + \omega_2)^2 - g|k_1 + k_2|} \right] \\
+ \omega_1 \left[ k_1 - k_2 + 2k_2 \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} \right] + \omega_2 (k_1 + k_2) \cos \theta \right\}. \tag{3.60}
\]
The surface displacement that corresponds to this case is found in a similar manner:

$$\frac{1}{g} \frac{\partial \phi_{21}}{\partial t} \bigg|_{x=0} = \frac{\phi_{(2,1)}^{(2,1)}}{2g} \sin[k_2 \cdot (xi + yj) - \omega_2 t].$$  \hspace{1cm} (3.61)

As we have seen in the third–order boundary value problem to the velocity potential of a single Stokes wave, secular behavior meant a change in the phase velocity of that component due to self–interactions. This was the well–known Stokes effect that was proportional to $(a_2 k_2)^2$. In a manner similar to equation 3.33, this term in equation 3.61 will introduce a change in the phase velocity of wave 2 resulting from its interaction with wave 1 (mutual–interactions). The surface elevation of the wave component with wavenumber $k_2$ can be written as:

$$\eta_2 = a_2 \cos \left( k_2 \cdot (xi + yj) - [1 + \frac{1}{2} a_2^2 k_2^2 + \phi_{(0,1)}^{(2,1)}(2ga_2 \omega_2)^{-1}] \omega_2 t \right).$$  \hspace{1cm} (3.62)

This equation demonstrates the change in the phase velocity of a wave component due to both self– and mutual–interactions. The nonlinear (corrected) wave frequency for this component is now given by:

$$\bar{\omega}_2 = \omega_2[1 + \frac{1}{2} a_2^2 k_2^2 + \phi_{(0,1)}^{(2,1)}(2ga_2 \omega_2)^{-1}].$$  \hspace{1cm} (3.63)

This is identical to equation 2.10 of Longuet–Higgins and Phillips (1962) except what appears to be a typographical error in their $\Phi_{(0,1)}^{(2,1)}$. We can now superpose the solutions we have in order to get the third–order velocity potential for the terms in $\alpha^2 \beta$ and then evaluate the surface wave height using equation 3.7 at $z = 0$:

$$g \zeta_{21} = \frac{\partial \phi_{21}}{\partial t} - \zeta_{20} \frac{\partial^2 \phi_{01}}{\partial z \partial t} - \zeta_{01} \frac{\partial^2 \phi_{20}}{\partial z \partial t} - \zeta_{10} \frac{\partial \phi_{11}}{\partial z \partial t} - \zeta_{11} \frac{\partial^2 \phi_{10}}{\partial z \partial t} - (\nabla \phi_{11} \cdot \nabla \phi_{10})$$

$$- \frac{1}{2} \zeta_{10} \frac{\partial^3 \phi_{01}}{\partial z^2 \partial t} - \zeta_{10} \zeta_{01} \frac{\partial^3 \phi_{10}}{\partial z^2 \partial t} - \zeta_{10} \frac{\partial}{\partial z} (\nabla \phi_{10} \cdot \nabla \phi_{01}) - \frac{1}{2} \zeta_{01} \frac{\partial}{\partial z} (\nabla \phi_{10})^2.$$

$$\hspace{1cm} (3.64)$$

The surface wave elevation is given in appendix B as all the individual terms in the
right-hand side are already known.

In equation 3.8, the terms in $\alpha \beta^2$ yield the following equation:

$$\begin{align*}
- \left( \frac{\partial^2 \phi_{12}}{\partial t^2} + g \frac{\partial \phi_{12}}{\partial z} \right) &= \zeta_{02} \frac{\partial^2 \phi_{10}}{\partial z^2} + g \frac{\partial \phi_{10}}{\partial z} + \zeta_{10} \frac{\partial^2 \phi_{02}}{\partial t^2} + g \frac{\partial \phi_{02}}{\partial z} \\
&+ \zeta_{11} \frac{\partial^2 \phi_{01}}{\partial t^2} + g \frac{\partial \phi_{01}}{\partial z} + \zeta_{01} \frac{\partial^2 \phi_{11}}{\partial t^2} + g \frac{\partial \phi_{11}}{\partial z} \\
&+ \frac{1}{2} \zeta_{01} \frac{\partial^2 \phi_{10}}{\partial z^2} + g \frac{\partial \phi_{10}}{\partial z} \\
&+ \frac{\partial}{\partial t} \left( 2 \nabla \phi_{02} \cdot \nabla \phi_{10} + 2 \nabla \phi_{11} \cdot \nabla \phi_{01} \right) \\
&+ \zeta_{01} \frac{\partial^2}{\partial z \partial t} (2 \nabla \phi_{01} \cdot \nabla \phi_{01}) + \zeta_{10} \frac{\partial^2}{\partial z \partial t} (\nabla \phi_{01})^2 \\
&+ \nabla \phi_{10} \cdot \nabla (\nabla \phi_{10} \cdot \nabla \phi_{01}) + \frac{1}{2} \nabla \phi_{10} \cdot \nabla (\nabla \phi_{01})^2 \quad (3.65)
\end{align*}$$

which is to be satisfied at $z = 0$. As exactly we had before for the equation in $\alpha^2 \beta$, some of the terms are zero due to same arguments. The only non-zero terms in the right-hand side are the following when evaluated at $z = 0$:

$$\begin{align*}
\zeta_{01} \frac{\partial}{\partial z} \left( \frac{\partial^2 \phi_{11}}{\partial t^2} + g \frac{\partial \phi_{11}}{\partial z} \right) &= \\
&- 2a_1 a_2 \omega_1 \omega_2 \left[ (\omega_2 - \omega_1) |k_1 - k_2| \cos^2 \frac{\theta}{2} \sin(\psi_2 - \psi_1) \cos \psi_2 \\
&- (\omega_1 + \omega_2) |k_1 + k_2| \sin^2 \frac{\theta}{2} \sin(\psi_1 + \psi_2) \cos \psi_2 \right], \quad (3.66)
\end{align*}$$

$$\begin{align*}
\frac{\partial}{\partial t} (2 \nabla \phi_{11} \cdot \nabla \phi_{01}) &= \\
&- 2 \Phi^{(1,1)}_{(1,1)} a_2 \omega_2 |k_1 - k_2| \left[ \omega_1 \sin^2 \frac{\theta}{2} \sin \psi_1 - (2 \omega_2 - \omega_1) \cos^2 \frac{\theta}{2} \sin(2 \psi_2 - \psi_1) \right] \\
&- 2 \Phi^{(1,1)}_{(1,1)} a_2 \omega_2 |k_1 + k_2| \left[ \omega_1 \sin^2 \frac{\theta}{2} \sin \psi_1 - (2 \omega_2 + \omega_1) \cos^2 \frac{\theta}{2} \sin(2 \psi_2 + \psi_1) \right], \quad (3.67)
\end{align*}$$

$$\begin{align*}
\zeta_{01} \frac{\partial^2}{\partial z \partial t} (2 \nabla \phi_{01} \cdot \nabla \phi_{10}) &= \\
&2a_1 a_2 \omega_1 \omega_2 (k_1 + k_2) \left[ (\omega_2 - \omega_1) \cos^2 \frac{\theta}{2} \sin(\psi_2 - \psi_1) \cos \psi_2 \\
&- (\omega_1 + \omega_2) \sin^2 \frac{\theta}{2} \sin(\psi_1 + \psi_2) \cos \psi_2 \right], \quad (3.68)
\end{align*}$$

$$\nabla \phi_{01} \cdot \nabla (\nabla \phi_{01} \cdot \nabla \phi_{10}) =$$
\[ a_1 a_2^2 \omega_1 \omega_2^2 \left[ (k_2 + 2k_1 \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2}) \sin \psi_1 + k_1 \cos^4 \frac{\theta}{2} \sin(2\psi_2 - \psi_1) - k_1 \sin^4 \frac{\theta}{2} \sin(2\psi_2 + \psi_1) \right], \] (3.69)

and

\[ \frac{1}{2} \nabla^2 \phi_{10} \cdot \nabla (\nabla \phi_{01})^2 = a_1 a_2^2 \omega_1 \omega_2^2 k_2 \sin \psi_1. \] (3.70)

Figure 3-2: Definition of angles \( \gamma_1 \) and \( \gamma_2 \)

The angles \( \gamma_1 \) and \( \gamma_2 \) are illustrated in Figure 3-2. After some lengthy algebra, equation 3.65 reduces to the following form:

\[- \left( \frac{\partial^2 \phi_{12}}{\partial t^2} + g \frac{\partial \phi_{12}}{\partial z} \right) = \Phi^{(1,2)}_{(-1,2)} \sin(2\psi_2 - \psi_1) + \Phi^{(1,2)}_{(1,2)} \sin(2\psi_2 + \psi_1) + \Phi^{(1,2)}_{(1,0)} \sin \psi_1. \] (3.71)

Again, the solution to \( \phi_{12} \) will be the superposition of the solutions to the terms in the right-hand side. Starting with the first term, we have the following solution:

\[ \phi_{12} = \frac{\Phi^{(1,2)}_{(-1,2)}}{(2\omega_2 - \omega_1)^2 - g|2k_2 - k_1|^2} e^{i|2k_2 - k_1|z} \sin(2\psi_2 - \psi_1) \] (3.72)

where

\[ \Phi^{(1,2)}_{(-1,2)} = a_1 a_2^2 \omega_1 \omega_2 (\omega_2 - \omega_1) \cos^2 \frac{\theta}{2} \left\{ -|k_1 - k_2| + (k_1 + k_2) \right\} \]
\[
\frac{\omega_2 k_1 \cos^2 \frac{\theta}{2}}{\omega_2 - \omega_1} - \frac{4\omega_2 (2\omega_2 - \omega_1)|k_1 - k_2| \cos^2 \frac{\theta}{2}}{(\omega_1 - \omega_2)^2 - g|k_1 - k_2|} \right\}.
\]

(3.73)

We get the solution to the second term in a similar manner:

\[
\phi_{12} = \frac{\Phi^{(1,2)}_{1,1}}{(2\omega_2 + \omega_1)^2 - g(2k_2 + k_1)^2} e^{i(2k_2 + k_1)|x|} \sin(2\psi_2 + \psi_1)
\]

(3.74)

where

\[
\Phi^{(1,2)}_{1,1} = a_1 a_2^2 \omega_1 \omega_2 (\omega_1 + \omega_2) \sin^2 \frac{\theta}{2} \left\{ |k_1 + k_2| - (k_1 + k_2) - \frac{\omega_2 k_1 \sin^2 \frac{\theta}{2}}{\omega_1 + \omega_2} + \frac{4\omega_2 (2\omega_2 + \omega_1)|k_1 + k_2| \cos^2 \frac{\theta}{2}}{(\omega_1 + \omega_2)^2 - g|k_1 + k_2|} \right\}.
\]

(3.75)

The last term in the right-hand side has a similar solution to equation 3.59:

\[
\phi_{12} = -\frac{\Phi^{(1,2)}_{1,0}}{2\omega_1} \cos \psi
\]

(3.76)

where

\[
\Phi^{(1,2)}_{1,0} = a_1 a_2^2 \omega_1 \omega_2 \left\{ (\omega_2 - \omega_1)|k_1 - k_2| \cos^2 \frac{\theta}{2} \left[ 1 + \frac{4\omega_1 \omega_2 \sin^2 \frac{\theta}{2}}{(\omega_1 - \omega_2)^2 - g|k_1 - k_2|} \right] (\omega_1 + \omega_2)|k_1 + k_2| \sin^2 \frac{\theta}{2} \left[ 1 - \frac{4\omega_1 \omega_2 \sin^2 \frac{\theta}{2}}{(\omega_1 + \omega_2)^2 - g|k_1 + k_2|} \right] + \omega_2 \left[ k_2 - k_1 + 2k_1 \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} + \omega_1 (k_1 + k_2) \cos \theta \right] \right\}.
\]

(3.77)

As we have seen in the solution to \(a^2 \beta\), this will influence only the phase speed of the first wave component due to mutual-interactions with the second primary wave. The nonlinear wave frequency for this component can be written in a similar manner to equation 3.63:

\[
\bar{\omega}_1 = \omega_1 [1 + \frac{1}{2} a_1^2 k_1^2 + \Phi^{(1,2)}_{1,0}(2g\alpha_1 \omega_1)^{-1}].
\]

(3.78)
Using equation 3.7, we can now evaluate the surface elevation for the terms in $\alpha \beta^2$:

$$
\begin{align*}
\zeta_{12} &= -\frac{\partial \phi_{12}}{\partial t} - \zeta_{02} \frac{\partial^2 \phi_{10}}{\partial z \partial t} - \zeta_{10} \frac{\partial^2 \phi_{02}}{\partial z \partial t} - \zeta_{01} \frac{\partial^2 \phi_{11}}{\partial z \partial t} - \zeta_{11} \frac{\partial^2 \phi_{01}}{\partial z \partial t} - (\nabla \phi_{11} \cdot \nabla \phi_{01}) \\
&- \frac{1}{2} \zeta_{01}^2 \frac{\partial^2 \phi_{10}}{\partial z^2 \partial t} - \zeta_{01} \zeta_{10} \frac{\partial^2 \phi_{01}}{\partial z^2 \partial t} - \zeta_{01} \frac{\partial}{\partial z} (\nabla \phi_{01} \cdot \nabla \phi_{10}) - \frac{1}{2} \zeta_{10} \frac{\partial}{\partial z} (\nabla \phi_{01})^2.
\end{align*}
$$

(3.79)

All the terms in the right-hand side are known from the lower order solutions.

We analyzed the third-order perturbation problem for the special triad. From equations 3.5 and 3.6, we can discard the small parameters used in the perturbation analysis and re-write the surface wave height and the velocity potential as follows:

$$
\zeta(x, y, t) = \zeta_{10} + \zeta_{01} + \zeta_{12} + \zeta_{20} + \zeta_{11} + \zeta_{02} + \zeta_{30} + \zeta_{21} + \zeta_{13} + \zeta_{03},
$$

(3.80)

$$
\phi(x, y, z, t) = \phi_{10} + \phi_{01} + \phi_{11} + \phi_{30} + \phi_{02} + \phi_{21} + \phi_{12}.
$$

(3.81)

The complete initial velocity potential and the surface wave elevation (at time $t = 0$) are presented in Appendix B.

### 3.2 Initial conditions for the special triad in the presence of a long wave

In the previous section, we derived the velocity potential and the surface wave elevation for a triad of waves using the classical theory. We will now extend the derivation for this triad when a long wave is present in the field. Figure 3-3 shows the wavenumber configuration for this case in which the long wave $k_3$ is propagating in the same direction as the second primary wave. As we did before, let us expand the surface wave elevation and the velocity potential including the long wave:

$$
\zeta(x, y, t) = (\alpha \zeta_{100} + \beta \zeta_{010} + \gamma \zeta_{001})
$$

$$
+(\alpha^2 \zeta_{200} + \beta^2 \zeta_{020} + \gamma^2 \zeta_{002} + \alpha \beta \zeta_{110} + \alpha \gamma \zeta_{101} + \beta \gamma \zeta_{011})
$$

50
Figure 3-3: The special triad in the presence of a long wave $k_3$.

\[ \begin{align*}
+(\alpha^3 \zeta_{300} + \beta^3 \zeta_{030} + \gamma^3 \zeta_{003} \\
+\alpha^2 \beta \zeta_{210} + \alpha \beta^2 \zeta_{120} + \alpha \gamma^2 \zeta_{102} + \alpha^2 \gamma \zeta_{201} + \beta \gamma^2 \zeta_{012} + \beta^2 \gamma \zeta_{021} \\
+\alpha \beta \gamma \zeta_{111}) + \ldots 
\end{align*} \]

\[ \phi(x, y, z, t) = (\alpha \phi_{100} + \beta \phi_{010} + \gamma \phi_{001}) \\
+(\alpha^2 \phi_{200} + \beta^2 \phi_{020} + \gamma^2 \phi_{002} + \alpha \beta \phi_{110} + \alpha \gamma \phi_{101} + \beta \gamma \phi_{011}) \\
+(\alpha^3 \phi_{300} + \beta^3 \phi_{030} + \gamma^3 \phi_{003} \\
+\alpha^2 \beta \phi_{210} + \alpha \beta^2 \phi_{120} + \alpha \gamma^2 \phi_{102} \\
+\alpha^2 \gamma \phi_{201} + \beta \gamma^2 \phi_{012} + \beta^2 \gamma \phi_{021} \\
+\alpha \beta \gamma \phi_{111}) + \ldots \] \quad (3.83)

in which $\gamma$ is a small parameter proportional to the slope of the long wave. The procedure is completely identical to the one we followed in Section 3.1. However, the inclusion of a separate wave component makes the mathematical analysis quite complicated signalling that the analysis we presented was the practical limit of hand calculations. We used MACSYMA, a software package that can handle symbolic algebra manipulations, to work out the three-wave interaction problem and also confirmed the results of the special-triad problem. We will not present a step-by-step analysis for this case as the procedure is identical to that of the previous section. The cross interaction of the long wave with the primary waves can indeed be deduced from the
results of the previous section involving only the two primary waves. We derived the interaction terms for two wave components traveling in arbitrary directions. Thus, the interaction of the long wave with either primary is covered by the earlier results when appropriate changes in the wavenumbers and wave frequencies are made. The only exception is due to the terms that are proportional to \( \alpha \beta \gamma \). In Appendix C, the initial conditions for the velocity potential and the surface wave height are given at \( t = 0 \).
Chapter 4

Numerical implementation

A numerical method is implemented for calculating the energy transfer due to non-linear interactions among waves. The evolution of the wave field is computed at each time step by solving an integral equation based on Green's theorem. This, in principle, is the solution to a boundary-value problem at each time step. Provided that the initial state of the field is described; this requires the knowledge of the velocity potential and the surface wave profile (a Dirichlet problem), we solve the numerical boundary-value problem for the normal velocity on the surface to update the free-surface shape and the potential in time by time-stepping the free-surface boundary conditions.

We will highlight the mathematical aspects of the method we will implement. The numerical description of the method will then follow. We will construct the algorithm and assess the accuracy and efficiency of the method with time-domain simulations of Stokes waves. A brief review on the spectral method is also given at end of this chapter.
4.1 Mathematical formulation

In Chapter 3, we have shown that the problem of an irrotational flow in an incompressible, inviscid fluid is governed by the Laplace equation:

$$\nabla^2 \phi(x, z, t) = 0 \quad \text{throughout the domain,} \quad (4.1)$$

where $\phi(x, z, t)$ represents the velocity potential. It is well known that the solution to Dirichlet and Neumann problems governed by the Laplace equation can be expressed in terms of boundary integrals of source and normal dipole distributions. A broad literature is available on the subject (Kellogg, 1954, Jaswon, 1963 and Burton and Miller, 1971).

A brief review of the solution to the field equation is given below. Assume that $\Omega$ is a simply-connected bounded domain and let $n$ be the unit outward normal vector to the boundary $\partial \Omega$ of $\Omega$. Let $\phi$ be any function of $C^2(\Omega)$. Upon the application of Green's second identity, the value of $\phi$ at any point on the boundary is given by the following integral equation (Zachmanoglou and Thoe, 1986 and Courant and Hilbert, 1953):

$$-2\pi \phi(p) = \iint \phi(q) \frac{\partial G(p, q)}{\partial n(q)} dS(q) - \int \int G(p, q) \frac{\partial \phi(q)}{\partial n(q)} dS(q) \quad (4.2)$$

where the first integral is in the sense of Cauchy principal value and excludes integration over the field point $p = (x, y, z)$. We define $q = (\xi, \eta, \zeta)$ to be the source point where the singularity is located. $G(p, q)$ is the three-dimensional free-space Green function defined as:

$$G(x, y, z, \xi, \eta, \zeta) = \frac{1}{|p - q|} = \left[ (x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2 \right]^{-1/2} \quad (4.3)$$

where $|p - q|$ is the distance between the source and field points. In equation (4.2), we constructed the induced velocity potential at an arbitrary point $p$ on the surface by distributions of sources $G$ with strength $\partial \phi/\partial n$ and normal dipoles $\partial G/\partial n$ with
strength $\phi$ over the boundary $\partial \Omega$ of domain $\Omega$. The boundary integral equation in (4.2) is a Fredholm integral equation of the first or second kind depending on the choice of the boundary conditions (Dirichlet or Neumann).

### 4.2 Numerical formulation

A standard tool in computation of the velocity potential or its normal gradient for potential–flow problems is the direct application of Green’s theorem. We have seen that the velocity potential at some arbitrary point on the surface can be constructed from distributions of sources and normal dipoles. This results in an integral equation over the surface of the boundary. The method is known by a variety of names such as: boundary integral equation method, boundary element method or panel method. To solve this integral equation numerically, the boundary $\partial \Omega$ is discretized into $N$ quadrilateral elements (panels), and the boundary conditions are satisfied at prescribed collocation points on each panel. Transforming the integrals in equation (4.2) into sums of integrals over each element, a discretized form of the continuous boundary integral equation can be written as:

$$
-2\pi \phi_i = \sum_{j=1 \atop (j \neq i)}^{N} H_{ij} - \sum_{j=1}^{N} G_{ij} \quad \text{for} \quad i = 1, N \tag{4.4}
$$

where

$$
H_{ij} = \iint_{\Delta s_j} \phi_j G_{ij} \ dS,
$$

$$
G_{ij} = \iint_{\Delta s_j} (\phi_n)_j \frac{\partial G_{ij}}{\partial n_j} \ dS. \tag{4.5}
$$

Applying (4.4) at each collocation point where the boundary conditions are imposed, a set of $N$ linear algebraic equations with $N$ unknown singularity strengths of $\phi(p)$ or $\partial \phi(p) / \partial n$ is obtained. There exist a number of standard methods to solve this
matrix equation.

This general technique was first introduced by Hess and Smith (1962) using constant–strength distributions on the plane quadrilateral panels. They evaluated the source and dipole integrals analytically over each panel surface and obtained the closed form expressions for $\mathcal{H}_{ij}$ and $G_{ij}$. In their formulation, the singularity distribution is discontinuous (a jump exists at the boundary of two elements), and it is not possible to construct a surface representation of plane quadrilateral panels whose all four corners match the corners of the adjacent panels. Consequently, an arbitrary surface constructed this way will have leaks between the neighboring elements. This has been criticized by Webster (1975). However, Hess (1975) discounts this argument indicating that the discretization is merely a mathematical approximation rather than a physical one. On the other hand, the use of triangular panels eliminates this problem completely (Bai and Yeung, 1974).

The accuracy of the boundary integral equation can be further refined by applying a higher order of approximation for the singularity distributions and for the surface geometry. Rather than constant singularity distribution, bilinear, biquadratic, bicubic distributions can be used in functional representations of $\phi(p)$ and $\partial\phi(p)/\partial n$. Each surface element can be approximated by curved panels. Especially in three-dimensional applications, the algorithmic complexities and the computational expense are the prices for these refinements. The additional complexity in the structure of the program may be a potential cause for the lack of robustness as well. For more information on the higher order panel methods, see Forrister (1978) and Romate (1989).

The boundary integral equation method (denoted hereafter as BIEM) has been successfully applied to water wave problems although almost all of the effort concentrated on two–dimensional applications. Longuet–Higgins and Cokelet (1976) implemented this method to simulate overturning waves in time domain. Among others who used similar techniques were Vinje and Brevig (1980,1981), Baker et al. (1982), Lin et al. (1984), Dold and Peregrine (1984), Dommermuth and Yue (1987c,1988), and Grilli et al. (1989). Examples of three–dimensional applications to water waves
are still very rare. Romate (1989) gives a detailed account of the recent developments in three-dimensional, high-order panel method applications.

### 4.2.1 Computational procedure

We consider the free-surface flow in a bounded region. The computational domain is composed of a free-surface and four vertical faces. As the deep water waves are of main interest, the computational domain might be freed of the bottom face provided that the height of the domain is deep enough to take account of any wave effect.

With prescribed wave elevation and the velocity potential on the free-surface, the boundary-value problem is not mathematically complete. Boundary conditions are to be invoked on the vertical sides of the domain. For all the numerical applications presented in this work, we will assume that the solution is spatially periodic (more on this in Section 4.3.1).

The next step is to evaluate the source and dipole integrals over the approximate representation of the surface. Each face of the domain is discretized into small elements. A geometric description of each element on the surface is needed for the evaluation of these integrals. Of many possible ways, we choose to approximate the surface by plane (flat) quadrilateral panels over each of which the singularity strength is assumed to be constant (Hess and Smith, 1962).

The boundary conditions are invoked at the collocation points which are the center points for each discretized element. On the top surface of the domain, free-surface movement is followed in time at these collocation points whose horizontal coordinates are fixed on the $z = 0$ plane (see Figure 4-1). The wave elevations that correspond to these discrete points describe the free-surface shape. We approximate the free-surface by plane, but tilted quadrilaterals. A plane quadrilateral is fully defined by its corner coordinates in space which we do not know a priori. The corner heights of each element can be determined through a bicubic interpolation of the wave elevations prescribed at the collocation points. Bicubic interpolation requires the gradients of the surface wave height, $\partial \zeta(x, y)/\partial x$, $\partial \zeta(x, y)/\partial y$ and the cross-derivative $\partial^2 \zeta(x, y)/\partial x \partial y$.
at each grid point. Then an interpolation function can be found such that the wave heights at the collocation points are exactly reproduced, and the wave height and its gradients change in a continuous manner from one grid point to the other. The numerical implementation in Press et al. (1986) is adopted for the above procedure. The gradients and the cross-derivatives of the surface elevation are found from a spline cubic function obtained by fitting it through a set of wave heights along a line in $x$ and $y$ directions. A slight difficulty arises in the computation of the corner heights that are located along the edges of the domain. An extrapolation to the edges seems to be a plausible approach. However, the use of spatial periodicity allows us to do better than that. Exploiting this feature, we can add more elements outside the physical domain (see Figure 4-1). The surface wave heights at the collocation points on these elements will be the same as those in the inner part of the domain on the opposite side. Repeating the same procedure outlined above, the corner heights along the edges can be computed more accurately. Having found the corner coordinates for each panel, these four points are used in forming the quadrilateral panel. In doing this, we will follow the approach used by Hess and Smith (1962) as summarized in appendix D.
Once we have the panel geometry completely defined, we can then evaluate the integrals. The analytic integration of singularities is preferred over numerical integration because of its simplicity in implementation for constant-strength singularity distributions. In our algorithm, we will follow the closed form expressions (recurrence relations) of Newman (1986) presented in a suitable format to program.

With evaluation of these integrals (influence coefficients), the discretized boundary integral equation can be re-arranged in a way to form a matrix equation. The solution to this set of equations yields the unknowns (the velocity potential or its normal gradient) on the boundaries of the domain. To update the potential and the free-surface shape on the surface, we need to integrate the free-surface boundary conditions, which are nonlinear, time-dependent partial differential equations which contain the gradients of the potential and the wave elevation. The algorithm that we implemented can be summarized as follows:

1. Prescribe the initial free-surface shape and the velocity potential at each collocation point at \( t = t_0 \).

2. Approximate the surface of the boundary by plane quadrilateral panels over which the singularity strength is assumed to be constant.

3. Invoke the boundary conditions at the centroid of each panel. An arrangement of the discretized integral equation leads to a matrix equation whose solution provides the normal gradient of the velocity potential on the surface.

4. Compute the horizontal gradients of the wave elevation and the velocity potential. Integrate the time-dependent free-surface boundary conditions to determine the wave height and the velocity potential at the collocation points at time \( t = t_0 + \Delta t \).

5. Repeat 2–4 until the desired time level is reached.
4.3 General Considerations

4.3.1 Boundary conditions

A boundary-value problem is correctly set for a field equation along with the appropriate boundary conditions on all physical boundaries. Free-surface boundary conditions are used in updating the free-surface shape and the velocity potential on the surface. In a well-posed problem, boundary conditions on the vertical sides of the computational domain are to be specified as well. In what follows, we will review the boundary conditions used in the time-marching procedure.

Free-surface boundary conditions

The motion of the free-surface waves is described by time-dependent, nonlinear, kinematic and dynamic boundary conditions. In time-domain simulations of nonlinear waves, fully nonlinear boundary conditions are to be used without any linearizing approximations. Free-surface boundary conditions have both Eulerian and Lagrangian descriptions. As the Lagrangian boundary conditions follow fluid particles on the free-surface, they are capable of representing multi-valued free-surface flows such as overturning waves. In our present work, an Eulerian formulation is used and several complexities of the Lagrangian approach are avoided. For example, in a Lagrangian description the collocation points on the surface tend to concentrate in certain regions of the surface during the time-stepping procedure. Therefore, a re-gridding scheme is necessary to re-distribute the Lagrangian points over the surface (Dommermuth and Yue, 1987c).

The kinematic boundary condition is that a fluid particle that is on the free-surface remains on the free-surface:

\[
\frac{\partial \zeta}{\partial t} + \nabla \phi \cdot \nabla \zeta - \frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = \zeta,
\]

(4.6)

where \( \nabla = (\partial/\partial x, \partial/\partial y) \) denotes the horizontal gradient. Bernoulli’s equation gives
the dynamic boundary condition:

\[
\frac{\partial \phi}{\partial t} + \frac{1}{2} \nabla \phi \cdot \nabla \phi + g \zeta = 0 \quad \text{on } z = \zeta.
\]  \hspace{1cm} (4.7)

In the Eulerian formulation, the velocity potential and the free-surface elevation are followed at selected collocation points. We will consider a point (not a fluid particle) on the free-surface which undergoes only vertical displacements without changing its horizontal coordinates (Faltinsen, 1978 and Sen et al., 1989). The rate of change in the velocity potential at this point can then be given by:

\[
d\phi = \frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial \zeta} d\zeta
\]  \hspace{1cm} (4.8)

where \(d\zeta = (\partial \zeta / \partial t) dt\). Substitution of \(d\zeta\), (4.6) and (4.7) into (4.8) gives the Eulerian form of the dynamic boundary condition:

\[
\frac{d\phi}{dt} = -g \zeta - \frac{1}{2} (\nabla \phi \cdot \nabla \phi) - \frac{\partial \phi}{\partial z} (\nabla \phi \cdot \nabla \zeta).
\]  \hspace{1cm} (4.9)

If we define \(\phi_*\) as the velocity potential on the free-surface:

\[
\phi_*(x, t) = \phi(x, \zeta(x, t), t),
\]  \hspace{1cm} (4.10)

we can write the kinematic and dynamic boundary conditions on the free-surface in terms of the surface potential. The vertical velocity on the surface is:

\[
w = \left. \frac{\partial \phi}{\partial z} \right|_{z = \zeta}.
\]  \hspace{1cm} (4.11)

Then by the chain rule, we have:

\[
\frac{\partial \phi_*}{\partial t} = \left. \frac{\partial \phi}{\partial t} \right|_{z = \zeta} + w \left. \frac{\partial \zeta}{\partial t} \right|_{z = \zeta},
\]

\[
\nabla \phi_* = \left. \nabla \phi \right|_{z = \zeta} + w \nabla \zeta.
\]  \hspace{1cm} (4.12)
Substitution of equation (4.12) into (4.6) and (4.9) yields the following equations:

$$\frac{\partial \zeta}{\partial t} = w(1 + \tilde{\nabla} \zeta \cdot \tilde{\nabla} \zeta) - \tilde{\nabla} \phi_e \cdot \tilde{\nabla} \zeta,$$ \hspace{1cm} (4.13)

$$\frac{\partial \phi_e}{\partial t} = -g \zeta - \frac{1}{2} \tilde{\nabla} \phi_e \cdot \tilde{\nabla} \phi_e + \frac{1}{2} \omega^2 (1 + \tilde{\nabla} \zeta \cdot \tilde{\nabla} \zeta)$$ \hspace{1cm} (4.14)

which are evaluated on $z = \zeta$. These equations, called the Zakharov equations, were first derived by Zakharov (1968) and used as the free-surface boundary conditions in our time-stepping scheme.

**End boundary conditions**

End boundary conditions pose a longstanding problem in applications of the BIEM to nonlinear water waves due to the lack of an appropriate open boundary or radiation condition. The open boundary can be located far away from the region of interest. This allows computations before the reflected waves from the far boundary contaminate the domain. If only the transient behavior is of interest, this approach avoids the complexities and problems of a radiation condition for nonlinear waves. This technique has been used by Isaacson (1982).

The application of an absorbing boundary condition to prevent the waves from reflecting back is used by Baker et al. (1981), Israeli and Orszag (1981), Larsen and Dancy (1983) and Chan (1975). This needs a fairly large damping zone and the solution is not completely free of reflection.

There are several other techniques that range from easy to difficult to implement. Romate (1989) gives an overall review on these methods. In our present work, we employ a technique that is specially suitable for wave-wave interactions. We assume that the solution is periodic in space. Then the boundary conditions on the vertical sides of the domain are periodic. On the free-surface, the velocity potential is known at any time, but not the normal gradient of the potential. On the vertical side of the domain, neither the potential nor its normal gradient is known. However, spatial periodicity condition assumes that both of these quantities are equal on the opposite
sides of the domain (with a sign change in the normal gradient) and thus reduces the number of unknowns in the integral equation so that the system of equations is not under-determined. This technique is easy to implement and has been used by several others including Longuet-Higgins and Cokelet (1976), Vinje and Brevig (1980, 1981) and Baker et al. (1982) in two-dimensional problems. The main disadvantage is that the dimensions of the domain must be an integer multiple of the wavelengths involved.

4.3.2 Spatial discretization

The discretized boundary integral equation is applied at selected points in space, and the free-surface boundary conditions are updated at these selected locations. The size of the domain is determined by the wavelengths involved due to the assumption of spatial periodicity. The free-surface is then discretized into plane quadrilateral elements. The free-surface elevation (which changes only vertically) is followed at the center point of these elements. There is not a rigorous rule governing the minimum grid spacing. Dommermuth and Yue (1988) report that the relative error in the discretized version of the field equation is less than 0.5% with 40 segments per wavelength when checked against known solutions. In all of the applications presented here, total number of elements per wavelength varied from 16 to 64 with uniform spacing. The effect of the spatial discretization in horizontal directions on the surface is investigated with the time-domain simulations of Stokes waves. The numerical applications are examined by changing the local grid spacing to confirm the convergence of the results.

The spacing along the $x$ and $y$ directions on the vertical sides of the domain are determined by the spacing used on the free-surface along the same directions. The depth of the domain is selected such that there is no significant wave motion beneath this depth. For linear deep water gravity waves, the amplitude of the velocity potential at different depths is given as percentage of the potential amplitude at the equilibrium level ($z = 0$) in Table 4.1. As can be seen from this table, the velocity potential reduces to less than 5% of its value on the quiescent free-surface at a depth
<table>
<thead>
<tr>
<th>Depth</th>
<th>Decay</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2 ( \lambda )</td>
<td>28 %</td>
</tr>
<tr>
<td>0.3 ( \lambda )</td>
<td>15 %</td>
</tr>
<tr>
<td>0.4 ( \lambda )</td>
<td>8 %</td>
</tr>
<tr>
<td>0.5 ( \lambda )</td>
<td>4 %</td>
</tr>
<tr>
<td>0.6 ( \lambda )</td>
<td>2 %</td>
</tr>
</tbody>
</table>

Table 4.1: Exponential decay in the velocity potential with depth

of half a wavelength from the equilibrium level. We test a uni-directional Stokes wave with a steepness of 0.1 and use 25 uniformly-spaced elements in each horizontal direction. This requires 625 elements on the free-surface of the domain. We have uniform spacing along the vertical direction in which the local grid size is selected to be 1/40th of the wavelength. Using different number of panels along the \( z \) direction, we compute the normal velocities on the free-surface and compare these with the exact, theoretical values. The RMS error on the free-surface is evaluated by:

\[
\epsilon_{RMS} = \frac{1}{M} \left( \sum_{n=1}^{M} \left[ \phi_{n}\text{num} - (\phi_{n})_{exact} \right]^2 \right)^{1/2}
\] (4.15)

where \( M \) is the total number of nodes on the free-surface, and this is plotted in Figure 4-2. The numbers in this figure indicate the total execution times on a Cray-II for the number of panels used along the depth. Given the spatial discretization on the free-surface, the RMS error drops to a nearly constant value at a water depth of 40% of the wavelength. In all of our applications, however, the domain depth is selected to be 50% of the longest wavelength involved in the computation. Having determined the depth of the computational domain, what kind of spacing should be used? A well-known problem in numerical applications of the boundary element method, though not studied in detail, is the rather poor accuracy along the edges of the boundaries where sharp discontinuities exist. The spacing along the vertical direction turns out to be the key factor for the accuracy in the corner elements. This is discussed in more details in Section 4.4.2.
Figure 4-2: RMS error for the normal velocity on the free-surface as a function of domain depth

4.3.3 Temporal discretization and time integrators

Having determined a suitable time step size, the integration of the free-surface boundary conditions is done numerically using a higher order explicit (e.g., Runge-Kutta) or a multi-step (e.g., Adams-Bashford-Moulton) method. The fully nonlinear free-surface boundary conditions (Zakharov's evolution equations) are integrated in time to update the wave elevation and the velocity potential at the collocation points. These equations can be represented by the following general form:

\[
\frac{dy}{dt} = f(x, y, z, t)
\]  \hspace{1cm} (4.16)

where \(f(x, y, z, t)\) denotes the right hand side. A variety of schemes are available for integrating the evolution equations. The first- and second-order explicit schemes are reported to show rather poor convergence characteristics (Sen et al., 1989). The most popular time integrators seem to be the fourth order multi-step Adams-Bashforth-
Moulton (ABM4) predictor-corrector and the fourth order Runge-Kutta (RK4) techniques (Gear, 1971). RK4 requires four operations to move one time step forward which means that we have to solve the integral equation four times. ABM4 is a prediction-correction scheme which requires only two evaluations of \( f(x, y, z, t) \) per time step, thus is twice as fast compared to RK4. For an analysis on the linear stability of both methods, see Dommermuth (1988). Unlike ABM4, RK4 is slightly dissipative which is a welcome feature in suppressing the potential instability on the free-surface which we will address later. Nevertheless, ABM4 is preferred due to computationally expensive nature of the problem. Since the ABM4 needs information (right-hand sides in the boundary conditions) from the previous three time steps, RK4 is used to time-march the evolution equations for the first three time steps from the initial conditions.

The step size that will be used in the above integrators should be chosen on the basis of accuracy and stability. As in the selection of the spatial discretization, temporal discretization is application-dependent as well. One practical consideration is the application of Courant condition based on wave celerity. This criterion requires that no fluid particle move more than the local grid spacing per time step. This could be formulated as:

\[
\frac{c \Delta t}{\Delta l} < 1
\]  

(4.17)

where \( c \) is the wave celerity, \( \Delta t \) is the time step, and \( \Delta l \) is the local grid spacing. Grilli and Svendsen (1990) report that the optimum relationship between spatial and temporal discretizations suggests a Courant number within the range of 0.3–0.5. In our numerical applications, this number varied from 0.2 to 0.8.

In Dommermuth and Yue (1988), the linear von Neumann stability analysis indicates that RK4 is conditionally stable when

\[
\Delta t^2 \leq \frac{8 \Delta l}{\pi g}.
\]  

(4.18)

This condition, in addition to the linear Courant condition, is also satisfied in the
time-stepping procedure.

4.3.4 Checks of accuracy

The global accuracy of the numerical scheme is checked by computing the total energy and the horizontal momenta. These quantities are typically maintained to within 1% for most cases. The relative change in these quantities from their initial values is plotted for each simulation to assess the global accuracy of that particular run.

The total energy per unit area is given by:

\[ E = E_k + E_p = \frac{1}{2} \int_{-h}^{h} \rho (\nabla \phi)^2 \, dz + \frac{1}{2} \rho g \zeta^2 \]  \hspace{1cm} (4.19)

where \( h \) is the water depth. The kinetic energy can also be written in the following form (Kinsman 1964):

\[ E_k = \frac{1}{2} \rho s \frac{\partial \zeta}{\partial t}. \] \hspace{1cm} (4.20)

Now we can write the total energy as:

\[ E = E_k + E_p = \frac{1}{2} \rho \int \int_s (\phi_s \frac{\partial \zeta}{\partial t} + g \zeta^2) \, dx \, dy \] \hspace{1cm} (4.21)

where \( S \) denotes the free-surface. Horizontal momenta are given by:

\[ H_x = \int \int_s \zeta \frac{\partial \phi}{\partial x} \, dx \, dy, \]

\[ H_y = \int \int_s \zeta \frac{\partial \phi}{\partial y} \, dx \, dy. \] \hspace{1cm} (4.22)

These are calculated at each time step and checked against the initial values. In case these quantities deviate from their initial values by an unacceptable amount, a smaller time step usually improves the global accuracy.
4.4 Specific considerations

4.4.1 Spatial derivatives on the surface

As can be seen from the evolution equations (4.13) and (4.14), time integration of the free-surface boundary conditions requires the evaluation of the right-hand sides which contain the spatial derivatives of the wave elevation and the velocity potential. The horizontal gradients of these quantities can be evaluated by ordinary finite difference schemes. We will, however, approximate $\phi$ and $\zeta$ as a function of the horizontal coordinates by fitting a cubic spline through the collocation points. The spatial derivatives can then be computed at selected points. In the formation of the spline, the curvature at the very ends could be extrapolated from the first two interior points. However, we exploit the spatial periodicity in the solution and add one more collocation point next to the end elements for a more accurate representation in the domain of interest (see Figure 4.1). The vertical gradient of the velocity potential is also involved in the evolution equations. The solution to the boundary integral equation yields the normal gradient of the potential on the surface which can be expressed in the following form:

$$\frac{\partial \phi}{\partial n} = \nabla \phi \cdot n$$  \hspace{1cm} (4.23)

where $n$ denotes the outward unit normal vector on the free-surface which is given by:

$$n = \frac{-\zeta_x i - \zeta_y j + k}{(1 + \zeta_x^2 + \zeta_y^2)^{1/2}}.$$  \hspace{1cm} (4.24)

The subscript denotes the partial derivative with respect to that variable. Given that the horizontal gradients of the potential are computed, the vertical gradient of the potential can be found by using equation (4.23). This completes the discussion for computation of the gradients of $\phi$ and $\zeta$ at the collocation points.
4.4.2 Influence and treatment of the edge elements

In numerical applications of the boundary integral equation, a problem arises for special geometries. This happens when the normal direction on the surface does not have a unique value as is the case for the corners. The discontinuity in the normal direction and the normal gradient of the potential impairs the accuracy along the edge panels located around the corners. This behavior of the corner elements are reported by several researchers. The accuracy in the solution to the normal gradient of the potential is poor along the edge elements. On the other hand, reasonably good accuracy (less than 0.5% relative error) is achieved for the velocity potential on the edge elements.

The importance of the accuracy along the edge panels on the free–surface lies in the fact that these errors will penetrate into the domain via the time–stepping process because of the hyperbolic nature of the free–surface boundary conditions. Romate (1989) used a strip of higher order singularities and panels along the edges of the domain to diminish the local error at these locations as these determine the overall accuracy of the time–domain simulation.

In search of a technique to diminish the local errors at the edge elements, we will re–analyze the sample problem we considered in Section 4.3.2. We learned that a domain depth of half the wavelength is sufficient for deep water waves. Now we solve the boundary integral equation for the same problem using only 20 uniformly–spaced elements along the depth. Then we compute the relative error in the normal gradient of the potential on the free–surface when checked against the theoretical values. Figure 4-3 shows the relative error on the free–surface. As can be seen from this figure, the relative error along the edges can be as high as 4.5% which is not acceptable. Increasing the number of elements along the depth improves the accuracy along the edges. However, for problems that require bigger domain sizes and more elements on the free–surface, this approach becomes impractical, and inhibitive in some cases due to increasing computational expense. We will implement a different technique without increasing the computation time. We mentioned that the accuracy
Figure 4-3: The relative error in the normal velocity on the free-surface for uniform spacing along the depth

in the solution to the velocity potential whenever it is to be computed, unlike its normal gradient, is within reasonable limits. As we assume spatial periodicity, both the potential and its normal gradient are computed on the vertical sides of the domain. One technique we will devise is the following: Since the velocity potential is known on the vertical sides of the domain, we can make use of the collocation points that are close to the free-surface to compute the vertical gradient of the potential on the surface along the edge. In Figure 4-4, using the potential computed at points \( V_1, V_2, V_3, \) etc., we find a spline fit for these points and compute the derivative of the potential along the depth. An extrapolation of these derivatives up to the free-surface gives the vertical gradient of the velocity potential on the free-surface at point \( V_0. \)
Having computed the vertical gradient of the velocity potential for all the inner points $S_2, S_3,$ etc. on the free-surface (see Section 4.4.1), we can now compute the vertical gradient of the potential at point $S_1,$ which is where the error is maximum, by a cubic interpolation using points $V_0, S_2, S_3,$ and so forth.

![Diagram](image)

Figure 4-4: Edge correction technique by an extrapolation of the potential gradient to the free-surface

A second technique we implemented does not use the potential on the vertical sides, but uses all the collocation points except the ones at the edges to form a bicubic spline for the normal velocity on the free-surface. As is seen in Figure 4-5, we extend the free-surface by making use of the periodicity assumption. Then we evaluate the normal velocity at points $S_1$ and $S_N$ by a cubic interpolation using the points $S_2$ through $S_{N-1}$.

Both of these edge correction techniques performed reasonably well in reducing the local error at the edges and provided an acceptable overall accuracy to maintain the time-marching procedure without a break-down. The application of the second technique to the sample problem above reduces the maximum error at the edges to
Figure 4-5: Edge correction technique by a cubic spline fit over discrete points on the free-surface

about 1.3% in Figure 4-6. Though this is a significant improvement, we will work on refining the accuracy even further without increasing the computational expense.

Considering the exponential decay in the velocity potential away from the free-surface, it is anticipated that a panel distribution scheme that allows more elements close to the surface might further improve the accuracy on the edge elements. We tried a cosine-spacing distribution along the depth. As can be seen in Figure 4-7, a quarter-cycle cosine-spacing is used for the elements on the sides of the domain. For the same computational effort, i.e., for 20 elements along the vertical direction, Figure 4-8 shows the relative error in the normal velocity on the free-surface. The maximum error, being about 1.1%, is already reduced by a significant amount compared to the results of the uniform-spacing. Now applying the edge-correction technique to this final run, the relative error in normal velocity on the free-surface is plotted in Figure 4-9. The maximum error at the edges is less than 0.5%. We consider this acceptable for our applications. The time-domain simulations of Stokes waves confirm this assertion.

Both edge correction techniques have been used in our applications. Displaying similar characteristics, one is not really superior to the other. The selection of the method depends more on the specific application. For example, the second technique is difficult to implement when only a few elements are used in one direction. This applies to the time-domain simulations of uni-directional Stokes waves. In this case, which is basically a two-dimensional problem, only a few elements are needed in the
Figure 4-6: The relative error in the normal velocity on the free-surface for uniform spacing along the depth (after the edge correction)

Figure 4-7: Quarter-cycle cosine-spacing along the depth of the domain
Figure 4-8: The relative error in the normal velocity on the free-surface for cosine-spacing along the depth

Figure 4-9: The relative error in the normal velocity on the free-surface for cosine-spacing along the depth (after the edge correction)
traverse direction which is not suitable for a cubic interpolation on the surface due to sparsity of the elements.

Before we move on to the next topic, let us see how the overall accuracy of the solution changes with respect to the number of panels used along the depth which is selected to be half the wavelength. Figure 4-10 shows the RMS error in the normal velocity on the free-surface when different numbers of panels are distributed in a cosine-spacing manner along the vertical direction. Although the global accuracy keeps improving with increasing number of panels, we find that using 20 elements along the depth is the optimum trade-off between the accuracy and the computational expense. In all of our applications, 20 elements are used along the depth with a quarter-cycle cosine-spacing distribution.

One last point we would like to bring up is the additional bonus of cosine-spacing distribution. Considering the exponential decay with depth for the deep water waves, cosine-spacing provides a better representation in regions of rapid variations in the
boundary conditions. In applications which involve several wavelengths, the depth of
the domain is determined by the maximum wavelength in the field. In the presence of
disparate length scales, a cosine-spacing type distribution increases the density of the
collocation points near the surface, and thus provides more elements for the smallest
wavelength involved.

4.4.3 Instability on the surface and smoothing

During the time-marching process, an instability on the free surface develops which,
if unsuppressed, eventually causes the numerical scheme to break down. A saw-
tooth type instability has been reported by several investigators including Longuet-
Higgins and Cpokelet (1976), Baker et al. (1982), Lin et al. (1984), Dommermuth and
Yue (1987c, 1988), Sen et al. (1989). This type of instability was not reported however

One case in which this type of instability appeared was one of the time evolution
runs (see the next chapter) for the triad without the long wave. We have used 32
elements in each direction along the surface. Along the depth 20 elements were
used with a cosine-spacing distribution. The time step was chosen to be 1/50th of
the shorter primary wave period. After 71 time steps, a saw-tooth type instability
appeared on the surface as can be seen in Figure 4-11. Unless suppressed, this high
wavenumber instability causes a numerical break-down as the solution progresses.

The work of Longuet-Higgins and Cpokelet (1976) was the first to discover this
type of instability. They used a 5-point smoothing formula to remove the saw-tooth
appearance from the free-surface profile. We adopted a fast Fourier transform (FFT)
technique by which all the high wavenumber instabilities are filtered out. The free-
surface wave height and the velocity potential are transformed into the Fourier space
by a two-dimensional FFT, and then all the higher order harmonics that are above the
wavenumber at which the instability is detected are filtered out. Then transforming
back into the physical space by an inverse FFT, the computations can be carried out
for the next time steps. The effect of the filtering process on the results is investigated
Figure 4-11: The saw-tooth instability on the free-surface after 71 time steps in the next chapter.

The exact cause of this instability is unknown. Although Longuet-Higgins and Cokelet (1976)’s contention was that this growth might be partly physical, it is nowadays believed that these are numerical (Dommermuth and Yue, 1987c).

4.5 Computer implementation

4.5.1 Vectorization of the influence coefficient computations

In the solution of the boundary integral equation, the most expensive parts are the evaluation of the source and dipole integrals (the matrix influence coefficients) and the solution to the matrix equation. The computer code was originally developed on a microcomputer, and the architecture of the program was not particularly suitable for vectorization. The computation of the influence coefficients was done in different modules. Although modularization of a computer code is a good programming practice in general, it does not lend itself to vectorization. The program was re-written in a way not to violate the basic rules of vectorization. The key point in formation of the matrix elements is to compute the induced velocity potential on all of the field points due to a given source. This part of the calculation is done within the inner most loop
of the computer code. The information needed for the field points is only the location of the centroid for each panel which is computed beforehand. For each source point, the geometry is defined for the execution of the integrals, and then all the field points on which the potential is computed are swept within the inner most loop. There should be no calls to any module within the loop to be vectorized. Considering the large number of calls to these modules in a highly modular, non-vectorized code, the vectorized version speeds up the execution even on a scalar computer due to reduced overhead of the calls to these modules.

4.5.2 Solution of the matrix equation

The discretized boundary integral equation yields a linear system of algebraic equations to be solved. Any method involving direct matrix inversion would require $O(N^3)$ operations ($N$ is the number of unknowns). An iterative technique requires $O(N^2)$ operations and is particularly important in three–dimensional applications because of the large number of unknowns to be solved. Unlike the finite element method, the boundary integral equation usually gives rise to a full matrix and makes it impossible to exploit some of the useful features applicable to sparse matrices.

An iterative solution technique is adopted in the present work. This method makes use of the classical Gauss-Seidel algorithm, but uses a new convergence acceleration technique (Clark, 1985). The computer implementation of the technique outlined in the above reference was done by Dr. Greeley of Atlantic Applied Research Corporation and made available to us. This code was originally developed for workstations with limited memory. We modified the code in a way to eliminate the input/output (I/O) processes utilizing the available memory on supercomputers. For larger numbers of panels, this made a significant improvement on the CPU performance. An initial guess for the solution vector is required to start the iteration process. Presently, a zero solution vector is used as the initial guess at each time step. For the type of problems in which one is tracing the evolution by a time-marching procedure, the solution from the previous time step might be a good initial guess. This, however, is
not done in the present work, and the zero solution vector is taken to be the initial guess. The iterative process is terminated based upon a specified tolerance. When the residual vector is smaller than this specified value, iteration is stopped. For the applications given in the present work, a converged solution required typically 15 to 20 iterations for a tolerance of $10^{-5}$. The CPU performance of the technique is given in the next section with a comparison to the classical Gaussian elimination.

### 4.5.3 Efficiency of the method

In this section, the efficiency of the panel code is considered. A Cray-II with four processors is used for the sample runs below. A fairly large number of panels, typically $O(1000)$ with $O(10^8)$ matrix elements to be evaluated, is required for most of the three-dimensional applications. Considering that Laplace’s equation is to be solved repeatedly in following the evolution of water waves, the need for an efficient computer code will become clear. The calculation of the influence coefficient matrix and the solution to this matrix equation are the most expensive parts of the procedure. A significant performance gain on a vector processor is possible if the application has a high vectorization ratio. This is defined as the ratio of the execution time of vectorizable instructions to that of scalar instructions. This measures how suitable the code is to vector processing. Our algorithm produces a vectorization ratio of 99% on a Cray-II processor.

Code optimization on vector–hardware supercomputers can improve the execution speed by a factor of 20. The computer code currently accommodates all of the operations in the memory of the system without resorting to any I/O processes. However, one may have to make a compromise among execution speed, memory requirements and the time spent in I/O for larger jobs that require excessive use of memory. Out–of–core applications (more use of I/O) might be the only choice for such large runs at the expense of execution speed. We will compare the CPU performance of an out–of–core application with reference to the total memory usage.

In Table 4.2, the performance of the scalar and vectorized versions of the programs
is given for 1325 panels. COFMAT is the subroutine that carries out the computation

<table>
<thead>
<tr>
<th>Case</th>
<th>COFMAT</th>
<th>SOLVER</th>
<th>Total CPU (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian Elimination No Multipole Expansion</td>
<td>50.0</td>
<td>33.0</td>
<td>83.0</td>
</tr>
<tr>
<td>Iterative Solution No Multipole Expansion</td>
<td>50.0</td>
<td>2.5</td>
<td>52.5</td>
</tr>
<tr>
<td>Iterative Solution With Multipole Expansion</td>
<td>42.0</td>
<td>2.5</td>
<td>44.5</td>
</tr>
<tr>
<td>Vectorized COFMAT Iterative Solution No multipole expansion</td>
<td>3.5</td>
<td>2.5</td>
<td>6.0</td>
</tr>
</tbody>
</table>

Table 4.2: CPU times for a typical three-dimensional free-surface problem

of the influence coefficients. The matrix equation is solved in SOLVER. The rest of the program deals with the computation of the panel geometry, updating of the boundary conditions, smoothing, etc. which are fully vectorized. The CPU time for this part never exceeded one second even in the largest runs. The use of an iterative matrix solution method has shifted most of the computational effort to the evaluation of the matrix elements which is not vectorized in the scalar code. This table also emphasizes the superiority of an iterative technique in CPU performance when compared to Gaussian elimination.

In evaluation of the source and dipole integrals, a multipole expansion technique can be employed for distant field points where the exact integration is not needed (Hess and Smith, 1964). As can be seen from Table 4.2, this could be quite beneficial in a scalar code. In this sample run, the multipole expansion was used when the distance between the field and the source point was more than 20 times the typical panel dimension (diagonal length) with no loss in accuracy. However, the use of the multipole expansions requires additional conditional statements which degrades the performance of the vectorization. Our benchmarks have shown that the multipole expansions did not improve the performance of the vectorized code. This was tested for panel numbers up to 4000. As pointed out by Romate (1989), a clustering tech-
nique can be used, and the field points can be grouped according to the distance to the source points before carrying out the integrations. This, though it might be rewarding, is not done in the present work.

Table 4.3 shows the CPU performance of the code for a sample run taken from the next chapter. On the free–surface, 32 elements are used in each direction. The depth is discretized into 20 elements. Total number of panels used on the boundaries of the domain is 3584. Table 4.3 is the performance of the vectorized code with and without

<table>
<thead>
<tr>
<th>Case</th>
<th>COFMAT (sec)</th>
<th>SOLVER (sec)</th>
<th>User CPU (sec)</th>
<th>System CPU (sec)</th>
<th>Memory usage (Mwords)</th>
</tr>
</thead>
<tbody>
<tr>
<td>In core (no I/O)</td>
<td>31.4+0.6</td>
<td>7.4+0.6</td>
<td>40.0</td>
<td>0.6</td>
<td>14.9</td>
</tr>
<tr>
<td>Out–of–core (with I/O)</td>
<td>31.2+2.0</td>
<td>6.2+17.6</td>
<td>57.0</td>
<td>4.2</td>
<td>0.47</td>
</tr>
</tbody>
</table>

Table 4.3: CPU performance of in–core and out–of–core applications

I/O in the iterative solution of the matrix equation. The first column is the user CPU time spent in COFMAT. The second column gives the CPU time for SOLVER. Two numbers are given in each of these two columns. The first one shows the CPU time for the floating point operations while the second one gives the total time spent in the I/O process. The last column shows the CPU time memory integral divided by the total CPU time which could be interpreted as the average memory usage for that run. We can see that the CPU performance can be improved by reducing the need to do I/O using the memory of the computer to store information if the space is available. For some applications that require fairly large number of panels, a trade–off between the execution speed and the memory requirements might be of prime importance due to unavailability of the memory space or cost of use of this extra memory. For pointing out the usefulness of vector operations, the vectorization in COFMAT is turned off. The CPU time in COFMAT rose to 330 seconds when compiled in the scalar mode. This indicates that, in addition to already available fast clock cycle, one should take
full advantage of the vector–hardware for better CPU performance by some tailoring of software. The effort is worth it.

Further improvements in performance can be achieved by making use of the multi–processors. This, however, requires possibly fundamental changes in the structure of the program and affects program portability. Because of the fact that this would improve merely the total wallclock time rather than the CPU, no effort is made in that direction.

4.6 Applications

To test the accuracy of the numerical implementation, we will conduct a series of runs for time–domain simulations of steadily progressing, nonlinear waves of permanent form. Numerical results are checked against the fifth–order theory for a Stokes wave given by Fenton (1985). Initial values of the surface wave elevation and the velocity potential on the surface are also calculated by the fifth–order theory. A Stokes wave with a wavelength of $2\pi$ is chosen for the numerical simulations. The steepness used varied from 0.1 to 0.2. The spatial resolution in the wave propagation direction is determined by 16 and 32 panels. Only 5 panels are used in the traverse direction as this is a uni–directional Stokes wave. The depth, which is half a wavelength, is discretized into 20 elements with cosine–spacing distribution for all runs. Table 4.4 below is a list of the runs we conducted. In this table, $N_i$ denotes the number of panels in $i$–direction, and $T$ refers to the nonlinear wave period.

In Figure 4-12, the numerical simulation of a Stokes wave with 16 panels per wavelength is carried out up to two wave periods. Time step size is selected to be $T/20$ with a Courant number of 0.8. Exact and computed wave profiles are plotted at quarters of the wave period. The solid lines are the numerical predictions, and the dashed lines are the fifth–order Stokes profiles. For the given temporal and spatial resolution, the numerical prediction is good. There is a slight phase shift, but the wave amplitude is in agreement with the theory.
<table>
<thead>
<tr>
<th>Run</th>
<th>Steepness</th>
<th>$N_x$</th>
<th>$N_y$</th>
<th>$N_z$</th>
<th>Time step</th>
<th>Courant number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.10</td>
<td>16</td>
<td>5</td>
<td>20</td>
<td>T/20</td>
<td>0.8</td>
</tr>
<tr>
<td>2</td>
<td>0.10</td>
<td>16</td>
<td>5</td>
<td>20</td>
<td>T/40</td>
<td>0.4</td>
</tr>
<tr>
<td>3</td>
<td>0.10</td>
<td>32</td>
<td>5</td>
<td>20</td>
<td>T/40</td>
<td>0.8</td>
</tr>
<tr>
<td>4</td>
<td>0.20</td>
<td>32</td>
<td>5</td>
<td>20</td>
<td>T/40</td>
<td>0.8</td>
</tr>
<tr>
<td>5</td>
<td>0.30</td>
<td>64</td>
<td>5</td>
<td>20</td>
<td>T/80</td>
<td>0.8</td>
</tr>
<tr>
<td>6</td>
<td>0.35</td>
<td>64</td>
<td>5</td>
<td>20</td>
<td>T/80</td>
<td>0.8</td>
</tr>
<tr>
<td>7</td>
<td>0.40</td>
<td>64</td>
<td>5</td>
<td>20</td>
<td>T/80</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Table 4.4: List of runs done for the Stokes wave simulations

We decrease the time step size and run the same simulation with the same number of panels per wavelength. A step size of T/40 is chosen for this run and the results are plotted in Figure 4-13. The Courant number for this run was 0.4. A comparison with Figure 4-12 reveals that a reduction in time step did not provide any further improvements.

The next set of runs is conducted with a finer resolution in space. We used 32 panels per wavelength with a time step of T/40. This case yields a Courant number of 0.8. As can be seen from Figure 4-14, the agreement between the theory and the numerical prediction is excellent up to two wave periods.

Figure 4-15 is the time-domain simulation of a Stokes wave of steepness 0.2 with the same spatial and temporal resolution as in the previous case. Although a moderately steep wave height is used for the present run, the numerical prediction is fairly good when checked against the theory.

To give the reader an idea about the limitations of the numerical technique, time-domain simulations are performed with steepnesses 0.3, 0.35 and 0.4. The last two steepnesses correspond to 80% and 90% of the limiting Stokes steepness. All the simulations are carried out for one wave period and shown in Figures 4-16, 4-17 and 4-18, respectively. The solid line is the numerical prediction and the dashed-line is the infinite-order Stokes profile. To maintain the same Courant number, we used 64 panels per wavelength and a time step of T/80. An infinite-order Stokes profile is used for the initial conditions, and the comparisons are made after one nonlinear
wave period. The amplitudes seem to be in agreement despite the existence of slight phase shifts. It would be interesting to see how the results improve with a smaller time step and more control points on the free-surface. However, this has not been done due to time constraints.

This section concludes the implementation of the numerical technique. Progressive, nonlinear waves of permanent form were used as a benchmark in assessing the accuracy of the method. These numerical tests clearly demonstrate that several nonlinear properties (including nonlinear wave-wave interactions) of water waves can be studied using this tool.

4.7 Spectral Method

Dommermuth and Yue (1987a, 1987b) have developed a high-order spectral method for the study of nonlinear gravity waves. Their formulation is based on the Zakharov (Zakharov, 1968) mode-coupling idea with a fixed number of free wave modes whose amplitudes are governed by the fully nonlinear evolution equations that are obtained by directly substituting a series of Fourier modes into the free-surface boundary conditions. In this method, the modes interact and evolve according to the expanded nonlinear evolution equations which are retained to an arbitrary order. See Dommermuth (1988) and Dommermuth and Yue (1987a) for details of the numerical implementation.

West et al. (1987) employed a similar technique to the one implemented by Dommermuth and Yue (1987a). In both numerical approaches though, the vertical component of the velocity potential is expanded in a Taylor series about the reference surface $z = 0$. As in the regular perturbation methods, a potential flaw in use of the spectral technique for problems involving disparate length scales has been noted by Zhang (1987). Brueckner and West (1988), however, discount this major criticism against the use of mode-coupled descriptions of multiple-scale water wave fields by showing that the modal description of water waves on the sea surface is independent.
Figure 4-12: Time-domain simulation of a Stokes wave of steepness 0.1 up to two wave periods with a time step of T/20 and 16 panels per wavelength (solid line: numerical prediction, dashed line: fifth-order theory)
Figure 4-13: Time-domain simulation of a Stokes wave of steepness 0.1 up to two wave periods with a time step of $T/40$ and 16 panels per wavelength (solid line: numerical prediction, dashed line: fifth-order theory)
Figure 4.14: Time-domain simulation of a Stokes wave of steepness 0.1 up to two wave periods with a time step of T/40 and 32 panels per wavelength (solid line: numerical prediction, dashed line: fifth-order theory)
Figure 4-15: Time-domain simulation of a Stokes wave of steepness 0.2 up to two wave periods with a time step of $T/40$ and 32 panels per wavelength (solid line: numerical prediction, dashed line: fifth-order theory)
Figure 4-16: Time-domain simulation of a Stokes wave of steepness 0.3 up to one wave period with a time step of T/80 and 64 panels per wavelength

Figure 4-17: Time-domain simulation of a Stokes wave of steepness 0.35 up to one wave period with a time step of T/80 and 64 panels per wavelength
Figure 4-18: Time-domain simulation of a Stokes wave of steepness 0.4 up to one wave period with a time step of T/80 and 64 panels per wavelength of any reference surface around which expansions of the velocity potential and the surface velocity are done.

In Chapter 1, we pointed out that a regular perturbation expansion around the equilibrium level might lead to divergent results for a wave whose wavelength is shorter than the displacement from the z = 0 plane. It is often argued that the modal expansions are only useful for a narrow-banded spectrum. Brueckner and West (1988) showed that although the expansion of the velocity potential and the normal velocity can be divergent when truncated at a finite order, these series can be reordered such that (Watson and West, 1975 and West et al., 1987) when substituted into the free-surface boundary conditions (Zakharov equations), the evolution equations turn out to be expansions in wave slopes for individual modes. They proved this for the second order and showed that when they substitute the truncated expansions into the Zakharov equations, the would-be divergent terms cancel. As the algebra becomes fairly complicated at the third-order, they made their statement above general for
all orders by relying on the formal proof for the second-order and claimed that the major criticism held against the use of mode-coupled equations does not hold in their formulations (see also Milder, 1990). The direct numerical approach will shed light on this issue as well.
Chapter 5

Numerical results and discussions

Nonlinear resonant interactions in the special triad are numerically studied using the technique implemented in the previous chapter. The growth of the resonant tertiary wave is computed for different steepnesses of the primary waves. First, a small steepness is used for benchmark purposes. Then, moderately steep waves are considered for testing the accuracy of the perturbation analyses. The numerical results are compared with the classical theory and with the Zakharov theory results. Finally, a long wave whose amplitude is of the order of the tertiary wavelength is added to the special triad, and the tertiary wave growth is computed under the influence of the long wave. Yue (1990) and Liu and Yue (1991) provided the numerical results of the spectral method for the test cases we studied. This made possible a direct comparison of the spectral method with the BIEM, which, to the best of our knowledge, is the first computational check on the spectral method for nonlinear wave problems in a comparison against a numerical approach that avoids perturbation expansion completely.

5.1 Numerical computations for the special triad

The special triad we will consider in this section is composed of two primary waves ($k_1$ and $k_2$) and the resonant tertiary wave as shown in Figure 2-1 of Chapter 2. To
determine the relations among the wavenumbers of the triad, we will make use of the resonance conditions (equation 2.3):

\[ 2k_1 = k_2 + k_{ter} \]  \hspace{1cm} (5.1)

\[ 2\omega_1 = \omega_2 + \omega_{ter} \]

where the subscript \( \text{ter} \) refers to the resonant tertiary wave. Using the linear dispersion relation, these two equations can be combined in the following form:

\[ \sqrt{k_{ter}} = 2\sqrt{k_1} - \sqrt{k_2} = [(k_{ter})_x^2 + (k_{ter})_y^2]^{1/4} \]  \hspace{1cm} (5.2)

where the subscripts \( x \) and \( y \) denote the components of the vector wavenumber for the tertiary wave. Assuming that the first primary wave is propagating in the \( x \) direction, we have \( (k_{ter})_x = 2k_1 \) and \( (k_{ter})_y = k_2 \). Introducing the following coefficient (see Chapter 2) for convenience:

\[ r = \frac{\omega_1}{\omega_2}, \]  \hspace{1cm} (5.3)

equation 5.2 gives rise to a nonlinear equation in \( r \):

\[ 3r^3 - 8r^2 + 6r - 2 = 0. \]  \hspace{1cm} (5.4)

Solution to this equation yields \( r = 1.73567 \). From the ratio of the primary wavenumbers, the tertiary wave is found to make an angle with \( k_1 \) given by 9.42 degrees. A prescribed wavenumber for one of the primaries defines the characteristics of the triad. For all of the numerical applications in the present work, \( k_1 \) is chosen to be 3.0 (rad/m). Table 5.1 summarizes the characteristics of each member of this special triad. For the present wave configuration, Longuet–Higgins’ analysis (the classical theory) gives the interaction (coupling) coefficient as:

\[ K = 2.51737687a_1^2a_2 \]  \hspace{1cm} (5.5)
<table>
<thead>
<tr>
<th>Wave</th>
<th>Wavenumber k (τ/m)</th>
<th>Frequency ω (τ/sec)</th>
<th>Wavelength λ (m)</th>
<th>Period T (sec)</th>
<th>Phase speed $C_p$ (m/s)</th>
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<td>6.309</td>
<td>2.011</td>
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</tr>
<tr>
<td>Tertiary</td>
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<td>7.723</td>
<td>1.033</td>
<td>0.814</td>
<td>1.270</td>
</tr>
</tbody>
</table>

Table 5.1: Data for the special triad

where $a_1$ and $a_2$ are the amplitudes of the primary waves. The coupling coefficients computed from equation 2.20 (with $T_{3211}=0.04486962$ from Appendix A) and from equation 3.53 are in agreement. The classical theory gives the growth rate in the tertiary wave amplitude by:

$$a_{ter}(t) = Kt.$$  \hspace{1cm} (5.6)

For most of the runs, the tertiary wave amplitude is plotted in the following normalized form;

$$a_N_{ter} = \frac{a_{ter} k_{ter}}{(a_1 k_1)^2 (a_2 k_2)}$$ \hspace{1cm} (5.7)

as a function of time or $t/T_{ter}$. As we reviewed in Chapter 2, the resonant tertiary growth can also be computed using Zakharov’s theory for surface gravity waves. Unlike the classical theory, this analysis of Zakharov is not limited by a short time period and can predict the long time evolution of water waves with changes in the first–order amplitudes of the wave components involved in the quartet interaction.

Two steepnesses are used for the primary waves of the special triad. The first set of runs is done with a small steepness of 0.1 for both primary waves. In the second set of runs, the steepness is raised to 0.2. The initial tertiary wave amplitude was always taken as zero except in the runs of the spectral method and of the BIEM when the third–order initial conditions are used. Using the Zakharov formulation, the changes in the amplitudes are computed as a function of the tertiary wave period. Figures 5-1 and 5-2 show the long time evolution of the wave field for steepnesses 0.1 and 0.2, respectively. For the smaller steepness, the agreement between the classical theory and the Zakharov theory is good up to nearly $20T_{ter}$. For the larger steepness,
Figure 5-1: Long time evolution of the special triad for steepness of 0.1

Figure 5-2: Long time evolution of the special triad for steepness of 0.2
the discrepancy settles in at an earlier time (about 5\(T_{\text{ter}}\)). The Zakharov theory is capable of predicting the changes in the first-order amplitudes of the primary waves. As can be seen from these figures, the amplitude of the first primary wave—this wave component is counted twice to form the interacting quartet—reduces to a minimum value when the amplitude of the tertiary wave reaches its maximum. This is a result of the energy sharing mechanism. For an ideal system in which there are no viscosity effects, these changes in the amplitudes occur in a periodic manner, and there is no tendency towards an equipartition of the energy. These periodic modulations (Fermi–Pasta–Ulam recurrence phenomenon, Yuen and Ferguson, 1978) are somewhat affected by dissipative mechanisms, wave breaking, etc. in realistic environments. In Chapter 2, we predicted the typical interaction time (time needed for the tertiary wave to be comparable to a primary wave in size) in equation 2.5 as:

\[
t_{\text{int}} = \frac{T_{\text{ter}}}{a_{1}k_{1}a_{2}k_{2}}. \tag{5.8}
\]

For the first run with steepness 0.1, this analysis yields an interaction time of about 100\(T_{\text{ter}}\). The interaction time for the second run with steepness 0.2 yields roughly 25\(T_{\text{ter}}\). Note that these quantities agree favorably well with Figures 5-1 and 5-2 for the appropriate steepnesses.

We now will examine the results of the spectral method for these runs. Yue (1990) considered a triad interaction with \(k_{1}=(3,0)\), \(k_{2}=(0,1)\) and \(k_{\text{ter}}=(6, -1)\) instead of the exact wavenumbers required. This does not satisfy the resonance conditions exactly. The error, however, is negligibly small (Yue, 1990). The initial conditions (doubly-periodic in physical space) are computed from Appendix B and posed for the exact conditions. The spectral method simulations are performed with a time step of \(T_{1}/40\) where \(T_{1}\) is the wave period of the first (shorter) primary wave. A square computational domain with \(L=W=3\lambda_{1}\) is chosen where \(\lambda_{1} = 2\pi/k_{1}\). Extensive convergence tests and the other numerical parameters of this particular implementation are given in Yue (1990).
In Appendix B, we have provided the initial velocity potential and the surface wave elevation up to and including third-order. We will consider both the second-order and the third-order initial conditions for the triad with $a_{ter} = 0$ for the first case, but including the steady-state, third-order bound part of the tertiary wave in the second case.

Figure 5-3 shows the growth rate of the resonant tertiary wave in comparison with the theoretical prediction of Longuet–Higgins for steepness of 0.1. The numerical computations are done with both the second-order and the third-order initial conditions. Note the highly oscillatory behavior of the normalized tertiary wave amplitude with the second-order initial conditions. The oscillations diminished significantly when the third-order initial conditions are used to start the evolution process. These oscillations were also experienced in our numerical computations with the direct method. Although the exact cause of this is not known with certainty, we shall briefly discuss this issue later. As can be seen from Figure 5-3, the tertiary wave has a non-zero amplitude at $t=0$. This is a steady-state (bound) component whose existence is due to cross interactions of the primary waves at the third-order. The unsteady (resonant) part of the tertiary wave has a zero initial amplitude. For the second-order initial conditions, the tertiary wave starts with a zero amplitude. The mean line for the growth rate with the second order initial conditions follows very much the growth rate with the third-order initial conditions. The classical theory compares reasonably well with the spectral method for a short period of time. Then, a discrepancy appears shortly after 5 seconds. This is obviously due to inability of the classical theory to predict the long time evolution of a wave field (Chapter 2) and is completely expected. Figure 5-4 illustrates the growth rate of the tertiary wave for steepness of 0.2. Similarly, the numerical computations are carried out using both the second-order and the third-order initial conditions. We note that the third-order initial conditions display slightly more oscillatory behavior for this higher steepness. The amplitudes of the oscillations in the second-order results are also slightly larger than those of the smaller steepness. The agreement between the spectral method and the classical
Figure 5-3: A comparison of the spectral method with the classical theory for steepness of 0.1

Figure 5-4: A comparison of the spectral method with the classical theory for steepness of 0.2
theory does not extend beyond 2.5 seconds (approximately $3T_{\text{err}}$).

A comparison of the Zakharov theory with the spectral method is given in Figures 5-5 and 5-6. Plotted in Figure 5-5 are the normalized tertiary wave amplitudes from the spectral method and from the Zakharov theory for steepness of 0.1. The predictions from both methods are in good agreement up to $20T_1$. For the higher steepness case in Figure 5-6, some local discrepancies are observed during the course of the simulation. The overall trend, however, is quite similar in both methods with reasonably good agreement for the peak amplitude and the peak location in time.

In Chapter 1, we examined the perturbation expansion of the velocity potential for two interacting waves widely separated in scale. When the ratio of the long wave amplitude to short wavelength increases, the Taylor expansion about the equilibrium level may result in a rather poor convergence when truncated at a finite order. Table 5.2 below gives the values that the expansion parameter takes for each member of the triad. For steepness 0.1, the expansion parameter is quite appropriate for a

<table>
<thead>
<tr>
<th>$a_{\text{max}}k_i$</th>
<th>$a_{\text{max}}k_i$</th>
</tr>
</thead>
<tbody>
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<td>$\epsilon = 0.1$</td>
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<tr>
<td>Tertiary</td>
<td>0.81</td>
</tr>
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</table>

Table 5.2: Expansion parameter for each member of the special triad in the absence of the long wave

perturbation–type analysis. For the tertiary wave component in the higher–steepness case of 0.2, the restriction set on the expansion parameter is clearly, though not severely, violated. This might be the cause of the minor discrepancies between the spectral method and the Zakharov theory. Nevertheless, the results of the two methods have the same global characteristics. Unfortunately, the spectral method runs did not cover a wide enough range in time for a comparison of the recurrence period from both methods. In the absence of a long wave, the perturbation theory performed reasonably well when compared to the spectral method. If we consider the fact that
Figure 5-5: A comparison of the spectral method with the Zakharov theory for steepness of 0.1

Figure 5-6: A comparison of the spectral method with the Zakharov theory for steepness of 0.2
the spectral method utilizes Taylor expansion about the equilibrium level for the normal velocity, one may then argue that the agreement between the two methods (although the order of approximation implied in the spectral method is not identical to the one in the standard perturbation technique because of the treatment of the free-surface) is not surprising after all. This remains to be seen. We shall now analyze these two cases with our direct numerical method and make a comparison to assess the accuracy of the perturbation approximations.

The numerical computations with the spectral method displayed regular oscillations (regular both in amplitude and in frequency) in the amplitude of the tertiary wave with the second-order initial conditions, and also with the third-order initial conditions to a lesser extent. This oscillatory behavior was also experienced in the BIEM. The results with the initial conditions involving only the first-order amplitudes had unacceptably large oscillations. Therefore, we were forced to improve the initial conditions to higher-order of approximations using the standard perturbation theory. As we have seen already, the third-order initial conditions reduced the amplitudes of these oscillations compared to the second-order initial conditions. The exact cause of these oscillations is not known. However, an analysis of the relative phases between the first-order velocity potential and the wave elevation hints at the possibility that these oscillations are due to standing waves. Though the fluid will respond to any set of initial conditions, starting with some initial conditions that are correct up to third-order and then using the fully nonlinear free-surface boundary conditions to update the free-surface quantities may not involve travelling waves only. If the oscillations are due to high-order imperfections in the representation of travelling waves in the initial conditions, caused by the finite order truncation of the velocity potential and the surface wave elevation, it is apparent that higher-order approximations are needed for the initial conditions to represent travelling waves only.

In all of the numerical applications for which we used the BIEM, a cosine-spacing distribution with 20 panels was applied along the depth which extends to half the wavelength of the longest wave component in the wave field. In the absence of the
long wave, the free-surface of the computational domain is approximated by 32 panels in each direction. In Figure 5-7 is the normalized amplitude of the tertiary wave component with the second-order initial conditions. This is equivalent of the run in Figure 5-3 with a steepness of 0.1 for the primary waves. The time domain simulation is plotted for about one wave period of the first primary. The normalized tertiary wave amplitude is plotted for time-step sizes $T_1/40$ and $T_1/50$. The results from the BIEM for two step sizes indicate that $T_1/40$ is quite satisfactory. However, in the expectation of steeper waves, we adopted the smaller time step $T_1/50$ for all the runs we conducted in the present work. As can be seen from this figure, the agreement between the spectral method and the BIEM is remarkably good for this steepness.

![Graph](image)

**Figure 5-7:** A comparison of the spectral method with the BIEM for steepness of 0.1 with the second-order initial conditions

The second set of runs were done with a steepness of 0.2. In Figure 5-8, the normalized tertiary wave amplitude is plotted along with that from the spectral method for about 4 wave periods of the shorter primary. Again, for the whole simulation period, quite good agreement is achieved between the two methods. For this rela-
Figure 5-8: A comparison of the spectral method with the BIEM for steepness of 0.2 with the second-order initial conditions.

Relatively high steepness, quantitative agreement between the two methods indicates that a perturbation scheme can be successful even at moderate steepnesses.

As we mentioned in Chapter 4, a saw-tooth instability appears on the surface as the time-domain simulation is in progress. These high-wavenumber instabilities grow in time and eventually cause a numerical break-down. When we ran the first case with steepness 0.1 without using any smoothing, these wiggles became quite visual on the free-surface after about 70 time steps (see Figure 4-11 in Chapter 4). Although the simulation was able to proceed beyond this step without any break-down, it is expected that this would eventually cause a numerical instability in the solution. One indication of this is the error in conservation of momentum in the z direction. Although the total energy and total momentum in the y direction were affected much less by the instability (because of relatively large total energy and momentum in the y direction), total momentum in the z direction, as can be seen from Figure 5-9, started to increase soon after 65 time steps. This also indicates that the spurious
wiggles are not physical as the basic conservation properties are violated. A filtering

![Graph showing total horizontal momentum in the x direction after 71 time steps with no filtering.](image)

Figure 5-9: Total horizontal momentum in the $x$ direction after 71 time steps with no filtering

scheme, as described in Chapter 4, is applied at each time step to remove the high-order instabilities. The convention is such that when the filtering is done, say at the $n$th-order, this implies that all the harmonics that are higher than the $n$th-order are removed. We applied 10th- and 8th-order filtering for the aforementioned runs. Some small-amplitude wiggles appeared on the wave profile also with the 10th-order filtering at later stages of the simulation. Therefore, filtering is done at the 8th-order for all the runs that do not involve the long wave. Figure 5-10 below shows the effect of filtering on the tertiary wave amplitude for the runs that we discussed. The difference between such small amplitudes that results from different filtering levels is negligibly small.

The growth rate of the tertiary wave amplitude with the third-order initial conditions is given in Figure 5-11 and 5-12 for two primary wave steepnesses 0.1 and 0.2, respectively. A comparison between the spectral method and the BIEM reveals that
both methods are in good agreement with each other. Figures 5-13 and 5-14 show the relative changes in the total energy and the horizontal momentum for both runs from the BIEM. These quantities are conserved reasonably well (within 0.1% for the higher-steepness) during the entire simulation.

A numerical study on wave-wave interactions in a special triad is conducted with various wave steepnesses. The nonlinear energy transfer is computed for waves of intermediate lengths in the absence of an underlying long wave. The classical theory was valid for small amplitudes for a short time period. For two steepness ranges we used, the Zakharov theory compared reasonably well with the direct numerical method. Zakharov's perturbation approximation, or Benney's for that matter, seems to be valid at moderate steepnesses for interactions amongst intermediate-length waves. At this point, we should perhaps mention some of the experiments done by Tomita (1989) (cf. Chapter 2). Tomita has done similar experiments to those of McGoldrick et al. (1966) and Longuet-Higgins and Smith (1966). With the advantage of having a large wave tank \((80 \times 80 \times 4.5 \text{ m.})\), he made measurements for
Figure 5-11: A comparison of the spectral method with the BIEM for steepness of 0.1 with the third-order initial conditions

Figure 5-12: A comparison of the spectral method with the BIEM for steepness of 0.2 with the third-order initial conditions
Figure 5-13: Relative changes in the total energy and horizontal momentum for steepness of 0.1

Figure 5-14: Relative changes in the total energy and horizontal momentum for steepness of 0.2
relatively higher steepnesses and studied the long-term behavior of wave-wave interactions in the special triad. This was especially useful in demonstrating the periodic nature of the interactions. The primary wave steepnesses used in his experiments varied from 0.02 to 0.16. In general, the experimental results were in good agreement with the Zakharov theory with the exception of one case in which the primary wave steepnesses were 0.16 and 0.14 for components $k_1$ and $k_2$, respectively. These are apparently smaller than the steepnesses we used in our numerical experiments. The measured amplitude of the tertiary wave from the experiment was smaller than the theoretical prediction of Zakharov almost by a factor of two. Relying on this case, it was concluded that the perturbation theory was valid for steepnesses less than 0.15 ($H/L < 0.05$). However, some of the observations stated in Tomita's paper raise a few questions about the applicability of this conclusion. Firstly, the reproducible experiments were conducted with steepnesses less than 0.15. Secondly, the primary wave amplitudes were affected intensely by diffraction along the tank edges. Tomita could not compare the measured primary wave amplitudes with the theoretical ones due to diffraction effects. The measurements were taken at selected locations in the tank. The first station in the tank was approximately 25 m. away from the wavemaker. The measurements were also taken at subsequent points towards the end of the tank. For the case we mentioned above, it takes about 70 wave periods for the tertiary wave to reach the first station in the tank. At this point, the tertiary wave amplitude was already close to its peak value. So the measurements do not provide any information concerning the initial growth rate of the tertiary wave component. Having traveled this far until the first station, it was not known to what extent the tertiary wave amplitude was affected by diffraction effects and such. The upshot of this argument is to call to reader's attention that there is some uncertainty concerning the validity of his conclusion based on this single case.
5.2 Numerical computations for the special triad in the presence of a long wave

In the previous section, we compared the results of the direct numerical method with those of the spectral method and perturbation theory for the tertiary wave growth in the absence of a long wave and found that they are in agreement. When a long wave whose amplitude is comparable to the tertiary wavelength is added to the wave field, the displacement from the equilibrium level for the tertiary wave will be on the order of its wavelength. This would clearly violate the size restriction on the expansion parameter we set in Chapter 2 for a perturbation-type analysis that is valid only up to third-order. The length of the long wave, however, is very much limited by the computational resources. An exceedingly long wavelength requires a big enough domain to accommodate the long wave. This, in turn, requires fine enough gridding for the other members of the triad to provide an acceptable resolution. Such a configuration is computationally impractical, perhaps inhibitive, for the direct numerical method in which a time-domain simulation is performed for several periods with tens of thousands of panels. Therefore, the long wave component was chosen to be not too long to keep the computational burden within reason, yet long enough to have an interesting case in terms of short-long wave interactions. Moreover, the long wave was not to undergo any genuine interaction to exchange energy with any of the members of the triad. The special triad interaction will just occur under the influence of the long wave component which is, indeed, what we are trying to investigate.

The long wave added to the special triad was chosen to be propagating in the same direction as $k_2$ with a wavelength of $4\lambda_2$. The configuration of the special triad with the long wave was shown in Figure 3-3 in Chapter 3. Third-order initial conditions for this case are presented in Appendix C. Table 5.3 gives the characteristics of the long wave component. The dimension of the computational domain in the $y$ direction is determined by the long wavelength along which we used 64 panels. This leaves 16 panels per wavelength for the longer primary wave which also travels in the same...
<table>
<thead>
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<th>Wavenumber $k$ (r/m)</th>
<th>Frequency $\omega$ (r/sec)</th>
<th>Wavelength $\lambda$ (m)</th>
<th>Period $T$ (sec)</th>
<th>Phase speed $C_p$ (m/s)</th>
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<tbody>
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<td>3</td>
<td>0.24895832</td>
<td>1.563</td>
<td>25.238</td>
<td>4.021</td>
<td>6.276</td>
</tr>
</tbody>
</table>

Table 5.3: Characteristics of the long wave component

direction. The presence of the long wave makes this computational task quite a demanding one. Therefore, we used only 16 panels in the $x$ direction keeping the same spatial resolution for the primary waves. As before, 20 panels were used along the depth. The depth of the domain is now chosen to be half the wavelength of the long wave. The effect of the spatial discretization is investigated for the convergence analysis. The steepnesses for both the long wave and the primary waves were 0.1. By this selection, a reasonably large long wave amplitude to tertiary wavelength ratio is maintained. For the present configuration, Table 5.4 below shows what values the expansion parameter takes for members of the triad in the presence of the long wave. As can be seen from this table, the size restriction on the expansion parameter for the

<table>
<thead>
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<th>i</th>
<th>$a_{max}k_i$ ($\epsilon = 0.1$)</th>
</tr>
</thead>
<tbody>
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<td>2</td>
<td>0.53</td>
</tr>
<tr>
<td>3</td>
<td>0.13</td>
</tr>
<tr>
<td>Tertiary</td>
<td>3.26</td>
</tr>
</tbody>
</table>

Table 5.4: Expansion parameter for each member of the special triad in the presence of the long wave

tertiary wave component is violated by a significant margin. Plotted in Figure 5-15 is the normalized amplitude of the growing tertiary wave under the influence of the long wave with the third-order initial conditions. This figure clearly shows that the tertiary wave growth is somewhat reduced when the long wave is added to the field. As can be seen from Figure 5-16, both the total energy and the momentum are conserved reasonably well during the entire simulation. This run has passed convergence tests.
Figure 5-15: Tertiary wave growth in the presence of the long wave with the third-order initial conditions

Figure 5-16: Relative changes in the total energy and horizontal momentum in the presence of the long wave with steepness 0.1
When we performed the simulation with twice as many panels on the free-surface, the growth rate, which is shown by the short dashed-lines, follows very much the one with fewer panels. We found this convincing as far as the convergence is concerned and did not perform the entire simulation because of the computationally expensive nature of this particular run. Another run with a steepness 0.05 for the long wave has shown that the growth rate increased due to weaker influence of the long wave on the interaction with notable similarities in the behavior of the oscillations.

Liu and Yue (1991) have calculated the tertiary wave growth in the presence of the long wave using the spectral method. All wave components involved in the interaction have a steepness of 0.1. The third-order initial conditions are computed from Appendix C. A comparison of the spectral method with the BIEM is given in Figure 5-17. Although there are some local discrepancies between the results for this particular configuration, the global behavior of the tertiary wave component is captured by both methods. The mean lines for the tertiary wave growth are in good agreement. See Liu and Yue (1991) for the convergence tests on the spectral method run.

It is clearly shown that the growth rate of the tertiary wave is reduced under the influence of the long wave. What are the possible causes of the apparent reduction in the growth rate? In the beginning of this chapter, we calculated the frequency ratio $r \approx 1.736$ for a resonant interaction to occur. This ratio will slightly change during the course of the interaction because of nonlinear effects. Self- and mutual-interactions will induce changes in the linear frequencies of the primary waves (amplitude dispersion effects). These are higher-order corrections which may detune the resonance conditions based on linear wave frequencies and thus cause a reduction in the tertiary wave growth. In Chapter 3, we examined the changes in linear frequencies via a third-order perturbation analysis. The corrections to the linear frequencies were also given by equation 2.15 in Chapter 2. Using these equations for the given wave configuration and amplitudes of the last case with the long wave, we find a frequency ratio of $r = 1.703$. This is slightly different from the theoretical ratio of 1.736. In
Figure 5-17: A comparison of the spectral method with the BIEM in the presence of the long wave

Figure 5-18, we plotted the tertiary wave growth based on modified ratio of the primary wave frequencies. The new ratio is obtained by reducing the frequency of the short primary wave, and the wave amplitudes are kept the same for a fair comparison. Although there is a small reduction in the growth rate, this is far from explaining the drastic reduction in the presence of the long wave. This is clearly not a detuning effect. Nevertheless, to illustrate the effect of off-resonance for the same configuration, we analyzed a rather extreme case with a frequency ratio of \( r = 1.5 \). The tertiary wave growth is significantly reduced due to detuning of the resonance.

In Chapter 3, we derived the third-order initial conditions for the interacting triad in the presence of the long wave. A close examination of the higher-order cross-interaction terms will reveal that some of these have the same phase as the first-order primary wave components. For example, the shorter primary wave has an
amplitude of the following form:

\[
\zeta_1 = a_1 \cos \psi_1 + \left( Z_{(1,0)}^{(1,2)} + Z_{(1,0,0)}^{(1,0,2)} - \frac{3}{8} a_1^3 k_1^2 \right) \cos \psi_1
\]  

(5.9)

where the first two terms in the parenthesis are due to cross-interactions of the shorter primary with the other primary wave and the long wave. Upon examining the numerical values of these terms, we find out that the interaction with the long wave is quite significant and accounts roughly for a 30% reduction in the first-order amplitude of the short primary wave. When we examine the velocity potential of the short primary, we note a similar reduction in the quantity that is in phase with the \( \sin \psi_1 \) term. A comparison of the wave amplitude and the velocity potential at the relevant phases indicates that this pair satisfies the linear free-surface boundary conditions within 4%. The implication of this is that the short primary wave "looks like" a smaller linear wave due to cross-interactions at the third-order. When we evaluate the theoretical growth rate using the third-order altered amplitude of the short primary wave in lieu
of the first–order component, the result of this “prescription” is in reasonable agreement with the numerical prediction as can be seen in Figure 5-19. Figure 5-20 shows the long–term behavior of the tertiary wave from the spectral method in the presence of the long wave. Similarly, when the first–order wave amplitudes are replaced by the total amplitudes, the perturbation theory prediction is in good agreement with the spectral method.

![Diagram](image)

Figure 5-19: A comparison of the perturbation theory (based on altered amplitudes) with the BIEM in the absence and presence of the long wave

An interesting experiment was done by Donelan (1987). He measured the wind induced short wave growth in a wind–wave tank. Then he repeated the measurements in the presence of a long (mechanically generated) wave. In the latter case, the short wave growth was inhibited. Donelan inferred the rate of working of the wind on the short waves from the quadrature spectrum between the wind pressure measured just above the waves and the surface elevation. He found out that the wind–input rate to short waves was insensitive to the presence of the long wave, but the short wave growth was inhibited anyway. In his experiment, the peak of the wind generated spectrum
Figure 5-20: A comparison of the perturbation theory (based on altered amplitudes) with the spectral method in the absence and presence of the long wave was at about 1.20 Hz. The long wave had a steepness of 0.067 and was generated at a frequency of 0.707 Hz. The long wave amplitude is very small compared to the wavelength at the peak of the spectrum of the wind–generated waves. This is quite a gentle condition from the standpoint of the perturbation theory especially when compared to those we used in our numerical applications. Donelan speculated that “the paddle waves may act to detune the sharp resonance at the heart of the weak nonlinear interactions among quartets of gravity waves.” However, if the long wave acts to detune the resonance conditions for a quartet, it might possibly tune in another quartet. If we look at the family of curves in the wavenumber plane defined by the resonance conditions in a spectrum of waves (Figure 3-8 in Phillips, 1977), we note that the detuning effect would be shifting the resonance conditions rather than destroying it. Despite its quite important implications, this phenomenon has not been addressed so far. The real cause of the growth inhibition will remain a mystery until some dedicated experiments are conducted to resolve the issue.
Chapter 6

Conclusions and recommendations

We developed a computer implementation of a direct numerical method to study the nonlinear wave–wave interactions for a quartet of waves. This has been accomplished by solving an integral equation based on Green’s theorem in three–dimensional space without using the perturbation theory approximation. The accuracy of the method has been confirmed with time–domain simulations of steadily progressing, nonlinear waves of permanent form. A direct comparison of the spectral method with the boundary integral equation method (BIEM) is done based on resonant nonlinear energy transfer in a quartet of waves.

The tertiary wave growth has been calculated in a special triad in the absence of the long wave. The numerical results are compared with the theoretical predictions of the classical perturbation theory and the Zakharov theory. Initial growth rates are found to be in agreement with the classical theory. The long term prediction of the growth rate by the Zakharov theory compared reasonably well with those of the spectral method and the BIEM. Two sets of runs with different steepnesses were done. First, relatively small amplitudes were used for benchmark purposes. Then moderately steep waves were considered to test the accuracy of the perturbation approximation. The higher–steepness run was especially helpful in demonstrating the extent to which the classical theory can be valid. As expected, the classical theory was only good for small amplitudes and valid for a short period of time. On the other
hand, the Zakharov theory was able to predict the long-term behavior of wave-wave interactions even at moderately high steepnesses. In the presence of a long underlying wave, the tertiary wave growth was somewhat reduced. A close examination of the third-order perturbation analysis revealed that strong third-order cross-interactions occur between the primary waves and the long wave. These interactions generate third-order bound components that are in phase with the first-order components of the triad and alter the first-order amplitudes. Tertiary wave growth is quite sensitive to the amplitude of the short primary wave and is affected by this modification at the third-order. Nevertheless, when the changes in the first-order amplitudes are taken into account by arbitrarily altering the wave amplitude used by the theory from the linear one to the actual one, the perturbation theory prediction is in reasonable agreement with the mean tertiary wave growth calculated by the numerical methods. This indicates that the perturbation theory is valid based on what would be measured on the ocean surface. We feel however that a word of caution is due here. This is by no means an extensive numerical study on the nonlinear energy transfer among waves. We confined our calculations to gravity waves and the diversity in scale of the interacting waves was limited by the computational resources due to the expensive nature of the numerical implementation. To what extent our findings could be applicable in a typical sea spectrum can be clarified by more extensive studies using a cost-effective numerical technique or by some experiments, perhaps by both. For example, the behavior of the nonlinear wave-wave interactions in the gravity-capillary range in the presence of much longer, larger waves (as in a typical sea-wave spectrum) might be quite different. Holliday (1977), based on his analysis, found that in a typical sea spectrum the perturbation expansion about $z = 0$ produces divergent results for wavenumbers much larger than 0.04 (1/cm).

Because of its intrinsic relation to perturbation theory, the accuracy of the spectral method calculations in the presence of disparate length scales was deemed uncertain. As we mentioned earlier, Brueckner and West (1988) discounted the major criticism held against the use of mode-coupled equations for the reasons we explained in Chap-
chapter 4. Based on our comparison of spectral method to the direct numerical method, their assertion seems to be correct. This provides an opportunity to use the spectral method for an extensive study of nonlinear interactions under a wide variety of cases, especially with much longer waves where it is computationally inhibitive for the direct numerical method. The reason for this is that the spectral method is more computationally efficient so that results for longer times and finer griddings can be accomplished with it for those problems where it is not compromised by the perturbation theory.

Experimental confirmation exists for growth of a short wave due to nonlinear interactions amongst intermediate length waves in the absence of long, large waves. The results are in reasonable agreement with the perturbation theory. We suggest a laboratory experiment in which both a long wave and resonantly interacting intermediate length waves will be generated and the growing short wave will be measured. The purpose of the experiment would be to determine the extent to which our theoretical and numerical approaches predict reality in which waves having diverse scales exist.

For the nonlinear numerical computations to be correct for purely travelling waves, the initial conditions must be appropriate to a set of nonlinear travelling waves. There are nonlinear components among all the waves in the system that have to be in the initial conditions. Even though the numerical computation itself is fully nonlinear, our best results for initial conditions have been to base them on high-order perturbation theory. Using a computer package that can handle symbolic algebra, we obtained initial conditions up to and including third-order. Refinement of initial conditions (an extension up to fifth-order) used as input to numerical evaluation could be rewarding for two reasons: One is that it should diminish unwanted standing waves in the system. Secondly, when numerical methods are extended to the stochastic case, we know fifth-order effects will be important so they will have to be in the initial conditions.
Appendix A

The coupling coefficient for an interacting quartet

The interaction (coupling) coefficient $T(k, k_1, k_2, k_3) = T_{0,1,2,3}$ is given here as it appears in the appendix of Crawford et al. (1981) for the sake of completeness:

$$T_{0,1,2,3} = -\frac{2V^{(-)}_{3,3,-1,1} V^{(-)}_{0,2,0,-2}}{\omega_{1-3} - \omega_3 + \omega_1} - \frac{2V^{(-)}_{2,0,2-0} V^{(-)}_{1,1-3,3}}{\omega_{1-3} - \omega_1 + \omega_3} - \frac{2V^{(-)}_{2,2-1,1} V^{(-)}_{0,3,0-3}}{\omega_{1-2} - \omega_2 + \omega_1}$$

$$-\frac{2V^{(-)}_{3,0,3-0} V^{(-)}_{1,1-2,2}}{\omega_{1-2} - \omega_1 + \omega_2} - \frac{2V^{(-)}_{0,1,0,1} V^{(-)}_{2,3,2,3}}{\omega_{2+3} - \omega_2 - \omega_3} - \frac{2V^{(+)}_{2,3,3-0,1} V^{(+)}_{0,1,0,1}}{\omega_{2+3} + \omega_2 + \omega_3}$$

$$+ W_{0,1,2,3} \quad (A.1)$$

where

$$V^{(\pm)}_{0,1,2} = \frac{1}{8\pi\sqrt{2}} \left\{ (k_0.k_1 \pm k_0k_1) \left( \frac{\omega_0\omega_1}{\omega} \left( \frac{k_2}{k_0k_1} \right) \right)^{1/2}$$

$$+ (k_0.k_2 \pm k_0k_2) \left( \frac{\omega_0\omega_2}{\omega_1} \left( \frac{k_1}{k_0k_2} \right) \right)^{1/2}$$

$$+ (k_1.k_2 + k_1k_2) \left( \frac{\omega_1\omega_2}{\omega_0} \left( \frac{k_0}{k_1k_2} \right) \right)^{1/2} \right\} \quad (A.2)$$
and

\[ W_{0,1,2,3} = W_{-0,-1,2,3} + W_{2,3,-0,-1} - W_{2,-1,-0,3} \]
\[ -W_{-0,2,-1,3} - W_{-0,3,2,-1} - W_{3,-1,2,-0} \]  \hspace{1cm} (A.3)

with

\[ \overline{W}_{0,1,2,3} = \frac{1}{64\pi^2} \left( \frac{\omega_0 \omega_1}{\omega_2 \omega_3} k_0 k_1 k_2 k_3 \right) \left[ 2(k_0 + k_1) - k_{1+3} - k_{1+2} - k_{0+3} - k_{0+2} \right] \]  \hspace{1cm} (A.4)

where \( k_{i\pm j} = |k_i \pm k_j| \), and \( \omega_{i\pm j} = \omega(k_{i\pm j}) \).
Appendix B

Initial conditions for the triad

For the special wave triad we analyzed in Chapter 3, we will combine all the findings of this analysis and provide the initial velocity potential and the surface wave elevation up to and including third order at $t = 0$. The velocity potential is given by:

\[
\phi = \frac{a_1 \omega_1}{k_1} e^{k_1 z} \sin \psi_1 + \frac{a_2 \omega_2}{k_2} e^{k_2 z} \sin \psi_2 \\
+ \Phi^{(1,1)}_{(1,-1)} e^{[k_1 - k_2] z} \sin(\psi_1 - \psi_2) - \Phi^{(1,1)}_{(1,1)} e^{[k_1 + k_2] z} \sin(\psi_1 + \psi_2) \\
- \frac{a_1 \omega_1}{2k_1} (a_1 k_1)^2 e^{k_1 z} \sin \psi_1 - \frac{a_2 \omega_2}{2k_2} (a_2 k_2)^2 e^{k_2 z} \sin \psi_2 \\
- \Phi^{(1,2)}_{(1,0)} e^{k_1 z} \sin \psi_1 - \Phi^{(2,1)}_{(0,1)} e^{k_2 z} \sin \psi_2 \\
+ \frac{\Phi^{(2,1)}_{(2,1)}}{2gk_1} e^{[2k_1 + k_2] z} \sin(2\psi_1 + \psi_2) \\
+ \frac{\Phi^{(1,2)}_{(-1,2)}}{2gk_2} e^{[2k_2 - k_1] z} \sin(2\psi_2 - \psi_1) \\
+ \frac{\Phi^{(1,2)}_{(1,2)}}{2gk_1} e^{[2k_2 + k_1] z} \sin(2\psi_2 + \psi_1) \\
\tag{B.1}
\]

where

\[
\Phi^{(1,1)}_{(1,-1)} = \frac{2a_1 a_2 \omega_1 \omega_2 (\omega_1 - \omega_2) \cos^2 \frac{\theta}{2}}{(\omega_1 - \omega_2)^2 - g|k_1 - k_2|} \\
\tag{B.2}
\]
\[
\Phi^{(1,1)}_{(1,1)} = \frac{2a_1a_2\omega_1\omega_2(\omega_1 + \omega_2)\sin^2 \frac{\theta}{2}}{\omega_1 + \omega_2} \left( \frac{\omega_1 \omega_2}{\omega_1 + \omega_2} - g|k_1 + k_2| \right) 
\]

(B.3)

\[
\Phi^{(2,1)}_{(2,1)} = a_1^2a_2\omega_1\omega_2(\omega_1 + \omega_2)\sin^2 \frac{\theta}{2} \left\{ |k_1 + k_2| - (k_1 + k_2) \right. \\
- \frac{\omega_1 k_2 \sin^2 \frac{\theta}{2}}{\omega_1 + \omega_2} \left( \frac{4\omega_1(2\omega_1 + \omega_2)|k_1 + k_2| \cos^2 \frac{\theta}{2}}{(\omega_1 + \omega_2)^2 - g|k_1 + k_2|} \right) 
\]

(B.4)

\[
\Phi^{(1,2)}_{(-1,2)} = a_1a_2^2\omega_1\omega_2(\omega_2 - \omega_1)\cos^2 \frac{\theta}{2} \left\{ |k_1 - k_2| + (k_1 + k_2) \right. \\
+ \frac{\omega_2 k_1 \cos^2 \frac{\theta}{2}}{\omega_2 - \omega_1} - \frac{4\omega_2(2\omega_2 - \omega_1)|k_1 - k_2| \cos^2 \frac{\theta}{2}}{(\omega_1 - \omega_2)^2 - g|k_1 - k_2|} \left\} 
\]

(B.5)

\[
\Phi^{(1,2)}_{(1,2)} = a_1a_2^2\omega_1\omega_2(\omega_1 + \omega_2)\sin^2 \frac{\theta}{2} \left\{ |k_1 + k_2| - (k_1 + k_2) \right. \\
- \frac{\omega_2 k_1 \sin^2 \frac{\theta}{2}}{\omega_1 + \omega_2} + \frac{4\omega_2(2\omega_2 + \omega_1)|k_1 + k_2| \cos^2 \frac{\theta}{2}}{(\omega_1 + \omega_2)^2 - g|k_1 + k_2|} \left\} 
\]

(B.6)

The surface wave height at \( t = 0 \) is:

\[
\zeta = (a_1 - \frac{3}{8}a_1^2k_1) \cos \psi_1 + (a_2 - \frac{3}{8}a_2^2k_2^2) \cos \psi_2 \\
+ \frac{1}{2}a_1^2k_1 \cos 2\psi_1 + \frac{1}{2}a_2^2k_2 \cos 2\psi_2 + \frac{3}{8}a_1^3k_1^2 \cos 3\psi_1 + \frac{3}{8}a_2^3k_2^2 \cos 3\psi_2 \\
+ Z^{(1,1)}_{(1,-1)} \cos(\psi_1 - \psi_2) + Z^{(1,1)}_{(1,1)} \cos(\psi_1 + \psi_2) \\
+ Z^{(2,1)}_{(2,-1)} \cos(2\psi_1 - \psi_2) + Z^{(2,1)}_{(2,1)} \cos(2\psi_1 + \psi_2) + Z^{(2,1)}_{(0,1)} \cos \psi_2 \\
+ Z^{(1,2)}_{(-1,-2)} \cos(2\psi_2 - \psi_1) + Z^{(1,2)}_{(1,2)} \cos(2\psi_2 + \psi_1) + Z^{(1,2)}_{(1,0)} \cos \psi_1 
\]

(B.7)

where

\[
Z^{(1,1)}_{(1,-1)} = \frac{a_1a_2\omega_1\omega_2 \cos^2 \frac{\theta}{2}}{g} \left[ \frac{2(\omega_1 - \omega_2)^2}{(\omega_1 - \omega_2)^2 - g|k_1 - k_2|^2} - 1 \right] + \frac{1}{2}a_1a_2(k_1 + k_2) 
\]

(B.8)

\[
Z^{(1,1)}_{(1,1)} = \frac{a_1a_2\omega_1\omega_2 \sin^2 \frac{\theta}{2}}{g} \left[ 1 - \frac{2(\omega_1 + \omega_2)^2}{(\omega_1 + \omega_2)^2 - g|k_1 + k_2|^2} \right] + \frac{1}{2}a_1a_2(k_1 + k_2) 
\]

(B.9)
\[ Z^{(2,1)}_{(2,-1)} = \frac{1}{2} a_1 k_1 Z^{(1,1)}_{(1,-1)} + \frac{\Phi^{(1,1)}_{(1,1)}}{g} a_1 |k_1 - k_2| \left[ \omega_1 \cos^2 \frac{\theta}{2} + \frac{1}{2} (\omega_1 - \omega_2) \right] \\
+ \frac{1}{8} a_1^2 a_2 (2k_1 k_2 + 2k_1^2 + k_2^2) - \frac{1}{2g} a_1^2 a_2 \omega_1 \omega_2 (k_1 + k_2) \cos^2 \theta \]

(B.10)

\[ Z^{(2,1)}_{(2,1)} = \frac{\Phi^{(1,1)}_{(2,1)} (2\omega_1 + \omega_2)}{g \left[ (2\omega_1 + \omega_2)^2 - g |2k_1 + k_2| \right]} + \frac{1}{2} a_1 k_1 Z^{(1,1)}_{(1,1)} \\
- \frac{\Phi^{(1,1)}_{(1,1)}}{g} a_1 |k_1 + k_2| \left[ \omega_1 \cos^2 \frac{\theta}{2} + \frac{1}{2} (\omega_1 + \omega_2) \right] \\
+ \frac{1}{8} a_1^2 a_2 (2k_1 k_2 + 2k_1^2 + k_2^2) + \frac{1}{2g} a_1^2 a_2 \omega_1 \omega_2 (k_1 + k_2) \sin^2 \frac{\theta}{2} \]

(B.11)

\[ Z^{(2,1)}_{(0,1)} = \frac{1}{2} a_1 k_1 \left[ Z^{(1,1)}_{(1,-1)} + Z^{(1,1)}_{(1,1)} \right] + \frac{\Phi^{(1,1)}_{(1,1)}}{g} a_1 |k_1 - k_2| \left[ \frac{1}{2} (\omega_1 - \omega_2) - \omega_1 \sin^2 \frac{\theta}{2} \right] \\
+ \frac{\Phi^{(2)}_{(1,1)}}{g} a_1 |k_1 + k_2| \left[ \omega_1 \sin^2 \frac{\theta}{2} - \frac{1}{2} (\omega_1 + \omega_2) \right] + \frac{1}{4} a_1^2 a_2 (k_2^2 - 2k_1^2) \\
- \frac{1}{2g} a_1^2 a_2 \omega_1 \omega_2 (k_1 + k_2) \cos \theta \]

(B.12)

\[ Z^{(1,2)}_{(-1,2)} = \frac{\Phi^{(1,2)}_{(-1,2)} (2\omega_2 - \omega_1)}{g \left[ (2\omega_2 - \omega_1)^2 - g |2k_2 - k_1| \right]} + \frac{1}{2} a_2 k_2 Z^{(1,1)}_{(1,-1)} \\
+ \frac{\Phi^{(1,1)}_{(1,1)}}{g} a_2 |k_1 - k_2| \left[ -\omega_2 \cos^2 \frac{\theta}{2} + \frac{1}{2} (\omega_1 - \omega_2) \right] \\
+ \frac{1}{8} a_1 a_2^2 (2k_1 k_2 + k_1^2 + 2k_2^2) - \frac{1}{2g} a_1 a_2^2 \omega_1 \omega_2 (k_1 + k_2) \cos^2 \theta \]

(B.13)

\[ Z^{(1,2)}_{(1,2)} = \frac{\Phi^{(1,2)}_{(1,2)} (2\omega_2 + \omega_1)}{g \left[ (2\omega_2 + \omega_1)^2 - g |2k_2 + k_1| \right]} + \frac{1}{2} a_2 k_2 Z^{(1,1)}_{(1,1)} \\
- \frac{\Phi^{(1,1)}_{(1,1)}}{g} a_2 |k_1 + k_2| \left[ \omega_2 \cos^2 \frac{\theta}{2} + \frac{1}{2} (\omega_1 + \omega_2) \right] \\
+ \frac{1}{8} a_1 a_2^2 (2k_1 k_2 + k_1^2 + 2k_2^2) + \frac{1}{2g} a_1 a_2^2 \omega_1 \omega_2 (k_1 + k_2) \sin^2 \frac{\theta}{2} \]

(B.14)

\[ Z^{(1,2)}_{(1,0)} = \frac{1}{2} a_2 k_2 \left[ Z^{(1,1)}_{(1,-1)} + Z^{(1,1)}_{(1,1)} \right] + \frac{\Phi^{(1,1)}_{(1,1)}}{g} a_2 |k_1 - k_2| \left[ \frac{1}{2} (\omega_1 - \omega_2) + \omega_2 \sin^2 \frac{\theta}{2} \right] \]
\[ + \frac{\Phi^{(1,1)}}{g} a_2 |k_1 + k_2| \left[ \omega_2 \sin^2 \frac{\pi}{2} - \frac{1}{2} (\omega_1 + \omega_2) \right] + \frac{1}{4} a_1 a_2^2 (k_1^2 - 2k_2^2) \\
- \frac{1}{2g} a_1 a_2^2 \omega_1 \omega_2 (k_1 + k_2) \cos \theta. \] (B.15)

Here we used \( \psi_m = k_m \cdot x \) for the initial conditions at \( t = 0 \). From figures 3-1 and 3-2, we can determine the following:

\[ \delta_1 = \pi - \tan^{-1} \left( \frac{\sin \theta}{r^2 - \cos \theta} \right), \] (B.16)

\[ \delta_2 = \pi - \tan^{-1} \left( \frac{\sin \theta}{r^2 + \cos \theta} \right), \] (B.17)

\[ \gamma_1 = \pi - \tan^{-1} \left( \frac{r^2 \sin \theta}{1 - r^2 \cos \theta} \right), \] (B.18)

and

\[ \gamma_2 = \pi - \tan^{-1} \left( \frac{r^2 \sin \theta}{1 + r^2 \cos \theta} \right) \] (B.19)

where \( r = \sqrt{k_1/k_2} \). For this special triad we analyzed, the angle \( \theta \) between the two primary wave components is \( \pi/2 \).
Appendix C

Initial conditions for the triad in the presence of a long wave

The initial velocity potential and the surface wave height in the presence of a long wave with \( k_3 \) are given. In what follows, \( \hat{\phi} \) and \( \zeta \) represent the initial velocity potential and the surface wave height given in Appendix B. As the typesetting for the formulae is done by a computer program which converts the ‘fortran’ output of MACSYM, no effort is made in simplifying or reducing the expressions presented herein. The velocity potential at \( t = 0 \) is:

\[
\phi = \hat{\phi} + \frac{a_3\omega_3}{k_3} e^{k_3z} \sin \psi_3 + \Phi^{(0,1,1)}_{(0,1,-1)} e^{\|k_3-k_3\|z} \sin(\psi_2 - \psi_3) \\
\Phi^{(1,0,1)}_{(1,0,-1)} e^{\|k_1-k_3\|z} \sin(\psi_1 - \psi_3) - \Phi^{(1,0,1)}_{(1,0,1)} e^{\|k_1+k_3\|z} \sin(\psi_1 + \psi_3) \\
- \frac{1}{2} G^{(1,0,2)}_{(1,0,0)} e^{k_3z} \sin \psi_1 - \frac{1}{2} G^{(0,1,2)}_{(0,1,0)} e^{k_3z} \sin \psi_2 \\
- \frac{1}{2} \left[ \frac{a_3\omega_3}{k_3} (a_3k_3)^2 + \frac{G^{(2,0,1)}_{(0,0,1)} + G^{(0,2,1)}_{(0,0,1)}}{gk_3} \right] e^{k_3z} \sin \psi_3 \\
+ \Phi^{(2,0,1)}_{(2,0,-1)} e^{\|2k_1-k_3\|z} \sin(2\psi_1 - \psi_3) + \Phi^{(2,0,1)}_{(2,0,1)} e^{\|2k_1+k_3\|z} \sin(2\psi_1 + \psi_3) \\
+ \Phi^{(1,0,2)}_{(-1,0,2)} e^{\|2k_3-k_1\|z} \sin(2\psi_3 - \psi_1) + \Phi^{(1,0,2)}_{(1,0,2)} e^{\|2k_3+k_1\|z} \sin(2\psi_3 + \psi_1) \\
+ \Phi^{(0,1,2)}_{(0,-1,2)} e^{\|2k_3-k_2\|z} \sin(2\psi_3 - \psi_2) + \Phi^{(0,2,1)}_{(0,2,-1)} e^{\|2k_3-k_3\|z} \sin(2\psi_2 - \psi_3) \\
+ \Phi^{(1,1,1)}_{(1,1,1)} e^{\|k_1+k_2+k_3\|z} \sin(\psi_1 + \psi_2 + \psi_3)
\]
\begin{align}
+ \Phi_{(1,1,-1)}^{(1,1,1)} e^{i|k_1 + k_3 - k_3|z} \sin(\psi_1 + \psi_2 - \psi_3), \\
+ \Phi_{(1,-1,1)}^{(1,1,1)} e^{i|k_1 - k_3 + k_3|z} \sin(\psi_1 - \psi_2 + \psi_3) \\
+ \Phi_{(1,-1,-1)}^{(1,1,1)} e^{i|k_1 - k_3 - k_3|z} \sin(\psi_1 - \psi_2 - \psi_3) 
\end{align} 
(C.1)

and the surface wave elevation at $t = 0$ is:

\begin{align}
\zeta &= \hat{\zeta} + (a_3 - \frac{3}{8} a_3^2 k_3^2) \cos \psi_3 + \frac{1}{2} a_3^2 k_3 \cos 2\psi_3 + \frac{3}{8} a_3^2 k_3^2 \cos 3\psi_3 \\
&+ Z_{(1,0,-1)}^{(1,0,1)} \cos(\psi_1 - \psi_3) + Z_{(1,0,1)}^{(1,0,1)} \cos(\psi_1 + \psi_3) \\
&+ Z_{(0,1,-1)}^{(0,1,1)} \cos(\psi_2 - \psi_3) + Z_{(0,1,1)}^{(0,1,1)} \cos(\psi_2 + \psi_3) \\
&+ Z_{(2,0,-1)}^{(2,0,1)} \cos(2\psi_1 - \psi_3) + Z_{(2,0,1)}^{(2,0,1)} \cos(2\psi_1 + \psi_3) + Z_{(0,0,1)}^{(2,0,1)} \cos \psi_3 \\
&+ Z_{(1,0,2)}^{(1,0,2)} \cos(2\psi_3 - \psi_1) + Z_{(1,0,2)}^{(1,0,2)} \cos(2\psi_3 + \psi_1) + Z_{(1,0,0)}^{(1,0,2)} \cos \psi_1 \\
&+ Z_{(0,2,-1)}^{(0,2,1)} \cos(2\psi_2 - \psi_3) + Z_{(0,2,1)}^{(0,2,1)} \cos(2\psi_2 + \psi_3) + Z_{(0,0,1)}^{(0,2,1)} \cos \psi_3 \\
&+ Z_{(0,0,2)}^{(0,1,2)} \cos(2\psi_2 - \psi_3) + Z_{(0,1,2)}^{(0,1,2)} \cos(2\psi_2 + \psi_3) + Z_{(0,1,0)}^{(0,1,2)} \cos \psi_2 \\
&+ \frac{1}{g} \left[ \zeta_{(1,1,1)}^{(1,1,1)} + \Phi_{(1,1,1),1}^{(1,1,1)} (\omega_1 + \omega_2 + \omega_3) \right] \cos(\psi_1 + \psi_2 + \psi_3) \\
&+ \frac{1}{g} \left[ \zeta_{(1,1,1)}^{(1,1,1)} - \Phi_{(1,1,1),1}^{(1,1,1)} (\omega_1 + \omega_2 - \omega_3) \right] \cos(\psi_1 + \psi_2 - \psi_3) \\
&+ \frac{1}{g} \left[ \zeta_{(1,1,1)}^{(1,1,1)} + \Phi_{(1,1,1),1}^{(1,1,1)} (\omega_1 - \omega_2 + \omega_3) \right] \cos(\psi_1 - \psi_2 + \psi_3) \\
&+ \frac{1}{g} \left[ \zeta_{(1,1,1)}^{(1,1,1)} - \Phi_{(1,1,1),1}^{(1,1,1)} (\omega_1 - \omega_2 - \omega_3) \right] \cos(\psi_1 - \psi_2 - \psi_3) 
\end{align} 
(C.2)

where

\begin{align}
\Phi_{(i,m,n)}^{(i,j,k)} &= \frac{G_{(i,m,n)}^{(i,j,k)}}{(i\omega_1 + m\omega_2 + n\omega_3)^2 - g|\omega_1 + m\omega_2 + n\omega_3|} 
\end{align} 
(C.3)

\begin{align}
Z_{(1,1,0)}^{(1,1,0)} &= \frac{1}{g} \left[ \Phi_{(1,-1,0)}^{(1,1,0)} (\omega_1 - \omega_2) + \frac{1}{2} a_1 a_2 (\omega_1^2 + \omega_2^2 - \omega_1 \omega_2) \right] 
\end{align} 
(C.4)

\begin{align}
Z_{(1,1,0)}^{(1,1,0)} &= -\frac{1}{g} \left[ \Phi_{(1,1,0)}^{(1,1,0)} (\omega_1 + \omega_2) - \frac{1}{2} a_1 a_2 (\omega_1^2 + \omega_2^2 + \omega_1 \omega_2) \right] 
\end{align} 
(C.5)
\[ Z_{(1,0,-1)}^{(1,0,1)} = \frac{1}{g} \left[ \Phi_{(1,0,-1)}^{(1,0,1)}(\omega_1 - \omega_3) + \frac{1}{2} a_1 a_3 (\omega_1^2 + \omega_3^2 - \omega_1 \omega_3) \right] \quad (C.6) \]

\[ Z_{(1,0,1)}^{(1,0,1)} = -\frac{1}{g} \left[ \Phi_{(1,0,1)}^{(1,0,1)}(\omega_1 + \omega_3) - \frac{1}{2} a_1 a_3 (\omega_1^2 + \omega_3^2 + \omega_1 \omega_3) \right] \quad (C.7) \]

\[ Z_{(0,1,-1)}^{(0,1,1)} = \frac{1}{g} \left[ \Phi_{(0,1,-1)}^{(0,1,1)}(\omega_2 - \omega_3) + a_2 a_3 \left[ \frac{1}{2} (\omega_2^2 + \omega_3^2) - \omega_2 \omega_3 \right] \right] \quad (C.8) \]

\[ Z_{(0,1,1)}^{(0,1,1)} = \frac{1}{2g} a_2 a_3 (\omega_2^2 + \omega_3^2) \quad (C.9) \]

\[ Z_{(2,0,-1)}^{(2,0,1)} = \frac{1}{8g} \left[ 4a_1 \omega_1^2 Z_{(1,0,-1)}^{(1,0,1)} + (8a_1 \omega_1 - 4a_1 \omega_3) \Phi_{(1,0,1)}^{(1,0,1)}(k_1^2 + k_3^2)^{1/2} 
- 4a_1 k_1 \omega_1 \Phi_{(1,0,1)}^{(1,0,1)} + (16\omega_1 - 8\omega_3) \Phi_{(2,0,-1)}^{(2,0,1)} + 2a_1^2 k_1 \omega_3^2 a_3 
-(2a_1^2 \omega_1 a_3 k_3 + 2a_1^2 k_1 \omega_1 a_3) \omega_3 + (a_1^2 a_3 k_3 + 2a_1^2 k_1 a_3) \omega_3 \right] \quad (C.10) \]

\[ Z_{(2,0,1)}^{(2,0,1)} = \frac{1}{8g} \left[ 4a_1 \omega_1^2 Z_{(1,0,1)}^{(1,0,1)} + (-4a_1 \omega_3 - 8a_1 \omega_1) \Phi_{(1,0,1)}^{(1,0,1)}(k_1^2 + k_3^2)^{1/2} 
+ 4a_1 k_1 \omega_1 \Phi_{(1,0,1)}^{(1,0,1)} + (8\omega_3 + 16\omega_1) \Phi_{(2,0,1)}^{(2,0,1)} + 2a_1^2 k_1 \omega_1^2 a_3 
+(2a_1^2 \omega_1 a_3 k_3 + 2a_1^2 k_1 \omega_1 a_3) \omega_3 + (a_1^2 a_3 k_3 + 2a_1^2 k_1 a_3) \omega_3 \right] \quad (C.11) \]

\[ Z_{(0,0,1)}^{(2,0,1)} = \frac{1}{4g} \left[ 2a_1 \omega_1^2 Z_{(1,0,1)}^{(1,0,1)} + 2a_1 \omega_1^2 Z_{(1,0,1)}^{(1,0,1)} 
-(2a_1 \omega_3 \Phi_{(1,0,1)}^{(1,0,1)} + 2a_1 \omega_3 \Phi_{(1,0,-1)}^{(1,0,1)})(k_1^2 + k_3^2)^{1/2} 
+ 2a_1 k_1 \omega_1 \Phi_{(1,0,1)}^{(1,0,1)} - 2a_1 k_1 \omega_1 \Phi_{(1,0,-1)}^{(1,0,1)} + a_1^2 a_3 k_3 \omega_3^2 - 2a_1^2 k_1 \omega_1^2 a_3 \right] \quad (C.12) \]

\[ Z_{(-1,0,2)}^{(1,0,2)} = \frac{1}{8g} \left[ 4a_3 \omega_3^2 Z_{(1,0,-1)}^{(1,0,1)} + (4\omega_1 a_3 - 8a_3 \omega_3) \Phi_{(1,0,1)}^{(1,0,1)}(k_1^2 + k_3^2)^{1/2} 
+(16\omega_3 - 8\omega_1) \Phi_{(-1,0,2)}^{(1,0,2)} + 4a_3 k_3 \omega_3 \Phi_{(1,0,-1)}^{(1,0,1)} + 2a_1 a_3^2 k_3 \omega_3^2 \right] \]
\[-(2a_1\omega_3a_3^2k_3 + 2a_1k_1\omega_1a_3^2)\omega_3 + 2a_1\omega_1^2a_3^2k_3 + a_1k_1\omega_1^2a_3^2\]  

(C.13)

\[Z^{(1,0,2)}_{(1,0,2)} = \frac{1}{8g} \left[ 4a_3\omega_3^2Z^{(1,0,1)}_{(1,0,1)} + (4\omega_3a_3 - 4\omega_1a_3)\Phi^{(1,0,1)}_{(1,0,1)}(k_1^2 + k_3^2)^{1/2} \right.\]
\[+ (16\omega_3 + 8\omega_1)\Phi^{(1,0,2)}_{(1,0,2)} + 4a_3k_3\omega_3\Phi^{(1,0,1)}_{(1,0,1)} + 2a_1a_3^2k_3\omega_3^2 \]
\[+ (2a_1\omega_1a_3^2k_3 + 2a_1k_1\omega_1a_3^2)\omega_3 + 2a_1\omega_1^2a_3^2k_3 + a_1k_1\omega_1^2a_3^2 \]

(C.14)

\[Z^{(1,0,2)}_{(1,0,0)} = \frac{1}{4g} \left[ 2a_3\omega_3^2Z^{(1,0,1)}_{(1,0,1)} + 2a_3\omega_3^2Z^{(1,0,1)}_{(1,0,-1)} \right.\]
\[+ (2\omega_1a_3^2\Phi^{(1,0,1)}_{(1,0,-1)} - 2\omega_1a_3\Phi^{(1,0,1)}_{(1,0,1)})(k_1^2 + k_3^2)^{1/2} \]
\[+ 2a_3k_2\omega_3\Phi^{(1,0,1)}_{(1,0,1)} + 2a_3k_3\omega_3\Phi^{(1,0,1)}_{(1,0,-1)} - 2a_1a_3^2k_3\omega_3^2 + a_1k_1\omega_1^2a_3^2 \]

(C.15)

\[Z^{(0,1,2)}_{(0,-1,2)} = \frac{1}{8g} \left[ 4a_3\omega_3^2Z^{(0,1,1)}_{(0,-1,-1)} + (4\omega_2a_3 - 8\omega_3a_3)\Phi^{(0,1,1)}_{(0,-1,-1)}|k_2 - k_3| \right.\]
\[+ (16\omega_3 - 8\omega_2)\Phi^{(0,1,2)}_{(0,-1,2)} + (4a_3k_3 - 4k_2a_3)\omega_3\Phi^{(0,1,1)}_{(0,1,-1)} \]
\[- (4a_2\omega_2a_3^2k_3 + 4a_2k_2\omega_2a_3^2)\omega_3 + (2a_2a_3^2k_3 + a_2k_2a_3^2)\omega_2^2 \]
\[+ 2a_2a_3^2k_3\omega_3^2 \]

(C.16)

\[Z^{(0,1,2)}_{(0,1,2)} = \frac{1}{8g} \left[ 4a_3\omega_3^2Z^{(0,1,1)}_{(0,1,1)} + 2a_2a_3^2k_3\omega_3^2 + 2a_2\omega_2a_3^2k_3 + a_2k_2\omega_2a_3^2 \right] \]

(C.17)

\[Z^{(0,1,2)}_{(1,0,0)} = \frac{1}{4g} \left[ 2a_3\omega_3^2Z^{(0,1,1)}_{(1,0,1)} + 2a_3\omega_3^2Z^{(0,1,1)}_{(0,1,-1)} + 2\omega_2a_3\Phi^{(0,1,1)}_{(0,1,-1)}|k_2 - k_3| \right.\]
\[+ (2a_3k_3 - 2k_2a_3)\omega_3\Phi^{(0,1,1)}_{(0,1,-1)} - 2a_2a_3^2k_3\omega_3^2 \]
\[+ (2a_2\omega_2a_3^2k_3 + 2a_2k_2\omega_2a_3^2)\omega_3 + a_2k_2\omega_2a_3^2 \]

(C.18)

\[Z^{(0,2,1)}_{(0,2,-1)} = \frac{1}{8g} \left[ 4a_2\omega_2a_3^2Z^{(0,1,1)}_{(0,-1,-1)} + (8a_2\omega_2 - 4\omega_3a_3)\Phi^{(0,1,1)}_{(0,-1,-1)}|k_2 - k_3| \right.\]
\[+ (4a_2\omega_2k_3 - 4a_2k_2\omega_2)\Phi^{(0,1,1)}_{(0,-1,-1)} + (16\omega_2 - 8\omega_3)\Phi^{(0,2,1)}_{(0,2,-1)} \]

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\begin{align}
Z^{(0,2,1)}_{(0,1,1)} &= \frac{1}{8g} \left[ 4a_2\omega_2^2 Z^{(0,1,1)}_{(0,1,1)} + (a_2^2 a_3 k_3 + 2a_2^2 k_2 a_3)\omega_3^2 + 2a_2^2 k_2^2 a_3 \right] \\
Z^{(0,2,1)}_{(0,0,1)} &= \frac{1}{4g} \left[ 2a_2\omega_2^2 Z^{(0,1,1)}_{(0,0,1)} + 2a_2\omega_2^2 Z^{(0,0,1)}_{(0,1,1)} - 2a_2\omega_3 \Phi^{(0,1,1)}_{(0,1,-1)} \|k_2 - k_3\| \\
&\quad + (2a_2\omega_2 k_3 - 2a_2 a_3 k_2) \Phi^{(0,1,1)}_{(0,1,-1)} + a_2^2 a_3 k_3\omega_3^2 \\
&\quad - (2a_2^2 \omega_2 a_3 k_3 + 2a_2^2 k_2 \omega_2 a_3)\omega_3 - 2a_2^2 k_2^2 a_3 \right] \\
Z^{(1,1,1)}_{(1,-1,-1)} &= \frac{1}{4g} \left[ 2a_1\omega_1^2 Z^{(0,1,1)}_{(0,1,1)} + 2a_2\omega_2^2 Z^{(1,0,1)}_{(1,1,-1)} \right.
\left. + 2a_3\omega_3^2 Z^{(1,1,0)}_{(1,-1,0)} \\
&\quad - (2a_2\omega_3 + 2a_2\omega_2 - 2a_1 a_2) \Phi^{(1,0,1)}_{(1,0,-1)}(k_1^2 + k_3^2)^{1/2} \\
&\quad + [(2\omega_1 - 2\omega_2)a_3 - 2a_3\omega_3] \Phi^{(1,1,0)}_{(1,-1,0)}(k_1^2 + k_2^2)^{1/2} + 2a_2 \omega_2 k_3 \Phi^{(1,0,1)}_{(1,0,-1)} \\
&\quad + 2k_2 a_3 \omega_3 \Phi^{(1,1,0)}_{(1,-1,0)} + a_1 a_2 a_3 k_3\omega_3 - (a_1 \omega_1 a_2 a_3 k_3 + a_1 k_1 \omega_1 a_2 a_3)\omega_3 \\
&\quad + [a_1 a_2 k_2 \omega_2^2 - (a_1 \omega_1 a_2 k_2 + a_1 k_1 \omega_1 a_2)\omega_2 + a_1 k_1 \omega_1^2 a_2] a_3 \right]
\end{align}

\begin{align}
Z^{(1,1,1)}_{(1,-1,1)} &= \frac{1}{4g} \left[ 2a_1\omega_1^2 Z^{(0,1,1)}_{(0,1,1)} + 2a_2\omega_2^2 Z^{(1,0,1)}_{(1,0,1)} \right.
\left. + 2a_3\omega_3^2 Z^{(1,1,0)}_{(1,-1,0)} \\
&\quad - (2a_1\omega_3 - 2a_1\omega_2 + 2a_1 a_1) \Phi^{(0,1,1)}_{(0,1,-1)} |k_2 - k_3| \\
&\quad - (2a_2\omega_3 - 2a_2\omega_2 + 2a_1 a_2) \Phi^{(1,0,1)}_{(1,0,1)}(k_1^2 + k_3^2)^{1/2} \\
&\quad + [2a_3 \omega_3 + (2\omega_1 - 2\omega_2)a_3] \Phi^{(1,1,0)}_{(1,1,0)}(k_1^2 + k_2^2)^{1/2} \\
&\quad + 2a_2 \omega_2 k_3 \Phi^{(1,0,1)}_{(1,0,1)} + 2k_2 a_3 \omega_3 \Phi^{(1,1,0)}_{(1,-1,0)} + a_1 a_2 a_3 k_3\omega_3^2 \\
&\quad + [(a_1 \omega_1 a_2 - 2a_1 a_2 \omega_2)a_3 k_3 + (a_1 k_1 \omega_1 a_2 - 2a_1 a_2 k_2 \omega_2) a_3] \omega_3 \\
&\quad + [a_1 a_2 k_2 \omega_2^2 - (a_1 \omega_1 a_2 k_2 + a_1 k_1 \omega_1 a_2)\omega_2 + a_1 k_1 \omega_1^2 a_2] a_3 \right]
\end{align}
\[ Z^{(1,1,1)}_{(1,1,1)} = \frac{1}{4g} \left[ 2a_1\omega_1^2 Z^{(0,1,1)}_{(0,1,1)} + 2a_2\omega_2^2 Z^{(1,0,1)}_{(1,0,1)} + 2a_3\omega_3^2 Z^{(1,1,0)}_{(1,1,0)} 

- (2a_2\omega_3 + 2a_2\omega_2 + 2a_1a_2)\Phi^{(1,0,1)}_{(1,0,1)}(k_1^2 + k_3^2)^{1/2} 

- [(2\omega_2 + 2\omega_1)a_3 + 2a_3\omega_3]\Phi^{(1,1,0)}_{(1,1,0)}(k_1^2 + k_2^2)^{1/2} + 2a_2\omega_2k_3\Phi^{(1,0,1)}_{(1,0,1)} 

+ 2k_2a_3\omega_3\Phi^{(1,1,0)}_{(1,1,0)} + a_1a_2a_3k_3^2 + (a_1\omega_1a_2a_3k_3 + a_1k_1\omega_1a_2a_3)\omega_3 

+ [a_1a_2k_2\omega_2^2 + (a_1\omega_1a_2k_2 + a_1k_1\omega_1a_2)\omega_2 + a_1k_1\omega_1^2a_2]a_3 \right] \] (C.24)

\[ Z^{(1,1,1)}_{(1,1,-1)} = \frac{1}{4g} \left[ 2a_1\omega_1^2 Z^{(0,1,1)}_{(0,1,-1)} + 2a_2\omega_2^2 Z^{(1,0,1)}_{(1,0,-1)} + 2a_3\omega_3^2 Z^{(1,1,0)}_{(1,1,0)} 

- (2a_1\omega_3 - 2a_1\omega_2 - 2a_1\omega_1)\Phi^{(0,1,1)}_{(0,1,-1)}|k_2 - k_3| 

- (2a_2\omega_3 - 2a_2\omega_2 - 2a_1a_2)\Phi^{(1,0,1)}_{(1,0,-1)}(k_1^2 + k_3^2)^{1/2} 

+ (2a_3\omega_3 - (2\omega_2 + 2\omega_1)a_3)\Phi^{(1,1,0)}_{(1,1,0)}(k_1^2 + k_2^2)^{1/2} 

+ 2a_2\omega_2k_3\Phi^{(1,0,1)}_{(1,0,-1)} + 2k_2a_3\omega_3\Phi^{(1,1,0)}_{(1,1,0)} + a_1a_2a_3k_3^2 

- [(2a_1a_2\omega_2 + a_1\omega_1a_2)a_3k_3 + (2a_1a_2k_2\omega_2 + a_1k_1\omega_1a_2)a_3]\omega_3 

+ [a_1a_2k_2\omega_2^2 + (a_1\omega_1a_2k_2 + a_1k_1\omega_1a_2)\omega_2 + a_1k_1\omega_1^2a_2]a_3 \right] \] (C.25)

\[ G^{(1,1,0)}_{(1,-1,0)} = a_1a_2\omega_1\omega_2(\omega_1 - \omega_2) \] (C.26)

\[ G^{(1,1,0)}_{(1,1,0)} = a_1a_2\omega_1\omega_2(\omega_1 + \omega_2) \] (C.27)

\[ G^{(1,0,1)}_{(1,0,-1)} = a_1a_3\omega_1\omega_3(\omega_1 - \omega_3) \] (C.28)

\[ G^{(1,0,1)}_{(1,0,1)} = a_1a_3\omega_1\omega_3(\omega_1 + \omega_3) \] (C.29)

\[ G^{(0,1,1)}_{(0,1,-1)} = 2a_2a_3\omega_2\omega_3(\omega_2 - \omega_3) \] (C.30)
\[ G_{(2,0,-1)}^{(2,0,1)} = -\frac{1}{4} \left[ 2a_1 G_{(1,0,,-1)}^{(1,0,1)} + (8a_1 \omega^2 - 4a_1 \omega_1 \omega_3) \Phi_{(1,0,,-1)}^{(1,0,1)}(k_1^2 + k_3^2)^{1/2} + (4a_1 k_1 \omega_1 \omega_3 - 8a_1 k_1 \omega_1^2) \Phi_{(1,0,,-1)}^{(1,0,1)} + (2a_1^2 \omega_1 a_3 k_3 + 2a_1^2 k_1 \omega_1 a_3) \omega_3^2 - (3a_1^2 \omega_1^2 a_3 k_3 + 2a_1^2 k_1 \omega_1 a_3^2) \omega_3 \right] \] (C.31)

\[ G_{(2,0,1)}^{(2,0,1)} = \frac{1}{4} \left[ 2a_1 G_{(1,0,1)}^{(1,0,1)} + (4a_1 \omega_1 \omega_3 + 8a_1 \omega_1^2) \Phi_{(1,0,1)}^{(1,0,1)}(k_1^2 + k_3^2)^{1/2} - (4a_1 k_1 \omega_1 \omega_3 + 8a_1 k_1 \omega_1^2) \Phi_{(1,0,1)}^{(1,0,1)} - (2a_1^2 \omega_1 a_3 k_3 + 2a_1^2 k_1 \omega_1 a_3) \omega_3^2 - (3a_1^2 \omega_1^2 a_3 k_3 + 2a_1^2 k_1 \omega_1 a_3^2) \omega_3 \right] \] (C.32)

\[ G_{(0,0,1)}^{(2,0,1)} = \frac{1}{2} \left[ a_1 (G_{(1,0,1)}^{(1,0,1)} + G_{(1,0,,-1)}^{(1,0,1)}) + 2a_1 \omega_1 \omega_3 (\Phi_{(1,0,,-1)}^{(1,0,1)} - \Phi_{(1,0,1)}^{(1,0,1)}) (k_1^2 + k_3^2)^{1/2} + 2a_1 k_1 \omega_1 \omega_3 (\Phi_{(1,0,,-1)}^{(1,0,1)} - \Phi_{(1,0,1)}^{(1,0,1)}) + (2a_1^2 k_1 \omega_1^2 a_3 - a_1^2 \omega_1 a_3^2) \omega_3 \right] \] (C.33)

\[ G_{(-1,0,2)}^{(1,0,2)} = \frac{1}{4} \left[ 2a_3 G_{(1,0,,-1)}^{(1,0,1)} + (8a_3 \omega_3^2 - 4a_3 \omega_3 \omega_1) \Phi_{(1,0,,-1)}^{(1,0,1)}(k_1^2 + k_3^2)^{1/2} + (4a_1 a_3 k_3 \omega_3 - 8a_3 k_3 \omega_3^2) \Phi_{(1,0,,-1)}^{(1,0,1)} + (2a_1 a_3 \omega_1^2 k_3 + 3a_1 k_1 \omega_1 a_3^2) \omega_3^2 - (2a_1 a_3 \omega_1^2 a_3 k_3 + 2a_1 k_1 \omega_1 a_3^2) \omega_3 \right] \] (C.34)

\[ G_{(1,0,2)}^{(1,0,2)} = \frac{1}{4} \left[ 2a_3 G_{(1,0,1)}^{(1,0,1)} + (8a_3 \omega_3^2 + 4a_3 \omega_3 \omega_1) \Phi_{(1,0,1)}^{(1,0,1)}(k_1^2 + k_3^2)^{1/2} - (8a_3 k_3 \omega_3^2 + 4a_3 a_3 k_3 \omega_3) \Phi_{(1,0,1)}^{(1,0,1)} - (2a_1 a_3 \omega_1^2 k_3 + 3a_1 k_1 \omega_1 a_3^2) \omega_3^2 - (2a_1 a_3 \omega_1^2 a_3 k_3 + 2a_1 k_1 \omega_1 a_3^2) \omega_3 \right] \] (C.35)
\[ G_{(1,0,0)}^{(1,0,2)} = \frac{1}{2} \left[ a_3 (G_{(1,0,1)}^{(1,0,1)} - G_{(1,0,-1)}^{(1,0,1)}) - 2a_3 \omega_1 \omega_3 (\Phi_{(1,0,1)}^{(1,0,1)} + \Phi_{(1,0,-1)}^{(1,0,1)}) \right] (k_2^2 + k_3^2)^{1/2} \\
-2a_3 k_2 \omega_1 \omega_3 (\Phi_{(1,0,1)}^{(1,0,1)} + \Phi_{(1,0,-1)}^{(1,0,1)}) + (2a_1 \omega_1 a_2^2 k_3 - a_1 k_1 \omega_1 a_2^2) \omega_3^2 \right] \quad (C.36) \]

\[ G_{(0,2,-1)}^{(0,2,1)} = -\frac{1}{2} \left[ a_2 G_{(0,1,-1)}^{(0,1,1)} + (4a_2 \omega_2^2 - 2a_2 \omega_2 \omega_3) \Phi_{(0,1,-1)}^{(0,1,1)} \right] k_2 - k_3 \]
\[ +[(2a_2 k_2 \omega_2 - 2a_2 \omega_2 k_3) \omega_3 + 4a_2 \omega_2^2 k_3 - 4a_2 k_2 \omega_2^2] \Phi_{(0,1,-1)}^{(0,1,1)} \]
\[ +(2a_2 \omega_2 a_3 k_3 + 2a_2^2 k_2 \omega_2 a_3) \omega_3^2 \\
-(4a_2^2 \omega_2^2 a_3 k_3 + 2a_2^2 k_2 \omega_2 a_3^2) \omega_3 \right] \quad (C.37) \]

\[ G_{(0,0,1)}^{(0,2,1)} = \frac{1}{2} \left[ (a_2 G_{(0,1,-1)}^{(0,1,1)} + 2a_2 \omega_2 \omega_3 \Phi_{(0,1,-1)}^{(0,1,1)}) k_2 - k_3 \right] + 2a_2 \omega_2 \omega_3 (k_2 - k_3) \Phi_{(0,1,-1)}^{(0,1,1)} \\
+[(2a_2 \omega_2 a_3 k_3 + 2a_2^2 k_2 \omega_2 a_3) \omega_3^2 + (2a_2^2 k_2 \omega_2^2 a_3 - 2a_2 \omega_2^2 a_3 k_3) \omega_3^2] \quad (C.38) \]

\[ G_{(0,-1,2)}^{(0,1,2)} = \frac{1}{2} \left[ (a_3 G_{(0,1,-1)}^{(0,1,1)} + (4a_3 \omega_3^2 - 2\omega_2 a_3 \omega_3) \Phi_{(0,1,-1)}^{(0,1,1)}) k_2 - k_3 \right] \\
+[(4k_2 a_3 - 4a_3 k_3) \omega_3^2 + (2\omega_2 a_3 k_3 - 2k_2 \omega_2 a_3) \omega_3] \Phi_{(0,1,-1)}^{(0,1,1)} \\
+(2a_3 \omega_2 a_3^2 k_3 + 4a_3 k_2 \omega_2 a_3^2) \omega_3^2 \\
-(2a_3 \omega_2^2 a_3^2 k_3 + 2a_3 k_2 \omega_2^2 a_3^2) \omega_3 \right] \quad (C.39) \]

\[ G_{(0,1,0)}^{(0,1,2)} = -\frac{1}{2} \left[ (a_3 G_{(0,1,-1)}^{(0,1,1)} + 2\omega_2 a_3 \omega_3 \Phi_{(0,1,-1)}^{(0,1,1)}) k_2 - k_3 \right] + 2a_3 \omega_2 \omega_3 (k_3 - k_2) \Phi_{(0,1,-1)}^{(0,1,1)} \\
+(2a_2 k_2 \omega_2 a_3^2 - 2a_2 \omega_2 a_3 k_3) \omega_3^2 - (2a_2 \omega_2^2 a_3^2 k_3 + 2a_2 k_2 \omega_2^2 a_3^2) \omega_3 \right] \quad (C.40) \]

\[ G_{(1,1,1)}^{(1,1,1)} = -\frac{1}{2} \left[ (a_2 G_{(1,0,-1)}^{(1,0,1)} + (2a_2 \omega_2 \omega_3 + a_2 \omega_2^2 - 2\omega_1 a_2 \omega_2) \Phi_{(1,0,-1)}^{(1,0,1)}) (k_1^2 + k_3^2)^{1/2} \\
+2a_3 G_{(1,1,0)}^{(1,1,0)} + (2\omega_2 a_3^2 + (2\omega_2 - 2\omega_1) a_3 \omega_3) \Phi_{(1,1,0)}^{(1,1,0)} (k_1^2 + k_3^2)^{1/2} \\
+(2\omega_2 a_3^2 - 2a_2 \omega_2 a_3 k_3 - 2a_2 \omega_2 k_3 \omega_3 b_i g r) \Phi_{(1,0,-1)}^{(1,1,1)} \\
+(2\omega_1 k_2 - 2k_2 \omega_2) a_3 \omega_3 - 2k_2 a_3 \omega_3^2 \Phi_{(1,-1,0)}^{(1,1,0)} \right] \]
\[ G_{(1,1,1)}^{(1,1,1)} = \frac{1}{2} \left[ a_2 G_{(1,1,1)}^{(0,1,1)} + (2a_2 a_3 k_3 + 2a_2 a_3 k_3) \Phi^{(1,1,1)}_{(1,1,1)}(k_1^2 + k_3^2)^{1/2} \right] \]

\[ + [(a_3 a_4 a_5 a_6 + a_3 a_4 a_5 a_6) a_3 - a_1 a_2^2 a_3 k_3] \omega_3 \]

\[ + [((a_1 a_2 a_3 a_4 a_5 a_6 + a_1 a_2 a_3 k_3 - a_1 a_2 a_3 k_3)] \omega_3 \]

\[ + [(a_1 a_2 k_2 + a_1 a_2) \omega_2^2 + (-a_1 a_2^2 a_2 k_2 - a_1 a_2 k_2^2) a_3]\]  \hspace{1cm} (C.41)
\[ +[(2\omega_2 + 2\omega_1)a_3\omega_3 - 2a_3\omega_3^2)\Phi^{(1,1,0)}_{(1,1,0)} - a_3G^{(1,1,0)}_{(1,1,0)}](k_1^2 + k_2^2)^{1/2} \\
+[(2a_2\omega_2^2 + 2\omega_1a_2\omega_2)k_3 - 2a_2\omega_2 k_3\omega_3]\Phi^{(1,0,1)}_{(1,0,-1)} \\
+[(2k_2\omega_2 + 2\omega_1k_2)a_3\omega_3 - 2k_2a_3\omega_3^2]\Phi^{(1,1,0)}_{(1,1,0)} \\
+[(2a_1a_2\omega_2 + a_1\omega_1a_2)a_3k_3 - (2a_1a_2k_2\omega_2 + a_1k_1\omega_1a_2)a_3]\omega_3^2 \\
+[-2a_1a_2\omega_2^2 - 2a_1\omega_1a_2\omega_2 - a_1\omega_1^2a_2)a_3k_3 \\
+(-2a_1a_2k_2\omega_2^2 + (-2a_1\omega_1a_2 k_2 - a_1k_1\omega_1a_2)\omega_2 - a_1k_1\omega_1^2a_2)a_3]\omega_3 \\
+[(a_1\omega_1a_2 k_2 + a_1k_1\omega_1a_2)\omega_2^2 + (a_1\omega_1^2a_2 k_2 + a_1k_1\omega_1^2a_2)\omega_2]a_3 \] \quad (C.44)

where the same subscript and superscript conventions apply.
Appendix D

Panel geometry

Having discretized the geometry into small panels (elements), we need to define a local coordinate system for each panel on which we will carry out the integrals in 4.5 to compute the source and dipole effect at selected field points. Given this coordinate system, we will find the direction cosines for a coordinate transformation between the local and global coordinate systems to provide the link between the source and field points. Some of the needed parameters such as the outward normal vector and the panel centroid are calculated in due process.

The free surface is approximated by flat, but tilted quadrilateral panels. One of these panels is shown in Figure D.1. Each panel has four vertices whose coordinates \( P_i(x_i, y_i, z_i) \) are known. These four points do not necessarily form a plane in space. We can, however, construct a plane if we know the normal vector and a point in space which this normal vector goes through. Hess and Smith (1962) takes this point as the average of the corner coordinates:

\[
\bar{x} = \frac{1}{4} \sum_{i=1}^{4} x_i \quad \bar{y} = \frac{1}{4} \sum_{i=1}^{4} y_i \quad \bar{z} = \frac{1}{4} \sum_{i=1}^{4} z_i.
\]  

(D.1)

Assuming this average point \( \bar{P}(\bar{x}, \bar{y}, \bar{z}) \) is on the plane, we will now compute the normal vector to define the plane completely. Cross product of two vectors, \( P_1 P_3 \)
and $P_2P_4$, connecting the corner points will yield the unit normal vector on the plane:

$$
n = \frac{P_1P_3 \times P_2P_4}{|P_1P_3 \times P_2P_4|}. \quad (D.2)$$

Then the corner coordinates are projected on this plane, and the centroid is calculated. New coordinates of the corner points when projected on the plane:

$$
\begin{align*}
  x_k' &= x_k + n_x d_k \\
  y_k' &= y_k + n_y d_k \\
  z_k' &= z_k + n_z d_k.
\end{align*} \quad (D.3)
$$

where $d_k$ is the distance between the plane and the original corner coordinates also known as the projection distance. We now will define the local (element) coordinate
system to perform the integrals. Local $x$–axis ($\xi$) is chosen to be in the $P_1P_3$ direction. If we call $t_1$ the unit normal vector in the local $z$ direction, that is,

$$t_1 = \frac{P_1P_3}{|P_1P_3|},$$

(D.4)

then the unit vector in local $y$–axis ($\zeta$) can be written as:

$$t_2 = n \times t_1.$$  

(D.5)

If a point has the coordinates $P(x_p, y_p, z_p)$ in the global coordinate system, its coordinates in the local coordinate system are found by the following transformation:

$$
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} = 
\begin{pmatrix}
  \chi_{11} & \chi_{12} & \chi_{13} \\
  \chi_{21} & \chi_{22} & \chi_{23} \\
  \chi_{31} & \chi_{32} & \chi_{33}
\end{pmatrix}
\begin{pmatrix}
  x_p - x_0 \\
  y_p - y_0 \\
  z_p - z_0
\end{pmatrix},
$$

(D.6)

where $\chi_{ij}$ are the direction cosines which are given by:

$$
\begin{pmatrix}
  \chi_{1i} \\
  \chi_{2i} \\
  \chi_{3i}
\end{pmatrix} = 
\begin{pmatrix}
  t_{i1} \\
  t_{i2} \\
  t_{i3}
\end{pmatrix}.
$$

(D.7)

If the corner coordinates $P_k'(x'_k, y'_k, z'_k)$ are given in the global coordinate system, they are given in the local coordinate system as follows:

$$
\begin{pmatrix}
  \xi_k^* \\
  \eta_k^* \\
  \zeta_k^*
\end{pmatrix} = 
\begin{pmatrix}
  \chi_{11} & \chi_{12} & \chi_{13} \\
  \chi_{21} & \chi_{22} & \chi_{23} \\
  0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
  x'_k - \bar{x} \\
  y'_k - \bar{y} \\
  z'_k - \bar{z}
\end{pmatrix}.
$$

(D.8)

Coordinates of the centroid in the local coordinate system with $(\xi_1, \eta_1) = (0, 0)$ as the origin:

$$\xi_0 = \frac{1}{3(\eta_2^* - \eta_4^*)}[\xi_4^*(\eta_1^* - \eta_2^*) + \xi_2^*(\eta_4^* - \eta_1^*)]$$

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\[ \eta_0 = -\frac{1}{3} \eta_1^*. \]  \hspace{1cm} \text{(D.9)}

Coordinates of the centroid in the global coordinate system are:

\[
\begin{pmatrix}
  x_c \\
  y_c \\
  z_c
\end{pmatrix} = \begin{pmatrix}
  \bar{x} \\
  \bar{y} \\
  \bar{z}
\end{pmatrix} + \begin{pmatrix}
  \chi_{11} & \chi_{21} \\
  \chi_{12} & \chi_{22} \\
  \chi_{13} & \chi_{23}
\end{pmatrix} \begin{pmatrix}
  \xi_0 \\
  \eta_0
\end{pmatrix}. \hspace{1cm} \text{(D.10)}
Bibliography


