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NEAR-OPTIMAL STOCHASTIC TERMINAL CONTROLLERS

by

David V. Stallard

Massachusetts Institute of Technology

Submitted in Partial Fulfillment
of the Requirements for the
Degree of Doctor of Science
at the
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__________________________
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February 11, 1971

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Thesis Supervisor

Accepted by: ____________________________

Chairman, Departmental Graduate Committee

Archives
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ABSTRACT

This thesis is concerned with problems in applying modern control
theory to the design of autopilots with engineering constraints and to the
formulation of computationally feasible guidance laws for the terminal
phase of flight of a homing interceptor missile. Kalman estimation of
states associated with target-missile geometry is assumed, and the
separation theorem is invoked so that the control subproblem may be
considered separately. Despite the specific nature of the problem, the
techniques herein are more widely applicable.

For a linear system with a quadratic performance index dominated
by the weighted final state, it is shown that the optimum control is pro-
portional to the similarly weighted, projected, zero-control terminal
state (including the effect of a known disturbance), which reduces to the
projected, zero-control miss distance in the case of an interceptor mis-
sile. This holds for discrete as well as continuous control.

The control subproblem is partitioned into the design of a fixed-gain
(for practical reasons) autopilot and time-varying guidance gains. Optim-
ization of the speed of response of the autopilot transfer function, in order
to reduce its sensitivity to mismatch, leads to a practical criterion for
specifying the transfer function as one RHP airframe zero over a compar-
able LHP pole. An additional benefit is the simple representation of the
autopilot in a matched guidance law.

A closed-form continuous guidance law for an autopilot with one pole
and one zero is derived, in terms of a time-varying effective navigation
ratio $\frac{N_g}{t}$, time to go, and the projected, zero-control miss distance.
This guidance law is suitable for on-line computation. The target maneu-
ver can be modelled as the output of a low-pass filter, and the contribu-
tion of missile axial acceleration to the relative geometry can be accounted
for as a known disturbance. A similar result is worked out for the dis-
crete-control case, except that the navigation ratio must be computed
off-line in inverse time and stored for the last few samples. The contin-
uous guidance law is suitable up to the last 10 or so sample intervals.

The autopilot is designed to have a fixed transfer function approx-
imating one zero over one pole. A separation principle holds for divid-
ing the autopilot design problem into an estimation subproblem and a
control subproblem, i.e., the placement of airframe poles to new loca-
tions and the cancellation of the unknown biases on the airframe. The
placement of poles is accomplished by applying the Crossley-Porter
formula for feedback of airframe modes, and so the estimator-controller
is designed in the plant modal space. The coupling of the high-frequency
fin-servo modes to the other modes is weak for fundamental reasons, and it facilitates a novel procedure for simplification of the controller structure from five states (the order of the original plant) to three. The resulting autopilot has the specified transfer function at the design point, acceptable off-design performance, cancellation of the bias and good filtering of noise. This combination of techniques is new, it closes somewhat the gap between modern and classical control theory, and it should be applicable to other types of systems.

Adjoint calculations of miss distance show that this new autopilot, in combination with the matched discrete guidance law, produces a miss distance approaching the theoretical minimum in the absence of radome distortion.

An investigation of methods is made for computing optimal control gains for continuous control of high-order plants. An improved transition-matrix approach, which avoids the catastrophic problem of growing exponentials versus decaying exponentials, has been rediscovered. Consideration of feedback from modes as well as states gives insight.

A new theory for the optimal control of selected, low-frequency modes is proposed as a possible step toward the simplification of structure of control systems.

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And of course, this doctoral program would scarcely have been possible without the loyal and cheerful support of the author's wife, Betsy, and his family.
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SYMBOLS

NOTE:

Dot over symbol, e.g., as in \( \dot{\lambda} \), denotes derivative of that quantity with respect to time.

The hat \( \hat{\lambda} \) over a symbol, e.g., as in \( \hat{x} \), denotes the estimate of that quantity.

The tilde \( \tilde{\lambda} \) over a symbol, e.g., as in \( \tilde{x} \), denotes the error in the estimate of that quantity.

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<td>Missile-acceleration state in one-pole, one zero model of autopilot</td>
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<td>A</td>
<td>Aspect angle of target velocity vector relative to line of sight</td>
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<td>A</td>
<td>Weighting matrix of state in loss function in performance index</td>
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<td>A_a</td>
<td>Missile axial acceleration</td>
<td>3</td>
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<td>A_c</td>
<td>Lateral-acceleration command to autopilot</td>
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<td>Adj ( )</td>
<td>Adjoint matrix for matrix in parenthesis</td>
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<td>A(i)</td>
<td>Weighting of state in performance index of linear discrete control problem</td>
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<td>A_m</td>
<td>Missile acceleration perpendicular to centerline</td>
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<td>A_t</td>
<td>Target acceleration perpendicular to original LOS</td>
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<tr>
<td>A_{ty}</td>
<td>Target acceleration perpendicular to the line of sight</td>
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<td>a_{11}</td>
<td>Coefficient of s in numerator of $G_1$</td>
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<td>a_{12}</td>
<td>Coefficient of $s^2$ in numerator of $G_1$</td>
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<td>a_{31}</td>
<td>&quot;Alpha over gamma dot&quot; time constant of airframe</td>
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CHAPTER 1

INTRODUCTION

1.1 The Choice and Relevance of This Subject

"What's in a name?" So asks Shakespeare's Juliet in her perplexity. In this case, the name of this thesis is chosen to avoid an undeserved opprobrium for the author and his family at the graduation ceremony, in the strange temper of the times. The original, and presently alternate, title is "Guidance of Homing Interceptor Missiles".

This choice of subject reflects the author's professional work at the Missile Systems Division of the Raytheon Company and his interest in applications, as manifested by his published papers. It is hoped that this work, and closely related work on radome compensation not contained in this volume, will be useful in missile engineering. This writer feels that it is still appropriate for a great university to be involved in some aspects of weapons research, for the common defense.

Despite its specific character, one may also hope that this thesis will have a more general appeal within the academic community, for some of the techniques herein may be applicable to other engineering problems, as indicated below and in Subsection 1.5 herein. A "near-optimal stochastic terminal controller" would have performance nearly as good as that of a truly optimal controller, and it would be less complex and costly to implement.

Homing interceptor guided missiles are chiefly defensive weapons (or at most, offensive weapons in a limited air-combat sense), rather than
carriers of mass destruction. In ground defense against aircraft, such missiles (for example, HAWK in the U.S. Army and Marine Corps) have largely supplanted anti-aircraft gunnery. Air-to-air guided missiles, such as the Sparrow, Falcon and infrared Sidewinder, are carried by many aircraft, but have not completely displaced guns.

Reference S1 is a reasonably adequate and still current summary of guidance techniques for homing interceptor missiles. In homing guidance, the missile tracks the target via radar or infrared energy and utilizes the information for guiding itself to the target. Command-guided missiles, such as the Nike HERCULES, are not included in this category. The homing missiles in this thesis could be of the active type with the transmitter in the missile, or, as is more usual, the semiactive type, with the transmitter in the launching site or aircraft.

Now that some 17 years have passed since the first successful flights of the Sparrow air-to-air missile, it is convenient to categorize homing-missile guidance as classical or modern. Classical guidance is based on proportional navigation and classical control-system theory, i.e., prior to Kalman's first major contributions in 1960-1961. Modern homing guidance makes increasing use of "modern control theory" (from 1960 to date) for time-varying, linear systems with stochastic inputs. Modern guidance appears at first to be sharply different from classical guidance, but nevertheless many problems, insights and techniques from the latter are still relevant. This thesis attacks some of the outstanding problems, mainly in terminal guidance, i.e., near the end of the flight.

Modern guidance is attractive because it appears to be capable of better homing accuracy than classical guidance. Unfortunately, a modern guidance system has appreciably more computational complexity, which
the guidance engineer must justify convincingly by improved performance, because the managers and customers of a missile-development organization are as pragmatic as most American businessmen. Parenthetically, it seems curious that although pragmatism is clearly a cornerstone of American engineering, it is almost never discussed, as if philosophy were not a respectable part of everyday life.

While this thesis may appear to be quite specialized, many ideas here may well be of interest to other control engineers, e.g., the development of closed-form solutions for on-line computation of optimal controls, computational techniques, optimal modal control of systems, and simplified estimator-controllers with specified fixed transfer functions.

1.2 Basic Theory of Classical Guidance

In compliance with the requirement to summarize the history of the problem, this subsection discusses classical guidance, which is based on proportional navigation and pre-1960 control theory. Because Reference S1 is still current with respect to classical guidance and is available in the Aeronautics and Astronautics Library, much of this subsection is taken from that document in verbatim or condensed form, so as to leave more time for current contributions.

Proportional navigation is preferred to other guidance laws such as pure pursuit (Reference H3), partly because proportional navigation tends to minimize the necessary lateral acceleration of the interceptor missile, which is particularly important at high altitudes. Moreover, classical control techniques and stationary analog filtering generally result in tolerable complexity of equipment.
1.2.1 The Criterion of Zero Line-of-Sight Angular Rate

Consider a target which is assumed to fly at constant speed in a constant direction. Let the interceptor missile be required to fly at a constant speed in a straight line, with no lateral acceleration, so that it hits the target.

The situation is shown in Figure 1-1, which is in the plane of the initial missile position, the initial target position and the target velocity vector \( \frac{dT}{dt} \). The use of vectors simplifies the derivation and facilitates an arbitrary coordinate system, although it will later be seen to be convenient to choose the X axis through the initial positions of the target and missile, i.e., the initial line-of-sight. At some particular instant, which is \( T_{go} \) seconds before intercept, the missile and target are respectively at positions \( \vec{M} \) and \( \vec{T} \), and the line-of-sight vector between them is \( \vec{T} - \vec{M} \). The condition for intercept is simply:

\[
(\vec{T} - \vec{M}) + T_{go} \frac{dT}{dt} = T_{go} \frac{d\vec{M}}{dt}
\]

(1.1)

These terms are the three sides of the "collision triangle". This is equivalent to:

\[
\frac{dT}{dt} - \frac{d\vec{M}}{dt} = \frac{d}{dt} (\vec{T} - \vec{M}) = - \frac{1}{T_{go}} (\vec{T} - \vec{M}),
\]

(1.2)

which shows that the relative velocity vector is proportional to the line-of-sight vector and therefore these vectors are collinear. Since the derivative of \( \vec{T} - \vec{M} \) is collinear with \( \vec{T} - \vec{M} \), the line-of-sight (LOS) does not rotate for the ideal collision course.

It may be concluded that if the line-of-sight from the interceptor to
Figure 1-1 - Geometry of Ideal Intercept
the target does not rotate in space, then the interceptor will hit the target (provided that the derivative of relative range $|\vec{T}-\vec{M}|$ is negative). This principle must have originated with mariners, who have long known that, if one ship sees another approaching it on a constant bearing, then they are on a collision course. A familiar everyday example is for one automobile driver entering a turnpike to observe the LOS angular rate of an approaching vehicle on the turnpike, but here the criterion of success is a bit different from that of this report.

An equivalent and useful form of the collision principle is apparent from Figure 1-1, in which the vector velocity difference perpendicular to the LOS is zero. With $V_m$ and $V_T$ denoting respectively the missile and target speeds ($|d\vec{M}/dt|$ and $|d\vec{T}/dt|$), it is evident that the components of their velocities perpendicular to the LOS ($\vec{T}-\vec{M}$) are equal:

$$V_m \sin L_c = V_t \sin A,$$

where $L_c$ and $A$ are respectively the correct lead angle and the aspect angle measured from the LOS. The closing velocity $V_c$ is:

$$V_c = -\frac{d}{dt} [\vec{T}-\vec{M}] = V_m \cos L + V_t \cos A$$

1.2.2 Miss Distance and LOS Angular Rate

In practice, a direct hit is uncommon because of many sources of error, to be explained in Subsection 2.2.1. On the other hand, if the miss distance (minimum range) is sufficiently small and if fuzing is timely, then the interceptor can still effect a kill with a non-nuclear warhead of moderate size. Of course, the missile has only one chance to hit the tar-
get, because it can neither track the target nor reverse course after passing the point of minimum range.

Figure 1-2 shows the geometry of an undershoot miss for straight-line, constant-speed courses of target and missile, with both velocity vectors in the same plane for simplicity. Note that the line-of-sight rotates counter-clockwise, with an LOS angle given by:

\[
\lambda = \tan^{-1} \left[ \frac{Y_t - Y_m}{X_t - X_m} \right] \\
\approx \frac{Y_t - Y_m}{X_t - X_m},
\]

(1.5)

where the small-angle approximation is valid until just before intercept. Figure 1-3 portrays the geometry of an overshoot miss.

Appendix A of Reference S1 shows that the vector miss distance is very nearly perpendicular to the original LOS. For the two-dimensional case:

\[
\text{Miss distance} = \text{Minimum } R_{mt} \cong (Y_t - Y_m)|_{X_t - X_m} = 0
\]

(1.6)

The concept of projected miss distance may be related to the LOS angular rate, which is obtained by taking the tangent of Equation (1.5) and differentiating:

\[
\sec^2 \lambda \left[ \dot{\lambda} \right] = \frac{\dot{Y}_t - \dot{Y}_m}{X_t - X_m} - (Y_t - Y_m) \frac{\dot{X}_t - \dot{X}_m}{(X_t - X_m)^2}
\]

(1.7)

For a small value of \( \lambda \) prior to intercept, \( \sec \lambda \) may be closely approximated by unity. From (1-4), the closing velocity \( V_c \) is very nearly
Figure 1-2 - Geometry of Undershoot Miss
Figure 1-3 - Geometry of Overshoot Miss
\(- (\dot{X}_t - \dot{X}_m)\) and is assumed to be constant. The time-to-go \(T_{go}\) is \((X_t - X_m)/V_c\), i.e., the time until \(X_t - X_m\) becomes zero. Hence, (1.7) can be approximated before intercept as:

\[
\dot{\lambda} = \frac{\dot{Y}_t - \dot{Y}_m}{X_t - X_m} + (Y_t - Y_m) \frac{V_c}{(X_t - X_m)^2}
\]

\[
= \frac{\dot{Y}_t - \dot{Y}_m}{V_c \ T_{go}} + (Y_t - Y_m) \frac{1}{V_c \ T_{go}^2}
\]

\[
= \frac{1}{V_c \ T_{go}^2} \left[ (Y_t - Y_m) + T_{go} (\ddot{Y}_t - \ddot{Y}_m) \right]
\]

\[
= \frac{1}{V_c \ T_{go}^2} \text{ [Projected zero-effort miss distance]}
\]

The term in square brackets is the miss distance that would result if neither \(\dot{Y}_t\) changes (nonmaneuvering target) nor \(\dot{Y}_m\) changes (zero effort by missile). Evidently, nulling \(\dot{\lambda}\) is particularly appropriate if the target has zero acceleration \(\ddot{Y}_t\) perpendicular to the LOS. See also References M4 and R5.

The more general applicability of the concept of "projected zero-effort miss distance" will be developed in Chapter 3.

1.2.3 The Proportional Navigation Law

The primary task of the guidance design engineer is a final-value control problem, i.e., he must minimize the miss distance (final value of \(Y_t - Y_m\)) with attainable values of maneuvering acceleration and thereby maximize kill probability. The previous material has shown that the miss distance will be zero if the LOS angular rate \(\dot{\lambda}\) is maintained at zero, and that the projected zero-effort miss distance at a particular value of
time-to-go is $V_c T^2_{\text{go}} \dot{\lambda}$. From this it seems plausible to take $\dot{\lambda}$ as an error signal, which hopefully can be measured and nulled in order to effect an intercept.

Referring to Figure 1-2 and (1.8), it is evident that a positive LOS rate $\dot{\lambda}$ results if the missile velocity $\dot{Y}_m$ perpendicular to the original LOS is too small. This situation is remedied if the missile then develops an acceleration $\ddot{Y}_m$ perpendicular to the LOS until $\dot{\lambda}$ is zero. In proportional navigation (Reference M4), the corrective acceleration perpendicular to the LOS is $N' V_c \dot{\lambda}$, where $N'$ is the "effective navigation ratio" which is held constant.

In some missiles, two accelerometers have been mounted on the seeker dish so as to measure directly the accelerations perpendicular to the LOS, but this case will not be considered here. In this paper, it is assumed that the missile has two body-mounted accelerometers with perpendicular input axes, so as to measure the lateral acceleration $A_m$ perpendicular to the centerline. Referring to the two-dimensional case in Figure 1-4, the acceleration perpendicular to the present LOS is:

$$A_m \cos \theta_h \approx A_m \cos (-\theta_h + \lambda) = \ddot{Y}_m$$

(1.9)

where $\ddot{Y}_m$ is the acceleration perpendicular to the original LOS direction, which is usually close to the present LOS. Thus, the desired acceleration perpendicular to the LOS is:

$$A_c \cos \theta_h = N' V_c \dot{\lambda}$$

(1.10)

where $A_c$ is the lateral-acceleration command to the autopilot. Comprehension can be simplified by considering the head-on attack with $\theta_h = 0$. 
Figure 1-4 - Corrective Acceleration for Positive LOS Rate $\dot{\lambda}$
From (1.5) and (1.8) through (1.10), Figures 1-5 and 1-6 are shown as two alternate forms of the guidance kinematic loop, the former being more conventional. The mathematical differentiation in Figure 1-5 is avoided in Figure 1-6 by the use of $\dot{Y}_t$ and (1.8). Both figures are merely ways of depicting the equations and both are simplified to the extent that the lags in measuring $\dot{\lambda}$ and developing the acceleration command $A_c$ are not shown. Also, $\lambda$ is neglected in the relation between $A_m$ and $\dot{Y}_m$. It is apparent that the kinematic gain $\cos \theta_h$ between $A_m$ and $\dot{Y}_m$ is cancelled by the $1/\cos \theta_h$ in the closing-velocity multiplier. Moreover, the kinematic gain $1/V_c$ in the block(s) ahead of $\dot{\lambda}$ is cancelled by the $V_c$ in the closing-velocity multiplier. These two features may be likened to gain compensation which maintains constant loop gain in a servomechanism despite some external variation. Their effect is to make the response of the guidance kinematic loop more nearly uniform and optimum for all engagements.

The gain of the guidance kinematic loop in Figures 1-5 and 1-6 does vary inversely with $T_{go}$, which makes the analysis of this loop rather interesting. The presence of the $1/T_{go}$ gain does not cause unacceptable behavior of the loop if the lags are of certain tolerable types. It does render closed-form analytic solutions impossible in all but a few restricted cases, such as that of a single lag in the guidance system.

Appendix B of Reference S1 derives the trajectory for a missile with proportional navigation, zero autopilot lag and an initial heading error, assuming a constant-velocity target and a constant-speed missile. It is shown that if the effective navigation ratio $N'$ is less than 2, an infinite lateral missile acceleration is required at the end of the intercept, while a value of $N'$ greater than 2 causes the acceleration to approach zero at the end. Furthermore, it is shown that, in order to minimize the integral
Figure 1-5 - The Guidance Kinematic Loop
Figure 1-6 - Alternate Form of the Guidance Kinematic Loop
of the squared missile acceleration, the $N'$ must be 3, which is also the value derived from optimal control theory, as described in Subsection 2.3.2.

If the target has a lateral acceleration $Y_t$ perpendicular to the LOS, then Reference 51 shows that a value of $N'$ larger than 3 is desirable in order to reduce terminal acceleration. Moreover, the parasitic attitude loop (Subsection 2.2.2) may increase or decrease $N'$ appreciably. Values of $N'$ such as 5 or more lead to excessive miss distance due to noise. A typical engineering compromise for best overall performance is to make $N'$ equal to 4.

1.3 Survey of Literature of Modern Guidance Prior to This Date

This subsection surveys the known literature to, and even concurrent with, this thesis. The various contributions are described in rough chronological order, but only qualitatively, so that a coherent analytic presentation of the more important ideas can be made in Chapter 2.

1.3.1 Early and Supporting Contributions

A rather early use of optimization techniques was made in 1959 by Stewart and Smith (Reference 57), who applied Wiener filtering to a missile system, in which the noise was necessarily restricted to a stationary case, in contrast with the marked nonstationarity in real cases.

Kalman's most noteworthy contributions (References 18 through 57) in optimal estimation and control were made in the period from 1960 through 1964. In the same period, the "separation theorem" for partitioning stochastic control problems into subproblems of estimation and control was stated in discrete form by Joseph and Tou (Reference 65) in
1961 and Gunckel and Franklin (Reference G6) in 1963. A proof for the continuous case was given by Potter (Ref. P3) in 1964, while a somewhat more accessible statement without proof of the separation theorem is found in Reference B1 under the name "certainty equivalence principle."

An interesting contribution of limited scope was made by Kishi and Bettwy (Reference K8) in 1965, citing the aforementioned References K3, K6, J6 and G6. Their paper assumed a lag-free measurement of line-of-sight angular rate with additive white noise, which is not realistic, and admitted the need for more realism in modelling dynamics.

A study on "Optimum Control of Air-to-Surface Missiles" was performed by Leistikow, McCorkle, Rishel et al of the Boeing Company in the year beginning March 1965; the final report (Reference L5) was submitted to the Air Force in September 1966, although the date of the printed document is March 1967. A partial summary appeared in Reference R4, which concluded in part:

"While the linearized optimal guidance system did have better performance than pursuit or proportional guidance the amount of the improvement was small. For the optimal system practically all the miss distance standard deviations could be attributed to filtering error."

From this one might suspect that errors were made in the Kalman filtering, partly because experience at Raytheon showed that optimal techniques improved miss distance greatly.

1.3.2 Work of Bryson and Associates

Under the guidance of Prof. Bryson at Harvard, Johansen wrote a doctoral thesis in 1964 (Reference J7) which is at least summarized in
the more accessible Reference J8. Commenting on the complexity of Bryson's estimator-controller solution to the interceptor problem (Reference B1, B8), as described below, Johansen sought to reduce complexity by starting with a somewhat classical guidance configuration, resembling Figure 2-7 herein but without autopilot lags or radome effects. He then sought to compute optimal time-varying solutions for four gains, one of which appears on inspection to be redundant. Widnall in Reference W3 has commented on some shortcomings of Johansen's work. Although the utility of Johansen's work for actual missile guidance design seems somewhat doubtful, nevertheless it is possible that his thesis would be useful in the optimization of a more well-conceived configuration for some other problem.

Ho, Bryson and Baron published in 1965 a paper (Reference H4) on differential games as related to pursuit and evasion, in which they assumed that all states of the system were known to both sides. For a general case without outside disturbances, they developed the concept of optimal control being proportional to predicted miss without control, although the derivation is not completely clear. For pure interception (with no target evasion), their solution reduced to pure proportional navigation.

The earliest major contribution to the development of modern guidance techniques for homing interceptor missiles appears to have been made by Widnall and Bryson in 1966 when they designed an optimal continuous guidance system with stochastic noise and target maneuver (References B8 and B1). As in the case of References J7 and J8, they used an efficient coordinate system, with one axis parallel to the original LOS and through the target, so that the first state variable was the component \( Y_d \) of missile-target separation perpendicular to that axis. The other two
states were $\dot{Y}_d$ and target acceleration $A_t$, perpendicular to the LOS, a zero-lag autopilot being assumed. The separation theorem was used to partition the problem into the design of an optimal estimator followed by a pure-gain optimal controller. The seeker was assumed to measure LOS angle plus additive white noise, which is much more realistic than the assumption of Reference K8. Reference B8 gave no explicit solution for the control gains, while Reference B1 gave the gains for $Y_d$ and $\dot{Y}_d$, but not for $A_t$. A very similar example, but with sampled data, is described in Widnall's doctoral thesis of 1967 (Reference W3).

1.3.3 Work at Raytheon in 1967 and 1968

At Raytheon, the most significant initial contributions to the application of optimal control to missile guidance were made by W. O'Halloran, after hearing Bryson repeat the Minta Martin Lecture (Reference B8) in early 1967 and before the appearance of Reference B1. O'Halloran also benefited from some comments and suggestions by Dr. F. W. Nesline (consultant), K. Tiernan, Dr. B. Hall and possibly this writer. In Reference O'H1, O'Halloran gave a closed-form optimal guidance law for the three-state problem of Bryson, after having solved the control gains numerically from the Riccati equation and having applied some insight and suggestions from others to fit the numerical solutions with the algebraic guidance law. Subsequently, J. Speyer put this guidance law on a sounder theoretical basis by deriving it as a closed-form solution to a linear two-point boundary-value problem in the state and costate vectors (Reference S8). In all the Raytheon work, the plant was linear and the performance index was quadratic, with squared miss distance plus the weighted integral of squared control effort (missile acceleration or command to the autopilot, depending on the plant).
The work of O'Halloran and his associates at Raytheon in 1967 on modern guidance systems is summarized in Reference W4, which is classified SECRET only because numbers from a then current missile contract were used to show potential applicability. The miss-distance performance for the optimal guidance system (with Kalman estimation and control gains, assuming a zero-lag autopilot and zero radome in the estimator) was superior to that of a prior neoclassical guidance system.

Also in Reference W4, this writer's theory of a "noise-averaged best-fit straight line" approximation to the nonlinear radome curve was developed. This straight line is the one which minimizes the mean-square error in fitting the nonlinear radome curve in the region about the mean gimbalm angle, and it is obtained by integrating over the ensemble probability density of the gimbalm angle. At the time of Reference W4, the use of this linear model in adjoint simulations did not give statistical miss-distance results consistently close to those of repeated analog simulations with the basic nonlinear radome curve. In related work, an attempt was made by the writer and R. Warren to estimate during a simulated intercept an unknown radome slope for a straight-line radome characteristic, but this was only partially successful, because it was not conceived quite properly.

The importance of autopilot lag was also realized. Unpublished work in 1968 by this author on optimizing speed of response of the Raytheon autopilot (Chapter 2, herein) led to realization of the importance of the airframe RHP zero and the use of a one-zero, one-pole autopilot model for modelling in the guidance system.

In 1968 O'Halloran attempted to develop an autopilot for the airframe, actuator servo and angle bias, using optimal control theory. It was assumed that a control integrator, as in the classical autopilot, Figure 2-2
herein, would be necessary because of the bias, but the optimal control solution always developed zero gains back to the integrator. After consideration of the deleterious effect of a small angle bias on missile performance at low altitude, the effort was abandoned. A different, successful approach is shown in Chapter 5 herein.

In a somewhat different vein, O'Halloran and McLaughlin were stimulated into an investigation of error bounds on the estimation of parameters, after the only partly successful attempts (Reference W4) of the writer and R. Warren to estimate radome slope in 1967. Their memo (Reference O'44) applied the Rao-Cramer bound (Reference V2) to the estimation of slope and intercept for a straight-line radome characteristic, as an isolated problem without the remainder of the guidance system and geometric state variables. Their verbal conclusion, which appears to have been accepted within Raytheon, was that there were not sufficient measurements in a typical intercept for adequate radome estimation. Consequently, interest in this approach languished.

Shortly before leaving Raytheon, O'Halloran wrote Reference O'H2, in which he utilized techniques similar to those of Reference S8 to solve for the optimum guidance law in the case of a missile with constant axial acceleration (contributing to the LOS angle because of the lead angle), a zero-lag autopilot, and a target with constant acceleration perpendicular to the LOS. He also gave a solution for the time of flight which depended on the component of missile axial acceleration along the LOS, as well as present range and closing velocity.

O'Halloran's final contribution to optimal guidance in Raytheon was Reference O'H3, in which he analyzed the optimal control gains for a four-
state model of the guidance system, in which the states were \( Y_d, \dot{Y}_d, A_t \) (the output of a low-pass filter) and autopilot acceleration \( A \) from a low-pass filter. The control variable was \( A_c \), the command to the autopilot. The control gain from \( A \) was seen to provide lead-lag compensation so as to speed up the autopilot response. This memo also presented a discrete form of an autopilot model for the Kalman estimator, beginning with an autopilot transfer function \((1 + s/\omega_2)/(1 + s/\omega_1)\).

1.3.4 Contributions by Others in 1967-1970

Reference T3, published by Teng and Phipps in May, 1967, is included in the list of references only for completeness.

The somewhat later paper (Reference N1) of W.L. Nelson of the Bell Telephone Laboratories was evidently concerned with command-guidance interception of a ballistic missile. As a first step, Nelson suggested "the determination of the nominal trajectory and the control necessary for keeping the missile on that trajectory." As an example, the nominal trajectory could be that which minimizes time to intercept. An analysis of the problem of tracking the ballistic target from the ground and estimating its state was given. Missile guidance to follow the nominal trajectory was achieved by estimating the missile state (perturbations from the nominal), by utilizing time-varying optimal control gains to develop the "steering control," and apparently by applying this latter as a command to a pitch-rate autopilot of negligible lag. Under "missile intercept strategy," Nelson proposed computing the "predicted (vector) miss distance" from the target and missile estimators. "The predicted miss distance is then resolved in two components, one which is parallel to the relative velocity vector and the other perpendicular to it. --The component in
the parallel direction is used to correct the time of intercept. The per-
pendicular component is used to develop the desired steering order." 
Account was also taken of the induced-drag deceleration and its effect on 
intercept time. An analysis of this guidance with predicted miss distance 
was not given, but overall, the paper is evidently quite worthwhile.

Also in 1967 there was published a report (Reference G7) by Garber, 
Flory and Dickson at U.S. Army Missile Command, Redstone Arsenal, 
Alabama. This considered an intercept problem with three-dimensional 
target-missile relative geometry, a zero-lag autopilot and a disturbance 
vector such as target acceleration. The analysis utilized the Riccati 
equation and an auxiliary equation including the disturbance. For a par-
ticular example with target acceleration and its constant rate of change, 
the authors noticed that the optimum missile acceleration was proportional 
"to the predicted miss distance at time $t_f$, if we were not to use any con-
trol." Apparently the utilization of the Riccati equation prevented the 
authors from generalizing this result beyond their specific example.

Reference G8 is a similar paper by Garber in which he speculated that the 
optimal control would be proportional to the projected miss distance "in 
general for more complex interceptor and target models."

Reference W5 by Willke was rather specialized, as the title would 
indicate, but it apparently has utility. The interceptor missile was con-
trolled to follow a trajectory which was computed rapidly from a series 
expansion of the true equations of motion.

Cunningham's paper (Reference C5) in early 1968 was a particular 
application of a technique of Bryson and Denham to the problem of con-
trolling a missile to reach a "desired final velocity and altitude in a fixed 
time in spite of disturbances." The lambda matrix was used to relate
the change in control to the deviation in state from the nominal, so that a precomputed trajectory would be flown.

Reference P12, published by Price in June 1968, proposed a closed-loop scheme for guiding a missile along a nominal optimum trajectory by continually recomputing the intercept time \( t_f \) and system control parameters \( \alpha \) to hold at zero the corresponding derivatives of cost. More clarity would have been desirable.

Reference B9 was a rather specialized paper which might be useful in an intercept problem in which propellant consumption is to be minimized.

In Reference W6, Willems initially sought to show that Speyer's solution (Reference S8) of the three-state intercept problem as a two-point boundary-value problem could also be solved by Ogata's method (Reference 01) using the Riccati equation. From an analytic viewpoint, this is not remarkable, because the two approaches are known (Reference A4, p. 756-761) to be equivalent methods of solving an optimal control problem with a quadratic performance index for a linear system. Of more interest was Willem's solution for the four-state optimal control problem solved earlier by O'Halloran in the unpublished Reference O'H3. The equivalence of the two results by different methods is encouraging evidence that algebraic errors were not made.

Willems continued this work in 1969 with Reference W7, which derived optimal control gains for a missile with two simple time lags. This work would be of more potential interest for a wing-controlled missile with only LHP zeroes than for a tail-controlled missile with one RHP zero, which is important near intercept.

Reference D3, published by Dickson and Garber in early 1969, represented a continuation and slight extension of References G7 and G8
for the case of a zero-lag autopilot.

In Reference A5, Axelband and Hardy utilized the results of Friedland's quasi-optimum feedback technique (Reference F3) to develop a guidance law for a missile with a large initial heading error. It was assumed that the seeker measures the LOS angular rate with no lag, and that the missile autopilot had zero lag. Noise or the need for estimation was not considered. A nonlinear feedback guidance law was developed, which reduced to a linear guidance law for small initial heading errors. Although the paper's results were not completely conclusive, the technique might warrant further investigation. Reference A6 was about the same work and it added very little.

In connection with research at the M.I.T. Lincoln Laboratory, Athans published Reference A7 in January 1970. He gave the following summary of his proposed method:

"a) At the initial time \( t_0 \) and on the basis of the available information one computes an open-loop optimal (nominal) interceptor control and interceptor trajectory which minimizes the generalized miss distance at time \( T \) and computes, if desired, the optimal value (nominal) of the intercept/rendezvous time \( T \).

b) As time goes on one uses the refined estimates of the target state vector and the nominal precomputed open-loop interceptor control and trajectory to deduce in a feedback form the appropriate corrections to the normal interceptor control vector so as to minimize the generalized miss distance at the nominal intercept/rendezvous time \( T \)."
Although Athans' contribution was scholarly and potentially useful as a point of departure, he did not give a worked-out example and his method appeared to have formidable computational requirements.

1.3.5 Work at Raytheon from 1969 to Present

In early 1969, Speyer (at this point a consultant to Raytheon) published an internal memo on discrete optimal control for a two-state system $(Y_d$ and $\dot{Y}_d$) for a quadratic performance index with weighted squared miss distance and summed squared control effort (missile acceleration). This derivation proved to be algebraically more difficult than that of Reference S8 and an error was detected by this writer in the fall of 1969. Subsequently Speyer published a corrected internal memo (Reference S10), in which he found that the optimal discrete navigation ratio $N'$ was somewhat lower than the corresponding optimal continuous $N'$, particularly just before intercept.

In the fall of 1969, P. Zarchan of this writer's department at Raytheon in Bedford began to derive the optimal control law for a system with four states i.e., $Y_d$, $\dot{Y}_d$, $A_T$ and $A$ (missile acceleration), in which an autopilot with an LHP pole and an RHP zero was used. This was stimulated in part by the realization of the importance of the RHP zero and by the earlier work of O'Halloran in Reference O'H3, which did not consider the RHP zero in the guidance-law formulation. Shortly afterward, this writer began independent work on the same problem. On October 15, Zarchan published a solution (Reference Z1) of this problem, but it contained an algebraic error which resulted in unsatisfactory miss distances during simulations. This writer's solution (Reference S9) appeared first on December 9, 1969 and was expanded on December 22. In between these dates, on December 16, Zarchan published Reference Z2, in which he
corrected his earlier mistake and noted that his new solution then agreed with that of Reference S9. Inasmuch as his work was done independently by a different method, the equivalence of the two results gives considerable credence to their accuracy. (It seems necessary to give these dates of publication in detail because the work in Reference S9 is offered in Chapter 3 herein as an original contribution.) An interesting technical result was that the effective navigation ratio $N'$ peaked and then reversed sign near intercept because of the RHP autopilot zero corresponding to the "wrong-way" tail force of the missile.

In References Z3 and Z4, Zarchan compared the miss distance and related performance for four guidance laws:

1) Augmented proportional navigation (APN), i.e., proportional navigation with a constant $N' = 3.5$ and a term proportional to target acceleration (perpendicular to the LOS), which was assumed to be exponentially decaying.

2) "Three-State Law" (3SL), which resembled APN except for the time-varying $N'$ developed by O'Halloran (Reference O'H1) and Speyer (Reference S8) for a zero-lag autopilot.

3) "Four-State Law" (4SL), which was O'Halloran's optimal guidance law (Reference O'H3) for an autopilot with one pole.

4) "Improved Four-State Law" (I4SL), which was the guidance law in References S9 and Z2 for the autopilot with one pole and one (typically RHP) zero.

Zarchan's evaluation tool was the Raytheon discrete adjoint program (Reference R3) for a sampled-data guidance system. Since the foregoing guidance laws were for a continuous system and were therefore
suboptimal in the discrete case, the weighting $\gamma = b$ on the integral of squared control effort was adjusted empirically for minimum miss. This weighting factor had the effect of reducing the effective navigation ratio near intercept. The explanation for the minimization of miss distance with optimum $b$ was that the discrete optimum guidance law with negligible $b$ probably had lower values of $N'$ near intercept than did the I4SL continuous guidance law, judging from Speyer's finding for the simpler case in Reference S10. The major point of Reference Z4 was that the most sophisticated law, I4SL, gave the best miss distance for an autopilot with one zero and one pole, and that it required lower missile acceleration (on an ensemble rms basis) than did APN.

Most of the literature surveyed and work reported in this thesis is for terminal guidance of a missile, i.e., guidance to intercept in the last few seconds of flight. On the other hand, the flight of a long-range interceptor missile could be divided into at least three phases: 1) launch, 2) midcourse and 3) terminal. The midcourse phase can be relatively long and its guidance might be rather different from that of the terminal phase, e.g., command guidance to follow a nominal trajectory. A typical requirement for this phase might be delivery of the missile to a certain position and heading relative to the target, so that terminal homing could proceed efficiently. Considering drag and gravitational effects, the optimum midcourse trajectory might temporarily go above the delivery (end of midcourse) altitude.

In connection with a study of the midcourse problem, Dr. R.J. Fitzgerald of Raytheon published Reference F4 in December 1969, in which he included gravity, drag and saturation, e.g., of normal force and angle of attack. Fitzgerald concluded in part:
"Although the (two-point boundary value) problems discussed here are difficult to solve when target trajectory and missile initial conditions are given, it is a relatively simple matter to generate optimum missile paths backwards in time from the intercept point. Then, after the solution has been generated, we find out what problem we have solved. Although not suitable as a feedback control technique, this approach is quite reasonable for parametric studies and for examination of the optimal trajectories."

Fitzgerald used a much different approach to midcourse guidance in Ref. F5.

Three memoranda (References H5 - H7) by Dr. B. Hall of Raytheon in early 1970 extended this writer's concept (Reference W4) of a "noise-averaged best-fit straight-line" approximation to the radome curve. Hall added the concept of considering the mean-square approximation error as additional range-independent noise and worked out specific examples. Using this equivalent linear radome slope and noise source in the adjoint program (Reference R3), T. Murray of Raytheon found fairly good agreement with repeated nonlinear simulations in terms of miss distance. This finding increased the utility of adjoint techniques.

The problem of knowing what process noise Q and measurement noise R to use in a realistic tactical situation was recognized early in the work at Raytheon, as stated in Reference S1, which suggested the possibility of using a "floating-window" calculation of the measurement-noise covariance from the n (e.g., about 10) most recent samples. This did not work particularly well, perhaps because it was not thoroughly analyzed. A more refined later approach (Reference K10) by Dr. I. Kliger and P. Zarchan of Raytheon was apparently more successful.
Kliger has very recently written two memos (Reference K11 and K12) in which he applied a little-known method of the Russian author Krasovskii (Reference K13) to optimal guidance. This method avoids adjoining the plant equation to the performance index by a costate vector or Lagrange multiplier vector. It appears to be limited to the case of a quadratic performance index and a linear system, which is the class of problem generally treated at Raytheon in the foregoing references. A key step in Reference K11 was the use of the integral form of the Schwartz inequality (Reference H2, p. 116) to show that the control variable should be proportional to the product of two variables: 1) the response of the weighted output state (differential position) for a control impulse, and 2) the projected zero-effort miss. Kliger claims that the method gives more insight than the calculus of variations and that its algebra is simpler. This writer would question the former assertion. The latter assertion may be true, but only simple examples were fully worked out in Reference K11; the most complicated example was approximated.

1.4 Problems Remaining Prior to this Thesis

1.4.1 Optimal or Near-Optimal Autopilot Design

The classical Raytheon autopilot (Subsection 2.1.2) is simple and easy to mechanize with fixed gains in each band, and prior to this thesis it has been retained in the design of modern guidance systems. On the other hand, there is no reason to suppose that it is optimum in any sense and indeed, its transfer function does not fit well the guidance laws of admissible tactical complexity (Subsection 1.3.5). There is need for an integrated autopilot design which will both meet practical criteria (e.g., fixed gains, simplicity, etc.) for an autopilot and will be compatible with admissible guidance laws so as to minimize miss distance.
This thesis develops a new type of autopilot with improved performance.

1.4.2 Optimum Guidance Law

An optimum guidance law of tolerable computational complexity is required. It should take account reasonably well of autopilot response and should use feedback from estimated current states, i.e., it should be "closed loop." Also, it should be optimum for the discrete computation which is performed in a digital computer on sampled radar data.

This thesis develops a new continuous guidance law in closed form and an algorithm for the discrete counterpart of this and other guidance laws.

1.4.3 Radome Compensation

Perhaps the worst guidance problem at high altitudes is the stochastic nonlinear feedback through the radome, which always increases miss distance in a modern guidance system; see also Subsection 2.2.2. Appr
ciable progress has been made on the problem for both linear (idealized) and nonlinear radome characteristics during this thesis work. Ironically, the limited academic appeal of this work and its potentially much greater appeal to the USSR make it inadvisable to describe much of it here.

1.4.4 Target Modelling

Reference S1 has discussed this problem adequately, and it still remains. In many cases, the single-lag model of evasive target acceleration perpendicular to the initial LOS is adequate if conservative values of \( \beta \) and \( \nu \) (Subsection 2.3.1.1) are used. One possible improvement would be to add an adaptive feature to change \( \beta \) depending on the current estimate of target acceleration. Another possible approach would be to predict future worst-case maneuvers and to be ready for them.
If future targets become able to track the interceptor missile, then a game-theoretic approach will be necessary on both sides.

1.4.5 Obtaining Measurement-Noise Statistics

The problem of obtaining an adequate model of the measurement-noise ensemble covariance has been recognized for some time (Reference S1). As Subsection 1.3.5 indicates, progress has been made currently at Raytheon on this problem (Reference K10).

1.5 The Chapters of This Thesis (Synthesis)

Although this thesis is intended to have fairly general applicability as indicated in Subsection 1.1, the specific technical problems are concerned with the application of modern estimation and control theory to the design of a missile guidance system in the terminal phase of flight, i.e., the last few (up to 15) seconds before intercept. The design of a prior midcourse phase is not considered.

Appreciable on-line computations are required for guidance, in order to utilize Kalman filtering and optimal control. On the other hand, the approach to autopilot design in Chapter 4 would utilize fixed gains and analog circuitry in each of several autopilot bands on the Mach-altitude plane, e.g., as in Figure 2-3. It is conceivable that a radio link could connect the missile with a guidance computer in the launching site (or launching aircraft, as the case may be), which would also contain the radar transmitter that illuminates and tracks the target. This arrangement would tend to minimize the complexity in the expendable missile.

1.5.1 Terminal Control and Guidance Laws for General and Specific Cases

Chapter 3 analyzes continuous terminal control (with a quadratic performance index) of a linear system with a known disturbance, and shows
that the optimal control is proportional to the projected, zero-control, weighted terminal state; the weighting is that of the performance index for the terminal state. For a guidance system in which only the miss distance is weighted at the terminal time, this result means that the optimal control is proportional to the projected, zero-control, miss distance with the effect of the known disturbance included. This is the most general proof of this principle known to the author, and it comes after years of conjecture and specialized examples by various workers.

Similar results are shown to apply to the discrete case. In contrast to optimal estimation (which is the dual problem), in optimal control it seems easier to this writer to find results first for the continuous case and then for the discrete case.

The foregoing general results are suitable for a plant which includes either the bare airframe, or alternatively an airframe with an autopilot. A useful result is obtained by applying parameter optimization to an autopilot transfer function with a cubic denominator, so as to minimize the integral of squared error in response to a step acceleration. The "fastest" autopilot in this sense has the transfer function \((1 - s/\omega_z)/(1 + s/\omega_z)\), and is limited by the RHP (right-half-plane) zero \(\omega_z\) of its airframe transfer function. This leads one to specify the autopilot transfer function to have one RHP zero over a dominant LHP pole of somewhat smaller magnitude.

Accordingly, a continuous guidance law for the foregoing autopilot transfer function is derived in closed form. The RHP zero causes the effective navigation ratio to reverse polarity shortly before intercept, so that the "wrong-way" tail force will decrease miss distance. An algorithm for computing the corresponding discrete \(N'\) is shown, for use in the last few sample periods of the intercept.
An algorithm for the time of flight is also shown.

1.5.2 Design of Autopilot as a Simplified Estimator Controller

In Chapter 6 it became evident that simultaneous guidance and autopilot design led to time-varying control gains for the airframe states as well as the kinematic states \( (Y_d, \dot{Y}_d, A_t) \), and so this technique was not attractive. Accordingly, it seemed better to design a suitable fixed-gain autopilot first, and then to use time-varying guidance gains. An autopilot transfer function, which can be approximated by one pole and one zero, is attractive for two reasons: 1) A closed-form guidance law of tolerable computational complexity (Chapter 3) may be used for it; 2) such an autopilot can approach the theoretically "fastest" type (limited by the RHP zero of the airframe), which should help to reduce the miss distance due to the inevitable mismatch of autopilot modelling, particularly as flight conditions change.

The autopilot design was chosen to be an estimator-controller, allowing for process noise (atmospheric turbulence) and measurement noise, but with certain innovations. Only the airframe poles were to be moved to specified locations, while the fin-servo poles were to be left unchanged, so that no modal feedback would be required from their modes. The inevitable tail-angle bias was cancelled by feedback from its estimate. The estimator-controller was expressed originally in plant modal space and it was found that the fin-servo modes were loosely coupled to the others. This simplified the reduction process, in which these controller modes were replaced essentially by certain gains which were equivalent at low frequency. The resulting simplified estimator-controller had the following features: 1) The specified transfer function, within close tolerances; 2) three states versus five (the order of the plant) for the conven-
tional estimator-controller; 3) cancellation of the bias, for the design point and all airframe changes with Mach number and altitude; 4) tolerable variations of its transfer function with changes in flight condition from the design point (Mach 2, S. L.); 5) near-optimal filtering of the noise.

Although this chapter treats a specific problem, nevertheless much of its techniques, concepts (both those directly applicable and those peripheral to the main problem) and literature should prove to be helpful in the design of simplified stochastic estimator-controllers. Perhaps this chapter will help somewhat to bridge the gap between classical and modern control design.

1.5.3 Influence of Autopilot Response, Guidance Law and Radome on Miss Distance

The penultimate criterion of a guidance system and autopilot is the ensemble rms miss distance, which is closely related to lethality. Chapter 5 compares the miss distance of the new autopilot and a classical design at the design point (Mach 2, sea level). The use of the discrete optimum guidance law with the new autopilot is found to reduce the miss distance obtained with the continuous guidance law. The influence of linear radome slope on miss distance is also examined; in all cases, an adjoint program (Reference R3) was used for computing miss distance because of its efficiency.

1.5.4 Compensation of Continuous Optimal Control Gains for High-Order Systems

Chapter 6 turns out to have only indirect utility in missile-guidance work. It was originally motivated by the desire to integrate guidance design and autopilot design so that both would be optimized simultaneously for a quadratic performance index. Although such a technique was not used in the final design, nevertheless this chapter should be of interest
for its comparison of computational techniques and its observations and analysis of the behavior of modal feedback gains.

A repeated observation throughout this thesis work has been that much insight and practical advantage are possible through the use of modal techniques, i.e., those related to the eigenvalues and eigenvectors of the plant matrix $F$.

It was found advisable to explore computational techniques for the combined airframe-guidance problem by starting with a simple model with a known closed-form solution for checking purposes, and then to expand the modelling until the full eight-state plant could be handled with confidence. Three major techniques were explored with varying success: 1) the transition-matrix method in state-costate space, 2) the method of Vaughn (Reference VI), which was expressly designed for accuracy near the steady-state, and 3) an improved transition-matrix method rediscovered by the author, but originally published in a somewhat obscure report (Reference K4) by Kalman et al.

Typically, optimal control gains were computed for the states of the system, but added insight was obtained by examining the behavior of the corresponding gains from the plant modes. In the steady-state (at a long time to go), feedback from only the unstable modes are finite, the gains from the stable modes being zero. An analysis for the case of one unstable mode is given.

1.5.5 Optimal Control of Selected Modes

Like the previous chapter, Chapter 7 is not of direct utility to the missile-guidance problem. Nevertheless, it has a novel idea which gives some insight and might be the basis for future research.

To illustrate the idea concretely, consider the problem of optimal
control of the airframe and guidance states, as before, with a general quadratic performance index. Using the plant eigenvalues and eigenvectors, the performance index may be converted into one in which the modes, rather than the states, of the plant are represented. The weightings on certain modes, e.g., the high-frequency actuator modes, may then be set equal to zero. Examination of the Riccati equation in modal space shows that the optimal control gains from these zero-weighted modes are zero throughout the problem, while the other modes have time-varying control gains. These gains tend to differ from the conventional solutions mainly at short times to go. Whether the new modal solution is actually useful (with a desirable performance index), or is easier to mechanize than the conventional solution, must be determined for the particular application.
CHAPTER 2

SUMMARY OF CURRENT PRACTICE IN GUIDANCE SYSTEM DESIGN

This chapter summarizes how the mechanization of a classical guidance system follows from the proportional navigation law, and what some of the major design problems are. Much of this mechanization and these general problems carry over into the application of modern control to guidance, which is also summarized. Modern guidance has more sophisticated and extensive computation.

2.1 Mechanization of Classical Guidance System

The proportional navigation law in (1.10) helps to show the reasons for the mechanization of a typical guidance system. The seeker and receiver subsystem are required to measure the LOS angular rate $\dot{\lambda}$ and the closing velocity $V_c$. Note that the LOS angle $\lambda$ and the relative range $R_{mt}$ need not be measured in a classical guidance system with proportional navigation, i.e., the relative vector position of the target need not be measured. With reference again to (1.10), the closing-velocity multiplier also utilizes the seeker gimbal angle $\theta_h$ in order to develop the lateral-acceleration command $A_c$ to the autopilot. Clearly, the basic function of the pitch and yaw autopilots is to control the lateral acceleration $A_m$ so that an intercept can be effected.

The foregoing discussion is obviously only elementary. The following subsections discuss the functions of some of the various portions of the missile, from the viewpoint of the guidance engineer, with reference to the schematic cross-section of an imaginary homing guided missile in Figure 2-1.
Figure 2-1 - Schematic of a Hypothetical Homing Guided Missile
Proportional navigation in two dimensions is considered from a different point of view in Reference A3, while References M4 and A2 analyze the problem in three dimensions.

2.1.1 Airframe

The airframe is usually symmetrical, with four fixed wings and four movable control surfaces. A cruciform configuration rather than an airplane configuration (Reference P10) permits lateral maneuvering in any direction without first rolling.

Consideration in this thesis will be restricted to the tail-controlled configuration, such as HAWK or Falcon, which have no downwash interference from the control surfaces as do the wing-controlled missiles such as Sparrow. Current trends seem to favor the former type, because of their lesser problems with induced roll moments. If the autopilot pitch and yaw axes are each 45 degrees from the planes of adjacent control surfaces, then all four control surfaces are deflected equally by the pitch (or yaw) autopilot. In some applications it is preferable to put the autopilot axes in the planes of the control surfaces, and so only two surfaces are deflected by the pitch autopilot and two by the yaw autopilot.

It is apparent that the tail-controlled airframe has a tail normal force opposite to the direction of the desired steady-state maneuver acceleration, thus causing a small initial airframe acceleration in the wrong direction. Analytically, this effect contributes a right-half-plane zero in the transfer function from control-surface deflection $\delta$ to a lateral acceleration $A_m$ at the missile c.g., thus tending to limit the speed of response of the guidance system. This "wrong-way" force is not present in a wing-controlled missile with movable wings slightly forward of the c.g.

In an aerodynamically maneuvering missile such as that of Figure 2-1,
the function of control surfaces is to exert a moment so that the missile can develop an angle of attack and thereby achieve lift from the body and wings (if any). As explained further in Subsection 2.2.2, the pitching of the airframe, together with radome distortion and imperfect seeker stabilization, causes the seeker to develop a spurious component of the measured LOS angular rate \( \dot{\lambda} \), which results in a "parasitic attitude loop" that interferes with guidance stability and may worsen miss distance. An important measure of the necessary pitching of the airframe is the "alpha over gamma dot" time constant of the linearized airframe response, defined (Appendix A) as:

\[
a_{31} = \frac{\Delta \alpha}{\Delta \dot{\gamma}} = \frac{2m}{\rho V_m S} \left[ \frac{\partial C_m}{\partial \delta} \left( \frac{\partial C_m}{\partial \alpha} - \frac{\partial C_L}{\partial \delta} \frac{\partial C_m}{\partial \alpha} \right) \right]
\]  

(2.1)

where \( M \) is mass, \( V_m \) is missile speed, \( S \) is the reference area, and the derivatives are conventionally defined; it is assumed that thrust and drag are negligible. At high altitudes, it is necessary for the airframe to have a relatively large angle of attack for a given lateral acceleration \( A_1 \) at the missile c.g., thus tending to limit the speed of response of the guidance system.

The airframe in this thesis is an obsolete design which was never built. This choice and that of the guidance parameters was made to avoid problems of security classification.

2.1.2 Autopilots

In the type of tail-controlled missile under investigation here, the roll autopilot sends commands to all four fin servos so as to hold the roll attitude of the missile nearly constant for two or more major reasons:

1) Because of the lags in the guidance system, rolling at moderate or high
frequencies may cause a lateral corrective acceleration to occur out of the proper plane and thereby increase miss distance; 2) severe continuous rolling may cause loss of tracking or aerodynamic control. Further discussion of roll autopilots can be found in References S1, S2 and S3.

The function of the pitch and yaw autopilots is to control the lateral acceleration of the missile in accordance with commands computed from the proportional navigation law (1.10). Accordingly, each autopilot must have feedback from an accelerometer. Additionally, one or usually two inner loops with feedback from a rate gyro are required for compensating the poles of the airframe response. The accelerometer and the rate gyro are typically spring-restrained instruments with moderately high natural frequencies.

The block diagram of one possible autopilot design is shown in Figure 2-2. The innermost rate-damping loop has a wide bandwidth for damping the poles of the airframe, while the synthetic stability loop stabilizes the poles of an unstable bare airframe (or improves the pole location of a stable bare airframe also). The analysis in this thesis is for a rigid-body model only, body-bending effects being neglected. Assuming constant speed $V_m$, the airframe has a linearized transfer function from fin angle $\delta$ to transformed pitch rate as $s\theta$ as follows (Appendix A):

$$G_3 = \frac{s\theta}{\delta} = \frac{K_3 (1 + a_{31}s)}{1 + b_{11}s + b_{12}s^2}$$

$$\approx \frac{M_{\delta} (s + 1/a_{31})}{s^2 + \frac{b_{11}}{b_{12}} s - M_{\alpha}}, \quad (2.2)$$

where the maneuvering time constant $a_{31}$ has been defined in (2.1).
Figure 2.2 - Block Diagram of Pitch (Yaw) Autopilot

ACCELERATION COMMAND

\[ A_c^+ \]

\[ K_{13} \]

INTEGRATOR

\[ \frac{K_{11}}{T_{11}S} \]

SYN. STAB. LOOP

\[ K_9 \]

\[ \frac{K_8}{1+T_8S} \]

FIN SERVOS

\[ \delta_c \]

FIN ANGLE

\[ \delta \]

RATE-DAMPING LOOP

\[ G_{12} \]

AIRFRAME

G_1

ACCELERATION AT C.G.

\[ A_m \]

ACCELEROMETER LOOP

\[ G_7 \approx 1 \]

\[ A_{1a} \]

ACCELEROMETER

\[ G_1 \approx G_2 \]

\[ G_2 \]

\[ G_3 \]

\[ G_6 \]

RATE GYRO

\[ S\theta \]

AIRFRAME

\[ G_3 \]

\[ G_2 \]

\[ G_1 \approx G_2 \]
The pitch-moment effectiveness $M_\delta$ and body-stability parameter $M_\alpha$ are:

$$M_\delta = \frac{QSD}{I_{yy}} \frac{\partial C_m}{\partial \delta} \quad (2.3)$$

$$M_\alpha = \frac{QSD}{I_{yy}} \frac{\partial C_m}{\partial \alpha} \quad (2.4)$$

while the damping coefficient $b_{11}$ is negligible. Let it be assumed for the moment that the bare airframe is incrementally stable with $M_\alpha$ negative. In the vicinity of the gain-crossover frequency $\omega_c$ (for unity loop gain) of the rate-damping loop, $G_3$ is approximated by its high-frequency asymptote, and the other blocks are approximated by their d-c gains, and so:

$$\omega_c = \left| \frac{M_\delta K_5 K_8 K_{12}}{M_\alpha} \right| \quad (2.5)$$

It is desirable to keep $\omega_c$ between a low value (with insufficient resultant damping) and too high a value at which the loop phase lags would cause instability. Hence, the gain $K_{12}$ (or another gain between $G_{12}$ and the innermost summing point in Figure 2-2) is band-switched between two or more values to compensate for the change in airframe response with Mach number and altitude. In some air-to-air applications the simplest parameter for band-switching is launch altitude. On the other hand, consideration of (2.5) indicates that it is better to band-switch the autopilot gain when the value of $M_\delta$ crosses a suitable boundary value. Figure 2-3 shows typical contours of constant $M_\delta$ for band-switching on the plane of altitude versus Mach number. In the case of an adaptive autopilot, References S2 and S3 indicate that the roll autopilot can
Figure 2-3 - Curves of Constant $M_0$ for Band-Switching Autopilot Gains in a Hypothetical Missile
adaptively keep constant the product of its electronic gain and the airframe roll-moment effectiveness, and that the same gain adjustment might then be used to hold approximately constant the $\omega_c$ of the pitch autopilot.

The synthetic stability loop in Figure 2-2 effectively feeds incremental pitch angle back to the fin servos and thereby moves the autopilot closed-loop poles, corresponding to the bare-airframe poles of (2.2), further from the origin of the complex plane. In general, it is well to keep the integral break frequency:

$$\omega_i = \frac{K_9 K_{11}}{K_8 T_{11}}$$  \hspace{1cm} (2.6)

below $0.6 \omega_c$. Although the foregoing discussion has been for an incrementally stable bare airframe (with $M_\alpha$ negative), the designer must often cope with the unstable condition in which $M_\alpha$ is positive, particularly at high Mach number, low altitude and the corresponding low angles of attack. In this case, the moment due to $\partial C_m/\partial \alpha$ is like that of a decentering spring, and so the centering-spring feedback of the synthetic stability loop aids in stabilizing the unstable bare airframe. Application of the Routh criterion has led to analytic limits on the positive value of $M_\alpha$. As a rule of thumb, the approximate limit for the tolerable $M_\alpha$ is:

$$\text{T tolerable } M_\alpha \approx \frac{1}{2} \frac{1}{\omega_i} \omega_c$$  \hspace{1cm} (2.7)

Both $\omega_c$ and $\omega_i$ are limited by the high-frequency lags, particularly in the actuator, which shows the need for fast actuator response.

In Figure 2-2, the transfer functions $G_1$ and $G_2$ are response from $\delta$ to the accelerations at the c.g. and the accelerometer location. For $G_1$: 
\[ G_1 = \frac{A_m}{\delta} = \frac{K_1(1 + a_{11}s + a_{12}s^2)}{1 + b_{11}s + b_{12}s^2} \] (2.8)

\[ K_1 = V_m K_3 \] (2.9)

The two zeroes in the transfer function are due to the tail-force derivative \( \partial C_n/\partial \delta \). As previously explained, at the beginning of a maneuver, a tail-controlled missile has an initial tail force and body acceleration opposite to the desired direction of acceleration. Analytically, this is manifested by one zero of (2.8) in the right-half-plane of \( s \). The transfer function \( G_2 \) is similar to \( G_1 \), except that the zeroes are altered by the body pitch-acceleration response, owing to the location of the accelerometers, usually forward of the c.g.

The outer accelerometer loop in Figure 2-2 has the lowest bandwidth of the three loops. A brief analysis of this classical autopilot is given in Appendix C.

It is assumed that each positional fin servo in this thesis is a hydraulic valve-controlled unit (Reference B6), because this type meets the general requirements in Reference S1 for the type of airframe, autopilot and system under consideration. Other types in previous use are summarized in Reference S1.

2.1.3 Seeker

The following discussion applies either to a passive infrared-seeking system or to one utilizing radar. An example of the latter is a semi-active radar system in which the target is illuminated by a radar transmitter in the launching site or aircraft and the missile seeker receives and tracks the reflected radiation. From the analytical viewpoint of the
guidance specialist, let the seeker be defined as the following:

1) The antenna for collecting the energy
2) The electronic receiver aboard the missile
3) The seeker gimballing and servo components
4) The angular-distortion effects of the IR dome or radome, to be discussed later

The functions of the seeker may be stated conveniently as follows:

1) It must track the target continuously after acquisition.
2) It must measure the LOS angular rate \( \dot{\lambda} \).
3) It must stabilize the seeker against a missile pitching rate \( \dot{\theta}_m \) (and yawing rate) that may be much larger than the LOS rate \( \dot{\lambda} \) to be measured.
4) It should measure the closing velocity \( V_c \), which is possible with some radars but difficult with infrared.

Consideration of the proportional-navigation law in three dimensions (References M4 and A2) shows that it is necessary to measure the LOS angular rate \( \dot{\lambda} \) in two seeker-instrument axes that are orthogonal to the seeker boresight axis (which is virtually coincident with the LOS to the target). Space-stabilization about these two instrument axes is necessary although a slow roll rate about the LOS itself is tolerable.

Reference S1 briefly describes some of the many possibilities for seeker gimballing and drives. The seeker configuration most relevant to this thesis has pitch and yaw axes of rotation, each with a servomotor and a gyro for stabilization and measurement of LOS rate \( \dot{\lambda} \). Some engineers favor d-c torque-motors because of their potential reliability, despite the low angular acceleration of a typical direct drive. Sometimes
it is necessary to have a small gear ratio (e.g., 5 to 1) for the sake of adequate angular acceleration with a practical choice of motor frame size, particularly in the outer axis. It is desirable for the seeker to be stabilized about each of two gyro axes, which are perpendicular to the seeker centerline. One gyro axis can be inertially stabilized if it lines up with a direct torquer drive on the inner axis. The other gyro axis will not line up with the outer-drive axis if the inner-gimbal angle is non-zero.

Another drive possibility is a valve-controlled hydraulic servomotor, which is a velocity source with no inertial stabilization; gyro feedback allows the drive to be space-stabilized at all frequencies of interest. Such hydraulic seeker drives (References S4 and S5) have very superior acceleration capability, high torque constant, wide speed range and good power-to-weight ratio.

Figure 2-4 shows several angles of interest in a two-dimensional intercept, while Figure 2-5 is a block diagram of one servo of a hypothetical seeker with electrical or hydraulic servo motors. A fast stabilization loop is provided to stabilize the seeker in space, while a much slower outer loop is employed to track the target. Either integrating or spring-restrained rate gyros may be used, depending on whether drift-rate performance or cost is the deciding factor. Two possible takeoff points for LOS rate signals are shown. (See also Reference J5). The radome error angle is shown in Figure 2-5 as a nonlinear function of gimbal angle $\theta_{h}$; it is discussed in the next subsection. The inevitable radar noise is shown as an equivalent radome angular input to the radar receiver in Figure 2-5.

In either a radar or an infrared system, noise tends to be a problem because it increases miss distance. In a radar system, the guidance designer would like to have high illumination power on the target, so as
Figure 2-4 - Basic Seeker Angles
Figure 2-5 - Block Diagram of Generalized Seeker
to reduce receiver gain and internally-generated noise. The antenna size is made a maximum within the constraint of missile body diameter, in order to maximize power reception and to minimize angular beamwidth (Reference S6). A passive infrared seeker is also designed to have a maximum aperture to maximize incoming power, and may utilize special optical modulation of the incoming radiation so as to amplify without drift the weak electrical signal of the infrared detector. In order to maximize the incoming infrared power, a tail attack against a jet pipe is preferred, as are moderate ranges and good weather.

A small wavelength in a radar reduces the angular beamwidth, (Reference S6), but the choice of wavelength is limited by problems of power generation and environmental absorption, etc. Consequently, a missile radar antenna usually has a relatively broad beamwidth and so it is unable to resolve two closely-spaced targets by their angular separation until the last moments of intercept. This classic problem may lead to a bad miss distance. Reference G5 discusses possible ways for a radar to resolve multiple targets. Because of its much higher ratio of aperture diameter to wavelength, an infrared seeker has a narrower "beamwidth" and much higher angular resolution.

Radar illumination may be continuous-wave (CW) or pulsed, depending on which factor of application is governing. In a simple CW radar, closing velocity $V_c$ is obtained from Doppler measurements.

### 2.1.4 Radome

A radome (or IR dome) is required in order to protect the seeker and to transmit the reflected radar (or infrared, as the case may be) energy from the target. The requirements (References Y1 and Y2) for the dome may be summarized as follows:
1) It must transmit the energy with minimum loss.

2) It must transmit the energy with minimum distortion; in particular, a change of angular distortion with seeker position causes a severe guidance problem with the parasitic attitude loop.

3) It must have minimum aerodynamic drag.

4) It must have satisfactory mechanical properties, such as sufficient strength, resistance to thermal shock (from rapid aerodynamic heating), resistance to rain erosion at high speeds, and minimum water absorption.

Figure 2-6 shows three conceivable shapes for a radome. For minimum angular distortion, a hemispherical shape (or hyperhemispherical as in a ground-based radar) would be ideal electromagnetically, but its drag would be excessive. The aerodynamicist would prefer the first shape in Figure 2-6 in order to minimize drag, but this shape tends to have rather high peak values of radome error slope $R$, i.e., the partial derivative of radome error angle $r$ with gimbal angle $\theta_h$. A typical compromise is the tangent-ogive shape with a length-to-diameter ratio of about 3, as shown in Figure 2-6. Some missiles use much blunter radomes despite the drag penalty.

The modelling, evaluation and compensation for radome-error-angle effects are among the most difficult problems of the guidance designer. For instance, each radome from a production run has a different characteristic which varies with plane of examination (defined by longitudinal axis and seeker boresight axis), frequency, and possibly even with time owing to environmental factors. Preliminary analytical models utilize fixed values of radome error slope $R$, which may be positive or negative.
Figure 2-6 - Three Conceivable Shapes for a Radome

(A) \( L/D \approx 5 \) IDEAL AERODYNAMICALLY

(B) \( L/D \approx 1/2 \) IDEAL ELECTROMAGNETICALLY

(C) \( L/D \approx 3 \) COMPROMISE RADOME
The slopes usually lie within the range from -0.1 to +0.1 degree/degree.

2.2 Two Major Design Problems

The engineer who designs a classical guidance system usually faces four or five major problems, which may be enumerated as follows:

1) The rms miss distance from all sources must be minimized.
2) Stability must be maintained in the parasitic attitude loop.
3) The autopilot must be kept stable and well-behaved over a wide range of altitude and Mach number, during and after motor burning.
4) The tendency for certain variables to saturate, such as angle and angular rate of control surfaces, must be kept within reasonable bounds.
5) In the case of a new missile design, the guidance designer is often asked to give specifications for the airframe.

The first two of these problems will be discussed briefly. Discussion of the other problems appears in Reference S1.

2.2.1 Minimization of Miss Distance

The many sources of miss distance may be divided into two main categories: 1) Noise (or "stochastic") sources; 2) Non-noise sources (sometimes called "deterministic" with reference to calculating the miss distance for a specified source). Usually, there are three main noise sources in a typical guidance system:

a) Receiver noise, with noise power varying with the square of illuminator-to-target range and with $R_{mt}^2$ (Reference S6)
b) Range-independent noise, due to roughness in the tracking servo and possible other effects

c) Scintillation noise, which is due to radar reflections varying over the target with time, as with sunlight "glinting" from the trim and windshield of an automobile; scintillation noise power varies inversely with $R_{mt}^2$ because it depends on the dimensions and configuration of the target.

The non-noise sources are chiefly:

d) Initial heading error, due mainly to kinematic conditions of launch or entry into the proportional-navigation phase

e) Acceleration bias within the autopilot, due to electronic drifts

f) LOS-rate bias, due to electronic drifts or gyro null offset in the seeker

g) Acceleration of the target perpendicular to the LOS

h) Gravitational pull, which should be compensated by an electronic bias on the autopilot acceleration command; this may be difficult to mechanize, inasmuch as good data on the vertical direction may not be available in the missile

i) Axial acceleration (usually deceleration) in the missile, which may be measured by an axial accelerometer and compensated by another electronic bias on the autopilot command

In general, sources (d) through (g) have a stochastic aspect, since each will have some probability distribution over a large ensemble of flights. Moreover, target acceleration may be random during one flight.

For preliminary design, the guidance system may be assumed to be linear and so superposition applies to the various sources of miss distance.
Within the limits of this assumption, the adjoint method (References P2, L4, and R3) is a useful and efficient tool for calculating (by digital or analog computer) the miss-distance components versus time of flight (adjoint time). Such a method is quicker and less expensive than many Monte Carlo simulations of a flight in forward time, each of which yields only a single miss distance for the given time of flight; the behavior before intercept is of less interest than the miss distance at the time of intercept.

It is also useful to have hand methods of calculating miss distance, so as to check for errors in computer adjoint results. Unfortunately, the complexity of analysis (caused chiefly by the $1/T_{go}$ term in Figure 1-5) makes closed-form hand solutions impossible for all but a limited class of linear guidance systems, of which one has the following transfer function:

$$\frac{A_m(s)}{\lambda(s)} = \frac{N' V_c}{(\cos \theta_n) (1 + Ts)} \quad (2.10)$$

where the effective navigation ratio $N'$ is restricted to integer values. Data on miss distance for this type of system is shown in Reference S1. See also Reference M5.

Actually, a guidance system with a single lag is somewhat idealized and oversimplified, but it approximately characterizes some systems. Inasmuch as a real guidance system must have many poles in its transfer function $A_m(s)/\lambda(s)$, it is interesting to see what types of poles give the best miss distance. Reference B7 has examined simulation results for different types of systems, each having the same coefficient of $s$ in the denominator of the foregoing transfer function, and has concluded that
the best transfer function has a single real pole as in (2.10). In order of increasing miss distance are systems with: 1) A real pole with another at higher frequency, 2) Two real poles at the same frequency, 3) A quadratic pole-pair with low damping, 4) Many identical small real poles.

From this one might conclude that the best realizable guidance-system transfer function would have one dominant real pole, with other poles at much higher frequencies and well-damped in the case of quadratic pole-pairs. Moreover, the dominant lag must be small for low miss distance due to most noise sources, except for scintillation noise.

2.2.2 Stability of Guidance System

A very significant problem in a classical guidance system is to maintain stability and good response of the parasitic attitude loop (Reference P11). Actually, this is very closely related to the problem of the previous subsection, because a bad attitude-loop characteristic causes a large miss distance.

Figure 2-7 is a block diagram of the parasitic attitude loop in slightly simplified form.

The direct path from \( \dot{\lambda} \) to \( A_m \) shows the mechanization of the proportional-navigation law, with a low-pass noise filter \( G_n \) in order to reduce the high-frequency noise, chiefly for the sake of the fin servos inside the autopilot. If only this direct path from \( \dot{\lambda} \) to \( A_m \) existed without the parasitic feedback, guidance design would be much easier than it actually is.

In the feedback path of Figure 2-7, the airframe transfer function relates the pitch rate to the lateral acceleration of the c.g. and is the ratio of (2.2) to (2.8). The denominator of this transfer function is cancelled within the parasitic attitude loop by the numerator of \( G_1 \) (2.8)
Figure 2.7 - The Parasitic Attitude Loop (inside Guidance Kinematic Loop)
inside the autopilot, but the term \((1 + a_{31} s)\) is not cancelled. The "alpha over gamma dot" time constant \(a_{31}\) in (2. 1) may be a fraction of a second at low altitude and may exceed 10 seconds at high altitude. The transfer function \(V_2/\dot{\theta}_m\) indicates the spurious contribution \(V_2\) of the missile pitch rate \(\dot{\theta}_m\) to the measurement of LOS rate \(\dot{\lambda}\). This transfer function is caused by the imperfect space-stabilization of the seeker, and also by the varying radome distortion of the LOS angle, the latter effect tending to be most troublesome in the guidance design. The transfer function \(V_2/\dot{\theta}_m\) can cause the parasitic attitude loop to be regenerative or degenerative at low frequencies, depending on the sign of radome error slope \(R\).

Physically, when the guidance system measures an LOS rate \(\dot{\lambda}\) and causes a corrective lateral acceleration \(A_m\), the missile must change its angle of attack. This causes the seeker to look at the target through a different part of the radome, with a different radome error angle \(r\) than before. Since the seeker must measure \(\dot{\lambda}\) by utilizing angular data that is distorted by the radome, its measurement of \(\dot{\lambda}\) is inevitably corrupted. The most significant parameter here appears to be the "effective radome slope \(R'\)" (in the curve of \(r\) versus gimbai angle) under the particular operating conditions, as discussed in Subsections 1.3.3 and 1.3.5.

At high altitudes, with a large \(a_{31}\), stability of the attitude loop can be a difficult problem. If the loop is degenerative at low frequencies because of a positive \(R\), the stability problem is experienced at high frequencies. On the other hand, if the parasitic attitude loop is regenerative at low frequencies because of a negative \(R\), it has a tendency toward under-damped (or even unstable) poles at low frequency; the guidance system then has a bad miss distance even if the loop is stable.

Stability of the attitude loop can always be achieved by increasing
certain major filtering time constants, but at the cost of making the guidance system slow. This increases most of the components of miss distance, as indicated in Reference S1 for the simplified case of one single lag in the overall transfer function $A_m/\lambda$. Therefore, the design of the parasitic attitude loop is crucial. Considering that factors of Mach number, altitude, radome modelling, design of the autopilot and design of the seeker all enter into the parasitic attitude loop, it is perhaps not surprising that different design approaches are utilized by each guidance engineer. One interesting possibility is the use of a digital computer to vary the major filtering time constants of the attitude loop, within constraints of stability and noise transmission.

2.3 Application of Modern Theory to Guidance

Subsection 1.3 summarized the literature of modern guidance of homing interceptor missiles, in roughly chronological order. The most significant contributions appear to have been made by Bryson and Widnall, and also by engineers at Raytheon, notably O'Halloran.

The following subsections summarize, without extensive derivation, the important results of applying estimation and control theory to interceptor missiles, both for continuous and discrete cases. The results are given in order of increasing complexity rather than chronological order.

The plant is assumed to be linear, with an autopilot having zero lag, or one pole or two in later subsections. The target is typically modelled as having a Poisson wave of evasive acceleration perpendicular to the line of sight (LOS). This random acceleration is fitted into the Kalman formulation by replacing it with the output of a low-pass filter driven by Gaussian white noise, such that both models of acceleration have the
same zero ensemble mean and same exponential autocorrelation function. The measurements of the LOS angle have additive white Gaussian non-stationary noise. The performance index is quadratic with the traditional mean-square miss distance plus the weighted integral of squared control effort:

\[
J = E \left[ \frac{1}{2} Y_d^2 (t_f) \right] + E \left[ \frac{b}{2} \int_{t_o}^{t_f} u^2 (\tau) \, d\tau \right]
\] (2.11)

This latter term makes the problem more mathematically tractable, but is physically justified since it reduces induced drag and increases the final missile velocity relative to less efficient guidance.

The foregoing conditions are sufficient to apply the separation theorem or "certainty equivalence principle" (References B1, S11, J6, G6), which allows the total control (intercept) problem to be broken into two sub-problems:

1) Optimal estimation (Kalman filtering) of the plant states from measurement data corrupted by white noise, with minimum mean-square error in the ensemble sense

2) Use of optimal control gains and the foregoing estimates as if they were perfect knowledge of the states, to develop missile accelerations to minimize the performance index.

Accordingly, the following discussion treats Kalman estimation and optimal control separately.

2.3.1 Estimation of States

Subsections 2.3.1.1 through 2.3.1.3 are adapted from Reference S1.
2.3.1.1 Geometry and Plant Modelling

Clearly, the overall problem is simplified by the proper choice of the coordinate system. Following Reference B1, one very suitable choice turns out to be a nonrotating, orthogonal set of three axes with its origin fixed to the missile seeker. The X axis is in the original boresight direction at the start of terminal phase, while the Y and Z axes are in the corresponding respective directions of the seeker pitch and yaw gyro axes. Therefore, Figures 1-2 and 2-8 are applicable to one axis of the estimator. A convenient set of variables for this axis is:

\[
\mathbf{x} = \begin{bmatrix}
Y_d \\
\dot{Y}_d \\
A_t
\end{bmatrix} \equiv \begin{bmatrix}
Y_t - Y_m \\
\dot{Y}_t - \dot{Y}_m \\
\ddot{Y}_t
\end{bmatrix} \quad (2.12)
\]

which is very pertinent, since the final value of \( Y_d \) is very nearly the miss distance, which is to be minimized. An alternate set of state variables would be:

\[
\mathbf{x}_{\text{alt}} = \begin{bmatrix}
\lambda \\
\dot{\lambda} \\
A_t
\end{bmatrix} \quad (2.13)
\]

but this leads to a time-varying essential matrix \( F \) of the plant, whereas the \( F \) matrix for the chosen formulation is constant, with a resultant simplification of computation. One channel of the optimal filter, therefore develops the estimate:
Figure 2-8 - Moving, Non-Rotating Coordinate System for Estimation
Most of the following discussion is restricted to this channel and, therefore, to the plane of Figure 2-8. In an actual application, the system must have another estimator channel to estimate $Z_d$, $\dot{Z}_d$, and $A_{tz}$ along the Z axis perpendicular to Figure 2-8. It has been found that cross-coupling between the two channels may be neglected. The estimator does not have to estimate quantities along the X axis, although it does require a fairly good measurement of:

$$X_{mt} = X_t - X_m \cong R_{mt}$$  \hspace{1cm} (2.15)

For generality, let it be assumed that the target has some initial unknown acceleration $A_{t_o}$ in Figure 2-8. For further generality, let it be assumed that the target may have random changes in its acceleration perpendicular to the LOS, such as the randomly-reversing Poisson square wave in Figure 2-9, with an average of $\nu$ zero-crossings per second and an rms level of $\beta$ in fps$^2$. This Poisson square wave for $A_t$ has an autocorrelation function (Reference L2), for observation times $t_1$ and $t_2$, equal to:

$$\phi_{pp}(t_1-t_2) = \beta^2 e^{-2\nu |t_1-t_2|}$$  \hspace{1cm} (2.16)

which shows that the mean-square value is indeed $\beta^2$. If $\nu$ approaches zero, the $A_t$ approaches a constant level physically and mathematically.
Figure 2-9 - Poisson Square Wave for Target Acceleration
The power spectral density of the Poisson wave of $A_t$ may be derived from (2.16) and is given (Reference L2) by:

$$\phi_{pp}(\omega) = \frac{\beta^2}{2\pi\nu} \left[ \frac{1}{1 + \left( \frac{\omega}{2\nu} \right)^2} \right]$$  \hspace{1cm} (2.17)

Since the Kalman filter is based on minimizing the mean-squared error of the state estimates, it is justifiable to replace the Poisson-wave model of target maneuver with one that has the same mean and autocorrelation function, so as to obtain the same quality of estimate with a mathematically more convenient model. It is readily shown from (2.17) that the target-maneuver model is equivalent to the output of a low-pass filter with a break frequency of $2\nu$ rad/s and an input that is white Gaussian noise with a power spectral density of $\beta^2/2\pi\nu$, double-sided. This is also the target-maneuver model that was used by Bryson in his discussion of the interceptor problem in References B1 and B8. See also Reference S7.

Figure 2-10 shows the model of the target maneuver and the system states $A_t$, $Y_d$, and $Y_d$ as shown in Figure 2-8 with corrective missile acceleration $A_m$. The initial missile heading error $L_e$ (with $L_{co}$ as the initially correct lead angle) makes a contribution to differential velocity $\dot{Y}_d$ which is included.

The vector differential equation of the plant in Figure 2-10 may be written as:

$$\dot{x} = F x + w + g u$$  \hspace{1cm} (2.18)
Figure 2-10 - Model of Target Maneuver and States of System with Corrective Missile Acceleration

WHITE NOISE WITH P.S.D.
\[ \frac{\beta^2}{2\pi v} \]

\[ 2v \]

INITIAL TARGET ACCELERATION

\[ A_{t_0} \]

INITIAL VELOCITY OF MISSILE PERPENDICULAR TO LOS BECAUSE OF INITIAL HEADING ERROR

\[ -v_m \cos \theta_{c0} \sin L_e \]

DIFFERENTIAL VELOCITY PERPENDICULAR TO ORIGINAL LOS

\[ Y_d \]

DIFFERENTIAL DISPLACEMENT PERPENDICULAR TO ORIGINAL LOS

\[ -\cos \theta_h \]

\[ A_m \]

AUTOPilot

\[ A_c \]
\[ \dot{\mathbf{x}} = \begin{bmatrix} \ddot{Y}_d \\ \ddot{Y}_d \\ \ddot{A}_t \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \nu \end{bmatrix} \begin{bmatrix} \ddot{Y}_d \\ \ddot{Y}_d \\ A_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \nu \sigma \end{bmatrix} u \] (2.19)

where the square $F$ matrix is the essential matrix of the plant, $w$ is the white "process" noise and $u$ is the corrective missile acceleration:

\[ u = \ddot{Y}_m = A_m \cos \theta_h \] (2.20)

### 2.3.1.2 Measurements

The formulation of the Kalman estimator requires that the measurement vector be a linear combination of the true states plus a white measurement-noise vector:

\[ \mathbf{y} = \mathbf{Hx} + \mathbf{v} \] (2.21)

In this case, it turns out to be appropriate to use the seeker as an angle-measuring device that delivers the measurement:

\[ \lambda_m = \lambda + \lambda_n \] (2.22)

where $\lambda_n$ is the nearly white noise having the three components in Subsection 2.2.1 and $\lambda$ is the angle between the present line of sight and the original LOS in Figure 2-8. Since only one measurement:

\[ y = \lambda + \lambda_n \] (2.23)
is considered in the present discussion, (2.21) reduces to the scalar form in (2.23) with:

\[
\lambda = H \mathbf{x} = \begin{bmatrix} \frac{1}{X_{mt}} & 0 & 0 \end{bmatrix} \begin{bmatrix} Y_d \\ \dot{Y}_d \\ A_t \end{bmatrix}
\]  

(2.24)

To mechanize (2.23), the measurement \( \lambda_m \) is obtained by adding the boresight error signal \( \epsilon \) to a signal proportional to dish angle \( D \) relative to the original LOS, this latter signal being obtained by integrating the dish-rate command. It is apparent that the true \( \lambda \) is:

\[
\lambda = \epsilon + D = \epsilon + \theta_h + \theta_m
\]  

(2.25)

and it is easily shown that the measurement \( \lambda_m \) will include white noise \( \lambda_n \). The Kalman estimation problem is rendered straightforward if the seeker is considered as a device for measuring \( \lambda \) (not LOS rate \( \dot{\lambda} \), as in a classical system) with additive white noise. The tracking function is simply a physical necessity which is not very germane to estimation as such. The present discussion neglects the effect of radome error angle \( r \), which is actually a serious problem.

2.3.1.3 The Continuous Kalman Estimator

Using the available noisy measurement, the estimation subproblem is to form an estimate \( \hat{\mathbf{x}} \) of the state vector in (2.12)

\[
\hat{\mathbf{x}} = \begin{bmatrix} \hat{Y}_d \\ \dot{\hat{Y}}_d \\ \hat{A}_t \end{bmatrix}
\]  

(2.26)

The error in the estimate of the state vector is denoted as \( \tilde{\mathbf{x}} \) and is:
\[
\hat{x} = \hat{x} - \hat{\hat{x}} = \begin{bmatrix}
\hat{Y}_d - \hat{\hat{Y}}_d \\
\hat{\hat{Y}}_d - \hat{Y}_d \\
\hat{A}_t - \hat{\hat{A}}_t
\end{bmatrix} = \begin{bmatrix}
\hat{\hat{Y}}_d \\
\hat{Y}_d \\
\hat{\hat{A}}_t
\end{bmatrix}
\] (2.27)

It is specified that the estimate is to be optimum in the sense of minimizing the expected square error, i.e., the mean squared error for an ensemble of many trial flights with the same conditions and random-noise statistics. Therefore, the following expected squared error is to be minimized:

\[
E \left[ (\hat{x})^T(\hat{x}) \right] = E \left[ (\hat{\hat{Y}}_d)^2 + (\hat{\hat{Y}}_d)^2 + (\hat{\hat{A}}_t)^2 \right]
\] (2.28)

Equation (2.28) is equivalent to the trace of the covariance matrix \( P \), i.e., the sum of its diagonal elements, where \( P \) is:

\[
P = E \left[ (\hat{x}) (\hat{x})^T \right] = E \left[ \begin{array}{c}
(\hat{\hat{Y}}_d)^2, (\hat{\hat{Y}}_d)(\hat{\hat{Y}}_d), (\hat{\hat{Y}}_d)(\hat{\hat{A}}_t) \\
(\hat{\hat{Y}}_d)(\hat{\hat{Y}}_d), (\hat{\hat{Y}}_d)^2, (\hat{\hat{Y}}_d)(\hat{\hat{A}}_t) \\
(\hat{\hat{Y}}_d)(\hat{\hat{A}}_t), (\hat{\hat{Y}}_d)(\hat{\hat{A}}_t), (\hat{\hat{A}}_t)^2
\end{array} \right]
\] (2.29)

From References B1 and S11, it is known that the estimate \( \hat{x} \) is the output of a filter with the differential equation:

\[
\dot{\hat{x}} = F\hat{x} + K(y^T - H\hat{x})
\] (2.30)

The filter gains \( K \) are found from:

\[
K = PH^T R^{-1}
\] (2.31)

and the solution of the matrix Riccati equation:
\[ \dot{P} = FP + PF^T + Q - PH^T R^{-1} HP \]  
(2.32)

with the initial condition:

\[ P_{t=0} = P_0 \]  
(2.33)

In (2.32), \( Q \) is the integral of the covariance of the process noise:

\[ E \left[ w(t_1)w(t_2)^T \right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\sigma^2}{\nu} \end{bmatrix} \delta(t_1 - t_2) = Q \delta(t_1 - t_2) \]  
(2.34)

and \( R \) is the integral of the covariance of the (scalar) measurement noise:

\[ E \left[ v(t_1)v(t_2) \right] = R \delta(t_1 - t_2) \]  
(2.35)

where the \( \delta \) function is an impulse of unit area. Simulations have shown that the Kalman gains \( K \) tend typically to increase during the first third of homing flight and then to decrease monotonically until intercept.

Figure 2-11 is a block diagram of the continuous Kalman estimator.

### 2.3.1.4 The Discrete Kalman Estimator

The use of a pulsed radar and a digital computer makes it natural to employ sampled data and discrete computations (References S11 and B1). The following two subsections give the equations for the general discrete estimator and then specifically for the "four-state" estimator.
Figure 2-11 - Continuous Kalman Estimator
for guidance. This material is adapted from Reference R3, which in turn, was based heavily on O'Halloran's work in References W4 and O'H3. This particular estimator has been used in Raytheon guidance simulations employing the "Improved Four-State Law" and sometimes the "Four-State Law" (Subsection 1.3.5 herein), i.e., those optimal control laws which modelled the autopilot respectively as having one zero and one pole or as having just the one pole.

2.3.1.4.1 General Discrete Equations

As in (2.18) the continuous plant equation is:

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{g}\mathbf{u} + \mathbf{w}$$  \hspace{1cm} (2.36)

The scalar control variable \( u \) is a weighted sum of the state estimate \( \hat{x} \):

$$u = \mathbf{c}^T \hat{x}$$  \hspace{1cm} (2.37)

Suppose there are available \( m \) measurements that are linearly related to the state and are corrupted by additive white Gaussian noise:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{v}$$  \hspace{1cm} (2.38)

The transition matrix for the system has the differential equation:

$$\dot{\mathbf{\phi}}(t, 0) = \mathbf{F}\mathbf{\phi}(t, 0), \hspace{0.5cm} \mathbf{\phi}(0, 0) = 1$$  \hspace{1cm} (2.39)

which for a constant \( F \) and a fixed sampling time \( T_s \) is equivalent to:
\[
\Phi(T_s, 0) = \mathcal{L}^{-1} \left[ (sI - F)^{-1} \right]_{t = T_s}
\] (2.40)

where \( s \) is the Laplace transform variable and \( \mathcal{L}^{-1} \) denotes "inverse transform of."

For the interval \( T_s \) between samples, the control \( u \) is held constant, while the white noise \( w \) produces a cumulative effect on \( x \). Therefore, the discrete form of (2.36) is:

\[
x_k = \Phi_k x_{k-1} + g_k u_{k-1} + w_k
\] (2.41)

where the subscript denotes the sample number, \( \Phi_k \) is \( \Phi(T_s, 0) \) and:

\[
g_k = \int_0^{T_s} \Phi(T_s - \tau, 0) g d\tau
\] (2.42)

\[
u_{k-1} = \mathcal{L}^{T_s} x_{k-1}
\] (2.43)

\[
w_k = \int_0^{T_s} \Phi(T_s - \tau, 0) w(\tau) d\tau
\] (2.44)

The discrete form of (2.38) is simply:

\[
y_k = Hx_k + v_k
\] (2.45)

The measurement noise has the following statistics:
\[ E(v_k) = 0 \] (2.46)

\[ E(v_k v_j^T) = R_k \delta_{kj} \] (2.47)

where:

\[ \delta_{kj} = 0, \quad j \neq k \] (2.48a)

\[ \delta_{kj} = 1, \quad j = k \] (2.48b)

The discrete effect \( w_k \) of the continuous process noise \( w(t) \) on the state \( x_k \) has the statistics:

\[ E(w_k) = 0 \] (2.49)

\[ E(w_k w_j^T) = Q_k \delta_{kj} \] (2.50)

\[ Q_k = \int_0^T \int_0^T \Phi(T_s - \tau, 0) E \left[ w(\tau) w(\tau)^T \right] \Phi^T(T_s - \tau, 0) \, d\tau \, d\tau \] (2.51)

where the covariance term \( E \left[ \right] \) is impulsive as in the case of (2.34).

Using the sampled measurements \( y_k \) in (2.45), the Kalman filter produces an estimate \( \hat{x}_k \) such that the sum of the mean square errors for the individual state estimates is a minimum:

\[ \text{Sum of MSE} = E \left[ (x_k - \hat{x}_k)^T (x_k - \hat{x}_k) \right] = \text{minimum} \] (2.52)

This quantity is the trace of the covariance of the error:
\[
E \left[ (\hat{x}_k - \bar{x}_k)^T (\bar{x}_k - \bar{x}_k) \right] = \text{Tr}(P_k) \tag{2.53}
\]
\[
P_k = E \left[ (x_k - \bar{x}_k) (x_k - \bar{x}_k)^T \right] \tag{2.54}
\]

The equation for the k-th estimate of \(x\), including the effect of the k-th measurement \(y_k\), is:
\[
\hat{x}_k = \Phi_k \hat{x}_{k-1} + g_k u_{k-1} + K_k \left[ y_k - H_k (\Phi_k \hat{x}_{k-1} + g_k u_{k-1}) \right] \tag{2.55}
\]

The Kalman gains \(K_k\) are obtained from the covariance equations as follows:
\[
M_k = \Phi_k P_{k-1} \Phi_k^T + Q_k \tag{2.56}
\]
\[
K_k = M_k H_k (H_k M_k H_k^T + R_k)^{-1} \tag{2.57}
\]
\[
P_k = (I - K_k H_k) M_k \tag{2.58}
\]

The \(n\) by \(n\) matrix \(M_k\) may be interpreted as the "projected forward" covariance of the error, from \(P_{k-1}\), including the effect of \(Q_k\) but not the k-th measurement \(y_k\). The gain \(K_k\) is expressible in terms of \(M_k\). The new covariance \(P_k\) is the result of incorporating the k-th measurement into the variance calculations.

A fuller discussion of discrete Kalman filtering, with alternate forms of the equations, may be found in References S11 and B1.
2.3.1.4.2 The Four-State Discrete Estimator

The discrete Kalman estimator in the Raytheon adjoint program (Reference R3) is based on the four-state plant for missile-target motion in Figure 2-12. From this figure, the vector plant equation corresponding to (2.36) is:

\[
\begin{bmatrix}
\dot{Y}_d \\
\dot{Y}_d \\
\dot{A}_t \\
\dot{A}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & -2\nu & 0 \\
0 & 0 & 0 & -\omega_1
\end{bmatrix}
\begin{bmatrix}
Y_d \\
\dot{Y}_d \\
A_t \\
\dot{A}
\end{bmatrix} +
\begin{bmatrix}
0 \\
-\omega_1/\omega_2 \\
0 \\
\omega_1(1-\omega_1/\omega_2)
\end{bmatrix}
A_c +
\begin{bmatrix}
0 \\
0 \\
2\nu\varphi(t) \\
0
\end{bmatrix}
\]  
(2.59)

Corresponding to (2.43), the control variable (acceleration command to the autopilot) is:

\[
A_c =
\begin{bmatrix}
^\wedge Y_d \\
^\wedge Y_d \\
^\wedge A_t \\
^\wedge A
\end{bmatrix}
\begin{bmatrix}
c_1 & c_2 & c_3 & c_4
\end{bmatrix}
\]  
(2.60)

With reference to (2.21), (2.23) and (2.45), the k-th measurement is:

\[
y_k = \lambda_m = \lambda_k + \lambda_n =
\begin{bmatrix}
1/\lambda_{mt} & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
Y_d \\
\dot{Y}_d \\
A_t \\
\dot{A}
\end{bmatrix} + \lambda_n
\]  
(2.61)
Figure 2-12 - Four-State Plant for Missile-Target Motion
The transition matrix can be found from (2.40) to be:

\[
\Phi(t, 0) = \\
\begin{bmatrix}
1, t, \frac{1}{2\nu} t - \frac{1}{4\nu^2} (1 - e^{-2\nu t}), -\frac{1}{\omega_1} t + \frac{1}{\omega_1^2} (1 - e^{-\omega_1 t}) \\
0, 1, \frac{1}{2\nu} (1 - e^{-2\nu t}) \\
0, 0, e^{-2\nu t} \\
0, 0, 0
\end{bmatrix}
\]

(2.62)

Assuming that:

\[
\nu T_s << 1 
\]

(2.63)

which is true in most cases so far, then the transition matrix for one sample interval can be approximated well by:

\[
\Phi_k = \\
\begin{bmatrix}
1, T_s, T_s^2/2, -\frac{1}{\omega_1} T_s + \frac{1}{\omega_1^2} (1 - e^{-\omega_1 T_s}) \\
0, 1, T_s, -\frac{1}{\omega_1} (1 - e^{-\omega_1 T_s}) \\
0, 0, 1-2\nu T_s, 0 \\
0, 0, 0, e^{-\omega_1 T_s}
\end{bmatrix}
\]

(2.64)

From (2.42), the discrete input distribution vector is:
\[
\begin{align*}
\mathbf{g}_k &= \begin{bmatrix}
-\frac{T_s^2}{2} - (1 - \frac{\omega_1}{\omega_2}) \left[ \frac{-T_s}{\omega_1} + \frac{1}{2} \left( 1 - e^{-\omega_1 T_s} \right) \right] \\
- T_s + \frac{1}{\omega_1} (1 - \frac{\omega_1}{\omega_2}) (1 - e^{-\omega_1 T_s}) \\
0 \\
(1 - \frac{\omega_1}{\omega_2}) (1 - e^{-\omega_1 T_s}) \\
\end{bmatrix} \\
\text{(2.65)}
\end{align*}
\]

Substitution of (2.64) and (2.65) into (2.55) gives:

\[
\begin{bmatrix}
\hat{Y}_{d_k} \\
\hat{A}_{t_k} \\
\hat{A}_k \\
\end{bmatrix} = \begin{bmatrix}
1 & T_s & T_s^2/2 & \Phi_{k_1, 4} \\
0 & 1 & T_s & \Phi_{k_2, 4} \\
0 & 0 & 1 - 2\nu T_s & 0 \\
0 & 0 & 0 & e^{-\omega_1 T_s} \\
\end{bmatrix} \cdot \begin{bmatrix}
\hat{Y}_{d_{k-1}} \\
\hat{A}_{t_{k-1}} \\
\hat{A}_{k-1} \\
\end{bmatrix} + \begin{bmatrix}
\hat{Y}_{d_{k-1}} \\
\hat{A}_{t_{k-1}} \\
\hat{A}_{k-1} \\
\end{bmatrix} + \begin{bmatrix}
g_{k_1} \\
g_{k_2} \\
g_{k_4} \\
\end{bmatrix} \cdot A_{c_{k-1}}
\]

\[
\begin{align*}
\begin{bmatrix}
K_1 \\
K_2 \\
K_3 \\
K_4 \\
\end{bmatrix}_k \cdot \left\{ y_k - \left[ 1 \right] R_{mt} 0 0 \right\} \\
\end{align*}
\]

\[
\begin{bmatrix}
1 & T_s & T_s^2/2 & \Phi_{k_1, 4} \\
0 & 1 & T_s & \Phi_{k_2, 4} \\
0 & 0 & 1 - 2\nu T_s & 0 \\
0 & 0 & 0 & e^{-\omega_1 T_s} \\
\end{bmatrix} \cdot \begin{bmatrix}
\hat{Y}_{d_{k-1}} \\
\hat{A}_{t_{k-1}} \\
\hat{A}_{k-1} \\
\end{bmatrix} + \begin{bmatrix}
\hat{Y}_{d_{k-1}} \\
\hat{A}_{t_{k-1}} \\
\hat{A}_{k-1} \\
\end{bmatrix} + \begin{bmatrix}
g_{k_1} \\
g_{k_2} \\
g_{k_4} \\
\end{bmatrix} \cdot A_{c_{k-1}}
\]
Rather than using their algebraic equivalents in (2.64) and (2.65), the elements \( \Phi_{k_1,4} \) and \( \Phi_{k_2,4} \), as well as the elements of \( \mathbf{g}_k \), are written symbolically for brevity and convenience. The control variable is:

\[
A_{c, k-1} = \begin{bmatrix}
^\wedge Y \\
^\wedge d_{k-1} \\
^\wedge Y_{d, k-1} \\
^\wedge A_{t, k-1} \\
^\wedge A_{k-1}
\end{bmatrix}
\]  \hspace{1cm} (2.67)

Further development of the estimator equations shows that \( K_4 \) is zero and that the largest matrix is then 3 by 3, with a saving of computation time. Referring to (2.59) and Figure 2-12, the continuous process-noise vector is:

\[
\mathbf{w}(t) = \begin{bmatrix}
0 \\
0 \\
2\nu w(t) \\
0
\end{bmatrix}
\]  \hspace{1cm} (2.68)

with covariance:

\[
E \left[ \mathbf{w}(\tau)\mathbf{w}(\sigma)^T \right] = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 4\nu^2 & 0 & \delta(\tau-\sigma) \\
0 & 0 & 0 & 0
\end{bmatrix}
\]  \hspace{1cm} (2.69)
In order to obtain $\Phi(T, 0)$ with $T \leq T_s$, substitute $T$ for $T_s$ in (2.64), and let the result, together with (2.69), be substituted into (2.51). After integration, the discrete process noise matrix $Q_k$ is found to be:

$$Q_k = 4 \nu \beta^2 \begin{bmatrix}
T_s^{5/20} & T_s^{4/8} & T_s^{3/6} & 0 \\
T_s^{4/8} & T_s^{3/3} & T_s^{2/2} & 0 \\
T_s^{3/6} & T_s^{2/2} & T_s & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$  \hspace{1cm} (2.70)

The measurement noise matrix $R_k$ in (2.47) and (2.57) is actually a scalar variance because only one variable (LOS angle) is measured. The classical model of LOS noise (Subsection 2.2.1), has the independent components which are receiver noise, range-independent noise and scintillation noise, the variances of which add:

$$R_k = \frac{\sigma_{rn}^2 R_{it}^2 R_{mt}^2}{R_o^4 (\sigma_t/\sigma_o)^2} + \sigma_{rin}^2 + \frac{\sigma_{sn}^2}{R_{mt}^2}$$  \hspace{1cm} (2.71)

where:

$$\sigma_{rn} = \text{standard deviation of receiver noise (rad)}$$
$$R_{mt} = R_o = R_{it} \quad \text{and} \quad \sigma_t = \sigma_o$$
$$R_{it} = \text{range from illuminator to target (ft)}$$
$$R_{mt} = \text{range from missile to target (ft)}$$
\[ R_o = \text{radar reference range (ft)} \]
\[ \sigma_t = \text{target cross section (ft}^2) \]
\[ \sigma_o = \text{reference cross section (ft}^2) \]
\[ \sigma_{\text{rin}} = \text{standard deviation of range-independent noise (rad)} \]
\[ \sigma_{\text{sn}} = \text{standard deviation of scintillation noise (ft)} \]

A suitable initial covariance matrix \( P_0 \) is necessary. For convenience, let the error in the estimate be represented by:

\[
\hat{\mathbf{x}}_k = (\mathbf{x}_k - \hat{\mathbf{x}}_{k}) = \begin{bmatrix}
    Y_{d_k} & \hat{Y}_{d_k} \\
    \hat{Y}_{d_k} & Y_{d_k} \\
    A_{t_k} & \hat{A}_{t_k} \\
    \hat{A}_{t_k} & A_{t_k}
\end{bmatrix} = \begin{bmatrix}
    \hat{Y}_{d_k} \\
    Y_{d_k} \\
    \hat{A}_{t_k} \\
    A_{t_k}
\end{bmatrix}
\]

(2.72)

\[
P_0 = \begin{bmatrix}
    P_{11} & \delta_{12} \sqrt{P_{11} P_{22}} & \delta_{13} \sqrt{P_{11} P_{33}} & 0 \\
    \delta_{12} \sqrt{P_{11} P_{22}} & P_{22} & \delta_{23} \sqrt{P_{22} P_{33}} & 0 \\
    \delta_{13} \sqrt{P_{11} P_{33}} & \delta_{23} \sqrt{P_{22} P_{33}} & P_{33} & 0 \\
    0 & 0 & 0 & 0
\end{bmatrix}
\]

(2.73)

where the internal subscript 0 as in \( P_{11} \) is omitted for brevity. The element \( P_{44} \) and the corresponding off-diagonal terms are set to zero because it is assumed that the autopilot acceleration \( A \) is known perfectly, even though knowledge of it cannot be perfect. The first diagonal term is related to the measurement-error variance:
\[ P_{110} = E \left[ \tilde{Y}_{d_o}^2 \right] = R_{m_t o} \left( \begin{array}{c} R_k \end{array} \right)_{k=0} \]

\[ (2.74) \]

The \( P_{220} \) term is proportional to the heading-error variance:

\[ P_{220} = E \left[ \tilde{Y}^2_{d_o} \right] = v^2_m \sigma_{he}^2 \]

\[ (2.75) \]

while the \( P_{330} \) term is simply the mean-square target acceleration:

\[ P_{330} = \beta^2 \]

\[ (2.76) \]

Typical Raytheon practice has been to use standard positive values for the correlation coefficients \( \delta_{12}, \delta_{23}, \delta_{13} \), which were apparently obtained by O'Halloran from a computation of the Riccati equation. This seems to be questionable because there is no computation of \( P_k \) or knowledge of these coefficients before an engagement, and so they have been set to zero in this thesis work.

If (2.70) and (2.73) are substituted into (2.56), it is apparent that \( M_k \) (with \( k = 1 \)) has zeroes in its fourth row and fourth column. This leads to the fourth element of \( K_k \) being zero in (2.57) and to the fourth row and column of \( P_k \) being zero in (2.58). These conditions persist throughout the rest of the computations. In other words, the uncertainties and Kalman estimation apply only to the variables \( Y_{d'}, \dot{Y}_{d'}, \) and \( A_t \), because \( A \) is computed deterministically. Accordingly, (2.66) can be partitioned so that the \( \Phi', Q', P' \) and \( M' \) matrices may be defined as 3 by 3 matrices:
\[
\Phi_k = \begin{bmatrix}
1 & T_s & T_s^2/2 \\
0 & 1 & T_s \\
0 & 0 & 1-2\nu T_s
\end{bmatrix}
\] (2.77)

\[
\Omega_k = 4\nu^2 \beta^2 \begin{bmatrix}
T_s^{5/20} & T_s^{4/8} & T_s^{3/6} \\
T_s^{4/8} & T_s^{3/3} & T_s^{2/2} \\
T_s^{3/6} & T_s^{2/2} & T_s
\end{bmatrix}
\] (2.78)

\[
H' = \begin{bmatrix}
1 & \frac{1}{R_{mt}} & 0 & 0
\end{bmatrix}
\] (2.79)

and the new \( P'_k \) and \( M'_k \) matrices are composed of the first three rows and columns of the old \( P_k \) and \( M_k \). Also, \( K_k' \) has the simple form:

\[
K_k' = \begin{bmatrix}
K_1 \\
K_2 \\
K_3
\end{bmatrix}
= \frac{R_{mt}}{M_{11} + R_k R_{mt}^2} \begin{bmatrix}
M_{11} \\
M_{21} \\
M_{31}
\end{bmatrix}_k
\] (2.80)

Finally, (2.66) simplifies to a vector equation with three elements:
\[
\begin{bmatrix}
\hat{Y}_{d_k} \\
\hat{Y}_{d_{k-1}} \\
\hat{A}_{t_k} \\
\hat{A}_{t_{k-1}}
\end{bmatrix}
= \begin{bmatrix}
1 & T_s & T_s^2/2 \\
0 & 1 & T_s \\
0 & 1 & 1-2\nu T_s
\end{bmatrix}
\begin{bmatrix}
\hat{Y}_{d_{k-1}} \\
\hat{A}_{t_{k-1}}
\end{bmatrix}
+ \begin{bmatrix}
a_1 \\
a_2 \\
0
\end{bmatrix}
\hat{A}_{k-1} + \begin{bmatrix}
g_{k_1} \\
g_{k_2} \\
0
\end{bmatrix}
A_{c_{k-1}}
\]

\[
+ \begin{bmatrix}
K_1 \\
K_2 \\
K_3
\end{bmatrix}
\begin{bmatrix}
1 \\
R_{mt}
\end{bmatrix}
\begin{bmatrix}
y_k \\
0 \\
0
\end{bmatrix}
\quad (2.81)
\]

and a scalar equation for the acceleration estimate \( \hat{A}_k \):

\[
\hat{A}_k = e^{-\omega_1 T_s} \hat{A}_{k-1} + b_1 A_{c_{k-1}}
\quad (2.82)
\]

where

\[
a_1 \equiv \frac{\Phi_{k_{1,4}}}{\omega_1} = -\frac{T_s}{\omega_1} + \frac{1}{\omega_2} (1 - e^{-\omega_1 T_s})
\quad (2.83)
\]

\[
a_2 \equiv \frac{\Phi_{k_{2,4}}}{\omega_1} = -\frac{1}{\omega_1} (1 - e^{-\omega_1 T_s})
\quad (2.84)
\]

\[
b_1 \equiv g_{k_4} = (1 - \frac{\omega_1}{\omega_2}) (1 - e^{-\omega_1 T_s})
\quad (2.85)
\]

2.3.2 Optimal Control Gains

This subsection briefly presents the more important guidance laws, prior to this thesis, that have been described in Subsection 1.3. Except in one simple case, these are all continuous guidance laws, because of the relative difficulty of deriving discrete control gains in closed form.
2.3.2.1 Case of Zero Autopilot Lag in Continuous System

In this case, the plant is that of Figure 2-10, with the autopilot having zero lag and $A_c = \frac{A_m}{m} = u$ being the control variable. The performance index is as shown in (2.11). The guidance law of O'Halloran (Reference O'H2) and Speyer (Reference S8) may be expressed as:

$$A_c = u = \frac{3}{2} \frac{N'_3}{t} \left[ e^{2\nu t \frac{g}{u}} + 2\nu t \frac{g}{u} - 1 \right]$$  \hspace{1cm} (2.86)

where:

$$N'_3 = \frac{3t}{3b + t}$$  \hspace{1cm} (2.87)

In this form of (2.86), which is suggested by this thesis work, the bracketed quantity is seen to be the (estimated) projected, zero-effort, miss distance. Using (1.8), (2.86) can be put into the form:

$$A_c = u = N'_3 \left[ e^{2\nu t \frac{g}{u}} + 2\nu t \frac{g}{u} - 1 \right]$$  \hspace{1cm} (2.88)

Certain special cases are of interest. If $\nu = 0$, then target acceleration is assumed to be constant. A double application of L'Hopital's theorem shows that the coefficient of $A_t$ in (2.86) and (2.88) is $1/2$. If the control weighting $b = 0$, then $N'_3 = 3$ in (2.87).

Equation (2.88) is evidently like proportional navigation. From another point of view, the classical proportional navigation law (1.10)
is optimal for the performance index (2.11) if \( b = 0 \) and \( \hat{A}_t = 0 \), i.e., it minimizes the expected square miss distance.

It is apparent that \( N_3' \) in (2.87) is a generalized effective navigation ratio, which goes to zero at intercept if \( b \neq 0 \).

References O'H2 and S8 tacitly assumed a head-on engagement. Otherwise, the gimbal angle \( \theta_h \neq 0 \), and the denominators of (2.86) and (2.88) should include the factor \( \cos \theta_h \).

2.3.2.2 Case of Zero Autopilot Lag in Discrete System

Reference S10 by Speyer treated the case in the previous subsection, except that discrete control (constant over the period \( T_s \) between samples of data) was used and no target maneuver was assumed. His performance index was equivalent to:

\[
J = E \left[ \frac{1}{2} Y_d^2 (t_f) \right] + E \left[ \frac{bT_s}{2} \sum_{k=1}^{t_f/T_s} A_{c,k}^2 \right] \tag{2.89}
\]

where the total number of samples, \( t_f/T_s \), is an integer, the last sample occurring \( T_s \) second before intercept.

Speyer's discrete guidance law can be expressed as:

\[
A_c = u = \frac{N_2'd}{t_g} \left[ \frac{\hat{Y}}{Y_d} + \frac{\hat{t}}{g Y_d} \right] \tag{2.90}
\]

where the effective navigation ratio is:

\[
N_2'd = \frac{3m^2 (m - 1/2)}{3b^3 + \frac{m}{T_s} (m^2 - 1/4)} \tag{2.91}
\]

and:
\[ m = \frac{t_g}{T_s} \] (2.92)

which is the number of samples before intercept. Again, the bracketed quantity in (2.90) is the projected zero-effort miss distance.

If (2.92) is substituted into (2.91) and \( T_s \to 0 \), then \( N_{2d}^1 \) approaches \( N_3^1 \) in (2.87) for a continuous system, as expected.

The navigation ratio in (2.91) is seen to be a function of two dimensionless variables, \( m \) and \( b/T_s^3 \). Figure 2-13, which is adapted from Reference S8, compares the discrete \( N_{2d}^1 \) and the continuous \( N_3^1 \) versus number of samples before intercept for two values of the normalized control weighting, \( b/T_s^3 \). It is apparent that \( N_{2d}^1 < N_3^1 \), but that they are virtually equal for \( m > 8 \). Moreover, all navigation ratios tend to be 3 for large values of \( m \).

2.3.2.3 Case of Autopilot with One Pole in Continuous System

For this case, let the plant be that of Figure 2-12, but with \( \omega_2 \to \infty \), so that the autopilot is characterized by only one pole. In this case the guidance law of O'Halloran (Reference O'H3) and the equivalent later solution by Willems (Reference W6) may be expressed as:

\[
A_c = u = \frac{N_4^1}{t_g^2} \left[ \frac{\Lambda}{Y_d} + \frac{\Lambda}{t_g Y_d} + \frac{e^{-2\nu t_g} + 2\nu t_g - 1}{4\nu^2} \frac{e^{-2\nu t_g} + 2\nu t_g - 1}{\Lambda t} \right]
\] (2.93)

where:

\[
\begin{align*}
\Lambda &= e^{-\omega_1 t_g} + \omega_1 t_g - 1 \\
\Lambda_m &= \frac{\omega_1^2 t_g}{\omega_1^2 t_g} \\
\end{align*}
\]
Figure 2-13 - Continuous and Discrete Navigation Ratios versus Number of Samples Before Intercept
\[ N'_4 = \frac{3 (\omega^t_{1g})^2 (e^{-\omega^t_{1g}} + \omega^t_{1g} - 1)}{(\omega^t_{1g})^3 + 3 \left[ b\omega^3_{1} + \omega^t_{1g} - (\omega^t_{1g})^2 - 2 (\omega^t_{1g}) e^{-\omega^t_{1g}} - \frac{1}{2} e^{-2\omega^t_{1g}} + \frac{1}{2} \right]} \] (2.94)

Again, the bracketed quantity in (2.93) is the projected, zero-effort, miss distance, although it was not so recognized in References O'H3 and W6.

The fourth term in the brackets of (2.93) is equivalent to feedback around the autopilot so as to speed up its response at small values of \( \omega^t_{1g} \), when the lag is important. The denominator of (2.93) should include the factor \( \cos \theta_h \) for other than head-on engagements.

It is apparent that if \( \omega^t_{1} \to \infty \), then \( N'_4 \) approaches \( N'_3 \) in (2.87) for the zero-lag autopilot and the fourth term in (2.93) approaches 0, which is to be expected. Similar behavior occurs for a large value of \( \omega^t_{1g} \), for which the autopilot lag is not important.

2.3.2.4 Case of Autopilot with Two Poles in Continuous System

In Reference W7, Willems derived the optimal control law for a guidance system with autopilot having the transfer function:

\[ \frac{A_m}{A_c} = \frac{1}{(1 + s/\omega_1)(1 + s/\omega_3)} \] (2.95)

He commented:

"The contribution of the target acceleration has, however, been previously computed (Reference W6) and, since it remains invariant with the order of the plant, it does not need to be explicitly considered herein."
The plant equation, with acceleration in units of fps² rather than in g's was:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & \omega_3 & \omega_3 \\
0 & 0 & 0 & \omega_1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
\omega_1
\end{bmatrix}
A_c
\]  
(2.96)

where \( x_1, x_2 \) and \( x_3 \) are \( Y_d, \dot{Y}_d \) and \( A_m \), respectively. The performance index was equivalent to (2.11). Willem's solution for the optimal acceleration command can be expressed as:

\[
A_c = \frac{N'_W}{t_g^2} \left\{ \frac{\hat{Y}_d + \frac{\hat{Y}_d}{t_g}}{t_g} - \frac{1}{\omega_3^2} (e^{-\omega_3 t_g} + \omega_3 t_g - 1) \hat{A}_m \right. \\
\left. - \frac{\omega_3}{\omega_1 - \omega_3} \left[ \frac{1}{\omega_1^2} (e^{-\omega_1 t_g} + \omega_1 t_g - 1) - \frac{1}{\omega_3^2} (e^{-\omega_3 t_g} + \omega_3 t_g - 1) \right] \hat{x}_4 \right\}
\]  
(2.97)

where the effective navigation ratio is:

\[
N'_W = \frac{3 (\omega_1 t_g)^2}{\omega_1^2 - \omega_3^2 \frac{e^{-\omega_1 t_g} + \omega_1 t_g - \omega_1^2}{\omega_3 (\omega_1 - \omega_3)^2}} \frac{e^{-\omega_3 t_g} + \omega_3 t_g - \frac{\omega_1}{\omega_2}}{D}
\]  
(2.98)

the denominator of which is:
\[ D = (\omega_1 t_g)^3 + 3 \left\{ b \omega_1^2 + \frac{\omega_1}{\omega_1 - \omega_3} (2 \omega_3 t_g - 1) e^{-\omega_1 t_g} \right\} \]

\[ - \frac{2 \omega_1^3 (\omega_1 t_g - 1)}{\omega_3^2 (\omega_1 - \omega_3)} e^{-\omega_3 t_g} - \frac{\omega_1^2}{2 (\omega_1 - \omega_3)^2} e^{-2 \omega_1 t_g} \]

\[ - \frac{\omega_1^5}{2 \omega_3^3 (\omega_1 - \omega_3)^2} e^{-2 \omega_3 t_g} + \frac{2 \omega_1^3 e^{-(\omega_1 + \omega_3) t_g}}{(\omega_1^2 - \omega_3^2) (\omega_1 - \omega_3)} \]

\[ + \frac{\omega_1^5}{2 \omega_3^3 (\omega_1^2 - \omega_3^2)} - \frac{\omega_3^2}{2 (\omega_1^2 - \omega_3^2)} - \frac{\omega_1 (\omega_1 + \omega_2)}{\omega_3^2} (\omega_1 t_g - 1) (\omega_3 t_g - 1) \right\} \]

(2.99)

On preliminary inspection, there appears to be a slight error in \( N_W^i \), because the term \( b \omega_1^2 \) should be \( b \omega_1^3 \) in order to be dimensionless like the other terms. Except for this anomaly, \( N_W^i \) reduces to \( N_4^i \) in (2.94) if \( \omega_3 \rightarrow \infty \). Another consequence of infinite \( \omega_3 \) is that the \( x_3 = A_m \) state in (2.96) disappears and \( x_4 \) becomes \( \hat{A}_m \). It is apparent that the \( \hat{A}_m \) term in (2.97) vanishes and that the \( \hat{x}_4 \) term then resembles the \( \hat{A}_m \) term in (2.93), except for the wrong sign.

The primary reason for giving the foregoing solution from Reference W7 is to show the complexity of closed-form guidance laws for more than one autopilot pole. If the term for \( \hat{A}_t \) were included in (2.97), the result would be somewhat more complex than the corresponding guidance law for an autopilot with one pole and one zero, as derived in the next chapter. For the sake of admissible tactical computational complexity and realism in modelling a tail-controlled missile, the latter guidance law is preferred in this thesis.
CHAPTER 3

TERMINAL CONTROL AND GUIDANCE LAWS FOR GENERAL AND SPECIFIC CASES

3.1 Introduction

Terminal guidance of a homing interceptor missile can be considered as a subset of a larger set of optimal terminal control problems, i.e., those for which the system is linear, it has a known disturbance vector, and the performance index is quadratic with a term in the final state. In addition to the known disturbance vector, there can be white process noise and white measurement noise, but these noises are relegated to the estimation subproblem by the separation theorem.

Subsection 3.2 solves this general terminal control problem for a system with continuous control and obtains a rather interesting result for the case in which the loss function in the performance index contains only the quadratically weighted control effort. This general result is then applied to the missile-guidance problem, where it shows that the optimal control is proportional to the projected, zero-control miss distance, including the effect of the known disturbance.

Subsection 3.3 gives an equivalent derivation of the same general result for a system with discrete (or multistage) control and applies it similarly to missile guidance.

The foregoing results could be applied to a guidance system with a given autopilot. Alternatively, they could be applied to the guidance kinematic states ($Y_d$, $\dot{Y}_d$ and $A_t$ if it is so modelled) and to a bare airframe,
in which case the autopilot would have time-varying gains. The latter alternative would not be suitable for designing an autopilot, for which practical requirements would dictate fixed gains.

Other criteria are needed to design a practical autopilot. One criterion is that it should be "fast," in order to minimize miss distance, as shown by many adjoint simulations. Subsection 3.4 optimizes a general autopilot transfer function with three poles so as to minimize the integral of square acceleration error in response to a step command. The resulting theoretical autopilot has a transfer function with one RHP zero (from the airframe) and an LHP pole of the same magnitude. Although this transfer function is not physically realizable in a real autopilot, it can usefully be approached in the actual design of a fast autopilot. Moreover, this means that the matching guidance law can model the autopilot as having only one state, which reduces computational complexity.

Accordingly, Subsection 3.5 gives in closed form the continuous guidance law for a one-pole, one-zero autopilot, with target acceleration and missile axial acceleration being treated as supposedly known disturbances. The behavior of the effective navigation ratio \( N_g \) for a particular example is explained.

Subsection 3.6 shows how to compute discrete optimal control gains for a variety of plants with a single flexible computer program. The effective navigation ratio \( N' \) for the discrete equivalent of the previous example is computed, interpreted and compared.

Under simplifying but generally valid assumptions that decouple the computation of time to go, \( t_g \), from the optimal control computations, Subsection 3.7 gives a general algorithm for computing \( t_g \).
3.2 The General Continuous Terminal Control Problem with a Known Disturbance Vector

3.2.1 Problem Statement

Let the plant equation be:

\[
\dot{x}(t) = F(t)x(t) + G(t)\bar{u}(t) + w(t) \tag{3.1a}
\]

\[
x(t_p) = x_p \tag{3.1b}
\]

where \(w(t)\) is an arbitrary disturbance vector which is a presumably known function of time. The dimensions of \(x(t)\) and \(w(t)\) are \(n\) by 1 while that of \(\bar{u}(t)\) is \(n_u\) by 1; \(t_p\) is "present time."

Initially, let the performance index be a fairly general quadratic:

\[
J = \frac{1}{2} x(t_f)^T S_f x(t_f) + \frac{1}{2} \int_{t_p}^{t_f} \left[ x(t)^T A(t) x(t) + \bar{u}(t)^T B(t) \bar{u}(t) \right] dt \tag{3.2}
\]

It is assumed that the final time is independent of the control \(\bar{u}(t)\) and is computed by other equations, as in Subsection 3.7.

3.2.2 Necessary Conditions for Optimality

The Hamiltonian is:

\[
H = \frac{1}{2} x^T A x + \frac{1}{2} \bar{u}^T B \bar{u} + p^T F x + p^T C \bar{u} + p^T w \tag{3.3}
\]

The functional dependence on time, e.g., \(x(t)\), is still understood, but the notation is simplified for the sake of clarity. From References B1 and A4, the necessary conditions for optimality (in addition to the plant equation (3.1)) are:

\[
\dot{p} = - \left[ \frac{\partial H}{\partial x} \right]^T = - A x - F^T p \tag{3.4}
\]
\[ p(t_f) = S_f x(t_f) \]  \hspace{1cm} (3.5)

\[
\begin{bmatrix}
\frac{\partial H}{\partial u} \\
\frac{\partial H}{\partial u}
\end{bmatrix}
\begin{bmatrix}
T \\
T
\end{bmatrix} = 0 = Bu + G^T p
\]  \hspace{1cm} (3.6)

from which

\[ u = -B^{-1}G^T p \]  \hspace{1cm} (3.7)

where \( p \) is called the Lagrange multiplier or costate vector.

### 3.2.3 Solution as a Two-Point Boundary Value Problem

The solution is somewhat generalized from pp. 758-760 of Reference A4. After substitution of (3.7) into (3.1), it may be combined with (3.4) into a 2n by 1 vector equation:

\[
\begin{bmatrix}
x \\
p
\end{bmatrix} =
\begin{bmatrix}
F & -GB^{-1}G^T \\
-A & -F^T
\end{bmatrix}
\begin{bmatrix}
x \\
p
\end{bmatrix} +
\begin{bmatrix}
w \\
0
\end{bmatrix}
\]  \hspace{1cm} (3.8)

for which there are boundary conditions at each of two points as given by (3.1b) and (3.5). In (3.8), the 2n by 2n matrix can be time-varying or fixed, and it has a 2n by 2n transition matrix which may be conveniently partitioned into four n by n submatrices:

\[
\Omega(t, t_p) =
\begin{bmatrix}
\Omega_{xx}(t, t_p) & \Omega_{xp}(t, t_p) \\
\Omega_{px}(t, t_p) & \Omega_{pp}(t, t_p)
\end{bmatrix}
\]  \hspace{1cm} (3.9)

This transition matrix has the differential equation and boundary condition:
\[
\frac{d}{dt} \Omega(t, t_p) = \begin{bmatrix} F & -GB^{-1}GT \\ -A & -FT \end{bmatrix} \Omega(t, t_p)
\]

(3.10)

\[
\Omega(t, t_p) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}
\]

(3.11)

The solution of (3.8) from "present time" conditions is then:

\[
\begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \Omega(t, t_p) \begin{bmatrix} x(t_p) \\ p(t_p) \end{bmatrix} + \int_{t_p}^{t} \Omega(t, \tau) \begin{bmatrix} w(\tau) \\ 0 \end{bmatrix} d\tau
\]

(3.12a)

where:

\[
t_p \leq t \leq t_f, \quad t_p \leq \tau \leq t
\]

(3.12b)

From (3.9) and (3.12a), the final value of the state vector is:

\[
x(t_f) = \Omega_{xx}(t_f, t_p)x(t_p) + \Omega_{xp}(t_f, t_p)p(t_p) + \int_{t_p}^{t_f} \Omega_{xx}(t_f, \tau)w(\tau) d\tau
\]

(3.13)

and the final value of the costate vector is:

\[
p(t_f) = \Omega_{px}(t_f, t_p)x(t_p) + \Omega_{pp}(t_f, t_p)p(t_p) + \int_{t_p}^{t_f} \Omega_{px}(t_f, \tau)w(\tau) d\tau
\]

(3.14)

Substitution of (3.13) and (3.14) into (3.5), together with collection of terms, gives:
\[
\left[ S_f \Omega_{xx}(t_f, t_p) - \Omega_{px}(t_f, t_p) \right] \mathbf{x}(t_p) \\
+ \int_{t_p}^{t_f} \left[ S_f \Omega_{xx}(t_f', \tau) - \Omega_{px}(t_f', \tau) \right] \mathbf{w}(\tau) d\tau \\
= \left[ \Omega_{pp}(t_f, t_p) - S_f \Omega_{xp}(t_f, t_p) \right] \mathbf{p}(t_p)
\]

from which \( \mathbf{p}(t_p) \) may be solved and substituted into (3.7) to give:

\[
\mathbf{u}(t_p) = -B^{-1}_G \mathbf{T} \left[ \Omega_{pp}(t_f, t_p) - S_f \Omega_{xp}(t_f, t_p) \right]^{-1}.
\]

\[
\left\{ \left[ S_f \Omega_{xx}(t_f, t_p) - \Omega_{px}(t_f, t_p) \right] \mathbf{x}(t_p) \\
+ \int_{t_p}^{t_f} \left[ S_f \Omega_{xx}(t_f', \tau) - \Omega_{px}(t_f', \tau) \right] \mathbf{w}(\tau) d\tau \right\}
\]

(3.16)

Reference A4 (p. 760) states that the indicated inverse exists. This equation gives the general solution for the optimal control vector \( \mathbf{u}(t_p) \) in terms of the present state \( \mathbf{x}(t_p) \) (which constitutes feedback and is desirable from a practical standpoint) and in terms of the presumably known disturbance \( \mathbf{w}(\tau) \).

References O'D1 and B1 (p. 157) discuss the symplectic aspects of the matrices in (3.8) and (3.9), respectively.

3.2.4 Solution with Control Vector Only in the Loss Function

3.2.4.1 General Result in Terms of Projected, Zero-Effort Terminal State

If there is no need to penalize the state \( \mathbf{x}(t) \) before the final time, then the matrix \( A \) in (3.2) may be set equal to zero. Then (3.10) and (3.11) can
be expressed as the following four matrix equations with corresponding boundary conditions:

\[
\frac{d}{dt} \Omega_{xx}(t, t_p) = F \Omega_{xx}(t, t_p) - GB^{-1} G^T \Omega_{px}(t, t_p) \tag{3.17a}
\]

\[
\Omega_{xx}(t_p', t_p) = I \tag{3.17b}
\]

\[
\frac{d}{dt} \Omega_{xp}(t, t_p) = F \Omega_{xp} - GB^{-1} G^T \Omega_{pp}(t, t_p) \tag{3.18a}
\]

\[
\Omega_{xp}(t_p', t_p) = 0 \tag{3.18b}
\]

\[
\frac{d}{dt} \Omega_{px}(t, t_p) = -F^T \Omega_{px} \tag{3.19a}
\]

\[
\Omega_{px}(t_p', t_p) = 0 \tag{3.19b}
\]

\[
\frac{d}{dt} \Omega_{pp}(t, t_p) = -F^T \Omega_{pp} \tag{3.20a}
\]

\[
\Omega_{pp}(t_p', t_p) = I \tag{3.20b}
\]

From this it follows directly that:

\[
\Omega_{px}(t, t_p) = 0 \tag{3.21}
\]

\[
\frac{d}{dt} \Omega_{xx}(t, t_p) = F \Omega_{xx}(t, t_p) \tag{3.22a}
\]

\[
\Omega_{xx}(t_p', t_p) = I \tag{3.22b}
\]
where \( \Omega_{xx}(t, t_p) \) is recognized as the transition matrix of the basic plant in (3.1a) without feedback. It is straightforward to show that:

\[
\Omega_{pp}(t, t_p) = \Omega_{xx}^{-T}(t, t_p) = \Omega_{xx}^{T}(t_p, t)
\]  

(3.23)

Substitution of (3.21) into (3.16) and factoring of the \( S_f \) matrix gives:

\[
\underline{u}(t_p) = -B^{-1}G^T \left[ \Omega_{pp}(t_f, t_p) - S_f \Omega_{xx}(t_f, t_p) \right]^{-1}.
\]  

(3.24)

\[
S_f \left\{ \Omega_{xx}(t_f, t_p)x(t_p) + \int_{t_p}^{t_f} \Omega_{xx}(t_f, \tau)w(\tau) \, d\tau \right\}
\]

This latter equation is rather significant for terminal controllers for fixed or time-varying plants. The expression in the curved braces is the projected state vector at terminal time in the presence of the disturbance vector \( \underline{w}(t) \) but in the absence of control effort \( \underline{u}(t) \), which is readily seen after reexamining (3.1a) and remembering that \( \Omega_{xx}(t, t_p) \) is the transition matrix for a fixed or time-varying plant matrix \( F \). Moreover, this projected state vector is weighted by \( S_f \), the weighting matrix of the terminal cost. To restate this general result:

For a linear system with a known disturbance and a quadratic performance index (that does not include the state vector in the loss function), the optimal control is proportional through a time-varying function to the terminally-weighted, projected, zero-control state vector (including the effect of the disturbance).
3.2.4.2 Application to Missile Guidance

The material in the previous section is readily applied to the guidance and control of an interceptor missile, in one plane. The same formulation applies to the perpendicular plane which intersects the first plane along a line through the missile parallel to the original line of sight. The coordinate system is that of Figure 3-1 (see also Reference Bl, p. 424), and the first two elements of the state vector $\mathbf{x}(t)$ are the differential position $Y_d$ and the differential velocity $V_d$ of the target and missile.

There are two possibilities for modelling the target acceleration $A_{ty}$ perpendicular to the line of sight:

1) It can be modelled as the third state of the system, coming from an integrator with feedback as in Figure 2-11.

2) Alternatively, $A_{ty}$ can be made part of the disturbance vector $\mathbf{w}(t)$, so that any of various plausible assumptions about its future behavior can be made; this choice also simplifies the analysis of the inverted matrix in (3.24).

Let it be assumed that the pitch angle $\theta$ of the missile is nearly constant and that the axial-acceleration component $A_a \sin \theta$ (which is perpendicular to the line of sight) is some known function of time, based on the thrust and zero-lift drag of the missile. Then, the disturbance vector $\mathbf{w}(t)$ contributes directly to the derivative $d (V_d)/dt$ of the second state and may be written as:

$$\mathbf{w}(t) = \begin{bmatrix} 0 \\ A_{ty} - A_a \sin \theta h \\ 0 \\ \vdots \end{bmatrix} \quad (3.25)$$
Figure 3-1 - Two-Dimensional Intercept with Moving, Nonrotating Coordinate System
The control \( u(t) \) will typically be a scalar for the planar intercept problem. There are various possibilities for \( u(t) \) and the remaining elements of the state vector:

1) In the simplest case, \( u(t) \) is \( A_n \), the lateral acceleration of the missile in Figure 3-1, and there are no other elements of \( x(t) \) besides \( Y_d \) and \( V_d \). This case of a zero-lag autopilot was analyzed in Reference O'H2 for constant values of \( A_{ty} \), \( A_a \) and \( \theta \).

2) Alternatively, the missile can have a fixed autopilot with \( n-2 \) states and \( u(t) \) can be the acceleration command. As a particular case, the guidance law for an autopilot with one pole and one zero (i.e., one state) is given in Subsection 3.5. Feedback from three states results.

3) Alternatively, the remaining \( n-2 \) states may be those of the bare airframe (without autopilot) and its fin servos and instruments; \( u(t) \) may be a scalar input (fin-angle command) to the fin servos or perhaps a vector input. Of course, time-varying feedback gains from all \( n \) states would result.

The performance index is then simply:

\[
J = \frac{1}{2} x_d^2 (t_f) + \frac{b}{2} \int_{t_p}^{t_f} u^2(t) \, dt \tag{3.26}
\]

where the terminal weighting matrix \( S_f \) has been chosen as:

\[
S_f = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & \\
& \cdots & \cdots & \\
0 & \cdots & \cdots & 0
\end{bmatrix} \tag{3.27}
\]
In any of the three foregoing representations of the system, (3.24) would apply and the expression in the curved braces would be the projected, zero-control final state. Moreover, the product of $S_f$ and this expression is simply:

$$S_f \left\{ \Omega_{xx}(t_f, t_p) x(t_p) + \int_{t_p}^{t_f} \Omega_{xx}(t_f, \tau) w(\tau) \, d\tau \right\} = \begin{bmatrix} M_{\text{pzC}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$  \hspace{1cm} (3.28)

where $M_{\text{pzC}}$ is the projected, zero-control miss distance, including the contribution of $w(t)$ but not the control $u(t)$. Therefore, the optimal control $u(t)$ is proportional to $M_{\text{pzC}}$, which seems reasonable and likely to minimize the contribution of the second term in (3.26) to $J$. Again, the plant in (3.1a) can be fixed or time-varying.

3.3 A General Discrete Terminal Control Problem with Control Vector Only in Loss Function, and a Known Disturbance Function

3.3.1 Necessary Conditions for Optimality

Following Reference B1, p. 43 somewhat in approach and notation, consider the general nonlinear discrete or "multistage" problem with a fixed number $N$ of stages:

$$x(i+1) = f^i \left[ x(i), u(i) \right] + z(i) \hspace{1cm} (3.29a)$$

$$x(0) = x_0 \hspace{1cm} (3.29b)$$

where $z(i)$ is a known disturbance corresponding to $w(t)$ in (3.1a). Let the performance index be:
\[ J = \Phi \left[ x \right] (N) + \sum_{i=0}^{N-1} L^i \left[ x(i), u(i) \right] \]  

(3.30)

The Hamiltonian at the i-th stage is:

\[ H^i = L^i \left[ x(i), u(i) \right] + P^T(i+1) \left\{ f^i \left[ x(i), u(i) \right] + z(i) \right\} \]  

(3.31)

The necessary conditions for optimality are:

\[ P^T(i) = \frac{\partial H^i}{\partial x(i)} = \frac{\partial L^i}{\partial x(i)} + P^T(i+1) \frac{\partial f^i}{\partial x(i)} \]  

(3.32)

\[ P^T(N) = \frac{\partial \Phi \left[ x(N) \right]}{\partial x(N)} \]  

(3.33)

\[ \frac{\partial H^i}{\partial u(i)} = 0 = \frac{\partial L^i}{\partial u(i)} + P^T(i+1) \frac{\partial f^i}{\partial u(i)} \]  

(3.34)

Equations (3.29a), (3.32) and (3.34) hold for steps 0 to N-1, the last one ending at the terminal time.

The foregoing result may be specialized to a linear system with a quadratic performance index:

\[ x(i+1) = \Phi (i) x(i) + \Gamma (i) u(i) + z(i) \]  

(3.35)

\[ J = \frac{1}{2} x^T(N)A(N)x(N) + \sum_{i=0}^{N-1} \left[ \frac{1}{2} x^T(i)A(i)x(i) + \frac{1}{2} u^T(i)B(i)u(i) \right] \]  

(3.36)

Equations (3.31) through (3.34) become:

\[ H^i = \frac{1}{2} x^T(i)A(i)x(i) + \frac{1}{2} u^T(i)B(i)u(i) \]  

\[ + P^T(i+1) \left[ \Phi (i)x(i) + \Gamma (i)u(i) + z(i) \right] \]  

(3.37)
\[ p^T(i) = x^T(i)A(i) + p^T(i+1) \Phi(i) \quad (3.38) \]

\[ p^T(N) = x^T(N)A(N) \quad (3.39) \]

\[ u^T(i)B(i) + p^T(i+1)\Gamma(i) = 0 \quad (3.40) \]

from which the optimal control at the i-th stage is:

\[ u(i) = -B^{-1}(i)\Gamma^T(i)p(i+1) \quad (3.41) \]

### 3.3.2 An Analytic Solution

The solution for \( A(i) = 0 \) proceeds as follows. Substitute (3.41) into (3.35) to obtain:

\[ x(i+1) = \Phi(i)x(i) - \Gamma(i)B^{-1}(i)\Gamma^T(i)p(i+1) + z(i) \quad (3.42) \]

With \( A(i) = 0 \), (3.38) becomes simply:

\[ p(i) = \Phi^T(i)p(i+1) \quad (3.43) \]

Now, let \( x(c) \) be the state at the current stage, corresponding to present time \( t_p \) in Subsection 3.2, such that:

\[ c \leq i < N \quad (3.44) \]

Recognizing the symmetry of \( A(N) \), it follows from (3.39) and (3.43) that the current costate is related to the terminal state by:
\[ p(c+1) = \Phi^T(c+1) \ldots \Phi^T(N-1)A(N)\bar{x}(N) \quad (3.45) \]

\[
= \left[ \prod_{i=c+1}^{N-1} \Phi^T(i) \right] A(N)\bar{x}(N)
\]

Substitution of (3.45) into (3.42) with \( i = c \) gives:

\[
\bar{x}(c+1) = \Phi(c)\bar{x}(c) - \Gamma(c)B^{-1}(c)\Gamma^T(c) \left[ \prod_{i=c+1}^{N-1} \Phi^T(i) \right] A(N)\bar{x}(N) + \bar{z}(c) \quad (3.46)
\]

The multiplication symbol \( \prod \) in (3.45) and (3.46) must be interpreted carefully. Reference to (3.39) shows that for \( c = N - 1 \) the square-bracketed quantity should be the identity matrix \( I \), while for \( c = N - 2 \) it should be \( \Phi^T(N-1) \). In general, the bottom index and the top index are respectively on the left and right in a matrix product.

The state \( x(c+1) \) in (3.46) may be extended forward stage by stage inductively. The next state is:

\[
x(c+2) = \Phi(c+1) \left\{ \Phi(c)\bar{x}(c) - \Gamma(c)B^{-1}(c)\Gamma^T(c) \left[ \prod_{i=c+1}^{N-1} \Phi^T(i) \right] A(N)\bar{x}(N) + \bar{z}(c) \right\}
\]

\[
- \Gamma(c+1)B^{-1}(c+1)\Gamma^T(c+1) \left[ \prod_{i=c+2}^{N-1} \Phi^T(i) \right] A(N)\bar{x}(N) + \bar{z}(c+1) \quad (3.47)
\]

which should be restricted to \( c \leq N-2 \) for best interpretation. By induction, the \( N \)-th state can be found from (3.47) as:

\[
\bar{x}(N) = \left[ \prod_{k=N-1}^{c} \Phi(k) \right] x(c) + \sum_{k=c}^{N-2} \left[ \prod_{i=N-1}^{k+1} \Phi(i) \right] \bar{z}(k) + \bar{z}(N-1)
\]
\[- \sum_{k = c}^{N-1} \prod_{j = N-1}^{k+1} \Phi(j) \Gamma(k)B^{-1}(k) \Gamma_T(k) \prod_{i = k+1}^{N-1} \Phi_T(i) A(N) \underline{x}(N) \]

(3.48)

which should also be restricted to \( c \leq N-2 \). The third multiplication symbol should be interpreted for \( k = N-1 \) as:

\[
\left[ \prod_{j = N-1}^{k+1} \Phi(j) \right]_{k = N-1} = I
\]

(3.49)

Equation (3.48) may be solved for the terminal state \( \underline{x}(N) \) in terms of the present state \( \underline{x}(c) \) and the disturbance function \( \underline{z}(k) \) as:

\[
\underline{x}(N) = \left[ I + \sum_{k = c}^{N-1} \prod_{j = N-1}^{k+1} \Phi(j) \Gamma(k)B^{-1}(k) \Gamma_T(k) \prod_{i = k+1}^{N-1} \Phi_T(i) A(N) \right]^{-1} \left\{ \left[ \prod_{k = N-1}^{c} \Phi(k) \right] \underline{x}(c) + \sum_{k = c}^{N-2} \prod_{i = N-1}^{k+1} \Phi(i) \underline{z}(k) + \underline{z}(N-1) \right\}
\]

(3.50)

In order to avoid confusion, the following equation should be used in place of (3.50) for \( c = N-1 \):

\[
\underline{x}(N) = \left[ I + \Gamma(N-1)B^{-1}(N-1) \Gamma_T(N-1)A(N) \right]^{-1} \left[ \Phi(N-1)\underline{x}(N-1) + \underline{z}(N-1) \right]
\]

(3.51)

which readily follows from (3.42).

Substitution of (3.45) into (3.41) with \( c = i \) gives the optimal control at the present time in terms of the final state \( \underline{x}(N) \):
\[ u(c) = -B^{-1}(c) \Gamma^T(c) \left[ \Phi^T(i) \right] A(N)x(N) \] (3.52)

Equation (3.50) could be substituted for \( x(N) \) in (3.52) to give the discrete optimal guidance in the desired feedback form, but the actual equation will be omitted for brevity. For interpretation, it should be noted that (3.50) gives the terminal state \( x(N) \), which is proportional to the projected, zero-control, terminal state (in curved braces) in the presence of the known disturbance \( z(k) \). Equations (3.50) and (3.52) were used in a digital computer program (DGL1) to compute guidance gains.

3.3.3 An Alternate Analytic Solution, with Application to Missile Guidance

Alternatively, (3.48) can be multiplied by \( A(N) \) and the result solved for \( A(N)x(N) \):

\[
A(N)x(N) = \left[ I + A(N) \sum_{k=c}^{N-1} \left[ \Phi^{(j)} \right] \Gamma(k)B^{-1}(k)\Gamma^T(k) \right]^{-1} \left[ \Phi^{(i)} \right] ^{-1} \] (3.53)

Then (3.53) can be substituted for \( A(N)x(N) \) in (3.52) to give an alternate form of the discrete optimal guidance law in feedback form; the square-bracketed product in (3.52) can be absorbed into the inverse in (3.53) so as to simplify its second term, if desired.

Again, the expression in curved braces in (3.53) is the projected, zero-control, terminal state, including the effect of the known disturbance.
$z(k)$, and this expression is now multiplied by the terminal weighting. Therefore, the optimal discrete control at the present stage in (3.52) is proportional to the terminally-weighted, projected, zero-control, final state, as in the continuous case in Subsection 3.2.4.1. For missile guidance, this means that the optimal control is proportional, through a time-varying factor, to the projected, zero-control miss distance (including the effect of the known disturbance), as in Subsection 3.2.4.2.

3.3.4 Efficient Tactical Computation during Missile Engagement

Initially, this writer made an attempt to obtain a closed-form, discrete guidance law for a system in which the autopilot had one pole and one zero, as in Subsections 1.3.1.5 and 3.5 herein for the continuous case. It quickly became apparent that a closed-form, discrete guidance law is extremely hard to derive for any case harder than the simple one worked out by J. Speyer in Reference S10, as shown in Subsection 2.3.2.2 herein. The principal difficulty is connected with the inverse in (3.53) or its alternate form as mentioned.

In a tactical engagement, it would appear to be disadvantageous to perform all the matrix algebra in (3.52) and (3.53) on line. The following more efficient alternative is possible. As in (2.93), the optimal control may be expressed in the following form:

$$u(c) = C_1(c)^\wedge Y_d(c) + C_2(c)^\wedge Y_d(c) + \ldots$$

$$= \frac{N_1}{2} g \left[ Y_d(c) + t g Y_d(c) + \ldots \right]$$

(3.54)

where the effect of the n controllable states plus the known disturbance
is included. The bracketed expression is the projected, zero-control, miss distance, as in the previous subsection. The discrete $N'$ can be solved as a function of the number of "samples-to-go", autopilot or airframe parameters and control weighting, with this computation being done off-line before the engagement; $N'$ can then be stored versus the number of "samples to go" for the last 10 or 15 samples of the engagement. Figure 2-3 would indicate that the use of a continuously-derived $N'$ would be quite sufficient for longer times to go. Therefore, the tactical computation of the discrete guidance law during the engagement would reduce to the computation of projected, zero-control miss distance, a table lookup for $N'$ and a multiplication to get the optimal control $u$.

3.4 Optimization of the Autopilot Transfer Function for Minimum Integral-Squared Error

3.4.1 Motivation

One might approach the autopilot design problem in at least four ways:

1) One could merely continue to utilize the classical autopilot in Figure 2-2 because of its simplicity and proven performance; however, this would foreclose opportunities for fundamental improvement.

2) One could attempt to design optimal control gains from the airframe and simultaneously from the guidance geometric variables to the fin-servo input, as is done in Chapter 6 herein; this has the disadvantage that the autopilot would have time-varying rather than fixed gains.

3) One could attempt to design time-varying control gains from the guidance geometric variables and fixed control gains
from the airframe, in the same optimization process; this appears to be rather difficult to do.

4) Finally, one could design the autopilot with fixed control gains for a specified transfer function, which is done in Chapter 4; however, at the outset it is not immediately clear how to specify this transfer function.

The research in this subsection originally grew out of an attempt to obtain the maximum speed of response of the Raytheon autopilot, in the sense of minimum of integral of squared acceleration error. Typically, faster autopilots produce lower miss distance, particularly when they are not perfectly matched to their guidance laws, because they permit effective corrections at the end of the intercept.

Fortunately, the cubic transfer function of the Raytheon autopilot can be applied to other potential configurations as well, and the investigation gave a rather fundamental result for a tail-controlled airframe. Furthermore, it turns out to be helpful in employing the approach no. 4 above.

3.4.2 Unconstrained Optimization

From Appendix C, one may utilize the following third-order transfer function for the Raytheon autopilot from command $A_c$ to output acceleration $A_m$:

$$\frac{A_m}{A_c} = \frac{1 + a_{11}s + a_{12}s^2}{1 + B_1s + B_2s^2 + B_3s^3} \quad (3.55)$$

where unity gain at zero frequency has been achieved by a simple gain ahead of the autopilot. The only lags are assumed to be in the airframe and the integrator. The numerator is due to airframe dynamics, with
a_{12} being related to tail force:

\[
a_{12} = -\frac{Z_\delta}{M' \alpha Z_\delta - M' \delta Z_\alpha}
\]  

(3.56)

which is negative in a tail-controlled airframe; the coefficient \(a_{11}\) is negligible. It is apparent that the transfer function in (3.55) could be applied to various other autopilot configurations with three poles.

Now, a plausible criterion for the speed of autopilot response is a low value of the integral of squared acceleration error in response to a unit step, as in Reference N2:

\[
I = \int_{-\infty}^{+\infty} \left[ A_c(t) - A_m(t) \right]^2 dt = \int_{-\infty}^{+\infty} E_1^2(t) dt
\]  

(3.57)

where \(u(t)\) is a unit step beginning at \(t = 0\). Reference N2, p. 43, shows that this is equal to:

\[
I = \frac{1}{2\pi j} \int_{-\infty}^{+\infty} E_1(s)E_1(-s) ds
\]  

(3.58)

where \(E_1(s)\) is the transform of \(E_1(t)\):

\[
E_1(s) = \frac{\left( B_1 - a_{11}\right) + (B_2 - a_{12})s + B_3s^2}{1 + B_1s + B_2s^2 + B_3s^3}
\]  

(3.59a)

\[
= \frac{c_0 + c_1s + c_2s^2}{d_0 + d_1s + d_2s^2 + d_3s^3}
\]  

(3.59b)

in which (3.59b) utilizes the notation of Reference N2, p. 371-372. This Reference gives the following algebraic equivalent for \(I\) in (3.57):
\[
I = \frac{c_2^2d_0d_1 + (c_1^2 - 2c_0c_2)d_0d_3 + c_0^2d_2d_3}{2d_0d_3 (-d_0d_3 + d_1d_2)} \tag{3.60a}
\]

\[
= \frac{B_3^2B_1 + [(B_2 - a_{12})^2 - 2(B_1 - a_{11})B_3]B_3 + (B_1 - a_{11})^2B_2B_3}{2B_3 (-B_3 + B_1B_2)} \tag{3.60b}
\]

\[
\approx \frac{(B_2 - a_{12})^2 - B_1B_3 + B_1^2B_2}{2(B_1B_2 - B_3)} \tag{3.60c}
\]

In (3.60c), \( B_3 \) was cancelled from the numerator and denominator of (3.60b) and then \( a_{11} \) was dropped because it is typically orders of magnitude less than \( B_1 \).

In the unconstrained minimization of \( I \), the parameters \( B_1, B_2 \) and \( B_3 \) are solved for a zero gradient:

\[
\frac{dI}{dB} = \left[ \frac{\partial I}{\partial B_1}, \frac{\partial I}{\partial B_2}, \frac{\partial I}{\partial B_3} \right] = \left[ 0 \quad 0 \quad 0 \right] \tag{3.61}
\]

The first partial derivative is:

\[
\frac{\partial I}{\partial B_1} = \frac{2(B_1B_2 - B_3)(2B_1B_2 - B_3) - \left[B_1(B_1B_2 - B_3)+(B_2 - a_{12})^2\right]2B_2}{4(B_1B_2 - B_3)^2} \tag{3.62}
\]

Setting this equal to zero results in:

\[
(B_1B_2 - B_3)^2 - B_2(B_2 - a_{12})^2 = 0 \tag{3.63}
\]

The second partial derivative is:

\[
\frac{\partial I}{\partial B_2} = \frac{2(B_1B_2 - B_3)\left[B_1^2 + 2(B_2 - a_{12})\right] - \left[B_1(B_1B_2 - B_3) + (B_2 - a_{12})^2\right]^2}{4(B_1B_2 - B_3)^2} \tag{3.64}
\]
which, when set equal to zero, results in:

$$2(B_1B_2 - B_3) - B_1(B_2 - a_{12}) = 0 \quad (3.65)$$

The third partial derivative in (2.61) is:

$$\frac{\partial^3 I}{\partial B_3^3} = \frac{2(B_1B_2 - B_3)(-B_1) - \left[B_1(B_1B_2 - B_3) + (B_2 - a_{12})^2\right](-2)}{4(B_1B_2 - B_3)^2} \quad (3.66a)$$

When (3.66a) is set equal to zero, it may be solved for $B_2$:

$$B_2 = a_{12} \quad (3.66b)$$

which is negative, causing the system to be unstable and (3.60) to be invalid. Although the impracticality of this solution is now emerging, nevertheless it is helpful to continue it. Substitution of (3.66b) into (3.65) results in:

$$B_3 = B_1a_{12} \quad (3.67)$$

It happens that substitution of (3.66b) into (3.63) also results in (3.67), and so a unique solution of $B_3$ and $B_1$ is not possible; a more complicated solution with the inclusion of $a_{11}$ led to a similar impasse. At any rate, if one assumes that $B_1$ must have a positive value, then (3.67) shows that $B_3$ is negative. Substitution of (3.66b) and (3.67) into (3.55) gives:

$$\frac{A_m}{A_c} = \frac{1 + a_{12}s^2}{1 + B_1s + a_{12}s^2 + B_1a_{12}s^3} \quad (3.68a)$$
\[
= \frac{1 + a_{12}s^2}{(1 + B_1s)(1 + a_{12}s^2)}
\]

(3.68b)

\[
= \frac{1}{1 + B_1s}
\]

(3.68c)

This shows that the unconstrained solution results in the cancellation of both the LHP and the RHP zeroes by poles, which is obviously impractical.

3.4.3 Constrained Optimization

Clearly, it is necessary to constrain the optimization such that all poles lie in the left half plane. Together with the requirement \( B_1 > 0 \), the Routh criterion (Reference G9) provides a necessary and sufficient set of inequality constraints:

\[
B_2 > 0
\]

(3.69)

\[
B_3 \geq 0
\]

(3.70)

\[
\frac{B_1B_2 - B_3}{B_3} > 0
\]

(3.71)

The following method of solving the optimization problem with inequality constraints is similar to that of pp. 24-28 of Reference B1, although this author first became aware of the technique from Reference V3. Since the unconstrained optimization violates at least one (in this case, both Equations (3.69) and (3.70)) of the inequality constraints, then the solution of the constrained optimization problem must lie on at least one of the constraint surfaces in parameter space, for which the vector is:
\[
B = \begin{bmatrix}
B_1 \\
B_2 \\
B_3
\end{bmatrix}
\]  \hspace{1cm} (3.72)

Suppose one hypothesizes, subject to verification, that the solution lies on the constraint surface of (3.70), i.e., \( B_3 = 0 \), which happens to eliminate (3.71) from consideration. It then follows from References B1 and V3 that the gradient in (3.61) must be normal to the constraint surface of (3.70):

\[
\frac{\partial I}{\partial B} = \begin{bmatrix}
0 & 0 & \frac{\partial I}{\partial B_3}
\end{bmatrix}
\]  \hspace{1cm} (3.73)

in which the third element is positive, which means that any reduction in the integral \( I \) can come only from violating the constraint (3.70). Then, the applicable equations are:

\[
B_3 = 0 \hspace{1cm} (3.74)
\]

and also (3.63) and (3.65). Substitution of (3.74) into (3.65) gives:

\[
B_2 = -a_{12} \hspace{1cm} (3.75)
\]

which is positive. Substitution of (3.74) and (3.75) into (3.63) results in:

\[
B_1 = 2 \sqrt{-a_{12}} \hspace{1cm} (3.76)
\]

after rejection of a negative root. The partial derivative in (3.66a) may be
simpli' directly to:

\[
\frac{\partial I}{\partial B_3} = \frac{(B_2 - a_{12})^2}{2(B_1 B_2 - B_3)^2}
\]  \hspace{1cm} (3.77)

and after substitution of (3.74) - (3.76) it becomes:

\[
\frac{\partial I}{\partial B_3} = \frac{1}{-2a_{12}}
\]  \hspace{1cm} (3.78)

which is positive. Therefore, the constraints in (3.69) - (3.71) and the gradient criterion in (3.73) are satisfied, and so (3.74) - (3.76) probably represents the constrained optimal solution. It seems unlikely that there are others.

3.4.4 Interpretation and Utilization of Result

The following interpretation of the solution is quite helpful conceptually. Substitution of (3.74) - (3.76) into (3.55) gives:

\[
\frac{A_m}{A_c} = \frac{1 + a_{12}s^2}{1 + 2\sqrt{-a_{12}s} - a_{12}s^2}
\]  \hspace{1cm} (3.79a)

\[
= \frac{(1 - \sqrt{-a_{12}s})(1 - \sqrt{-a_{12}s})}{(1 + \sqrt{-a_{12}s})^2}
\]  \hspace{1cm} (3.79b)

\[
= \frac{1 - \sqrt{-a_{12}s}}{1 + \sqrt{-a_{12}s}}
\]  \hspace{1cm} (3.79c)

It is apparent that the optimal transfer function has only two poles, one of which cancels the LHP pole, leaving an identical pole symmetrically arrayed in the complex plane with the RHP zero. This is the transfer function of an "all-pass" filter, and it has the step response in Figure 3-2.
Figure 3-2 - Step Response of Autopilot with Minimum Integral of Squared Error
Finally, substitution of (3.74) - (3.76) into (3.60c) shows that the constrained minimum integral of squared acceleration error is:

\[ I = 2 \sqrt{-a_{12}} \]  

(3.80)

which can be numerically evaluated from (3.56).

Therefore, it is evident that the "wrong-way" tail force results in an ultimate lower limit on the integral of squared error in (3.57). Of course, other practical limits like the speed of response of the actuators, stability and saturation would come into play considering that the initial response in Figure 3-2 would correspond to an instantaneous deflection of the tail surfaces.

This result also throws some light on approach no. 4 in Subsection 3.4.1, i.e., how to specify an autopilot with a fixed transfer function. Ideally, one would like to have the transfer function in (3.79c), since this gives the fastest possible response in the sense of (3.57); this would help to overcome the problems of mismatch between the autopilot response and the model thereof in the guidance system, and thereby reduce miss distance. Unfortunately, the transfer function in (3.79a) is not physically realizable for a missile.

On the other hand, one could design an autopilot with a transfer function approximating that of (3.79c), such that one pole is dominant, another pole cancels the LHP airframe zero and the remaining poles have much larger magnitudes. The dominant pole could probably not be as large as \(-1/ \sqrt{-a_{12}}\) because of problems of stability and noise transmission, but could approach it. Such an autopilot would be fast in the sense of (3.57).
Since its transfer function could be approximated as one RHP zero over one LHP pole, a closed-form guidance law of tolerable complexity can be developed for it, as in the next subsection. The design of such an autopilot is worked out in Chapter 4.

3.5 Continuous Guidance Law for an Autopilot with One Pole and One Zero

The previous subsection has motivated the design of the autopilot such that its net transfer function can be approximated by one RHP airframe zero and one LHP pole. Among other considerations, this matches the autopilot design to a guidance law of reasonable complexity. Matching of the autopilot and its guidance law is desirable in order to approach the minimum miss distance. The purpose of this subsection is to sketch the development of this guidance law, state it explicitly in closed algebraic form, and to show its behavior. It should be noted that this guidance law is intended for the terminal part of the flight and therefore does not need to account for moderate drag or varying air density, as would a midcourse guidance law.

3.5.1 Plant Model and Performance Index

For a short homing time (e.g., 15 seconds or less) in a horizontal glide condition, the autopilot transfer function may be considered as nearly time-invariant. It is approximated as:

$$\frac{A_m}{A_c} = \frac{1 + s/\omega_2}{1 + s/\omega_1}$$  \hspace{1cm} (3.81)

where $\omega_2$ is typically a fixed negative number (corresponding to a RHP zero) and $\omega_1$ is always positive. The basic plant model is shown in Figure 3-3. It is straightforward to show that an additional feedback
Figure 3-3 - Basic Plant Model in State Variable Form
from the state $A$ (or output $A_m$) to $A_c$ will not affect the zero at $-\omega_2$, but merely the pole and closed-loop gain of the autopilot; see also Reference B2.

Target acceleration $A_{ty}$ and missile acceleration $A_a$ in Figure 3-1 are shown as disturbance inputs in Figure 3-3. For the sake of simplicity, the angle $\theta_h$ is assumed to be constant, which is typically a reasonable assumption for the terminal phase of many engagements. The plant equation may be expressed in the form of (3.1a) as:

$$\dot{x} = Fx + gu + w \quad (3.82a)$$

$$\begin{bmatrix} \dot{Y}_d \\ \dot{V}_d \\ \dot{A} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -\cos\theta_h \\ 0 & 0 & -\omega_1 \end{bmatrix} \begin{bmatrix} Y_d \\ V_d \\ A \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{\omega_1}{\omega_2} \cos\theta_h \\ \omega_1 \left(1 - \frac{\omega_1}{\omega_2}\right) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ -A_{ty} - A_a \sin\theta_h \\ 0 \end{bmatrix} \quad (3.82b)$$

The performance index is that of (3.26):

$$J = \frac{1}{2} Y_d^2(t_f) + \frac{b}{2} \int_{t_p}^{t_f} u^2(t) \, dt \quad (3.83)$$

Even for a low value of control weighting $b$, this performance index and the resulting guidance law tend to minimize the integral of $u^2$, which helps
to reduce velocity loss because induced drag is approximately proportional to the square of lift.

3.5.2 Solution of the Optimal Control Problem

The actual detailed solution of this problem was long and tedious, and will be omitted here for the sake of brevity in a thesis that is already rather long. In summary, the solution utilized the methods of Subsections 3.2.3 and 3.2.4. Calculation of the 2n by 2n transition matrix in (3.9) proceeded by Laplace-transforming (3.10), which has a fixed matrix, and by utilizing partitioning methods. Of course, the major work was in obtaining the matrix inverse in (3.2.4).

3.5.2.1 The General Solution

The optimal acceleration command \( A_c(t_p) \) at "present time" \( t_p \) was found to be:

\[
A_c(t_p) = \frac{N_g' \hat{M}_{pz}\cos \theta_h}{t_g^2} \quad (3.84)
\]

where \( t_g \) is "time to go" until intercept:

\[
t_g = t_f - t_p \quad (3.85)
\]

\( \hat{M}_{pz} \) is the estimate of the projected, zero-control miss distance and \( N_g' \) is a generalized effective navigation ratio given by:
\[ N_g' = \left[ \frac{\omega_1}{\omega_2} \right] (\omega_1 t_g)^3 - \left[ 1 - \frac{\omega_1}{\omega_2} \right] (\omega_1 t_g)^2 \left[ 1 + \omega_1 t_g - e^{-\omega_1 t_g} \right] \]
\[ \frac{\frac{b\omega_1}{\cos^2 \theta_h} + \frac{1}{3} (\omega_1 t_g)^3 + \left[ \frac{\omega_1}{\omega_2} - 1 \right] \left\{ \omega_1 t_g \left[ 2e^{-\omega_1 t_g} + \omega_1 t_g + \frac{\omega_1}{\omega_2} - 1 \right] \right\} - 2 \left[ \frac{\omega_1}{\omega_2} \right] e^{-\omega_1 t_g} + \frac{1}{2} \left[ 1 - \frac{\omega_1}{\omega_2} \right] e^{-2\omega_1 t_g} - \frac{1}{2} \left[ 1 + 3 \frac{\omega_1}{\omega_2} \right] \} \]  

(3.86)

This is a dimensionless function of three normalized variables, i.e., 
\( \omega_1 t_g \), \( \omega_1/\omega_2 \), and \( b\omega_1^3/\cos^2 \theta_h \). The latter may now be set equal to zero if desired, but it was necessary in the beginning of the problem.

The estimate of projected, zero-control, miss distance may be expressed as:

\[ \hat{M}_{pzc} = Y_d(t_p) + t_g Y_d(t_p) - (\cos \theta_h) \left[ \frac{e^{-\omega_1 t_g} + \omega_1 t_g - 1}{\omega_1^2} \right] \hat{A}(t_p) \]
\[ + \int_{t_f}^{t_f} \int_{t_p}^{t_2} \left[ \hat{A}_{ty}(t_1) - \hat{A}_a(t_1) \sin \theta_h \right] dt_1 dt_2 \]  

(3.87)

From Reference H2, p. 225, the fourth term of (3.87) may be expressed as:

\[ \int_{t_f}^{t_f} \int_{t_p}^{t_2} \left[ \hat{A}_{ty}(t_1) - \hat{A}_a(t_1) \sin \theta_h \right] dt_1 dt_2 = \]
\[ \int_{t_p}^{t_f} (t_f - \tau) \left[ \hat{A}_{ty}(\tau) - \hat{A}_a(\tau) \sin \theta_h \right] d\tau \]  

(3.88)
From a comparison of (3.84) and (3.87) it should be noted that, except for \( \hat{Y}_d(t_p) \), the control gains for the estimated variables are factored, so that the convenient form \( \hat{M}_{pzc} \) may be utilized. Figure 3-4 is a block diagram of the guidance system without estimation, which portrays its feedback nature and the importance of \( \hat{M}_{pzc} \).

3.5.2.2 Target Acceleration with Exponential Correlation

Suppose that the future target acceleration \( A_{ty}(t) \) has an exponential decay, as would that of Figure 2-12 in the absence of process noise; this latter point is in keeping with the application of the separation theorem. Then, the estimated future target acceleration \( \hat{A}_{ty}(t) \) and its contribution to \( \hat{M}_{pzc} \) in (3.88) are:

\[
A_{ty}(t) = A_{ty}(t_p) e^{-2\nu(t - t_p)}
\]

(3.89)

\[
\int_{t_p}^{t_f} (t_f - \tau) A_{ty}(\tau) d\tau = \left[ \frac{e^{-2\nu t_f g} + 2\nu t_f g - 1}{(2\nu)^2} \right] A_{ty}(t_p)
\]

(3.90)

3.5.2.3 Special Case of Zero Autopilot Lag

Suppose that in (3.81):

\[
\frac{\omega_1}{\omega_2} = 1
\]

(3.91)

which makes the autopilot have a transfer function of 1 and opens the input of the integrator for \( A(t_p) \) in Figure 3-3. In this case, \( N_{g'} \) in (3.86) reduces to:

\[
N_{g'} = \frac{3t^3}{3b + t^3} \frac{g}{\cos^2 \theta h}
\]

(3.92)
Figure 3-4 - Block Diagram of Guidance System without Estimator
which is the same as (2.87), except that (2.87) and (2.86) do not have appropriate factors in $\cos \theta_h$ for other than a head-on engagement. In addition to (3.91), $\omega_1$ can approach infinity, so as to eliminate mathematically the effect of the now-isolated integrator for $A(t_p)$. This value for $\omega_1$ forces the third term in (3.87) to be zero and brings it into further agreement with (2.86), the remaining difference being the more flexible modelling of $A_{ty}$ and the inclusion of $A_a$.

3.5.2.4 Special Case of an Autopilot with One Pole and No Zero

In this case, let $\omega_2$ approach infinity in (3.81). It is then apparent by inspection that (3.86) reduces to (2.94), as it should. Moreover, (2.93) and (3.87) would agree if (3.88) and (3.90) are used in (3.87) and $A_a$ is set equal to zero.

3.5.3 Behavior of $N_g'$ for an Autopilot with One Pole and One RHP Zero

In order to illustrate the behavior of this guidance law, the following transfer function was used as a close approximation of the autopilot response in Chapter 4:

$$\frac{A_m}{A_c} = \frac{1 - s/33.4}{1 + s/10.2}$$  \hspace{1cm} (3.93)

The step response for this transfer function would qualitatively resemble Figure 3-2, but the response at zero time would be $-10.2/33.4$ or $-0.306$ rather than $-1$.

Figures 3-5 and 3-6 show the effective navigation ratio $N_g'$ versus time-to-go for this autopilot transfer function, $\theta = 0$, and two values of control weighting $b$ in the latter figure. At long values of time-to-go, the lag of the autopilot is not important and so $N_g'$ is nearly 3.0, the optimum
Figure 3-5 - Effective Navigation Ratio versus Time To Go for Two Optimum Guidance Laws
Figure 3-6 - Effective Navigation Ratio versus Time to Go for Continuous Guidance Law and Two Values of Control Weighting
value for the zero-lag case. At short values of time-to-go, the lag of the autopilot is significant, and $N_g'$ peaks in order to speed it up.

The reversal of $N_g'$ and hence $A_c$ in (3.84) at very short times to go is appropriate to the initially negative response of the autopilot to a step (compare Figure 3-2), when only the "wrong-way" tail force is effective. As illustrated by Figure 3-6, a small value of $b$ has an important effect only at the end of the intercept, when it acts to reduce $N_g'$ to zero. For the case of $b = 0$, E. Greenberg of Raytheon pointed out to the writer (Reference S9) that it appeared that the $N_g'$ at intercept times the high-frequency gain of the autopilot was about 3, i.e.:

$$\left(\frac{\omega}{\omega_2} N_g' \right)_{b = 0, t_g = 0} = 3 \quad (3.94)$$

For the $b = 0$ case in Figure 3-6, a direct computation from (3.86) shows that:

$$\left[\frac{\omega}{\omega_2} N_g' \right]_{b = 0, t_g = 6.64 \times 10^{-4}} = \frac{10.2}{(-33.4)}(-9.936) = 3.03 \quad (3.95)$$

which is close to the conjectured value of 3. Analytically, if $b = 0$ and $t_g \to 0$ in (3.86), it approaches 0/0. After three successive applications of L'Hopital's theorem, E. Greenberg's conjecture in (3.94) was found to be correct.

After examining a curve in Reference S9 similar to Figure 3-5 herein, Dr. R. Fitzgerald of Raytheon commented that the high positive
peak of \( N'_g \) might occur at that value of \( t_g \) at which an impulse of autopilot command \( A_c \) would produce a zero contribution to miss distance, in the absence of all feedbacks through \( M_{pzc} \) in Figure 3-4. This conjecture was not quite borne out numerically in Reference S9, because this particular value of \( t_g \) was somewhat less than that of the peak of \( N'_g \). Noting this result, Dr. B. Hall of Raytheon commented that perhaps the open-loop response of miss distance goes to zero for a step of autopilot command applied at that value of \( t_g \) for a peak \( N'_g \). The analytical expression for this contribution to miss distance in the absence of guidance feedback is:

\[
Y_d(t) = \frac{1}{\omega_1^2} \left\{ \left[ 1 - \frac{\omega_1}{\omega_2} \right] \left[ 1 - \omega_1 t_g e^{-\omega_1 t_g} \right] + \frac{1}{2} (\omega_1 t_g)^2 \right\}
\]

(3.96)

This latter conjecture was numerically verified in Reference S9. For Figure 3-6 herein, \( N'_g \) peaks at about \( t_g = 0.0795 \text{ sec} \); slide-rule calculations indicate that \( Y_d(t) \) in (3.96) is very close to zero for this \( t_g \). Physically, this result might be explained by an attempt of the guidance law to drive the autopilot with a large command in the vicinity of this \( t_g \), so as to overcome the temporarily low contribution of the autopilot to miss distance.

3.6 Discrete Guidance for an Autopilot with One Pole and One Zero

Subsection 3.3 has shown the general analytic solution for the discrete terminal guidance problem. Subsection 3.3.4 has commented on the difficulty of obtaining a closed-form expression for the optimal discrete guidance law for an autopilot with one pole and one zero, and has recommended off-line computation of the discrete effective navigation ratio \( N' \). This subsection deals with the computation of the optimal discrete \( N' \) for the
one-pole, one-zero case, as illustrated numerically by the example in Subsection 3.5.3.

3.6.1 Computational Approach

Once the analytical expressions for discrete optimal control have been worked out, for example, those in Subsection 3.3.2, it should be straightforward to program them for digital computation in a specific case, at least in theory. In practice, however, there are many opportunities for error and it is not obvious how to check the program. This may be the reason why an attempt of C. Graves of Raytheon to extend Speyer's result (Reference S10, which is discussed in Subsection 2.3.2.2 herein) to the one-zero, one-pole autopilot was apparently not successful, as judged by comparative miss-distance results.

This writer's approach to the design of a digital program (DGL1) for computing discrete guidance gains produced more successful results and may be summarized as follows:

1) Make the program able to handle a general plant matrix of order \( n \) by \( n \), with \( n \) between 2 and some reasonable upper limit.

2) Compute the transition matrix from the plant matrix as a truncated exponential series, as explained below.

3) Compute the discrete input distribution vector by an extension of the method in (2), as explained below.

4) Compute the optimal discrete gains from (3.50) - (3.52) by iterative matrix calculations.

5) Using (3.54), compute \( N' \) at each step.

6) Check the program with a known simple case, i.e., Reference S10.
In connection with steps 2) and 3), the problem is to derive $\Phi(i)$ and
$\Gamma(i)$, as in (3.35):

$$x(i+1) = \Phi(i)x(i) + \Gamma(i)u(i) \quad (3.97)$$

where a constant sample interval $T_s$, a fixed plant and a scalar control
input $u$ will now be assumed. As in Subsection 3.2.1, let the continuous
plant equation be:

$$\dot{x}(t) = Fx(t) + gu \quad (3.98)$$

The transition matrix for an interval $T$ may be found as a truncated
exponential series:

$$\Phi(T, 0) = I + \sum_{k=1}^{m} \frac{T^k}{k!} F^k \quad (3.99)$$

where $m$ is large enough so that the $m$-th term in the series is negligibly
small. The interval $T$ should be chosen so that the product of $T$ times
the magnitude of the largest eigenvalue of $F$ is less than 1, for best
convergence of (3.99). If necessary, $T$ is chosen as an integral sub-
multiple of $T_s$ and the desired $\Phi(T_s, 0)$ is found as:

$$\Phi(T_s, 0) = \left[ \Phi(T, 0) \right]^c \quad (3.100)$$

where $c$ is an integer such that:

$$T_s = cT \quad (3.101)$$
With respect to the discrete input distribution vector $\Gamma (i)$, the last term in (2.97) may be found from:

$$\Gamma (i)u(i) = \int_0^{T_s} \Phi (T_s, \tau) g(\tau) u(\tau) d\tau$$

(3.102a)

$$= \left[ \int_0^{T_s} \Phi (T_s, \tau) d\tau \right] g_u(i)$$

(3.102b)

where it is recognized that $g$ is constant, and $u(i)$ is a constant input over the sample interval $T_s$. As a convenient conceptual and computational artifice, let $u(i)$ be considered as the $(n+1)$th state variable coming from an integrator with no input during the interval $T_s$, such that:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{u}(t) \end{bmatrix} = \begin{bmatrix} F & g \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

(3.103)

For this augmented plant matrix, the transition matrix may be calculated as in (3.99) - (3.100), and then $\Gamma (i)$ is found as the elements in rows 1 through $n$ of the $(n+1)$st column of this new transition matrix. These purely numerical computations of $\Phi (i)$ and $\Gamma (i)$ eliminate much algebra and chance of error.

3.6.2 Numerical Example

Because of its flexibility, as indicated in step (i) of the previous subsection, it was possible to check out program DGL1 with Speyer's closed-form solution (Reference S10) for a zero-lag autopilot ($n = 2$); his solution and the particular two numerical examples are discussed in Subsection
2.3.2.2. It was then easy to change the plant matrix \( F \) and input distribution vector \( g \) to the case for a one-zero, one-pole autopilot, with target acceleration as the fourth state as in Figure 2-12; no programming changes for (3.50) - (3.52) were necessary.

Figure 3-5 shows the discrete effective navigation ratio \( N' \) versus time to go as computed by program DGL1 using the foregoing method. The sample interval was:

\[
T_s = 0.06640625 = \frac{34}{512} \approx \frac{1}{15} \text{ second}
\]

(3.104)

which is a number compatible with the Raytheon adjoint program (Reference R3) that was later used to evaluate miss distance.

It is apparent from Figure 3-5 that, for more than a few samples to go, the discrete optimal \( N' \) is very close to the continuous optimal \( N_g' \), which is to be expected from physical reasoning. Near intercept, the discrete \( N' \) is significantly less than the continuous \( N_g' \), and the reversal at the last sample to go is more dramatic than the reversal of the smooth curve of \( N_g' \). This might be explained physically as follows. Consider two identical intercepts with no noise and the same given initial heading error, one using continuous guidance with \( N_g' \) and the other using discrete guidance with the \( N' \) points in Figure 3-5. The two values of projected zero-effort miss distance \( \hat{M}_{pzc} \) would continuously decrease as \( t_g \) decreases, and may be hypothesized to remain nearly equal down to about \( t_g = 0.4 \) second. Since \( \hat{M}_{pzc} \) decreases between the sample points in Figure 3-5, the continuous command \( A_c \) would behave proportionally, but the discrete command would be held at the last, relatively large, sampled value of \( \hat{M}_{pzc} \). In order to compensate for this sampled data and to produce a correctly calibrated acceleration command for virtually zero miss
distance, it is appropriate for the discrete $N'$ to be smaller than the continuous $N_g'$, particularly near intercept, when $T_s$ is comparable to $t_g$.

From another point of view, it is evident that the continuous guidance law could be used in a discrete system (circled points in Figure 3-5) quite adequately up to a short time to go, but that it would be definitely sub-optimal near intercept, particularly at the last sample.

### 3.7 The Computation of Time to Go

In all the closed-form guidance laws for the terminal phase that have been discussed, the time to go, $t_g$, appears explicitly. It is usually assumed, as stated in Subsection 3.2.1, that $t_g$ is independent of the control variable and can be computed separately. This subsection discusses three such methods.

#### 3.7.1 Previous Methods

A typical assumption is that the closing velocity $V_c$ will remain constant until intercept. Referring to Subsection 1.2.2 and Figure 1-2, the time to go is then:

$$t_g = \frac{X_t - X_m}{V_c} \approx \frac{R_{mt}}{V_c} \quad (3.105)$$

Obviously, this calculation of $t_g$ can be significantly in error if $A_a$ or $A_{tx}$ in Figure 3-1 are significantly large. In Reference O'H2, O'Halloran gave a formula for $t_g$ which was equivalent to the following:

$$t_g = \frac{V_c(t_p)}{A_a \cos \theta_h} \left[ \sqrt{1 + \frac{2(A_a \cos \theta_h)R_{mt}(t_p)}{V_c(t_p)^2} - 1} \right] \quad (3.106)$$

under the assumption that $A_a$ and $\theta_h$ were constant and that the effect of
lateral acceleration on $t_\text{g}$ was negligible. It is readily shown that (3.106) reduces to (3.105) if $A_a = 0$.

### 3.7.2 A General Algorithm for Time To Go

In certain critical cases, e.g., intercept just after the boost phase with $A_a$ going from large positive to large negative values, an improved method of computing $t_\text{g}$ may help to decrease miss distance significantly. This subsection proposes a generalized iterative scheme for three dimensions, which is portrayed graphically in two dimensions for clarity.

Figure 3-7 shows in two dimensions the present position $X_m(t_p)$, $Y_m(t_p)$ of the missile and the present position $X_t(t_p)$, $Y_t(t_p)$ of the target, which is assumed to have the future dashed trajectory up to the intercept point at time $t_f$. The actual future trajectory of the missile is shown by the dashed trajectory, which is assumed to be not far off the "ideal" trajectory, i.e., the dashed-dot-dashed straight line to the point of intercept. A central assumption is that these two trajectories (ideal straight and actual curved) have nearly the same arc length. To test the plausibility of this, suppose that the actual curved trajectory is a circular arc of radius $R$ and subtended angle $\theta$, for which the ideal straight trajectory is the chord. The normalized error in length is:

$$\frac{\text{Arc length} - \text{chord length}}{\text{arc length}} = \frac{R \theta - 2R \sin (\theta/2)}{R \theta} = \frac{\theta - 2 \sin (\theta/2)}{\theta} \quad \text{(3.107)}$$

For example, if $\theta = 45$ degrees $= 0.785$ radian (which corresponds to a quite curved trajectory), then the normalized error is only $0.0268$.

A further assumption is that the missile speed $V_m$ is not significantly affected by its lateral acceleration, and only the zero-lift drag is important. Even if this assumption is somewhat in error, the feedback nature
Figure 3-7 - Coordinates for Time-to-Go Calculation in Fixed Coordinate System
of the guidance law and the proposed iterative calculation of $t_g$ during the engagement should correct matters. These two assumptions make the calculation of $t_g$ independent of lateral acceleration.

Now, suppose that the future missile velocity $V_m(t)$ is known or can be computed readily from thrust data and drag or axial coefficient. The distance that the missile can travel in the fixed coordinate frame in Figure 3-7 is simply:

$$
\int_{t_p}^{t_f} V_m(t) \, dt
$$

(3.108)

and this is assumed to be virtually equal to both the length of the actual missile trajectory and the length of the ideal straight trajectory. Equating the target and missile position at intercept and invoking the Pythagorean theorem gives as the length of the ideal straight trajectory:

$$
\int_{t_p}^{t_f} V_m(t) \, dt = \sqrt{\left[ X_t(t_f) - X_m(t_p) \right]^2 + \left[ Y_t(t_f) - Y_m(t_p) \right]^2 + \left[ Z_t(t_f) - Z_m(t_p) \right]^2}
$$

(3.109)

Again, an assumption is made as to the target's future trajectory, so that $X_t$, $Y_t$, $Z_t$ and their derivatives are calculable. For future convenience, denote the radical in (3.109) as the range to intercept, $R(t_f)$.

Now, (3.109) is an implicit equation in the unknown time of intercept, $t_f$, which would not be easy to solve directly in the general case. Suppose that the solution is iterated during the engagement, and that the $j$-th iterat-
tion gives a value $t_{fj}$, resulting in a slightly erroneous intercept point. The corresponding "error distance" is:

$$f(t_{fj}) = R(t_{fj}) - \int_{t_p}^{t_{fj}} V_m(t) \, dt$$  \hspace{1cm} (3.110)

Newton's method of iterative solution may be utilized to find the $(j+1)$th iterative value of $t_f$:

$$t_{fj+1} = t_{fj} - \frac{f(t_{fj})}{\frac{df}{dt_f}} \bigg|_{t_f = t_{fj}}$$  \hspace{1cm} (3.111)

Using the radical equivalent to $R(t_{fj})$, the derivative in (3.111) may be computed from (3.110) and put in the inner-product form:

$$\left[ \begin{array}{c} X_t(t_{fj}) - X_m(t_p) \\ R(t_{fj}) \\ Y_t(t_{fj}) - Y_m(t_p) \\ R(t_{fj}) \\ Z_t(t_{fj}) - Z_m(t_p) \\ R(t_{fj}) \end{array} \right]$$

$$- V_m(t_{fj}) \hspace{1cm} (3.112)$$

The inner product may be recognized as that component of target velocity at time $t_{fj}$ which is along the line of sight from the present missile pos-
tion to the j-th computed intercept position, \(X_t(t_{fj}), \ Y_t(t_{fj}), \ Z_t(t_{fj})\).

The physical interpretation of (3.111) and (3.112) then becomes clear.

The obvious first guess at the intercept time would be directly related to (3.105):

\[
t_{f1} = t_p + t_g = t_p + \frac{R_{mt}}{V_C}
\]

(3.113)

where:

\[
R_{mt} = \sqrt{\left[ X_t(t_p) - X_m(t_p) \right]^2 + \left[ Y_t(t_p) - Y_m(t_p) \right]^2 + \left[ Z_t(t_p) - Z_m(t_p) \right]^2}
\]

(3.114)

\[
V_C = \frac{1}{R_{mt}} \left[ (X_t - X_m)(\dot{X}_m - \dot{X}_t) + (Y_t - Y_m)(\dot{Y}_m - \dot{Y}_t) + (Z_t - Z_m)(\dot{Z}_m - \dot{Z}_t) \right] \bigg|_{t = t_p}
\]

(3.115)

It may be shown that, in an idealized intercept with constant velocities, the correct time to go is the first-guess value \(R_{mt}/V_C\), and it results in the correction in (3.111) being zero as desired.
CHAPTER 4

DESIGN OF AUTOPilot AS A SIMPLIFIED ESTIMATOR-CONTROLLER

4.1 Introduction

The requirements on the autopilot design draw on successful Raytheon experience with a simple autopilot configuration, and include: simplicity of mechanization with constant gains, use of hydraulic fin servos with position feedback, feedback instrumentation limited primarily to rate gyro and accelerometer, tolerance of biases on effective fin angle and tolerances to airframe-response variation with flight condition. Additionally, in order to match the tolerably complex guidance law in Subsection 3.5, the autopilot should have a transfer function which is approximated by one pole and one zero; it should have fast response in order to minimize miss distance. Various types of configurations are considered, including the classical Raytheon autopilot and one with an estimator-controller.

All autopilot design was performed for Mach 2, sea level, in order to minimize the radome problem. In order to compare miss distances obtainable with old and new autopilot techniques, two classical autopilots were designed.

Most of this chapter is concerned with a new type of autopilot utilizing a simplified estimator-controller. A separation principle is shown to hold in designing for specified poles of the autopilot closed-loop transfer function, if the model of the airframe matches the airframe. The control design for selected poles (leaving the fin-servo poles fixed) utilizes modal-control theory and the formula of Crossley-Porter (Ref. C2). If an estimate of lumped bias on the fin angle is available, it may be used to cancel
the effect of the bias on the output acceleration without using an additional classical integrator in the controller, despite variations in the airframe response.

A quasi-steady-state solution of the covariance matrix is computed by a little-known but effective algorithm to find the estimator gains. The behavior of the estimator eigenvalues for steady-state covariance at various noise levels is examined and some misconceptions regarding estimators and observers are discussed.

The preliminary estimator-controller design is in the plant modal space for convenience and has as many states as the plant, namely 5, which necessitates simplification. It is shown that for fundamental reasons the actuator modes in the estimator-controller are only weakly coupled to the other modes. By eliminating these modes, a new technique of simplification permits reduction of the estimator-controller states from 5 to 3, with virtually the same closed-loop transfer function, which can be approximated at the design point (Mach 2, S.L.) by an airframe zero at +33.4 rad/sec and a single pole at -10.2 rad/sec, nearly as originally specified. The transfer function varies with flight condition, but within acceptable limits.

This new autopilot has the following features: 1) Its transfer function is virtually one zero over one pole; 2) its cancellation of tail-angle bias is independent of flight condition; 3) noise filtering is near the optimum; 4) its complexity is tolerable; 5) performance changes with flight conditions are acceptable; 6) its miss distance is good, as shown in the next chapter.

This chapter includes a survey of the literature on modal control and discusses the approach of Johnston (Ref. J4) to a regulator design that has
specified poles and overcomes a bias without estimating it. A derivation is also given of Jameson's technique (Ref. J1) for obtaining specified roots by using pure feedback from a limited output vector; his result is simplified and its relation to the Crossley-Porter formula (Ref. C2) is shown.

4.2 General Approach to Autopilot Design

In the overall problem of design of missile pitch autopilots, there are many facets - some obvious and some rather subtle. The following statement of the problem draws heavily on Raytheon's experience in designing pitch autopilots by 'classical' techniques (Ref. S1). It seems wise to do this even though improvements are desired, because examples in other fields suggest that to ignore an established body of successful experience may result in stumbling anew into old pitfalls. This should be particularly true in attempting to apply modern control theory, partly because of its implicit assumption of perfect knowledge of the 'plant' and because of the necessary truncation of high-frequency dynamics in the plant for the sake of feasible matrix computation.

4.2.1 Statement of Problem

The following list summarizes the important aspects, which will be discussed briefly in separate subsections below:

- Simplicity of Mechanization
- Transfer Function
- Fin Servos with Position Feedback
- Instrumentation
- Structural Modes
- Bias on Fin Position
- Constraints on Feedback
- Tolerance to Plant Variation
4.2.1.1 Simplicity of Mechanization

A homing-guided missile is an expendable weapon which must be reasonably simple for the sake of reliability and acceptable cost. Therefore, most 'classical' autopilot designs (Ref. S1) have had constant electronic gains within a given autopilot band. This requirement is carried over to the problem statement here, despite the fact that Chapter 6 has shown that the simultaneous design of the autopilot and guidance law to optimize a quadratic performance index results in time-varying feedback gains from all states. Of course, it is technically possible to build time-varying gains by using analog multipliers (e.g., as high-frequency pulse-width modulators) and programmed time-varying voltages for compatibility with the customary available analog gyros, accelerometers and fin servos, but the resulting complexity would be inadmissible.

In addition to requiring constant electronic gains within a given band, it is desirable to attempt to minimize the number of amplifiers in the autopilot, particularly integrators, which must have better drift performance than simple amplifiers for summing, inversion or pure gain. The number of electronic integrators in the autopilot will be taken as a rough but effective measure of complexity in comparing candidate autopilots.

The requirement for simplicity must extend also to the tactical computations for estimation and control in the whole guidance system.

4.2.1.2 Transfer Function

The approach here is to design the autopilot with fixed gains for an appropriate transfer function, and then to treat the autopilot as part of the fixed 'plant', to which the overall guidance law is applied. In order to restrict the complexity of tactical computation in the overall guidance, it seems desirable to use the guidance law in Subsection 3.5 for an autopilot
with one pole and one zero. This guidance law will allow for the right-half-plane zero (resulting from the 'wrong-way' tail force) of an autopilot for a tail-controlled missile, and it seems to be about at the upper limit of tolerable complexity for rapid tactical computation at a sufficiently high repetition rate. The autopilot should be fast in order to minimize miss distance.

Alternatively, one might choose the guidance law of Willems (Ref. W3) for an autopilot with two real poles, but this law ignores the apparently important right-half-plane zero and is somewhat more complex than that of Subsection 3.5. It is believed that a guidance law for an autopilot with two poles and a zero would entail too much computation to be tactically feasible.

Therefore, it is desirable to be able to approximate the autopilot transfer function as having one dominant pole at fairly high frequency and one zero. The other poles should have large negative real parts, and one of them might conceivably be chosen to cancel the left-half-plane zero. The guidance law would work with a one-pole, one-zero autopilot model, which would probably have to be trimmed slightly to account for the other actual poles in some appropriate way.

It turns out that design of the autopilot to have a dominant real pole at low altitude is not a trivial matter, because the classical Raytheon autopilot typically has a dominant quadratic pole-pair at such flight conditions but never a dominant high-frequency pole.

4.2.1.3 Hydraulic Fin Servos with Position Feedback

Reference S1 touches on the variety of fin servos which have been used and proposed for homing missiles. Raytheon has always successfully used hydraulic fin servos with position feedback. Such fin servos have numerous advantages, including good frequency response and high stiffness to hinge moments. In order to enhance the potential usefulness of the candidate autopilot herein, this type of fin servo will be adopted.
As in Chapter 6, the fin-servo transfer function will be approximated by a quadratic pole-pair with a natural frequency of 100 rad/sec and a damping ratio of 0.5, although faster servos can be built. Hence, the states of the fin servo(s) are fin angle $\delta$ and fin rate $\dot{\delta}$. Realistically, such a transfer function represents a truncation of a number of lags, including the important hydraulic-mechanical resonance (Ref. S4).

4.2.1.4 Instrumentation

Traditional feedback instrumentation for a Raytheon pitch autopilot has been a spring-restrained rate gyro with a high natural frequency and a spring-restrained penduluous accelerometer with a somewhat lower natural frequency. These will be used to measure the states $A_m$ and $\dot{\theta}_m$ and it will be assumed that the lags of the instruments are negligible, so as to keep down the number of states in the airframe subplant. The contribution of $\ddot{\theta}$ to the accelerometer output owing to a location off the c.g., will be neglected.

D-C potentiometers have been used for the fin servos in Raytheon pitch autopilots, and it will be assumed that their signals are available for the measurement of total elevator angle $\delta$. This does not include the effect of certain biases, to be discussed. It will not be assumed that a measurement of $\delta$ is available, since no tachometers are present on current fin servos and their adoption would involve extra complexity, risk and cost.

4.2.1.5 Structural Modes

Because of their smaller size, homing interceptor missiles can be built with bending-mode natural frequencies which are much higher than those of space boosters. Nevertheless, these modes are very lightly damped, (e.g., the damping ratio is usually assumed to be only 0.02) and they tend to destabilize conventional autopilots. Indeed, a forerunner
(built by a well-known company) of the Sparrow missile suffered a flight failure because of structurally-induced instability that had not been anticipated in the autopilot design.

For the moment, it will be assumed that the structural bending modes can be dealt with by a combination of traditional techniques, i.e., by locating the gyros at the antinode (zero-slope point) of the first bending mode, by ad hoc high-frequency filters and by constraints on the feedback, to be discussed below. It is often, but not always, possible to locate the gyros and accelerometers near this antinode, thus minimizing the contribution of the first bending mode to the gyro output. Of course, this location tends to maximize the first-bending-mode content in the accelerometer output, but a separate accelerometer location (e.g., a node of the first bending mode) would incur extra packaging problems in a tightly-packed missile, as a glance at the schematic cross section of a hypothetical missile in Figure 2-1 will indicate. Clearly, it is not practical to locate an instrument in the propulsion section or warhead.

For the sake of initial analytical simplicity, the bending modes will not be represented by state variables.

4.2.1.6 Bias on Fin Position

Inevitably, there are sources of bias on the effective elevator angle $\delta$, such as bias in the fin-servo amplifiers and misalignment of the fin-shaft feedback potentiometers. Also, slight asymmetry of the control fins and wings (if any) can also cause pitching moments. Indeed, a typical aerodynamic tolerance is a maximum of 1 degree of elevator angle for trimming out the pitching-moment bias. If the autopilot does not overcome the total effective bias on tail angle $\delta$, there will be an acceleration bias which may lead to a bad miss distance. As an example, W. O'Halloran (formerly of
Raytheon) attempted to apply modern control theory to autopilot design in 1968 and found that, for the airframe under study, one degree of bias on $\delta$ produced a 5g acceleration bias in the airframe response at a flight condition near maximum dynamic pressure. This and other findings terminated interest in that particular autopilot design, which did not include an integrator ahead of the fin servo.

It is an accepted lesson of classical control theory that proportional plus integral control is often effective in stabilizing a system and in countering the effect of bias. This is one advantage of the integrator in a classical Raytheon autopilot (Figure 2-2). Therefore, it will be tentatively assumed that the plant may include an integrator in addition to the airframe and fin servo, thus providing a second control input $u_2$ and an extra degree of design freedom.

Figure 4-1 is a preliminary block diagram of the plant, including a bias $\delta_1$ from voltage offset into the fin servo (with a gain such that $\delta = \delta_1$ in the absence of other inputs), a measurement bias $\delta_2$, and an aerodynamic bias $\delta_3$. The added integrator ($x_1$) includes a small feedback of $f_{55} \approx -0.01$ rad/sec because it is difficult to build a perfect integrator and because of the computational difficulties associated with a zero eigenvalue. Such a small feedback would make no practical difference in the design.

It is desirable to combine the bias states and thereby to simplify the plant model. The variable $\delta_{\text{eff}}$ is the effective elevator angle and has the following relationships:

$$\delta_{\text{eff}} = \delta + \delta_3$$  \hspace{1cm} (4.1)

$$\dot{\delta}_{\text{eff}} = \dot{\delta} + \dot{\delta}_3 = \ddot{\delta}$$  \hspace{1cm} (4.2)
Figure 4-1 - Preliminary Block Diagram of the Plant, Including Airframe, Fin Servo, Control Integrator and Three Tail-Angle Biases
For the summing point ahead of the $x_4$ integrator:

$$
\dot{x}_4 = \frac{d}{dt} \delta = g_{41}u_1 - F_{43} \delta_1 + F_{43} \delta + F_{44} \delta_2 + F_{44} \dot{\delta} + F_{45} x_i
$$

Substituting (4.1) and (4.2) gives:

$$
\frac{d}{dt}(\delta_{eff}) = g_{41}u_1 - F_{43} \delta_1 + F_{43} (\delta_{eff} - \delta_3) + F_{43} \delta_2 + F_{44} \dot{\delta}_{eff} + F_{45} x_i
$$

$$
= F_{43} \delta_{eff} + F_{44} \dot{\delta}_{eff} - F_{43} (\delta_1 + \delta_3 - \delta_2) + g_{41}u_1 + F_{45} x_i
$$

Therefore, it is possible to combine the three biases into one:

$$
\delta_b = \delta_1 + \delta_3 - \delta_2
$$

Figure 4-2, which is equivalent to Figure 4-1, shows $\delta_b$ as the fifth state, with a negligible feedback gain such as $-0.01 \text{ rad/sec}$ around its integrator for convenience in computing eigenvalues. The state variables $x_3$ and $x_4$ have been changed to $\delta_{eff}$ and $\dot{\delta}_{eff}$ in accordance with (4.4). The potentiometer output is unfortunately not $\delta_{eff}$, as might be desired for control and compensation, but is instead $\delta_{eff} + \delta_2 - \delta_3$. If potentiometer feedback is not used, then the input $\delta_2 - \delta_3$ does not need to be modelled as a separate state variable.

4.2.1.7 Constraints on Feedback

Assuming that the instruments can be practically located together near the antinode of the first bending mode (Subsection 4.2.1.5), it would appear to be practical to feed the $q = \dot{\theta}_m$ signal back to the control input $u_1$ driving the fin servo directly. On the other hand, the high structural mode content on the accelerometer signal would probably preclude direct feedback to $u_1$; some form of ad hoc high-frequency filter would be necessary.
Figure 4-2 - Block Diagram of the Plant, with Three Tail-Angle Biases Combined into $\delta_b$
in the feedback path. Moreover, the neglected high-frequency dynamics of the fin servo would preclude direct feedback of \( \delta \) (or the potentiometer output \( \delta + \delta_2 \)) and \( \delta \) to \( u_1 \).

The acceleration \( A_m \) could be fed back to the integrator input \( u_2 \) with less chance of instability in the actual physical system, and this would tend to counteract the total bias \( \delta_b \). Feedback of the potentiometer signal \( \delta + \delta_2 \) to the integrator would tend to counteract only the electronic bias \( \delta_1 \), and not the biases \( \delta_2 \) or \( \delta_3 \).

4.2.1.8 Tolerance to Plant Variations

The airframe is required to operate over a variety of flight conditions even within one autopilot band (defined by some contours on the altitude-Mach plane), and it has some nonlinear dependency on angle-of-attack. Structural dynamics also vary over a flight. Therefore, it is necessary for the autopilot to tolerate reasonable plant variations without gross changes in its transfer function or instability. This is an important consideration in applying modern control theory, which tacitly assumes perfect knowledge of the plant.

4.2.2 Configuration Possibilities

4.2.2.1 Classical Autopilot

The classical Raytheon pitch autopilot (Figure 4-3) has rate feedback to the fin-servo input (rate-damping loop) and to an integrator (synthetic stability loop), as well as acceleration feedback to the integrator. This autopilot has the virtues of stability and many successful flights, but the achievable pole locations are limited by the feedbacks of only two states. As previously mentioned, the autopilot transfer function usually has a dominant low-frequency pole-pair at low-altitude flight conditions.
Figure 4-3 - Block Diagram of Classical Raytheon Pitch Autopilot
4.2.2.2 Estimator and Controller

Alternatively, one may design the autopilot with an estimator to deliver optimum estimates of the unmeasured states (e.g., \( \delta_{\text{eff}} \)) and then a fixed-gain controller to feed back these estimates to the appropriate inputs. Unfortunately, the five-state autopilot representation would require a five-state estimator with five integrators, which is rather complex for a missile autopilot. Conceivably, the optimum design can be simplified to a near-optimal design, but such simplification is not an easy task.

Another possibility, which is widely cited in the literature, is to use a Luenberger observer (Ref. L1) of order \( n - m \), where \( n \) is the order (5) of the system and \( m \) is the number of measurements (typically 2, as in the classical autopilot, e.g., \( A_m \) and \( \theta_m \)). Unfortunately, Reference L1 assumes that each measurement is completely noise-free and leaves open the choice of the observer poles. Therefore, it is quite conceivable that the observer would have intolerable noise transmission.

4.2.2.3 State Feedback Only

Another possibility is to use state feedback directly, as in the case of the classical autopilot. An extra degree of design freedom would be gained by utilizing the measurement of \( \delta \) (assuming \( \delta_2 = 0 = \delta_3 \)) for feedback to the integrator. This minimizes the number of integrators and has the attraction of simplicity. Here, the recently published paper of Jameson (Ref. J1) has been found to be quite helpful. The difficulty is that feedback of fewer than all the states limits the attainable poles and may result in changing some to undesirable or even unstable values.
4.2.2.4 Hybrid Designs

The possibility of new combinations should also be considered, such as a partial or approximate estimator, together with state feedback.

4.3 Analysis and Design of Candidate Autopilots

This section considers the theory and design of some candidate autopilots, of the types mentioned in Subsection 4.2.2, "Configuration Possibilities".

4.3.1 Classical Autopilot Design

The classical theory of design of the Raytheon pitch autopilot in terms of transfer functions is discussed in Appendix C. This subsection will focus on two numerical examples at Mach 2, sea level (Ref. S5) and will relate them to the state-space block diagram in Figure 4-2.

4.3.1.1 Slow Autopilot

Let the gain-crossover frequency of the rate loop (from $\dot{\theta}$ to $u_1$ to $\dot{\theta}$) be chosen at 25 rad/sec, based on the actuator frequency response and previous experience:

$$\omega_{cr} = -K_{22}K_{12}K_6M_\delta$$  \hspace{1cm} (4.6)

With $K_{22} = 1$, $K_{12} = g_{41}/(-f_{43}) = 1$, $K_6 = 1$ and $M_\delta = -105.2$ sec$^{-2}$, we find:

$$K_8 = 0.238 = C_{12}$$  \hspace{1cm} (4.7)

In former design practice, an old rule of thumb was that the "integral break frequency" $\omega_i$ for the synthetic stability loop (from $\dot{\theta}$ back to the integrator) should be about one-third $\omega_{cr}$. Let:
\[ \omega_i = \frac{K_9 K_{11}}{K_8 T_{11}} = 8 \text{ rad/sec} \] (4.8)

The d-c loop gain of the rate loop is found to be:

\[ Y_{ro} = -\omega_{cr} K_3 / M_\delta = 0.267 \] (4.9)

for which the airframe gain \( K_3 = 1.1243 \text{ sec}^{-1} \).

For the synthetic stability loop, a significant gain number is:

\[ \frac{Y_{so}}{T_{11}} = \omega_i Y_{ro} = 2.14 \text{ sec}^{-1} \] (4.10)

It remains to choose the accelerometer loop gain. For the cubic model of the autopilot in Appendix C, the coefficient of \( s^2 \) in the characteristic equation is:

\[ B_2 = \frac{2 \zeta_2}{\omega_2} + \frac{1}{\omega_2} = \frac{b_{12} + b_{11} T_{11} - Y_{ro} A_{31} T_{11} - Y_{ao} a_{22}}{1 - Y_{ro} - Y_{so} - Y_{ao}} \] (4.11)

A good low-altitude approximation is:

\[ \frac{1}{\omega_2} \approx \frac{-Y_{ro} A_{31} - (Y_{ao} / T_{11}) a_{22}}{-Y_{so} / T_{11} - Y_{ao} / T_{11}} \] (4.12)

with the assumption (to be verified) of a dominant quadratic pole-pair.

Choosing an initial value of \( \omega_2 = 5 \text{ rad/sec} \), we have:

\[ 0.04 = \frac{-0.267(0.67345) - (Y_{ao} / T_{11})(-0.000304)}{-2.14 - Y_{ao} / T_{11}} \] (4.13)

from which:
\[
\frac{Y_{ao}}{T_{11}} = 2.33 \text{ sec}^{-1} = K_{22} K_{12} K_K K_{69} (K_{11}/T_{11}) \tag{4.14}
\]

The integrator transfer function is:

\[
\frac{K_{11}/T_{11}}{s + 1/T_{11}} = \frac{g_{52} f_{45}}{s + f_{55}} = \frac{10^4}{s + 0.01} \tag{4.15}
\]

Letting \(K_{69} = 1\) and using the previous values of \(K_{22}\) and \(K_{12}\), together with the airframe gain \(K_2 = 2520 \text{ ft/sec}^2 \text{-rad}\), it is found that:

\[
K_7 = 9.26 \times 10^{-4} \text{ rad-sec}^2/\text{ft} = C_{21} \tag{4.16}
\]

Let the plant matrix after feedback be \(F_f\), where:

\[
\dot{x} =Fx + Gu = Fx + GCx = F_fx \tag{4.17}
\]

Then \(F_f\) is:

\[
F_f = \begin{bmatrix}
-1.304 & 2910 & 0 & 302 & 0 \\
0.0482 & -1.016 & -119.6 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 2380 & -1 \times 10^4 & -100 & 1 \times 10^4 \\
K_7 & 1.9 & 0 & 0 & -0.01 \\
\end{bmatrix} \tag{4.18}
\]

where the element \(F_{51}^f\) is left as \(K_7\), which will be chosen as different values.

Table 4-1 shows the effects of varying parameters (chiefly \(K_7\)) on the poles of the autopilot.
Table 4-1 - Poles of Classical Autopilot

<table>
<thead>
<tr>
<th>$K_7$</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$\zeta_2$</th>
<th>Actuator Poles</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-28.7</td>
<td>3.67</td>
<td>0.525</td>
<td>$-34.9 \pm j80.5$</td>
</tr>
<tr>
<td>$9.26 \times 10^{-4}$</td>
<td>-28.9</td>
<td>5.25</td>
<td>0.302</td>
<td>$-35.1 \pm j80.5$</td>
</tr>
<tr>
<td>$2.0 \times 10^{-3}$</td>
<td>-29.1</td>
<td>6.61</td>
<td>0.181</td>
<td>$-35.4 \pm j80.6$</td>
</tr>
<tr>
<td>$5.0 \times 10^{-3}$</td>
<td>-29.6</td>
<td>9.56</td>
<td>0.200</td>
<td>$-36.2 \pm j80.7$</td>
</tr>
<tr>
<td>$1.0 \times 10^{-2}$</td>
<td>-30.2</td>
<td>Unstable poles at $+1.361 \pm j12.45$</td>
<td></td>
<td>$-37.4 \pm j81.0$</td>
</tr>
</tbody>
</table>

The initial design came close to the value of $\omega_2 = 5$ rad/sec. It is apparent that the response is dominated by a quadratic pole-pair, which goes into the right-half-plane at too large a value of accelerometer gain $K_7$, and that $\omega_1$ is always larger than $\omega_2$.

For comparison, the original eigenvalues of the plant (without any feedback) were $-0.01$, $+10.684$, $-13.004$ and $-50 \pm j86.6$ rad/sec, where the first is due to the integrator with a time constant $T_{11} = 100$ sec. With rate feedback only, the eigenvalues were $-0.01$, $+3.45$, $-37.4$ and $-34.2 \pm j82$ rad/sec. The first entry in Table 4-1 corresponds to the use of the rate feedback loop and synthetic stability feedback loops only. The progression of the bare-airframe eigenvalue of $-13.004$ rad/sec to become the non-dominant real pole is evident.

Chapter 5 shows that for a plausible set of noise and target parameters at Mach 2, sea level, the miss distance in a single plane (containing the original line-of-sight) was found to be 44.4 feet rms. This rather large value was attributed to the low frequency (5.25 rad/sec) of the dominant pole pair and to its slightly low damping ratio.
4.3.1.2 Faster Autopilot

The autopilot was then redesigned so as to have higher gain in the accelerometer loop and synthetic stability loop, but the same gain in the innermost rate-damping loop. In (4.17), the matrix $F_f$ after feedback was then:

$$F_f = \begin{bmatrix}
-1.304 & 2910 & 0 & 302 & 0 \\
0.0482 & -1.016 & -119.6 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 2380 & -1. \times 10^4 & -100 & 1. \times 10^4 \\
4.125 \times 10^{-3} & 4.09 & 0 & 0 & -0.01
\end{bmatrix} \quad (4.19)$$

The eigenvalues of this matrix were: -13.3, -7.46 ± j12.23, -37.1 ± j78.9 rad/sec, which shows that this autopilot was appreciably faster. For the same conditions of evaluation, the single-plane miss distance improved to 13.8 feet rms, which fits the typical experience that miss distance decreases with increased speed of response of the autopilot.

As previously indicated, classical autopilot designs do not have a dominant real pole at low altitude, and so they do not fit too well the guidance law of moderate complexity in Subsection 3.5, as required. It will become apparent in later subsections that feedback from more states, or ideally, feedback from certain modes, would permit the autopilot to be characterized by a single, dominant real pole. The configuration of the classical autopilot, although successful in many missile flights, appears to be somewhat too limited for the purpose here. Moreover, the latter miss distance is still considerably larger than the ultimate lower limit (Ref. B1, p. 425), which is the square root of the final covariance of the error in the differential-position estimate, or 4.6 feet rms.
4.3.2 Estimation and Control

4.3.2.1 Separation of Estimator (Observer) Design from Control Design

The Separation Theorem or "Certainty Equivalence Principle", as discussed in Reference P3 and B1, is commonly applied to optimal (for a quadratic performance index) control of a linear system with additive white Gaussian process and measurement noise. It allows one to design a Kalman estimator to furnish an optimum estimate \( \hat{x} \) of the system state from limited measurements, and then to compute the optimum feedback gains to operate on these estimates as if they were perfect knowledge of the state \( x \).

In the case of fixed control gains for specified poles and a fixed-gain estimator or Luenberger observer for operation on incomplete measurements, References B3 and L1 show that a similar separation procedure applies. The subsequent transfer-function derivation follows Bryson’s approach in Reference B3, and is herein generalized for mismatch in the estimator’s model of the plant, which in this case comprises the airframe, etc., in Figure 4-2. Figure 4-4 shows a generalized estimator-controller. Mismatch can arise from nonlinearity in the airframe, imperfect knowledge of its characteristics and variations with flight condition in a given autopilot band wherein the estimator-controller gains would be fixed.

Let the plant equation without process noise be:

\[
x = Fx + Gu + G_r u_r
\]  

(4.20)

where \( u_r \) is the external command (typically scalar) and \( u \) is the control variable which is related to the state estimate by a constant gain matrix \( C \):

\[
u = C\hat{x}
\]  

(4.21)
Figure 4-4 - A Generalized Estimator-Controller
The estimator's model of the plant is assumed to be in error, with matrices $F + F_e$, $G + G_e$ and $G_r + G_{re}$. Hence, the estimator equation is:

$$\dot{\hat{x}} = (F + F_e)\hat{x} + (G + G_e)C\hat{x} + K(H\hat{x} - H\hat{z}) + (G_r + G_{re})u_r$$  \hspace{1cm} (4.22)

where it is assumed that the estimator's knowledge of the measurement matrix $H$ is not in error. After defining the error in the estimate as:

$$x_e = x - \hat{x}$$  \hspace{1cm} (4.23)

let (4.22) be subtracted from (4.20), and let $x$ and $u$ be eliminated, so as to obtain:

$$\dot{x}_e = (F_e - KH + G_e C)x_e - (F_e + G_e C)x - G_{re}u_r$$  \hspace{1cm} (4.24)

Equations (4.21) and (4.23) may be substituted into (4.20) to obtain:

$$\dot{x} = (F + GC)x - GCx_e + G_r u_r$$  \hspace{1cm} (4.25)

With the assumption of constant matrices, (4.24) and (4.25) can be Laplace transformed without difficulty to give:

$$\begin{align*}
(sI - F - F_e + KH - G_e C)x_e(s) &= x_e(0^+) - (F_e + G_e C)x(s) - G_{re}u_r \hspace{1cm} (4.26) \\
(sI - F - GC)x(s) - x(0^+) - GCx_e + G_r u_r(s) &= \hspace{1cm} (4.27)
\end{align*}$$

where $x(0^+)$ and $x_e(0^+)$ are the vector values at the start of operation and $x(s)$ and $x_e(s)$ are the transforms of the time-varying vector values. Equation (4.26) may then be solved for $x_e(s)$ and substituted into (4.27), after which the result may be solved for $x(s)$:
\[
\bar{x}(s) = \left\{ \left( sI - F - GC \right) - GC(sI - F - F_e + KH - G_e C)^{-1}(F_e + G_e C) \right\}^{-1} \times \\
\left\{ G_r + GC(sI - F - F_e + KH - G_e C)^{-1}G_{re} \right\} u_r(s) + \bar{x}(0+) \\
- GC(sI - F - F_e + KH - G_e C)^{-1}x_e(0+) \right\}
\]

Equation (4.28) may be viewed as containing the transfer functions relating the command \( u_r \) and the initial conditions \( \bar{x}(0+) \) and \( x_e(0+) \) to the transformed state vector \( \bar{x}(s) \) for the general case including mismatches.

If there are no mismatches (i.e., if \( F_e \), \( G_e \) and \( G_{re} \) are null matrices), then (4.26) and (4.28) are drastically simplified:

\[
(sI - F + KH)x_e(s) = x_e(0+) \tag{4.29}
\]

\[
\bar{x}(s) = (sI - F - GC)^{-1} \left[ G_r u_r(s) + \bar{x}(0+) - GC(sI - F + KH)x_e(0+) \right] \tag{4.30}
\]

Equation (4.29) shows that, in the matched case, the estimator error decays with poles which are the eigenvalues of \( (F - KH) \) and is unaffected by the state \( \bar{x}(s) \) or the external command \( u_r \). Similarly, (4.30) shows that the transfer function from \( u_r \) and \( \bar{x}(0+) \) has poles which are the eigenvalues of \( F + GC \) only, and that the initial condition on \( x_e \) but not its later values affects \( \bar{x}(s) \). Hence, the "control poles" may be chosen separately from the "estimator poles", the latter being chosen optimally from Kalman filtering theory if desired. References B3 and L1 give somewhat similar results for the matched case.
4.3.2.2 Control Design for Selected Poles

4.3.2.2.1 Feedback for Pole Placement

4.3.2.2.1.1 Historical Development

Many authors have made contributions to the theory of feedback for pole placement, and quite often they have been unaware of related contributions by others. This section will summarize chronologically the important historical developments.

The earliest contribution appears to have been made in 1962 by Rosenbrock in England (Ref. R2) for a multi-input system, with idealized assumptions on the input distribution matrix $G$ and the use of modal feedback.

A different approach was taken by Brockett in 1965 (Ref. B2), who treated a controllable, single-input system with a plant matrix in companion form, i.e., with each upper-diagonal element unity, with the bottom row as -1 times the coefficients of the characteristic equation and zeroes elsewhere. For this coordinate system, he showed readily that feedback of all states to the single input could be found such that the poles would have any desired location. Unfortunately, extra computational work is required to go into the companion-matrix space and back to the original state space, and not much analytical insight would be gained. Brockett also credits an earlier contribution along this line by Morgan. Brockett's paper is important for other reasons as well, such as his method of finding the zeroes of a transfer function from state-variable information.

Ellis and White in England wrote a lengthy series of papers (Ref. E1) in 1965, in which the main contribution of interest here was an explicit formula (similar to equation (4.42) later herein) for changing one eigenvalue by modal feedback, for the case of a single input and a general (not modal) input distribution vector $g$. 
Gould and Murray-Lasso contributed a paper (Ref. G1) in 1966, based on the latter's doctoral thesis, in which modal control was applied to distributed (e.g., process-control) systems. Gould's book (Ref. G2) in 1969 presented a tutorial summary of References R2 and G1.

Wonham (Ref. W2) contributed an important but difficult paper in 1967 on assigning poles in multi-input systems. Using a set of state variables based in part on the companion matrix, he showed how to construct an inner vector feedback loop "which renders the system cyclic, that is, controllable by a single input". This single (scalar) input $u$ is converted to a vector input $gu$ of the original control dimension by the choice of a suitable vector $g$, and $u$ is set equal to $c^T x$. The proper choice of $g$ and $c$ achieve the required poles.

Porter and Carter of England published a paper (Ref. P2) in 1968 showing how to "alter one pair of conjugate eigenvalues at a time by the provision of a suitable pair of control loops" with modal feedback (underline added). The assumption that "two feedback loops are required" was later found not to be necessary.

A paper (Ref. T1) on modal control in a multi-input system was published by Takahashi and others of the University of California in 1968. Although it is a contribution, this paper suffers from the disadvantage that the analysis is restricted to the original modal space of the plant without feedback, which unnecessarily restricts the specification on the matrix of the system after feedback. It would appear that some flexibility inherent in multiple inputs is thereby lost.

Crossley and Porter of England contributed in 1968 a paper (Ref. C1) which concisely showed how to shift one real eigenvalue at a time in a single-input system, with aircraft control systems as examples.
Later in 1968 Simon and Mitter published a lengthy paper (Ref. S12) on modal control for a multi-input system, reading in part: "To algorithmize the procedure for deriving a modal control law, a restriction is placed on the control which in effect reduces the system to a scalar-input system". It comments on the difficulty of moving all the specified eigenvalues at one time, and recommends a recursive procedure to shift "one or two eigenvalues at a time". Although this paper makes theoretical contributions, it appears that the recursive procedure could prove to be time-consuming if a number of eigenvalues are to be changed.

Potter and Vander Velde's paper (Ref. P6) in early 1969 showed that stabilization of an unstable plant is possible if only the unstable modes are controllable and are fed back appropriately through a dynamic controller.

The most useful paper of all prior to early 1969 was published in a somewhat obscure English journal by Crossley and Porter (Ref. C2). For a single-input system it gave a concise formula for simultaneously placing r eigenvalues of controllable modes by modal feedback. This formula has been used successfully herein. The paper suffers from the disadvantage of a difficult derivation, in which two key steps are hard for the reader to verify.

Porter (Ref. P4), and Crossley and Porter (Ref. C3 and C4) published papers in 1969 on sensitivities of eigenvalues and eigenvectors to parameter (e.g., loop gain) variations. Reference P4 treats a multi-input system and defines an eigenvalue sensitivity index I:

\[ I = \sum_{i=1}^{n} \left( \frac{\Delta \lambda_i}{\lambda_i} \right)^2 \]  \hspace{1cm} (4.31)
for a 1 percent variation in the absolute value of each feedback gain. It suggests that: "The preferred system is then that with the smallest value of \( l \)". As an example, it shows four different control systems for a two-state plant with two control inputs, all four having the same two eigenvalues. The preferred design has an eigenvalue sensitivity index which about 0.16 times that of the most sensitive design. This is a potentially important way to make use of the design freedom inherent in multi-input plants, and one which appears to have been overlooked by other authors, who are chiefly concerned with reducing the plant to a single-input plant for convenience.

Porter and Crossley contributed another paper (Ref. P5) in 1970 on eigenvalue placement in a system with proportional, integral and derivative inputs.

In early 1970, Mayne and Murdoch in England published a paper (Ref. M1), which they had submitted in December 1968, in which they derived the Crossley-Porter formula for shifting simultaneously a number of eigenvalues in a single-input system by modal feedback. Judging from the dates and citations, Mayne and Murdoch appear to have been unaware of the earlier work of Crossley and Porter (Ref. C2), and they give a somewhat clearer derivation.

The most recent known paper on the exact placement of poles by modal feedback in a single-input system appears to be Reference G3, published by Gould, Murphy and Berkman in April 1970. The multiple-input case is initially reduced to a single-input case. The chief contribution is a relatively clear derivation of the Crossley-Porter formula in Reference C2, with
extension to the case of repeated eigenvalues of the open-loop system. This is accomplished partly by considering the final characteristic equation and not the final eigenvectors.

In all the previously discussed literature, it is assumed that any necessary state or modal feedback is available; feedback of even a few modes typically requires measurement of all state variables. If this measurement requirement is not met, the use of a Kalman estimator or Luenberger observer, together with the separation principle in Subsection 4.3.2.1, is generally assumed. For the case where the configuration is restricted to incomplete measurements and pure feedback gains (no estimator or observer), two recent papers (Ref. J1 and D1) show how to place a limited number of poles, although some movement of others must be accepted. These papers will be discussed in a later subsection herein.

4.3.2.2.1.2 Elementary Aspects of Modal Feedback

An elementary treatment of modal feedback, similar to that of Reference C1, will illustrate the basic principles. Consider a time-invariant system with one input variable $u$ and feedback from the states:

$$\dot{x} = Fx + gu = \left[ F + gc^T \right] x \quad (4.32)$$

Let $v_i$ and $q_i$ be the right and left eigenvectors for the eigenvalue $\lambda_i$, as in Reference W1:

$$Fv_i = \lambda_i v_i \quad (4.33)$$

$$q_i^TF = \lambda_i q_i^T \quad (4.34)$$

with:
\[ v_i^T q_j = 1 \quad j = i \] (4.35)
\[ v_i^T q_j = 0 \quad j \neq i \] (4.36)

Suppose that \( q_j^T \mathbf{g} = 0 \). Then the \( j \)-th mode is uncontrollable, which may be seen from (4.55) and (4.58) later in this chapter, and:

\[ q_j^T \left[ \mathbf{F} + \mathbf{g} \mathbf{c}^T \right] = q_j^T \mathbf{F} = q_j^T \mathbf{F} = \lambda_j q_i^T \] (4.37)

so that any feedback row vector \( c^T \) leaves the \( j \)-th eigenvalue unchanged.

Now, let the row vector \( c^T \) be decomposed into the \( n \) left eigenvectors of \( \mathbf{F} \) (which span the space if the \( n \) eigenvalues are distinct), so that:

\[ \dot{x} = Fx + \mathbf{g} \left[ \sum_{i=1}^{n} c_{zi} q_i^T \right] x \] (4.38)

where the \( c_{zi} \) are scalar gains. If the complete system matrix is multiplied by a particular \( v_k \):

\[ \left[ \mathbf{F} + \mathbf{g} \sum_{i=1}^{n} c_{zi} q_i^T \right] v_k = \lambda_k v_k + \mathbf{g} c_{zk} \] (4.39)

If \( c_{zk} = 0 \), corresponding to "unobservability" of \( v_k \) in \( u \), then this right eigenvector and eigenvalue are unchanged.

Now, let only \( c_{zk} \) be nonzero, so that only one mode is fed back and:

\[ \dot{x} = Fx + c_{zk} q_k^T x \] (4.40)

Multiplication of the complete system matrix by \( q_k^T \) leads to:
\[ a_k^T \left[ F + c_{zk} g c_k^T \right] = \lambda_k a_k^T + c_{zk} (a_k^T g) c_k^T = \mu_k a_k^T \]  

(4.41)

where \( \mu_k \) is a scalar that turns out to be the new eigenvalue of the \( k \)-th mode. It is related to the gain \( c_{zk} \) by:

\[ c_{zk} = \frac{\mu_k - \lambda_k}{a_k^T g} \]  

(4.42)

By the result of the previous paragraph, the other eigenvalues are unchanged. From another viewpoint, if it is desired to shift only one eigenvalue from \( \lambda_k \) to \( \mu_k \), then (4.42) gives the necessary gain by which that one mode must be fed back.

For this simple case, the sensitivity of \( \mu_k \) to \( c_{zk} \) is seen to be:

\[ \frac{d \mu_k}{d c_{zk}} = a_k^T g \]  

(4.43)

For an uncontrollable \( k \)-th mode, \( a_k^T g \) and this sensitivity are zero. This type of sensitivity affords a quantitative measure of controllability rather than the "yes or no" qualitative answer with the commonly cited criterion (Ref. B1, p. 164) involving \( g, Fg, F^2g \), etc.

4.3.2.1.3 Derivation of Crossley-Porter Formula for Placing Poles in a Single-Input System

The Crossley-Porter formula (Ref. C2) is sufficiently important in this work to warrant a derivation here. For the sake of clarity, the subsequent derivation of Gould et al (Ref. G3) is presented with four modifications:

1) A single input is assumed at the outset, so that multiple inputs can be dealt with independently; 2) Distinct eigenvalues of the open-loop plant are assumed for added clarity, because the complication of repeated eigenvalues
can usually be avoided readily in practice; 3) Movement of \( r \leq n \) eigenvalues is treated; 4) The initial treatment is general enough to encompass a measurement vector \( y = Hx \) of order \( n_y \leq n \). This latter modification will facilitate the later derivation of Jameson's result (Ref. J1) and help to show its relation to the modal feedback literature (e.g., Ref. C2).

Let the plant and measurement equations be:

\[
\dot{x} = Fx + gu
\]

\[
y = Hx
\]

where the order of \( y \) is \( n_y \leq n \). The scalar input consists of a feedback control \( u_f \) and a 'command' or 'reference input' control \( u_r \), such that:

\[
u = u_f + u_r
\]

\[
u = c_y^T y = c_y^T Hx
\]

where \( c_y \) is a measurement vector of order \( n_y \).

The Laplace transform of (4.44) can be expressed as:

\[
(sI - F)x(s) = x(0+) + gu_f + gu_r
\]

\[
x(s) = (sI - F)^{-1} x(0+) + (sI - F)^{-1} gu_f + (sI - F)^{-1} gu_r
\]

where \( x(0+) \) is the initial state. Equation (4.49) can be substituted in the transform of (4.47) to give:

\[
\left[ 1 - c_y^T H (sI - F)^{-1} g \right] u_f = c_y^T H (sI - F)^{-1} x(0+) + c_y^T H (sI - F)^{-1} gu_r
\]

Now assume that the initial-condition term in (4.50) has died away (which implies LHP eigenvalues) or else assume that \( x(0+) \) is zero in the first
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place. Assume also that the command \( u_r \) is zero, and so (4.50) reduces to the closed-loop characteristic equation:

\[
1 - \frac{c^T H}{y} (sI - F)^{-1} g = 0
\]  

(4.51)

Up to now, the derivation is suitable either for the Crossley-Porter formula (Ref. C2 and C3) or Jameson's result (Ref. J1), and it will now be specialized for the first one. Digressing for a moment, the matrix of eigenvector equations is:

\[
F \overline{V} = \overline{V} \Lambda
\]  

(4.52)

where \( V \) and \( \Lambda \) are respectively the matrix of right eigenvectors and the matrix of eigenvalues. It is readily found that:

\[
(sI - F)^{-1} = V (sI - \Lambda)^{-1} V^{-1}
\]  

(4.53)

With:

\[
\overline{x} = V \overline{z}
\]  

(4.54)

the modal form of (4.44) and (4.45) is:

\[
\dot{\overline{z}} = \Lambda \overline{z} + V^{-1} g u
\]  

(4.55)

\[
y = H \overline{x} = HV \overline{z}
\]  

(4.56)

Equations (4.35) and (4.36) may be expressed as:

\[
QV = I
\]  

(4.57)

\[
Q = V^{-1}
\]  

(4.58)

where \( Q \) may be interpreted as the \( n \) by \( n \) matrix of transposed left eigen-
vectors. The modal measurement matrix $H_z$ and modal input-distribution vector $g_z$ are:

$$H_z = HV$$  \hspace{1cm} (4.59)$$

$$g_z = V^{-1}g = Qg$$  \hspace{1cm} (4.60)$$

Substitution of (4.53), (4.59) and (4.60) into (4.51) gives:

$$1 - c_y^T H_z (sI - \Lambda)^{-1} g_z = 0$$  \hspace{1cm} (4.61)$$

At this point, the measurement vector $y$ will be set equal to the modal vector $z$ of order $n$, so that $H_z = I$, $H = Q$, and $c_z$ is used instead of $c_y$. The matrix in (4.61) is expressible as:

$$(sI - \Lambda)^{-1} = \frac{\text{Adj} (sI - \Lambda)}{|sI - \Lambda|}$$  \hspace{1cm} (4.62)$$

in which the determinant is also the characteristic equation of the open-loop system when it is set equal to zero:

$$|sI - \Lambda| = \prod_{i=1}^{n} (s - \lambda_i) = A(s) = 0$$  \hspace{1cm} (4.63)$$

Then (4.61) can be expressed as:

$$1 - c_z^T [1/A(s)] g_z = 0$$  \hspace{1cm} (4.64)$$

in which the $j$-$j$-th element of the matrix does not include the factor $(s-\lambda_j)$. 
Note that after division of the matrix by \( A(s) \), this element would be merely \( 1/(s - \lambda_j) \).

At this point, the treatment diverges slightly from that of Ref. C3, because the Crossley-Porter formula (Ref. C2) is derived for the change of the first \( r \) eigenvalues, where \( 1 \leq r \leq n \). Hence, let:

\[
p_j \neq \lambda_j \quad j \leq r
\]

\[
p_j = \lambda_j \quad j > r
\]

where the \( p_j \) are the desired closed-loop poles. Subsection 4.3.2.2.1.2 has already shown that the latter condition may be accomplished by not feeding back modes \( r + 1 \) to \( n \), so that \( c_{z, r+1} \) to \( c_{z, n} \) in (4.64) should be set equal to zero. Because the matrix is diagonal, (4.64) can be expressed as:

\[
1 - \left[ 1/A(s) \right] \sum_{j=1}^{r} c_{zj} g_j \prod_{i=1 \atop i \neq j}^{n} (s - \lambda_i) = 0
\]

\[
1 - \sum_{j=1}^{r} \frac{c_{zj} g_j}{s - \lambda_j} = 0
\]

Now, this is the closed-loop characteristic equation, which must be zero for \( s = p_k \), where \( p_k \) is any one of the first \( r \) closed-loop poles. Therefore, (4.68) must be the following ratio of polynomials:

\[
\frac{B(s)}{A(s)} = \frac{\prod_{i=1}^{n} (s - p_i)}{\prod_{i=1}^{n} (s - \lambda_i)} = 1 - \sum_{j=1}^{r} \frac{c_{zj} g_j}{s - \lambda_j}
\]

(4.69)
Note that if $s$ approaches infinity, each side approaches 1. Equation (4.66) means that the factors $(s - \lambda_i)$ for $i$ greater than $r$ will cancel in the numerator and denominator of (4.69), so that:

$$\prod_{i=1}^{r} \frac{(s - p_i)}{(s - \lambda_i)} = 1 - \sum_{j=1}^{r} \frac{c_{zj}g_{zj}}{s - \lambda_j}$$  \hspace{1cm} (4.70)$$

Each product $c_{zj}g_{zj}$ is a residue which may be found by the partial-fraction technique as:

$$c_{zj}g_{zj} = -\left[\frac{\prod_{i=1}^{r} (s - p_i)}{\prod_{i=1}^{r} (s - \lambda_i)}\right]_{s=\lambda_j}$$  \hspace{1cm} (4.71)$$

$$= \frac{\prod_{i=1}^{r} (p_i - \lambda_i)}{\prod_{i=1, i\neq j}^{r} (\lambda_i - \lambda_j)}$$

Recognizing from (4.60) that:

$$g_{zj} = g_j^{T}g$$  \hspace{1cm} (4.72)$$

it is clear from considering both (4.71) and Subsection 4.3.2.2.1.2 that if $g_{zj} = 0$, this mode is uncontrollable and $p_j = \lambda_j$.

Finally, (4.71) is equivalent to:
\[ c_{zj} = \frac{\prod_{i=1}^{r} (p_i - \lambda_j)}{\prod_{i=1}^{r} (\lambda_i - \lambda_j)} \]  \hspace{1cm} (4.73)

which is the Crossley-Porter formula (Ref. C2) with different symbols.

If only one mode is to be changed, \( r = 1 \) and (4.73) reduces to (4.42).

4.3.2.2.1.4 Design Example

The following example utilizes the Crossley-Porter formula (4.73), for feedback only to the \( u_1 \) input in Figure 4-2. At this point in the research, the integrator with input \( u_2 \) is not justified and so \( x_i = 0 \). The bias \( \delta_b \) was represented as shown, with the addition of a negligible feedback of \( -0.01 \) around its integrator, so as to move the associated eigenvalue from 0 to \( -0.01 \) rad/sec for numerical convenience. Note that the \( \delta_b \) integrator was not driven and that it develops a mode which is uncontrollable from \( u_1 \). The example uses the same airframe as before, at Mach 2, sea level.

The state equation was:

\[
\begin{align*}
\dot{x} &= Fx + gu \\
\begin{bmatrix}
A_m \\
\dot{\theta} \\
\dot{\delta} \\
\ddot{\delta} \\
\dddot{\delta}_1
\end{bmatrix} &= \begin{bmatrix}
-1.304 & 2910 & 0 & 302 & 0 \\
0.0482 & -1.016 & -119.6 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 \times 10^4 & -100 & 1 \times 10^4 \\
0 & 0 & 0 & 0 & -0.01
\end{bmatrix}
\begin{bmatrix}
A_m \\
\dot{\theta} \\
\dot{\delta} \\
\ddot{\delta} \\
\dddot{\delta}_1
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
\delta_b \\
1 \times 10^4 \\
0
\end{bmatrix} u_1 \\
\end{align*}
\]  \hspace{1cm} (4.74a)

For the sake of completeness and later examples as well, the block-diagonal eigenvalue matrix and the two corresponding eigenvector matrices
will now be shown. The first of these is:

\[
\Lambda_b = \begin{bmatrix}
10.684 & 0 & 0 & 0 & 0 \\
0 & -13.004 & 0 & 0 & 0 \\
0 & 0 & -50.0 & 86.602 & 0 \\
0 & 0 & -86.602 & -50.0 & 0 \\
0 & 0 & 0 & 0 & -0.01
\end{bmatrix}
\] (4.75)

The corresponding eigenvector matrix \( V_b \), which satisfies:

\[
FV_b = V_b \Lambda_b
\] (4.76)

was computed as:

\[
V_b = \begin{bmatrix}
1 & 1 & 1 & 0 & 1 \\
4.12 \times 10^{-3} & -4.02 \times 10^{-3} & 1.37 \times 10^{-3} & 2.96 \times 10^{-3} & 4.45 \times 10^{-4} \\
0 & 0 & 3.11 \times 10^{-3} & 2.20 \times 10^{-4} & 3.99 \times 10^{-4} \\
0 & 0 & -0.174 & -0.258 & -3.99 \times 10^{-6} \\
0 & 0 & 0 & 0 & 3.99 \times 10^{-4}
\end{bmatrix}
\] (4.77)

Initially, computations were made for the diagonal form of \( \Lambda \) and its corresponding non-normalized eigenvector matrix \( V \), which was then normalized so that the largest element in each column (which happened to be the first) was +1. Then \( V_b \) was found by making its third and fourth columns be the real and imaginary parts respectively of the third column of \( V \), and retaining the other columns (eigenvectors of the real eigenvalues). Then the matrix \( Q_b \) was computed as:
\[ Q_b = V_b^{-1} = \]
\[
\begin{bmatrix}
0.493 & 123 & -279 & -1.17 & -1100 \\
0.506 & -123 & -28.2 & -1.43 & -1100 \\
0 & 0 & 307 & -0.261 & -307 \\
0 & 0 & 207 & 3.70 & -207 \\
0 & 0 & 0 & 0 & 2500
\end{bmatrix}
\]

(4.78)

It is readily shown that the first, second and fifth rows of \( Q_b \) are the transposed left eigenvectors of the real eigenvalues of \( F \), satisfying:

\[ q^T F = \lambda \ q^T \]

(4.79)

In this design example, it was desired to shift the airframe eigenvalues (+10.684 and -13.004) to -10. and -30. rad/sec. The eigenvalue at -0.01 rad/sec cannot be shifted by feedback to the \( x_4 \) integrator and it was desired to leave the actuator eigenvalues fixed, so that the stability of the real system with more actuator lags would not be impaired. Hence, feedback of the first two modes was required.

The Crossley-Porter formula (4.73) requires the following elements of the modal input distribution vector (see also equation (4.60)):

\[ g_{z1} = q_1^T g = -1.17 \times 10^4 \]

(4.80)

\[ g_{z2} = q_2^T g = 1.43 \times 10^4 \]

(4.81)

Equation (4.73) gave the two modal feedback gains as:

\[ c_{z1} = 3.03 \times 10^{-3} \]

(4.82)

\[ c_{z2} = -1.502 \times 10^{-4} \]

(4.83)
and of course the modal gains $c_{z3}$ through $c_{z5}$ were zero.

The plant equation with feedback was:

$$\dot{x} = Fx + gc_z^T Hx = \left[F + g (c_{z1}q_1^T + c_{z2}q_2^T)\right] x = F_f x \quad (4.84)$$

$$F_f = \begin{bmatrix}
-1.304 & 2910 & 0 & 302 & 0 \\
0.0482 & -1.016 & -119.6 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
14.218 & 3910.5 & -18413 & -137.68 & -21560 \\
0 & 0 & 0 & 0 & -0.01 \\
\end{bmatrix} \quad (4.85)$$

Note that $H = Q$ from the previous subsection, and that scaling $q_1$ and $q_2$ does not affect (4.84). The eigenvalues of $F_f$ were found to be:

-10.00, -30.00, -0.01, and -50. $\pm j86.6$ rad/sec, confirming the Crossley-Porter formula.

It should be noted that this preliminary design does not eliminate the effect of the bias, which is dealt with in the next subsection. Of course, an estimator would be required, because some of the states are not directly measurable.

### 4.3.2.2.2 Treatment of Bias Problem

The problem of biases in regulators is a surprisingly neglected subject in the literature of modern control, as Johnson has very recently pointed out (Ref. J4).

### 4.3.2.2.2.1 Estimation of Bias

A bias such as $\delta_b$ in Figure 4-2 is seldom measurable directly. The usual treatment, as shown in Ref. A1, is to treat the bias as another state variable which is the output of an undriven integrator, and to estimate the bias with a Kalman estimator.
Friedland (Ref. F2) has developed a scheme in which the estimation of the biases is uncoupled from the estimation of the bias-free states, except for a final vector addition. Reference F2 claims that the main advantage of the new method is in avoiding computations with large vectors and matrices when the number of biases approaches the number of bias-free states. In the case at hand, the biases can be lumped as one bias $\delta_b$ entering the fin-servo input, and so Reference F2 does not seem to be needed.

Reference P7 treats an interesting problem of optimal design of an estimation and control scheme for a star tracker mounted on a gyro-stabilized platform. The star tracker has virtually white noise, while the gyro angular drift rate is modelled as a "random walk", i.e., integrated white noise, which is called "Brownian motion" in Reference D2. Reference P7 states on page 9 that: "The optimum compensator is seen to be a parallel connection of a proportional and an integral path. -- The integral path results from the estimate of the gyro drift rate".

4.3.2.2.2 Cancellation of Bias If Estimator is Used

It has been suggested to the writer by Professors Potter and Vander-Velde that the effect of the bias in Figure 4-2 on the other states can be cancelled by feeding a matched gain from an estimate of $\delta_b$ into the $u_1$ terminal, and that a separate integrator (producing $x_i$ in Figure 2) is not required.

A generalization of this technique of cancellation follows, for the case of a correct plant model in the estimator, to which (4.29) and (4.30) apply. The contribution of $x_e(0+)$ to $x(s)$ in (4.30) dies away because the eigenvalues of $F-KH$ for a proper estimator design are in the left half-plane. The bias term may be represented as element $x_b$ of $x(0+)$, assuming a negligible
feedback around the bias integrator. Suppose that it is required that state $x_1(t)$ be unaffected by $x_b$. For $u_r$ and $x_e(0^+)$ both zero, (4.27) can be inverse transformed to give:

$$x(t) = \mathcal{L}^{-1} \left[ \frac{J(s)}{[sI - F - GC]} x(0^+) \right]$$  \hspace{1cm} (4.86)

where $J(s)$ is the adjoint matrix for $(sI - F - GC)$. Element $x_b$ (the bias) cannot affect $x_1(t)$ if the $i$-$b$-th element of $J(s)$ is zero:

$$J_{ib}(s) = 0$$  \hspace{1cm} (4.87)

This is the equivalent to requiring that the minor determinant, formed by deleting the $b$-th row and $i$-th column of $(sI - F - GC)$, be zero.

For the case at hand, the output variable of interest is $A_m = x_1 = x_i$ and the bias variable is $\delta_b = x_5 = x_b$. From (4.75) this means that:

$$\begin{bmatrix} 2910 & 0 & 302 & 0 \\ -1.016 & -119.6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 \times 10^4 & -100 & 1 \times 10^4 (1+c_b) \end{bmatrix} = 0$$  \hspace{1cm} (4.88)

where $c_b$ is the bias gain from $\delta_b$ to the input of the $\delta$ integrator. Obviously, this determinant is zero if $c_b = -1$, which is the matching gain previously suggested for cancellation. (There may be applications which are not so simple, in which this determinant technique would be helpful.)

It is desired both to cancel the bias and to achieve the required poles, i.e., $-10.0$, $-30.0$, $-0.01$ (which cannot be changed by feedback to $u_1$) and $-50. \pm j86.6 \text{ rad/sec}$. Notice from (4.85) that the amount added by
modal feedback to $F_{45}$ was $-31560$. Suppose this feedback is conceptually cancelled by a suitable feedback of $c_5^T q_5^T x$, thus restoring $F_{45} = 10000$.

The actual net feedback that is necessary for bias cancellation has already been shown to be:

$$u = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \end{bmatrix} x$$

(4.89)

Thus, referring to (4.85), the combination of net modal feedback and bias-cancellation feedback is:

$$u_f = 10^{-4} \begin{bmatrix} 14.218 & 3910.5 & -8413 & -37.68 & -10000 \end{bmatrix} x$$

(4.90)

and the plant after modal feedback and bias cancellation has the following matrix:

$$F_{fbc} = \begin{bmatrix}
-1.304 & 2910 & 0 & 302 & 0 \\
0.0482 & -1.016 & -119.6 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
14.218 & 3910.5 & -18413 & -137.68 & F_{fbc(4,5)} \\
0 & 0 & 0 & 0 & -0.01 \\
\end{bmatrix}$$

(4.91)

where $F_{fbc(4,5)} = 0$. Now, the eigenvalues of $F_f$ in (4.85) were calculated to be the specified values. A little further thought will show that the determinant $|F_f - \lambda I|$ will be (-0.01 - $\lambda$) times the upper left 4 by 4 minor determinant, irrespective of the value of the element $F_f(4,5)$. Therefore, the matrix $F_{fbc}$ will have the specified eigenvalues, whether $F_{fbc}(4,5)$ is -21560 as in (4.85) or is 0, which is desired for bias cancellation. Of course, eigenvalue computations have confirmed this.

In the current example, the use of modal feedback followed by bias cancellation led to a conceptual difficulty with element $F_{45}$. A simpler
approach is to perform the bias cancellation first, because it then turns out that the subsequent modal feedback does not influence the element $F_{45}$.

The basic plant matrix with bias cancellation is:

$$F_{bc} = \begin{bmatrix}
-1.304 & 2910 & 0 & 302 & 0 \\
0.0482 & -1.016 & -119.6 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1.1 \times 10^4 & -100 & 0 \\
0 & 0 & 0 & 0 & -0.01 \\
\end{bmatrix} \tag{4.92}$$

which is seen to be the combination of two uncoupled plants, the second one being the bias integrator with numerically convenient feedback -0.01. Clearly, $F_{bc}$ has the same block-diagonal eigenvalue matrix $\Lambda_b$ as (4.76), but the matrices $V_b$ and $Q_b$ are now somewhat simplified:

$$V_{bbc} = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
4.12 \times 10^{-3} & -4.02 \times 10^{-3} & 1.37 \times 10^{-3} & 2.96 \times 10^{-3} & 0 \\
0 & 0 & 3.11 \times 10^{-3} & 2.20 \times 10^{-4} & 0 \\
0 & 0 & -0.174 & 0.258 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \tag{4.93}$$

$$Q_{bbc} = V_{bbc}^{-1} = \begin{bmatrix}
0.493 & 123 & -279 & -1.17 & 0 \\
0.506 & -123 & -28.2 & 1.43 & 0 \\
0 & 0 & 307 & -0.261 & 0 \\
0 & 0 & 207 & 3.70 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \tag{4.94}$$
It is apparent that only the fifth column of $Q_{b,bc}$ is different from that of $Q_b$ in (4.78). In applying modal feedback to $F_{b,c}$ for the required shift of the eigenvalues, it is evident that $g_{z1}$, $g_{z2}$, $c_{z1}$ and $c_{z2}$ are the same as in (4.80) - (4.83). Applying (4.84) then leads to the same result as in (4.91), with the element $F_{f,bc}(4,5) = 0$ and without any conceptual difficulty.

The conclusion to be drawn from these two methods is that it is simpler to perform the bias cancellation first, thus uncoupling the bias sources from the rest of the plant, and then to apply modal feedback to obtain the required pole distribution.

Of course, the bias or sum of biases is seldom measurable directly, and so the estimator must estimate it from available measurements, which is to be discussed in a subsequent subsection.

4.3.2.2.3 Integral Compensation without Estimator

Suppose that the designer seeks an initially simpler configuration than one using an estimator-controller. In order to deal with the bias problem, he recalls that the classical technique is to use an integrator in the forward path so as to "buck off" the bias. A useful way to reconcile the classical and modern view of integral control has been found by the author.

In Figure 4-2, let the control integrator ($x_1$) be utilized, and let the feedback around it and around the bias integrator be either zero or some small number such as -0.01 rad/sec. Unfortunately, the estimator cannot distinguish between the bias $\delta_b$ and the initial condition $x_1(0^+)$ on the control integrator if it is restricted to using the gyro and accelerometer for measurements, i.e., the system is "unobservable". Furthermore, the system is "uncontrollable", because the bias state cannot be driven. These are conceptual difficulties which can be overcome if it is noted that $x_1$ and $\delta_b$ are both the outputs of simple integrators which makes it possible
to define a new state:

\[ x_5 = x_1 + \delta_b \]  \hspace{1cm} (4.95)

In Figure 4-5, the new state \( x_5 \) has replaced the two old states, and the system of five states is now both observable from \( A_m \) and \( \theta \) and controllable by inputs from \( u_1 \) and \( u_2 \), which constitute the "proportional plus integral" control inputs. (A minor feedback of -0.01 rad/sec added around \( x_5 \) purely for computational convenience.) An appropriate regulator design will now drive all five states down to zero, including \( x_5 \), and so the output \( x_1 \) of the electronic integrator will be -1 times the bias, thus "bucking off" the bias as one thinks of it in classical control theory.

It is important to remember that \( x_5 \) is the output of a conceptual integrator, not a physical electronic integrator, and is not measurable directly. The open-loop plant has a pole at the origin resulting from the integrator, and this has an important effect on the closed-loop poles, as in the case of the classical Raytheon autopilot. Since the use of an estimator is assumed to be disallowed, then there may be difficulty in placing the poles of the closed-loop plant. Certain papers by Johnson (Ref. J4) and Jameson (Ref. J1), to be discussed subsequently, are relevant to this problem.

4.3.2.2.4 A Servo Problem Reducible to a Regulator with a Bias on the Input

A suitable redefinition of states, as in the previous subsection, can simplify the approach to another type of problem, namely, a servo with a constant-speed input.

Figure 4-6a represents an amplifier and a servomotor such as a d-c torque motor with damping and inertia load, having the state equation:
Figure 4-5 - Proportional Plus Integral Compensation without Estimator
-229-

a) AMPLIFIER AND SERVOMOTOR TO BE CONTROLLED (PLANT)

b) COMMAND-INPUT GENERATOR

c) PLANT AND COMMAND-INPUT GENERATOR COMBINED INTO A REGULATOR

Figure 4-6 - Reduction of a Servo Problem to a Regulator with a Bias on the Input
\[
\begin{bmatrix}
\dot{\theta} \\
\dot{\Omega}
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
0 & F_{22}
\end{bmatrix}
\begin{bmatrix}
\theta \\
\Omega
\end{bmatrix} + 
\begin{bmatrix}
0 \\
g_2
\end{bmatrix} u
\] (4.96)

Suppose that the servo is required to run at a constant speed \( \Omega_{in} \) with zero position error relative to the reference input \( \theta_{in} \) in Figure 4-6b, which has the equation:

\[
\begin{bmatrix}
\dot{\theta}_{in} \\
\dot{\Omega}_{in}
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\theta_{in} \\
\Omega_{in}
\end{bmatrix}
\] (4.97)

Let new state variables \( E_\theta \) and \( E_\Omega \) be defined as the position error \( \theta_{in} - \theta \) and velocity error \( \Omega_{in} - \Omega \), with the state equation:

\[
\begin{bmatrix}
\dot{E}_\theta \\
\dot{E}_\Omega
\end{bmatrix} = 
\begin{bmatrix}
\dot{\theta}_{in} - \dot{\theta} \\
\dot{\Omega}_{in} - \dot{\Omega}
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
0 & F_{22}
\end{bmatrix}
\begin{bmatrix}
E_\theta \\
E_\Omega
\end{bmatrix} + 
\begin{bmatrix}
0 \\
1
\end{bmatrix} b + 
\begin{bmatrix}
0 \\
-g_2
\end{bmatrix} u
\] (4.98)

where \( b \) is a constant bias quantity \(-F_{22} \Omega_{in}\). The block diagram for (4.97) in Figure 4-6c is that of a regulator with a bias \( b \) at the input; the state variables are the error quantities which are to be driven to zero. There are evidently two possible ways to proceed: 1) Cancel the bias with an input \( b/g_2 = -F_{22} \Omega_{in}/g_2 \) to \( u \) and add feedback from \( E_\Omega \) and \( E_\theta \); 2) If some problem such as imprecise knowledge of \( F_{22} \) (which is plausible for a torque-motor) makes (1) impractical, then add an electronic integrator producing \( x_1 \) and define \( x_1 + b \) as the third state which is to be driven to zero by proportional plus integral compensation as in the previous subsection.
4.3.2.2.5 Johnson's Method

Three papers by Johnson (Refs. J2, J3, and J4) appear to be significant contributions to the scarce literature on biases and other disturbances. The first of these treats the problem of optimal control of a plant with a constant unmeasurable bias disturbance, so as to minimize a special quadratic performance index over an infinite time period. The scalar control input and the bias are required to enter the same integrator, and an "auxiliary state variable" is then defined as the sum of the control input plus the bias:

\[ x_{n+1} = u(t) + \text{bias} \quad (4.99) \]

and a new control input is defined as:

\[ v(t) = \dot{x}_{n+1} \quad (4.100) \]

Johnson then shows that the optimal control solution has proportional control (a weighted sum of states fed into the u input) plus integral control (another weighted sum of states fed into the new v input). All states except \( x_{n+1} \) may be needed for these feedbacks. The choice of a new integrator to produce \( x_{n+1} \) as shown in (4.99) was independently conceived by this writer in the manner described in the previous subsection.

Johnson's second paper (Ref. J3) in this series elaborates on the initial concept, but his third paper (Ref. J4) is of more interest here. He discusses a theorem for stabilization of a linear dynamical system with a constant disturbance. Changing the notation to conform to this report, let the plant equation be:

\[ \dot{x}(t) = Fx(t) + gu(t) + gw \quad (4.101) \]
where \( F \) is the bias-free plant matrix and \( gw \) is a constant vector disturbance that is collinear with the control input vector \( gu(t) \). Quoting from the paper with notation changes: "Suppose further, that \( c \) is any real constant \( n \) vector which stabilizes the matrix \( (F + gc^T) \), and let \( k \) be any real negative scalar constant. Then, if the scalar control \( u(t) \) in (4.101) is chosen as:

\[
\begin{align*}
     u(t) &= (c^T + kg^T)x - k \int_0^t g^T(F + gc^T)x(\tau)d\tau + \mu \\
     \mu &= \text{constant}
\end{align*}
\]

- - the system (4.101) will be globally asymptotically stable to the origin \( x = 0 \) for all finite values of the unknown constant disturbance \( w \) and the parameter \( \mu \). In fact, the \( (n+1) \) closed-loop eigenvalues of "the system "will be the \( n \) eigenvalues of \( (F + gc^T) \) and the number \( k \)." For a proof of the theorem, "in a somewhat more general format", Johnson refers to his previous paper (Ref. J3).

This rather ingenious result merits some discussion and a slight correction. In order to compute and correct the closed-loop eigenvalues, first define the additional state variable:

\[
    x_{n+1} = \int_0^t g^T(F + gc^T)x(\tau)d\tau
\]

Substitution of (4.102) and (4.104) into (4.101) gives the following equation of the closed-loop system:
\[
\begin{bmatrix}
\dot{x} \\
\dot{x}_{n+1}
\end{bmatrix} =
\begin{bmatrix}
F + gc^T + kgg^T \\
g^T (F + gc^T)
\end{bmatrix}
\begin{bmatrix}
x \\
x_{n+1}
\end{bmatrix} +
\begin{bmatrix}
g \\
0
\end{bmatrix} w +
\begin{bmatrix}
g \\
0
\end{bmatrix} \mu
\] (4.105)

The closed-loop eigenvalues are found from the following determinant:

\[
\begin{bmatrix}
\lambda I - F - gc^T - kgg^T & kg \\
-g^T (F + gc^T) & \lambda
\end{bmatrix} = 0
\] (4.106)

The following elementary property of a determinant (Ref. H2, p. 10) is now utilized: "If to the elements of any row (column) are added k times the corresponding elements of any other row (column), the determinant is unchanged". This may be extended to appropriate vector or matrix operations on the determinant of a partitioned matrix (for example, see p. 46 in Ref. G4). In (4.106), postmultiply the last column by \( g^T \) and add the \( n \) results to the other columns in the two left-hand partitions:

\[
\begin{bmatrix}
\lambda I - F - gc^T & kg \\
-g^T (F + gc^T) + \lambda g^T & \lambda
\end{bmatrix} = 0
\] (4.107)

Now, premultiply the upper two partitions by \( g^T \) and subtract them from the lowest row, giving:

\[
\begin{bmatrix}
\lambda I - F - gc^T & kg \\
0^T & \lambda - kgg^T \ge
\end{bmatrix} = 0
\] (4.108)
It is then immediately apparent that the eigenvalues of the closed-loop system are those of the matrix \((F + gc^T)\) and the number \(kg^Tg\), not \(k\) as claimed in Ref. J4.

The applicability of Johnson's paper requires some clarification. It is not limited to mere "stabilization" in the sense of placing eigenvalues in the left half-plane, but rather, \(c\) and \(k\) can be chosen for eigenvalues which are both stable and numerically suited to the task at hand. One objective of the paper and its predecessors was the design of regulators to overcome disturbances without requiring their measurement or estimation. On the other hand, consider a case in which the row vector \(c^T\) must have certain zero elements because certain state variables cannot be measured directly; (a later subsection discusses Jameson's contribution (Ref. J1) to feedback of a restricted output vector). Suppose that a suitable \(c\) is found and that \((F + gc^T)\) has acceptable eigenvalues. Now, Johnson's feedback in (4.102) is to be applied so as to overcome the bias and to produce an added eigenvalue \(kg^T\). In the integral in (4.102), the term \(g^T gc^T x(T)\) can be mechanized, but the term \(g^T F x(T)\) may well require unmeasured state variables. Continuing the current example, the row vector \(g^T F\) for the bias-free plant is:

\[
g^T F = \begin{bmatrix} 0 & 0 & 0 & 1 \times 10^4 \end{bmatrix} \begin{bmatrix} -1.304 & 2910 & 0 & 302 \\ 0.0482 & -1.016 & -119.6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 \times 10^4 & -100 \end{bmatrix}
\]

\[
= \begin{bmatrix} 0 & 0 & -1 \times 10^8 & -1 \times 10^6 \end{bmatrix}
\] (4.109)

The fourth element of this row vector indicates that feedback proportional
to \( \dot{\delta} \) is required, but this would necessitate additional instrumentation such as a tachometer(s) that is undesirable, as previously stated. Moreover, the third element corresponds to feedback from \( \delta_{\text{eff}} \) which is made difficult by the possible potentiometer-misalignment bias \( \delta_2 \); a possible aero-moment bias \( \delta_3 \) would further complicate matters.

Therefore, although it is true that Johnson's method does not require a measurement or estimate of the bias, it does require feedback of other variables (via \( g^T_F \)) than are required by the basic proportional feedback via \( gc^T \). Hence, the convenience of Johnson's method is apparently limited to special applications, wherein virtually every state variable (other than the integrator output) is available for feedback.
4.3.2.3 Estimation

Continuing the current design example, it is necessary to feed back the bias state \( x_5 \) and the two airframe modes, or actually their estimates, inasmuch as practical measurements are limited to acceleration \( \dot{A}_m \) and pitch rate \( \dot{\theta} \). The separation principle in Subsection 4.3.2.1 makes it possible to design the estimator or observer separately without pole interaction, as long as the model of the plant in the estimator-controller matches the plant. A Kalman estimator with constant (nearly steady-state) gains rather than a Luenberger observer (Ref. L1) was chosen to provide near-optimum filtering.

4.3.2.3.1 Modal Estimation

Widnall (Ref. W3) has pointed out that the state space of the estimator-controller does not have to be that of the plant, but instead it can be chosen for convenience, as long as the input-output relationships of the estimator-controller are correct. In the case at hand, the experience of Chapter 6 indicated that it was most convenient to compute the Kalman gains in the plant space and then to convert to plant-modal space by using the appropriate eigenvector matrix \( V_b \) and its inverse \( Q_b \). The modal choice was convenient partly because zero control gains were required from the estimates of the actuator modes, inasmuch as their poles were not to be changed. Moreover, it turned out that the Kalman gains for the two actuator modes were virtually zero. These two aspects of the modal gains appreciably simplified the subsequent simplification process on the estimator-controller.

4.3.2.3.2 Computation of Kalman Filter Gains by a Little-Known Algorithm

The derivation of the statistical model for estimation is
continued in Appendix B and will be briefly summarized. Atmospheric
turbulence was slightly approximated so as to provide white process
noise into the integrators for \( A_m \) and \( \theta \), while a plausible assumption
was made for electronic white noise driving the bias integrator. Other
plausible assumptions were made for the measurement noise of the
gyro and accelerometer, based somewhat on current specifications.

In the usual solution (Ref. Bl, p. 365) for the Kalman estimator
gains, the covariance of the estimator error is computed from the
Riccati equation:

\[
\frac{dP}{dt} = FP + PF^T + G_nQG_n^T - PHTR^{-1}HP
\]  
(4.110)

with the boundary condition \( P(0) \) at zero time. Here \( G_n \) is the input
distribution matrix for the process noise with covariance \( Q \), \( R \) is the
covariance of the measurement noise, and \( H \) is the measurement
matrix. The Kalman gain matrix is:

\[
K = PHTR^{-1}
\]  
(4.111)

Now, it may be possible to integrate (4.110) numerically, but the diff-
culties of this are well known; Ref. O'Dl mentions this problem in
describing an eigenvector method for obtaining the asymptotic solu-
tion. Severe numerical difficulties in integrating the Riccati equation
for optimal control were encountered by this writer (Chapter 6),
although admittedly the initial numerical changes for the \( P(t) \) matrix
would be less than those of the control-gain counterpart \( S(t) \) for very
low control-effort weighting in the performance index.

As an alternative to the Riccati equation for Kalman estimation,
Kliger (Ref. K9) has derived an equivalent set of \( 2n \) linear equations
by posing an equivalent pseudo-problem in optimal control with state vector \( x(t) \) and costate vector \( p(t) \). See also Reference K4. This writer proposes deriving a similar set of linear matrix equations on the basis of matrix substitution. Postulate the solution of \( P(t) \) as:

\[
P = ZY^{-1}
\]

(4.112)

where all three matrices are \( n \) by \( n \) and time-varying, with \( Z \) and \( Y \) being the solutions of linear equations:

\[
\begin{bmatrix}
\frac{dY}{dt} \\
\vdots \\
\frac{dZ}{dt}
\end{bmatrix}
= W
\begin{bmatrix}
Y \\
\vdots \\
Z
\end{bmatrix}
= 
\begin{bmatrix}
W_{11} & W_{12} \\
\vdots & \vdots \\
W_{21} & W_{22}
\end{bmatrix}
\begin{bmatrix}
Y \\
\vdots \\
Z
\end{bmatrix}
\]

(4.113)

\( W \) is a \( 2n \) by \( 2n \) matrix which is partitioned into four \( n \) by \( n \) matrices.

The boundary conditions are:

\[
Z(O) = P(O)
\]  
(4.114)

\[
Y(O) = I
\]  
(4.115)

From (4.113) it follows directly that:

\[
\frac{dY}{dt} = W_{11}Y + W_{12}Z
\]  
(4.116)

\[
\frac{dZ}{dt} = W_{21}Y + W_{22}Z
\]  
(4.117)

Now, differentiating (4.112) gives:

\[
\frac{dP}{dt} = \frac{dZ}{dt}Y^{-1} - ZY^{-1}\frac{dY}{dt}Y^{-1}
\]  
(4.118)

Substitution of (4.110), (4.112), (4.114) and (4.117) into (4.118) gives:

\[
FZY^{-1} + ZY^{-1}F + G_nQG_n^{T} - ZY^{-1}HTR^{-1}HZY^{-1} = W_{21} + W_{22}ZY^{-1} - ZY^{-1}W_{11} - ZY^{-1}W_{12}Z
\]  
(4.119)

Equating like terms in (4.119) gives:
\[ W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} = \begin{bmatrix} -F^T & H^T R^{-1} H \\ G_n Q G_n^T & F \end{bmatrix} \] (4.120)

which completes the conversion of the Riccati equation (4.110) to a set of 2n linear equations.

This formulation of the estimation equations is the dual of the 2n formulation of the optimal-control problem, as mentioned in Ref. O'Dl, which shows that the matrix in (4.120) is both symplectic and Hamiltonian, with n RHP and n LHP eigenvalues symmetrically arrayed around the imaginary axis. Both References P1 and O'Dl give a solution for the asymptotic value of P(t), i.e. for t approaching infinity, in terms of the eigenvectors of W for RHP eigenvalues.

The computation of the time-varying P(t) in (4.112) can be approached in the following way. In (4.113), consider the first column (2n by 1) on the left as a vector which can be computed from W times the first column of \( \begin{bmatrix} Y \\ Z \end{bmatrix} \). The same holds for the c-th column on the left, which is computed as W times the c-th column of \( \begin{bmatrix} Y \\ Z \end{bmatrix} \). Hence the solution of (4.113) is conceptually like the solution of n vector differential equations by a transition matrix. The 2n by 2n transition matrix of the constant W matrix can be approximated closely as:

\[ \Omega (D, 0) = I + \sum_{k=1}^{j} \frac{D^k}{k!} W^k \] (4.121)

for a suitably small time interval D and a large enough (ideally infinite) value of j. If \( \Lambda_w \) and \( V_w \) are the eigenvalue and right-eigenvector matrices of W:

\[ WV_w = V_w \Lambda_w \] (4.122)
it is easy to show that:

\[
\Omega(D, 0) = V_w^{-1} e^{\Lambda_w D} V_w
\]

For best convergence, the largest product \(|\text{Re}(\lambda_i)|D\) should be less than unity and the number of terms \(k\) in the series (4.121) should be appropriately large.

If the desired time interval for the transition matrix \(\tau\) is an integer \(m\) times \(D\), it follows that the desired transition matrix is:

\[
\Omega(\tau, 0) = \Omega(mD, 0) = \left[\Omega(D, 0)\right]^m
\]

Parenthetically, the computational accuracy of these methods (Ref. K4) is a potentially useful topic for another thesis. Up to this point, Ref. M2 would probably also be helpful.

In the usual transition-matrix method of computation, one computes:

\[
\begin{bmatrix}
Y(\tau) \\
Z(\tau)
\end{bmatrix} = \Omega(\tau, 0) \begin{bmatrix}
I \\
P(0)
\end{bmatrix}
\]

\[
P(\tau) = Z(\tau)Y^{-1}(\tau)
\]

\[
\begin{bmatrix}
Y(2\tau) \\
Z(2\tau)
\end{bmatrix} = \Omega(\tau, 0) \begin{bmatrix}
Y(\tau) \\
Z(\tau)
\end{bmatrix}
\]

\[
P(2\tau) = Z(2\tau)Y^{-1}(2\tau)
\]

and so forth, the \(s\)-th step being:
\[
\begin{bmatrix}
Y(s\tau) \\
Z(s\tau)
\end{bmatrix}
= \Omega(\tau, 0) \begin{bmatrix}
Y(s-1\tau) \\
Z(s-1\tau)
\end{bmatrix}
= \begin{bmatrix}
\Omega(\tau, 0)
\end{bmatrix}^s \begin{bmatrix}
I \\
P(0)
\end{bmatrix}
\] (4.129)

\[
P(s\tau) = Z(s\tau)Y^{-1}(s\tau)
\] (4.130)

The classic computational difficulty with this method now becomes apparent. In (4.129), the \( s \)-th power of the transition matrix is analytically equivalent to raising each element of the diagonal (matrix in) (4.123) to the power \( ms \). Therefore the RHP eigenvalues of \( W \) with the largest real parts give rise to very large numbers in \( Z \) and \( Y \) which tend to swamp out numerically the corresponding numbers for the LHP eigenvalues. But the LHP eigenvalues are also important in the solution of \( P(t) \), and so at some point the computation of (4.127) to (4.130) exhibits growing asymmetry of \( P(t) \) and finally catastrophic inaccuracy. This problem is also explained in Refs. O'DI and VI; in the latter case, Vaughn seemed to be unaware of the following algorithm.

We finally come to a little-known algorithm which was discovered independently by the writer, only to learn to his chagrin that Kalman and others (Ref. K4, p. 103) had devised it in 1962 but had not publicized it very widely. Suppose that the transition matrix in (4.124) and the first step in (4.125) and (4.126) have been appropriately computed with only small errors. In order to stop any tendency of \( P(\tau) \) toward asymmetry, one computes:

\[
P_{\text{sym}}(\tau) = \frac{1}{2} \begin{bmatrix}
P(\tau) + P^T(\tau)
\end{bmatrix}
\] (4.131)

In a sense, the problem now "starts over" by re-invoking (4.125):

\[
\begin{bmatrix}
Y(2\tau) \\
Z(2\tau)
\end{bmatrix}
= \Omega(\tau, 0) \begin{bmatrix}
I \\
P_{\text{sym}}(\tau)
\end{bmatrix}
\] (4.132)
\[
P(2\tau) = Z(2\tau)Y^{-1}(2\tau) \quad (4.133)
\]
\[
P_{\text{sym}}(2\tau) = \frac{1}{2} \left[ P(2\tau) + P^T(2\tau) \right] \quad (4.134)
\]

The computational cycle of (4.132) to (4.134) is repeated at each step. The terms in \( P(t) \) are quite bounded and free of indefinitely growing exponentials if the physical problem is properly posed, and so the problem of enormous components in \( Z \) and \( Y \) does not arise. Equations (4.132) and (4.133) are the major feature of this method, while the "symmetrizing" operation in (4.134) is a secondary but helpful feature. Typically, the most rapid changes in \( P(t) \) occur at the beginning of the problem; this is particularly true for \( S(t) \) in an optimal control problem with a very small weighting of integral of squared control effort in the performance index (Chapter 6). Therefore, a small step size \( \tau \) should be used in the beginning, followed by larger steps as \( P(t) \) approaches steady-state.

Continuing the current example, the initial value of the covariance matrix in the state space of the 5-state plant was set at:

\[
P(0) = \begin{bmatrix}
53.5 & 0 & -0.0148 & 0 & -0.0148 \\
0 & 1 \times 10^{-4} & 0 & 0 & 0 \\
-0.0148 & 0 & 49 \times 10^{-6} & 0 & -0.0148 \\
0 & 0 & 0 & 0 & 0 \\
-0.0148 & 0 & -0.0148 & 49 \times 10^{-6}
\end{bmatrix} \quad (4.135)
\]

Previous calculations had indicated that the largest RHP eigenvalues of \( W \) were about \(+50 + j36.6\), and so the basic transition matrix in (4.121) was calculated with \( D = .005 \) sec and \( k = 20 \) terms. The number \( m \) of cascaded matrices was set at 10, and so the first 10 steps were each 0.05 sec long. After 0.5 seconds, the \( P \) matrix had a much smaller \( \dot{P} \) than in the beginning, and the step size was changed to 0.1 second by
making \( m = 20 \), with \( D \) and \( k \) remaining the same. The computations were stopped at the quasi-steady-state time of 1.5 seconds, at which:

\[
P(1.5) = \begin{bmatrix}
41.67 & 0.143 & 2.31 \times 10^{-3} & -1.436 \times 10^{-5} & 2.314 \times 10^{-3} \\
0.143 & 6.23 \times 10^{-4} & -4.73 \times 10^{-6} & 4.17 \times 10^{-8} & -4.73 \times 10^{-6} \\
2.31 \times 10^{-3} & -4.73 \times 10^{-6} & 1.55 \times 10^{-6} & 1.50 \times 10^{-8} & 1.55 \times 10^{-6} \\
-1.44 \times 10^{-5} & 4.17 \times 10^{-8} & 1.50 \times 10^{-8} & 3.05 \times 10^{-6} & 4.55 \times 10^{-8} \\
2.31 \times 10^{-3} & -4.73 \times 10^{-6} & 1.55 \times 10^{-6} & 4.55 \times 10^{-8} & 1.55 \times 10^{-6}
\end{bmatrix}
\]

and

\[
K = \begin{bmatrix}
2.71 & 4549. \\
9.30 \times 10^{-3} & 19.8 \\
1.50 \times 10^{-4} & -0.150 \\
-9.33 \times 10^{-7} & 1.33 \times 10^{-3} \\
1.50 \times 10^{-4} & -0.150
\end{bmatrix}
\]

Of course, the measurement matrix \( H \) for the two outputs \( (A_m, \theta) \) was:

\[
H = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

4.3.2.3.3 Eigenvalues of \( F - KH \)

The eigenvalues of \( F - KH \) were found to be: -0.69, -12.1 ± j1.7, -50 ± j86.6 rad/sec. The first of these is evidently related to the bias eigenvalue at -0.01 rad/sec and would be lower if the computation had proceeded further. The second and third eigenvalues are related to the airframe eigenvalues at 10.684 and -13.004, and have about the same magnitude.

The fourth and fifth eigenvalues of \( F - KH \) are those of the fin servo. This observation can be related to the nature of \( KH \) by the
The following computation:

\[
V_b^{-1}KH = Q_bKH = \begin{bmatrix}
2.29 & 4880. \\
.072 & 34. \\
-1.2 \times 10^{-7} & -2.5 \times 10^{-3} \\
-2.1 \times 10^{-6} & -2.4 \times 10^{-3} \\
.37 & -380. \\
\end{bmatrix}
\]  \hspace{1cm} (4.139)

The relatively tiny elements in the third and fourth rows mean that for the actuator eigenvalues:

\[
q_i^T KH \approx 0 \hspace{1cm} (4.140)
\]

and:

\[
q_i^T (F - KH) \approx q_i^T F = q_i^T \lambda_i \hspace{1cm} (4.141)
\]

This shows that if each column of \( K \) is decomposed into a weighted sum of right eigenvectors \( v \) of \( F \), then neither column contains large components of the two actuator eigenvectors, and so the estimator does not have large loop gain in these modes.

The reason for this situation may be found from the eigenvalues of \((F - KH)_{ss}\) for the steady-state \( K \) resulting from the solution of \( P \) at infinite time, using the technique of References P1 and O'Dl. The Hamiltonian \( W \) matrix (4.120) is evaluated from Appendix B and shown in (4.142) on the next page. The 2n eigenvalues of this matrix are symmetrical about the imaginary axis; those in the left half plane are the eigenvalues of \((F - KH)_{ss}\). In the particular case of (4.142), it is helpful to partition the matrix as shown, and to observe that the only non-zero element in the lower left partition is:

\[
W_{10,5} = Q_1 = 6.1 \times 10^{-8} \text{ rad}^3/\text{sec} \hspace{1cm} (4.143)
\]
which is the integrated autocorrelation of the process noise into the bias integrator. If this element \( W_{10,5} \) were replaced by 0, then the eigenvalues of (4.142) would be those of the upper left 7 by 7 sub-matrix and the lower right 3 by 3 sub-matrix, which has the eigenvalues \(-.01, -50. + j86.6 \text{ rad/sec}\). This latter point is confirmed by the bottom entry of Table 4-2, which shows the eigenvalues of \((F - KH)_{ss}\); of course the eigenvalues of \(W\) are those in the table and also \(-1\) times them.

The actual matrix in (4.142) includes the very small element \( W_{10,5} \), which serves to couple the left eigenvectors of the aforementioned three eigenvalues of the lower 3 by 3 sub-matrix into the remainder of the \(W\) matrix. The result is shown in the top entry of Table 4-2, which indicates that the bias-integrator eigenvalue at \(-.01 \text{ rad/sec}\) has been shifted to \(-0.164\), while those at \(-50.0 \pm j86.6 \text{ rad/sec}\) are virtually unchanged. This difference in effect is apparently related to the relative magnitude of the numbers \(6.1 \times 10^{-8}\), \(-.01\) and \(-10,000\). The error covariance for the low-frequency bias state \(x_5\) is appreciably affected by the noise \(Q_1\), but the actuator integrators (\(x_4\) and \(x_5\)) do not have direct white-noise inputs and their eigenvalues with large negative real parts result in disturbances dying out quickly. The constancy of the actuator eigenvalues is apparently the result of their modes not being substantially excited by the process noise.
Table 4-2 - Effect of Variable Noise Power on Eigenvalues of \((F - KH)_{ss}\) for Estimator with Steady-State Gain Matrix K

<table>
<thead>
<tr>
<th>Measurement Noise Power</th>
<th>Process Noise into Bias Integrator</th>
<th>Eigenvalues of ((F - KH)_{ss}) rad/sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bandwidth (\omega_b) rad/sec</td>
<td>(</td>
<td>R</td>
</tr>
<tr>
<td>10</td>
<td>1 (ref.)</td>
<td>Normal</td>
</tr>
<tr>
<td>100</td>
<td>10^{-1}</td>
<td>Normal</td>
</tr>
<tr>
<td>1000</td>
<td>10^{-2}</td>
<td>Normal</td>
</tr>
<tr>
<td>(10^4)</td>
<td>10^{-3}</td>
<td>Normal</td>
</tr>
<tr>
<td>(10^5)</td>
<td>10^{-4}</td>
<td>Normal</td>
</tr>
<tr>
<td>(10^6)</td>
<td>10^{-5}</td>
<td>Normal</td>
</tr>
<tr>
<td>(10^7)</td>
<td>10^{-6}</td>
<td>Normal</td>
</tr>
<tr>
<td>10</td>
<td>1 (ref.)</td>
<td>None (W_{10,5} = 0)</td>
</tr>
</tbody>
</table>
It is apparent that all eigenvalues (except the lowest) of $F - KH$ at 1.5 sec are virtually identical to those of $(F - KH)_{ss}$ for the same measurement noise. It is observed in Chapter 6 that the control gains for modes with the largest negative real parts settled to their steady-state values in the shortest time, and a plausible analytic argument was shown for this behavior. The principle of duality (Ref. B1, p. 370) between control gains and estimator gains suggests that the estimator gains for the modes of $F - KH$ should have a similar relative behavior, and so it is quite plausible that the lowest eigenvalue should change most slowly.

Based on engineering judgement, the fixed-gain estimator was designed from the quasi-steady solution of $K$ at 1.5 seconds, for which the eigenvalues were $-0.69, -12.1 \pm j1.7$ and $-50. \pm j86.6$ rad/sec. Because of the short time of flight of the missile, modelling uncertainties, etc., it was judged better to have the slowest error mode in the estimator decay with an eigenvalue of $-0.69$ rad/sec rather than the steady-state value of $-0.164$ rad/sec from $(F - KH)_{ss}$. It will become apparent from Table 4-2 and the next subsection that the eigenvalues of the chosen $F - KH$ approximate those of $(F - KH)_{ss}$ in the steady-state for a somewhat lower measurement-noise level.

4.3.2.3.4 Estimators versus Observers

Subsection 4.3.2.1 has indicated that the separation principle allows the control poles and gains to be chosen separately from those of the Kalman estimator, which effectively reconstructs the unmeasured states or modes. It also indicated that an alternative to the Kalman estimator is a Luenberger observer, for which References B3 and L1 indicate that the separation principle also applies.
Unfortunately there appears to be a flaw in the discussion of "Supplemental Observers - General" on p. 9 of Ref. B3, in which a portion of the M matrix is to be chosen so as to make a submatrix $F_{vv}$ of the matrix $MFM^{-1}$ have eigenvalues with negative real parts. Since $MFM^{-1}$ is a similarity transformation on the original plant matrix $F$, it follows (Ref. H2, p. 54) that $F$ and $MFM^{-1}$ have the same eigenvalues. It may be difficult to constrain $F_{vv}$ to have suitable eigenvalues.

On the other hand, Luenberger's paper (Ref. L1), which appears to be more general than Ref. B3, shows that the general Luenberger observer should have an observer square plant matrix $B$ with no eigenvalues in common with those of the original plant matrix. Ref. T2 is a recent thesis which discusses estimators and observers for systems with both noisy and noise-free measurements.

This writer takes some exception to the basic idea of building an observer of order less than $n$ to operate on supposedly noise-free measurements, and also to the applicability of the end of the following statement from Ref. L1:

"Therefore, optimal estimators can be regarded as observers with their pole locations determined by the statistical properties of the noise. In many practical situations (namely, those in which the noise level is significant), statistically optimal estimators offer excellent advantages over other estimation schemes. As the noise level decreases, however, the optimal pole locations move toward $-\infty$ and in the limiting case of perfect (noise-free) measurements, the statistically optimal estimator consists of a number of differentiators (Ref. B5)."
In order to test this latter idea, the measurement noise power of the estimator in the previous subsection was scaled (Appendix B) by the following relationship:

$$R^{-1} = \frac{\omega_b}{10} \begin{bmatrix} 0.065 & 0 \\ 0 & 31800 \end{bmatrix}$$  \hspace{1cm} (4.144)$$

where $\omega_b$ was the bandwidth for a fixed mean-square value of measurement noise. Increasing $\omega_b$ by a factor of 10 corresponded to lowering the power density of the measurement noise by a factor of 10. The eigenvalues of $(F - KH)_{ss}$ for the steady-state estimator were calculated for 7 different levels of measurement-noise power and are shown in Table 4-2. Certain observations are of interest:

1) Rather than moving toward $-\infty$ as speculated in Ref. L1, the eigenvalues move orders of magnitude less than the factor of $10^6$ in reduction of measurement-noise power density.

2) The actuator eigenvalues do not move significantly; there were apparently some numerical difficulties in computing those for $\omega_b = 10^6$ and $10^7$ rad/sec.

3) Even for the largest values of $\omega_b$, there are two eigenvalues with magnitudes comparable to those of the airframe (10.7 and -13.0 rad/sec).

4) For the first 4 or 5 entries in Table 4-2, the lowest eigenvalue increases by a factor close to $\sqrt{10}$ in each decade reduction of power; an analytic explanation has been found for this, but not tested.
5) For \( \omega_b = 10^5, 10^6 \) and \( 10^7 \) rad/sec, the largest eigenvalue in each case increases by a factor of \( \sqrt{10} \); an analysis of this behavior has been found and tested successfully.

It is concluded that it is better to design an optimal or near-optimal estimator and then to simplify the total estimator-controller, rather than to design an observer with arbitrary poles and to hope that noise will not be a problem. Moreover, the estimator has low-pass filtering for all inputs and, therefore, should probably cope better with the unmodelled, poorly damped, structural modes than an observer with arbitrary poles and parallel direct feed-forward of measurements.

4.3.2.4 Combined Estimator-Controller

As stated in Subsection 4.3.2.3.1, it appears to be most convenient to compute the estimator gains in plant space and then to convert them to modal space. It would appear that questions of mismatch between the plant and its model in the estimator-controller, owing to changes in flight condition, may be analyzed most conveniently if the estimator model of the plant is in plant space. Numerical comparisons may then be made for the actual estimator-controller in modal space.

4.3.2.4.1 Block Diagram in State Space and Mismatch Problem

Figure 4-7 is a block diagram of the plant and its estimator-controller in plant state space. In order to analyze mismatch, it appears to be desirable to alter the treatment of Subsection 4.3.2.1, which was concerned with deriving transfer functions, chiefly for the matched case. Repeating (4.25), but with scalar control \( u \) and reference input \( u_r \):
Figure 4-7 - Plant and Estimator-Controller in Plant State Space
\[
\dot{x} = F_x \ddot{x} + \frac{g_c}{T} \ddot{x} + g_r u_r \\
= (F + \frac{g_c}{T})x - \frac{g_c}{T} x_e + g_r u_r
\]

(4.145)

where \(x_e = x - \dot{x}\), as before.

Now, let \(F_c\), \(g_c\) and \(g_{rc}\) be the estimator-controller's counterparts of the quantities in (4.145); no separate \(c_c\) is required because there is only one set of control gains. For the estimator-controller:

\[
\dot{x} = F_c \ddot{x} + g_c \frac{T}{T} \ddot{x} + KH (x - \dot{x}) + g_{rc} u_r
\]

(4.146)

Subtracting (4.146) from (4.145):

\[
\dot{x}_e = \left[ \begin{array}{cc} F_c - KH + (g_c - \frac{g_c}{T}) & (g_c - \frac{g_c}{T}) \end{array} \right] x_e + \left[ \begin{array}{cc} F - F_c + (g - \frac{g_c}{T}) & (g - \frac{g_c}{T}) \end{array} \right] x + \left[ \begin{array}{cc} g_r - g_{rc} \end{array} \right] u_r
\]

(4.147)

For the combined plant and estimator:

\[
\begin{bmatrix} \dot{x} \\ \dot{x}_e \end{bmatrix} = \begin{bmatrix} F + g_c \frac{T}{T} \\ -g_c \frac{T}{T} \\ F - F_c + (g - \frac{g_c}{T}) \end{bmatrix} \begin{bmatrix} x \\ x_e \end{bmatrix} + \begin{bmatrix} g_r \\ g_r - g_{rc} \end{bmatrix} u_r
\]

(4.148)

In the matched case, the lower left submatrix in (4.148) is 0 and it is apparent that the eigenvalues of the system are those of \(F + \frac{g_c}{T}\) and \(F - KH\), as shown in Subsection 4.3.2.1.

In the unmatched case, let it be assumed that the eigenvalues of the total system matrix in (4.148) lie in the left half plane. It is of interest to examine whether mismatches can cause the bias cancellation to be ineffective, as the writer originally thought. In Figure 4-2, note that the bias \(\delta_b\) enters the system with a gain \(F_{45} = -F_{43}\), which was done as a matter of analytical convenience. The element \((g_c \frac{T}{T})_{45}\) is supposed to cancel \(F_{45}\), but if there is a slight mismatch, then one would redefine \(F_{45}\) as:
\[ F_{45} = - (gc^T)_{45} \]  \hspace{1cm} (4.149) 

for bias cancellation, with the result that \( x_5 \) would now be the original \( \delta_b \) times an adjustment factor near unity; this is possible because \( x_5 \) is only an equivalent input to the rest of the system. Now consider whether \( \dot{x} \) in (4.148) can reach equilibrium with \( x^T = (0 \ 0 \ 0 \ 0 \ \delta_b)^T \) and \( x_e = 0 \):

\[
\dot{x} = (F + gc^T) x - gc^T x_e
\]

\[
\begin{bmatrix}
F_{11} & F_{12} & 0 & F_{14} & 0 \\
F_{21} & F_{22} & F_{23} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
10^4 c_1 & 10^4 c_2 & 10^4(c_3-1) & 10^4(c_4-0.01) & 0 \\
0 & 0 & 0 & 0 & F_{55}
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\delta_b
\end{bmatrix}
\]

(4.150)

Taking \( F_{55} = 0 \) rather than the usual -0.01 rad/sec that is used for numerical convenience in computing eigenvalues, it is apparent that \( \dot{x} = 0 \) for the assumed equilibrium values of \( x \) and \( x_e \). Now, examining the lower half of (4.148), the element \( (F + gc^T)_{45} \) is still zero and the element \( (F_c + gc_c^T)_{45} \) can be assumed to be zero or virtually so, because the use of resistors with 1 percent tolerance is common in missile electronics design. Moreover, \( g - gc \) can be considered as negligible for the same reason. There can be substantial differences in the first two rows of \( F \) and \( F_c \) owing to a flight condition off the design point. The lower half of (4.148) then has the form:
\[ \dot{x}_e = \left[ F - F_c + (g - g_c)c^T \right] x + \left[ F_c - KH + (g_c - g)c^T \right] x_e \]

\[
\begin{bmatrix}
* & * & 0 & * & 0 \\
* & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \delta_b \end{bmatrix}
\]

(4.151)

\[
\begin{bmatrix}
* & * & 0 & * & 0 \\
* & * & * & 0 & 0 \\
* & * & 0 & 1 & 0 \\
* & * & -10^4 & -100 & 10^4 \\
* & * & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

where the asterisks are either numbers which are dependent on flight condition and/or elements of KH. Equation (4.151) indicates that equilibrium is reached for the assumed final-condition vectors \( x \) and \( x_e \) and that the bias \( x_5 = \delta_b \) is not transmitted to the output acceleration \( A \).

### 4.3.2.4.2 Block Diagram in Modal Space and Mismatch Behavior

In order to express the estimator-controller in the plant modal space, let:

\[
\hat{x} = V \hat{z}
\]

(4.152)

where \( V \) is the right-eigenvector matrix that corresponds to the diagonal form \( \Lambda \) of the eigenvalue matrix.

Substituting (4.152) into (4.145) and (4.146) leads to:
\[
\dot{\mathbf{x}} = \mathbf{F} \mathbf{x} + \mathbf{g}_c^T \mathbf{V} \dot{\mathbf{z}} + \mathbf{g}_r \mathbf{u}_r \tag{4.153}
\]
\[
\dot{\mathbf{z}} = \mathbf{V}^{-1} (\mathbf{F}_c + \mathbf{g}_c^T \mathbf{V} - \mathbf{K} \mathbf{H}) \mathbf{V} \dot{\mathbf{z}} + \mathbf{V}^{-1} \mathbf{K} \mathbf{H} \mathbf{x} - \mathbf{V}^{-1} \mathbf{g}_r \mathbf{u}_r \tag{4.154}
\]
\[
= \left[ \Lambda_c + \mathbf{g}_z \mathbf{c}_z^T - \mathbf{K}_z \mathbf{H}_z \right] \dot{\mathbf{z}} + \mathbf{K}_z \mathbf{H} \mathbf{x} + \mathbf{g}_{rz} \mathbf{u}_r
\]

where \( \mathbf{g}_c \) and \( \mathbf{g}_{rc} \) are respectively assumed to be equal to \( \mathbf{g} \) and \( \mathbf{g}_r \) by the argument in the previous subsection, and the following modal gains are defined:

\[
\mathbf{g}_z = \mathbf{V}^{-1} \mathbf{g} \tag{4.155}
\]
\[
\mathbf{c}_z^T = \mathbf{c}^T \mathbf{V} \tag{4.156}
\]
\[
\mathbf{K}_z = \mathbf{V}^{-1} \mathbf{K} \tag{4.157}
\]
\[
\mathbf{H}_z = \mathbf{H} \mathbf{V} \tag{4.158}
\]
\[
\mathbf{g}_{rz} = \mathbf{V}^{-1} \mathbf{g}_r \tag{4.159}
\]

Since simple linear transformations relate the present total system vector to that of the previous subsection, it is apparent that the eigenvalues and at least the first-order mismatch characteristics of the two representations are the same.

Figure 4-8 is the block diagram of the system with the estimator-controller in the modal space (diagonal form) of the plant. This figure is useful because it shows the relative isolation of the actuator modal integrators, inasmuch as \( K_{z31}, K_{z32}, K_{z41} \) and \( K_{z42} \) are virtually zero. The conceptual importance of this representation will be apparent later.

The disadvantage of the diagonal form of \( \Lambda \) is that it introduces complex numbers, which are a bit of a nuisance computationally. The
Figure 4-8 - Plant and Estimator-Controller in Plant Modal Space
actual computer runs were carried out with the block-diagonal form of $\Lambda_b$ and the corresponding $V_b$. Consider the further transformation:

$$\hat{z} = T \hat{w}$$  \hspace{1cm} (4.160)  

where:

$$
T = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1/2 & -j/2 & 0 \\
0 & 0 & 1/2 & j/2 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$  \hspace{1cm} (4.161)  

Note that the first, second and fifth modes (two airframe modes and the bias mode) of $\hat{z}$ are identical to those of $\hat{w}$. It will become apparent that the two actuator modes have been given a "local rotation", so to speak.

Substituting (4.161) into (4.152):

$$\hat{x} = VT \hat{w} = V_b \hat{w}$$  \hspace{1cm} (4.162)  

Since the fourth column of $V$ can be expressed as the complex conjugate of its third column, it follows that:
\[ V_b = \begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ v_1 & v_2 & \text{Re}(v_3) & \text{Im}(v_3) & v_5 \end{bmatrix} \]

(4.163)

which has only real elements. The inverse \( V_b^{-1} \) is also required, and is found from (4.162) to be:

\[ V_b^{-1} = T^{-1}V^{-1} = T^{-1}Q \]  

(4.164)

where:

\[
T^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & j & -j & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

(4.165)

and \( Q \) is the matrix of transposed left eigenvectors:
\[ Q = \begin{bmatrix}
q_1^T \\
q_2^T \\
q_3^T \\
q_4^T \\
q_5^T
\end{bmatrix} \quad (4.166) \]

Defining \( Q_b \) as \( V_b^{-1} \), it is apparent from (4.164) through (4.166) that:

\[ Q_b = \begin{bmatrix}
q_1^T \\
q_2^T \\
2 \text{Re}(q_3^T) \\
-2 \text{Im}(q_3^T) \\
q_5^T
\end{bmatrix} \quad (4.167) \]

The conventional diagonal eigenvalue matrix is equivalent to:

\[ \Lambda = V^{-1} F V \quad (4.168) \]

Use of \( V_b \) rather than \( V \) in (4.154) and (4.168) is equivalent to:
\[ V_b^{-1} F V_b = T^{-1} V^{-1} F V T = T^{-1} \Lambda_T \]

\[
\begin{bmatrix}
\lambda_1 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 & 0 \\
0 & 0 & \text{Re} \lambda_3 & \text{Im} \lambda_3 & 0 \\
0 & 0 & -\text{Im} \lambda_3 & \text{Re} \lambda_3 & 0 \\
0 & 0 & 0 & 0 & \lambda_5
\end{bmatrix} = \Lambda_b
\]

which is the block-diagonal form of \( \Lambda \).

It should be noted that only the third and fourth rows and/or columns of \( V_b \), \( Q_b \) and \( \Lambda_b \) (as is evident from their form) differ from those of \( V \), \( Q \) and \( \Lambda \) and that real numbers are used throughout. Also, it is apparent from (4.160), (4.161) and (4.165) that \( z_1 \), \( z_2 \) and \( z_5 \) are respectively equal to \( w_1 \), \( w_2 \) and \( w_5 \), and that \( w_3 \) and \( w_4 \) are linear combinations of \( z_3 \) and \( z_4 \).

The plant-estimator equations for the block-diagonal \( \Lambda_b \) are similar to (4.153) and (4.154):

\[ \dot{x} = FX + g_c^T v_b \dot{w} + g_r u_r \quad (4.170) \]

\[ \dot{\hat{w}} = V_b^{-1} (F_c + g_c^T - KH) V_b \dot{w} + V_b^{-1} KHx + V_b^{-1} g_r u_r \quad (4.171) \]

\[ = \left[ \Lambda_{bc} + \frac{g_w c_w}{w_h} - K_w H_w \right] \dot{\hat{w}} + K_w H x + g_{rw} u_r \]

where the following (block-diagonal) modal gains are defined:
\[ g_w = V_b^{-1} g = T^{-1} g_z \]  
(4.172)

\[ c_w^T = c_z^T V_b = c_z^T T \]  
(4.173)

\[ K_w = V_b^{-1} K = T^{-1} K_z \]  
(4.174)

\[ H_w = HV_b = H_z T \]  
(4.175)

\[ g_{rw} = V_b^{-1} g_r = T^{-1} g_{rz} \]  
(4.176)

In \[ c_w^T \] and \[ H_w \], only the third and fourth columns differ from those of \[ c_z^T \] and \[ H_z \]. Likewise, in \[ g_w \], \[ K_w \] and \[ g_{rw} \], only the third and fourth rows differ from those of \[ g_z \], \[ K_z \] and \[ g_{rz} \].

Figure 4-9 shows how only the actuator modes of the estimator in the block-diagonal case differ from those in Figure 4-8 for the diagonal form of \( \Lambda \). The control gains from modes \( w_3 \) and \( w_4 \) are virtually zero (within computational accuracy), while the estimator modal gains \( K_{w31} \), \( K_{w41} \), \( K_{w32} \) and \( K_{w42} \) are very low but non-zero, as indicated in Subsection 4.3.2.3.3. It is important to note that therefore the actuator modes in the estimator are weakly coupled to the rest of the system. This will be important in the reduction process.

In (4.171), the estimator-controller's plant matrix may be evaluated partially at the design point (Mach 2, S.L.) from (4.91), (4.137) and (4.138) as:
Figure 4-9 - Actuator Modes in Estimator-Controller for Block-Diagonal $\Lambda_b$ (Replaces part of Figure (4-8))
\[
F_{ecw}^{-1} = V_b^{-1} \begin{bmatrix}
-4.026 & -1621 & 0 & 302 & 0 \\
3.894 \times 10^{-2} & -20.80 & -119.6 & 0 & 0 \\
-1.49 \times 10^{-4} & 0.151 & 0 & 1 & 0 \\
14.2 & 3910 & -18400 & -138 & 0 \\
-1.49 \times 10^{-4} & 0.151 & 0 & 0 & -0.01 \\
\end{bmatrix} V_b
\]

Because the premultiplication by \( V_b^{-1} \) and postmultiplication by \( V_b \) are a similarity transformation on the central (numerical) matrix, the eigenvalues of the latter are those of \( F_{ecw} \). The bias cancellation (Subsection 4.3.2.2.2.2) in the 4, 5 element of the central matrix results in all terms of the fifth column being zero, except the bottom one, which therefore must be an eigenvalue of the estimator-controller. This results in virtual integral compensation, which is important in suppressing the effect of the original lumped bias \( \delta_b \). The eigenvalues of \( F_{ecw} \) were found by direct computation to be: -0.01, -2.28, -70.0, and -45.1 \( \pm \) j94.8 rad/sec. The latter pair is slightly shifted from the actuator pole-pair at -50. \( \pm \) j86.6 rad/sec.

Table 4-3 shows the poles and zeroes of the transfer functions \( A_m/u_r \) and \( A_m/\delta_b \), where \( u_r \) and \( \delta_b \) are the command input and lumped bias angle respectively, at the design point and two other flight conditions. It is evident that at the design point there are certain poles and zeroes which cancel each other. These are the eigenvalues of \( F - KH \) from Subsection 4.3.2.3.3, namely: -0.693, -12.1 \( \pm \) j1.66, -50. \( \pm \) j86.6 rad/sec. The pole at -0.01 is cancelled by a zero at -0.01 rad/sec by virtue of the bias cancellation in Subsection 4.3.2.2.2.2. This pole and the uncanceled remaining poles are those of \( F + gc^T \), which is predicted by the separation
Table 4-3 - Poles and Zeroes of Transfer Functions $A_m/u_r$ and $A_m/\delta_b$ for Plant and Estimator-Controller

Having Modal Feedback and Bias Cancellation, at Three Flight Conditions

<table>
<thead>
<tr>
<th>Flight Condition</th>
<th>(1) Mach 2, S. L. (design)</th>
<th>(2) Mach 1.75, 5000 ft</th>
<th>(3) Mach 3, S. L.</th>
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</thead>
<tbody>
<tr>
<td>$M_\delta$, sec$^{-2}$</td>
<td>-105.2</td>
<td>-71.41</td>
<td>-182.4</td>
</tr>
<tr>
<td>$M_\delta/M_\delta$ des</td>
<td>1</td>
<td>0.677</td>
<td>1.735</td>
</tr>
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<td>Poles</td>
<td>-0.01</td>
<td>-0.01</td>
<td>-0.01</td>
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<tr>
<td></td>
<td>-0.693</td>
<td>-0.824</td>
<td>-0.569</td>
</tr>
<tr>
<td></td>
<td>-12.07 ± j1.66</td>
<td>-2.48</td>
<td>-6.64 ± j7.41</td>
</tr>
<tr>
<td></td>
<td>-10.0</td>
<td>-6.97 ± j9.31</td>
<td>-18.5 ± j35.3</td>
</tr>
<tr>
<td></td>
<td>-30.0</td>
<td>-51.1</td>
<td>-50.0 ± j86.6</td>
</tr>
<tr>
<td></td>
<td>-50.0 ± j86.6</td>
<td>-48.0 ± j89.4</td>
<td>-57.4 ± j81.1</td>
</tr>
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<td>Gain of $A_m/u_r$ at zero frequency, ft/sec$^2$</td>
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<td>-1066</td>
<td>-1455</td>
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<tr>
<td>Zer0es of $A_m/u_r$</td>
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<td>-0.01</td>
<td>-0.01</td>
</tr>
<tr>
<td></td>
<td>-0.693</td>
<td>-0.693</td>
<td>-0.693</td>
</tr>
<tr>
<td></td>
<td>-12.07 ± j1.66</td>
<td>-12.07 ± j1.66</td>
<td>-12.07 ± j1.66</td>
</tr>
<tr>
<td></td>
<td>+33.4 (a/f)</td>
<td>+26.5 (a/f)</td>
<td>+48.1 (a/f)</td>
</tr>
<tr>
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<td>-27.2 (a/f)</td>
<td>-49.4 (a/f)</td>
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<td>-50.0 ± j86.6</td>
<td>-50.0 ± j86.6</td>
<td>-50.0 ± j86.6</td>
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<td>1815</td>
<td>2480</td>
</tr>
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<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>-0.0103</td>
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<td>-0.009968</td>
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<td>-2.28</td>
<td>-2.28</td>
</tr>
<tr>
<td></td>
<td>+33.4 (a/f)</td>
<td>+26.5 (a/f)</td>
<td>+48.1 (a/f)</td>
</tr>
<tr>
<td></td>
<td>-34.4 (a/f)</td>
<td>-27.2 (a/f)</td>
<td>-49.4 (a/f)</td>
</tr>
<tr>
<td></td>
<td>-45.1 ± j94.8</td>
<td>-45.1 ± j94.8</td>
<td>-45.1 ± j94.8</td>
</tr>
<tr>
<td></td>
<td>-70.0</td>
<td>-70.0</td>
<td>-70.0</td>
</tr>
</tbody>
</table>
principle for the case of matched plant model and plant, in Subsection 4.3.2.1. Of course, the uncancelled zeroes at +33.4 and -34.4 rad/sec are those of the airframe.

Table 4-3 also shows the poles and zeroes of these transfer functions for two flight conditions off the design point, at higher and lower values of tail-moment effectiveness $M_6$. The mismatch between the actual plant (airframe) and the fixed model of it (at Mach 2, S.L.) in the estimator-controller is responsible for the movement of some of the poles. The only zeroes which change significantly are those of the airframe. The less responsive airframe (Mach 1.75, S.L.) causes a somewhat slower system response, which is to be expected; at this flight condition, the plant matrix $F$ is:

$$
F = \begin{bmatrix}
-0.963 & 1880 & 0 & 209 & 0 \\
-0.0417 & -0.784 & -80.1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1. \times 10^4 & -100 & 1. \times 10^4 \\
0 & 0 & 0 & 0 & -0.01 
\end{bmatrix}
$$

The more responsive airframe (Mach 3, S.L.) appears to cause a somewhat faster system response than that of the design point, judging from the lead effect of the zeroes at -12.07 ± j1.66 relative to the corresponding poles at -6.64 ± j7.41 and judging also from the poles at -18.5 ± j35.3 rad/sec. The plant matrix at this airframe condition is:
The most convenient way to compare the response of the system at these three flight conditions is to compare the three step responses, i.e., the response of $A_m$ to a step command $u_r$. This is particularly pertinent for an overall guidance system in which a digital computer receives discrete radar target returns, performs discrete estimation and issues a series of step commands to the autopilot, based on a discrete guidance law. This comparison of autopilot step responses will be deferred until Subsection 4.3.2.4.4.

4.3.2.4.3 Simplification of Modal Estimator-Controller

The modal estimator-controller in the previous subsection appears to have near-optimal filtering of the process and measurement noise and to produce the required poles of $A_m/u_r$, at least at the design flight condition. Unfortunately, it has as many states as those of the plant, namely 5. If one makes a rough comparison of autopilot complexity based on the number of added integrators (each of which requires a low-drift operational amplifier), then this design with 5 added integrators is appreciably more complex than the classical Raytheon autopilot with only one integrator. Hence, simplification is highly desirable. It is hoped that the following discussion of simplification methods will prove to be applicable to other systems as well as this one; such a possibility appears to be rather likely.
4.3.2.4.3.1 Simple Truncation of Poles and Zeroes of the Estimator-Controller

Reference A1 shows a straightforward method of designing a combined estimator-regulator, with fixed gains and no input. It speculates that simplification of such an estimator-regulator can be accomplished by merely truncating its highest modes, if the residues associated with those modes in a partial-fraction expansion are small enough.

A similar approach has been suggested to the author and it is best explained by continuing the current example at the design point. Referring to the lower half of Figure 4-7, the transfer functions from measured states and from the command signal $u_r$ to the scalar feedback variable $u_f$ were found by Brockett's method (Ref. B2) to be:

$$\frac{u_f}{A_m} = \frac{(0.00722)(s + 0.353)(s + 13.45)(s + 50 - j86.6)(s + 50 + j86.6)}{D} \quad (4.180)$$

$$\frac{u_f}{\dot{\theta}} = \frac{14.5(s + 0.651)(s + 13.07)(s + 50 + j86.6)(s + 50 - j86.6)}{D} \quad (4.181)$$

$$\frac{u_f}{u_r} = \frac{-37.7(s - 1.259)(s + 1.472)(s + 66.97 + j99.98)(s + 66.97 - j99.98)}{D} \quad (4.182)$$

where the denominator polynomial is:

$$D = (s+0.01)(s+2.28)(s+70.0)(s+45.1 + j94.8)(s+45.1 - j94.8) \quad (4.183)$$

These poles are, of course, the eigenvalues of $F_{ecw}$ that were listed in the previous subsection. In each transfer function there is a zero-pair with a magnitude close to the pole-pair at $-45.1 \pm j94.8$, which originated with the actuator pole-pair at $-50. \pm j86.6$.

Because it appears that the gains, zeroes and poles of these transfer functions are significant mainly at lower frequencies, it was suggested
that the largest zero-pair and pole-pair in each transfer function could be truncated. This was accomplished by dropping the Laplace-transform variables in each of the four corresponding parentheses in each transfer function, and then by absorbing the resulting four numerical factors into the gain. The result was:

\[
\frac{u_f}{A_m} \approx \frac{0.00655(s + 0.353)(s + 13.45)}{D_t} \tag{4.184}
\]

\[
\frac{u_f}{\dot{\theta}} \approx \frac{13.1(s + 0.651)(s + 13.07)}{D_t} \tag{4.185}
\]

\[
\frac{u_f}{u_r} \approx \frac{-49.5(s - 1.259)(s + 1.472)}{D_t} \tag{4.186}
\]

where:

\[
D_t = (s + 0.01)(s + 2.28)(s + 70.0) \tag{4.187}
\]

For each transfer function, a partial-fraction expansion was performed so as to convert it to a sum of three single-lag terms, each of which could be represented by a separate integrator with a feedback gain equal to the particular pole. Figure 4-10 is a block diagram of the plant and the reduced estimator-controller with the three new integrators for the new estimator-controller states \(x_6\), \(x_7\) and \(x_8\). Of course, the feedback gains \(F_{66}\), \(F_{77}\) and \(F_{88}\) are -0.01, -2.28 and -70.0 rad/sec.

The results of this reduction technique are summarized in columns 1 and 4 of Table 4-4, which shows the poles and zeroes of the transfer function of the original 10-state system and the reduced 8-state system. It would appear that there has somehow been an interaction between four original poles (-10., -30. and -12.07 ± j1.66 rad/sec), so as to produce four new
Figure 4-10 - Block Diagram of Plant and Reduced Estimator-Controller
<table>
<thead>
<tr>
<th></th>
<th>(1) Original 10-State System</th>
<th>(2) System (1) w/ Mult. by 100</th>
<th>(3) System (2) After Reduction</th>
<th>(4) System (1) After Red. w/o Mult.</th>
<th>(5) Est. - Contr. Only from (1) 9/25/70</th>
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<td><strong>Poles Date</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9/4/70</td>
<td>9/15/70</td>
<td>9/21/70</td>
<td>10/1/70</td>
<td>Poles</td>
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</tr>
<tr>
<td></td>
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<td>-0.693</td>
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<td>-12.05 ± j1.67</td>
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<tr>
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<td>-57.28 ± j76.57</td>
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<td></td>
<td></td>
<td></td>
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<tr>
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<td>9/15/70</td>
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<tr>
<td><strong>Zeroes ( A_m/\delta b ) Date</strong></td>
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<td></td>
<td></td>
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<td>+33.4</td>
<td>+33.44</td>
<td>+33.44</td>
<td>+33.44</td>
<td>+33.44</td>
</tr>
<tr>
<td></td>
<td>-70.0</td>
<td>-34.46</td>
<td>-34.46</td>
<td>-34.46</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-45.1 ± j94.8</td>
<td>-4991 ± j8664</td>
<td>-4991 ± j8664</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
poles (-8.30, -18.83 ± j4.93, -13.68 rad/sec). The former cancellation of the poles from \( F - KH \) does not now apply and the new pole structure does not include the originally specified poles at -10. and -30. rad/sec.

Obviously, an improvement in the reduction technique is desirable.

4.3.2.4.3.2 Multiplication of Actuator Modal Rows in Estimator-Controller and Subsequent Truncation

Evidently, the method in the previous subsection would have worked if cancellation of the largest zero-pair and pole-pair in each transfer function had been exact.

A more successful reduction technique has been found. It has worked quite well in the current example, as explained below, and nearly as well in a similar example that will not be detailed here. The reasons for its success are plausible but only partly analytical and not rigorous at this stage.

Referring to Figure 4-8, one reason for the choice of the plant modal space for the estimator-controller is that it makes evident the partial decoupling of the actuator modes. Figure 4-8 is analytically equivalent to Figure 4-9 but is superior for the following conceptual purpose. By design, the actuator modal control gains \( C_{z3} \) and \( C_{z4} \) are zero (within computational errors after matrix transformations), and it happens for fundamental reasons that the estimator gains for these modes are near zero. Of course, the command \( u_r \) drives these modes through the gains \( g_{z3} \) and \( g_{z4} \), but this does not alter the feedback structure of the estimator-controller itself.

It is hypothesized that the most important effect of the integrators with high-frequency feedback for \( z_3 \) and \( z_4 \) is their d-c gains, and that these gains could replace the integrators and feedback without significantly
altering the system transfer functions. Consider the isolated transfer function \( \frac{z_3}{x_1} \) with \( H_{z13} \) and \( H_{z23} \) set equal to zero:

\[
\frac{z_3}{x_1} = \frac{K_{z31}}{s - \lambda_3} = \frac{K_{z31}}{s + 50 - j86.6}
\] (4.188)

The d-c gain of \( \frac{z_3}{x_1} \) is simply \( \frac{K_{z31}}{(-\lambda_3)} \). Now, a convenient computational technique for replacing these integrators and their feedbacks by the appropriate gains begins by multiplying each actuator modal row of the estimator-controller in the original 2n by 2n system matrix by some large factor \( k \), such as \( k = 100 \). The effect on each isolated transfer function through the estimator-controller is illustrated by the previous example:

\[
\frac{z_3}{x_1} = \frac{kK_{z31}}{s - k\lambda_3}
\] (4.189)

Evidently, the d-c gain of \( \frac{z_3}{x_1} \) is still \( \frac{K_{z31}}{(-\lambda_3)} \), but the pole has been raised to \( k\lambda_3 \). It seems very likely that the largest zero-pair and pole-pair of each transfer function through the estimator-controller, like (4.124) through (4.126), will be raised to the vicinity of \( k\lambda_3 \) and \( k\lambda_3 \). If so, then the cancellation of these poles and zeroes without interfering with the others will be possible. A partial-fraction expansion and realization as in the previous subsection would complete the design.

Referring again to Figure 4-8, the estimator-controller actuator-modal rows (8 and 9) of the total system matrix were multiplied by 100, and the following transfer functions for the estimator-controller only were then calculated:

\[
\frac{u_f}{A_m} = \frac{0.00722(s + 0.353)(s + 13.5)(s + 5000 + j8660)(s + 5000 - j8660)}{D_m}
\] (4.190)
\[
\frac{u_f}{\dot{\theta}} = \frac{14.5(s + 0.651)(s + 13.07)(s + 5000 + j8660)(s + 5000 - j8660)}{D_m} \tag{4.191}
\]

\[
\frac{u_f}{u_r} = \frac{-37.7(s-1.255)(s+1.469)(s+7003 + j9826)(s+7003-j9826)}{D_m} \tag{4.192}
\]

where the denominator is:

\[
D_m = (s + 0.01)(s + 2.28)(s + 77.4)(s + 4991 + j8664)(s + 4991 - j8664) \tag{4.193}
\]

Here it is apparent that cancellation of the highest pole-pair and zero-pair is quite good except for the latter transfer function.

Just as in the previous subsection, the procedure of transfer-function truncation, partial-fraction expansion and realization as in Figure 4-10 was applied. Of course, the feedback gains \( F_{66} \), \( F_{77} \) and \( F_{88} \) are now -0.01, -2.28 and -77.4 rad/sec. The 8 by 8 total system matrix for the plant and the reduced estimator-controller is shown on the next page.

In Table 4-4, columns 1 and 2 show the poles and zeroes for the original system, and the original system after the modal multiplication but before reduction. In addition to the expected change in two poles and two zeroes, there are some slight shifts of other poles and zeroes. After reduction, the poles and zeroes of the system transfer functions in column 3 are quite close to those of column 2, except for the obvious truncation of the largest pairs. Cancellation of the poles at -0.693 and -11.97 ± j1.67 rad/sec from \( F - KH \) is good, and bias cancellation of the pole at -0.01 rad/sec persists. A control pole at -30. has shifted to -32.6 rad/sec, but this improves the near-cancellation with the airframe zero at -34.46 rad/sec. The actuator pole-pair (from the plant, not the estimator) at -50. ± j86.6 rad/sec has actually shifted to a more stable position at -57.3 ± j76.6 rad/sec. It is evident that a good first-order approximation to the transfer function \( A_m/u_r \) has one pole at -10.2 rad/sec and one zero at
$$F_{sys} =$$

$$\begin{bmatrix}
-1.304 & 2910 & 0 & 302 & 0 & | & 0 & 0 & 0 & 0 \\
0.0482 & -1.016 & -119.6 & 0 & 0 & | & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & 0 \\
0 & 0 & -1 \times 10^4 & -100 & 1 \times 10^4 & | & 1 \times 10^4 & 1 \times 10^4 & 1 \times 10^4 & | & 1 \times 10^4 \\
0 & 0 & 0 & 0 & -0.01 & | & 0 & 0 & 0 & 0 \\
1.896 \times 10^{-4} & 0.6904 & 0 & 0 & 0 & | & -0.10 & 0 & 0 & 0 \\
9.112 \times 10^{-4} & 1.489 & 0 & 0 & 0 & | & 0 & -2.275 & 0 & 0 \\
6.117 \times 10^{-3} & 12.27 & 0 & 0 & 0 & | & 0 & 0 & -77.42 & 0 \\
\end{bmatrix}$$

(4.194)
33.4 rad/sec.

As in previous versions of the system, the transfer function $A_m/\delta_b$ has a zero at the origin and another zero very close to the bias eigenvalue of -0.01 rad/sec, which indicates that the transient response of the system to the bias after startup should be quite low.

The block diagram in Figure 4-11 has an interesting interpretation in classical terms. There are three summing points (ahead of the $x_6$, $x_7$ and $x_8$ integrators) for the command signal $u_r$ and the feedback variables $A_m$ and $\dot{\theta}$. The $x_6$ integrator with virtually zero feedback constitutes integral compensation in the forward path (from command $u_r$ to output $A_m$). At frequencies well below 77.4 rad/sec, there is approximately a transfer function $-56.3/77.4$ from $u_r$ to $x_8$ which is of opposite sign to the feedforward gain of +1 from $u_r$ to the $g_4$ input. The path through $g_7$ and $x_7$ provides additional positive forward path gain at low frequencies. The fact that airframe gain $F_{23} = -119.6 \text{ sec}^{-2}$ is negative helps to make the feedback around the airframe degenerative and stable. Lead networks are not used for compensation. Rather, all feedback is through integrators which provide near-optimal noise filtering.

4.3.2.4.4 Autopilot Response Off Design Point

Table 4-3 in Subsection 4.3.2.4.2 showed the transfer functions of the plant and original 5-mode estimator-controller at three flight conditions, including the design point. Table 4-5 shows these transfer functions for the final autopilot, consisting of the airframe and reduced (3-state) estimator-controller, and it is apparent that they are quite similar to those of Table 4-3.

These flight conditions were chosen for a reasonable range in pitch-moment effectiveness $M_\delta$, i.e., $1.735/0.677 = 2.56$, which follows an
Figure 4-11 - Block Diagram of Plant and Reduced Estimator-Controller with Chosen Gains
Table 4-5 - Poles and Zeroes of Transfer Functions $A_m/u_r$ and $A_m/\delta_b$ for Plant and Simplified Estimator-Controller, at Three Flight Conditions

<table>
<thead>
<tr>
<th>Flight Condition</th>
<th>Mach 2, S.L. (design)</th>
<th>Mach 1.75, 5000 ft</th>
<th>Mach 3, S.L.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_\delta$, sec$^{-2}$</td>
<td>-105.2</td>
<td>-71.41</td>
<td>-182.4</td>
</tr>
<tr>
<td>$M_\delta/M_\delta$ at design</td>
<td>1.0</td>
<td>0.677</td>
<td>1.735</td>
</tr>
<tr>
<td>Poles</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-0.01$</td>
<td>$-0.01$</td>
<td>$-0.01$</td>
<td></td>
</tr>
<tr>
<td>$-0.693$</td>
<td>$-0.824$</td>
<td>$-0.569$</td>
<td></td>
</tr>
<tr>
<td>$-11.97 \pm j1.67$</td>
<td>$-2.48$</td>
<td>$-6.67 \pm j7.43$</td>
<td></td>
</tr>
<tr>
<td>$-10.2$</td>
<td>$-6.94 \pm j9.35$</td>
<td>$-16.49 \pm j36.10$</td>
<td></td>
</tr>
<tr>
<td>$-32.6$</td>
<td>$-56.1$</td>
<td>$-67.94 \pm j72.67$</td>
<td></td>
</tr>
<tr>
<td>$-57.27 \pm j76.58$</td>
<td>$-54.07 \pm j79.89$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gain of $A_m/u_r$ at zero frequency, ft/sec$^2$</td>
<td>-1163</td>
<td>-1072</td>
<td>-1456</td>
</tr>
<tr>
<td>Zeros of $A_m/u_r$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-0.01$</td>
<td>$-0.01$</td>
<td>$-0.01$</td>
<td></td>
</tr>
<tr>
<td>$-0.693$</td>
<td>$-0.693$</td>
<td>$-0.693$</td>
<td></td>
</tr>
<tr>
<td>$-12.08 \pm j1.57$</td>
<td>$-12.07 \pm j1.65$</td>
<td>$-12.07 \pm j1.65$</td>
<td></td>
</tr>
<tr>
<td>$-34.46$ (a/f)</td>
<td>$+26.47$ (a/f)</td>
<td>$+48.16$ (a/f)</td>
<td></td>
</tr>
<tr>
<td>$+33.44$ (a/f)</td>
<td>$-27.25$ (a/f)</td>
<td>$-49.42$ (a/f)</td>
<td></td>
</tr>
<tr>
<td>Gain of $A_m/\delta_b$ at zero frequency</td>
<td>2012</td>
<td>1851</td>
<td>2486</td>
</tr>
<tr>
<td>Zeros of $A_m/\delta_b$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$-0.01008$</td>
<td>$-0.01006$</td>
<td>$-0.00995$</td>
<td></td>
</tr>
<tr>
<td>$-2.28$</td>
<td>$-2.28$</td>
<td>$-2.28$</td>
<td></td>
</tr>
<tr>
<td>$-34.46$ (a/f)</td>
<td>$+26.47$ (a/f)</td>
<td>$+48.16$ (a/f)</td>
<td></td>
</tr>
<tr>
<td>$+33.44$ (a/f)</td>
<td>$-27.25$ (a/f)</td>
<td>$-49.42$ (a/f)</td>
<td></td>
</tr>
<tr>
<td>$-77.42$</td>
<td>77.42</td>
<td>-77.42</td>
<td></td>
</tr>
</tbody>
</table>
accepted design practice for classical autopilots. From (A-35) in Appendix A, it is apparent that the transfer function $s\theta/\delta$ can be approximated by $M_\delta/s$ for frequencies above $\sqrt{M_\alpha}$. In order to keep the gain variation tolerable within the important, wide-bandwidth, rate-damping loop of the classical autopilot, the range of $M_\delta$ is restricted typically to perhaps 2.0 or 2.5 to 1 within an autopilot band for which the electronic gains are fixed.

As stated previously, the most convenient way to compare autopilot responses is to compare their step responses, which is done in Figure 4-12 for the three flight conditions, using an input command that produces unit response at the design flight condition. The variation in response is typical of, or even less than that of, classical autopilot designs over an autopilot band, and is quite tolerable. Of course, the guidance-law computation should utilize an autopilot pole and zero appropriate to the flight condition. E. L. Greenberg has advised the writer, that on the basis of tradeoff studies with the adjoint miss program, it is satisfactory to use the airframe RHP zero and to truncate the actual denominator of the autopilot transfer function to two terms, so as to obtain the "equivalent" pole for guidance computation.

Based on work related to Ref. S3, it was thought that the responses in Figure 4-12 could be made more uniform by an adaptive technique, in which the gain of 10,000 ahead of the actuator in Figure 4-11 is replaced by a gain equal to 10,000 ($M_{\delta\text{des}}/M_\delta$), where $M_{\delta\text{des}}$ is the $M_\delta$ at the design point. It turned out that this technique worsened the two off-design step responses appreciably, and it is thought that this is due to the abnormally high $M_\alpha$, which is 291. sec$^{-2}$ at Mach 3, sea level. Parenthetically, this high an $M_\alpha$ is one reason why this very unstable airframe was considered unsuitable in actual practice, but it makes an interesting
Figure 4-12 - Step Response of Autopilot for Three Flight Conditions
challenge for a new autopilot design.

Such an adaptive technique, or a technique of gain adjustment over much smaller ranges of $M_0$ (suggested by Professor Vander Velde), would probably reduce autopilot response variation over a band at higher altitudes, where the magnitude of $M_\alpha$ is much smaller.

**4.3.3 Pure Feedback of Restricted Output Vector**

The papers summarized in Subsection 4.3.2.2.1.1 either assumed the measurement of any required modes or states, or else the use of an estimator or observer. There may be cases in which the simplicity characteristic of a classical system is required, so that an estimator is ruled out and pure feedback is permitted only from a given output vector. In a very worthwhile recent paper, (Ref. J1, June 1970), Jameson shows how to feed back an output vector $y$ of order $r$ in a single-input system, so as to place $r$ poles in required positions, subject to certain conditions and a consequential (hopefully slight) motion of the other (usually higher-frequency) poles. Jameson's contribution, although not perfect, is all the more remarkable when one considers that he was apparently unaware of the previous literature in Subsection 4.3.2.2.1.1, judging from his references.

In the same journal, Davison (Ref. D1) attacked the same problem, except that he considered a multi-input system. He was able to generalize Wonham's result (Ref. W2) to the case of limited output feedback, but he also reduced the multi-input system to a single-input system with input $u$ by using a vector input $gu$ with fixed (and somewhat arbitrary) $g$.

The purpose of this subsection is to clarify and simplify Jameson's contribution, and to show how it can be related to the Crossley-Porter formula (Ref. C2) for modal feedback to a single input.
4.3.3.1 Derivation of Jameson's Result

Consider (4.51) in Subsection 4.3.2.2.1.3, which is a useful point of departure that avoids an unclear step in Jameson's derivation:

\[ 1 - c^T H(sI - F)^{-1} g = 0 \]  

(4.195)

The matrix inversion can be expressed as:

\[ (sI - F)^{-1} = \frac{J(s)}{A(s)} \]  

(4.196)

where \( J(s) \) is the adjoint matrix for \( (sI - F) \) and \( A(s) \) is the characteristic open-loop equation. Considering (4.196), the characteristic equation (4.195) must be expressible as the ratio of two polynomials:

\[ \frac{h(s)}{A(s)} = 1 - c^T H J(s) g \left[ 1/A(s) \right] \]  

(4.197)

where \( h(s) \) is an n-order polynomial whose zeroes are the closed-loop poles. This equation is also expressible as:

\[ h(s) = A(s) - c^T H J(s) g \]  

(4.198)

This is the same as Jameson's equation (8), but with different symbols and the use of the Laplace-transform variable \( s \) instead of \( \lambda \). In order to reach this point, Jameson invoked an unclear matrix identity without a reference, which can be replaced by:

\[ |1 + ab^T| = 1 + b^T a \]  

(4.199)

where \( a \) and \( b \) are n-vectors. The proof of this is given in Reference M1.

Unfortunately, Jameson chose to remain in the state space of \( F \),
instead of utilizing its modal space. He expressed the adjoint $J(s)$ as a matrix series (Ref. G4):

$$J(s) = I s^{n-1} + s^{n-2} J_1 + \ldots + J_{n-1} \quad (4.200)$$

Define the $n_y$ by $n$ matrix $W$ as:

$$W = H \begin{bmatrix} g_1 & J_1 g_2 & \ldots & J_{n-1} g \end{bmatrix} \quad (4.201)$$

By using (4.200) and (4.201), the vector $HJ(s)g$ may be expressed as:

$$HJ(s)g = W \begin{bmatrix} s^{n-1} \\ s^{n-2} \\ \vdots \\ \vdots \\ 1 \end{bmatrix} \quad (4.202)$$

Now, let $r$ closed-loop poles $p_1$ through $p_r$ be specified. The closed-loop characteristic equation (4.198) must vanish at each of these, and the result may be expressed from (4.202) as the following row-vector equation of dimension $r$:

$$c^T W L = e^T \quad (4.203a)$$

where $L$ and $e^T$ are the new matrix and row vector in:

$$c^T H \begin{bmatrix} g_1 & J_1 g_2 & \ldots & J_{n-1} g \end{bmatrix} \begin{bmatrix} p_1^{n-1} & p_r^{n-1} \\ p_1^{n-2} & \ldots & p_r^{n-2} \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{bmatrix} \quad (4.203b)$$

$= \begin{bmatrix} A(p_1) & \ldots & A(p_r) \end{bmatrix}$

If the number of measurements $n_y$ equals the number of specified poles $r$, then $WL$ is square. The vector of feedback gains $c$ is then solved from
(4.203) by inverting \( WL \), if it is nonsingular. Jameson shows that \( WL \) is nonsingular if the system is controllable and observable, with \( H \) having the rank \( n_y = r \).

Jameson also shows that an approximate solution can be obtained if the number of specified roots \( r \) is greater than the number of measurements \( n_y \).

4.3.3.2 Clarification and Extension of Jameson's Result

Jameson's result will now be clarified and extended somewhat by utilizing the modal space of \( F \). By performing the matrix product \( WL \) in (4.203) and utilizing (4.200):

\[
L^T H \begin{bmatrix} J(p_1)g & \ldots & J(p_r)g \end{bmatrix} = \begin{bmatrix} A(p_1) & \ldots & A(p_r) \end{bmatrix} \quad (4.204)
\]

From (4.196) and (4.53), each adjoint matrix in (4.204) can be replaced to give:

\[
L^T H \begin{bmatrix} A(p_1)V(p_1I - \Lambda)^{-1}V^{-1}g & \ldots & A(p_r)V(p_rI - \Lambda)^{-1}V^{-1}g \end{bmatrix} = \begin{bmatrix} A(p_1) & \ldots & A(p_r) \end{bmatrix} \quad (4.205)
\]

As in (4.59) and (4.60), let \( HV \) and \( V^{-1}g \) be replaced by \( H_z \) and \( g_z \), the modal measurement matrix and modal input-distribution vector. Furthermore, let (4.205) be post-multiplied by the normalizing matrix:

\[
\begin{bmatrix}
\frac{1}{A(p_1)} & 0 & \cdots & 0 \\
0 & \frac{1}{A(p_2)} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \frac{1}{A(p_r)} \\
\end{bmatrix} \quad (4.206)
\]

in which each diagonal element is assumed to be nonsingular. The result is:
\[-285-\]

\[c^T H_z \left[ (p_1 I - \Lambda)^{-1} g_z \ldots \ldots (p_r I - \Lambda)^{-1} g_z \right] = [1, 1, \ldots, 1] \]

(4.207a)

\[= c^T H_z \left[ \begin{array}{ccc}
\frac{g_{z1}}{(p_1 - \lambda_1)} & - & - & \frac{g_{z1}}{(p_r - \lambda_1)} \\
& \ddots & & \\
& & \frac{g_{zn}}{(p_1 - \lambda_n)} & - & - & \frac{g_{zn}}{(p_r - \lambda_n)} \\
\end{array} \right] \]

(4.207b)

\[= c^T \left[ \begin{array}{ccc}
H_{z11} & - & - & - & H_{zn1} \\
& \ddots & & \ddots & \\
& & H_{znn} & - & - & - & H_{zn1} \\
\end{array} \right] \left[ \begin{array}{ccc}
g_{z1} & 0 & 0 \\
0 & - & - & - & g_{zn} \\
0 & & 1 & - & - & - & 1 \\
\end{array} \right] \]

(4.207c)

\[= c^T \left[ \begin{array}{ccc}
H_{z11} g_{z1} & - & - & - & H_{zn1} g_{zn} \\
& \ddots & & \ddots & \\
& & H_{znn} g_{z1} & - & - & - & H_{zn1} g_{zn} \\
\end{array} \right] \left[ \begin{array}{ccc}
\frac{1}{(p_1 - \lambda_1)} & - & - & - & \frac{1}{(p_r - \lambda_1)} \\
& \ddots & & \ddots & \\
& & \frac{1}{(p_1 - \lambda_n)} & - & - & - & \frac{1}{(p_r - \lambda_n)} \\
\end{array} \right] \]

(4.207d)

This equation is simpler for computation and interpretation than Jameson's result (4.203). A program utilizing (4.207b) has been used successfully for placing poles. Evidently (4.207a) could have been derived directly from (4.61), without utilizing Jameson's difficult matrix inverse.

4.3.3.2.1 Case where Number of Measurements Equals Number of Specified Poles

In this case \(n_y = r\), and the matrix product in (4.207d) is square. If this matrix product is nonsingular, then it may be inverted to solve for \(c^T\). Let the original measurement \(H\) be of rank \(n_y\) (no redundancy in
measurements) and let the system be observable, which means that $H_z$ has no zero columns. Also, let the system be controllable, which means that $g_{z_1}$ through $g_{z_n}$ are nonzero. Since $H_z = HV$ and the square matrix $V$ of eigenvectors is nonsingular (for distinct eigenvalues), then $H_z$ must also be of rank $n_y$ (Ref. B4, p. 308). The matrix with diagonal elements $g_{z_1}$ through $g_{z_n}$ in (4.207c) is clearly invertible by the assumption of controllability, and so the matrix product with upper left element $H_{z_{11}} g_{z_1}$ in (4.207d) is also of rank $n_y$ (Ref. B4, p. 308). Assuming that the specified poles $p_1$ through $p_n$ are distinct, then there are $r$ independent columns in the right-hand matrix of (4.207d). For $n_y = r$, then the matrix product in (4.207d) has a maximum rank of $r$, and it is at least plausible that this rank is $r$, and that the matrix is invertible. This corresponds approximately to Jameson's conditions (Ref. J1) for solving for $c^T$ in the case where $n_y = r$.

It is evident that complete observability and controllability are not required for the invertibility of the matrix product in (4.207d) and the solution of $c^T$. Suppose that the $n$-th mode is either unobservable (which means that the $n$-th column of $H_z$ is zero) or uncontrollable (which means that $g_{z_n}$ is zero). From Subsection 4.3.2.2.1.2 it is known that the $n$-th open-loop eigenvalue $\lambda_n$ will not be changed by the feedback. Moreover, either supposition about the $n$-th mode leads to the $n$-th column $\begin{bmatrix} H_{z_{in}} & g_{z_n} & \ldots & H_{z_{n_{y'}}, n_z} \\ g_{z_{n_{y'}}} & \end{bmatrix}$ in (4.207d) being zero. If there are $r$ modes which are both observable and controllable, then this matrix will have $r$ nonzero corresponding columns and will almost certainly be of rank $r$. Then the matrix product in (4.207d) will be of rank $r$ and therefore invertible.

4.3.3.2.2 Specialization to Crossley-Porter Formula

Again, suppose that $n_y = r$, and now let the measurement matrix be:
with \( r \) rows and \( n \) columns, corresponding to feedback of the first \( r \) modes.

In performing the matrix multiplication in (4.207d), the elements with \( \lambda_{r+1} \) through \( \lambda_n \) will vanish. The result may be simplified and transposed to give:

\[
\begin{bmatrix}
\frac{1}{(p_1 - \lambda_1)} & \cdots & \frac{1}{(p_1 - \lambda_n)} \\
\frac{1}{(p_2 - \lambda_1)} & \cdots & \frac{1}{(p_2 - \lambda_n)} \\
\vdots & \ddots & \vdots \\
\frac{1}{(p_r - \lambda_1)} & \cdots & \frac{1}{(p_r - \lambda_n)}
\end{bmatrix}
\begin{bmatrix}
c_1 g_{z_1} \\
c_r g_{z_r}
\end{bmatrix}
= \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}
\]  

(4.209)

Now, the Crossley-Porter paper (Ref. C2) was concerned with the same conditions of measurement and has the following equation (Equation 28 in Ref. C2) with different notation:

\[
\sum_{k=1}^{r} c_k g_{z_k} = 1 \quad (j = 1, 2, \ldots, r)
\]  

(4.210)

From this, Reference C2 jumps in an unclear step to (4.73) herein, but fortunately (4.73) has been verified by the derivations in Reference G3 and M1 and by numerical calculations. It is apparent that (4.210) leads directly to (4.209), which therefore, must lead to (4.73). Hence, the Crossley-Porter formula (4.73) is seen to be the result of the special case in Jameson's formula wherein \( r \) pure modes are fed back and \( r \) closed-loop poles are specified.
4.3.3.3 Applicability to Autopilot Design

Although Jameson's contribution (Ref. J1) is important, it appears to be only indirectly applicable to the current problem of autopilot design.

Examining the bias problem first, it does not seem possible to obtain an estimate of $\delta_b$ for cancellation purposes by Jameson's method. If the alternative approach of a control integrator (for proportional plus integral control) is used instead, then one must cope with the additional pole introduced by the integrator and specify its closed-loop location. This pole and the two airframe poles to be moved make a total of three, compared to the two feedbacks available (acceleration and pitch rate). Hence, the three specified poles could not be achieved exactly. Of course, the latter part of Jameson's paper shows how to specify more poles than available feedbacks, but their closed-loop location can only be roughly approximated at best.

It is possible to imagine the following applicability of Jameson's paper to the autopilot in Figure 4-11, or to a conceptually similar system. Suppose that the final pole locations of this autopilot are moderately off specification, owing to the effects of the simplification process (Subsection 4.3.2.4.3.2). One may then cut the output lines from integrators $x_6$, $x_7$ and $x_8$ and define the resulting block diagram as the basic plant to which Jameson's method is to be applied. This plant has 8 eigenvalues, of which 5 are hopefully satisfactory and 3 are to be moved. The three outputs of $x_6$, $x_7$ and $x_8$ are available as the output vector. Jameson's method may then be applied to determine the gains from these integrators to the input into integrator $x_4$, and presumably these gains will not be greatly different than those obtained by the process in Subsection 4.3.2.4.3.2, but the specified poles will be achieved.
Another conceivable way to apply Jameson's method would be to add integral control to Figure 4-2, combining the bias state and the output of the real control integrator into state $x_5$, as in Subsection 4.3.2.2.2.3. After providing a suitable gain into the integrator input, it would be connected in parallel with $u_1$ to a new single input $u$. The sum of the two outputs into integrator $x_4$ could be defined as $u_1'$, with the transfer function:

$$\frac{u_1'}{u} = g_{41} \frac{(s + \omega_1)}{s} \quad (4.211)$$

where $\omega_1$ is a suitable integral break frequency, such as 5 rad/sec, that would determine the integrator path gain.

Now let the closed-loop system be required to have a pole at $-\omega_1$ (thus cancelling the new zero in the forward transmission) and two others, say at -10 and -30 rad/sec, as before. This requires one additional measurement besides the two now available. Let this be obtained from the output of a high-pass filter attached to the gyro measurement of $\dot{\theta}_m$, so as to approximate $\dot{\theta}_m'$ or almost equivalently $\delta_{eff}$, because the airframe gains $F_{21}$ and $F_{22}$ in Figure 4-2 are weak. Conceivably then, the three poles could be placed as required, using Jameson's method. One problem would be the noise transmission of the high-pass filter, which should have sufficiently high break frequency so that its pole will not be troublesome.
CHAPTER 5

INFLUENCE OF AUTOPilot RESPONSE, GUIDANCE LAW AND RADOME ON MISS DISTANCE

5.1 Introduction

A most important criterion of a candidate pitch autopilot is statistical miss distance for a specified set of noise conditions, target maneuver, biases, etc. The standard deviation of miss distance may be compared to that of another candidate design and (for a modern guidance system) ultimately against the square root of the covariance of the differential-position estimate at terminal time, which is an absolute lower bound (Reference B1, pp. 422, 425).

Subsection 2.2.1 has briefly described the major sources of miss distance. In preliminary design and evaluation, the guidance system (including the airframe and radome) are always assumed to be linear, and so superposition applies to the calculation of miss distance from the various sources. Accordingly, it is possible to use the adjoint method (References P2, L4 and R3) to calculate quickly and reasonably inexpensively the components of miss distance versus time of flight (adjoint time). In later stages of the design process, it is advisable to utilize digital or hybrid nonlinear simulations with six degrees of freedom to obtain Monte Carlo simulations of a flight in forward time.

This chapter describes briefly the Raytheon discrete adjoint simulation (Reference R3) and the plausible input numbers that have been used in this thesis research.

Subsection 5.3 compares the miss distances of the two classical
autopilots and the new autopilot in Chapter 4, as well as a hypothetical faster autopilot at the design point of Mach 2, sea level. It turns out that the continuous guidance law (Subsection 3.5) with small control weighting \( b \) is appreciably suboptimal because the guidance system is discrete, but that miss distance can be improved by empirically determining the optimum (nonzero) \( b \). The miss distance of the new autopilot is then superior to that of both classical autopilots and even approaches the theoretical lower bound when the discrete optimal guidance law is used. Miss-distance data is also presented for various linear radome slopes and changes in flight condition.

For another new autopilot, miss-distance data is presented for the flight conditions of Mach 2, 50,000 ft. Because of the increase of "alpha over gamma dot" with altitude (Subsections 2.1.1 and 2.2.2), the miss distance increases more markedly with radome slope at the higher altitudes than at sea level.

5.2 Computation of RMS Miss Distance

This chapter is concerned with statistical miss distance in one plane of a given engagement, on an ensemble basis. To simplify matters, it will be assumed that no source of miss distance has a mean value; this is a reasonably valid assumption, but it does imply that gravitational acceleration is perfectly biased out by a correctly calibrated command to the pitch autopilot. All sources of miss distance are assumed to have a Gaussian probability density distribution, which should therefore hold true for the distribution of miss distance. The zero mean of the sources implies zero mean of the miss distance, and so the root-mean-square is equal to the standard deviation.
5.2.1 Discrete Adjoint Simulation

This subsection very briefly describes the Raytheon discrete adjoint simulation (Reference R3).

The model represents the behavior of the guidance system in a single plane containing the initial line of sight. In forward time (Figure 5-1), sampled radar data on the LOS angle is fed to a discrete estimator-controller which generates discrete acceleration commands for the autopilot. The representation of the autopilot, missile-target geometry and seeker is essentially continuous, because the digital computations use a rather short integration time, the maximum being typically \(1/512\) second. The adjoint program has the following major modules:

1) Seeker
2) Kalman filter and optimum controller
3) Autopilot
4) Geometry
5) Target model
6) Noise model

The Kalman gains are first computed in forward time and stored. Then the adjoint computations are carried out in inverse time so as to yield the sensitivity of miss distance (or the specific variable being evaluated) with respect to each of the several sources, as indicated in Figure 5-2. Figures 5-3 through 5-6 give the forward and adjoint models of the estimator-controller and the cubic model of the autopilot.

Various modelling options are available. For example, the autopilot can be represented as having: 1) zero lag, 2) a transfer function with one pole and one zero, 3) a cubic transfer function with two zeroes and three
Figure 5-3 - Forward Model of Discrete Kalman Filter and Optimal Controller
Figure 5-4 - Adjoint Model of Discrete Kalman Filter and Optimal Controller
Figure 5-5 - Forward Model of Cubic Autopilot/Airframe
Figure 5-6 - Adjoint Model of Cubic Autopilot/Airframe
poles, or 4) an explicit three-loop representation of the classical Raytheon autopilot (Subsection 4.2.2.1). Of course, the model of the autopilot in the estimator-controller is more limited (Subsection 2.3.1.4.2). Most of the options for the guidance law in the estimator-controller utilize closed-form optimal continuous control gains, computed at each sampling instant; this continuous law is suboptimal at the end of the intercept (Subsections 3.5.3 and 3.6.2). A limited option makes it possible to insert a table of optimal discrete control gains for 30 samples, which restricts the time of flight to $29T_s$.

Instead of computing the components of miss distance, it is alternatively possible to compute the components of lateral acceleration, pitch angle, pitch rate, fin deflection or fin rate at a specified time to go.

The summary printout of the program lists the sources, the sensitivity of the output variable (e.g., miss distance) to each source, the standard deviation and mean of each component of the output variable, and the total standard deviation and mean.

5.2.2 Equivalent Radome for Adjoint Program

Fundamentally, the adjoint method of analysis requires the system to be linear (References P2, L4 and R3). Realistically, all physical radomes have nonlinear curves of radome error angle $r$ ($\theta_h$) versus gimbal angle $\theta_h$. Subsection 1.3.5 describes briefly how this problem of radome modelling in the adjoint program was overcome by Hall and others, who utilized an equivalent radome slope and additional range-independent noise. Briefly let $\theta'$ be the variation of $\theta_h$ from its ensemble mean:

$$\theta' = \theta_h - \bar{\theta}_h$$  (5.1)
and let \( p(\theta') \) be its ensemble probability density, which is usually assumed to be Gaussian, but does not have to be in the following expression. The equivalent radome slope is readily shown to be:

\[
R_{eq} = \frac{1}{\sigma^2} \int_{\theta'_\text{min}}^{\theta'_\text{max}} \theta' r(\theta') p(\theta') \, d\theta'
\]  

(5.2)

where \( \sigma \) is the ensemble standard deviation of \( \theta_h \).

Proofs of this simple result may be found in References W4, G10 and H5, in order of publication. If \( \bar{\theta}_h \) is at the worst tangent slope, \( dr/d\theta_h \), of a typical realistic radome curve, it turns out that \( R_{eq} \) is usually much less in magnitude than this tangent slope. Reference H5 also gives a formula for the mean-square error in fitting the radome curve, which is utilized as additional range-independent noise in adjoint calculations.

5.2.3 Inputs to Adjoint Program

Except for those cases noted in later subsections, computations were for the components of rms miss distance. The diagonal elements of the initial covariance matrix of the Kalman filter were computed from initial uncertainties; miss distance did not appear to be sensitive to the use of zero or standard nonzero correlation coefficients. Control gains were computed either from (3.84), (3.86), (3.87) and (3.90) in Subsection 3.5.2 or were taken from a table of gains, such as one related to the discrete example in Subsection 3.6.2. The model of the autopilot was typically a cubic transfer function, as noted below. The statistical sources of miss distance were modelled correctly in the Kalman filter, because mismatch studies were not of interest, except for radome slope \( R \).
The target was assumed to be roughly comparable to the F4 aircraft, approaching head-on with aspect angle $A = 0$ in Figure 1-1, with other parameters shown in Table 5-1. By Raytheon convention, the target was assumed to have a Poisson random acceleration $A_t$ perpendicular to the LOS with an rms value $\beta$ and an average frequency $\nu$ of zero-crossing. Following the argument of Subsection 2.3.1.1, one may replace the Poisson model with the low-pass model in Figure 2-10, such that both statistical models of target acceleration have the same mean (zero) and autocorrelation function (double-sided exponential).

Some parameters of the missile are shown in Table 5-1, the transfer functions being left to the following subsections.

As previously mentioned, the sampling time was 0.06640625 second. For most simulations with continuous guidance, the time of flight was set at 15 seconds. As explained in the previous subsection, the limitations of the program (Reference R3) made it necessary to restrict the time of flight to $29T_s = 1,925$ seconds in the cases utilizing a table of discrete optimum control gains. Since almost all the miss distance is incurred in the last part of the flight, it is still meaningful to compare the miss performance for a long flight with that of a short flight.

The distance downrange (horizontally) from the battery to the intercept point was arbitrarily set at 10 miles. The variance of the radar noise was modelled as in (2.71), which is repeated below for convenience:

$$R_k = \frac{\sigma^2}{R_o^4} \left( \frac{R^2}{\sigma_t/\sigma_o} \right) + \sigma^2_{\text{in}} + \frac{\sigma^2_{\text{sn}}}{R_{\text{mt}}^2}$$  \hspace{1cm} (5.3)

For computational convenience, $R_{\text{mt}}$ (range from illuminator to target) was
Table 5-1
Input Parameters for the Discrete Adjoint Program

<table>
<thead>
<tr>
<th>Target</th>
<th>Sea Level</th>
<th>50,000 feet</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aspect angle, degrees</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Mach number</td>
<td>0.8</td>
<td>1.5</td>
</tr>
<tr>
<td>Rms random acceleration, fps²</td>
<td>200</td>
<td>200 or 100.</td>
</tr>
<tr>
<td>Poisson parameter ν, sec⁻¹ or 0.5/(time constant)</td>
<td>0.125</td>
<td>0.125</td>
</tr>
<tr>
<td>Missile</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mach number</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>RMS heading error (long flight), degrees</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>RMS heading error (short flight), degrees</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>Transfer function of autopilot</td>
<td>text</td>
<td>text</td>
</tr>
<tr>
<td>Time of flight (continuous guidance), sec (discrete guidance), sec</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>Intercept downrange from battery, mi.</td>
<td>1.925</td>
<td>1.925</td>
</tr>
<tr>
<td>Average range from battery to target, feet</td>
<td>10.</td>
<td>10.</td>
</tr>
<tr>
<td></td>
<td>69500.</td>
<td>89000.</td>
</tr>
<tr>
<td>Control weighting b in performance index, sec³</td>
<td>text</td>
<td>text</td>
</tr>
<tr>
<td>Measurement noise (see Equation (5.3))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Standard deviation of scintillation noise, feet</td>
<td>7.</td>
<td>7.</td>
</tr>
<tr>
<td>Standard deviation of range-independent noise, rad.</td>
<td>0.002</td>
<td>0.002</td>
</tr>
<tr>
<td>Receiver-noise parameter</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sigma_{rn}^2 \sigma_o R_o^{-4} \sigma_t ), rad²/ft⁴</td>
<td>5.9 x 10⁻²⁵</td>
<td>5.9 x 10⁻²⁵</td>
</tr>
</tbody>
</table>
held constant at its value midway through the intercept, this value being shown in Table 5-1. The factor multiplying $R^2_{it}R^2_{mt}$ in the first term of (5.3) is shown in Table 5-1 also, as are the standard deviations pertinent to the other two terms.

It should be emphasized that Table 5-1 shows plausible values only, none being from an actual guidance system.

5.3 Results at Mach 2, Sea Level

5.3.1 Various Autopilots with Continuous Guidance Law

As indicated by the table in Subsection 4.2.4.3.2, the "new autopilot" (as it will be called here) at its design point had the following approximate transfer function:

$$\frac{A_m}{A_c} = \frac{1 - \frac{s}{33.44}}{1 + \frac{s}{10.2}} \left(1 + \frac{s}{32.6} \right) \left[ 1 + 2(0.6) \frac{s}{95.4} + \left(\frac{s}{95.4}\right)^2 \right]$$

which, since the LHP zero nearly cancels a pole, is nearly the same is:

$$\frac{A_m}{A_c} = \frac{1 - \frac{s}{33.44}}{1 + \frac{s}{10.2}} \left[ 1 + 2(0.6) \frac{s}{95.4} + \left(\frac{s}{95.4}\right)^2 \right]$$

An attenuation stage with a gain of $-1/1163$ has been introduced between the command $A_c$ and $u_r$ in Chapter 4, so that the autopilot transfer function has a gain of unity at zero frequency. Equation (5.5) has been used for the autopilot proper in the adjoint simulation, while its model in the guidance portion of the simulation utilized only the zero at $+33.44$ and the pole at $-10.2$ rad/sec.
Subsection 4.3.1.1 describes the "slow classical autopilot," which is the name convenient for this purpose, with the following transfer function:

\[
\frac{A_m}{A_c} = \frac{\left[ 1 - \frac{s}{33.4} \right] \left[ 1 + \frac{s}{34.5} \right]}{\left[ 1 + 2(0.3) \frac{s}{5.3} + \left( \frac{s}{5.3} \right)^2 \right] \left[ 1 + \frac{s}{28.1} \right]}
\] (5.6)

where the actuator poles have been omitted because they clearly contribute so little to the step response. This cubic model was utilized in the adjoint program.

Entries 1 and 2 of Table 5-2 compare the rms miss distances of the slow classical and the new autopilot; the continuous guidance law with a very small control weighting \( b \) of \( 1 \times 10^{-8} \text{ sec}^3 \) was used. Since the slow classical autopilot performed so badly, it was apparent that a better classical design was needed, so as to have a more honest performance comparison between the new and classical techniques of autopilot design. Accordingly, the "faster classical autopilot," described in Subsection 4.3.1.2, was designed with the following approximate transfer function:

\[
\frac{A_m}{A_c} = \frac{\left[ 1 - \frac{s}{33.4} \right] \left[ 1 + \frac{s}{34.5} \right]}{\left[ 1 + \frac{s}{13.3} \right] \left[ 1 + (0.52) \frac{s}{14.3} + \left( \frac{s}{14.3} \right)^2 \right]}
\] (5.7)

Under the same conditions of evaluation, this autopilot had a miss distance (entry 3 of Table 5-2) of 13.92 ft rms, which was not significantly worse than the 13.08 feet rms miss of the new autopilot.

At this point, it seemed possible that the new autopilot simply was not fast enough. To test this, the following elementary transfer function of a "hypothetical autopilot" was used in the adjoint program:
<table>
<thead>
<tr>
<th>Entry</th>
<th>Type of Autopilot</th>
<th>Date of Adjoint</th>
<th>Guidance Law</th>
<th>Control wt. b, sec$^3$</th>
<th>Time of Flight</th>
<th>Components of Miss, ft. rms</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Slow Classical</td>
<td>9/28/70</td>
<td>Cont.</td>
<td>1. x 10^{-8}</td>
<td>15</td>
<td>6.93</td>
</tr>
<tr>
<td></td>
<td>New (Chapter 4)</td>
<td>9/29/70</td>
<td>Cont.</td>
<td>1. x 10^{-8}</td>
<td>15</td>
<td>0.16</td>
</tr>
<tr>
<td></td>
<td>Faster Classical</td>
<td>10/5/70</td>
<td>Cont.</td>
<td>1. x 10^{-8}</td>
<td>15</td>
<td>0.71</td>
</tr>
<tr>
<td></td>
<td>Hypothetical</td>
<td>10/5/70</td>
<td>Cont.</td>
<td>3. x 10^{-5}</td>
<td>15</td>
<td>0.09</td>
</tr>
<tr>
<td></td>
<td>Hypothetical</td>
<td>10/7/70</td>
<td>Cont.</td>
<td>1. x 10^{-5}</td>
<td>15</td>
<td>0.15</td>
</tr>
<tr>
<td></td>
<td>New (Chapter 4)</td>
<td>1/5/71</td>
<td>Cont.</td>
<td>3. x 10^{-5}</td>
<td>15</td>
<td>0.71</td>
</tr>
<tr>
<td></td>
<td>Faster Classical</td>
<td>10/9/70</td>
<td>Cont.</td>
<td>3. x 10^{-5}</td>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>
\[
\frac{A_m}{A_c} = \frac{1 - \frac{s}{33.4}}{1 + \frac{s}{15}}
\]  
(5.8)

Contrary to expectation, the miss distance for this autopilot (entry 4 of Table 5-2) was not less than that of the new autopilot (which obviously had a slower step response); instead, it was slightly larger.

These aforementioned adjoint computations were made before the discrete guidance gains in Figure 3-5 were computed and plotted, with resulting enhanced insight. After the computations of entries 1 through 4 in Table 5-2, it seemed likely that the guidance law itself was limiting the miss-distance performance. Zarchan and other associates of the writer had discovered empirically that the performance of the continuous guidance law in the discrete adjoint simulation could be improved if the control weighting parameter \( b \) (or \( \gamma \) in Reference R3) were adjusted empirically for the minimum miss distance. Accordingly, this parameter was varied in adjoint simulations with the hypothetical autopilot, with the results shown in Figure 5-7. The minimum miss distance of 8.11 feet occurred for \( b = 3. \times 10^{-5} \text{ sec}^3 \), which was therefore used for preliminary evaluations of later autopilots; see also entry 5 in Table 5-2. Later simulations with the new autopilot (having the more realistic transfer function in Equation (5.5)) produced the other data in Figure 5-7 with a slightly higher minimum miss distance as expected; more data on the minimum-miss case is shown in entry 6 of Table 5-2.

Although the optimum value of \( b = 3. \times 10^{-5} \text{ sec}^3 \) markedly improved the performance of the hypothetical autopilot (cf. entries 4 and 5 in Table 5-2), it did not significantly improve the miss distance of the faster classical autopilot, as shown by a comparison of entries 3 and 7.
Figure 5.7 - Miss Distance versus Control Weighting b in Continuous Guidance Law, for Two Autopilots at Mach 2, Sea Level
Although further data necessary for an exhaustive comparison between the faster classical autopilot and the new autopilot was not gathered, it seemed clear at this point (from entries 3, 6 and 7 of Table 5-2) that the new autopilot produced better miss distance with the practical guidance law of Subsection 3.5. Accordingly, effort was then concentrated on the new autopilot.

Table 5-3 shows the significant components of miss distance for the new autopilot, with the transfer function in (5.4). In entries 1 and 2, which are repeated from Table 5-2 for convenience, the bottom line shows the square root of the final covariance in the estimate of $\mathbf{Y}_d$; it is apparent that even entry 2 with the optimum $b$ for the continuous guidance law (Figure 5-7) does not have a miss distance very close to this ultimate lower bound. Entries 3 and 4 show the miss components for the continuous and discrete guidance laws, the time of flight of each case being restricted to the table-size limit for the latter, i.e., $29T_s = 1.925$ sec. Heading error was semi-arbitrarily reduced to 1 degree so as to keep this component of miss from being artificially large; the increase in this component of miss distance still did not affect the total miss very significantly, because of the root-sum-squaring. Entry 3 closely resembles entry 1 in guidance law, control weighting $b$, miss distance and final covariance. Entry 4 used the optimal discrete control gains that are partially shown in Figure 3-5 and 5-8; the miss distance was dramatically improved and it even approached the ultimate lower limit.

The explanation for the comparative miss distances in entries 2 through 4 of Table 5-3 may be seen from Figure 5-8, part of which is reproduced from Figure 3-5. The larger value of $b$ with the continuous guidance law produces values of $N'$ which are significantly closer to
Table 5-3
Miss Distance of New Autopilot (Chapter 4) for Variations in Guidance Law, at Mach 2, Sea Level with Zero Radome Slope

<table>
<thead>
<tr>
<th>Entry</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Date of Adjcint</td>
<td>9/29/70</td>
<td>1/5/71</td>
<td>10/19/70</td>
<td>10/20/70</td>
</tr>
<tr>
<td>Control wt. b, sec$^3$</td>
<td>1. x 10$^{-8}$</td>
<td>1. x 10$^{-5}$</td>
<td>1. x 10$^{-8}$</td>
<td>1. x 10$^{-8}$</td>
</tr>
<tr>
<td>Time of flight, sec</td>
<td>15.</td>
<td>15.</td>
<td>1.925</td>
<td>1.925</td>
</tr>
<tr>
<td>Rms Initial Heading Error, degrees</td>
<td>5.</td>
<td>5.</td>
<td>1.</td>
<td>1.</td>
</tr>
<tr>
<td>Components of Miss, ft. rms</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Heading error</td>
<td>0.0</td>
<td>0.0</td>
<td>1.02</td>
<td>1.02</td>
</tr>
<tr>
<td>LOS rate</td>
<td>0.18</td>
<td>0.16</td>
<td>0.50</td>
<td>0.28</td>
</tr>
<tr>
<td>Receiver noise</td>
<td>0.16</td>
<td>0.15</td>
<td>0.15</td>
<td>0.09</td>
</tr>
<tr>
<td>Range-independent noise</td>
<td>2.24</td>
<td>1.89</td>
<td>2.27</td>
<td>1.22</td>
</tr>
<tr>
<td>Scintillation noise</td>
<td>11.33</td>
<td>6.12</td>
<td>11.34</td>
<td>5.30</td>
</tr>
<tr>
<td>Target maneuver</td>
<td>6.14</td>
<td>5.63</td>
<td>6.17</td>
<td>3.80</td>
</tr>
<tr>
<td>Rms total miss, ft.</td>
<td>13.08</td>
<td>8.53</td>
<td>13.16</td>
<td>6.66</td>
</tr>
<tr>
<td>$\sqrt{P_{11}(t_f)}$, ft.</td>
<td>4.58</td>
<td>4.58</td>
<td>4.58</td>
<td>4.58</td>
</tr>
</tbody>
</table>
Figure 5-8 - Effective Navigation Ratio versus Time to Go for Three Optimum Guidance Laws
those of the truly optimum discrete guidance law (for this system with
discrete autopilot commands) in the last two critical commands. For
times to go greater than 0.2 second, the two continuous guidance laws
give very similar results, and all three guidance laws give merging values
of $N_g$, asymptotic to 3.0, at longer times to go.

Figure 5-9 shows adjoint data for the rms missile acceleration with
two different guidance laws. Evidently it is more advantageous in terms
of required missile acceleration to use the optimum discrete guidance law,
although realistically there would be saturation in both cases at the end
of the intercept, with a consequent small increase in miss distance.

5.3.2 Effect of Linear Radome

Figure 5-10 examines the case of entry 4, Table 5-3 at various
values of linear radome slope R. From theoretical consideration it would
be expected that the minimum miss would be at R = 0; the fact that the
calculated rms miss is 0.18 feet less at R = -0.01 is probably due to
some computational anomaly and is not of particular interest. The miss
distance is fairly insensitive to radome slopes above -0.02 because of
the small value of the airframe $\alpha/\dot{\gamma}$ (Figure 5-10); see also Subsection
2.2.2. At large negative values of R the miss increases abruptly
because of the strong regeneration in the parasitic attitude loop at low
frequencies.

5.4 Results at Mach 2, 50,000 Feet

As mentioned in Subsection 1.4.3, appreciable progress was made
on the problem of compensating the parasitic feedback through the non-
linear radome characteristic at high altitude. Unfortunately, it seems advisible not to describe this work in detail here; it may appear in a
classified Raytheon report. Some related results of interest follow.
Figure 5-9 - RMS Normalized Missile Acceleration versus Time to Go at Mach 2, S.L. for Two Guidance Laws
Figure 5-10 - Miss Distance versus Radome Slope at Sea Level for New Autopilot and Optimal Discrete Guidance Gains
A flight condition of Mach 2, 50,000 feet (with trim $\alpha = 7.5$ degrees) was selected. The $\alpha/\gamma$ of 2.81 seconds was about 4.2 times that of Mach 2, sea level, and the airframe was stable. After a preliminary process of selecting the dominant pole of the autopilot as described in the next section, the autopilot was designed according to the methods of Chapter 4 with the following net transfer function after cancellation:

$$\frac{A_m}{A_c} = \frac{1 - \frac{s}{15,3}}{\left[1 + \frac{s}{5}\right]\left[1 + 2(0.50)\frac{s}{100} + \frac{s^2}{100}\right]} \quad (5.9)$$

5.4.1 Continuous Guidance Law

From the results of Figure 5-7 for the hypothetical autopilot, it was assumed initially that the best value of $b$ for the continuous guidance law would be about $3 \times 10^{-5}$ sec$^3$. Figure 5-11 shows the variation with radome slope of rms miss distance for two different autopilots at Mach 2, 50,000 feet with a continuous guidance law, the aforementioned value of $b$ and an rms target acceleration $\beta = 200$ fps$^2$. It is obvious that these two cases are much more sensitive to radome slope than is the roughly comparable case in Figure 5-10 at sea level. This is due to the relative sizes of $\alpha/\gamma$, i.e., at high altitude the missile must have a larger angle of attack for a given lateral acceleration than at low altitude and so the change in radome error angle $r$ for a given slope $R$ is larger. On the other hand, the miss-distance values for $R = 0$ in Figure 5-11 are not much larger than the miss distance of the new autopilot at $b = 3 \times 10^{-5}$ sec$^3$ in Figure 5-7 for the sea level condition.

Although the miss-distance data for the faster autopilot in Figure 5-11 is admittedly only five points, nevertheless this autopilot appears to be
Figure 5-11 - Miss Distance versus Radome Slope at 50,000 Feet for Two Autopilots
significantly more sensitive to radome slope than the slower one. In order to lessen the chance of an occasional large miss distance, the slower transfer function with only a slight increase in miss at $R = 0$ was chosen; the final autopilot design had the transfer function in (5.9).

5.4.2 Discrete Guidance Law

On reflection, it appeared that the target would not be capable of high lateral accelerations at 50,000 feet, and so the value of $\beta$ was reduced to 100 fps$^2$, both for the statistical target acceleration and its model in the Kalman filter. Figure 5-12 shows rms miss distance versus radome slope for the continuous guidance law ($t_f = 15$ sec) and the discrete guidance law ($t_f = 1.75$ seconds). In the discrete-guidance case, the miss distance of 5.79 ft rms at $R = 0$ approaches the square root of the final covariance of position error, i.e., 4.25 feet. For the continuous-guidance case, the reduction in miss distance at $R = 0$ relative to the corresponding point in Figure 5-11 is due to the reduction in $\beta$ and consequent reduction in bandwidth of the Kalman filter.
Figure 5-12 - Rms Miss Distance versus Radome Slope for Two Guidance Laws at Mach 2, 50,000 Ft.
CHAPTER 6

COMPUTATION OF CONTINUOUS OPTIMAL CONTROL GAINS
FOR HIGH-ORDER LINEAR SYSTEMS

6.1 Introduction

Before the idea of autopilot design as in Chapter 4 was developed, it was attempted to compute optimal control gains simultaneously for the guidance kinematic states and the airframe states, in the hope of gaining insight into the simultaneous design of the guidance law and the autopilot. Although these techniques are outside the main stream of the potentially useful work in guidance of interceptor missiles, nevertheless this chapter should be of interest to those concerned with practical computation of optimal control gains for continuous control and optimal gains for continuous estimation.

Various digital-computer programs have been written and used for the computation of optimal control gains. Of these, one of the most reliable and generally useful programs is SGARAF, which utilizes a 2n by 2n transition matrix in a conventional manner, but it has fundamental numerical difficulties at long times-to-go. Program STGVAU, which utilizes the algorithm of Vaughan (Reference V1), successfully complemented SGARAF in the computation of gains for a 5-state plant at medium and long times-to-go. STGVAU appears to have further potential usefulness if more attention is given to its numerical problems. Program SGIMP is like SGARAF except that it uses an improved transition-matrix method and gives better results at long times-to-go. The theory of each of these programs is discussed briefly.

In order to verify results and to check out new programs, computation
proceeded from the known case in Subsection 3.5 for a simple 3-state plant through successively more complicated models up to an 8-state plant with missile-target kinematics, an unstable pitch airframe (at Mach 2, sea level), a fin servo and a bias integrator. The results appear to be correct.

Modal feedback gains as well as state feedback gains were computed for the 8-state plant. It was found that the actuator modal gains decayed to virtually zero most quickly (at a small time-to-go), and a plausible explanation is given. Professor Potter's theory for the steady-state feedback from a plant with one unstable mode was extended somewhat and applied successfully to this 8-state plant. Only the unstable mode has feedback in the steady-state and its RHP eigenvalue $\lambda_4$ is changed to $-\lambda_4$ by the steady-state feedback, the other (LHP) eigenvalues being unchanged.

The latter theory could be generalized somewhat further to the case of $k$ unstable modes in the original plant and could perhaps be unified with the theory of Crossley and Porter (Subsection 4.3.2.2.1.3) for placing poles in a feedback system. Moreover, there may be interesting possibilities for further research into the accurate computation of optimal control gains (or optimal estimator gains, the dual problem) over wide ranges of time; modal interpretations and techniques might be useful here.

6.2 Theory of Computation of Optimal Control Gains by a Transition Matrix

This subsection summarizes the theory of two transition-matrix approaches, one being conventional and the other being somewhat improved. A similar, slightly fuller, discussion of these methods for the dual estimation problem is contained in Subsection 4.3.2.3.2, with which the reader should be familiar.
6.2.1 Conventional Transition-Matrix Approach

There are at least two ways to derive the transition-matrix approach to the problem of calculating optimal control gains:

1) One can manipulate the necessary conditions into the matrix Riccati equation, with the transversality condition giving the matrix boundary condition, as in Reference B1, pp. 151-152. Then the treatment can follow from equation (4.110), as in estimation, which is the dual problem.

2) Alternatively, one may avoid the Riccati equation and derive a 2n by n matrix equation, somewhat in the manner of Reference B1, pp. 150-151. This approach soon merges with that of Subsection 4.3.2.3.2.

Following the second approach, one may utilize the following plant equation and necessary conditions from Subsection 3.2.1:

\[
\frac{dx(t)}{dt} = Fx(t) + Gu(t) \quad (6.1)
\]

\[
x(t_c) = x_o \quad (6.2)
\]

\[
\frac{dp(t)}{dt} = -Ax(t) - FTp(t) \quad (6.3)
\]

\[
p(t_f) = S_f x(t_f) \quad (6.4)
\]

\[
u(t) = -B^{-1}GTp(t) \quad (6.5)
\]

Here \(t_o\) is the initial time of the problem.

Inasmuch as the calculation will proceed from final time \(t_f\) backwards, it is convenient to define "inverse time" (or "time to go" in the case of
missile interception) as:

\[ \tau = t_f - t \]  

(6.6)

Functional dependence on inverse time \( \tau \) will be denoted by small square brackets, and derivatives will be taken with respect to it, e.g.:

\[ \frac{d\mathbf{x}[\tau]}{d\tau} = -\frac{d\mathbf{x}(t)}{dt} \]  

(6.7)

Equations (6.1), (6.3) and (6.5) may be combined into one vector equation in inverse time:

\[
\begin{bmatrix}
\frac{d\mathbf{x}[\tau]}{d\tau} \\
\frac{d\mathbf{p}[\tau]}{d\tau}
\end{bmatrix} =
\begin{bmatrix}
-F & GB^{-1}G^T \\
-A & F^T
\end{bmatrix}
\begin{bmatrix}
\mathbf{x}[\tau] \\
\mathbf{p}[\tau]
\end{bmatrix}
\]  

(6.8)

Then, (6.4) may be expressed as:

\[
\begin{bmatrix}
\mathbf{x}[0] \\
\mathbf{p}[0]
\end{bmatrix} =
\begin{bmatrix}
I \\
S_f
\end{bmatrix}
\mathbf{x}[0]
\]  

(6.9)

Now, postulate the solution of (6.8) with (6.9) as a boundary condition, in the form:

\[
\begin{bmatrix}
\mathbf{x}[\tau] \\
\mathbf{p}[\tau]
\end{bmatrix} =
\begin{bmatrix}
\mathbf{Y}[\tau] \\
\mathbf{Z}[\tau]
\end{bmatrix}
\mathbf{x}[0]
\]  

(6.10)

where \( \mathbf{Z}[\tau] \) and \( \mathbf{Y}[\tau] \) are time-varying \( n \) by \( n \) matrices, with boundary conditions:

\[ \mathbf{Z}[0] = S_f \]  

(6.11)
\[ Y[0] = I \] \hfill (6.12)

It is readily shown that solution of \( Z[\tau] \) and \( Y[\tau] \) leads directly to the optimal control gains, for from (6.10):

\[
\begin{align*}
\bar{x}[0] &= \left[Y[\tau]\right]^{-1} x[\tau] \quad (6.13) \\
\pi[\tau] &= Z[\tau] x[0] \\
&= Z[\tau] \left[Y[\tau]\right]^{-1} x[\tau] \quad (6.14a)
\end{align*}
\]

Substituting (6.14a) into (6.5):

\[
\begin{align*}
\bar{u}(t) &= -B^{-1}G^T Z(t) \left[Y(t)\right]^{-1} \bar{x}(t) \quad (6.15)
\end{align*}
\]

where the independent variable is forward time rather than inverse time.

For convenience, denote the \( S(t) \) matrix:

\[
\begin{align*}
S(t) &= Z(t) \left[Y(t)\right]^{-1} \quad (6.16)
\end{align*}
\]

which is seen from (6.14b) to be a generalization of \( S[t] \) in (6.4).

It remains to show a means for computing \( Z[\tau] \) and \( Y[\tau] \), first by forming a suitable matrix differential equation and then by using a transition matrix.

Substitution of (6.10) into (6.8) gives:

\[
\begin{align*}
\begin{bmatrix}
\frac{dY[\tau]}{d\tau} \\
\frac{dZ[\tau]}{d\tau}
\end{bmatrix}
\begin{bmatrix}
\bar{x}[0]
\end{bmatrix}
\begin{bmatrix}
-F & GB^{-1}G^T \\
A & F^T
\end{bmatrix}
\begin{bmatrix}
Y[\tau] \\
Z[\tau]
\end{bmatrix}
\end{align*}
\]

It is possible to put both terms of (6.17) on one side of the equation (with zero on the other side) and to factor out \( \bar{x}[0] \), leaving a 2n by n matrix as the
other factor. Since \( x[0] \) could take on values which span all of \( n \)-space, the matrix factor must be a null matrix, which is equivalent to the matrix differential equation:

\[
\begin{bmatrix}
\frac{dY[\tau]}{d\tau} \\
\cdots \\
\frac{dZ[\tau]}{d\tau}
\end{bmatrix} = W_c \begin{bmatrix}
Y[\tau] \\
\cdots \\
Z[\tau]
\end{bmatrix}
\]  

(6.18)

where \( W_c \) is the Hamiltonian matrix for the optimal control problem (Reference O'DI):

\[
W_c = \begin{bmatrix}
-F & GB^{-1}GT \\
\cdots & \cdots \\
A & F^T
\end{bmatrix}
\]  

(6.19)

Equations (6.11) and (6.12) are the boundary conditions for (6.18). It should be noted that (6.17) and (6.18) imply that the solution of the latter is independent of the state \( x[0] \) at the end of the problem.

Equation (6.18) and its boundary conditions are in the same form as (4.113) - (4.115) of Chapter 4 and so it has the same transition-matrix solution, which will only be summarized here. Up to this point, the treatment is applicable both to fixed and time-varying systems. For the fixed case which is of interest here, the transition matrix for \( W_c \) can be conveniently computed for a suitable small interval \( D \) from:

\[
\Omega(D, 0) = I + \sum_{k=1}^{j} \frac{D^k}{k!} W_c^k
\]  

(6.20)

provided \( j \) is large enough. As with (4.124), the transition matrix for a time step \( t_s \), which is an integer multiple of \( D \) can be found from:
\[ \Omega (t_s, 0) = \Omega (mD, 0) = [\Omega (D, 0)]^m \]  \tag{6.21} 

Now, proceeding from the terminal condition, one computes:

\[
\begin{bmatrix}
Y (t_s) \\
Z (t_s)
\end{bmatrix} = \Omega (t_s, 0)
\begin{bmatrix}
I \\
S_f
\end{bmatrix} \tag{6.22}
\]

\[ S[t_s] = Z[t_s] \left[ Y[t_s] \right]^{-1} \tag{6.23} \]

In the conventional method, the next step is:

\[
\begin{bmatrix}
Y [2t_s] \\
Y [2t_s]
\end{bmatrix} = \Omega (t_s, 0)
\begin{bmatrix}
Y [t_s] \\
Z [t_s]
\end{bmatrix} \tag{6.24}
\]

\[ S[2t_s] = Z[2t_s] \left[ Y[2t_s] \right]^{-1} \tag{6.25} \]

and so forth.

Assuming appropriate computation of the transition matrix in (6.20) and (6.21), this method has the advantage that a moderately large step \( t_s \) can be taken with good accuracy, in contrast to the much smaller step size that is required when integrating the Riccati equation.

Unfortunately, this method has the same limitation at a large value of inverse time \( \tau \) as has the corresponding method of computing estimator gains at a large value of forward time (Subsection 4.3.2.3.2). The Hamiltonian matrix in (6.19) has eigenvalues symmetrically arrayed about the imaginary axis and so the transition matrix in (6.21) has both growing and decaying exponentials. Examination of (6.22) and (6.24) shows that
Z [τ] and Y [τ] have both growing and decaying exponentials, and clearly
the latter can be swamped by the former at some critical time τ in machine
computation, depending partly on the computer word length. After reaching
this critical time, the result is a catastrophic inaccuracy in computing
S(t) and the control gains in (6.16) and (6.15).

6.2.2 Computation of Components of Performance Index

It has been found that the conventional transition-matrix method can
be slightly extended so as to compute the two major components of the
performance index. As in (3.2), the performance index is:

\[ J(t_p) = \frac{1}{2} \mathbf{x}(t_f)^T S_f \mathbf{x}(t_f) + \frac{1}{2} \int_{t_p}^{t_f} \left[ \mathbf{x}(t)^T A(t) \mathbf{x}(t) + \mathbf{u}(t)^T B(t) \mathbf{u}(t) \right] dt \]

(6.26)

where functional notation for J emphasizes that it is a function of time \( t_p \)
and (by implication) of the state \( \mathbf{x}(t_p) \). Now, it is known (Reference A4)
that:

\[ J(t_p) = \frac{1}{2} \mathbf{x}(t_f)^T S(t_f) \mathbf{x}(t_f) \]

(6.27)

This is readily computed from the known state and the matrix \( S(t_p) \) which
is computed for the optimal gains. With appropriate changes in time-
dependent notation from inverse to forward time, (6.13) becomes:

\[ \mathbf{x}(t_f) = [Y(t_p)]^{-1} \mathbf{x}(t_p) \]

(6.28)

Substitution of (6.28) into the first term of (6.26) gives:

\[ \frac{1}{2} \mathbf{x}(t_f)^T S_f \mathbf{x}(t_f) = \frac{1}{2} \mathbf{x}(t_p)^T [Y(t_p)]^{-T} S_f [Y(t_p)]^{-1} \mathbf{x}(t_p) \]

(6.29)
By the conventional transition-matrix method, $Y(t_p)$ is explicitly computed and so (6.29) may be computed to give the first term in (6.26). Subtraction of this term from the total cost (computed in (6.27)) gives the second term in (6.26).

Therefore, use of the conventional transition-matrix approach makes it easy to compute and compare the final-state term and the integral term in the performance index (6.26). In some problems, this might be useful in determining suitable weighting matrices, which seems to be something of an art at present.

6.2.3 Improved Transition-Matrix Approach

This improved transition-matrix approach overcomes the problems of growing versus decaying exponentials in the conventional approach. As explained in Subsection 4.3.2.3.2, it was discovered independently by this writer but it was originally devised by Kalman and others in 1962 who published it somewhat obscurely (Reference K4, p. 103).

Unless one is interested in computing $Y(t_p)$ explicitly as in the previous subsection, one may limit one's interest to $S(t)$ itself (cf. (6.15) and (6.16)). In a properly posed problem, $S[T]$ should be a well-behaved matrix without very large elements, although its elements tend to change very rapidly at low values of inverse time $T$ if the scalar control weighting $b$ is very small and the state weighting $A$ is zero, as in the computations of Subsection 6.4.

Suppose that one has computed $S[t_s]$ in the first step, as in (6.23), with reasonable accuracy. In order to stop any asymmetry arising from computational errors, this improved method now computes:

$$S_{sym}[t_s] = \frac{1}{2} \left[ S[t_s] + S[t_s]^T \right]$$  \hspace{1cm} (6.30)
The computational cycle of (6.22), (6.23) and (6.30) is now repeated:

\[
\begin{bmatrix}
Y [2t_s]' \\
- - - - - -
\end{bmatrix}
= \Omega (t_s, 0)
\begin{bmatrix}
I \\
- - - - - \\
S_{sym} [t_s]
\end{bmatrix}
\]  
\hspace{5cm} (6.31)

\[
S [2t_s] = \left[ Z [2t_s]' \right][Y [2t_s]']^{-1}
\]  
\hspace{5cm} (6.32)

where the primes emphasize that these Y and Z matrices are not identical to those of (6.24).

\[
S_{sym} [2t_s] = \frac{1}{2} \left[ S [2t_s] + S [2t_s]^T \right]
\]  
\hspace{5cm} (6.33)

For a situation in which \( S [\tau] \) changes rapidly at the beginning (small inverse time \( \tau \)), the initial time-step \( t_s \) should be small, followed by larger steps.

This algorithm worked quite well for computing \( P(t) \) for estimator gains as in Subsection 4.3.2.3.2 and it was an improvement over the conventional approach in Subsection 6.4.2. In the latter case, the decay of modal feedback gains (Subsection 6.5.1) for the stable modes was the apparent cause of some rather small elements in the \( S [\tau] \) matrix, which had a few very small negative elements on the diagonal; this is contrary to theoretical considerations, which require \( S [\tau] \) to be positive definite.

This latter numerical difficulty could be overcome by replacing each diagonal element of \( S_{sym} [\tau] \) with its absolute value, but this does not guarantee accuracy.

From another point of view, at a moderately large \( \tau \) only certain modal control gains may be important, such as that of the unstable mode
(for \( A = 0 \)) in Subsection 6.5. Let \( F' \) be the new system matrix with the feedback gains absorbed into \( F \), as in (4.39). The eigenvalues of \( F' \) will be somewhat different from those of \( F \); in the case of Subsection 6.5, at steady-state the RHP eigenvalue of the unstable mode in \( F \) will be replaced by an LHP eigenvalue of equal magnitude in \( F' \). As the theory of Subsections 4.3, 2.2.1, 2 and 4.3, 2.2.1, 3 indicates, weak feedback of another mode will not change its own eigenvalue substantially. Hence, this latter mode (identified computationally by its low eigenvalue shift) is operating essentially "open-loop" and therefore high accuracy in computing its modal feedback gain is not of practical interest.

6.3 Digital Programs for Computing Optimal Control Gains

Because of the fundamental and well-known difficulties of computing optimal control gains reliably, it has been found necessary to proceed in computation from a simple known case (i.e., the closed-form gains in (3.84) - (3.87)) step by step to more complicated plant models. This method has proved to be successful in checking results and in reducing the possibility of unknown numerical difficulties.

Various digital programs for computing optimal control gains and navigation ratios as a function of "time to go," for plants of varying complexity, are summarized in Tables 6-1A and 6-1B. A brief description of the more successful programs will now be given.

Program DVSRIC (Table 6-1A) utilized the Riccati equation with fourth-order Runge-Kutta integration (Reference H1), and was unsuccessful, apparently because of fundamental step-size problems. Professor J.E. Potter has commented that the product of the time step and the magnitude of the largest eigenvalue of \( F \) should be much smaller than unity. Only a little effort was expended on program DVSRIC and no further discussion
### Table 6-1A
Summary of Digital Programs for Computing Optimal Control Gains
(All programs on this page are in BASIC, on GE 430 with 11 digits)

<table>
<thead>
<tr>
<th>Name</th>
<th>Function</th>
<th>Airframe or Autopilot Model</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>NPRIME</td>
<td>Computes $N_\text{i}$' and autopilot feedback gain as in (3.86) and (6.34).</td>
<td>Fixed autopilot, one pole and one zero; no $F_{11}$ or $F_{22}$.</td>
<td>Simple and successful Very limited as to plant model.</td>
</tr>
<tr>
<td>OPGAIN</td>
<td>Computes optimal control gains in modal space with no feedback from unwanted modes. Uses 2m by 2m transition matrix.</td>
<td>8 states - kin., pitch airframe, actuator. Actuator modes dropped in gain computation.</td>
<td>Requires further development and evaluation, so that state gains can be checked against simpler programs.</td>
</tr>
<tr>
<td>OPOZGA</td>
<td>Computes optimal control gains in state space by 2n by 2n transition matrix.</td>
<td>Autopilot with one pole and one zero. One-state target model.</td>
<td>Checked results of NPRIME very closely, with $b = 1 \times 10^{-8}$ sec$^{-3}$.</td>
</tr>
<tr>
<td>STGVAU</td>
<td>Optimal control gains in state space by Vaughan's method, Reference VI. Uses eigenvectors of Hamiltonian matrix. Always $b = 1$, variable $S_{f11}$.</td>
<td>Fixed autopilot, one pole, one zero. $F_{11}$ and $F_{22}$ as in OPGAIN.</td>
<td>Modified for $F_{11}$ and $F_{22}$ in STGVAU. This plant and $b = 1 \times 10^{-2}$ and $1 \times 10^{-3}$ gives results close to those of NPRIME. Agreed with OPOZGA for $S_{f11} = 1$. Good for $S_{f11} = 100$, except for intermediate time; worse for $S_{f11} = 1 \times 10^3$, good at long time. Bad for $1 \times 10^6$ and $1 \times 10^8$ at intermediate time. Critical $R$ matrix insensitive to $S_{f11}$. Overlapped SGARAF runs (on GE 430 comp.) from 0.2 to 0.8 second; plausible answers at long time; bad answers below 0.2 sec because of numerical difficulties.</td>
</tr>
</tbody>
</table>

- $S_{f11}$: A parameter in the system.
Table 6-1B
Summary of Digital Programs for Computing Optimal Control Gains (Cont.)

<table>
<thead>
<tr>
<th>Name</th>
<th>Function</th>
<th>Airframe or Autopilot Model</th>
<th>Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>SGARAF</td>
<td>Computes optimal control gains in state space by conventional transition-matrix approach.</td>
<td>5 states: Kinematics, target, rudimentary pitch airframe, usually unstable, Mach 2, S.L.</td>
<td>Correct at short t&lt;sub&gt;go&lt;/sub&gt; Qualitatively plausible gains at long t&lt;sub&gt;go&lt;/sub&gt; for unstable or stable airframe. Usually numerical troubles beyond 0.8 sec. b = 1.x10&lt;sup&gt;-8&lt;/sup&gt;, usually.</td>
</tr>
<tr>
<td>(on CALL/360)</td>
<td></td>
<td></td>
<td>Comparison of effect of word length: Good to 0.3 sec with app. 8 digits on 360 Good to 0.7 sec with app. 11 dig. on GE 430 Good to 1.2 sec with app. 15 dig. on 360, using BASIC long-form.</td>
</tr>
<tr>
<td>(360 in BASICL)</td>
<td></td>
<td></td>
<td>Gains very close to those of last run in BASICL, but addition of gain from bias integrator.</td>
</tr>
<tr>
<td>Also modal control gains</td>
<td></td>
<td>Same model as above, but with bias integrator added for tail-angle bias.</td>
<td>Gains reasonably similar to those of last run, particularly at long times to go; but higher N&lt;sub&gt;a&lt;/sub&gt; at short times to go, indicating forcing to overcome actuator lags.</td>
</tr>
<tr>
<td>SGIMP</td>
<td>Computes optimal control gains in state space by improved transition-matrix method; also modal-control gains.</td>
<td>8 states: Kinematics, target, correct airframe with actuator servo and bias on tail-angle command.</td>
<td>Results similar to SGARAF, but anomalies starting at t&lt;sub&gt;g&lt;/sub&gt; = 1.1 sec. Some negative diagonal elements in S. Correct bias-mode gain at large t&lt;sub&gt;g&lt;/sub&gt;.</td>
</tr>
<tr>
<td>(on Tym-share)</td>
<td></td>
<td>Full 8 states as above.</td>
<td>Agreed with significant results of programs SGARAF and STGVAU without divergence. Some negative elements in S at large t&lt;sub&gt;g&lt;/sub&gt;.</td>
</tr>
<tr>
<td>SGIMP2</td>
<td>Computes optimal control gains in state space by improved transition-matrix method. No modal-control gains.</td>
<td>5 states: Kinematics, target, rudimentary pitch airframe, unstable at Mach 2, S.L.</td>
<td></td>
</tr>
</tbody>
</table>
of it is planned.

6.3.1 Programs for Guidance System with Autopilot Having One Pole and One Zero

Subsection 3.5, with its closed-form solution for this simple system, was the starting point when the computational effort was reoriented in accordance with the methodology of Subsection 6.3. Figure 6-1 shows the block diagram for this guidance system with variations as noted.

6.3.1.1 Program with Closed-Form Solution

Referring to Figure 6-1, the state vector and state equation for program NPRIME are:

\[
\dot{x}^T = [x_1, x_2, x_3]^T = [\dot{Y}_d, V_d, A]^T
\]

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & -1 \\
0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + 
\begin{bmatrix}
0 \\
0.2 \\
1.2
\end{bmatrix} u
\]

This corresponds to an autopilot transfer function:

\[
\frac{A_m}{A_c} = \frac{1 - \frac{s}{5}}{1 + \frac{s}{1}}
\]

This zero-to-pole ratio was fairly typical of some past designs at high altitude. A head-on intercept with \(\theta_h = 0\) assumed.

Program NPRIME simply computes \(N'_g\) from (3.86) and also the feedback gain \(C_3\) for state \(x_3 = A\) (cf. (3.84) and (3.87)):

\[
C_3 = \frac{u}{A} = \frac{1 - \omega_{1g}t - e^{-\omega_{1g}t}}{(\omega_{1g}t)^2} \frac{-N'_g}{N'_g}
\]
Figure 6-1 - Guidance System with Autopilot Having One Pole and One Zero
It should be noted from Figure 6-1 that this state A is different from the total lateral acceleration $A_m$ of the missile.

### 6.3.1.2 Program with Conventional Transition-Matrix Approach

In program OPOZGA, all four state variables in Figure 6-1 were modelled with the following state vector and state equation:

$$\mathbf{x}^T = [x_1, x_2, x_3, x_4]^T = [Y_d, V_d, A_t, A]^T$$

$$\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
F_{11} & 1 & 0 & 0 \\
0 & F_{22} & 1 & -1 \\
0 & 0 & -0.25 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} +
\begin{bmatrix}
0 \\
0.2 \\
0 \\
1.2
\end{bmatrix} u$$

(6.39)

For initial comparisons the results of NPRIME, $F_{11} = 0 = F_{22}$ in OPOZGA.

For later comparisons between computations with OPOZGA and STGVAU the following values were used:

$$F_{11} = -0.001 \text{ rad/sec}$$

(6.40)

$$F_{22} = -0.0485 \text{ rad/sec}$$

(6.41)

Program OPOZGA uses the conventional transition-matrix approach in Subsection 6.2.1 to compute the four optimal control gains in the vector $\mathbf{c}$ (cf. (6.15) and (6.16)):

$$u = A_c = - \left( \frac{1}{b} \right) \mathbf{g}^T S [\tau] \mathbf{x} [\tau]$$

(6.42)

$$= \mathbf{c} [\tau]^T \mathbf{x} [\tau]$$
where $T$ is inverse time, or time to go $t_g$ until intercept. Using $\omega = 1$ rad/sec, computations were carried out at discrete time points, typically 0.1, 0.2, ..., 1.1, 2.1, ..., 11 seconds to go, in succession.

OPOZGA computes the effective navigation ratio:

$$N_g' = c_1 T^2$$  \hspace{1cm} (6.43)

the validity of which may be seen by comparing (6.42), (3.84) and (3.87) with $\theta_h = 0$. Examination of these equations and (3.87) in particular shows that the gains should have the following ratios:

$$\frac{c_2}{c_1} = \frac{u/V_d}{u/Y_d} = T$$  \hspace{1cm} (6.44)

$$\frac{c_4}{c_1} = \frac{u/A}{u/Y_d} = \frac{1 - \omega_1 T - e^{-\omega_1 T}}{\omega_1^2}$$  \hspace{1cm} (6.45)

The ratio $c_3/c_1$ should be the same as (6.45), if $-F_{33}$ is substituted for $\omega_1$. Program OPOZGA computes these ratios as a check on computational accuracy with repeated matrix operations.

This program was devised in order to have a simple check of hitherto unproved (in the writer's experience) transition-matrix techniques against the known closed-form solutions of Subsection 3.5. The excellent computational agreement between programs OPOZGA and NPRIME is described in Subsection 6.4.1.

6.3.2 Transition-Matrix Programs for Plants with Eight States or Less

6.3.2.1 Program with Conventional Transition-Matrix Approach

SGARAF, one of the most useful programs in Tables 6-1A and 6-1B, utilizes a conventional transition-matrix approach (Subsection 6.2.1)
very similar to that of OPOZGA, but with more flexibility. It has been used for plants with 5 to 8 states, and could be modified for a larger number of states, limited presumably by computer storage or some other practical limit.

In addition to the features of OPOZGA, SGARAF can multiply 2 to 20 basic transition matrices of interval D together so as to compute a transition matrix for a time step of 2D to 20D, which overcomes the problem of step size D being limited by series convergence for a large plant eigenvalue, e.g., from the actuator modes. Also, SGARAF computes modal feedback gains as well as conventional feedback gains in state space, and it has an option to modify the original $S_f$ matrix so as to prevent feedback from the actuator modes; the theory of this approach is explained in Subsection 7.1.

SGARAF appears to be a reliable program for computing feedback gains with plants of the types in Subsection 6.4 for the following two conditions: 1) Times-to-go between 0 and about 1 second at which a quasi-steady state condition is usually reached; 2) a conventional $S_f$ matrix with $S_f = 1$ and zeroes elsewhere, rather than one modified to exclude the actuator modes. SGARAF has given results that partially overlap those of STGVAAI, which is a very different type of program that is explained in the next section. The quasi-steady-state results of SGARAF have also been checked by program SSGOUM, which utilizes the theory of Potter (Reference P1) and O'Donnell (Reference O'D1) for steady-state solutions of the Riccati equation.

Beyond some limit, such as about 1 second for an 8-state plant with an actuator model, SGARAF develops an asymmetric $S$ matrix and anomalous gains. This is believed to be due to two causes: 1) Accumulated
numerical errors from repeated matrix multiplication; and 2) The presence of implicit growing exponentials which swamp decaying but important exponentials in the transition matrix, as explained by Vaughan (Reference V1). The effect of computer word length on this problem is briefly shown in Table 6-1B for a 5-state plant.

Although dependence of each computation of $Z$, $Y$ and $S$ upon previous computations of $Z$ and $Y$ has the disadvantage of accumulating numerical errors, it has the advantage of giving some credence to previous computations when the quasi-steady-state is checked against other theory and computations for the true steady-state gains.

As a parenthetical note for possible future investigation, it appears that a fundamental source of difficulty in computation is the very rapid rate of change of $S[\tau]$ for small values of time to go, $\tau$. For instance, a typical computation for an 8-state plant (Subsection 6.4.4) with control weighting $b = 1. \times 10^{-8} \text{ sec}^3$ (virtually negligible) and $S_{f11} = 1$ (other elements of $S_f$ being zero) has the following result at 0.01 second:

$S_{11}[0.01] = 4.529 \times 10^{-5}$, with other elements of $S[0.01]$ being of lesser magnitude but significant. This very rapid change of $S[\tau]$ is difficult for many programs, notably DVSRTC, to compute. A modification of $S_f$ to exclude the actuator modes in SGARAF, or a similar modification plus computation in modal space as in OPGAIN, apparently worsens the computational accuracy at this critical time.

6.3.2.2 Programs with Improved Transition-Matrix Approach

Programs SGIMP and SGIMP2 are very similar to program SGARAF (Subsection 6.3.2) but with the improved transition-matrix approach in Subsection 6.2.3. Program SGIMP2 differs from SGIMP in that it omits the additional computation of modal control gains.
6.3.3 Program Using the 2N-Modal Space Approach of Vaughan

This program, which is based on the work of Vaughan in Reference V1, was devised because Vaughan's technique appeared to overcome the numerical problems of the conventional transition-matrix approach.

Vaughan has expanded the theory of References P1 and O'D1 to develop an algorithm for computing $S[\tau]$ for constant matrices $F$ and $G$ in the plant model and constant $A$ and $B$ in the performance index. This method gives particular attention to the classically troublesome problem of computing the $S$ matrix at long times to go and in the steady state. The theory is well explained in Reference V1 and will be only summarized here. The Hamiltonian $W_c$ matrix is converted to a $2n$ matrix of eigenvalues, which are symmetrically arrayed around the imaginary axis in the eigenvalue plane; for the case of zero state weighting $A$ in the performance index (of interest in this interception problem), half of these $2n$ eigenvalues are precisely the eigenvalues of the bare plant. The corresponding $2n$ by $2n$ eigenvector matrix is used to transform $x$ and $p$ (the costate vector) to this $2n$ modal space. The equations are then manipulated so that only decaying exponentials (from LHP eigenvalues) are computed, thus avoiding the classic problem wherein growing exponentials swamp out significant decaying exponentials. The matrix $S[\tau]$ is computed from these decaying exponentials, four square partitioned elements of the $2n$ by $2n$ eigenvector matrix $V_w$ and a matrix $R$ which is a function of $S_f$ and the four partitioned sub-matrices of $V_w$.

It is strongly suggested that the reader consult Vaughan's paper (Reference V1), which illuminates well the difficulty of the conventional transition-matrix method as well as explaining his own method. A
summary of the important equations in Vaughan's method follows. Recognizing the symplectic character of the Hamiltonian matrix $W_c$ in (6.19), let the $2n$ by $2n$ matrix of its eigenvalues be arranged such that $\Lambda_1$ is diagonal with only the right-half-plane eigenvalues:

$$\Lambda_c = \begin{bmatrix} \Lambda_1 & 0 \\ \vdots & \vdots \\ 0 & -\Lambda_1 \end{bmatrix}$$ \hspace{1cm} (6.46)

and let $V_w$ be the matrix of right eigenvectors, such that:

$$W_c V_w = V_w \Lambda_c$$ \hspace{1cm} (6.47)

with partitioning into four $n$ by $n$ submatrices:

$$V_w = \begin{bmatrix} V_{w11} & V_{w12} \\ \vdots & \vdots \\ V_{w21} & V_{w22} \end{bmatrix}$$ \hspace{1cm} (6.48)

Define the $n$ by $n$ $R$ matrix:

$$R = - \left[ V_{w22} - S_f V_{w12} \right]^{-1} \left[ V_{w21} - S_f V_{w11} \right]$$ \hspace{1cm} (6.49)

Let $G[\tau]$ be an $n$ by $n$ time-varying matrix:

$$G(\tau) = e^{-\Lambda_1 \tau} R e^{-\Lambda_1 \tau}$$ \hspace{1cm} (6.50)

where the exponentials denote diagonal $n$ by $n$ matrices. Vaughan shows that the $n$ by $n$ $S[\tau]$ matrix is given by:
\[
S[\tau] = \left[ V_{w21} + V_{w22} G[\tau] \right] \left[ V_{w11} + V_{w21} G[\tau] \right]^{-1} \quad (6.51)
\]

Since \( G[\tau] \) decays to zero for large inverse time \( \tau \), it is obvious that \( S[\tau] \) must converge to:

\[
S[\infty] = V_{w21} (V_{w11})^{-1} \quad (6.52)
\]

which is known (References Pl and O'D1) to be the "asymptotic" or steady-state value. Another advantage of Vaughan's method is that each value of \( S[\tau] \) is computed separately from each other value, without accumulating errors. Both of these features contrast with the disadvantages of the conventional transition-matrix method in the previous subsection.

Vaughan's method has been incorporated directly into program STGVAU. Unfortunately, it has proven to have unforseen numerical difficulties; Vaughan commented in Reverence V1 that his method was "not yet numerically tested." In the cases tried (Subsections 6.3.1 and 6.3.2 herein), Vaughan's method did not work at short times to go for small values of \( b/S_{f11} \). Answers were better at longer times to go. Good agreement was obtained between STGVAU and SGARAF for the 5-state plant (Subsection 6.3.2) at intermediate times to go. Some of the difficulties are thought to be due to:

1) The important R matrix is insensitive to order-of-magnitude changes in \( S_{f11} \), indicating a numerically ill-conditioned situation; analytically, for \( \tau = 0 \), \( S[\tau] \) in (6.51) must equal \( S_f \), but consideration of the matrix operations in (6.49) - (6.51) indicates considerable chance for numerical error;
2) Appreciable matrix addition, subtraction, inversion and multiplication are involved;

3) Initial accuracy of input eigenvectors was only fair (limited in part by 5 significant figures);

4) About 11 digits were carried in the computations on the time-shared GE 430. Improvement in these latter two areas would probably improve the accuracy of STGVAU.

6.4 Results with Different Plant Models

This section discusses the results of computing optimal control gains for plant models of increasing complexity, beginning with known results from Subsection 3.6 for the 3-state or 4-state guidance system with an autopilot having one pole and one zero. The objective was to bridge over from these known results to the optimal control gains for a realistic complicated plant with 8 states (Subsection 6.4.4) with assurance. This approach has been fruitful in checking out computer programs, improving confidence in the numerical results and in gaining insight.

6.4.1 Autopilot with One Pole and One Zero

Subsection 6.3.1 has discussed this plant model, for which the block diagram and state equation are shown in Figure 6-1. For the simpler version of this figure, program NPRIME was utilized to compute the $N_g'$ and autopilot gain $C_3$. Figure 6-2 shows $N_g'$ versus $\omega t_g$ for the case where $\omega_1/\omega_2 = -0.2$ and control weighting $b = 0$, which was the starting point for the current computational work. The polarity reversal of $N_g'$ at short $t_g$ is due to the initial negative portion of the autopilot step response.

For comparison, the transition-matrix program, OPOZGA, utilized the following parameters:
Figure 6-2 - Effective Navigation Ratio $N_g^*$ versus Normalized Time to Go, $\omega_1 t_g$, for Autopilot with One Pole and One Zero
\[ \omega_1 = 1 \text{ rad/sec} \quad (6.53) \]
\[ \omega_2 = -5 \text{ rad/sec} \quad (6.54) \]
\[ b = 1 \times 10^{-8} \text{ sec}^3 \quad (6.55) \]

the latter being a virtually zero weighting of integral of squared control effort. Target correlation time \((-1/F_{33})\) was taken as 4.0 seconds, instead of 2.0 seconds as shown on p. 425 of Reference B1. The plant eigenvalues were then 0, 0, -0.25 and -1 rad/sec.

The initial time step in OPOZGA was 0.1 second and \( j = 5 \) terms were used in the series expansion of (6.20); it should be recalled that thus the time step multiplied by the largest eigenvalue magnitude in this plant was only 0.1. Ten steps were computed up to a time-to-go \( \tau = 1 \) second. Because of the diminishing rate of change of \( N^l_g \), the time step was then changed to 1 second after recomputing \( \Omega \) in (6.20) with \( j = 10 \) terms. The gains were then computed out to 11 sec, or 11 on the dimensionless abscissa in Figure 6-2.

The effective navigation ratio \( N^l_g \) calculated by OPOZGA agreed generally to four significant figures with points computed (Figure 6-2) by program NPRIME from the closed-form expression in (3.86). Moreover, OPOZGA gave correct results for the four control gains. This gave confidence in both programs and in the analysis leading to (3.86). Gratification over this result was tempered by the realization that the largest eigenvalue magnitude times the largest time-to-go was only 11, corresponding to a term \( e^{11} \) which is implicit in \( \Omega \) (11, 0) as indicated in Subsections 4.3.2, 3.2 and 6.2.1. The spectral form of the backward transition matrix is shown rather clearly in Reference V1 to have diagonal elements of the form \( e^{\lambda_1 \tau} \) and \( e^{-\lambda_1 \tau} \).
After confidence was established in program OPOZGA, an attempt was made to check out program STGVAU with a similar case. In order to avoid two plant eigenvalues at the origin with consequent computational difficulties, feedback gains $F_{11}$ and $F_{22}$ were added as shown in (6.40) and (6.41). The value for $F_{22}$ was computed from a typical drag situation at Mach 2, sea level, although its true applicability here is debatable. The eigenvalues of this modified plant were $-0.001$, $-0.0485$, $-0.25$ and $-1$. rad/sec. Typically, OPOZGA utilized various values of $b$, with $S_{f11} = 1$, while the opposite was usually true in STGVAU. However, consideration of the performance index $J$ with $A = 0$ in (6.26) shows that only the ratio $b/S_{f11}$ is significant. The following table shows a summary of the major results with STGVAU.

Table 6-2

Major Results with STGVAU for Full Plant in Figure 6-1

<table>
<thead>
<tr>
<th>$b/S_{f11}$, sec$^3$</th>
<th>Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Excellent agreement with OPOZGA for $0 &lt; \tau &lt; 1.6$ sec</td>
</tr>
<tr>
<td>$1 \times 10^{-2}$ sec$^3$</td>
<td>Good agreement with OPOZGA up to 3.5 sec, except for moderate errors in band $0.4 &lt; \tau &lt; 1.0$ sec</td>
</tr>
<tr>
<td>$1 \times 10^{-3}$ sec$^3$</td>
<td>Agreement with OPOZGA fair at 0.1 sec, poor for $0.2 \leq \tau \leq 1.5$ sec, good for $1.5 \leq \tau \leq 3.5$ sec</td>
</tr>
<tr>
<td>$1 \times 10^{-4}$ sec$^3$</td>
<td>Directly comparable OPOZGA case not run. Answers compare well with similar OPOZGA case for $2 \leq \tau \leq 4$ sec. Anomalous results for $0 &lt; \tau \leq 1$ sec</td>
</tr>
<tr>
<td>$1 \times 10^{-8}$ sec$^3$</td>
<td>Agreement good for OPOZGA case (with $F_{11} = 0 = F_{22}$) for $2 \leq \tau \leq 4$ sec. Bad anomalies for $\tau &lt; 1$ sec</td>
</tr>
</tbody>
</table>

It is also worth noting that $S_{11}[\tau]$ changes quite slowly for large $b/S_{f11}$ (and vice versa), and that Vaughan's important R matrix (Reference VI)
appears to be relatively insensitive to large changes in $S_{f11}$. These observations and Table 6-2 indicate that the difficulties with Vaughan's method and program STGVAU are due to inherent numerical difficulties in the method and also the particular numerical inputs here.

6.4.2 Plant with Rudimentary Pitch Airframe

The next level of complexity was a five-state plant with a rudimentary model of an unstable pitch airframe at Mach 2, sea level, which was simplified from the fuller correct model in Subsection 6.4.4.

The derivation of this model is given in Subsection A-3.0 of Appendix A and its block diagram is shown in Figure 6-3. Its state vector and state equation are:

$$x^T = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^T$$

$$= \begin{bmatrix} Y_d & V_d & A_t & A_q \end{bmatrix}^T$$

(6.56)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} F_{11} & 1 & 0 & 0 & 0 \\ 0 & F_{22} & 1 & -1 & 0 \\ 0 & 0 & -0.25 & 0 & 0 \\ 0 & 0 & 0 & -1.304 & 2910 \\ 0 & 0 & 0 & 0.0482 & -1.016 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \delta_e \end{bmatrix}$$

(6.57)

For SGARAF, $F_{11} = 0 = F_{22}$, while for STGVAU the values in (6.40) and (6.41) were used. A tail-angle bias and actuator model were not included.
Figure 6-3 - Block Diagram for System with Differential Geometry, Evasive Target Acceleration and Rudimentary Pitch Airframe
The eigenvalues of this plant are \(-0.01, -0.0485, -0.25, +10.684\) and \(-13.004 \text{ rad/sec}\), which are associated with \(Y_d, V_d\), target acceleration, the unstable pitch mode and stable pitch mode, respectively.

Figure 6-4 shows the apparent effective navigation ratio \(N_a\), defined as shown, with \(t_g\) being the time-to-go and \(c_1\) being the control gain from \(Y_d\) back to control input \(u = \delta_e\). The polarity reversal, as in Figure 6-2, is due to the right-half plane zero. It should be noted that the results of SGARAF and STGVAU overlap well from 0.2 through 0.7 second. This band of agreement is limited by the numerical difficulties inherent in each program, by the word length (and accuracy) of the inputted eigenvalues and eigenvectors and by the word length (evidently 11 digits) of the GE 430 computer. Table 6-1B shows the effect of word length on the maximum feasible time-to-go, using SGARAF with this plan: model. For these runs, \(S_{f11} = 1\) and control weighting \(b = 1 \times 10^{-8} \text{ sec}^3\). It is apparent that program SGIMP2 gives credible results for the largest ratio of maximum to minimum \(t_g\).

For this unstable bare airframe, the effective navigation ratio goes to zero at infinite time-to-go. On the other hand, Figures 6-5 and 6-6 show that the gains \(c_4\) (from acceleration due to body lift) and \(c_5\) (from pitch rate back to \(u = \delta_e\)) go to nonzero steady-state values at long times-to-go. Later theoretical work (Subsection 6.5) showed that these gains are required for the unstable mode, so as to minimize the performance index. Again, program SGIMP2 gives credible results for the largest ratio of maximum to minimum \(t_g\).

In an early experimental computation with the plant in Figure 6-3, the airframe was artificially made stable by reversing the sign of \(M_\alpha\) (proportional to \(C_{m\alpha}\)) and the sign of element \(F_{54}\) in the plant matrix) in
Figure 6-4 - Apparent Navigation Ratio $N_a$ versus Time-to-Go $T_{a0}$ for Rudimentary Unstable Bare Airframe at Mach 2, Sea Level

$N_a$, rad/sec²/ft

0.030

0.025

0.020

0.015

0.010

0.005

0.000

0.001

0.002

0.003

0.004

0.005

0.006

0.007

0.008

0.009

0.010

0.015

0.020

0.025

0.030

0.035

0.040

0.045

0.050

0.055

0.060

0.065

0.070

0.075

0.080

0.085

0.090

0.095

0.100

T_{a0}(sec)

NOTE: $N_a = C_1T_g^2$

+ CALC. BY SGARAF (WITH 2n by 2n TRANSITION MATRIX)

○ CALC. BY STGVAU (VAUGHAN'S METHOD WITH HAMILTONIAN EIGENVECTORS)

□ CALC. BY SGIMP2 (IMPROVED TRANSITION MATRIX)

(POINTS OFF CURVE NOT RELIABLE)
Figure 6-5 - Gain from Acceleration ($x_4$ due to body lift) to Tail-Angle Input, versus Time to Go, for Rudimentary Unstable Bare Airframe at Mach 2, Sea Level
Figure 6-6 - Gain from Pitch Rate to Tail-Angle Input versus Time to Go, for Rudimentary Unstable Bare Airframe at Mach 2, Sea Level
(6.57). It turned out that the apparent navigation ratio $N_a = c_1 t^2 g$ was at a nearly steady-state value such that the product of $N_a$ times the low-frequency airframe gain (meaningful only for a stable bare airframe) was 3.04, or about 3. This agreed with the result in Figure 6-2 and with prior expectations. It happened that element $g_2 = -V_m Z_δ$ had the wrong sign in this early computation, but this was allowed for in the computation of low-frequency airframe gain; this error in high-frequency modelling did not alter the validity of this result at a long time to go but did invalidate the calculation at short times to go.

6.4.3 Plant with Rudimentary Pitch Airframe and Tail-Angle Bias

Figure 6-3, with inclusion of the bias integrator, is the block diagram of this plant. With reference to Subsection A-3.0 of Appendix A, the state vector and state equations are:

$$\mathbf{x}^T = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}^T$$

$$= \begin{bmatrix} Y_d & V_d & A_t & A_b & \dot{q} & \dot{\delta}_b \end{bmatrix} \mathbf{1}'$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.0485 & 1 & -1 & 0 & -302 \\ 0 & 0 & -0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.304 & 2910 & -394 \\ 0 & 0 & 0 & 0.0482 & -1.016 & -105 \\ 0 & 0 & 0 & 0 & 0 & -0.01 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 \\ -302 \\ 0 \\ -394 \\ -105 \\ 0 \end{bmatrix} u$$

(6.59)
where $\delta_b$ is the tail-angle bias. As indicated in Table 6-1B, the optimal control gains were computed by SGARAF and found to be very close to those for the plant without the bias integrator, but with the addition of a gain $c_6$ from the bias $\delta_b$ back to $u = \delta_e$. In addition to the eigenvalues of the previous section, this plant has an eigenvalue at $-0.01 \text{ rad/sec}$ from the bias integrator. As before, $S_{f11} = 1$ and control weighting $b = 1 \times 10^{-8} \text{ sec}^{-3}$.

Figure 6-7 shows this gain $c_6$, which has a reversal near 0.1 second, apparently related to the peak in $N_a$ (Figure 6-4). At long times to go, this gain is almost $-2.0$, instead of the value $-1.0$ which would exactly cancel the bias. The reason is undoubtedly the same as for the 8-state plant, for which it is known that this state gain is a direct consequence of the modal gain from the unstable mode, which is the only significant modal gain at long times to go. Modal feedback gains are discussed in Subsection 6.5.

6.4.4 Plant with Pitch Airframe, Fin Servo and Tail-Angle Bias

Figure 6-3 is a simplified version of the true plant, with an unstable pitch airframe at Mach 2, sea level but without an actuator servo. The true plant is depicted in Figure 6-8; see also Subsection A-1.0 of Appendix A. The airframe poles and zeroes are the same in Figures 6-3 and 6-8. The next two equations are the state vector and the state equation, while Table 6-3 shows the plant eigenvalues.

$$\begin{align*}
\mathbf{x}^T &= \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{bmatrix}^T \\
&= \begin{bmatrix} Y_d & V_d & A_t & A_m & q & \delta_e & \dot{\delta}_e & \delta_b \end{bmatrix}^T
\end{align*} \tag{6.60}$$
Figure 6-7 - Gain $c_6$ from Bias to Tail-Angle Input, and Gain $c_8$ from Bias to Tail-Command Input (for System with Actuator), versus Time to Go
Figure 6-8 - Block Diagram for Full System Model with Differential Geometry, Evasive Target Acceleration, Correct Pitch Airframe, Actuator Servo and Bias on Tail-Angle Command
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5 \\
\dot{x}_6 \\
\dot{x}_7 \\
\dot{x}_8 \\
\end{bmatrix} = 
\begin{bmatrix}
-0.001 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.0485 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.25 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1.304 & 2910 & 0 & 302 & 0 \\
0 & 0 & 0 & 0.0482 & -1.016 & -119.6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -10^4 & -100 \ F_{78} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.01 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
x_8 \\
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} u - 355
\]

Note: In most of the work, \( F_{78} \) was \( 10^4 \) sec\(^{-1} \)
### Table 6-3
Eigenvalues of the Eight-State Plant and Their Physical Interpretation

<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 ) = -0.001 rad/sec</td>
<td>( Y_d )</td>
</tr>
<tr>
<td>( \lambda_2 ) = -0.0485 rad/sec</td>
<td>( V_d )</td>
</tr>
<tr>
<td>( \lambda_3 ) = -0.25 rad/sec</td>
<td>( A_t ) (target acceleration)</td>
</tr>
<tr>
<td>( \lambda_4 ) = +10.68 rad/sec</td>
<td>Unstable mode of pitch airframe</td>
</tr>
<tr>
<td>( \lambda_5 ) = -13.004 rad/sec</td>
<td>Stable mode of pitch airframe</td>
</tr>
<tr>
<td>( \lambda_6 ) = -0.01 rad/sec</td>
<td>Bias integrator</td>
</tr>
<tr>
<td>( \lambda_7 ) = -50. + j 86.6 rad/sec</td>
<td>(Fin servo with natural frequency of 100 rad/sec and damping ratio = 0.5)</td>
</tr>
<tr>
<td>( \lambda_8 ) = -50. - j 86.6 rad/sec</td>
<td></td>
</tr>
</tbody>
</table>

Figure 6-9 shows the apparent navigation ratio \( N_a \) for this plant as computed by SGARAF, together with \( N_a \) for the previous plant (Figure 6-3), which has the same steady-state gain but no fin-servo lags. The polarity reversal in Figure 6-9 is again due to the right-half-plane airframe zero (Subsection B-2.0). For a time-to-go greater than 0.1 second, the two plots of \( N_a \) are reasonably close, which is to be expected, since the 6 lower-frequency eigenvalues of the 8-state plant are the eigenvalues of the 6-state plant. At a shorter time-to-go, the plant with the fin servo has appreciably higher \( N_a \) and related gains, due to the need for compensating the added lag of the fin servo. The left-hand dashed portion of the curve for this plant is to be read on the farthest-left scale with a maximum ordinate of 0.2. As before, the final state weighting matrix \( S_f \) weighted only \( Y_d \) (\( S_{f11} = 1 \)), and the control weighting was \( b = 1 \times 10^{-8} \text{ sec}^{-3} \).
Figure 6-9 - Apparent Navigation Ratio $N_a$ versus Time-to-Go for Unstable Bare Airframe at Mach 2, Sea Level
The gain $c_8$ for the bias, as shown in Figure 6-7, is close to that of the plant without the fin servo.

Figures 6-10 and 6-11 are respectively the gains from acceleration and pitch rate back to the tail-angle command $u$. At times to go greater than 0.1 second, they are similar to Figures 6-5 and 6-6, which are corresponding gains for the five-state system with rudimentary bare airframe, but Figures 6-10 and 6-11 have larger gains at short times to go. The same explanation as for the comparative behavior of $N_a$ applies here.

These computations for the eight-state plant in Figures 6-7 and 6-9 through 6-11 were originally made on March 13, 1970 with an IBM 360 computer via the "SBC Call/360" service; "Basic Long Form" with approximately 15 decimal digits in each word was used. The element $F_{78}$ was $-10^4$ sec$^{-1}$ by mistake instead of $10^4$, but this merely caused a polarity reversal of $c_8$ which was corrected in a later computation and in Figure 6-7; state gains but not modal gains were computed. At an inverse time $\tau$ (i.e., time to go) of 1.2 seconds, the $S$ matrix developed appreciable asymmetry; later computations failed the test on the ratio $c_2/c_1$ and are not thought to be reliable. A run on the same computer on March 27, 1970 utilized the correct $F_{78}$ and also computed the modal control gains. In this case the noticeable difficulties began at $\tau = 1.1$ seconds. At $\tau = 1.2$ sec the modal control gain $c_{z4}$ for the unstable mode was beginning to diverge from the correct value in Subsection 6.5.3. The other 7 modal gains were orders of magnitude less.

For reasons connected with reliability and convenience, later computations were performed with the Tymshare, Inc. service, utilizing the XDS-940 computer with 11 significant digits for each number. Figures 6-7 and 6-9 through 6-11 show computations made on this machine with
Figure 6-10 - Gain from Acceleration $A_m$ to Tail-Angle Command, versus Time to Go, for Unstable Bare Airframe at Mach 2, Sea Level, with Fin Servo and Bias on Tail Command.
Figure 6-11 - Gain from Pitch Rate to Tail-Angle Command versus Time to Go, for Unstable Bare Airframe at Mach 2, Sea Level, with Fin Servo and Bias on Tail Command
program SGIMP, i.e., the version of SGARAF utilizing the improved transition-matrix approach. Computations of $N_a'$ and state gains $c_1$ through $c_3$ did not appear to be reliable past $\tau = 1$ second. On the other hand, the only significant modal gain, $c_{z_4}$, appeared to be correct out to the end of the run at $\tau = 1.7$ seconds. Hence, although this comparison is a bit mixed, (with two different computers), program SGIMP appears to be better than SGARAF, at least for the only significant modal gain at large $\tau$.

It will be shown in the next section that the steady-state values of gains $c_4$, $c_5$ and $c_8$ for acceleration, pitch rate and tail-angle-command bias are due to steady-state feedback of the unstable mode of the bare plant. At a time to go of 1.0 second, the nearly-steady-state gain from the tail angle $\delta_e$ was $c_6 = -0.509$ rad/rad and that from the tail angular rate $\dot{\delta}_e$ was $c_7 = -2.139 \times 10^{-3}$ sec.

6.5 Behavior of Modal Feedback Gains

6.5.1 Eigenvalues, Eigenvectors and Modal Representation

It is quite useful to consider the system and its optimal control gains in terms of the plant modes, which are readily related to the state variables by a transformation that may be considered as a coordinate rotation. The following treatment is given in some detail for the sake of clarity, even though part of it overlaps Subsections 4.3.2.1.2 and 4.3.2.2.1.3.

Let the plant matrix $F$ have $n$ distinct eigenvalues $\lambda_1, \ldots \lambda_i \ldots \lambda_n'$ which are the solution of:

$$\det [F - \lambda I] = 0 \quad (6.62)$$

The equation for $i$-th eigenvector and eigenvalue:
\[ Fv_i = \lambda_i v_i \quad (6.63) \]

may be generalized to an equation involving the matrix \( V_d \) of the \( n \) eigenvectors and the matrix \( \Lambda_d \) of the \( n \) eigenvalues:

\[ FV_d = V_d \Lambda_d \quad (6.64) \]

where:

\[ V_d = \begin{bmatrix} \uparrow \cdots \uparrow \cdots \uparrow \downarrow \cdots \downarrow \cdots \downarrow \end{bmatrix} \begin{bmatrix} v_1 \cdots v_i \cdots v_n \end{bmatrix} \quad (6.65) \]

\[ \Lambda_d = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \quad (6.66) \]

where the subscript \( d \) denotes the diagonal conventional form of \( \Lambda \). For the 8-state plant with a complex conjugate pair of eigenvalues for the fin servo:

\[ \lambda_8 = \overline{\lambda}_7 \quad (6.67) \]

\[ v_8 = \overline{v}_7 \quad (6.68) \]

where the super-bar denotes "complex conjugate of." In this case, the following form (Reference T1) is used, so as to avoid computation with complex numbers:

\[ V_b = \begin{bmatrix} \uparrow \cdots \uparrow \cdots \uparrow \Re (v_7) \Im (v_7) \downarrow \cdots \downarrow \cdots \downarrow \end{bmatrix} \begin{bmatrix} v_1 \cdots v_6 \end{bmatrix} \quad (6.69) \]
\[
A_b = \begin{bmatrix}
\lambda_1 & \cdots & 0 \\
0 & \ddots & \cdots \\
0 & \cdots & \lambda_\sigma \\
\end{bmatrix}
\]

\[(6.70)\]

\[
FV_b = V_b A_b
\]

\[(6.71)\]

where \(b\) denotes the block-diagonal form of \(\Lambda\). For normalization, it is convenient to let the largest element of the eigenvectors in (6.65) be \(1 + j0\).

Reference H2 shows that the \(n\) eigenvectors in (6.65) span \(n\)-space if the \(n\) eigenvalues are distinct. Hence, \(V_d\) should be invertible. The same should be true of \(V_b\) in (6.69). Equation (6.64) or (6.71) can be expressed as:

\[
V^{-1} F V = \Lambda
\]

\[(6.72)\]

From Reference H2, the following is easily shown. The matrix \(F^T\) has the same \(n\) eigenvalues as \(F\); their eigenvectors are different unless \(F\) is symmetric, which is at least uncommon for a plant matrix of a physical system. For an eigenvector of \(F^T\):

\[
F^T q_i = \lambda_i q_i
\]

\[(6.73)\]

it can be shown that:

\[
q_i^T v_j = 0 \quad \text{if} \; i \neq j
\]

\[(6.74)\]

With suitable adjustment of the magnitude of each \(q_i\):

\[
q_i^T v_i = 1
\]

\[(6.75)\]

Defining a matrix \(Q\) as indicated, (6.74) and (6.75) are equivalent to:
\[ QV = \begin{bmatrix}
q_1^T \\
q_2^T \\
\vdots \\
q_n^T
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{bmatrix} = I \quad (6.76) \]

It is then apparent, although not usually stated in texts, that:

\[ Q = V^{-1} \quad (6.77) \]

An alternate formulation of (6.73) is:

\[ q_i^T F = \lambda_i q_i^T \quad (6.78) \]

In Reference W1, which is the most widely cited computational reference in its field, Wilkinson calls \( q \) a "left eigenvector" and \( v \) in (6.63) a "right eigenvector," which are convenient names for the purpose here.

When (6.69) and (6.70) are used in order to avoid complex numbers, it is convenient to define \( Q_b \) as:

\[ Q_b = V_b^{-1} \quad (6.79) \]

Only in the case of a real eigenvalue \( \lambda_i \), the \( i \)-th row of \( Q_b \) is \( q_i^T \). For the conjugate complex pair of eigenvalues \( \lambda_7 \) and \( \lambda_8 \), the corresponding rows of \( Q_b \) may be derived from (6.79) and (6.69) as:

\[ q_{b7}^T = q_7^T + q_8^T \quad (6.80) \]

\[ q_{b8}^T = j (q_7^T - q_8^T) \quad (6.81) \]
Let the plant equation have constant $F$ and $G$ matrices:

$$\dot{\mathbf{x}} = F\mathbf{x} + G\mathbf{u} \quad (6.82)$$

Now define a modal vector $\mathbf{z}$ which is linearly related to the state vector $\mathbf{x}$ by the simple transformation:

$$\mathbf{x} = \mathbf{Vz} \quad (6.83)$$

Substitute (6.83) into (6.82) and pre-multiply the result by $\mathbf{V}^{-1}$, thus obtaining:

$$\dot{\mathbf{z}} = \mathbf{V}^{-1}F\mathbf{Vz} + \mathbf{V}^{-1}G\mathbf{u} \quad (6.84)$$

$$= \mathbf{\Lambda z} + \mathbf{QG} \mathbf{u}$$

If $\mathbf{V}$ and $\mathbf{\Lambda}$ are of the form in (6.65) and (6.66), then (6.84) shows that each mode $\mathbf{z}_i$ has feedback only around itself, with no feedback to or from other modes. This statement is only slightly modified if (6.69) and (6.70) are used, i.e., the two modes of a conjugate complex pair of eigenvalues (e.g., the actuator eigenvalues) are intercoupled; in this case (6.84) would be replaced by:

$$\dot{\mathbf{z}} = \mathbf{\Lambda_b z} + \mathbf{Q_b G} \mathbf{u} \quad (6.85)$$
The block-diagonal eigenvalue matrix for the eight-state plant is shown in (6.85), which may be compared with the conventional eigenvalues in Table 6-3.

\[
\Lambda_b = \begin{bmatrix}
-0.001 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.0485 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.25 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 10.684 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -13.004 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.01 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -50. & 86.603 \\
0 & 0 & 0 & 0 & 0 & 0 & -86.603 & -50.
\end{bmatrix}
\]
The eigenvector matrix $V_b$ corresponding to the block-diagonal $\Lambda_b$ is given by (6.87) below, with elements rounded off to two significant figures, although actual computations with $V_b$ were much more accurate. See also (6.69) and (6.70).

$$V_b =$$

$$
\begin{bmatrix}
1 & 1 & 1 & -8.7 \times 10^{-3} & -5.9 \times 10^{-3} & 1 & 5.0 \times 10^{-5} & -8.7 \times 10^{-5} \\
0 & -0.0475 & -0.249 & -0.093 & +0.077 & 9.0 \times 10^{-3} & 5.0 \times 10^{-3} & 8.7 \times 10^{-5} \\
0 & 0 & 0.050 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 3.4 \times 10^{-4} & 1 & 0 \\
0 & 0 & 0 & 4.1 \times 10^{-3} & -4.0 \times 10^{-3} & 1.5 \times 10^{-7} & 1.4 \times 10^{-3} & 3.0 \times 10^{-3} \\
0 & 0 & 0 & 0 & 0 & 1.4 \times 10^{-7} & 3.1 \times 10^{-3} & 2.2 \times 10^{-4} \\
0 & 0 & 0 & 0 & 0 & -1.4 \times 10^{-9} & -0.17 & 0.26 \\
0 & 0 & 0 & 0 & 0 & 1.4 \times 10^{-7} & 0 & 0
\end{bmatrix}
$$

(6.87)
With the use of $V_b$, (6.83) may be expressed as in (6.88) below. Reference to (6.87) helps to show the modal content of each state variable. For instance, only the bias mode $z_6$ is contained in $x_8 = \delta_b$, while all modes are contained in $x_1$, which is surely plausible from Figure 6-8.

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
  x_6 \\
  x_7 \\
  x_8 \\
\end{bmatrix}
= \begin{bmatrix}
  Y_d \\
  V_d \\
  A_t \\
  A_m \\
  q \\
  \delta_e \\
  \delta_e \\
  \delta_b \\
\end{bmatrix}
= \begin{bmatrix}
  z_1 \\
  z_2 \\
  z_3 \\
  z_4 \\
  z_5 \\
  z_6 \\
  z_7 \\
  z_8 \\
\end{bmatrix}
= \begin{bmatrix}
  V_b \\
  V_b \\
  V_b \\
  V_b \\
  V_b \\
  V_b \\
  V_b \\
\end{bmatrix}
\]

- Mode associated with $Y_d$ integrator
- Mode associated with $V_d$ integrator
- Mode associated with target maneuver
- Unstable mode of airframe
- Stable mode of airframe
- Mode associated with bias integrator
- First of two coupled actuator modes
- Second of two coupled actuator modes

(6.88)
\( Q_b \) below is the inverse of \( V_b \) in (6.87).

\[
Q_b = \begin{bmatrix}
-5.9 \times 10^6 & -5.3 & -570. & 440. \\
-1.4 \times 10^6 & 5.3 & 570. & 0 \\
0 & 0 & -1100. & -280. \\
0 & 0 & -1100. & 120. \\
0 & 0 & 7.2 \times 10^6 & 0 \\
0 & 0 & -310. & 3.7 \\
0 & 0 & 210 & 0
\end{bmatrix}
\]

\[
V_b^{-1} = Q_b
\]
Multiplication of (6.88) by \( Q_b = V_b^{-1} \) yields (6.90) below. Reference to (6.89) shows the state-variable content of each mode. For instance, only the target-maneuver acceleration \( x_3 \) is contained in the target-maneuver mode \( z_3 \), while all states contribute to the \( z_1 \) mode. See also Figure 6-8.

\[
\begin{bmatrix}
  z_1 \\
  z_2 \\
  z_3 \\
  z_4 \\
  z_5 \\
  z_6 \\
  z_7 \\
  z_8
\end{bmatrix}
\begin{aligned}
\text{Mode associated with } Y_d \text{ integrator} \\
\text{Mode associated with } V_d \text{ integrator} \\
\text{Mode associated with target maneuver} \\
\text{Unstable mode of airframe} \\
\text{Stable mode of airframe} \\
\text{Mode associated with bias integrator} \\
\text{First of two coupled actuator modes} \\
\text{Second of two coupled actuator modes}
\end{aligned}
= Q_b
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
  x_6 \\
  x_7 \\
  x_8
\end{bmatrix}

= Q_b
\begin{bmatrix}
  Y_d \\
  V_d \\
  A_t \\
  A_m \\
  q \\
  \delta_e \\
  \delta_e \\
  \delta_b
\end{bmatrix}

(6.90)
6.5.2 Theory of Steady-State Optimal Gains for One Unstable Mode

6.5.2.1 Derivation for Vector Control Case

The theory in this subsection was derived by the author for a vector control $u$ with an original approach, after Professor Potter communicated verbally his result for a scalar $u$. For the optimal control problem in Subsections 3.2.1 and 6.2.1, let the performance index be:

$$ J = \frac{1}{2} x^T(t_f) S_f x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} (x^T A x + u^T B u) \, dt $$

(6.91)

Then the optimal control vector is:

$$ u = - B^{-1} G^T S(t) \bar{x}(t) $$

(6.92)

For constant matrices in (6.91), the time-varying matrix $S(t)$ can be computed as indicated in Subsections 6.2 and 6.3 herein. For the interceptor missile problem, it is sensible to let $A = 0$ and to let $B$ be small, because miss distance is the penultimate criterion of success.

References Pl and O'Dl have shown that the steady-state solution of $S[\tau]$ for very large inverse time $\tau$ is given by (6.52):

$$ S[\infty] = V_{w21} (V_{w11})^{-1} $$

(6.93)

where $V_{w21}$ and $V_{w11}$ are $n$ by $n$ submatrices of the eigenvector matrix $V_w$ (see (6.48)) of the Hamiltonian matrix $W_c$. The problem of finding the steady-state optimal gains then reduces to finding these two submatrices.

Let $v_{wi}$ and $\mu_i$ be the $i$-th eigenvector ($2n$ by 1) and $i$-th eigenvalue of $W_c$ in (6.19), and so:
\[ W_c w_i = \begin{bmatrix} -F & \cdots & GB^{-1}\Gamma^T \\ \cdots & \cdots & \cdots \\ 0 & \cdots & F^T \end{bmatrix} v_{wi} = \mu_i v_{wi} \quad (6.94) \]

where \( A = 0 \). By inspection, \( n \) of the eigenvector-eigenvalue pairs are given by:

\[ \begin{bmatrix} -F & \cdots & D \\ \cdots & \cdots & \cdots \\ 0 & \cdots & F^T \end{bmatrix} \begin{bmatrix} v_i \\ \cdots \\ 0 \end{bmatrix} = -\lambda_i \begin{bmatrix} v_i \\ \cdots \\ 0 \end{bmatrix} \quad (6.95) \]

where \( D \) denotes \( GB^{-1}\Gamma^T \) for convenience and where \( v_i \) and \( \lambda_i \) are the \( i \)-th eigenvector and \( i \)-th eigenvalue of \( F \); note that this depends on \( A \) being zero.

Hence \( n \) of the \( \mu_i \) are -1 times the plant eigenvalues and so the other \( n \) must be the plant eigenvalues themselves (Subsection 6.2.1); for these, (6.94) becomes:

\[ \begin{bmatrix} F & \cdots & D \\ \cdots & \cdots & \cdots \\ 0 & \cdots & F^T \end{bmatrix} \begin{bmatrix} r_j \\ \cdots \\ q_j \end{bmatrix} = \lambda_j \begin{bmatrix} r_j \\ \cdots \\ q_j \end{bmatrix} \quad (6.96) \]

where \( v_{wj} \) is partitioned for convenience. Its lower half is seen to be \( q_j \), the \( j \)-th eigenvector of \( F^T \) as in (6.73) or the \( j \)-th left eigenvector of \( F \) as in (6.78). From (6.96), \( r_j \) must obey the equations:

\[ -F r_j + D q_j = \lambda_j r_j \quad (6.97) \]

\[ [F + \lambda_j I] r_j = D q_j \quad (6.98) \]
\[ r_j = (F + \lambda_j I)^{-1} Dq_j \]  
(6.99)

Clearly, \((F - \lambda_j I)\) is singular because \(\lambda_j\) is an eigenvalue of \(F\), but \((F + \lambda_j I)\) will not be singular unless \(-\lambda_j\) is also an eigenvalue of \(F\). Therefore, a restriction on this theory is that there must be no pairs \(\lambda_j, -\lambda_j\) of eigenvalues of \(F\).

A further relationship of interest is found by forming the inner product of (6.98) and \(q_j\):

\[ q_j^T (F + \lambda_j I) r_j = q_j^T Dq_j \]  
(6.100)

\[ 2 \lambda_j q_j^T r_j = q_j^T Dq_j \]  
(6.101)

\[ q_j^T r_j = \frac{q_j^T \Gamma_{\cdots}}{2 \lambda_j} \]  
(6.102)

Now, as in Vaughan's method and as in (6.46) in particular, let the eigenvalue matrix of the Hamiltonian be arranged with RHP eigenvalues in the upper left corner and LHP eigenvalues in the lower right corner:

\[
\Lambda_c = \begin{bmatrix}
\lambda_1 & -\lambda_2 & \cdots & 0 \\
0 & \lambda_2 & \cdots & -\lambda_n \\
0 & 0 & \ddots & \lambda_1 \\
\end{bmatrix} 
\]  
(6.103)

where \(\lambda_1\) is the single RHP eigenvalue of the plant, which is assumed to have only one unstable mode as the title of this subsection indicates. Then the eigenvector matrix is, from (6.95) and (6.96):
\[
V_w = \begin{bmatrix}
V_{w11} & V_{w12} \\
\vdots & \vdots \\
V_{w21} & V_{w22}
\end{bmatrix}
\]

\[= \begin{bmatrix}
\uparrow & \uparrow & \ldots & \uparrow & \uparrow & \ldots & \uparrow \\
\downarrow & \downarrow & \ldots & \downarrow & \downarrow & \ldots & \downarrow \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\uparrow & \uparrow & \ldots & \uparrow & \uparrow & \ldots & \uparrow \\
\downarrow & \downarrow & \ldots & \downarrow & \downarrow & \ldots & \downarrow \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\uparrow & \uparrow & \ldots & \uparrow & \uparrow & \ldots & \uparrow \\
\downarrow & \downarrow & \ldots & \downarrow & \downarrow & \ldots & \downarrow \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}
\]

(6.104)

Then, from (6.93):

\[
S[\infty] = V_{w21} \left[ V_{w11} \right]^{-1} = \begin{bmatrix}
\uparrow & \uparrow & \ldots & \uparrow \\
\downarrow & \downarrow & \ldots & \downarrow \\
\vdots & \vdots & \ddots & \vdots \\
\uparrow & \uparrow & \ldots & \uparrow \\
\downarrow & \downarrow & \ldots & \downarrow \\
\vdots & \vdots & \ddots & \vdots \\
\uparrow & \uparrow & \ldots & \uparrow \\
\downarrow & \downarrow & \ldots & \downarrow \\
\vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}^{-1}
\]

(6.105)

Let \( M \) be the indicated inverse:

\[
\begin{bmatrix}
\uparrow & \uparrow & \ldots & \uparrow \\
\downarrow & \downarrow & \ldots & \downarrow \\
\vdots & \vdots & \ddots & \vdots \\
\uparrow & \uparrow & \ldots & \uparrow \\
\downarrow & \downarrow & \ldots & \downarrow \\
\vdots & \vdots & \ddots & \vdots \\
\uparrow & \uparrow & \ldots & \uparrow \\
\downarrow & \downarrow & \ldots & \downarrow \\
\vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}^{-1} = M = \left[ \begin{bmatrix}
m_1^T \\
m_2^T \\
\vdots \\
m_n^T \\
\end{bmatrix}
\right]
\]

(6.106)

from which:

\[
\left[ \begin{bmatrix}
m_1^T \\
m_2^T \\
\vdots \\
m_n^T \\
\end{bmatrix}
\right] \left[ \begin{bmatrix}
\uparrow & \uparrow & \ldots & \uparrow \\
\downarrow & \downarrow & \ldots & \downarrow \\
\vdots & \vdots & \ddots & \vdots \\
\uparrow & \uparrow & \ldots & \uparrow \\
\downarrow & \downarrow & \ldots & \downarrow \\
\vdots & \vdots & \ddots & \vdots \\
\uparrow & \uparrow & \ldots & \uparrow \\
\downarrow & \downarrow & \ldots & \downarrow \\
\vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}
\right] = I
\]

(6.107)
\[ m_1^T r_1 = 1 \]  \hspace{1cm} (6.108)

\[ m_1^T v_i = 0 \quad \text{for} \ 2 \leq i \leq n \]  \hspace{1cm} (6.109)

With the assumption that the eigenvalues \( \lambda \) of \( F \) and \( F^T \) are distinct, the eigenvectors \( q \) of \( F^T \) must span all of \( n \)-space (Reference H2), and so the unknown \( m_1^T \) could be expressed as:

\[ m_1^T = \gamma_1 q_1^T + \gamma_2 q_2^T + \ldots + \gamma_n q_n^T \]  \hspace{1cm} (6.110)

From (6.74), (6.75) and (6.109):

\[ m_1^T v_i = 0 \quad \Rightarrow \quad \gamma_i \quad 2 \leq i \leq n \]  \hspace{1cm} (6.111)

which leaves:

\[ m_1^T = \gamma_1 q_1^T \]  \hspace{1cm} (6.112)

Taking the inner product of this with \( r_1 \) and invoking (6.108) and (6.102):

\[ 1 = m_1^T r_1 = \gamma_1 q_1^T r_1 = \gamma_1 \frac{q_1^T D q_1}{2 \lambda_1} \]  \hspace{1cm} (6.113)

Then (6.112) becomes:

\[ m_1^T = \begin{bmatrix} \frac{2 \lambda_1}{q_1^T D q_1} \end{bmatrix} q_1^T \]  \hspace{1cm} (6.114)

Substitution of (6.106) and (6.114) into (6.105) gives:
\[ S[\infty] = \begin{bmatrix} 0 & 0 \\ m_1^T & \begin{bmatrix} 2\lambda_1 \\ q_1^T D q_1 \end{bmatrix} \end{bmatrix} q_1 q_1^T \]

which is a scalar (in brackets) times the symmetric matrix \( q_1 q_1^T \).

Then, the near-steady-state optimal control for a long time to go, i.e., for forward time \( t << t_f \), is found by substituting (6.115) into (6.92):

\[ u(t) = -B^{-1}G^T \begin{bmatrix} 2\lambda_1 \\ q_1^T D q_1 \end{bmatrix} q_1 q_1^T \begin{bmatrix} x(t) \end{bmatrix} \]  

(6.116)

### 6.5.2.2 Interpretation

Let \( x(t) \) be expressed in terms of the modes, as in (6.83):

\[ x = Vz = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \]

(6.117)

Substituting (6.117) into (6.116) and invoking (6.74) and (6.75) gives:

\[ u(t) = -B^{-1}G^T \begin{bmatrix} 2\lambda_1 \\ q_1^T D q_1 \end{bmatrix} q_1 \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \]  

(6.118)

which shows that only the unstable mode \( z_1 \) is fed back at a long time to
go. This is reasonable because the remaining modes, which are stable and will die away without contributing much to the first term in (6.91), do not have to be fed back at a long time to go. Hence, feedback of only the unstable mode at a long time to go conserves the second term in (6.91).

Reverting to state space, let (6.116) be substituted into (6.1):

\[
\dot{x} = \left[ F - D \begin{bmatrix} 2\lambda_1 & 0 \\ \frac{2\lambda_1}{q_1^T D q_1} & 0 \end{bmatrix} q_1 q_1^T \right] x
\]

\[
= F_2 x
\]

where \( D \) is substituted for \( GB^{-1}G^T \) as in (6.95) and \( F_2 \) denotes the new plant matrix which effectively results with the steady-state feedback in (6.116). Multiplication of \( F_2 \) by the left eigenvector of the original unstable mode gives:

\[
q_1^T F_2 = q_1^T F - \frac{q_1^T D q_1}{q_1^T D q_1} (2\lambda_1) q_1^T
\]

\[
= \lambda_1 q_1^T - 2\lambda_1 q_1^T
\]

\[
= -\lambda_1 q_1^T
\]

Therefore, the left eigenvector \( q_1 \) of the unstable mode of the original plant is also a left eigenvector of the new plant with steady-state feedback, but the original right-half-plane \( \lambda_1 \) has been replaced by a symmetrically opposite eigenvalue \(-\lambda_1\) in the left half plane. Now consider any other (stable) mode of the original plant and multiply \( F_2 \) from (6.119) by \( v_i \) for that mode:
\[ F_2 \mathbf{v}_i = F \mathbf{v}_i - 0 = \lambda_i \mathbf{v}_i \quad (6.122) \]

which follows from (6.63) and (6.74). Hence the eigenvalues and right eigenvectors of the stable modes are unchanged.

This behavior of the eigenvalues may be understood from the following:

a) The original plant had one RHP eigenvalue, denoted for convenience as \( \lambda_1 \).

b) From References Pl and O'D1, it is known that the eigenvalues of the Hamiltonian matrix \( W_c \) in (6.19) are symmetrically arrayed around the imaginary axis, and that the eigenvalues of \( F_2 = F + GC \) for the steady-state optimal control gains must be the LHP eigenvalues of \( W_c \).

c) It was shown in (6.95) that \( n \) eigenvalues of \( W_c \) were \(-1\) times the eigenvalues of \( F \), because \( A = 0 \), and so the other \( n \) eigenvalues of \( W_c \) must be the eigenvalues of \( F \).

d) Hence, the LHP eigenvalues of \( W_c \) must be \(-\lambda_1\) and the original plant eigenvalues \( \lambda_2 \) through \( \lambda_n \), which must also be the closed-loop eigenvalues of the plant \( F_2 \).

This argument should hold for any \( m \leq n \) RHP plant eigenvalues.

6.5.2.3 Case of Scalar Control

For the case of a scalar \( u \), the matrix \( B \) in (6.91) is replaced by the scalar \( b \). Then (6.116) for the control in the near-steady-state becomes:

\[ u(t) = -b^{-1} (2\lambda_1) \frac{\mathbf{g}^T \mathbf{q}_1}{b^{-1} \mathbf{q}_1^T \mathbf{g} \mathbf{q}_1} \mathbf{q}_1^T \mathbf{x}(t) \]
\[ -379- \]

\[ \begin{align*}
&= - \frac{2\lambda_1}{q_1 \mathbf{K}} q_1 \mathbf{T} \mathbf{x}(t)
\end{align*} \tag{6.123} \]

which is essentially the result communicated by Professor Potter.

In another context, suppose that the modal-control methods of Subsection 4.3.2.2.1.2 were used to move the unstable eigenvalue \( \lambda_1 \) to \(-\lambda_1\). Equation (4.42) in that subsection would give the required feedback gain for that mode, and is clearly equivalent to (6.123).

Equation (6.123) may be applied to the unstable mode of the eight-state plant in Subsection 6.4.4. The different mode numbers are simply for convenience in their respective subsections.

### 6.5.2.4 Possibilities for Further Research

For the case of more than one unstable mode in the plant, the reasoning of Subsection 6.5.2.2 will show that the unstable eigenvalues, which may be labelled \( \lambda_1 \) through \( \lambda_m \), will be replaced by eigenvalues \(-\lambda_1\) through \(-\lambda_m\) in the steady-state closed-loop plant matrix \( F_2(t) \) for \( t \ll t_f \).

For the scalar control case, the Crossley-Porter formula (4.73) must apply; moreover, there is interaction between the gains of the unstable modes. Feedback would not be required from the stable modes.

An attempt was made to extend the research of Subsection 6.5.2.1 to the case of vector control and \( m \) unstable modes, which represents a possibility for more research. Although some progress was made, the algebra proved to be difficult, which is not surprising, considering the difficult algebra in Reference C2 for the placement of poles in the scalar-control case.

### 6.5.3 Steady-State Modal Feedback from Plant with Eight States

The theory of Subsections 6.5.2.1 through 6.5.2.3 was applied to the
eight-state plant and then checked successfully against quasi-steady-state gains computed by programs SGARAF and SGIMP.

Eigenvalues and eigenvectors of the eight-state plant (Figure 6-8 and Equation (6.61)) were computed (Table 6-3 and equation (6.87)) by a program (Reference F1) on the CDC 6600 computer to 5 significant figures, and then inputted to a special-purpose time-sharing program, SSGOUM. (The printout subroutine of the program in Reference F1 was later altered to give answers to 14 significant figures, although there is no guarantee that accuracy improved correspondingly.) SSGOUM obtained $Q_{b}$ by inverting $V_{b}$ as shown in (6.79). Note that the left eigenvector, $\varphi_{4}$, of the unstable mode is the transpose of the fourth row in (6.89). The theoretical steady-state feedback gains from the 8 states were then computed from (6.123). These gains are compared in Table 6-4 with the quasi-steady-state gains computed by SGARAF (March 27, 1970) at a time-to-go of 1 second.

Table 6-4
State Feedback Gains from Eight-State Plant

<table>
<thead>
<tr>
<th>State</th>
<th>Quasi-Steady-State (SGARAF)</th>
<th>Theoretical Steady State (SSGOUM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 = Y_d$</td>
<td>$-4.794 \times 10^{-6}$ rad/ft</td>
<td>$4.605 \times 10^{-32}$ (analytically zero)</td>
</tr>
<tr>
<td>$x_2 = V_d$</td>
<td>$-4.684 \times 10^{-6}$ rad/sec/ft</td>
<td>$-1.215 \times 10^{-18}$ (analytically zero)</td>
</tr>
<tr>
<td>$x_3 = A_t$</td>
<td>$-2.176 \times 10^{-6}$ rad/sec^2/ft</td>
<td>$-1.296 \times 10^{-17}$ (analytically zero)</td>
</tr>
<tr>
<td>$x_4 = A_m$</td>
<td>$9.023 \times 10^{-4}$ rad/sec^2/ft</td>
<td>$9.010 \times 10^{-4}$</td>
</tr>
<tr>
<td>$x_5 = q$</td>
<td>$2.243 \times 10^{-1}$ rad/sec/rad</td>
<td>$2.241 \times 10^{-1}$</td>
</tr>
<tr>
<td>$x_6 = \delta_e$</td>
<td>$-5.092 \times 10^{-1}$ rad/rad</td>
<td>$-5.086 \times 10^{-1}$</td>
</tr>
<tr>
<td>$x_7 = \dot{\delta}_e$</td>
<td>$-2.139 \times 10^{-3}$ rad/sec/rad</td>
<td>$-2.137 \times 10^{-3}$</td>
</tr>
<tr>
<td>$x_8 = \delta_b$</td>
<td>$-1.995$ rad/rad</td>
<td>$-1.998$ rad/rad</td>
</tr>
</tbody>
</table>
The fact that steady-state gains from $x_1$ through $x_3$ should be analytically zero is inferred from the theoretically zero elements of $q_4$ (i.e., elements $Q_{b41}$, $Q_{b42}$, and $Q_{b43}$ in (6.89) and from the fact that only mode 4 is fed back.

The corresponding SSGOUM gains are close enough to zero within accuracy of input data and computation, while the larger SGARAF gains are due in part to the transient nature of these gains (associated with the slow modes $\lambda_1$, $\lambda_2$ and $\lambda_3$) and the time-to-go of only 1 second; incipient numerical difficulties did not permit use of a larger time to go. On the other hand, agreement is excellent between the two sets of gains for $x_4$ through $x_8$.

Modal feedback gains were found by post-multiplying the state feedback gains by $V$, which is equivalent in the SSGOUM case to substituting (6.83) into (6.123). Table 6-5 shows the modal feedback gains as computed by SGARAF (at 1 second to go) and by SSGOUM.

Table 6-5

<table>
<thead>
<tr>
<th>Mode</th>
<th>Quasi-Steady State (SGARAF)</th>
<th>Theoretical Steady State (SSGOUM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_1$</td>
<td>$-4.794 \times 10^{-6}$</td>
<td>$2.047 \times 10^{-32}$ (analytically zero)</td>
</tr>
<tr>
<td>$z_2$</td>
<td>$-4.571 \times 10^{-6}$</td>
<td>$5.772 \times 10^{-20}$ (analytically zero)</td>
</tr>
<tr>
<td>$z_3$</td>
<td>$-3.737 \times 10^{-6}$</td>
<td>$-3.478 \times 10^{-19}$ (analytically zero)</td>
</tr>
<tr>
<td>$z_4$</td>
<td>$1.827 \times 10^{-3}$</td>
<td>$1.824 \times 10^{-3}$ unstable mode</td>
</tr>
<tr>
<td>$z_5$</td>
<td>$-1.046 \times 10^{-8}$</td>
<td>$6.458 \times 10^{-20}$ (analytically zero)</td>
</tr>
<tr>
<td>$z_6$</td>
<td>$-4.751 \times 10^{-6}$</td>
<td>$-6.021 \times 10^{-20}$ (analytically zero)</td>
</tr>
<tr>
<td>$z_7$</td>
<td>$-1.308 \times 10^{-5}$</td>
<td>$2.168 \times 10^{-19}$ (analytically zero)</td>
</tr>
<tr>
<td>$z_8$</td>
<td>$5.614 \times 10^{-9}$</td>
<td>$0$ (analytically zero)</td>
</tr>
</tbody>
</table>
Agreement is excellent for the two computations of gain for mode 4, which should theoretically have the only feedback gain in the steady state. The SSGOUM gains for the other modes are numerically essentially zero, while the larger SGARAF gains reflect the limited time to go. At any rate, the SGARAF gain for mode 4 has a magnitude more than 100 times larger than that of the next largest gain.

By now, it is quite apparent that feedback exists from only the unstable mode in the steady state, and that it accounts for the gain of \(-1.998\) rad/rad from the \(\mathbf{\delta}_b\) bias state.

As indicated in Subsection 6.5.2.2, the left eigenvector \(q_4\) of the unstable mode in the original unstable plant \(F\) is also the new left eigenvector of the new stable plant \(F_2\) with steady-state optimal feedback. It is readily shown from (6.119) that the new right eigenvector \(v_4'\) for the newly stable eigenvalue \(-\lambda_4\) is:

\[
v_4' = (F + \lambda_4 I)^{-1} D \left[ \begin{array}{c} 2\lambda_4 \\ q_4^T D q_4 \\ \end{array} \right] q_4
\]  

(6.124)

This was computed by SSGOUM as:

\[
v_4' = \begin{bmatrix} -0.11 \\ 1.2 \\ 0 \\ 13. \\ -0.039 \\ 2.0 \times 10^{-3} \\ -0.022 \\ -3.1 \times 10^{-19} \end{bmatrix}
\]

(6.125)
This new right eigenvector (rounded off here to two significant figures) may be compared with \( v_4 \), which is the fourth column of \( V_b \) in (6.87). As a check, the following vector product was computed:

\[
(a'_4)^T v'_4 = 1.000056522, \tag{6.126}
\]

which is quite close to the theoretical value of unity.

As a final check, the matrix difference \( F_2 V_{b2} - V_{b2} \Lambda_{b2} \) was computed for the plant \( F_2 \) with steady-state optimal feedback. It was quite close to the null matrix, as would be expected from (6.71).

These checks, and the good agreement between the results of SSGOUM and SGARAF, appear to validate this theory of steady-state optimal feedback gains for the case of one unstable mode.

### 6.5.4 Partial Explanation of Dynamic Behavior of Modal Feedback Gains

Figure 6-12 (the points marked by squares) shows the variation of feedback gain \( C_{z_4} \) from the unstable mode versus time to go. In order to accommodate the wide dynamic range, semilogarithmic paper with a dashed line for negative values of \( C_{z_4} \) has been used. The steady-state value of \( C_{z_4} \) is reached by 0.7 second.

By contrast, the actuator modal gains settle much more quickly, as indicated by Table 6-6.

<table>
<thead>
<tr>
<th>Time to Go, sec</th>
<th>Gain from Mode ( z_7 )</th>
<th>Gain from Mode ( z_8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>( 1.02 \times 10^{-1} )</td>
<td>(-1.87 \times 10^{-2} )</td>
</tr>
<tr>
<td>0.02</td>
<td>( 3.88 \times 10^{-3} )</td>
<td>( 3.14 \times 10^{-3} )</td>
</tr>
<tr>
<td>0.03</td>
<td>( 4.07 \times 10^{-5} )</td>
<td>( 7.64 \times 10^{-5} )</td>
</tr>
<tr>
<td>0.04</td>
<td>(-1.17 \times 10^{-4} )</td>
<td>( 1.25 \times 10^{-5} )</td>
</tr>
<tr>
<td>0.05</td>
<td>(-3.40 \times 10^{-5} )</td>
<td>(-8.54 \times 10^{-6} )</td>
</tr>
</tbody>
</table>
Figure 6-12 - Feedback Gain $c_{z4}$ from Unstable Mode versus Time-to-Go, for Unstable Bare Airframe at Mach 2, Sea Level, with Fin Servo and Bias on Tail Command.
Figure 6-13 shows the behavior of modal feedback gain $c_{z_1}$ from 0.01 to 0.08 second, which happens to coincide in this interval with the modal gains $c_{z_2}$, $c_{z_3}$, and $c_{z_6}$ and with state feedback gain $c_1$. The first of two reasons for this coincidence is that numerically the state gain $c_1$ is larger than $c_2$ or $c_3$ in this interval, which may be partly explained by the fact that $c_2 \approx \frac{t}{g} c_1$ as in (6.44). The second reason is that the $j$-th modal gain is given by:

$$c_{z_j} = \begin{bmatrix} c_1, c_2 \ldots c_n \end{bmatrix} \frac{v_j}{v_j}, \quad (6.127)$$

as indicated by the development of (6.118), where $v_j$ is the $j$-th eigenvector. Now, from (6.87), the first element of $v_1$, $v_2$, $v_3$ and $v_6$ is 1, which is much larger than the next largest element of the eigenvector in each case. It is then apparent from (6.127) that these four modal gains should be about $c_1$. Note that Figure 6-9 has a plot of $N_a = c_1 t^2$ over a longer time interval.

Figure 6-14 shows the modal gain $c_{z_5}$ for the stable airframe mode, which appears to settle faster than the gain $c_{z_4}$ in Figure 6-13. It is interesting to note that the modal gain $c_{z_4}$ (for a RHP eigenvalue) settles most slowly, while the gains in Table 6-6 (for the eigenvalues with most negative real parts) settle most quickly. This leads to the following conjecture:

**Conjecture:** The settling time of modal feedback gains for stable modes is shorter for eigenvalues with large negative real parts than for eigenvalues with algebraically larger real parts, unless the modal gains are closely coupled by similar eigenvectors as in the case of $c_{z_1}$, $c_{z_2}$, $c_{z_3}$ and $c_{z_6}$. 
Figure 6-13 - Modal Gains $c_{z1}$, $c_{z2}$, $c_{z3}$, $c_{z6}$ and State Gain $c_1$ versus Time to Go, for Unstable Bare Airframe at Mach 2, Sea Level, with Fin Servo and Bias on Tail Command
Figure 6-14 - Modal Gain $c_{Z5}$ for Stable Airframe Mode versus Time to Go for Unstable Bare Airframe at Mach 2, Sea Level with Fin Servo and Bias on Tail Command
The following plausibility argument (not a proof) is set forth in partial support of this conjecture. Consider a plant with all its eigenvalues in the left half plane, for simplicity. If $x = Vz$ is substituted into (6.91), it becomes obvious that the final weighting matrix for modes is:

$$S_{fz} = V^T S_V V_f$$  \hspace{1cm} (6.128)

As a definition consistent with this:

$$S_z(t) = V^T S(t) V$$  \hspace{1cm} (6.129)

Substitution of (6.92) and (6.83) into (6.84) leads to:

$$\dot{z} = \Lambda z - V^{-1} G B^{-1} G^T S(t) V z$$

$$= \Lambda z - (V^{-1} G B^{-1} G^T V V^T) S_z(t) z$$  \hspace{1cm} (6.130)

Let only the purely diagonal form of $\Lambda$ be used.

For convenience of notation, define as before:

$$D = GB^{-1} G^T$$  \hspace{1cm} (6.131)

Correspondingly, define the modal form of $D$ as:

$$D_z = V^{-1} G B^{-1} G^T V V^T$$  \hspace{1cm} (6.132)

The equation for $S_z(t)$ and the modal feedback gains may be solved for $t \ll t_f$ by the method in Reference V1. The $2n$ by $2n$ Hamiltonian matrix for the plant modes is:

$$W_z = \begin{bmatrix} -\Lambda & D_z \\ - & - \\ 0 & \Lambda \end{bmatrix}$$  \hspace{1cm} (6.133)
where $\Lambda$ is the plant eigenvalue matrix, which is strictly diagonal. Now, by inspection, $W_z$ has the eigenvalues of $-\Lambda$ (all RHP) and of $\Lambda$ (all LHP). The matrix of eigenvalues for the $W_z$ matrix satisfies:

\[
\begin{bmatrix}
-\Lambda & D_z \\
0 & \Lambda
\end{bmatrix}
\begin{bmatrix}
I_n & r_1 & \cdots & r_n \\
0 & I_n
\end{bmatrix}
= 
\begin{bmatrix}
I_n & r_1 & \cdots & r_n \\
0 & I_n
\end{bmatrix}
\begin{bmatrix}
-\Lambda & 0_n \\
0_n & \Lambda
\end{bmatrix}
\]

(6.134)

where:

\[
-\Lambda r_{1} + D_z \begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix} = r_{1} \lambda_{1}
\]

(6.135)

For the modal form of the plant equation as in (6.130), denote $V_{wz}$ as the 2n by 2n eigenvector matrix corresponding to (6.48) and define $R_z$ and $G_z[\tau]$ for inverse time $\tau$ to correspond to (6.49) and (6.50):

\[
R_z = -\left[ V_{wz22} - S_{fz} V_{wz12} \right]^{-1} \left[ V_{wz21} - S_{fz} V_{wz11} \right]
\]

(6.136)

\[
G_z[\tau] = e^{\Lambda \tau} R_z e^{\Lambda \tau}
\]

(6.137)

Then the modal form of the S matrix is:

\[
S_z[\tau] = \left[ V_{wz21} + V_{wz22} G_z[\tau] \right] \left[ V_{wz11} + V_{wz12} G_z[\tau] \right]^{-1}
\]

(6.138)

All elements of $G_z[\tau]$ must decay with increasing time-to-go $\tau$, because all the eigenvalues of $\Lambda$ are in the left half plane, by assumption. At
some time the matrix to be inverted in (6.138) is virtually $I_n$; note that
$V_{wz}$ appears twice in (6.134). Then $S_z[\tau]$ behaves as $G_z[\tau]$. The left
and right exponentials respectively in (6.137) show that the rows and col-
umns of $G_z[\tau]$ for a high-frequency mode decay very quickly. Therefore,
so should those of $S_z[\tau]$ (unless the matrix to be inverted in (6.138) causes
some anomaly, which is an admitted weakness of this plausibility argu-
ment). Assuming from this argument that the rows and columns for the
high-frequency modes of $S_z[\tau]$ do indeed decay quickly, then it is apparent
from (6.130) that the high-frequency modal gains decay more quickly than
the low-frequency modal gains.
CHAPTER 7

OPTIMAL CONTROL OF SELECTED MODES

7.1 Introduction

Like Chapter 6, this chapter is somewhat outside the main stream of the potentially useful work in guidance of interceptor missiles. Nevertheless, this chapter may be of intellectual interest and it might conceivably have some future application.

The general continuous terminal control problem in Subsections 3.2 and 6.2.1 generally has feedback from all the state variables of the state vector \( \mathbf{x} \) as in (6.15). An attempt to build such a system, at least in the case of a plant such as that of Figure 6-8, would surely be fraught with difficulty. The two-state representation of the fin servo is just an approximation that neglects higher-frequency poles such as those of the servo-valve, the hydraulic-mechanical resonance of the actuator and fin, etc. An attempt to feed back from the actuator modes would probably destabilize the real actuator, which is usually designed for highest bandwidth consistent with stability. Moreover, feedback from the high-frequency modes of the plant probably would not affect the mean-square miss distance much anyway. It should also be recognized that other high-frequency lags (with corresponding state variables) have been neglected in Figure 6-8, e.g., the lags of the accelerometer, rate gyro, first and second body-bending modes, etc.

The question then arises as to how to simplify the optimal control solution to a near-optimal solution which will not destabilize the real
plant and will not be unacceptably complicated. At present, this appears to be an art rather than a science.

It is possible to approach this goal by modifying the performance index so as to weight only specified modes of the bare plant. The remaining modes, typically the high-frequency modes such as those of the fin servo, are unweighted and uncontrolled. Therefore, their stability will not be affected. Moreover, the high-frequency nature of these modes makes it likely that lack of feedback control for them will not change the important terms in the original performance index, e.g., mean-square miss distance.

Potentially, simplification might result from such a scheme, because fewer time-varying feedback gains would be required. Admittedly, modal feedback typically requires the equivalent of knowledge of each state, from measurements or an estimate. Conceivably a reduction scheme as in Subsection 4.3.2.4.3.2 might be applied.

This chapter appears to represent a theoretical contribution, because previous modal-control schemes, summarized in Subsection 4.3.2.2.1.1, have been concerned with constant gains so as to place poles, not with optimal control to minimize a performance index. Curiously, this is the only instance in the writer's experience where insight has been obtained from the Riccati equation.

7.2 Theoretical Development

7.2.1 Conventional Riccati Equation in State Space

Consideration of the Riccati equation in optimal control has previously been suppressed in this thesis because it was not found to be useful for closed-form solutions or for practical computation (Subsection 6.3).
Nevertheless, the Riccati equation represents an analytic alternative to the two-point boundary-value approach to solving the optimal regulator problem, (Subsections 3.2 and 6.2.1), as shown in References B1 and A4 (p. 761). Although there may be slightly easier ways of deriving the Riccati equation, it will be derived from (6.15) through (6.17) for the sake of unifying this and the previous chapter.

To motivate the development, note that the optimal control vector in (6.15) may be written directly in terms of the $S(t)$ matrix as:

$$ u(t) = -B^{-1}G^T S(t)x(t) $$  \hspace{2cm} (7.1)

where the variables are expressed in terms of forward time $t$. From (6.4) and (6.5), it is apparent that the final value of $S(t)$ is:

$$ S(t_f) = S_f $$  \hspace{2cm} (7.2)

A differential equation for $S(t)$, with (7.2) as a boundary condition, may be derived from (6.18), which is expressed in inverse time $\tau$. Using the partitioned form of $W_c$ in (6.17), with $D = GB^{-1}G^T$ for convenience, perform the indicated multiplication to obtain:

$$ \frac{dY[\tau]}{d\tau} = -FY[\tau] + DZ[\tau] $$  \hspace{2cm} (7.3)

$$ \frac{dZ[\tau]}{d\tau} = AY[\tau] + F^TZ[\tau] $$  \hspace{2cm} (7.4)

From the definition of $S$ in (6.16):

$$ S[\tau] = Z[\tau] \left[ Y[\tau] \right]^{-1} $$  \hspace{2cm} (7.5)
its derivative with respect to inverse time is:

\[
\frac{dS}{d\tau} = Y^{-1} - ZY^{-1} \left[ \frac{dY}{d\tau} \right] Y^{-1}
\]  \hspace{1cm} (7.6)

where the functional notation has been dropped for simplicity. The first and second terms of (7.6) may be found from (7.4) and (7.3) respectively. Postmultiply (7.4) by \(Y^{-1}\):

\[
\frac{dZ}{d\tau} Y^{-1} = A + F^T Z Y^{-1}
\]  \hspace{1cm} (7.7)

Postmultiply (7.3) by \(Y^{-1}\) and premultiply it by \(ZY^{-1}\):

\[
ZY^{-1} \frac{dY}{d\tau} Y^{-1} = -ZY^{-1} F + ZY^{-1} D Y^{-1}
\]  \hspace{1cm} (7.8)

Subtract (7.8) from (7.7) to obtain (7.6) and substitute (7.5):

\[
\frac{dS}{d\tau} = A + F^T S + SF - SDS
\]  \hspace{1cm} (7.9)

which is the conventional Riccati equation in inverse time, with boundary condition from (7.2):

\[
S[0] = S_f
\]  \hspace{1cm} (7.10)

Note that various elements of \(S\) will couple into an element of \(dS/d\tau\) through the terms in \(F^T\) and \(D\) in (7.9).

### 7.2.2 Modal Form of the Riccati Equation

Now, proceeding as in Subsection 6.5.4, substitute \(x = Vz\) into (6.91) which becomes:
\[
J = \left( \frac{1}{2} \right) z^T(t_f) V^T S_f V z(t_f) + \left( \frac{1}{2} \right) \int_{t_0}^{t_f} (z^T V^T A z + u^T B u) \, dt
\]

(7.11)

\[
= \left( \frac{1}{2} \right) z^T(t_f) S_{fz1} z(t_f) + \left( \frac{1}{2} \right) \int_{t_0}^{t_f} (z^T A z + u^T B u) \, dt
\]

where:

\[
S_{fz1} = V^T S_f V
\]

(7.12)

\[
A_{z1} = V^T A V
\]

(7.13)

As a definition consistent with (7.12):

\[
S_{z1}(t) = V^T S(t) V
\]

(7.14)

If (7.9) is premultiplied by \(V^T\) and postmultiplied by \(V\), it may be written as:

\[
\frac{dS_{z1}}{dT} = A_{z1} + \Lambda^T S_{z1} + S_{z1} \Lambda - S_{z1} D_z S_{z1}
\]

(7.15)

where:

\[
\Lambda = V^{-1} F V
\]

(7.16)

\[
D_z = V^{-1} D V^{-T}
\]

(7.17)

Now, (7.15) is the modal form of the Riccati equation, with (7.12) as the boundary condition at \(\tau = 0\). The matrix \(\Lambda\) may be either purely diagonal, or at most, block-diagonal in the case of conjugate complex eigen-
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values, depending on which form is chosen. For the moment, let it be purely diagonal.

7.2.3 Modal Control

Suppose that $S_{fz1}$ in (7.12) is replaced by $S_{fz}$:

\[
S_{fz} = \begin{bmatrix}
I_{n-2} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

That is, $S_{fz}$ is formed by making zero the last two rows and columns of $S_{fz1}$, corresponding to two high-frequency modes, such as the actuator modes in Subsection 6.5.2. Let the same thing be done to $A_{z1}$. Let the variable $S_{z1}$ in (7.15) be replaced by $S_z$, which is the solution with $S_{fz}$ as the boundary condition at inverse time 0.

Then, at inverse time 0, each term in (7.15) has zeroes in the last two columns and rows, which is easily verifiable. Therefore, $dS_z/d\tau$ has zeroes in its last two rows and columns and so $S_z[d\tau]$ does also. Therefore, the last two rows and columns of $S_z[\tau]$ remain zero throughout the problem. Since the optimal control vector in (7.1) may be written as:

\[
u(t) = -B^{-1}G^TV^{-T}S_z(t)z(t),
\]

it is apparent that the last two modes ($z_{n-1}$ and $z_n$) in $z(t)$ are never fed back to $u(t)$.

Clearly, this development makes use of the diagonal (or block-diagonal) form of $A$ in (7.15), so that the unwanted modes are not coupled
with the other modes, which are controlled by feedback. This latter group of modes appears to have interdependence in its control gains from the $D_z$ term in (7.15), even if $A$ and $A_z$ are zero.

This recasting of the problem to avoid control of unwanted high-frequency modes may be interpreted in state space as follows. From (7.12) the new final state weighting matrix may be expressed in terms of the new $S_{fz}$ (7.18) as:

$$ S'_f = V^{-T} S_{fz} V^{-1} $$  \hspace{1cm} (7.20)

Now, instead of a matrix $S_f$ which is a null matrix with $S_{f11} = 1$, $S'_f$ has small elements in place of the former zeroes. This means that the optimal control system attempts to minimize a performance index which has a different quadratic term in the final state vector $x(t_f)$, rather than the simple mean-square miss distance. Thus, one should not be surprised if the control gains at short times to go are altered by this revision of $S_f$ and $S_{fz1}$ to $S'_f$ and $S_{fz}$ respectively. Further design (e.g., the estimator) and evaluation of miss distance with disturbances would be required to determine whether this type of control system is acceptable.

7.3 Computational Results for Eight-State Plant, without Control of Actuator Modes

Program "OPGAIN" was designed to compute optimal control gains for the 6 low-frequency modes $z_1$ through $z_6$ (i.e., the non-actuator modes) for the 8-state plant (Subsections 6.4.4 and 6.5). A transition-matrix approach similar to that of SGARAF in Subsection 6.3.2.1 was used, except that computations were in modal space, using only the 6 wanted modes. Therefore, the 8 by 8 $F$ matrix and 16 by 16 $W_c$ matrix in (6.17) were replaced by a 6 by 6 matrix (with $\lambda_1$ through $\lambda_6$ on the diagonal) and the corresponding 12 by 12 $W_c$ matrix. A 6 by 6 $S_z$ matrix and a
l by 6 modal gain matrix were then computed, followed by the corresponding state gain matrix.

As in the past, the chosen value of control weighting scalar was 
\[ b = 1 \times 10^{-8} \text{ sec}^3 \], which was negligible at all times-to-go of interest. Program OPGAIN failed because of numerical difficulties. As an alternative, a variation on program SGARAF was used, by employing (7.20) to convert the altered modal weighting matrix to a new state weighting matrix \( S_{f}' \), which also excluded the actuator modes. Parenthetically, it should be noted that analytically, the Riccati equation (7.15), program OPGAIN and program SGARAF are equivalent and should theoretically give the same results for a given problem.

The apparent navigation ratio \( N_a \), as computed by SGARAF, is shown in Figure 7-1, which shows that it is initially different from \( N_a \) for a conventional \( S_{f} \) but essentially the same after 0.08 second. The initial difference has already been explained in the previous subsection. As for the later coincidence, Subsection 6.5.4 has shown that for a conventional \( S_{f} \) the actuator modal gains settle to negligible values at a very short time to go anyway. Hence this new computation of \( N_a \) seems quite plausible.

Figure 6-12 compares the modal gain \( C_{z4} \) (for the unstable mode) for the special \( S_{f}' \) with that for the conventional \( S_{f} \). Again, they are initially quite different (at short times to go) but virtually coincident after 0.08 second.

As a partial test of suspected numerical difficulties, the problem was rerun on OPGAIN out to \( \tau = 0.01 \) sec in steps of 0.002 second, with \( b = 1 \text{ sec}^3 \), which is not negligible. Here, results agreed rather well with those of SGARAF except for the state gain \( c_8 \), which was small
Figure 7-1 - Apparent Navigation Ratio $N_a$ versus Time-to-Go for Unstable Bare Airframe at Mach 2, Sea Level, with Fin Servo and Bias on Tail Command
anyway. This relatively large value of $b$ caused the control gains at short $\tau$ to be smaller than those of the former case, but after $\tau = 0.2$ second they were essentially equal. It is significant that the $S_z$ matrix was changing rather slowly.

The same comparison between OPGAIN and SGARAF was made for $b = 1 \times 10^{-4}$ sec$^3$, with only partial agreement; here the $S_z$ matrix changed much more rapidly than previously. SGARAF computed essentially the same gains for this case as for $b = 1 \times 10^{-8}$ sec$^3$, indicating that $1 \times 10^{-4}$ sec$^3$ is close enough to zero for all times-to-go of interest.

Therefore, it is apparent that the theory of program OPGAIN is probably sound, but that it has numerical difficulties. Further work would be required to make this program truly useful. It should have two advantages over SGARAF: 1) OPGAIN computes in a reduced modal space with smaller matrices; 2) by eliminating the larger eigenvalues, OPGAIN should be able to take relatively large time steps more easily and to forestall large exponentials longer than SGARAF.
CHAPTER 8

CONCLUSIONS

It is convenient to divide guidance of homing interceptor missiles into two categories: 1) Classical guidance, which utilizes proportional navigation and control theory developed prior to 1960; and 2) Modern guidance, which utilizes post-1960 control theory, particularly Kalman estimation and optimal control. The classical guidance of Chapter 2 has been rather successful, partly because it is straightforward to mechanize, but it is inherently limited in performance. Modern guidance is computationally more complex, but it has great promise for improved miss distance.

This thesis is concerned with problems in applying modern control theory to the design of practical autopilots and computationally feasible guidance laws. Although these problems are rather specific, nevertheless, they can be and are generalized into larger problems outside the missile area with wider applicability, i.e., stochastic control to minimize a performance index in which the weighted terminal state predominates, near-optimal fixed-gain controllers, etc.

Chapters 1 and 2 summarize the current state of the art in guidance of homing interceptor missiles.

Subsection 1.5 has a fuller condensation of the total thesis than the space-limited Abstract.

8.1 A General Result for Terminal Control with a Known Disturbance Vector

Since the optimal control problems here are concerned with a linear system, quadratic performance index and additive white noise, the
separation theorem is immediately invoked to divide the total stochastic
control problem into the subproblems of optimal estimation and noise-
free optimal control from the estimate. Chapter 3 formulates a general
continuous terminal control problem with a known disturbance vector.
For the case of zero weighting of the state in the loss function, the optimal
control is proportional (through a time-varying gain) to the projected,
weighted, zero-control terminal state, including the effect of the known
disturbance but not the control; the weighting on the terminal state is the
matrix $S_f$ in the performance index. This new result is rather general
and it is immediately applicable to the missile guidance problem, whatever
the autopilot or lack thereof; it means that the optimal control is
proportional to the projected, zero-control miss distance.

The same principles are shown to hold in the general discrete-
control problem, with the same applications to missile guidance, both
conceptually and computationally.

8.2 Partitioning the Total System Control Subproblem into Design of a "Fast" Autopilot and Suitable Guidance

As in other optimal regulator problems, the optimal control is propor-
tional through the time-varying $S$ matrix to the state vector, although
the presence of a known disturbance adds another control term. By
itself, this theory is hard to apply to the integrated optimal design of
guidance and the autopilot, because practical mechanization considera-
tions for the latter require it to have fixed analog gains in a given auto-
pilot band on the Mach-altitude plane. The "autopilot" may be defined as
the closed-loop subsystem (airframe plus instrumentation) which produces
a lateral acceleration in response to a command.

The total approach to this problem is summarized in Figure 8-1.

In Chapter 3, the problem is resolved with an insight from extensive
Figure 8-1 - Chart Summarizing the Solution to the Terminal Guidance Problem
simulation experience, from which it is well known that "fast" autopilots have less miss distance than do "slow" ones. This is probably due to the fact that computationally practical guidance laws are almost always mismatched to the autopilots, due to multiple significant poles and changes with flight condition. The output of a fast autopilot settles quickly and it affords a better opportunity for a calibrated correction at the end of the intercept.

Accordingly, an optimization is performed on the coefficients of a standard cubic autopilot transfer function (applicable both to the classical Raytheon autopilot and to various others), so as to minimize the integral of the squared acceleration error in response to a step command of acceleration. The resulting autopilot has an "all-pass" transfer function with an RHP zero from the airframe and an LHP pole of the same magnitude, the LHP zero of the airframe being cancelled by one of the poles. Although this transfer function is not practically realizable, it can be approached in designing a practical autopilot. Not only is such an autopilot "fast" in the sense of a small integral of squared error, but it can be approximated by one pole and one zero in a guidance law that is computationally feasible in a tactical situation.

Accordingly, a continuous guidance law is derived in closed form for an autopilot with one pole and one zero. The guidance law can be expressed in terms of a time-varying effective navigation ratio $N_g$ and the projected, zero-control, miss distance. "Forcing" via $N_g$ is provided for the autopilot lag. This projected, zero-control miss distance can include the effect of missile axial acceleration and a flexible modelling of the target acceleration $A_t$ normal to the LOS, either as a decaying exponential or some other model of the target maneuver.
A discrete optimal solution for the control gains is computed. Tactically, the continuous effective navigation ratio \( N_e \) can be computed and used up to the last 10 or 15 data samples in the engagement. Thereafter, a table of stored discrete effective navigation ratio together with the general law (optimal control proportional to projected, zero-control miss distance) can be used.

An algorithm for the tactical computation of time-to-go is developed. Although somewhat complicated, this algorithm might be useful in a difficult engagement, particularly at minimum range and with extreme target maneuver.

A conclusion from the foregoing work is that conventional optimal control theory is not always enough for the practical design of a system. In this case, the constraints of constant autopilot gains and tolerable complexity of guidance computation were met partly by choosing an autopilot transfer function which approaches the optimal transfer function having a minimum integral of squared acceleration error. Hence, the system subproblem of optimal control is further partitioned into the subproblem of design of a fixed-gain autopilot with a specified transfer function and design of a guidance law of tolerable computational complexity.

8.3 Design of Autopilot as an Estimator-Controller in Modal Space with Simplification

In Chapter 4, the subproblem of the autopilot design to meet a specified transfer function is further divided into the subproblems of estimator (or observer) design and control design for specified poles, by deriving a separation principle which is shown to hold if the model of the airframe in the estimator-controller matches the true airframe. The subproblem of pole placement should be approached from the modal point of view, which
has advantages conceptually, computationally and in terms of ease of later structural simplification; the formula of Crossley and Porter is suitable for the modal gains. In order to overcome the unknown bias, the availability of an estimate makes it possible to cancel the effects of the bias; other schemes are examined.

As to the other autopilot subproblem, it is believed to be better to build an estimator, specifically for the modes, rather than an observer, partly because the former can give better filtering of the noises. Measurement noise and process noise (atmospheric turbulence) are modelled.

In order to simplify the five-state modal estimator-controller in the autopilot, advantage is taken of the fact that the two actuator modes are loosely coupled because no control gains are necessary from them and because they are not highly excited by the process noise. In the reduction process, the integrators with feedback for these two modes are essentially replaced by their zero-frequency gains. The resulting three-state controller and the airframe together have the specified transfer function at the design point (Mach 2, sea level), which may be closely approximated by the airframe RHP zero and one pole. Filtering of noise is good and off-design performance is acceptable. Another autopilot design was made for Mach 2 at 50,000 feet, but it is not described in detail.

This combination of modal estimation, modal control, bias cancellation and simplification of structure via modes is believed to be new, and it appears to help to bridge the gap between classical and modern methods of control design. The controller may be interpreted as having appropriate frequency compensation including proportional plus integral compensation.

Jameson's result for pure feedback of a restricted output vector is derived and fitted into the related literature of modal control.
8.4 Effect of Autopilot Response, Guidance Law and Radome on Miss Distance

In order to evaluate the foregoing new autopilot for Mach 2, sea level, its statistical miss-distance performance is compared in Chapter 5 with that of two classical autopilots and a hypothetical faster autopilot, using a simulation of a discrete guidance system in a discrete adjoint program. The continuous guidance law for the one-zero, one-pole autopilot is somewhat suboptimal in the discrete guidance system near the end of the interception, but the miss performance is improved by a suitable choice of weighting coefficient for the integral of squared control effort. With the use of the proper discrete guidance law, the miss distance is reduced further and it approaches the theoretical lower bound. The new autopilot has better miss distance than the classical autopilots, because it is faster and better matched to the guidance law.

A suitable linear model for the radome is explained briefly. Radome slope, particularly if negative, always acts to increase the rms miss distance, although the effect is not too severe at low altitude.

At Mach 2, 50,000 feet, the discrete guidance law is again superior to the continuous law, and the miss distance is appreciably more sensitive to the radome slope. Other work, not reported here, shows that it is possible to compensate an unknown nonlinear radome characteristic so as to improve the miss distance at high altitude.

8.5 Computation of Continuous Optimal-Control Gains for High-Order Linear Systems

Outside the mainstream of the guidance work, Chapter 6 describes research on computation of optimal control gains to minimize a performance index. It was found desirable to proceed in modest steps from the known case of the closed-form guidance law for the one-pole, one-zero
autopilot up to more complex cases. A conventional transition-matrix approach in 2n state-costate space worked fairly well in program SGARAF, but it had the inevitable trouble with growing and decaying exponentials at long time-to-go. It was complemented by program STGVAU (based on the work of Vaughan), which gave reliable results only at medium and long times to go.

An improved transition-matrix approach with reduced numerical trouble was rediscovered; it "symmetrizes" the S matrix and uses it to compute the next Z and Y matrix at each step. This latter approach gave fairly good results both in this optimal control problem and in the computation of Kalman-estimator gains for the autopilot.

Insight is gained by considering the modal feedback gains as well as the more conventional state feedback gains; in an unstable plant, only the unstable modes are fed back significantly at a long time to go. The feedback gains of stable high-frequency modes are seen to settle to zero faster than those of stable lower frequency modes. More work on accurate algorithms for computing optimal control gains would be desirable.

A theory of steady-state optimal control gains for one unstable mode is derived and related to the Crossley-Porter formula for pole placement. This theory is checked against the computed modal gain for an eight-state plant (airframe and guidance variables) with one unstable mode. It might be interesting to extend this work to include more than one unstable mode and to unify it with the pole-placement work of Crossley and Porter.

8.6 Optimal Control of Selected Modes

Also outside the mainstream of the missile-guidance work, a novel scheme for optimal control of selected modes is presented in Chapter 7. The quadratic performance index is converted to modal space and then
altered so that no weighting is put on stable, high-frequency modes, e.g.,
the actuator modes. From an examination of the Riccati equation in modal
space, it is apparent that the $S_z$ matrix always has zero elements in the
rows and columns corresponding to the unweighted (actuator) modes, and
so these modes always have zero feedback gains. This property may aid
in the stability of a physical system, in which other, unmodelled high-
frequency lags might otherwise destabilize the system. Moreover, this
technique would aid in simplification by eliminating certain time-varying
modal feedback gains. Again, work is needed on improved methods of
computing such gains.

8.7 Other Conclusions

Clearly, there are advantages in modal techniques for purposes of
conception, computation and simplification, particularly in dealing with
high-frequency (e.g., fin-servo) modes versus more important (e.g.,
rigid-body) modes.

As in other engineering work, the total design problem should be
divided rationally into manageable pieces. This is done here not only
with the usual separation theorem of optimal stochastic control, but
also with the separate design of the autopilot as a near-optimal, fixed-gain
controller which is then treated as part of the fixed plant from a guidance
point of view. This procedure takes advantage of the inherent flexibility
in modelling the plant and choosing the performance index.

The designer should be sensitive to the practical constraints on his
system and should not automatically apply a conventional, complex,
estimator-controller with time-varying gains for many states.
8.8 Suggestions for Future Work

The missile-guidance problems in this thesis are restricted to the terminal phase of flight and do not include the midcourse phase, which can be much longer. It would be useful to pose and solve a suitable midcourse problem at high altitude and far downrange from the battery, taking account of drag, variable air density, etc., and requiring the on-line computation to be moderate.

If closed-form analytical solutions to an optimal control problem can be obtained, they can be quite useful for on-line computation as in the missile-guidance case, but the derivation is quite difficult algebraically for moderately realistic plants. The methods of Kliger and Krasovskii in Subsection 1.3.5 should be explored further to determine whether they can provide closed-form solutions more reliably and simply; challenging problems would be to derive the closed-form guidance law for the one-pole, one-zero autopilot in the continuous and particularly the discrete control case.

The design and structural simplification of a fixed-gain autopilot in Chapter 4 is effective for the 5-state model shown. It should be tested with a fuller dynamic model, including the first two body-bending modes, so as to attempt to solve the classical problem of maintaining stability despite uncertainties and shifts in these bending characteristics. Moreover, it would be desirable to simplify the rather complex multi-step design process. Perhaps a near-optimal estimator of simple structure could be designed to model only these modes (e.g., the airframe rigid-body modes and the bias mode) which are to have control gains.

The methods of Chapter 4 could be applied to other plants with limited measurements if the desired transfer function can be meaningfully
specified. Other system considerations might be brought to bear on the latter problem, as in the case of optimizing the autopilot transfer function for minimum integral of squared error.

Further work would be desirable on the problem of reliably computing optimal control gains and optimal estimator gains. The improved transition-matrix method could be further improved, e.g., with the optimal computation of the transition matrix itself for a given plant, desired time step and computer word length. Improvements to Vaughan's method at short times-to-go would be desirable. A fundamental investigation of the behavior of the $S[\tau]$ matrix at small $\tau$ with small control weighting $b$ would be of associated interest.

Suggestions for the extension of the theory of steady-state optimal control gains for more than one unstable mode have been made in Subsection 8.5.

The behavior of eigenvalues of $F\cdot KH$ for measurement-noise-power levels approaching zero might be of interest (Subsection 4.3.2.3.3). Also research into the other problem of process-noise-power levels approaching zero would be of interest, as would a combination of these two problems.

In general, there is a need for simplifying the structure of optimal-control solutions while preserving near-optimality of a basic performance index. To this end, the optimal control of selected modes (Chapter 7) might be extended further and expanded into a truly unified theory of modal control. For real utility, a simplified means of estimating the modes to be controlled would be needed, as previously mentioned.
APPENDIX A

DERIVATION OF PITCH AIRFRAME EQUATIONS OF MOTION, TRANSFER FUNCTIONS AND TWO STATE-VARIABLE MODELS

1.0 Pitch Airframe Equations of Motion and State-Variable Model

Assume that the missile has constant velocity \( V_m \), zero roll rate and small perturbations in pitch about trim. Newton's second law for acceleration \( A_m \) (positive up) perpendicular to the velocity vector and for pitch angular acceleration \( q \) [positive nose up] may be expressed as:

\[
A_m = V_m \begin{bmatrix} -Z_\alpha \\ -Z_\delta \end{bmatrix} \tag{A-1}
\]

\[
\dot{q} = M_q q + M_\alpha \alpha + M_\cdot \cdot \alpha + M_\delta \delta \tag{A-2}
\]

where

\[\alpha = \text{Angle of attack, radians, positive with nose up} \quad (A-3)\]

\[\delta = \text{Tail or fin angle, radians, positive with trailing edge down} \quad (A-4)\]

\[Z_\alpha = -\frac{\rho V_m S}{2m} \frac{\partial C_L}{\partial \alpha} = -1.304 \text{ sec}^{-1} \quad (A-5)\]

\[z_\delta = -\frac{\rho V_m S}{2m} \frac{\partial C_L}{\partial \delta} = -0.1357 \text{ sec}^{-1} \quad (A-6)\]

\[M_q = \frac{d^2 \rho V_m S}{4 I_{yy}} \begin{bmatrix} \partial C_m \\ \partial \left( \frac{dq}{2V_m} \right) \end{bmatrix} = -1.016 \text{ sec}^{-1} \quad (A-7)\]
\[ M_\alpha = \frac{d\rho V_m^2 S}{2 I_{yy}} \left[ \frac{\partial C_m}{\partial \alpha} \right] = 140.3 \text{ sec}^{-2} \quad (A-8) \]

\[ M_\dot{\alpha} = \frac{d^2 \rho V_m S}{4 I_{yy}} \left[ \frac{\partial C_m}{\partial (\frac{d\alpha}{2V_m})} \right] = 0 \text{ sec}^{-1} \quad (A-9) \]

\[ M_\delta = \frac{d\rho V_m^2 S}{2 I_{yy}} \left[ \frac{\partial C_m}{\partial \delta_e} \right] = -105.2 \text{ sec}^{-2} \quad (A-10) \]

\( \rho \) = mass density of air = \( 2.3769 \times 10^{-3} \text{ lb-sec}^2/\text{ft}^4 \) \quad (A-11)

\( V_m \) = missile velocity = 2232.8 ft/sec \quad (A-12)

\( m \) = missile mass, \( \text{lb-ft}^{-1}\text{sec}^2 \) \quad (A-13)

\( d \) = reference dimension, usually body diameter, ft \quad (A-14)

\( S \) = reference area, usually body cross-sectional area, \( \text{ft}^2 \) \quad (A-15)

\( I_{yy} \) = pitch moment of inertia about c.g., \( \text{lb-ft-sec}^2 \) \quad (A-16)

For convenience, numerical parameters for the \( Z \) and \( M \) parameters (Reference R1) are given for this flight condition, which is Mach 2 at sea level. An obsolete airframe design, which is rather unstable with positive \( C_m \), has been utilized. Numbers are not given for the last four parameters because they would be indicative of current classified airframe designs, but this omission does not interfere with the developments in this unclassified thesis.

From kinematics, \( A_m \) and the rate of change of flight-path-angle \( \dot{\gamma} \) are related by:

\[ A_m = V_m \dot{\gamma} = V_m (q - \dot{\alpha}) \quad (A-17) \]
Equations (A-1) and (A-17) may be solved respectively for $\alpha$ and $\dot{\alpha}$ and substituted into (A-2), which may then be put into the state-variable form:

$$
\dot{q} = M_q \dot{q} - \frac{M_\alpha}{V m Z_\alpha} A_m + \left[ M_\delta \dot{Z}_\delta - \frac{Z_\delta}{Z_\alpha} M_\alpha \right] \delta e
$$  \hspace{1cm} (A-18)

where the following auxiliary variables (Reference R1) are used:

$$
M_q' = M_q + M_\alpha = -1.016 \text{ sec}^{-1} \hspace{1cm} (A-19)
$$

$$
M_\alpha' = M_\alpha + M_\alpha Z_\alpha = 140.3 \text{ sec}^{-2} \hspace{1cm} (A-20)
$$

$$
M_\delta' = M_\delta + M_\alpha M_\delta = -105.2 \text{ sec}^{-2} \hspace{1cm} (A-21)
$$

The other state-variable equation is found by differentiating (A-1) and then substituting for $\dot{\alpha}$ from (A-17):

$$
\dot{A}_m = Z_\alpha A_m - V m Z_\alpha q - V m Z_\delta \delta_e \hspace{1cm} (A-22)
$$

Equations (A-18) and (A-22) are incorporated into the block diagram in Figures 4-2 and 6-8. In this model, the actuator has a quadratic transfer function with unity gain at zero frequency, a damping ratio of 0.5 and a natural frequency of 100 rad/sec.

2.0 Pitch Transfer Functions

Equation (A-15) may be Laplace-transformed (without initial conditions) and solved for the transform of (A-1) to give:

$$
A_m(s) = \frac{Z_\alpha}{s} A_m(s) - \frac{V m Z_\alpha}{s} q(s) - V m Z_\delta \delta_e(s) \hspace{1cm} (A-23)
$$
Equation (A-18) may be solved for \( q(s) \):

\[
q(s) = -\frac{1}{s-M'_q} \left[ \frac{-M'\alpha}{V M'Z\alpha} \right] A_m(s) + \left[ \frac{M'\delta}{Z\alpha} - \frac{M\alpha}{s-M'_q} \right] \delta_e(s)
\]  

(A-24)

If (A-24) is substituted into (A-23), the transfer function from elevator angle to lateral acceleration may be found and put into the form:

\[
\frac{A_m(s)}{\delta_e(s)} = K_1 \frac{1 + a_{11}s + a_{12}s^2}{1 + b_{11}s + b_{12}s^2}
\]

(A-25)

where (Reference R1):

\[
K_1 = V \frac{M'\alpha Z_\delta - M'\delta Z_\alpha}{M_q ^\alpha Z_\alpha - M^{\prime\prime}_\alpha Z_\alpha} = 2510 \ \text{ft/sec}^2 \ \text{rad}
\]

(A-26)

\[
a_{11} = \frac{M_q ^\prime Z_\delta}{M^{\prime\prime}_\alpha Z_\alpha - M^{\prime\prime}_\delta Z_\alpha} = -0.00088 \ \text{sec}
\]

(A-27)

\[
a_{12} = \frac{-Z_\delta}{M^{\prime\prime}_\alpha Z_\alpha - M^{\prime\prime}_\delta Z_\alpha} = -0.000869 \ \text{sec}^{-2}
\]

(A-28)

\[
b_{11} = \frac{M_q ^\prime + Z_\alpha}{M_q ^\prime Z_\alpha - M^{\prime\prime}_\alpha} = -0.01670 \ \text{sec}
\]

(A-29)

\[
b_{12} = \frac{1}{M_q ^\prime Z_\alpha - M^{\prime\prime}_\alpha} = -0.007197 \ \text{sec}^2
\]

(A-30)

For these numbers, (A-25) may be factored as follows:
\[
\frac{A_m(s)}{\delta_e(s)} = (2510 \text{ ft/sec}^2) \frac{1 + \frac{s}{34.44}}{1 + \frac{s}{13.004}} \frac{1 - \frac{s}{33.42}}{1 - \frac{s}{10.684}}
\]  

(A-31)

The poles at 10.684 and -13.004 rad/sec are also two eigenvalues of the total plant in Figure 6-8. In (A-31), the right-half plane zero at 33.42 rad/sec is responsible for the polarity reversal of \(N_a'\) in Figure 6-9 at a short time to go.

It should be noted that \(A_m\) is the total lateral acceleration of the missile c.g., resulting from both \(\alpha\) and \(\delta_e\) in (A-1).

In order to obtain the transfer function from fin angle to pitch rate, \(A_m\) may be eliminated from (A-23) and (A-24) and the result may be manipulated to give:

\[
\frac{q(s)}{\delta_e(s)} = K_3 \frac{1 + a_{31}s}{1 + b_{11}s + b_{12}s^2}
\]  

(A-32)

where:

\[
K_3 = \frac{M_\alpha'Z_\delta - M_\delta'Z_\alpha}{M_q'Z_\alpha - M_\alpha'}
\]  

(A-33)

and \(a_{31}\) is the "alpha over gamma dot" time constant:

\[
a_{31} = \frac{M_\delta'}{M_\alpha'Z_\delta - M_\delta'Z_\alpha}
\]  

(A-34)

At high frequencies, the asymptotic approximation to (A-32) is evidently:
Table A-1 (next page) gives the various coefficients for the transfer functions in (A-25) and (A-35), at each of the three flight conditions in Subsection 4.3.2.4.4.

3.0 State Equations for Rudimentary Pitch Airframe Model

Equation (A-22) and Figure 6-8 require the availability of $\dot{\delta}_e$, the tail-angle rate. Partly for this reason, and for added realism, Figure 6-8 incorporated an actuator model. Nevertheless, in building up to this level of modelling realism, it was desirable at one point to have a rudimentary pitch airframe model with some realism but without the two states in the actuator model, for the computations summarized in Subsections 6.4.2 and 6.4.3.

This was accomplished by utilizing the alpha component of lateral acceleration $-V_m Z \alpha$ (right-hand side of (A-1) as a new state variable $A_m$, and by redefining $A_{m\dot{}}$ as an output variable rather than a state variable, so that (A-1) became:

\[
A_{m\dot{}} = A - V_m Z \dot{\delta}_e
\]  

(A-36)

\[
\dot{A}_{m\dot{}} = \dot{A} - V_m Z \ddot{\delta}_e
\]  

(A-37)

Substitution of these two equations into (A-22) results in a new state equation for $A$:
Table A-1
Airframe Parameters at Four Flight Conditions

<table>
<thead>
<tr>
<th>Case</th>
<th>(1) (design point)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4) (design point)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Flight Condition</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mach No.</td>
<td>2.</td>
<td>1.75</td>
<td>3.</td>
<td>2</td>
</tr>
<tr>
<td>Altitude, feet</td>
<td>0</td>
<td>5000</td>
<td>0</td>
<td>50,000</td>
</tr>
<tr>
<td>Angle of attack, deg</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>7.5</td>
</tr>
<tr>
<td><strong>Airframe Parameters</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M_\beta^\prime$, sec^{-2}</td>
<td>-105.2</td>
<td>-71.4</td>
<td>-182.4</td>
<td>-10.05</td>
</tr>
<tr>
<td>$M_\alpha^\prime$, sec^{-2}</td>
<td>140.3</td>
<td>78.38</td>
<td>291.</td>
<td>-8.47</td>
</tr>
<tr>
<td>$M_q^\prime$, sec^{-1}</td>
<td>-1.016</td>
<td>-0.7835</td>
<td>-1.266</td>
<td>-0.1345</td>
</tr>
<tr>
<td>$M_a^\prime$, sec^{-1}</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$Z_\delta$, sec^{-1}</td>
<td>-0.1357</td>
<td>-0.1070</td>
<td>-0.157</td>
<td>-0.01501</td>
</tr>
<tr>
<td>$Z_\alpha$, sec^{-1}</td>
<td>-1.304</td>
<td>-0.9631</td>
<td>-1.798</td>
<td>-0.368</td>
</tr>
<tr>
<td>$K_2$, sec^{-1}</td>
<td>1.1243</td>
<td>0.9941</td>
<td>1.2939</td>
<td>-0.4194</td>
</tr>
<tr>
<td>$a_{31}$, sec</td>
<td>0.674</td>
<td>0.926</td>
<td>0.489</td>
<td>2.814</td>
</tr>
<tr>
<td>$K_1$, ft/sec^2</td>
<td>2510.</td>
<td>1910.</td>
<td>4340.</td>
<td>-815.</td>
</tr>
<tr>
<td>$a_{11}$, sec</td>
<td>-0.00088</td>
<td>-0.00109</td>
<td>-0.00053</td>
<td>-0.00057</td>
</tr>
<tr>
<td>$a_{12}$, sec^{-2}</td>
<td>-0.000869</td>
<td>-0.001387</td>
<td>-0.00042</td>
<td>-0.0042</td>
</tr>
<tr>
<td>Zeroes of $\frac{A_m}{\delta}$, rad/sec</td>
<td>33.42</td>
<td>26.47</td>
<td>48.15</td>
<td>15.36</td>
</tr>
<tr>
<td></td>
<td>-34.44</td>
<td>-27.25</td>
<td>-49.41</td>
<td>-15.50</td>
</tr>
<tr>
<td>$b_{11}$</td>
<td>-0.01670</td>
<td>-0.02250</td>
<td>-0.01061</td>
<td>0.05898</td>
</tr>
<tr>
<td>$b_{12}$</td>
<td>-0.007197</td>
<td>-0.012882</td>
<td>-0.003463</td>
<td>0.1174</td>
</tr>
<tr>
<td>Poles, rad/sec</td>
<td>10.684</td>
<td>7.98</td>
<td>15.53</td>
<td>-0.251 ± j 2.908</td>
</tr>
<tr>
<td></td>
<td>-13.004</td>
<td>-9.73</td>
<td>-18.59</td>
<td></td>
</tr>
</tbody>
</table>
\[ \dot{A} = Z_{\alpha} A - V m Z_{\delta} \dot{\alpha} e - V m Z_{\alpha} q \]  \hspace{1cm} (A-38)

Substitution of (A-36) into (A-18) results in a new state equation for q:

\[ \dot{q} = M_q q - \frac{M_q^\prime}{V m Z_{\alpha}} A + M_q^\prime \dot{\delta} e \]  \hspace{1cm} (A-39)

Equations (A-38) and (A-39) are incorporated into the block diagram in Figure 6-3. This change of variable has not altered the differential equations relating \( A_m(t) \) to \( \delta_e(t) \). Therefore, the transfer function \( A_m(s)/\delta_e(s) \) for this airframe is unchanged. Nevertheless, in interpreting optimal control-gain results, it should be noted that it would be impossible to measure A directly. An accelerometer at the c.g. would respond to \( A_m \). Nevertheless, this rudimentary pitch airframe model has been found to be useful for its interim purpose.
APPENDIX B

NOISE MODELS

1.0 Process Noise Model

1.1 Atmospheric Turbulence as Process Noise for the Airframe

1.1.1 Statistical Turbulence Model

The approach here is based on the statistical treatment in Reference E2, which says on p. 311:

"Enough data have now been gathered that a reasonably adequate picture of the structure of the turbulence can be given. From the information presented by Press and Meadows (Ref. P8, 1955) it appears reasonable to assume that it takes the form of individual patches, in each of which the turbulence is approximately random, homogeneous, and isotropic. The feature that distinguishes one patch from another is the intensity $\sigma$, defined as the root mean square (space average at fixed time) of any one of the velocity components. The spectrum of the turbulence in each patch is related to the intensity".

References P8 and P9 may also be of interest to the reader.

Reference E2 (p. 320) gives the following power spectral density for the vertical component of random air velocity, with respect to space frequency $\Omega_1$ in the horizontal plane:

$$
\Phi_3 (\Omega_1) = \frac{\sigma^2 L}{2\pi} \left[ \frac{1 + 3\Omega_1^2 L^2}{(1 + \Omega_1^2 L^2)^2} \right]
$$

(B-1)

where $L$ is a characteristic length, usually taken as 1000 feet, that would
correspond to a time constant for a conventional power spectral density. As a check, the following integral (equivalent to one on p. 58 of Ref. L2) was performed:

\[ 2 \int_{0}^{\infty} \Phi_3 (\Omega_1) \, d\Omega_1 = \sigma^2 \]  

(B-2)

which is the mean-square vertical velocity, as required for this power spectral density.

Following Ref. E2, consider a sinusoidal variation of vertical velocity \( -w_g \) (Figure B-1c) with a horizontal space frequency \( \Omega_1 \) (radians per foot) and a wavelength in feet of:

\[ \lambda_1 = \frac{2\pi}{\Omega_1} \]  

(B-3)

An aircraft flying horizontally with velocity \( V_m \) would traverse this wavelength in time:

\[ T = \frac{\lambda_1}{V_m} \]  

(B-4)

This time \( T \) is the period \( 2\pi/\omega \) of the sinusoidal variation in vertical gust velocity that the aircraft experiences, with \( \omega \) being the frequency in radians per second. It follows directly that the time frequency is:

\[ \omega = \Omega_1 V_m \]  

(B-5a)

Examining the denominator of (B-1), it is apparent that the asymptotic (space) break frequency is:

\[ \Omega_1 = 0.001 \text{ radians/foot} \]  

(B-5b)

For the flight condition of Mach 2 at sea level, \( V_m \) is 2232. feet/sec, and so the corresponding (time) break frequency is:
Figure B-1 - Velocities and Angles in Vertical Plane, With and Without Vertical Gust
ω = 2.23 rad/sec \hspace{1cm} (B-6)

The asymptotic break frequency corresponding to the numerator of (B-1) is only $2.23/\sqrt{3} = 1.29$ rad/sec.

In order to simplify the modelling problem, (B-1) is approximated as follows:

$$\Phi_3''(\Omega_1) = \frac{aL}{1 + \Omega_1^2L^2} \hspace{1cm} (B-7)$$

where $a$ is a constant to be determined. In order to make (B-7) correspond to the same mean-square gust velocity, its integral in (B-2) is set equal to $\sigma^2$, from which:

$$a = \frac{\sigma^2}{\pi} \hspace{1cm} (B-8)$$

$$\Phi_3(\Omega_1) = \frac{\sigma^2L}{\pi} \frac{1}{1 + \Omega_1^2L^2} \hspace{1cm} (B-9)$$

Equation B-9 has an asymptotic zero slope at low frequency and an asymptotic slope of -40 db/decade above 2.23 rad/sec, as does (B-1), and appears to be a quite adequate approximation to the latter.

In order to arrive at a number for the rms vertical gust velocity $\sigma$, the "feeder" curve for probability density $f(\sigma)$ on p. 312 of Ref. E2 was fitted with the following straight line:

$$\frac{\log_{10} f(\sigma) - \log_{10}(0.9)}{\sigma - 0} = -0.278 \text{ sec/ft} \hspace{1cm} (B-10)$$

which is equivalent to:
\[ f(\sigma) = 0.9e^{-0.638\sigma} \text{ sec/ft} \]  
\[ (B-11) \]

This curve is similar to another for low altitude on p. 313 of Ref. E3. As a check, the following integral was performed:
\[ \int_{0}^{\infty} f(\sigma) d\sigma = 1.41 \]  
\[ (B-12) \]
which indicated a discrepancy, because the integral of a probability density over all values of the argument must be unity. The discrepancy is thought to be due to an error in reading the difficult scale of the curve in the reference and also to a slight error in the curve itself. Accordingly, the ordinate rather than slope of the curve was adjusted to make the integral unity, with the result:
\[ f(\sigma) = 0.638 e^{-0.638\sigma} \]  
\[ (B-13) \]

Equation B-13 then approximates the probability density of the rms gust velocity over various turbulence patches. In order to have one overall value of \( \sigma \), the mean square was found by integration over all turbulence patches:
\[
\overline{\sigma^2} = \int_0^\infty \sigma^2 f(\sigma) d\sigma
\]

\[
= -\sigma^2 e^{-0.638\sigma} \left. \int_0^\infty \sigma e^{-0.638\sigma} d\sigma \right|_0^\infty + 2 \int_0^\infty \sigma e^{-0.638\sigma} d\sigma
\]

\[
= \frac{2}{(0.638)^2} e^{-0.638\sigma}(-0.638\sigma - 1) \left|_0^\infty \right.
\]

\[
= 4.91 \text{ ft}^2/\text{sec}^2
\]

Hence, the overall value of \( \sigma \) was taken as 2.22 ft/sec.

1.1.2 Pitch Airframe Equations of Motion

Following p. 321 of Ref. E2, consider a sinusoidal gust component (Figure B-1b, c):

\[
w_g = A\sin (\Omega_1 x)
\]

(B-15)

where \( x \) is the horizontal distance and \( w_g \) is positive downward. The gust component of angle-of-attack for horizontal ground velocity \( V_m \) is:

\[
\alpha_g = -\frac{w_g}{V_m} - \frac{A}{V_m} \sin (\Omega_1 x)
\]

(B-16)

The sinusoidal variation of \( w_g \) also causes the same moment on the aircraft (via \( M_q \)) that would be caused by the following equivalent pitch rate relative to a still air mass:
\[
q_g = \frac{\partial w_g}{\partial x} = \Omega_1 A \cos (\Omega_1 x) \\
= \frac{w_g}{V_m}
\]  

(B-17)

where the dot denotes time derivative.

The pitch equations of motion in Appendix A herein are modified to include the gust component \( q_g \) of angle of attack and the equivalent pitch rate \( q_g \) in (B-12):

\[
A_m = V_m \left[ -Z_x \alpha - Z_\delta \delta_e \right] 
\]  

(B-18)

\[
\dot{\theta} = \dot{q} = M q + M q_g + M \alpha + M \dot{\alpha} + M \delta_e 
\]  

(B-19)

\[
\alpha = \alpha_o + \alpha_g 
\]  

(B-20)

Referring to Figure B-1b, \( \alpha_o \) is the angle-of-attack for a still air mass and \( \alpha \) is the total instantaneous angle-of-attack, including the gust contribution \( \alpha_g \), and \( \alpha_o \) is given by:

\[
\dot{\alpha}_o = q - \gamma = q - \frac{A_m}{V_m} 
\]  

(B-21)

with the assumption of constant velocity magnitude \( V_m \).

Equation (B-18) may be solved for \( \alpha \):

\[
\alpha = -\frac{A_m}{V_m Z_{\alpha}} - \frac{Z_\delta}{Z_{\alpha}} \delta_e 
\]  

(B-22)

Substitution of (B-21) and (B-22) into (B-19) gives:
\[
\dot{\mathbf{q}} = (M_q + M_{\dot{\alpha}})q - \left( \frac{M_{\dot{\alpha}}}{V_m Z_{\dot{\alpha}}} + \frac{M_{\dot{\alpha}} Z_{\dot{\alpha}}}{V_m Z_{\dot{\alpha}}} \right) A_m + \left( M_{\dot{\delta}} - \frac{M_{\dot{\alpha}} Z_{\dot{\delta}}}{Z_{\dot{\alpha}}} \right) \delta e 
+ M_q \dot{q} g + M_{\dot{\alpha}} \dot{\alpha} g 
\]

\( \text{As in (A-19) through (A-21), define the auxiliary variables:} \)

\[
M_{\dot{\alpha}} = M_{\dot{\alpha}} + M_{\dot{\alpha}} 
\]

\[
M_{\dot{\alpha}} = M_{\dot{\alpha}} + M_{\dot{\alpha}} Z_{\dot{\alpha}} 
\]

\[
M_{\dot{\delta}} = M_{\dot{\delta}} + M_{\dot{\alpha}} Z_{\dot{\delta}} 
\]

\( \text{Combining (B-24) through (B-26) with (B-23) gives:} \)

\[
\dot{\mathbf{q}} = M_q \dot{q} - \frac{M_{\dot{\alpha}}}{V_m Z_{\dot{\alpha}}} A_m + \left( M_{\dot{\delta}} \dot{\delta} - \frac{M_{\dot{\alpha}} Z_{\dot{\delta}}}{Z_{\dot{\alpha}}} \right) \delta e 
+ M_q \dot{q} g + M_{\dot{\alpha}} \dot{\alpha} g 
\]

\( \text{where the last two terms are due to the vertical gust.} \)

\( \text{Substitution of (B-20) into (B-18) and differentation gives:} \)

\[
\dot{A}_m = - \frac{V_m Z_{\dot{\alpha}} (\dot{\alpha}_g + \dot{\alpha}) - V_m Z_{\dot{\delta}} \delta e}{V_m Z_{\dot{\alpha}}} \]

\( \text{Substitution of (B-21) gives:} \)

\[
\dot{A}_m = Z_{\dot{\alpha}} A_m - \frac{V_m Z_{\dot{\alpha}} q - V_m Z_{\dot{\delta}} \delta e - V_m Z_{\dot{\alpha}} \dot{\alpha}_g}{V_m Z_{\dot{\alpha}}} \]

\( \text{where the last term is due to the vertical gust.} \)

\( \text{The gust terms in (B-27) are found from (B-11) and (B-12) to be proportional to the derivative of gust velocity:} \)

\[
M_q \dot{q} g + M_{\dot{\alpha}} \dot{\alpha} g = \frac{M_q - M_{\dot{\alpha}}}{V_m} w_g \]

\( \text{The gust term in (B-29) is also found from (B-11) to be proportional to gust velocity:} \)
1.1.3 Simplified Model of Process Noise

The equations of motion (B-27) and (B-29), together with (B-30) and (B-31), are represented pictorially in the state-variable block diagram in Figure B-2. It remains to complete the modelling of the turbulence-induced process noise and to introduce a final simplification.

The power spectral density with respect to space frequency in (B-9) may be converted to a conventional power spectral density by substitution of (B-5):

\[ \Phi_{W_g}(\omega) = K \frac{1}{1 + \omega^2 \left( \frac{L}{V_m} \right)^2} \]  \hspace{1cm} (B-32)

If (B-32) is integrated over all frequencies and set equal to \( \sigma^2 \), the constant \( K \) is found to be:

\[ K = \frac{\sigma^2 L}{\pi V_m} \]  \hspace{1cm} (B-33)

Hence, the vertical gust velocity \( w_g \) is modelled in Figure B-2 as the output of a low-pass filter with a break frequency \( V_m/L \) and an input which is gaussian white noise with an autocorrelation function:

\[ \mathbb{E}[w'_g(t)w'_g(t')] = Q_g(t) \delta(t - t') \]  \hspace{1cm} (B-34)

where \( Q(t) \) is assumed to be a constant which is now to be determined. It may be shown that the output autocorrelation is:

\[ \mathbb{E}[w_g(t)w_g(t')] = \frac{Q_g}{2V_m} e^{-\frac{V_m}{L} |t-t'|} \]  \hspace{1cm} (B-35)
Figure B-2 - Plant with Process-Noise Inputs to Airframe and Bias Integrator
For zero time displacement t - t' between samples, (B-35) is just the mean square gust velocity $\sigma^2$ and so:

$$Q_g = \frac{2V_m \sigma^2}{L}$$  \hspace{1cm} (B-36)

In order to check the foregoing, the sample-autocorrelation is set equal to the time-autocorrelation function for this stationary base by invoking the ergodic hypothesis (Ref. L2, p. 212):

$$E \left[ w_g(t)w_g(t+\tau) \right] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} w_g(t)w_g(t+\tau) d\tau \cong \Phi_{w_g}(\tau)$$  \hspace{1cm} (B-37)

Reference L2, p. 57, shows that the power spectral density may be found from the autocorrelation function by:

$$\Phi_{w_g}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\omega \tau} \phi(\tau) d\tau$$  \hspace{1cm} (B-38)

Substitution of (B-37) and (B-35) leads to:

$$\Phi_{w_g}(\omega) = \frac{\sigma^2 L}{\pi V_m} \left[ \frac{1}{1 + \omega^2 \left( \frac{L}{V_m} \right)^2} \right]$$  \hspace{1cm} (B-39)

which agrees with (B-32) and (B-33).

Referring to Figure B-2, it is the variable $\dot{w}_g$, not $w_g$, which drives the airframe model. The power spectral density for $\dot{w}_g$ is:

$$\Phi_{\dot{w}_g}(\omega) = \frac{\sigma^2 L}{\pi V_m} \left[ \frac{\omega^2}{1 + \omega^2 \left( \frac{L}{V_m} \right)^2} \right]$$  \hspace{1cm} (B-40)

Asymptotically, this power spectral density is flat above the break frequency $V_m/L = 2.23$ rad/sec. Figure B-2 is now approximated by replacing
the feedback \(-V_m/L\) by zero, which results in the elimination of the added state variable \(w_g\), and conservatively increases the turbulence excitation of the airframe near and below 2.23 rad/sec. This results in approximating \(\dot{w}_g\) by the white noise \(\nu_g\), for which \(Q_g\) from (B-36) and (B-14) is:

\[
Q_g = 2(2.23 \text{ rad/sec})(4.91 \text{ ft}^2/\text{sec}^2) = 22 \text{ ft}^2/\text{sec}^3
\]  
(B-41)

Referring to Figure B-2 with this simplification in mind, the following may be considered as gains in a noise "input distribution matrix":

\[
\frac{M_q - M_{q\alpha}}{V_m} = \frac{-1.016 - 0}{2232} = -0.455 \times 10^{-3} \text{ ft}^{-1}
\]  
(B-42)

\[
Z_{\alpha} = -1.304 \text{ sec}^{-1}
\]  
(B-43)

where the airframe numbers are taken from Appendix A.

1.2 Process Noise into Bias Integrator

In order to allow for changes in the bias variable \(x_5 = \delta_b\) in Figure B-2, owing to electronic drifts, etc., some white process noise \(\nu_i(t)\) was included at the input to the \(x_5\) integrator. Equations very similar to (B-34) through (B-36) apply here, and so the integral of the autocorrelation of \(\nu_i(t)\) is:

\[
Q_i = 2|F_{55}|\sigma_b^2
\]  
(B-44)

where \(\sigma_b\) is the standard deviation of the random variable \(\delta_b\) and is assumed to be 0.1 degree or \(1.745 \times 10^{-3}\) radian. Since \(F_{55}\) is -0.01 rad/sec, it follows that:

\[
Q_i = 6.1 \times 10^{-8} \text{ rad}^3/\text{sec}
\]  
(B-45)
After this noise level was chosen, it was found that it did not excite the actuator modes very significantly, i.e., the eigenvalues for the steady-state estimator matrix \((F - KH)_{ss}\) included a pair very close to the original actuator eigenvalues.

1.3 Matrices for Process Noise

The equation of the bare plant with process noise is:

\[
\dot{x} = Fx + gu + G_n \eta
\]  

(B-46)

where the process noise vector is:

\[
\eta = \begin{bmatrix} \eta_g \\ \eta_i \end{bmatrix}
\]

(B-47)

with autocorrelation:

\[
E[\eta(t) \eta(t')^T] = Q(t) \delta(t - t')
\]

(B-48)

\[
= \begin{bmatrix} 22 & 0 \\ 0 & 6.18 \times 10^{-8} \end{bmatrix} \delta(t - t')
\]

The input noise distribution matrix is:

\[
G_n = \begin{bmatrix} -1.304 & 0 \\ -0.455 \times 10^{-3} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

(B-49)

The following matrix is useful as a submatrix in the Hamiltonian \(W\) matrix for the estimator (Subsection 4.3.2.3.2):


\[
G_n Q G_n^T = \\
\begin{bmatrix}
37.4 & 0.01305 & 0 & 0 & 0 \\
0.01305 & 4.55 \times 10^{-6} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 6.1 \times 10^{-8}
\end{bmatrix}
\]

(B-50)

2.0 Measurement-Noise Model

2.1 Instrument Errors

Based on current spring-restrained gyros, the following specification values were assumed:

- Maximum zero offset: 0.3 deg/sec
- Maximum hysteresis: 0.2 deg/sec
- Maximum linearity error: 1.0 deg/sec

RMS of above: 1.06 deg/sec

This was taken as a 2σ value, so that in about 95 percent of all cases the instrument error should lie between +1.06 and -1.06 deg/sec. From this, σ should be \(9.25 \times 10^{-3}\) rad/sec, and a rounded-off figure of \(1. \times 10^{-2}\) rad/sec was then chosen.

The following specification values were assumed for the accelerometer:

- Maximum zero offset: 0.1g
- Maximum hysteresis: 0.0385g
- Maximum linearity error: 0.35g

RMS of above: 0.364g = 11.8 ft/sec^2

This was taken to be a 2σ figure, and σ was chosen to be 7. ft/sec^2.
2.2 White-Noise Modelling

It is admittedly difficult to model the errors of these instruments as the additive white noise which is required in the Kalman filter formulation. A reasonable plausibility argument is offered as follows. Radar noise perturbs the missile and causes pitching constantly. If this random motion were temporarily approximated as a sine wave, it would cause linearity errors at higher frequencies. Since the linearity error is predominant in each instrument, the error output versus time would appear to be random and broad-band in spectral content.

Next, the power spectral density was imagined to be flat at all frequencies for this white-noise model, and the noise power \( \sigma^2 \) was imagined to lie in a square bandwidth between \(-\omega_b\) and \(+\omega_b\) rad/sec. Based on observations of missile spectra, the writer and a colleague (E.L. Greenberg) agreed on the conservative figure \( \omega_b = 10 \) rad/sec.

Now, the autocorrelation function for white noise is:

\[
E[v_1(t)v_1(t')] = \phi(t-t') = R_1(t)\delta(t-t') \quad (B-51)
\]

From (B-38) the power spectral density is:

\[
\Phi_{v_1}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\omega T} \phi(T)dT
\]

\[
= \frac{R_1}{2\pi} \quad (B-52)
\]

In order to have a power of \( \sigma^2 \) in the double-sided bandwidth of \( \omega_b \) rad/sec:

\[
\sigma^2 = \int_{-\omega_b}^{+\omega_b} \Phi_{v_1}(\omega)d\omega = \frac{R_1\omega_b}{\pi} \quad (B-53)
\]

\[
R_1 = \frac{\pi}{\omega_b} \sigma^2 \quad (B-54)
\]
2.3 Matrices for Measurement Noise

The measurement vector is:

\[
\begin{bmatrix}
\text{Accelerometer output} \\
\text{Gyro output}
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{v}_1 \\
\dot{v}_2
\end{bmatrix}
\]

From (B-54) and the values for \( \sigma \), the measurement noise matrix is:

\[
R = \frac{e^{-2\pi}}{\omega_b} \begin{bmatrix}
49 \text{ ft}^2/\text{s}^4 & 0 \\
0 & 1 \times 10^{-4} \text{ rad}^2/\text{s}^2
\end{bmatrix}
\]

Other matrices which are useful in the estimator calculations (Subsection 4.3.2.3.2) are:

\[
R^{-1} = \frac{\omega_b}{0.0065}
\]

\[
H^T R^{-1} H = \frac{\omega_b}{0.0065} 
\]

Normally, the bandwidth \( \omega_b = 10 \text{ rad/sec} \), but \( \omega_b \) is shown literally in the preceding equations for conceptual convenience in connection with the noise-variation experiment in Subsection 4.3.2.3.3.
APPENDIX C

SUMMARY OF THEORY OF CLASSICAL RAYTHEON AUTOPILOT

The classical Raytheon autopilot in Figure 4-3 has three elementary loops, each containing the fin servo \( G_{12} \) as a forward element, in terms of Mason's loop convention (Reference M3). From Mason's formula, the closed-loop transfer function of the autopilot is:

\[
\frac{A_m}{A_c} = \frac{K_{69} \left( \frac{1}{T_{11}} \right) K_{22} K_{12} \left[ \frac{1}{s + 1/T_{11}} \right]}{1 - Y_r - Y_s - Y_a} \quad (C-1)
\]

where the fin-servo transfer function is approximated by its closed-loop d-c gain \( K_{12} \). The loop ratios are:

Rate loop: \( Y_r = K_6 K_9 K_{22} K_{12} G_3 \quad (C-2) \)

Synthetic stability loop: \( Y_s = K_6 K_9 \left[ \frac{K_{11} T_{11}}{s + 1/T_{11}} \right] K_{22} K_{12} G_3 \quad (C-3) \)

Accelerometer loop: \( Y_a = K_7 K_6 \left[ \frac{K_{11} T_{11}}{s + 1/T_{11}} \right] K_{22} K_{12} G_2 \quad (C-4) \)

and the airframe transfer functions are:

\[
\text{Acceleration at c.g.} = \frac{\text{Fin Angle}}{G_1} = K_1 \left[ \frac{1 + a_{11}s + a_{12}s^2}{1 + b_{11}s + b_{12}s^2} \right] \quad (C-5)
\]
\[
\text{Accel. at accel. station at Fin Angle} = G_2 = K_2 \left[ \frac{1 + a_{21}s + a_{22}s^2}{1 + b_{11}s + b_{12}s^2} \right] \quad (C-6)
\]

\[
\text{Pitch rate at Fin Angle} = G_3 = K_3 \left[ \frac{1 + a_{31}s}{1 + b_{11}s + b_{12}s^2} \right] \quad (C-7)
\]

Of course, \(K_1 = K_2\) in any case. In the numerical work of this report, it is assumed that the accelerometer is close enough to the missile c.g. so that \(G_2\) may be taken as \(G_1\).

In the cubic autopilot model, all lags other than those of the airframe and the integrator are neglected.

After substitution of (C-5) through (C-7) into (C-1), it may be reduced to:

\[
\frac{A_m}{A_c} = \frac{1}{K_7} \left[ \frac{1}{1 - Y_{ro} - Y_{so} - Y_{ao}} \right]\left[ \frac{1 + a_{11}s + a_{12}s^2}{1 + B_1s + B_2s^2 + B_3s^3} \right] \quad (C-8)
\]

where:

\[
Y_{ro} = K_{22}K_{12}K_3K_6K_9 \quad (C-9)
\]

\[
Y_{so} = K_{22}K_{12}K_36911 \quad (C-10)
\]

\[
Y_{ao} = K_{22}K_{12}K_3K_6K_11 \quad (C-11)
\]

\[
B_1 = \frac{b_{11} + T_{1i} - Y_{ro}T_{11} - Y_{ro}a_{31} - Y_{so}a_{31} - Y_{ao}a_{21}}{1 - Y_{ro} - Y_{so} - Y_{ao}} \quad (C-12)
\]
\[ B_2 = \frac{b_{12} + b_{11}T_{11} - Y_{ro}a_{21}T_{11} - Y_{ao}a_{22}}{1 - Y_{ro} - Y_{so} - Y_{ao}} \]  

\[ B_3 = \frac{b_{12}T_{11}}{1 - Y_{ro} - Y_{so} - Y_{ao}} \]  

For the pitch-rate transfer function, it may be shown that:

\[ G_3 = \frac{K_3a_{31}}{b_{12}} \left[ \frac{s + 1/a_{31}}{s^2 + b_{11}s/b_{12} + 1/b_{12}} \right] \]  

\[ = \frac{M'_\delta(s + 1/a_{31})}{s^2 + b_{11}s/b_{12} - M'_\alpha} \]  

At frequencies above the magnitudes of the poles and zero of (C-16), the asymptotic approximation for \( G_3 \) is \( M'_\delta/s \), \( M'_\delta \) being negative for a tail-controlled missile. The asymptotic gain-crossover for the rate loop is then seen from (C-2) to be:

\[ \omega_{cr} = K_6K_8K_{22}K_{12}(-M'_\delta) \]  

Substitution of (C-17) into (C-9) gives:

\[ Y_{ro} = -\omega_{cr}K_3/M'_\delta \]  

The cubic denominator of (C-8) may be factored in each numerical case:

\[ 1 + B_1s + B_2s^2 + B_3s^3 = \left[ 1 + \frac{s}{\omega_1} \right] \left[ 1 + 2\xi_2 \left( \frac{s}{\omega_2} \right) + \left( \frac{s}{\omega_2} \right)^2 \right] \]  

\[ (C-19) \]
from which:

\[ B_1 = \frac{1}{\omega_1} + \frac{2\zeta_2}{\omega_2} \]  \hspace{1cm} (C-20)

\[ B_2 = \frac{2\zeta_2}{\omega_1\omega_2} + \frac{1}{\omega_2^2} \]  \hspace{1cm} (C-21)

\[ B_3 = \frac{1}{\omega_1\omega_2^2} \]  \hspace{1cm} (C-22)
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K7 Kalman, R. E., "When Is a Linear Control System Optimal?", Journal of Basic Engineering, (Transactions ASME), March 1964, pp. 51-60.


BIOGRAPHICAL NOTE

David V. Stallard was born in Evanston, Illinois on January 22, 1928. Later he moved to Yonkers, New York, attended Riverdale Country School in Fieldston, New York and was Valedictorian of his class in June 1945.

At M.I.T. in the difficult postwar transitional period, he studied in the Cooperative Course in Electrical Engineering and had three assignments at plants of the General Electric Company; he was fortunate to have the wise counsel of Professor E.W. Boehne, who helped him to find his graduate field of specialty. He was a member of Sigma Chi social fraternity, participated in the Glee Club, Technique and Public Relations Committee, and was elected in his junior year to Tau Beta Pi and Eta Kappa Nu. His graduate specialty was automatic control systems and he received classroom instruction and thesis supervision from that outstanding control engineer, Professor W.M. Pease. He graduated in June 1950 with an S.B.E.E., S.M.E.E. and an R.O.T.C. commission as Second Lieutenant.

He was employed as a trainee and engineer by Westinghouse Electric Corporation in Pittsburgh, Buffalo and Baltimore from August 1950 until March 1953. From then until July 1956 he worked as a Research Engineer in the M.I.T. Servomechanisms Laboratory (now known as the Electronic Systems Laboratory) on automatic control research, which led to contributions and papers in hydraulic control systems. At Feedback Controls Inc. from 1956 until September 1960 he performed and supervised developmental work on a variety of electrical and hydraulic control
systems. For a short period from September 1960 until June 1962 he was
associated with Sylvania Electric Systems in Waltham. From July 1962
through the present he has been a Principal Engineer and (part of the time)
Section Head at the Systems Analysis Department in the Missile Systems
Division of Raytheon Company, where he has contributed to a number of
guidance and control projects, notably the SAM-D (Surface to Air Missile
Defense) system.

With academic credits accumulated over the years, Mr. Stallard was
granted the degree of Electrical Engineer at M.I.T. in September 1968.
Feeling the need for more advanced training in the increasingly complex
field of automatic control, he was and is grateful for the opportunity to
enter the doctoral program of the Department of Aeronautics and Astro-
nautics in 1969, and for the opportunity to be associated with Professor
W. E. Vander Velde. He has been affiliated with the I. E. E. E. and the old
A. I. E. E. since undergraduate school and is a Registered Professional
Engineer in Massachusetts. He was recently elected to Sigma Xi.

Mr. Stallard is very fortunate to be married to the former Elizabeth L.
Gysi of Port Washington, New York and to have three lively sons, one of
whom seems quite likely to become a scientist.

This biographical note concludes with the obligatory list of published
papers:

1) "A Series Method of Calculating Control-System Transient
Response from the Frequency Response," AIEE Transactions,

2) "Analysis and Performance of a Valve-Controlled Hydraulic
Servomechanism," AIEE Transactions, Applications and
Industry, No. 24, May 1956, pp. 75-85.


