GEOMETRY AND THE KINEMATICS OF THE NORMAL HUMAN KNEE

by

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ABSTRACT

Understanding how the human body uses a combination of passive constraints, active controls and multiple sensors to control motion across complex, multi-degree-of-freedom kinematic couplings is a topic of interest to designers of advanced machines and clinicians. A necessary prerequisite for studying the control and observation of these couplings is a complete evaluation of the possible kinematics. This research has focused on elucidating the kinematics of the normal human knee. The work may be divided into two general parts: experimental measurements of the in vivo kinematics of the knee and the development of a model-based approach to examining the roles of the passive constraints in determining those kinematics.

In vivo kinematic data for the lower extremity of a single subject were recorded in a series of experiments. All data were acquired with the TRACK system in the Newman Laboratory. The first set of experiments were direct measurements of skeletal motion, with TRACK marker arrays mounted on skeletal pins implanted in the subject's femur (thigh) and tibia (shank). The primary objective of these measurements was to establish the number of independent degrees-of-freedom in the knee for different tasks. Instantaneous helical axes (IHAs) were calculated from the data for each task. Distinctly different loci of IHAs were found for three movements: a voluntary swing, normal gait and a pivot maneuver. Motion for all three tasks was spatial, but the number of degrees-of-freedom differed for each task. There is no simple mechanical analog for the knee coupling.

Many more measurements are necessary for continued study of the knee. Frequent use of skeletal pins use is neither practical nor desirable. In the second set of experiments data were acquired for the gait and swing tests with marker arrays taped to the skin. Skin-mounted markers are widely used in gait analysis and human movement studies. The skin-mounted array data did not accurately reflect the underlying skeletal motion. Large transients were seen in both displacements and orientations at heel-strike in normal gait. Rotations about all axes were attenuated by more than fifty percent for both tasks.

Kinematic data characterizes the coupling but reveals nothing about specific anatomical structures such as the ligaments of the knee or other passive constraints. A geometry-based method for predicting the potential kinematics of complicated couplings was developed for application to the knee. The approach combines solution of both the geometric compatibility, or finite kinematic, problem and the instantaneous kinematic problem to determine the full
range of possible movements for a coupling. Evaluating the instantaneous total freedom of the knee required a new application of extended screw theory to a constraint model including both contact and extensible constraints acting in concert.

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This dissertation has been a long time in the making. Twice as long as I anticipated at the start. Believe it or not, I have enjoyed the work and even learned a fair amount. Although a Ph.D. seemed to be a singularly solitary quest at times, it was only possible with the unstinting support of many, many people.

My parents were a constant source of strength and support, even though I am not sure that they ever completely understood what I was doing or why. Their forbearance over the years is amazing. My mom was always a teacher; I only wish she had been able to see her slowest child finally finish school. My dad is still a source of strength, even as he struggles with his own loss and adversity. I could not have done it without them.

I am fortunate, not that I will always admit it, to have one brother and two sisters. Beyond their normal generous encouragement they have shouldered some of my responsibilities as I slogged toward the finish. Thanks to all three.

I had a near perfect match in Professor Robert Mann as my thesis adviser and committee chairman. He demonstrated admirable patience in providing the time, space and resources for me to learn how to perform and supervise research. He also gave me ample freedom to define my own project and approach it in the way I thought best. I could not have asked for any more.

The final thesis also reflects the vital contributions of the other members of my committee: Dr. Bertram Zarins, Professor Neville Hogan and Professor David Hardt. Bert Zarins contributed his knee and clinical insight to the project, while Neville and Dave kept me honest throughout the process.

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It has been said that the denizens of the biomechanics lab do not graduate quickly because they actually enjoy working there. True enough. The result was a very special
environment to work in. The senior members of the lab who I spent the most time with — Bill, Dov, Tim and Pete — became good friends and helped each other along as we tried to discover the essence of good engineering. My 'brother' Greg, JB, Pat, Keita, Eric, Crispin, Sylvain and many others helped make the lab different from most labs at M.I.T. I hope it will continue to be a place where learning is fun and the free exchange of ideas commonplace.

I sought temporary relief from the academic pressure in weightlifting and rugby. A lifting partner is part goad, part nursemaid and absolute trust in his judgement is essential. My partners, Jim Hubbard and Luis-Filipe Martins, have been all that and good friends, too.

Rugby is about making friends, both at home and on tour, and intense competition. From my start with the Psychopathic Samoans, I have been fortunate enough to meet some kindred souls on the rugby pitch. Conversations with Leo spanned every topic imaginable over the years. Jim Culliton opened both his office and home as occasional refuges and was a great companion on tour. Tiehie and Marcel, Ben, George, Kwas, Joe, Brian, Bruce, Suber, Shawn, Chevy and everyone else made life at M.I.T. a little more enjoyable. Any mention of rugby is not complete without acknowledging Dr. Ron Geiger and M.I.T. Sports Medicine — Paul, Gary, Kathy and the rest — for their efforts in putting me back together after each new injury.

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One last comment on the whole experience. Years ago, after I made my first unsuccessful attempt to take on an immovable object with my head, Tom Muench told me I would never know if I could do it unless I gave it at least one more try. Well, I kept trying and I think I finally moved one.

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Michael Charles Murphy
To Mom and Dad and my orthopedists
For keeping me together all these years
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Chapter 1

INTRODUCTION

1.1 A Brief Description

Observation of the in vivo human knee may be misleading. Outwardly the kinematics of the knee appear to be simple, representative of a single degree of freedom hinge or perhaps a planar joint at most. Concealed under layers of skin and soft tissue is a remarkably involved mechanical system. The knee is the articulation of the two largest bones in the human lower extremity, the tibia and the femur (Figure 1.1). Incongruent articular surfaces mate with essentially zero friction over a wide range of static and dynamic loading conditions. Four major ligaments, bands of collagenous tissue, are the primary members of a convoluted passive constraint system that envelops the bearing surfaces. Voluntary control forces are generated by an apparently redundant network of muscles. Three quarters of the muscles are biarticular, exerting forces and moments about one of the other joints in the lower extremity in addition to the knee. Embedded throughout the system is an incompletely identified array of sensors [29, 175]. All of the components of these subsystems are comprised of living tissue, making time a structural variable as well as a motion parameter.

Mechanically, the knees are two of the six principal joints in the kinematic chain (Figure 1.2) of the lower extremity. The combination of the bearing and passive con-
Figure 1.1: Knee anatomy (from Zarins and Adams [297]).

Restraint subsystems form a frictionless joint that simultaneously connects and constrains the relative motion between two rigid bodies. Only the degree and nature of the constraint provided by the kinematic coupling can be evaluated independently of the rest of the kinematic chain. The demands placed on the knee by motion, load-bearing or the application of external forces require consideration of the configuration of the entire kinematic chain and of the states of the actuator and sensor subsystems.

1.2 Motivation

It is remarkable that the subsystems of the knee typically interact flawlessly, and unnoticed, throughout a person’s lifetime. Like many mechanisms in an age in which machines are ubiquitous, knees are underappreciated until failure occurs. The reduction in or the loss of mobility that accompanies traumatic or degenerative changes in the knee mechanism is a jarring reminder of the vital role of the knee in locomotion.

Incentive to study the movement of the knee centers on two principal goals: improvement of the current state of knowledge of the kinematics and control of the knee
system and improvement in the methods for treatment and restoration of damaged knees. Expanded understanding of the function of the knee may aid in explaining the overall function of the locomotor system. There is the additional promise of being a source of ideas for improving the design of mechanisms and machines. From bearings to complicated mechanical couplings to the coordination of interacting systems, the body is an existence proof of some intriguing possibilities for improved machine design. Current treatment of knee injuries focuses on restoring the structural geometry of the system in order to recover normal joint function. With enhanced knowledge of the higher order properties of the mechanical system, the kinematic geometry of the knee, clinical practice could concentrate on modalities which directly affect the function of the joint.

Some of the specific motivating questions and incentives are outlined in the following sections.
1.2.1 State of knowledge

Form follows function in well-engineered mechanical systems. This implies that the structural intricacy of the knee is indicative of complex kinematics rather than a simple underlying mechanism. The problem of deducing the constraint properties of the knee as a mechanical coupling reduces to three fundamental questions: (1) Does the joint allow spatial motion?, (2) How many independent degrees-of-freedom does the permissible motion have?, and (3) What roles do the individual structural members of the knee system have in determining the kinematics properties of the joint?

Resolution of all three questions requires accurate measurements of the skeletal kinematics of the knee. There is a surprisingly small amount of data based on direct measurements, with markers attached directly to the bones, of the in vivo skeletal kinematics of the knee [138, 150]. These data clearly demonstrate that the relative motion at the knee is spatial, and also confirm in vitro [26] and clinical observations [297] of 'laxity' in the normal joint. There is insufficient information to conclude how many independent degrees-of-freedom underly knee motion. Most of the available in vivo kinematic data were recorded with markers or instruments [43, 231, 232] not directly attached to the bone, so that intervening soft tissue may have affected how accurately the measurements reflected the skeletal motion, which is of primary interest. Studies of skin motion [157, 271], indicate that there is a significant amount of skin and tissue motion relative to the bone.

Fifty years ago, Brantigan and Voshell [26] observed that, "To discuss fully the contradictory statements with regard to ligament function would necessitate a long paper in itself...." Unfortunately, neither their experiments nor the many succeeding efforts have clarified the roles of the individual ligaments. Most of the experiments have been in vitro observations of the state of the joint under applied loads or during a specified motion. The principal limitations of these experiments are the tacit assumptions about the
kinematics, based on inadequate measurements, involved in choosing the specified input forces or motions. The net result has been an increase in the amount of contradictory data presented in the literature.

Two of the three basic questions about the kinematics of the knee remain unanswered, and there is only limited data to support conclusions on the third. If the fundamental kinematic properties of the knee are poorly understood, then the control of motion about the knee is a complete mystery. The lack of measurements is again the crux of the problem. Individual muscle forces and firing patterns are the quantities of interest. No method exists for in vivo measurement of muscle forces in the human. Muscle activity can be monitored by recording the myoelectric signals (MES), but the activity of anything but the large muscles close to the surface is difficult to detect under dynamic conditions and even those signals may be contaminated by crosstalk. Correlations of MES signal with force are available for only a limited set of isometric activities [199, 198, 228]. Records of muscle firing patterns in the lower extremity are available for gait [152, 178, 229, 230] and some athletic activities [41]. Without the option of direct measurements there have been many attempts to estimate individual muscle forces from measured kinematics and estimated net joint forces using optimization theory [53, 65, 101, 225]. All of these models assumed that the knee acted as a simple revolute joint. Mixed results, including reasonable predictions of firing patterns for some muscles, were obtained using a variety of different optimization criteria. Patriarco et. al. [188] found that the predicted muscle activity was more sensitive to the quality of the input kinematics, forces and joint models than to the variation of the optimization criterion.

1.2.2 State of clinical practice

Immediate motivation for achieving an understanding of the behavior of the knee in detail lies in the pathologies of the joint's subsystems. Failures can be separated into
three categories, corresponding to the different subsystems: bearing, passive constraints and actuators. Common threads link many of the treatment modalities for the different subsystems. Treatments tend to be 'zeroth order'; the focus is on restoring or modifying the structural geometry of the system to affect change in the function, or kinematic geometry. The emphasis on this indirect approach is partially attributable to the inadequate information on the higher order properties of the system, such as kinematics and muscle firing patterns.

Degenerative diseases are the greatest cause of disability in the United States for those over the age of 49. Arthritis of the lower extremity occurs at a frequency of more than 400 cases per 1000 over the age of 69 [144]. Approximately 250,000 total knees are implanted in the world each year, with half of those in the United States [164]. This figure does not represent the magnitude of the problem since design limitations of current artificial knees prevent many who have impaired function from receiving them. The surgical alternatives are fusion of the joint or a tibial osteotomy [42], an attempt to realign the load bearing surface so that healthy cartilage is carrying the load. Chondromalacia patella and meniscal tears are the prevalent forms of damage to the bearing system in the younger population. Until the advent of arthroscopy, meniscal tears were treated by excision of the entire damaged meniscus. This turned out to be a precursor for future degenerative changes in the bearing surfaces and present arthroscopic treatment attempts to remove as small a portion of the damaged cartilage as is possible. Chondromalacia is not understood; many of the approximately 150 different surgical procedures in use to relieve its symptoms are attempts to 'realign' the patella so that healthy cartilage is bearing the load.

Failure of the passive constraint system entails the rupture of one or more of the major ligaments. Loss of the ligament is generally considered to lead to 'instability' and requires surgical intervention to restore the system [297]. There is a limited amount
of empirical evidence that suggests ligament reconstruction is not always necessary and that the musculature and sensing systems can compensate for the change in the passive constraints, even for those participating in strenuous athletic activities [40]. Surgical "reconstruction" typically involves the replacement of the ligament with a tendinous autograft. Portions of the patellar and semitendinosus tendons are the most commonly used. There is considerable interest in the development of ligament prostheses, but none are in widespread use at this time. A major emphasis with cruciate ligament reconstruction is placed on the selection of attachment points that maintain isometric or nearly isometric ligament length [235, 236, 238, 237]. This is consistent with the planar four-bar linkage model of knee kinematics. Even after reconstruction, the use of cumbersome orthotics and reduced activity are often recommended. In the interest of reducing knee, primarily ligament, injuries in football, there was a rapid increase in the past five years in the use of prophylactic knee bracing in college and high school football. Results have been mixed, but there is some indication that the number and severity of injuries actually increased slightly in the braced population [89, 259].

Control system pathologies affecting the knee cover a wide range of problems. Probably the most compelling cases are the 150,000 paralyzed or partially paralyzed people in the United States [1]. The hope for the future is that neural prostheses, using functional electrical stimulation (FES) of the muscles, can restore some level of function to those with lower extremity paralysis. There is already some use of stimulation in riding exercise bicycles using FES to maintain and build muscle in the legs. Among the many problems that need to be considered before practical application of FES is reality is whether the muscle stimulation patterns used will adversely affect the joints from which there is no sensory feedback. Stimulation patterns must be chosen that do not induce degenerative changes in the joint.
1.3 Hypothesis and Objectives

1.3.1 Hypothesis

A prerequisite to delineating the normal mechanical function of the knee is the realization that the knee is one component in the locomotor system and that its design is influenced by the requirements of the larger system. To understand how the knee fails mechanically and how to restore its kinematic, not just structural, characteristics it is necessary to know what constitutes normal mechanical function of the knee. To characterize normal function of the knee, the fundamental kinematic properties of the joint must be elucidated first. Much of the prior work on the knee has suffered from having too narrow a focus, considering the knee only in isolation, and has been based on unsupported assumptions about the kinematics of the joint, leading to the morass of contradictory experimental and analytical results extant.

A simple thought experiment demonstrates the importance of recognizing the knee as part of a larger system. Consider the lower extremity as a kinematic chain. With either one or neither foot on the ground the chain is open (Figure 1.3a). With both feet on the ground a closed spatial polygon is formed (Figure 1.3b). It is a simple exercise to show that the mobility, or number of degrees of freedom, of the open chain is greater than for the closed polygon\(^1\). Based on the kinematics of the chain an ideal knee design would have the characteristics of a hinge during swing, when the chain is open and extra degrees of freedom at the knee are unneeded, and provide multiple degrees of freedom during double stance, when the closing of the chain restricts the mobility of the entire lower extremity. It is significant that the principal kinematic design constraint on the knee arises when the lower extremity forms a closed loop mechanism, not when the lower extremity is in single support or unsupported states which constitute a significant percentage of gait and

\(^1\)For definitions of chains and mobility refer to Chapter 2. The generalised mobility criterion defined there is the simplest approach to evaluating the mobility of the lower extremity.
Figure 1.3: The lower extremity is: (a) An open kinematic chain in single support stance, and (b) a closed kinematic chain in double support stance.
many other activities.

The actuation and sensing system in the lower extremity is capable of separately initiating movements and modulating stiffness around each joint. A possible 'natural' knee design would allow six degree-of-freedom motion with the controller 'selecting' the instantaneous helical axis by varying the stiffness and the net forces and moments about the joint according to the demands of the current task. A possible disadvantage is the energy needed to maintain active stiffness in the joint, the metabolic cost. A compromise design could be achieved by a passive constraint system that provided a desired level of stiffness and still allowed multiple degree-of-freedom motion. In effect, the passive constraints would define a 'movement region', a section of six-dimensional screw space. Active control would then select a trajectory through the region depending on the boundary conditions. A simple example is a door with leather hinges. A conventional door is mounted on revolute pair hinges and has only a single degree-of-freedom. With the revolutes replaced by leather hinges, the door can be moved around a locus of instantaneous axes constrained by the hinges. A particular axis is 'fixed' instantaneously by the forces and moments applied to the door.

The passive constraint system in the knee is comprised entirely of soft tissue with viscoelastic characteristics. The human body is capable of mineralizing tissue to create bony material, in response to physical demands on the system. If the ligaments were performing the function of essentially rigid, isometric members the body presumably would remodel to reflect that configuration. Instead, the geometry of the attachments and the strain rate dependence of the constraints suggest a system designed to collaborate with the active control of motion at the knee and permit multiple degree-of-freedom motion with reduced metabolic cost.

This leads to a simple statement of the proposal which is the basis of this investigation:

\footnote{Wolfe's Law of bone response to stress is the classic example of this capability.}
The passive constraint system of the knee, primarily the ligaments, defines a controllable 'movement region' in six-dimensional screw space for the relative motion of the two bones of the knee.

The corollary is that the passive constraints do not constitute a fixed linkage, such as a planar four-bar mechanism.

1.3.2 Approach

A unified, systematic study of the kinematics of the knee is required to evaluate the validity of the proposed hypothesis. The foundation for the investigation is the recognition that the knee acts as a kinematic pair in a kinematic chain and is a subsystem of the locomotor system. Its design is a reflection of the role the knee plays in locomotion and the design cannot be analyzed in compete isolation from the other components of the locomotor system. Initial efforts will be directed toward establishing the kinematics of the 'normal' knee, where normal is taken to mean the undamaged knee of a young, fit individual.

There are different approaches to the kinematic analysis of mechanical systems. The most straightforward requires only measurements. Sufficient kinematic data are recorded to characterize the system. This is efficient for simple mechanical systems, but prodigious amounts of data are necessary to fully define the behavior of more complicated systems. An alternative approach is to use a combination of experiments and a mathematical model of the kinematics to study a complicated mechanical system. Experiments are designed to define and verify the boundaries of the mathematical model, and the model is in turn used to exhaustively evaluate the possible kinematics. There must be no a priori assumptions about the kinematics of the system in either measurement or modelling, or else the approach becomes self-defeating and fails to provide clarification of the system kinematics.
Kinematic measurements are the essential first stage of the analysis. Precise, accurate, rigorous measurements of the skeletal motion. Experiments will be in vivo, because of the difficulty in properly defining the boundary conditions for in vitro experiments. The full spatial kinematics of each rigid link in the kinematic chain are required, with relative kinematics calculated from those. Tasks will be simple and well-defined. Since additional experiments are a part of the plan, the tasks must be easily repeatable yet illuminating. Results will be presented in a format that allows comparison of data at all levels: between subjects, between experiments, between tasks.

Within the bounds defined by the kinematic experiments, a method for evaluating constraint activity and predicting will be described. The approach is based on a mathematical model of the knee, with a precondition that no a priori assumptions about the kinematics are inherent in the model. Elements will be considered for inclusion in the model on the basis of potential influence on joint kinematics and on measureability. Emphasis is placed on parameters that may be measured with confidence because of the biological composition of the system. Material properties of ligaments and other tissues vary significantly from person to person, or at least from experiment to experiment, so their role in the model should be minimized. This leads to a model based on the physical geometry of the constraints. A two stage kinematic analysis may be built on a primarily geometrical base. First, the finite kinematics of the system may be defined using geometric compatibility. A method for identifying the boundaries of relative tibial and femoral displacements in a rigid body configuration state space will be presented. The constraint configurations thus determined will serve as inputs into the second phase of the approach which evaluates the complete range of possible motions permitted by a set of constraints. Extended screw theory [182] is an elegant tool for predicting the total freedom\(^3\) a rigid body from the geometry of the configuration of applied constraint forces at any instant.

\(^3\)Total freedom encompasses all possible motions of a rigid body.
It will be demonstrated how extended screw theory may be used to evaluate the relative motions of the tibia and the femur under the influence of different types of constraints. Initial emphasis will be on the passive constraint system, but the model will be developed so that contact constraints and external forces can be accommodated as data becomes available.

1.3.3 Thesis objectives

The combination of measurements and mathematical model are the initial stages of an extended analysis. Consequently the goals of the thesis, while oriented to evaluating the proposed hypothesis, are also directed toward developing a set of tools for use in that extended analysis. The broad objectives of the thesis are:

- Characterize the kinematics of the knee as determined by the passive constraint system in six-dimensional screw space.

- Develop a set of tasks and tools that help elucidate the kinematics of the knee.

- Develop a method for assessing the role of individual constraints in determining the kinematics of the knee using an extendable mathematical model of the knee based on a reliably measurable set of physical parameters.

1.4 Organization

The central focus of the main body of this document is the kinematics of the normal knee and methods for evaluating the role of the passive constraints in determining them. To
maintain this focus, much of the background material on spatial kinematics, numerical differentiation and other topics important to defining the kinematics of the knee has been placed in the appendices rather than in the main body of the text. The reader may wish to scan the appendices prior to reading the text to determine if any unfamiliar topics are reviewed.

The text is organized sequentially. Chapter 2 contains a review of the anatomy of the knee, a review of the significant and relevant work in the literature and a very brief introduction to kinematic terminology. The kinematic measurement problem is described in Chapter 3. Included are the results of a set of direct measurements of skeletal kinematics, a set of measurements made with skin-mounted markers and a comparison of the two with a discussion of the implications for future studies of knee kinematics. Chapter 4 presents work on assessing the contributions of different constraints to both the relative displacements and motions of the knee. The final chapter is a discussion of the overall results and a presentation of suggestions for future directions for knee research.
Chapter 2

Background

2.1 Introduction

The material in the following chapters is presented with the assumption that the reader is comfortable, if not conversant, with several diverse topics. In order to provide the rudimentary background, and establish an appropriate tone, for the ensuing sections this chapter provides background information in three principle areas. First and most important is the language of kinematic analysis, in particular instantaneous kinematics and the mathematics of screws. Until experimental evidence is available to prove it is not, the knee must be assumed to be a spatial coupling. The basic terminology and tools necessary to address the problem precisely are outlined. Second, the anatomy of the knee and lower extremity are introduced and described from both the engineering perspective and the conventional descriptive anatomy or surgical viewpoint. Finally, a brief review and evaluation of previous studies of the knee and lower extremity give the current work historical perspective.
2.2 Kinematic Fundamentals

This section introduces key kinematic concepts that will be used throughout the dissertation. It is not intended as a comprehensive review. More detail on some topics may be found in the appendices (Appendix A and Appendix B in particular), and in the references. Although spatial motion is treated only briefly in most kinematics texts, good references include Hunt [113], a review of geometry and spatial movement which will be cited frequently in the ensuing section; a new offering by McCarthy [162], which covers the mathematical aspects of kinematics; and two review papers [210, 293] which provide a broad introduction to most of the major ideas. For screw theory and applications of screws to dynamics and kinematics, Ball's original treatise [14] is still the best reference.

2.2.1 Definitions

Dynamics is the ,"...branch of theoretical mechanics dealing with the motion and equilibrium of bodies and mechanical systems under the action of forces." [22] If only the motion is of interest kinematic analysis is sufficient. Kinematics deals "...with the geometry of motion, irrespective of the causes producing the motion." [22] Kinematic geometry [113] is the study of the geometry of displacements.

In biomechanics and machine design most of the kinematic analysis focusses on the motion of rigid bodies. A rigid body is a locus of points which do not displace relative to each other. It is an idealized concept, particularly in biomechanics where most of the components involved in movement are composed of living cells and soft tissue. While this does not invalidate the rigid body assumption in biomechanical systems, it emphasizes the central importance of the time scale of a movement or an applied force.

The motion of two or more connected bodies is of more interest than the movement of a single rigid body. Assemblages of bodies are prevalent in biomechanical systems. Rigid
bodies are linked together to form *kinematic chains*. Chains may be composed of any number of *links* or rigid bodies, with each link having an unrestricted number of *joints* connecting it to other members of the chain. The simplest kinematic chain is an *open chain* (Figure 2.1a), distinguished by at least one link carrying a single kinematic pair. Starting at any link there is only a single path to any other link without tracing through any member of the chain more than once. In single leg stance the human lower extremity is an open chain (Figure 2.1b). In a *closed chain* (Figure 2.2a) every link is connected to at least two other members of the chain. From any starting point on a closed chain there are parallel paths to every other member. Closed chains form spatial polygons [292]. With both feet in contact with the ground, the lower extremity chain forms a closed kinematic chain (Figure 2.2b). Complex chains are formed by connecting open and closed chains. When assembled so as to "... control, transmit or constrain relative movement,"[113] a chain becomes a mechanism. Assemblages of mechanisms, actuators, sensors and structures - e.g. electrical or hydraulic - form mechanical systems, also known as, machines.

Joints, or *kinematic pairs*, are the connections between two rigid bodies in a chain which prescribe the movement between the two links. Generally, a kinematic pair consists of a single contact between the two links; connections that comprise multiple contacts, usually series and parallel combinations of simpler pairs, are called *couplings* [113]. If one variable completely describes the relative displacement of two rigid bodies across a joint, it is a single degree of freedom pair. Kinematic pairs are classified as either higher or lower pairs. Originally only single degree of freedom pairs were classified as lower pairs. The classification evolved so that any pair formed by surface contact is considered a lower pair [113]. Higher kinematic pairs are formed by point, line or curve contacts. There are six lower pairs (Table 2.1): the screw (H), the revolute (R) or hinge, the prismatic (P), the cylindrical (C), the planar (E) and the spherical (S). All the lower pairs are special
Figure 2.1: (a) An open kinematic chain. (b) The human lower extremity is an open chain in single leg stance.
Figure 2.2: (a) A closed kinematic chain. (b) The human lower extremity is a closed chain in double support stance.
Table 2.1: Lower Kinematic Pairs (after Hunt [113]).

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Freedoms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Screw</td>
<td>H</td>
<td>1</td>
</tr>
<tr>
<td>Revolute</td>
<td>R</td>
<td>1</td>
</tr>
<tr>
<td>Prismatic</td>
<td>P</td>
<td>1</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>C</td>
<td>2</td>
</tr>
<tr>
<td>Planar</td>
<td>E</td>
<td>3</td>
</tr>
<tr>
<td>Spherical</td>
<td>S</td>
<td>3</td>
</tr>
</tbody>
</table>

cases of the screw pair, or can be replaced by a substitute mechanism comprised of a combination of screw pairs. Higher pairs include gear trains, cams and roller bearings. While lower pairs have analytical advantages, both classes of joints have wide application.

2.2.2 Movement Screws (Twists)

Instantaneous screws

There are many methods for describing the instantaneous spatial motion of a rigid body. Mozzi [82] first proposed that general spatial motion was equivalent to an infinitesimal twist about a screw. Chasles [113] later demonstrated that the instantaneous screw or helical axis (IHA) is the most general representation of rigid body spatial motion.

The velocity of a rigid body may be represented as a screw in the form of a dual vector (Appendix A), composed of the angular and translational velocity vectors of a moving body[293]:

$$\vec{v} = \Omega + \epsilon \vec{r} \tag{2.1}$$

where

- $\vec{v}$ = Velocity screw
- $\Omega$ = Angular velocity vector
- $\vec{r}$ = Translational velocity vector
- $\epsilon$ = Dual unit ($\epsilon^2 = 0$)
Any dual vector may be rewritten as the product of a scalar dual magnitude and a unit dual vector [24]:

\[ \bar{V} = |\Omega| (1 + \epsilon h)(u + \epsilon u^0) \]  

(2.2)

where

- \(|\Omega|\) = magnitude of the angular velocity vector
- \(h\) = pitch of the screw
- \(u\) = real component of unit velocity screw
- \(u^0\) = dual component of unit velocity screw

The pitch is invariant\(^1\) with the point of reference, characteristic of the mechanism generating the helical axis, and completely describes the motion geometry to first order[131]. The pitch,

\[ h = \frac{\Omega \cdot \dot{r}}{|\Omega|^2} \]  

(2.3)

is the ratio of the translational speed along the screw axis to the rotational speed around the axis. Since the magnitude of the angular velocity about the screw is \(|\Omega|\), the translational speed along the helical axis is:

\[ V = \frac{\Omega \cdot \dot{r}}{|\Omega|} \]  

(2.4)

The real component of the unit screw defines the orientation of the line the helical axis lies on. It is simply the normalized angular velocity:

\[ u = \frac{\Omega}{|\Omega|} \]  

(2.5)

and is invariant with a change in origin. The dual component of the helical axis locates the screw in space. It is the moment of the screw axis about the origin. With respect to a coordinate system fixed in the moving body, the dual component of the unit screw is:

\[ u^0_m = \frac{\Omega \times \dot{r} \times \Omega}{|\Omega|^3} \]  

(2.6)

\(^1\)Considerable effort has been expended in deriving first order and higher instantaneous invariants of spatial motions. Two thorough references are Kirson and Yang [131], which approaches the problem from the perspective of line geometry, and Veldkamp [273], which starts from point coordinates.
The corresponding perpendicular vector from the origin of the moving coordinate system to the helical axis is:

\[ p_m = \frac{\Omega \times \dot{r}}{|\Omega|^2} \]  \hspace{1cm} (2.7)

Shifting the reference to the fixed origin is a simple correction to the perpendicular vector:

\[ p_f = \frac{\Omega \times \dot{r}}{|\Omega|^2} + r \]  \hspace{1cm} (2.8)

and to the dual component of the unit screw:

\[ u^0_f = \frac{\Omega \times \dot{r} \times \Omega}{|\Omega|^3} + r \times \frac{\Omega}{|\Omega|} \]  \hspace{1cm} (2.9)

When there is zero angular but not zero translational velocity, the screw axis is undefined. Every point in the body moves with the same translational velocity so that direct designation of a specific axis is impossible.

**Instantaneous invariants**

The theory of instantaneous invariants is based on work in higher order curvature theory for planar mechanisms by Freudenstein [76]. Veldkamp [273] developed invariants for spatial motion using point coordinates. Extension of the theory to line coordinates was performed by Kirson and Yang [131]. The fundamental principle is that the geometric properties of the motion of a rigid body at any instant may be described in terms of a finite set of scalar parameters. To first-order the motion is completely described by a single scalar, the pitch. For a second-order description of the motion four scalar parameters are required. Third or higher orders require an additional six parameters for each order. In addition to describing a motion, it has been shown that instantaneous invariants may be readily applied to the synthesis of mechanisms [211].
Axodes

A single IHA provides only a snapshot of the workings of a mechanism. For planar mechanisms it is well known that a locus of instantaneous centers define fixed and moving centrodes, plane curves characteristic of a mechanism [233]. Observation of a sequence of instantaneous screws should offer more insight into the movement of spatial mechanisms.

The spatial analog to centrodes are the axodes of a mechanical system. The IHA is defined relative to both the fixed and moving coordinate frames at any instant. In fact, the instantaneous screw is the coincidence of two lines, one fixed in each coordinate system. The locus of lines in each frame defines a ruled surface, called the axodes of the mechanism. The two axodes roll about and slide along each other, with only one generator of each surface coincident at any instant. Translation perpendicular to the IHA is indicated by the separation of adjacent rulings. Axodes are time independent and uniquely characterize the kinematics of a motion [113, 129].

Ruled surfaces are analogous to space curves. Through the principle of transference [293], all the algebraic properties for space curves can be extended to ruled surfaces using dual numbers. Dual curvature, torsion and geodesic curvature of a ruled surface are defined and interrelated by a set of dual Frénet formulae. A spatial extension of Burmester theory has been developed using these basic quantities [163, 216]. Karlsson [121] derived equations for curvature and torsion in terms of the relative motion of a rigid body based on work in Dimentberg [59]. Practical use of the curvature properties of axodes in biomechanics has been limited by the inability to obtain sufficiently high quality estimates of the higher derivatives.
2.2.3 Constraint Screws (Wrenches)

Screws have been introduced as a tool for kinematic analysis. The geometrical nature of the amplitude-pitch-unit screw representation is applicable to other systems. Wrenches, or force screws, are one such application\(^2\). Any conservative force system may be resolved into a resultant force vector and a couple acting parallel to it. Plücker [194] and Ball [14] recognized that the resultant force-couple pair could be represented by a screw, which Plücker referred to as a dyname and Ball a wrench. The development of the equations for the wrench is identical to the velocity screw parameter derivations. The starting point is a screw:

\[
\mathbf{F} = \mathbf{F} + \epsilon \mathbf{C} \quad \text{(2.10)}
\]

It is interesting to note that force is analogous to angular velocity and the moment, or couple, to the linear velocity. This is often counter-intuitive initially.

A representation for a screw may be approached from the physical geometry of the system. Woo and Freudenstein [287] developed a set of screw coordinates based on Plücker's homogeneous coordinates for a line [194]. Two points on a line may be represented in three-dimensional projective space with homogeneous coordinates \((x_0, x_1, x_2, x_3)\) and \((y_0, y_1, y_2, y_3)\). The line coordinates, a point in five-dimensional projective space, are:

\[
\begin{align*}
 l_1 &= x_1 y_0 - y_1 x_0 \\
 l_2 &= x_2 y_0 - y_2 x_0 \\
 l_3 &= x_3 y_0 - y_3 x_0 \\
 l_4 &= x_1 y_3 - y_1 x_3 \\
 l_5 &= x_2 y_3 - y_2 x_3 \\
 l_6 &= x_3 y_2 - y_3 x_2
\end{align*}
\]

\(^2\)Yang [292, 293] also develops relationships for a momentum screw.
Two conditions reduce the six coordinates to the four independent quantities necessary to describe all $\infty^4$ lines in space. The first is an orthogonality condition,

$$l_1l_4 + l_2l_5 + l_3l_6 = 0,$$

(2.12)

and the second,

$$l_1^2 + l_2^2 + l_3^2 = 1,$$

(2.13)

reduces the system to a unit line vector, since any amplitude still specifies the same line. By introducing a scalar, $h_\alpha$, a new set of coordinates, $\alpha_i$, based on the line coordinates may be defined as follows:

$$\alpha_1 = l_1$$

(2.14)
$$\alpha_2 = l_2$$
$$\alpha_3 = l_3$$
$$\alpha_4 = l_4 + h_\alpha l_1$$
$$\alpha_5 = l_5 + h_\alpha l_2$$
$$\alpha_6 = l_6 + h_\alpha l_3$$

The $\alpha_i$ are equivalent to the unit velocity screw components in Equation 2.2. A representation of screws based on physical geometry is particularly useful for the description of constraint systems. A simple example is a frictionless, point contact. The contact force (there is no moment) lies along the line of the normal to the tangent surface at the point of contact. The constraint can be represented by a zero-pitch screw coincident with the normal.
2.2.4 Screw Systems

Reciprocal screws and screw theory

By combining the concept of infinitesimal movements described by a twist about an IHA with that of net forces and moments expressed in the form of a wrench, Ball [14] made his best-known contribution to the dynamics of rigid bodies. This is the theory of reciprocal screws. Given a screw \( \alpha \) with a twist about it, and a second arbitrary screw \( \beta \) with a wrench acting on it (see Figure 2.3), the virtual work done by the wrench in producing the instantaneous twist is:

\[
W_{\alpha \beta} = \alpha \beta ((p_\alpha + p_\beta) \cos \theta - d \sin \theta) \tag{2.15}
\]

\[
= \alpha \beta \omega_{\alpha \beta},
\]
where

\[ W_{\alpha \beta} = \text{Virtual work done by } \dot{\beta} \]
\[ \alpha = \text{Twist magnitude} \]
\[ \beta = \text{Wrench intensity} \]
\[ p_\alpha = \text{Pitch of twist} \]
\[ p_\beta = \text{Pitch of wrench} \]
\[ \theta = \text{Angle between } \dot{\alpha} \text{ and } \dot{\beta} \]
\[ d = \text{Perpendicular distance between } \dot{\alpha} \text{ and } \dot{\beta} \]
\[ \omega_{\alpha \beta} = \text{Virtual coefficient for } \dot{\alpha} \text{ and } \dot{\beta}. \]

If the virtual work is zero, the two screws form a reciprocal pair. In this case, two features stand out. First, the wrench and the twist may be applied to either screw because the reciprocal screw pair is symmetrical. Second, the reciprocity condition is independent of the wrench intensity and the twist amplitude, since it is reduced to the value of the virtual coefficient, \( \omega_{\alpha \beta} \), for any twist amplitude and wrench intensity. It is dependent only on the pitches of the two screws and the angle and distance between them. If the wrench and twist are expressed in screw coordinates, the reciprocity condition is a linear equation:

\[ \omega_{\alpha \beta} = \beta_1 \alpha_4 + \beta_2 \alpha_5 + \beta_3 \alpha_6 + \beta_4 \alpha_1 + \beta_5 \alpha_2 + \beta_6 \alpha_3. \] (2.16)

Given a constraining wrench, all possible workless movements may be found by solving a linear equation.

Extended screw theory

Only a subset of the possible movements of a rigid body, defined by the requirement of zero virtual work, satisfy the conditions of reciprocity. An example of the limitations of reciprocal screws is found in considering the contact problem shown in Figure 2.4. For two bodies in frictionless point contact, the contact constraint is the force acting along the normal to the surfaces at the point of contact. This constraint, \( \dot{\beta} \), is a zero-pitch wrench. The only twists about some screw, \( \dot{\alpha} \), that satisfy the reciprocity condition result in either movement perpendicular to the normal force or pivoting about the point
Figure 2.4: Two contacting bodies, (from Ohwovoriole [181]).
of contact. Any twist that results in a separation of the two contacting bodies would not be revealed, since it requires positive work. Twists that violate the contact constraint by attempting to force the bodies together are not part of the reciprocal solution either. Ohwovoriode [181] studied the contact problem, recognizing that the cases of negative and positive virtual work were solvable using the theory of linear inequalities [79]. Contrary screw pairs are defined as pairs for which the virtual coefficient is negative. In the case of contact these are the physically unrealizable solutions where the bodies are forced together. Repelling screw pairs are pairs with a positive virtual coefficient, which for the contact problem results in a loss of both contact and constraint. The fundamental relationships of extended screw theory are the original reciprocity condition and two inequalities, one for the repelling and the other for the contrary screw case:

\[ \omega = 0 \quad \text{Reciprocal} \]
\[ \omega > 0 \quad \text{Repelling} \]
\[ \omega < 0 \quad \text{Contrary}. \]

**Linear algebra of screw systems**

A pair of screws, \( \alpha \) and \( \beta \), are linearly independent when the following equation is satisfied:

\[ a_1 \alpha + a_2 \beta = 0 \]

where \( a_i \) are scalar constants. Linear algebra may be applied to the testing of the independence of several screws. A straightforward approach is possible when the screws are represented in the form of screw coordinates. For \( k \) screws, each set of screw coordinates composes one row of a \( k \times 6 \) matrix. The rank of the matrix indicates the number of independent screws in the set [113]. The rank of a matrix may be estimated using singular value decomposition [196]. This approach becomes awkward for large numbers of screws.
and screws calculated from noisy data.

An alternative approach is to calculate an orthogonal basis for a screw system [253]. Two screws, $\hat{\alpha}_1$ and $\hat{\alpha}_2$, are orthogonal and linearly independent if the inner product of the pair vanishes:

$$\hat{\alpha}_1 \cdot \hat{\alpha}_2 = \alpha_1 \cdot \alpha_2 + \alpha_1^0 \cdot \alpha_2^0$$

$$= 0$$

(2.19)

Extending this to a system of screws, Sugimoto [253] showed that $n$ screws are linearly dependent if the Gramian is zero:

$$\begin{vmatrix}
\hat{\alpha}_1 \cdot \hat{\alpha}_1 & \hat{\alpha}_1 \cdot \hat{\alpha}_2 & \cdots & \hat{\alpha}_1 \cdot \hat{\alpha}_n \\
\hat{\alpha}_2 \cdot \hat{\alpha}_1 & \hat{\alpha}_2 \cdot \hat{\alpha}_2 & \cdots & \hat{\alpha}_2 \cdot \hat{\alpha}_n \\
\vdots & \vdots & \ddots & \vdots \\
\hat{\alpha}_n \cdot \hat{\alpha}_1 & \hat{\alpha}_n \cdot \hat{\alpha}_2 & \cdots & \hat{\alpha}_n \cdot \hat{\alpha}_n \\
\end{vmatrix} = 0$$

(2.20)

This equation led to an equation for the $n$th member of an orthogonal basis:

$$\hat{\beta}_n = \begin{vmatrix}
\hat{\alpha}_1 & \hat{\alpha}_2 & \cdots & \hat{\alpha}_n \\
\hat{\alpha}_1 \cdot \hat{\alpha}_1 & \hat{\alpha}_1 \cdot \hat{\alpha}_2 & \cdots & \hat{\alpha}_1 \cdot \hat{\alpha}_n \\
\hat{\alpha}_2 \cdot \hat{\alpha}_1 & \hat{\alpha}_2 \cdot \hat{\alpha}_2 & \cdots & \hat{\alpha}_2 \cdot \hat{\alpha}_n \\
\vdots & \vdots & \ddots & \vdots \\
\hat{\alpha}_{n-1} \cdot \hat{\alpha}_1 & \hat{\alpha}_{n-1} \cdot \hat{\alpha}_2 & \cdots & \hat{\alpha}_{n-1} \cdot \hat{\alpha}_n \\
\end{vmatrix}$$

(2.21)

Using Equation 2.21, Sugimoto developed an algorithm for calculating an orthogonal basis for a system of screws. In Sugimoto's procedure a screw from the system is selected to be the first member of the basis. The second basis screw, if it exists, is found by sequentially evaluating Equation 2.21 for each screw in the system. If a null basis screw results, the screw is linearly dependent on the selected basis screw. If not, the resulting screw is added to the basis and the procedure repeated until the calculation produces only null screws. A simple modification to the algorithm enables reliable estimation for screw systems generated from noisy measurements. Instead of evaluating the screws in sequence and taking the first non-null result as the next basis screw, Gramians (Equation 2.20 are calculated for each remaining screw in the system. The screw with the maximum
Gramian exceeding a threshold based on the data quality, the most independent screw in the system, is used to calculate the next basis screw. When no Gramian exceeds the threshold, the basis is complete. The dimension of the resulting basis is the number of independent screws in the screw system.

2.2.5 Screw theory and screw systems

All the screws which a rigid body can twist about constitute a screw system [14]. Among the screws of the screw system will be \( n \) independent screws. All the other screws in the system will be composed of linear combinations of the \( n \) independent screws. The number of degrees of freedom of the body correspond directly with the number of independent screws. An unconstrained rigid body will have a sixth order screw system.

The screws for a set of constraining wrenches form a screw system. The order of this "constraint" system is equal to the number of independent screws to which the wrenches are applied. If a wrench acting about a screw is reciprocal to \( n \) independent screws, it is reciprocal to all linear combinations of those screws or the \( n \)th order screw system. The set of screws reciprocal to a given system of order \( n \) form a reciprocal screw system of order \((6 - n)\).

For repelling and contrary screws there is a definite solution only for positive combinations of the independent screws. If a screw is repelling to \( n \) independent screws, it is repelling to any positive linear combination of those screws. For any linear combinations of the independent screws with negative coefficients the screw may be repelling, contrary or reciprocal. The same condition holds for contrary screw systems [181].

2.2.6 Mobility

In biomechanics and machine design it is necessary to know the mobility or number of degrees of freedom of chains of rigid bodies. The classic approach to estimating the
number of degrees of freedom in a chain is the Grüberl/Kutzbach criterion [113]:

\[ M = 6(n - g - 1) + \sum_{i=1}^{g} f_i \]  

(2.22)

where

\[ n \] = Number of links in the chain
\[ g \] = Number of joints
\[ f_i \] = Number of degrees of freedom of the \( i \)th joint.

For mechanisms with multiple independent loops, this can be rewritten as:

\[ M = \sum_{i=1}^{g} f_i - 6l \]  

(2.23)

where

\[ l \] = Number of independent loops in the chain
\[ g \] = Number of joints
\[ f_i \] = Number of degrees of freedom of the \( i \)th joint

While effective for many applications, this is a general mobility criterion, and Hunt [113] cites many cases where does not hold. For multiloop mechanisms such as in Figure 2.5, where two loops share a side of a parallelogram, the criterion fails. The technique also fails to predict transitory mobilities, when the mobility of a mechanism varies with position. Waldron [276] developed an approach to determining mobility using screw systems which avoids most of the problems of the general criterion and offers improved geometrical insight.

2.3 Anatomy

There are two motives for offering a brief description of the anatomy of the knee system. First, and simplest, is to provide an introduction to the major components of the knee coupling as background for the ensuing discussion. More important is to address the locomotor system as a mechanical system and describe where the knee fits into that system. In particular, it is essential to understand the highest level at which control of
Figure 2.5: The loop formulation of the general mobility criterion fails when two loops in a multiloop mechanism share the side of a parallelogram.

the knee may be treated in isolation from the rest of the lower extremity and locomotor system.

More complete descriptions of the anatomy may be found in general anatomy texts [6, 90]. Some interesting comparisons of human and other animal bones is contained in Schmid [218]. Detailed discussions of nerve and muscle physiology are found in several places including Mountcastle [175], McMahon [167] and Brooks [29].

2.3.1 Anatomy of the Lower Extremity Mechanical System

The purpose of this section is to describe the anatomy of the lower extremities, the 'end-effectors' of locomotion, from a mechanical system perspective. The lower extremity is comprised of three major subsystems: the structural system, the passive constraint system and the active control system. In addition to the physical interleaving of the different components, the same tissue component may have a role in more than one of these functional subsystems. This complexity makes the assignment of causality in the
system much more difficult than cursory examination would suggest.

**Structural subsystem**

There are four major structural units in the lower extremity: the pelvic girdle, the thigh, the shank and the foot. Three couplings link the units into a kinematic chain: the hip, the knee and the ankle.

The pelvic girdle is the uppermost portion of the lower extremity, serving as the intersection between the trunk and the lower extremity. There is one structural member, a large bone known as the os innominatum [90] or pelvis. The hip bone is usually described in terms of three regions: the ilium, the ischium and the pubis. There are articulations with the thigh and the other bones of the pelvic girdle, the opposite pelvis and the sacrum. The socket of the hip joint, the acetabulum, is centrally located in the outer surface of the bone. Rushfeldt [214] showed that the acetabular surface was essentially spherical. From the standpoint of knee function the pelvic girdle is important primarily as the point of origin for several muscles.

A single bone, the femur, provides the structure for the thigh. It is the longest and largest bone in the human body. At the proximal end the femoral neck and head comprise the ball portion of the hip joint. Tepic [260] used spherical harmonics to show that the femoral head was spherical to within one hundred microns. The femoral condyles, one half of the tibio-femoral joint and the principle articulation of the knee, form the distal end. The condyles blend into an anterior groove, the trochlea, which is the articulation for the patella or knee-cap.

Joining the thigh at the knee is the shank, or leg. It is constructed from two bones. The tibia is the primary structural member and the second longest and largest bone in the body after the femur. It has a long, straight shaft. On the proximal surface of
the tibia are the two tibial condyles, which articulate with the femur. The condyles are separated by an intercondylar eminence which serves as an attachment point for the anterior cruciate ligament and the menisci. The tibial condyles are also asymmetric, with the lateral condyle the smaller of the two. Adjoined to the tibia on the lateral side is the fibula. It is tied to the tibia along its length by a dense membrane and at the ends by cartilaginous joints supported by ligaments. The connection of the bones permits very little relative movement [118]. The proximal tibio-fibular joint is below the level of the tibio-femoral articulation, while the distal ends of the two bones form the proximal end of the ankle joint. The fibula serves as an attachment site for more muscles than the tibia.

Acting as the interface of the lower extremity and the locomotor system with the ground is the foot. It is a complex assemblage of twenty-six bones. From the standpoint of a mechanical analysis of the lower extremity the foot is treated as a single unit. The details of the ground/foot interface are assumed to be secondary to the relative motion at the ankle. It has recently been found that the ankle joint includes two principal articulations: the tibio-talar and the talo-calcaneal. Each contributes a different degree of freedom to the overall ankle joint [44, 239].

All three of the major couplings in the lower extremity are synovial joints. The articulating surfaces of the bones are lined with a layer of hyaline cartilage of varying thickness [96, 261]. Synovial fluid is the lubricating medium, but the precise mechanics of synovial joint lubrication are not fully understood. Several mechanisms for lubrication have been proposed. Current experimental results have not resolved the issue but do support either ‘weeping’ lubrication or elastohydrodynamic lubrication as the two most probable processes [261]. Experiments have shown that the formation of an interarticular seal and the restricted flow of fluid laterally in the cartilage layers are critical to successful load-bearing. Estimates based on a model of the in situ cartilage layer show the entrapped
fluid supporting as much as ninety percent of the load, while the hydrostatically swollen cartilage matrix supports the remaining ten percent [156]. Measurements have shown that the consolidation time for the cartilage, the time to squeeze all the trapped fluid out under a steady load, is on the order of twenty minutes.

Passive constraint system

The first functional layer of the lower extremity control system is the set of passive constraints which help bind the links of the kinematic chain together. These include components designed primarily as passive constraints and others, such as muscle, that have secondary passive constraining roles.

The most basic passive constraints are the 'hard' constraints imposed by the bony geometry of the major joints. Tepic [260] found the acetabulum and the femoral hip are essentially spherical and conforming. In the ankle, the talo-tibial joint, with the talus moving between the malleoli of the tibia and fibula, has the rudimentary characteristics of a planar kinematic pair. Only the knee lacks significant fixed constraining structures. The separation of the tibial condyles by the intercondylar eminence, and the corresponding intercondylar notch of the femur, are a minimal form of 'hard' constraint compared to what is found in the other two joints.

At least two of the joints have soft tissue components which directly complement the conformity of the bony constraints. In the hip the labrum provides a ring seal around the rim of the acetabulum, holding the femoral head in the socket. The menisci of the knee ride on the tibial plateau and significantly increase the load-bearing area in the joint [31].

Each of the joints is supported by an elaborate network of ligamentous constraints [90]. The articulating surfaces of the joints are encased in thin, membranous joint capsules, which are in turn supported by systems of ligaments. Ligaments are parallel-fibered
connective tissues with nonlinear stress-strain characteristics. Inspection shows that the ligaments are positioned and oriented to resist specific motions. There is no ready explanation for the arrangements, but it is probable that one function of the ligaments is to relieve the demand on the active constraints. It has also been suggested that the geometry of ligament positioning is related to a sensory role.

All of the remaining passive constraints have another primary role in the locomotor system and serve as passive constraints only in a secondary capacity. Muscles and fascia are oriented tissues which cross both single and multiple joints. Without activation the tissue bulk serves as an additional restraint to the motion which the muscle also actively modifies. Skin and subcutaneous fascia layers provide a degree of isotropic constraint. Haut showed that the presence of the secondary tissues significantly increased the resistance of the knee to tibial displacements.

Active control system

Muscles serve as the actuators in the active control system of the lower extremity. Approximately forty (see Table 2.2) muscles cross the three major joints in a complex pattern of interwoven lines of action. Many are biarticular, crossing two of the joints. Of the fourteen muscles crossing the knee, nine cross either the hip or the ankle too. The muscles identified in the table are composites of the true fundamental actuators, the motor units. Each motor unit may contain from a few to thousands of three types of muscle fibers, which are differentiated by energy source. McMahon presents a thorough discussion of the current understanding of the physiology of muscles and locomotion.

The actuation system has a remarkable level of apparent redundancy. Ignoring the composite nature of each muscle-actuator and the role of the passive constraints, so that each joint is assumed to have six degrees-of-freedom, there are twice the minimum num-

A more detailed discussion of the mechanical properties of ligaments is presented in Chapter 4.
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ber of actuators required. At least some of the redundancy can be explained. Muscles are not all aligned to modulate single degrees-of-freedom, there is a geometric overlap and crossover between separate muscles. Some muscles share insertions and may cooperate in some, as yet unidentified, manner. Hogan [109] has shown that for a planar, two-link arm biarticular and mono-articular muscles permit different trajectories, making different portions of the manipulator configuration space accessible. Co-contraction of agonist-antagonist pairs is also used to modulate the impedance of joints. In vivo pressure measurements in the hip have demonstrated significant levels of co-contraction in activities such as stair-climbing and rising from a chair [107].

The second part of the active control system present in the lower extremity is the network of proprioceptors, or sensors. There are two classes involved in the control of lower extremity movements.

One class are embedded in the actuators. These include stretch receptors, or spindle organs, and the Golgi tendon organs. The stretch receptors are sensitive to changes in muscle fiber length and the rate of length change. Sensitivity of the spindle organs is controlled by the $\gamma$-motoneurons. The distribution density of stretch receptors varies with the role of the muscle. Muscles used in fine control tasks, such as controlling the finger, have significantly higher density of receptors per unit muscle mass than muscles used primarily for power generation, such as the gastrocnemius [167]. Golgi tendon organs are embedded at the connection of the muscles and tendons and are sensitive to changes in force. There are generally one-third to one-half the number of Golgi tendon organs as spindle organs in a muscle.

The second class of receptors is resident in the passive constraints of the joints. There is evidence of both spindle and Golgi type nerve endings in the joint capsule. Receptors are distributed so that specific neurons are at maximum sensitivity at specific joint positions [175]. McMahon [167] cites evidence that some reflex loops reverse depending on
joint position. Spindle-like endings have been found on the anterior cruciate ligament of human knees [125, 221, 243]. There is speculation that these endings are involved in maintaining joint stability [243], but no conclusive evidence too date.

2.3.2 Anatomy of the Knee Coupling

The following sections are an introduction to the major components of the knee coupling. It is important to remember that,

...to discuss the function of each element of the knee joint as an isolated entity is to fail to appreciate the close interrelationship of structure and function of this whole complex.[280]

More detailed descriptions are available in the literature [13, 27, 125, 145, 201, 279, 280] or classical anatomy texts [6, 90].

Articulations

The knee coupling incorporates two articulations: the previously mentioned tibio-femoral joint and the patello-femoral joint.

The tibio-femoral joint is the connection between the thigh and shank links of the lower extremity kinematic chain. The articular surface of the femur (see Figure 2.6) is a continuous composite of the condyles of the tibio-femoral joint and the patellar groove. The condyles are asymmetrical, with the medial condyle longer and narrower of the two. Each condyle of the femur mates with a separate portion of the tibial plateau (see Figure 2.7a). It may be seen that the bony surfaces of the tibio-femoral joint are noncongruent.

The absence of bony conformity, or hard kinematic constraint, in the knee is a direct contrast with the other lower extremity couplings. In the knee there is only the separation
of the two tibial condyles. As a partial replacement for absent bony constraints the menisci, or semilunar cartilages, are attached to the tibial plateau as shown in Figure 2.7b. The menisci are C-shaped in plan view, with the medial meniscus being more open and having a larger radius. The femoral surface of each meniscus is concave, tapering from its thickest point at the periphery to thinnest at its inner edge. The tibial surfaces are flat, so the menisci form receptacles for the femoral condyles. Collagen fibers in the interior of the menisci are arranged circumferentially. Attachments are complex and described in detail in Brown [31]. Brown was able to show that displacements of the meniscus under load were very small (less than one millimeter), supporting the assumption that a primary function of the menisci is to enhance the conformity of the tibio-femoral joint.

The patella (Figure 2.8) is a sesamoid bone embedded in the quadriceps tendon-patellar ligament complex in the anterior portion of the knee coupling. It slides in the the patellar groove, or trochlea, on the femur where the two condyles blend together. The tendons of the quadriceps muscles (rectus femoris, vastus intermedius, vastus lateralis and vastus medialis) insert into the superior edge of the patella. The patellar ligament, the
Figure 2.7: The tibial plateau: (a) Without menisci, and (b) with menisci, (after Anderson [6]).
thickest ligament in the lower extremity, anchors the inferior edge of the patella to the tibia. Although the patella has no structural role in the kinematic chain, it does transmit a posterior force to the femur. This acts as a supplement to the other passive constraints which resist anterior sliding of the femur relative to the tibia.

Ligaments and capsule

A thin, fibrous capsule completely encloses the articulations of the knee. Its attachments essentially follow the boundaries of the articular surfaces. Warren and Marshall [279] warn that the separation from other structures, particularly the retinaculum of the quadriceps tendon, is difficult and may lead to the mistaken conclusion that the capsule functions as a stabilizing ligament. Lining the inside of the capsule is the synovial membrane, which produces the synovial fluid instrumental in the lubrication of the joint.

Supporting the capsule is an elaborate network of ligaments which provide passive resistance to relative displacements of the tibia and femur. The ligaments are grouped into the cruciates, located between the condyles, and the collaterals, which flank the
joint.

There are two cruciate ligaments, the anterior cruciate (ACL) and the posterior cruciate (PCL), named for the relative locations of their tibial attachments. Figure 2.9 shows the points of attachment of the cruciate ligaments to the tibia and femur. The cruciates are inside the capsule, but exterior to the synovial membrane. The anterior cruciate is attached to the intercondylar eminence posterior to the transverse ligament joining the two menisci. The line across the width of the ligament is nearly parallel to the line of the tibial spine. Converging from the tibial attachment the fibers are attached to the lateral side of the intercondylar notch on the femur in the pattern of a segment of a circle [13]. The chord of the segment is tilted anteriorly. Fibers from anterior part of the tibial attachment are attached to the superior portion of the femoral attachment. The tibial attachment of the posterior cruciate ligament is at the posterior end of the tibial spine, with its fibers aligned nearly perpendicular to the axis of the spine. The attachment is actually slightly below the plane of the tibial condyles. The PCL is broader and shorter than the ACL. From the tibia the fibers converge and are attached to the medial side of the intercondylar notch of the femur with the axis of the fibers nearly vertical. Fibers from the medial part of the tibial attachment are attached to the superior portion of the femoral attachment resulting in a twisting of the band.

The lateral, or fibular, collateral ligament is the narrowest of the major ligaments surrounding the knee. It extends from an attachment on the lateral epicondyle of the femur to an attachment in the anterior aspect of the head of the fibula (see Figure 2.10).

Support of the medial side of the knee is more complex [279]. At the level of the capsule there is the mid-third medial capsule, which is sometimes called the deep portion of the tibial collateral ligament. It is attached to the femur on the medial epicondyle and just below the joint line on the tibia. The medial, or tibial, collateral ligament arises
Figure 2.9: Attachments of the cruciate ligaments: (a) Tibial, and (b) Femoral.
Figure 2.10: Attachment points of the lateral, or fibular, collateral ligament (after Anderson [6]).
Figure 2.11: Insertions of the medial, or tibial, collateral ligament.

from the media epicondyle also. The femoral attachments of the two ligaments may
or may not be distinct. It is a broad band which is attached to the tibia well below
the joint line, in the region of the pes anserinus [27]. The attachments are shown in
Figure 2.11. Extensions of the semimembranosus tendon sheath are often mistaken for
additional portions of the tibial collateral ligament [279].

2.4 Previous Work

There is a vast amount of published research on all aspects of the human knee. The
following is a review of works that are pertinent to this dissertation.
2.4.1 Estimating the Instantaneous Kinematics of the Knee

The interest in understanding the motion of the human knee extends back to the work of the early anatomists [153]. A significant portion of what is known today has been learned in the past century, primarily a consequence of improved measurement equipment and techniques.

Kinematic data of knee movement have been collected using a variety of techniques including interrupted light photography [178], electrogoniometry [43, 128, 231, 232], serial radiography [75, 77, 97, 242], roentgen stereophotogrammetry [21, 112, 123], cine film [138, 150, 173] and an isokinetic dynamometer [298]. Only data acquired using ionizing radiation or markers attached directly to the bone by some invasive technique [138, 150] were not corrupted by soft tissue motion and could claim to represent the motion of skeletal members within the accuracy and resolution limits of the measurement systems used. Two of the most important results were the demonstration that the rotation of the tibia about its longitudinal axis in the terminal phase of extension was not automatic and that the magnitude of rotation about other than the flexion axis was greater than anticipated. The former motion, the so-called 'screw-home' movement, had been hypothesized on the basis of the asymmetry of the condyles of the femur. Lafortune [138] found that the rotation appeared inconsistently in gait complementing the findings of Hallen and Lindahl [97] for a knee extension task.

While the growing body of data was providing interesting, and surprising, insights into the kinematics of the knee, some limitations in the methodology were evident. A significant problem was the use of Euler angles, or other three-parameter set, to represent the relative rotations measured at the knee. The approach was found to be very sensitive to small variations in the definition of the reference Cartesian coordinate systems. This was particularly true when the principal rotation, flexion, covered a significantly larger range than the other rotations. Since the probability of exactly reproducing the
alignment of reference coordinate systems from experiment to experiment or laboratory to laboratory is near zero, comparison of kinematic data was essentially impossible. This led to an interest in estimating the instantaneous kinematics of the knee, instead of just the finite kinematic representation of the angles. The principal advantage in identifying the centrole of a planar motion or axode of a spatial motion is that it is characteristic of the mechanism producing the motion and its shape is invariant with changes in coordinate system. Additional benefits would depend on whether a single, or closely related group, of centrodes or axodes were sufficient to represent the kinematics of the knee. If true, changes in the centrole or axode could be used as a diagnostic and post-surgical evaluation tool. It would also greatly simplify the design of knee orthoses and prostheses.

Early efforts attempted to determine the centrole or centrodes of knee motion under the assumption that the motion was planar or nearly planar [75, 77, 242]. Instantaneous centers of rotation were generally estimated using the graphical method of Reuleaux which did not require the direct calculation of relative velocities, but used the finitely separated displacements of two points on the moving body [9, 33]. The drawback with the method is that as the displacements of the two points decreases, making the estimate closer to the true instantaneous center of rotation, the accuracy of the estimate decreases. The estimate is also very sensitive to the accuracy of the measurement of point coordinates [33, 64, 185, 186].

As acceptance of the idea that the assumption of planar kinematics was restrictive, investigators attempted to estimate the instantaneous helical axis of the knee. If the motion was planar all the helical axes would be parallel and the special case easily recognized, while an a priori assumption of planar kinematics precluded recognition of spatial motion. Some of the first investigators to estimate the helical axes for the knee [18, 165] used a simple extension of the planar methods. Motion was projected onto two parallel planes and the corresponding estimates of the instantaneous center in each
plane connected to give an estimate of the helical axis. Soudan et. al. [244] showed that there was no method analogous to the planar techniques for estimating the helical axis of a motion. More important was the finding that a simple extrapolation of the planar method for estimating instantaneous centers to determination of the helical axes was subject to much greater inaccuracies than those already documented for the planar method.

Subsequent investigations employed spatial kinematic techniques to estimate the instantaneous helical axis. The most common approach was to calculate a finite helical axis based on the relative displacements and orientations of the two tibia and femur [21, 112, 231, 232], rather than make a direct estimate of the instantaneous helical axis from velocities. The primary reason is the difficulty of calculating velocities from noisy position data using numerical differentiation. Peterson and Erdman showed that the finite helical axis was a good estimator of the instantaneous helical axis under some conditions [189]. The finite helical axes for flexion motions in vitro [21, 112] showed the axes tightly clustered and indicating only a small amount of tibial rotation. Application of a tibial exorotation during the flexion resulted in a slightly different set of axes, but no straightforward physical interpretation [112]. Shiavi et. al. [231, 232] examined both normal gait and pivoting movements, finding distinctly different distributions of helical axes for each movement.

Finite helical axes are sensitive to measurement errors [245, 284, 285]. In particular, the estimates of the orientation of the helical axis are very poor with noisy data. This has spurred the refinement of techniques for differentiating noisy data [283] so that the instantaneous helical axis may be estimated more directly. One published result for the instantaneous helical axis of a flexion motion [123] shows a distribution of axes similar to the finite helical axis estimates of Huiskes et. al. [112].
2.4.2 Assessing Passive Constraint Activity

There is both clinical and engineering interest in the assessment of which passive constraints are active during a given movement. Clinicians need to evaluate the urgency and level of intervention necessary in the case of disruption of a ligament. Knowing the activity level of different constraints will provide insight into the control of motion at the knee.

There are no reliable techniques for in vivo imaging and identification of the passive constraint structures available yet. That has left two approaches to the problem of assessing passive constraint activity: in vitro studies and mathematical models.

The greatest amount of effort has been expended on in vitro evaluations of the roles of ligaments. One of the seminal investigations was performed by Brantigan and Voshell [26]. This paper is notable for its listing of the contradictory statements available in the literature up to that time, as well as the experimental conclusions. The behavior of the ligaments was observed by examination of their states in various positions and orientations of the joint. One of the more enduring observations in the study was that the passive constraints in a fresh cadaveric specimen permitted ranges of rotation about the non-flexion axes.

Most of the subsequent studies were similar in intent, but attempted to make more quantitative observations. Two general approaches are used. The first is the stiffness method [191], in which known displacements are applied to either the tibia or femur while the resultant torques and forces on the other bone are measured [37, 191, 192, 193, 224]. Motions may either be constrained [37] and a standard materials testing machine load cell used to estimate the loads or a six degree-of-freedom dynamometer used to monitor all the forces and moments [191]. After measurements on the intact knee, ligaments are cut in sequence and the resulting change in the stiffness of the joint recorded. The order
of ligament sectioning is varied to ensure that a general observation is made. Ligaments are then identified as primary or secondary constraints to a particular motion based on how the stiffness changes for a particular sequence of cutting. The second approach is the flexibility method [191]. In this method a known load is applied and the relative displacements measured [88, 92, 93, 159, 234, 255]. Sequential ligament sectioning is then performed in the same manner to get an estimate of the relative constraint properties of each ligament. Piziali et al. [191] contended that application of a single known force to a system like the knee was more difficult than application of a known displacement. One of the arguments for flexibility testing is that application of a load or moment is consistent with clinical evaluations of ligament integrity.

There are several problems inherent with both approaches. Assuming Brantigan and Voshell’s [26] initial observation and supporting clinical evidence of laxity in the joint is correct, the possible relative displacements of the tibia and femur define a region in a six degree-of-freedom configuration space. The configuration space is a geometric entity, defined by the geometry of the constraints. Neither method can assure that the induced ligament strains are within a physiological range or not. Although points are being found on the boundary of a configuration space, it is not clear that it is a consistently defined configuration space. Added to the variability expected between specimens, the quantitative approaches offer little more than a qualitative assessment of ligament function.

Mathematical models have been used to mimic the in vitro approaches [54, 80, 281]. The models of the constraint geometry are subjected to displacements [54] or loads [281] and the resulting ligament length patterns observed. The latter approach requires a model of the stiffness characteristics of the ligament as well as the geometry.

Another model-based approach to the analysis of constraint activity was proposed by Storace and Wolf [249] in a study of the role of finger tendons. A configuration space
for the finite kinematics of the model system was defined. Then the limits imposed on motion within that configuration space by each tendon were determined. The resulting map of the constrained configuration space contained all possible relative displacements of the system for each tendon strain level. This generated a geometrically consistent set of motion boundaries and identified the regions where the different constraints were effective.
Chapter 3

Kinematic Measurements

3.1 Introduction

An enormous amount of research over the past fifty years has done little to advance understanding of the kinematics of the human knee. The primary cause for the persistent confusion reflected in the literature is a failure to appreciate the fundamental kinematic properties of the system. This is directly attributable to the absence of adequate kinematic data. A comprehensive program of measurements of skeletal knee movement must underly any investigation of knee kinematics. Only when sufficient data are available to identify the screw systems of the knee will progress in the prevention and treatment of the loss or degradation of normal knee function become possible.

There are two immediate applications for knee kinematic measurements. First, clinical practice will benefit from insights into characteristic knee behavior that may be drawn directly from the analysis of kinematic data. Simple experiments could identify whether motion at the knee varies with task and load, and whether this movement is spatial or predominantly planar. However, kinematic measurements alone cannot reveal individual ligament activity in a designated task or how changes in muscle activation patterns affect joint stability. A second and more powerful use of the measurements is in aiding the development of a mathematical model of the knee. Measurements are needed initially
to identify the minimum model order that will accurately represent knee motion. Once established, a knee model may be used to explore the roles that passive and active control elements take in determining the motion at the knee joint. Additional kinematic measurements are required to evaluate the accuracy of the predicted motions and behavior of the control elements.

An initial set of kinematic experiments was performed. The primary objective of these measurements was to provide a foundation for the development of a mathematical model of normal knee motion. Experiments were designed to evaluate whether the motion was planar or spatial, how many kinematic degrees of freedom (DOF) the knee constraint system allowed and whether the motion changed with load and activity. Since these measurements were exploratory, and many more will be necessary to fully characterize the knee, a second goal was a careful examination of the problem of measuring skeletal kinematics. This would lead to selection of kinematic tools that would facilitate further measurements. Design issues such as selection of experimental tasks, data processing algorithms and methods for data presentation were evaluated on the basis of ease of comparison with kinematic data from different experiments and with model predictions of joint motion. Additional experiments were carried out to test different methods of acquiring skeletal motion data.

The chapter begins with a discussion of the planning of the kinematic measurements of the knee. Emphasis is placed on the unique problems present in designing a set of experiments to characterize the kinematics of the knee or any other biomechanical system. Results of the primary kinematic experiments are presented next, followed by selected results of the additional measurements. The discussion covers the significant features of the primary kinematic data and the complications affecting making additional measurements to better characterize the knee.
3.2 Experiment Design Issues

To obtain accurate and useful results, experiments must be designed with the same care with which a good machine is designed. Lewis Thomas made a succinct summary of the fundamental issues of experiment design when he noted that:

... measurement works when the instruments work, and when you have a fairly clear idea of what is being measured, and when you know what to do with the numbers when they tumble out. [263]

All three points require elaboration, but the pivotal requirement from the perspective of kinematic measurements of a biomechanical system is to establish what is to be measured. Decisions about how to make the measurements, data processing and data presentation are simplified if this is clearly defined.

3.2.1 What to measure

It is disingenuous to simply identify the skeletal kinematics of the lower extremity as the physical quantity to be measured. In order to plan an effective set of measurements the object of the experiments must be defined in greater detail.

The skeletal members are the frame of the locomotor system and it is the motion of that frame, not the surrounding soft tissue, that is of interest. Most of the available means of measuring kinematics do not readily access the bone. An assumption is generally made that when markers or instruments are placed on the skin over bony landmarks the measured motion is an accurate reflection of the kinematics of the bone despite the intervening soft tissue. Although there is only limited data available, recent results indicate that this is a poor assumption [157, 271].

The objective of the experiments is to record the full, three-dimensional relative
motion of the tibia and the femur. It is essential that the measurements are completely unbiased with no \textit{a priori} assumptions about the kinematics. Many measurements of lower extremity kinematics are made with markers placed at approximate joint centers to define segment endpoints [157]. This assumes planar motion and consequently out of plane motions are detectable only by changes in segment length, negating the purpose of the measurements.

The kinematics of the knee are not independent of the movements of the rest of the lower extremity. The knees are two joints in a kinematic chain (see Chapter 2). Determination of the mobility of the lower extremity chain demonstrates that very different demands are placed on the knee depending on whether the chain is open or closed. Consider the lower extremity in single support stance as in Figure 3.1. Assuming one foot to be fixed to ground, there are seven links and six joints in the chain. The hips are essentially spherical joints [260] with three degrees of freedom. Measurements indicate
that the ankles have two degrees of freedom [66, 149, 239]. For an unspecified knee joint, the general mobility criterion gives the mobility of the chain as:

\[ M = 10 + 2f_{\text{knee}} \]  \hspace{1cm} (3.1)

In double support the lower extremity forms a closed polygon with ground and the mobility of the chain is reduced to:

\[ M = 4 + 2f_{\text{knee}} \]  \hspace{1cm} (3.2)

Overall mobility of the lower extremity is dependent on the number of degrees of freedom available at the knee, particularly in double support stance.

In addition to the cross-coupling between the joints and limbs of the lower extremity due to the kinematics of the chain, there is coupling through the actuation system. Active control of the joint is provided by fourteen muscles; nine of them are biarticular. Measurements of knee kinematics must either include concurrent measurements of motion at the hip and ankle or incorporate additional restrictions on the movements of those joints.

All knees are not created equal. There are catalogued anatomical differences in bone size, relative orientation of the kinematic pairs, the presence of some ligaments [28], and other features. Differences are a function of height, sex, age, and a host of other factors. There is no established correlation between variations in kinematics and any of the nonpathological anatomical variations. For kinematic measurements to have any general application, a 'normal' knee must be defined. One possible approach is to attempt to establish a statistical kinematic norm. Chao [43] initiated a normative model for the knee in gait. The kinematic data were obtained using externally mounted goniometers, and so are not unquestionably of skeletal kinematics. None of the reported studies of strictly skeletal motion [138, 150] have included a sufficient number of subjects to establish a statistical norm. An alternative is to assume that the anatomical variations do not
impose any more than minor perturbations on a distinguishable pattern of characteristic knee motion. Normal is then defined as equivalent to being free from any prior history of pathological changes.

The feasible set of kinematic configurations for the lower extremity and the knee is enormous. Each additional degree of freedom multiplies the set of possible motions by a factor of infinity. Any realistic measurement protocol must rely on selected experiments to characterize the significant features of the space of possible kinematic configurations. This requires identification of a set of tasks for which measurements will be made. Much of the prior work has concentrated on monitoring tasks related to activities of daily living [231] or athletics. Activities of daily living include gait, stair climbing and rising from a chair. These are all complex movements involving interaction and coordination of movements about all the joints in the lower extremity. From the standpoint of developing a mathematical model, choosing a set of simpler actions would be more valuable. Important features of simple tasks would be the use of a full or large range of motion, the nominal restriction of the motions of the other joints, and variation of the load and weight bearing requirements. Tasks should be selected so that as many of the boundary conditions as possible are defined by the experimenter.

3.2.2 How to measure it

In vivo versus in vitro testing

Selecting appropriate instrumentation is only half of the problem of how to measure the kinematics of the knee and lower extremity. Before opting for a particular measurement system, a difficult choice between in vivo and in vitro testing must be made. Evaluated in terms of the experimental objectives, each approach has distinct advantages and disadvantages.
Monitoring skeletal motion in vitro is straightforward. Instrumentation may be attached directly to the bones. Alignment and placement of transducers continue to be important, but there is no question that bone motion is being measured. In vivo skeletal motion cannot, as yet, be measured directly with any confidence without resorting to either exposure to ionizing radiation [119] or fixation of markers or instruments to the bone through an invasive procedure [139, 150, 177]. Woltring reported [286] measurements of the relative positions of two static limb segments using magnetic resonance imaging, but the elapsed time required for each scan still precludes the recording of relative segmental motion.

Until noninvasive methods of measuring skeletal motion become practical, ethical constraints will prevent acquisition of sufficient in vivo kinematic data to establish a statistical model of normal knee motion. There is a greater likelihood of testing significant numbers of joints in vitro, but the composition of the available subject population becomes an important consideration. The pool of knees available at autopsy is drawn from predominantly elderly donors. There is a much greater risk of prior pathological change to donor joints from the elderly, and effects on the kinematics due to age-related changes to soft tissue mechanical properties are currently undetermined [180, 288].

The potential for age-related changes in soft tissue material properties raises a more general concern with the validity of an in vitro preparation as a model for in vivo kinematics. Soft tissue (ligaments, tendons, etc.) material properties are a function of temperature, humidity, and several other parameters in addition to age. Black [19] demonstrated that some properties also depend on the length of time following autopsy. Other authors reported that tensile properties of ligaments [74, 290] and articular cartilage [264] are not changed from immediately after autopsy by repeated freezing and thawing cycles over extended periods of time. Seering [224] detected no difference in whole knee joint force-displacement characteristics immediately following autopsy and after twenty-one days of
storage at -20 C. Post-mortem change in tissue mechanical properties is essentially an unexplored area, particularly with reference to the possible effects on kinematics.

Selection and application of appropriate boundary conditions poses another problem for measurements. Relative motion of the thigh and shank can occur around an entire family of instantaneous screw axes. To select motion about a particular axis or locus of axes, the motion of the limbs must be constrained or appropriate control forces and moments applied. Specific tasks may be defined in vivo, but only within limits. Due to lack of direct access to the bone, tasks requiring constraint of a particular limb can only be executed approximately. Measurement of forces in vivo is even more difficult than measuring kinematics. Net external forces and moments may be measured using force platforms and other transducers, but instrumentation does not exist to monitor individual muscle forces during a specified task. The myoelectric signal (MES) is an indicator of muscle activity [58], but correlations of MES with individual muscle forces exist for only a few isometric activities [199, 198, 228]. Several groups have attempted to estimate individual muscle forces from input kinematics and net joint forces and moments using linear and nonlinear programming algorithms [53, 66, 101, 225]. Success was limited by several factors including inaccurate kinematic data, inadequate joint models and the inability to predict co-contraction of agonist-antagonist muscle pairs [188].

Application of boundary conditions is a more severe problem for in vitro testing. Any experimental apparatus for mounting the in vitro joint preparation must be designed so that none of the relative degrees of freedom of the joint are artificially constrained. It must also be designed so that a range of appropriate forces and moments, reflecting the individual muscle actions, can be applied to the preparation. These required forces and moments are not known. By collecting muscles into functional groups, the number of actuators in the equations of motion may be reduced until a linear system of equations results [173]. This approach yields an estimate of the net force in a muscle group, for
example the quadriceps, which may be used as a boundary condition for an in vitro experiment.

Measurement system selection criteria

Any experiment can be ruined by the selection of an inappropriate measurement system. There are several criteria which may be used to identify desirable measurement system characteristics. The demands specific to a particular experiment will affect which criteria are weighted the most heavily. Among the more important considerations affecting the selection of a kinematic measurement system are: (1) resolution, precision and accuracy; (2) bandwidth and sampling rate; (3) degree of operator intervention necessary; (4) requirements for a special environment; (5) exposure of the subject to potential hazards; and (6) the amount of equipment worn by the subject. All of the criteria apply to in vivo testing; the first four are most important for in vitro experiments.

The first two items define the limits of the measurement system. Resolution is the smallest quantity that the system can distinguish, while precision and accuracy represent how well the measurement can be made [60]. Bandwidth represents the range of frequencies that an instrument will respond to, making it the temporal equivalent of resolution. For digital systems, the theoretical bandwidth is determined by the Sampling Theroem [248] to be one-half the sampling rate. In practice, particularly when differentiating data [170], the sampling frequency should be five to ten times the highest frequency component expected in the data. Antonsson and Mann [10] showed that in gait eighty percent of the frequency content was below ten Hertz, so that the preferred sampling frequency is in the range from fifty to one hundred Hertz.

Ideally a measurement system will be fully automated, with all data manipulation from the raw data to the final output performed without user involvement. Operator intervention slows the data processing and is a conduit for the introduction of error. In
practice it is difficult to achieve fully automated processing and many of the systems used for kinematic data acquisition require some user interaction [138, 258].

The final three criteria assess how the measurement system may limit or modify the planned protocol. Safety and comfort affect the ability of a subject to perform the assigned tasks reliably. Exposure to hazards such as ionizing radiation reduce the amount of data that may be collected since exposure time must be limited. Special environments, including limited viewing volumes or darkened rooms, restrict the tasks that may be monitored. Similarly, if encumbered by bulky markers or electronics the subject will not be able to execute tasks in a natural manner. The choice of instrumentation can affect the scope of the experimental protocol.

Measurement system options

The technology for measuring human motion has advanced significantly since Muybridge first used sequences of still photographs to record human and animal gait in the late nineteenth century [179]. There are a wide variety of different systems available capable of measuring three-dimensional skeletal joint kinematics, each having a different blend of positive and negative features. One drawback common to all systems except those based on radiography is the requirement that instruments or markers be attached directly to the bone in order to monitor skeletal motion. With that in mind a brief exposition of some of the attributes of the different approaches follows. More complete reviews of the different technologies are given in Antonsson [9], Krag [134] and Rowell and Mann [212].

Measurement systems may be classified as either recording joint kinematics or segmental kinematics. Joint kinematics measurement systems directly sample the relative motion of two rigid bodies. By recording the target motion directly there are fewer potential sources of error, but there is no inertial reference for the data. The most common in vivo joint measurement systems are multi-axial goniometers [43] or instrumented spatial
linkages [130, 231]. These devices are bulky, massive and uncomfortable for the subject. Typically they are used at the knee due to their bulk and since it is difficult to find accessible bony reference points at the other joints in the lower extremity.

Segmental kinematic measurement systems have an inherent inertial reference. Since joint kinematics are calculated from the measured kinematics of the two adjoining segments, there are more potential error sources than if the joint kinematics were measured directly. Most of the systems measure the positions of markers and calculate orientations and kinematic derivatives from that base. Some type of camera is used in a majority of the systems to capture the kinematic information, with the implication that at least two cameras are required to obtain the three-dimensional kinematics of a rigid body and the viewing volume is restricted to the intersection of the fields of view of all the cameras.

Systems may be further subdivided into passive and active marker groups. Passive marker systems require no umbilical connection between the subject and the data acquisition equipment. Markers can be lightweight and relatively unobtrusive. They are ideal for use in activities where the subject must remain unencumbered, such as athletic events. Cinematographic cameras [138] have high frame rates, but require manual digitization of the film. Video cameras [70, 157] may have automatic digitization, although markers must be initially manually identified and then manually reidentified when marker trajectories cross or approach each other, but they have lower sampling rates than are typically available with cine film. Markers are also reflective and generally need to be of large diameter to get reliable images. Rotating mirror systems [212] avoid operator intervention with passive prismatic reflectors, identifying the different markers automatically by the color of light reflected. Currently available systems are restricted to fewer than ten markers, allowing tracking of only three rigid bodies under ideal conditions.

Active marker systems cover a variety of technologies. Among the available systems are optoelectronic cameras [11], laser detectors [155], accelerometry [151] and acoustic
systems [134]. Accelerometry possesses the advantage of measuring acceleration and thus integrating, a noise suppressing process, to obtain velocity and position. However, there is no inherent inertial reference. Some additional measurements are required in order to identify the gravity vector and provide the constants of integration. Optoelectronic systems have high sampling rates and automatic digitization but require an umbilical and are very sensitive to reflections.

3.2.3 What to do with the data

Presentation of the experimental data is as critical as any of the preceding stages of designing kinematic experiments. If the results of experiments cannot be effectively communicated to the appropriate audiences or compared to data from other experiments, the principal goal of the measurements is lost.

The diversity of the audience is a particular problem. Clinicians describe joint orientations in terms of angles with specific anatomical references. Engineers are familiar with discussing the translational and angular displacements of rigid bodies in terms of the relative movements of Cartesian coordinate frames attached to rigid bodies and have a variety of mathematical tools available to them for that purpose. Confusion frequently arises when data presented in one form are compared to similar data presented in another form.

There are two general approaches to presenting the kinematics of rigid bodies. One is to present the kinematics of finite movements, the displacements and changes in orientation of the moving segments. The alternative is to compute and display the instantaneous kinematics, based on the loci of instantaneous screw axes.
Finite kinematics

Presentation of the relative displacements of two rigid bodies has several apparent advantages. Most of the measurement systems used to acquire human movement data record marker displacements. Resolving rigid body translations and orientations from the three-dimensional marker coordinates is a relatively straightforward procedure. Several algorithms are available for that purpose [246, 219, 158]. End users of the data also tend to be more comfortable with the concept of displacements, which are directly observable, than with the abstractions of differential geometry associated with instantaneous kinematic analysis.

The spatial displacements of a rigid body may be described by tracking the movement of points or lines in the body. Rooney [208] gives a thorough summary of both classes of transformation.

Point transformations are the most commonly used. The fundamental form is:

\[ R = Mr + p \]  

(3.3)

where

- \( R \) = New position vector of the moving point
- \( r \) = Original position vector of the moving point
- \( p \) = Position vector of the origin of the moving coordinate system
- \( M \) = Orthogonal rotation matrix

An alternative form, widely used in robot analysis, is based on the use of homogeneous coordinates for three-dimensional marker locations [227]. In that case, the transformation becomes:

\[ R = Sr \]  

(3.4)

where

- \( R \) = New position vector of the moving point
- \( r \) = Original position vector of the moving point
- \( S \) = Transformation matrix incorporating \( p \) and \( M \)
In both cases, the rotational displacements are described by the nine elements of the orthogonal rotation matrix $M$. In order to obtain a physically interpretable representation of orientation changes, many different reduced order parameterizations of the rotation matrix have been proposed.

A rigid body has three independent rotational degrees of freedom, consequently three parameter representations of the orientation of a body are favored. Although there are no three parameter one-to-one mappings of the space of three-dimensional rotations [252], singularities may be avoided by restricting the range of the movements being studied. One approach widely used in engineering practice, particularly in aerospace, is to define a set of three Euler angles [87]. Euler angles are combinations of three sequential rotations about the axes of intermediate coordinate frames. There are many combinations, with the sequence dictated by the requirements of the problem. Physical interpretation, particularly of the third angle, is difficult because of the order dependence of the rotations.

Grood and Suntay developed a set of 'clinical' angles and translations [91] in an attempt to avoid some of the difficulties inherent in the use of Euler angles and achieve a representation that would be understandable and acceptable to both engineers and clinicians. The flexion axis is fixed in the femur and the internal/external rotation axis is fixed in the tibia, with the third rotation axis constrained only to being perpendicular to both of the other axes. By fixing one rotation axis in the moving body they obtained a similarity transform [209] and eliminated order dependence but not singularities.

A two-to-one mapping of the rotation space is possible with the four parameter unit quaternion (see Appendices A and B). The components of the unit quaternion, also known as Euler parameters [247], define a rotation axis through the coordinate frame origin in a manner consistent with Euler's Theroem of Finite Rotations [45]. As an alternative to parameterizing the rotation matrix in terms of the quaternion, a unit
quaternion rotation operator may be defined, such that:

\[ R = M(q)r \]
\[ = qrq^{-1} \]  

(3.5)  

(3.6)

where

\[ R = \text{New position vector of the moving point} \]
\[ r = \text{Original position vector of the moving point} \]
\[ q = \text{Unit quaternion (} q = d + a_i + b_j + ck \text{)} \]
\[ q^{-1} = \text{Reciprocal unit quaternion (} q^{-1} = d - a_i - b_j - ck \text{)} \]
\[ M(q) = \text{Rotation matrix parameterized in terms of } q \]

Application of the unit quaternion operator follows the rules of quaternion algebra (see Appendix A). Comparisons have shown the quaternion operators to be computationally more efficient than rotation matrices [78] and Euler angles [48].

An alternative approach to observing the motion of points on a rigid body is to track the movement of lines embedded in the body. Lines may be represented in terms of Plücker coordinates [296], which are a set of homogeneous coordinates for a line. Matrix transformations for lines are similar in form to the point transformations [208]. One form is the dual\(^1\) orthogonal transformation matrix. Alternatively, a dual quaternion operator [292], which embodies the finite helical axis (FHA), may be defined. In the limit the FHA between two successive positions is the instantaneous helical axis (IHA). Chasles [113] demonstrated that the most general representation of rigid body spatial motion is a screwing action about a line in space. Peterson [189] showed that for angular excursions of up to twenty degrees the FHA was a good approximation to the IHA. Several algorithms for computing the FHA from three-dimensional marker coordinates are available [189, 275, 285]. There have been extensive studies of the error sources and sensitivities of FHA calculations [57, 245, 284]. While the estimate of the magnitude of

\(^1\) Dual numbers are described briefly in Appendix A.
the twist about the FHA is robust, the calculated axis orientation is very sensitive to small changes in the input data.

All the finite orientation schemes share a common problems of sensitivity to the orientation of the chosen reference coordinate system and the varying definitions of a neutral position and orientation. For the Euler angle parameterization of orientations, Ramakrishnan \textit{et. al.} [197] found that slight misalignments of the reference coordinate system did not appreciably change the flexion angle but resulted in the introduction of significant components to the two succeeding angles. These problems affect any chance of making reliable comparisons between data from different experiments or basing clinical decisions on measured changes. Unless the reference system is aligned for every set of experiments, quantitative comparison of data becomes impossible.

**Instantaneous kinematics**

Instantaneous kinematic analysis is required to study the movement, as opposed to just the displacement, of a rigid body in three-dimensional space. The movement of both points [273] and lines [131, 216, 253] of a rigid body may be analyzed. A more extensive set of tools is available for examining the motion of the lines of a body, in particular the instantaneous helical axes (IHAs). As shown in Chapter 2, the IHA and its parameters are functions of the translational and angular velocity vectors of the moving body.

The IHA is the coincidence of two lines, one embedded in the fixed body and the other in the moving body. In a general constrained spatial movement the loci of lines in the two bodies will define a pair of surfaces, called the fixed and moving axodes of the system [131, 293]. For a given configuration of a mechanical system the axodes are characteristic of the system and do not change as long as the configuration of the system is fixed. If facsimiles of the axodes were constructed and the moving axode rolled, without slip, over the fixed axode the movement of the generating mechanical system would be reproduced.
exactly. By fixing the reference coordinate system to one of the surface generators, coinciding with a particular event such as heel-strike in the gait cycle, a representation of the movement independent of specific experimental conditions results. If adequate measurements of relative joint velocities are available the axodes are an ideal means of comparing joint motions. As a simple example, the fixed axodes for a wheel rolling on a flat surface without slip and with slip are shown in Figures 3.2a and 3.2b, respectively.

The geometry of the axodes facilitates quantitative comparison of movements. Since the surfaces are generated by straight lines, the IHAs, the axodes are ruled surfaces [69]. The differential geometry of ruled surfaces is well-developed [251]. Dual vectors were applied to the analysis of the properties recently [163, 216]. Surfaces may be described in terms of the dual curvature, dual torsion, distribution parameter and striction curve [216, 293]. All these parameters may be calculated from the generators of the surface, which are functions of the velocity of the moving body.

The kinematic geometry of many mechanical systems cannot be described by axodes. Either the IHA is undefined ($|\Omega| = 0$) or the loci of IHAs do not define surfaces. Examples include mechanisms containing simple revolute kinematic pairs such as a grinding wheel (Figure 3.3a) and a single degree of freedom pendulum (Figure 3.3b). A more general approach for describing the kinematic geometry of rigid bodies in spatial motion are instantaneous invariants [131, 273]. The geometric properties of a spatial movement are completely described to first order by a single scalar parameter, the pitch. A second scalar parameter, the pitch rate, and a dual constant are needed to describe movements to second order. Kirson and Yang [131] show that to characterize a spatial motion beyond the second order requires an additional six parameters for each order above the second. Schaaf [216] showed that the dual curvature and torsion functions, as well as other geometric properties, for a ruled surface may be expressed in terms of the instantaneous invariants of a motion.
Figure 3.2: Fixed axode for a wheel rolling on a flat surface, (a) without slip ($V_s = 0$) and (b) with slip ($V_s \neq 0$).
Figure 3.3: IHA loci for systems that have degenerate axodes. (a) Grinding wheel, and (b) Single degree of freedom pendulum.
Instantaneous kinematic analysis is not restricted to characterizing the kinematic geometry of the movements. Ball [14] introduced the concept of screw systems in his seminal work on the theory of screws (see Chapter 2). Several authors have made significant additional contributions to understanding screw systems [82, 83, 113, 154]. The underlying principle is that constrained movements of a rigid body may be described in terms of a finite set of screw systems. These screw systems have a precise relationship with each other that are important for understanding the motion of the body and how that motion may be controlled [154]. For a body moving in three-dimensional Euclidean space, the number of independent kinematic freedoms and the number of independent constraints must sum to a maximum of six. Given a system of \( n \) independent kinematic screws extended screw theory [182] can be used to study the possible sets of \( (6 - n) \) independent constraints [189]. The converse is also true. With a measured set of \( k \) IHA's, or screws, the problem is to identify the \( n \) independent kinematic screws that form a basis for the set. Hunt [113] showed that the number of independent screws was equal to the rank of the \((k \times 6)\) matrix formed by taking each IHA expressed in screw coordinates as a row. Linear algebra techniques have proved effective in analyzing systems of screws [253, 254]. A technique for determining an orthogonal basis for a set of screws using the definition of the inner product of two screws has been developed [253].

The critical problem with widespread application of instantaneous kinematic analysis is data quality. In order to generate the axodes, the angular and translational velocity of the moving body must be determined. In most cases this involves differentiation, a noise amplifying process, of the position data supplied by the measurement system. The typical measurement system available for use in human movement studies does not provide smooth enough data to allow accurate calculation of derivatives. Calculation of the axodes is difficult, but the calculation of surface curvatures and torsion from experimental data is currently impossible. Curvature and torsion are functions of the second and third
derivatives of the generators of the ruled surface, which are functions of the velocities of the moving body. Calculation of these derivatives is not practical with the quality of most data available. The linear algebra techniques for determining the orthogonal basis of a set of screws require calculation of a \( m \)th order Gramian (see Appendix A) to check for the independence of \( m \) screws. The necessary screw products amplify the uncertainty of the individual IHAs and introduce thresholds for valid evaluation of screw independence.

3.3 Experiment Design

3.3.1 Objectives

Two principal goals were set for the initial experimental phase of the project:

- Obtain direct measurements of the three-dimensional, skeletal kinematics of the knee in different tasks.

- Obtain measurements of knee kinematics using skin-mounted markers for comparison with the data from the first part.

The primary objective of the kinematic experiments was to obtain data on the relative skeletal movements at the knee. Direct skeletal measurements were necessary for two reasons. First, exploratory kinematic measurements were needed as a prerequisite to the development of a mathematical model of the knee. The limited amount of skeletal movement data in the literature indicated that motion at the knee is spatial [20, 112, 138, 150]. Data were primarily of normal gait and did not look at differences in loading between tasks. For future studies, a set of precise, accurate skeletal kinematic data corresponding
to specific tasks was needed as a standard for comparison for in vivo movement studies with skin-mounted markers. This would allow a quantitative evaluation of how well data collected using different external marker mounting schemes reflected the underlying skeletal kinematics. There have been no such quantitative comparisons made for any of the kinematic measurement systems in use. The viability of the movement of the externally mounted markers as a model of the skeletal motion is usually a tacit assumption in most human movement experiments.

Skin-mounted markers were used in a second set of experiments as the initial step in evaluating different external marker mounting schemes. Direct attachment to the skin was chosen for evaluation because it is potentially the easiest method to implement and probably the least obtrusive. Many laboratories currently use a similar system of attaching markers\(^2\).

### 3.3.2 Subject

Exploratory kinematic data could be obtained either in vivo or in vitro, but for the standard it was essential to have skeletal kinematics for a particular set of tasks. Protocols for human movement or gait experiments center around the performance of a prescribed set of tasks. Selecting appropriate forces and moments to produce task-related movements in vitro would be difficult, if not impossible, due to the scarcity of reliable joint force and moment data. Consequently, the direct kinematic measurements were made in vivo.

A single subject performed all the experiments. The study was limited because of the emphasis on measuring skeletal kinematics. This required either direct attachment of markers to the bone or exposure to ionizing radiation. Prior [150] and concurrent [138] studies have recorded data from multiple subjects using markers mounted on skeletal

\(^{2}\text{Most laboratories do not use arrays of markers, but attempt to a priori identify so-called "joint centers" or "axes of rotation" on the basis of identifiable bony landmarks.}\)
pins. Obtaining a reference set of kinematic data for evaluating less drastic procedures for acquiring knee movement information better served the long-term needs of this research. The exploratory nature of the measurements made a better understanding of the data more important than planning for multiple subjects at this time.

The subject was a volunteer, Caucasian male with no prior history of knee pathology. At the time of experiment he was approximately 1.91 meters tall and weighed approximately 91 kilograms. As an orthopedic surgeon he was fully aware of the potential hazards and contributions of the experiments. A copy of the informed consent document is included in Appendix D. No compensation was offered for participating in the experiments.

3.3.3 Measurement System

Selection

At the time of the experiments, no kinematic measurement system available was proven capable of acquiring skeletal kinematic data noninvasively. Measurement system options fell into two classifications: (1) Large-scale X-ray stereophotogrammetry, or (2) any other segmental measurement system. The X-ray approach offered more direct reference to bony landmarks, but had several disadvantages in addition to the inherent problem of radiation exposure. All of the X-ray stereophotogrammetric systems described in the literature [270, 123, 213, 257] had small viewing volumes, unsuitable for monitoring gait or other large-excursion motions of the lower extremity. A cinematographic X-ray system was used in animal studies [119], but with only a single camera and unspecified accuracy and resolution. Considerable development work would have been necessary to use an X-ray measurement system. Bony landmark identification on X-rays is too imprecise for high accuracy measurements [123, 257], so radio-opaque markers must be implanted to define a body-fixed system in each limb segment. Acquisition of accurate data would
then require both an invasive procedure and the exposure to radiation.

The other options were similar in their requirement of an invasive procedure for attaching the markers or transducers to the bone. Surgical bone pins had been used for mounting markers for a cinematographic measurement system [150] and accelerometers [151] on bone. Accelerometry was ruled out as the sole measurement system because of the need for additional measurements to ascertain the orientation of the gravitational acceleration vector. Video and film cinematography were eliminated due to the necessity of operator intervention. Available video systems had sampling rates limited to the 60 Hertz bandwidth of television. An optoelectronic system offered the best possible overall performance. In addition, an optoelectronic based system had been under development in the Newman Laboratory for over a decade [50, 8, 9] and considerable practical experience had been acquired with the idiosyncracies of the system.

Hardware

Human movement data were recorded using the TRACK (Telemetered Acquisition and Calculation of Kinematic Data) system in the Eric P. and Evelyn E. Newman Laboratory of Biomechanics and Human Rehabilitation at M.I.T.

Several different computer systems were used in the collection and processing of the kinematic data. The rapidly expanding capabilities of the machines is testament to both the improvements in available hardware and the computational capacity required for proper study of the kinematics of complex systems. Data collection and processing was controlled by a PDP 11/60 minicomputer (Digital Equipment Corp., Maynard, MA). Most of the data processing was performed on a VAXStation II minicomputer (Digital Equipment Corp., Maynard, MA). The resulting instantaneous kinematics were displayed on a Personal IRIS graphics workstation (Silicon Graphics Corp., Mountainview, CA).
Position data were recorded using the Selspot I kinematic measurement system (Selcom AB, Partille, Sweden). Selspot I includes two optoelectronic cameras, up to thirty infrared light emitting diodes (LEDs) for markers, a camera coordinating unit and an LED control unit. LEDs are fired sequentially at 10 kiloHertz; for thirty LEDs the frame rate is 315 Hertz. Each camera contains a lateral photo-effect diode. The centroid of an LED image on the photodiode is converted to a pair of analog camera coordinates, which are digitized at the camera coordinating unit. Camera resolution is 1024 x 1024.

For the experiments, the cameras were mounted on optical benches (Klinger Scientific Corp., Lake Success, NY) as shown in Figure 3.4. The cameras were mounted so that the viewing volume was centered over the force platform. Camera mounting angles and positions are detailed in Appendix C. With the above arrangement, the system had a theoretical resolution of approximately one millimeter (0.001 m) for translational and 20 milliradians for angular displacements [9].

Synchronous with the kinematic measurements, foot-floor reaction forces and moments could be recorded with a Kistler piezoelectric force platform (Kistler Instrumente AG, Winterthur, Switzerland).

Algorithms

Data acquisition and processing was completely automated under the control of the TRACK software package. The TRACK software and hardware design was initiated by Conati [50]. Several people have made contributions as the system evolved in the ensuing years [8, 9, 184, 262]. The form of the package, TRACK III, used at the time of the experiments is described in Anttonsson’s doctoral thesis [9]. Prior to processing the data from these experiments significant additions to and modifications of the organization of the code were made.
Figure 3.4: TRACK III system hardware in the Eric P. and Evelyn E. Newman Laboratory for Biomechanics at M.I.T.
There are no *a priori* assumptions about the underlying kinematics of the system being studied using TRACK. When a rigid body is in spatial motion, the kinematics of the body in absolute or inertial coordinates are completely described if the locations of any three noncolinear points on the body are known. For measurement purposes TRACK assumes that each segment in a kinematic chain, such as the shank of the lower extremity, is an independent rigid body. Arrays of noncolinear LEDs are attached to each segment and used to define a body-fixed Cartesian coordinate system. The position and orientation of the segment, with the segment represented by its embedded Cartesian coordinate system, are then calculated in inertial space. By measuring the kinematics of each segment independently, the relative kinematics of any two segments may be estimated without any *a priori* assumptions about the mechanical coupling between them.

Central to the TRACK data processing algorithm are the error elimination routines. Raw camera coordinates and three-dimensional LED coordinate data are passed through five stages of screening before the rigid body position and orientation for a segment are calculated. A copy of the original raw data is used in the processing, so that the original data remains unchanged and available for multiple evaluations.

The first two stages of the screening are applied to the raw camera coordinate data. The Selspot system flags LEDs not seen by one or both of the cameras and LEDs which saturate the photodiode. Flagged points are eliminated prior to any further processing. In the second stage, camera nonlinearities are corrected for in the raw camera coordinate data. Lens and photodiode nonlinearities are accounted for in an internal calibration table for each camera [11].

The corrected camera coordinates are used to generate the three-dimensional spatial coordinates of each LED. As part of this procedure, the data are screened for skew-ray errors. Ideally, the rays traced from the image of an LED on each photodiode through the focal point of the camera will intersect in the viewing volume, with the point of
Figure 3.5: Rays from the two cameras are typically skew. The midpoint of their mutual perpendicular is used as the LED position.

intersection being the location of the LED. In practice, rays from the two cameras are usually skew. TRACK allows the experimenter to select an acceptable level of skewness, as defined by the length of the common perpendicular of the two rays, and assumes that the LED is located at the midpoint of the mutual perpendicular (see Figure 3.5). This is particularly useful for eliminating points skewed due to reflections which tend to distribute the intensity of the LED image over the photo-diode and shift the centroid.

An additional check is placed on the LED positions to minimize distortion. The user supplies TRACK with a segment file defining the locations of the LEDs in the body-fixed coordinate systems. TRACK then compares the measured relative positions of the markers to those supplied in the segment file. All LEDs with relative distances exceeding

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a user supplied error bound are eliminated. If fewer than three markers remain at the 
end of this process the segment data is flagged as bad.

All the preceding procedures address measurement errors, but only incidentally affect 
measurement noise, if at all. All experimental data are inherently noisy, with unwanted 
information contaminating the desired signal. Instantaneous kinematic or dynamic analyses require derivatives of experimental data. Numerical differentiation is a noise amplifying process [100]. In order to obtain meaningful derivatives the noise must be attenuated or stripped from the signal before numerical differentiation. Noise attenuation may be addressed in either the frequency domain or the time domain. A time domain approach, data smoothing, was implemented in the TRACK program primarily because of the uncertain frequency characteristics of both human movement data and the measurement system noise. Frequency domain approaches typically assume that the signal and noise power spectra are separate and distinct. Insufficient data on human lower extremity movements are available to make any conclusive statements about frequency content. Antonsson and Mann [10] assumed that the maximum frequencies in gait would be associated with heel strike. Analyses of forceplate data for a heel drop test, modeling heel strike, showed that eighty percent of the signal power was at frequencies below ten Hertz. No other data are available. Measurement system noise for most available systems is also poorly characterized in the frequency domain. Furthermore, there is not enough information to determine if the noise and signal power spectra are distinct or overlap.

Smoothing assumes that the measurement is the sum of the signal of interest and measurement system noise:

$$z_k = y_k + n_k$$  \hspace{1cm} (3.7)

where

$$z_k = \text{Measurement}$$

$$y_k = \text{Signal}$$

$$n_k = \text{Noise}$$

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This assumption allows for overlapping noise and signal frequency spectra. A number of smoothing algorithms have been proposed [7, 56, 62, 102, 267, 272, 282, 291]. Many are based on the work of Reinsch [202, 203] with smoothing splines. Reinsch tried to find a function to minimize:

$$J(f) = \int_{-\infty}^{\infty} (D^m f)^2 dz + p \sum_{i=1}^{n} \left[ \frac{z_i - f(x_i)}{\delta z_i} \right]^2, \quad 0 \leq p < \infty$$

(3.8)

where

- $z_i$ = Measurements
- $f$ = Smoothing spline function
- $m$ = Derivative order criterion
- $p$ = Smoothing parameter

The smoothing parameter, $p$, controls the degree of smoothing. The solution is a compromise between the smoothing and the closeness of fit to the input data. Reinsch's algorithm required selection of the smoothing parameter. Craven and Wahba [52] introduced regularization of the generalized cross-validation (GCV) parameter as a means of automatically selecting the optimal, minimum mean squared error, smoothing parameter. They also demonstrated that, to first order, the smoothing spline had the properties of a Butterworth low-pass filter with a cutoff frequency dependent upon the smoothing parameter and spline order, the advantage over a conventional filter being that the GCV algorithm chooses the cutoff frequency automatically based on an evaluation of all the data. Algorithms combining these results and other developments have been written by Dohrmann [62] and Woltring [282]. Appendix H contains a comparison of the two smoothing algorithms using the approach proposed by Ladin [137]. Performance of the two approaches was essentially identical. Dohrmann's algorithm was chosen for implementation because of greater flexibility and faster execution times.

Smoothing was inserted in the TRACK data processing stream after the inter-LED length error check and prior to the computation of the rigid body parameters for each segment. The nonlinearity of the algorithm for conversion of the marker coordinates to
rigid body positions and orientations was the primary reason for smoothing at this point. All operations on the raw data prior to this stage of the processing were linear. If the central assumption of additive measurement noise is correct, the three-dimensional LED coordinates are the sum of a noise term and a signal at this stage. In addition, all the 'bad' measurements, within the specified limits, have been stripped out of the data. The algorithm serves both as a smoother and as an interpolator, providing a complete set of three-dimensional coordinates from which to determine the rigid body position and orientation. If the additive noise is processed through the nonlinear rigid body parameter calculation algorithm, the resultant positions and orientations will contain products of the noise and signal which cannot be eliminated by the smoothing algorithm.

Smoothed and interpolated three-dimensional coordinates for the LEDs are used to calculate the rigid body position vector and orientation for each segment. Conati [50] incorporated a modified [148] version of an algorithm developed by Schut [219, 220] into TRACK. Schut derived the necessary equations starting from both the orthogonal matrix representation of the finite rotation of a rigid body and the unit quaternion representation. TRACK was modified to include Schut's original unit quaternion formulation. Unit quaternions are a singularity free, 2 $\mapsto$ 1 mapping$^3$ of the space of three-dimensional rotations. Only four values are needed to describe the orientation of a rigid body in an

\[ q = a\sin b + c + d \quad \text{(3.9)} \]

where

\[ a = \cos \alpha \sin \beta \]
\[ b = \cos \beta \sin \gamma \]
\[ c = \cos \gamma \sin \beta \]
\[ d = \cos \alpha \]

\((\cos \alpha, \cos \beta, \cos \gamma) = \text{direction cosines of rotation axis},\)
\(\theta = \text{rotation angle},\)

produces the same rotation as the operator \(-q,\)

\[ -q = -a - b\gamma - c\beta - d \quad \text{(3.10)} \]

giving the two-to-one mapping.

$^3$The unit quaternion operator $q$, $q = a\sin b + c + d$
ing frame. Orientations may be displayed as quaternions, Euler angles or Grood/Suntay anatomical angles. Many of the equations presented in Chapter 2 for estimating the fundamental instantaneous kinematic quantities was incorporated into a second kinematic analysis program. Components of of the unit velocity screw, an estimate of the striction curve for the axode, and the first order instantaneous invariant are the geometric parameters that may be calculated and plotted using NCAR routines. An option to calculate an orthogonal basis for the screw system in each data set is also included. Data for the axodes and moment surfaces, the dual components of the unit velocity screw, may be ported to a Personal IRIS graphics workstation where a three-dimensional color display of the surfaces can be generated. The three-dimensional display includes a movie option which is invaluable in examining the axodes.

LED Arrays

Only the subject’s right leg was instrumented for the experiments. There are four segments in the lower extremity: the foot, the shank (tibia), the thigh (femur) and the pelvis. Five arrays of LEDs were used in each experiment, with two mounted on the thigh and one on each of the remaining limbs.

The LEDs used in making the arrays were 0.004 meter diameter and with a half-power solid angle of 60 degrees (Telefunken AG, Somerville, NJ). LEDs were sorted by intensity and only the strongest ones used.

To define the body-fixed coordinate systems, the LEDs were mounted in planar, rigid, plexiglas arrays. Isotropic distributions of markers have been shown to be statistically optimal for the computation of rigid body motion from three-dimensional point coordinates [285], while planar marker distribution topologies have been shown to produce a bias error. Morris [172] concluded that due to the difficulty in achieving the required

\[4\text{Thanks to Patrick Lord for writing the display program.}\]
manufacturing tolerances for isotropic distributions, the small bias errors induced by using planar arrays were acceptable. The arrays used for marking the pelvis and foot in all the experiments were designed by Antonsson\(^8\) [9]. All the other arrays were constructed for the experiments. Each array contained six LEDs mounted in a circular pattern with a 0.04 meter radius. Excess material was milled out to keep the array mass below fifteen grams. A drawing of an array is shown in Figure 3.6.

The frames for the pelvis and foot arrays referenced bony landmarks and were attached to the limbs with elastic straps. For the direct kinematic measurements, the thigh and shank arrays were mounted on threaded 0.00357 meter (\(\frac{9}{64}\) inch) diameter skeletal pins. The pin diameter was selected after a parametric evaluation of bending due to soft tissue loading\(^6\) and vibration frequencies. Arrays were mounted at the greater trochanter and the lateral femoral condyle on the thigh and at mid-tibia on the shank (see Figure 3.7). In order to assess skin motion effects the arrays for the external kinematic measurements were attached in the same locations on the lower extremity (see Figure 3.8). The skin mounted arrays were attached to the skin using 3M 1512 double-sided adhesive tape\(^7\) (3M, Minneapolis, MN).

### 3.3.4 Tasks

Lower extremity movement was measured during four different activities. Tasks were selected that were representative of fundamental motions, activities of daily living, athletics and clinical testing. Experiments were planned to include voluntary muscular loading across the joint, body weight loading and a relaxation of loading to assess the importance of load on knee kinematics. The primary constraint on the task selection was the limited

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\(^8\)Pictures of the two arrays and corresponding segment files are given in Antonsson's thesis [9]. Due to the limit of a total of thirty LEDs, only seven LEDs (of eight) on the pelvis array and five LEDs (of six) on the foot array were connected. LED 5 on the pelvis array and LED 3 of the foot array were not used.

\(^6\)Tibio-tibial band load estimates were based on the optimisation studies of Patriarco et. al. [188].

\(^7\)The tape was donated by the 3M Medical Adhesives Division.
Figure 3.6: Thigh and shank array LED distribution pattern.
Figure 3.7: Skeletal pin mounted arrays.
Figure 3.8: Skin mounted arrays.
time available with the surgical pins inserted.

The first set of movements were meant to exercise the voluntary range of motion of each joint in the lower extremity. While standing erect in the center of the measurement system viewing volume, the subject was asked to move each joint of the lower extremity through its full range while holding the other two joints on the instrumented leg fixed.

Normal gait was the second task studied. The subject was asked to walk at a natural pace with the instrumented foot contacting the force platform. Both forceplate and kinematic data were recorded. Due to the limited size of the viewing volume, the instrumented leg was typically in view of for only a portion of the gait cycle.

A third task was related to standard clinical tests of the laxity in the knee [297]. The subject was seated on a flat table, with the thigh of his instrumented extremity held by a strap just proximal to the knee. The lower leg was free to swing over the edge of the table. The subject was instructed to completely relax the muscles around the knee. At different flexion angles, the experimenter manually rotated the shank through its full range of movement about a rotation axis approximately aligned with the long axis of the tibia.

The final activity was related to athletic movements. The subject jogged to the force platform, then planted the instrumented leg and pivoted ninety degrees away from the cameras. Many ligament injuries occur during pivoting movements where the foot is planted.

3.4 Data Acquisition and Processing

3.4.1 Data Acquisition

The measurements with skin-mounted and surgical pin mounted arrays were carried out on the same day. There were two sessions, each of approximately two and a half hours,
with a one hour break in between. All data were recorded at a sampling rate of 315 Hertz. Each trial lasted two seconds.

Prior to carrying out any of the planned tasks described in the previous section, static data files were recorded. The subject was instructed to stand erect in the center of the viewing volume (on the forceplate) while marker position data were recorded. The data were processed and checked to ensure that all the LEDs were functioning correctly and in view. Static trials were recorded until all the LEDs were in view. The first three trials were range of movement measurements for the ankle, knee and hip. Five trials of normal gait kinematics and forceplate data were recorded. Laxity data were recorded at flexion angles of approximately 5, 30, 60 and 90 degrees. Two sets of data were recorded at each angle. Next with the pin-mounted arrays but not the skin-mounted arrays, lower extremity kinematics in pivoting were measured in three trials. The final three sets of data were range of movement measurements for each of the joints of the lower extremity.

This sequence of tasks was executed with the subject instrumented with the skin-mounted transducers in the first session. After a short break the surgical pins were implanted and the arrays mounted on them. An injection of 0.1% Lidocaine was administered at the pin insertion sites before the pins were implanted. The subject noted no discomfort due to either the pin or the anaesthetic and neither had any apparent effect on his ability to execute the requested tasks in a normal manner.

### 3.4.2 Data Processing

The provision of several pieces of information to TRACK were necessary to enable processing of the data. Segment files for the arrays used in both portions of the experiment, the skew ray error limit, the maximum acceptable inter-LED length error allowable and the smoother order and derivative criterion all had to be selected.

There are two approaches to the definition of the segment files. The first is to use the
design drawing of the array as the basis for the file [9]. A second, empirical, method has been used in several experiments employing the TRACK system [116, 169]. LED three-dimensional coordinate data are collected for a static experiment, typically with the subject standing erect in the center of the cameras' viewing volume. Mean coordinates are calculated for each LED and each array of LEDs. The mean LED coordinates point of reference is then transformed from the laboratory coordinate system origin to the array mean coordinate, establishing a body-fixed coordinate system aligned with the laboratory coordinate system on each array. This approach has two principal advantages: (1) manufacturing errors and tolerances become immaterial, since the segment file is based on what the cameras actually see; (2) the segment file defines a set of 'neutral' orientations and positions in the laboratory coordinate frame. The empirical method was used to generate the segment files used in processing all the data from these experiments. Details are presented in Appendix E.

Empirical approaches were used to select the combination of skew ray error limit and maximum inter-LED length error. Each skeletal pin-mounted array data set was processed with a matrix of different combinations of the two parameters. Some of the comparisons are presented in Appendix G. For the pin-mounted arrays data a skew ray error limit of 20 Selspot units and a maximum inter-LED length error of 25% were used in processing the data. For the skin-mounted array data a skew ray error limit of 30 Selspot units and a maximum inter-LED length error of 30% were selected.

Smoothing parameters were selected on the basis of the first derivative quality and the resulting power spectrum of the smoothed data. Power spectra of the data showed that most of the signal power was contained at frequencies less than three Hertz. Dohrmann's algorithm for the natural cubic spline, with derivative criterion of 2, was found to consistently yield the best results for the pin-mounted array data. This set of smoothing parameters was found to oversmooth the gait data obtained with the skin-mounted ar-
rays. The raw marker and rigid body data clearly showed a significant disturbance corresponding to heel strike which was not evident with the natural cubic spline smoother. Improved fidelity resulted when the derivative criterion was increased to three, but there was still oversmoothing. In order to use a derivative criterion of four, a quintic spline was required. This reproduced the principle characteristics of the raw gait data most accurately. A review of the selection process is given in Appendix I.

The gait experiments were windowed to contain only the portion of the data when all the LEDs were within the camera range. Choosing the initial and final frames in the windows was an iterative procedure. Tables containing the final window values are presented in Appendix F.

### 3.5 Results

The presentation of the results will be divided into two sections, one for the pin-mounted array data and one for the skin-mounted array data. The former stand alone as novel measurements and estimates of the instantaneous kinematics of the normal knee, while the latter must be viewed as the first step toward resolving the problem of making repeatable, reliable estimates of the kinematics of the knee without invasive means.

The kinematic data represent the motions of the embedded coordinate systems attached to each array or segment. Figure 3.9 shows the locations and orientations of the body-fixed coordinate systems for each of the five segments.

#### 3.5.1 Knee Kinematics (Pin data)

**Data fidelity**

Smoothing is essential to the calculation of the instantaneous kinematics from the measured markers. In order to preempt questions about oversmoothing of the data, the
Figure 3.9: Location and orientation of the embedded segmental coordinate systems.
following figures will show unsmoothed and smoothed relative rigid body displacements and orientations and demonstrate that the smoothed data include all of the significant features of the unsmoothed data. Figures 3.10 and 3.11 compare the unsmoothed and smoothed relative displacements and orientations for the first trial of the voluntary swing task. The relative displacement vector and quaternion for a representative gait trial are shown in Figures 3.12 and 3.13. It is interesting that there is no evidence of a heel strike transient in either the displacement or orientation data although it is very apparent in the forceplate data (see Figure 3.32a). Finally, data for a typical trial of the pivot step experiment is presented in Figures 3.14 and 3.15. There is no evidence of oversmoothing of the rigid body displacement and orientation data for any of the three tasks, so that the estimates of the kinematic parameters may be presumed to be representative of the actual movements.

Frequency content

One of the persistent questions in the measurement of human movement concerns the bandwidth of skeletal kinematics. The dearth of skeletal kinematic data for frequency domain analysis has contributed to the duration of the debate.

Several investigators have examined the heel-strike transient of normal gait, with varying results. Gilbert et. al. [84] mounted pairs of accelerometers on aluminum brackets which were strapped to the thigh and shank. Acceleration peaks with frequency components between 20 and 30 Hertz were measured. In related work, Maxwell [161]\(^8\) reported that some gait parameters have frequency components at 60 Hertz or greater. The authors dismissed the possibility of soft tissue effects based on a comparison of their results with those of Light et. al. [151]. In that study, heel-strike accelerations on the tibia were measured with one accelerometer mounted on skeletal pins and a second one

\(^8\)As reported by Gilbert et. al. [84].
Figure 3.10: Relative displacement vector at the knee for Trial 1 of the voluntary swing task: (a) Unsmoothed, (b) Smoothed (Natural cubic spline).
Figure 3.11: Relative quaternion at the knee for Trial 1 of the voluntary swing task: (a) Unsmoothed, (b) Smoothed (Natural cubic spline).
Figure 3.12: Relative displacement vector at the knee for Trial 1 of the normal gait task: (a) Unsmoothed, (b) Smoothed (Natural cubic spline).
Figure 3.13: Relative quaternion at the knee for Trial 1 of the normal gait task: (a) Unsmoothed, (b) Smoothed (Natural cubic spline).
Figure 3.14: Relative displacement vector at the knee for Trial 2 of the pivot step task: (a) Unsmoothed, (b) Smoothed (Natural cubic spline).
Figure 3.15: Relative quaternion at the knee for Trial 2 of the pivot step task: (a) Unsmoothed, (b) Smoothed (Natural cubic spline).
mounted on a molded polyethylene plate strapped to the skin over the bone. Accelerations measured with both transducers were similar in shape and amplitude. Simon et. al. [240] and Antonsson and Mann [10] both looked at the frequency content of heel-strike using data from high bandwidth force platforms. In the former study, Fourier analysis of foot-floor force data showed components up to 55 Hertz for shod subjects and 75 Hertz for unshod subjects. Employing similar techniques, Antonsson and Mann found that 99% of the signal power in gait was contained in frequencies below 15 Hertz.

The pin-mounted array data may help resolve the questions about bandwidth. The data are unquestionably measurements of the skeletal kinematics and the bandwidth of the TRACK system, 315 Hertz, is sufficiently wide to measure the entire range of probable frequencies in normal gait without aliasing.

Power spectral densities were calculated for the unsmoothed rigid body position and orientation data using a modified periodogram with a Hanning window [190]. Representative spectra for gait are presented in Figures 3.16-3.19. Spectra for the components of the displacement vectors for the body-fixed coordinated systems on the shank and thigh are shown in Figures 3.16a-3.16c and Figures 3.17a-3.17c, respectively. The most significant peaks for all components are in the range of one to two Hertz. There is very little observable, and no significant, activity beyond 8 Hertz for any of displacement vector components.

The frequency composition of the quaternions describing the orientation of shank and thigh are shown in Figures 3.18a-3.18d and Figures 3.19a-3.19d, respectively. The vector components of the quaternion represent the orientation of the rotation axis. There is no significant activity at frequencies above 5 Hertz. The scalar component of the quaternion is a function of the magnitude of the rotation. Both the thigh and shank rotations show

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9Significance is based strictly on observation of relative magnitude in this case. A rigorous approach to evaluating the significance of a peak is presented in Adelstein [3], pp. 148-149.
Figure 3.16: Power spectrum for the unsmoothed displacement vector to the shank in Trial 1 of the gait experiments: (a) X component of displacement, (b) Y component of displacement, (c) Z component of displacement.
Figure 3.17: Power spectrum for the unsmoothed displacement vector to the thigh in Trial 1 of the gait experiments: (a) X component of displacement, (b) Y component of displacement, (c) Z component of displacement.
Figure 3.18: Power spectrum for the unsmoothed quaternion of the shank in Trial 1 of the gait experiments: (a) X vector component of quaternion, (b) Y vector component of quaternion, (c) Z vector component of quaternion, and (d) Scalar component of quaternion (rotation magnitude).
Figure 3.19: Power spectrum for the unsmoothed quaternion of the thigh in Trial 1 of the gait experiments: (a) X vector component of quaternion, (b) Y vector component of quaternion, (c) Z vector component of quaternion, and (d) Scalar component of quaternion (rotation magnitude).
Figure 3.20: Flexion angle as a function of time for voluntary swing trial 1.

essentially no activity above 8 Hertz.

The preceding results are representative of the five gait trials. Similar frequency composition was observed for the pivot step displacements and orientations.

Voluntary swing

Kinematic data were recorded for two trials of the voluntary swing task. A two second duration snapshot of the movement was collected as the subject swung the tibia through its full range of motion. No other boundary conditions, such as hip flexion angle, were specified in the instructions.

Figure 3.20 shows the anatomical flexion angle\textsuperscript{10} as a function of time for the first trial. A complete extension swing was captured with a portion of the subsequent flexion. There was a slight hyperextension of the knee, approximately five degrees.

The axode\textsuperscript{11} for the movement is shown in Figure 3.21 in a coordinate system fixed

\textsuperscript{10}Calculated using the method of Grood and Suntay [91].
\textsuperscript{11}The IHAs shown in the axode were thresholded to show only those with a standard deviation of

135
to the array mounted on the lateral condyle of the knee. The positive x-axis is in the posterior direction, the positive y-axis points proximally and the positive z-axis medially. Figure 3.21a contains the extension phase of the swing movement, with the complete axode presented in Figure 3.21b. In both flexion and extension the movement is primarily a rotation about an axis nearly parallel to the z-axis. The other components of rotation appear to be small, although more is evident during extension than during the portion of flexion measured in this trial. Viewing the axode from the posterior (Figure 3.22a) and superior directions (Figure 3.22b) shows that the mean flexion and extension axis is symmetrical, with only a change in the sign of the rotation readily apparent. The distribution of IHAs is remarkably close to what would be measured for a simple revolute [121, 269].

A record of the flexion angle for the second trial of the voluntary swing task is displayed in Figure 3.23. A complete swing cycle of flexion and extension was recorded. Instead of hyperextension in this trial, there is extension only to about twenty degrees of flexion.

Figure 3.24 presents the axode for the second voluntary swing trial. Examined from an oblique angle, the general distribution of IHAs in the two trials is similar. Upon consideration of the posterior and superior views (Figure 3.25), subtle differences between the two trials become more apparent. The flexion-extension axes in the second trial show less inclination relative to the z-axis of the coordinate system. The components of the IHAs along the other axes are smaller in the second trial than in the first. This particularly evident in the flexion phase of the two trials. This is surprising since the greater range of movement was measured in the second trial.

Confirmation of the observed differences in the axodes is obtained from calculations

less than 0.1 for the components of the unit vector. Threshold was set based on the estimated standard deviations calculated using the methods of Appendix J.
Figure 3.21: Axode for Trial 1 of the voluntary swing task: (a) Extension phase of the swing movement, (b) Complete axode.
Figure 3.22: Alternate views of the axode for Trial 1 of the voluntary swing task show flexion-extension symmetry: (a) Posterior view, (b) Superior view.
Figure 3.23: Flexion angle as a function of time for voluntary swing trial 2.

Table 3.1: Orthogonal Basis Screws for Voluntary Swing Task Trials

<table>
<thead>
<tr>
<th>Trial</th>
<th>Basis Screw</th>
<th>Real Part</th>
<th>Dual Part</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>x</td>
<td>y</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.073</td>
<td>0.218</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>-0.451</td>
<td>-0.122</td>
</tr>
</tbody>
</table>

of orthogonal bases for the two trials. The components of the basis screws are listed in Table 3.1. A single screw is sufficient to represent all of the IHAs for the second trial. The principal component of the basis screw is aligned with the z-axis, with small components along the other two axes. Although the primary motion is a rotation about the flexion-extension axis, there is coupled abduction-adduction and internal-external rotation. For the first trial the orthogonal basis contains two screws, representing two independent rotations. The first screw has its principal component along the z-axis, with a coupled rotation component about the y-axis. The second rotation is primarily about the x-axis. Rotation about the x-axis, essentially abduction-adduction for small flexion angles, is decoupled from flexion in the first trial and coupled in the second.
Figure 3.24: Axode for Trial 2 of the voluntary swing task: (a) Extension phase of the swing movement, (b) Complete axode.
Figure 3.25: Alternate views of the axode for Trial 2 of the voluntary swing task show flexion-extension symmetry: (a) Posterior view, (b) Superior view.
Figure 3.26: Internal-external rotation angle as a function of flexion angle for Trial 1 of the voluntary swing task.

Shifting from instantaneous kinematics to finite kinematics, there is additional evidence of the different rotational behavior in the two trials of the same task. The internal-external rotation angle is shown as a function of flexion in Figure 3.26. Extending from approximately 130 to 70 degrees of flexion there is essentially no accompanying internal-external rotation. As the movement continues to a slight hyperextension, there is a linear relationship between the amount of extension and the amount of external rotation. The same coupling between the two rotations is evident through the first fifty degrees of flexion. A completely different pattern is seen in a plot of the same two rotations for the second trial displayed in Figure 3.27. In extension from 130 to 20 degrees of flexion, the internal-external rotation angle varies sinusoidally, although it is internally rotated throughout the movement. When the motion is reversed and the leg flexed, the pattern of internal-external rotation changes. There is a small internal rotation through 70 degrees of flexion, followed by a sharp external rotation over the next fifty degrees of flexion. The rotation again reverses in the final thirty degrees of flexion. Overall the pattern appears to be following a closed cycle over the full range of flexion and extension, consistent with
Figure 3.27: Internal-external rotation angle as a function of flexion angle for Trial 2 of the voluntary swing task.

The single basis screw. The range of change in internal-external rotation is one half the 22 degrees of the first trial of the same task.

Figure 3.28 compares the degree of abduction-adduction as a function of flexion angle for the two trials. There is a deceptive superficial similarity between the two curves, particularly at large flexion angles. Both trials show approximately six degrees of abduction at a flexion angle of 110 degrees, but in the first trial it occurs during extension and in the second trial during flexion. From twenty degrees of flexion to hyperextension there is a linear increase in abduction and adduction coupled to flexion and extension, respectively, in the first trial.

The biarticular muscles crossing the hip and ankle, as well as the knee, may have contributed to the differences observed in the two trials. The subject was only instructed to swing the knee joint through its full range of motion. No restriction was placed on the position or movement of the hip and ankle. Figure 3.29 displays the anatomical angles at the hip for the two trials. Orientation at the hip was clearly different in the two trials.
Figure 3.28: Abduction-adduction as a function of flexion angle for the two trials of the voluntary swing task: (a) Trial 1, (b) Trial 2.
Figure 3.29: Anatomical angles at the hip for the voluntary swing task: (a) Trial 1, (b) Trial 2.
Differences between the two trials are also seen in the anatomical angles at the ankle shown in Figure 3.30. The variations in the relative rotations at all three joints of the lower extremity in the two trials may be evidence of differences in either the passive or the active constraint behavior of the biarticular muscles.

There has been no discussion of any relative translation. Translation may be resolved into components along and perpendicular to the IHA. The latter is indicated by magnitude of the first order instantaneous invariant, the screw pitch, while the former is reflected in the distribution of successive IHAs\textsuperscript{12}. If there is no translation perpendicular to the IHA, successive IHAs will intersect at a point. Figure 3.31 shows the pitch for the portions of the two trials during which the IHA is defined. The magnitude of the pitch indicates no significant translation along the screw axis during either trial of the voluntary swing movement.

Normal gait

Five normal gait trials were performed. Both kinematic and forceplate data were recorded as the subject walked through the field of view of the TRACK system at a self-selected normal pace.

Vertical foot-floor force is shown as a function of time for the first gait trial in Figure 3.32a. Due to the centering of the camera's viewing volume on the forceplate and the fixed camera positions limiting the size of the viewing volume, data were obtained for all of the stance phase but a portion of the swing phase was missed. Figure 3.32b presents the flexion angle as a function of time for the same trial. The curve is consistent with the results reported by Chao et. al. [43] and demonstrates that only the extreme range of the swing phase was not recorded. The total flexion angle excursion in gait is

\textsuperscript{12}See Equation 2.7 and 2.8 for the vector to the screw axis. If there is no translational velocity perpendicular to the screw axis the cross product is zero.
Figure 3.30: Anatomical angles at the ankle for the voluntary swing task: (a) Trial 1, (b) Trial 2.
Figure 3.31: First order instantaneous invariant – screw pitch – for the voluntary swing task: (a) Trial 1, (b) Trial 2.
Figure 3.32: Trial 1 of the normal gait task: (a) Vertical foot-floor reaction force, and (b) Knee flexion angle as functions of time.
Table 3.2: Orthogonal Basis Screws for Trial 1 of Normal Gait Task

| Trial | Basis Screw | Real Part | | | Dual Part |
|-------|-------------|-----------|-----------| |-----------|
| 1     | 1           | 0.230     | 0.006     | -0.973 | 0.023     | 0.088 | 0.006   |
| 2     | 2           | 0.938     | 0.191     | 0.224  | -0.010    | 0.001 | 0.165   |

less than half the range observed in the voluntary swing task.

The axode for the first gait trial, shown in Figures 3.33 through 3.35, differs significantly from the axode of either of the voluntary swing task trials. Figure 3.33a shows the initial portion of the axode which represents the movement of the knee during the initial extension portion of the swing phase. The IHAs are distributed nearly parallel to the z-axis in the same pattern seen in voluntary swing. Prior to heel strike (Figure 3.33b) the pattern of the IHAs for gait diverges from anything seen in the other task with a steady transition from an extension dominated motion to a combination of extension and internal rotation. After heel strike the principal rotation reverses and the knee undergoes a movement comprised of all three components of rotation (Figure 3.34a). With the foot flat the rotation again reverses into a mix of extension and the other two rotational components as shown in Figure 3.34b. A brief adduction occurs at full extension (Figure 3.35a) followed by the push-off and toe-off which are marked by a flexion dominated movement (Figure 3.35b). Views of the complete axode are displayed in Figure 3.36a looking directly along the y-axis in the distal direction and in Figure 3.36b along the x-axis in the anterior direction. The distinct segments of the movement are clearly distinguished.

An orthogonal basis for the system of screws comprising the axode is shown in Table 3.2. The basis contains two screws and is different from either of the bases calculated for the voluntary swing trials. There are rotational components around all three axes. The dual portions of the two basis screws include significant components indicating some
Figure 3.33: Axode for normal gait task Trial 1: (a) Extension portion of the swing phase—similar to voluntary swing task pattern, and (b) Swing prior to heel strike showing a combination of extension and internal rotation in a pattern differing from that seen in the voluntary swing task.
Figure 3.34: Axode for normal gait task Trial 1 (continued): (a) Flexion dominated movement follows heel strike until the foot is flat on the floor, then (b) the knee extends and rotates.
Figure 3.35: Axode for normal gait task Trial 1 (concluded): (a) A brief adduction at full extension is followed by (b) flexion dominated movement through push-off and toe-off.
Figure 3.36: Internal/external rotation versus flexion angle for Trial 1 of the normal gait task.

translational movement perpendicular to the screw axis coupled with the rotations.

The finite rotations for the gait show a pattern distinctly different from those seen in either voluntary swing trial. Figure 3.36 presents the internal/external rotation versus flexion angle with significant points in the gait cycle indicated. There are eleven degrees of internal rotation in the final twenty degrees of extension prior to heel strike. Once the foot is in contact with the floor the range of rotation about the tibial shaft is greatly reduced. The overall pattern appears to be cyclic. Abduction/adduction is shown in Figure 3.37. Abduction of six degrees precedes heel strike with insignificant movement while the heel is in contact with the floor. During the push-off phase, prior to toe-off, there was nearly twelve degrees of knee adduction.

There was a remarkable similarity between all five trials of the gait task. This is most clearly evident in the anatomical angles. Figures 3.38 and 3.39 show the internal/external rotation angle and abduction/adduction angle versus flexion angle histories for the four remaining gait trials. The general pattern of the rotations is repeated in each trial. In
Figure 3.37: Abduction/adduction versus flexion angle for Trial 1 of the normal gait task.

comparison, two completely different patterns were observed in two trials of the voluntary swing task.

Pivot step

The final movement task performed was the pivot step maneuver. This was included to simulate movements common to many sports and frequently associated with knee injury. The subject jogged to the forceplate, planted the foot of his instrumented leg on the forceplate and pivoted ninety degrees. Three trials of the task were recorded: one fast and two at a slower, 'normal' pace. Although there were gross similarities in the finite and instantaneous kinematics for the different trials there was not a consistent pattern like the one seen in the gait task. Changing the speed on the maneuver resulted in distinct changes in the movement patterns.

The locus of IHAs for the second trial of the pivot task is presented in Figures 3.40 through 3.42. Initially the movement is an external rotation of the tibia and abduction at the knee, with almost no flexion component (Figure 3.40a). This progresses into a
Figure 3.38: Internal/external rotation angle versus flexion angle for: (1) Gait trial 2, (2) Gait trial 3, (3) Gait trial 4, (4) Gait trial 5.
Figure 3.39: Abduction/adduction angle versus flexion angle for: (1) Gait trial 2, (2) Gait trial 3, (3) Gait trial 4, (4) Gait trial 5.
Figure 3.40: Axode for Trial 2 of the pivot step maneuver: (a) Initial external rotation and abduction, (b) Extension combined with other rotational components.
Figure 3.41: Axode for Trial 2 of the pivot step maneuver showing progression to an internal rotation.

combined rotation with extension as the principal component (Figure 3.40b). The next phase of the movement, shown in Figure 3.41, includes an internal rotation of the tibia with smaller components of adduction and flexion. The conclusion of the movement is an extension effecting the push-off from the pivot (Figure 3.42). Several distinct phases of the movement are identifiable in the locus of IHAs. In the pivoting task the IHAs are more widely distributed than in gait or voluntary swing, with components about all three axes throughout the movement. Figure 3.43 presents two orthographic projections of the complete axode which show the distribution of the IHAs.
Figure 3.42: Axode for Trial 2 of the pivot step maneuver showing the extension leading to push-off.
Figure 3.43: Orthographic views of the axode for Trial 2 of the pivot step maneuver: (a) View down the y-axis of the reference coordinates from the proximal direction, and (b) View along the x-axis from the posterior direction.
Table 3.3: Orthogonal Basis Screws for Trial 2 of Pivot Step Task

<table>
<thead>
<tr>
<th>Trial</th>
<th>Basis Screw</th>
<th>Real Part</th>
<th>Dual Part</th>
</tr>
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</tr>
<tr>
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<td>0.202</td>
</tr>
</tbody>
</table>

Figure 3.44: Internal/external rotation angle versus flexion angle for Trial 2 of the pivot step task.

An orthogonal basis calculated for the IHAs in the axode contains two screws (Table 3.3). As expected from the geometry of the axode, the basis screws have rotational components about all three axes.

The finite, anatomical rotations show the same distinct phases that are observable in the axode. Figures 3.44 and 3.45 display internal/external rotation and abduction/adduction versus flexion angle for the second trial of the pivot task, respectively.

Both curves show distinct phases with essentially no activity mixed with periods of significant rotation. The limited range of the internal/external rotation, a total of eight degrees, is significantly less than seen in the other two tasks. There is a total range of
Figure 3.45: Abduction/adduction angle versus flexion angle for Trial 2 of the pivot step task.

thirty-five degrees of abduction/adduction. The pivot step maneuver is achieved with more abduction/adduction of the knee than internal/external rotation.

3.5.2 Measurement of Knee Kinematics (Skin data)

Data fidelity and processing considerations

It is important that the processed data accurately reflect the content of the raw data. This is particularly true in the case of the data obtained with skin-mounted arrays, where the objective is to identify whether or not the measurements are an accurate representation of the underlying skeletal motion.

Unsmoothed and smoothed relative displacement vectors and quaternions for motion at the knee in the first trial of the voluntary swing task are shown in Figures 3.46 and 3.47, respectively. The data were smoothed using the same cubic spline smoother used for processing the pin-mounted array data. All of the significant features of the unsmoothed data are present in the smoothed curves.
Figure 3.46: Relative displacement vector at the knee for Trial 1 of the voluntary swing task measured using skin-mounted arrays: (a) Unsmoothed, (b) Smoothed (natural cubic spline, \( p=3, m=2 \)).
Figure 3.47: Relative quaternion at the knee for Trial 1 of the voluntary swing task measured using skin-mounted arrays: (a) Unsmoothed, (b) Smoothed with natural cubic spline, \( p = 3, m = 2 \).
Figure 3.47: Relative quaternion at the knee for Trial 1 of the voluntary swing task measured using skin-mounted arrays: (a) Unsmoothed, (b) Smoothed with natural cubic spline, \((p = 3, m = 2)\).
Results for the normal gait task were not as straightforward. The unsmoothed relative displacement vector for the first gait trial is shown in Figure 3.48a. A distinct transient is seen in all components of the displacement vector between 0.20 and 0.25 seconds. The same data smoothed with the natural cubic spline is shown in Figure 3.48b. The effective cutoff frequency of the smoother is too low to capture the transient. Experimentation with the smoother parameters indicated that the fidelity with the unsmoothed data increased with the selection of a higher derivative criterion, m. Figure 3.48c displays the displacement vector smoothed with a quintic\textsuperscript{13} spline and a derivative criterion of four. Similar results were obtained for the relative quaternion and are presented in Figure 3.49.

Frequency content

One hypothesis suggests that soft tissue motion will introduce high frequency artifacts. If this was the case, the power spectra for the rigid body kinematic parameters measured with the skin-mounted arrays would be expected to be broader based than the spectra from the pin-mounted arrays. Comparison of the spectra for the most difficult parameters for the TRACK system to estimate, the z vector component of the quaternion and z displacement, for two representative gait trials does not support this hypothesis. The spectra for the z displacement in the first gait trial of each set of measurements are presented in Figure 3.50. There are no significant differences in the spectra for either the shank or thigh segment. Figure 3.51 displays the estimated power spectra for the z vector component of the quaternion, which is proportional to the angle between the z axis of the laboratory coordinate frame and the quaternion axis, for each segment. There are no major differences in the power spectra for either segment. The similarity in power spectra is evident for all the rigid body kinematic parameters for both segments. The lack of an oscillatory soft tissue artifact is consistent with the well-damped response to

\textsuperscript{13}The spline order had to be increased to 5 because the derivative criterion may not be greater than the spline order using Dohrmann's algorithm.
Figure 3.48: Relative displacement vector at the knee for Trial 1 of the normal gait task measured using the skin-mounted arrays: (a) Unsmoothed, (b) Smoothed with natural cubic spline \((p = 3, m = 2)\), (c) Smoothed with quintic spline \((p = 5, m = 4)\).
Figure 3.49: Relative quaternion at the knee for Trial 1 of the normal gait task measured using the skin-mounted arrays: (a) Unsmoothed, (b) Smoothed with natural cubic spline ($p = 3, m = 2$), (c) Smoothed with quintic spline ($p = 5, m = 4$).
Figure 3.50: Power spectral estimates for the z displacement of the origin: (a) Segment 2 (shank) in Trial 1 of gait task measured with pin-mounted arrays, (b) Segment 3 (thigh) in Trial 1 of gait task measured with pin-mounted arrays, (c) Segment 2 in Trial 1 of gait task measured with skin-mounted arrays, and (d) Segment 3 in Trial 1 of gait task measured with skin-mounted arrays.
Figure 3.51: Power spectral estimates for the z vector component of the quaternion: (a) Segment 2 (shank) in Trial 1 of gait task measured with pin-mounted arrays, (b) Segment 3 (thigh) in Trial 1 of gait task measured with pin-mounted arrays, (c) Segment 2 in Trial 1 of gait task measured with skin-mounted arrays, and (d) Segment 3 in Trial 1 of gait task measured with skin-mounted arrays.
the heel-strike transient observed in the gait data shown in the preceding section (see Figures 3.48 and 3.49).

Voluntary swing

Kinematic data were recorded for two trials of the voluntary swing task with the skin-mounted arrays. Instructions to the subject were identical to those used in the pin-mounted array experiments.

The flexion angle is presented as a function of time for the first trial of the voluntary swing task in Figure 3.52a. For comparison the flexion angle record for the second voluntary swing trial from the pin-mounted array experiments is shown in Figure 3.52b. Although the general shape of the two curves is similar there is approximately a fifty percent difference in the overall range of the flexion angle. Several explanations are possible for the discrepancy. Among the possibilities are:

- subject did not follow instructions.

- the movement was slower so that only the portion near full extension was recorded.

- array rotations are not tracking the bone rotations accurately.

A simple calculation based on the relative positions of the array-fixed coordinate system origins eliminates the first two possibilities. Recall that there were two arrays mounted in the femur, at the greater trochanter and the lateral femoral condyle, and one in the tibia (see Figure 3.8). An estimate of the flexion angle may be obtained by taking the
Figure 3.52: Flexion angle versus time for: (a) Trial 1 of the voluntary swing task measured with skin-mounted arrays, and (b) Trial 2 of the voluntary swing task measured with pin-mounted arrays.
Figure 3.53: Schematic of vectors used in estimating the flexion angle at the knee based on the positions of the array-fixed coordinate system origins.

The scalar product of the vector from the lateral condyle array (Segment 3) to the greater trochanter array (Segment 4) and the vector from the lateral condyle array to the tibial array (Segment 2). Figure 3.53 shows a schematic of the array fixed coordinate systems and vectors. The changes in the estimated angle should match the changes in the true flexion angle. The estimate for the first voluntary swing trial is shown in Figure 3.54. The one hundred degrees of extension and eighty degrees of flexion are consistent with the results of the pin-mounted array measurements and do not support either of the first two possible explanations for the flexion angle discrepancy. The measured orientations of the two skin-mounted arrays used to calculate the anatomical angles at the knee were attenuated by nearly fifty percent. Soft tissue motion or inadequate fixation of the array to the skin probable causes of the attenuation.

Generation of instantaneous kinematic parameters is predicated on the acquisition of accurate rigid body velocities. A comparison of the components of the relative translational velocity of the origin of the tibial coordinate system to the femoral coordinate system for Trial 2 of the pin-mounted voluntary swing experiment and the first trial of
Figure 3.54: Estimated flexion angle at the knee for Trial 1 of the voluntary swing task in the skin-mounted array experiments.

The skin-mounted swing task is presented in Figure 3.55. The time history of the $x$ component of velocity is the only one which shows any similarity. The skin-mounted array translational velocity measurements do not reproduce the patterns of the pin-mounted array velocities. The angular velocity is used to determine the orientation of the IHA. In the pin-mounted array voluntary swing experiments, the motion was pure flexion. Relative angular velocities for the second pin-mounted trial and the first skin-mounted trial are displayed in Figure 3.56. The skin-mounted data are for a complicated rotation with components on all three axes. The initial rotation at the flexed position is measured as an abduction. The angular velocity pattern measured with the skin-mounted array is not in agreement with the much simpler motion measured with the bone-fixed arrays.

**Normal gait**

Seven sets of normal gait data were acquired using the skin-mounted arrays. The complete stance phase and a significant portion of the swing phase were recorded in each
Figure 3.55: Relative translational velocity vector for the voluntary swing task: (a) Trial 2 of the pin-mounted array experiments, and (b) Trial 1 of the skin-mounted array experiments.
Figure 3.56: Relative angular velocity vector for the voluntary swing task: (a) Trial 2 of the pin-mounted array experiments, and (b) Trial 1 of the skin-mounted array experiments.
case.

The presence of a pronounced heel strike transient in the data was introduced in the discussion of data processing parameter selection. The relative displacement vector x and z components (Figure 3.57a) show a transient with an amplitude on the order of two to three centimeters. In the relative quaternion (Figure 3.57b), the transient is most clearly observed in the y vector component which is proportional to the y direction cosine for the rotation axis.

A distinct, repeatable pattern was evident in the angular and translational velocity vector components for gait measured with the pin-mounted arrays. No such pattern is distinguishable in the data from the skin-mounted arrays. Pin-mounted and skin-mounted data are compared in Figure 3.58. There is no evidence in the skin-mounted data that the principal motion is about the flexion axis. The largest components of angular velocity are about the y-axis, corresponding to internal/external rotation and abduction/adduction. Generation of the axode based on the velocity data obtained in the skin-mounted array experiments would offer no useful information.

The same attenuation of the flexion angle observed in the voluntary swing task is seen in Figure 3.59. Compared to both the estimated flexion angle and the flexion angle measured in the first normal gait trial for the pin-mounted experiments, the flexion angle is attenuated by approximately fifty percent. The heel-strike transient appears as a spurious hyperextension of approximately six degrees.

A repeatable pattern of internal/external rotation and abduction/adduction was observed in the gait trials of pin-mounted array experiment. Internal/external rotation angle curves are shown for two gait trials from the skin-mounted array experiment in Figure 3.60 and for abduction/adduction in Figure 3.61. There are gross similarities in the angular histories for the two trials. It is also evident that the angular excursions do not match those measured in the pin-mounted experiment (Figures 3.60c and 3.61c).
Figure 3.57: Trial 7 of the normal gait task measured with skin-mounted arrays showing the heel-strike transient: (a) Relative displacement vector components, and (b) Relative quaternion components.
Figure 3.58: Comparison of velocity vector components in normal gait task for the pin-mounted arrays and skin-mounted arrays: (a) Relative translational velocity vector for Trial 1 of the pin-mounted data, (b) Relative translational velocity vector for Trial 7 of the skin-mounted data, (c) Relative angular velocity vector for Trial 1 of the pin-mounted data, and (d) Relative angular velocity for Trial 7 of the skin-mounted data.
Figure 3.59: Comparison of the flexion angle measurements for the normal gait task: (a) Flexion angle from Trial 1 of the pin-mounted array experiments, (b) Flexion angle from Trial 7 of the skin-mounted array experiments, and (c) Estimated flexion angle from Trial 7 of the skin-mounted array experiments.
Figure 3.60: Internal/external rotation angle for the normal gait task: (a) Trial 1 of the skin-mounted array experiment, (b) Trial 7 of the skin-mounted array experiment, and (c) Trial 1 of the pin-mounted array experiment.
Figure 3.61: Abduction/adduction angle for the normal gait task: (a) Trial 1 of the skin-mounted array experiment, (b) Trial 7 of the skin-mounted array experiment, and (c) Trial 1 of the pin-mounted array experiment.
3.6 Discussion

3.6.1 Kinematics of the Normal Knee

Two goals for the pin-mounted array experiments were stated in the beginning of this chapter. The first, with implications for both clinical treatment and research on knee function, was to evaluate the kinematics of the knee in different tasks. A principal objective of selecting the different tasks was to study the knee under the action of different types of loading. The second purpose for the experiments was to determine whether the kinematics of the joint were planar, spherical or spatial and, if possible, estimate the number of degrees of freedom. This is a necessary prerequisite to the development of adequate mathematical models of the knee.

The data provided partial answers to both questions. The single most important result is that the knee is not a simple revolute joint. It is not a planar four-bar linkage. It is a geometrically complicated mechanical coupling capable of a broad variety of movements, including single degree-of-freedom rotations as one of many subsets. All of the motions measured were spatial, a fact that must be accounted for in any model. Estimates of the number of degrees-of-freedom available in the joint varied with task indicating that either there is a significant level of active constraint in the movements studied or that there are regions of the joint configuration space where some passive constraints are disengaged.

Each of the different tasks contributes to understanding that the coupling is adaptable. The axodes for the two trials of the active swing tasks were similar but, as indicated by the orthogonal bases, represented a single degree-of-freedom movement in the first trial and a two degree-of-freedom joint in the second. In the first trial the single degree-of-freedom and near linear relationship between the flexion and internal/external rotation angles are consistent with an absence of hard passive constraints such as the articular surfaces. The relationship between the two angles could be the result of either the
lines-of-action of the active constraints or coupling due to a ligament or ligaments. The cyclic relation between the same two angles in the second trial is more characteristic of specific contact surface geometry. This indicates that joint stiffness and the presence or absence of contact are instrumental in determining the kinematics of the joint. In active swing, the joint stiffness is most probably being modulated through co-contraction of the muscles surrounding the joint. The differences in the hip and ankle angle imply that the biarticular muscles may play a significant role in the stiffness variation.

A topic of persistent interest in knee kinematics is the presence of an automatic terminal rotation of the tibia in the last few degrees of extension, called the ‘screw-home’ movement. There was no evidence of a specific terminal rotation in the first trial of the voluntary swing when the subject slightly hyperextended the joint. In the second trial, there was an obvious cyclic relationship between flexion and internal/external rotation but the movement was incomplete, only reaching twenty degrees of flexion. This agrees with Hallen and Lindahl [97], who found that the screw-home was not automatic.

Only a single set of IHAs are found in the knee literature due to the difficulties in calculating joint velocities. Kärrholm [123] presents axodes for an active flexion of the knee which qualitatively appear to agree with the results of the pin-mounted array experiments. However, variation of internal/external rotation and abduction/adduction with flexion corresponding to the IHAs differs from both the current trials. Distinctly different finite helical axes for normal gait and a pivot step are presented by Shiavi et. al. [232], which is consistent with the different IHAs for the two tasks in the current results.

In contrast to the voluntary swing results, the gait trial data revealed a consistent sequence of movement in all the trials. Each portion of the axode was well-defined. The movements had two degrees-of-freedom. Particularly interesting was the internal rotation of the tibia in the final stage of extension prior to heel-strike. The rotation was clearly evident in both the axode and the anatomical angles. In a previous study of knee
kinematics using skeletal pins, LaFortune [138] found no evidence of such a rotation in normal gait. Although the rotation was present in conjunction with the extension prior to heel-strike, there was no combined rotation in the extension over the same range during stance phase. There was also no unscrewing motion entering the swing phase at toe-off. The presence of the purported screw-home movement appears to depend on more than just contact of the articular surfaces – either muscular action or the engagement of the soft tissue passive constraints could be responsible.

Estimates of the frequency content in normal gait supported the contention of Antonsson and Mann [10] that most of the signal was contained at frequencies below twenty Hertz. Most of the knee kinematic data showed only small components above five Hertz.

The pivot step demonstrated the extreme versatility of the joint. At different phases the movement was dominated by rotations about each of the three principal axes. There was no outstanding evidence of any compulsory coupling of the different rotations.

3.6.2 Measurement of Knee Kinematics

An inescapable fact of these experiments is that there was a single subject. The findings must be validated with additional subjects and the scope of the experiments expanded. There is also the hope that kinematic analysis will become a significant part of the orthopedic clinician's toolkit. Unfortunately, the technique used for ensuring that skeletal kinematic data were acquired is too invasive to be used on a widespread basis. The salient question is whether it is possible to use the same equipment and analytical tools, without the invasive mounting, to accurately measure the skeletal kinematics of the lower extremity through the surrounding soft tissue.

The initial results for the simple noninvasive mounting technique used in these experiments are not encouraging. Despite placement of the arrays on bony prominences, where there is a minimum of soft tissue, the kinematics measured with the skin-mounted
arrays did not match the skeletal motions described by the pin-mounted array data. The pin-mounted array data conclusively demonstrated the value of instantaneous kinematic calculations for identifying the underlying kinematic geometry of a movement. Distinct patterns were observed for each task. Using the skin-mounted arrays the estimated relative translational and angular velocity vectors were essentially devoid of any recognizable patterns. This was particularly true in the normal gait results. A substantial heel-strike transient was the most prominent feature of the rigid body displacement and orientation data. The disturbance dominated the velocity estimates, obscuring any possible skeletal movement. The technique of calculating rigid body velocities based on the motion of taped-on arrays of markers is not suitable for observing the underlying skeletal kinematics of the lower extremity.

More encouraging, although still poor, were the results of the finite kinematic analysis based on the same array motions. The flexion angle was severely attenuated in both the active swing and gait tasks, but the shape of the curve mimicked the results from the pin-mounted array experiments. The heel-strike transient appeared as a spurious hyperextension in the gait task data. For the other anatomical angles the results are not as clear. In the gait data, there was a discernible pattern in abduction/adduction and the internal/external rotation angle. In stance there was essentially no non-flexion activity. The trends for both angles did not match the skeletal rotations estimated from the pin-mounted array data. The presence of a repeated motion and the attenuating effect of the soft tissue might support the development of a ‘correction’ scheme for skin-mounted data similar to the approach proposed by van den Bogert et. al. [268] with horses. A substantial amount of additional data are needed to assess the practicality of doing this for just the gait task in humans, without addressing other activities where no obvious patterns were distinguishable.
3.6.3 Presentation of Kinematics

Instantaneous kinematic analysis is a powerful tool for visualizing the underlying kinematic geometry of complex mechanical systems. The adaptable kinematics of the knee were clearly identified by generating the three-dimensional images of the axodes for different tasks. This was accomplished despite the limitations of the data. Recall that all the axodes were thresholded for angular speeds of 0.8 radians per second or greater. The resolution of the calculations was limited by the differentiation of discrete data, even though these data were very high quality in comparison to typical human movement data. To continue to take advantage of instantaneous kinematic analysis, and take advantage of properties such as the invariance of axode shape with coordinate changes, improved estimates of the relative rigid body velocities are required. Then more of the tools of linear algebra and differential geometry may be applied with confidence to understanding the complex behavior of mechanical couplings such as the knee.

Although the arguments in favor of instantaneous kinematic analysis are strong, that does not eliminate the value of and need for finite kinematic analysis. The combination of the two tools was essential for describing the phenomena observed in the performance of the knee in different tasks. The active swing axodes were difficult to differentiate, but the completely different anatomical angle patterns demonstrated that the kinematics of the two trials were not the same. The single most important restriction on the widespread use of anatomical angles and other finite kinematic tools is the absence of a uniform definition of reference coordinate systems and corresponding neutral positions and orientations. Until such standards are established comparison of finite kinematic parameters will remain problematic.
3.7 Conclusions

Several important conclusions may be drawn from the experiments described in this chapter:

- There is no evidence of a single mechanical analog joint for the knee coupling. The kinematic geometry is distinctly different from task to task, and even between different trials of the same task. Differences in hip and ankle angles for the same task support the possibility that biarticular muscles are playing a significant role in determining knee kinematics.

- Knee motion is spatial. Orthogonal components of rotation were present in all three tasks and there was evidence of translation in both normal gait and the pivot movement.

- Although the motion is spatial, calculated orthogonal bases for the axodes showed that the movements had two independent degrees-of-freedom.

- The knee joint may have sufficient laxity so that the articular surfaces are not in intimate contact in some motions. Contact is typically assumed in simulations of knee function and in vitro experiments.

- There is no compulsory, automatic ‘screw-home’ rotation of the tibia in the terminal portion of extension. A rotation was observed under some conditions, but not for all occurrences of terminal extension. The rotation may be due to the either muscle action or the engagement of soft tissue constraints, but does not appear to
depend solely on contact of the articular surfaces.

- Arrays taped to the skin over bony prominences do not accurately reflect the relative orientations of the underlying skeletal members. Estimates of the joint angle could be made using the position vectors to the array origins, equivalent to the use of a putative 'joint center marker' approach. This sacrifices the power of the rigid body assumption and does not provide any useful information on the instantaneous kinematics of the motion. More work on referencing skin-mounted arrays to the bone is required if sufficient measurements necessary to fully characterize the kinematics of the knee are to be made.
Chapter 4

A Method for Assessing Constraint Activity

4.1 Introduction

The *in vivo* kinematic data of the previous chapter demonstrate that the normal human knee is a spatial coupling. The degree of constraint is variable, changing with the task and load across the joint. This implies that the passive constraint system performs a bounding or guidance function rather than specifying a fixed trajectory for relative motion at the knee. A more detailed analysis is necessary to understand the behavior of the normal knee and to have the potential to predict the response to pathological changes in the physical geometry of the joint.

A mathematical model is an invaluable tool for fully exploring the motion of a complex joint like the knee. Modeling offers the opportunity to examine the behavior of anatomical structures which are difficult to observe experimentally. Since it appears that there is no simple mechanical analog for the knee, a two stage kinematic analysis is required. The first step is solution of the geometric compatibility problem, an evaluation of the possible relative displacements and orientations permitted by the constraints. However, finite kinematic analysis is not sufficient for a complete understanding of knee movement.
There may be positions and orientations that are acceptable from a geometric perspective, but not attainable from any other permissible configurations of the system. In order to study relative movement of the knee components, an instantaneous kinematic analysis is necessary. Using the theory of screw systems and reciprocal screws movements of the bodies may be studied, not only displacements.

A model must be based on reliable input parameters if it is to produce meaningful results. Mathematical elegance is not a substitute for good data. This is particularly true in modeling biological systems, where many of the material dimensions and properties either have not or cannot be measured with confidence. The physical dimensions of the knee anatomy – e.g. ligament attachment point locations and articular surface geometry – are the most readily measurable parameters in the knee.

A method for investigating the finite and instantaneous kinematics of the knee is presented. The principal objectives are to minimize the number of kinematic assumptions and reduce reliance on difficult to measure parameters. The initial focus is on examining the role of the passive constraints in determining the kinematics of the knee, although the approach is general. The first section discusses the elements of the passive constraint system and the measurement of appropriate descriptive parameters. This is followed by a classical mobility analysis, which provides some insight into the interaction of the different types of passive constraints. A technique for examination of the possible displacements at the knee and how different ligament models affect them is introduced in the next section. The final section contains an application of extended screw theory to the mixed constraint problem found in the knee and presents an approach for determining the instantaneous total freedom of the knee.
4.2 Model Elements

Physical systems modeling may be approached in two ways. In the first the mathematical models are composed initially from a minimal set of elements, adding elements and detail as required to explain different phenomena. The primary risk is that in early stages the model will not predict experimentally observed behavior. An alternative is to construct a detailed homologue of the physical system, to attempt to recreate not just its behavior but rather the physical system mathematically. The latter approach is particularly tempting in analyzing biomechanical systems, since physical geometry is obviously a dominant factor. The primary risk, since all results of the model must be accepted or explained, is the introduction of experimentally unsupportable results into the modeling process.

An additional concern with any mathematical model of a biological system is whether the necessary input parameters may be measured with sufficient resolution. This is neither a new problem nor unique to biological systems, although the measurements in biomechanical systems may be more difficult to make than in the typical engineering system. T.H. Huxley commented on this issue during the nineteenth century debate on the theory of evolution, noting that Lord Kelvin's underestimation of the earth's age was:

"...one of the many cases in which the admitted accuracy of mathematical processes is allowed to throw a wholly inadmissible appearance of authority over the results obtained by them ... As the grandest mill in the world will not extract wheat flour from peascods, so pages of formulas will not get a definite result from loose data."[263]

Equations must be formulated with the measurablity of the necessary parameters in mind.
4.2.1 Included Elements

The focus of this work is on how the passive constraint system affects the kinematics of the knee. There are two general sources of passive restraining forces in the knee, surface contacts and extensible (e.g. tissue) elements.

Three bones are in contact at the knee: the femur, the tibia and the patella. The tibia and femur are the principal structural members of the lower extremity. The patella is a sesamoid bone, embedded in the patellar ligament and quadriceps tendon. The primary role of the patella is to transmit the active constraining forces of the muscles. It will not be considered as part of the passive constraint system in this study. Only tibio-femoral contacts will be evaluated.

Interposed between the tibia and femur are the menisci, or semilunar cartilages. Although several hypotheses of meniscal function have been proposed [120], the role of the menisci in normal knee function is unclear. There is no in vivo information on the kinematics of the menisci, and only a small amount of in vitro data. Passive kinematic experiments [111] suggest that the menisci may be insignificant kinematic constraints. Meniscectomy resulted in no significant changes in the passive equilibrium positions of the knee. Other in vitro studies measured only small displacements of the menisci under axial loading [39, 31], implying that one role of the menisci is to increase the effective tibio-femoral contact area. This may be modeled as a modification of the tibial surface geometry, and included in a model of the joint kinematics.

The extensible constraints may be divided into two groups: joint and chain constraints. Muscle and tendon, skin, fascia and other superficial tissues which cross multiple joints in the lower extremity constitute the chain constraints. While Haut [103] found that these tissues contributed as much as twenty percent of the restraint to an applied tibial drawer displacement, the combined passive effect on the fundamental kinematic
behavior of the knee coupling is assumed to be small. In addition, an analysis of the
constraint behavior of the multi-joint tissues would have to incorporate the kinematics
of the other involved joints and is beyond the scope of this investigation.

The remaining joint constraints act only across the knee. There are four primary
ligaments: the anterior cruciate, the posterior cruciate, the medial collateral and the
lateral collateral. Other ligaments in the knee – e.g. the ligaments of Wrisberg and
Humphrey – are omitted from consideration because they are not present in all knees [28].
Previous models of the knee [55, 281] incorporated portions of the posterior joint capsule
in the same manner as ligaments. In vitro experiments [192, 193, 224, 234] indicated that
the capsule contributes only a minor portion of the total knee joint stiffness. Studies on
the posterior capsule of the cat [108] measured only very small loads with the knee in
extension and hyperextension. Maximum loads were found to be less than four percent of
the applied moment. The capsule would be expected to make any important contribution
to the constraint forces near full extension. In the absence of significant forces it will be
assumed that the posterior capsule does not affect the primary kinematic characteristics
of the knee.

4.2.2 Ligaments

The following sections review the important properties of ligaments from the perspective
of developing a mathematical model of ligaments as kinematic constraints.

Ligament Morphology

Ligaments are dense connective tissues linking bone to bone¹. Superficially ligaments
and tendons, which attach muscle to bone, are similar. Both belong to the same family

¹There are, of course, exceptions. For example, in the knee ligaments anchor the menisci to the tibial
plateau.
Figure 4.1: Hierarchical organization of tendon and ligament (from Kastelic and Baer [124], p. 398).

Of connective tissues, which includes skin and articular cartilage also. Both are parallel-fibered collagen-based tissues with the hierarchical organization shown in Figure 4.1. The collagen fascicles exhibit a characteristic periodic crimp pattern, which varies in period along the length of the fascicle. Extrastitial water contributes 60–70 % of the total weight of either tissue. Solid components include collagen, elastin, proteoglycans and attached glycosaminoglycans, and cellular material. Differences between tendons and ligaments appear in the histology and biochemical composition of solid components [5]. Table 4.1 contains the approximate fat-free, dry weight composition of the main biochemical components of knee ligaments.

Tendons and ligaments attach to bone in a similar manner [51]. Superficial ligament fibers blend into the periosteum, while the deeper fibers penetrate directly into the bone. The depth of the attachment zone is typically less than one millimeter in total. The attachment for the deeper fibers can be divided into four zones: ligament, with the collagen fibers essentially parallel and elongated cells interspersed between the fascicles; fibrocartilage (150-400 μm), with parallel fibers and cells arranged in rows contained
Table 4.1: Percent fat-free, dry weight biochemical composition of ligaments (from Woo [288] and Amiel et. al. [5]).

<table>
<thead>
<tr>
<th>Constituent</th>
<th>Collaterals</th>
<th>Cruciates</th>
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<tbody>
<tr>
<td>Collagen - Type I</td>
<td>66 - 71</td>
<td>66 - 71</td>
</tr>
<tr>
<td>Collagen - Type III</td>
<td>9 - 10</td>
<td>9 - 10</td>
</tr>
<tr>
<td>Elastin</td>
<td>&lt; 5</td>
<td>&lt; 5</td>
</tr>
<tr>
<td>Proteoglycans</td>
<td>1 - 1.5</td>
<td>2.5 - 3</td>
</tr>
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</table>

In lacunae of extracellular material between separated fibers, mineralized fibrocartilage (100–300 μm), matrix mineralized and some cells disintegrated; bone, with lamellae conforming to the contours of the mineralized fibrocartilage. It has been suggested that the bone behind ligament attachments is more dense than the surrounding bone, in keeping with Wolff’s Law [140]. So far, no supporting quantitative evidence has been presented.

Ligament Tensile Properties

Investigations of ligament and tendon tensile properties have followed two paths. First is the use of standard uniaxial test protocols to attempt to elucidate the stress-strain relationship for both whole tissues and portions of whole tissues. This is often coupled with microscopic studies of the tissue at different points along the yield curve. Second is the direct measurement or mapping of local strains, usually surface, on ligaments and tendons.

Initial efforts to determine the uniaxial yield curve for parallel fibered collagenous materials assumed that the superficial similarities in tendon and ligament extended to their tensile properties. Typically tendon fibers were tested [2, 205, 104] because it was easier to obtain test specimens of a consistent age and length, usually from rat tail tendons. Although the critical stress and strain values are different, the characteristic
Figure 4.2: Typical yield curve for a parallel-fibered tissue.

tensile test curves for tendon and ligament are similar in shape. A typical yield curve for a parallel-fibered tissue is shown in Figure 4.2 and a loading-unloading curve for a parallel-fibered tissue in cyclic loading is shown in Figure 4.3. There are several salient features on each curve, some common to both of them. The initial “toe” region of both curves is characterized by a low stiffness up to about 4–6 % strain. Strains in this region have been shown to correlate with the uncrimping of the fibrils [136, 250] and straightening of the collagen fibers. The primary contributor to the stiffness is thought to be the matrix of elastin fibers. In the linear portion of the curve all the collagen fibers are aligned and any additional elongation is due to strain of the collagen. It has been proposed that there are fibers with different lengths in the matrix, so that some fibers start to fail at lower overall strains than others [136]. This leads to the reduction of slope in the yield curve, followed by a maximum stress and ultimately rupture of the tissue. If the tissue is loaded cyclically in the linear portion of the curve, Figure 4.3 shows that there is a net energy loss in the unloading segment of each cycle.

Very few human knee specific ligament tensile test results have been reported in the
Figure 4.3: Loading-unloading curve for a parallel-fibered tissue in a uniaxial test.

literature [36, 126, 180, 265]. There has been little consistency in the data published. Lanzendorf [143] found that there was agreement only in the maximum stresses measured. Some of the differences may be attributable to the age differences of the specimens and normal inter-subject variability, but these cannot explain the large disparities in reported tensile properties. The major differences are due to differences in specimen preparation and experimental protocol. Error sources include variations in test specimen preparation (excised tissue [126] versus bone-ligament-bone preparations [289]), different methods of tissue storage, varied experiment temperatures, different techniques for cross-sectional area measurement and inconsistent definitions of a ligament "rest" length [143, 288]. The latter two are major problems in the presentation of stress-strain data, particularly the cross-sectional area. Several new approaches to cross-sectional area measurement of soft tissue specimens have been developed [94, 115, 143, 146]. Many are based on non-invasive optical techniques. Lee et. al. [146] used an image reconstruction technique to demonstrate that conventional constant-pressure area micrometer measurements underestimated areas by an average of 18%.

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Butler et al. [36] performed a series of uniaxial tension tests on knee ligament and patellar tendon fascicles. They found that the ligament fascicles had similar failure stress and strain, regardless of the ligament from which the fascicle was extracted. Similarly, tendon fascicles had the same maximum stress and strain. Failure strains for both tendon and ligament fascicles were nearly the same, while tendon failure stresses were higher than those for the ligament fascicles. Other data have shown differences in failure stress and strain for different ligaments and tendons. This implies that the differentiation in whole ligament failure properties is a function of the structure above the fascicle level and may be a function of the distribution of fibers across the cross-section of the tissue.

Several techniques have been used to measure ligament and tendon strains. Direct methods include buckle transducers [4], Hall effect transducers [12, 63], liquid metal strain gauges [67] and conventional strain gauges [73]. These approaches all share the requirement that the transducer be attached to the tissue being measured, introducing the possibility of interference with the normal tissue response. Other methods attempt to avoid this potential limitation by using video [288] or high-speed cinematographic [38, 35, 299] monitoring of landmarks on the tissue surface. Several of these studies have shown that the strain varies along the length and width of ligaments and tendons. Stouffer et al. [250] confirmed this variation and demonstrated that the mid-substance strains were substantially less than the grip-to-grip strains in uniaxial testing, with much higher strains in the vicinity of the ligament attachment or tendon insertion.

Ligament Insertion Site Coordinates

Ligaments are extensible elements, elongating under a tensile load and providing no resistance to a compressive force. The elongation is a function of the applied load. Changes in ligament length may be monitored during knee motion [85, 277] by tracking the relative displacements of the distal and proximal attachments of the ligament. The
instantaneous length of the ligament is an indicator of whether a constraint is active and, if appropriate tensile properties are known, the level of constraint force.

Coordinates of the ligament attachment sites are essential for a model intended to evaluate the activity of the passive constraints in the knee. Ideally, the proximal and distal attachment site coordinates would be measured in separate bone-fixed coordinate systems. This would avoid many of the problems of an incompletely specified reference anatomical position.

There is very little attachment point location data available in the literature [55, 85], although several authors have referred to making measurements [140, 281]. Measurement techniques have ranged from computerized tomography [140] to the use of a spatial linkage to designate anatomical points of interest [281]. The most complete set of data available are in Crowninshield [55]. A set of mean tibial and femoral attachment point coordinates for selected ligament filaments, measured from seven knees, are provided in a Cartesian frame centered on the the tibial plateau. The attachment point locations were measured with the knee in full extension. No more detailed specification of the reference position is provided.

A set of trial measurements of ligament attachment site coordinates was carried out as part of this knee model development. Computerized tomography was used, in an approach based on Langrana et al. [140]. Insertion site boundaries for the ACL, PCL, superficial MCL and LCL on the disarticulated femur and tibia were dissected out and marked with one millimeter diameter stainless steel balls. CT scans of the two bones were obtained and the marker coordinates extracted from the scans. Details of the procedures are presented in Appendix K.
Ligament “Rest” Lengths

Knowledge of the ligament attachment point coordinates in any configuration of the joint is sufficient to estimate the current length of the constraints. However, it is not enough information to deduce the activity level of the ligament. In order to estimate the absolute change in a parameter, such as the length of a ligament, a reference value is required.

The reference length corresponds to the unstressed length of the constraint. In uniaxial testing of tendons and ligaments, the rest length is typically defined as the bone-to-bone or grip-to-grip distance when the smallest measurable load is imposed on the tissue. This may underestimate the length because of the sensitivity of the load cell, the relatively high compliance of the tissue at low loads and variations in strain behavior across the tissue. The use of a single length for each ligament is a problem. For displacement analyses, Bartel et al. [15] showed that a minimum of two filaments on the outer borders of a ligament was required to effectively model how a ligament constrained planar displacements. Since ligament attachments are not uniform (see Figure 4.4) across the width of the ligament, separate reference lengths are required for different filaments. The very definition of what is ligament introduces another difficulty. Each ligament attachment site includes a length of up to one millimeter of fibrocartilage [51]. There is no information on how loading affects strains in the vicinity of the attachment. The underlying assumption is generally that the bone is rigid and everything else is ligament. If the high strains seen near the attachments of ligaments are also present in the fibrocartilage region of the attachments, the reference length must account for the fibrocartilage as part of the ligament. To understand why differences of a millimeter may be important, consider that Girgis [85] measured an average ACL length of 38.2 millimeters. Five percent strain, at the beginning of the linear strain region for the whole ligament, of an ACL is only 1.9 millimeters. Careful definition, and measurement, of the reference length is essential.

As with the attachment point coordinates, definition and measurement of a “rest”
length should be independent of the joint and any particular reference orientation and position. It is a geometric property of the constraint, not the joint.

Based on the preceding discussion, there are essentially no useable data on knee ligament reference lengths available in the literature. The primary limitation of the small amount of data extant is the measurement of a ligament reference length in situ in a given reference orientation and position of the knee. Crowninshield et. al. [55] and Clement et. al. both present data with this limitation.

Lanzendorf [143] developed a technique for measuring a ligament rest length which avoids many of the limitations of other measurement protocols, but still has significant inaccuracies. Bone-ligament-bone preparations were used and the alignment loads were less than 0.05 Newtons. The visible perimeters of the bone-ligament interfaces were marked and the separation measured using a high resolution video imaging system. Inherent problems of the method, primarily staining the bone ligament boundary and the
neglect of the fibrocartilage region of the attachment site, resulted in an estimated error in ligament rest length of four millimeters.

4.2.3 Articular Surface and Bone Geometry

The second type of passive joint constraint influencing movement of the knee is contact of the tibial and femoral condyles. There is certainly contact in weight bearing activities, the repeatability of the stance phase kinematics in the previous chapter is indicative of that. However, it is not clear that contact is present under all loading conditions. The variability of the swing data suggest an absence of contact, with the nearly revolute motion resembling the simpler modes of a bifilar pendulum.

Precise articular surface geometry data are required to evaluate the presence of and constraint effects of tibio-femoral contact. The geometry of the surfaces influences the kinematics both directly and indirectly. Contact of the condyles imposes a direct restraint on movement, adding to the number of active constraints at any instant. It is less obvious that changes in the effective radius of the surfaces with respect to the instantaneous helical axis affect the estimated length of the ligamentous constraints. Inaccuracies in surface geometry affect all constraints, not just the number of active contact constraints.

A number of methods for mapping articular surface geometry of widely varying resolution and accuracy have been described in the literature. Techniques include roentgenstereophotogrammetry [111], digitization of serial slices [80, 135, 190, 200], successive resection of surfaces with a milling machine [166, 241], electro-optical stereophotogrammetry [71], a dial gauge and three-dimensional pointer positioning rig [281], instrumented spatial linkages [217], and pulse-echo ultrasound [214, 215, 260, 261]. Other measurements of principal condylar dimensions have been made in conjunction with knee prosthesis designs [68, 168, 223], usually on the basis of X-ray images. The ultrasound and milling machine techniques provide data on the articular cartilage thickness over the surfaces,
which is needed for more complicated models of the contact surface interface dynamics.

In conjunction with the work in this thesis, design and development of a joint geometry scanner for mapping the articular cartilage surface geometry and thickness of an arbitrary joint using pulse-echo ultrasound was intitiated [117, 222]. The ultrasound approach was selected for several reasons, including the extensive experience in scanning the geometry of the hip in this laboratory [214, 260, 261]. Resolution at least an order of magnitude better than the other methods has been reported. The approach also produces a combination of surface geometry and cartilage thickness data from a single scan. The design and status of the six degree-of-freedom scanner are presented in Appendix L.

4.3 Classical Mobility Analysis

A first step in analyzing the kinematics of the knee is to perform a mobility analysis to estimate the number of degrees of freedom available at the joint, using the general mobility criterion introduced in Chapter 2. An analog model of the knee coupling, with the active constraints replaced by equivalent linkages, is used. Although the approach is simple, some interesting insights into the number of constraints active at a particular instant may be gained.

For a ligament or some portion of it to be an active constraint, it must be taut. Consider two rigid bodies, one fixed and the other moving, tethered together by a single filament. An approximate kinematic equivalent to a taut string or filament (Figure 4.5a) is a rigid link with a spherical (-S-) pair at either end (Figure 4.5b). This equivalence is easy to visualize, the moving link pivoting around the end of the taut filament is identical in effect to the -S- pair, but the model retains a spurious degree of freedom not present in the string. The link with the -S- pairs is free to spin about it's central axis without affecting the relative movement of the two bodies. A better model, without
Table 4.2: Multi-constraint general mobility assuming taut filament or point contact constraints.

<table>
<thead>
<tr>
<th>Constraints</th>
<th>Links</th>
<th>Pairs</th>
<th>Mobility</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>14</td>
<td>15</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>18</td>
<td>20</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>22</td>
<td>25</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>26</td>
<td>30</td>
<td>0</td>
</tr>
</tbody>
</table>

the extraneous freedom, is the five revolute (-5R-) mechanism show in Figure 4.5c [113]. This equivalent may be substituted for each active ligament or ligament band in a model of the knee coupling.

The frictionless point contact of two rigid bodies, shown in Figure 4.6a, may be described in terms of the normal to the tangent plane at the point of contact. Hunt [113] shows that a kinematic equivalent to the point contact is a taut string on the line of the normal. A kinematic equivalent to the taut string is the -5R- linkage of the preceding paragraph. Any transitory configuration of passive constraints in the model knee coupling may be replaced by a combination of -5R- kinematic equivalents.

Consider the two bodies restricted by a single passive constraint, either a contact or a ligament band. The Grübler/Kutzbach mobility criterion estimates the number of degrees of freedom of a kinematic chain given the number of links and lower kinematic pairs, and the freedoms allowed by the pairs, in the chain. The equivalent constraint has four links and five single degree-of-freedom revolute pairs. Adding the fixed and moving bodies for two additional links in the chain and substituting in Equation 2.22 yields a mobility of five, as expected. If a second constraint restricts the motion of the two bodies there are four additional links and five more pairs. The mobility of the assembly is reduced to four. As shown in Table 4.2, each additional constraint reduces the transitory
Figure 4.5: Kinematic equivalent linkages for (a) a taut string or filament are (b) a rigid link with an -S- pair at either end – which has an extraneous degree of freedom – or (c) a four link, five revolute assembly (from Hunt [113], p. 334).
Figure 4.6: Kinematic equivalent linkages for (a) a frictionless point contact are (b) a taut string or filament or (c) a rigid link with an -S- pair at either end – which has an extraneous degree of freedom – or (d) a four link, five revolute assembly (from Hunt [113], p. 334).
mobility of the body by one. With six constraints, the kinematic chain has been reduced to a structure.

On the basis of the kinematic data for three different tasks presented in the previous chapter the knee is a multiple degree-of-freedom joint. In the tasks where there was clearly contact between the tibial and femoral condyles, the stance phase of gait and the pivot step, there is at least one equivalent point contact constraint. On the basis of the simple model introduced, that leaves the bodies with four, and more likely less, relative degrees-of-freedom before considering the possibility of any active ligaments.

Within the limits of the data resolution and accuracy the fixed axes for the voluntary swing experiments are for a planar or possibly even single degree-of-freedom motion. Now, hypothesize that the articular surfaces are either not in contact, or not sufficiently so to act as a kinematic constraint. The simple kinematic equivalent model implies that some combination of four ligament bands could be active and have this motion result. One possibility is that the tibia is a behaving like a bifilar pendulum with the four peripheral bands of the collaterals as the active constraints.

Previous investigators have proposed, based on planar analyses of radiographs, that the cruciates form a four-bar linkage with the tibial and femoral condyles and that this linkage determines the kinematics of the knee [114, 176]. Muller [176] proposed that the spatial movement of the knee, particularly the internal-external rotation, could be modeled by a pair of four-bars comprised of the different bands of the cruciate ligaments. In light of the previous model this seems implausible since the assumed articular contacts and active bands of the ligaments would result in a structure incapable of movement.

The classical kinematic equivalent model of the knee coupling is simple to an extreme. It does imply that the idea of the ligaments as controllers or primary determinants of knee motion may be suspect. Analysis with the model is straightforward and accounts for the spatial kinematics of the coupling. If nothing more, the questions raised are sufficient
incentive for a more detailed analysis of the potential kinematics.

4.4 Finite Kinematics

*In vivo* experimental data show that the knee coupling permits spatial relative movements of the tibia and femur. The order of the screw subsystem varies between tasks and for different trials of what appears to be the same movement. A classical mobility analysis of the knee, using simple mechanical equivalents for the transitorily active passive constraints, indicates that there are limits on the number of constraints that may be active at any instant. It will be shown in the next section that, given the geometric configuration of the active constraints at any instant, the complete screw system encompassing all possible relative movements of the two bones may be predicted. In order to determine the active constraint configuration at any instant, a complete finite kinematic analysis of the joint is required.

Displacement analysis is standard in the design and synthesis of mechanisms. Several iterative algorithms are available for determining the displacement trajectory of a mechanism resulting from a given set of initial conditions [95, 266]. However, the displacement problem for the knee differs from the typical mechanism analysis. In a conventional mechanism analysis, the freedoms of the mechanism are well-defined in terms of its kinematic pairs. The constraints in the knee do not have a clearly defined substructure. One objective of the study is to determine if the constraints, acting in concert, exhibit the characteristics of any of the fundamental kinematic pairs. The knee problem has more in common with workspace mapping for manipulators and robots [132, 147, 295] than conventional displacement analysis.

An analysis of the finite kinematics of the knee is the solution of the geometric compatibility problem, the enumeration of all the permissible relative displacements of the
tibia and femur. By performing the analysis on a mathematical model of the constraints of the knee it is possible to identify which constraints are active for any relative position and orientation and map out their geometry, and map out the geometry of the constraint configuration.

The approach used is based on the method presented by Storace and Wolf [249] for an analysis of the roles of finger tendons. A brief review of their approach to the simpler problem is presented first. This is followed by a discussion of the assumptions underlying application of the method to the knee and the corresponding limitations imposed on any possible conclusions. Next, the configuration state space for the knee is defined and the boundary functions for extensible and contact constraints in the state space are examined in detail. Finally, an implementation of the analysis using a simple computer search to identify the constraint boundaries for a given displacement is presented with some results.

4.4.1 Displacement analysis of constraints

Storace and Wolf [249] devised a scheme for evaluating how effectively the three tendons in the figure constrain motion of the finger. The study was limited to planar displacements and the finger was modeled as a three link kinematic chain with two revolute pairs connecting the links (see Figure 4.7). The three tendons, all attached to the end link, were modeled as inextensible.

A two-dimensional configuration state space, with the two revolute angles as members, was defined and the tendon displacements were taken to be the inputs to the system. For each tendon a displacement function of the form,

$$g_i(\theta_1, \theta_2, x_i) \geq 0$$  \hspace{1cm} (4.1)

was derived. The displacement functions delimited regions in the configuration state space where the constraint allowed movement. Graphically, a schematic of a displace-
Figure 4.7: Planar kinematic chain model of the finger, from Storace and Wolf [249].

Diagram showing the kinematic chain model with labels $g_1$, $g_2$, and $g_3$, and points $X_1$ and $X_2$.

ment function and the region of attainable states for a constraint is presented in Figure 4.8. When the displacement functions for all the tendons were plotted in the state space, the full set of attainable states could be determined. If a given set of tendon displacements permitted only a single pair of states, Storace and Wolf said that the finger was constrained (Figure 4.9). If a region of the state space was attainable, the finger was unconstrained (Figure 4.10).

4.4.2 Assumptions

The single most important aspect of this analysis, and a recurring theme throughout the dissertation, is that there are no \textit{a priori} assumptions concerning the relative motion of the tibia and femur. No kinematics need to be specified to the model. The principal objective is to examine \textit{all} possible displacements.

All of the assumptions are related to how the constraints are modeled. Initially simple models of the contact and ligament geometry are employed. Complexity can always be increased if the model does not accurately reflect the measured kinematics of the physical system.
Figure 4.8: Displacement boundary and region of attainable states for a single constraint in the configuration state space (after Storace and Wolf [249]).

Figure 4.9: Configuration state space and displacement boundaries for a constrained finger (after Storace and Wolf [249]).
Figure 4.10: Configuration state space and displacement boundaries for an unconstrained finger (after Storace and Wolf [249]).

The knee is a synovial joint. Synovial joints are remarkable bearings permitting essentially frictionless movement [204] under forces resulting in high pressures [107] and a wide range of velocities. While the exact mechanism of lubrication is not known, experiments and modeling [156, 261] have shown that approximately ninety percent of the load is carried by the synovial fluid. The fluid swells the articular cartilage matrix. Under load consolidation compressive strain reaches equilibrium, at 10 – 20%, with squeezing of the fluid out of the cartilage. The process takes on the order of twenty minutes. Underlying the cartilage is a layer of cancellous bone. Finite element models of the femoral head [32] and the knee [31] predict deflections of the bone of less than one half millimeter under presumed physiological loads. The movements being studied with this model occur at frequencies significantly higher than those associated with the consolidation of cartilage. Due to the combination of negligible bone displacements, the low coefficient of friction and hydrostatically swollen cartilage the contact surfaces are assumed to be incompressible and all contact frictionless. Under these assumptions the normal to the surface at the center of pressure completely represents the contact constraint.
The most common ligament model is to assume that the whole tissue may be represented by one or more bands or filaments. Each filament or band is assumed to independent of the others in a multifilament model. An additional, either explicit or implicit, assumption is that the 'length' of a filament is the distance between the proximal and distal attachments [55, 270, 106, 265, 277, 281]. This straight line model of the ligament presumes that a taut ligament band is not significantly deflected from the line connecting its attachments by interference with bone or other tissues. On the basis of in vitro observations some models [80, 105] have tried to account for interference of the cruciate ligaments and the bending of ligaments around bone. Garg and Walker [80] reported incremental strains of up to 3.1% in the ACL due to interference with the PCL. The observations for the Garg and Walker model were based on an 'average' knee trajectory. Initially, straight line ligament fiber models will be used, but the algorithm will flag cases of potential interference and sensitivity studies will be carried out to evaluate the efficacy of the model.

4.4.3 Configuration state space definition

A Cartesian coordinate frame is fixed in each bone representing the rigid body. To ease possible future coordination of the model with TRACK kinematic data the coordinate systems are oriented approximately in the manner of the pin arrays in Figure 3.4.

A seven component configuration state vector, \( \mathbf{X} \),

\[
\mathbf{X} = (r_x, r_y, r_z, a, b, c, d)
\]  

(4.2)

where

\[
(r_x, r_y, r_z) = \mathbf{r}, \text{ the relative position vector.}
\]

\[
(a, b, c, d) = \mathbf{q}, \text{ the relative unit quaternion.}
\]

completely describes relative displacements of the tibia and femur. Unit quaternions are used for describing the rotations because they are a two-to-one, singularity free map of
the complete set of three-dimensional rotations and have a clear physical interpretation
in terms of Euler's Theorem (see Appendix A).

4.4.4 Tissue constraints

The ligaments are the principal passive tissue constraints in the knee and will be the focus
of the following discussion. A ligament applies a constraining force over a narrow band
of displacements, ranging between its rest length and length at rupture. Based on the
failure strains of 15% measured by Butler et. al. [36] and the rest length measurements
of Lanzendorf [143], the range of constrained displacements for the ACL is on the order
of 3-4 millimeters. The functional range of extension is certainly less than that. For the
remainder of the discussion the ligament fibers will be assumed to be inextensible with
a length, $L_c$, lying between the rest and rupture lengths.

Single constraint case

It is instructive to build the finite kinematic analysis up from the single constraint case.
The geometry of the single constraint case is simple and the fundamental concepts are
the same as for multiple constraints.

The input geometry for each constraint consists of two vectors and a scalar. The
scalar is the critical constraint length, $L_c$. The vectors locate the point of constraint
attachment to each bone, in the local Cartesian coordinate systems. Taking the tibia, or
distal body, to be the moving body and the femur, or proximal body, to be the reference
coordinate system the transformation of the distal attachment point vector to reference
coordinates is simply,

$$r_{dp} = q^{-1}r_{dx}q + r_{mf}$$  \hspace{1cm} (4.3)

where

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\[ \mathbf{r}_{dp} = \text{Distal attachment point vector in fixed coordinates.} \]
\[ \mathbf{r}_{dm} = \text{Distal attachment point vector in moving coordinates.} \]
\[ \mathbf{r}_{mp} = \text{Vector to moving origin in fixed coordinates.} \]
\[ q = \text{Relative unit quaternion.} \]

Figure 4.11 presents the different vectors.

A displacement function may be defined in terms of the state variables:

\[
\left[ (\mathbf{r}_{dp} - \mathbf{r}_{pp})^T (\mathbf{r}_{dp} - \mathbf{r}_{pp}) \right]^{\frac{1}{2}} - L_c \leq 0
\]

(4.4)

where
\[ \mathbf{r}_{dp} = \text{Distal attachment point vector in fixed coordinates,} \]
\[ \mathbf{r}_{pp} = \text{Proximal attachment point vector in fixed coordinates,} \]
\[ L_c = \text{Critical constraint length.} \]

As long as the current 'length' of the constraint, determined by the straight line model or some other method, is less than the critical length of the constraint, \( L_c \), a movement is permissible.

In order to understand how the constraint affects the motion of the body in state space, first consider its effect on the orientation of the moving body. There is only a point attachment to the moving body, so the constraint cannot restrict the rotational displacement of the tibia. All values of \( q \) which satisfy the unit quaternion condition that,

\[
d^2 + a^2 + b^2 + c^2 = 1
\]

(4.5)

describe allowable rotations of the tibia with respect to the femur. The unit quaternion condition defines a hypersphere in the orientation subspace of the state space. All points on the surface of the hypersphere are valid rotations.

A simple reorganization of the displacement function clarifies the effect of the single fiber constraint on the translational states. Substituting for \( \mathbf{r}_{dp} \) yields,

\[
\left[ (\mathbf{r}_{mp} - (\mathbf{r}_{pp} - q^{-1} \mathbf{r}_{dm} q))^T (\mathbf{r}_{mp} - (\mathbf{r}_{pp} - q^{-1} \mathbf{r}_{dm} q)) \right]^{\frac{1}{2}} - L_c \leq 0
\]

(4.6)

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Figure 4.11: Single constraint input geometry.
Figure 4.12: Single constraint displacement function boundary in translational state subspace for a fixed orientation.

For a specified orientation this equation defines a sphere in the translational subspace. The radius is the critical length of the constraint, $L_c$, and the coordinates of the center of the sphere are a function of the constraint attachment point vector coordinates and the orientation,

$$ r_C = r_{pp} - q^{-1} r_{dm} q $$  \hspace{1cm} (4.7) 

Figure 4.12 shows a schematic of the displacement function in the translational state subspace for a fixed orientation. Two special cases occur when the constraint attachment points in the fixed and moving bodies and the origin of the moving coordinate frame are all collinear. One corresponds to the maximum displacement of the origin of the moving coordinate system from the proximal attachment site while the constraint is active.
Figure 4.13: Geometry of the maximum and minimum displacements for a fixed orientation.

The second case is the minimum displacement of the moving origin from the proximal attachment point under active constraint. The geometry of the maximum and minimum displacement cases is presented in Figure 4.13. The collinearity condition results in simple equations for the vector to the points of maximum displacement (Equation 4.8),

$$ r_{MP} = r_{PP} - q^{-1}r_{dM}q \left( \frac{L_c + |r_{dM}|}{|r_{dM}|} \right) \quad (4.8) $$

and minimum displacement (Equation 4.9),

$$ r_{MP} = r_{PP} + q^{-1}r_{dM}q \left( \frac{L_c - |r_{dM}|}{|r_{dM}|} \right) \quad (4.9) $$

The range of allowable movements in the Cartesian coordinate system is a composite of the permissible regions in the configuration state space. As intuition suggests, the
composite region is spherical also. Its center is at the point of attachment of the constraint on the fixed body and its radius is the sum \((L_c + |r_{de}|)\). Each point on the surface of the sphere is the point of maximum displacement for one of the permitted orientations. Figure 4.14 shows the region of allowable displacement in the reference coordinate system. It is important to distinguish the spherical region in Cartesian space from the spherical regions defined in the translational state subspace for each permitted orientation. The region of Figure 4.14 encompasses all possible relative displacements of the two bones under the action of the constraint. At interior points, the moving body could have any one of many different allowable orientations.

**Two or more constraints**

Evaluation of multiple fiber constraints is conceptually simple. The same set of geometric inputs are required for each additional filament. Displacement functions are specified for each constraint. For a given orientation, the set of permissible translational displacement states is delineated by the intersection of the spheres for the individual constraints. If there is no intersection, then the constraints prevent movement at that orientation and the corresponding unit quaternion is an inaccessible point on the surface of the unit hypersphere. If all the spheres intersect at a single point, then the movement is fully constrained at that orientation.

Some simple examples will help illustrate the process. Table 4.3 lists the geometric inputs – attachment point coordinates and critical lengths – for a pair of constraints. Three orientations, with the unit quaternion components listed in Table 4.4, were evaluated. The first case is for no relative rotation. Orthographic views of the displacement function boundaries in the translational displacement state subspace for the two constraints are presented in Figure 4.15. The allowable sets of displacements under the action of each constraint are equivalent. Addition of the second constraint does not affect modify the
Figure 4.14: The region of allowable displacements in the reference coordinate system under a single constraint is a sphere of radius \((L_c + |r_{iM}|)\) centered on the point of attachment of the constraint in the reference system.
Figure 4.15: Orthographic views of the displacement function boundaries for the no rotation case of the two constraint example: (a) Top view, and (b) Side view.
Table 4.3: Geometric inputs for two constraint example.

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Fixed Insertion</th>
<th>Moving Insertion</th>
<th>$L_c$</th>
</tr>
</thead>
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<tr>
<td></td>
<td>$x_{pp}$</td>
<td>$y_{pp}$</td>
<td>$z_{pp}$</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>-10</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4.4: Orientation inputs for three two constraint examples.

<table>
<thead>
<tr>
<th>Case</th>
<th>$\theta$</th>
<th>Direction Cosines</th>
<th>Unit Quaternion</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$l$</td>
<td>$m$</td>
</tr>
<tr>
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<td>0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
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</tr>
<tr>
<td>3</td>
<td>45</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

degree of constraint for this orientation.

In the second case, the selected orientation is a small, five degree, rotation about the vertical (Y) axis of the reference coordinate system. This is a crude model of internal/external rotation. Now the two sets of allowable translational displacement states, shown in Figure 4.16, are no longer equivalent. There is a set of common permitted translations, delineated by the shaded area of intersection of the spherical displacement boundaries for the two constraints.

A forty-five degree rotation about the same axis is the input for the final case. Figure 4.17 displays orthographic projections of the translational displacement boundaries for the two constraints. The intersection of the two sets is the null set, so there are no translational displacements satisfying both constraints at this orientation. As a result, the orientation is not an accessible point in the orientational displacement state subspace.

The simple two constraint demonstrates both the power of the geometric approach for studying constraint behavior and the magnitude of the task of identifying the full set
Figure 4.16: Orthographic views of the displacement function boundaries for the second rotation case of the two constraint example: (a) Top view, and (b) Side view.
Figure 4.17: Orthographic views of the displacement function boundaries for the second rotation case of the two constraint example: (a) Top view, and (b) Side view.
of permissible displacement states.

### 4.4.5 Contact constraints

Articular surface contact is the second major type of passive constraint in the knee. The tibio-femoral joint is comprised of two articulations: medial femoral condyle on medial tibial condyle and lateral femoral condyle on lateral tibial condyle. The tibial articular surfaces are physically separated by the intercondylar notch, while the femoral condyles are distinct regions in the single layer of articular cartilage covering the distal end of the femur.

A mapping of the contact regions is needed for at least two purposes. First, constraint analysis requires an estimate of the contact geometry for evaluating both the finite and instantaneous kinematics. Also, studies of synovial joint lubrication in the hip have shown that a significant level of congruence is instrumental in maintaining effective lubrication. The hip has a high inherent level of congruence due to its bony geometry, which is not present in the knee. Contact evaluation with the meniscal geometry included could clarify some of the issues pertinent to determining the mechanism of joint lubrication.

In the study of the finite kinematics of the knee, the contact constraints will introduce additional boundaries in the displacement configuration state space. None of the geometrical elegance found in the preceding study of ligament constraints is present in the evaluation of the displacement boundaries introduced by articular surface contact. Available procedures are essentially systematic, and computationally ponderous, point by point searches for interference or intersection of the mating surfaces. For that reason the contact constraint evaluation is performed on the reduced set of permissible configurations delineated by the tissue constraint analysis. It is also the principal justification for using a heirarchical search scheme. The objective is to eliminate as many cases of noncontacting surfaces as possible without resorting to exhaustive calculations.
Articular surface data are assumed to be in the form of a regular grid of discrete points such that the surface may be represented as a composite of topologically rectangular patches [69]. The type of patch fit to a set of four vertices will vary depending on the stage of the search.

Contact identification is a special case of the static object interference problem [23], which has been treated in the computer-aided design (CAD) and computer-aided manufacturing (CAM) literature. Algorithms for evaluating the interference of polyhedra include those presented by Boyse [23] and Maruyama [160]. Interference of more complex surfaces has investigated by Timmer [174] and Comba [49]. The polyhedral interference algorithms are particularly useful in the early stages of the search, but more precise models of the surface geometry are required for the final evaluation.

A three stage procedure is used to identify contact regions. The first two steps draw on Maruyama's polyhedral object interference detection algorithm. The principal objective of these initial stages is to reduce the number of candidate pairs [160]. Candidate pairs are patches on the two mating surfaces which may be in contact.

The first phase is used to eliminate cases of gross interference or noncontact. Following Maruyama several definitions are required. An object, \( \Omega \), is defined by a four-tuple

\[
\Omega = \Omega(P, E, F; \bar{p})
\]  
(4.10)

where

- \( P = \) the set of vertices,
- \( E = \) the set of edges,
- \( F = \) the set of faces,
- \( \bar{p} = \) the c-point.

The c-point is defined as the mean of all the vertices on a given object,

\[
\bar{p} = \frac{1}{n} \sum_{i=1}^{n} p_i
\]  
(4.11)
A minimal sphere, \( \hat{S}(R) \), is defined as the minimum radius sphere centered at the c-point which contains all the vertices of an object. The maximal sphere, \( \hat{S}(r) \), for an object is the smallest radius sphere with an origin at the c-point which is contained within the faces of the object. Two tests are defined using the distance, \( D \), separating the c-points of the mating objects. The objects are said to be strongly separated if the sum of the radii of the associated minimal spheres is greater than the separation distance:

\[
D > R_F + R_M
\]  \hspace{1cm} (4.12)

If the distance is less than the sum of the radii of the maximal spheres for the two objects,

\[
D < r_F + r_M
\]  \hspace{1cm} (4.13)

the two objects are said to be strongly intersecting.

Condylar contact in the knee imposes some particular limitations on this screening technique. The tibia, with meniscal geometry included, presents a concave surface. For a concave surface, the c-point will generally lie above the surface and the corresponding maximal sphere is undefined. In practice the maximal sphere test is modified, comparing only the radius of the maximal sphere for the convex femoral condyle to the separation distance for the c-points:

\[
D > r_F
\]  \hspace{1cm} (4.14)

If this test is satisfied there is no contact between the two condyles, although it is not a good indicator of interference.

Tests are applied to successively smaller regions of the condylar surfaces, reducing the number of candidate pairs with each step. A simple example is presented in Figure 4.18 showing a series of minimal spheres for the tibia used to investigate a potential contact.

The second stage relies on the definition of a minimal box containing the object. A minimal box is the smallest rectangular parallelepiped with its faces parallel to the
Figure 4.18: Example of first stage contact constraint identification.
reference coordinate planes containing the object. An example of a minimal box for a patch is displayed in Figure 4.19. The box may be defined with a tolerance on its dimensions. Intersection of the boxes for a candidate pair is checked by comparing the vertex coordinates. If an intersection exists the common region is called the solution box (see Figure 4.20). It is only necessary to check for the intersection of object faces which penetrate the solution box. If no object faces are contained in the solution box, or only the faces of one object, the corresponding surfaces may be eliminated from the list of candidate pairs.

The final stage involves the identification of the contact regions on the mating surfaces. A three phase approach for dealing with complex surfaces was developed by Timmer and is described in detail in Mortenson [174]. The three steps are: the hunting phase, the tracing phase, and the sorting phase. In the hunting phase, intersections of the moving surface with patch boundaries on the target surface are located. These boundary parts are connected into curve segments in the tracing phase. Finally, in the sorting phase, the curve segments are reordered and joined to complete the map of the contact region.
Figure 4.20: Solution box for a candidate pair of patches.

Unless the number of candidate pairs is significantly reduced prior to this stage, mapping of the contact region will be difficult and tedious.

4.4.6 Implementation

A partial implementation of the constraint assessment algorithms has been made on a Personal IRIS graphics workstation (Silicon Graphics, Mountainview, CA). The contact constraint algorithm is not included in the implementation at this time.

The algorithms were implemented with the aim of providing several different outputs for use in analyzing the kinematics of the knee. The primary objective was to develop a tool for visualizing constraint activity, showing which regions of the configuration state space the different constraints affect. Although visualization is valuable, there is a need for a quantitative measure of constraint to assess the effect of changes in the constraint model. Finally, constraint geometry, in the form of wrenches, is required for input to the instantaneous kinematic analysis.

An approach similar in concept to the graphical analysis used in describing the lig-
4.5 Instantaneous Kinematics

Finite kinematic analysis can produce a wealth of valuable information about the displacement of the knee. However, the knee is part of a kinematic system, not a structure, and the real interest lies in how the passive constraints contribute to the control of movement at the joint. Allowable displacements may prove to be inaccessible from many other perspectives.
Figure 4.21: Accessible translational displacement states for a fixed orientation under the influence of a single tissue constraint displayed in three-dimensions on the Personal IRIS.
configurations. Only an instantaneous kinematic analysis of the system can clarify these issues.

Extended screw theory is an elegant approach to relating the wrenches of constraint in a mechanical system to the permitted twist, or instantaneous helical, axes. Developed for the study of the contact of rigid bodies, application of the concepts to extensible constraints, such as the ligaments of the knee, is straightforward. This enables solution for the possible movements of systems constrained by a composite of contact and extensible constraints.

The ensuing sections develop the application of screw theory to a study of the instantaneous kinematics of composite systems such as the knee. A review of the essential ideas of classical and extended screw theory is presented first, followed by the extension of the approach to the study of tissue constraints. A simple example is then presented to demonstrate the power and utility of the approach for studying the movements of complex mechanical systems.

4.5.1 A review of extended screw theory

Basics

Classical and extended screw theory were briefly introduced in Chapter 2. The fundamental concepts are based on the fact that the virtual work for a pair of screws reduces to a geometric quantity, the virtual coefficient. The virtual coefficient was defined (Equation 2.16) in terms of screw coordinates of the screw pair \((\dot{\alpha}, \dot{\beta})\) as:

\[
\omega_{\Delta\beta} = \alpha_1 \beta_4 + \alpha_2 \beta_5 + \alpha_3 \beta_6 + \alpha_4 \beta_1 + \alpha_5 \beta_2 + \alpha_6 \beta_3
\]  

(4.15)

The constraint may be acting along either \(\dot{\alpha}\) or \(\dot{\beta}\).

Classical screw theory [14] showed that all screws reciprocal to a given screw were solutions of Equation 2.16 with a virtual coefficient of zero. It was also shown that if a
screw was reciprocal to a system of screws, it was reciprocal to all linear combinations of those screws.

Ohwovoriole [181, 182] extended screw theory by considering cases of nonzero virtual work in a study of contacting bodies. If the virtual coefficient was positive ($\varpi > 0$), a screw pair were defined to be repelling. A negative virtual coefficient determined pairs of contrary screws. Figure 4.22, a repeat of Figure 2.4, illustrates the physical interpretation of these inequalities. Positive virtual work corresponds to the two contacting bodies separating, or repelling. Negative virtual work results when the movement would require one body to penetrate the other. If a screw is repelling or contrary to a system of screws, it is repelling or contrary to all positive linear combinations of the screws in that system.

Ohwovoriole [181] called the composite solution of the reciprocal, repelling and contrary screw systems the 'total freedom' of a system. All possible movements of the system under the action of the constraints were described by the screw systems.
Method of solution

For a body acted upon by a system of several constraints, there is one equation or inequality for each constraint. Reciprocal, repelling and contrary screws are found by solving the resulting systems of equations or inequalities. The set of solutions to a system of linear inequalities is a finite convex cone [79]. A convex cone is the intersection of a set of linear halfspaces, where each inequality defines a halfspace. Ohwoviriole [181] presented a detailed procedure for finding the general solution of a system of linear inequalities which will be reviewed briefly here.

For a system of \( k \) constraints the contrary and repelling screw problems may be formulated as a set of linear equalities of the form:

\[
A\alpha \leq 0
\]  

(4.16)

where

\[
A = k \times 6 \text{ matrix of wrench coordinates},
\]

\[
\alpha = k \times 1 \text{ vector of screw coordinates}.
\]

The rows of \( A \) are the screw coordinates of the \( k \) constraining wrenches at a specific point in the configuration state space.

It can be shown [86] that the complete solution to the system of inequalities in Equation 4.16 is given by either the homogeneous solution, or \( d \)-dimensional face of its convex cone, alone or the convex combination of the homogeneous solution and all of the particular solutions, or \( d+1 \)-dimensional faces of the convex cone. The dimension of the solution, \( d \), is found by first determining the number of independent constraints by evaluating the rank, \( r \), of \( A \). Then \( d \) is,

\[
d = 6 - r
\]  

(4.17)

since the number of independent constraints and acting on a body and twist freedoms available to the body must always add up to six.
After checking the equations for consistency the homogeneous solution to the equations may be found. The \(d\)-dimensional face of the finite convex cone is the solution of the system of linear equalities,

\[
A\alpha = 0
\] (4.18)

The solution to Equation 4.18 defines the reciprocal screw system for the set of constraints and has the form:

\[
\alpha_H = a_1\alpha_{1H} + a_2\alpha_{2H} + \ldots + a_d\alpha_{dH}
\] (4.19)

The \(d+1\)-dimensional faces of the cone are particular solutions to the set of inequalities. Particular solutions are obtained by evaluating the set of \(kC_{(k-(r-1))}\) subproblems of the form:

\[
M\alpha_p < 0
\] (4.20)

\[
N\alpha_p = 0
\]

where

\[
M = (k-(r-1)) \times 6 \text{ submatrix of } A,
\]

\[
N = (r-1) \times 6 \text{ submatrix of } A \text{ containing the rows not included in } M,
\]

\[
\alpha_p \quad \text{a particular solution to the inequalities.}
\]

Not all of the subproblems will have solutions. The general solution of the system of inequalities will be the linear combination of the homogeneous solution and the particular solutions:

\[
\alpha = \alpha_H + b_1\alpha_{1p} + b_2\alpha_{2p} + \ldots + b_j\alpha_{jp}
\] (4.21)

\[
0 \leq b_1, \ldots, b_j
\]

where

\[
j \leq k-(r-1).
\]
Figure 4.23: Twist axis, $\hat{\beta}$, and wrench, $\hat{\alpha}$ acting along the line of action of an extensible constraint such as a ligament.

4.5.2 Application to extensible constraints

Ohwovorlole developed extended screw theory to evaluate contact problems associated with automatic parts mating. In order to apply the theory to analysis of the instantaneous kinematics of the knee it is necessary to determine whether tissue constraints may be treated in the same manner as contact constraints.

Consider the situation displayed in Figure 4.23, where $\hat{\alpha}$ is the wrench axis along the line of action of an extensible constraint such as a ligament. Twists about $\hat{\beta}$ resulting in positive virtual work must induce motion along the positive direction of the wrench. Since the extensible constraint is attached to the fixed body, the wrench axis will in general be directed toward the fixed body. Movement along the wrench axis in this sense will relax the strain on the tissue, easing the constraint. This is directly analogous to the
repelling contact solution defined by Ohwovoriole [181].

In the opposite case, a twist inducing movement away from the fixed body, in the negative direction along the wrench axis, would produce negative virtual work. For a tissue constraint such a movement would increase the strain and intensify the constraint. For the contact problem, negative virtual work is the result of a contrary solution which would violate geometric compatibility. Viewed from a different perspective, the violation of geometric compatibility is an intensification of the contact constraint. In this manner the sense of contrariness for the extensible is the same as for the contact case. The lengthening of the constraint is an intensification and physically permissible, but undesirable if it proceeds to the point of failure of the constraint.

As long as extensible constraints are anchored on the fixed reference body, the definitions of extended screw theory are appropriate. It is only necessary to remember that contrary solutions may be physically permissible for extensible constraints, but not for contacts.

4.5.3 Example

A simple example will demonstrate the use of extended screw theory. A sample rigid body and constraint system is shown in Figure 4.24. There are four ligament constraints $L_i$, with attachment point geometry approximating the major knee ligaments, and two contact constraints $C_i$. Choosing the coordinate system origin to be between the two contact constraints as shown, the wrench coordinates for the constraints are presented in Table 4.5. The wrench coordinates for each constraint are written, in the form of
Figure 4.24: Sample case: Rigid body with ligamentous and contact wrenches.

Equation 2.16, as a row in a matrix:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{4} & 0 \\
0 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{4} & 0 \\
-\frac{1}{2} & 0 & 0 & 1 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]  

(4.22)

Table 4.5: Wrench coordinates for extended screw theory example.

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Real Part</th>
<th>Dual Part</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta_1$</td>
<td>$\beta_2$</td>
</tr>
<tr>
<td>$L_1$</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$L_2$</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$L_3$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$-\frac{\sqrt{2}}{2}$</td>
</tr>
<tr>
<td>$L_4$</td>
<td>$-\frac{\sqrt{2}}{2}$</td>
<td>$-\frac{\sqrt{2}}{2}$</td>
</tr>
<tr>
<td>$C_1$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$C_2$</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

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Calculation of the rank of the matrix indicates that only three of the wrenches are independent.

The system of equations to be solved for the reciprocal screw problem is:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2} & 0 \\
0 & 0 & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2} & 0 \\
-\frac{1}{2} & 0 & 0 & 0 & 1 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\alpha_6
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]  \hspace{1cm} (4.23)

A screw reciprocal to a system of constraints must be workless. Solution for the system of screws reciprocal to the set of wrenches in the example is:

\[
\alpha_H = \begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\alpha_2H + \begin{bmatrix}
0 \\
0 \\
1 \\
-\frac{1}{2} \\
0 \\
0
\end{bmatrix}
\alpha_3H + \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
1
\end{bmatrix}
\alpha_6H
\]  \hspace{1cm} (4.24)

Any screw which is a linear combination of the three basis screws, will produce a motion reciprocal to the given set of constraining wrenches.

Using the definitions of screw and line coordinates the motion allowed by the basis screws can be interpreted physically. The first twist is zero pitch, a pure rotation, about an axis parallel to the \textit{Y}-axis. Since there is no moment, the screw passes through the origin of the coordinate system. The rotation corresponds to a relative internal/external rotation of the bones. The second basis screw is also a pure rotation, but the axis does not pass through the origin since the screw has a dual, or moment, component. The twist is about an axis parallel to the \textit{Z}-axis, passing through the intersection of the lines of action of \( L_3 \) and \( L_4 \). This is essentially a flexion axis. The last of the basis twists has infinite pitch. It corresponds to a translation parallel to the \textit{Z}-axis. Except for the translational component, which is probably due to the simplicity of the contact constraints defined in the model, the reciprocal screw system for this simple model would explain many of the
IHAs determined for the experimental swing and gait data of Chapter 3.

Solution for the repelling and contrary screw systems requires evaluating particular solutions to the inequalities associated with the constraints. Following the procedures developed by Ohwovoriole and outlined above, no particular solutions were found for either the repelling or contrary problems.

4.5.4 Mixed constraint problem

A closer examination of the combination of constraints in the sample configuration may offer an explanation. Recall that a twist about a screw repelling to a contact wrench breaks the contact. A twist repelling to a ligamentous constraint relaxes the tension in the ligament. Achieving the former requires motion of the moving body away from the fixed body, while relaxing an extensible constraint requires the movement to be towards the fixed body. Typically, a twist that is repelling to a contact constraint will not be repelling to a ligamentous constraint. The same situation holds for the contrary screw system. The solution procedure seeks to find screws that are repelling or contrary to the entire system of constraints, without differentiating between the mixed constraints.

It is not clear that the repelling, reciprocal and contrary screw systems produced by following Ohwovoriole's solution procedure for the contact constraints are describing the total freedom of a configuration with mixed constraint types. Since the physical reasoning supporting application of screw theory to the two types of constraints separately appears sound, future work should focus on evaluating and generalizing the solution procedure.

4.6 Summary

A method for assessing constraint activity in a general coupling, with application to the human knee, has been developed. Several aspects of the approach distinguish it from
previous models of the knee:

- The model clearly elucidates the connection between the geometry of the wrenches constraining the system and the range of instantaneous kinematics possible. Prior models have not considered instantaneous kinematics.

- A general approach to the finite kinematics has been developed which does not rely on any a priori assumptions about equivalent linkages in the system. Many previous models have either made assumptions about equivalent linkages or the nature of the constraints.

- The finite kinematic analysis produces a unique visualization of constraint activity in the configuration state space.

- Model inputs are essentially geometric quantities. Current technology allows reliable and relatively precise measurements of geometric quantities from in vitro human specimens, with the potential for in vivo measurements high. The limiting factor is how precisely the relevant structures may be identified. Previous models have relied on estimates of soft tissue mechanical properties based on unreliable measurement techniques and a miniscule human database.

- Model inputs are easily modified, so that the robustness of the model to input data changes and the kinematic effects of constraint configuration modifications may be investigated.

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Chapter 5

Conclusions and Recommendations

5.1 Conclusions

Three principal goals were established at the beginning of this dissertation. Substantial progress toward all three objectives was achieved. The primary accomplishments were:

- The knee is a spatial coupling. There was clear evidence of three orthogonal rotational components and indications of translation in some movements.

- The number of degrees-of-freedom of the coupling varies with task and load. The influence of contact surface geometry on joint kinematics was clearly observed. Concurrent measurements of the kinematics of the other joints of the lower extremity support the contention that the biarticular muscles strongly influence knee kinematics.

- No evidence of any automatic rotation or screw-home movement directly coupled to flexion angle was observed in the data. Combined rotations were present under some conditions, but not universally.

- Data unequivocally demonstrate that the movements of arrays of markers taped to the skin over bony prominences do not accurately reproduce the underlying skeletal
kinematics.

• An approach to assessing the activity of constraints and how the constraints moderate both the finite and instantaneous kinematics of the joint was developed. The mathematical model of the knee requires no a priori assumptions about the kinematics of the joint, and is based on measurements of constraint geometry.

5.2 Recommendations for Further Work

All research sets out to answer questions and, in the end, generates more questions to answer. Some of the more important new, and old, unanswered questions are:

• Acquire more lower extremity kinematic data from more subjects performing more trials of more tasks. Data in this dissertation were acquired from a single subject, and until proven otherwise may be assumed to be representative, but cannot be considered conclusive until additional evidence is collected.

• In order to acquire significant amounts of data a method of mounting arrays non-invasively on the skin must be found. Skeletal pin mounted arrays cannot be used extensively. Additional efforts to evaluate some different mounting schemes have [122] shown some improvements over the tape mounting, but still have not produced an array mounting scheme which mimics the underlying skeletal kinematics.

• Improve kinematic data accuracy and resolution, in particular the derivative estimates. One approach already being investigated combines direct measurements of acceleration and displacement. This should be pursued to the fullest extent. High quality derivative data are essential for application of the differential geometry of axodes for comparison of kinematic data from different subjects and laboratories.
• Obtain real knee constraint geometry data for input to the mathematical model. The model has not yet been exercised with actual skeletal geometry inputs.

• Explore solution space of the instantaneous kinematic problem with mixed constraint types and determine whether or not Ohwovoriole's solution generates the complete set of possible twist axes. One avenue for studying this would be to approach the problem through the differential geometry of screw systems.

• Improve visualization of constraint activity. The current method presents constraint activity for a fixed orientation. Alternative techniques for displaying a full set of configuration states and constraint activity simultaneously are needed.

• Improve the efficiency of the constraint assessment search algorithms.
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Appendix A

Mathematical Tools

Many mathematical tools are available for describing the motion of rigid bodies in space. In choosing the approach to use for a particular task some of the important considerations are: the compactness of the representation, the presence of singularities, whether there is any straightforward physical interpretation for the parameters used, and whether the rotational and general spatial motion are represented in a similar manner.

Quaternions, for spherical motion [206, 247, 256], and dual quaternions, for spatial motion [207, 294], are elegant tools which satisfy all of the criteria. Although they were developed separately in the nineteenth century, subsequent developments have shown that the algebra of quaternions and dual quaternions are subsets of the Clifford algebras [195].

Quaternions are used extensively in the versions of the TRACK programs developed for this thesis and dual quaternions are used in the ultrasound scanning machine control algorithms (see Appendix L). This appendix will outline the fundamental properties of quaternions, dual numbers, motors and dual quaternions applicable to the description of spherical and spatial motion of a rigid body.

A.1 Quaternions

Sir William Rowan Hamilton developed quaternion algebra to extend the algebra of three-dimensional vectors to include multiplication and division. His last twenty years work, up to his death in 1865, were devoted to the problem. Unfortunately, his treatise on the subject [98] was not published until the end of the century and much of the work has lingered in obscurity. Hamilton's treatise is the definitive reference on the subject, but excellent summaries of the fundamental properties of quaternions may be found in Brand [24] and Yang [292].
Table A.1: Quaternionic Unit Multiplication Rules.

<table>
<thead>
<tr>
<th></th>
<th>i</th>
<th>j</th>
<th>k</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>-1</td>
<td>k</td>
<td>-j</td>
<td>i</td>
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<td>j</td>
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<td>-1</td>
<td>i</td>
<td>j</td>
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<tr>
<td>k</td>
<td>j</td>
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<td>k</td>
</tr>
<tr>
<td>1</td>
<td>i</td>
<td>j</td>
<td>k</td>
<td>1</td>
</tr>
</tbody>
</table>

A.1.1 Definition

Multiplication and division of three-dimensional vectors is not closed; in general, the
product of two vectors is not another vector. Gibbs circumvented this by defining the
familiar scalar and vector products. Hamilton was interested in developing an algebra
for the direct calculation of products and quotients. He found that, to obtain closure,
it was necessary to develop an algebra for sets of four numbers. These sets are called
quaternions and have the general form:

\[ q = a_1 + b_j + c_k + d \]  \hspace{1cm} (A.1)

where the components \( a, b, c, d \) are ordinary numbers and \( i, j, k \) are the quaternionic units.
The quaternionic units are right versors and satisfy the rules shown in Table A.1. These
properties are similar to those given to the unit basis vectors of Cartesian coordinate
systems, particularly with regard to cross products. It is evident that while components
\( a, b, c \) are multiplied by unit vectors and, consequently, constitute a three-dimensional
vector, the fourth component, \( d \), is a scalar with multiplier 1. This leads to an alterna-
tive representation for quaternions as the sum of a three-dimensional vector and a scalar:

\[ q = Vq + Sq \]  \hspace{1cm} (A.2)

where \( Vq \) and \( Sq \) are

\[ Vq = a_1 + b_j + c_k \]
\[ Sq = d \]  \hspace{1cm} (A.3)

respectively. Cartesian vectors are quaternions with zero scalar components and scalars
are quaternions with zero vector components. Quaternions may also be represented as
quadruples of the form:

\[ q = (a, b, c, d) \]  \hspace{1cm} (A.4)

with the caution, apparent from the properties of the quaternionic units, that the order
must be maintained.
Geometrically, the dividend and divisor vectors define a plane. Hamilton referred to this as the plane of the quaternion \( q \). The axis of the quaternion is at a right angle to the plane and is given by the a unit vector in the direction of the vector component of the quaternion.

### A.1.2 Quaternion algebra

Quaternions obey the normal rules of vector algebra for addition, multiplication by a scalar and subtraction. Some of the fundamental definitions follow.

Consider two quaternions, \( q_1 \) and \( q_2 \):

\[
q_1 = a_1 i + b_1 j + c_1 k + d_1 \\
q_2 = a_2 i + b_2 j + c_2 k + d_2
\]

(A.5)

The two quaternions will be considered equal \((q_1 = q_2)\) if and only if the components multiplying each of the quaternionic units and the scalar term are equal.

\[
a_1 = a_2 \\
b_1 = b_2 \\
c_1 = c_2 \\
d_1 = d_2
\]

(A.6)

Addition is distributed over the quaternionic units, so that the sum of two quaternions is the sum of the components of the two quaternions. Similarly, given the product of a scalar \( \lambda \) and a quaternion, the result is the distribution of the scalar multiplication over each component of the quaternion. This leads directly to the definition of a negative quaternion

\[
\lambda q = \lambda a + \lambda b j + \lambda c k + \lambda d
\]

(A.8)

as simply a quaternion multiplied by the scalar \(-1\). Subtraction of a quaternion, \( q_2 \), from a second quaternion, \( q_1 \), may then be

\[
q_1 - q_2 = q_1 + (-1)q_2
\]

(A.10)

defined as the sum of \( q_1 \) and the negative of \( q_2 \). Quaternions are associative, commutative and distributive under scalar multiplication and addition.

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For multiplication of two quaternions, the operation is again distributed over

\[ q_1 q_2 = (a_1 i + b_1 j + c_1 k + d_1)(a_2 i + b_2 j + c_2 k + d_2) \]  
(A.11)

the components. Due to the multiplicative properties of the quaternionic units order 
must be preserved. The source of the order dependence may be clarified by recasting the 
two quaternions as the sums of scalars and vectors. The distribution of the multiplication 
is equivalent to the following:

\[ q_1 q_2 = S q_1 S q_2 - V q_1 \cdot V q_2 + V q_1 \times V q_2 + S q_1 V q_2 + S q_2 V q_1 \]  
(A.12)

The order dependence is reflected in the cross product of the two vector components. 
This leads directly to the result that multiplication of two

\[ q_1 q_2 \neq q_2 q_1 \]  
(A.13)

quaternions is, in general, not commutative. If and only if the two vector components are 
equal or proportional to one another will the product of two quaternions with non-zero 
vector parts be commutative. Also evident in this equation is a general product of two 
vec tors (or vector quaternions) that is the difference of the two types of Gibbs vector 
products, the dot product and cross product. The distributive and associative properties 
hold for multiplication of two quaternions.

The conjugate of a quaternion is defined as:

\[ K q = S q - V q \]  
(A.14)

Geometrically, the conjugate is a reversal of the quaternion axis. For a sum of quater-
nions, the conjugate is the sum of the conjugates of each quaternion in the sum.

\[ K (q_1 + q_2 + \ldots + q_n) = K q_1 + K q_2 + \ldots + K q_n \]  
(A.15)

However, the conjugate of a quaternion product is the product of the conjugates

\[ K (q_1 q_2 \ldots q_n) = K q_n K q_{n-1} \ldots K q_1 \]  
(A.16)

of the individual multiplicands in reverse order.

The norm of a quaternion is defined as the product of the quaternion and its conjugate.

\[ N q = q (K q) = (K q) q = a^2 + b^2 + c^2 + d^2 \]  
(A.17)

A product of quaternions has a norm equal to the products of the norms of the

\[ N (q_1 q_2 \ldots q_n) = N q_1 N q_2 \ldots N q_n \]  
(A.18)

individual multiplicands.
If the norm of a quaternion is equal to one, the quaternion is a unit quaternion. Any unit quaternion may be written as

\[ q = \cos \theta + e \sin \theta \quad (0 \leq \theta \leq \pi) \]  
(A.19)

where \( e \) is given by

\[ e = \frac{a + b + c k}{\sqrt{a^2 + b^2 + c^2}} \]  
(A.20)

The angle \( \theta \) is the magnitude of a rotation about the quaternion axis defined by \( e \). The quaternion components have the following values in this form

\[ a = l \sin \theta \]
\[ b = m \sin \theta \]  
(A.21)
\[ c = n \sin \theta \]
\[ d = \cos \theta \]

where \( l, m \) and \( n \) are the direction cosines of the quaternion axis. The components of a unit quaternion are sometimes referred to as Euler parameters.

For a nonzero \( q \), the norm of \( q \) is nonzero, and the reciprocal of the quaternion is defined as:

\[ q^{-1} = \frac{Kq}{Nq} \]  
(A.22)

This leads to

\[ qq^{-1} = 1 = q^{-1}q \]  
(A.23)

Since the reciprocal is the ratio of the conjugate and the norm of a quaternion, for a unit quaternion the reciprocal and conjugate are equal. The reciprocal of a product of quaternions is the product of the reciprocals of the multiplicands taken in reverse order, a consequence of the inclusion of the

\[ (q_1 q_2 \ldots q_n)^{-1} = q_n^{-1} q_{n-1}^{-1} \ldots q_1^{-1} \]  
(A.24)

A.1.3 Differentiation of quaternions

For describing the motion of a rigid body, time derivatives of quaternions are necessary. Given a quaternion,

\[ q = f(t) \]  
(A.25)
which is a function of time, the first derivative with respect to time is given by:

\[
\begin{align*}
\frac{d}{dt}(q(t)) &= \dot{q}(t) \\
&= \frac{d}{dt}(a) + \frac{d}{dt}(b)j + \frac{d}{dt}(c)k + \frac{d}{dt}(d) \\
&= \dot{a} + \dot{b}j + \dot{c}k + \dot{d}
\end{align*}
\]  

(A.26)

Since the derivative is a quaternion itself, higher derivatives are obtained by repeated application of the same process.

The differential for a function of a quaternion, \(f(q)\), is:

\[
d(f(q)) = f'(d(q))
\]  

(A.27)

This leads to differentiation rules for the first, second and third derivatives of the product of two functions of a quaternion:

\[
\begin{align*}
\frac{d}{dt}(f(q)\phi(q)) &= \frac{d}{dt}(f(q))\phi(q) + f(q)d(\phi(q)) \\
\frac{d^2}{dt^2}(f(q)\phi(q)) &= \frac{d^2}{dt^2}(f(q))\phi(q) + 2\frac{d}{dt}(f(q))d(\phi(q)) \\
&\quad+ f(q)d^2(\phi(q)) \\
\frac{d^3}{dt^3}(f(q)\phi(q)) &= \frac{d^3}{dt^3}(f(q))\phi(q) + 3\frac{d^2}{dt^2}(f(q))d(\phi(q)) \\
&\quad+ 3\frac{d}{dt}(f(q))d^2(\phi(q)) + f(q)d^3(\phi(q))
\end{align*}
\]  

(A.28)

As an example consider the first, second and third differentials of the reciprocal quaternion, \(q^{-1}\):

\[
\begin{align*}
\frac{d}{dt}(q^{-1}) &= -q^{-1}\dot{q}q^{-1} \\
\frac{d^2}{dt^2}(q^{-1}) &= 2(q^{-1}\frac{d}{dt}(q))q^{-1} - q^{-1}\frac{d^2}{dt^2}(q)q^{-1} \\
\frac{d^3}{dt^3}(q^{-1}) &= 3q^{-1}\frac{d^2}{dt^2}(q)q^{-1}d(q)q^{-1} + 3q^{-1}\frac{d}{dt}(q)q^{-1}\frac{d^2}{dt^2}(q)q^{-1} \\
&\quad- 6(q^{-1}\frac{d}{dt}(q))q^{-1}\frac{d^2}{dt^2}(q)q^{-1} - q^{-1}\frac{d^3}{dt^3}(q)q^{-1}
\end{align*}
\]  

(A.29)

These will be particularly useful in the discussion of rigid body rotational motion.

### A.2 Dual Numbers

Dual numbers are a type of complex number introduced by Clifford in the late nineteenth century [47]. A detailed exposition of the algebra and calculus of dual numbers is given in Dimentberg [59], while several authors have reviewed the basic properties [24, 274, 293]. Rooney compared the geometrical characteristics of dual numbers and other types of complex number [208].
Table A.2: Dual Number Unit Multiplication Rules.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>(\varepsilon)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>(\varepsilon)</td>
</tr>
<tr>
<td>(\varepsilon)</td>
<td>(\varepsilon)</td>
<td>0</td>
</tr>
</tbody>
</table>

A.2.1 Definition

A dual number, \(\hat{x}\), has the form

\[
\hat{x} = x + \varepsilon x_0
\]  
(A.30)

where both \(x\) and \(x_0\) are real numbers. The principal or primary unit is one (1) and the dual unit, \(\varepsilon\). Multiplication rules for the dual and principal units are shown in Table A.2. Special dual numbers may be defined depending on the values of the principal and dual parts. The most important are:

\[
x \neq 0 \quad x_0 \neq 0 \quad \text{Proper Dual Number} \tag{A.31}
\]
\[
x \neq 0 \quad x_0 = 0 \quad \text{Real Number}
\]
\[
x = 0 \quad x_0 \neq 0 \quad \text{Pure Dual Number}
\]
\[
x = 0 \quad x_0 = 0 \quad \text{Zero}
\]

A.2.2 Algebra

The algebra for dual numbers is essentially that of real numbers distributed over the two parts of the dual number, keeping in mind the multiplication rules for the dual unit. This is straightforward for the equality of two dual numbers, addition, multiplication by a scalar and subtraction.

Addition and subtraction of two dual numbers \(\hat{x}\) and \(\hat{y}\) is given by

\[
\hat{x} \pm \hat{y} = (x \pm y) + \varepsilon(x_0 \pm y_0)
\]  
(A.32)

For multiplication the rules for the dual unit multiplication must be accounted for \((\varepsilon^2 = 0)\) yielding

\[
\hat{x}\hat{y} = xy + \varepsilon(x_0 y + x y_0)
\]  
(A.33)

It is interesting to note that the product of two pure dual numbers is always zero. One of the multiplicands does not have to be zero to result in a zero product.

The quotient of two dual numbers is

\[
\frac{\hat{x}}{\hat{y}} = \frac{x}{y} + \varepsilon \frac{x_0 y - xy_0}{x^2}
\]  
(A.34)

\(^1\)The symbol \(\omega\) is sometimes used for the dual unit \([47, 59]\)
This is undefined for pure dual numbers.

A function of a dual number, \( f(\hat{x}) \), is
\[
f(\hat{x}) = f(x) + \varepsilon x_0 \frac{df(x)}{dx}
\]  
(A.35)

This shows that a function of a dual number is defined by a function of the principal part of the dual number alone. Two useful functions are for the \( n \)th power and \( n \)th root of a dual number
\[
(\hat{x})^n = x^n + \varepsilon n x_0 x^{n-1} 
\]  
(A.36)
\[
\sqrt[\varepsilon]{\hat{x}} = \sqrt[x_0]{x} + \varepsilon \frac{x_0}{n \sqrt[x_0]{x^{n-1}}} 
\]  
(A.37)

A useful dual number is the dual angle, which is a measure of the skewness of two lines in space. Figure A.1 shows two skew lines and their mutual perpendicular. A dual angle is defined as:
\[
\hat{\theta} = \theta + \varepsilon s 
\]  
(A.38)

where
\[
\begin{align*}
\theta &= \text{The angle between the two lines} \\
\varepsilon s &= \text{The length of the mutual perpendicular.}
\end{align*}
\]

By applying the rules for functions of dual numbers, it can be shown that the fundamental trigonometric functions of a dual angle are:
\[
\begin{align*}
\cos \hat{\theta} &= \cos \theta - \varepsilon s \sin \theta \\
\sin \hat{\theta} &= \sin \theta + \varepsilon \cos \theta 
\end{align*}
\]  
(A.39)
All trigonometric identities which hold for real angles are valid for dual angles.

A.3 Dual Vectors

Dual vectors are a natural extension of dual numbers. Two subsets of dual vectors, line vectors and screws, have applications in geometry and mechanics. McAulay is given credit for the first application of dual vectors for the problems of mechanics and von Mises viewed screw calculus as a significant advance [293]. Brand [24] has a chapter on motor algebra in his text and Yang provides a good overview of the properties of dual vectors [292, 293].

A.3.1 Definitions

Dual vectors, also known as screws or motors, may be viewed in several ways. First consider a vector composed of dual rather than real numbers:

\[ \mathbf{M} = \hat{x} + \hat{y} \hat{z} k \]
\[ = (x + \varepsilon x_0) \hat{x} + (y + \varepsilon y_0) \hat{y} + (z + \varepsilon z_0) k \]

This can be expanded and rearranged to give an alternative form:

\[ \mathbf{M} = (x \hat{x} + y \hat{y} + z \hat{k}) + \varepsilon (x_0 \hat{x} + y_0 \hat{y} + z_0 k) \]
\[ = m + \varepsilon m_0 \]

A motor is always referenced to an arbitrary point in space, usually a coordinate system origin in mechanics. If the reference point is shifted the real component of the dual vector is unaffected, but the dual portion is changed by addition of the term:

\[ m_{0*} = m_0 + r_{0*} \times m \]

where \( r \) is the vector from the new reference point to the original reference point.

A dual magnitude, or length, may be computed for a dual vector. The length is,

\[ \alpha = \alpha + \varepsilon \alpha_0 \]
\[ = \sqrt{m \cdot m} + \varepsilon \frac{m \cdot m_0}{\sqrt{m \cdot m}} \]

It can be shown that the dual length is invariant with a change of origin. A scalar real parameter is defined in terms of the components of the dual magnitude. This parameter is the pitch of the motor, given by:

\[ \mu = \frac{\alpha_0}{\alpha} \]
\[ = \frac{m \cdot m_0}{m \cdot m} \]
Table A.3: Classification of Dual Vectors.

<table>
<thead>
<tr>
<th>Type</th>
<th>Dual Magnitude</th>
<th>Pitch</th>
<th>Proper</th>
</tr>
</thead>
<tbody>
<tr>
<td>Motor or Screw</td>
<td>( \alpha + \varepsilon \alpha_0 )</td>
<td>( \frac{\alpha}{\alpha} )</td>
<td>Yes</td>
</tr>
<tr>
<td>Line Vector</td>
<td>( \alpha )</td>
<td>0</td>
<td>Yes</td>
</tr>
<tr>
<td>Couple</td>
<td>( \varepsilon \alpha_0 )</td>
<td>( \mu \to \infty )</td>
<td>No</td>
</tr>
<tr>
<td>Zero</td>
<td>0</td>
<td>( - )</td>
<td>No</td>
</tr>
</tbody>
</table>

A dual vector may be normalized by dividing by the dual magnitude, the result being a unit dual vector,

\[
\hat{\mathbf{A}} = \frac{\hat{\mathbf{M}}}{\hat{\alpha}} = \mathbf{a} + \varepsilon \mathbf{a}_0
\]  

(A.45)

Any dual vector may be written as the product of a dual magnitude and a unit dual vector:

\[
\hat{\mathbf{M}} = (\alpha + \varepsilon \alpha_0) (\mathbf{a} + \varepsilon \mathbf{a}_0)
\]  

(A.46)

\[
= \alpha (1 + \varepsilon \mu) (\mathbf{a} + \varepsilon \mathbf{a}_0)
\]

There are several special classifications of dual vectors based on the values of the pitch and dual magnitude. A summary is shown in Table A.3. A dual vector is proper if \( \alpha \) is nonzero.

### A.3.2 Algebra

Study, in Germany, and Kotelnikov, in Russia, are given credit for independently proving the principle of transference [293]. Briefly, the principle of transference states that the rules of vector algebra hold for dual vectors and the rules of vector calculus may be extended to the screw calculus. More detailed discussions of transference may be found in Hsia and Yang [110] and Selig [226].

The sum of two dual vectors \( \hat{\mathbf{M}} \) and \( \hat{\mathbf{N}} \) is given by:

\[
\hat{\mathbf{M}} + \hat{\mathbf{N}} = (\mathbf{m} + \mathbf{n}) + \varepsilon (\mathbf{m}_0 + \mathbf{n}_0)
\]  

(A.47)

The scalar product of two motors is defined as,

\[
\hat{\mathbf{M}} \cdot \hat{\mathbf{N}} = (\mathbf{m} \cdot \mathbf{n}) + \varepsilon (\mathbf{m} \cdot \mathbf{n}_0 + \mathbf{n} \cdot \mathbf{m}_0)
\]  

(A.48)
It can be shown that this dual number is independent of the reference point. The scalar product distributes over addition,

\[ \mathbf{M} \cdot (\mathbf{L} + \mathbf{N}) = \mathbf{M} \cdot \mathbf{L} + \mathbf{M} \cdot \mathbf{N} \]  
(A.49)

The condition for two proper motors intersecting each other at right angles is the scalar product being zero. Von Mises defined the scalar product differently, taking only the dual portion of the above definition as the scalar product,

\[ \mathbf{M} \circ \mathbf{N} = \mathbf{m} \cdot \mathbf{n}_0 + \mathbf{n} \cdot \mathbf{m}_0 \]  
(A.50)

The von Mises product is commutative and distributive with respect to addition like the full scalar product. A zero von Mises product is a necessary and sufficient condition for two line vectors to be coplanar [24]. Two screws are reciprocal if their von Mises product is zero [127].

The inner product of a pair of motors is given by:

\[ \mathbf{M} \ast \mathbf{N} = \mathbf{m} \cdot \mathbf{n} + \mathbf{m}_0 \cdot \mathbf{n}_0 \]  
(A.51)

Two screws are orthogonal if their inner product is zero.

A pair of proper dual vectors are coaxial if their motor product vanishes. The motor product is defined to be:

\[ \mathbf{M} \times \mathbf{N} = \mathbf{m} \times \mathbf{n} + \epsilon (\mathbf{m} \times \mathbf{n}_0 + \mathbf{m}_0 \times \mathbf{n}) \]  
(A.52)

### A.4 Dual Quaternions

Dual quaternions were introduced by Clifford [47] as a tool for investigating non-Euclidean geometry. McAulay [293] applied them, as octonions, to investigations in mechanics and deformable bodies. More recently, Yang [292] applied dual quaternions to the finite displacements of kinematic mechanisms. Rooney [207] concluded that the dual quaternion is the best form of transforming line coordinates available.

#### A.4.1 Definitions

As with dual vectors, a dual quaternion may be viewed as quaternion with dual numbers substituted for its components or as the sum of real and dual quaternions. The different forms are:

\[ Q = \hat{d} + \hat{a}s + \hat{b}j + \hat{c}k \]  
(A.53)

\[ = (d + as + bj + ck) + \epsilon (d_0 + a_0s + b_0j + c_0k) \]

\[ = q + \epsilon q_0 \]

Real numbers, vectors and quaternions and dual numbers and vectors are just special cases of the dual quaternion.
A.4.2 Algebra

The rules for quaternion algebra hold for dual quaternions with dual components substituted everywhere for the real components.

Of particular interest is the unit dual quaternion, a dual quaternion with a norm of unity. Given a dual quaternion, \( Q \), with

\[
Q = (d + a_1 + b_1 + c_1 k) + \epsilon(d_0 + a_0 + b_0 + c_0 k)
\]  
(A.54)

its norm must satisfy,

\[
NQ = (d^2 + a^2 + b^2 + c^2) + \epsilon(d_0 a_0 + a_0 b_0 + b_0 c_0)
\]  
(A.55)

\[
= 1 + \epsilon 0
\]

for \( Q \) to be a unit dual quaternion. If the requirement is satisfied \( Q \) may be rewritten as,

\[
Q = \cos \hat{\theta} + \hat{\mathbf{S}} \sin \hat{\theta}
\]  
(A.56)

where

\[
\hat{\theta} = \text{The dual angle of the unit dual quaternion.}
\]

\[
\hat{\mathbf{S}} = \text{The unit line vector defining the quaternion axis.}
\]

The dual angle may be expressed in terms of the real components of the dual quaternion as,

\[
\hat{\theta} = \cos^{-1} \hat{d}
\]  
(A.57)

\[
= \cos^{-1} d - \epsilon \frac{d_0}{\sqrt{a^2 + b^2 + c^2}}
\]

\[
= \theta + \epsilon \delta
\]

Similarly, the real component of the unit line vector on the quaternion axis is,

\[
\mathbf{S} = \frac{a_1 + b_1 + c_1 k}{\sqrt{a^2 + b^2 + c^2}}
\]  
(A.58)

and the dual component is,

\[
S_0 = \frac{a_0 + b_0 + c_0 k}{\sqrt{a^2 + b^2 + c^2}} + \frac{dd_0 + a_0 b_0 + b_0 c_0 + c_0 d_0}{\sqrt{a^2 + b^2 + c^2}}
\]  
(A.59)

A dual unit quaternion satisfying this form is a line transformation operator. When \( Q \) operates on any dual vector \( \mathbf{M} \) that intersects the dual quaternion axis at right angles from the left,

\[
\tilde{\mathbf{N}} = Q \mathbf{M}
\]  
(A.60)
$\mathbf{M}$ is rotated about $\hat{S}$ by a positive angle $\theta$ and displaced along the axis of $\hat{S}$ by a positive distance $s$. If $Q$ operates on $\mathbf{N}$ from the right, $\mathbf{N}$ is rotated through a negative angle $\theta$ about the axis $\hat{S}$ and displaced a distance $s$ in the negative direction along the axis. Yang [292] calls $Q$ a screw operator. In order to avoid the restriction that the $Q$ may only operate on line vectors intersecting its axis at right angles, the operator must be modified in manner similar to the quaternion operator for finite rotations (see Appendix B). To obtain a displacement of any line through a dual angle of $\hat{\theta}$, a dual unit quaternion $Q'$, with

$$Q' = \cos \frac{\hat{\theta}}{2} + \hat{S} \sin \frac{\hat{\theta}}{2}$$

(A.61)

must be used. The transformation is now defined as,

$$\mathbf{N} = Q'^{-1} \mathbf{M} Q'$$

(A.62)

This is a general line transformation analogous to the real quaternion rotation operator.
Appendix B

Finite Rotations of a Rigid Body

The fundamental theorem of finite rigid body rotations, proven by Euler in the eighteenth century [45], is:

The general displacement of a rigid body with one point fixed is a rotation about an axis through the fixed point.

Many mathematical tools are available for describing the spherical motion of a rigid body. In choosing the approach to use for a particular task some of the important considerations are: the compactness of the representation, the presence of singularities, the existence of any straightforward physical interpretation for the parameters used, and whether the rotational and general spatial motion are represented in a similar manner.

Quaternions are an elegant method of representing spherical motion [208, 247, 256] which satisfies all of the criteria.

B.1 Orthogonal Transformations

The most commonly used method of describing the finite rotations of a rigid body employs an orthogonal transformation matrix [16, 87].

Euler's theorem states that any displacement of a rigid body with one fixed point is a rotation about an axis through that point. There are several methods for describing spherical motion of a rigid body [208]. The most common approach is through use of the rotation matrix, an orthogonal transformation.

Consider two orthogonal triads of unit vectors, (I, J, K) and (i, j, k). The relative orientation of the two triads is given by the matrix equation:

\[
\begin{bmatrix}
  I \\
  J \\
  K 
\end{bmatrix} = M \begin{bmatrix}
  i \\
  j \\
  k 
\end{bmatrix} 
\]  

(B.1)
The matrix, \( M \), is given by:

\[
M = \begin{bmatrix}
I \cdot i & I \cdot j & I \cdot k \\
J \cdot i & J \cdot j & J \cdot k \\
K \cdot i & K \cdot j & K \cdot k 
\end{bmatrix}
\]  

(B.2)

and is proper orthogonal.

Associate a Cartesian coordinate system with each triad. The vector from the origin of the \( (i, j, k) \) system to a point \( P \) is given by,

\[
r_1 = x_1i + y_1j + z_1k
\]  

(B.3)

and the vector in the \( (I, J, K) \) system is:

\[
r_2 = x_2I + y_2J + z_2K
\]  

(B.4)

It can be shown that the two vectors are related to each other by:

\[
r_2 = Mr_1
\]  

(B.5)

So \( M \) is a coordinate transformation matrix.

Again take a point \( P \). Initially, the two coordinate systems are coincident so that:

\[
r_2(t_1) = r_1(t_1)
\]  

(B.6)

Rotate the \( (I, J, K) \) system relative to the \( (i, j, k) \) system, keeping \( P \) fixed in the \( (I, J, K) \) system, so that:

\[
r_2(t_2) = r_2(t_1) = r_1(t_1)
\]  

(B.7)

From the coordinate transformation in Equation B.5,

\[
r_2(t_2) = Mr_1(t_2)
\]  

(B.8)

or,

\[
r_1(t_2) = M^T r_2(t_2)
\]  

(B.9)

Substituting gives:

\[
r_1(t_2) = M^T r_1(t_1)
\]  

(B.10)

The transpose of the coordinate transformation matrix is a rotation operator in a Cartesian frame, rotating the coordinates of a point \( P \) from one position to another.
B.2 Unit Quaternion Transformations

B.2.1 Rotations and Coordinate Transformations

The angular displacement of a point in a fixed coordinate system can be expressed as a unit quaternion transformation. If the initial position vector for the point is \( r_1 \), the rotated position vector will be:

\[
r_1(t_2) = q[r_1(t_1)]q^{-1}
\]

(B.11)

The quaternion operator, \( q(\cdots)q^{-1} \), is equivalent to the rotation matrix, \( M^T \) of Equation B.10.

A coordinate transformation is defined by the transpose of the rotation matrix:

\[
r_2 = Mr_1
\]

(B.12)

Similarly, the reciprocal of the quaternion operator may be shown to be a coordinate transformation:

\[
r_2 = q^{-1}r_1q
\]

(B.13)

B.2.2 Angular Velocities and Higher Derivatives

There is a compact representation for the angular velocity of a body undergoing a rotation described by the unit quaternion operator defined in the previous section. The angular velocity of the body with respect to the inertial frame is:

\[
\Omega = 2\dot{q}q^{-1}
\]

(B.14)

This is simply the quaternion product (see Appendix A) of the quaternion rate and the reciprocal quaternion. The latter is equal to the conjugate quaternion for a unit quaternion. Defined relative to a coordinate system fixed in the moving body, the angular velocity is:

\[
\omega = 2q^{-1}\dot{q}
\]

(B.15)

The angular velocity in the fixed frame is related to the angular velocity in the moving frame by the unit quaternion transformation:

\[
\Omega = q^{-1}\omega q
\]

(B.16)

Extension to the higher time derivatives is straightforward using the quaternion differentiation rules given in Appendix A. The angular acceleration relative to the fixed coordinate frame is:

\[
\ddot{\Omega} = 2\dot{q}q^{-1} - 2(\dot{q}q^{-1})^2
\]

(B.17)
A further differentiation produces the angular jerk relative to the inertial frame:

\[
\ddot{\Omega} = 2\frac{d^3q}{dt^3}q^{-1} + 4(\dot{q}q^{-1})^3 - 2\dot{q}q^{-1}q^{-1}q^{-1} - 4\ddot{q}q^{-1}\dot{q}q^{-1}
\]

(B.18)

Similar relationships for the acceleration and jerk relative to the moving frame are easily derived by application of the differentiation rules to equation B.15.

B.3 Relative Rotational Motion

Application of the quaternion relationships to the relative motion of two bodies in an inertial coordinate system produces some interesting results. Consider the two rigid bodies shown in Figure B.1 with their respective embedded coordinate systems. At any instant the unit quaternions describing the orientation of each body in the fixed coordinate system (System 1) are known and given by \(q_{12}\) and \(q_{13}\).

Given the unit quaternions for the two bodies with respect to the fixed frame, a relative unit quaternion describing the orientation of the two moving bodies with respect
to each other may be derived. For any point $P$, a vector may be drawn from the origin of each coordinate system (see Figure B.2). The vectors from the moving origins to $P$ are functions of the unit quaternions and the vector from the origin of the fixed coordinate system:

$$r_{P_2} = q_{12}^{-1}r_{P_1}q_{12}$$
$$r_{P_3} = q_{13}^{-1}r_{P_1}q_{13} \quad (B.19)$$

By rearranging the first equation, an expression for the vector from the origin of the fixed system as a function of the vector from the origin of the moving system and the unit quaternions may be found. Substituting for $r_{P_1}$ in the second equation leaves:

$$r_{P_3} = q_{13}^{-1}q_{12}r_{P_2}q_{12}^{-1}q_{13} \quad (B.20)$$

Clearly, the unit quaternion relating the rotations of the two moving systems is given by:

$$q_{23} = q_{12}^{-1}q_{13} \quad (B.21)$$
In the previous section, the angular velocity of a moving body relative to the fixed coordinate frame was defined as:

$$\Omega_{31} = 2q_{13}q_{13}^{-1}$$  \hspace{1cm} (B.22)

It is not obvious that the relative unit quaternion can be used in the same manner to determine a relative angular velocity:

$$\Omega_{32} = 2q_{23}q_{23}^{-1}$$  \hspace{1cm} (B.23)

Substituting for the relative quaternion and performing some basic quaternion manipulations produces the following:

$$\Omega_{32} = 2q_{23}q_{23}^{-1}$$
$$= 2(q_{12}^{-1}q_{13} - q_{12}^{-1}q_{12}q_{12}^{-1}q_{13})(q_{13}^{-1}q_{12})$$
$$= 2(q_{12}^{-1}q_{13}q_{13}^{-1}q_{12} - q_{12}^{-1}q_{12})$$
$$= q_{12}^{-1}\Omega_{31}q_{12} - \omega_{21}$$  \hspace{1cm} (B.24)
$$= q_{12}^{-1}(\Omega_{31} - \Omega_{21})q_{12}$$

Equation B.23 is exactly equivalent to the classical expression for relative angular velocity (see for example Beggs [16]), incorporating the coordinate transformation from the fixed system to the moving system. Computationally this makes implementation of relative angular velocities simple. Only a relative quaternion product needs to be calculated, then the angular velocity may be calculated using the same equations. There is no confusion about the coordinate system, since the transformation results automatically.

It is simple to show that the same results may be derived for higher derivatives, further enhancing the utility of the quaternion representation of finite rotations. The only additional calculation needed for relative motion is the computation of the quaternion product to generate the relative quaternion. The properties of the quaternion take care of the remaining details.
Appendix C

Experimental Set-Up

The relative positioning and orientation of the cameras in the TRACK III system is shown in Figure C.1. Parameter values used in the experiments are presented in Table C.1. Figure C.2 displays the viewing volume of the cameras relative to the forceplate for the given set of camera position and orientation parameters.
Figure C.1: Relative position and orientation of the two Selspot 1 cameras in the TRACK III system.
Figure C.2: Viewing volume (shaded region) of the Selspot I cameras for the parameters of Table ??¿. The position of the forceplate is outlined in the viewing volume.
Table C.1: Selspot camera position and orientation parameter values.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$ (meters)</td>
<td>3.353</td>
</tr>
<tr>
<td>$H$ (meters)</td>
<td>0.617</td>
</tr>
<tr>
<td>$\theta_1$ (degrees)</td>
<td>-26.</td>
</tr>
<tr>
<td>$\beta_1$ (degrees)</td>
<td>15.</td>
</tr>
<tr>
<td>$\theta_2$ (degrees)</td>
<td>26.</td>
</tr>
<tr>
<td>$\beta_2$ (degrees)</td>
<td>15.</td>
</tr>
</tbody>
</table>
Appendix D

Informed Consent Document

The experiments in Chapter 3 were approved by the Committee on the Use of Humans as Experimental Subjects under application number 1302. The informed consent document for the kinematic experiments is presented in the following pages.
INFORMED CONSENT STATEMENT

Project Title: Measurement of the Relative Motion Between Skin and Bone at the Bony Landmarks of the Lower Extremity

These experiments will play a vital role in gait analysis and in the detailed study of joint motions. Without an accurate assessment of the relative motion between the bone and skin at bony landmarks of the lower extremity, data on the motion of the skeletal segments gathered using gait-analysis techniques would be of uncertain accuracy and limited value. The understanding of the relative motion might also be applied directly to the design of orthopaedic braces or other rehabilitative equipment.

The experiment will be performed in three approximately three-hour sessions at the Eric P. and Evelyn E. Newman Laboratory for Biomechanics and Rehabilitation at M.I.T. The experiments are all gait experiments and similar in nature. The TRACK system, a gait-analysis system, will be used to make measurements of the kinematics of gait in each session. The TRACK system follows the motion of small, infrared light-emitting diodes (LEDs), fixed in rigid frames and attached to the body. Since the LEDs emit in the nonvisible range, they will not be distracting in any way. The three sessions will differ only in the manner in which the LEDs are attached to your body. The three sessions will be as follows:

1. You will be asked to wear lightweight Plexiglas frames which support the LED frames on your foot, shank, thigh and pelvic girdle. The arrays will be attached to your body with elastic straps only. You will asked to move each joint of your lower extremity through its full range of motion while standing in front of the TRACK cameras. You will then be asked to walk in front of the cameras in a normal manner. During this portion of the experiment, your foot-floor contact force will be measured using a force-platform embedded in the floor; combined with the three-dimensional kinematics measured by the TRACK cameras, this force measurement will allow calculation of the forces and moments at the joints of the lower extremity during the stance phase of walking. In the next portion of the experiment, your tibia will be manually rotated about its axis relative to the femur at several different angles of flexion while you are seated. Finally, you will be asked to repeat the first portion of the experiment, moving the joints through their ranges of motion.

2. In the second session the rigid Plexiglas frames will be attached in a different manner; the remainder of the experiment will be the same as in the first session. Frames will be attached to the pelvic girdle and foot in the same manner as above. Three additional frames will be attached to the skin over the greater trochanter and lateral epicondyle on your thigh and at approximately the middle of the tibia.
The frames will be attached using double-sided tape with 3M 1512 adhesive. This is an adhesive designed specifically for and commonly used in medical applications. The tape is easily removable and will be removed immediately following completion of the experiment.

3. In the third and final session, the rigid Plexiglas frames on the shank and thigh will be attached so the motion of the bone may be measured directly. The frames at the pelvic girdle and foot will be attached in the same manner as above. The frames for the shank and thigh will be mounted on small Kirschner wires; Kirschner wires are commonly used in orthopaedic surgery for the fixation of fractures. The pins will be inserted into the greater trochanter and the lateral epicondyle on the thigh and parallel to the tibial plateau at approximately mid-shank. The insertion will be performed under sterile conditions by a trained surgeon and with a local anesthetic, Lidocaine, used at the insertion points. There is less than 0.001 percent chance of infection due to this common orthopaedic procedure. The same series of experiments will be performed, and the pins will be removed immediately.

There are no immediate benefits to you from this experiment beyond helping to improve the understanding of the motion of the joints of the lower extremity. The bone pins should not cause any pain or discomfort, and there is a very small chance of infection (less than 0.001 percent) since the pins are being inserted extra-articularly. You are welcome to discuss the risks, benefits, and objectives of the experiments with the experimenters at any time before, during or after the experiments.

I understand that in the event of injury resulting from the research procedure, medical care is available from the M.I.T. Medical Department. The costs of that care will be borne by my own health insurance or other personal resources. Information about the resources available at the M.I.T. Medical Department is available from Lawrence Bishoff at 253-1744.

There is no compensation for possible injury, either financial or insurance, furnished to research subjects merely because they are research subjects. Further information may be obtained by calling Kimball Valentine on 253-2822.

I have read and understand the above and and agree to participate in the experiments as described above.

__________________________________________
Subject/Parent/Guardian

__________________________  __________________________
Date                       Witness to Signature

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Appendix E
Segment File Generation

Segment files, containing the three-dimensional coordinates for each LED in a body-fixed segmental coordinate frame, are a required input to the TRACK algorithms. The segment file is used in two parts of the algorithm, the inter-LED length error check and in the calculation of segmental positions and orientations from LED global three-dimensional coordinates using the Schut algorithm.

Two approaches are used to generate segment files. The simplest is to define the LED geometry for each segment based on the design drawings. This approach presumes accurate machining and mounting of the LEDs and neglects any other effects due to equipment or set-up variations. An empirical approach [116, 169] has been developed which results in a reduction in inter-LED length errors.

A variation of the empirical method was used to generate the segment files for processing all the skeletal pin and skin-mounted array data. Input data were from two second duration static trials for each type of array mounting with the subject standing in an anatomically neutral position\(^1\). Sample means were calculated for the three-dimensional coordinates of each LED in the laboratory coordinate system,

\[
\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \tag{E.1}
\]

\[
\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n}
\]

\[
\bar{z} = \frac{\sum_{i=1}^{n} z_i}{n}
\]

where

\(n\) = number of frames for each channel,

\(^1\)For the skeletal pin mounted arrays the static file was trial 14MA30 and for the skin-mounted arrays trial 14MA05 was used.
and for each segment,

\[
\bar{z}_{sk} = \sum_{j=1}^{n_k} \sum_{i=1}^{n} \frac{z_{ji}}{n \cdot n_k}
\]

\[
\bar{y}_{sk} = \sum_{j=1}^{n_k} \sum_{i=1}^{n} \frac{y_{ji}}{n \cdot n_k}
\]

\[
\bar{x}_{sk} = \sum_{j=1}^{n_k} \sum_{i=1}^{n} \frac{x_{ji}}{n \cdot n_k}
\]

where

\[n = \text{number of frames for each channel},\]

\[n_k = \text{number of channels for the } k\text{th segment},\]

\[(x_{ji}, y_{ji}, z_{ji}) = \text{global coordinates for the } i\text{th frame of the } j\text{th channel of the } k\text{th segment},\]

\[(\bar{x}_{sk}, \bar{y}_{sk}, \bar{z}_{sk}) = \text{mean global coordinates for the } k\text{th segment},\]

The mean coordinate for each segment was taken as the origin for the body-fixed segmental coordinate system. Each segmental triad was oriented so as to be aligned with the fixed laboratory frame. Segmental coordinates for each channel were the difference of the mean global coordinates for each channel and the mean for each segment,

\[
x_{sf} = \bar{x} - \bar{z}_{sk}
\]

\[
y_{sf} = \bar{y} - \bar{y}_{sk}
\]

\[
z_{sf} = \bar{z} - \bar{z}_{sk}
\]

The resulting segmental coordinates were written into appropriate segment files to be used for processing kinematic data.
Appendix F

Data Window Selection

The sampling period for each trial was fixed at two seconds. Valid data were recorded for the entire sampling period for the joint range-of-motion and static trials, but generally not for the gait and pivot trials. In order to insure that the subject was moving with a natural cadence in the viewing volume, gait and pivot trials were started outside the volume. The experimenter triggered the sampling by hand just prior to the subject’s entering the viewing volume and a two second sample was recorded. The subject often exited the viewing volume of the cameras before the end of the sampling period. Consequently, there were periods of invalid data at either end of each gait and pivot sample. As LEDs came into and went out of view of the two cameras, sharp transients appeared in the kinematic data. The smoothing procedures are particularly sensitive to transients so it was essential to apply a time window to the data prior to processing in order to obtain only valid data, with all the LEDs in view, for each trial. The resulting windows are summarized in Table F.1 for the skeletal pin data and in Table F.2 for the skin-mounted array data.
Table F.1: Windowing for Skeletal Pin Data Files

<table>
<thead>
<tr>
<th>File</th>
<th>Task</th>
<th>Start Time (Seconds)</th>
<th>End Time (Seconds)</th>
<th>Total Time (Seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>14MA30</td>
<td>Static</td>
<td>0.00</td>
<td>2.00</td>
<td>2.00</td>
</tr>
<tr>
<td>14MA31</td>
<td>Ankle ROM</td>
<td>0.00</td>
<td>2.00</td>
<td>2.00</td>
</tr>
<tr>
<td>14MA32</td>
<td>Knee ROM</td>
<td>0.00</td>
<td>2.00</td>
<td>2.00</td>
</tr>
<tr>
<td>14MA33</td>
<td>Hip ROM</td>
<td>0.00</td>
<td>2.00</td>
<td>2.00</td>
</tr>
<tr>
<td>14MA34</td>
<td>Gait</td>
<td>0.25</td>
<td>1.30</td>
<td>1.05</td>
</tr>
<tr>
<td>14MA35</td>
<td>Gait</td>
<td>0.50</td>
<td>1.62</td>
<td>1.12</td>
</tr>
<tr>
<td>14MA36</td>
<td>Gait</td>
<td>0.72</td>
<td>1.90</td>
<td>1.18</td>
</tr>
<tr>
<td>14MA37</td>
<td>Gait</td>
<td>0.20</td>
<td>1.35</td>
<td>1.15</td>
</tr>
<tr>
<td>14MA38</td>
<td>Gait</td>
<td>0.00</td>
<td>1.10</td>
<td>1.10</td>
</tr>
<tr>
<td>14MA47</td>
<td>Pivot</td>
<td>0.10</td>
<td>2.00</td>
<td>1.90</td>
</tr>
<tr>
<td>14MA48</td>
<td>Pivot</td>
<td>0.00</td>
<td>2.00</td>
<td>2.00</td>
</tr>
<tr>
<td>14MA49</td>
<td>Pivot</td>
<td>1.10</td>
<td>2.00</td>
<td>0.90</td>
</tr>
<tr>
<td>14MA50</td>
<td>Ankle ROM</td>
<td>0.00</td>
<td>2.00</td>
<td>2.00</td>
</tr>
<tr>
<td>14MA51</td>
<td>Knee ROM</td>
<td>0.00</td>
<td>2.00</td>
<td>2.00</td>
</tr>
<tr>
<td>14MA52</td>
<td>Hip ROM</td>
<td>0.00</td>
<td>2.00</td>
<td>2.00</td>
</tr>
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</table>

Table F.2: Windowing for Skin-Mounted LED Array Data Files

<table>
<thead>
<tr>
<th>File</th>
<th>Task</th>
<th>Start Time (Seconds)</th>
<th>End Time (Seconds)</th>
<th>Total Time (Seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>14MA06</td>
<td>Ankle ROM</td>
<td>0.00</td>
<td>2.00</td>
<td>2.00</td>
</tr>
<tr>
<td>14MA07</td>
<td>Knee ROM</td>
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<td>2.00</td>
</tr>
<tr>
<td>14MA08</td>
<td>Hip ROM</td>
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<td>2.00</td>
<td>2.00</td>
</tr>
<tr>
<td>14MA09</td>
<td>Gait</td>
<td>0.50</td>
<td>1.55</td>
<td>1.05</td>
</tr>
<tr>
<td>14MA11</td>
<td>Gait</td>
<td>0.59</td>
<td>1.65</td>
<td>1.06</td>
</tr>
<tr>
<td>14MA12</td>
<td>Gait</td>
<td>0.48</td>
<td>1.55</td>
<td>1.07</td>
</tr>
<tr>
<td>14MA21</td>
<td>Gait</td>
<td>0.47</td>
<td>1.60</td>
<td>1.13</td>
</tr>
<tr>
<td>14MA22</td>
<td>Gait</td>
<td>0.54</td>
<td>1.55</td>
<td>1.01</td>
</tr>
<tr>
<td>14MA23</td>
<td>Gait</td>
<td>0.27</td>
<td>1.30</td>
<td>1.03</td>
</tr>
<tr>
<td>14MA24</td>
<td>Gait</td>
<td>0.20</td>
<td>1.35</td>
<td>1.15</td>
</tr>
<tr>
<td>14MA25</td>
<td>Ankle ROM</td>
<td>0.00</td>
<td>2.00</td>
<td>2.00</td>
</tr>
<tr>
<td>14MA26</td>
<td>Knee ROM</td>
<td>0.00</td>
<td>2.00</td>
<td>2.00</td>
</tr>
<tr>
<td>14MA27</td>
<td>Hip ROM</td>
<td>0.00</td>
<td>2.00</td>
<td>2.00</td>
</tr>
</tbody>
</table>
Appendix G

TRACK Processing Parameter Selection

Two error checking parameters, the maximum allowable skew ray error and the maximum inter-LED length error, are required inputs to the TRACK algorithms. The maximum skew ray error is the maximum allowable length in Selspot units of the common perpendicular between the rays to an LED from the two cameras (see Figure 3.5). For the camera set-up used in the experiments, shown in Appendix C, a Selspot unit was approximately one millimeter at the forceplate. The maximum allowable inter-LED length error is defined as a percentage of the difference in the length between two LEDs specified in the segment file and measured at a given time step.

There is an obvious relationship between the two parameters. An empirical method was used to evaluate the most effective combination of the two parameters to use in processing the data. Each of the skeletal pin-mounted data files was processed with a matrix of different combinations of the parameters in order to select the combination to be used for processing both the pin- and skin-mounted array data. Table G.1 lists the

<table>
<thead>
<tr>
<th>Maximum skew ray error (Selspot units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
</tr>
<tr>
<td>20</td>
</tr>
<tr>
<td>30</td>
</tr>
<tr>
<td>40</td>
</tr>
</tbody>
</table>

Table G.1: Maximum skew ray error parameter values in evaluation grid.
Table G.2: Maximum inter-LED length error parameter values in evaluation grid.

<table>
<thead>
<tr>
<th>Maximum inter-LED length error</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
</tr>
<tr>
<td>15</td>
</tr>
<tr>
<td>20</td>
</tr>
<tr>
<td>25</td>
</tr>
<tr>
<td>30</td>
</tr>
<tr>
<td>40</td>
</tr>
</tbody>
</table>

skew ray error limits used in the evaluation grid and Table G.2 contains the different maximum inter-LED length error values considered.

Decreasing the maximum allowable skew ray error results in a significant loss of data. The progression is seen clearly in Figures G.1 and G.2, which present the displacement vector and quaternion components for the first gait trial processed with the four skew ray error limits of Table G.1 at a fixed maximum inter-LED length error of 25%. The data loss between the skew error limits of 20 and 10 Selspot units is significant, with only small modifications visible for higher values.

A typical sequence of variation of the maximum inter-LED length error at a fixed maximum skew ray error is shown in Figures G.3 and G.4. Data are for the first gait trial and a skew ray error limit of 20 Selspot units was used. Increases in the inter-LED length error limit appear to reduce the overall noise amplitude (evaluated by visual inspection) up to about 20%, but further changes only add data at the fringes of the data set. The latter additions are suspect since the data is recorded near the edges of the camera viewing volume.

On the basis of inspection of data from all the skeletal pin-mounted array trials a skew ray error limit of 20 Selspot units and a maximum inter-LED length error of 25% were selected for processing all the kinematic data.
Figure G.1: The effect of varying the maximum skew ray error for a fixed maximum inter-LED length error. Displacement vector data from the first gait trial processed with a maximum inter-LED length error of 25% and maximum skew ray error of: (a) 10 Selspot units, (b) 20 Selspot units, (c) 30 Selspot units and (d) 40 Selspot units.
Figure G.2: The effect of varying the maximum skew ray error for a fixed maximum inter-LED length error. Relative quaternion data from the first gait trial processed with a maximum inter-LED length error of 25% and maximum skew ray error of: (a) 10 Selspot units, (b) 20 Selspot units, (c) 30 Selspot units and (d) 40 Selspot units.
Figure G.3: The effect of varying the maximum inter-LED length error with a fixed maximum skew ray error. Displacement vector data for the first gait trial processed with a maximum skew ray error of 20 Selspot units and maximum inter-LED length errors of: (a) 10%, (b) 15%, (c) 20%, (d) 25%, (e) 30%, and (f) 40%.
Figure G.4: The effect of varying the maximum inter-LED length error with a fixed maximum skew ray error. Relative quaternion data for the first gait trial processed with a maximum skew ray error of 20 Selspot units and maximum inter-LED length errors of: (a) 10%, (b) 15%, (c) 20%, (d) 25%, (e) 30%, and (f) 40%. 
Appendix H

Differentiation of Noisy Kinematic Data

H.1 Introduction

Instantaneous kinematic analysis requires high-quality velocity data. Velocity data may be used to calculate the instantaneous helical axes (IHA) of a motion and to generate the fixed and moving axodes, ruled surfaces characteristic of the mechanism generating a motion. Using higher derivatives curvature theory can be used to characterize the surfaces completely [293].

Velocity is difficult to measure accurately. Most gait and human motion data, including the data obtained with the TRACK system, are acquired with position measurement systems [11]. Numerical differentiation must be used to obtain the derivatives necessary for application of instantaneous kinematic analysis and curvature theory to movement studies. Several algorithms for numerical differentiation are available, including finite difference methods [81], polynomial curve fits [133], differentiating filters [100] and smoothing splines [7, 56, 267, 282]. Unfortunately, all experimental data is inherently noisy and numerical differentiation is a noise amplifying process [100]. Most of the numerical differentiation algorithms are designed to work on smooth, not noisy, data. Lanshammar [142] and Miller [170] have shown that the overwhelming factor in selecting a differentiation technique was the quality of the data being differentiated. Numerical differentiation is a problem of obtaining a high quality signal.

Two features of human movement data compound the difficulty associated with differentiation of noisy data. Neither the frequency characteristics of the motions being studied nor the noise contaminating the data are well-known a priori. There is a wide variation in the estimated bandwidth of even normal gait [10, 84, 240]. A second problem is that the data records from motion measurement systems tend to be short. This is partially a restriction of the measurement systems, which have limited viewing volumes,
and partially due to the transient nature of the phenomena of interest. Both features impose significant restriction on the selection of a method for obtaining differentiable data.

The most direct approach to obtaining better data is to improve the data acquisition techniques. Improved procedures for calibration of measurement systems, as well as advanced hardware, are proliferating [158]. There are limits on the marginal returns from this approach. Further improvements are possible through post-processing of the motion data to attenuate or remove the noise.

Both frequency and time domain techniques are available for processing and differentiating position data. Digital filtering [183, 248] may be used to attenuate noise over a known frequency range. This approach is particularly efficacious when the frequency spectra of the noise and the signal of interest are well defined. Filters may be designed to emphasize a particular set of characteristics. The designer is in control of the pass-band and stop-band characteristics. There are two classes of filters, finite impulse response (FIR) and infinite impulse response (IIR). In general, designing a filter with characteristics close to an ideal filter involves trade-offs. For FIR filters flattening the pass-band requires adding coefficients, requiring the loss of data at the beginning and end of a data record or the use of padding. Fewer terms may be used to obtain near ideal pass-band characteristics with IIR designs, but the result is typically poor transient response. A wide selection of filter designs, one of the more common being the Butterworth type [11, 278], are in use.

Time domain approaches are best when the frequency characteristics of the signal and noise are not well-defined. Kalman smoothing [72] and optimally regularized natural b-splines [52, 62, 282] are two time domain data smoothing methods that have been reported in the literature. Both models assume that the discrete data is composed of a signal component, the desired value, and an additive noise component,

\[ z_k = y_k + n_k \]  

(H.1)

where

\[ z_k \] = measurement,
\[ y_k \] = signal,
\[ n_k \] = additive noise.

The smoothing splines based on generalized cross-validation [52] assume that the signal is narrow-band, but otherwise undefined, and that the noise is broad-band or white. The principal drawbacks of smoothing are the requirement that the entire data set be available for processing, preventing use in any real-time applications, and the large amount of memory required to store needed arrays of dimension \( m \times m \) for \( m \) samples.

Choosing an appropriate method of attenuating measurement noise is difficult. Different approaches to evaluating the variety of algorithm choices have been proposed.
Lanshammar [141] suggested the comparative processing of analytical waveforms as a means of selecting the best algorithm. While analytical waveforms have the advantage of known solutions which serve as a standard of comparison for numerically produced results, it is difficult to imbue analytically generated data with the characteristics of a particular measurement system. The algorithm which is most effective for a given analytically produced signal may perform miserably when applied to actual measurements. Ladin et. al [137] recognized this inadequacy and developed a device to electromechanically generate sinusoidal kinematics. The motion of the device could be recorded with any motion measurement system, providing actual system-specific experimental measurements of a motion with a well-known analytical solution for corroboration.

The following section presents the results of a comparison, using the method developed by Ladin et. al., of digital Butterworth filtering with two smoothing algorithms for processing TRACK data.

### H.2 Evaluation

#### H.2.1 Algorithms

The filtering/differentiation algorithm evaluated was a sixth order, two-pass, low pass, digital Butterworth filter. Butterworth filters are maximally flat in the pass and stop bands. Cutoff frequencies were chosen based on estimates of the frequency content of normal gait. Antonsson and Mann [10] determined that eighty percent of the signal power in normal gait was below fifteen Hertz, with ninety-eight percent within twenty Hertz. A 20 Hertz cutoff frequency with the selected filter design has less than one percent attenuation at frequencies below 12 Hertz, and a 30 Hertz cutoff attenuates all information below 20 Hertz at less than one percent. Both values of cutoff frequency were evaluated. Derivatives were calculated using a Lagrangian five-point differentiation scheme. Position and orientation data were filtered, double differentiated and the derivatives filtered using the same filter.

Two smoothing algorithms were evaluated: one developed by Woltring [282] and the other by Dohrmann and Busby [61, 62]. Woltring uses natural b-splines with optimization of the generalized cross-validation parameter to give a least-squares fit of the spline to the data. The algorithm is based on the work of Utreras [267] with improvements by several other authors included.

Dohrmann and Busby's algorithm is an extension of work by Busby and Trujillo [34]. The original work was based on a state space formulation of the smoothing and differentiation problem for cubic splines. The method required the user to select a smoothing parameter. Dohrmann incorporated a solution for the optimum smoothing parameter, based on the generalized cross-validation, and expanded the algorithm to permit use of derivative criteria and splines of any order.
Table H.1: Smoothing spline parameter sets for comparison.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>p</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
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<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Dohrmann</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Woltring</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

The smoothing splines evaluated are listed in Table H.1.

H.2.2 Data

The electromechanical device capable of generating pure sinusoidal kinematics developed and described by Ladin et. al. [137] was used to generate sinusoidal kinematics. Frequencies ranged from one to ten Hertz and were varied in approximately one half Hertz increments. The position data was acquired using the TRACK system in the Eric P. and Evelyn E. Newman Laboratory for Biomechanics and Human Rehabilitation at MIT. An array of eight LEDs was attached to the sinusoid generator. All data records were two seconds long and the sampling rate was 286 Hertz.

H.2.3 Method of comparison

Given a sinusoidal position signal of the form

\[ x(t) = C_1 + C_2 \sin(2\pi ft + \phi) \]  \hspace{1cm} (H.2)

expressions for the velocity and acceleration are easily obtained. Velocity is,

\[ \dot{x}(t) = 2\pi f C_2 \cos(2\pi ft + \phi), \]  \hspace{1cm} (H.3)

and the acceleration is given by:

\[ \ddot{x}(t) = -(2\pi f)^3 C_2 \sin(2\pi ft + \phi) \]  \hspace{1cm} (H.4)

Four parameters define the sinusoid: the offset, \( C_1 \); the amplitude, \( C_2 \); the frequency, \( f \); and the phase angle, \( \phi \).

In order to evaluate the effectiveness of the noise attenuation on the original position data and the numerically generated derivatives a nonlinear least squares algorithm [99] was applied to find the best-fit sinusoid for each experimental record. Estimation of the best-fit sinusoid produced four parameters to compare to the values based on the analytical solution, the known frequency of the generated sinusoids and the physical
measurements of the sinusoid generator. A fifth parameter, the signal-to-noise ration (S/N), was calculated as a measure of the overall quality of the curvefit. The S/N was the ratio of the estimated sinusoid amplitude ($C_2$) to the standard deviation of the difference between the experimental data and the best-fit sinusoid.

H.2.4 Results

Representative smoothed and filtered position and acceleration data are shown in Figure H.1. Although the quality of the smoothed and filtered position data are comparable, the estimate of acceleration based on the smoothed data is superior to the estimate from the filtered data.

Both the smoothing and the filter were able to accurately estimate the frequency of the original data and its derivatives over the entire range of input frequencies (Figure H.2). Errors were usually on the order of 0.01 Hertz; less than one percent of the input frequency.

The phase angle estimates, presented in Figure H.3, varied more with the noise attenuation method than the frequency, particularly for acceleration. Both the filtered and the smoothed data yielded excellent phase angle estimates for velocity. For a 2.1 Hertz input sine wave, the smoothed and filtered data estimated the phase angle for velocity within 0.1 percent of the exact value of $\pi/2$. For acceleration, the estimate from the smoothed data was within five percent, while the thirty Hertz cutoff filter was within 6.5 percent and the twenty Hertz cutoff filter within eight percent.

For a given sinusoid, the amplitudes of the velocity and acceleration data should be exact functions of the amplitude and frequency of the input position data. With the geometry of the apparatus and the input frequency known, expected amplitudes could be calculated for the velocity and acceleration. Estimated sinusoid amplitudes are shown in Figure H.4. The smoothed data gave the best estimates of amplitude for the entire input frequency range. There was little change in the accuracy of the amplitude estimate with each derivative. For a 6.46 Hertz input sine wave, amplitude estimates from the smoothed data and its derivatives were within one percent of the expected value. In comparison, the accuracy of the amplitude estimates from the filtered data decreased with each successive differentiation. The filter cutoff frequency did not play a strong role.

Figure H.5 presents estimates of the offset of the sinusoid for the position data and calculated velocities and accelerations. All the algorithms produced good estimates of the offset for the position data, but the filter performance degraded with each differentiation. Smoother performance started to fall off at higher frequencies, but not as much as for the filter.

The S/N ratio for each data record was a good indicator of the overall performance of each method. Values of the S/N ratio are show in Figure H.6. For a 2.1 Hertz input sine
Figure H.1: Representative smoothed and filtered position data and acceleration estimates: (a) Smoothed (Woltring) position data, (b) Filtered (20 Hertz cutoff) position data, (c) Smoothed acceleration estimate, and (d) Filtered acceleration estimate.
Figure H.2: Estimated frequencies for the best-fit sinusoids for smoothed and filtered data. Frequency estimates for: (a) Position data, (b) Velocity estimates, and (c) Acceleration estimates.
Figure H.3: Estimated phase angles for the best-fit sinusoids for smoothed and filtered data. Phase angle estimates for: (a) Position data, (b) Velocity estimates, and (c) Acceleration estimates.
Figure H.4: Estimated amplitude for the best-fit sinusoids for smoothed and filtered data. Amplitude estimates for: (a) Position data, (b) Velocity estimates, and (c) Acceleration estimates.
Figure H.5: Estimated offset for the best-fit sinusoids for smoothed and filtered data. Offset estimates for: (a) Position data, (b) Velocity estimates, and (c) Acceleration estimates.
Figure H.6: Estimated signal-to-noise ratio for the best-fit sinusoids for smoothed and filtered data. Signal-to-noise ratio estimates for: (a) Position data, (b) Velocity estimates, and (c) Acceleration estimates.
wave, the smoothed position data had a S/N ratio of over one hundred. It dropped by forty percent with the first differentiation and sixty percent with the second. The data filtered with the thirty Hertz cutoff filter had a S/N ratio approximately twelve percent lower than the smoothed position data. With a single differentiation this dropped by fifty percent, giving a S/N ratio for the velocity estimate twenty percent less than the smoothed velocity estimate S/N ratio. A second differentiation, dropped the S/N ratio for the filtered estimate eighty percent from the velocity value. The S/N ratio for the acceleration estimated by the filter was sixty percent less than the S/N ratio for the acceleration estimated by the smoothing algorithm. The performance of the lower cutoff frequency filter was somewhat worse than that of the thirty Hertz cutoff filter. At higher frequencies the S/N ratios of both smoothed and filtered position data dropped significantly. There was a proportionate decrease in the S/N ratio for the velocity and acceleration estimates of the filtered data, but not the smoothed data. The performance of the smoothing algorithm was significantly better than the filter/differentiating polynomial algorithm. Values of S/N for the smoothed acceleration estimates were comparable to those obtained by Ladin et. al. [137] for directly measured accelerations.

H.3 Conclusions

The overall performance of the smoothing algorithms was superior to the performance of the Butterworth filter/Lagrangian differentiating polynomial method for estimating derivatives from noisy data. Both methods were able to estimate the least-squares sinusoid parameters for velocity well, but the smoothing algorithm gave better estimates for acceleration. The signal-to-noise ratio of the derivatives produced by the smoothing algorithm was significantly higher than that produced by the other method. The optimally regularized splines are a better method of calculating the higher derivatives required for instantaneous kinematic analysis and dynamic calculations.
Appendix I

TRACK Smoothing Parameter Selection

Based on the comparisons described in Appendix H, time-domain smoothing was selected as the approach for attenuating noise in the TRACK data. It was still necessary to choose a smoothing algorithm and determine which spline order and derivative criterion to use.

Two smoothing algorithms using generalized cross-validation were available: one developed by Woltring [282] and the second by Dohrmann and Busby [62]. Only one spline parameter, the derivative criterion \( m \), is required input to Woltring's algorithm. The spline order, \( p \), for the natural b-splines used in the algorithm is related to the derivative criterion by:

\[
p = 2m - 1
\]  

(I.1)

The Dohrmann and Busby formulation allows the user to specify both the spline order and the derivative criterion of the smoothing spline.

Both algorithms produced essentially identical results when used to process the mechanically generated sinusoid data of Appendix H. A significant difference was found in the CPU time required for processing the same file. As the final step in the selection procedure the two algorithms, with the four different sets of input spline parameters shown in Table I.1, were used to process the skeletal pin-mounted kinematic data and the results compared.

Results for the first gait trial are presented in Figures I.1 and I.5. The relative displacement vector components are shown in Figure I.1. Small differences were observed between the results of the different spline fits. The smoothest output was produced by Dohrmann's natural cubic spline \( p = 3, m = 2 \). Increasing the derivative criterion with the same algorithm introduces some higher frequency ripple, particularly in the \( z \)-displacement component of the vector. Processing with Woltring's natural cubic spline leaves more higher frequency components than either of of the Dohrmann cubic splines. Use of Dohrmann's formulation with natural quintic spline parameters does not result
Figure I.1: Comparison of the smoothing algorithm effects on the components of the relative displacement vector. Relative displacement vector components for the first gait trial smoothed with: (a) Dohrmann, $p = 3, m = 2$; (b) Dohrmann, $p = 3, m = 3$; (c) Woltring, $p = 3, m = 2$; and (d) Dohrmann, $p = 5, m = 3$. 

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Table I.1: Smoothing spline parameter sets for comparison.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>p</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
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<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Dohrmann</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Dohrmann</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>Woltring</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

in noticeable change of the displacement vector from smoothing with the higher derivative criterion cubic spline. The relative quaternion components for the same cases are displayed in Figure I.2. Behavior is similar to that observed for the relative displacement vector, except the higher frequency ripple is apparent in both the $z$ and $y$ vector quaternion components. These components of the quaternion and the $z$ component of the displacement vector all represent movement out of the $zy$ plane of the laboratory coordinate system. It is likely that the smoothing is picking up variations in measurement resolution and quality, such as changes in LED image intensity, rather than actual relative movement at the knee. Results for the second swing trial presented in Figures I.3 and I.4 exhibit the same characteristic observed in the gait trial results.

A comparison of the frequency domain characteristics of some of the smoothed data demonstrates the differences between the different smoothers. Power spectral densities for the $y$ vector quaternion component of the relative quaternion for the second swing trial are shown in Figure I.5. All the smoothing splines reproduce the dominant low frequency peak at approximately one Hertz. Above three Hertz varying degrees of attenuation of the next three peaks are observed. Woltring's algorithm attenuates the peaks the least and has a notably sharper cutoff frequency, at approximately seven Hertz, than any of the Dohrmann smoothing splines. The similarity of the higher derivative criterion Dohrmann cubic spline result and the output of the natural quintic spline smoother is also evident in the frequency domain.

Comparison of the different smoothing splines shows identical performance for the important low frequency peak. Since the secondary peaks, between three and seven Hertz, are probably not significant the smoothing spline which best attenuated them while preserving the initial peak was selected for processing the kinematic data.
Figure I.2: Comparison of the smoothing algorithm effects on the components of the relative quaternion. Relative quaternion components for the first gait trial smoothed with: (a) Dohrmann, $p = 3$, $m = 2$; (b) Dohrmann, $p = 3$, $m = 3$; (c) Woltring, $p = 3$, $m = 2$; and (d) Dohrmann, $p = 5$, $m = 3$. 
Figure I.3: Comparison of the smoothing algorithm effects on the components of the relative displacement vector. Relative displacement vector components for the second swing trial smoothed with: (a) Dohrmann, $p = 3$, $m = 2$; (b) Dohrmann, $p = 3$, $m = 3$; (c) Woltring, $p = 3$, $m = 2$; and (d) Dohrmann, $p = 5$, $m = 3$. 
Figure I.4: Comparison of the smoothing algorithm effects on the components of the relative quaternion. Relative quaternion components for the second swing trial smoothed with: (a) Dohrmann, $p = 3$, $m = 2$; (b) Dohrmann, $p = 3$, $m = 3$; (c) Woltring, $p = 3$, $m = 2$; and (d) Dohrmann, $p = 5$, $m = 3$. 

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Figure I.5: Comparison of the smoothing algorithm effects on the $y$ vector component of the relative quaternion. Power spectral densities for the relative $y$ vector quaternion component for the second swing trial smoothed with: (a) Dohrmann, $p = 3$, $m = 2$; (b) Dohrmann, $p = 3$, $m = 3$; (c) Woltring, $p = 3$, $m = 2$; and (d) Dohrmann, $p = 5$, $m = 3$. 
Appendix J

KINEMATIC ERROR ESTIMATION

J.1 Background

Despite the different error elimination schemes and data smoothing the final rigid body kinematic data used in calculating instantaneous helical axis parameters are corrupted by residual noise. A portion of this is due to the generalized resolution of the system; measurement discretization and noise which the various error checking processes are not designed to or capable of removing form a lower bound on what kinematics the system may actually resolve. An estimate of how the generalized error propagates through the kinematic calculations is necessary to establish thresholds for subsequent calculation and display of the kinematic information. This appendix is a brief discussion of the approach used to estimate the effective resolution of the TRACK data obtained in the experiments reported in this dissertation.

J.2 Method

J.2.1 Approach and Assumptions

Ideally, an error analysis of the kinematic information produced by the TRACK system would start from the original measurements, the individual camera coordinates \((u_1, v_1), (u_2, v_2)\) for each LED. The process of propagating variances in the camera coordinates through to the rigid body kinematics and helical axis parameters is a formidable task. Most difficult would be characterizing the intermediate error elimination schemes, skew ray error elimination and inter-LED error elimination, in a suitable form.

An alternative approach is to define a generalized resolution. This treats the rigid
body position and orientation in the laboratory coordinate system of each array as the base measurements. Variances calculated for these data reflect discretization limits of the cameras in addition to errors that have survived the skew ray error elimination, the inter-LED length check and data smoothing. It should be an effective lower bound on what displacements the measurement system can successfully detect.

The relative motion of any two segments, including the anatomical angles and all of the screw parameters, may be expressed to first order as functions of the fourteen displacement parameters for the two segments and the first derivatives of those parameters. Means and variances of the displacement parameters and their derivatives may be calculated if the displacements and velocities are random variables. Using a Taylor series expansion the mean and variance of any analytic function of the random variables may be approximated as functions of the means and variances of the displacements and velocities. If the displacements and velocities can be assumed to be independent random variables the approximation is simplified considerably. With the assumption of the independent measurements, the variances of the instantaneous helical axis parameters may be calculated using the following approximate equations [171]:

\[
\sigma_f^2 = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \sigma_x^2 \right) + \frac{1}{2} \sum_{i=1}^{n} \left( \frac{\partial^2 f}{\partial x_i^2} \right) \sigma_x^4
\]  

(J.1)

The derivation for a function of one random variable is given in Papoulis [187].

In TRACK displacements of a rigid body are described by seven parameters: the three components of the displacement vector and the four components of the unit quaternion. The components of the unit quaternion are constrained by the relationship:

\[
a^2 + b^2 + c^2 + d^2 = 1
\]  

(J.2)

and are not independent. Each segment is considered as an independent rigid body by the TRACK algorithms. In some cases that may be true physically, but for the current experiments the different segments were links in the same mechanism with the relative motion between them constrained.

Despite these reservations, the simplified analysis should illuminate the importance of improved measurement resolution to future estimates of the instantaneous kinematics of constrained rigid bodies. A complete analysis of the errors in the calculated instantaneous helical axis parameters would need to evaluate the full Taylor series expansion of each function without the assuming that the random variables are independent.

### J.2.2 Base Kinematic Data

The base standard deviation for each of the five segments was estimated from static data recorded as part of the protocol for both the experiments with the pin-mounted arrays and the skin-mounted arrays. In the static trial the subject stood erect with

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straight legs in the center of the force platform (essentially the center of the viewing volume). Two seconds of data were recorded at a 315 Hertz sampling frequency. The three-dimensional LED point coordinates were screened for skew-ray errors and inter-LED length discrepancies. Data surviving the elimination procedures were smoothed and interpolated using a GCV-based cubic spline algorithm [282]. Rigid body positions and orientations were calculated from the LED coordinates and the parameter velocities were obtained using a Lanczos differentiating filter [100]. Figures J.1 and J.2 present the displacement vector, quaternion, translational velocity and quaternion rate for the shank and thigh segments in the pin-mounted array experiments. The same data for the skin-mounted array static trial are shown in Figures J.3 and J.4.

Estimates of the sample means, variances and standard errors for the rigid body parameters of each segment were calculated using [17]:

\[ \hat{\mu}_x = \frac{1}{N} \sum_{i=1}^{N} x_i \]  \hspace{1cm} (J.3)

\[ \hat{\sigma}_x^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \hat{\mu}_x)^2 \]  \hspace{1cm} (J.4)

\[ \hat{s}_x = +\sqrt{\hat{\sigma}_x^2} \]  \hspace{1cm} (J.5)

Resultant standard errors for data collected with the pin-mounted arrays are listed in Table J.1 and for data from the skin-mounted arrays in Table J.2.

**J.3 Instantaneous Helical Axis Parameter Variance Estimation**

**J.3.1 Definitions and Notation**

The instantaneous helical axis represents the relative motion of two rigid bodies. Given the position, orientation and velocities of the two bodies, or more correctly of the Cartesian coordinate systems fixed in each body, relative to a global Cartesian reference frame all the quantities relevant to the helical axis may be calculated. Figure J.5 shows the relationships of the different coordinate systems, with the global coordinate frame designated as system 1, the moving reference frame as system 2 and the frame attached to the second moving body as system 3.

Twenty-eight variables constitute the input kinematic data. The position vectors from the origin of the global system to the origins of each of the body-fixed systems are:

\[ \mathbf{r}_{12} = x_{12i} + y_{12j} + z_{12k} \]  \hspace{1cm} (J.6)

\[ = (x_{12}, y_{12}, z_{12}) \]  \hspace{1cm} (J.7)
Figure J.1: Static data for the shank segment with the pin-mounted arrays (14MA30): (a) Position vector for segment 2, (b) Quaternion for segment 2, (c) Translational velocity vector for segment 2, and (d) Quaternion rate for segment 2.
Figure J.2: Static data for the thigh segment with the pin-mounted arrays (14MA30): (a) Position vector for segment 3, (b) Quaternion for segment 3, (c) Translational velocity vector for segment 3, and (d) Quaternion rate for segment 3.
Figure J.3: Static data for the shank segment with the skin-mounted arrays (14MA05): (a) Position vector for segment 2, (b) Quaternion for segment 2, (c) Translational velocity vector for segment 2, and (d) Quaternion rate for segment 2.
Figure J.4: Static data for the thigh segment with the skin-mounted arrays (14MA05): (a) Position vector for segment 3, (b) Quaternion for segment 3, (c) Translational velocity vector for segment 3, and (d) Quaternion rate for segment 3.
Figure J.5: Coordinate systems for relative motion.
Table J.1: Pin Data: Static File (14MA30) Standard Errors.

<table>
<thead>
<tr>
<th></th>
<th>Segment 1</th>
<th>Segment 2</th>
<th>Segment 3</th>
<th>Segment 4</th>
<th>Segment 5</th>
</tr>
</thead>
<tbody>
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<td>(x) (m)</td>
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<td>0.001</td>
<td>0.003</td>
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</tr>
<tr>
<td>(y) (m)</td>
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<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>(z) (m)</td>
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<td>0.003</td>
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<td>0.0025</td>
</tr>
<tr>
<td>(\dot{y}) (m/s)</td>
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<td>0.0008</td>
<td>0.0014</td>
<td>0.0006</td>
</tr>
<tr>
<td>(\dot{z}) (m/s)</td>
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<td>0.0031</td>
<td>0.0029</td>
<td>0.0056</td>
<td>0.0054</td>
</tr>
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<td>(a)</td>
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<td>0.00305</td>
<td>0.00187</td>
</tr>
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<td>(b)</td>
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<td>0.00083</td>
</tr>
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<td>(c)</td>
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<td>0.00317</td>
</tr>
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<td>0.00001</td>
<td>0.00001</td>
<td>0.00001</td>
<td>0.00001</td>
</tr>
<tr>
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<td>0.0266</td>
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<td>0.0039</td>
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<tr>
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<td>0.0002</td>
<td>0.0002</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

\[
r_{13} = x_{13} + y_{13} + z_{13} \]

\[
= (x_{13}, y_{13}, z_{13})
\]

The subscripts indicate the reference coordinate system first, and the moving body second. The vector \(r_{13}\) is the position of frame 2 in global coordinates. Similarly, the orientations for each moving body are described by unit quaternions:

\[
g_{12} = a_{12} + b_{12} + c_{12}k + d_{12}
\]

\[
= (a_{12}, b_{12}, c_{12}, d_{12})
\]

\[
g_{13} = a_{13} + b_{13} + c_{13}k + d_{13}
\]

\[
= (a_{13}, b_{13}, c_{13}, d_{13})
\]

The base kinematic velocities are component-by-component derivatives of each of these quantities.

### J.3.2 Partial derivatives of relative kinematic functions

Estimation of the variances of the instantaneous helical axis parameters using Equation J.1 requires the first and second partial derivatives of each parameter with respect to the base kinematic variables. The following sections present the equations for the
Table J.2: Skin Data: Static File (14MA05) Standard Errors.

<table>
<thead>
<tr>
<th></th>
<th>Segment 1</th>
<th>Segment 2</th>
<th>Segment 3</th>
<th>Segment 4</th>
<th>Segment 5</th>
</tr>
</thead>
<tbody>
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<td>$x$ (m)</td>
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<td>0.001</td>
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<td>0.003</td>
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<td>$y$ (m)</td>
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<td>0.000</td>
<td>0.000</td>
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</tr>
<tr>
<td>$z$ (m)</td>
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<td>0.001</td>
<td>0.002</td>
<td>0.003</td>
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<tr>
<td>$\dot{z}$ (m/s)</td>
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<td>$\dot{y}$ (m/s)</td>
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<td>$\dot{z}$ (m/s)</td>
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</tr>
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<td>$\dot{b}$</td>
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<td>0.0000</td>
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</tbody>
</table>

Required partial derivatives. In the interest of saving space an indicial notation was used where possible. The appearance of an index on both sides of an equation indicates that there are actually $n$ equations, where $n$ is four for derivatives with respect to the quaternion or quaternion rate components and $n$ is three for derivatives with respect to the components of the position vector or translational velocity.

Relative quaternion, $q_{23}$

The relative orientation of two rigid bodies in terms of the orientations of each body in a global coordinate system is given by the unit quaternion:

$$ q_{23} = q^{-1}_{12} q_{13} \tag{J.14} $$

The relative quaternion is a linear function of the eight unit quaternion components. Eight first partial derivatives exist:

$$ \frac{\partial q_{23}}{\partial a_{12}} = (-d_{13}, c_{13}, -b_{13}, a_{13}) \tag{J.15} $$

$$ \frac{\partial q_{23}}{\partial b_{12}} = (-c_{13}, -d_{13}, a_{13}, b_{13}) \tag{J.16} $$

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\[
\frac{\partial q_{23}}{\partial c_{12}} = (b_{13}, -a_{13}, -d_{13}, c_{13}) \quad \text{(J.17)}
\]
\[
\frac{\partial q_{23}}{\partial d_{12}} = (a_{13}, b_{13}, c_{13}, d_{13}) \quad \text{(J.18)}
\]
\[
\frac{\partial q_{23}}{\partial a_{13}} = (d_{12}, -c_{12}, b_{12}, a_{12}) \quad \text{(J.19)}
\]
\[
\frac{\partial q_{23}}{\partial b_{13}} = (c_{12}, d_{12}, -a_{12}, b_{12}) \quad \text{(J.20)}
\]
\[
\frac{\partial q_{23}}{\partial c_{13}} = (-b_{12}, a_{12}, d_{12}, c_{12}) \quad \text{(J.21)}
\]
\[
\frac{\partial q_{23}}{\partial d_{13}} = (-a_{12}, -b_{12}, -c_{12}, d_{12}) \quad \text{(J.22)}
\]

Relative quaternion rate, \( \dot{q}_{23} \)

The relative quaternion rate is given by:

\[
\dot{q}_{23} = q_{12}^{-1} \dot{q}_{13} - q_{12}^{-1} \dot{q}_{12} q_{12}^{1} q_{13} \quad \text{(J.23)}
\]

This is a linear function of the eight body-fixed quaternion components and the eight body-fixed quaternion rates. There are sixteen first partial derivatives and no second partial derivatives:

\[
\frac{\partial \dot{q}_{23}}{\partial a_{12}} = (-\dot{d}_{13}, \dot{c}_{13}, -\dot{b}_{13}, \dot{a}_{13}) \quad \text{(J.24)}
\]
\[
\frac{\partial \dot{q}_{23}}{\partial b_{12}} = (-\dot{c}_{13}, -\dot{d}_{13}, \dot{a}_{13}, \dot{b}_{13}) \quad \text{(J.25)}
\]
\[
\frac{\partial \dot{q}_{23}}{\partial c_{12}} = (\dot{b}_{13}, -\dot{a}_{13}, -\dot{d}_{13}, \dot{c}_{13}) \quad \text{(J.26)}
\]
\[
\frac{\partial \dot{q}_{23}}{\partial d_{12}} = (\dot{a}_{13}, \dot{b}_{13}, \dot{c}_{13}, \dot{d}_{13}) \quad \text{(J.27)}
\]
\[
\frac{\partial \dot{q}_{23}}{\partial a_{13}} = (\dot{d}_{12}, -\dot{c}_{12}, \dot{b}_{12}, \dot{a}_{12}) \quad \text{(J.28)}
\]
\[
\frac{\partial \dot{q}_{23}}{\partial b_{13}} = (\dot{c}_{12}, \dot{d}_{12}, -\dot{a}_{12}, \dot{b}_{12}) \quad \text{(J.29)}
\]
\[
\frac{\partial \dot{q}_{23}}{\partial c_{13}} = (-\dot{b}_{12}, \dot{a}_{12}, \dot{d}_{12}, \dot{c}_{12}) \quad \text{(J.30)}
\]
\[
\frac{\partial \dot{q}_{23}}{\partial d_{13}} = (-\dot{a}_{12}, -\dot{b}_{12}, -\dot{c}_{12}, \dot{d}_{12}) \quad \text{(J.31)}
\]
\[
\frac{\partial \dot{q}_{23}}{\partial a_{12}} = (-\dot{d}_{13}, \dot{c}_{13}, -\dot{b}_{13}, \dot{a}_{13}) \quad \text{(J.32)}
\]

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\[ \frac{\partial \dot{q}_{23}}{\partial b_{12}} = (-c_{13}, -d_{13}, a_{13}, b_{13}) \]  
(J.33)

\[ \frac{\partial \dot{q}_{23}}{\partial c_{12}} = (b_{13}, -a_{13}, -d_{13}, c_{13}) \]  
(J.34)

\[ \frac{\partial \dot{q}_{23}}{\partial d_{12}} = (a_{13}, b_{13}, c_{13}, d_{13}) \]  
(J.35)

\[ \frac{\partial \dot{q}_{23}}{\partial d_{13}} = (d_{12}, -c_{12}, b_{12}, a_{12}) \]  
(J.36)

\[ \frac{\partial \dot{q}_{23}}{\partial d_{13}} = (c_{12}, b_{13}, a_{12}, -d_{12}) \]  
(J.37)

\[ \frac{\partial \dot{q}_{23}}{\partial c_{13}} = (-b_{12}, a_{12}, b_{12}, c_{12}) \]  
(J.38)

\[ \frac{\partial \dot{q}_{23}}{\partial d_{13}} = (-a_{12}, -b_{12}, -c_{12}, d_{12}) \]  
(J.39)

**Relative position vector, \( r_{23} \)**

The relative position vector in the moving reference system is:

\[ r_{23} = q_{12}^{-1}(r_{13} - r_{12})q_{12} \]  
(J.40)

It is a linear function of the segment position vectors and a function of the unit quaternion for the moving reference. The first partial derivatives are:

\[ \frac{\partial r_{23}}{\partial x_{12}} = ((b_{12}^2 + c_{12}^2 - a_{12}^2 - d_{12}^2), \]

\[ 2(c_{12}d_{12} - a_{12}b_{12}), \]

\[ -2(a_{12}c_{12} + b_{12}d_{12})) \]  
(J.41)

\[ \frac{\partial r_{23}}{\partial y_{12}} = ((-2(a_{12}b_{12} + c_{12}d_{12}), \]

\[ (a_{12}^2 + c_{12}^2 - b_{12}^2 - d_{12}^2), \]

\[ 2(a_{12}d_{12} - b_{12}c_{12})) \]  
(J.42)

\[ \frac{\partial r_{23}}{\partial z_{12}} = (2(b_{12}d_{12} - a_{12}c_{12}), \]

\[ -2(b_{12}c_{12} + a_{12}d_{12}), \]

\[ (a_{12}^2 + b_{12}^2 - c_{12}^2 - d_{12}^2)) \]  
(J.43)

\[ \frac{\partial r_{23}}{\partial x_{13}} = -\frac{\partial r_{23}}{\partial x_{12}} \]  
(J.44)

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\[
\frac{\partial r_{23}}{\partial y_{13}} = -\frac{\partial r_{23}}{\partial y_{12}} \\
\frac{\partial r_{23}}{\partial x_{13}} = -\frac{\partial r_{23}}{\partial x_{12}} \\
\frac{\partial r_{23}}{\partial a_{12}} = (2(a_{12}(x_{13} - x_{12}) + b_{12}(y_{13} - y_{12}) + c_{12}(z_{13} - z_{12})), \\
2(b_{12}(x_{13} - x_{12}) - a_{12}(y_{13} - y_{12}) + d_{12}(z_{13} - z_{12})), \\
2(c_{12}(x_{13} - x_{12}) - d_{12}(y_{13} - y_{12}) - a_{12}(z_{13} - z_{12}))) \\
\frac{\partial r_{23}}{\partial b_{12}} = (2(-b_{12}(x_{13} - x_{12}) + a_{12}(y_{13} - y_{12}) - d_{12}(z_{13} - z_{12})), \\
2(a_{12}(x_{13} - x_{12}) + b_{12}(y_{13} - y_{12}) + c_{12}(z_{13} - z_{12})), \\
2(d_{12}(x_{13} - x_{12}) + c_{12}(y_{13} - y_{12}) - b_{12}(z_{13} - z_{12}))) \\
\frac{\partial r_{23}}{\partial c_{12}} = (2(-c_{12}(x_{13} - x_{12}) + d_{12}(y_{13} - y_{12}) + a_{12}(z_{13} - z_{12})), \\
2(-d_{12}(x_{13} - x_{12}) - c_{12}(y_{13} - y_{12}) + b_{12}(z_{13} - z_{12})), \\
2(a_{12}(x_{13} - x_{12}) + b_{12}(y_{13} - y_{12}) + c_{12}(z_{13} - z_{12}))) \\
\frac{\partial r_{23}}{\partial d_{12}} = (2(d_{12}(x_{13} - x_{12}) + c_{12}(y_{13} - y_{12}) - b_{12}(z_{13} - z_{12})), \\
2(-c_{12}(x_{13} - x_{12}) + d_{12}(y_{13} - y_{12}) + a_{12}(z_{13} - z_{12})), \\
2(b_{12}(x_{13} - x_{12}) - a_{12}(y_{13} - y_{12}) + d_{12}(z_{13} - z_{12})))
\]

There are second partial derivatives with respect to the quaternion components:

\[
\frac{\partial^2 r_{23}}{\partial a_{12}^2} = (2(x_{13} - x_{12}), -2(y_{13} - y_{12}), -2(z_{13} - z_{12})) \\
\frac{\partial^2 r_{23}}{\partial b_{12}^2} = (-2(x_{13} - x_{12}), 2(y_{13} - y_{12}), -2(z_{13} - z_{12})) \\
\frac{\partial^2 r_{23}}{\partial c_{12}^2} = (-2(x_{13} - x_{12}), -2(y_{13} - y_{12}), 2(z_{13} - z_{12})) \\
\frac{\partial^2 r_{23}}{\partial d_{12}^2} = (2(x_{13} - x_{12}), 2(y_{13} - y_{12}), 2(z_{13} - z_{12}))
\]

Relative translational velocity, \( \hat{r}_{23} \)

The relative translational velocity is:

\[
\hat{r}_{23} = q_{12}^{-1}(\hat{r}_{13} - \hat{r}_{12} - 2\dot{q}_{12}q_{12}^{-1} \times (r_{13} - r_{12}))q_{12} \\
\]

First partial derivatives are:
\[
\frac{\partial \mathbf{r}_{23}}{\partial x_{12}} = (4(b_{12}d_{12} + c_{12}c_{12}), 2(c_{12}d_{12} + d_{12}c_{12} - a_{12}b_{12} - b_{12}a_{12}),
\]
\[
-2(b_{12}d_{12} + d_{12}b_{12} + a_{12}c_{12} + c_{12}b_{12} + c_{12}d_{12})), \tag{J.56}
\]
\[
\frac{\partial \mathbf{r}_{23}}{\partial y_{12}} = -2(c_{12}d_{12} + d_{12}c_{12} + a_{12}b_{12} + b_{12}a_{12}, (4(a_{12}b_{12} + c_{12}c_{12}),
\]
\[
2(a_{12}d_{12} + d_{12}a_{12} - c_{12}b_{12} - b_{12}c_{12})), \tag{J.57}
\]
\[
\frac{\partial \mathbf{r}_{23}}{\partial z_{12}} = 2(b_{12}d_{12} + d_{12}b_{12} - c_{12}a_{12} - a_{12}c_{12}),
\]
\[
-2(a_{12}d_{12} + d_{12}a_{12} + c_{12}b_{12} + b_{12}c_{12}), (4(a_{12}b_{12} + b_{12}b_{12})), \tag{J.58}
\]
\[
\frac{\partial \mathbf{r}_{23}}{\partial x_{13}} = -\frac{\partial \mathbf{r}_{23}}{\partial x_{12}}, \tag{J.59}
\]
\[
\frac{\partial \mathbf{r}_{23}}{\partial y_{13}} = -\frac{\partial \mathbf{r}_{23}}{\partial y_{12}}, \tag{J.60}
\]
\[
\frac{\partial \mathbf{r}_{23}}{\partial z_{13}} = -\frac{\partial \mathbf{r}_{23}}{\partial z_{12}}, \tag{J.61}
\]
\[
\frac{\partial \mathbf{r}_{23}}{\partial x_{12}} = ((b_{12} + c_{12} - a_{12}^2 - d_{12}^2), 2(c_{12}d_{12} - a_{12}b_{12}), -2(a_{12}c_{12} + b_{12}d_{12})), \tag{J.62}
\]
\[
\frac{\partial \mathbf{r}_{23}}{\partial y_{12}} = ((-2(a_{12}b_{12} + c_{12}d_{12}), (a_{12}^2 + c_{12} - b_{12}^2 - d_{12}^2), 2(a_{12}d_{12} - b_{12}c_{12})), \tag{J.63}
\]
\[
\frac{\partial \mathbf{r}_{23}}{\partial z_{12}} = (2(b_{12}d_{12} - a_{12}c_{12}), -2(b_{12}c_{12} + a_{12}d_{12}), (a_{12}^2 + b_{12}^2 - c_{12}^2 - d_{12}^2)), \tag{J.64}
\]
\[
\frac{\partial \mathbf{r}_{23}}{\partial x_{13}} = -\frac{\partial \mathbf{r}_{23}}{\partial x_{12}}, \tag{J.65}
\]
\[
\frac{\partial \mathbf{r}_{23}}{\partial y_{13}} = -\frac{\partial \mathbf{r}_{23}}{\partial y_{12}}, \tag{J.66}
\]
\[
\frac{\partial \mathbf{r}_{23}}{\partial z_{13}} = -\frac{\partial \mathbf{r}_{23}}{\partial z_{12}}, \tag{J.67}
\]
\[
\frac{\partial \mathbf{r}_{23}}{\partial a_{12}} = (2(a_{12}(z_{13} - \hat{z}_{13}) + b_{12}d_{13}(y_{13} - \hat{y}_{13}) + c_{12}(z_{13} - \hat{z}_{13} +
\]
\[
2a_{12}(x_{13} - x_{12}) + b_{12}(y_{13} - y_{12}) + c_{12}(z_{13} - z_{12})),
\]
\[
2(b_{12}(x_{13} - x_{12}) - a_{12}(y_{13} - y_{12}) + d_{12}(z_{13} - z_{12})
\]
\[
+ b_{12}(x_{13} - x_{12} + d_{12}(y_{13} - y_{12})),
\]
\[
2(c_{12}(x_{13} - x_{12}) - d_{12}(y_{13} - y_{12}) - a_{12}(z_{13} - z_{12})
\]
\[
+ c_{12}(x_{13} - x_{12} - d_{12}(y_{13} - y_{12})))) \tag{J.68}
\]
\[
\frac{\partial \mathbf{r}_{23}}{\partial b_{12}} = (2(-b_{12}(x_{13} - x_{12}) + a_{12}(y_{13} - y_{12}) - d_{12}(z_{13} - z_{12})
\]
\[
+ a_{12}(y_{13} - y_{12}) - d_{12}(x_{13} - x_{12})), \tag{J.69}
\]
\[
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\]
\[
\frac{\partial \hat{r}_{23}}{\partial c_{12}} = (2(-c_{12}(\dot{x}_{13} - \dot{x}_{12}) + d_{12}(\dot{y}_{13} - \dot{y}_{12}) + a_{12}(\dot{z}_{13} - \dot{z}_{12}) - \dot{d}_{12}(y_{13} - y_{12})) + \dot{a}_{12}(x_{13} - x_{12} + \dot{c}_{12}(y_{13} - y_{12}))
\]

\[
\frac{\partial \hat{r}_{23}}{\partial d_{12}} = (2(d_{12}(\dot{x}_{13} - \dot{x}_{12}) + c_{12}(\dot{y}_{13} - \dot{y}_{12}) - b_{12}(\dot{z}_{13} - \dot{z}_{12}) + 2d_{12}(y_{13} - y_{12}) + 2c_{12}(x_{13} - x_{12}))
\]

\[
\frac{\partial \hat{r}_{23}}{\partial a_{12}} = (2(2a_{12}(x_{13} - x_{12}) + y_{13} - y_{12}) + c_{12}(\dot{z}_{13} - \dot{z}_{12}) + 2c_{12}(y_{13} - y_{12}))
\]

\[
\frac{\partial \hat{r}_{23}}{\partial b_{12}} = (2(a_{12}(x_{13} - x_{12}) + y_{13} - y_{12}) + c_{12}(\dot{z}_{13} - \dot{z}_{12}) + 2d_{12}(y_{13} - y_{12}))
\]

\[
\frac{\partial \hat{r}_{23}}{\partial c_{12}} = (2(-d_{12}(x_{13} - x_{12}) + b_{12}(y_{13} - y_{12}) + c_{12}(\dot{z}_{13} - \dot{z}_{12}) + 2b_{12}(y_{13} - y_{12}) + 2c_{12}(x_{13} - x_{12}))
\]
There are second partial derivatives with respect to the quaternion components:

\[
\frac{\partial^2 \mathbf{r}_{23}}{\partial a_{12}^2} = (2(\dot{x}_{13} - \dot{x}_{12}), -2(\dot{y}_{13} - \dot{y}_{12}), -2(\dot{z}_{13} - \dot{z}_{12})) \quad (J.76)
\]

\[
\frac{\partial^2 \mathbf{r}_{23}}{\partial b_{12}^2} = (-2(\dot{x}_{13} - \dot{x}_{12}), 2(\dot{y}_{13} - \dot{y}_{12}), -2(\dot{z}_{13} - \dot{z}_{12})) \quad (J.77)
\]

\[
\frac{\partial^2 \mathbf{r}_{23}}{\partial c_{12}^2} = (-2(\dot{x}_{13} - \dot{x}_{12}), -2(\dot{y}_{13} - \dot{y}_{12}), 2(\dot{z}_{13} - \dot{z}_{12})) \quad (J.78)
\]

\[
\frac{\partial^2 \mathbf{r}_{23}}{\partial d_{12}^2} = (2(\dot{x}_{13} - \dot{x}_{12}), 2(\dot{y}_{13} - \dot{y}_{12}), 2(\dot{z}_{13} - \dot{z}_{12})) \quad (J.79)
\]

Relative angular velocity, \( \Omega_{23} \)

The angular velocity is a simple function of the quaternion and quaternion rate:

\[
\Omega_{23} = 2 \dot{q}_{23} q_{23}^{-1} \quad (J.80)
\]

The first partial derivatives of the angular velocity with respect to the global segment quaternion components are:

\[
\frac{\partial \Omega_{23}}{\partial q_{12}(i)} = 2(\frac{\partial \dot{q}_{23}}{\partial q_{12}(i)} q_{23}^{-1} + \dot{q}_{23} q_{23}^{-1}) \frac{\partial q_{23}}{\partial q_{12}(i)} \quad (J.81)
\]

For derivatives with respect to the segment quaternion rates the second term vanishes, leaving an equation of the form:

\[
\frac{\partial \Omega_{23}}{\partial q_{12}(i)} = 2 \frac{\partial \dot{q}_{23}}{\partial q_{12}(i)} q_{23}^{-1} \quad (J.82)
\]

The relative quaternion has no second partial derivatives, but the relative angular velocity does due to the distribution rules for quaternion differentiation. The form of the second derivatives is:

\[
\frac{\partial^2 \Omega_{23}}{\partial q_{12}(i)^2} = 4 \frac{\partial \dot{q}_{23}}{\partial q_{12}(i)} \frac{\partial q_{23}^{-1}}{\partial q_{12}(i)} \quad (J.83)
\]

This exists only for derivatives with respect to the quaternion components, not the quaternion rates, since the second term is zero for derivatives with respect to the quaternion rates.

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Screw invariants: Angular speed about the screw axis ($\Omega_{23}$)

The magnitude of the angular velocity vector, the angular speed, is one of the invariants of a screw. It is simply:

$$\Omega_{23} = \sqrt{\Omega_{23} \cdot \Omega_{23}}$$  \hfill (J.84)

First partial derivatives with respect to the quaternion components and rates are given by:

$$\frac{\partial \Omega_{23}}{\partial q_{12}(i)} = \frac{1}{\Omega_{23}} \left( \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \Omega_{23} \right)$$  \hfill (J.85)

Second partial derivatives are:

$$\frac{\partial^2 \Omega_{23}}{\partial q_{12}^2(i)} = \frac{1}{\Omega_{23}} \left( \frac{\partial^2 \Omega_{23}}{\partial q_{12}^2(i)} \cdot \Omega_{23} + \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \right) - \frac{1}{\Omega_{23}^2} \left( \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \Omega_{23} \right)^2$$  \hfill (J.86)

Screw invariants: Translational speed along screw axis ($V_{23}$)

The translational speed along the screw axis (by definition there is no component perpendicular to the screw axis) is given by:

$$V_{23} = \frac{\Omega_{23} \cdot \dot{r}_{23}}{\Omega_{23}}$$  \hfill (J.87)

First partial derivatives with respect to the segment position and translational velocity vector components are:

$$\frac{\partial V_{23}}{\partial r_{12}(i)} = \frac{\Omega_{23} \cdot \dot{r}_{23}}{\Omega_{23}} \cdot \frac{\partial \dot{r}_{23}}{\partial r_{12}(i)}$$  \hfill (J.88)

Derivatives with respect to the reference segment quaternion and quaternion rate components are of the form:

$$\frac{\partial V_{23}}{\partial q_{12}(i)} = \frac{1}{\Omega_{23}} \left( \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \dot{r}_{23} + \Omega_{23} \cdot \frac{\partial \dot{r}_{23}}{\partial q_{12}(i)} \right) - \frac{V_{23}}{\Omega_{23}^2} \left( \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \Omega_{23} \right)$$  \hfill (J.89)
First partials with respect to the moving segment quaternion are simplified by the relative translational velocity not being a function of the components of the moving segment quaternion or quaternion rate:

\[
\frac{\partial V_{23}}{\partial \bar{q}_{13}(i)} = \frac{1}{\Omega_{23}} (\dot{r}_{23} \cdot \frac{\partial \Omega_{23}}{\partial \bar{q}_{13}(i)}) - \frac{V_{23}}{\Omega_{23}^2} (\Omega_{23} \cdot \frac{\partial \Omega_{23}}{\partial \bar{q}_{13}(i)}) \tag{J.90}
\]

All second partials with respect to the components of the position and translational velocity vectors are zero. Second partials with respect to the components of the quaternion of the moving reference segment, the rate of the moving reference quaternion, the quaternion of the moving body and the rate of that quaternion are of the form:

\[
\frac{\partial^2 V_{23}}{\partial \bar{q}_{13}(i)^2} = \frac{1}{\Omega_{23}} \left( \frac{\partial^2 \Omega_{23}}{\partial \bar{q}_{13}(i)^2} \cdot \dot{r}_{23} + 2 \frac{\partial \Omega_{23}}{\partial \bar{q}_{13}(i)} \cdot \frac{\partial \dot{r}_{23}}{\partial \bar{q}_{13}(i)} \right) + \frac{V_{23}}{\Omega_{23}} \left( \frac{\partial^2 \Omega_{23}}{\partial \bar{q}_{13}(i)^2} \cdot \Omega_{23} + \frac{\partial \Omega_{23}}{\partial \bar{q}_{13}(i)} \cdot \frac{\partial \Omega_{23}}{\partial \bar{q}_{13}(i)} \right) + \frac{V_{23}^2}{\Omega_{23}^2} \cdot \Omega_{23}^2 \tag{J.91}
\]

\[
\frac{\partial^2 V_{23}}{\partial \bar{q}_{13}(i)^2} = \frac{2}{\Omega_{23}} \left( \frac{\partial \Omega_{23}}{\partial \bar{q}_{13}(i)} \cdot \frac{\partial \dot{r}_{23}}{\partial \bar{q}_{13}(i)} \right) + \frac{V_{23}}{\Omega_{23}} \left( \frac{\partial \Omega_{23}}{\partial \bar{q}_{13}(i)} \cdot \Omega_{23} + \frac{\partial \Omega_{23}}{\partial \bar{q}_{13}(i)} \cdot \frac{\partial \Omega_{23}}{\partial \bar{q}_{13}(i)} \right) + \frac{V_{23}^2}{\Omega_{23}^2} \cdot \Omega_{23} \tag{J.92}
\]

\[
\frac{\partial^2 V_{23}}{\partial \bar{q}_{13}(i)^2} = \frac{1}{\Omega_{23}^2} \left( \frac{\partial \Omega_{23}}{\partial \bar{q}_{13}(i)^2} \cdot \dot{r}_{23} - 2 \frac{\partial \Omega_{23}}{\partial \bar{q}_{13}(i)} \cdot \frac{\partial \Omega_{23}}{\partial \bar{q}_{13}(i)} \right) \tag{J.93}
\]

\[
\frac{\partial^2 V_{23}}{\partial \bar{q}_{13}(i)^2} = -\frac{2}{\Omega_{23}^2} \left( \frac{\partial \Omega_{23}}{\partial \bar{q}_{13}(i)} \cdot \dot{r}_{23} \right) \cdot \frac{\partial \Omega_{23}}{\partial \bar{q}_{13}(i)} \cdot \Omega_{23} - \frac{V_{23}^2}{\Omega_{23}^2} \cdot \Omega_{23} \tag{J.94}
\]

Screw invariants: Helical axis pitch ($h_{23}$)

The pitch of the helical axis at any instant is:

\[
h_{23} = \frac{\Omega_{23} \cdot \dot{r}_{23}}{\Omega_{23}^2} \tag{J.95}
\]

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First partial derivatives with respect to the components of the position vectors for the two moving segments and their translational velocity vectors are of the form:

\[
\frac{\partial h_{23}}{\partial r_{12}(i)} = \frac{1}{\Omega_{23}^2} (\Omega_{23} \cdot \frac{\partial r_{23}}{\partial r_{12}(i)}) \tag{J.96}
\]

With respect to the quaternion and quaternion rate for the moving reference frame:

\[
\frac{\partial h_{23}}{\partial q_{12}(i)} = \frac{1}{\Omega_{23}^2} (\frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \dot{r}_{23} + \Omega_{23} \cdot \frac{\partial \dot{r}_{23}}{\partial q_{12}(i)}) - 2 \frac{h_{23}}{\Omega_{23}^2} (\Omega_{23} \cdot \frac{\partial \Omega_{23}}{\partial q_{12}(i)}) \tag{J.97}
\]

With respect to the quaternion and quaternion rate of the other moving segment:

\[
\frac{\partial h_{23}}{\partial q_{13}(i)} = \frac{1}{\Omega_{23}^2} (\frac{\partial \Omega_{23}}{\partial q_{13}(i)} \cdot \dot{r}_{23}) - 2 \frac{h_{23}}{\Omega_{23}^2} (\frac{\partial \Omega_{23}}{\partial q_{13}(i)} \cdot \Omega_{23}) \tag{J.98}
\]

All second partials with respect to position and translational velocity vector components are zero. Second partials with respect to the quaternions and quaternion rates are of the form:

\[
\frac{\partial^2 h_{23}}{\partial q_{12}(i)^2} = \frac{1}{\Omega_{23}^2} (\frac{\partial^2 \Omega_{23}}{\partial q_{12}(i)} \cdot \dot{r}_{23} + 2 \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \frac{\partial \dot{r}_{23}}{\partial q_{12}(i)} + \Omega_{23} \cdot \frac{\partial^2 \dot{r}_{23}}{\partial q_{12}(i)^2}) \tag{J.99}
\]

\[
- 2 \frac{h_{23}}{\Omega_{23}^2} (\frac{\partial^2 \Omega_{23}}{\partial q_{12}(i)^2} \cdot \Omega_{23} + \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \frac{\partial \Omega_{23}}{\partial q_{12}(i)})
\]

\[
- \frac{4}{\Omega_{23}^2} (\frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \dot{r}_{23} + \Omega_{23} \cdot \frac{\partial \dot{r}_{23}}{\partial q_{12}(i)})(\Omega_{23} \cdot \frac{\partial \Omega_{23}}{\partial q_{12}(i)})
\]

\[
+ \frac{4}{\Omega_{23}^4} (\Omega_{23} \cdot \frac{\partial \Omega_{23}}{\partial q_{12}(i)})^2
\]

\[
\frac{\partial^2 h_{23}}{\partial q_{13}(i)^2} = \frac{2}{\Omega_{23}^2} (\frac{\partial \Omega_{23}}{\partial q_{13}(i)} \cdot \frac{\partial \dot{r}_{23}}{\partial q_{13}(i)}) - 2 \frac{h_{23}}{\Omega_{23}^2} (\frac{\partial \Omega_{23}}{\partial q_{13}(i)} \cdot \frac{\partial \Omega_{23}}{\partial q_{13}(i)}) \tag{J.100}
\]

\[
- \frac{4}{\Omega_{23}^2} (\frac{\partial \Omega_{23}}{\partial q_{13}(i)} \cdot \dot{r}_{23} + \Omega_{23} \cdot \frac{\partial \dot{r}_{23}}{\partial q_{13}(i)})(\Omega_{23} \cdot \frac{\partial \Omega_{23}}{\partial q_{13}(i)})
\]

\[
+ \frac{4}{\Omega_{23}^4} (\Omega_{23} \cdot \frac{\partial \Omega_{23}}{\partial q_{13}(i)})^2
\]

\[
\frac{\partial^2 h_{23}}{\partial q_{12}(i) \partial q_{13}(i)} = \frac{1}{\Omega_{23}^2} (\frac{\partial^2 \Omega_{23}}{\partial q_{12}(i)} \cdot \dot{r}_{23} - 2 \frac{h_{23}}{\Omega_{23}^2} (\frac{\partial^2 \Omega_{23}}{\partial q_{12}(i) \partial q_{13}(i)} \cdot \Omega_{23} + \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \frac{\partial \Omega_{23}}{\partial q_{13}(i)})) \tag{J.101}
\]

\[
- \frac{4}{\Omega_{23}^2} (\frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \dot{r}_{23})(\Omega_{23} \cdot \frac{\partial \Omega_{23}}{\partial q_{13}(i)}) + \frac{4}{\Omega_{23}^4} (\Omega_{23} \cdot \frac{\partial \Omega_{23}}{\partial q_{13}(i)})^2
\]

\[
\frac{\partial^2 h_{23}}{\partial q_{13}^2} = -2 \frac{h_{23}}{\Omega_{23}^2} (\frac{\partial \Omega_{23}}{\partial q_{13}} \cdot \frac{\partial \Omega_{23}}{\partial q_{13}}) - \frac{4}{\Omega_{23}^2} (\frac{\partial \Omega_{23}}{\partial q_{13}} \cdot \dot{r}_{23})(\Omega_{23} \cdot \frac{\partial \Omega_{23}}{\partial q_{13}}) \tag{J.102}
\]
\[ + 4 \frac{h_{23}}{\Omega_{23}^4} (\Omega_{23} \cdot \frac{\partial \Omega_{23}}{\partial \Omega_{23}})^3 \]

Screw parameters: Real part of unit velocity screw \((u_{23})\)

The real part of the unit velocity screw is the normalized angular velocity, a unit vector in the direction of the angular velocity:

\[ u_{23} = \frac{\Omega_{23}}{\Omega_{23}} \]  \[ (J.103) \]

The first partial derivatives of the real part of the screw are defined with respect to both moving segment quaternions and quaternion rates:

\[ \frac{\partial u_{23}}{\partial q_{12}(i)} = \frac{1}{\Omega_{23}} \frac{\partial \Omega_{23}}{\partial q_{12}(i)} - \frac{\Omega_{23}}{\Omega_{23}^3} (\Omega_{23} \cdot \frac{\partial \Omega_{23}}{\partial q_{12}(i)}) \]  \[ (J.104) \]

Second partial derivatives are nonzero for components of both quaternions and the quaternion rates. The form for each is:

\[ \frac{\partial^2 u_{23}}{\partial q_{12}^2(i)} = \frac{1}{\Omega_{23}} \frac{\partial^2 \Omega_{23}}{\partial q_{12}^2(i)} - \frac{2}{\Omega_{23}^3} (\frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \frac{\partial \Omega_{23}}{\partial q_{12}(i)}) \frac{\Omega_{23}}{\partial q_{12}(i)} \]

\[ - \frac{\Omega_{23}}{\Omega_{23}^3} (\frac{\partial^2 \Omega_{23}}{\partial q_{12}(i)} \cdot \Omega_{23} + \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \frac{\partial \Omega_{23}}{\partial q_{12}(i)}) + 3 \frac{\Omega_{23}}{\Omega_{23}^5} (\frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \Omega_{23})^2 \]

\[ \frac{\partial^2 u_{23}}{\partial q_{12}(i)} = - \frac{2}{\Omega_{23}^3} (\frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \Omega_{23}) \frac{\Omega_{23}}{\partial q_{12}(i)} - \frac{u_{23}}{\Omega_{23}^2} \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \]

\[ + 3 \frac{u_{23}}{\Omega_{23}^2} (\frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \Omega_{23})^2 \]  \[ (J.105) \]

\[ (J.106) \]

Screw parameters: Vector from fixed reference frame origin to the screw axis \((p_{23})\)

A vector perpendicular to the screw axis may be drawn from the origin of the fixed reference frame to the screw axis. It is given by:

\[ p_{23} = \frac{\Omega_{23} \times \dot{r}_{23}}{\Omega_{23}^2} + r_{23} \]  \[ (J.107) \]

First partials with respect to the position and translational velocity vector components are:
\[
\frac{\partial p_{23}}{\partial r_{12}(i)} = \frac{1}{\Omega_{23}^2}(\Omega_{23} \times \frac{\partial r_{23}}{\partial r_{12}(i)}) + \frac{\partial r_{23}}{\partial r_{12}(i)} \quad (J.108)
\]

\[
\frac{\partial p_{23}}{\partial r_{13}(i)} = -\frac{\partial p_{23}}{\partial r_{13}(i)} \quad (J.109)
\]

\[
\frac{\partial p_{23}}{\partial r_{12}(i)} = \frac{1}{\Omega_{23}^2}(\Omega_{23} \times \frac{\partial r_{23}}{\partial r_{12}(i)}) \quad (J.110)
\]

\[
\frac{\partial p_{23}}{\partial r_{13}(i)} = -\frac{\partial p_{23}}{\partial r_{13}(i)} \quad (J.111)
\]

For the quaternion and quaternion rate components:

\[
\frac{\partial p_{23}}{\partial q_{12}(i)} = \frac{1}{\Omega_{23}^2} \left( \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \times \dot{r}_{23} + \Omega_{23} \times \frac{\partial \dot{r}_{23}}{\partial q_{12}(i)} \right) - 2p_{23} \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \Omega_{23} \quad (J.112)
\]

\[
\frac{\partial p_{23}}{\partial q_{13}(i)} = \frac{1}{\Omega_{23}^2} \left( \frac{\partial \Omega_{23}}{\partial q_{13}(i)} \times \dot{r}_{23} - 2p_{23} \frac{\partial \Omega_{23}}{\partial q_{13}(i)} \cdot \Omega_{23} \right) \quad (J.113)
\]

\[
\frac{\partial p_{23}}{\partial q_{12}(i)} = \frac{1}{\Omega_{23}^2} \left( \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \times \dot{r}_{23} \right) - 2p_{23} \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \Omega_{23} \quad (J.114)
\]

\[
\frac{\partial p_{23}}{\partial q_{13}(i)} = \frac{1}{\Omega_{23}^2} \left( \frac{\partial \Omega_{23}}{\partial q_{13}(i)} \times \dot{r}_{23} \right) - 2p_{23} \frac{\partial \Omega_{23}}{\partial q_{13}(i)} \cdot \Omega_{23} \quad (J.115)
\]

Second partials with respect to the position and translational velocity vector components are all zero. With respect to the quaternion and quaternion rate components they are:

\[
\frac{\partial^2 p_{23}}{\partial q_{12}(i)^2} = \frac{1}{\Omega_{23}^2} \left( \frac{\partial^2 \Omega_{23}}{\partial q_{13}(i)} \times \dot{r}_{23} + 2 \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \right) \frac{\partial \dot{r}_{23}}{\partial q_{13}(i)} + \Omega_{23} \times \frac{\partial^2 \dot{r}_{23}}{\partial q_{13}(i)^2} \quad (J.116)
\]

\[
-2p_{23} \sqrt{\Omega_{23}^2} \left( \frac{\partial^2 \Omega_{23}}{\partial q_{12}(i)} \cdot \Omega_{23} + \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \right)
\]

\[
+ \frac{8p_{23}^2}{\Omega_{23}^4} \left( \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \Omega_{23} \right)^2 - 6 \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \times \dot{r}_{23} + \Omega_{23} \times \frac{\partial \dot{r}_{23}}{\partial q_{12}(i)} \right)
\]

\[
\frac{\partial^2 p_{23}}{\partial q_{12}(i)} = \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \Omega_{23} \quad (J.117)
\]

\[
\frac{\partial^2 p_{23}}{\partial q_{13}(i)^2} = \frac{1}{\Omega_{23}^2} \left( \frac{\partial^2 \Omega_{23}}{\partial q_{13}(i)} \times \dot{r}_{23} \right) - 2p_{23} \left( \frac{\partial^2 \Omega_{23}}{\partial q_{12}(i)} \right) \cdot \Omega_{23} \quad (J.117)
\]

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\[ \frac{\partial^2 p_{23}}{\partial q_{12}^2(i)} = \frac{1}{\Omega_{23}^2} \left( \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \times \frac{\partial \dot{r}_{23}}{\partial q_{12}(i)} \right) - 2 \frac{p_{23}}{\Omega_{23}^2} \left( \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \right) 
+ 8 \frac{p_{23}}{\Omega_{23}^4} \left( \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \Omega_{23} \right)^2 \]

\[ \frac{\partial^2 p_{23}}{\partial q_{13}^2(i)} = -2 \frac{p_{23}}{\Omega_{23}^2} \left( \frac{\partial \Omega_{23}}{\partial q_{13}(i)} \cdot \frac{\partial \Omega_{23}}{\partial q_{13}(i)} \right) - 4 \frac{p_{23}}{\Omega_{23}^4} \left( \frac{\partial \Omega_{23}}{\partial q_{13}(i)} \times \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \Omega_{23} \right) \]

Screw parameters: Dual part of unit velocity screw \( (u_{23}^0) \)

The dual part of the unit velocity screw, or moment of the screw, is:

\[ u_{23}^0 = \frac{\Omega_{23} \times \dot{r}_{23} \times \Omega_{23}}{\Omega_{23}^3} + r_{23} \times \frac{\Omega_{23}}{\Omega_{23}^3} \]

The first partials with respect to the position and translational velocity vector components are:

\[ \frac{\partial u_{23}^0}{\partial r_{12}(i)} = \frac{1}{\Omega_{23}^2} \left( \Omega_{23} \times \frac{\partial \dot{r}_{23}}{\partial r_{12}(i)} \times \Omega_{23} \right) + \frac{\partial r_{23}}{\partial r_{12}(i)} \times \frac{\Omega_{23}}{\Omega_{23}^3} \]

\[ \frac{\partial u_{23}^0}{\partial r_{13}(i)} = -\frac{\partial u_{23}^0}{\partial r_{12}(i)} \]

First partials with respect to the moving segment quaternions and quaternion rates are:

\[ \frac{\partial u_{23}^0}{\partial q_{12}(i)} = \frac{1}{\Omega_{23}^2} \left( \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \times \dot{r}_{23} \times \Omega_{23} + \Omega_{23} \times \frac{\partial \dot{r}_{23}}{\partial q_{12}(i)} \times \Omega_{23} \right) \]
\[
\frac{\partial u_0^{\omega}}{\partial q_{13}(i)} = \frac{1}{\Omega_{23}} \left( \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \times \dot{r}_{23} \times \Omega_{23} + \Omega_{23} \times \dot{r}_{23} \times \Omega_{23} \right) - 3 \left( \frac{\Omega_{23} \times \dot{r}_{23} \times \Omega_{23}}{\Omega_{23}^2} \right) \left( \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \Omega_{23} \right) + r_{23} \left( \frac{1}{\Omega_{23}} \frac{\partial \Omega_{23}}{\partial q_{12}(i)} - \frac{\partial \Omega_{23}}{\partial q_{13}(i)} \cdot \Omega_{23} \right) \]
\[
+ \Omega_{23} \times \dot{r}_{23} \times \frac{\partial \Omega_{23}}{\partial q_{13}(i)} - 3 \left( \frac{\Omega_{23} \times \dot{r}_{23} \times \Omega_{23}}{\Omega_{23}^2} \right) \left( \frac{\partial \Omega_{23}}{\partial q_{13}(i)} \cdot \Omega_{23} \right) + r_{23} \left( \frac{1}{\Omega_{23}} \frac{\partial \Omega_{23}}{\partial q_{13}(i)} - \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \Omega_{23} \right) \]
\[
\frac{\partial u_0^{\omega}}{\partial q_{12}(i)} = \frac{1}{\Omega_{23}} \left( \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \times \dot{r}_{23} \times \Omega_{23} + \Omega_{23} \times \dot{r}_{23} \times \dot{r}_{23} \times \Omega_{23} \right) - 3 \left( \frac{\Omega_{23} \times \dot{r}_{23} \times \Omega_{23}}{\Omega_{23}^2} \right) \left( \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \Omega_{23} \right) + r_{23} \left( \frac{1}{\Omega_{23}} \frac{\partial \Omega_{23}}{\partial q_{12}(i)} - \frac{\partial \Omega_{23}}{\partial q_{13}(i)} \cdot \Omega_{23} \right) \]
\[
\frac{\partial u_0^{\omega}}{\partial q_{13}(i)} = \frac{1}{\Omega_{23}} \left( \frac{\partial \Omega_{23}}{\partial q_{13}(i)} \times \dot{r}_{23} \times \Omega_{23} + \Omega_{23} \times \dot{r}_{23} \times \Omega_{23} \right) - 3 \left( \frac{\Omega_{23} \times \dot{r}_{23} \times \Omega_{23}}{\Omega_{23}^2} \right) \left( \frac{\partial \Omega_{23}}{\partial q_{13}(i)} \cdot \Omega_{23} \right) + r_{23} \left( \frac{1}{\Omega_{23}} \frac{\partial \Omega_{23}}{\partial q_{13}(i)} - \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \Omega_{23} \right) \]
\]

Second partials with respect to the position and translation vector components are zero. With respect to the quaternion and quaternion rates:

\[
\frac{\partial^2 u_0^{\omega}}{\partial q_{12}(i)} = \frac{1}{\Omega_{23}^2} \left( \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \times \dot{r}_{23} \times \Omega_{23} + \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \times \dot{r}_{23} \times \Omega_{23} \right) + 2 \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \times \dot{r}_{23} \times \dot{r}_{23} \times \Omega_{23} + \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \times \dot{r}_{23} \times \Omega_{23} \times \Omega_{23}
\]
\[
+ \Omega_{23} \times \frac{\partial^2 \Omega_{23}}{\partial q_{12}(i)} \times \Omega_{23} + \Omega_{23} \times \dot{r}_{23} \times \dot{r}_{23} \times \Omega_{23} + \frac{\partial^2 \Omega_{23}}{\partial q_{12}(i)} \times \dot{r}_{23} \times \Omega_{23} \times \Omega_{23}
\]
\[
- \frac{6}{\Omega_{23}^5} \left( \frac{\partial \Omega_{23}}{\partial q_{13}(i)} \cdot \Omega_{23} \right) \left( \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \cdot \Omega_{23} \right) \times \dot{r}_{23} \times \Omega_{23} + \Omega_{23} \times \dot{r}_{23} \times \Omega_{23} \times \Omega_{23} \times \dot{r}_{23} \times \Omega_{23} \times \Omega_{23}
\]

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\[
\frac{\partial^3 u_{33}}{\partial q_{13}^2(i)} = \frac{1}{\Omega_{23}^3}\left(2 \frac{\partial \Omega_{33}}{\partial q_{13}(i)} \times \Omega_{23} + \frac{\partial \Omega_{23}}{\partial q_{13}(i)} \times \Omega_{23} \times \frac{\partial \Omega_{23}}{\partial q_{13}(i)}ightharpoonup \right) + 2 \frac{\partial \Omega_{23}}{\partial q_{13}(i)} \times \Omega_{23} + 2 \frac{\partial \Omega_{23}}{\partial q_{13}(i)} \times \Omega_{23} \times \frac{\partial \Omega_{23}}{\partial q_{13}(i)}
\]

\[
\frac{\partial^2 u_{33}}{\partial q_{12}^2(i)} = \frac{1}{\Omega_{23}^3}\left(2 \frac{\partial \Omega_{33}}{\partial q_{12}(i)} \times \Omega_{23} + \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \times \Omega_{23} \times \frac{\partial \Omega_{23}}{\partial q_{12}(i)}ightharpoonup \right) + 2 \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \times \Omega_{23} + 2 \frac{\partial \Omega_{23}}{\partial q_{12}(i)} \times \Omega_{23} \times \frac{\partial \Omega_{23}}{\partial q_{12}(i)}
\]

\[
\frac{\partial^2 u_{23}}{\partial q_{13}^2(i)} = \frac{1}{\Omega_{23}^3}\left(2 \frac{\partial \Omega_{33}}{\partial q_{13}(i)} \times \Omega_{23} + \frac{\partial \Omega_{23}}{\partial q_{13}(i)} \times \Omega_{23} \times \frac{\partial \Omega_{23}}{\partial q_{13}(i)}ightharpoonup \right) + 2 \frac{\partial \Omega_{23}}{\partial q_{13}(i)} \times \Omega_{23} + 2 \frac{\partial \Omega_{23}}{\partial q_{13}(i)} \times \Omega_{23} \times \frac{\partial \Omega_{23}}{\partial q_{13}(i)}
\]
\[ + \Omega_{23} \times \dot{r}_{23} \times \frac{\partial \Omega_{23}}{\partial \dot{q}_{13}(i)} - 3 \frac{\Omega_{23} \times \dot{r}_{23} \times \Omega_{23}}{\Omega_{23}^{2}} \left( \frac{\partial \Omega_{23}}{\partial \dot{q}_{13}(i)} \cdot \frac{\partial \Omega_{23}}{\partial \dot{q}_{13}(i)} \right) \\
+ 15 \frac{\Omega_{23} \times \dot{r}_{23} \times \Omega_{23}}{\Omega_{23}^{2}} \left( \frac{\partial \Omega_{23}}{\partial \dot{q}_{13}(i)} \cdot \Omega_{23} \right)^{2} + \left( \frac{2}{\Omega_{23}^{2}} \left( \frac{\partial \Omega_{23}}{\partial \dot{q}_{13}(i)} \cdot \Omega_{23} \right) \frac{\partial \Omega_{23}}{\partial \dot{q}_{13}(i)} \right) \\
- \frac{\Omega_{23}}{\Omega_{23}^{2}} \left( \frac{\partial \Omega_{23}}{\partial \dot{q}_{13}(i)} \cdot \frac{\partial \Omega_{23}}{\partial \dot{q}_{13}(i)} \right) - 3 \frac{\Omega_{23}}{\Omega_{23}^{2}} \left( \frac{\partial \Omega_{23}}{\partial \dot{q}_{13}(i)} \cdot \Omega_{23} \right)^{2} \times \dot{r}_{23} \]

### J.4 Estimated Standard Deviations

The partial derivatives of the preceding section were used in Equation J.1 to estimate variances and standard deviations for the relative kinematic variables and screw parameters at the knee. Representative standard deviations for the first swing trial with the pin-mounted arrays are shown in Figures J.6 and J.7. The standard deviations of the screw parameters in Figure J.7 are highest as the angular velocity approaches zero, since all screw parameters are normalized by the magnitude of the angular velocity.
Figure J.6: Estimated standard deviations of the relative kinematic variables at the knee for the first voluntary swing trial with the pin-mounted arrays. Shown are standard deviations for the: (a) Relative position vector, $\mathbf{r}_{23}$; (b) Relative quaternion, $\mathbf{q}_{23}$; (c) Relative translational velocity, $\mathbf{v}_{23}$; and (d) Relative angular velocity, $\Omega_{23}$. 

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Figure J.7: Estimated standard deviations of the instantaneous helical axis parameters at the knee for the first voluntary swing trial with the pin-mounted arrays. Shown are standard deviations for the: (a) Unit vector along the instantaneous helical axis, \( \mathbf{u}_{23} \); (b) Vector from the origin of the reference coordinate system to the IHA, \( \mathbf{p}_{23} \); (c) First-order instantaneous invariant – pitch, \( h_{23} \); and (d) Magnitude of the angular velocity about the IHA, \( \Omega_{23} \).
Appendix K

Ligament Geometry Measurements

Ligament attachment and rest length data are essential inputs to the kinematic model for assessing constraint activity. The first objective is to develop techniques for acquiring accurate data in vitro. In the long-term, methods for obtaining patient specific geometry data in vivo are needed to enable clinical application of the analysis.

Developmental work on ligament geometry data acquisition was carried out in conjunction with this dissertation. Brief descriptions of the ligament attachment geometry and rest length measurements are presented in the following sections.

K.1 Subject

All of the measurements reported on in this dissertation were made on a pair of knees obtained at autopsy from a single subject. Subject information is summarized in Table K.1. The right knee was used for the attachment measurement tests and the left knee was used in ligament rest length and material properties tests [143].

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Leg</th>
<th>Age</th>
<th>Sex</th>
<th>Weight (kg)</th>
<th>Cause of Death</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>left</td>
<td>27</td>
<td>male</td>
<td>102</td>
<td>metastatic melanoma</td>
</tr>
<tr>
<td>2</td>
<td>right</td>
<td>27</td>
<td>male</td>
<td>102</td>
<td>metastatic melanoma</td>
</tr>
</tbody>
</table>

Table K.1: Knee donor information.
K.2 Ligament Attachment Sites

K.2.1 Background

The three-dimensional geometry of the ligament attachment site is required for analyses of ligament constraint behavior. A number of techniques have been used to determine the three-dimensional coordinates of ligament attachments. The most promising for in vivo mapping of ligament attachment geometry from large numbers of subjects was reported by Langrana and Bronfeld [140]. After marking attachment sites on a cadaver knee with Kirschner wires, computer tomographic (CT) scans of the joint were obtained for different angles of flexion. CT number is proportional to density [30] and the images showed that there were regions of higher density bone in the neighborhood of the attachment site markers. Although no further quantitative corroboration was presented Langrana and Bronfeld asserted that the regions of higher density bone corresponded to the physical attachment sites and that CT scans could be used to map attachment site coordinates.

Development of CT, or preferably magnetic resonance imaging (MRI), techniques for reliable and accurate mapping of ligament attachment site coordinates would be a significant advance toward clinical application of knee models. However, quantitative proof of the technique is necessary. Several questions needed to studied in detail, including how consistently higher density bone regions appeared near attachment sites and whether or not there was a recognizable distribution of high density bone for each ligament.

K.2.2 Marker implantation and CT scans

As a first step toward evaluating the CT technique for determining ligament attachment point coordinates, the cadaver's right knee was CT scanned with and without the ligament attachment sites marked. Scan data were obtained for approximately 12 centimeters above and below the joint line. Prior to the first set of CT scans, all extraneous soft tissue was dissected away leaving the capsule and major ligaments. Each ligament was sectioned at mid-substance, separating the femur and the tibia-fibula.

The CT scans were performed on an EMI 7070 CT scanner at the Massachusetts General Hospital (MGH) Radiation Medicine Department. Jane Pardy donated her time to operate the scanner. Michael Stracher of the Orthopedic Research Laboratory at MGH arranged use of the scanner and converted the raw scanner data to a VAX (Digital Equipment Corporation, Maynard, MA) format for further processing.

The disarticulated bones were suspended in a Plexiglas water bath and the unmarked set of CT scans taken. The scan axis was aligned with the long axes of the two bones. In the neighborhood of the attachments the slices were nominally separated by two millimeters. The total number of slices for each scan is summarized in Table K.2.

Following completion of the unmarked scans, the ligaments were resected down to the
Table K.2: Number of slices per CT scan.

<table>
<thead>
<tr>
<th>Bone</th>
<th>Marked</th>
<th>Number of Slices</th>
<th>Number of Good Slices</th>
</tr>
</thead>
<tbody>
<tr>
<td>Femur</td>
<td>No</td>
<td>32</td>
<td>31</td>
</tr>
<tr>
<td>Tibia</td>
<td>No</td>
<td>62</td>
<td>61</td>
</tr>
<tr>
<td>Femur</td>
<td>Yes</td>
<td>45</td>
<td>43</td>
</tr>
<tr>
<td>Tibia</td>
<td>Yes</td>
<td>65</td>
<td>64</td>
</tr>
</tbody>
</table>

Table K.3: Attachment site marker distribution.

<table>
<thead>
<tr>
<th>Ligament</th>
<th>Tibia</th>
<th>Femur</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tibial Collateral</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>Fibular Collateral</td>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>Anterior Cruciate</td>
<td>10</td>
<td>9</td>
</tr>
<tr>
<td>Posterior Cruciate</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

attachments on each bone. The physical boundary of each insertion was marked with one millimeter diameter 316 stainless steel balls (New England Miniature Ball). The stainless steel balls would show up as very high density spots on the CT image. Holes were predrilled in the cortical shell using a #62 drill bit, 0.97 millimeter (0.0381 inches) diameter, and the balls were press fit into the surface of the bone. Attachments for the anterior cruciate, posterior cruciate, fibular collateral and tibial collateral were marked. The number of markers used to identify the boundary of each attachment site is presented in Table K.3. The bones with the attachment sites marked were CT scanned using the same procedure as in the first set of images. The nominal spacing between all scan slices was two millimeters and the resulting number of slices is summarized in Table K.2.

The CT data was supplied in a VAX file format with one slice per data file. Each slice was 320 by 320 pixels, with each pixel 0.47 millimeters on a side.

K.2.3 Marker coordinate identification

Image processing was performed using the facilities of the Whitaker College Medical Image Processing Laboratory at the Massachusetts Institute of Technology. One of the resources of the laboratory is a Sun 4/280 (Sun Microsystems Inc., Milpitas, CA) graphics workstation with a TAAC-1 image accelerator board.

An extensive package of demonstration software (TADEMO) was provided with the
image accelerator board. The output of one three-dimensional reconstruction program is presented in Figure K.1. A composite image of the marked femur is shown and the locations of the markers are clearly identifiable. Another program generated a variable-sized, mouse-driven, zoom magnification window that could be used to examine details of a single CT slice. The demonstration program was modified\(^1\) so that the coordinates of the center of the zoom window were displayed on the workstation screen. Since the markers appeared as discrete clusters of high intensity pixels on the slice images, the zoom window could be placed around each high density region on an image and the coordinates of the center of the cluster recorded. Figure K.2 shows the workstation screen during a typical session with a slice in the left-hand side of the screen and the magnified image of a marker on the right-hand side.

Three different users sequentially searched the marked scans for the tibia and femur and recorded the coordinates of every 'marker' image that could be identified. The total number of marker images identified exceeded the actual number of stainless steel balls implanted, primarily because balls could lie on slice boundaries and appear in multiple slices. A program was written to consolidate the marker images into a reduced set of three-dimensional marker coordinates.

**K.3 Ligament Rest Lengths**

Ligament rest lengths for the subject's left knee were measured using the procedure described in detail in Lanzendorf [143].

\(^1\)A set of utilities for converting the VAX format data to the proper format and the modifications to the demonstration program were made by Professor Derek Rowell, director of the imaging laboratory.
Figure K.1: Three-dimensional reconstruction of the marked femur from CT scans. The locations of the stainless steel markers are seen clearly.
Figure K.2: Workstation screen during the identification of the coordinates of ligament attachment markers. Markers are bright spots in the left-hand display of the slice. A magnified image of the pixels contained in the magnification window is on the right-hand side.
Appendix L

Joint Geometry Scanning Machine

L.1 Design Objectives

Articular surface geometry is a critical input to any kinematic model of the knee. Techniques for mapping the surface geometry and articular cartilage thickness in the hip using pulse-echo ultrasound were developed initially by Rushfeldt [214] and then extended by Tepic [260, 261]. The apparatus used in scanning the hip joint, which swept out circular arcs with a focussed transducer, assumed a nearly spherical target surface and proved to be inadequate for the asymmetric and noncongruent articular surfaces in the knee.

The new machine was designed with two principal objectives:

- Be capable of scanning an arbitrary surface with sufficient speed so that tissue degeneration was not a problem and with accuracy equal to or better than with the old apparatus.

- Be capable of duplicating the scanning algorithm from the old machine.

Repeated computer failures during the design and construction phase added a third objective of designing the scanner controls to be as machine independent as possible.

L.2 Mechanical Design

The mechanical design and component selection for the scanner are described in a Bachelor's thesis by Ito [117]. Detailed drawings of the assembly and parts specifications are included in the document. The assembled machine is shown in Figure L.1 and a detail of the transducer drive box is presented in Figure L.2.

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Figure L.1: Joint geometry scanning machine in the Eric P. and Evelyn E. Newman Laboratory for Biomechanics and Rehabilitation.
Figure L.2: Transducer orientation mechanism.
Table L.1: Scanner Control Electronics.

<table>
<thead>
<tr>
<th>Display Driver Cage</th>
<th>Motor Control Cage</th>
</tr>
</thead>
<tbody>
<tr>
<td>θ display driver</td>
<td>θ motor control</td>
</tr>
<tr>
<td>ψ display driver</td>
<td>ψ motor control</td>
</tr>
<tr>
<td>Reset control</td>
<td>Microstep translator interface</td>
</tr>
<tr>
<td>φ display driver</td>
<td>Manual motor control interface</td>
</tr>
<tr>
<td>X display driver</td>
<td>Slo-Syn translator interface</td>
</tr>
<tr>
<td>Y display driver</td>
<td>φ motor control</td>
</tr>
<tr>
<td>Z display driver</td>
<td>X motor control</td>
</tr>
<tr>
<td></td>
<td>Y motor control</td>
</tr>
<tr>
<td></td>
<td>Z motor control</td>
</tr>
</tbody>
</table>

L.3 Control Design

L.3.1 Hardware

Most of the control and interface electronics are mounted in the rack shown in Figure L.3. There are five panels on the front of the rack. From top to bottom they are: a motor position display, a computer interface, a manual control interface, a DC power supply for the electronics and the power supply/translator units for the microstepped motors driving the transducer. Two card cages are mounted in the upper rear portion of the rack (see Figure L.4). The upper cage contains the drivers for the LCD motor position displays and the lower contains all the cards for the stepper motor controls. There are a total of sixteen cards in the cages, seven in the display driver cage and nine in the motor control cage. Table L.1 lists the cards in the two cages by function.

Actuators

The scanner is a bifurcated structure. The main branch carries and positions the ultrasound transducer and has five independently controlled degrees-of-freedom. A chain-driven triad of ball screws vertically translate the transducer platform. Suspended from the platform are an orthogonal pair of lead screw slides giving two additional translational degrees of freedom. The transducer drive box is mounted at the center of the lower stage. It is directly driven in rotation about a vertical axis through the center of the translational stage. A second motor drives the transducer arm around an axis perpendicular to the vertical rotation axis, with the transducer sweeping through a plane containing the vertical rotation axis. The second branch carries the target specimen. It has one automatically actuated degree-of-freedom, rotation about a vertical axis in the center of the ball screw triad. An orthogonal pair of manually positioned translational slides is mounted on the rotary stage for alignment of the specimen in the machine coordinate
Figure L.3: Rack for control and interface hardware for the joint geometry scanning machine.
Figure L.4: Card cages for the scanner control electronics. The upper cage contains the display drivers and the lower cage all the cards needed for the stepper motor controls.
Table L.2: Stage Ranges of Travel.

<table>
<thead>
<tr>
<th>Branch</th>
<th>Variable</th>
<th>Control</th>
<th>Drive</th>
<th>Positive Range</th>
<th>Negative Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Z</td>
<td>Stepper</td>
<td>Ball Screw</td>
<td>44.45 cm</td>
<td>0.0 cm</td>
</tr>
<tr>
<td></td>
<td>X</td>
<td>Stepper</td>
<td>Lead Screw</td>
<td>15.24 cm</td>
<td>15.24 cm</td>
</tr>
<tr>
<td></td>
<td>Y</td>
<td>Stepper</td>
<td>Lead Screw</td>
<td>15.24 cm</td>
<td>15.24 cm</td>
</tr>
<tr>
<td></td>
<td>θ</td>
<td>Stepper</td>
<td>Direct</td>
<td>180 deg</td>
<td>180 deg</td>
</tr>
<tr>
<td></td>
<td>ψ</td>
<td>Stepper</td>
<td>Direct</td>
<td>180 deg</td>
<td>180 deg</td>
</tr>
<tr>
<td></td>
<td>φ</td>
<td>Stepper</td>
<td>Gear</td>
<td>180 deg</td>
<td>180 deg</td>
</tr>
<tr>
<td>2</td>
<td>z</td>
<td>Manual</td>
<td>Lead Screw</td>
<td>3.81 cm</td>
<td>3.81 cm</td>
</tr>
<tr>
<td></td>
<td>y</td>
<td>Manual</td>
<td>Lead Screw</td>
<td>3.81 cm</td>
<td>3.81 cm</td>
</tr>
</tbody>
</table>

Table L.3: Stage Resolutions.

<table>
<thead>
<tr>
<th>Branch</th>
<th>Variable</th>
<th>Motor</th>
<th>Motor Resolution (deg/step)</th>
<th>Motor Accuracy (%)</th>
<th>Resolution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Z</td>
<td>Slo-Syn M111-FD12</td>
<td>0.9</td>
<td>5</td>
<td>15.9 μm</td>
</tr>
<tr>
<td></td>
<td>X</td>
<td>Slo-Syn M062-FC09</td>
<td>0.9</td>
<td>3</td>
<td>1.6 μm</td>
</tr>
<tr>
<td></td>
<td>Y</td>
<td>Slo-Syn M062-FC09</td>
<td>0.9</td>
<td>3</td>
<td>1.6 μm</td>
</tr>
<tr>
<td></td>
<td>θ</td>
<td>PMI USS-72M</td>
<td>0.058</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ψ</td>
<td>PMI USS-72M</td>
<td>0.058</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>φ</td>
<td>Slo-Syn M062-FD09</td>
<td>.9</td>
<td>5</td>
<td>0.005 deg</td>
</tr>
</tbody>
</table>

system. The water bath holding the specimen sits on top of the crossed translational stages. Table L.2 presents the type of actuator, the drive mechanism and the ranges of travel for each motion axis (including the manually operated stages).

All of the automatically actuated stages are driven by stepper motors. The motor models, resolutions and accuracies are listed in Table L.3. Slo-Syn motors (Superior Electric Co., Bristol, CT.) drive the three translational stages and the rotary table. Each motor has a separate Slo-Syn MPS3000A Power Supply (Superior Electric Co., Bristol, CT) and Slo-Syn STM-103 Translator Module (Superior Electric Co., Bristol, CT). The power supply/translator pairs are all mounted on the bottom of the machine frame. All four motors are driven in half-step mode. The two motors driving the transducer arm are microsteppable disc motors by PMI (PMI Motors, Inc., Syosset, NY). Translator and power supply functions for each motor are combined in a Whedco SMD-1150 Stepping Motor Translator (Whedco, Inc., Ann Arbor, MI). Both motors are set to be driven in microstep mode with 32 steps-per-step yielding the angular resolution shown in Table L.3. The two microstepping translators are mounted on the back of a panel at the bottom of the electronics rack for the scanner.

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### Table L.4: Peripheral Device Installation Information.

<table>
<thead>
<tr>
<th>Device</th>
<th>Manufacturer</th>
<th>Part</th>
<th>Base Address</th>
<th>DMA Channel</th>
<th>Interrupt</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hard Disk Controller</td>
<td>Western Digital</td>
<td>Etherlink 3C501</td>
<td>300H</td>
<td>3</td>
<td>IRQ3</td>
</tr>
<tr>
<td>Ethernet</td>
<td>3Com</td>
<td>PDMA-16</td>
<td>310H</td>
<td>1</td>
<td>NA</td>
</tr>
<tr>
<td>DMA</td>
<td>Metabyte</td>
<td>PXB-721</td>
<td>330H</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>Digital I/O</td>
<td>QuaTech</td>
<td>PXB-721</td>
<td>340H</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>Digital I/O</td>
<td>QuaTech</td>
<td>RTI-815</td>
<td>350H</td>
<td>NA</td>
<td>IRQ2</td>
</tr>
<tr>
<td>A/D-D/A</td>
<td>Analog Devices</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Computer**

An AT&T 6300+ personal computer is used to control the scanner. It is based on the Intel 80286 microprocessor with the Intel 80287 floating point math coprocessor and one megabyte (1 Mb) of random access memory. There is a 40 Mb hard disk drive, divided into logical system, user and data partitions, and a 1.2 Mb floppy disk drive.

There are seven slots on the bus for peripherals. Three are nominally 16-bit slots for IBM PC-AT compatible peripherals, but the edge connector configuration is different from the PC-AT and it is difficult to find 16-bit boards to fill the slots. The sixteen bit slots may be used for standard 8-bit bus boards without any compatibility problems. Table L.4 shows the current set of peripherals in the slots with addresses and interrupt vectors.

One slot is used for the hard disk controller and one for the Ethernet communications board, which connects the computer to the local laboratory network. Communication is through a 3Com 3C501 Etherlink board (3Com Corp., Mountain View, CA), using TC/PIP software.

The DMA board (Metabyte Corp., Taunton, MA) is installed for transferring ultrasound data from the waveform recorder to the computer for display and processing. Maximum rate of transfer for an 8-bit word is 250000 Kb/sec. The handshaking scheme is discussed in the ultrasound hardware section below.

Two digital I/O boards are used for controlling both the ultrasound sampling and the motors. Each board has three Intel 8255 digital I/O chips. Each chip has three 8-bit ports, designated A, B and C. Addresses for each port and the control word for each 8255 are shown in Table L.5. Port A of each 8255 is configured as a bidirectional port with the upper half of port C (bits 5-8) used for handshaking to control access to the port. The lower half of port C and port B on each chip are available for use as ordinary digital inputs or outputs. The lower half of port C is configured as a digital input and used to transmit status signals from the stepper motor control board to the computer. On the first two 8255s (addresses 330H and 336H), port B is configured as a digital output and
Table L.5: Digital I/O Port and Control Word Addresses.

<table>
<thead>
<tr>
<th>Board</th>
<th>Chip</th>
<th>Port</th>
<th>Base Address</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>A</td>
<td>330H</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B</td>
<td>331H</td>
</tr>
<tr>
<td></td>
<td></td>
<td>C</td>
<td>332H</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>A</td>
<td>334H</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B</td>
<td>335H</td>
</tr>
<tr>
<td></td>
<td></td>
<td>C</td>
<td>336H</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>A</td>
<td>338H</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B</td>
<td>339H</td>
</tr>
<tr>
<td></td>
<td></td>
<td>C</td>
<td>33AH</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>A</td>
<td>340H</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B</td>
<td>341H</td>
</tr>
<tr>
<td></td>
<td></td>
<td>C</td>
<td>342H</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>A</td>
<td>344H</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B</td>
<td>345H</td>
</tr>
<tr>
<td></td>
<td></td>
<td>C</td>
<td>346H</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>A</td>
<td>348H</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B</td>
<td>349H</td>
</tr>
<tr>
<td></td>
<td></td>
<td>C</td>
<td>34AH</td>
</tr>
</tbody>
</table>

used to transmit the 16-bit command word to the waveform analyzer.

Stepper motor controls

The secondary objective of the design was to design scanner controls that were computer, in the sense of bus architecture, independent. There are several manufacturers supplying stepper motor control boards for specific bus architectures, especially the PC bus. Using such controllers would be easier to interface in the short-term, but would either tie all future computer upgrades to the PC bus or require a new investment in motor controllers and interface wiring with each improvement in computational power. In order to make the scanner controls independent, the lowest level of communication with the motor translators had to be separated from the computer.

Independence was achieved by designing stepper motor control boards centered around a commercially available stepper motor control chip, the Anaheim Automation SMC-25 (Anaheim Automation Co., Anaheim, CA). The SMC-25 is a 64-pin compressed DIP package. Either parallel or serial communication with a host are available; parallel communication was selected. Parallel communication is through an 8-bit bi-directional port. Jogging movement may be initiated manually. The chip also has an 8-bit input port and an 8-bit output port for communication with other devices. There are pins for inputs from both hard and soft limit switches.
For normal movements, the SMC-25 has a set of commands which may be transmitted through the parallel interface in the form of ASCII characters. The command set is summarized in Table L.6. Individual commands may be transmitted by the host or programs of up to 128 instructions downloaded to the chip’s onboard memory.

Once a move is initiated outputs from the SMC-25 are a direction bit and a train of rectangular pulses, with one pulse per step. Interface boards convert this to the format of the Whedco or Slo-Syn translators.

The stepper motor control boards are in the debugging stage. There is currently a problem in the communication between the boards and the computer.

Ultrasound

Most of the ultrasound equipment was inherited from the geometry scanning apparatus on the hip simulator [261]. Several focussed transducers are available. The one currently installed on the machine is a three millimeter diameter transducer (Panametrics, Waltham, MA). Pulse damping and intensity are controlled by a Panametrics PR5052 (Panametrics, Waltham MA). Pulse repetition rates are clocked with a square wave from a Tektronix FG502 Function Generator (Tektronix, Beaverton, OR). The analog echo waveforms are digitized with a Biomat 8100 (Gould Test & Instruments, Cupertino, CA). The Biomat can sample at up to 100 megahertz and has two kilobytes of onboard memory. All functions on the Biomat may be controlled either digitally, with a 16-bit input command word, or through front panel switches. Data from the Biomat are transmitted to the computer through the DMA interface. The connection is through a fifty pin shielded flat ribbon cable. The cable is terminated and signals buffered in the control box, which is connected to the DMA via a 37-pin cable with D-type connectors.

L.3.2 Kinematic Model

The structure of the scanner and the nature of the ultrasound scan are conducive to the development of a kinematic model of the system based on line and screw geometry. All of the actuators may be modeled as revolute or prismatic joints, which have simple instantaneous screw models. The relative position and orientation of the different actuator screws is known. The scan itself may be reduced to the alignment of two line vectors in space. The objective of the scan is to align the axis of the transducer and the local normal to the specimen surface, with the face of the transducer displaced from the target surface by the focal distance.

Yang [292, 294] introduced the notion of spatial polygons and developed a kinematic notation for describing spatial linkages in terms of screw and line geometry. This notation was applied to the development of a kinematic model of the scanner by Schwartz [222].
Table L.6: Stepper Motor Control (SMC-25) Command Set.

<table>
<thead>
<tr>
<th>Command</th>
<th>Number of Parameters</th>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>Set acceleration</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>Set base speed</td>
</tr>
<tr>
<td>CH</td>
<td>1</td>
<td>Set time to power save</td>
</tr>
<tr>
<td>CS</td>
<td>1</td>
<td>Save internal program</td>
</tr>
<tr>
<td>CR</td>
<td>1</td>
<td>Restore program</td>
</tr>
<tr>
<td>CX</td>
<td>1</td>
<td>Set crystal frequency</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>Set deceleration</td>
</tr>
<tr>
<td>E</td>
<td>1</td>
<td>Enter program</td>
</tr>
<tr>
<td>F</td>
<td>0</td>
<td>Wait for motor idle</td>
</tr>
<tr>
<td>G</td>
<td>0</td>
<td>Start indexing (move)</td>
</tr>
<tr>
<td>H</td>
<td>1</td>
<td>Look for home position</td>
</tr>
<tr>
<td>I/J</td>
<td>3</td>
<td>Jump to program step</td>
</tr>
<tr>
<td>L</td>
<td>1</td>
<td>Set slow jog speed</td>
</tr>
<tr>
<td>M</td>
<td>2</td>
<td>Loop</td>
</tr>
<tr>
<td>N</td>
<td>1</td>
<td>Set maximum speed</td>
</tr>
<tr>
<td>O</td>
<td>1</td>
<td>Set number of steps for move</td>
</tr>
<tr>
<td>P</td>
<td>2</td>
<td>Set output port</td>
</tr>
<tr>
<td>Q/R</td>
<td>1</td>
<td>Set absolute move</td>
</tr>
<tr>
<td>S</td>
<td>0</td>
<td>Ramp down motor and stop</td>
</tr>
<tr>
<td>T/U</td>
<td>0</td>
<td>Start program</td>
</tr>
<tr>
<td>V</td>
<td>0</td>
<td>Slew at max. speed</td>
</tr>
<tr>
<td>W</td>
<td>3</td>
<td>Trace program step</td>
</tr>
<tr>
<td>X</td>
<td>1</td>
<td>Branch until stopped</td>
</tr>
<tr>
<td>Z</td>
<td>1</td>
<td>Verify register contents</td>
</tr>
<tr>
<td>@/</td>
<td>n</td>
<td>Pause</td>
</tr>
<tr>
<td>+</td>
<td>0</td>
<td>Continue program</td>
</tr>
<tr>
<td>-</td>
<td>0</td>
<td>Set internal position counter</td>
</tr>
<tr>
<td>%</td>
<td>0</td>
<td>Deselect all but n axes</td>
</tr>
<tr>
<td>?/ .</td>
<td>0</td>
<td>Set direction positive</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>Set direction negative</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>Poll for messages</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>Device id and revision</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>Hard stop</td>
</tr>
</tbody>
</table>
Table L.7: Actuator Dual Angle Parameters.

<table>
<thead>
<tr>
<th>Branch</th>
<th>Actuator</th>
<th>( \theta_i )</th>
<th>( S_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0°</td>
<td>( S_{01} + n_1 p_{11} \Delta \theta_{11} )</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0°</td>
<td>( n_2 p_{22} \Delta \theta_{22} )</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0°</td>
<td>( n_3 p_{33} \Delta \theta_{33} )</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>( n_4 \Delta \theta_{44} )</td>
<td>( S_{44} )</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>( n_5 \Delta \theta_{55} )</td>
<td>( S_{55} )</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>( n_6 \Delta \theta_{66} N_{66} )</td>
<td>( S_{66} )</td>
</tr>
<tr>
<td></td>
<td>T</td>
<td>0°</td>
<td>( S_{TT} )</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>( n_T \Delta \theta_{TT} )</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>0°</td>
<td>( S_{8} )</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>( \theta_{99} )</td>
<td>( S_{9} )</td>
</tr>
<tr>
<td></td>
<td>N</td>
<td>0°</td>
<td>( S_{N} )</td>
</tr>
</tbody>
</table>

A schematic of the screw geometry of the scanner is shown in Figure L.5. Actuators\(^1\) are represented by screws, \( \hat{S}_i \), and the relative geometry of the links between any two screws is described by a screw along their common normal, \( \hat{a}_i \). The dual angle between two successive screws characterizes the link joining them. Similarly, each actuator is parameterized by the dual angle describing the allowable movement of the joint. Tables L.7 and L.8 list the dual angle components for the actuators and links, respectively.

Two adjacent actuator screws, \( \hat{S}_i \) and \( \hat{S}_j \), are related by a biquaternion:

\[
\hat{S}_j = Q_{ij} \hat{S}_i \tag{L.1}
\]

The biquaternion, \( Q_{ij} \), is a function of the dual angle parameterizing the intervening link:

\[
Q_{ij} = \cos \hat{a}_{ij} + \hat{a}_{ij}^* \sin \hat{a}_{ij} \tag{L.2}
\]

Similarly, the relative position and orientation of successive links are described by a biquaternion that is a function of the dual angle of the actuator connecting them:

\[
\hat{a}_{ik}^* = Q_{ik} \hat{a}_{ik} \tag{L.3}
\]

### L.3.3 Control Algorithm

The purpose of the scanner is to map the three-dimensional geometry of a general surface. All the actuators are open-loop. The only source of feedback information available in the

---

\(^1\)On examination the model seems to have one more screw than is needed to represent the six motor controlled pairs, two manually controlled pairs, surface normal and the transducer axis. The transducer position angle, \( \psi \), is driven through a bevel gear arrangement which was best represented by a pair of screws.
Figure L.5: Kinematic model of the joint geometry scanner.
Table L.8: Link Dual Angle Parameters.

<table>
<thead>
<tr>
<th>Branch</th>
<th>Link</th>
<th>$\alpha_{1}$</th>
<th>$\alpha_{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>01</td>
<td>$0^\circ$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>$90^\circ$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>23</td>
<td>$90^\circ$</td>
<td>$a_{23}$</td>
</tr>
<tr>
<td></td>
<td>34</td>
<td>$90^\circ$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>45</td>
<td>$0^\circ$</td>
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The present system is the ultrasound transducer. The proposed control algorithm is essentially open-loop, with the feedback used only in the final alignment of the transducer with the surface.

The scanning problem reduces to the alignment of the transducer axis and a normal to the surface. In terms of the kinematic notation, the objective is to close the spatial polygon by connecting the ends of the two branches, driving the dual angle between the transducer and the normal to $(-180^\circ + \epsilon 0)$. Two sets of measurements are essential to this statement of the positioning problem: the orientation and position of the transducer and the surface normal at the start of each move. The transducer can be located in the scanner coordinate system with careful bookkeeping. Determination of the normal to the surface at a scan point is more complicated.

In order to carry out the scan a method of estimating the normal to the surface at each scan point is needed. The proposed approach is to manually scan the surface geometry initially, sampling along the boundary and at the intersections of a coarse grid over the surface. All the surfaces of interest in the knee are generally smooth. The data points of the coarse scan would then be fit with a surface or surface patches. For the detailed scan, the normal to the surface at a particular point in the scan grid would then be estimated using the normal to the fit surface at the same point. With a means of estimating the positions and orientation of the surface normals an automatic, detailed scan can be performed.

A detailed mapping of the surface geometry of a joint may be broken into four phases: trajectory planning, moving, alignment and recording geometry data. The first phase is finding a solution to the equation

$$Q_{ON} \dot{S}_{N} = Q_{OT} \dot{S}_{T}$$

where the biquaternions are function of the number of steps for each motor. Some of the
criteria for a solution are minimum time to preserve the specimen and avoiding collision with the specimen or sides of the saline bath. Once the move is complete alignment may be required because the surface normal used in planning the move was estimated from the coarse scan. At the scan point the ultrasound can be used as feedback to insure that the estimated normal is the true normal and to check the height of the transducer above the cartilage surface.

L.3.4 Software

Several layers of software are required to implement the control algorithm and ultrasound scanning. Implementation has been hampered by the problems with the stepper motor controllers. There has been no working hardware on which to test the programs.

The most basic level of software is used for communication with the stepper motor control boards. An 80286 assembly language subroutine has been written for each of the SMC-25 commands (see Table L.6) and included in a library. There are both Fortran and C callable versions.

One layer above the stepper motor control functions are the actuator movement initiation routines. These convert a command to execute a movement of an actuator to a sequence of stepper motor control commands. There is one program for each of the motors in the system.

The next level of motor control software is task oriented. Programs are required to perform several tasks including: the preliminary coarse mapping of the surface and surface patch fitting, the trajectory planning and collision avoidance evaluation, and the execution of a movement. Instrumental to the trajectory planning and movement execution is a library of C functions to perform vector, quaternion and dual number manipulations. The function library has been written, but the higher level programs are unwritten at this writing.

A similar set of layered software governs the collection of ultrasound data and conversion of the ultrasound information to a mapping of the surface geometry. The basic level is an assembly language subroutine to control the generation of ultrasound data and the transfer of the data to the computer by DMA. Functionally this program is a duplicate of programs used with previous computers, although there are significant internal differences due to the difference in computer hardware. One level above this is a set of routines used to manipulate the ultrasound data and extract geometry information from it. Finally, a program is needed to control the scanning procedure, from sampling the ultrasound to generation of a three-dimensional point in scanner coordinates for the cartilage surface at each grid location.

Overseeing the task performance routines is a supervisory program. It is needed to coordinate all the stages of a complete surface mapping, from calibration to output of the surface geometry map.