A MODIFIED GRAMME-RING ARMATURE FOR A HIGH VOLTAGE
SUPERCONDUCTING ALTERNATOR

by

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ABSTRACT

The electrical design of a high voltage armature winding
for a superconducting alternator is discussed. Insulation
problems associated with more conventional winding geometries
are pointed out, and the toroidal geometry of the modified
Gramme-Ring winding is presented as a possible solution to
some of these problems. It is shown that high power density
machines of this type will require ferromagnetic cores,
which may be placed electrically near the potential of the
armature winding, through utilization of the insulated core
principle. Design equations for this class of toroidally
wound machines are developed from fundamental principles,
and the desirability of a highly conductive, image-current
magnetic shield is shown. These equations make use of the
results of a two-dimensional magnetic field analysis for an
alternator with a ring-wound armature. Expressions for
magnetic fields and machine inductances are listed in tables
in the appendix of this paper. The magnetic loading on the
core is also included in the appendix.

Design curves showing power rating, reactances, and
various operating losses are given for a 5 MVA machine.
The results of a conceptual design for a 2000 MVA machine,
developed to size a ring-wound machine of large rating,
are listed in a table.

An estimate of air gap electric fields in the spacing
between the rotor and armature, and the armature and shield,
is made by modeling the machine as a cylindrical capacitor.
The spatial distribution of armature potential is approximated
from a space phasor diagram, and this distribution is used
as a boundary condition in the determination of air gap fields.
Tables for air gap electric fields on both sides of the armature
are presented.

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TITLE: Assistant Professor of Electrical Engineering
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GLOSSARY OF SYMBOLS

Subscripts:

\( a \) armature winding
\( a \) phase A
\( b \) phase B
\( c \) phase C
\( c \) core
\( d \) damper shield
\( d \) direct axis
\( f \) field winding
\( i \) inner
\( o \) outer
\( s \) magnetic shield

Symbols:

\( B \) magnetic flux density
\( d_c \) core loss per unit mass
\( d_w \) armature wire diameter
\( E_f \) voltage behind synchronous reactance
\( E'_f \) voltage behind transient reactance
\( E''_f \) voltage behind subtransient reactance
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<tr>
<td>$E_{fr}$</td>
<td>rated voltage behind synchronous reactance</td>
</tr>
<tr>
<td>$F$</td>
<td>force density</td>
</tr>
<tr>
<td>$H$</td>
<td>magnetic field intensity</td>
</tr>
<tr>
<td>$i$</td>
<td>current</td>
</tr>
<tr>
<td>$I_a$</td>
<td>rated armature current (RMS)</td>
</tr>
<tr>
<td>$I_{foc}$</td>
<td>open (armature) circuit field current</td>
</tr>
<tr>
<td>$J$</td>
<td>current density</td>
</tr>
<tr>
<td>$J_{foc}$</td>
<td>open circuit field current density</td>
</tr>
<tr>
<td>$J_{fr}$</td>
<td>rated field current density</td>
</tr>
<tr>
<td>$K$</td>
<td>surface current density</td>
</tr>
<tr>
<td>$k$</td>
<td>wavenumber (integer)</td>
</tr>
<tr>
<td>$l$</td>
<td>straight section length</td>
</tr>
<tr>
<td>$l_a$</td>
<td>armature electrical length</td>
</tr>
<tr>
<td>$l_f$</td>
<td>field electrical length</td>
</tr>
<tr>
<td>$l_m$</td>
<td>effective length for field-to-armature mutual inductance</td>
</tr>
<tr>
<td>$l_d$</td>
<td>damper electrical length</td>
</tr>
<tr>
<td>$l_{ad}$</td>
<td>effective length for armature-to-damper mutual inductance</td>
</tr>
<tr>
<td>$l_{fd}$</td>
<td>effective length for field-to-damper mutual inductance</td>
</tr>
<tr>
<td>$L_{ai}$</td>
<td>inner armature self inductance</td>
</tr>
<tr>
<td>$L_{ao}$</td>
<td>outer armature self inductance</td>
</tr>
<tr>
<td>$L_a$</td>
<td>armature self inductance</td>
</tr>
<tr>
<td>$L_{ab}$</td>
<td>armature phase-to-phase mutual inductance</td>
</tr>
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<td>Symbol</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
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<tr>
<td>$L_{adn}$</td>
<td>armature-to-damper mutual inductance</td>
</tr>
<tr>
<td>$L_f$</td>
<td>field self inductance</td>
</tr>
<tr>
<td>$L_{fd}$</td>
<td>field-to-damper mutual inductance</td>
</tr>
<tr>
<td>$L_{sd}$</td>
<td>damper self inductance</td>
</tr>
<tr>
<td>$M$</td>
<td>armature-to-field mutual inductance (for two parallel connected armature windings)</td>
</tr>
<tr>
<td>$M_w$</td>
<td>armature-to-field mutual inductance (for a single armature winding)</td>
</tr>
<tr>
<td>$N_{at}$</td>
<td>number of turns in armature winding</td>
</tr>
<tr>
<td>$N_{ft}$</td>
<td>number of turns in field winding</td>
</tr>
<tr>
<td>$p$</td>
<td>number of pole pairs</td>
</tr>
<tr>
<td>$P_{va}$</td>
<td>machine rating in volt-amperes</td>
</tr>
<tr>
<td>$P_a$</td>
<td>conduction losses in the armature</td>
</tr>
<tr>
<td>$P_c$</td>
<td>core loss</td>
</tr>
<tr>
<td>$P_{ec}$</td>
<td>armature eddy-current loss</td>
</tr>
<tr>
<td>$P_{sh}$</td>
<td>magnetic shield dissipation</td>
</tr>
<tr>
<td>$r$</td>
<td>radius</td>
</tr>
<tr>
<td>$R$</td>
<td>radius (current sheets)</td>
</tr>
<tr>
<td>$R_{fi}$</td>
<td>field winding inner radius</td>
</tr>
<tr>
<td>$R_{fo}$</td>
<td>field winding outer radius</td>
</tr>
<tr>
<td>$R_t$</td>
<td>damper shield outer radius</td>
</tr>
<tr>
<td>$R_{ai1}$</td>
<td>inner armature inner radius</td>
</tr>
<tr>
<td>$R_{ao1}$</td>
<td>inner armature outer radius</td>
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\( R_{ci} \) core inner radius
\( R_{co} \) core outer radius
\( R_{aio} \) outer armature inner radius
\( R_{aoo} \) outer armature outer radius
\( R_s \) magnetic shield radius
\( t \) time
\( t_c \) core thickness
\( T_{mn} \) \( mn \) th component of Maxwell Stress Tensor
\( v_a \) phase A armature voltage
\( V_{toe} \) rated terminal voltage
\( V_n \) normalized armature voltage
\( X \) reactance (ohms)
\( x_d \) per unit direct axis synchronous reactance
\( x_d' \) per unit direct axis transient reactance
\( x_d'' \) per unit direct axis subtransient reactance
\( x_q \) per unit quadrature axis synchronous reactance
\( x_q' \) per unit quadrature axis transient reactance
\( x_q'' \) per unit quadrature axis subtransient reactance
\( x_i = R_{aii}/R_{aoo} \) inner armature radius ratio
\( x_o = R_{aio}/R_{aoo} \) outer armature radius ratio
\( y = R_{fi}/R_{fo} \) field radius ratio
\( z = (R_{co}/R_s)^{2np} \) boundary radius ratio (outer armature)
Greek symbols:

\( \alpha \)  angular measure from the field winding axis

\( \gamma \)  initial rotor angle measured from the axis of phase A

\( \delta \)  correction factor for electrical length

\( \delta_s \)  skin depth of the magnetic shield

\( \varepsilon \)  permittivity

\( \eta \)  efficiency

\( \Theta \)  angular measure

\( \Theta_{wfe} = p \Theta_{wf} \)  field winding angle

\( \Theta_{vae} = p \Theta_{va} \)  armature winding angle

\( \lambda \)  flux linkage

\( \lambda_p \)  armature packing factor

\( \mu \)  permeability

\( \rho_c \)  core mass density

\( \sigma \)  armature conductivity

\( \sigma_s \)  magnetic shield conductivity

\( \tau \)  traction

\( \phi \)  angular displacement of the field winding axis from the axis of phase A

\( \Phi \)  potential

\( \phi \)  flux

\( \psi \)  armature current phase angle

\( \omega \)  angular frequency
I. INTRODUCTION

The need for a high permeability magnetic circuit in conventional turbine generators limits the space available to armature conductors. These conductors are placed in slots along the inner surface of the stator iron, requiring each conductor to be insulated for full machine potential. Since the number of armature turns is constrained by the space available to conductors, terminal voltages for large machines (1000 MVA) are relatively low, on the order of 24-32 kV.

The application of high field superconductors to the field windings of turbine generators eliminates the need for a stator magnetic circuit in most alternator designs. \((1,2,4)\) By utilizing a controlled gradient winding scheme, in which bars physically close together are made electrically close together, conducting bars no longer have to be insulated for full machine potential. Bars must be insulated only from neighboring bars, allowing for a significant increase in the conductor to insulation ratio.

One benefit of an increased terminal voltage is the reduction of heating losses in the isolated phase bus ducts and power transformers. Another, more important, benefit may be the elimination of unit transformers altogether, by designing for terminal voltages equal to standard
transmission line voltages. High voltage machines may be tied directly to generating station switchgear, where the machine takes on the duties of both a generator and a power transformer. Such a machine would be favorable in situations where the size or weight of conventional apparatus is of paramount importance. Also, the stability characteristics of a high voltage machine may be better than those of a conventional generator-transformer set.

Problems in the design of a high voltage armature center around the choice of armature materials and their spatial configuration. Armature materials will be subjected to high potential gradients and periodic electromagnetic loadings. The spatial nature of the resulting electrical and mechanical stresses depends to a large degree upon armature geometry. For instance, magnetic forces on the armature windings tend to structurally weaken the armature insulation, which can lead to eventual dielectric failure. It is important to choose a geometry for a high voltage armature so that the duty on insulation is minimized as much as possible.

The geometry of a high voltage winding must satisfy certain basic electrical rules, if the winding is to be operational for long periods of time. First, the winding should be able to distribute voltages among turns in such
a way that bar-to-bar insulation is never subjected to excessive electrical stresses. Second, conducting bars should have relatively smooth surfaces, so that bar insulation sees as uniform an electric field as possible. Third, abrupt bends in the winding, resulting in sharp corners, should be avoided. And finally, crossing conductors of large potential difference should also be avoided. Otherwise, corona resulting from electric field concentrations will result in the degradation and eventual failure of armature insulation.

An air core armature for a superconducting alternator \((1,2)\) has been developed at M.I.T. In this armature, a two layer straight section is split into a four layer end turn region. Each layer in the end turns is bent in a helical path, and at one end of the machine, opposing layers spiral in opposite directions. This means that conductors of high potential difference cross each other in different layers. Crossing points are regions of high electric field concentration, and hence, possible failure points. A high voltage armature winding of this configuration will have large insulation requirements in the end turn regions. This fact has led to the study of alternative armature geometries, one of which is the subject of this thesis.
II. A MODIFIED GRAMME-RING ARMATURE

An effort to reduce the insulation problems associated with more conventional winding geometries has led to a consideration of a variation on the Gramme-Ring spiral winding. The original Gramme-Ring geometry is illustrated in figure 1.

A delta-connected high voltage armature, wound in a toroidal configuration, has advantages over more conventional winding geometries, which arise from the simplicity of its end turns. Perhaps the most significant advantage is the lack of crossing conductors; conductors see only their neighbors, which are at slight potential differences.

The winding to be discussed in this paper differs from the standard Gramme-Ring, and it is appropriate to clarify the differences between a "modified" Gramme-Ring and a conventional Gramme-Ring winding.

The winding under consideration is modified by the manner in which armature phase belts are wrapped around the core. In fact, this winding is not a Gramme-Ring at all, since each phase of the modified version is made up of opposing spiral windings. That is, the sense of helical direction changes from clockwise to counterclockwise, and vice-versa, as one proceeds around the armature in an azimuthal direction. A Gramme-Ring is spoken of in the
FIG. 1. GRAMME-RING SPIRAL WINDING
sense of a classification, not an actual geometry.

The high voltage armature under discussion will be wound and connected as shown in figure 2. This winding is delta-connected, and phase belts 180° apart in space are connected in parallel. Magnetic flux from the field winding links the armature winding in an azimuthal sense, being guided along the low reluctance path of the iron core. It should be noted, however, that a modified Gramme-Ring will work with or without a ferromagnetic core. The purpose of the ferromagnetic core is to provide a high power density machine, by increasing armature-field mutual inductance.

The geometry of the modified Gramme-Ring has advantages over that of a conventional spiral winding. For example, a three phase machine could be built by symmetrically tapping a conventional Gramme-Ring at points 120° apart in space. However, one disadvantage of this machine would be its poor breadth factor. Another disadvantage is the fact that the asymmetry of the phase belts about the longitudinal axis would produce unsymmetrical mechanical side loadings under conditions of electrical unbalance.

There are other possible winding schemes, but future discussions will be limited to the parallel-delta-connected, modified Gramme-Ring shown in figure 2. This winding
FIG. 2. A PARALLEL-DELTA CONNECTED MODIFIED GRAMME-RING WINDING.
satisfies most of the standards set forth for a high voltage armature: it has no crossing conductors, few corners, and adjacent conductors are electrically close together.

III. A FERROMAGNETIC CORE FOR A HIGH VOLTAGE ARMATURE WINDING

A ferromagnetic core for a high voltage winding enhances armature-field magnetic coupling and allows for more compact machines. Trial calculations have shown that air core machines will have two to three times the volume of iron core machines of equivalent rating. Unless otherwise stated, future discussions will be restricted to higher power density iron core machines.

There are two basic core configurations: grounded or floating. Both are feasible, but a disadvantage of a grounded core is the amount of insulation necessary to insulate the core from the high voltage armature. A floating core, on the other hand, shows promise for more compact and possibly more reliable machines, if the insulated core principle is used. This principle involves the slicing of an iron core into segments, between which are sandwiched layers of insulation. The insulated core principle has been applied successfully in an experimental 100 MVA, 765 kV (8,9) shunt reactor on the American Electric Power 765 kV system.
It should be possible to build a cylindrical core using insulated core segments. A core for a high voltage machine could be built from stacks of thin wedge-shaped laminations, the stacks being electrically isolated from one another by planar insulating strips. Conducting bars could be placed on the stacks of core segments, and the segments brought to bar potential by electrically shorting the bars to the core segments. Figure 3 illustrates the concept.

The insulating strips shown in figure 3 run the axial length of the core, separating stacks of laminated segments. Each segment in a stack is shorted to an armature bar, bringing the entire stack up to bar potential. For example, series connections are made between bars 1-2-3-4. By tying bar 2 to each segment in a stack, the amount of insulation between each of the four bars and the stack would be practically negligible, compared to the amount necessary if the stack were grounded. Also, the electric fields in the planar insulating strips between core stacks will be nearly uniform, if the gaps between stacks are of uniform dimension.

The number of segments necessary to build this type of core depends upon the size of the machine and the size of each segment. For a modified Gramme-Ring with ten turns per phase belt, there would be a total of sixty armature turns.
FIG. 3. END VIEW OF A CIRCUMFERENTIALLY SEGMENTED INSULATED CORE.
This would yield thirty core segments per annular ring, if an arrangement of two turns per segment were chosen. The number of segments per stack would depend upon the length of the machine. A machine ten feet long, having segments 200 mils thick, would require 600 segments per stack. Thirty stacks of 600 segments each gives a total of 18,000 segments per machine.

Careful consideration will have to be given to the binding of this core. It has been proposed to tightly wrap the core structure with a fiberglass banding, so that the core segments are always under compression, and the fiberglass always under tension. The core would be held together like a barrel, with the staves analogous to the stacks of laminations, and the hoops analogous to the fiberglass wrapping.

The core structure will be subjected to two disruptive loadings: gravitational and magnetic. Gravitational loading is constant with time, and would tend to cause sagging of the core. The magnetic loading is a double power frequency attractive loading, and would tend to ovalise the core. That is, magnetization forces on the core would tend to give it an elliptical shape, such that the minor axis of the ellipse would be in line with the direct axis of the field winding. This elliptical shape would rotate at synchronous
speed with the rotor, subjecting the insulating sheets between core stacks to double power frequency bending stresses. A preliminary analysis of the magnetic core loading is made in Appendix II.

IV. A MODEL FOR A HIGH VOLTAGE MACHINE

The electrical performance of a superconducting alternator with a modified Gramme-Ring armature may be studied by modeling the machine in two dimensions, neglecting end effects. For a toroidal winding, this is probably a good approximation, because end turn lengths are small in comparison to the active length of the machine. The physical configuration of the machine is shown in figures 4 and 5. A two pole machine is shown, although multipolar machines are certainly possible.

Figure 4 represents an end-view cross section of the machine. Its electrical performance is characterized by a field winding, an electrical damper shield, an "inner" armature, a ferromagnetic core, an "outer" armature; and a magnetic shield. These machine elements are concentrically nested, one inside the other. It is assumed that the field and armature current densities are uniform over their respective phase belts, and that the core has infinite
permeability. The outer magnetic shield will be taken as a smooth, diamagnetic "image" shield of perfect conductivity, since a ferromagnetic shield produces a high reactance machine. That is, the best location for an iron magnetic shield, from a reactance viewpoint, is at infinity. The best location for an image shield is as close to the outer armature as possible, the two constraints on location being eddy-current heating in the shield, and possible dielectric breakdown of the insulation between the shield and the outer armature.

The constraints on the inner armature, electrical shield, and field winding dimensions are mostly mechanical. Strength and cooling requirements restrict these radial dimensions to certain allowed intervals. Another constraint, imposed by the high voltage armature winding, is on the air gap between the rotor surface and the inner armature. The size of the air gap is constrained by the electric gradient between the high voltage armature and grounded rotor. This means that high voltage machines will tend to have large air gaps, lowering armature-field magnetic coupling.

Figure 5 shows a longitudinal cross section of the machine. The end turns have been rounded to reduce the electric field strengths at the corners.
FIG. 5. LATERAL CROSS-SECTION OF A MODIFIED GRAMME-RING ALTERNATOR.

- Image shield
- Outer armature
- Inner armature
- Core lamination
- End turn
- Field winding
V. MACHINE DESIGN FORMULAS

Given the geometry of a machine, the next step in the analysis of its electrical performance is to obtain expressions for machine inductances. Inductances are calculated from a knowledge of the magnetic field distribution, which is a function of the current distribution. In the two-dimensional model under consideration, machine currents are uniformly distributed in both the field and armature windings. The magnetic field intensities due to current flow patterns can be found by solving a magnetostatics problem. Under the assumptions of boundaries with infinite conductivity and permeability, the fields problem is linear, and superposition principles may be applied.

It should be noted that a two-dimensional model of a modified Gramme-Ring machine does not differ very much from a two-dimensional model of an air core machine with a ferromagnetic shield. In fact, the only difference is the presence of the outer armature, sandwiched between the core and image shield. This may seem like a significant difference, but it isn't, because the primary effect of the outer armature is to increase the self inductance of the inner armature. That is, for purposes of analysis the outer armature can be thought of as and turns.
As far as the inner armature and field winding are concerned, the outer armature is a separate, decoupled winding, adding only to the inner armature self inductance. The infinitely permeable core prevents inductive coupling between inner and outer windings on opposite sides of the core. The field analysis of this machine is similar to that of a conventional air core machine, if care is taken to treat the core as a ferromagnetic shield and the outer armature as end turns. An outline of the field analysis, along with tables of magnetic field expressions and inductances, is included in Appendix I.

Figure 6 is a schematic electrical representation of a superconducting alternator with a modified Gramme-Ring armature. This representation is identical to that of a conventional air core machine, in that it contains a superconducting field winding, three armature windings, and two damper windings which represent the electrical shield. Each of the three armature windings represents a pair of diametrically opposite, parallel-connected windings on the modified Gramme-Ring. Rather than analyze a six winding machine, it is easier to treat opposite pairs as the branches of a single winding.
FIG. 6. SCHEMATIC ELECTRICAL REPRESENTATION OF A SUPERCONDUCTING ALTERNATOR WITH A MODIFIED GRAMME-RING ARMATURE.
Since there are no effects of saliency in this machine, self inductances are independent of rotor position. The angle $\phi$ in figure 6 represents the instantaneous angle between the direct axis of the field winding and the magnetic axis of phase A. Motor variables are shown.

Assuming that mutual inductances between armature and rotor coils vary sinusoidally with angle $\phi$, a flux linkage-current relationship for the machine may be written in matrix form:

$$\begin{bmatrix}
\lambda_a \\
\lambda_b \\
\lambda_c \\
\lambda_f \\
\lambda_{sd} \\
\lambda_{sq}
\end{bmatrix} = \begin{bmatrix} M \end{bmatrix} \begin{bmatrix}
i_a \\
i_b \\
i_c \\
i_f \\
i_{sd} \\
i_{sq}
\end{bmatrix} \quad (1)$$

where the matrix $[M]$ is given by

$$\begin{bmatrix}
L_a & L_{ab} & L_{ab} & M \cos \phi & L_{dn} \cos \phi & L_{dn} \sin \phi \\
L_{ab} & L_a & L_{ab} & M \cos(\phi - \frac{2\pi}{3}) & L_{dn} \cos(\phi - \frac{2\pi}{3}) & L_{dn} \sin(\phi - \frac{2\pi}{3}) \\
L_{ab} & L_{ab} & L_a & M \cos(\phi + \frac{2\pi}{3}) & L_{dn} \cos(\phi + \frac{2\pi}{3}) & L_{dn} \sin(\phi + \frac{2\pi}{3}) \\
M \cos \phi & M \cos(\phi - \frac{2\pi}{3}) & M \cos(\phi + \frac{2\pi}{3}) & L_f & L_{fd} & 0 \\
L_{dn} \cos \phi & L_{dn} \cos(\phi - \frac{2\pi}{3}) & L_{dn} \cos(\phi + \frac{2\pi}{3}) & L_{fd} & L_{sd} & 0 \\
L_{dn} \sin \phi & L_{dn} \sin(\phi - \frac{2\pi}{3}) & L_{dn} \sin(\phi + \frac{2\pi}{3}) & 0 & 0 & L_{sd}
\end{bmatrix}$$
The inductances found from the magnetic field analysis are the coefficients of this relationship. The self inductance of each armature phase, \( L_a \), is equal to one half of the self inductance of either of the parallel windings comprising that phase.

The mutual inductance between the armature and field has been calculated under the assumption that the inner armature forms a coil which couples to the field winding. When calculating mutual flux linked by this coil, an open surface bounded by the inner armature and inner surface of the core is chosen. Only the radial component of the magnetic field intensity passes through this surface. When the inner armature and field windings are in alignment, it is reasonable to assume that the flux linking the inner armature divides evenly between branches of the core. Hence, it can be argued that \( M \), the mutual inductance between the inner armature and field, is twice the mutual inductance between either component winding and the field.

Under balanced polyphase conditions, the machine may be represented by a single phase equivalent circuit using some of the inductances listed in (1). This equivalent circuit will be developed using phase \( A \) variables.

In the steady state, armature currents form a balanced three phase set, and the rotor increases at a uniform rate.
Thus,

\[ i_a = \sqrt{2} I_a \cos(\omega t - \psi) \]
\[ i_b = \sqrt{2} I_a \cos(\omega t - \frac{2\pi}{3} - \psi) \]
\[ i_c = \sqrt{2} I_a \cos(\omega t + \frac{2\pi}{3} - \psi) \]
\[ \phi = \omega t + \gamma \]

where \( I \) is the RMS value of the current, and \( \gamma \) is the initial rotor angle. From (1), the flux linkage with phase A becomes

\[ \lambda_a = (L_a - L_{ab}) \sqrt{2} I_a \cos(\omega t - \psi) + M I_f \cos(\omega t + \gamma) \]

(3)

Phase A consists of two parallel windings. The flux linking either half of phase A is

\[ \lambda_{va} = \frac{1}{2} \lambda_a \]

(4)

Switching to generator variables by replacing \( I \) by \(-I_a\), ignoring resistance drops, and differentiating (4) to find terminal voltage yields

\[ \mathcal{V}_a = \omega \frac{1}{2}(L_a - L_{ab}) \sqrt{2} I_a \sin(\omega t - \psi) - \frac{1}{2} \omega M I_f \sin(\omega t + \gamma) \]

(5)

Equation (5) is the basis of a single phase steady state circuit model for the machine. The internal voltage, generated by the rotation of the field winding, has an amplitude given by \( \frac{1}{2} \omega M I_f \). The effects of armature reaction
on one winding are given by the reactive voltage drop
\[ \omega (L_a - L_{ab}) \sqrt{2} (\frac{\Psi}{2}) \].

Defining the magnitude of RMS internal voltage as
\[ E_r = \frac{\omega M_w I_f}{\sqrt{2}} \]  \hspace{1cm} (6)
where \( M_w \) is the mutual inductance between one armature winding and the field winding, and synchronous reactance as
\[ X_d = \omega (L_a - L_{ab}) \]  \hspace{1cm} (7)

the model of a synchronous machine as a voltage source behind a reactance becomes apparent. Figure 7 shows the single phase equivalent circuit, and figure 8 the steady state phasor diagram representing this circuit.

If a machine is running open circuited, then the terminal voltage is equal to the internal voltage. With this in mind, rated terminal voltage can be defined in terms of an open circuit field current as
\[ V_{toc} = \frac{\omega M_w I_{foc}}{\sqrt{2}} \]  \hspace{1cm} (8)
Based on experience with presently available superconductors, the open circuit field current necessary to produce rated terminal voltage will be on the order of 600 amperes.

Since the field current density is given by
FIG. 7. SINGLE PHASE EQUIVALENT CIRCUIT.

FIG. 8. STEADY STATE PHASOR DIAGRAM
\[ J_f = \frac{2N_{ft} I_f}{\theta_{wfe} R_{fo}(1-y^2)} \]  

(9)

(8) becomes

\[ V_{toc} = \frac{\omega M_w J_{foc} \theta_{wfe} R_{fo}^2 (1-y^2)}{2(2) N_{ft}} \]  

(10)

The three phase rating in volt-amperes is given by

\[ P_{va} = 3V_{toc} I_a \]  

(11)

The armature current density is limited by armature heating, and for phase A is given by

\[ J_a = \frac{2 N_{at} I_a}{\theta_{wa} R_{aoi}^2 (1-x_1^2)} \]  

(12)

Equation (12) has been written in terms of inner armature dimensions. Outer armature dimensions also could be used if inner and outer armature current densities are taken to be the same.

Combining (10), (11), and (12), and taking the fundamental component of the mutual inductance listed in Table VII, the volt-ampere rating becomes

\[ P_{va} = \frac{24\omega \mu_0 J_a J_{foc} R_{fo}^{2+ph} R_{aoi} 2-ml}{\sqrt{2} p \pi} \]  

(13)
The rated field current density can be related to the field current density necessary to produce rated open circuit terminal voltage by using the phasor diagram of figure 8. That is,

$$J_{fr} = \left( \frac{E_{fr}}{V_{toc}} \right) J_{foc}$$  \hspace{1cm} (14)

where $E_{fr}$ is rated internal voltage, and $J_{fr}$ is rated field current density. From figure 8,

$$E_{fr}^2 = (V_{toc}^2 + X_d I_a \sin \psi)^2 + (X_d I_a \cos \psi)^2$$  \hspace{1cm} (15)

Dividing both sides by $V_{toc}^2$, and recognizing that $V_{toc}/I_a$ defines a base impedance for this machine, i.e.,

$$\frac{X_d I_a}{V_{toc}} = x_d \quad \text{ (per unit)}$$  \hspace{1cm} (16)

one may write (14) as

$$J_{fr} = J_{foc} \sqrt{(1 + x_d \sin \psi)^2 + (x_d \cos \psi)^2}$$  \hspace{1cm} (17)

This equation defines the operational field current density in terms of the open circuit field current density, the per unit synchronous reactance, and the armature current angle $\psi$. The power factor of the machine is defined by $\cos \psi$. 
The synchronous reactance of this machine is given in (6). It is proportional to the difference between the self inductance of an armature phase and the mutual inductance between armature phases. For a three phase machine, the mutual inductance $L_{ab}$ can be approximated by

$$L_{ab} = -\frac{1}{2} L_a$$  \hspace{1cm} (18)

This can be seen by exciting a given armature phase and calculating the flux linking an adjacent armature phase. It is assumed that flux produced by each phase leaves the core and travels some distance through the air, accounting for the imperfect coupling between phases (two diametrically opposite windings constitute a phase).

Using (18), the expression for synchronous reactance becomes

$$X_d = \frac{3}{2} \omega L_a$$  \hspace{1cm} (19)

The self inductance $L_a$ is taken to be the sum of the self inductances of the inner and outer armatures. That is,

$$L_a = L_{a1} + L_{a0}$$  \hspace{1cm} (20)

where $L_{a1}$ and $L_{a0}$ can be found in Table VI. Through manipulation, (20) can be put in the form:

$$L_a = L_{a1} (1 + \delta)$$  \hspace{1cm} (21)
where $\delta$ is a rather complex constant whose value depends upon the geometry of the entire armature and magnetic shield. Equation (21) supports the idea of the outer armature as end turns in the two dimensional model. The ring-wound machine can be modeled in a conventional sense, and the factor $(1 + \delta)$ used as a correction factor for length. If $l$ is the physical straight section length of the inner armature, then $l(1 + \delta)$ can be taken as the electrical length due to the effects of the outer armature. Writing an expression for electrical length as

$$l_a = l + \Delta l$$  \hspace{1cm} (22)

it can be seen that the incremental length $\Delta l$ is given by

$$\Delta l = l\delta$$  \hspace{1cm} (23)

This can be taken as a rule for the effective length of the end turns of a modified Gramme-Ring armature.

The parameter $\delta$ can be used to illustrate the advantages of magnetic shielding using induced eddy-currents. An expression for $\delta$ is given by

$$\delta = \left(\frac{1 - x_1^2}{1 - x_o^2}\right)^2 \frac{C_{xo}}{C_{x1}} \frac{1}{s + 1}$$  \hspace{1cm} (24)

where $C_{x1}$ and $C_{xo}$ are constants depending upon armature
The $\mp$ signs in the expression $(s \mp 1)$ denote the alternatives of ferromagnetic versus diamagnetic shielding. The negative sign is for an iron shield, while the positive sign is for an image shield. For reasonable values of shield radius, $R_s$, the number $z$ is fairly close to 1. This says that $\delta$ for iron shielded machines will be considerably larger than $\delta$ for image shielded machines; on the order of 20 to 1.

The reactance of a winding depends upon the nature of the material near the winding. Surrounding the outer armature with iron boundaries increases the flux linkage with this winding, and hence its reactance. However, if one of the boundaries is a highly conductive material, such as aluminum, the flux linkage and reactance of the winding are reduced.

Considering an image shielded machine, the per unit synchronous reactance can be made less than one, through proper design. This reactance has been defined by (16). Substituting the value for $X_d$ given in (19) yields

$$x_d = \frac{(3/2) \omega L_a I}{V_{toe}}$$

or, using (21)

$$x_d = \frac{3 \omega L_a L_{ai}(1 + \delta)}{2 V_{toe}}$$

Making use of (10), (12), and expressions for $L_a$ and $M_w$
found in Table VI, the per unit synchronous reactance becomes \((n=1)\)

\[
x_d = \frac{3\sqrt{2}}{4} \frac{J_o C_s p \sin(\theta_{wfe}/2)}{J_{loc} C_p \sin(\theta_{wfe}/2)} (1+\delta) \left( \frac{R_{aoi}}{R_{fo}} \right)^2 \rho
\]  

(28)

where

\[
\frac{4}{p} \frac{1-x_1^2 p + \left( \frac{2-p}{2+p} \right)}{4 - p^2} \frac{(1-x_1^2 p^2) (R_{aoi}/R_{ci})^{2p}}
\]

(29)

\[
C_{s p} = \frac{(p-2) + 4x_1^{p+2}-(p+2)x_1^4 + 2 \left( \frac{p-2}{p+2} \right)}{p^2 - 4} \frac{(R_{aoi}/R_{ci})^{2p}}
\]

(30)

Expressions for transient and subtransient reactances can be obtained from a d-q transformation of the three phase flux linkage-current relationship given in (1). That is, the transient analysis of a machine is greatly simplified by transforming armature variables into a synchronously rotating reference frame. One possible transform pair is given in reference (2).

Transient reactance may be calculated by assuming that the field winding traps flux. The direct axis transient reactance reads

\[
x'_d = \left[ (L_a - L_{ab}) - (3/2) \frac{M^2}{L_f} \right] \omega
\]

(31)

The quadrature axis transient reactance must equal the
direct axis synchronous reactance.

\[ X'_q = X'_d = X_d \quad (32) \]

The per unit direct axis transient reactance is found by normalizing (31) to the base impedance of the machine.

\[ x'_d = x'_d I_a / V_{toe} = x_d \left[ 1 - \frac{2}{L_a L_p} \right] \quad (33) \]

Making the appropriate substitutions, (33) becomes

\[ x'_d = x_d \left[ 1 - \frac{4(1-y^3)^2 C_p^2}{C_{sp} C_{fp} (1+\delta)} \left( \frac{R_{fo}}{R_{aoi}} \right)^2 \right] \quad (34) \]

where \( C_{fp} \) is identical to \( C_{sp} \) when the substitution of \( y \) for \( x_1 \) is made. Subtransient reactance is calculated by assuming that the damper windings, and hence all rotor windings, trap flux. An expression for the direct axis subtransient reactance is given by

\[ X''_d = \left[ (L_a - L_{ab}) - (3/2) \frac{L_{adn}}{L_{sd}} \right] \omega \quad (35) \]

Since the electrical shield is a cylindrical tube, the damper winding is symmetrical, and

\[ X''_q = X''_d \quad (36) \]

The per unit subtransient reactance is given by
\[ x''_d = \frac{3\omega L I_{a\alpha}}{2 V_{toc} \left( 1 - \frac{L_{adn}}{L_a L_{sd}} \right)} \]  \hspace{1cm} (37)

or, for \( n = 1 \)

\[ x''_d = x_d \left[ 1 - \frac{2(2p)^2 c_p^2}{(1+\delta) c_{sp} c_{dl}} \right] \]  \hspace{1cm} (38)

where

\[ c_{dl} = 1 + (R_t/R_s)^{2p} \]  \hspace{1cm} (39)

The voltages behind the transient and subtransient reactances can be found from a phasor diagram showing transient and subtransient internal voltages and reactances. Defining \( E'_t \) as the voltage behind the transient reactance and \( E''_t \) as the voltage behind the subtransient reactance, one may write

\[ E'_t = V_t \sqrt{(1-x'_d \sin \psi')^2 + (x'_d \cos \psi')^2} \]  \hspace{1cm} (40)

\[ E''_t = V_t \sqrt{(1-x''_d \sin \psi'')^2 + (x''_d \cos \psi'')^2} \]  \hspace{1cm} (41)

An evaluation of machine performance is incomplete without estimates of power losses in the armature, core, and shield. Armature losses have three components: circulating current losses, eddy current losses, and Joule heating.
losses. Since the transposition scheme for the bars of a modified Gramme-Ring has yet to be worked out, an estimate of circulating current losses is impractical. Effective determination of these losses requires experimental measurement. Joule heating, on the other hand, can be calculated from a knowledge of armature geometry, rated current density, and conductivity. That is, the heating in the armature may be written as

$$P_a = \frac{3J_a^2 \ell \theta_{\text{vac}} (1-x_i^2) R_{aoi}^2}{\sigma \lambda_p} \tag{42}$$

In (42), $\ell$ is the total length of one armature winding. For a winding with square corners:

$$\ell = 2(l + R_{a00} + R_{a01} + R_{a01} - R_{a11}) \tag{43}$$

Heating by means of conduction currents may be the dominant loss mode in the armature.

Eddy current losses can be estimated from a knowledge of both armature geometry and flux density. The power dissipated in a straight circular wire, whose diameter is less than a skin depth, when subjected to a time varying magnetic field is

$$P_{\text{eddy/wire}} = \frac{\pi \sigma \omega B_{av}^2 d_w^4 \ell}{128} \tag{44}$$

where $B_{av}^2$ is the mean squared flux density. When
estimating eddy current losses for a toroidal winding, it must be remembered that $B_{av}$ is different on different sides of the core. Assuming that losses in the end turn regions (connections between the inner and outer armatures) are small, the total eddy current loss in the armature will be

$$P_{ec} = \frac{3 \theta \omega^2 \lambda_p \rho \ell d^2_w (B_{aoi}^2 + B_{aio}^2)}{32}$$  \hspace{1cm} (45)$$

where $B_{aoi}^2$ and $B_{aio}^2$ are the mean squared flux densities at the inner and outer armatures respectively.

Losses in the ferromagnetic core are due to hysteresis and eddy currents. If the loss per unit mass is given by $d_c$, and the mass density by $\rho_c$, then the core loss may be written

$$P_c = \pi (R_{ec}^2 - R_{ci}^2) \ell d_c \rho_c$$  \hspace{1cm} (46)$$

Losses in the image shield are due to induced eddy currents and are given by

$$P_{sh} = \pi \frac{B_{a_{s}}^2 R_{s} \ell}{\sigma_{s} \sigma_{s} \delta_{s}}$$  \hspace{1cm} (47)$$

where $\delta_{s}$ is the skin depth of the shield material, and $B_{a_{s}}$ is the azimuthal component of the flux density at the shield surface.
The radial dimensions of the core may be estimated by requiring that the ferromagnetic core material remain in an unsaturated state. The insulation on the core is ignored, and it is assumed that the core is made of a uniform material of infinite permeability. Applying the value of field current necessary to produce rated open circuit voltage establishes the operating level of flux. Using Gauss's Law (conservation of flux through a closed surface) on a quarter section of the core results in the following formula, if it is assumed that the flux density in the core is uniform.

\[
\begin{align*}
t_c &= \frac{4 \mu J_f \sin \left( \frac{\theta_{mR}}{2} \right)}{\tau (2+p)B_c s} \frac{2}{R_{ci}} \left( \frac{R_{fo}}{R_{ci}} \right)^{p+2} \\
\end{align*}
\]

where \( t_c \) is the radial thickness of the core, and \( s \) is the core stacking factor.

Design curves for a machine whose rating is on the order of 3 MVA are given in figures 9-17. These curves illustrate the functional relationships expressed in the previously developed design formulas and serve to quantify various parameters of the machine. The constraints under which the curves were developed are that the armature "fit" both the rotor and eddy current shield of an experimental \((1,2)\) machine already existing at MIT. The curves allow
for a conceptual design of a small ring-wound armature; dimensional constraints were imposed to provide a comparison with the existing MIT machine. A summary of dimensional constraints is presented below.

\[ \lambda = 30 \text{ inches} \]
\[ R_{fi} = 3 \text{ inches} \]
\[ R_{fo} = 4 \text{ inches} \]
\[ R_{c} = 4.5 \text{ inches} \]
\[ R_{s} = 18 \text{ inches} \]

These dimensions are those of the existing machine.

Another dimensional constraint, which was self imposed, is that the inner radius of the inner armature be 5 inches, the same as that of the armature of the existing machine.

\[ R_{all} = 5 \text{ inches} \]

The design current densities were taken to be

\[ J_{a} = 1900 \text{ amperes/inch}^2 \]
\[ J_{foc} = 60,000 \text{ amperes/inch}^2 \]

These values are typical numbers for armature and field current densities.
Implicit in the design formulation and curves is the assumption that all electrical lengths in a modified Gramme-Ring armature are equal to the straight section length of the machine. It is hoped that the absence of complicated end turns justifies this approximation.

Plotted in figures 9-11 are the synchronous, transient, and subtransient reactances of the machine as a function of the outer radius of the inner armature (outside edge of the inner winding). It has been assumed that the insulated core principle will be adopted, so that the approximations

\[ R_{cl} = R_{ao1} \quad \text{and} \quad R_{co} = R_{aio} \]

can be made. Also, core thickness has been taken as a parameter. By specifying \( R_{ao1} \) and core thickness, armature dimensions are fixed, if inner and outer armature current densities are required to be the same. That is, if the phase belt cross sectional area of the inner armature equals that of the outer armature, then \( R_{ao0} \) is a function of \( R_{ao1} \) and core thickness.

The graphs of reactances versus \( R_{ao1} \) show that all three reactances increase as the armature gets thicker. This is reasonable because the self inductance of the armature increases with increases in armature thickness. For a given current, a thick winding links more self flux than a thin one. However, increases in core thickness tend
to reduce reactances for a fixed value of $R_{aoi}$, since a thick core places the outer armature nearer to the image shield, thus reducing armature self inductance.

Figure 12 is a plot of rating versus $R_{aoi}$, and figure 13 compares reactances and efficiency with rating, for a machine with a six inch iron core. As the rating is increased by making the armature thicker, reactances also increase and efficiency drops. The drop in efficiency is due primarily to the combined effect of increases in armature and shield $I^2R$ heating losses. High voltage machines may possibly be allowed higher ratings than conventional machines, because the increase in reactance with rating can be compensated with the removal of the step-up transformer between the machine and the high voltage bus.

Rated field current density as a function of synchronous reactance, with power factor as a parameter, has been plotted in figure 14. As indicated, unity power factor operation would require the lowest values of rated field current.

In figure 15, core flux density is plotted against $R_{aoi}$ with core thickness as a parameter. This graph was made under the assumption of a fixed terminal voltage, and hence, a fixed air gap flux. As the armature gets thicker
FIG. 10. TRANSIENT REACTANCE VERSUS $R_{aoi}$ WITH CORE THICKNESS AS A PARAMETER.
FIG. 11. SUBTRANSIENT REACTANCE VERSUS $R_{aoi}$ WITH CORE THICKNESS AS A PARAMETER.
FIG. 13. SYNCHRONOUS, TRANSIENT, AND SUBTRANSIENT REACTANCES, 
AND (1 - EFFICIENCY) VERSUS MACHINE RATING.
FIG. 14. RATED FIELD CURRENT DENSITY VERSUS $x_d$ WITH POWER FACTOR AS A PARAMETER.
FIG. 15. CORE FLUX DENSITY VERSUS $R_{ao1}$ WITH CORE THICKNESS AS A PARAMETER.
FIG. 16. IMAGE SHIELD DISSIPATION VERSUS $R_{ao1}$ WITH CORE THICKNESS AS A PARAMETER.
Fig. 17. Armature Dissipation Versus $R_{aol}$ with Core Thickness as a Parameter.
by increasing $R_{ao1}$, the flux density drops, because less flux travels through the low reluctance core. In a sense, more leakage flux is generated as the armature gets thicker, because the core moves away from the field winding.

Figure 16 is a plot of image shield dissipation, while figure 17 shows $I^2R$ losses in the armature. Image shield losses are dominated by conduction current losses for small values of $R_{ao1}$. Core losses and armature eddy current losses are considerably smaller; core losses are on the order of 3 kW and eddy current losses are about 2 kW.

The advantages of superconducting machinery become more apparent with increases in rating. Therefore, following the tradition of previous investigators, a bench mark design of a 2000 MVA modified Gramme-Ring alternator is (3,4) presented in Table I. The design is given mostly to "size" the armature of a large ring-wound machine. Rotor dimensions and the inner armature radius were chosen so that the maximum radial loading on the damper shield during a three phase fault was less than 4000 psi. Other mechanical considerations, such as fault forces on the armature and core, were not taken into account. The possible terminal voltages and the required number of turns to produce them are listed; the limitation on voltage being the insulation scheme adopted.
### TABLE I. DESIGN DIMENSIONS, ASSUMED PARAMETERS, RULES OF THUMB, AND EXPECTED PROPERTIES OF A 2000 MVA MODIFIED GRAMME-RING ALTERNATOR.

**Design dimensions:**

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Field winding inner radius</td>
<td>R₁ᵢ = 10&quot;</td>
</tr>
<tr>
<td>Field winding outer radius</td>
<td>R₂₀ = 12&quot;</td>
</tr>
<tr>
<td>Damper shield outer radius</td>
<td>Rₚ = 18&quot;</td>
</tr>
<tr>
<td>Inner armature inner radius</td>
<td>R⁺₁ᵪ = 22&quot;</td>
</tr>
<tr>
<td>Inner armature outer radius</td>
<td>R⁺₂₀ = 29&quot;</td>
</tr>
<tr>
<td>Core inner radius</td>
<td>R⁺₂₁ = 29&quot;</td>
</tr>
<tr>
<td>Core outer radius</td>
<td>R⁺₂₀ = 59&quot;</td>
</tr>
<tr>
<td>Outer armature inner radius</td>
<td>R⁺₂₁₀ = 59&quot;</td>
</tr>
<tr>
<td>Outer armature outer radius</td>
<td>R⁺₂₂₀ = 62&quot;</td>
</tr>
<tr>
<td>Shield inner radius</td>
<td>Rₛ = 64&quot;</td>
</tr>
<tr>
<td>Field winding angle</td>
<td>θₑ₀ = 2π/3</td>
</tr>
<tr>
<td>Armature winding angle</td>
<td>θₐ₀ = π/3</td>
</tr>
<tr>
<td>Machine length</td>
<td>l = 18&quot;</td>
</tr>
<tr>
<td>Core thickness</td>
<td>t = 30&quot;</td>
</tr>
</tbody>
</table>

**Assumed parameters:**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Open circuit field current density</td>
<td>Jₑ₀ = 60,000 A/in²</td>
</tr>
<tr>
<td>Armature current density</td>
<td>Jₐ = 1900 A/in²</td>
</tr>
<tr>
<td>Power factor</td>
<td>cos ψ = 0.85 lagging</td>
</tr>
</tbody>
</table>
Number of pole pairs \[ p = 1 \]
Armature space factor \[ \lambda_f = 0.33 \]
Armature conductivity \[ \sigma = 6 \times 10^7 \text{ mhos/m} \]
Shield conductivity \[ \sigma_s = 3.6 \times 10^7 \text{ mhos/m} \]
Core mass density \[ \rho_c = 0.27 \text{ lb/in}^3 \]
Core dissipation factor \[ c_c = 0.63 \text{ w/lb} \]

Rules of thumb:

Armature current densities \[ J_{a1} = J_{ao} \]
All electrical lengths \[ L \]
Insulated core \[ R_{ci} = R_{aco}, \quad R_{co} = R_{aio} \]

Expected properties:

Rated field current density \[ J_r = 91,000 \text{ A/in}^2 \]
Rating \[ P_{va} = 2092 \text{ MVA} \]
Delta factor \[ \delta = 0.057 \]
Average core flux density \[ B_c = 1.0 \text{ tesla} \]
Synchronous reactance \[ x_d = 0.73 \]
Transient reactance \[ x_d' = 0.51 \]
Subtransient reactance \[ x_d'' = 0.27 \]
Armature conduction losses \[ P_a = 3.8 \text{ MW} \]
Armature eddy current losses (estimate) \[ P_{ec} = 0.2 \text{ MW} \]
Shield losses

\[ P_{sh} = 3.2 \text{ MW} \]

Core losses

\[ P_c = 0.3 \text{ MW} \]

Total losses

\[ P_{\text{total}} = 7.5 \text{ MW} \]

Percent rating

\[ \% P = 0.36 \]

Voltage per turn

1st harmonic = 2945 volts/turn

3rd harmonic = 0 volts/turn

Rated voltage (line-to-line) | Turns/winding
---|---
69 kv | 24
115 kv | 39
138 kv | 47
230 kv | 78
345 kv | 117
VI, ELECTRIC FIELD MODELING

One consideration in the design of a high voltage alternator is the prevention of electrical breakdown from the high voltage armature to the grounded rotor or to the image shield. Although the most critical electric field concentrations will appear at the core ends, it is important to realize that air gap sizes in high voltage machines cannot be determined by mechanical considerations alone. Assuming that end turn electric fields can be managed by special methods, such as rounding the core ends, air gap electric fields will be one of the constraints on gap size. It is the purpose of this section to determine the nature of the gap fields and to recommend possible insulation schemes for high voltage machinery.

The first step in the analysis is to obtain a mathematical model from which the electrical potential in both the rotor-armature and armature-shield air gaps may be obtained. There are several possible insulation schemes, but the model for this analysis will be gaps filled with two concentric dielectrics. The electric field will divide between the dielectrics in an inverse proportion to their dielectric constants. Knowledge of local field concentrations may be useful in determining rated voltage, gap size, or both; since local concentrations are the most...
probable cause of dielectric failure. Another interesting number may be the voltage drop across each dielectric, since the voltage drop and thickness of the dielectric may be related to a parameter known as dielectric strength (potential gradient at which breakdown occurs). That is, the voltage drop across an insulator, and the thickness of an insulator, when taken together, yield an average electric field strength, which may be compared to the dielectric strength of a gap insulator.

The electric field intensities and the distribution of fields depend upon rotor-armature-shield radial dimensions. It is expected that the rotor-armature electric fields will be more nonuniform than armature-shield electric fields, and the strongest electrical stress concentrations probably will be at the rotor surface. If the gap surfaces are kept relatively smooth, it is expected that corona will not start in the straight section gaps for the operating voltages. Voltages causing local stress concentrations high enough to initiate corona should be avoided, since voltages of this magnitude will probably lead to destructive arcs.

The problem of finding gap fields is not a simple one, since rotating machines are essentially magnetic field systems. The time variation of armature flux linkage, brought about by the rotation of the field winding, gives
rise to an electric field in the space around the armature. The nature of this field depends upon the nature of the boundary condition presented by the armature winding.

An approximate method of finding gap fields involves the determination of the armature voltage distribution. That is, the spatial distribution of voltage around the armature can be used as a boundary condition on gap potentials. Specification of the voltage distribution inherently takes into account winding geometry, and hopefully will yield more accurate expressions for gap electric fields. It is expected that the geometry of a modified Gramme-Ring will give rise to a four pole voltage distribution.

IV, I ARMATURE VOLTAGE DISTRIBUTION

The purpose of what immediately follows is to find the voltage distribution around the armature of a modified Gramme-Ring alternator. The following assumptions are made:

(1) The winding has six phase belts with diametrically opposite belts connected in parallel.

(2) Adjacent phase belts are counter-wound, i.e., clockwise winding, counterclockwise winding, clockwise winding, etc.

(3) The winding is delta connected.
(4) Under balanced steady state conditions, the winding can be described by time/space phasor diagrams.

Figure 18 is a space phasor diagram for one half of the armature (180 mechanical degrees), drawn for an instant of time when the voltage in one winding is at a maximum. A large number of turns per phase belt is assumed, so that the voltage distribution among the turns of a phase belt becomes smooth in the azimuthal direction. The arcs between phasor tips approximate the actual voltage distribution due to the breadth of the winding. Under balanced steady state conditions, the neutral point is at the center of the delta, and line to neutral voltages are read off of the real time axis, which coincides with the phase $V_{an}$.

The quantity of interest is the winding voltage to neutral, $V_{pn}$. The projection of $V_{pn}$ on the real time axis yields $V_{tn}$, the time variation of the winding voltage at point $P$. It is believed that the winding voltage will be a wave traveling at synchronous speed around the armature; the spatial nature of this wave is to be determined. From the phasor diagram, defining the line-to-line voltage as $V$, the following must hold true:

$$V_{ab} = V_{pb} = V$$

$$V_{an} = V/\sqrt{3}$$
FIG. 18. SPACE PHASOR DIAGRAM FOR ONE HALF OF A MODIFIED GRAMME-RING ARMATURE.
Writing the law of cosines for \( V_{pn} \) yields

\[
V_{pn}^2 = V_{pb}^2 + V_{bn}^2 - 2V_{pb}V_{bn}\cos \theta
\]  

(49)

or, solving for \( V_{pn} \),

\[
V_{pn} = V \sqrt{\frac{4}{3}} - (2/\sqrt{3})\cos \theta
\]  

(50)

where \( \theta \) is a measure of spatial angle around the periphery of a phase belt, taken from the center of the phase belt.

The projection of \( V_{pn} \) on the real time axis gives \( V_{tn} \).

\[
V_{tn} = V \sqrt{\frac{4}{3}} - (2/\sqrt{3})\cos \theta \cos \phi
\]  

(51)

The angle \( \phi \) is given by

\[
\phi = \omega t + \phi_o
\]  

(52)

where \( \phi_o \) is the initial time phase angle of the voltage at point \( P \). Note that changes in \( \phi_o \) are brought about by changing the position of point \( P \). Making a transformation of variables by defining the spatial angle \( \theta_m \), measured from the edge of a phase belt

\[
\theta_m = \frac{\pi}{6} - \theta
\]  

(53)

and noting that

\[
\phi_o = 2 \theta_m
\]  

(54)
equation (51) becomes

\[ V_{tn} = V \sqrt{(4/3) - (2/\sqrt{3}) \cos\left(\frac{\pi}{6} - \Theta_m\right) \cos(2\Theta_m + \omega t)} \] (55)

which holds on the interval \((0 < \Theta_m < \pi/3)\), i.e., across phase A. For phase B, the line-to-neutral voltage is given by

\[ V_{tn} = V \sqrt{(4/3) - (2/\sqrt{3}) \cos\left(\frac{\pi}{2} - \Theta_m\right) \cos(2\Theta_m + \omega t)} \] (56)

for \((\pi/3 < \Theta_m < 2\pi/3)\). And for phase C,

\[ V_{tn} = V \sqrt{(4/3) - (2/\sqrt{3}) \cos\left(\frac{5\pi}{6} - \Theta_m\right) \cos(2\Theta_m + \omega t)} \] (57)

which holds on the interval \((2\pi/3 < \Theta_m < \pi)\). The voltage distribution represented above is repeated over the other half of the winding.

Now consider the normalized voltage distribution along one half of the winding:

\[ V_n = \frac{V_{tn}}{\sqrt{\cos(\phi_m - \Theta_m)} \cos(2\Theta_m + \omega t)} = \sqrt{4 - 2\sqrt{3} \cos(\phi_m - \Theta_m)} \cos(2\Theta_m + \omega t) \] (58)

where

\[ \phi_m = \begin{cases} 
\pi/6 \text{ for } 0 < \Theta_m < \pi/3 \\
\pi/2 \text{ for } \pi/3 < \Theta_m < 2\pi/3 \\
5\pi/6 \text{ for } 2\pi/3 < \Theta_m < \pi 
\end{cases} \]

The term under the radical sign in (58) may be expanded in a Fourier series.
\[ V_n = [a_0 + a_1 \cos \theta_m + b_1 \sin \theta_m + a_2 \cos 2\theta_m + \]
\[ b_2 \sin 2\theta_m + \ldots] \cos (2\theta_m + \omega t) \]  

where

\[ a_0 = \frac{1}{T} \int_0^T (4 - 2\sqrt{3} \cos (\phi_m - \theta_m))^\frac{1}{2} d\theta_m \]

\[ a_1 = \frac{2}{T} \int_0^T (4 - 2\sqrt{3} \cos (\phi_m - \theta_m))^\frac{1}{2} \cos \theta_m d\theta_m \]

\[ b_1 = \frac{2}{T} \int_0^T (4 - 2\sqrt{3} \cos (\phi_m - \theta_m))^\frac{1}{2} \sin \theta_m d\theta_m \]

and \( T = \pi \).

A numerical analysis of the Fourier coefficients provides the result

\[ (4 - 2\sqrt{3} \cos (\phi_m - \theta_m))^\frac{1}{2} = 0.83 + 0.11 \cos 6\theta_m + 0.02 \cos 12\theta_m \]  

Substitution of this result into (59) gives, after some manipulation, an approximate expression for \( V_n \), the normalized voltage over the interval \( 0 < \theta_m < \pi \).

\[ V_n = 0.83 \cos (2\theta_m + \omega t) + 0.055 \cos (4\theta_m + \omega t) + 0.055 \cos (8\theta_m + \omega t) \]
\[ + 0.01 \cos (10\theta_m + \omega t) + 0.01 \cos (14\theta_m + \omega t) \]  

The point of this exercise has been to establish the approximate form of the armature voltage distribution. If
the method attempted is correct, then the voltage distribution is predominantly sinusoidal, with a wavelength equal to one half of the armature circumference. This result seems reasonable, since opposite armature windings are in phase with each other. In what follows, attention will be focused on the spatial distribution of voltage; it will be assumed that the voltage distribution is static and may be written

\[ V = \text{Re} \, V e^{-jz^2 \theta} \quad (62) \]

where \( V = V/\sqrt{3} \). In reality, the voltage distribution moves as a traveling wave around the armature.

VI,II GAP ELECTRIC FIELDS

The approximate form of the electric field distribution in the gap between the rotor and armature can be found by treating the rotor and armature surfaces as the plates of a cylindrical capacitor. The region between the plates is occupied by two concentric dielectrics, each of which is homogeneous and characterized by a particular dielectric constant. In an actual high voltage machine, the dielectrics might be a porcelain cylinder placed against the inner armature, with air between the porcelain cylinder and rotor shaft. It is assumed that the rotor shaft is a perfectly conducting smooth cylinder held at ground potential, while
the potential at the armature surface is sinusoidally
distributed in the azimuthal direction. The problem is to
find the two dimensional fields for the model shown below.
\( \theta_m = \theta \)

\[ \phi_a = \frac{R_0 \pi}{2} e^{-j2\theta} \]

In the model, \( \phi_r \) represents the rotor potential, and
\( \phi_a \) the armature potential. The dielectric constants
of the inner and outer insulators are \( \varepsilon_1 \) and \( \varepsilon_2 \) respectively.
The smooth dielectric boundary is represented by the radius
\( R_d \).

In a transverse plane between the armature and rotor
surfaces, the curl of the electric field must be zero,
since the magnetic field in the straight section of the machine
has no \( z \)-component. There are no sources of free charge
in the system, so the divergence of the electric field
also must be zero. For a field with zero divergence and curl,
the electrical potential must satisfy Laplace's equation.

\[ \nabla^2 \phi = 0 \]  \hspace{1cm} (63)
Solutions to (63) take the form of the driving potential.

For \( R_t < r < R_d \) \[ \Phi_1 = \text{Re} \Phi_1 e^{-j \Theta} \] (64)

For \( R_d < r < R_{ai} \) \[ \Phi_2 = \text{Re} \Phi_2 e^{-j \Theta} \] (65)

where

\[ \Phi_1 = A_1 r^2 + B_1 \] (66)

\[ \Phi_2 = A_2 r^2 + B_2 \] (67)

The boundary conditions at the dielectric interface are that the tangential component of the electric field is continuous, since the s-directed magnetic field is zero; and, the normal component of the displacement vector is continuous, since no free charge resides on the dielectric interface.

Thus,

\[ \Phi_1 = \Phi_2 \text{ at } r = R_d \] (68)

and

\[ \varepsilon_1 (A_1 - B_1 R_d^{-4}) = \varepsilon_2 (A_2 - B_2 R_d^{-4}) \] (69)

The boundary condition at the rotor surface is

\[ 0 = A_1 R_t^2 + B_1 R_t^{-2} \] (70)

and the boundary condition at the stator surface is
\[ V = A_2 R_{a11}^2 + B_2 R_{a11}^{-2} \]  

(71)

Therefore, the potential in each dielectric is given by

\[ \Phi_1 = \text{Re} \left[ A_1 \left( r^2 - (R_t^4 / r^2) \right) e^{-j2\theta} \right] \]  

(72)

\[ \Phi_2 = \text{Re} \left[ A_2 \left( r^2 - (R_{a11}^4 / r^2) \right) + \frac{V(R_{a11}^2 / r^2)}{D} \right] e^{-j2\theta} \]  

(73)

where

\[ A_1 = \frac{(2VR_{a11}^2 R_d^{-4})}{D} \]  

(74)

\[ A_2 = \frac{VR_{a11}^2 R_d^{-4}}{D} \left[ \left( 1 - (R_t / R_d)^4 \right) + (\varepsilon_1 / \varepsilon_2)(1 + (R_t / R_d)^4) \right] \]  

(75)

\[ D = \left[ 1 - (R_t / R_d)^4 \right] \left[ 1 + (R_{a11} / R_d)^4 \right] + (\varepsilon_1 / \varepsilon_2) \left[ (R_{a11} / R_d)^4 - 1 \right] \left[ (R_t / R_d)^4 + 1 \right] \]  

(76)

The electric fields in the gap can be estimated by taking the gradient of the potential. This calculation should reveal the presence of localized stress concentrations, which will have a determining influence on dielectric lifetimes. The radial and azimuthal field components are listed in Table II. Also listed in Table II are the voltage drops across each dielectric.
Table II. ARMATURE-ROTOR ELECTRIC FIELDS (fundamental components).

Radial fields:

\[ E_r1 = -2A_1r \left[ 1 + \frac{R_c}{r} \right] \cos 2\theta \]

\[ E_r2 = \left[ -2A_2r(1+\frac{R_{al1}}{r}) + \frac{2Vr_{a11}}{\sqrt{3}r^3} \right] \cos 2\theta \]

Azimuthal fields:

\[ E_{\theta1} = 2A_1r \left[ 1 - \left( \frac{R_c}{r} \right)^4 \right] \sin 2\theta \]

\[ E_{\theta2} = \left[ 2A_2r \left[ 1-\left( \frac{R_{al1}}{r} \right)^4 \right] + \frac{2Vr_{a11}}{\sqrt{3}r^3} \right] \sin 2\theta \]

Voltage drops:

\[ V_1 = \left( \frac{2Vr_{a11}}{\sqrt{3}R_d^2} \right) \left[ 1-\left( \frac{R_c}{R_d} \right)^4 \right] \cos 2\theta \]

\[ V_2 = \left( \frac{V}{\sqrt{3}} \right) \left[ 1-\left( \frac{2R_{al1}^2}{R_d^2} \right) \left[ 1-\left( \frac{R_c}{R_d} \right)^4 \right] \right] \cos 2\theta \]

The constants \( A_1 \) and \( A_2 \) are real and are defined by equations (74)-(76), (it has been assumed that \( V = v \)).
Expressions for the armature-shield fields are found in a similar manner. The shield is assumed to be a perfectly conducting cylinder separated from the high voltage armature by two concentric dielectrics. The armature potential is sinusoidally distributed in the azimuthal direction. The model for this calculation is sketched below, and armature-shield fields are listed in Table III.
Table III. ARMATURE-SHIELD ELECTRIC FIELDS (fundamental components).

Radial fields:

\[ E_{r4} = -2A_4 r [1 + (R_s/r)^4] \cos 2\theta \]

\[ E_{r3} = [2A_3 r(1+(R_{aoo}/r)^4 + (2VR_{aoo}^2/\sqrt{3} r^3)] \cos 2\theta \]

Asimuthal fields:

\[ E_{\theta 4} = 2A_4 r [1 - (R_s/r)^4] \sin 2\theta \]

\[ E_{\theta 3} = [2A_3 r(1-(R_{aoo}/r)^4 + (2VR_{aoo}^2/\sqrt{3} r^3)] \sin 2\theta \]

Voltage drops:

\[ V_4 = ((2VR_{aoo}^2)/\sqrt{3} R_w^4) (1 - (R_s/R_w)^4) \cos 2\theta \]

\[ V_3 = (\sqrt{3})(1-(2R_{aoo}^2/R_w^2)) (1-(R_s/R_w)^4) \cos 2\theta \]

where

\[ A_4 = (2VR_{aoo}^2)/\sqrt{3} R_w^4 \]

\[ A_3 = (VR_{aoo}^2)/\sqrt{3} R_w^4 \]

\[ A_3 = [(1-(R_s/R_w)^4)^4 + \varepsilon_4/\varepsilon_3] \]

\[ D = [1-(R_s/R_w)^4] \]

\[ (R_{aoo}/R_w)^4 - 1] [R_s/R_w]^4 + 1 \]
The equations for the electric fields may be used to estimate the electrical stresses to which air gap insulators will be subjected. Estimates of the steady state electrical loadings require that the voltage, $V$, be a steady state line-to-line value; however, insulation thickness should not be determined by steady state armature voltages, especially in a machine which may act as both a generator and a transformer. Lightning and switching surge overvoltages, three to four times steady state values, are not uncommon; and these values should be used for $V$ when considering electrical stress concentrations in air gap insulators.

In choosing an insulation system for a high voltage winding, it should be remembered that the highest electrical stress concentration is in the material of lowest dielectric constant. Since the dielectric constants of most gases are near $\varepsilon_0$, a composite system of a cylindrical solid dielectric and a gas would place the highest stress concentrations in the gas. For air, gap electric fields above 20 kv/inch in the steady state, under normal atmospheric conditions (standard conditions of temperature and pressure), would be excessive, since this number only allows a safety factor of about 3.5 for overvoltages. That is, the breakdown potential of air is about 70 kv/inch. If steady state electric fields were on the order of 20 kv/inch, overvoltages
in excess of 70 kv/inch could be obtained under transient conditions possibly leading to flashovers across the gap. For an air-solid dielectric system, gap size will have to increase with increases in terminal voltage, because most of the electrical stress concentrations will be in the air. As gap size increases, machine rating falls for a given machine length.

A gas such as sulfur hexafluoride, on the other hand, can have a dielectric strength two to three times that of air, depending, of course, on operating conditions such as pressure, purity, temperature, etc. It should be possible to reduce the size of the gap for a given terminal voltage by using a gas of this nature. That is, for a composite gas-solid insulation system, the highest electrical stress concentrations will probably be in the gas. It is desirable to have a gas with the highest dielectric strength possible, since this allows for the smallest gap sizes, and hence, highest power densities in the machine.

The solid portion of the insulation system under discussion might be a material such as porcelain, if the mechanical duty upon this material is not overly severe; i.e., it would be undesirable to shatter the porcelain insulating cylinder due to the transfer of fault forces from the armature and core to the insulating cylinder.
Assuming that a porcelain cylinder would be mechanically acceptable, its high dielectric strength would make it an ideal choice for gap insulation, especially if most of the gap electric fields could be confined inside the porcelain.

VII. DISCUSSION AND SUGGESTIONS FOR FUTURE WORK

The object of this work has been to present a possible design configuration for a high voltage superconducting alternator. The scheme adopted is that of a modified Gramme-Ring armature surrounding a superconducting field winding. This configuration solves some of the electric field problems associated with the armatures of more conventionally wound machines. It is believed that a toroidal geometry may be superior to a more conventional geometry in high voltage applications; whether or not this is true requires experimental verification.

Several suggestions for future work can be made. First, a method of managing core end fields will have to be devised. This is basically a problem in geometry, where the fundamental rule will be to avoid sharp points and corners, if at all possible. Maybe all that is needed is a set of curved caps for each segment of the core end; perhaps a more elaborate scheme is necessary. Further
study of the end region fields should provide the answer.

Building a ferromagnetic core for a modified Graeme-Ring winding may prove to be a difficult task. Mechanical stresses on the core and core insulation will have to be evaluated before the insulated core principle may be applied effectively. Also, the steady state and fault forces on the armature will have to be found. A method of binding together both the core and armature will have to be developed. And finally, a means for cooling both the armature and core will have to be determined.
APPENDIX I. MAGNETIC FIELD ANALYSIS

In order to calculate the inductances of a ring-wound machine, it is necessary to have expressions for the magnetic fields in the machine. The inner air gap fields (between the rotor and core) have been calculated in reference (1). It is the purpose of this section to calculate the outer air gap fields (between the core and image shield).

The problem is to find expressions for the magnetic fields in an annular region made up of free space and a thick winding, surrounded by an infinitely conducting boundary on one side and an infinitely permeable boundary on the other side. In order to find the total field, the current in the outer armature is divided into component current sheets. The magnetic field of each current sheet is found, and the resultant field is taken as the sum of the component fields, using the principle of superposition.

The first step in the analysis is to find the field due to a single current sheet. Figure A.1 illustrates the geometry of the problem. It is assumed that the current sheet at radius \( R \) is described by the equation

\[
\vec{B} = I \vec{R}_0 (K e^{-jk\theta})
\]  

(A-1)

where \( K \) is the complex amplitude, and \( k \) is an integer.
FIG. A.1. A SINUSOIDAL CURRENT SHEET SURROUNDED BY INFINITELY CONDUCTING AND INFINITELY PERMEABLE BOUNDARIES.
There are two regions of interest: the inner region for which $R_C < r < R$, and the outer regions for which $R < r < R_s$. These regions share the current sheet as a common boundary. In each region, Ampere's Law becomes

$$\nabla \times \vec{H} = 0$$  \hspace{1cm} (A-2)

allowing the field intensity to be written as the negative gradient of a scalar potential.

$$\vec{H} = -\nabla \phi$$  \hspace{1cm} (A-3)

This leads to Laplace's Equation for the magnetic potential.

$$\nabla^2 \phi = 0$$  \hspace{1cm} (A-4)

Solutions to this equation take the form

$$\phi = \text{Re}\left\{A e^{-jkr} + B e^{jk\theta}\right\}$$  \hspace{1cm} (A-5)

where $A$ and $B$ are complex constants determined by the boundary conditions. Let $\phi_i$ be the potential for $R_C < r < R$, and $\phi_o$ be the potential for $R < r < R_s$. Then

$$\phi_i = \text{Re}\left\{A_i e^{-jkr} + B_i e^{jk\theta}\right\}$$  \hspace{1cm} (A-6)

$$\phi_o = \text{Re}\left\{A_o e^{-jkr} + B_o e^{jk\theta}\right\}$$  \hspace{1cm} (A-7)

There are four unknown constants and four boundary conditions.
which must be satisfied.

At \( r = R_c \), the tangential (azimuthal) component of the magnetic field intensity must be zero, since the iron is assumed to have infinite permeability. Therefore,

\[
A_1^k R_{c0} + B_1^k R_{c0} = 0 \quad (A-8)
\]

At \( r = R_s \), the discontinuity in the tangential component of the magnetic field intensity must equal the surface current. Therefore,

\[
(A_1 - A_0) R^{k-1} + (B_1 - B_0) R^{-k-1} = jk/k \quad (A-9)
\]

Also, at \( r = R_s \), the normal component of the magnetic flux density must be continuous. Therefore,

\[
\mu : (A_1^k R^{-k-1} = B_1^k R^{-k-1}) = \mu \left( A_0^k R^{-k-1} = B_0^k R^{-k-1} \right) \quad (A-10)
\]

And finally, at \( r = R_s \), the normal component of the flux density must be zero. Therefore,

\[
A_o^k R_s + B_o^{-k} R_s = 0 \quad (A-11)
\]

Using boundary conditions (A-6) through (A-11) and the definition of the field intensity as the negative gradient of a scalar potential, one may find expressions for the complex fields of a current sheet. They are
for \( R_{co} < r < R \)

\[
H_{\tau r} = \frac{k}{j k} \left[ \frac{r^{k-1} + R_{co}^{2k} r^{-k-1}}{R_{s}^{2k} + R_{co}^{2k}} \right] \left[ R^{k+1} - \frac{2k}{R_{s}^{2k} R_{co}^{2k}} \right] \quad (A-12)
\]

\[
H_{\theta r} = \frac{k}{j k} \left[ \frac{r^{k-1} - R_{co}^{2k} r^{-k-1}}{R_{s}^{2k} + R_{co}^{2k}} \right] \left[ R^{k+1} - \frac{2k}{R_{s}^{2k} R_{co}^{2k}} \right] \quad (A-13)
\]

for \( R < r < R_{s} \)

\[
H_{\tau o} = \frac{k}{j k} \left[ \frac{r^{k-1} - R_{s}^{2k} r^{-k-1}}{R_{s}^{2k} + R_{co}^{2k}} \right] \left[ R^{k+1} + \frac{2k}{R_{s}^{2k} R_{co}^{2k}} \right] \quad (A-14)
\]

\[
H_{\theta o} = \frac{k}{j k} \left[ \frac{r^{k-1} + R_{s}^{2k} r^{-k-1}}{R_{s}^{2k} + R_{co}^{2k}} \right] \left[ R^{k+1} + \frac{2k}{R_{s}^{2k} R_{co}^{2k}} \right] \quad (A-15)
\]

The total magnetic field is found by adding the fields of each current sheet, where current sheets are taken as the Fourier components of the spatial current distribution. That is, each phase belt in the armature subtends an angle of \( \theta_{w} \) degrees and has a radial thickness of \( (R_{a0} - R_{a10}) \). The angular dependence of the current density for the excitation of a single armature phase is shown in figure A.2.

A Fourier analysis of this periodic waveform yields

\[
J(\theta) = \sum_{n=1}^{\infty} J_{n} \cos n\theta \quad (A-16)
\]

where

\[
J_{n} = \begin{cases} 
\left(4J_{o}/n\pi\right) \sin (n\theta_{w}/2) & \text{n odd} \\
0 & \text{n even}
\end{cases} \quad (A-17)
\]
FIG. A.2. ANGULAR DEPENDENCE OF ARMATURE CURRENT DENSITY

IN PHASE A.
where \( p \) is the number of pole pairs. Writing (A-16) in complex notation, and dividing \( J \) into an infinite number of current sheets yields

\[
K = \text{Re} \left( J_n \, dR \, e^{-j k \theta} \right) \quad (A-18)
\]

from which it can be concluded that

\[
K = J_n \, dR \quad (A-19)
\]

The geometrical arrangement of the thick winding under consideration is illustrated in figure A.3.

The field of a thick winding is found by integrating over the radius \( R \).

For \( R_{co} < r < R_{a1o} \)

\[
H = \int_{R_{a1o}}^{R_{aoo}} H_1(r) \, dR \quad (A-20)
\]

For \( R_{a1o} < r < R_{aoo} \)

\[
H = \int_{R_{a1o}}^{r} H_0(r) \, dR + \int_{r}^{R_{aoo}} H_1(r) \, dR \quad (A-21)
\]

For \( R_{aoo} < r < R_s \)

\[
H = \int_{R_{aoo}}^{R_s} H_0(r) \, dR \quad (A-22)
\]

where \( H_1 \) and \( H_0 \) represent the complex amplitudes of either component (radial or azimuthal) of the inner and outer field intensities of a current sheet.
FIG. A.3. CROSS SECTION OF A THICK WINDING SURROUNDED BY INFINITELY CONDUCTING AND INFINITELY PERMEABLE BOUNDARIES.
Carrying out the integrations provides expressions for the complex amplitudes of the fields of the outer armature. These expressions are listed in Table IV. The fields between the inner armature and rotor are listed (1) in Table V.

Winding inductance is taken to be the ratio of flux linkage to current. It is defined as self inductance when the flux linkage is due to self current, and mutual inductance when the flux linkage is due to current in a nearby coil. The flux linked by a thick winding is found by summing the flux linked by incremental elements of the winding.

The flux linking a pair of incremental elements is given by

$$\Phi = \int_S \vec{B} \cdot \vec{n} \, dA \quad (A-23)$$

where $S$ defines a surface bounded by the elements. A pair of elements represents one turn, so the incremental flux linkage is

$$d^2 \lambda = \Phi \, d^2 N \quad (A-24)$$

where

$$d^2 N = \frac{2 \, N_t \, dA}{\theta_{we} \left( R_{wo}^2 - R_{wi}^2 \right)} \quad (A-25)$$
In (A-25), \( N_t \) represents winding turns, and \( R_{wo} \) and \( R_{wi} \) are outer and inner winding radii. A pair of differential winding elements are shown in figure A.4.

The flux \( \Phi \), defined by the surface integral of (A-23), is calculated over a cylindrical surface of radius \( r \) and length \( l \) in the \( z \)-direction. The total flux linkage is found by summing the flux linked by all the differential elements in the winding.

\[
\lambda = l p \left[ \int_{-\frac{\theta_w}{2}}^{\frac{\theta_w}{2}} \int_{R_{wi}}^{R_{wo}} \frac{2 N_t}{\theta_w (R_{wo}^2 - R_{wi}^2)} \int_{\psi - \frac{\pi}{p}}^{\psi} \mu_0 H_r \, r \, d\Theta \, dr \, d\psi \right] \quad (A-26)
\]

Machine inductances are given in Table VI.
FIG. A.4. FLUX LINKAGE DIFFERENTIAL ELEMENTS
TABLE IV. ARMATURE PHASE A MAGNETIC FIELDS - OUTER GAP

For \( R_{co} < r < R_{aio} \) (between the outer surface of the core and the inner surface of the outer armature)

Radial component:

\[
H_r = \sum_{n=1 \text{ odd}}^{\infty} \frac{2 J_a}{n \pi} \sin\left(\frac{n \Theta_{wae}}{2}\right) \sin(n \Theta) \left\{ \frac{2-np}{R_{aoo}} - \frac{2-np}{R_{aio}} \right\} \left\{ \frac{2-np}{R_{s}} - \frac{2-np}{R_{co}} \right\} \times
\]

\[
\left\{ \frac{2+np}{R_{aoo}} - \frac{2+np}{R_{aio}} \right\} \frac{2np}{r} - \frac{np-1}{R_{co} r} \frac{2np}{R_{s}} + \frac{2np}{R_{co} r} \right\}
\]

, \( np \neq 2 \)

\[
H_r = \frac{2 J_a}{\pi} \sin\left(\frac{\Theta_{wae}}{2}\right) \sin(2\Theta) \left\{ \frac{r + R_{co}}{R_{s}} + \frac{r}{R_{co}} \right\} \times
\]

\[
\left\{ \frac{R_{aoo} - R_{aio}}{4} - \frac{R_{aoo}}{R_{aio}} \ln \frac{R_{aoo}}{R_{aio}} \right\} \]

, \( n = 1, p = 2 \)

Azimuthal component:

\[
H_\theta = \sum_{n=1 \text{ odd}}^{\infty} \frac{2 J_a}{n \pi} \sin\left(\frac{n \Theta_{wae}}{2}\right) \cos(n \Theta) \left\{ \frac{np-1}{R_{co}} - \frac{2np}{R_{s}} - \frac{np-1}{R_{co}} \right\} \times
\]

\[
\left\{ \frac{2+np}{R_{aoo}} - \frac{2+np}{R_{aio}} \right\} \left\{ \frac{2-np}{R_{s}} - \frac{2-np}{R_{co}} \right\} \]

, \( np \neq 2 \)

\[
H_\theta = \frac{2 J_a}{\pi} \sin\left(\frac{\Theta_{wae}}{2}\right) \cos(2\Theta) \left\{ \frac{r - R_{co}}{R_{s}} + \frac{r}{R_{co}} \right\} \times
\]

\[
\left\{ \frac{R_{aoo} - R_{aio}}{4} - \frac{R_{aoo}}{R_{aio}} \ln \frac{R_{aoo}}{R_{aio}} \right\} \]

, \( n = 1, p = 2 \)
for \( R_{aio} < r < R_{aoo} \) (within the winding) Radial component:

\[
H_r = \sum_{n=1}^{\infty} \frac{2 J_s}{n \pi} \sin(n \Theta_{wa}/2) \sin(n \phi \Theta) \left[ \left\{ \frac{r^{np-1} - 2^{np} r^{-np-1}}{R_s + 2np} \right\} \times \right.
\]

\[
\left. \left\{ \frac{r^{2np} - R_{aio}^{2np}}{2 + np} + \frac{r^{2np} - R_{aio}^{2np}}{R_{co}^{2np} + 2np} \right\} \times \left\{ \frac{r^{np-1} 2^{np} r^{-np-1}}{R_s^{2np} + R_{co}^{2np}} \right\} \right]
\]

\[
\left\{ \frac{R_{aoo}^{2np} - r^{2np}}{2 + np} - \frac{R_{aoo}^{2np} - r^{2np}}{R_s^{2np} + 2np} \right\} \right\} \right], \text{np} \neq 2
\]

Azimuthal component:

\[
H_\theta = \sum_{n=1}^{\infty} \frac{2 J_s}{n \pi} \sin(n \Theta_{wa}/2) \cos(n \phi \Theta) \left[ \left\{ \frac{r^{4} - R_{s}^{4}}{R_{s}^{4} + R_{co}^{4}} \right\} \times \right.
\]

\[
\left. \left\{ \frac{r^{4} - 2^{4} R_{aio}^{4}}{R_{co}^{4}} + \frac{r^{4} - 2^{4} R_{aio}^{4}}{R_{s}^{4} + R_{co}^{4}} \right\} \times \left\{ \frac{4^{4} - R_{aoo}^{4}}{4} - \frac{4^{4} - R_{aoo}^{4}}{R_{s}^{4} + R_{co}^{4}} \right\} \right], n = 1, p = 2
\]
For \( R_{ao} < r < R_{co} \) (between the outer surface of the outer armature and the inner surface of the image shield)

**Radial component:**

\[
H_r = \sum_{n=1}^{\infty} \frac{2 J_a}{n \pi} \frac{\sin(n \Theta)}{\sin(2 \Theta)} \sin(n \Theta) \left\{ \frac{r}{2np} + \frac{2np}{R_{ao} - R_{aio}} \right\} \left\{ \frac{r - R_s}{2np} + \frac{2np}{R_{co}} \right\}
\]

\[
\left\{ \frac{R_{ao} - R_{aio}}{2np} + \frac{2np}{R_{co}} \right\}, \text{np} \neq 2
\]

\[
H_r = \frac{2 J_a}{\pi} \frac{\sin(\Theta)}{\sin(2 \Theta)} \sin(2 \Theta) \left\{ \frac{r - 4}{2np} \right\} \left\{ \frac{4}{R_s + \frac{4}{R_{co}}} \right\}
\]

\[
\left\{ \frac{R_{ao} - R_{aio}}{4} + \frac{4}{R_{co}} \ln \frac{R_{ao}}{R_{aio}} \right\}, \text{n = 1, p = 2}
\]

**Azimuthal component:**

\[
H_\theta = \sum_{n=1}^{\infty} \frac{2 J_a}{n \pi} \frac{\sin(n \Theta)}{\sin(2 \Theta)} \cos(n \Theta) \left\{ \frac{r}{2np} + \frac{2np}{R_{ao} - R_{aio}} \right\} \left\{ \frac{r - R_s}{2np} + \frac{2np}{R_{co}} \right\}
\]

\[
\left\{ \frac{R_{ao} - R_{aio}}{2np} + \frac{2np}{R_{co}} \right\}, \text{np} \neq 2
\]

\[
H_\theta = \frac{2 J_a}{\pi} \frac{\sin(\Theta)}{\sin(2 \Theta)} \cos(2 \Theta) \left\{ \frac{r + \frac{4}{R_s}}{R_{ao} - R_{aio}} \right\} \left\{ \frac{4}{R_s + \frac{4}{R_{co}}} \right\}
\]

\[
\left\{ \frac{R_{ao} - R_{aio}}{4} + \frac{4}{R_{co}} \ln \frac{R_{ao}}{R_{aio}} \right\}, \text{n = 1, p = 2}
\]
\[ H_0 = \frac{2}{\gamma} J_{a} \sin(\theta_{vae}/2) \cos(2\Theta) \left[ \left\{ \frac{r + R_{a}}{4} \right\} \left\{ \frac{4 - R_{a}}{R_{a0o} - r} \right\} \left\{ \frac{4 - R_{a}}{R_{a0o} \ln \frac{r}{R_{a0o}}} \right\} \right] \]

\[ + \left\{ \frac{r - R_{co}}{4} \right\} \left\{ \frac{R_{a0o} - r}{4} \right\} \left\{ \frac{4}{R_{s} \ln \frac{R_{a0o}}{r}} \right\} \right] , n = 1, p = 2 \]

**TABLE V. INNER GAP MAGNETIC FIELDS**

Field Winding Magnetic Fields (\( \phi \) is the angle between the axis of the field winding and the axis of phase A)

For \( r < R_{ci} \)

Radial component:

\[ H_r = - \sum_{n=1}^{\infty} \frac{2 J_{f} \sin(n \theta_{wfe}/2) \sin np(\Theta - \phi)}{n \gamma (2 - np)} \frac{r^{np=2}}{r^{R_{fo}}} \cdot \left[ 1 - y^{2-np} + \frac{2 - np}{2 + np} \left( \frac{R_{fo}}{R_{ci}} \right)^{2np} (1 - y^{2+np}) \right] \]

\[ n = 1, p = 2 \]
Azimuthal component:

\[ H_\theta = - \sum_{n=1, \text{ odd}}^{\infty} \frac{2 J_f \sin(n \theta_{wfe}/2) \cos np(\Theta - \phi)}{n \pi (2 - np)} \frac{r}{R_{fo}}^{np-2} \]
\[ \left[ 1 - y^{2-np} \left( \frac{2 - np}{2 + np} \right) \left( \frac{R_{fo}}{R_{ci}} \right)^{2np} (1 - y^{2+np}) \right], \quad np \neq 2 \]

\[ H_\theta = - \frac{J_f \sin(\theta_{wfe}/2) \cos 2(\Theta - \phi)}{\pi} \quad r \left[ \ln y + \frac{1}{2} (1 - y^4) \left( \frac{R_{fo}}{R_{ci}} \right)^4 \right] \]

For \( R_{fi} < r < R_{fo} \)

Radial component:

\[ H_r = - \sum_{n=1, \text{ odd}}^{\infty} \frac{2 J_f \sin(n \theta_{wfe}/2) \sin np(\Theta - \phi)}{n \pi (4 - np^2)} \frac{r}{R_{fo}}^{np-2} + (2 + np) \left( \frac{r}{R_{fo}} \right)^{np-2} + (2 - np) \left( \frac{r}{R_{ci}} \right)^{np+2} \left( \frac{R_{fo}}{R_{ci}} \right)^{np+2} (1 - y^{np+2}) \]

\[ H_r = - \frac{J_f \sin(\theta_{wfe}/2) \sin 2(\Theta - \phi)}{4 \pi} \quad r \left[ 1 - \left( \frac{R_{fi}}{r} \right)^4 + \ln \left( \frac{R_{fo}}{r} \right) + \left( \frac{R_{fo}}{R_{ci}} \right)^4 (1 - y^4) \right] \]

\( n = 1, \quad p = 2 \)
Asimuthal component:

\[ H = - \sum_{n=1}^{\infty} \frac{2 J_f \sin(n \theta \frac{w_f}{2}) \cos n \phi}{n \pi} \frac{np(\theta - \phi)}{(4 - np^2)} \left[ \frac{\pi}{R_{fo}} \right]^{np+2} \frac{r}{R_{fo}} \left( \frac{R_{fo}}{R_{ci}} \right)^n \left( 1 + \frac{R_{fo}}{R_{ci}} \right)^{np+2} \left( 1 - y^{np+2} \right) \]

np \neq 2

For \( R_{fo} < r < R_{ci} \)

Radial component:

\[ H_r = - \sum_{n=1}^{\infty} \frac{2 J_f \sin(n \theta \frac{w_f}{2}) \sin np(\theta - \phi)}{n \pi (2 + np)} \left[ \frac{\pi}{R_{fo}} \right]^{np+2} \frac{r}{R_{fo}} \left( \frac{R_{fo}}{R_{ci}} \right)^{2np} \]

\[ \left( 1 - y^{np+2} \right) \left[ 1 + \left( \frac{r}{R_{ci}} \right)^{2np} \right] \]
Anisothal component:

\[ H = - \sum_{n=1 \text{ odd}}^{\infty} \frac{2 J_f \sin(n \theta \text{wfe} / 2) \cos np(\theta - \phi)}{n \gamma (2 + np)} \times r \left( \frac{R_{fo}}{s} \right)^{np+2} \]

\[ (1 - y^{np+2}) \left[ 1 - \left( \frac{r}{R_{ci}} \right)^{2np} \right] \]

To get armature phase A fields:

Replace
\[ \theta - \phi \]
\[ J_f \]
\[ y \]
\[ \theta \text{wfe} \]
\[ R_{f1} \]
\[ R_{fo} \]

By
\[ \theta \]
\[ J_a \]
\[ x_{1} \]
\[ \theta \text{wae} \]
\[ R_{aii} \]
\[ R_{aci} \]
TABLE VI. INDUCTANCES

Field winding self inductance:

\[ L_f = \sum_{n=1, \text{odd}}^{\infty} \frac{16 \mu_0 \mu_r N_t^2 \sin^2(n \frac{w_{fe}}{2})}{n^3 p \gamma \Theta \frac{2}{w_{fe}} (n^2 p^2 - 4)(1-y^2)^2} \left[ (np-2) + 4 y^{np+2} \right] \]

\[-(np+2)y^4 + 2 \left( \frac{np-2}{np+2} \right) (1-y^{np+2})^2 \left( \frac{R_{fo}}{R_{ci}} \right)^{2np} \]

\[ n = 1, p = 2 \]

\[ L_f = \frac{8 \mu_0 \mu_r N_t^2 \sin^2(\frac{w_{fe}}{2})}{\gamma \Theta w_{fe}(1-y^2)^2} \left[ 4 \ln y + \frac{1-y^4}{4} + \frac{(1-y^4)^2}{8} \left( \frac{R_{fo}}{R_{ci}} \right)^4 \right] \]

Inner armature self inductance (for the two parallel windings comprising phase A):

\[ L_{ai} = \sum_{n=1, \text{odd}}^{\infty} \frac{16 \mu_0 \mu_r N_t^2 \sin^2(n \frac{w_{ae}}{2})}{n^3 p \gamma \Theta \frac{2}{w_{ae}} (n^2 p^2 - 4)(1-x_1^2)^2} \left[ (np-2) + 4 x_1^{np+2} \right] \]

\[-(np+2)x_1^4 + 2 \left( \frac{np-2}{np+2} \right) (1-x_1^{np+2})^2 \left( \frac{R_{ao1}}{R_{ci}} \right)^{2np} \]
\[ L_{ai} = \frac{8 \mu \mu_0 N_{at}^2 \sin^2(\theta_{wae}/2)}{\pi \theta_w^2 \omega_{wae}^2} \left[ x_1^4 \ln x_1 + \frac{1-x_1^4}{4} + \frac{(1-x_1^4)^2}{8} \left( \frac{R_{ao}}{R_{c1}} \right)^4 \right] \]

For a single winding:

\[ L_{ais} = 2 L_{ai} \]

Outer armature self inductance (parallel windings):

\[ L_{ao} = \sum_{n=1, \text{odd}}^{\infty} \frac{16 \mu \mu_0 N_{at}^2 \sin^2(n \theta_{wae}/2)}{n^3 \pi \theta_w^2 \omega_{wae}^2 (n^2 \omega_{wae}^2 - 4)(1-x_o^2)^2(1+\varepsilon)} \left[ \frac{np(1+\varepsilon)(1-x_o^4)}{2} \right. \]

\[ + 2 \left( \frac{2-np}{2+np} \right) (1-x_o^{2+np})^2 \left( \frac{R_{ao}}{R_s} \right)^{2np} - 2(1-x_o^{2+np})(1+2x_o^{2-np}) \]

\[ - 2 \left( \frac{2+np}{2-np} \right) (1-x_o^{2-np})^2 \left( \frac{R_{ao}}{R_{ao}} \right)^{2np} + 2(1-x_o^{2-np})(x_o^{2+np}(1+\varepsilon)) \]

\[ n = 1, p = 2 \]

\[ L_{ao} = \frac{8 \mu \mu_0 N_{at}^2 \sin^2(\theta_{wae}/2)}{\pi \theta_w^2 \omega_{wae}^2 (1-x_o^2)^2(1+\varepsilon)} \left[ - \frac{(1-x_o^4)^2}{8} \left( \frac{R_{ao}}{R_s} \right)^2 \right] \]
\[ + \frac{1-x^4}{4} + x_0^4 \ln x_0 + z \left( \frac{1-x_0^4}{4} + \ln x_0 \right) + 2 \left( \frac{R_{co}}{R_{coo}} \right)^4 (\ln x_0)^2 \]

For a single winding:

\[ L_{aos} = 2 L_{ao} \]

Total armature phase A self inductance (parallel windings):

\[ np \neq 2 \]

\[ L_a = \sum_{n=1, \text{odd}}^{\infty} \frac{16 \mu_0 N_{at}^2 \sin^2(n \Theta_{\text{wae}}/2)}{3 n p \pi \Theta_{\text{wae}}^2 (n^2 p^2 - 4)} \left[ \frac{C_{x_1}}{(1-x_1^2)^2} + \frac{C_{x_0}}{(1-x_0^2)^2 (1+z)} \right] \]

\[ n = 1, \ p = 2 \]

\[ L_a = \frac{8 \mu_0 N_{at}^2 \sin^2(n \Theta_{\text{wae}}/2)}{7 \pi \Theta_{\text{wae}}^2} \left[ \frac{C_{x_1}}{(1-x_1^2)^2} + \frac{C_{x_0}}{(1-x_0^2)^2 (1+z)} \right] \]

where, for \( np \neq 2 \)

\[ C_{x_1} = (np-2) + 4x_1^{np+2} - (np+2)x_1^4 + 2 \left( \frac{np-2}{np+2} \right) (1-x_1^{np+2}) (\frac{R_{aoi}}{R_{c1}})^{2np} \]
For \( n = 1, \ p = 2 \)

\[
C_{x_1} = x_1 \ln x_1 + \frac{1-x_1^4}{4} + \frac{(1-x_1^4)^2}{8} \left( \frac{R_{ao}}{R_{c1}} \right)^4
\]

And, for \( np \neq 2 \)

\[
C_{x_0} = np(1+x)(1-x_0^4) + 2 \left( \frac{2-np}{2+np} \right) (1-x_0^-2np)^2 \left( \frac{R_{ao}}{R_a} \right)^{2np} - 2(1-x_0^-2np)X
\]

\[
(1+x_0^-2np) = 2 \left( \frac{2-np}{2+np} \right) (1-x_0^-2np)^2 \left( \frac{R_{ao}}{R_{ao}} \right)^{2np} + 2(1-x_0^-2np)(x_0^- + s)
\]

For \( n = 1, \ p = 2 \)

\[
C_{x_0} = -\frac{(1-x_0^-4)^2}{8} \left( \frac{R_{ao}}{R_s} \right)^2 + \frac{1-x_0^-4}{4} + x_0^-4 \ln x_0 + x \left( \frac{1-x_0^-4}{4} + \ln x_0 \right)
\]

\[
+ 2 \left( \frac{R_{co}}{R_{ao}} \right)^4 (\ln x_0)^2
\]

Armature phase A self inductance for a single winding:

\[
L_{as} = 2 L_a
\]

Armature phase-to-phase mutual inductance:

\[
L_{ab} = L_a \cos (2n \pi/3)
\]
Armature-field mutual inductance (parallel windings):

\[ M = \sum_{n=1}^{\infty} \frac{32 l m \mu_0 N^2 N \cos np \phi}{p \pi \Theta \Theta (1-y^2)(1-x_1^2)} \left( \frac{R_{fo}}{R_{ao1}} \right)^{np} \]

where, for \( np \neq 2 \)

\[ C_{mn} = \frac{\sin(n \Theta) \sin(n \Theta)}{n^3 (4 - n^2 p^2)} \left[ 1 - x_1^{2-np} \right. \]

\[ + \left. \left( \frac{2-np}{2+np} \right) (1-x_1^{2+np}) \left( \frac{R_{ao1}}{R_{ci}} \right)^{2np} \right] \]

and for \( n = 1, p = 2 \)

\[ C_{mn} = \frac{\sin(\Theta \Theta/2) \sin(\Theta \Theta/2)}{8} \left[ -\ln x_1 + \frac{1-x_1^4}{4} \left( \frac{R_{ao1}}{R_{ci}} \right)^4 \right] \]

Single winding: \( M_v = \frac{1}{2} M \)

Damper winding self inductance:

\[ L_{dn} = \frac{\mu_0 l d N^2}{8 np} \left[ 1 + \left( \frac{R_t}{R_{ci}} \right)^{2np} \right] \]
Armature phase $A$-damper mutual inducance (parallel windings):

For $np \neq 2$

\[
L_{ad1} = \frac{2\mu_0 L_{ad1}N_{at}N_{np} \sin(n\theta_{vae}/2) \cos np\phi}{np \theta_{vae} (1-x_i^2)(2-np)} \left(\frac{R_t}{R_{aoi}}\right)^{np} \left[ 1 + \frac{2-np}{2+np} \left(\frac{R_{aoi}}{R_{ci}}\right)^{2np} \right]
\]

$n = 1, p = 2$

\[
L_{ad1} = \frac{\mu_0 L_{ad1}N_{at}N_{np} \sin(\theta_{vae}/2) \cos 2\phi}{\theta_{vae} (1-x_i^2)} \left[ -\ln x_i + \frac{1-x_i^4}{4} \left(\frac{R_{aoi}}{R_{ci}}\right)^4 \right]
\]

Damper-field mutual inductance:

\[
L_{fdn} = \frac{2\mu_0 L_{fdn}N_{ft}N_{np} \sin(n\theta_{wfe}/2)}{\theta_{wfe} (1-y^2)n^2p} \left(\frac{R_{fo}}{R_t}\right)^{np} \left(\frac{1-y^{np+2}}{np+2}\right) \times \left[ 1 + \left(\frac{R_t}{R_{ci}}\right)^{2np} \right]
\]
APPENDIX II. A CORE LOADING ESTIMATE

The ferromagnetic core of a modified Gramme-Ring armature will be subjected to an inward attractive loading due to the presence of the strong magnetic field of the field winding currents. The spatial distribution of the loading depends upon the number of magnetic poles, with the number of "induced poles" on the surface of the core material equaling the number of poles on the field winding surface. The loading problem is simply that due to a magnet inside a cylinder of iron. In what follows, it will be assumed that the core is made of solid iron, although this will not be the case if the insulated core principle is adopted.

The force density on a magnetisable medium is proportional to the negative gradient of the permeability of the medium.

\[ \vec{F} = -\nabla/\mu \]  \hspace{1cm} (3-1)

Moving outward from the centerline of the machine, no changes in permeability are encountered until the first core-air gap interface is met. At this boundary the rate of change of permeability with radial distance from the centerline increases rapidly; the gradient of the permeability is positive, and the force on the inner core surface is attractive, as would be expected. On the outer core surface,
however, the gradient of the permeability is negative, and the force due to leakage fields is repulsive. The net surface force will be inward, and it is the purpose of this section to estimate this inward loading on the core.

The inward loading will be estimated using the Maxwell Stress Tensor. The model for this calculation is shown below,

![Figure B.1 Core loading estimate model.](image)

The surface $S$, shown in B.1, encloses the upper half of the core and has unit depth in the $s$-direction. It will be assumed that the radial repulsive force on the outer core surface (surface 4) is small enough to be negligible. With this in mind, the magnetic loading on the core can be found by calculating the radial traction on the inner core surface (surface 1). The radial traction is given by

$$\tau_r = T_{rr} n_r + T_{r\theta} n_\theta$$  \hspace{1cm} (B-2)
where, at surface 1

\[ T_{mn} = \mu_0 M_m H_n - \frac{1}{2} \mu_0 \delta_{mn} H_k H_k \]  

(B-3)

Therefore,

\[ T_{rr} = \frac{1}{2} \mu_0 \left( H_r^2 - H_\theta^2 \right) = \frac{1}{2} \mu_0 H_r^2 \]  

(B-4)

since \( H_\theta = 0 \) at \( r = R_{ci} \), the radial position of surface 1.

Hence the traction is given by

\[ \gamma_r = T_{rr} n_r = -\frac{1}{2} \mu_0 H_r^2 \]  

(B-5)

The negative sign represents the fact that the unit normal \( n_r \) points in the negative radial direction (attractive force).

The magnetic field intensity at the core surface is
given in Table VI. Using the \( n = 1 \) component in (B-5),
and noting that the angle \( \alpha \) is measured from the axis
of the field winding, yields

\[ \gamma_r = -\frac{1}{2} \mu_0 \frac{\sin^2(\Theta/m/2) \cos^2 \alpha}{9 \gamma} \left( \frac{R_{ifo}}{r} \right)^6 \left( 1 - y^2 \right)^2 \left( 1 + \left( \frac{r}{R_{ci}} \right)^2 \right)^2 \]  

(B-6)

The core loading varies as \( \cos^2 \alpha \) around the gap. For the
2000 MVA machine described in Table 1, the core loading
is on the order of

\[ \gamma_r = 50 \cos^2 \alpha \quad \text{(psi)} \]  

(B-7)


