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A PRIORI BOUNDS ON THE PERFORMANCE OF
OPTIMAL SYSTEMS

by

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ABSTRACT

The problem of bounding the optimal cost of nonlinear dynamical systems is studied. Conditions are given under which it is possible to upper bound the optimal performance of a nonlinear system with a quadratic cost functional by the optimal performance of a linear system with the same cost functional. Results are stated both in the frequency and the time domain. A sharp lower bound for the optimal performance of a linear system with quadratic penalty function is given in terms of the performance of a class of suboptimal systems. The application of this result to direct evaluation of the degree of suboptimality of a given design is studied. Two numerical examples illustrate the usefulness of the results in design procedures when computational and/or structural restrictions are present.

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CHAPTER I
INTRODUCTION

"El peor enemigo de lo bueno es lo perfecto"

1. **Motivation**

Some optimization problems—although they can be formulated in a precise mathematical form—are too large to solve by analytical or iterative techniques. Moreover, even if the optimal control can be found it may be very difficult to implement.

Linear programming, for instance, has been used to solve many problems of resource allocation (transportation routing, machine scheduling, product mix, oil refinery operations, etc.), but some problems are just too large for it—job scheduling for instance. Conceptually, linear programming could lead to an optimum assignment of start times for the thousands of jobs to be scheduled in a large shop, given some criterion, like "minimizing the idle machine time," but the number of steps necessary to reach the optimum solution—though finite—is so large as to render the method useless. In this application, linear programming is computationally inefficient since by the time the optimal solution has been reached, enough time has elapsed, even with the aid of the fastest computers, as to make the solution obsolete.

It can be said, in general, that the usual tools available for system optimization (Dynamic Programming, Calculus of Variations, Pontryagin's Maximum Principle etc.), are not useful

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† Superscripts refer to the items in the Bibliography.
on most "large problems"\textsuperscript{†} because of computational infeasibility.\textsuperscript{B-5} For this reason it is necessary to find short-cuts. The price paid, however, is that the solutions obtained are not optimal according to the minimization criterion. This type of solution, called suboptimal, is then a trade-off between a guaranteed optimal solution and a "good" one, if the computational effort to find the latter is considerably reduced.

Little attention has been paid to the problem of predicting "how far" from optimal a given suboptimal system is. When either because of computational infeasibility or because of easier or more convenient implementation (real time computation, for example) it is necessary to resort to suboptimal solutions, it becomes imperative to have at hand a way of determining the "quality" of the suboptimal system. A way of measuring the suboptimality of a system is to compare its performance with the optimal one.

In conclusion, because either

\begin{itemize}
  \item a. a computation of the optimal control is not feasible (large problems), or
  \item b. the implementation of the control is not practical (off-line computation) or
  \item c. initial guesses of the optimal performance are needed (iterative solutions) then, it is necessary to find suboptimal controls.
\end{itemize}

The aim of the present research is to find ways to evaluate the performance of suboptimal solutions without either computing the optimal one or using simulation.

\textsuperscript{†} A large problem is defined as one in which $10^{12}$ or more operations are required for its solution.
When the optimal solution is impossible to compute there is, as a rule, no knowledge of the value of its performance, and, therefore, it is impossible to judge beforehand how close a suboptimal control comes to the optimal control.

Upper and lower bounds can be used to fill this gap. An a priori lower bound will put a limit on the optimal performance of the system, and therefore enables one to compute a bound on the "efficiency" of a given control. An upper bound supplies information to the effect that the optimal performance is better than certain value. The difference between the upper and lower bounds defines an interval and the search for suboptimal controls can be stopped when the performance falls within this range, if the two bounds are close enough to each other. With this criterion the "efficiency" of the controls that belong to the suboptimal interval is guaranteed to be acceptable according to the definition of optimality.

Another important area in which a priori bounds play an important role is that of determining the effects on the performance when finite changes are made in some parameters of the system. This aspect of sensitivity analysis, from the practical point of view, is more relevant than the usual problem treated in the literature, i.e., when infinitesimal changes occur in the plant; the reason being that in actual situations the parameters used in the model of a physical system are nothing else but a rough approximation of the actual parameters. Therefore, it is important to know how the performance of an optimal system changes when a given parameter varies between some given limits. By means of upper and lower bounds we are able to guarantee
that for a certain range of parameter changes the optimal performance will remain within certain values.

2. Organization

The results of the present research are divided into two fairly independent parts. The first one, Chapters II and III, deals with the problem of determining a priori upper bounds on the performance of nonlinear dynamical systems for quadratic cost functionals.

Chapter II treats the problem entirely in the frequency domain and the analysis is restricted to scalar systems, that is, systems in which both the control and the output are scalar functions of time. There are two main ideas involved in the analysis presented in Chapter II. The first one is to control a nonlinear system with an optimal feedback law derived for a linear system (model) and then to express the performance of the nonlinear system associated with the chosen control as the sum of two terms, one of which is the optimal performance of the model. Conditions under which the second term is negative are found; when those conditions are satisfied then it is possible to give an upper bound on the optimal performance of the nonlinear system in terms of the optimal performance of a linear system.

The second idea used in Chapter II is to find a control for a nonlinear system such that the trajectories thus generated coincide with the optimal trajectories of a linear system with respect to the same cost functional. As before the performance of the nonlinear system for that given control is expressed as the sum of two terms, one of which is the optimal performance of the linear system and conditions under which the second term is nonpositive are given.
One of the interesting features of the analysis described above is that results dealing with the problem of determining when the product of two operators is a positive operator, previously restricted to stability analysis, can be used in the determination of upper bounds.

Chapter III deals with the extensions of the results given in Chapter II to systems in which the control and the output are not necessarily scalar functions. The results in this chapter are stated in the time domain.

The second part of this thesis, Chapter IV, treats the problem of determining lower bounds on the performance of linear systems with respect to quadratic cost functionals. The main result of this chapter is a sharp lower bound on the optimal performance of a system in terms of a suboptimal performance. One of the advantages of that result is that it is possible to give a qualitative measure of the degree of suboptimality of a given (suboptimal) design entirely in terms of the design on hand. Applications of the above ideas are presented in two specific examples; one of them calling for the design of feedback control in which only part of the state vector is allowed to be fed back and the other one deals with the evaluation of a suboptimal design for a large system.

Two appendices are included. Appendix A presents some theorems about positive real functions relevant to some derivations in Chapter II. In Appendix B the proof of one lemma used in several other proofs of Chapter II is given.

Each chapter is divided into sections. Section 2.5 is the fifth section of Chapter II. Definitions, theorems, lemmas, corollaries and equations are numbered separately within each section, thus theorem 3.4.1 is the first theorem of Section 3.4. When reference is made say, to equation 9,
it refers to the ninth equation of the current section, equation 3.4, is the fourth equation of Section 3 of the current chapter.
CHAPTER II
SCALAR SYSTEMS

In this chapter conditions under which it is possible to find a linear system with an optimal performance greater than the optimal performance of a given nonlinear system are derived.

1. Preliminary Concepts

In reference to dynamical systems the optimization problem is defined as follows:

Given the elements:

1. A system that satisfies a given Dynamical Equation

\[ \frac{dx(t)}{dt} = f[x(t), u(t), t] \]

where

- \( t \) is the current time
- \( x(t) \) is an \( n \) vector representing the state of the system at time \( t \).
- \( u(t) \) is an \( r \) vector representing the control of the system at time \( t \).
- \( f \) is a vector valued function of the state, the control and the current time.

2. A constraint set \( \Omega \) in \( r \)-space to which the control \( u(t) \) should belong.

3. \( x(t_0) \) a set of initial conditions.

4. A target set \( T \), in \( \mathbb{R}^n \), to which it is desired to drive the system, and

5. A cost functional

\[ J = \int_{t_0}^{t_f} L[x(t), u(t), t] \, dt \]
Find a control \( u^* \) that satisfies the constraint \( u^* \in \Omega \), transfers the system from the initial condition to the target set, and minimizes the performance \( J \).

Our aim is to be able to answer the following question:

How can we give bounds on \( \min_u J \) without computing \( u^* \)?

A coarser classification of the elements of the optimization problem is the following:

1. A control \( u \) and a set \( \Omega \) such that \( u \in \Omega \)
2. The state \( x \), the initial condition \( x_0 \) and the target set \( T \)
3. The cost functional \( J \)
4. The dynamic equation

One general approach to finding bounds on the optimal performance is to make simplifying assumptions on the above elements in order to reduce the computational effort and to make the changes in such a way as to be able to guarantee that the performance of the modified system fulfills the requirements of being less (or greater) than that of the original system.

In the present research only the type of modifications dealing with the dynamical equation have been treated. In this case the general statement of the problem is: Given the elements of the optimization problem then:

To find a lower bound

replace the vector valued function \( f \) by \( f_* \) such that

\[
\min_u J(u) \leq \min_u J(u)
\]

To find upper bound

replace the vector valued function \( f \) by \( f^* \) such that

\[
\min_u J(u) \geq \min_u J(u)
\]
then it can be said that in some sense a system described by the differential equation

\[
\frac{dx}{dt} = f_{\alpha}[x(t), u(t), t]
\]  

(1)
is "better" than a system described by the differential equation

\[
\frac{dx}{dt} = f[x(t), u(t), t]
\]  

(2)
and in a similar fashion a system described by the differential equation (2) is in some sense "better" than the system described by the differential equation

\[
\frac{dx}{dt} = f^*[x(t), u(t), t]
\]  

(3)

From the above considerations the problem of finding bounds on the optimal performance \( J \) is substituted by the problem of determining conditions under which it is possible to find a better system than the given one and such that the computations needed to find the optimal performance value of the modified system are simpler than those of the original system.

In the problems treated in the present research it will be assumed that \( \Omega \) is \( \mathbb{R}^r \) and that the target set is \( \mathbb{R}^n \).

The following definitions give a more precise meaning to the ideas presented in the paragraphs above:

**Definition 1.1.1** A system \( \Sigma_2 \) is said to be **better** than a system \( \Sigma_1 \) in some region \( S \) of the state space with respect to \( J \) if

\[\text{When reference is made to the two systems } \Sigma_1 \text{ and } \Sigma_2, \text{ it is assumed that the differential equations governing the motion of both systems is of the same order.}\]
\[ J_1^* = J_1(u_1^*, x_0) = \inf_{u_1} J(u, x_0) \geq \inf_u J(u, x_0) = J_2(u_2^*, x_0) = J_2^* \]

for all \( x_0 \in S \).

**Definition 1.1.2** A system \( \Sigma_2 \) is said to be **globally better** than a system \( \Sigma_1 \) with respect to \( J \) if it is better than system \( \Sigma_1 \) in \( \mathbb{R}^n \) with respect to \( J \).

**Definition 1.1.3** A system \( \Sigma_2 \) is said to be **universally better** than a system \( \Sigma_1 \) with respect to a class of cost functionals \( \mathcal{M} \) if system \( \Sigma_2 \) is globally better than system \( \Sigma_1 \) with respect to any \( J \in \mathcal{M} \).

The only problem treated in this investigation is the so called **Regulator Problem**. This is an optimization problem in which the cost functional is of the form \( J = \int_{t_0}^{t_f} \left[ u'(t)u(t) + x'(t)C(t)x(t) \right] dt \).

The only Dynamical Systems for which a solution to the regulator problem is known are those in which the dynamics are linear (Ref. A-2, K-4, K-3), and it will not be included here since there exist an extensive literature in the subject.

2. **Notation and Assumptions**

Denote by \( D \) the operator \( \frac{d}{dt} \). Let \( p \) be a polynomial of degree \( n \) with real coefficients, \( q \) and \( h \) be polynomials of degree less than \( n \). Let \( f(t, \cdot) \) be a nonlinear time varying mapping from the real line into the real line. \( x(t) \) an element of \( \mathbb{R}^n \) associated with the \( (n-1) \)-times differentiable function \( x(t) \) such that

\( \dagger \) There exists more general forms of the cost functional but they can be reduced to the form presented above. (See reference K-4)
\( x(t) = \text{col}[x(t), Dx(t), \ldots, D^{n-1}x(t)] \). A function \( x(t) \) is said to belong to \( C^n[0, \infty) \) if it is \( n \) times differentiable with respect to \( t \) and

\[
\lim_{t \to \infty} \|x(t)\|_E = 0,
\]

where \( \| \cdot \|_E \) denotes the Euclidian norm. Let \( S \) be a subset of \( \mathbb{R}^n \). Let \( \Sigma_1 \) be the system described by the equations (see Fig. 1)

\[
p(D)x(t) = u(t) \quad ; \quad h(D)x(t) = y(t)
\]

![Fig. 1 The System \( \Sigma_1 \)]

and let \( \Sigma_2 \) be the system described by the equations (see Fig. 2)

\[
p(D)x(t) + f[t, q(D)x(t)] = u(t) \quad ; \quad h(D)x(t) = y(t)
\]

![Fig. 2 The System \( \Sigma_2 \)]

Let \( J \) be the cost functional given by

\[
J(u, x_0) = \int_0^\infty \{u^2(t) + y^2(t)\} \, dt
\]
It will be assumed that the polynomials $p(s)$ and $h(s)$ do not have common factors and that the nonlinearity $f(t, \sigma)$ satisfies the condition $f(t, 0) = 0$.

3. **Purpose**

The object of the present chapter is to give some sufficiency conditions on $p$, $q$, $h$ and $f(t, \cdot)$ that will guarantee that the system $\Sigma_2$ is better, in a region $S$ with respect to $J$, than the system $\Sigma_1$.

It should be recalled that the optimal feedback law for the system $\Sigma_1$ (denoted by $k_1^*(D)x(t)$) is linear and time invariant $K^{-1}$ and there are methods to compute it $B^{-6}, L^{-1}, L^{-2}$ while in general the optimal feedback control law for a system of the form of $\Sigma_2$ is not known when the cost functional is quadratic in $u$ and $y$. If the sufficiency conditions to be derived in this chapter are satisfied then it is possible to obtain an upper bound to the optimal performance of the system $\Sigma_2$ by simply computing the optimal performance of the linear system $\Sigma_1$.

Intuitively it would be expected that if $f(t, q(D)x)$ is "close" to or in the "direction" of the optimal control law of the system $\Sigma_1$ then the system $\Sigma_2$ is better than the system $\Sigma_1$. To illustrate this point take the following example: If it is assumed that $f(t, q(D)x)$ is identically equal to the negative feedback law $k_1^*(D)x$ (the optimal feedback law for the linear system $\Sigma_1$) then, the performance of $\Sigma_2$ when no control is applied (i.e., $u(t) = 0$) coincides with the optimal performance of the system $\Sigma_1$. However, as it will be shown, under less restrictive conditions, the system $\Sigma_2$ is better than the system $\Sigma_1$.

The study of the problem in the present chapter is carried out in the frequency domain, the reason being that many concepts, relevant to the solution of our problem such as passivity, are displayed more
explicitly in that framework. Once the underlying ideas are understood, the extensions to multiple input-output systems becomes a less difficult task.

4. **The Two Main Theorems**

In this section two of the main theorems of the present chapter are derived. Theorem 2.4.1 was motivated by the observation made in Section 3. Essentially, the optimal feedback law for the linear system \(\Sigma_1\) (i.e., \(k_1^*(D)x\)) is applied to the nonlinear system \(\Sigma_2\). The performance of the system \(\Sigma_2\), generated by the feedback law \(k_1^*\), denoted by \(J_2(k_1^*)\), is obtained and conditions under which

\[ J_2(k_1^*) \leq J_1(k_1^*) = J_1^* \]

are derived. But, since by the optimality of \(k_2^*\),

\[ J_2^* = J_2(k_2^*) \leq J_2(k_1^*) \]

we want to guarantee that

\[ J_2^* \leq J_1^* \]

The above idea is not new in the literature and has been exploited by Rissanen\(^{R-2}\) and McClamrock\(^{M-1}\).

In the theorem 2.4.2, on the other hand, a different approach is used. In this case a feedback control law \((k_\beta(x(t)))\) (in general time varying and nonlinear) is applied to the system \(\Sigma_2\) in order to generate trajectories that are identical to the optimal trajectories of the system \(\Sigma_1\). Using the same type of arguments as in theorem 2.4.1 conditions are given as to guarantee that

\[ J_1^* \geq J_2(k_\beta) \geq J_2^* \]
Theorem 2.4.1: The system $\Sigma_2$ is better than the system $\Sigma_1$ in $S$ with respect to $J$ if

\[ \int_0^\infty f(t, q(D)x) [p(D)x - (pp + hh)^+(D)x] \, dt \leq 0 \]  \hspace{1cm} (3)

along trajectories satisfying

\[ f(t, q(D)x) + ([pp + hh]^+(D)x) = 0 \]  \hspace{1cm} (4)

b. All solutions of (4) with $x_0 \in S$ belong to $\mathbb{R}_0^+ \cup [0, \infty]$

Proof:

Assume that $x_0 \in S$. Apply the optimal feedback law of system $\Sigma_1$, namely $B^{-1}$

\[ k_1^*(D)x = p(D)x - ([pp + hh]^+(D)x) \] \hspace{1cm} (5)

to the system $\Sigma_2$. Then the trajectories thus generated satisfy the differential equation (4).

The performance of system $\Sigma_2$ associated with the feedback law $k_1$ is given by

\[ J_2(k_1) = \int_0^\infty \{ [p(D)x + f(t, q(D)x)]^2 + [h(D)x]^2 \} \, dt \]  \hspace{1cm} (6)

with $x(t)$ satisfying the differential equation (4).

Therefore, from equations (6) and (4) it follows that

\[ J_2(k_1^*) = \int_0^\infty \{ [p(D)x + f(t, q(D)x)]^2 + [h(D)x]^2 - [f(t, q(D)x) + [pp + hh]^+(D)x]^2 \} \, dt \] \hspace{1cm} (7)

$[g]^+$ denotes the left half-plane spectral factor of the polynomial $g$.

$\overline{p}$ is the polynomial such that $\overline{p}(s) = p(-s)$. 
From the above equations it follows that

$$J_2(k_1^*) = \int_0^\infty \left( \left[ p(D)x \right]^2 + \left[ h(D)x \right]^2 - \left[ p(D)x \right]^2 \right) dt$$

$$+ 2 \int_0^\infty f(t, q(D)x) \left\{ p(D)x - \left[ p + h\right]^+(D)x \right\} dt$$

(8)

The first integral of the right hand side of (8) is independent of $x(t)$ and is only a function of $x_0$ and $\lim_{t \to \infty} x(t)$. Also if $\lim_{t \to \infty} x(t) = 0$, that integral is identical to the optimal performance of system $\Sigma_1$ from the initial condition $x(0) = x_0$. But by assumption (b) if $x(0) \in S$ and $x(t)$ is a solution of (4) then $x(t) \in S^n[0, \infty)$, therefore

$$J_2(k_1^*) = J_1^* + \int_0^\infty f(t, q(D)x) \left\{ p(D)x - \left[ p + h\right]^+(D)x \right\} dt$$

(9)

and by assumption (a) it follows that

$$J_2^* \leq J_2(k_1^*) \leq J_1^*$$

(10)

therefore $\Sigma_2$ is better than $\Sigma_1$ in $S$ with respect to $J$.

Corollary 2.4.1: The system $\Sigma_2$ is better than the system $\Sigma_1$ in $S$ with respect to $J$ if

a. $\int_0^\infty f\left[ t, q(D)x \right] \left[ p(D)x + f(t, q(D)x) \right] dt \leq 0$

(11)

along trajectories satisfying (4), and

b. all solutions of (4) with $x(0) \in S$ belong to $S^n[0, \infty)$. 
Corollary 2.4.2: The system $\Sigma_2$ is globally better than the system $\Sigma_1$ with respect to $J$ if

a. Inequality (11) holds

b. The null solution of (4) is asymptotically stable in the large.

Comment:

As it was mentioned in Section 3, if $f(t, \cdot)$ is the "direction" of $-k_1(D)x(t)$ then it would be expected that the system $\Sigma_2$ is better than the system $\Sigma_1$. Equation (3) gives in some way a more precise meaning to the word "direction". Note that if $k_1(D)x(t) = -f(t, q(D)x)$ for all $x(t)$ then conditions (a) and (b) of theorem 2.4.1 are satisfied.

One of the drawbacks of the conditions given by theorem 2.4.1 and corollaries 2.4.1 and 2.4.2 is that besides testing for inequality (3) it is necessary to investigate the stability characteristics of the differential equation (4). To avoid this problem there are two alternatives: Either find conditions under which if inequality (3) is satisfied then stability condition (b) will automatically hold, or find an alternative feedback control law that will insure that the trajectories thus generated will approach zero as $t \to \infty$. We take the second alternative.

Theorem 2.4.2: System $\Sigma_2$ is better than system $\Sigma_1$ in $S$ with respect to $J$ if

\[ \int_0^\infty f(t, q(D)x) \left\{ 2p(D)x + f(t, q(D)x) \right\} dt \leq 0 \quad (12) \]

along trajectories satisfying

\[ \left[ pP + hH \right]^+(D)x(t) = 0 \quad (13) \]

and $x_0 \in S$. 
Proof:

When the feedback control law

\[ k_\beta (D)x = p(D)x + f(t, q(D)x) - \left[ p\bar{p} + h\bar{h} \right]^+(D)x \]  

(14)
is applied to system \( \Sigma_2 \) the trajectories thus generated satisfy the differential equation (13). The value of the performance of system \( \Sigma_2 \) associated with \( k_\beta \) is given by

\[ J_2(k_\beta) = \int_0^\infty \left\{ \left[ p(D)x + f(t, q(D)x) \right]^2 + \left[ h(D)x \right]^2 \right\} dt \]  

(15)

where \( x(t) \) satisfies (13) and \( x(0) \in S \). Since the trajectories satisfying (13) and the initial condition are the optimal trajectories of the system \( \Sigma_1 \) then

\[ J_2(k_\beta) = J_1^* + \int_0^\infty f(t, q(D)x) \left[ 2p(D)x + f(t, q(D)x) \right] dt \]

(16)

but by hypothesis the second integral of the right hand side of (16) is nonpositive then

\[ J_2^* \leq J_2[k_\beta] \leq J_1^* \]

and it can be concluded that the system \( \Sigma_2 \) is better than the system \( \Sigma_1 \) in \( S \) with respect to \( J \).

**Corollary 2.4.3:** System \( \Sigma_2 \) is globally better than the system \( \Sigma_1 \) if inequality (12) is satisfied along all trajectories satisfying (13).
5. **On Universally Better Systems**

In the previous section some sufficiency conditions that guarantee that a nonlinear system is better than a linear system have been derived. In this section specific conditions under which a nonlinear system is universally better than a linear system for a wide class of quadratic cost functionals are presented. For that purpose the class of nonlinearities that will be treated are classified as follows:

1. \( f(t, \cdot) = f(\cdot) \) and \( f \) is a first and third quadrant nonlinearity.
2. \( f(t, \cdot) = f(\cdot) \) and \( f \) is a monotone or odd monotone nonlinearity.
3. \( f(t, \cdot) = k(t)(\cdot) \) and \( k(t) \) is restricted.
4. \( f(t, \cdot) = (\cdot)k \).

In this section the results derived in stability theory about positive operators will be used extensively.

As was mentioned in definition 1.1.3, a system \( \Sigma_2 \) is universally better than a system \( \Sigma_1 \) with respect to certain classes of cost functionals \( M \). In the remainder of this chapter the class of cost functionals \( M \) will be any quadratic functional of the form

\[
J = \int_0^\infty \{u^2(t) + y^2(t)\} dt
\]

and \( h(s) \) and \( p(s) \) do not have common factors.

It is not clear if it is possible to find a system \( \Sigma_2 \) that is universally better than another system \( \Sigma_1 \) with respect to \( J \). To gain some insight into what class of systems might be universally better than another consider the following problem:

Given a linear time invariant system \( \Sigma_1 \) when does there exist another system \( \Sigma_2 \) taken among the linear and time invariant systems
of the same degree as $\Sigma_1$, such that the system $\Sigma_2$ is universally better than the system $\Sigma_1$?

The question of existence is completely solved by the following theorem:

**Theorem 2.5.1:** Given the system

$$\Sigma_1 : p(D)x = u$$

then there exists another system of the form

$$\Sigma_2 : p(D)x + q(D)x = u ; q(s) \neq 0$$

that is universally better than $\Sigma_1$ if and only if $p(s)$ has one or more zeros in $\text{Re}[s] > 0$.

**Proof: Sufficiency**

Assume that $p(s)$ has one or more zeros in $\text{Re}[s] > 0$ then factor $p(s)$ as follows

$$p(s) = r(s)m(s)$$

such that $m(-s)$ is a Hurwitz polynomial. Make

$$q(s) = r(s)m(-s) - p(s)$$

and define

$$p^*(s) \triangleq p(s) + q(s)$$

The difference between the optimal performance of the system $\Sigma_1$ and $\Sigma_2$ is given by

$$J^*_2 - J^*_1 = \int_0^\infty \left\{ \left[ p^*(D)x \right]^2 + [h(D)x]^2 - \left[ (p^*p^* + hh^+) + (D)x \right]^2 \right\} dt$$

$$- \int_0^\infty \left\{ \left[ p^*(D)x \right]^2 + [h(D)x]^2 - \left[ (p^*p^* + hh^+) + (D)x \right]^2 \right\} dt$$

(5)
\[ \begin{align*}
&= \int_0^\infty \left\{ \left[ p^*(D)x \right]^2 - \left[ p(D)x \right]^2 - \left[ \left[ p^*p^* + h\overline{h} \right]^+ (D)x \right]^2 \right. \\
&\quad + \left. \left[ \left[ p\overline{p} + h\overline{h} \right]^+ (D)x \right]^2 \right\} \, dt \\
&\text{but} \\
&\quad \overline{p}(D) = p^*(D) \overline{p}^*(D) \\
&\text{by construction, then} \\
&J^*_2 - J^*_1 = \int_0^\infty \left\{ \left[ p^*(D)x \right]^2 - \left[ p(D)x \right]^2 \right\} \, dt \\
\end{align*} \] 

However, the integral of the right hand side of equation (8) is independent of path (i.e., depends only on the value of \( x \) and its first \((n-1)\) derivatives at \( t=0 \) then

\[ J^*_2 - J^*_1 = \int_0^\infty \left\{ \left[ m(-D)r(D)x \right]^2 - \left[ m(D)r(D)x \right]^2 \right\} \, dt \] 

Denoting by

\[ z(t) \overset{\Delta}{=} r(D)x(t) \] 

the integral in the right hand side of equation (9) depends on \( z(t) \) and its first \((\beta-1)\) derivatives where \( \beta \) is the degree of \( m \). Then since \( m(-s) \) is a Hurwitz polynomial we can choose a trajectory such that

\[ m(-D)z(t) = 0 \] 

then

\[ J^*_2 - J^*_1 \leq 0 \]

with the inequality holding for a space of \( \beta \) degrees of freedom.

**Necessity:**

As in the sufficiency proof we will define
\[ p^*(s) = p(s) + q(s) \]

We will show first that if \( p^*(s) \) does not have the same zeros of \( p(s) \) that are in \( \text{Re}[s] < 0 \) then there exists initial conditions and quadratic cost functional for which

\[ J_2^* > J_1^* \quad (12) \]

Once the above statement has been proved we will consider systems \( \Sigma_2 \) where \( p^*(s) \) differs from \( p(s) \) only on the zeros with \( \text{Re}[s] = 0 \), then we will show that in the latter cases also there exist initial conditions and quadratic cost functionals for which inequality (12) holds.

Assume that there exists an \( s_0 \) with \( \text{Re}[s_0] < 0 \) and

\[ p(s_0) = 0 \quad \text{while} \quad p^*(s_0) \neq 0 \]

The difference between the cost functionals of the systems \( \Sigma_1 \) and \( \Sigma_2 \) is given by (6), and the integral of the right hand side of (6) is independent of path.

But by virtue of lemma 1B (Appendix B) it is possible to choose \( h^*(s) \) such that

\[ [p^*p^* + h^*h^*]^+(s_0) = 0 \quad (13) \]

Now if \( x(t) \) is chosen as

\[ x(t) = \text{Re} e^{s_0 t} \quad (14) \]

it follows that

\[ J_2^* - J_1^* = \int_0^\infty \left\{ [p^*(D)x]^2 + ([pp^* + hh^*]^+(D)x)^2 \right\} dt > 0 \quad (15) \]

The last part of the theorem is to show that if \( p(j\omega_0) = 0 \) and \( p^*(j\omega_0) \neq 0 \) then there exist initial conditions and quadratic cost functionals for which inequality (12) holds.
In a similar manner as the proof above choose
\[ x(t) = e^{-\sigma t} \text{Re}[e^{j\omega_0 t}] \]  
(16)
and \( h \) can be chosen (by virtue of lemma 1B) in such a way as to satisfy the condition
\[ [p^*p + hh^*]^{+(\sigma+j\omega_0)} = 0 \]  
(17)

If \( \sigma \) is made small enough it is possible to make the term
\[-\int_0^\infty [p(D)x]^2 dt \] of equation (6) as small as desired, therefore
\[ \int_0^\infty \left\{ [p^*(D)x]^2 + ([p^* - h\bar{h} + h^*h]^{+(D)x})^2 - [p(D)x]^2 \right\} dt > 0 \]  
(18)
for \( \sigma \) sufficiently small, then
\[ \dot{J}_2^* - \dot{J}_1^* > 0 \text{ for some } \sigma. \]

Most of the result that follow are based largely on corollary 2.4.3.

As it was mentioned before, many of the results in stability theory dealing with the problem of determining when the product of two operators is a positive operator will be used in this section. The reason is the following: Since the trajectories satisfying equation (4.13) approach zero as \( t \to \infty \), then in order to guarantee that a system \( \Sigma_2 \) is universally better than a system \( \Sigma_1 \) it is only necessary to verify inequality (4.12) for all \( x(t) \) that approach zero as \( t \to \infty \). With this idea in mind the next theorem follows trivially:

**Theorem 2.5.2:** System \( \Sigma_2 \) is universally better than system \( \Sigma_1 \) if
\[ \int_0^\infty f(t, q(D)x)\{2p(D)x + f(t, q(D)x)\} dt \leq 0 \]  
(19)
for all \( x(t) \in \mathcal{S}_o^n[0, \infty) \).
It should be made clear that if inequality (19) holds for all ⋂(t) ∈ {0, ∞} then inequality (4.11) is automatically satisfied. However, if (4.11) is satisfied for all ⋂(t) ∈ {0, ∞} it cannot be concluded that the system Σ₂ is universally better than the system Σ₁ because of the additional stability condition on the solution of the differential equation (4.4).

Let us consider now the nonlinear systems mentioned at the beginning of this section:

The Class Aₖ

Definition 2.5.1: A nonlinear function -f( )- is said to belong to the class Aₖ if (see Fig. 3)

\[ 0 ≤ \frac{f(\sigma)}{\sigma} ≤ k \quad \text{for all } \sigma \]

(20)

Fig. 3 An Element of the Class Aₖ
The following two properties of the nonlinearities of the class $A_k$ will be used

i) $\int_0^\infty (f[z(t)] \frac{dz(t)}{dt}) \, dt \leq 0$ provided that $z(t) \in \mathcal{C}^1[0, \infty)$ and $f \in A_\infty$ 

$$2 \int_0^\infty \left( f[z(t)] \frac{dz(t)}{dt} \right) \, dt \leq 0 \quad \text{provided that } f \in A_\infty$$

(20a)

ii) $\int_0^\infty z(t)f[z(t)] \left[ -k + \frac{f[z(t)]}{z(t)} \right] \, dt \leq 0$ provided that $f \in A_k$ and $z(t) \in L_2$ (square integrable function) 

(20b)

Lemma 2.5.1: Inequality (19) holds for all $x(t) \in \mathcal{C}^n[0, \infty]$ if

$$2p(s) = [cs-k]q(s)$$

(21)

where $c$ is a positive constant.

Proof:

Denote by

$$a[x(t)] \triangleq \int_0^\infty f[q(D)x] \left\{ 2p(D)x + f[q(D)x] \right\} \, dt$$

but by equation (21)

$$a[x(t)] = \int_0^\infty f[q(D)x] \left\{ cDq(D)x - kq(D)x + f[q(D)x] \right\} \, dt$$

(22)

Denoting by

$$z(t) = q(D)x(t)$$

then $z(t) \in \mathcal{C}^1[0, \infty)$ and $z(t) \in L_2$, therefore

$$a[x(t)] = c \int_0^\infty f[z(t)] \, Dz(t) \, dt + \int_0^\infty z(t)f[z(t)] \left\{ -k + \frac{f[z(t)]}{z(t)} \right\} \, dt$$

(24)
but by properties (i) and (ii) it follows that
\[ a[x(t)] \leq 0 \]

Theorem 2.5.3: System \( \Sigma_2 \) is universally better than system \( \Sigma_1 \) if \( f \in A_k \) and
\[ 2p(s) = [cs-k]q(s) \text{ with } c \geq 0 \]

Proof:
Follows immediately from lemma 2.5.1 and theorem 2.5.2.

Comment:
There exist a very close relationship between theorem 2.5.3 and Popov's criterion. In a similar manner as Popov's criteria theorem 2.5.3 has the same kind of limitations. From equation (21) two important restrictions of the class of systems for which theorem 2.5.3 could give information are the following:

a. The difference in degree of the polynomials \( p(s) \) and \( q(s) \) should not exceed unity.

b. \( p(s) \) should have at least one zero in \( \text{Re}[s] > 0 \).
This condition, however, is a trivial consequence of theorem 2.5.1.

Example 2.5.1: Assume that system \( \Sigma_2 \) is given by
\[ \frac{d^2x(t)}{dt^2} + f[\frac{dx(t)}{dt} + x(t)] - x(t) = u(t) \]
where \( f \in A_2 \). Then, given any quadratic cost functional it is possible to upper bound the performance of the system \( \Sigma_2 \) by simply computing the optimal performance of the linear system \( \Sigma_1 \),
\[ \frac{d^2x(t)}{dt^2} - x(t) = u(t) \]
The Classes $M_k$ and $O_k$

**Definition 2.5.2:** A nonlinear function $f(\cdot)$ is said to belong to the class $M_k$ if (see Fig. 4)

a. $f \in A_k$

b. $(\sigma_1 - \sigma_2)[f(\sigma_1) - f(\sigma_2)] \geq 0$ for all $\sigma_1$ and $\sigma_2$.

![Fig. 4 An Element of the Class $M_k$](image)

**Properties of the class $M_k$**

Of course the class $M_k$ has properties (i) and (ii) of the class $A_k$. The following two additional properties will be needed for later developments:

iii) $F(y) - F(x) + (x-y)f(x) \geq 0$ for all $x$ and $y$

where $F(\sigma) = \int_{0}^{\sigma} f(u)du$
Proof:

(See Fig. 5) The result is immediate from the graph. However it follows also very easily from the fact that

\[ \int_{x}^{y} [f(u) - f(x)] \, du \geq 0 \]

\[ \int_{0}^{\infty} x(t + \tau) f[x(t)] \, dt \leq \int_{0}^{\infty} x(t) f[x(t)] \, dt \] (26)

Fig. 5 If \( f \in M_k \) then \( F(y) - F(x) \geq (y - x)f(x) \)

Proof:

From property (iii) when \( y = x(t + \tau) \) and \( x = x(t) \) it follows that

\[ F[x(t + \tau)] - F[x(t)] + [x(t) - x(t + \tau)] f[x(t)] \geq 0 \] (27)

Integrating with respect to \( t \) from \(-\infty\) to \( x \) it is obtained that
\[
\int_{-\infty}^{\infty} F[x(t+\tau)] \, dt - \int_{-\infty}^{\infty} F[x(t)] \, dt + \int_{-\infty}^{\infty} x(t) f[x(t)] \, dt
\]

(28)

\[-\int_{-\infty}^{\infty} x(t+\tau) f[x(t)] \, dt \geq 0\]

From (28) inequality (26) follows trivially.

**Lemma 2.5.2:** Inequality (19) holds for any \( x(t) \in \mathcal{D}_{\emptyset}[0, \infty) \) if \( f \in M_k \) and

\[
2p(s) = [-c_0 - k + c_1 s] q(s) + r(s)
\]

(29)

where \( r(s) \) is a polynomial of degree strictly less than the degree of \( q(s) \) and if \( [A, b, c] \) is a realization \( \gamma^{-1} \) of \( \frac{r(s)}{q(s)} \) then

\[
w(t) = c'e^{At}b \geq 0 \quad \text{for all} \quad t \in [0, \infty)
\]

(30)

\[
\int_{0}^{\infty} w(t) \, dt - \frac{r(0)}{q(0)} \leq c_0
\]

(31)

and \( c_0, c_1 \) are nonnegative constants

**Proof:**

Define

\[
a[x(t)] = \int_{0}^{\infty} f[q(D)x] \{2p(D)x + f[q(D)x]\} \, dt
\]

(32)

but since \( f \in M_k \) then

\[
a[x(t)] \leq \int_{0}^{\infty} f[q(D)x] \{2p(D)x + kq(D)x\} \, dt
\]

(33)
Now, substitute equation (29) into (33) to obtain

\[ a[x(t)] \leq \int_0^\infty f[q(D)x] \left[ (-c_0 + c_1 D)q(D)x + r(D)x \right] dt \]  

(34)

Define

\[ z(t) = q(D)x(t) \quad \text{then} \]

(35)

\[ r(D)x(t) = \int_0^\infty w(\tau) z(t-\tau) d\tau \]

Therefore

\[ a[x(t)] \leq c_1 \int_0^\infty f[z(t)] Dz(t) dt + \int_0^\infty f[z(t)] \left\{ -c_0 z(t) + \int_0^t w(\tau) z(t-\tau) d\tau \right\} dt \]  

(36)

The first integral on the right hand side of inequality (36) is negative by property (i) the fact \( c_1 \geq 0 \) and \( z(t) \in [0, \infty) \). Define now

\[ g_\sigma(t) = \begin{cases} 0 & \text{for } t < 0 \\ g(t) & \text{for } 0 \leq t \leq \sigma \\ 0 & \text{for } t > \sigma \end{cases} \]

then,

\[ a[x(t)] \leq \lim_{\sigma \to \infty} \int_0^\infty f[z(t)] \left\{ -c_0 z(t) + \int_0^t w(\tau) z(t-\tau) d\tau \right\} dt \]

\[ = \lim_{\sigma \to \infty} \int_0^\infty f[z_\sigma(t)] \left\{ -c_0 z_\sigma(t) + \int_0^t w(\tau) z_\sigma(t-\tau) d\tau \right\} dt \]

\[ = \lim_{\sigma \to \infty} \left[ \int_0^\infty f[z_\sigma(t)] \left[ -c_0 z_\sigma(t) \right] dt + \int_0^\infty w(\tau) \int_0^\infty f[z_\sigma(t)] z_\sigma(t-\tau) d\tau dt \right] \]  

(37)
by using property (iv) and condition (30) it follows that

\[
a[x(t)] \leq \lim_{\sigma \to -\infty} \left[ \int_{-\infty}^{\infty} f[z_{\sigma}(t)] \left[ -c_0 x_{\sigma}(t) \right] \, dt \right.
\]

\[
+ \int_{0}^{\infty} w(\tau) \, d\tau \int_{-\infty}^{\infty} f[z_{\sigma}(t)] z_{\sigma}(t) \, dt \right]
\]

\[
\left. \to -\infty \right) \tag{38}
\]

then

\[
a[x(t)] \leq \int_{-\infty}^{\infty} f[z(t)] z(t) [-c_0 + \int_{0}^{\infty} w(\tau) \, d\tau] \, dt
\]

and finally, by inequality (31) it can be concluded that

\[
a[x(t)] \leq 0
\]

**Theorem 2.5.4:** System $\Sigma_2$ is universally better than system $\Sigma_1$ if $f \in M_k$ and

i) there exist positive constants $c_0$ and $c_1$ such that

\[
2p(s) = \left[ -c_0 - k + c_1 s \right] q(s) + r(s)
\]

ii) the degree of $r(s)$ is strictly less than the degree of $q(s)$ and if $[A, b, c]$ is a realization of $r(s)/q(s)$ then

\[
w(t) = ce^{At}b > 0 \text{ for all } t \in [0, \infty)
\]

iii) $\int_{0}^{\infty} w(t) \, dt = r(0)/q(0) \leq c_0$

**Proof:**

The above statement is an immediate consequence of lemma 2.5.2 and theorem 2.5.2.
Definition 2.5.3: A nonlinear function \(-f(\cdot)\) is said to be long to the class \(O_k\) if

\[
d) \quad f \in M_k
\]
\[
e) \quad f(\sigma) = -f(-\sigma)
\]
The additional property of the class \(O_k\) that will be used in lemma 2.5.3 is the following

\[
v) \quad \left| \int_{-\infty}^{\infty} x(t+\tau) \cdot f[x(t)] \, dt \right| \leq \int_{-\infty}^{\infty} x(t)f[x(t)] \, dt \tag{39}
\]

Proof:
In inequality (28) replace \(x(t+\tau)\) by \(-x(t+\tau)\) then

\[
\int_{-\infty}^{\infty} F[-x(t+\tau)] \, dt - \int_{-\infty}^{\infty} F[x(t)] + \int_{-\infty}^{\infty} x(t)f[x(t)] \, dt \geq 0 \tag{40}
\]

but by property (e), since \(F[-x(t+\tau)] = F[x(t+\tau)]\) then

\[
-\int_{-\infty}^{\infty} x(t+\tau)f[x(t)] \, dt \leq \int_{-\infty}^{\infty} x(t)f[x(t)] \, dt \tag{40a}
\]
Therefore inequality (40a) in conjunction with inequality (26) gives inequality (39).

With the additional property of nonlinearities of the class \(O_k\) less restrictive conditions on \(q(s)\) and \(p(s)\) can be derived in order to satisfy inequality (19). The following lemma gives those results:

Lemma 2.5.3: Inequality (19) holds for any \(x(t) \in \mathcal{D}_0^{+}[0, \infty)\) if \(f \in O_k\) and
\[ 2p(s) = [-c_0 - k + c_1 s]q(s) + r(s) \]  

(41)

where \( r(s) \) is a polynomial of degree strictly less than the degree of \( q(s) \) and if \( [A, b, c] \) is a realization of \( \frac{r(s)}{q(s)} \) then

\[
\int_0^\infty |c'te^{A_t}b|dt \leq c_0
\]  

(42)

where \( c_0 \) and \( c_1 \) are nonnegative constants.

**Proof:**

The proof of this lemma follows the same lines as the proof of lemma 2.5.2 with the difference that after inequality (3.7) the fact that

\[
\int_0^\infty \int_{-\infty}^\infty |c'te^{A_t}b| |z'(t)| z'(t-\tau)dt d\tau
\]  

(43)

is used. The remainder of the proof is the same.

**Theorem 2.5.5:** A system \( \Sigma_2 \) is universally better than a system \( \Sigma_1 \) if \( f \in \Omega_k \) and conditions (41) and (42) hold.

**Proof:**

Direct consequence of lemma 2.5.3 and theorem 2.5.2.

**The Time Varying Class**

In this section sufficient conditions that guarantee that a system \( \Sigma_2 \) in which \( f(t, q(D)x) = k(t)q(D)x \) is universally better than a system \( \Sigma_1 \) are derived.

**Lemma 2.5.4:** [Gruber and Willems] \( G^{-1} \) The integral

\[
a[x(t)] = \int_0^\infty k(t)m(D)x(t)n(D)x(t)dt
\]  

is nonpositive for all
a) \( x(t) \in \mathcal{D}^n_0(0, \infty) \)

b) \( 0 \leq k(t) < \infty \) for all \( t > 0 \)

c) \( \frac{dk(t)}{dt} \geq 0 \) for all \( t > 0 \) \hspace{1cm} (44)

if and only if

\[
\frac{-m(-s)}{n(-s)} \text{ is positive real} \hspace{1cm} (45)
\]

Proof:

Define

\[
\alpha_T[x(t)] = \int_0^T k(t)m(D)x(t)n(D)x(t)\,dt
\]

then integrating by parts the above integral it follows that

\[
\alpha_T[x(t)] = -k(t) \int_0^T m(D)x(t)n(D)x(t)\,dt + k(T)\int_0^T m(D)x(t)n(D)x(t)\,dt + \frac{dk(t)}{dt} \int_0^T m(D)x(t)n(D)x(t)\,dt \hspace{1cm} (46)
\]

or

\[
\alpha_T[x(t)] = -k(T) \int_0^T m(D)x(t)n(D)x(t)\,dt + k(0)\int_0^T m(D)x(t)n(D)x(t)\,dt + \frac{dk(t)}{dt} \int_0^T m(D)x(t)n(D)x(t)\,dt \hspace{1cm} (47)
\]

since

\[
\alpha[x(t)] = \lim_{T \to \infty} \alpha_T[x(t)]
\]
then

\[
a[x(t)] = \lim_{T \to \infty} \left\{ -k(T) \int_{0}^{T} m(D)x(\tau)n(D)x(\tau) d\tau \right\} + \\
k(0) \int_{0}^{T} m(D)x(t)n(D)x(t) dt + \int_{0}^{T} \frac{dk(t)}{dt} \int_{0}^{T} m(D)x(\tau)n(D)x(\tau) d\tau dt
\]

(48)

The first term in the above expression is zero since by condition (44b) \(k(T)\) is bounded for all \(T > 0\); the second term is nonpositive from (44a), (44b) and theorem 2A (Appendix). The third term is also nonpositive since its integrand is, by use of (44a), (44b) and theorem (2A) also nonpositive. The only if part follows from theorem 2A.

**Lemma 2.5.5:** The integral

\[
a[x(t)] = \int_{0}^{\infty} k(t)m(D)x(t)n(D)x(t) dt
\]

is nonpositive for all

a) \(x(t) \in \mathcal{S}_0^n[0, \infty)\)

b) \(0 \leq k(t) < \infty\) (49)

c) \(\frac{dk(t)}{dt} \geq -2\gamma k(t)\)

if and only if

\[
-\frac{m(-s+\gamma)}{n(-s+\gamma)} \text{ is positive real}
\]

(50)

**Proof:**

Let us introduce some preliminary facts

i) \(m(D)[x(t)e^{-\beta t}] = [m(D-\beta)x(t)]e^{-\beta t}\) (51)
ii) there exists a constant matrix $M$ such that

$$(m(D+a)[x(t)e^{-\gamma t}](n(D+a)[x(t)e^{-\gamma t}]) = e^{-2\gamma t}x'(t)Mx(t)$$  

(52)

Let

$$a[x(t)] = \int_0^\infty k(t)e^{2\gamma t}[m(D)x(t)] e^{-\gamma t}[n(D)x(t)] e^{-\gamma t}dt$$

by virtue of (51) it follows that

$$a[x(t)] = \int_0^\infty k(t)e^{2\gamma t}(m(D+a)[x(t)e^{-\gamma t}](n(D+a)[x(t)e^{-\gamma t}])dt$$  

(53)

proceeding in a similar fashion as in the proof of the previous lemma, after integration by parts of (53) and by taking the limit as $T \to \infty$ it is obtained that

$$a[x(t)] = \lim_{T \to \infty} \{ -k(T)e^{2\gamma t} \int_0^T [m(D+\gamma)(x(t)e^{-\gamma t})][n(D+\gamma)(x(t)e^{-\gamma t})]dt \}$$

$$+ k(0) \int_0^\infty [m(D+\gamma)(x(t)e^{-\gamma t})][n(D+\gamma)(x(t)e^{-\gamma t})]dt$$

$$+ \frac{dk(t)}{dt} + 2\gamma k(t)]e^{2\gamma t} \int_0^\infty [m(D+\gamma)(xe^{-\gamma \sigma})][n(D+\gamma)(xe^{-\gamma \sigma})]d\sigma dt$$  

(54)

The last two terms in the expression above are nonpositive, since $x(t)e^{-\gamma t} \in L^1_{\infty}[0, \infty)$ for $\gamma \geq 0$, (49b), (49c) and theorem 2A. It remains to be shown that the first term of the right hand side of (54) is zero. That fact can be proved as follows:

$$|k(T)e^{2\gamma t} \int_0^T [m(D+\gamma)(x(t)e^{-\gamma t})][n(D+\gamma)(x(t)e^{-\gamma t})]dt|$$

........... Continued on next page
\[ \leq k(T)e^{2\gamma t} \int_0^\infty \left[ m(D+\gamma)(x(t)e^{-\gamma t}) \right] \left[ n(D+\gamma)(x(t)e^{-\gamma t}) \right] \, dt \]

\[ \leq k(T)e^{2\gamma t} \int_0^\infty e^{-2\gamma t} \left| x'(t)Mx(t) \right| \, dt \text{ by virtue of (ii)} \]

\[ \leq k(T)e^{2\gamma t} \int_0^\infty e^{-2(\gamma+\epsilon)t} \left\| M \right\|_E dt \text{ for some } \epsilon > 0 \text{ by virtue of (44a)} \]

\[ = \frac{k(T)e^{-2\epsilon T}}{2(\gamma+\epsilon)} \left\| M \right\|_E \]

and

\[ \lim_{T \to \infty} \frac{k(T)\left\| M \right\|_E^2}{2(\gamma+\epsilon)} e^{-2\epsilon t} = 0 \]

**Theorem 2.5.6:** The system \( \Sigma_2 \) is universally better than the system \( \Sigma_1 \) for all \( f(t,q(D)x) = k(t)q(D)x \) if

a) \( 0 < k(t) < k^* \)

b) \( \frac{dk(t)}{dt} \geq -2\gamma \)

c) \( \frac{-2p(-s+\gamma)}{q(-s+\gamma)} - k^* \) is positive real

**Proof:**

The inequality

\[ \int_0^\infty k(t)q(D)x[2p(D)x+k(t)q(D)x] \, dt \leq 0 \]

in conjunction with theorem 2.5.2 leads to the claim of the theorem.
The Linear Time Invariant Class

In this section it will be assumed that \( f(t, q(D)x) \) is of the form \( kq(D)x \).

**Lemma 2.5.6:** Inequality (19) holds for all \( x(t) \in \mathcal{X}[0, \infty] \) if and only if

\[
\frac{2p(s)}{q(s)} = -Z(-s) - k \tag{54a}
\]

and \( Z(s) \) is positive real.

**Proof:**

Let us define

\[
\alpha[x(t)] = \int_0^\infty kq(D)x(t)[2p(D)x(t) + kq(D)x(t)] \, dt \tag{55}
\]

and

\[
\frac{m(-s)}{q(-s)} = Z(-s) \tag{56}
\]

Then

\[
\alpha[x(t)] = -\int_0^\infty [kq(D)x(t)][m(-D)x(t)] \, dt
\]

But by virtue of theorem 2A, \( \alpha[x(t)] \) is negative if and only if \( -Z(-s) \) is positive real.

**Theorem 2.5.8:** The system \( \Sigma_2 \) is universally better than the system \( \Sigma_1 \) for \( f(t, q(D)x) = kq(D)x \) if and only if condition (54a) is satisfied.

**Proof:**

Use lemma 2.5.6 and theorem 2.5.2.

**Example 2.5.2:** Consider the following specific case

\[
\Sigma_2: D^2x + b_2Dx + a_2x = u
\]

\[
\Sigma_1: D^2x + b_1Dx + a_1x = u
\]
Find conditions on $a_1$, $a_2$, $b_1$ and $b_2$ such that $\Sigma_2$ is universally better than $\Sigma_1$.

Solution:

Apply Theorem 2.5.8. In this case

$$p(s) = s^2 + b_1 s + a_1$$

and

$$q(s) = (b_2 - b_1) s + (a_2 - a_1)$$

then we want to find conditions under which

$$z(s) = \frac{2s^2 - 2b_1 s + 2a_1 - (b_2 - b_1)s + a_2 - a_1}{-(b_2 - b_1)s + a_1 - a_2}$$

is p.r., or

$$z(s) = \frac{2s^2 - (b_2 + b_1)s + a_2 + a_1}{(b_2 - b_1)s + a_1 - a_2}$$

These conditions are

$$-b_2 - b_1 > 0$$

$$b_2 - b_1 > 0$$

$$a_1 + a_2 > 0$$

$$a_1 - a_2 > 0$$

and they can be reduced to

$$a_1 > 0 \quad b_1 < 0$$

$$|a_2| < a_1 \quad |b_2| < -b_1$$

The above result is apparently new in the literature and it simply says that a second order system, with a damping less in magnitude and with a spring constant less in magnitude than a system with negative damping and positive spring constant, will have a lower quadratic criteria independent of the penalty on $x$ and $\frac{dx}{dt}$.

6. On Globally Better Systems

The conditions under which a nonlinear system $\Sigma_2$ is universally
better than a linear system derived in the previous section are easy to apply. The drawback of course is that the upper bounds obtained by computing the performance of the linear system are in general not sharp. Improvements of the upper bounds can be expected if the quadratic cost functional is specified (i.e., \(-h(s)\) is fixed) since it is required that the nonlinear system performs better than the linear only with respect to a specific cost functional. Nevertheless the problem becomes more difficult because, in contrast with the results of Section 5, it is necessary to determine the negativity of an operator not in all \(\mathcal{B}_Q^N(0,\infty)\) but for a manifold of it. That manifold in our case is the one form by the solutions of a linear time invariant differential equation. We will present here only an extension of theorem 2.5.3.

**Lemma 2.6.1:** Inequality (3.12) holds for all \(f \in A_k\) along trajectories satisfying (3.14) if there exist a nonnegative constant \(c_1\) and \(c_2\) arbitrary such that

\[
2p(s) = [c_1 s - k]q(s) + c_2[p + h] + (s) \quad (1)
\]

**Proof:**

Denote by

\[
a[x(t)] = \int_0^\infty f(q(D)x)\{2p(D)x + f(q(D)x)\}dt \quad (2)
\]

and by using equation (1), it follows that

\[
a[x(t)] = \int_0^\infty f(q(D)x)\{(c_1 D - k)q(D)x + c_2[p + h] + (D)x + f(q(D)x)\}dt
\]
but since $x(t)$ satisfies (2.13) then

$$a[x(t)] = \int_{0}^{\infty} f[q(D)x] \{(c_1 D - k)q(D)x + f(q(D)x)\} \, dt \quad (3)$$

Denoting by $z(t) = q(D)x(t)$ then $z(t) \in \mathcal{D}_{1}[0, \infty)$ and $z(t) \in \mathcal{L}_{2}[0, \infty)$ therefore

$$a[x(t)] = c_1 \int_{0}^{\infty} f[z(t)] \, Dz(t) + \int_{0}^{\infty} z(t) f[z(t)] \left[-k + \frac{f[z(t)]}{z(t)}\right] \, dt \quad (4)$$

but by virtue of properties (5.20a) and (5.20b) of the class $A_k$ it follows that

$$a[x(t)] \leq 0$$

Theorem 2.6.1: The system $\Sigma_2$ is globally better than the system $\Sigma_1$ with respect to $J$ if condition (1) is satisfied.

Proof:

Immediate consequence of lemma 2.6.1 and corollary 2.3.3

Example 2.6.1: Consider the following second order systems

$$\Sigma_2: \frac{d^2 x(t)}{dt^2} + f\left(\frac{dx(t)}{dt}\right) + b_1 \frac{dx(t)}{dt} + a_1 x(t) = u(t)$$

$$\Sigma_2: \frac{d^2 x(t)}{dt^2} + b_1 \frac{dx(t)}{dt} + a_1 x(t) = u(t)$$

If $f \in A_k$ find conditions on $a_1$, $b_1$, and $k$ such that the system $\Sigma_2$ is universally better than the system $\Sigma_1$.

Solution:

Apply theorem 2.6.1. In the present case

$$p(s) = s^2 + b_1 s + a_1$$

$$q(s) = s$$
then, we want to find constants $c_1$ and $c_2$ referred to in theorem 2.6.1 such that

$$2s^2 + 2b_1 s + 2a_1 = [c_1 s - k] s + c_2 (s^2 + \beta_2 s + \beta_1)$$

where

\begin{align*}
\beta_1 &> 1 \quad \beta_2 > 0 \\
\beta_1^2 - a_1^2 &\geq 0 \\
\beta_2^2 - b_1^2 - 2(\beta_1 - a_1) &\geq 0
\end{align*}

(see Kalman (K-1))

The equations that the constants $c_1$, $c_2$ and $k$ should satisfy are the following:

$$c_1 + c_2 = 2$$

$$-k + c_2 \beta_2 = 2b_1$$

$$c_2 \beta_1 = 2a_1$$

therefore $c_2 = \frac{2a_1}{\beta_1}$ and $c_1 = 2(1 - \frac{a_1}{\beta_1})$.

We can guarantee that $c_1$ is positive by condition (ii) above. On the other hand

$$k = c_2 \beta_2 - 2b_1 = \frac{2a_1 \beta_2}{\beta_1} - 2b_1$$

then, if we restrict $b_1 < 0$ and $a_1 > 0$ we have

$$k \leq |2b_1|$$

(as it can be seen the above example is an improvement over example 2.5.2 of the previous section). The conditions are then

$$f \in A |2b_1| \quad a_1 > 0 \quad b_1 < 0$$

Sharper results will be obtained if we specified a given cost functional.

Assume for example, the following numerical values:
then in this case

\[ f \in \mathcal{A}_2 \]
CHAPTER III
GENERALIZATIONS TO THE VECTOR CASE

1. Preliminaries and Notation

In the present chapter the results presented in Chapter II will be
generalized to multivariable systems. In the sequel when reference
is made to system $\Sigma_1$ it will be understood to be the system described
by the equations

$$\Sigma_1: \quad \frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t)$$

$$y(t) = C'(t)x(t)$$

and when reference is made to system $\Sigma_2$ it will be understood to be
the system described by:

$$\Sigma_2: \quad \frac{dx(t)}{dt} = A(t)x(t) + f(t, x) + B(t)u(t)$$

$$y(t) = C'(t)x(t)$$

The cost functional is of the quadratic type

$$J(u, x_0) = \int_{t_0}^{T} \left[ y'(t)y(t) + u'(t)u(t) \right] dt$$

It will be assumed that the system $\Sigma_1$ is uniformly completely
controllable and uniformly completely observable.

The nonlinearity is assumed to have the property

$$f(t, 0) = 0$$
2. The Main Theorems

The theorems derived in the present section follow the same ideas as the theorems of Section 4 of the previous chapter.

**Theorem 3.2.1**: System $\Sigma_2$ is better than system $\Sigma_1$ is $S$ with respect to $J$ if

i) \[ \int_{t_0}^{T} f'[t, x(t)] K(t) x(t) dt \leq 0 \] (1)

along solutions of

\[ \frac{dx(t)}{dt} = [A(t) - B(t)B'(t)K(t)] x(t) + f(t, x(t)) \] (2)

with \[ x(t_0) = x_0 \in S \]

ii) the solutions of (2) are bounded in the interval \[ [t_0, T] \] for all \[ x_0 \in S \]

where $K(t)$ is the unique (positive semidefinite) solution of the Riccati equation

\[ \frac{dK(t)}{dt} = -K(t)A(t) - A'(t)K(t) - C(t)C'(t) + K(t)B(t)B'(t)K(t) \] (3)

with the boundary condition \[ K(T) = 0 \]

**Proof:**

Observe that the optimal feedback solution of system with respect to $J$ is given by $u_1^*(x) = -B'(t)K(t)x(t)$

Apply this control law to the system $\Sigma_2$ to obtain the equation
\[ \frac{dx(t)}{dt} = [A(t) - B(t)B'(t)K(t)]x(t) + f(t, x(t)) \] (6)

The performance of system $\Sigma_2$ associated with the feedback law (5) is

\[ J_2[u_1^*(x)] = \int_{t_0}^{T} [x'(t)C(t)C'(t)x(t) + x'(t)K(t)B(t)B'(t)x(t)] dt \] (7)

along solutions of (6) or

\[ J_2[u_1^*(x)] = \int_{t_0}^{T} \{x'(t)[C(t)C'(t) + K(t)B(t)B'(t)K(t)]x(t) \}
-2 \frac{dx'(t)}{dt} - A(t)x(t) - f(t, x(t)) + B(t)B'(t)K(t)x(t) \}K(t)x(t)} dt \] (8)

\[ = \int_{t_0}^{T} \{x'(t)[C(t)C'(t) + K(t)B(t)B'(t)K(t)]x(t) - 2 \frac{dx'(t)}{dt} - K(t)x(t) \}
+2x'(t)A'(t)K(t)x(t) + 2f'(t, x(t))K(t)x(t) - 2x'(t)K(t)B(t)B'(t)K(t)x(t) \} dt \] (9)

\[ = \int_{t_0}^{T} \{x'(t)[C(t)C'(t) - K(t)B(t)B'(t)K(t)]x(t) + K(t)A(t) + A'(t)K(t)]x(t) \} dt \]

\[ - \int_{t_0}^{T} \{x'(t)K(t)x(t) + x'(t)K(t)x(t) - 2f[t, x(t)]K(t)x(t) \} dt \] (10)

By using now the fact that $K(t)$ satisfies the differential equation (3) we can write
\[
J_2(u_1^*) = \int_{t_0}^{T} \{ -x'(t)K(t) \frac{dx(t)}{dt} - \frac{dx'(t)}{dt} K(t)x(t) - x(t) \frac{dK(t)}{dt} K(t)x(t) \} dt
\]

\[
+ 2 \int_{t_0}^{T} \{ f'(t, x(t))K(t)x(t) \} dt
\]

(11)

The integrand of the first term of the right-hand side of (11) is a perfect differential, therefore

\[
J_2(u_1^*) = \int_{t_0}^{T} \left\{ -x'(t)K(t) \frac{dx(t)}{dt} \right\} dt + 2 \int_{t_0}^{T} \{ f'(t, x(t))K(t)x(t) \} dt
\]

= \int_{t_0}^{T} x'(t_0)K(t_0)x(t_0) - x'(T)K(T)x(t)

\[
+ 2 \int_{t_0}^{T} \{ f'(t, x(t))K(t)x(t) \} dt
\]

(12)

but from boundary condition (4) and from hypothesis (ii) it follows that

\[
J_2(u_1^*) = x'(t_0)K(t_0)x(t_0) + 2 \int_{t_0}^{T} \{ f'(t, x(t))K(t)x(t) \} dt
\]

(13)

but the first term of the right-hand side of (13) is exactly equal to $J_1^*(u_1) = J_1^*(u_1^*)K^{-3}$ then

\[
J_2(u_1^*) = J_1^*(u_1^*) + 2 \int_{t_0}^{T} \{ f'(t, x(t))K(t)x(t) \} dt
\]

(14)

therefore if condition (i) holds then

\[
J_2^* \leq J_1^* = J_1^*(u_1^*)
\]

and system $\Sigma_2$ is better than system $\Sigma_1$. | |
One of the motivations of the definitions and study of "better" systems was to find bounds on the performance of optimal systems. For that purpose some improvements over the previous theorem can be achieved by finding conditions under which

\[ J_2^* \leq q J_1^* \quad \text{where} \quad q > 0 \]

Those conditions can be given without too much effort from theorem 3.2.1 as follows:

**Collorary 3.2.1:** An upper bound for the optimal performance of the system \( \Sigma_2 \) is given by

\[ J_2^*(u, x_0) \leq q \frac{1}{2} x_0' K(t_0) x_0 \quad (15) \]

if

\[ t_0 \int_{t_0}^T \{ t'[t, x(t)] K(t)x(t) \} dt \leq \frac{q-1}{2} x_0' K(t_0) x_0 \quad (16) \]

where \( K(t) \) is the unique solution of the differential equation (3) with the boundary condition (4) and \( x(t) \) is the solution of (2) with \( x(t_0) = x_0 \) and all solution of (2) are bounded in the interval \([t_0, T]\).

**Proof:**

As in the proof of the theorem after equation (13) use inequality (16) then, \( J_2^* \leq x'(t_0) K(t_0) x_0 + (q-1) \frac{1}{2} x_0' K(t_0) x_0 = qx_0' K(t_0) x_0 \quad (15) \)

Of great interest is the case in which the system \( \Sigma_1 \) is time invariant and \( T \to \infty \) and should be considered separately. The following theorem furnishes conditions similar to those given in theorem 3.2.1.
Theorem 3.2. If $A, B$ and $C$ are constant matrices and $T \to \infty$ system $\Sigma_2$ is better than system $\Sigma_1$, in $S$ with respect to $J$ if
\[ \int_{t_0}^{\infty} f'[t, \dot{x}(t)] K x(t) dt \leq 0 \] (17)
along solutions of
\[ \frac{dx(t)}{dt} = [A - BB'K] x(t) + f[t, x(t)] \] (18)
with $x(t_0) = x_0 \in S$

ii) the null solutions of (18) is asymptotically stable, $S \subset R_a$ where $R_a$ is the region of attraction of the null solution of (18) and $K$ is the unique positive definite solution of the algebraic equation
\[ KA + A'K - KBB'K + C'C' = 0 \] (19)

Proof:

The proof of this theorem follows the same lines as the proof of theorem 3.2.1 and will be omitted. The only difference in the argument is that after equation (12) instead of using the boundary condition (4) in order to show that
\[ x'(T)K(T)x(T) = 0 \]
the fact that $x(t) \to 0$ as $T \to \infty$ is used based on the assumption (ii) in the statement of the theorem.

The counterpart of corollary 3.2.1 is the following:

**Corollary 3.2.2**: An upper bound for the optimal performance of the system $\Sigma_2$ is given by
\[ J^*_2(u_2, x_0) \leq q x_0 K x_0 \]
if

\[
\int_{t_0}^{\infty} \{f'[t, x(t)]Kx(t)\}dt \leq \frac{q-1}{2} x_0^T K x_0
\]  

(20)

where \( K \) is the unique positive solution of (19) and \( x(t) \) is the 
solution of (18) with \( x(t_0) = x_0 \) and the null solution of (18) 
is asymptotically stable in the large. The proof of this corollary 
will be omitted.

Comment:

The usefulness of theorems 3.2.1 and 3.2.2 is limited by the fact 
that in order to apply them it is necessary to find the required stability 
characteristics of equations (2) and (18). That might be a difficult 
task. The following theorem is a simplified version of theorem 3.2.1 
that overcomes the difficulty of condition (ii).

**Theorem 3.2.3:** If the matrices \( A, B \) and \( C \) are constant 
and

\[ f'(t, x)Kx < 0 \quad \text{for} \quad x \neq 0 
\]  

(21)

then the system \( \Sigma_2 \) is globally better than the system \( \Sigma_1 \).

**Proof:**

Condition (i) of theorem 3.2.2 is trivially satisfied. To show that 
condition (21) implies that condition (ii) is automatically satisfied, 
consider the following Lyapunov function

\[ v(x) = x^T K x \]  

(22)

then \( \frac{dv(x)}{dt} \) along solutions of (18) is given by

\[
\frac{dv(x)}{dt} = -x'C'x - x'^T K B B' K x + f'(t, x)Kx
\]

which is negative definite, therefore the null solution of (18) is asymptotically stable.
Another important class of nonlinear time varying systems that will be considered is the following:

\[ \dot{\Sigma}_2: \frac{dx(t)}{dt} = A(t)x(t) + B(t)g(t, x(t)) + B(t)u(t) \]  \hspace{1cm} (23)

\[ Y(t) = C'(t)x(t) \]  \hspace{1cm} (24)

Even though the above class of systems is a subclass of systems considered in theorem 3.2.1 additional conditions under which a system \( \Sigma_2 \) (as described by equations (23) and (24)) is better than system \( \Sigma_1 \) can be derived. The results are contained in the next theorem:

**Theorem 3.2.4:** System \( \Sigma_2 \) is better than system \( \Sigma_1 \) in \( S \) with respect to \( J \) if

\[ \int_{t_0}^{T} g'(t, x(t))\left[ g(t, x(t)) + 2B'(t)K(t)x(t)\right] dt \leq 0 \]  \hspace{1cm} (25)

along solutions of

\[ \frac{dx(t)}{dt} = \left[ A(t) - B(t)B'(t)K(t)\right]x(t) \]  \hspace{1cm} (26)

where \( K(t) \) is the unique positive solution of (3) with boundary condition (4), and \( x_0 = x(t_0) \in S \).

**Proof:**

A feedback law will be chosen in such a way that the trajectories of system \( \Sigma_2 \) generated by that feedback law coincide with the optimal trajectories of system \( \Sigma_1 \). The appropriate feedback law is given by

\[ u(x) = -B'(t)K(t)x(t) - g'[t, x(t)] \]  \hspace{1cm} (27)

then the trajectories of systems \( \Sigma_2 \) generated by the feedback law (27) satisfy (26).
The performance of system $\Sigma_2$ associated with the feedback law (27) is given by

$$J_2(u) = \int_{t_0}^{T} \left\{ x'(t)C(t)C'(t)x(t) + \left[ B'(t)K(t)x(t) + g(t,x(t)) \right]' \left[ B'(t)K(t)x(t) + g(t,x(t)) \right] \right\} dt$$

(28)

$$= \int_{t_0}^{T} \left\{ x'(t)C(t)C'(t)x(t) + x'(t)K(t)B(t)B'(t)K(t)x(t) \right\} dt$$

$$+ \int_{t_0}^{T} \left\{ 2x'(t)K(t)B(t)g(t,x(t)) + g'(t,x(t))g(t,x(t)) \right\} dt$$

(29)

The first integral of the above expression is identically equal to the optimal performance of the system $\Sigma_1$ then

$$J_2^* \leq J_2(u) = J_1^* + \int_{t_0}^{T} g'(t,x(t))[g(t,x(t)) + 2B'(t)K(t)x(t)] dt$$

(30)

therefore when inequality (25) is satisfied along solutions of (26) system $\Sigma_2$ is better than system $\Sigma_1$.

**Collorary 3.2.3**: An upper bound for the optimal performance of the system $\Sigma_2$ is

$$J_2^* \leq q x_0'[t_0,K(t_0)x_0]$$
if
\[ \int_{t_0}^{T} \left[ g(t, x(t)) + 2B(t)K(t)x(t) \right] dt \leq (q-1)x_0K(t_0)x_0 \] (31)
along solutions of (26).

3. On Universally Better Systems

It will be assumed in this section the class \( \mathcal{M} \) in the definition of universally better systems is the class of cost functionals

\[ J(u, x_0) = \int_{0}^{\infty} \{ u' \mathbf{u} + x' \mathbf{C} \mathbf{x} \} dt \] (1)

for all \( \mathbf{C} \)'s such that \( [\mathbf{A}, \mathbf{C}] \) is an observable pair. The existence of universally better systems is guaranteed by the following theorem:

**Theorem 3.3.1:** Given any time invariant linear system \( \Sigma_1 \) of the form

\[ \frac{d\mathbf{x}(t)}{dt} = \mathbf{A} \mathbf{x}(t) + \mathbf{B} \mathbf{u}(t) \]

and a cost functional (1), there always exists another linear system \( \Sigma_2 \) of the form

\[ \frac{d\mathbf{x}(t)}{dt} = \mathbf{A} \mathbf{x}(t) + \mathbf{D}(t)\mathbf{x}(t) + \mathbf{B} \mathbf{u} \]

universally better than system \( \Sigma_1 \)

**Proof:**

Consider the system \( \Sigma_2 \) given by

\[ \frac{d\mathbf{x}(t)}{dt} = (\mathbf{A} + k(t)I) \mathbf{x}(t) + \mathbf{B} \mathbf{u}(t) \]

with \( k(t) \leq \epsilon < 0 \). Since \( k(t)\mathbf{x'} K \mathbf{x} < 0 \) and \( K \) is positive definite for all \( \mathbf{J} \in \mathcal{M} \), then theorem 3.3.1 follows immediately from theorem 3.2.3.
The result of the above theorem seems to contradict the result of theorem 2.5.1, however it should be kept in mind that the form of the permissible class of systems $\Sigma_2$ in theorem 2.5.1 was more restrictive than the one allowed in this chapter. In contrast with the result there it is not necessary to require that the system $\Sigma_1$ be unstable in order to find a universally better system.

4. Globally Better Systems

In this section we will focus our attention to linear systems. Let us assume that $f(t,x) = D(t)x$ then the following theorem is an immediate consequence of theorem 3.3.1.

**Theorem 3.4.1:** The system $\Sigma_2$ is globally better than the system $\Sigma_1$ with respect to $J$ if $D(t)K(t)$ is positive semidefinite, where $K(t)$ is the solution of (23) with the boundary condition (24).

For the time invariant case an upper bound for the optimal cost of the system $\Sigma_2$ can be found by using the result of colorary 2.2.

**Theorem 3.4.2:** An upper bound for the optimal performance of the system $\Sigma_2$ is given by

$$J_2^*(u,x_0) \leq q x_0^T K x_0 ; \quad q > 0$$

if

$$(q-1)(CC' + KB'B') - q[D'K + KD] = HH'$$

for some matrix $H$ and if the pair $[A + D, C]$ is observable.

**Proof:**

The proof is in two parts. First it will be shown that condition (2) implies that the null solution of

$$\frac{dx(t)}{dt} = (A - B B'K + D)x(t)$$
is asymptotically stable in the large.

For that purpose consider the following Lyapunov function:
\[ v(x) = x'Kx \]
then along solutions of (3) \( \frac{dv(x)}{dt} \) is given by
\[
\frac{dv(x)}{dt} = x'[A'K - KB'B'K + D'K + KD + KA - KB'B'K]x
\]
but since \( A'K + KA - KB'B'K = -CC' \) (4)
then
\[
\frac{dv(x)}{dt} = x'[-CC' - KB'B'K + D'K + KD]x
\]
now, by using equation (1)
\[
\frac{dv(x)}{dt} = x'[-\frac{1}{q}HH' + (\frac{q-1}{q} - 1)(CC' + KB'B'K)]x
\]
but since \( q \) is assumed to be positive then \( \frac{dv(x)}{dt} \leq 0 \); for \( \frac{dv(x)}{dt} \) to vanish identically it is necessary to have \( B'Kx(t) = 0 \) which implies that (1) becomes
\[
\frac{dx}{dt} = (A + D)x
\]
but then \( C'x(t) \) can not be identically zero, since the pair \([A+D, C]\) was assumed to be observable. Then \( \frac{dv(x)}{dt} \leq 0 \) and not identically zero along solutions of (1). It should be noted here that if we define
\[
PP' = CC' + HH'
\]
the observability assumption can be weakened by requiring only that the pair \([A+D, P]\) be observable. The second part of the proof consists in showing that inequality (2.2.1) is satisfied, that is
\[
\int_{0}^{\infty} e^{(A+D - BB'K)t} (D'K + KD)e^{(A + D - BB'K)t} dt \leq 0
\]
if equation (2) holds for some \( H \).
From equations (2) and (4) it follows that

\[(q-1)[ -A'K - KA + 2KB'B'K ] - q[D'K + KD] = HH' \]

or

\[(q-1)[ -A'K - KA + 2KB'B'K - D'K - KD] = HH' + D'K + KD \]

\[(q-1)[ (A - BB'K + D)'K + K(A - BB'K + D) ] = HH' + D'K + KD \]

Pre- and post-multiplying the above equation by \(e(A - BB'K + D)'t\) and \(e(A - BB'K + D)t\) respectively, and by integrating from 0 to \(\infty\) it follows that

\[-(q-1) \int_0^\infty \left\{ \frac{d}{dt} (A - BB'K + D)'t \right\} K e(A - BB'K + D)t dt \]

\[= \int_0^\infty e(A - BB'K + D)'t HH' e(A - BB'K + D)t dt \]

\[+ \int_0^\infty e(A - BB'K + D)'t (D'K + KD) e(A - BB'K + D)t dt \]

Performing the integration of the left-hand side and using the fact proved in the first part, that the eigenvalues of \((A - BB'K + D)\) are in \(\text{Re}[s] < 0\) it is obtained that

\[(q-1)K = \int_0^\infty e(A - BB'K + D)'t HH' e(A - BB'K + D)t dt \]

\[+ \int_0^\infty e(A + D - BB'K)'t (D'K + KD) e(A + D - BB'K)t dt \]

which implies that inequality (6) is satisfied, and by colllorary 3.2.2 inequality (1) holds.
A very similar result to the above theorem was presented in Ref. M-1.
CHAPTER IV
ON LOWER BOUNDS ON THE PERFORMANCE OF
THE REGULATOR PROBLEM

In this chapter the problem of determining a priori lower bounds
on the optimal performance of linear systems is analyzed. Then the
problem of evaluating the degree of suboptimality of a given design is
studied. Two examples illustrating the advantages of the obtained re-
results are given.

1. Notation and Preliminaries

Consider the linear time varying system described by the equation
\[
\Sigma_1: \quad \frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) \tag{1}
\]
\[
\psi(t) = C'(t)x(t) \quad ; \quad x(t_0) = x_0 \tag{2}
\]
and the cost functional
\[
J(u, x_0) = \int_{t_0}^{T} [\psi'(t)\psi(t) + u'(t)u(t)] dt \quad ; \quad \infty > T > t_0 \tag{3}
\]
also denote by
\[
\|u\|^2 \triangleq \int_{t_0}^{T} u'(t)u(t)dt \tag{4}
\]
\[
\|\psi\|^2 \triangleq \int_{0}^{T} \psi'(t)\psi(t)dt \tag{5}
\]
and the operator $G$ mapping $R^{rx}[t_0, T]$ into $R^{mx}[t_0, T]$ as

$$G_u \triangleq \int_{t_0}^{t} C'(t) \bar{\Phi}_A(t, \sigma) R(\sigma) u(\sigma) d\sigma ; \quad t_0 \leq t \leq T$$  \hfill (6)

where $\bar{\Phi}_A(t, t_0)$ is the transition matrix associated with the matrix $A(t)$ that is $\bar{\Phi}_A$ should satisfy the following two conditions

i) $\bar{\Phi}_A(t_0, t_0) = I$

ii) $\frac{d\bar{\Phi}_A}{dt}(t, t_0) = A(t)\bar{\Phi}_A(t, t_0)$ \hfill (8)

The gain of the operator $G$ is defined as

$$g = \| G \| = \sup_{u \in L^2[t_0, T]} \| \int_G u \|$$ \hfill (9)

The inner product of two vectors functions of time, $\Psi(t)$ and $\Gamma(t)$ is defined as

$$<\Psi(t), \Gamma(t)> = \int_{t_0}^{T} \Psi'(t)\Gamma(t)dt$$ \hfill (10)

$\mathcal{X}_0$ will denote the homogeneous output of system $\Sigma_1$, that is

$$\mathcal{X}_0(t) = C'(t) \bar{\Phi}_A(t, t_0)x_0$$ \hfill (11)

The performance associated with the control $u(t) \equiv 0$ will be called the homogeneous performance and is denoted by

$$J_0 \triangleq \| \mathcal{X}_0 \|^2$$ \hfill (12)

Note that for any $u(t)$ the performance associated with that control, by virtue of linearity, can be expressed alternatively as

$$J(u, x_0) = \| u \|^2 + \| Gu + \mathcal{X}_0 \|^2$$ \hfill (13)
2. **A Fundamental Inequality**

In this section we will derive an inequality that is essential for the development of latter sections. First we will derive a theorem giving some characterization of optimal feedback controls.

**Lemma 4.2.1**: (A characterization of optimal controls). If \( u^* \) is the optimal control for the system \( \Sigma_1 \) and \( y^* \) is the corresponding output then

\[
J^* = \langle y^*, y_0 \rangle = \int_{t_0}^{T} y_{\text{out}}^*(t) y_0(t) dt
\]

**Proof:**

It has been determined \( K \geq 0 \) that the optimal trajectories generated by the optimal control satisfy the equation

\[
\frac{dx(t)}{dt} = [A(t) - B(t)B'(t)K(t)] x(t)
\]

where \( K(t) \) is the unique (positive semidefinite) solution of the Matrix Riccati equation

\[
A'(t)K(t) + K(t)A(t) - K(t)B(t)B'(t)K(t) + \frac{dK(t)}{dt} = -C(t)C'(t)
\]

and \( K(T) = 0 \). In addition, the optimal cost is given by

\[
J^* = x_0' K(t_0) x_0
\]

Denote by \( \Phi_A(t, t_0) \) the transition matrix associated with the matrix \( A(t) - B(t)B'(t)K(t) \) then, after pre- and post-multiplying \( (3) \) by \( x_0' \) \( \Phi_A^{-1}(t, t_0) \) and \( \Phi_K(t, t_0) x_0 \) respectively, one obtains that
\[ x_0' \Phi_A(t, t_0) A(t) K(t) \Phi_K(t, t_0) x_0 + x_0' \Phi_A(t, t_0) K(t) \left[ A(t) - B(t) B'(t) K(t) \right] \Phi_K(t, t_0) x_0 \]

\[ + x_0' \Phi_A(t, t_0) \frac{dK(t)}{dt} \Phi_K(t, t_0) x_0 = -x_0' \Phi_A(t, t_0) C(t) C'(t) \Phi_K(t, t_0) x_0 \]

However, the left-hand side of equation (5) is a perfect differential, then equation (5) can be written as

\[ \frac{d}{dt} \left( \Phi_A(t, x_0, t_0) K(t) \Phi_K(t, x_0, t_0) \right) = -x_0' \Phi_A(t, x_0, t_0) C(t) C'(t) \Phi_K(t, x_0, t_0) \]

(6)

where we have used the following definitions:

\[ \Phi_A(t, x_0, t_0) \equiv \Phi_A(t, t_0)x_0 \]  

(7)

\[ \Phi_K(t, x_0, t_0) \equiv \Phi_K(t, t_0)x_0 \]  

(8)

Integrating both sides of (6) from \( t_0 \) to \( T \) it is obtained that

\[ -x_0' K(t_0)x_0 + \Phi_A(T, x_0, t_0) K(T) \Phi_K(T, x_0, t_0) = -<y_0, y^*> \]

(9)

since

\[ y_0(t) = C'(t) \Phi_A(t, x_0, t_0) \]

and

\[ y^*(t) = C'(t) \Phi_K(t, x_0, t_0) \]

From (4), (6) and the boundary condition on \( K(t) \) it follows that

\[ J^* = <y_0, y^*> \]

Collorary 4.2.1: If \( J_0 \) and \( J^* \) are the optimal performance and the homogeneous performance of the system \( \Sigma_1 \) respectively, and if \( u^* \) is the optimal control then

\[ a) \quad J^* = J_0 - \| Gu^* \|^2 - \| u^* \|^2 \]  

(10)

\[ b) \quad J^* = J_0 + <Gu^*, y_0> \]  

(11)
Proof:
Because of linearity of the system, by relation (1.13) and equation (1) it is obtained that
\[\| u^* \|^2 + \| Gu^* \|^2 + 2\langle Gu^*, y_0 \rangle + \| y_0 \|^2 = \| y_0 \|^2 + \langle Gu^*, y_0 \rangle \]
(12)
or
\[\| u^* \|^2 + \| Gu^* \|^2 + \langle Gu^*, y_0 \rangle = 0 \]
(13)
then, by using equation (13) and equation (1.13) both equations (10) and (11) follow immediately.

Another interesting, nontrivial by-product of lemma 4.2.1 is the following:

**Collorary 4.2.2:** An upper bound for the optimal energy is
\[\| u^* \|^2 \leq \frac{J_0}{4} \]
(14)
and equality holds if and only if \( y^* = 2y_0 \).
Proof:
From equation (1) it follows that
\[\| u^* \|^2 + \| y^* \|^2 = \langle y^*, y_0 \rangle \]
(15)
then, by completing the square, it is obtained that
\[\| u^* \|^2 + \| y^* \|^2 - \langle y^*, y_0 \rangle + \| y_0 \|^2/4 = \| y_0 \|^2/4 \]
(16)
therefore
\[\| u^* \|^2 + \| y_0/2 - y^* \|^2 = J_0/4 \]
(17)
from which the statement of the collorary follows immediately.

**Theorem 4.2.1:** The ratio of the optimal performance to the homogeneous performance of system \( \Sigma_1 \) obeys the inequality
\[\frac{\text{optimal performance}}{\text{homogeneous performance}} \leq \frac{J_0}{4} \]

\[\text{The original statement and proof of this particular theorem is due to R.W. Brockett. His proof however is different to the one presented here.}\]
\[
\frac{J^*}{J_0} \geq \frac{1}{1+g^2}
\]  
(18)

where \( g = \| G\| \cdot B^{-4} \)

**Proof:**

By using equation (1) and the fact that \( \| G u \| \leq \| G \| \| u \| \) it follows that

\[
J^* \leq J_0 - \| G u^* \|^2 (1 + \frac{1}{g^2})
\]  
(19)

Similarly by using equation (11) and the Schwarz inequality, (i.e.,

\[
< G u^*, y_0 > \geq - \| G u^* \| \| y_0 \|
\]  
(20)

then

\[
J^* \geq J_0 - \| G u^* \| \| y_0 \|
\]  
(21)

therefore

\[
\| G u^* \| \geq \frac{J_0 - J^*}{\| y_0 \|}
\]

but since both sides of the inequality are nonnegative, then

\[
\| G u^* \|^2 \geq \frac{(J_0 - J^*)^2}{J_0}
\]  
(22)

By making the definition

\[
R = \frac{J^*}{J_0}
\]

it follows from (22) that

\[
\frac{\| G u^* \|^2}{J_0} \geq (1-R)^2
\]  
(23)

by dividing both sides of inequality (19) by \( J_0 \) it is obtained that

\[
R \leq 1 - \frac{\| G u^* \|^2}{J_0} \left( 1 + \frac{1}{g^2} \right)
\]  
(24)

By substituting inequality (23) into inequality (24) it follows that

\[
R \leq 1 - (1-R)^2 \left( 1 + \frac{1}{g^2} \right)
\]
or

\[(1-R) \geq (1-R)^2 \left(1 + \frac{1}{g^2}\right)\]  \hspace{1cm} (25)

Inequality (25) is satisfied (see Fig. 6) if and only if \( R \) satisfies

\[1 \geq R \geq \frac{1}{1+g^2}\]

**Fig. 6** Region for which inequality (25) is satisfied.

therefore

\[1 \geq \frac{J*}{J_0} \geq \frac{1}{1+g^2}\]

**Comment:**

The proof of the theorem above displays explicitly the two approximations made. For completeness we repeat them here:
a) \[ \| G u^* \|^2 \leq \| G \| \| u^* \| \] \hspace{1cm} (26)

b) \[ < G u^*, x_0 > \geq -\| G u^* \| \| x_0 \| \] \hspace{1cm} (27)

The difference between the right- and left-hand sides of inequality (26) becomes arbitrarily small for some \( u^* \). This fact follows immediately from the definition of the gain of the operator \( G \). Inequality (27), on the other hand, becomes equality whenever \( G u^* = cy_0 \) where \( c \) is a negative constant. Therefore it can be said that inequality (18) is the best of its kind, that is, given that only known parameter of the system \( \Sigma_1 \) is the gain of the operator \( G \) then there does not exist a function \( f(\| G \|) \) such that

\[ \frac{J^*}{J_0} \geq f(\| G \|) > \frac{1}{1 + \| G \|^2} \] \hspace{1cm} (28)

One of the natural questions that arise at this point is the following: Given that the only known parameter of the system is the gain of the operator \( G \) it is possible to find an upper bound strictly less than unity for the ratio \( \frac{J^*}{J_0} \)? Posing the problem in a more precise manner, does there exist a function \( f(\| G \|) \) such that

\[ 1 > f(\| G \|) \geq \frac{J^*}{J_0} \]

The answer to this question is negative as shown in the following theorem.

**Theorem 4.2.2:** Given any positive real number \( g \) then there exist a linear time invariant system with a transfer function \( G^{B-3} \) such that

\[ g = \| G \| \]

and a set of initial conditions such that

\[ J^* = J_0 \] \hspace{1cm} (29)
Proof:

Assume that

\[ G(s) = \frac{aq(s)}{p(s)} \]

and that \( p(s) \) is a Hurwitz polynomial. Without loss of generality it can be assumed that \( p(s) \) is chosen with all its roots in the real axis and \( a \) will be adjusted accordingly in order to make \( \| G \| = g \).

As it was shown earlier, the optimal performance of the system

\[ p(D)x(t) = u(t) \]
\[ aq(D)x(t) = y(t) \]

for the cost functional

\[ J(u) = \int_0^\infty \{ u^2(t) + y^2(t) \} \, dt \]

is given by

\[ J^* = \int_0^\infty \left\{ [p(D)x(t)]^2 + [aq(D)x(t)]^2 - [p\overline{p} + a^2 q\overline{q} + (D)x(t)^2] \right\} \, dt \]

(30)

where \( x(t) \) satisfies some prescribed initial conditions.

Let us assume that \( s_0 \) is a root of \( p(s) \). \( q(s) \) is chosen such that

\[ q(-s_0) = 0 \quad \text{but} \quad q(s_0) \neq 0 \]

(31)

therefore

\[ (p\overline{p} + a^2 q\overline{q})(s_0) = 0 \]

(32)

Since the integral of the right-hand side of (30) is independent of path, it can be evaluated along solutions of the equation

\[ p(D)x(t) = 0 \]

(33)
then

\[
J^* = \int_0^\infty \left\{ [a_1(D)x(t)]^2 - \left[ (p_0 + a_0^2q_0)(D)x(t) \right]^2 \right\} dt
\]

whenever \( x(t) \) satisfies (33).

If the initial conditions are chosen in such a way that

\[
x(t) = e^{s_0t}
\]

then by using (32) it follows that

\[
J^* = \int_0^\infty [a_q(s_0)]^2 e^{2s_0t} dt
\]

but on the other hand

\[
J_0 = \int_0^\infty [a_q(s_0)]^2 e^{2s_0t} dt
\]

therefore \( J^* = J_0 \).

Now based on theorem 4.2.1 the fundamental inequality of the section will be derived.

Consider the system \( \Sigma_2 \) described by the equations

\[
\begin{align*}
\dot{x}(t) &= F(t)x(t) + B(t)\dot{u}(t) \\
x(t) &= D(t)x(t)
\end{align*}
\]

and in a similar manner as in Section 1, define the operator \( H \) by the relation

\[
H_\dot{u} = \int_{t_0}^t \tilde{\Phi}(t, \sigma)B(\sigma)\dot{u}(\sigma) \, d\sigma \quad t_0 \leq t \leq T
\]

where \( \tilde{\Phi} \) is the transition matrix associated with the matrix \( F(t) \).
Theorem 4.2.3: The inequality

\[ \mathcal{X}_0 \int_{t_0}^{T} \Phi_F^{-1}(t, t_0)K(t)B(t)B'(t)K(t)\Phi_F(t, t_0)dt \leq (1 + \|K(t)\|^2_2) \times \mathcal{X}_0 \int_{t_0}^{T} J_P(t, t_0)D(t)D'(t)\Phi_F(t, t_0)dt \times \mathcal{X}_0 \]

holds for all \( \mathcal{X}_0 \) where \( K(t) \) is the unique (positive semi-definite) solution of the differential equation

\[ \frac{dK(t)}{dt} + F'(t)K(t) + K(t)F(t) - K(t)B(t)B'(t)K(t) = -D(t)D'(t) \]

with \( K(T) = 0 \)

Proof:

Pre- and post-multiplying equation (40) by \( \mathcal{X}_0 \Phi_F^{-1}(t, t_0) \) and \( \Phi_F(t, t_0)\mathcal{X}_0 \) respectively, and by noticing that the left-hand side becomes a perfect differential, it follows that

\[ \frac{d}{dt} \left( \mathcal{X}_0 \Phi_F^{-1}(t, t_0)K(t)\Phi_F(t, t_0)\mathcal{X}_0 \right) \]

by integrating equation (42) from \( t_0 \) to \( T \) and use of the boundary condition (41) it follows that

\[ \mathcal{X}_0 K(t_0)\mathcal{X}_0 + \int_{t_0}^{T} \mathcal{X}_0 \Phi_F^{-1}(t, t_0)K(t)B(t)B'(t)K(t)\Phi_F(t, t_0)\mathcal{X}_0 dt \]

by integrating equation (42) from \( t_0 \) to \( T \) and use of the boundary condition (41) it follows that

\[ \int_{t_0}^{T} \mathcal{X}_0 \Phi_F^{-1}(t, t_0)D(t)D'(t)\Phi_F(t, t_0)\mathcal{X}_0 dt \]
However, $x_0^t K(t_0) x_0$ is the optimal cost of the system $\Sigma_2$ with respect to the cost functional

$$J(u, x_0) = \int_{t_0}^{T} \{ u'(t) u(t) + y'(t) y(t) \} dt$$

and

$$\int_{t_0}^{T} x_0^t \Phi^t_F(t, t_0) D(t) D'(t) \Phi_F(t, t_0) x_0 dt$$

is the homogeneous performance of the system $\Sigma_2$, then

$$J^* + \int_{t_0}^{T} x_0^t \Phi^t_F(t, t_0) K(t) B(t) B'(t) K(t) \Phi_F(t, t_0) x_o dt = J_0$$

(44)

but from theorem 4.2.1

$$J^* \geq \frac{1}{1 + \| H \|^2} J_0 = J_0 - \left[ \frac{\| H \|^2}{1 + \| H \|^2} \right] J_0$$

(45)

then from equation (44) and inequality (45) the statement of the theorem follows immediately.

3. A Measure of the Degree of Sub-Optimality

In theorem 4.2.1 a lower bound for the ratio of the optimal to the homogeneous performance of a system has been given as a function of the gain the operator $G$. In this section a lower bound for the ratio of the optimal to some sub-optimal design will be given. We will restrict our study to those sub-optimal feedback laws given by

$$u(t) = -B(t) L(t) x(t)$$

(1)

where $L(t)$ is a positive semidefinite matrix that satisfies a differential equation of the form

$$\frac{dL(t)}{dt} + A(t)L(t) + L(t)A(t) - L(t)B(t)B'(t)L(t) + C'(t)C(t) = D'(t)D(t)$$

(2)

and $L(T) = 0$. 
Theorem 4.3.1: Given a suboptimal feedback law (1) where \( L(t) \) satisfies equation (2) then

\[
\frac{J^*_L}{J_L} \geq \frac{1}{1+g^2} + \frac{1}{1+g^2} \frac{x_0^T L(t_0) x_0}{J_L}
\]

(3)

where \( J_L \) is the performance associated with the feedback law (1) and \( g \) is the gain of the operator defined by

\[
GW = \int_{t_0}^{t} D'(t) \Phi_L(t, \sigma) B(\sigma) w(\sigma) d\sigma
\]

(4)

and \( \Phi_L \) is the transition matrix associated with the matrix \( A(t) - B(t)B'(t)L(t) \)

Proof:

By adding to both sides of equation (2.3) the matrix

\[
- K(t) B(t)B'(t)L(t) - L(t)B(t)B'(t)K(t)
\]

it is obtained that

\[
\frac{dK(t)}{dt} + K(t) [A(t) - B(t)B'(t)L(t)] + [A(t) - B(t)B'(t)L(t)]' K(t)
\]

\[- K(t)B(t)B'(t)K(t)\]

\[-C(t)C'(t) - K(t)B(t)B'(t)L(t) - L(t)B(t)B'(t)K(t) \]

(5)

Subtracting from both sides of equation (5) the matrix

\( L(t)B(t)B'(t)L(t) \) and rearranging terms it follows that

\[
\frac{dK(t)}{dt} + K(t) [A(t) - B(t)B'(t)L(t)] + [A(t) - B(t)B'(t)L(t)]' K(t)
\]

\[- [K(t) - L(t)] B(t)B'(t) [K(t) - L(t)] = -C(t)C'(t) - L(t)B(t)B'(t)L(t) \]

(6)
Now by pre- and post-multiplying the above equation by $x'_0 \Phi_L(t,t_0)$ and $\Phi_L(t,t_0)x_0$ respectively and integrating from $t_0$ to $T$, the following relation is obtained:

$$\int_{t_0}^{T} x_0^t K(t_0)x_0 + x'_0 \int_{t_0}^{T} \Phi_L(t,t_0)[K(t)-L(t)]B(t)B'(t)[K(t)-L(t)]\Phi_L(t,t_0)x_0 dt = J_L$$

therefore

$$J^* + x'_0 \int_{t_0}^{T} \Phi_L(t,t_0)[K(t)-L(t)]B(t)B'(t)[K(t)-L(t)]\Phi_L(t,t_0)x_0 dt = J_L$$

Since the matrix $M(t) = [K(t)-L(t)]$ satisfies the differential equation

$$\frac{dM(t)}{dt} + F'(t)M(t) + M(t)F(t) - M(t)B(t)B'(t)M(t) = -L(t)L'$$

where $F(t) = [A(t)-B(t)B'(t)L(t)]$ (9)

then by using theorem 4. 2. 3 and equation (8) it follows that

$$J_L - J^* \leq \|G\| \frac{2}{1+\|G\|} \left( \int_{t_0}^{T} x_0' \Phi_L(t,t_0)D'(t)D(t)\Phi_L(t,t_0)x_0 dt \right)$$

Now the left-hand side of inequality (10) can again be expressed in terms of $J_L$ and $x_0' \Phi_L(t_0)x_0$. In order to do this consider equation (2), by adding $L(t)B(t)B'(t)L(t)$ to both sides of the equation it is found that

$$\frac{dL(t)}{dt} + [A(t)-B(t)B'(t)L(t)]'L(t) + L(t)[A(t)-B(t)B'(t)L(t)]$$

...Continuted on next page

\[ \text{To verify the above statement simply substitute } M(t) \text{ by } [K(t)-L(t)] \] and use the differential equations that $K(t)$ and $L(t)$ satisfy.
After pre- and post-multiplying the terms of equation (11) by 
\( x_0 \Phi_L(t, t_0) \) and \( \Phi_L(t, t_0) x_0 \) respectively and by integrating every term, it follows that

\[
x_0 L x_0 = J_L - \int_{t_0}^{T} \Phi_L(t, t_0) D(t) D'(t) \Phi_L(t, t_0) x_0 \, dt
\]  

(12)

By using equation (12) in conjunction with inequality (10) the following relation is obtained

\[
J_L - J^* \leq \frac{\| G \|^2}{1 + \| G \|^2} \left( J_L - x_0 L(t_0) x_0 \right)
\]  

(13)

by dividing both sides of (13) by \( J_L \) then

\[
1 - \frac{J^*}{J_L} \leq \frac{\| G \|^2}{1 + \| G \|^2} \left( 1 - \frac{x_0 L(t_0) x_0}{J_L} \right)
\]

then

\[
\frac{J^*}{J_L} \geq \frac{1}{1 + \| G \|^2} + \frac{\| G \|^2}{1 + \| G \|^2} \left( \frac{x_0 L(t_0) x_0}{J_L} \right)
\]

Collorary 4.3.1

\[
\frac{J^*}{J_L} \geq \frac{1}{1 + \| G \|^2}
\]

The importance of both collorary 4.3.1 and theorem 4.3.1 are obvious for the following reasons:

a) The lower bound of the ratio \( J^* / J_L \) is given entirely as a function of \( L \),

b) The bound given in theorem 4.3.1 is sharp, that is, there exist initial conditions for which equality is attained.

c) The bound given in collorary 4.3.1 is independent of the initial conditions, and last
d) It is possible to give a quantitative measure of the degree of suboptimality of a large class of designs by evaluation of the lower bound on $J^*_L$.

It has been assumed that the optimization interval (i.e., $[t_0, T]$) is finite, and results given so far are based very strongly on the finiteness of $T$.

As has been mentioned before, time invariant systems are of great importance in conjunction with the optimization interval $[0, \infty]$. For this case the optimal feedback control is linear and time invariant and therefore very attractive from the implementation point of view.

It would be expected that the results given in theorem 4.3.1 generalizes for the case in which the system $\Sigma_1$ is time invariant and $T \to \infty$. However, some problems of existence and uniqueness are introduced that should be taken into account, for example, existence of integrals of the form $\int_{t_0}^{\infty} e^{-A(t)} R R'^t e^{-A(t)} x_0 \, dt$ is not guaranteed, unless specific assumptions on the location of the eigenvalues of the matrix $A$ are made. Also uniqueness of positive semidefinite solutions of the algebraic Riccati equation is not guaranteed. In order to overcome these difficulties several assumptions will be made and some introductory results will be proved before stating the version of theorem 4.3.1 for the case in which $T \to \infty$.

The system $\Sigma_1$ under consideration is the system described by the equations

$$\dot{x}(t) = A x(t) + B u(t) \tag{14}$$

$$\dot{y}(t) = C' x(t) ; \quad x(0) = x_0 \tag{15}$$
with the qualification that the system $\Sigma_1$ is both completely controllable and completely observable. The cost functional is given by

$$J(u, x_0) = \int_0^\infty \{u'(t)u(t) + y(t)y(t)\} dt \quad (16)$$

The matrix $L$ is assumed to be positive semidefinite and there exists a matrix $D$ such that

$$A'L + LA - LBB'L + C'C = D'D \quad (17)$$

It will be assumed that the suboptimal feedback law $u = -B'Lx$ is such that the eigenvalues of $(A - BB'L)$ lie entirely in $\text{Re}[s] < 0$.

The following theorem establishes the positive semidefiniteness of $K - L$.

**Theorem 4.3.2:** If the matrix $L$ satisfies equation (17) then $K - L$ is positive semidefinite where $K$ is the unique positive definite solution of the algebraic Riccati equation

$$A'K + KA - KBB'K = -CC' \quad (18)$$

**Proof:**

After some algebraic manipulation from equations (17) and (18) the following relation is obtained:


and since the eigenvalues of $[A - BB'K]$ are in $\text{Re}[s] < 0$ it follows that $K - L$ is positive semidefinite.

The following theorem guarantees the existence of a unique positive semidefinite solution of the algebraic Riccati equation.
Theorem 4.3.3: If the matrix $A$ is stable (all the eigenvalues are in $\text{Re}(s)<0$) and the pair $[A,B]$ is controllable, then there exists a unique positive semidefinite solution to the algebraic equation

$$A K + K A - K B B' K = -D'D$$

(20)

(Note here that the usual observability assumption on the pair $[A,D]$ is absent.)

Proof:

First we will show that if two solutions of (20), say $K_1$ and $K_2$ are such that the matrices are $[A - B B'K_1]$ and $[A' - B B'K_2]$ are stable then $K_1$ should be identical to $K_2$. So if $K_1$ and $K_2$ are solutions of (20) then

$$A K_1 + K_1 A - K_1 B B' K_1 = -D'D'$$
$$A'K_2 + K_2 A - K_2 B B' K_2 = -D'D'$$

by subtracting the two equations and rearranging terms it is obtained that

$$[A - B B'K_2][K_1 - K_2]^T + [K_1 - K_2][A - B B'K_1] = 0$$

However the equation $A X + X'B = 0$ has a unique solution $X = 0$ if $A$ and $-B'$ do not have common eigenvalues, then it can be concluded that

$$K_1 = K_2$$

By using the result obtained by Wonham (Theorem 4.1 of Ref. W-2) that if any positive semidefinite solution of (20), say $P$, with $A$ stable and $[A,B]$ controllable, has the property that the matrix $A - B B'P$ is stable. Then by the first part of the theorem it follows that equation (20) has a unique positive semidefinite solution.
For the system under consideration with the assumption that the matrix $A$ is stable, both lemma 4.2.2 and theorem 4.2.2 still hold. The proofs for the time invariant case and $T \to \infty$ are identical to the ones presented in the previous section, and will be omitted. By assuming stability of $A$, the existence of the terms in equations (2.9) and (2.22) are assured. Theorem 2.3 also holds for the time invariant case under the assumption that the matrix $F$ is stable and the pair $[F, B]$ is controllable. With the above observation we can state the following theorem:

**Theorem 4.3.4:** Given a suboptimal control law

$$u(t) = -B'Lx(t)$$  \hspace{1cm} (21)

with $L$ positive semidefinite, satisfying the algebraic equation (17) and $(A - BB'L)$ stable, then

$$\frac{J^*}{J_L} > \frac{1}{1+g^2} + \frac{g^2}{1+g^2} \frac{x_0'Lx_0}{J_L}$$  \hspace{1cm} (22)

where $g$ is the gain of the operator†

$$G_w = \frac{D'e^{(A-BB'L)(t-\sigma)}Bw(\sigma)d\sigma}{0}$$  \hspace{1cm} (23)

† It is not difficult to verify that the gain of the operator $G$ is given by

$$g^2 = \|G\|^2 = \sup_{\omega} \sup_{\mathbf{v}} \frac{\tilde{\mathbf{v}}'D'(-Ij\omega - A + BB'L)^{-1}BB'(Ij\omega - A + BB'L)^{-1}\tilde{\mathbf{v}}}{\tilde{\mathbf{v}}'\tilde{\mathbf{v}}}$$

where $\tilde{\mathbf{v}}$ is the complex conjugate of $\mathbf{v}$.
Proof:

Proceed as in theorem 4.3.1. The equation (27) becomes

\[ x_0'Kx_0 + x_0' \int_0^\infty e^{(A-BB'L)t} [K-L]BB'[K-L] e^{(A-BB'L)t} x_0 \, dt = J_L \]  

(24)

By virtue of the fact that feedback does not destroy controllability, theorem 4.3.2 and theorem 4.3.3, guarantee that the matrix \( M \) is the unique positive semidefinite solution of the algebraic equation

\[ [A-BB'L]'M + M[A-BB'L] - MBB'M = -DD' \]  

(25)

then by theorem 4.2.3

\[ J_L - J^* \leq 2 \left( \frac{\|G\|}{1 + \|G\|^2} \left( \int_0^\infty x_0'e^{(A-BB'L)t} DD'e^{(A-BB'L)t} x_0 \, dt \right) \right) \]  

(26)

By using the fact that the eigenvalues of \( A-BB'L \) are in the left-half plane then

\[ x_0' L x_0 = J_L - \int_0^\infty x_0'e^{(A-BB'L)t} DD'e^{(A-BB'L)t} x_0 \, dt \]  

(27)

therefore from equations (26) and (27) the claim of the theorem follows.

4. Frequency Domain Interpretation

One of the advantages of treating some problems in the frequency domain is that a great deal of insight is gained and new interpretation of results can be found that otherwise, in the time domain, would be very difficult to give. Above all, the most important advantages of the frequency domain analysis is that fundamental properties, such as the input output relations, are coordinate free (i.e., they do not depend upon the particular choice of the state variables).
Consider the algebraic equation that the matrix $L$ satisfies (i.e., 4.3.17), then, by adding and subtracting $L$'s to the left-hand side and rearranging terms it follows that

$$(-sI - A + BB' L)' L + L(\mathbb{I} - A + BB' L) = CC' - DD' + LB B' L$$

(1)

Defining now

$$\Phi_L = (\mathbb{I} - A + BB' L)^{-1}$$

(2)

then, by pre- and post-multiplying the terms of equation (1) by

$$B' \Phi_L'(-s) \text{ and } \Phi_L(s)B$$

respectively, it follows that

$$B' \Phi_L(s)B + B' \Phi_L'(-s)LB$$

(3)

$$= B' \Phi_L'(-s)CC' \Phi_L(s)B - B' \Phi_L'(-s)DD' \Phi_L(s)B + B' \Phi_L'(-s)LBB' L \Phi_L'(-s)B$$

By calling $K_L = LB$ (the suboptimal feedback gain matrix) then

$$B' \Phi_L'(-s)DD' \Phi_L'(+s)B$$

$$= B' \Phi_L'(-s)CC' \Phi_L(s)B + [I - B' \Phi_L'(-s)K_L][I - K_L \Phi_L(s)B]$$

(4)

It should be noted that $G(s) = D' \Phi_L(s)B$ is the Laplace transform associated with the convolution operator $G$ of equation (3.23). Similarly $H_K(s) = C' \Phi_L(s)B$ is the transfer function of the close loop system with the feedback law $u = -K_L x$ then equation (4) takes the simple form

$$G'(-s)G(s) = H'_K(-s)H_K(s) + [I - B' \Phi_L'(-s)K_L][I - K_L \Phi_L(s)B] - I$$

(5)

by replacing $s = j\omega$ then

$$G'(-j\omega)G(-j\omega) = H'_K(-j\omega)H_K(j\omega) + [I - B' \Phi_L'(-j\omega)K_L][I - K_L \Phi_L(j\omega)B] - I$$

(6)
From equation (5) it follows that if the matrix \( L \) satisfies an equation of the form of (3.17) implies (and is implied by) the condition\(^\dagger\)

\[
\left| H_K(j\omega) \right| + \left| 1 - K_L \Phi_L(j\omega)B \right| - I \geq 0
\]  

(7)

Note that only under optimality of the feedback law the equality holds, and in that case inequality (7) becomes the main result of Kalman.\( ^K-1 \)

In order to interpret inequality (3.3) in its simplest form assume that the system \( \Sigma_1 \) is a system with a scalar transfer function, then \( B, K_L \) and \( C \) become vectors and will be denoted by \( b, k \) and \( c \) respectively. Under these circumstances equation (5) becomes

\[
\left| G(j\omega) \right| = \left| h_k(j\omega) \right|^2 + \left| 1 - k L \Phi_L(j\omega)b \right|^2 - 1
\]  

(8)

Now, since \( G(j\omega) \) is exactly the operator \( G \) of theorem 4.3.4 in the frequency domain,\(^\ddagger\) then in order to find a bound for the suboptimal control law it is necessary to have

\[
\left| h_k(j\omega) \right|^2 + \left| 1 - k L \Phi_L(j\omega)b \right|^2 \geq 1
\]  

(9)

and if

\[
\Delta = \sup_{\omega} \left\{ \left| h_k(j\omega) \right|^2 + \left| 1 - k L \Phi_L(j\omega)b \right|^2 \right\}
\]  

(10)

then

\[
\frac{J^*}{J_L} \geq \frac{1}{\Delta}
\]

Notice that \( 1 - k L \Phi_L(j\omega)b \) in "classical control" language is nothing else but the inverse of the "return difference" of the feedback law \( -k L \Phi_L^{-2} \).

\(^\dagger\) \( \| A \| \) denotes \( \tilde{A} A \) where \( \tilde{A} \) is the adjoint (complex conjugate transpose) of \( A \).

\(^\ddagger\) \( G(j\omega) \) is the Fourier transform of the \( L_1 \) function \( \langle A - bk^t \rangle_t L^t b \).
If the return difference is denoted by $T_k(j\omega)$ the conditions for optimality becomes simply

$$|h_k(j\omega)|^2 + \frac{1}{|T_k(j\omega)|^2} = 1 \quad (11)$$

which says that the square of the magnitude of transfer function plus the inverse square of the return difference should be unity for all frequencies. The theorem of Kalman $K^{-1}$ which states that a feedback law is optimal with respect to some quadratic cost functional if and only if $|T_k(j\omega)| \geq 1$, follows immediately from the relation (11). In addition, we obtain the condition that the gain of the closed loop transfer function should be less than unity. Going back to our problem of suboptimality of given feedback law, in order to be able to apply our criterion of suboptimality we require that

$$|h_k(j\omega)|^2 + \frac{1}{|T_k(j\omega)|^2} \geq 1 \quad (12)$$

then

$$\frac{J^*}{J} \geq \frac{1}{\sup \{ |h_k(j\omega)|^2 + \frac{1}{|T_k(j\omega)|^2} \}} \quad (13)$$

The importance of the above result stems from its simplicity since once a feedback law is given then condition (12) is easily checked and the left-hand side of (13) is not difficult to compute, it is possible to give a quick measure of the degree of suboptimality of a given feedback law.

5. **Applications**

Suboptimal design with restricted feedback structure. One realistic assumption from the practical point of view is that not all the state variables of a system to be optimized can be measured directly.
Another restriction from the implementation point of view is that the feedback structure is restricted by the available instrumentation. With these constraints in mind the following example illustrates an application of some of the inequalities presented in this chapter.

**Problem:**

Given a system described by the equations (see Fig. 7)

\[
\begin{align*}
\frac{d^2}{dt^2} x(t) - \frac{dx(t)}{dt} + 3x(t) &= u(t) \\
y(t) &= \frac{dx(t)}{dt} + x(t)
\end{align*}
\]

![Fig. 7 System to be Optimized](image)

the restriction is that \( u(t) = -a \frac{dx}{dt} \) and the cost functional

\[
J = \int_{0}^{\infty} \left\{ u^2(t) + y^2(t) \right\} dt
\]

The problem is to determine a value of \( a \) such that the performance of the system is "close" to the optimal performance (if all the states were measurable and the structure of the feedback were not restricted).

**Solution:**

The closed loop transfer function of the system with a feedback of the form \( u(t) = -a \frac{dx(t)}{dt} \) is given by

\[
h(s) = \frac{s+1}{s^2+(a-1)s+3}
\]
and the return difference is given by

\[ T_a = \frac{s^2 + (a-1)s + 1}{s^2 - s + 3} \]

In order to have stability of the feedback system it is necessary to restrict \( a \) to be larger than unity. On the other hand, to satisfy condition (4.12) it is necessary to have

\[ \frac{(-\omega^2 + 3)^2 + 2\omega^2 + 1}{(-\omega^2 + 3)^2 + (a-1)^2 \omega^2} \geq 1 \]

which implies that \( a \leq \sqrt{2} + 1 \), therefore

\[ 1 < a \leq \sqrt{2} + 1 \]

If \( a \) is chosen to be \( \sqrt{2} + 1 \), then \( \Delta \) (as given by equation 4.10) is

\[ \Delta = \frac{5}{4} \]

therefore

\[ \frac{J^*}{J_s} \geq \frac{4}{5} \]

In Fig. 8 a lower bound for the ratio of \( \frac{J^*}{J_s} \) is displayed as a function of \( a \), for the allowable range of \( a \).

Fig. 8 Region where the Performance Ratio should lie as a function of the parameter \( a \).
However, the above lower bound is quite conservative since it is based on corollary 4.1. A sharper result is obtained if full advantage of the lower bound derived in theorem 4.4 is taken, that is, when the term \( \frac{2}{1+g_i} \frac{x_0 L x_0}{J_L} \) is included. Of course more computations are needed but still they are quite simple.

Let us assume, based on the graph of Fig. 8, that it has been decided to make

\[
u(t) = -(\sqrt{2} + 1) \frac{dx}{dt}
\]

We will compute the corresponding \( L \) matrix and the cost functional \( J_L \). The matrix \( L \) should be positive definite and satisfy the equation

\[
A'L + LA - LBB'L + CC' = 0
\]

In our case

\[
A = \begin{bmatrix} 0 & 1 \\ -3 & 1 \end{bmatrix}; \quad B'L = \begin{bmatrix} 0 \\ \sqrt{2}+1 \end{bmatrix}; \quad CC' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\]

then the value obtained for \( L \) is given by

\[
L = \begin{bmatrix} 6.23 & 0 \\ 0 & 2.41 \end{bmatrix}
\]

In order to obtain \( J_L = x_0'Qx_0 \) it is necessary to solve the linear equation

\[
(A - BB'L)Q + Q(A - BB'L) = -CC' - LBB'L
\]

the corresponding value of the matrix \( Q \) is

\[
Q = \begin{bmatrix} 6.83 & 0.166 \\ 0.166 & 2.53 \end{bmatrix}
\]
therefore

\[
\frac{\dot{x}_0^1 \cdot \frac{1}{J_L} \cdot \dot{x}_0}{J_L} = \frac{6.23 \beta^2 + 2.41}{6.83 \beta^2 + 0.33 \beta + 2.53}
\]

where

\[
\beta = \frac{x_1(0)}{x_2(0)}
\]

then a lower bound for the performance of the feedback

\[
u(t) = -\left(\sqrt{2} + 1\right) \frac{dx(t)}{dt}
\]

given by

\[
\frac{J^*}{J_L} \geq 0.8 + 0.2 \frac{6.23 \beta^2 + 2.41}{6.83 \beta^2 + 0.33 \beta + 2.53}
\]

In Fig. 9, a graph of the lower bound on the performance ratio versus the ratio of the initial position to the initial velocity is given. The result is quite impressive since the minimum of that lower bound occurs in the neighborhood of \(\beta = 1\) and its value is larger than \(0.977\), therefore

\[
\frac{J^*}{J_s} \geq 0.977
\]

then, no matter what type of feedback is used (even by allowing the feedback of \(x(t)\)) the performance cannot be improved by more than two percent! The optimal feedback solution, as computed by digital computer was obtained to be

\[
u(t) = -0.158 \cdot x(t) - 2.521 \frac{dx(t)}{dt}
\]

However, when simulation was made for the worst initial condition, that is, \(x(0) = 1\) and \(\frac{dx(0)}{dt} = 1\) it was difficult to distinguish between the optimal and suboptimal trajectories (see Figs. 10a and 10b).
Fig. 9 Lower bound on the suboptimal performance as a function of the ratio of the initial position to the initial velocity.
Fig. 10 Optimal and Suboptimal Trajectories for the "Worst" Initial Condition
Suboptimal Design of Weakly-Coupled Systems

The objective of this application is to demonstrate that by the use of the bounds derived in the present chapter it is possible to find suboptimal controls for large systems, composed of two (or more) weakly coupled systems, with a guaranteed performance ratio, by finding the optimal feedback controls of the individual systems that make up the large system. In order to present the above idea in a more precise context consider, for simplicity, that the large system is composed of only two weakly coupled systems.

Assume that a system, denoted by $S_1$ is given by

$$\dot{x}_1 = A_1 x_1 + B_1 u_1$$
$$y_1 = C_1^t x_1$$

and a system $S_2$ given by

$$\dot{x}_2 = A_2 x_2 + B_2 u_2$$
$$y_2 = C_2^t x_2$$

and the large scale system $S_L$ given by

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_{12} \\ A_{21} & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

The $S_L$ is said to be weakly coupled if the matrices $A_{12}$ and $A_{21}$ are in some sense "small" relative to the matrices $A_1$ and $A_2$.

Intuitively it would be expected that the optimal feedback law that minimizes the cost functional
\[ J = \int_{0}^{\infty} \{u_1' u_1 + u_2' u_2 + y_1' y_1 + y_2' y_2 \} dt \]

will be "close" to the decoupled feedback law generated when the systems \( S_1 \) and \( S_2 \) are assumed to be decoupled (i.e., when \( A_{12} \) and \( A_{21} \) are identically zero).

With the aid of the results presented in this chapter it is possible to give a qualitative measure of the degree of suboptimality when the above approximation is made.

In order to be able to apply the results derived in Section 4 certain preliminary considerations are in order.

Assume that \( K_1 \) and \( K_2 \) are respectively the unique positive definite solutions of the equations

\[ A_1' K_1 + K_1 A_1 - K_1 B_1 B_1' K_1 = -C_1 C_1' \] \( \ldots \) \( 1 \)

\[ A_2' K_2 + K_2 A_2 - K_2 B_2 B_2' K_2 = -C_2 C_2' \] \( \ldots \) \( 2 \)

the matrix

\[ L \triangleq \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \]

does not necessarily satisfy the condition imposed by equation 4.3.17, that is the matrix

\[ M = \begin{bmatrix} A_1' & A_{12}' \\ A_{12}' & A_2' \end{bmatrix} L + L \begin{bmatrix} A_1 & A_{12} \\ A_{12} & A_2 \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} B_1' & 0 \\ 0 & B_2' \end{bmatrix} \begin{bmatrix} C_1 C_1' & 0 \\ 0 & C_2 C_2' \end{bmatrix} L' \]

\[ = \begin{bmatrix} 0 & A_{21}' K_2 + K_1 A_{12} \\ K_2 A_{21} + A_{12}' K_1 & 0 \end{bmatrix} \]
is not always positive semidefinite. However if we consider instead
\[
L(\epsilon, \delta) = \begin{bmatrix}
(1-\epsilon)K_1 & 0 \\
0 & (1-\delta)K_1
\end{bmatrix}
\]
then
\[
M(\epsilon, \delta) = \begin{bmatrix}
\epsilon C_1C_1^T + (1-\epsilon)K_1B_1B_1^T & (1-\delta)A_2^TK_2 + (1-\epsilon)K_1A_{12} \\
(1-\delta)K_2A_{21} + (1-\epsilon)A_1K_1 & \delta C_2C_2^T + (1-\delta)K_2B_2B_2^T
\end{bmatrix}
\]
can be made positive semidefinite if \( \delta \) and \( \epsilon \) are made sufficiently large \((0 < \epsilon < 1, 0 < \delta < 1)\). It should be kept in mind that for our criterion of suboptimality to be applicable it is required to have the suboptimal closed loop system stable.

If \( M(\epsilon, \delta) \) is positive semidefinite then
\[
\frac{J^*}{J_L} \geq \frac{1}{1+g^2}
\]
where \( g \) is the appropriate gain defined by equation (3.23).

In order to demonstrate the procedure in detail consider the following specific system:
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5 \\
\dot{x}_6
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0.05 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-1 & -2 & -3 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -2 & 1 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
1 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\]
and the cost functional
\[
Y = x
\]
\[
J = \int_0^\infty \{ \sum_{i=1}^{2} u_i^2 + \sum_{i=1}^{6} y_i^2 \} \, dt
\]

If we consider the two systems

\[
S_1: \frac{dx}{dt} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad u_1 ; \quad y_1 = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}
\]

\[
S_2: \frac{dx}{dt} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad u_2 ; \quad y_2 = \begin{bmatrix} x_5 \\ x_6 \end{bmatrix}
\]

then the positive definite solutions of equations (1) and (2) are respectively

\[
K_1 = \begin{bmatrix} 2.19 & 1.83 & .41 \\ 1.83 & 3.81 & .96 \\ .41 & .96 & .45 \end{bmatrix}
\]

\[
K_2 = \begin{bmatrix} .921 & 4.2 & .24 \\ 4.20 & 1.36 & 4.22 \\ .24 & 4.22 & 1.67 \end{bmatrix}
\]

At this point it is necessary to pick the values of \( \epsilon \) and \( \delta \) in order to guarantee that the matrix \( M \) is positive semidefinite. One way of doing so is to choose \( \epsilon \) and \( \delta \) large enough as to make the matrix \( M \) diagonal dominant, that is that for every row the diagonal element is larger in magnitude than the sum of the absolute value of the off diagonal elements of that row. By Gersagorin theorem all the eigenvalues of \( M \) will be positive. By this method the tedious task of determining the nonnegativity (Sylvester test) of \( M \) will be avoided. However on the other hand the degree of suboptimality will be increased.

For our particular example the values of the parameters \( \delta \) and \( \epsilon \)
chosen are .01 and .1 respectively. For these values of $\epsilon$ and $\delta$ the corresponding $M$ matrix is

$$M = \begin{bmatrix}
.115 & .035 & .016 & 0 & .060 & 0 \\
.035 & .185 & .038 & 0 & -.04 & 0 \\
.016 & .038 & .118 & 0 & -.021 & 0 \\
0 & 0 & 0 & .011 & .010 & .040 \\
.060 & -.040 & -.021 & .010 & .188 & .070 \\
0 & 0 & 0 & .04 & .070 & .028
\end{bmatrix}$$

which is indeed positive definite. The closed loop system with the suboptimal feedback corresponding to the values of $\epsilon$ and $\delta$ becomes

$$\begin{align*}
\frac{dx}{dt} &= \begin{bmatrix}
0 & 1 & 0 & 0 & .05 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-1.38 & -2.86 & -3.40 & 0 & -.1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -2.24 & -3.22 & -3.67
\end{bmatrix}x
\end{align*}$$

and the suboptimal feedback law is

$$\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix} = \begin{bmatrix}
.38 & .86 & -.40 & 0 & 0 & 0 \\
0 & 0 & 0 & .24 & 4.22 & 1.67
\end{bmatrix}x$$

In order to have a measure of the degree of suboptimality of the given control law it is necessary to compute the gain of the operator $G$ defined by (4.23). After some algebraic manipulations the following relation is obtained:

$$G(s)G(-s) = \begin{bmatrix}
.115 -.153s^2 + .118s^4 & .06s -.04s^2 + .021s^3 \\
\frac{p_1(s)p_1(s)}{p_1(-s)p_1(s)} & \frac{p_1(-s)p_2(+s)}{p_1(-s)p_2(+s)} \\
-.06s + .04s^2 -.021s^3 & .011 -.108s^2 + .028s^4 \\
\frac{p_1(s)p_2(-s)}{p_1(-s)p_2(-s)} & \frac{p_2(s)p_2(-s)}{p_2(s)p_2(-s)}
\end{bmatrix}$$

where $p_1(s) = (s^3 + 3.40s^2 + 2.86s + 1.38)$
and \( p_2(s) = (s^3 + 3.67s^2 + 3.22s + 2.24) \)

the value of \( \|G\|^2 \) is 0.092

therefore \( \frac{J^*}{J_s} \geq \frac{1}{1.092} = 0.91 \)

then, the suboptimal feedback law has a guaranteed performance ratio of 0.91. However as it was demonstrated vividly in the previous example this lower bound is quite conservative since the term due to the initial conditions has been ignored.
CHAPTER V
CONCLUSIONS

The research presented in the first part of this thesis was motivated by the lack of systematic methods in finding optimal feedback control laws for systems with nonlinear dynamics and quadratic cost functionals. The contribution of the present work to the field of optimal control has been the study of the possibility of being able to give upper bounds to the optimal performance of a nonlinear system by computing the optimal performance of a linear system. One of the methods used was to derive the optimal feedback law of a linearized version of the nonlinear system and apply that control law to the nonlinear system. However this method had the disadvantage that in order to apply it, it is necessary to determine the closed loop stability characteristic of the nonlinear system with linear feedback. In order to overcome this difficulty a second method was investigated--namely, use of a nonlinear feedback control law such that the trajectories thus generated coincide with the optimal trajectories of a linear system, known a priori to be stable. In the development of this second method extensive use was made of the known results of the problem of determining when the product of two operators is a positive operator, thus bringing results already widely used in stability theory to the area of optimal control. Specific conditions were given under which the second method can be applied.

The second part of the research dealt with the problem of obtaining lower bounds on the performance of linear systems with quadratic cost
functionals. An attainable lower bound was derived in terms of a suboptimal performance, thus obtaining, a qualitative measure of the degree of suboptimality of a given design. The lower bound was given in terms of a measure of the deviation of a given control law from satisfying a certain optimality condition (the Riccati equation). Two numerical examples were presented in order to illustrate the usefulness of the derived lower bound in the design of suboptimal systems, when, either it is desired to avoid a large number of computations or, due to practical limitations, there exists a structural constraint on the class of allowable feedback controls. The main contribution of the second part of this research has been the derivation of a simple measure of the degree in which a given suboptimal design can be improved. This measure is specified entirely in terms of the given design hence avoiding the need for computation of the optimal performance.
APPENDIX A

POSITIVE REAL FUNCTIONS

**Definition 1A**
A function $f$ of a complex variable $s - f(s)$ is said to be **positive real** (p.r. in short) if

$$\text{Re}[f(s)] > 0$$

throughout the region

$$\text{Re}[s] > 0$$

of the complex $s$-plane where $f(s)$ is defined and

$$f(\bar{s}) = \bar{f}(s) \quad \text{for all complex } s.$$

**Theorem 1A** [Weinberg and Slepian]

If $f(s)$ is a rational function of $s - f(s) = \frac{q(s)}{p(s)}$ where $q(s)$ and $p(s)$ do not have common factors, then a necessary and sufficient condition for $f(s)$ to be positive real is that

a) \( \text{Re}\left[\frac{q(j\omega)}{p(j\omega)}\right] \geq 0 \) for all real $\omega$

b) $q(s) + p(s)$ has all its zeros in $\text{Re}[s] < 0$

**Proof: Necessity**

Assume that $f(s)$ is positive real, then (a) follows immediately from the definition of positive realness.

If $q(s_0) + p(s_0) = 0$ then

$$\frac{q(s_0)}{p(s_0)} = -1$$

$^\dagger$\textbf{a} indicates the complex conjugate of $a$. 

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therefore

\[ \text{Re} \left( \frac{q(s_0)}{p(s_0)} \right) = -1 < 0 \]

but since \( f(s) \) is positive real it follows that \( \text{Re}[s_0] < 0 \)

**Sufficiency** Define

\[ w(s) = \frac{1 - \frac{q(s)}{p(s)}}{1 + \frac{q(s)}{p(s)}} = \frac{p(s) - q(s)}{p(s) + q(s)} \]

then

\[ \text{Re} \left( \frac{q(s)}{p(s)} \right) \geq 0 \quad \text{if and only if} \quad |w(s)| \leq 0 \]

Apply now the maximum modulus theorem \( R-1 \) to the restriction \( w(s) \) to the closure of the right-half plane and use the fact that \( w(s) \) is analytic in \( \text{Re}[s] \geq 0 \) and \( \text{Re}[f(j\omega)] \geq 0 \) in order to conclude that

\[ |w(s)| \leq 1 \quad \text{for} \quad \text{Re}[s] \geq 0, \quad \text{then} \quad \text{Re} \left( \frac{q(s)}{p(s)} \right) \geq 0 \quad \text{for} \quad \text{Re}[s] > 0 \]

**Theorem 2A** (on integral properties of p. r. functions)

\[ \int_0^\infty p(D)x(t)q(D)x(t)dt \leq 0 \quad \text{for all} \quad x(t) \in \mathcal{C}_0^\infty[0, \infty) \]

if and only if \( -\frac{p(-s)}{q(-s)} \) is p. r.

**Proof: Sufficiency**

\[ \int_0^\infty p(D)x(t)q(D)x(t)dt = \int_0^\infty \{p(D)x(t)q(D)x(t) + [(Ev[pq]^{-1}(D)x(t))^2] \} dt \]

\[ - \int_0^\infty \{ (Ev[pq])^{-1}(D)x(t) \}^2 dt \]
The second integral is negative. The first integral is independent of path $B^{-1}$ and (depends only on $x(0)$ and $\lim_{t \to \infty} x(t)$) but since $x(t) \in \mathcal{C}^n_0[0, \infty)$ then

$$\lim_{t \to \infty} x(t) = 0$$

therefore we can write

$$F[x(0)] = \int_0^\infty \{p(D)x(t)q(D)x(t) + [(E\nu[pq])^{-1}(D)x(t)]^2\} dt$$

$$= \int_{t[x(0)]}^{t[0]} \{p(D)z(t)q(D)z(t) + [(E\nu[pq])^{-1}(D)z(t)]^2\} dt$$

where

$$z[t(x(0))] = x(0) \quad \text{and} \quad z[t(0)] = 0.$$  

Changing now $t$ by $-t$, it is obtained that

$$F[x(0)] = \int_{-t[0]}^{-t(0)} \{p(-D)z(-t)q(-D)z(-t) - [(E\nu[pq])^+(+D)z(-t)]^2\} dt$$

Choose a $z(t)$ such that

$$p(-D)z(-t) - q(-D)z(-t) = 0$$

with $z[t(x(0))] = x(0)$. But since $\frac{p(-s)}{q(-s)}$ is p. r. $z[t(0)] = 0$ then

$$F[x(0)] = \int_{-t[0]}^{-t(0)} \{-[p(-D)z(-t)]^2 - [(E\nu[pq])^+(+D)z(-t)]^2\} dt$$

which is certainly nonpositive.
Necessity: This is more complicated. Here we want to show that if \(- \frac{p(-s)}{q(-s)}\) is not p. r. then

\[
\int_{0}^{\infty} p(D)x(t)q(D)x(t)dt > 0 \quad \text{for some} \quad x(t) \in \mathcal{D}[0, \infty)
\]

If \(- \frac{q(-s)}{p(-s)}\) is not p. r. then, either

a) there exists an \(s_0\) with positive real part such that

\[
q(-s_0) - p(-s_0) = 0
\]

and

\[
\text{Re} \left[ -\frac{q(-j\omega)}{p(-j\omega)} \right] > 0 \quad \text{for all real} \ \omega, \quad \text{or} \quad (2)
\]

b) \(\text{Re} \left[ -\frac{q(-j\omega_0)}{p(-j\omega_0)} \right] < 0 \quad \text{for some real} \ \omega_0 \quad (3)
\]

Consider the first case. Define

\[
x_0(t) = a_0 e^{-s_0 t} + \tilde{a}_0 e^{-\tilde{s}_0 t} \in \mathcal{D}[0, \infty)
\]

then

\[
a[x_0(t)] = \int_{0}^{\infty} p(D)x_0(t)q(D)x_0(t)dt
\]

\[
= \int_{0}^{\infty} \left[ p(-s_0)a_0 e^{-s_0 t} + p(-\tilde{s}_0)\tilde{a}_0 e^{-\tilde{s}_0 t} \right] q(-s_0)a_0 e^{-s_0 t}
\]

\[
+ q(-\tilde{s}_0)\tilde{a}_0 e^{-\tilde{s}_0 t} \right] dt
\]
but by virtue of (1)

\[ a[x_0(t)] = \int_0^\infty \left[ p(-s_0) a_0 e^{-s_0 t} + p(-\tilde{s}_0) \tilde{a}_0 e^{-\tilde{s}_0 t} \right]^2 \, dt \geq 0 \]

Consider now the second case. Inequality (3) can be written as

\[ q(-j\omega_0)p(j\omega_0) + p(-j\omega_0)q(j\omega_0) > 0 \]

observe that by continuity in \( \omega \) it follows that there exists an \( \epsilon > 0 \) such that

\[ q(-j\omega_0 - \sigma_0)p(+j\omega_0 - \sigma_0) + p(-j\omega_0 - \sigma_0)q(j\omega_0 - \sigma_0) \geq 0 \]

for all \( |\sigma_0| < \epsilon \)

take now \( x(t) = e^{-\sigma_0 t} \left[ e^{-j\omega_0 t} + e^{j\omega_0 t} \right] e^{\sigma_0 t} \) if \( \sigma_0 > 0 \)
then

\[ \sigma[x(t)] = \int_0^\infty p(D)x(t)q(D)x(t)dt \]

\[ = \int_0^\infty e^{-2\sigma_0 t} \left[ p(-j\omega_0 - \sigma_0)e^{-j\omega_0} + p(j\omega_0 - \sigma_0)e^{j\omega_0} \right]\left[ q(-j\omega_0 - \sigma_0)e^{-j\omega_0} \right] \]

\[ + q(j\omega_0 - \sigma_0)e^{j\omega_0} \] \, dt

\[ = \int_0^\infty \left\{ e^{-2\sigma_0 t} \left[ p(-j\omega_0 - \sigma_0)q(-j\omega_0 - \sigma_0)e^{-2j\omega_0} + p(j\omega_0 - \sigma_0)q(j\omega_0 - \sigma_0)e^{2j\omega_0} \right] \right. \]

\[ + \left. \left[ p(-j\omega_0 - \sigma_0)q(j\omega_0 - \sigma_0) + q(-j\omega_0 - \sigma_0)p(j\omega_0 - \sigma_0) \right] e^{-2\sigma_0 t} \right\} dt \]
After performing the integration it is obtained that

\[ a[x(t)] = \text{Re}\left[ \frac{p(j\omega_0 - \sigma_0)q(j\omega_0 - \sigma_0)}{2(-j\omega_0 - \sigma_0)} \right] + \text{Re}\left[ \frac{p(j\omega_0 - \sigma_0)q(-j\omega_0 - \sigma_0)}{2\sigma_0} \right] \]

the term on the right is positive. Choose \( \sigma_0 \) small enough such that

\[ \left| \text{Re}\left[ \frac{p(j\omega_0 - \sigma_0)q(j\omega_0 - \sigma_0)}{2(-j\omega_0 - \sigma_0)} \right] \right| \leq \text{Re}\left[ \frac{p(j\omega_0 - \sigma_0)q(-j\omega_0 - \sigma_0)}{2\sigma_0} \right] \]

therefore \( a[x(t)] > 0 \).
APPENDIX B

Lemma 1B  Given any monic real polynomial $p(s)$ of order $n$ ($n > 2$) and any complex number $s_0$ with $\text{Re}[s_0] \neq 0$ then there exists another real polynomial $h(s)$ of order less than $n$ such that

$$p(s_0)p(-s_0) + h(s_0)h(-s_0) = 0$$

Furthermore there exist a polynomial $h$ of second order that satisfies the above equality.

Proof: i) Assume that $s_0$ is real. Then we will show that for any real number $x$ there exist real numbers $a$ and $b$ such that:

$$x = (as_0 + b)(-as_0 + b) = -a^2s_0^2 + b^2$$  \hspace{1cm} (1)

Simply choose $a$ such that

$$a^2 > \frac{|x|}{s_0^2}$$  \hspace{1cm} (2)

For that choice of $a$, $b$ is guaranteed to be real.

ii) The proof when $s_0$ is complex [$\text{Im}(s_0) \neq 0$] is more involved. We will show that for any complex number $x + jy$ then exists real constants $a$, $b$ and $c$ such that

$$(as_0^2 + bs_0 + c)(as_0^2 - bs_0 + c) = x + iy$$  \hspace{1cm} (3)

or

$$(as_0^2 + c)^2 - b^2s_0^2 = x + iy$$  \hspace{1cm} (4)

$\dagger \ j = \sqrt{-1}$
then in order to satisfy equation (4) for any \( s_0 \), given that \( s_0^2 = \sigma_0^2 + j\omega_0 \)
it is necessary to have

\[
a^2(\sigma_0^2 - \omega_0^2) + (2ac - b^2)\sigma_0 + c^2 = x \tag{5}
\]

\[
2a^2\sigma_0 \sigma_0 + (2ac - b^2)\omega_0 = y \tag{6}
\]

Substituting (6) into (5) it follows that

\[
a^2 = \frac{-x + \frac{y}{\omega_0} + c^2}{\sigma_0^2 + \omega_0^2} \tag{7}
\]

therefore if \( c \) is chosen such that

\[
c^2 > -x + \frac{y}{\omega_0} \tag{8}
\]
a will be real.

The problem now is to determine if it is possible to choose \( b \) real
and satisfy (6). The expression for \( b^2 \) as a function \( c \) is given by

\[
b^2 = \left( \frac{-x + \frac{y}{\omega_0} + c^2}{\sigma_0^2 + \omega_0^2} \right) \sigma_0^2 + 2\left( \frac{-x + \frac{y}{\omega_0} + c^2}{\sigma_0^2 + \omega_0^2} \right)^{1/2} c - \frac{y}{\omega_0}
\]

If \( c \to \infty \) then \( b \) approaches asymptotically to

\[
b^2 \to 2c^2 \left( \frac{\sigma_0^2}{\sigma_0^2 + \omega_0^2} + \frac{1}{\sqrt{\sigma_0^2 + \omega_0^2}} \right) - \frac{y}{\omega_0}
\]

however

\[
\frac{\sigma_0^2}{\sigma_0^2 + \omega_0^2} + \frac{1}{\sqrt{\sigma_0^2 + \omega_0^2}} > 0 \text{ for all } \sigma_0, \text{ then}
\]

\( b^2 \) \to positive number, therefore for \( c \) large enough both \( a \) and \( b \)
will be real.
REFERENCES


REFERENCES (Contd.)


REFERENCES (Contd.)
