Numerical and Analytical Modelling of Downhole Seismic Sources: The Near and Far Field

by

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Abstract

The physical and mathematical description of seismic sources is well developed in the fields of earthquake seismology and reflection seismology. In contrast, the description of radiation from seismic sources that have been placed in boreholes is poorly developed. Most of the work describing downhole seismic sources has consisted of the evaluation of integrals in the far field using the method of stationary phase. These far field descriptions, however, fail to describe long-standing experimental observations made in low velocity sediments surrounding fluid-filled boreholes. In particular, the far field descriptions fail to duplicate the relative increase in shear wave amplitude in the vertical direction for low velocity sediments. However, the far field representations do describe radiation into high velocity sediments fairly well.

Two approaches, one analytical and one numerical, were used in this thesis to enhance the description of radiation from downhole seismic sources. The numerical approach involved development of a Thomson-Haskell type algorithm to investigate radiation from axial, torsional, radial and volume point sources. The analytical approach was to verify, explain and most importantly extend the far field analysis for the empty and fluid-filled borehole.

Using the analytical and numerical approaches and comparison between the two techniques proved to be a powerful tool for the description of radiation from downhole seismic sources. The presence of fluid in the borehole, the presence of casing surrounding the borehole, and the velocities of the lithology surrounding the borehole were shown to be important elements governing the observed radiation. One exception, is that in the case of axial and torsional sources, radiation was found to be independent of borehole effects.

For radial and point sources, $P$ wave radiation is somewhat affected by the borehole but not significantly by the presence of fluid in the borehole. The $P$ wave radiation is essentially spherical with a slight perturbation along the borehole axis. The effect of casing on the $P$ wave radiation pattern is limited to reducing the amplitude.
Radiation of $S$ waves from radial and point sources was most significantly affected by the borehole. Radiation of $S$ waves from empty boreholes and into high velocity sediments surrounding fluid-filled boreholes was found to be qualitatively similar and well understood. Radiation of $S$ waves into low velocity sediments surrounding boreholes was found to be controlled by the tube wave travelling up the borehole. In fact, the tube wave generates conical wave fronts which produce part of the radiated $S$ wave field. This has been hypothesized on the basis of experimental evidence and here the physics and mathematics of the phenomena is thoroughly explored. It is shown that these conical wavefronts are in fact Mach waves analogous to their counterparts in aerodynamics and seismology. Mathematically, Mach waves are governed by a tube wave pole and the proximity of the integration path to this pole governs the behavior of the radiated $S$ wave field. The dependence differs according to the velocity range

- $\beta > \alpha_f$ – the shear wave velocity is greater than the fluid velocity and the $S$ wave radiation is essentially similar to that for an empty borehole. As $\beta$ approaches $\alpha_f$ a greater effect due to the tube wave pole is observed.

- $\alpha_f > \beta > C_T$ – when the shear wave velocity is in between the fluid velocity and tube wave velocity the existence of shear wave lobes is confirmed.

- $\alpha_f > C_T > \beta$ – when the shear wave velocity is less than the tube wave velocity, a supershear condition exists and Mach waves are radiated away from the borehole. If the borehole is cased $C_T$ will increase to be closer to $\alpha_f$ and this velocity range is more likely to be encountered.

Calculation of the residue for the tube wave pole allows the contribution of the Mach wave to be isolated and calculated. Through this residue calculation, analogous to the calculation of the residue of the Rayleigh wave in seismology, the existence of the necessary phase delays for Mach wave radiation are derived and this phase delay was seen to be equivalent to that determined by geometric arguments. Additionally, a geometric decay of $\frac{1}{r^2}$ could be derived from calculation of the residue, where $r$ is the horizontal distance between the source and receiver not the total distance.

Comparison of seismograms generated with the numerical algorithm and those generated with the radiation pattern formulas showed good agreement when lithologies with high velocities surrounded the source borehole. Comparison of the algorithm results with two data sets from published experiments measuring radiation from point sources (dynamite) into low velocity sediments surrounding the borehole showed excellent agreement and helped resolve long standing differences with the theoretical predictions. A third data set showing radiation from a downhole axial source into a low velocity sediment also agreed with the algorithm results quite well.

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## Contents

1 Introduction .................................................. 7
   1.1 Introduction ........................................... 7
   1.2 Present state of understanding ....................... 8
   1.3 Thesis outline ......................................... 12

2 Near Field Radiation ........................................... 19
   2.1 Introduction ........................................... 19
   2.2 General model: P-SV radiation ......................... 21
       2.2.1 Radial layering ................................... 24
       2.2.2 Radial, axial and volume point sources ........... 28
   2.3 Particular models: P-SV radiation .................... 31
       2.3.1 Empty borehole surrounded by half space ........ 32
       2.3.2 Fluid-filled borehole surrounded by half space .. 32
       2.3.3 Comparison of particular models ................. 36
   2.4 General and particular model: SH radiation .......... 36
   2.5 Calculation using the discrete wavenumber method ... 38
   2.6 Numerical results - synthetic seismograms .......... 39
       2.6.1 Testing of the algorithm ......................... 40
       2.6.2 P-SV radiation into high and low velocity sediments 42
       2.6.3 Numerical Examples ............................... 43
       2.6.4 The physics of Mach waves ........................ 46

6
2.6.5 more on P-SV radiation ................. 51
2.6.6 SH radiation .......................... 51
2.7 Conclusions ......................... 52

3 Far Field Radiation - Analytic Approximations 86
3.1 Introduction .......................... 86
3.1.1 Previous work of Heelan ............... 87
3.1.2 Previous work of Lee and Balch .......... 90
3.2 Calculations for the empty borehole: P-SV .......... 91
3.2.1 Expansions for small argument .......... 92
3.2.2 Expansions for large argument .......... 94
3.2.3 Application of the method of stationary phase .......... 94
3.3 Calculations for the fluid-filled borehole: P-SV .......... 98
3.3.1 Calculation of residues: Mach waves .......... 103
3.4 Table of results for P-SV case .......... 104
3.5 Calculations for SH ........................ 106
3.6 Comparison of radiation pattern formulas .......... 106
3.6.1 Radiation from torsional and axial sources .......... 107
3.6.2 Radiation from a radial source in an empty borehole .......... 109
3.6.3 Radiation from a radial source in a fluid-filled borehole .......... 111
3.6.4 Radiation from a point source in a fluid-filled borehole .......... 112
3.7 Comparison between numerical and analytical results .......... 112
3.7.1 Torsional and axial sources .......... 113
3.7.2 Radial source ........................ 114
3.7.3 Volume point source .................. 115
3.8 Conclusions .......................... 116

4 Modelling of Field Data Sets 135
4.1 Introduction .......................... 135
4.2 White and Sengbush (1963) experiment ........................................... 136
4.3 de Bruin and Huizer (1989) experiment .......................................... 138
4.4 Chevron experiment ................................................................. 140
  4.4.1 Well logs ............................................................................. 142
  4.4.2 Modelling of data and discussion .......................................... 143
4.5 Conclusions ............................................................................. 144

5 Conclusions ............................................................................. 164
  5.1 Practical implications ............................................................... 167
  5.2 Future Work ........................................................................... 168

Bibliography ............................................................................. 170

A Wave Equation Relationships in Cylindrical Coordinates .......... 180
  A.1 Introduction ........................................................................... 180
    A.1.1 Types of symmetry assumptions ....................................... 181
  A.2 Displacement potential and wave equation relationships .......... 182
    A.2.1 No symmetry assumptions ................................................ 183
    A.2.2 Axisymmetry – transformation of $\psi$ ................................. 187
    A.2.3 Axisymmetry – no transformation of $\psi$ ............................. 191
    A.2.4 Axisymmetry – torsional motion ....................................... 192
    A.2.5 No Axisymmetry – symmetry in $z$ ..................................... 193
  A.3 Transforming the resultant wave equations using separation of variables 194
    A.3.1 Separation of variables - no symmetry assumptions .......... 194
    A.3.2 Use of radiation conditions ................................................ 199
    A.3.3 Separation of variables - transformation of $\psi$ .................... 201
    A.3.4 Separation of variables, no transformation of $\psi$ ............... 203
    A.3.5 Separation of variables, axisymmetry, torsional motion .......... 204
    A.3.6 Separation of variables, symmetry in $z$ .............................. 204
  A.4 Conclusions ........................................................................... 205
B Mathematics and physics of radiation from empty boreholes 210
  B.1 Heelan’s Analysis ........................................... 211
    B.1.1 Separation of variables procedure .................. 212
    B.1.2 Development of the displacement potential relations .... 213
    B.1.3 Relationship to the Sommerfeld integral .......... 215
    B.1.4 Boundary conditions for radial and axial sources .... 216
    B.1.5 P-SV case .............................................. 218
    B.1.6 SH case .............................................. 222
  B.2 Criticisms of Heelan’s results ........................ 222

C Brekhovskikh’s analysis and notation 225
  C.1 Development of the displacement potential relations ...... 226
    C.1.1 Weyl integral formulation ............................ 228
  C.2 Properties of the $\Gamma_1$ Contour ....................... 231
  C.3 Displacement potential relations continued ................ 232
  C.4 P-SV case .............................................. 233
    C.4.1 Solution by the method of stationary phase .......... 235
  C.5 SH case .............................................. 238
  C.6 Conclusions ........................................... 240

D Modifications of the Bouchon Schmitt boundary integral technique 243

E Hansen Vector Theory and Cylindrical Coordinates 248

F Algebra for Hankel functions 253
  F.1 Particular models: P-SV .................................. 255
  F.2 Particular models: SH .................................... 258

G Uniform Asymptotic Expansion of the Integral 259
Chapter 1

Numerical and Analytical
Modelling of Downhole Seismic
Sources: An Introduction

1.1 Introduction

In this thesis, we study radiation from seismic sources emplaced in boreholes. The chief use of this study will be to help unravel the full wavefield observed in crosshole tomography and to a lesser extent reverse vertical seismic profiling experiments.

Reflection seismology is the principal seismic technique used in delineation of the structure of the earth’s crust and in the search for petroleum resources. However, there are limits to the resolution obtainable in seismic data and therefore complementary techniques such as crosshole tomography have been developed to try to provide a more complete description of the in-situ geology.

An advantage of crosshole tomography over reflection seismology is that both the source and receiver are closer to the target and the attenuating features of the near surface are avoided. However, substantial disadvantages of crosshole tomography over reflection seismology are also apparent including the fixed location of boreholes and
usually the lack of three dimensional coverage.

The data derived from crosshole tomography experiments typically is a set of first break $P$ wave traveltimes which are inverted to produce images of the velocity field, tomograms. Thus the effort in evaluating the full waveform including the $S$ wave and the magnitudes has not proceeded apace to that in reflection seismology. It is hoped that studies such as the present one will help lead to utilization of both $P$ and $S$ waves and their amplitudes in crosshole tomography experiments.

1.2 Present state of understanding

To date, the majority of theoretical studies in modelling radiation from downhole seismic sources have been far field studies. In these studies, integrals representing displacements have been evaluated with the well-known method of stationary phase. Typically, the magnitude of the resulting displacements are presented in a graphical form as a function of angle away from the source and this graphical representation of displacement is known as a radiation pattern.

The first far field study applicable to downhole seismic sources was performed by Heelan (1952, 1953a,b) for sources in empty boreholes. Heelan studied radiation into an elastic medium emanating from stress discontinuities applied to the wall of the borehole. The goal of this research was to simulate the radiation due to the explosion of dynamite in a borehole. In Heelan’s paper, various sources were considered: a) axisymmetric axial sources applied to the wall of the borehole in a vertical direction, b) axisymmetric radial sources applied to the wall of the borehole in a radial direction and c) axisymmetric torsional sources applying a twisting force on the borehole wall. Fig. 1-1 shows the radiation patterns developed by Heelan for radial, axial, and torsional sources – these radiation patterns will be more fully explained in Chapter 3. The most important information derived from Heelan’s work was the prediction of the direct generation of shear waves from radial stresses applied to the walls of
boreholes as seen in the upper left half of Fig. 1-1. Heelan’s work has been criticized by Jordan (1962) and Abo-Zena (1977) but it is shown later that his work is correct.

More recent work on radiation from empty boreholes has included a description of radiation from a point force on a borehole wall conducted by Greenfield (1978) which required a more complex nonaxisymmetric treatment of wave propagation. Additionally, White (1983) described the effects of anisotropy on radiation from an empty borehole. Fehler and Pearson (1981, 1984) using a moment tensor representation found radiation pattern results identical to Heelan’s.

Lee and Balch (1982) extended Heelan’s work by considering a fluid in the borehole and by allowing the modelling of a fourth type of axisymmetric source, a volumetric point source on the borehole axis. Adding a third boundary condition, made the algebra more complicated but the results of Lee and Balch are not fundamentally different from those of Heelan for high velocity sediments and for $P$ waves in all sediments. A major exception to this agreement was the case when the shear wave velocity was less than the tube wave velocity for which no radiation pattern could be determined for $S$ waves. The reason no radiation pattern could be determined in this case was due to the presence of a tube wave pole. When the shear wave velocity was greater than the tube wave velocity but less than the fluid velocity, nodal planes were predicted for the radial source and a vertical skewness for a volume source. The observation of nodal planes in the shear wave radiation pattern is shown in Fig. 1-2 which is taken from Fig. 4 of Lee and Balch (1982). Lee and Balch’s observation of nodal planes in the shear wave radiation pattern and the inability to calculate a radiation pattern for certain velocities are the first theoretical indications that shear wave radiation from boreholes embedded in low velocity sediments is unusual.

Recent theoretical and numerical work (Samec and Kostov, 1988; Winbow, 1989) concerning radiation from fluid-filled boreholes has verified the numerous observations that the tube wave travelling inside the borehole is the predominant radiation product from downhole sources and contains more than 99% of the acoustic energy emanating
from the source.

Complementary to the theoretical work describing radiation from boreholes are theoretical investigations describing radiation incident on boreholes. Such work has been accomplished by White (1953, 1965) utilizing the reciprocity theorem, Schoenberg (1986), and Lee (1986, 1987) with power series expansions. Also complementary to this work is the substantial volume of work done in describing radiation inside a borehole for acoustic well logging and vertical seismic profiling purposes. Good reviews of seismic wave propagation in a borehole are given in a review paper by Toksöz and Cheng (1984) and in books on vertical seismic profiling (e.g. Balch and Lee, 1984).

One of the first experiments describing radiation from downhole seismic sources was published by Riggs (1955) who showed the dominance of tube waves on propagation in a borehole at seismic frequencies and was also able to show effects of tube waves on a small-scale crosshole tomography experiment. Experimental work in high velocity, hard rock domains such as that by Fehler and Pearson (1981, 1984) showed good agreement between the shape of the radiation pattern predicted by Heelan’s formulas and experimental data taken in granite at a hot dry rock geothermal site. However, experiments performed in low velocity sediment environments such as that of White and Sengbush (1963) which was designed to test Heelan’s results showed poor agreement. A more recent experiment by de Bruin and Huizer (1989) exciting a source in a fluid-filled cased borehole also shows disagreement with Heelan’s predictions for low velocity sediments and shows secondary arrivals moving out at velocities much higher than the shear wave velocity. The disagreement between theory and experiment in low velocity sediments has been hypothesized to be due to the influence of tube waves and the generation of conical wavefronts (de Bruin and Huizer, 1989), (Mach waves). The aforementioned experiments will be discussed in Chapter 4 and the generation of Mach waves in Chapters 2-4.

Experimental work has also shown that obstructions in or around the borehole
can create secondary sources which can be just as energetic as the arrivals from the primary sources (Lee et al., 1984) due to the passage of tube waves. Such obstructions are due to the end of the borehole, changes in casing thickness, bed boundaries and fractures. The same phenomena have also been observed in VSP experiments (Beydoun et al., 1984a,b; Cheng and Toksöz, 1984; Cicerone et al., 1988).

Complementary to experimental work describing radiation of downhole seismic sources, there has been a substantial effort in developing different types of sources. The use of airguns as seismic sources is well established from work in the marine geophysics community and is being adapted to the borehole environment. The advantage of this type of source is the reliability and repeatability of the waveform. Another advantage is that as the source is fired deeper in the well, the pressure from the airgun is greater creating a more energetic source (Lee et al., 1984). Another type of point source is the piezoelectric source. These sources are rings made up of piezoelectric materials and are excited when a current passes through them causing them to expand and contract (e.g. Balogh et al. 1988). Sparker sources have been developed and can also be modelled as point sources (Owen et al., 1988). Explosives can be used as a point source if the amount of dynamite is so small that the borehole is not damaged. Experimental results with explosives have been conducted by White and Sengbush (1963), Chen and Eriksen (1988) and de Bruin and Huizer (1989). The work of Chen and Eriksen (1988) shows that explosives may force the formations nonlinearly. Implosive sources have also been developed (e.g. Taylor and Brooks, 1989).

Axial sources have been predominantly developed as downhole vibrators (Hardee et al., 1987; Paulsson, 1988; Elbring et al., 1989). Formulation of these axial sources as a vibrator allows control of the frequency content and time duration of the source with the disadvantage that deconvolution is required to yield the impulse response. Another disadvantage is that axial sources theoretically require an open hole or a good casing bond since they are dependent on the propagation of axial stresses (Winbow,
Torsional sources impart a twisting motion to the borehole wall which only produces $SH$ waves. Torsional sources are independent of the fluid inside the borehole and only dependent on the shear modulus of the formation. Like axial sources, torsional sources require the presence of a good casing bond. As with radial sources, the development of torsional downhole seismic sources has not been published to date in the literature known to the author.

1.3 Thesis outline

As was just shown, the major contribution to the description of radiation from downhole seismic sources to date has been the calculation of far field radiation patterns by Heelan (1952, 1953a,b) for radiation from empty boreholes and by Lee and Balch (1982) for radiation from fluid-filled boreholes. However, despite the utility of the far field radiation patterns, there are many unanswered experimental and theoretical observations which require the development of numerical techniques for the analysis of the near and far field. This thesis goes a long way toward providing complete solutions and answering the questions related to the unexplained observations.

The next three chapters of this thesis cover the numerical solution, the analytic solution, and the application of these results to experimental data. The numerical results are perfectly capable of calculating radiation from downhole seismic sources in the near and far field and the analytic results can be used for calculation of radiation in the far field. This immediately brings up the question, “what is the distinction in the borehole environment between near and far field”? The answer is not as elementary as a few wavelengths or the discarding of terms which have little effect in the far field (e.g. Aki and Richards, 1980) as is often done in seismology. Instead the distinction is the borehole's influence on the radiation itself. If the borehole itself continues to have influence beyond perturbing the initial wavefield geometry we define this to be
in the near field. Such is the case with shear wave radiation in low velocity sediments. If the borehole has no influence beyond the initial geometrical effect we are in the far field and the analytic results can be used for describing radiation. Thus the far field may be quite close to the source borehole for the $P$ waves but very far away for the $S$ waves.

There has been much work in the field of acoustic well logging describing radiation inside a radially layered borehole medium (e.g. Schoenberg et al., 1981; Baker, 1984; Tubman, 1984; Tubman et al., 1984, 1986; Schmitt and Bouchon, 1985). In acoustic well logging, a Thomson-Haskell algorithm for the calculation of the pressure response inside a fluid-filled borehole surrounded by a one-dimensional radially layered medium was developed by Tubman (1984), Tubman et al., (1984,1986) and refined by Schmitt and Bouchon (1985). A contribution of this thesis is the use of the Thomson-Haskell algorithm for the calculation of the displacement fields outside a radially layered borehole and all of the examples in this thesis will be for the calculation of radiation outside the borehole. The algorithm developed here includes an important simplification of the inverse of the layer matrices provided by Schmitt and Bouchon (1985) and includes the addition of the capability of studying radiation from axial, radial, torsional, as well as volume point sources from either empty or fluid-filled boreholes. Once the algorithm is developed it is first used to investigate the differences between radiation into high velocity sediments surrounding boreholes versus radiation into low velocity sediments surrounding boreholes. The difference is due to the generation of conical wavefronts, hypothesized on the basis of experimental evidence (de Bruin and Huizer, 1989), which is demonstrated with the algorithm very well. These conical wave fronts are Mach waves (Ben-Menahem and Singh, 1987) completely analogous to Mach waves in aerodynamics (e.g. Ferrari and Tricomi, 1968). Besides the numerical results confirming their existence, the physics of the Mach waves and a discussion of the related hypotheses of de Bruin and Huizer (1989) and White and Sengbush (1963) is also thoroughly explained in Chapter 2.
In Chapter 3, radiation in the far field is studied analytically including a detailed analysis of the work of Heelan (1952, 1953a,b) and Lee and Balch (1982) in calculating far field radiation patterns using the method of stationary phase. Heelan’s results have been very influential (e.g. White and Sengbush, 1963; White, 1965, 1983; Paulsson, 1988) but also criticized and the proof that Heelan’s algebraic results are correct is introduced in Chapter 3 and completed in Appendix B and Appendix C. Lee and Balch’s result for the fluid-filled borehole is also analyzed and the limitations due to the presence of Mach waves are identified. The chief limitation is the presence of the tube wave pole which intersects the path of steepest descent at a particular value of \((r, z)\) in a low velocity sediment. Additionally, in the vicinity of the tube wave pole the regularity assumption necessary for the use of the stationary phase method is no longer valid. It is possible to calculate an integral where a pole intersects the steepest descent path (Bleistein, 1966; Felsen and Marcuvitz, 1959; 1973; Bleistein and Handelsman, 1976) and the steps are outlined in Appendix G. Instead of using these cumbersome expressions it is shown that by calculating the residue of the tube wave pole as a separate arrival, the Mach wave expressions for the geometric decay and travel time can be obtained. This procedure is very similar to the treatment of the Rayleigh wave pole in seismology. Chapter 3 will also present far field radiation patterns for a wide variety of sources and lithologies surrounding the borehole. It is found that the tube wave pole disappears in the case of axial sources and that radiation from axial and torsional sources are virtually identical to the radiation patterns in infinite media (e.g. White, 1983). Also in Chapter 3, a comparison is made between synthetic seismograms calculated using the numerical results from Chapter 2 and the far field radiation pattern formulas. The agreement was found to be excellent when high velocity sediments surrounded the borehole. This shows that radiation into high velocity sediments is well predicted by the use of the radiation pattern formulas.

Chapter 4 shows the comparison of the theoretical results to data sets. The
first two data sets used are those published by White and Sengbush (1963) and de Bruin and Huizer (1989). A third data set was provided by Chevron Oil Field Research Co. (e.g. Winsterstein and Paulsson, 1990). A shot gather from this crosshole tomography experiment is analyzed. In general, excellent agreement is seen between the theoretical predictions and the data and the agreement is far superior to traditional far field analysis for the de Bruin and Huizer (1989) and White and Sengbush (1963) experiment which were taken in low velocity sediments and are affected by the generation of Mach waves.

Conclusions are provided in Chapter 5 along with applications of the thesis to the design of downhole seismic sources. In order to keep the main body of the thesis coherence, for ease of reading, and yet maintain completeness, many of the mathematical details are placed in Appendices. The appendices include: Appendix A which provides a detailed description of the conventions used for wave propagation in cylindrical coordinates; Appendix B and Appendix C which are devoted to analysis of the algebra and mathematics needed to justify the veracity of Heelan's and Brekhovskikh's work and include extensions of Brekhovskikh's work to axial sources; Appendix D which details the modifications to the Bouchon and Schmitt (1989a,b) boundary integral technique to allow calculation of radiation outside the borehole for testing purposes; Appendix E which details the role Hansen vector theory plays in analysis using the cylindrical coordinate system; Appendix F which duplicates the algebra presented in Chapter 2 when Hankel functions instead of modified Bessel functions are used as eigenfunctions and is a perquisite of the analysis presented in Chapter 3; and Appendix G which outlines steps to evaluate an integral which has a pole coincident with a saddle point.

The principal contributions of this thesis are as follows

- Development of the Thomson-Haskell algorithm for numerical investigation of radiation from downhole seismic sources;

- Identification of the limitations of existing far field asymptotic results due to
the presence of singularities and the generation of Mach waves;

- Identification of the implications of the singularities on the radiation from a borehole by development of the Mach wave hypothesis;

- A complete comparison of numerical and analytical results showing that the agreement is excellent for high velocity sediments;

- Use of the numerical results to answer long-standing unresolved experimental observations;

- Extension of the far field asymptotic results to include a description of the geometric and arrival time properties of Mach waves;
Figure 1-1: Radiation patterns of Heelan (1953a) determined by the method of stationary phase and presented in two dimensions. Qualitative graphic description only. The three dimensional description would involve rotating these patterns about the $z$ axis.
Figure 1-2: Shear wave radiation patterns of Lee and Balch (1982) determined by the method of stationary phase and presented in two dimensions. Part of Fig. 4 from Lee and Balch’s paper. $S$ wave velocity is greater than the tube wave velocity but less than the fluid velocity. Volume point source pattern is skewed towards the vertical but radial source actually exhibits nodal planes. Qualitative graphic description only. The three dimensional description would involve rotating these patterns about the $z$ axis.
Chapter 2

Description of Near Field Radiation from Downhole Seismic Sources

2.1 Introduction

In this chapter, an algorithm is developed for the calculation of near and far field radiation from axisymmetric downhole seismic sources in empty and fluid-filled boreholes. The algorithm is used here to calculate radiation in the near field to help define the near field vs. far field domains, and to investigate the effect of the velocity of the material surrounding the borehole on the radiated wave field. The types of sources considered include a volume point source fired in a fluid-filled borehole plus radial, axial, and torsional sources applied to the walls of empty or fluid-filled boreholes.

For the general model, an infinitely long cylindrical borehole is surrounded by a radially layered, axisymmetric medium embedded in an otherwise infinite homogeneous space (Fig. 2-1 and Fig. 2-2). The radial layers surrounding the borehole are homogeneous, axisymmetric and isotropic cylindrical shells and may be fluid or solid. The Thomson-Haskell method as developed by Tubman (1984), Schmitt and
Bouchon (1985), Tubman et al. (1984, 1986) for borehole geometries will be utilized for calculating the Green's functions for torsional, radial and axial sources. For the limiting case of boreholes surrounded by an otherwise infinite space with no intermediate layers, Cramer's rule solutions for the coefficient functions of the integrals will also be written for illustration and discussion.

The general model presented here provides a complete solution for the calculation of radiation outside a radially layered medium but with the restriction that the layered media must consist of cylindrical shells. To test this algorithm, a boundary integral technique presented in Bouchon and Schmitt (1989a,b) was modified for the calculation of displacements outside the borehole with details given in Appendix D. Agreement was excellent between displacements calculated with both the boundary integral and the Thomson-Haskell algorithms.

There are a number of interesting questions to explore by examining radiation from a radially layered medium. It is known, for instance, that when a volume point source is excited inside a fluid-filled borehole more than 99%, of energy is trapped as tube waves which travel up and down the borehole wall (Winbow, 1989). Thus the tube wave is the dominant radiation product of excitation by a point source. Both White and Sengbush (1963) and de Bruin and Huizer (1989) hypothesized from experimental observation that the energetic tube wave would in fact lead to the radiation of S waves, particularly in low velocity sediments, as the tube wave travelled up and down the borehole and this chapter will help confirm this hypothesis with numerical examples and will discuss the physics of the phenomena.

The discrete wavenumber method (White and Zechman, 1968; Bouchon and Aki, 1977; Cheng and Toksöz, 1981) plus the extensions for radially layered media (Tubman, 1984; Schmitt and Bouchon, 1985; Tubman et al., 1984, 1986) are techniques particularly well suited to near field solutions and will be used for the calculations in this chapter. Modifications introduced here will be used to calculate the Green's function for radiation outside the borehole for many different combinations of lay-
ers and sources. However, limitations of the treatment presented here are that no fluid-fluid boundaries or empty external layers will be allowed and the calculations described in this thesis will only consider at most four layers.

To produce the time domain synthetic seismograms, the well known Ricker wavelet will be convolved with the Green's function.

### 2.2 General model: P-SV radiation

Consider an elastic medium to be described in cylindrical coordinates. The displacement field can be represented by three displacement potentials $\phi, \psi, \chi$ corresponding to a longitudinal $\phi$ and transverse $\psi, \chi$ displacement potentials. Assuming axisymmetry, the $P-SV$ and $SH$ problems can be solved independently as will be done here.

Modified Bessel functions $(K_{\{0,1\}}, I_{\{0,1\}})$ will be used as the eigenfunctions (e.g. White, 1983; also Appendix 1) in this chapter. The advantage of using modified Bessel functions here is to maintain the precedent set in the literature (Tubman, 1984; Tubman et al., 1984, 1986; Schmitt and Bouchon, 1985) for numerical calculations, equivalent results using Hankel and Bessel functions as the eigenfunctions are presented in Appendix F.

Fig. 2-1 shows the general configuration of the radially layered model. The distance $r_1$ is the radius of the fluid-filled, empty or solid borehole and there are 1 to $n$ solid or fluid layers of radius $r_i$.

The displacement potential solutions for any layer $i$ and the fluid-filled borehole are for the $P-SV$ problem (Eq. A.72)

$$\phi_f = \frac{1}{(2\pi)^2} \int \int_{-\infty}^{\infty} B_f I_0(f r_1) e^{i k z z} e^{-i \omega t} d k_z \ d\omega$$

$$\phi_i = \frac{1}{(2\pi)^2} \int \int_{-\infty}^{\infty} (A_i K_0(l_i r_i) + B_i I_0(l_i r_i)) e^{i k z z} e^{-i \omega t} d k_z \ d\omega$$

$$\psi_i = \frac{1}{(2\pi)^2} \int \int_{-\infty}^{\infty} (C_i K_1(m_i r_i) + D_i I_1(m_i r_i)) e^{i k z z} e^{-i \omega t} d k_z \ d\omega$$
where

\[ f = \sqrt{k_z^2 - \frac{\omega^2}{\alpha_i^2}}, \quad l_i = \sqrt{k_z^2 - \frac{\omega^2}{\beta_i^2}}, \quad m_i = \sqrt{k_z^2 - \frac{\omega^2}{\beta_i^2}} \]  

(2.2)

are the fluid, compressional and shear radial wave numbers for each layer, respectively. More information concerning the choice of eigenfunctions, displacement potential and time dependences used may be found in Appendix A.

The \( A_i, B_i, C_i, D_i \) terms are coefficient functions of \( k_z \) and \( \omega \) and the integrals are recognized as Fourier transforms over \( k_z \) and \( \omega \) (e.g. White, 1983). For notational convenience the integral signs and \( dk_z, d\omega \) will often be set aside in subsequent development.

At each boundary, a boundary condition matrix \( \mathbf{D} \) is calculated which multiplies a coefficient vector \( \mathbf{a} \) - the coefficient vector \( \mathbf{a}_i \) remains constant within each layer \( i \). The product of the matrix \( \mathbf{D} \) and the vector \( \mathbf{a} \) is the displacement-stress vector \( \mathbf{u} \) (e.g. Thomson, 1950; Haskell, 1953; Tubman, 1984; Tubman et al., 1984, 1986)

\[ \mathbf{D}_i(r_i)\mathbf{a}_i = \mathbf{u}_i(r_i) \]  

(2.3)

The coefficient and displacement stress vectors can be written

\[
\mathbf{a}_i = \begin{bmatrix} A_i \\ B_i \\ C_i \\ D_i \end{bmatrix}, \quad \mathbf{u}_i = \begin{bmatrix} U_r \\ U_z \\ p_r \\ p_{rz} \end{bmatrix}
\]  

(2.4)

However, many times elements of the coefficient vectors are zero such as for a fluid-filled borehole \( \mathbf{a}_f \), an intermediate fluid layer \( \mathbf{a}_l \), the infinite half space \( \mathbf{a}_n \) and of academic interest only the solid borehole \( \mathbf{a}_0 \), which are

\[
\mathbf{a}_0 = \begin{bmatrix} 0 \\ B_0 \\ 0 \\ D_0 \end{bmatrix}, \quad \mathbf{a}_f = \begin{bmatrix} 0 \\ B_f \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{a}_l = \begin{bmatrix} A_l \\ B_l \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{a}_n = \begin{bmatrix} A_n \\ 0 \\ C_n \\ 0 \end{bmatrix}
\]  

(2.5)
The reasons for the behavior in \( a_n \) is the absence of the \( I_{(0,1)} \) potentials, which increase exponentially at infinity, in the half space. Additionally, for \( a_0, a_f \) inside the borehole only solutions in \( I_{(0,1)} \), which are finite at the origin, are allowed so \( A_0, A_f, C_0 \) and \( C_f \) are zero. \( D_f, C_i, \) and \( D_l \) for \( a_i \) and \( a_f \) in Eq. 2.5 are zero because no shear potential is supported in a fluid layer.

Once \( a_n \) is determined, calculation of the radial and vertical displacements in the infinite half space proceeds from the definition of displacements (Eq. A.19)

\[
U_r = -A_n l_n K_1(l_n r) - ik_z C_n K_1(m_n r) \\
U_z = ik_z A_n K_0(l_n r) - m_n C_n K_0(m_n r)
\]  

(2.6)

where \( r \) is the radius of investigation in the infinite half space.

The boundary condition matrix \( D_i \) can be written

\[
D_i = \begin{pmatrix}
D_{11} & D_{12} & D_{13} & D_{14} \\
D_{21} & D_{22} & D_{23} & D_{24} \\
D_{31} & D_{32} & D_{33} & D_{34} \\
D_{41} & D_{42} & D_{43} & D_{44}
\end{pmatrix}
\]

(2.7)

where the equation for the first row is the normal displacement (Eq. A.18), the second row is the tangential displacement (Eq. A.18), the third row is the normal stress equation (Eq. A.27) and the fourth row is the tangential stress equation (Eq. A.27). By substituting the definitions for the displacement potentials (Eq. 2.1) into the displacement stress relations (Eq. A.18, A.27) the elements of \( D_i \) (Tubman, 1984) are the following

\[
D_{11} = -l_i K_1(l_i r_i) \\
D_{12} = l_i I_1(l_i r_i) \\
D_{13} = -ik_z K_1(m_i r_i) \\
D_{14} = -ik_z I_1(m_i r_i) \\
D_{21} = ik_z K_0(l_i r_i)
\]  

(2.8)
\[
D_{22} = i k_z I_0(l; r_i) \\
D_{23} = -m_i K_0(m; r_i) \\
D_{24} = m_i I_0(m; r_i) \\
D_{31} = (-\rho_i \omega^2 + 2\mu_i k_z^2) K_0(l; r_i) + \frac{2\mu_i l_i}{r_i} K_1(l; r_i) \\
D_{32} = (-\rho_i \omega^2 + 2\mu_i k_z^2) I_0(l; r_i) - \frac{2\mu_i l_i}{r_i} I_1(l; r_i) \\
D_{33} = 2\mu_i k_z [m_i K_0(m; r_i) + \frac{K_1(m; r_i)}{r_i}] \\
D_{34} = -2\mu_i k_z [m_i I_0(m; r_i) - \frac{I_1(m; r_i)}{r_i}] \\
D_{41} = -2\mu_i k_z l_i K_1(l; r_i) \\
D_{42} = 2\mu_i k_z l_i I_1(l; r_i) \\
D_{43} = (-\rho_i \omega^2 + 2\mu_i k_z^2) K_1(m; r_i) \\
D_{44} = (-\rho_i \omega^2 + 2\mu_i k_z^2) I_1(m; r_i)
\]

**2.2.1 Radial layering**

Radial layering is an integral part of the general model provided by Eq. 2.3. One important type of boundary is the solid-solid interface at which continuity of the two stresses and two displacements in the vector \( u \) is satisfied so the displacement-stress vectors from both solid layers can be equated (e.g. Thomson, 1950; Haskell, 1953; Tubman, 1984; Tubman et al. 1984, 1986)

\[
D_i(r_{i+1}) a_i = D_{i+1}(r_{i+1}) a_{i+1} = u_{i+1} \quad (2.9)
\]

At a solid-fluid interface or at the wall of an empty borehole and the solid borehole wall, however, only a subset of the stress-displacement components of \( u \) can be equated across boundaries.

The equality presented in Eq. 2.9 is satisfied for a solid-solid boundary but there are 4 equations and 8 unknowns and thus an underdetermined system. Therefore, to construct a properly determined system it is required to add more equations by
propagating the solution from the innermost layer to the outermost one. Thus the character of the borehole itself, the innermost layer, is now addressed.

**Solid Borehole**

Although of only academic interest, it is useful for the full treatment of the wave propagation problem in radially layered media to consider the case of a solid borehole. If a solid central layer is surrounded by a \( n \) layered medium where all the external layers are solid the governing equation is

\[
\left( \prod_{i=1}^{n} \mathbf{D}_i(r_i) \mathbf{D}_i^{-1}(r_{i+1}) \right) \mathbf{a}_n = \mathbf{D}_0(r_1) \mathbf{a}_0
\]  

(2.10)

The coefficient matrix \( \mathbf{a}_n \) is the desired solution for calculating propagation outside the borehole and the term \( \mathbf{D}^{-1}(r_{n+1}) \) is set equal to the identity matrix \( I \). \( \prod \) is the product symbol. The fact that the last element of the product matrix equals the identity matrix is a convention that will be followed subsequently. There are no outgoing waves from the central layer so \( \mathbf{a}_0 \) has only two elements (Eq. 2.5). Therefore, Eq. 2.10 represents a \( 4 \times 4 \) system of equations in \( A_n, C_n, B_0, D_0 \) that is properly determined and the solution to the homogeneous problem.

A notational simplification is introduced by equating a matrix \( \mathbf{G} \) to the product matrix such that

\[
\mathbf{G} = \prod_{i=1}^{n} \mathbf{D}_i(r_i) \mathbf{D}_i^{-1}(r_{i+1})
\]  

(2.11)

Therefore, the matrix equation for the solid borehole Eq. 2.10 can be rewritten

\[
\mathbf{G} \mathbf{a}_n = \mathbf{D}_0(r_1) \mathbf{a}_0
\]  

(2.12)

Another simplification is due to Schmitt and Bouchon (1985) which is an analytic expression for the inverse of the \( \mathbf{D}_i \) matrices

\[
\mathbf{D}_i^{-1} = \frac{r_i}{\omega^2 \rho_i} \left|
\begin{array}{cccc}
D_{32} & -D_{42} & -D_{12} & D_{22} \\
-D_{31} & D_{41} & D_{11} & -D_{21} \\
-D_{34} & D_{44} & D_{14} & -D_{24} \\
D_{33} & -D_{43} & -D_{13} & D_{23}
\end{array}
\right|
\]  

(2.13)
This formula decreases the number of computations required and adds stability (sign changes have been made from that of Schmitt and Bouchon due to different rotation).

**Empty Borehole**

For an empty borehole the boundary conditions are limited to the vanishing of normal and tangential stress. Consequently, a subset of Eq. 2.11 (G) is used such that

\[
\begin{bmatrix}
G_{31} & G_{33} \\
G_{41} & G_{43}
\end{bmatrix}
\begin{bmatrix}
A_n \\
C_n
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]  

(2.14)

**Fluid-Filled Borehole**

At the interface of the fluid-filled borehole and the borehole wall, the boundary conditions are continuity of normal displacement and stress and vanishing of tangential stress. As in Eq. 2.14, a subset of Eq. 2.11 is used

\[
\begin{bmatrix}
G_{11} & G_{13} \\
G_{31} & G_{33} \\
G_{41} & G_{43}
\end{bmatrix}
\begin{bmatrix}
A_n \\
C_n
\end{bmatrix}
=
\begin{bmatrix}
D_{f12} \\
D_{f32} \\
0
\end{bmatrix}
\begin{bmatrix}
B_f
\end{bmatrix}
\]  

(2.15)

which is a $3 \times 3$ system of equations that excludes the axial displacement boundary condition allowing axial displacement to be discontinuous. This is the homogeneous solution for a fluid-filled borehole.

**Intermediate Fluid Layers**

If the external layered medium includes intermediate fluid layers such as a water layer behind casing special consideration must be made (Tubman, 1984; Tubman et al., 1986). If layer $l$ is a fluid layer, the procedure is to first propagate the solution out to the layer $l - 1$. Calculating $G$

\[
G = \prod_{i=1}^{l-1} D_i(r_i)D_i^{-1}(r_{i+1})
\]  

(2.16)
where as in Eq. 2.10, $D_l^{-1}(r_l)$ equals the identity matrix and Eq. 2.16 is used to construct the following solution

$$
\begin{vmatrix}
G_{11} & G_{12} & G_{13} & G_{14} \\
G_{31} & G_{32} & G_{33} & G_{34} \\
G_{41} & G_{42} & G_{43} & G_{44}
\end{vmatrix}
\begin{bmatrix}
A_{l-1} \\
B_{l-1} \\
C_{l-1} \\
D_{l-1}
\end{bmatrix}
= 
\begin{bmatrix}
D_{f12} \\
D_{f32} \\
0
\end{bmatrix}
\begin{bmatrix}
B_f
\end{bmatrix}
\quad (2.17)
$$

Eq. 2.17 represents an underdetermined system of 3 equations and 5 unknowns. To achieve a fully determined system, it is again necessary to propagate out to the infinite half-space $n$. The first step is to solve for the coefficients $a_l$ of the fluid layer $l$ from the solutions for $a_{l-1}$

$$
\begin{vmatrix}
D_{11} & D_{12} \\
D_{31} & D_{32} \\
0 & 0
\end{vmatrix}
\begin{bmatrix}
A_l \\
B_l
\end{bmatrix}
= 
\begin{bmatrix}
D_{11} & D_{12} & D_{13} & D_{14} \\
D_{31} & D_{32} & D_{33} & D_{34} \\
D_{41} & D_{42} & D_{43} & D_{44}
\end{bmatrix}
\begin{bmatrix}
A_{l-1} \\
B_{l-1} \\
C_{l-1} \\
D_{l-1}
\end{bmatrix}
\quad (2.18)
$$

where the left-hand side boundary condition matrix is $D_l(r_l)$ and the right-hand side is $D_{l-1}(r_l)$. To solve from the intermediate boundary layers outwards another system of equations is introduced

$$
\begin{vmatrix}
G_{11} & G_{13} \\
G_{31} & G_{33} \\
G_{41} & G_{43}
\end{vmatrix}
\begin{bmatrix}
A_n \\
C_n
\end{bmatrix}
= 
\begin{bmatrix}
D_{11} & D_{12} \\
D_{31} & D_{32} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
A_l \\
B_l
\end{bmatrix}
\quad (2.19)
$$

where the boundary condition matrix on the right-hand side is $D_l(r_{l+1})$ and the $G$ matrix on the left is the product from the infinite half space $n$ inwards to the layer $l$ calculated using

$$
G = \prod_{i=l+1}^{n} D_i(r_i)D_i^{-1}(r_{i+1})
\quad (2.20)
$$

Eq. 2.17, Eq. 2.18, and Eq. 2.19 represent nine equations which can be used to solve for the 9 unknowns ($B_f, A_{l-1}, B_{l-1}, C_{l-1}, D_{l-1}, A_l, B_l, A_n, C_n$). This procedure can be
Radial and Axial Sources

The radial and axial sources are considered as discontinuities in normal and tangential stress respectively so can be written in terms of the stress displacement vectors as

\[
\mathbf{S}_{radial} = \begin{bmatrix}
0 \\
G(\omega)F(k_z) \\
0
\end{bmatrix}
\quad \begin{bmatrix}
0 \\
0 \\
G(\omega)F(k_z)
\end{bmatrix}
\mathbf{S}_{axial} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

(2.23)

where \(G(\omega)\) is the frequency transform of the time function governing the source and \(F(k_z)\) is the wavenumber transform of the function which governs the distribution of the source along the \(z\) axis. Only axisymmetric sources are considered here so there is no dependence on \(\theta\) and \(r\). Additionally, emphasis in this thesis is on describing each source individually since that is how downhole seismic sources would most likely be used in practice. If the description of radiation from a combination of sources is desired (e.g. Heelan (1953a,b)), it can be achieved through simple superposition.

To model a source applied over a distance \(2l\) use is made of a rectangle or boxcar function \(\Pi(z)\) (e.g. Bracewell, 1978)

\[
G(t)F(z) = G(t)\Pi(z)
\]

(2.24)

where the rectangle function is defined as

\[
\Pi(z) = 1 \quad |z| < l
\]

\[
\Pi(z) = 0 \quad |z| > l
\]

(2.25)

The transform in the frequency, axial wavenumber domain is the “sinc” function

\[
\Pi(k_z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} G(\omega) \frac{2\sin lk_z}{k_z} e^{ik_zz} e^{-i\omega t} dk_z d\omega
\]

The units of stress are newtons per square meter (pascals). However, a pascal is a very small quantity in the borehole environment and therefore for the calculation of numerical results the quantity \(10^5\) pascals or 1 bar (100 kPa or 14.7 psi) which is approximately atmospheric pressure will be used.
generalized to any number of intermediate fluid layers. For a more general treatment of intermediate fluid-layers and the treatment of fluid-fluid layers, it is recommended that the Tubman (1984) or Tubman et al. (1984, 1986) references be consulted.

2.2.2 Radial, axial and volume point sources

The solution for the homogeneous problem of wave propagation outside a radially layered medium surrounding a borehole has been developed for the \( P-SV \) problem. Therefore, to solve the inhomogeneous problem, sources are now introduced.

In general, there are two classes of sources governing \( P-SV \) wave propagation: axial and radial sources which are axisymmetrically applied to the borehole wall and are considered as stress discontinuities; and volume point sources which are excited in the center of fluid-filled boreholes (see Fig. 2-2).

If a stress discontinuity is added to the borehole wall at \( r_1 \), the stress displacement vector is transformed in the following simple manner

\[
\mathbf{G}_n \rightarrow \mathbf{G}_n + \mathbf{S}
\]

(2.21)

A more complicated model is constructed for a volume point source located in the center of the borehole. In this model, the source becomes part of the fluid displacement potential (e.g. Tubman, 1984) such that

\[
\mathbf{D}_f(r_1) \mathbf{a}_f = \begin{bmatrix} \mathbf{D}_{12} & \mathbf{D}_{12} \\ \mathbf{D}_{32} & \mathbf{D}_{32} \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{31} & \mathbf{D}_{32} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{S}_f \\ \mathbf{B}_f \end{bmatrix}
\]

(2.22)
Volume Point Source

For a volume point source, the classic hertzian oscillator representation (e.g. Brekhovskikh, 1960, 1980; Lee and Balch, 1982) is used in which the source contributes to the acoustic potential $\phi_f$

$$
\phi_f = -\frac{V_0}{4\pi} \frac{e^{-i\omega t} R}{R}
$$

(2.26)

$V_0$ is the volume displacement of the source (pg. 192, White, 1965; Lee and Balch, 1982) and $R$ is the distance from the source to the receiver. The negative sign is a convention that insures point source radiation will have the same polarity as a positive radial stress source (Lee and Balch, 1982). In the sonic well logging literature, the opposite sign convention is sometimes used.

Eq. 2.26 is strictly an idealization of a complex phenomena. Eq. 2.26 models a source region that is so small in comparison to its surrounding that all of the energy radiates spherically outward and additionally the energy has no reactive component just radiative (Swenson, 1965; Morse and Ingard, 1968) so the energy radiated is completely in phase. The electric analog would be a DC circuit with only a resistive element with no capacitive or inductive elements. However, the point source model is quite a useful model for many experiments. When working with displacement potentials as is done here $V_0$ is considered the volume displacement whereas when working with velocity potentials $V_0$ is considered the volume velocity, the product of the surface area of the spherical source multiplied by the rate of expansion of the source (Brekhovskikh, 1960, 1980).

If the source is located at the origin which is assumed here, Eq. 2.26 is equated to the modified Bessel function $K_0$ (Eq. 6.677.5, Gradshteyn and Ryzhik, 1980) through the relationship

$$
-\frac{V_0}{4\pi} \frac{e^{-i\omega t} R}{R} = -\frac{V_0}{4\pi^2} \int_{-\infty}^{\infty} K_0(fr)e^{ikz}e^{-i\omega t}dk_zd\omega
$$

(2.27)

thus $S_f$ of Eq. 2.22 equals

$$
S_f = -\frac{G(\omega)V_0}{2\pi}
$$

(2.28)
The standard unit for a volume point source is a cubic meter which is quite large. Therefore, for the numerical calculations a value of 1600 cubic centimeters or approximately 100 cubic inches (97.6) will be used. This value was used because of the frequent use of airguns with 100 cubic inch displacement in the marine geophysics community and recognizing the efforts of individuals to adopt these sources for borehole use.

The pressure on the borehole wall due to a point source of this magnitude will be quite large and will in fact exert nonlinear stresses on the borehole wall. Additionally, the displacements for such a large volume point source will be orders of magnitude greater than those for a radial, axial or torsional stress source at $10^5$ pascals. Nonetheless, the stress calculations will continue to use the $10^5$ pascal value because as Paulsson (1988) for instance has pointed out, American Petroleum Institute recommendations are that shearing stress at the cement bond should not exceed 20 psi (approximately $1.3 \times 10^5$ pascals). Determination of the “efficiency” or effective volume of volume point sources whether they be dynamite or airguns, although a fascinating topic is beyond the scope of this thesis. Some insight into this topic could be derived from White and Sengbush’s (1963) and Chen and Eriksen’s (1988) papers among others. Additionally, the consideration of nonlinear effects is also beyond the scope of this thesis although it is shown in Chapter 4 that the linear treatment developed here agrees well with experimental results.

2.3 Particular models: P-SV radiation

The development of the radially layered model just presented is completely general but to better visualize the mathematics and physics of wave propagation it is very common in the literature to consider the simplification of a borehole surrounded by an infinite halfspace. By doing so, $2 \times 2$ or $3 \times 3$ systems of equations are produced which can be solved by Cramer’s rule to yield the coefficient functions $A$ and $C$. 

34
2.3.1 Empty borehole surrounded by infinite half space

The most elementary case of radiation from downhole sources is radiation from sources applied to the borehole wall of an empty borehole. For instance, far field solutions to this problem were addressed by Heelan (1952, 1953a,b) who considered axial and radial stresses in combination and following Heelan, by Brekhovskikh (1960, 1980) who considered radial stresses only. Additionally, discrete wavenumber solutions were calculated by White (1982) for an empty borehole surrounded by an anisotropic solid for which this isotropic case is a limiting case.

Numerous simplifications occur in this case. The boundary conditions for an empty borehole are the vanishing of normal stress and tangential stress. Writing the general solution by combining Eq. 2.14 and Eq. 2.21 the following expression is obtained

\[
\begin{pmatrix}
D_{31} & D_{33} \\
D_{41} & D_{43}
\end{pmatrix}
\begin{pmatrix}
A \\
C
\end{pmatrix}
+ S = 0
\]  

(2.29)

Cramer's rule solutions are quotients with the denominator being the determinant of the matrix \( D \) in Eq. 2.29 and equalling

\[
(-\rho \omega^2 + 2\mu k_z^2)^2 K_0(l_1 a)K_1(m_1 a) - \frac{2\mu l_1 \rho \omega^2}{a} K_1(l_1 a)K_1(m_1 a)
- 4\mu^2 k_z^2 l_1 m_1 K_0(m_1 a)K_1(l_1 a)
\]  

(2.30)

The numerators for the different types of sources are presented in Table 2.1 below. To obtain the complete coefficient functions \( A(k_z, \omega) \) or \( C(k_z, \omega) \) these numerators are simply divided by the denominator Eq. 2.30.

2.3.2 Fluid-filled borehole surrounded by an infinite half space

In exploration practice it is much more common to have boreholes that are fluid-filled than empty and consequently the fluid-filled borehole problem has been treated in greater detail. Most of the development has been carried out for a point source
centered in the borehole and solutions for the coefficient functions were provided by Peterson (1974), Schoenberg et al. (1981), and Lee and Balch (1982). Lee and Balch (1982) additionally considered a radial source in their treatment of the problem.

For a fluid-filled borehole surrounded by an infinite half space there are three boundary conditions: continuity of normal displacement and stress and vanishing of tangential stress. Axial displacement can be discontinuous. The source formulation is written using Eq. 2.21 and Eq. 2.15 for a source applied to the borehole wall

$$
\begin{vmatrix}
D_{11} & D_{13} \\
D_{31} & D_{33}
\end{vmatrix}
A
\begin{vmatrix}
D_{f12}B_f \\
D_{f32}B_f
\end{vmatrix} + \mathbf{S} =
\begin{vmatrix}
D_{f12}B_f \\
D_{f32}B_f
\end{vmatrix}
$$

and for a volume point source in the fluid using Eq. 2.22 and Eq. 2.15

$$
\begin{vmatrix}
D_{11} & D_{13} \\
D_{31} & D_{33}
\end{vmatrix}
A
\begin{vmatrix}
D_{f11} & D_{f12} \\
D_{f31} & D_{f32}
\end{vmatrix} + \mathbf{S} =
\begin{vmatrix}
-G(\omega)V_0 \\
\frac{2\pi}{B_f}
\end{vmatrix}
$$

The Cramer’s rule quotient has the denominator

$$
-f[(-\rho\omega^2 + 2\mu k_0^2)^2 K_0(l_1a)K_1(m_1a) - \frac{2\mu l_1\rho_1\omega^2}{a} K_1(l_1a)K_1(m_1a)] - 4\mu^2 k_2^2 l_1 m_1 K_1(l_1a)K_0(m_1a) I_1(fa) - \rho \rho_f \omega^4 l_1 K_1(l_1a)K_1(m_1a) I_0(fa)
$$

The numerator and denominators are not completely simplified in order that the same denominator will be valid for each solution, (a particularly elegant simplification of this denominator was provided by Cheng et al. (1982) in the calculation of $B_f$ for a point source). The numerators for the Cramer’s rule solution are presented in Table 2.2 below. One important note is that use has been made of the following Wronskian relationship (i.e. Abramowitz and Stegun, 1964)

$$
K_0(z)I_1(z) + K_1(z)I_0(z) = \frac{1}{z}
$$

for the solution of $A$ and $C$ for a volume point source.
<table>
<thead>
<tr>
<th>Source Type</th>
<th>Cramer’s Rule Numerator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radial Source (2l)</td>
<td>$A = -G(\omega)(-\rho \omega^2 + 2\mu k_z^2)K_1(m_1a) \frac{2 \sin lk_z}{k_z}$</td>
</tr>
<tr>
<td></td>
<td>$C = -G(\omega)2\mu ik_z l_1 K_1(l_1a) \frac{2 \sin lk_z}{k_z}$</td>
</tr>
<tr>
<td>Axial Source (2l)</td>
<td>$A = G(\omega)2\mu ik_z [m_1 K_0(m_1a) + \frac{K_1(m_1a)}{a}] \frac{2 \sin lk_z}{k_z}$</td>
</tr>
<tr>
<td></td>
<td>$C = -G(\omega)\left[(-\rho \omega^2 + 2\mu k_z^2)K_0(l_1a) + \frac{2\mu l_1}{a} K_1(l_1a)\right] \frac{2 \sin lk_z}{k_z}$</td>
</tr>
</tbody>
</table>

Table 2.1: Cramer’s rule numerators for an empty borehole.
<table>
<thead>
<tr>
<th>Source Type</th>
<th>Cramer's rule Numerator</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Radial Source (2l)</strong></td>
<td>$B_f = G(\omega)\frac{2\sin lk_z}{k_z}\rho \omega^2 l_1 K_1(l_1 a) K_1(m_1 a)$</td>
</tr>
<tr>
<td></td>
<td>$A = G(\omega)\frac{2\sin lk_z}{k_z} f(-\rho \omega^2 + 2\mu k_z^2) K_1(m_1 a) I_1(fa)$</td>
</tr>
<tr>
<td></td>
<td>$C = G(\omega)\frac{2\sin lk_z}{k_z} 2\mu ik_z f l_1 K_1(l_1 a) I_1(fa)$</td>
</tr>
</tbody>
</table>
| **Axial Source (2l)**  | $B_f = G(\omega)\frac{2\sin lk_z}{k_z} (2\mu ik_z l_1 m_1 K_0(m_1 a) K_1(l_1 a)\
|                        | $- ik_z(-\rho \omega^2 + 2\mu k_z^2) K_0(l_1 a) K_1(m_1 a))$                        |
|                        | $A = -G(\omega)\frac{2\sin lk_z}{k_z} (2f \mu ik_z [m_1 K_0(m_1 a) + \frac{K_1(m_1 a)}{a}] I_1(fa)$ |
|                        | $- \rho f \omega^2 i k_z I_0(fa) K_1(m_1 a))$                                        |
|                        | $C = G(\omega)\frac{2\sin lk_z}{k_z} (f(-\rho \omega^2 + 2\mu k_z^2) K_0(l_1 a)$   |
|                        | $+\frac{2\mu l_1}{a} K_1(l_1 a)) I_1(fa) - \rho f \omega^2 l_1 I_0(fa) K_1(l_1 a))$ |
| **Volume point source**| $B_f = -G(\omega)\frac{V_0}{2\pi}(-f[(-\rho \omega^2 + 2\mu k_z^2)^2 K_0(l_1 a) K_1(m_1 a)$ |
|                        | $- \frac{2\mu l_1 \rho \omega^2}{a} K_1(l_1 a) K_1(m_1 a) - 4\mu^2 k_z^2 l_1 m_1 K_1(l_1 a) K_0(m_1 a))$ |
|                        | $K_1(fa) + \rho \rho f \omega^4 l_1 K_1(l_1 a) K_1(m_1 a) K_0(fa)$                  |
|                        | $A = -G(\omega)\frac{V_0}{2\pi} \rho f \omega^2 (-\rho \omega^2 + 2\mu k_z^2) K_1(m_1 a)$ |
|                        | $C = -G(\omega)\frac{V_0}{2\pi} \rho f \mu ik_z \omega^2 l_1 K_1(l_1 a)$           |

Table 2.2: Cramer's rule numerators for a fluid-filled borehole.
2.3.3 Comparison of particular models

Fundamentally, the solutions presented for the empty borehole in Table 2.1 and the fluid-filled borehole in Table 2.2 for the radial and axial sources are qualitatively similar. The major difference in the numerators and the denominators are the presence of the terms \( J_0(fa) \) and \( f I_1(fa) \) in the fluid. For small argument the term \( f I_1(fa) \) equals \( \frac{L^2}{2} \). Thus for a radial source, the solutions for \( A \) governing the compressional displacement potential \( \phi \) and \( C \) governing the shear displacement potential \( \psi \) are equal to the empty borehole solution multiplied by \( \frac{L^2}{2} = \frac{k_f^2 - \frac{\omega^2}{c_f^2}}{2} \) which will always be less than 1 or imaginary. The decrease in efficiency of the radial source for a fluid-filled borehole is due to the presence of a tube wave.

Additional factors are terms of the form \( \rho p_f \) which would be zero if \( \rho_f \) were zero as in an empty borehole. Therefore the empty borehole case can be seen as the zero density limit of the quotients for the fluid-filled borehole which is to be expected. It is also seen in comparing Table 2.1 and Table 2.2 that radial and axial sources do in fact produce a contribution to the pressure response \( (B_f \neq 0) \). However, in the case of a radial source both modified Bessel functions are of order 1 which predominate over those of order 0 as in the axial source case for small argument. Thus the fluid potential for an axial source would be expected to be vanishingly small.

2.4 General and particular model: SH radiation

Torsional sources produce \( SH \) wave radiation. This is a problem much simpler than for the \( P-SV \) and was treated by White (1982, 1984a,b). Far field solutions were discussed by Heelan (1952, 1953a) and Brekhovskikh (1960, 1980). For completeness, this problem is addressed here including the full generality of radial layering.

The torsional source solution is valid for both empty and fluid-filled boreholes and is simply a two equations two unknowns type solution. Intermediate fluid layers are not allowed because fluid layers do not support equivoluminal motion and thus would

39
decouple the boundaries.

The integral formulation for layer $i$ is written as (Eq. A.80)

$$
\chi = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} (E_i K_0(m_i r_i) + F_i I_0(m_i r_i)) e^{ik_\theta z} e^{-i\omega t} dk_z d\omega
$$

(2.35)

The azimuthal stress and displacement involved in boundary condition calculations are written in terms of $\chi$ (Eq. A.35)

$$
U_\theta = -\frac{\partial \chi}{\partial r}
$$

(2.36)

$$
p_{\theta \theta} = \mu (-\frac{\partial}{\partial r} + \frac{1}{r}) \frac{\partial \chi}{\partial r}
$$

In its most general form corresponding to a solid-solid boundary, the boundary condition matrix can be written

$$
\mathbf{D}_i = \begin{pmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{pmatrix}
$$

(2.37)

where the first row is the equation for azimuthal displacement and the second row is the equation for azimuthal stress. Writing the elements explicitly

$$
D_{11} = m_i K_1(m_i r_i)
$$

(2.38)

$$
D_{12} = -m_i I_1(m_i r_i)
$$

$$
D_{21} = -\mu_i (m_i^2 K_0(m_i r_i) + \frac{2m_i K_1(m_i r_i)}{r_i})
$$

$$
D_{22} = -\mu_i (r_i^2 I_0(m_i r_i) - \frac{2m_i I_1(m_i r_i)}{r_i})
$$

(2.39)

The continuity of azimuthal stress is the only boundary condition at the borehole wall. The inverse of the matrix $\mathbf{D}$ can be written explicitly (e.g. Schmitt and Bouchon, 1985)

$$
\mathbf{D}_i^{-1} = \begin{pmatrix}
\frac{-r_i}{\mu_i m_i^2} & D_{22} & -D_{12} \\
-D_{21} & D_{22} & D_{12}
\end{pmatrix}
$$

(2.40)

to reduce the number of computations required and improve stability.
For the infinite half space, only solutions that decay to infinity are allowed so $a_n$ is written

$$a_n = \begin{vmatrix} E_n \\ 0 \end{vmatrix}$$

(2.41)

For a torsional source of length $2l$ the source term is a discontinuity in azimuthal stress given by

$$\begin{vmatrix} 0 \\ G(\omega) \frac{2\sin lk_z}{k_z} \end{vmatrix}$$

(2.42)

Solving the equation for a torsional source in a radially layered medium governed only by the single boundary condition of vanishing azimuthal stress, the following development is constructed

$$G_{21}E_n + S = 0$$

(2.43)

where as before $G$ is defined by Eq. 2.16.

Eq. 2.43 is the complete solution for radiation from a torsional source into a radially layered medium. A closed form solution for a torsional source of length $2l$ applied in a borehole of radius $a$ surrounded by an infinite half space is given by

$$E = \frac{G(\omega)}{\mu(m^2 K_0(ma) + \frac{2mK_1(ma)}{a})} \frac{2\sin lk_z}{k_z}$$

(2.44)

2.5 Calculation using the discrete wavenumber method

To this point, the matrix equations have been specified without consideration of the method used to numerically calculate them. In fact, for the calculations presented in this thesis, each of the preceding integrals is evaluated using the discrete wavenumber method (White and Zechman, 1968; Bouchon and Aki, 1977; Bouchon, 1979, 1980, 1981; Cheng and Toksöz, 1981; Tubman, 1984; Tubman et al., 1984, 1986). Briefly, the discrete wavenumber technique can be considered a transformation of the
wavenumber integrals. The transformation is accomplished by replacing the source with a periodic array of sources. This replacement discretizes the integral in \( k_z \) space using a sampling theorem of Fourier analysis. The interval between the sources in the periodic array is a distance \( L \) and is chosen such that arrivals from the fictitious adjacent sources do not occur in the time interval of interest. Thus, integrals of the form

\[
\int_{-\infty}^{\infty} F(k_z) e^{ik_z z} dk_z
\]

are transformed into sums of the form

\[
\frac{2\pi}{L} \sum_{m=-M}^{M} F(k_{zm}) e^{ik_{zm} z}
\]

(2.46)

where \( k_{zm} \) equals \( \frac{2\pi m}{L} \), \( L \) is the discretization length and \( M \) is chosen by a suitable convergence criteria. In the preceding and subsequent development, only the functions \( F(k_z) \) are listed. For more details concerning the discrete wavenumber method it is recommended to consult the Bouchon and Aki (1977) and Cheng and Toksöz (1981) papers.

### 2.6 Numerical results - synthetic seismograms

A series of synthetic seismograms generated with the just described Thomson-Haskell algorithm will now be presented. An important goal of this exposition is to convey information about the properties of the borehole and the borehole environment which affect radiation outside the borehole.

From the physical properties literature, three lithologies were chosen to represent typical media for the infinite half space surrounding the borehole. A representative from each one of the three most common sedimentary rock types, sand, shale and limestone, was selected. The lithologies are: Pierre Shale, a low velocity material; Berea Sandstone, a well-cemented sandstone with low Poisson’s ratio; and Solenhofen Limestone, a high velocity limestone. It should be noted that Pierre Shale is known
to be anisotropic but inclusion of anisotropy awaits further work. A fourth set of physical properties was used to show the existence of shear wave lobes in the radiation pattern for certain instances. It is a model from a paper by Lee and Balch (1982) and is entitled in this thesis the Lee and Balch sediment. The properties of this lithology are that the shear wave velocity is in between the fluid velocity and tube wave velocity, the material is a Poisson solid, and the fluid density to bulk density ratio is .5. Table 2.3 lists the physical properties of these materials along with references to their origin. These properties will be extensively referred to in this and subsequent chapters. In addition, Table 2.3 gives the physical parameters of steel casing and an ideal drilling fluid, essentially water, used in the modelling for a fluid-filled borehole.

The geometry of the synthetic seismograms is presented in Fig. 2-3 and consists of $N$ point receivers equally spaced along an array of height $z$. The array is placed a distance $r$ outside and parallel to the source borehole. Radial, axial, and torsional sources applied to the borehole wall are used as sources and, if the borehole is fluid-filled, volume point sources located on the borehole axis can also be used.

All of the seismograms were convolved with the Ricker (1953) wavelet which has a representation

$$\frac{\omega^2}{\omega_c^2} e^{-\frac{\omega^2}{\omega_c^2}}$$

(2.47)

where $\omega_c$ is the center frequency. The Ricker wavelet is non-causal and therefore for short time windows the onset of the wavelet will develop slightly before zero time. Where the non-causality caused confusion in interpretation, a constant time shift proportional to the inverse of the center frequency was applied and was duly noted in the captions for the respective figures.

2.6.1 Testing of the algorithm

The algorithm was thoroughly checked but one test is worth mentioning. This test consisted of a comparison between displacements calculated using the Thomson-Haskell algorithm and the boundary integral technique of Bouchon and Schmitt
(1989a,b) including the corrections (Bouchon and Schmitt, 1989b).

The Bouchon-Schmitt boundary integral technique, which was developed for calculating the pressure response inside an irregular borehole, can be modified to calculate wave propagation outside a borehole. The modifications necessary are described in Appendix D and by performing these modifications, the boundary integral calculation can be directly compared to the Thomson-Haskell calculation for regular boreholes surrounded by an infinite half space.

For the test, the lithology of the infinite halfspace surrounding the fluid-filled borehole was Berea Sandstone (see Table 2.3) and the radius of the borehole was 10cm. Both programs were run without consideration of $Q$ values for this test. The receiver array was located a distance of .33m from the center of the borehole for the calculation of the radial and vertical displacements. The source was a 1600 cubic centimeter volume point source located on the borehole axis and the Ricker wavelet had a center frequency of 1000 Hz.

Excellent agreement ($\sim 1 - 2\%$) was seen between the radial and vertical components of displacement and the pressure response inside the borehole calculated with both techniques. Only the radial components are shown and are presented in Fig. 2-4 confirming the excellent agreement. A time shift of .75 milliseconds has been applied to both radial components due to the non-causality of the Ricker wavelet.

Boundary integral techniques are computationally intensive though powerful and the computation time for this test was approximately 500 minutes for the boundary integral technique and 3 seconds for the Thomson-Haskell algorithm on a minicomputer. The identical results achieved with both techniques give confidence in the correctness of the new algorithm, as well as in the accuracy of the boundary integral-discrete wavenumber method.
2.6.2 P-SV radiation into high and low velocity sediments

From both an experimental and theoretical point of view, the most interesting distinction worth demonstrating is the contrast between radiation emanating from boreholes surrounded by high velocity sediments such as the Berea Sandstone and radiation emanating from boreholes surrounded by low velocity sediments such as the Pierre Shale. High velocity sediments are defined as sediments with shear wave velocities greater than the fluid velocity. The difference in radiation is due to the influence of the dominant radiation product, the tube wave. The definition of low velocity sediments requires a subdivision on the basis of the tube wave velocity which is defined below.

The tube wave velocity is always less than the fluid velocity and the zero frequency phase velocity of the tube wave for an uncased borehole is given by the equation (e.g. Biot, 1952; White, 1965, 1983; Cheng and Toksöz, 1984)

\[
C_T = \frac{\alpha_f}{\sqrt{1 + \frac{\rho_f \alpha_f^2}{M}}}
\]

where \(M\) is the shear modulus \(\mu\) of the formation. For a cased borehole, \(M\) is given by Marzetta and Schoenberg (1985) as

\[
M = \frac{\mu + (\mu_c - \mu)(1 - \frac{\rho_a^2}{\rho_c^2})(1 - \frac{r_a^2}{r_c^2})}{\mu_c - (\mu_c - \mu)\frac{\rho_a^2}{\rho_c^2}(1 - \frac{r_a^2}{r_c^2})}
\]  

(2.49)

where \(r_a\) is the inner radius of the casing, \(r_c\) the outer radius of the casing, and the subscript \(c\) refers to casing velocities and moduli. The formula of Norris (1990) for a rigid casing and no tool is identical to Eq. 2.49 but Eq. 2.49 is slightly different from that of White (Eq. 5.6, 1983; Marzetta and Schoenberg, 1985).

The two types of low velocity sediments are defined as sediments which have shear wave velocity less than the fluid velocity but greater than the tube wave velocity such as the Lee and Balch sediment (Table 2.3); and sediments with shear wave velocities less than tube wave velocities such as the Pierre Shale. The latter case is by far the most prevalent because boreholes surrounded by low velocity sediments are usually
cased with steel casing to prevent collapse. The effect of steel casing is to increase the tube wave velocity which restricts the range of shear wave velocity for a Lee and Balch type sediment.

In the first low velocity case, Lee and Balch noticed lobes in the shear wave radiation pattern (See Fig. 1-2). In the second low velocity case, a supershear condition arises which leads to the emanation of radiation from the upgoing and downgoing tube wave (White and Sengbush, 1963; de Bruin and Huizer, 1989). This radiation is emitted as a conical wavefront (de Bruin and Huizer, 1989) and is called a Mach wave in this thesis (Ben-Menahem and Singh, 1981, 1987). The analogy of this borehole Mach wave and Mach waves from earthquake seismology will be thoroughly demonstrated in a later section.

2.6.3 Numerical Examples

The first numerical example will describe the radiation when the infinite half space surrounding a fluid-filled borehole is comprised of a high velocity sediment, Berea Sandstone.

Other details of the experiment include that the source is a point source (1600 cc) at the center of a 10cm radius borehole, the point receiver array is located 5m away from the source borehole, and the receiver array is 20m in height with 1 meter interreceiver spacing. Q values of 1000, essentially no attenuation, was considered in the modelling for the Berea Sandstone because it is a very competent sediment. From Table 2.3 and Eq. 2.48 the tube wave velocity in this case is 1400 m/sec which is far less than the shear wave velocity of Berea sandstone of 2664 m/sec.

Fig. 2-5 shows the vertical and radial components for the uncased borehole and clearly discernible P and S wave arrivals are seen with characteristic hyperbolic moveouts. It is seen that the vertical component on the horizontal axis is zero due to symmetry considerations. Fig. 2-6 also shows radial and vertical components but for this example a 0.5 cm layer of steel casing has been inserted and it is seen that these
seismograms are nearly identical to those presented in Fig. 2-5. The tube wave velocity for the cased borehole is calculated using Eq. 2.48 and Eq. 2.49 as 1420 m/sec—a slight increase. A direct comparison between the cased and uncased seismograms is displayed in Fig. 2-7 which shows that the radial components for the two cases are qualitatively identical. The one difference between the two radial components is a slight loss in amplitude for the cased borehole configuration because a cased borehole is a more efficient waveguide and better traps the radiation as a tube wave inside the borehole compared to an uncased borehole. Particle motion diagrams for the cased and uncased boreholes are shown in Fig. 2-8 and also seen to be virtually identical. The particle motion diagrams well exhibit the orthogonal behavior expected between $P$ and $S$ waves and as one proceeds up the array the $P$ wave motion becomes more vertical and the $S$ wave more horizontal.

Thus in the case when a high velocity sediment surrounds the borehole there is nothing unusual encountered. There is no evidence for the influence of the borehole or the tube wave besides perturbation of the initial geometry of the wavefield. Thus even as close as 5m away from the borehole, the radiation is substantially far field.

However, unusual behavior is encountered when low velocity sediments, such as the Pierre Shale, surround the borehole. For this experiment the geometry of the source, the source borehole and the receiver array are identical to those just presented for Fig. 2-5-Fig. 2-8. Attenuation values were determined from the work of McDonal et al. (1958) who give Q values of 10 for $S$ waves and 32 for $P$ waves (also Toksöz and Johnston, 1981) for Pierre Shale. Fig. 2-9 shows the vertical and radial components for an uncased borehole and Fig. 2-10 shows these same components for a cased borehole. By comparing the two it is seen that there is a dramatic difference in radiation into a low velocity sediment due to the presence of casing, the difference being limited to the $S$ wave primarily. The $S$ wave seen in Fig. 2-10 arrives before that in Fig. 2-9 and is thus seen to be travelling at a faster velocity, a velocity higher than the shear wave velocity. Fig. 2-11 shows the radial components for the cased
and uncased borehole plotted on top of each other. This figure shows that the S wave for the cased borehole is much more energetic relative to the P wave than for the uncased borehole. Fig. 2-12 shows particle motion diagrams for the cased and uncased boreholes and it is seen that besides being very different the particle motion for the cased borehole no longer exhibits orthogonal P and S wave behavior. Instead the P wave motion is nearly identical to that seen for the uncased borehole but the S wave motion is closer to vertical as one proceeds up the array. If the motion was true S coming from the source then its direction would become more horizontal as one proceeded up the array as seen in Fig. 2-8.

A similar set of comparisons was performed for an uncased and cased borehole surrounded by a Lee and Balch sediment. However, for brevity only the radial components are plotted as dashed and solid lines on top of each other in Fig. 2-13. For the uncased borehole, the tube wave velocity 1,130 meters per second is less than the shear wave velocity of 1,212 and for the cased borehole, the reverse is true, the tube wave velocity of 1,310 m/sec is greater than the shear wave velocity of 1,212. Thus this case meets velocity criteria regarding the tube wave in between the Pierre Shale (Fig. 2-9-Fig. 2-12) and Berea Sandstone (Fig. 2-5-Fig. 2-8) cases presented above. The uncased borehole behavior is similar to that for the Berea Sandstone figures but the cased borehole resembles the seismograms seen in the Pierre Shale figures. Thus there's a slight velocity discrepancy between the two but quite a large amplitude discrepancy in the shear wave.

It is fundamental to seismology that in an isotropic, infinite, homogeneous medium there are only two body waves which travel at compressional and shear wave velocity respectively. Thus the change in moveout velocity of the shear waves for the cased borehole provides quite an interesting dilemma that has been seen in field experiments (White and Sengbush, 1963; de Bruin and Huizer, 1989). The fundamental significance of this observation is that borehole properties have an influence on the radiation into the surrounding medium. The distinction is due to the fact that when
the tube wave velocities are greater than the shear wave velocities, direct radiation from the up and downgoing tube waves (White and Sengbush, 1963) in the form of conical waves (de Bruin and Huizer, 1989), which will be referred to as Mach waves here, is present.

Now is a good opportunity to explain the physics behind Mach waves and draw the parallels to the counterparts of Mach waves in earthquake seismology and aerodynamics.

2.6.4 The physics of Mach waves

The name of Mach is well known from aerodynamics where for instance an airplane travelling at a supersonic velocity of twice the speed of sound is said to be travelling at Mach 2. The ratio of the speed of the aircraft to the speed of sound is called the Mach number ($M$) (e.g. Ferrari and Tricomi, 1968) and for the supersonic aircraft cited above $M$ equals 2.

When an object is travelling at subsonic velocity, sound generated by that object exhibits no constructive interference, as shown in the top half of Fig. 2-14 from Howarth (1953), and the sound decays rapidly away from the object. Sound emitted at a time $t_0$ at a point $x_0$ has bypassed $x_1$ by the time the object reaches $x_1$ and emits sound at a time $t_1$. When an object travels at a supersonic velocity, however, the sound generated by that object exhibits constructive interference and a Mach cone is generated as shown in the bottom half of Fig. 2-14. In this case, the sound emitted at time $t_0$ is coincident with sound emitted at time $t_1$ in a plane wrapped around the borehole or the Mach cone. In aerodynamics, the outside of this Mach cone is manifest as a sonic boom.

Now in seismology, there are two velocities, a shear wave velocity and a compressional wave velocity and thus there are two possible Mach waves. The case of fault rupture proceeding at a rate faster than body wave velocities has been examined by Ben-Menahem and Singh (1981) and especially by Ben-Menahem and Singh (1987)
and others. The \textit{Ma-h} waves in seismology are also sometimes referred to as shock waves. Examples from the literature of highly energetic seismograms due to the influence of Mach waves can be seen in the reference list of Ben-Menahem and Singh (1987).

In the borehole case, the causative agent for Mach wave generation, the moving source, is the tube wave and not fault rupture. Except in very rare instances, the tube wave velocity is less than the compressional wave velocity in the surrounding medium. Therefore, only the supershear case is of interest for the borehole. The analogy of constructive and destructive interference producing a conical wave front for the tube wave was advanced by de Bruin and Huizer (1989) as seen in Fig. 2-15 from de Bruin and Huizer's paper (1989).

The Mach number for the tube wave is the zero frequency tube wave velocity over the shear wave velocity of the surrounding medium

\[ M = \frac{C_T}{\beta} \]  \hspace{1cm} (2.50)

and if the Mach number is greater than one, Mach waves are generated. In the borehole case, there are upgoing and downgoing tube waves but the physical and mathematical development will only consider the upgoing wave - the theory holding equally well for the downgoing one.

A diagram of the Mach cone is displayed in Fig. 2-16 and it is seen that the Mach cone is actually the intersection of two cones. One of the cones travels with the tube wave and is governed by the Mach angle $\phi$ seen in Fig. 2-16. The other cone begins from the source and the angle governing this cone is a measure of co-latitude which is complementary to the Mach angle and hence will be called the complementary Mach angle $\phi_c$. $\phi_c$ equals

\[ \cos \phi_c = \frac{\beta}{C_T} = \frac{1}{M} \]  \hspace{1cm} (2.51)

or the sine of the Mach angle $\phi$. The outside of this Mach cone, the Mach wave front, has a normal vector in the $\phi_c$ direction as seen in Fig. 2-16. Therefore, the angle of propagation of the Mach wave does not change with increased moveout.
Tangent to the Mach cone will be the direct shear spherical wavefront. This is demonstrated with a series of seismograms in Fig. 2-19. The geometry is outlined in Fig. 2-18 and consists of a series of parallel vertical receiver arrays that are spaced 1, 2, 5, 10 and 20m away from the source borehole. The height of the receiver arrays is 50m with a 1m interreceiver spacing and no attenuation is considered. A Ricker wavelet of 200 Hz center frequency is used as the source wavelet. The angle of the Mach wave is seen in these figures to be 27.5 degrees and the spherical wavefront is tangent to these seismograms and grows as the arrays get progressively further out. The $P$ wave decays as $\frac{1}{R}$ and is not affected by the presence of the Mach waves. The decay of the Mach waves up the borehole is seen to be less and in the next chapter is shown to be $\frac{1}{\sqrt{r}}$. As the source-receiver array distance increases to 5m, 10m and 20m the separation of the $P$ wave from the $S$ wave is seen.

From Fig. 2-16, the traveltime of the Mach waves can be determined. If a point $(r, z)$ is inside the Mach cone, then the traveltime will be a hybrid traveltime consisting of travel first as a tube wave up to a point $O1$ and then as a Mach wave with a normal vector in the $\phi_c$ direction to the point $(r, z)$. The hybrid nature of this traveltime was first recognized by White and Sengbush (1963) as shown in Fig. 2-15. It is seen in Fig. 2-16 that this travel time from the source origin to a point $(r, z)$ is given by the equation

$$ t = t1 - t2 + t3 $$

(2.52)

where

$$ t1 = \frac{z}{C_T} \quad t2 = \frac{r}{\sqrt{M^2 - 1}C_T} \quad t3 = \frac{rM}{\sqrt{M^2 - 1}\beta} $$

(2.53)

t1 is travel to the height $z$ at tube wave velocity. $t_2$ is a correction factor $(z - O1)$ because the Mach wave emanates at height $O1$ and $t_3$ is the traveltime as a Mach wave from the point $O1$ to the point $(r, z)$. Substituting Eq. 2.53 into Eq. 2.52 and using Eq. 2.50 it is seen that the total travel time after some simplification is given by

$$ t = \frac{z}{C_T} + \frac{r\sqrt{M^2 - 1}}{C_T} $$

(2.54)

51
A similar equation was found by Ben-Menahem and Singh (1987) in modelling radiation from a line source. This is an important result and it is shown in the next chapter that by calculating the residue of the tube wave pole that this identical time delay is reproduced.

An important implication of this time delay is that receivers inside the Mach cone will measure the tube wave velocity of the source borehole. This is because the $r$ in Eq. 2.54 will remain constant and traveltime variations along the array of receivers will be seen due to the variations in $z$ exclusively. Therefore, instead of exhibiting just a hyperbolic moveout across an array of receivers, both a hyperbolic and tangent to it a linear moveout will be seen as in Fig. 2-10.

Although White and Sengbush were the first to recognize the hybrid nature of the traveltime, their formulation of the traveltime equation was too general. For instance, White and Sengbush considered both $P$ and $S$ waves and simultaneously considered the radiation contributions from upgoing and downgoing waves. However, only $S$ waves needed to be considered and the contributions from the upgoing and downgoing waves could be isolated. Moreover, White and Sengbush did not realize that the propagation of the wavefront would travel at a constant angle, the Mach angle, so White and Sengbush required an integral over all $z$ along the borehole. Hence, the formulation developed here is simpler.

de Bruin and Huizer (1989) stated that the tube wave in essence acts as a Rayleigh wave that leaks energy, conical (Mach) waves, in the super shear case. Although this is a useful analogy, the mathematical formulation did not include the borehole but instead only a cartesian half space. However, like the Rayleigh wave, qualitative information concerning the Mach wave is achieved by evaluating the residue of the tube wave pole in the next chapter.

The most compelling evidence for the existence of the Mach waves can be seen by the use of time domain snapshots which tie together the constructive interference, time delay, and geometric properties of the Mach waves. These snapshots of the
magnitude of the displacement wavefield are presented in Fig. 2-20 through Fig. 2-22 and are contour plots. This type of representation is used most often when presenting finite difference results but through the calculation of many seismograms can also help represent discrete wavenumber results.

For the snapshots 100 vertical arrays of receivers were placed between .5 and 50m away from the borehole with a .5m interarray spacing. Each array consisted of 100 receivers located from 0 to 49.5 m in height (Fig. 2-3). This produced a 50m by 50m grid of 10000 points. A 200 Hz Ricker wavelet was used as the source wavelet throughout.

The first pair of snapshots, Fig. 2-20, is for an uncased fluid-filled borehole of 10cm radius surrounded by Pierre Shale. As with Fig. 2-19 no attenuation was considered in this modelling. The features of this snapshot include a P wave which is seen to be little affected by the borehole, a S wave and a high amplitude Mach wave generated from the tube wave and a clearly developed Mach cone. It can be seen that the genesis of the Mach wave is due to the constructive interference along the tube wave. It is also clearly evident that the Mach wave front is tangent to the S wave. The complementary Mach angle is calculated using the tube wave velocity of 980 m/sec and the shear wave velocity of 869 m/sec to be \( \phi_c = 63 \) degrees and fits the observed Mach wave propagation well. The Mach wave has grown from 20msecs, part a) to 30msecs part b) but the Mach angle has not changed.

The next pair of snapshots, Fig. 2-21, is identical to that for Fig. 2-20 except that a .5 cm layer of casing has been inserted between the borehole and the Pierre Shale medium. The same qualitative features are present but now it is seen that the tube wave travels faster and that the Mach wave propagates at a steeper angle. The new complementary Mach angle is calculated for a tube wave velocity of 1280 m/sec and a shear wave velocity of 869 m/sec to be \( \phi_c = 43 \) degrees which well matches the observed result. As a consequence of the smaller \( \phi_c \) the spherical portion of the wavefront is less evident and the Mach wave exists for a greater aperture.
Additionally, it can be seen that the P wave is affected by the presence of the casing but only slightly. The location of the P wave and the S wave are identical at 20 and 30ms in comparison with Fig. 2-21 as would be expected.

A final pair of snapshots, Fig. 2-22, is again similar to that of Fig. 2-20 but now Berea Sandstone surrounds the borehole. At 20ms, the P wave has already travelled out of the grid but the S wave is clearly seen along with the slower moving tube wave along the edge. The S wave amplitude is slightly skewed from a maximum of 45 degrees along the borehole axis due to the presence of the fluid in the borehole but there is no direct interaction between the tube wave and the S wave and there is clearly no Mach wave. The decay of the tube wave is very gradual, there is no constructive interference and its influence only extends to about 5m away from the borehole.

2.6.5 more on P-SV radiation

Radial source propagation is nearly identical to the point source radiation just described and since seismograms from radial sources will be presented in Chapter 3 they will not be discussed here.

In Fig. 2-23 radial component seismograms are presented which compare radiation from an axial source in a cased and uncased borehole surrounded by Pierre Shale. The comparison between these two is excellent, demonstrating that there are no Mach wave effects on the axial source radiation. The good agreement between the two further corroborates the observation that axial source radiation is not significantly affected by the presence of casing in the borehole.

2.6.6 SH radiation

Fig. 2-24 presents azimuthal component seismograms describing radiation from a torsional source in an uncased borehole surrounded by Pierre Shale (top) and a cased borehole surrounded by Pierre Shale (bottom). The length of the torsional source is
.5 m and the strength is 1 bar. Again the source-receiver array separation was 5m and the array height was 20m with 1m interreceiver spacing.

The identical behavior of radiation from torsional sources in cased and uncased boreholes allows the echoing of the result for the axial source. Namely, radiation from torsional sources is little affected by the presence of casing.

2.7 Conclusions

The Thomson-Haskell algorithm developed here for calculation of displacement fields in a radially layered media provided a powerful tool for the modelling of radiation from downhole seismic sources. Additionally, it complements well the algorithms of Tubman, (1984), Schmitt and Bouchon (1985), Tubman et al. (1984, 1986) for calculation of wave propagation inside a radially layered borehole. Testing of the algorithm using a modification of boundary integral technique of Bouchon and Schmitt (1989a,b) was very successful.

In particular, the algorithm was able to demonstrate the differences seen between radiation in low velocity sediments versus high velocity sediments that has been seen in experimental data. This difference is due to the presence of the Mach waves which radiate into the formation from the tube wave when the tube wave travels at super shear velocities. The implication of this fact is that receivers inside the Mach cone will measure the tube wave velocity and not the true shear wave velocity. The effect of adding steel casing is to expand the Mach wave cone and make the direct detection of the Mach wave more likely.

The Mach wave concept is similar to the "conical wavefront" hypothesis of de Bruin and Huizer (1989) although they did not include the contribution of the spherical wavefront nor recognize the propagation at the Mach angle. The hybrid travel time nature was noticed by White and Sengbush (1963) but they did not recognize the propagation of a Mach wave at a constant angle. The unification of these
two paradigms is an important piece in understanding radiation from boreholes surrounded by low velocity sediments.

Finally, \( P \) wave radiation is not affected by the presence of Mach waves nor is radiation from axial or torsional sources. The radiation of these sources is not affected by direct radiation from the borehole and thus is nearly in the far field only 5m away from the borehole.
Physical Properties Table

<table>
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<tr>
<th>Lithology/Property</th>
<th>$V_p$</th>
<th>$V_s$</th>
<th>$\sigma$</th>
<th>$\rho$</th>
<th>$C_T$</th>
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<tr>
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<td>.308</td>
<td>2,656</td>
<td>1,430</td>
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<tr>
<td>Berea Sandstone</td>
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<td>2,664</td>
<td>.165</td>
<td>2,140</td>
<td>1,400</td>
</tr>
<tr>
<td>Pierre Shale</td>
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<td>869</td>
<td>.394</td>
<td>2,250</td>
<td>980</td>
</tr>
<tr>
<td>Lee and Balch Sediment</td>
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<td>1,212</td>
<td>.250</td>
<td>2,000</td>
<td>1,130</td>
</tr>
<tr>
<td>Steel (Casing)</td>
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<td>3,228</td>
<td>.290</td>
<td>6,268</td>
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<td>0</td>
<td>.500</td>
<td>1,000</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.3: Physical properties of lithologies and materials used in this thesis. Physical properties include P and S wave velocities, densities, Poisson ratios ($\sigma$), zero frequency tube wave velocities ($C_T$) for uncased boreholes from Biot's formula. Velocities in $m/sec$, densities in $kg/m^3$. References, not the original, are for the Solenhofen Limestone and Steel (Press, 1966) for the Berea Sandstone and Pierre Shale (Thomsen, 1986) where the density of steel was calculated using American Petroleum Institute weight per foot designations (Austin, 1983). Lee and Balch sediment is from examples used in the paper by Lee and Balch (1982).
Figure 2-1: Geometry of the radially layered borehole. Figure also shows layer numbering and radii numbering conventions. Infinite half space is terminating layer.
Figure 2.2: Geometry of the sources for the radially layered borehole. One alternative for the source is a centrally located volume point source in the center of the borehole if it is fluid-filled which is not shown.
Figure 2-3: Geometry of the synthetic seismograms. The source is either: a) a volume point source in a fluid-filled borehole, b) axial and radial sources in either fluid-filled or empty boreholes, or c) torsional sources in a dry boreholes since the presence of fluid is irrelevant to radiation from a torsional source. A point receiver array of height $z$ with $N$ equally spaced receivers is separated from the source borehole by a radial distance $r$, and total distance $R$. The distance $r$ is typically 0-5m for Chapter 2 models and 20-90m for Chapter 3 and 4 models.
Figure 2-4: A comparison of seismograms calculated with the Thomson-Haskell method (DW - Dashed) and the Bouchon-Schmitt boundary-integral (BI - Solid) method for a regular borehole. 1600 cc volume point source in fluid-filled borehole surrounded by Berea Sandstone (See Table 2.4 for properties). Vertical array of receivers is .33 meters away from borehole. This figure shows the two radial components of displacement measured in meters and excellent agreement is seen (~1%). Similar agreement was seen with the pressure response and vertical displacement component. 1000 Hz. Ricker Wavelet with .75 msec time shift added to compensate for non-causality of the Ricker wavelet. DW result is flipped in polarity to account for the differing sign convention used in the Bouchon and Schmitt (1989a) BI technique.
Figure 2-5: a) Seismograms calculated using the discrete wavenumber implementation of the Thomson-Haskell technique for a fluid-filled borehole surrounded by Berea Sandstone, a high velocity sediment. Radial component of displacement measured in meters. Source to vertical receiver array distance is 5m with receivers equally spaced between 0 and 20m in height at 1m intervals. 1000 Hz Ricker Wavelet. No time shift applied to data. 1600 cc volume point source.
Figure 2-5: b) Same as a) but vertical component of displacement.
Figure 2-6: a) Seismograms calculated using the discrete wavenumber implementation of the Thomson-Haskell technique for a fluid-filled borehole surrounded by Berea Sandstone, a high velocity sediment. A .5 cm layer of steel casing is in between the formation and the borehole. Radial component of displacement measured in meters (bottom) and vertical component (top). Source to vertical receiver array distance is 5m with receivers equally spaced between 0 and 20m in height at 1m intervals. 1000 Hz Ricker Wavelet. No time shift applied to data. 1600 cc volume point source.
Lithology: Berea Sandstone
Purpose: Cased Borehole Test
Component: Vertical
Maximum Amplitude: 0.6489e-03
Gain: 1.500
Clipping Factor: None

Figure 2.6: b) Same as a) but vertical component of displacement.
Figure 2-7: Radial components of displacement from Fig. 2-6 and Fig. 2-5. Models are fluid-filled borehole surrounded by Berea Sandstone (dashed line) and fluid-filled borehole surrounded by Berea Sandstone with intervening casing (solid line). Peak amplitude for uncased borehole is \(0.738\times10^{-3}\) m and for cased borehole \(0.618\times10^{-3}\) m. Source to vertical receiver array distance is 5 m with receivers equally spaced between 0 and 20 m in height at 1 m intervals.
Figure 2-8: Particle motion diagrams for the seismograms presented in Fig. 2-6 and Fig. 2-5. Radial component is in $x$ direction and vertical component in $y$ direction. Receivers are numbered from bottom to top of receiver array in Fig. 2-6 and Fig. 2-5. Clear rotation of the particle motion with increasing azimuth is discernible. Excellent agreement is seen between the two sets of particle motions which will contrast with low velocity sediment case. Particle motion is purely radial for receiver one.
Lithology: Pierre Shale
Purpose: Uncased Borehole Test
Component: Radial
Maximum Amplitude: 0.4942e-02
Gain: 1.500
Clipping Factor: None

Figure 2-9: a) Seismograms calculated using the discrete wavenumber implementation of the Thomson-Haskell technique for a fluid-filled borehole surrounded by Pierre Shale, a low velocity sediment. Radial component of displacement measured in meters. Source to vertical receiver array distance is 5m with receivers equally spaced between 0 and 20m in height at 1m intervals. 1000 Hz Ricker Wavelet. No time shift applied to data. 1600 cc volume point source.
Lithology: Pierre Shale
Purpose: Uncased Borehole Test
Component: Vertical
Maximum Amplitude: 0.2155e-02
Gain: 1.500
Clipping Factor: None

Figure 2-9: b) Same as a) but for vertical component of displacement.
Lithology: Pierre Shale
Purpose: Cased Borehole Test
Component: Radial
Maximum Amplitude: 0.2336e-02
Gain: 1.500
Clipping Factor: None

Figure 2-10: a) Seismograms calculated using the discrete wavenumber implementation of the Thomson-Haskell technique for a fluid-filled borehole surrounded by Pierre Shale, a low velocity sediment with a .5cm layer of steel casing inserted. Radial component of displacement measured in meters. Source to vertical receiver array distance is 5m with receivers equally spaced between 0 and 20m in height at 1m intervals. 1000 Hz Ricker Wavelet. No time shift applied to data. 1600 cc volume point source.
Lithology: Pierre Shale
Purpose: Cased Borehole Test
Component: Vertical
Maximum Amplitude: 0.1667e-02
Gain: 1.500
Clipping Factor: None

Figure 2-10: b) Same as a) but for vertical component of displacement.
Figure 2-11: Radial components of displacement from Fig. 2-10 and Fig. 2-9. Models are fluid-filled borehole surrounded by Pierre Shale (dashed line) and surrounded by Pierre Shale with intervening casing (solid line). Source to vertical receiver array distance is 5m with receivers equally spaced between 0 and 20m in height at 1m intervals. Peak amplitude for uncased borehole is .494e-02 m and for cased borehole .2336e-02 m. These two radial components are not identical as they were with the hard sediments. Different moveout in S wave is seen but P wave moveout is identical.
Figure 2-12: Particle motion diagrams for the seismograms presented in Fig. 2-10 and Fig. 2-9. Radial component is in x direction and vertical component in y direction. Receivers are numbered from bottom to top of receiver array in Fig. 2-10 and Fig. 2-9. Clear rotation of the particle motion with increasing azimuth is discernible. In contrast to Fig. 2-8, only good agreement is seen in the P particle motion. At higher angles, the cased borehole particle motion starts deviating from horizontal and becomes more vertically directed. 1600 cc volume point source. Particle motion is purely radial for receiver one.
Lithology: Pierre Shale
Purpose: Cased and Uncased Borehole Test
Component: Radial
Maximum Amplitude: 
Gain: 1.500
Clipping Factor: None

Figure 2-13: Radial components of displacement for cased and uncased boreholes surrounded by Lee and Balch sediment (See Table 2.4). This figure shows that for uncased borehole (dashed, .73 e-02 m peak displacement), where the tube wave velocity is less than the shear wave velocity, a hyperbolic moveout is seen in S wave but with cased borehole (solid, .019 m peak displacement) the S wave arrival is much more dominant. The P wave is identical for the two except the scaling is different due to the S wave difference.
Figure 2-14: Pictorial diagrams used to explain the concept of Mach wave propagation from Howarth (1953) and essentially the reasoning used by Mach and in geophysics recently by de Bruin and Huizer (1989). In the top figure, a point sound-emitting source is moving from point 03 to 0 at a subsonic velocity. The spherical wavefront emitted by the source at 01 is completely contained in the spherical wavefront emitted at an earlier time by 03 and thus there is no constructive interference. Conversely in the bottom figure, the sound-emitting source is moving at supersonic velocity and constructive interference occurs generating a conical wave front or Mach Cone. In the case of the borehole the cone's tip would be blunt but this is irrelevant. The sine of the angle of the Mach cone is called the Mach number and equals the ratio of the source speed over the speed of sound. If the source was a supersonic aircraft the Mach wave would be often manifested as a sonic boom.
Figure 2-15: Display of Fig. 3 of de Bruin and Huizer (1989) and Fig. 11 from White and Sengbush (1963) demonstrating their hypotheses governing radiation into the formation. The de Bruin and Huizer hypothesis is physically identical to the Mach wave concept explained in the text and in the last figure. The White and Sengbush prediction of a hybrid traveltime consisting of a combination of tube wave propagation in the borehole and propagation at a shear wave velocity outside the borehole is correct. White and Sengbush did not realize the wave would propagate at the complementary Mach angle whose cosine was the inverse Mach number. This observation allows reconstruction of the position of emanation of the Mach wave on the z axis definitively.
Figure 2-16: Figure displays more about the geometry and the physics of the Mach wave. The Mach Cone consists of the intersection of two cones. One at the Mach angle $\phi$ emanating from the tube wave and one at the complementary Mach angle $\phi_c$ emanating from the source. Direct propagation from the source at the origin governs the radiation outside the Mach cone.
Figure 2.17: Figure displays information about the travel time for Mach waves. The total travel time \( t \) consists of the time to travel from the origin \( O \) to the point \( O_1 \) and the travel as a Mach wave from \( O_1 \) to the point \((r,z)\). This travel time can be constructed as \( t_1 \)-\( t_2 \)+\( t_3 \) in the figure which simplifies to \( \frac{z}{C_T} + \frac{r \sqrt{M^2-1}}{C_T} \).
Figure 2-18: Diagram explaining next figure which is a series of seismograms recorded at varying distances away from uncased source borehole. Receiver arrays are respectively 1, 2, 5, 10, and 20m away from the source borehole. The receiver arrays are 50m in height with 1m interreceiver spacing. Lithology surrounding borehole is Pierre Shale. Complementary Mach angle is 27.5 degrees.
Figure 2-19: Series of seismograms (see Fig. 2-18 for description of the geometry) of receiver arrays progressively greater distance away from the receiver borehole. Radial component models of fluid-filled borehole surrounded by Pierre Shale. 1600 cc volume point source. Receiver arrays of 50m height and 1m interreceiver spacing are spaced 1,2,5,10, and 20m away from the source borehole. Development of Mach cone and propagation of Mach wave with a normal vector at the complementary Mach angle of about 27.5 degrees is clearly seen. No attenuation added and 200 Hz center frequency Ricker wavelet used as source.
Figure 2-20: a) A snapshot at 20 msec. This is a contour plot of the magnitude of the displacement at 20 msec calculated on a 100 by 100 array of receiver locations using the algorithm developed in this chapter. The borehole is to the left and the leftmost vertical array is a half meter away from the borehole. Pierre Shale surrounds the uncased borehole. Tube wave velocity is calculated to be 980 m/sec and shear wave velocity is 869 m/sec. Mach angle, $\phi$, is thus 63 degrees. The radiation of the Mach Wave from the tube wave travelling up the borehole is clearly evident along with the generation of the spherical P wave front. 200 Hz Ricker Wavelet. Contours: minimum 2e-05m, maximum 7.02e-3m, interval 1e-3m b) shows a snapshot of the same geometry at 30 msec.
Figure 2-20: b) A snapshot at 30 msec for an uncased borehole surrounded by Pierre Shale. Contours: minimum 1e-05m, maximum 7.01e-3m, interval 1e-3m Same geometry as in a)
Figure 2.21: a) The following pair of snapshots is essentially identical to that presented in Fig. 2.20 but now the borehole is cased with steel. Snapshot at 20msec. Casing inner radius is .1m and outer radius .105m. The tube wave velocity is 1280 m/sec and combined with the shear wave velocity of 869 m/sec yields a Mach angle of 43 degrees. 200 Hz Ricker wavelet. Contours: minimum 1e-5m, maximum 4.01e-3m, interval 1e-3m b) shows a snapshot with this same geometry at 30 msecs.
Figure 2-21: b) A snapshot at 30 msec for a cased borehole surrounded by Pierre Shale. Contours: minimum 1e-5m, maximum 4.01e-3m, interval 1e-3m. Same geometry as in a).
Figure 2-22: a) A final pair of snapshots which is also essentially identical to that presented in Fig. 2-20 but now the borehole is uncased and surrounded by Berea Sandstone, a high velocity sediment. There is no longer any Mach wave and the influence of the borehole has decayed substantially by 5m radial distance. The $P$ wave has passed out of the grid. 200 Hz Ricker wavelet. Contours: minimum 5e-06m, maximum 1.005e-3m, interval 2e-4m. b) shows a snapshot at 30 msec.
Figure 2-22: b) A snapshot at 30 msec for an uncased borehole surrounded by Berea Sandstone. Contours: minimum $5e-06m$, maximum $1.005e-3m$, interval $2e-4m$. Same geometry as in a)
Lithology: Pierre Shale
Purpose: Axial Source Cased and Uncased Borehole Test
Component: Radial
Gain: 1.500
Maximum Amplitude: .178, 166e-05
Clipping Factor: None

Figure 2-23: Radial component seismograms calculated using the Thomson-Haskell algorithm for a fluid-filled borehole surrounded by Pierre Shale. 1 bar axial source of .5m length. Uncased borehole (dashed line – .178 e-5 m displacement) and borehole with .5cm thick casing layer (solid line – .166e-5 m displacement). Demonstrates no difference for these two cases. 5m source-receiver array distance, 20m vertical array height. 1000 Hz Ricker wavelet.
Lithology: Pierre Shale
Purpose: Torsional Source Uncased and Cased Borehole Test
Component: Azimuthal
Gain: 4.000
Maximum Amplitude: 0.182, 185e-05
Clipping Factor: 2.000

Figure 2-24: Azimuthal component seismograms calculated using the Thomson-Haskell algorithm for a fluid-filled or empty borehole surrounded by Pierre Shale. 1 bar torsional source of 0.5m length. Uncased borehole (dashed line) and borehole with 0.5cm thick casing layer (solid line) plotted on top of each other. Demonstrates no difference for these two cases. Rapid decay in vertical direction is a radiation pattern effect. 5m source-receiver array distance, 20m vertical array height. 1000 Hz Ricker Wavelet.
Chapter 3

Description of Far Field Radiation from Downhole Seismic Sources - Analytic Approximations and Comparison to Numerical Results

3.1 Introduction

Chapter 2 described radiation from downhole seismic sources where the integral expressions developed were evaluated using the Thomson-Haskell algorithm. In this chapter, radiation from downhole seismic sources is studied using analytic approximations. A direct comparison of the numeric and analytic techniques is also made in this chapter along with extensions to the analytic results.

As explained in Chapter 1, the distinction between near and far field in the borehole environment is complex. Near field in this thesis refers to radiation that is influenced by guided waves, the Mach waves, generated from the borehole itself. Since the evaluation of the integral by the method of stationary phase does not consider the effect of the Mach wave, the method of stationary phase is considered far field for low
velocity sediments. However, by extension of these analytic techniques, important information about the Mach waves and therefore, the near field can be obtained.

The algebra of the integral expressions presented in Chapter 2 can be simplified by expanding functions of borehole radius with small argument type expansions and expanding functions of the radial distance r which goes to infinity with asymptotic expansions (e.g. Abramowitz and Stegun, 1964). In other words, when approximating the far field, both small and large argument expansions are made. When these arguments are valid the resulting integrals are amenable to calculation by the method of stationary phase as first done by Heelan (1952, 1953a,b) and after Heelan by Brekhovskikh (1960,1980), Abo-Zena (1977), and Fehler and Pearson (1981,1984) for the empty borehole and by Lee and Balch (1982) for the fluid-filled borehole.

The result of the asymptotic evaluation of the integrals by the method of stationary phase, when possible, is a radiation pattern formula which is a description of the amplitude of the P, SV, and SH components as a function of azimuth and radius. A radiation pattern is a graphical snapshot of a radiation pattern formula at a fixed distance which shows the amplitude of the radiation as a function of direction. See Fig. 1.1 for an example. The radiation pattern formula contains the expression for the source wavelet and thus can be used to calculate displacement which is done later in this chapter.

This chapter will show the development of far field expressions for radiation patterns in empty and fluid-filled boreholes, will explore the limitations of the far field expressions thus far obtained in the literature and will isolate the Mach wave contribution through the calculation of the residues allowing a qualitative discussion of its properties.

3.1.1 Previous work of Heelan

Heelan published two papers (1953a,b) based on thesis work (1952) which have been very influential in the description of downhole seismic sources. The first paper entitled
"Radiation from a cylindrical source of finite length" (1953a) calculated radiation from axisymmetric radial, axial and torsional stresses applied to a short length of an infinite cylindrical cavity embedded in an infinite elastic medium. Heelan's work was one of the first in geophysics to examine radiation from cylindrical objects. The intent of Heelan's work was to investigate the radiation from dynamite exploded in cylindrical shot holes and Heelan believed that applying stresses over a short length of an infinite cavity was a good approximation to this radiation. The work is also directly applicable to the description of the radiation from downhole seismic sources (e.g. White, 1965).

Heelan's second paper (1953b) utilized the integral expressions from the first paper to calculate head waves propagated along an interface. The source of the head waves were cylindrical waves generated from the dynamite source in a cylindrical shothole. However, only the results from Heelan's first paper will be addressed here.

One major limitation of Heelan's work when discussed in the context of modelling downhole seismic sources is that the cavity Heelan used was empty while it is most often the case in practice that boreholes are fluid-filled. Because the cavity considered was empty, no centrally located point source could be modelled. Nonetheless, Heelan's study is important for its historical precedence, and also because for source development (Paulsson, 1988) and also in the near surface the borehole is often dry.

Heelan's results have been cited in numerous references and many different journals since 1953 including recent work (Lee and Balch, 1982; White, 1983; Paulsson, 1988). One of Heelan's major results was that shear waves could be induced by an artificial compressional source (dynamite) in a borehole although at the time many researchers believed only $P$ waves would be generated from such a source or that $S$ waves observed from explosions were in fact of secondary or converted origin. Another major contribution Heelan made was that the far field radiation patterns calculated had simple geometric interpretations. For instance, a four-leaved rose with an angular dependence of $\sin 2\theta$ was found for the shear wave radiation pattern cross section
and for the compressional wave radiation pattern cross section a peanut shape was obtained.

Heelan’s work has been used by others in further developments and research. For instance, both of Heelan’s papers were referenced heavily in Brekhovskikh’s book “Waves in layered media” (1960, 1980) who rederived Heelan’s results. Additionally, there are many important references to Heelan’s work by White (1960, 1965, 1983) for instance who used the reciprocity theorem and results from White’s earlier paper (1953) to duplicate Heelan’s radiation patterns. Thus White claimed confidence in the results from both techniques.

To date the radiation patterns produced by Heelan have strictly been used for their geometric description of the far field radiation and not for the magnitudes. The only experiment that has truly tested the magnitude properties of Heelan’s results has been the experiment presented in White and Sengbush (1963) which will be discussed in Chapter 4. In general, the agreement between the White and Sengbush experimental results and the results of Heelan are not good but the boreholes used by White and Sengbush were fluid-filled so are causes for substantial difference.

Despite the importance of Heelan’s work it has come under severe criticism due to omissions and misprints in the papers (1953a,b) and possibly because of the unique treatment of the algebra, separation of variables procedure and contour integration presented by Heelan. The criticism levied against Heelan’s work comes in different forms, for instance, Jordan (1962) dismissed Heelan’s work as mathematically unsound while Hazebroek (1966) pointed out that Heelan’s analysis was improperly entitled since the cylinder was not closed. The most severe criticism has come from Abo-Zena who wrote a paper with a title similar to Heelan’s first which was entitled “Radiation from a finite cylindrical explosive source” (Abo-Zena, 1977). In this paper, Abo-Zena devotes the appendix to criticism of Heelan’s work although White has shown that Abo-Zena’s results are equivalent in the far field (although this is only true if a correction is made to Abo-Zena’s work, see Appendix B).
One major problem in answering the extensive criticism of Heelan’s work was that Heelan did not adequately specify the complex contour $C$ used in the contour analysis in either the papers (1953a,b) or thesis (1952). Therefore, because of the importance of Heelan’s work, a parallel contour integration strategy developed by Brekhovskikh (1960, 1980) is explained and expanded to include axial sources in Appendix C. The integration strategy used by Brekhovskikh amounts to an expansion of the displacement potentials in terms of a superposition of homogeneous and inhomogeneous plane waves, the Weyl integral, which is well known in electromagnetics (e.g. Stratton, 1941; Brekhovskikh, 1960, 1980; Felsen and Marcuvitz, 1973). It is further shown in Appendix C when introducing the notation of Brekhovskikh that Heelan’s integrals are integrals of cylindrical waves, a Sommerfeld integral type problem. The details of Heelan’s algebra is considered and thoroughly explained in Appendix B and shown to be correct.

3.1.2 Previous work of Lee and Balch

Lee and Balch (1982) used the same borehole model as Heelan except for the inclusion of fluid in the infinite cavity. The addition of fluid in the cavity meant that there were now three boundary conditions to contend with, continuity of normal displacement, continuity of normal stress and vanishing of tangential stress plus a volume point source in the center of the borehole could be considered. Lee and Balch provided far field radiation pattern solutions calculated with the method of stationary phase for a volume point source on the borehole axis and for a radial source of finite length applied to the borehole wall.

Lee and Balch found the existence of a tube wave pole due to the inclusion of the fluid in the cavity. But the resulting radiation patterns for hard sediments were substantially similar to those of Heelan barring some slight distortion of the radiation pattern. However, one exception was the radiation pattern for shear waves in sediments that had shear wave velocities lower than the tube wave velocity. Shales
and other poorly consolidated sediments, for instance, often meet this criteria. In this case, the tube wave pole would dominate and the radiation pattern could only be calculated in certain ranges of $r$ and $z$, otherwise, singularities would occur in the expressions.

In fact, Mach waves are generated when the shear wave velocity is less than the tube wave velocity. When this occurs, the radiation patterns of Lee and Balch and Heelan for shear waves no longer govern radiation for all azimuths. These radiation patterns were dependent on a uniform geometric decay ($\frac{1}{r}$) which is no longer valid when inside the Mach cone. As will be shown, Mach waves decay as $\frac{1}{\sqrt{r}}$ where $r$ is the radial distance of investigation. Thus shear wave radiation into a low velocity sediment will be dependent on both the coordinates $r$ and $z$ and will exhibit nonuniform geometric decay. Therefore, no single radiation pattern can suffice in describing the wavefield as in the high velocity sediment case.

### 3.2 Calculations for the empty borehole: P-SV

The extensive verification of Heelan's results presented in Appendices B and C are primarily of historical importance. To maintain the continuity of the thesis, far field radiation from an empty borehole will be analyzed in this chapter using the eigenfunctions (Hankel functions) of Lee and Balch (1982). Hence this treatment for the empty borehole can be considered a subset of the calculations presented by Lee and Balch. The algebraic results presented in Chapter 2 for modified Bessel functions are duplicated for Hankel functions in a cursory manner in Appendix F.

Sources are modeled as stress discontinuities applied over a finite length of the borehole wall as presented in Chapter 2. The boundary conditions are vanishing of normal and tangential stress for the $P$-$SV$ problem and azimuthal stress for the $SH$ problem.

The displacement potential relations of Lee and Balch (1982) (also Eq. A.72) for
a borehole surrounded by just an infinite half space is utilized as follows

\[
\phi_f = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} B_f J_0(f r) e^{-ikxz} e^{i\omega t} dk_z \, d\omega
\]  

(3.1)

\[
\phi_i = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} A H_0^{(2)}(l_1 r) e^{-ikxz} e^{i\omega t} dk_z \, d\omega
\]

\[
\psi_i = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} C H_1^{(2)}(m_1 r) e^{-ikxz} e^{i\omega t} dk_z \, d\omega
\]

where \( J_0 \) is the eigenfunction for the fluid potential used to avoid the logarithmic singularity of the Hankel function \( H_0^{(1)} \) at the origin. Because Hankel functions are used as eigenfunctions the definitions of radial wavenumbers have changed from those of Chapter 2 and Appendix A and these new definitions are

\[
f = \sqrt{\frac{\omega^2}{\alpha^2} - k_z^2} \quad l_1 = \sqrt{\frac{\omega^2}{\alpha^2} - k_z^2} \quad m_1 = \sqrt{\frac{\omega^2}{\beta^2} - k_z^2}
\]  

(3.2)

For the empty borehole \( \phi_f \) equals zero by definition.

From Table F.1, the solution for the numerator of the coefficients \( A \) and \( C \) for a radial source of length \( 2l \) applied to the wall of an empty borehole of radius \( a \) is

\[
A = -G(\omega)(-\rho \omega^2 + 2\mu k_z^2) H_1^{(2)}(m_1 a) \frac{2 \sin(lk_z)}{k_z}
\]  

(3.3)

\[
C = G(\omega)2\mu ik_z l_1 H_1^{(2)}(l_1 a) \frac{2 \sin(lk_z)}{k_z}
\]

and the denominator is Eq. F.7

\[
(-\rho \omega^2 + 2\mu k_z^2)^2 H_0^{(2)}(l_1 a) H_1^{(2)}(m_1 a) - \frac{2\mu l_1 \rho \omega^2}{a} H_1^{(2)}(l_1 a) H_1^{(2)}(m_1 a) + 4\mu^2 k_z^2 l_1 m_1 H_0^{(2)}(m_1 a) H_1^{(2)}(l_1 a)
\]  

(3.4)

where the index has been kept on the wavenumber components to avoid confusion with the length \( 2l \) of the source.

### 3.2.1 Expansions for small argument

When evaluating radiation in the far field it is assumed that since the borehole radius, \( a \), is small in comparison to the radial distance of investigation, \( r \), the terms as a
function of \( a \) can be expanded for small argument. With such an assumption terms in \( H_1^{(2)}(z) \) predominate over those of \( H_0^{(2)}(z) \).

The predominance of the \( H_1^{(2)}(z) \) terms over \( H_0^{(2)}(z) \) is analogous to the predominance of \( K_1(z) \) over \( K_0(z) \) (e.g. Abo-Zena, 1977) which can be seen in Fig. A-4. Mathematically this can be seen by recalling that the limiting form for small argument for \( H_0^{(2)}(z) \) and \( H_1^{(2)}(z) \) are (Abramowitz and Stegun, 1964)

\[
H_0^{(2)}(z) = \frac{2}{i\pi} \ln z
\]

\[
H_1^{(2)}(z) = \frac{-2}{i\pi z}
\]

and therefore the ratio of \( \frac{H_0^{(2)}(z)}{H_1^{(2)}(z)} \) approaches zero as \( z \) approaches zero by L’Hôpital’s rule.

Thus terms in \( H_0^{(2)}(z) \) are discarded in Eq. 3.4 leaving for the denominator

\[
-\frac{2\mu l_1 \rho \omega^2}{a} H_1^{(2)}(l_1 a) H_1^{(2)}(m_1 a)
\]

Combining numerator and denominator and applying the transformations for small argument

\[
\sin z \rightarrow z
\]

and Eq. 3.5 produces the following expressions for \( A \) and \( C \)

\[
A = -G(\omega) \frac{i \pi a^2 l (-\rho \omega^2 + 2\mu k_z^2)}{2\mu \rho \omega^2}
\]

\[
C = -G(\omega) \frac{k_z m_1 \pi a^2 l}{\rho \omega^2}
\]

It is seen that \( A \) which governs the dilatational potential is independent of the shear radial wavenumber \( m_1 \) and \( C \) governing the shear potential is independent of the compressional radial wavenumber \( l_1 \). Thus in the far field, it is important to observe that the \( P \) and \( S \) solutions have been decoupled.

96
3.2.2 Expansions for large argument

Recall that the displacement components in the infinite half space surrounding the borehole are written (Eq. 2.6) in their full integral form as

\[
U_r = \frac{1}{(2\pi)^2} \int \int \int_{-\infty}^{\infty} [-l_1 AH_1^{(2)}(l_1 r) + ik_z CH_1^{(2)}(m_1 r)] e^{-ik_z z} e^{i\omega t} dk_z d\omega
\]

\[
U_z = \frac{1}{(2\pi)^2} \int \int \int_{-\infty}^{\infty} [-ik_z AH_0^{(2)}(l_1 r) + m_1 CH_0^{(2)}(m_1 r)] e^{-ik_z z} e^{i\omega t} dk_z d\omega
\]

Substituting in the approximations for \( A \) (Eq. 3.8) and \( C \) (Eq. 3.9) and setting aside the integral with respect to \( \omega \) provides the following integrand over \( k_z \)

\[
U_r = \frac{G(\omega)}{2\pi} \left[ \frac{i l_1 \pi a^2 l (-\rho \omega^2 + 2 \mu k_z^2)}{2 \mu \rho \omega^2} H_1^{(2)}(l_1 r) - \frac{ik_z^2 m_1 \pi a^2 l}{\rho \omega^2} H_1^{(2)}(m_1 r) \right] e^{-ik_z z}
\]

\[
U_z = \frac{G(\omega)}{2\pi} \left[ -\frac{k_z \pi a^2 l (-\rho \omega^2 + 2 \mu k_z^2)}{2 \mu \rho \omega^2} H_0^{(2)}(l_1 r) - \frac{k_z^2 m_1 \pi a^2 l}{\rho \omega^2} H_0^{(2)}(m_1 r) \right] e^{-ik_z z}
\]

The Hankel functions are evaluated using a principal asymptotic expansion for large argument (Abramowitz and Stegun, 1964) which is

\[
H_0^{(2)}(z) = \sqrt{\frac{2}{\pi z}} e^{-i(z - \frac{\pi}{4})}
\]

\[
H_1^{(2)}(z) = \sqrt{\frac{2}{\pi z}} e^{-i(z - \frac{3\pi}{4})}
\]

Substituting in these values yields

\[
U_r = \frac{G(\omega)}{2\pi} \left[ \frac{i l_1 \pi a^2 l (-\rho \omega^2 + 2 \mu k_z^2)}{\mu \rho \omega^2} \sqrt{\frac{\pi}{2l_1 r}} e^{-i l_1 r} - \frac{ik_z^2 m_1 \pi a^2 l}{\rho \omega^2} \sqrt{\frac{\pi}{2m_1 r}} e^{-im_1 r} e^{-ik_z z} e^{i\frac{3\pi}{4}} \right]
\]

\[
U_z = \frac{G(\omega)}{2\pi} \left[ -\frac{k_z \pi a^2 l (-\rho \omega^2 + 2 \mu k_z^2)}{2 \mu \rho \omega^2} \sqrt{\frac{\pi}{2l_1 r}} e^{-i l_1 r} - \frac{k_z^2 m_1 \pi a^2 l}{\rho \omega^2} \sqrt{\frac{\pi}{2m_1 r}} e^{-im_1 r} e^{-ik_z z} e^{i\frac{3\pi}{4}} \right]
\]

(3.13)

3.2.3 Application of the method of stationary phase

The integrals represented by Eq. 3.13 are to be evaluated using the method of stationary phase. To do so, the integrals over \( k_z \) are formally decoupled as follows

\[
U_{r1} = U_{rP} = \frac{G(\omega)}{2\pi} \left[ \frac{i l_1 \pi a^2 l (-\rho \omega^2 + 2 \mu k_z^2)}{\mu \rho \omega^2} \sqrt{\frac{\pi}{2l_1 r}} e^{-i l_1 r} e^{-ik_z z} e^{i\frac{3\pi}{4}} \right]
\]

(3.14)
\begin{align*}
U_{r2} = U_{rSV} &= \frac{G(\omega)}{2\pi} \frac{-ik_z^2 m_1 \pi a^2 l}{\rho \omega^2} \sqrt{\frac{\pi}{2m_1 r}} e^{-im_1 r} e^{-ik_z z} e^{i\frac{\pi}{4}} \\
U_{z1} = U_{zP} &= \frac{G(\omega)}{2\pi} \frac{-k_z \pi a^2 l (-\rho \omega^2 + 2 \mu k_z^2)}{\mu \rho \omega^2} \sqrt{\frac{\pi}{2l_1 r}} e^{-il_1 r} e^{-ik_z z} e^{i\frac{\pi}{4}} \\
U_{z2} = U_{zSV} &= \frac{G(\omega)}{2\pi} \frac{-k_z m_1 \pi a^2 l}{\rho \omega^2} \sqrt{\frac{\pi}{2m_1 r}} e^{-im_1 r} e^{-ik_z z} e^{i\frac{\pi}{4}}
\end{align*}

In the far field the distance \( R = \sqrt{r^2 + z^2} \) will approach infinity as seen in Fig. 3-1.

The predominance of a parameter in an exponential as that parameter approaches infinity is the key to the method of stationary phase or sometimes referred to as Kelvin’s Formula (e.g. Ben-Menahem and Singh, 1981). The highlights of the method will be briefly stated with further details available in many excellent textbooks (e.g. Mathews and Walker, 1970; Bleistein and Handelsman, 1976, Bender and Orszag, 1978). The integrals in Eq. 3.14 are members of the class of integrals given by the general form

\[
\int_C e^{i\rho f(\zeta)} F(\zeta) d\zeta \tag{3.15}
\]

where \( \rho \) is the parameter at large values and in this particular case \( R \). Since integration is fundamentally the computation of an area under a curve, the rapid oscillations produced by the large parameter times the imaginary exponential will tend to cancel each other. This is true except for a limited region surrounding the point where \( f'(\zeta) \) is zero at a value \( \zeta = \zeta_0 \) – the saddle point. At the saddle point the oscillations are much less frequent and the integral can be approximated by a short integration path near the saddle point. The method of stationary phase allows the approximation of the integral (Eq. 3.15) by the function

\[
\sqrt{\frac{2\pi}{\rho |f''(\zeta_0)|}} e^{i\frac{\pi}{4}} e^{i\rho f(\zeta_0)} F(\zeta_0) \tag{3.16}
\]

where the sign in front of the first exponential is the sign of the second derivative \( f''(\zeta_0) \) and the absolute value sign under the square root signifies the magnitude of the second derivative.

For the problem presented in this chapter the function \( f \) is given by

\[
i\rho f(k_z) = iR(k_z \cos \phi - l_1 \sin \phi) = -ik_z z - il_1 r \tag{3.17}
\]

98
or equivalently

\[ f(k_z) = k_z \cos \phi - \sqrt{\frac{\omega^2}{\alpha^2} - k_z^2 \sin \phi} \]  \hspace{1cm} (3.18)

with \( z = -R \cos \phi \) and \( r = R \sin \phi \) (Fig. 3-1).

The derivative function is given by

\[ f'(k_z) = \cos \phi + \frac{k_z \sin \phi}{\sqrt{\frac{\omega^2}{\alpha^2} - k_z^2}} \]  \hspace{1cm} (3.19)

which equals zero at the saddle point \( k_{z0} \) or

\[ f'(k_{z0}) = 0 \quad k_{z0} = \frac{-\omega \cos \phi}{\alpha} \]  \hspace{1cm} (3.20)

and the second derivative \( f''(k_{z0}) \) is

\[ f''(k_{z0}) = \frac{\alpha}{\omega \sin^2 \phi} \]  \hspace{1cm} (3.21)

which is positive so the phase term is \( e^{i\pi} \).

The saddle point is at \( k_z = k_{z0} \) for each of the integrals presented in Eq. 3.14. The following substitutions are made in the integrals.

\[ l_1 \rightarrow \frac{\omega r}{\alpha R} = \frac{\omega}{\alpha} \sin \phi \]  \hspace{1cm} (3.22)

\[ k_z \rightarrow \frac{\omega z}{\alpha R} = \frac{-\omega}{\alpha} \cos \phi \]

When integrals involving \( e^{i(-m_1 r - k_z z)} \) are calculated the same transformations apply but with \( \beta \) substituted for \( \alpha \) in Eq. 3.23, Eq. 3.20, and Eq. 3.21.

One should take note that for \( k_{z0}, \frac{\omega z}{\alpha R} < \frac{\omega}{\beta}, \) and \( \frac{\omega z}{\alpha R} < \frac{\omega}{\alpha} \) and thus \( l_1 \) and \( m_1 \) will be greater than zero and real (Eq. 3.2) at the stationary point. This however precludes consideration of cases where \( r = 0 \) which is the borehole. The relationship of the saddle point to the singularities for the empty borehole cases can be seen in the top half of Fig. 3-2 and Fig. 3-3 and implies that no branch cuts will be crossed when constructing the path of steepest descent. Additionally, since there are no poles for the empty borehole, there are no poles to cross. Therefore, the method of stationary phase approximation can be applied directly.
Applying the method of stationary phase to Eq. 3.14 generates the following expressions.

\[
U_{r1} = G(\omega)\left[\frac{-i\omega \sin \phi a^2 l(-1 + \frac{2\beta^2 \cos^2 \phi}{a^2})}{2\mu \alpha} e^{-i\frac{\omega}{\beta} R}\right] \frac{e^{-i\frac{\omega}{\alpha} R}}{R} \quad (3.23)
\]

\[
U_{r2} = G(\omega)\left[\frac{i\omega \sin 2\phi \cos \phi a^2 l}{2\mu \beta} e^{-i\frac{\omega}{\beta} R}\right] \frac{e^{-i\frac{\omega}{\alpha} R}}{R}
\]

\[
U_{z1} = G(\omega)\left[\frac{i\omega \cos \phi a^2 l(-1 + \frac{2\beta^2 \cos^2 \phi}{a^2})}{2\mu \alpha} e^{-i\frac{\omega}{\alpha} R}\right] \frac{e^{-i\frac{\omega}{\beta} R}}{R}
\]

\[
U_{z2} = G(\omega)\left[\frac{i\omega \sin 2\phi \sin \phi a^2 l}{2\mu \beta} e^{-i\frac{\omega}{\alpha} R}\right] \frac{e^{-i\frac{\omega}{\beta} R}}{R}
\]

The integrals over \(\omega\) are now reintroduced and the following operators are understood

\[
G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)e^{i\omega t} d\omega \quad (3.24)
\]

\[
G(t - \frac{R}{v}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)e^{i\omega t} e^{-i\frac{\omega}{v} R} d\omega
\]

\[
G'(t - \frac{R}{v}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} i\omega G(\omega)e^{i\omega t} e^{-i\frac{\omega}{v} R} d\omega
\]

where \(v\) is either \(\alpha\) or \(\beta\). Factoring in Eq. 3.24 into Eq. 3.23 and rearranging yields

\[
U_{r1} = G'(t - \frac{R}{\alpha})\left[\frac{\sin \phi a^2 l(1 - \frac{2\beta^2 \cos^2 \phi}{a^2})}{2\mu \alpha R}\right] \quad (3.25)
\]

\[
U_{r2} = G'(t - \frac{R}{\beta})\left[\frac{\sin 2\phi \cos \phi a^2 l}{2\mu \beta R}\right]
\]

\[
U_{z1} = -G'(t - \frac{R}{\alpha})\left[\frac{a^2 l \cos \phi(1 - \frac{2\beta^2 \cos^2 \phi}{a^2})}{2\mu \alpha R}\right]
\]

\[
U_{z2} = G'(t - \frac{R}{\beta})\left[\frac{\sin 2\phi \sin \phi a^2 l}{2\mu \beta R}\right]
\]

Heelan presented the radiation pattern results in terms of spherical coordinates \(R\) and \(\phi\) which will be done here. Heelan also broke down the radiation pattern formulas into one governing \(P\) waves, \(F1\) and one governing \(SV\) waves, \(F2\). Thus the following redefinition of Eq. 3.25 is required

\[
U_{r1} = U_{rP} = F_1(\phi) \sin \phi \quad (3.26)
\]
\[ U_{s1} = U_{zP} = -F_1(\phi) \cos \phi \]
\[ U_{s2} = U_{zSV} = F_2(\phi) \cos \phi \]
\[ U_{z2} = U_{zSV} = F_2(\phi) \sin \phi \]

where
\[ F_1(\phi) = G'(t - \frac{R}{\alpha}) \left[ \frac{a^2 l (1 - \frac{2\beta \cos^2 \phi}{\sigma^2})}{2\mu \alpha R} \right] \]
\[ F_2(\phi) = G'(t - \frac{R}{\beta}) \left[ \frac{\sin 2\phi a^2 l}{2\mu \beta R} \right] \quad (3.27) \]

The same analysis was performed for the axial source of finite length and the results are presented along with these radial source results in Table 3.1. One note is that in Table 3.1, the symbol \( \Delta \) represents \( 2\pi a^2 l \), the volume of the cylinder of length \( 2l \) over which the source is applied, and \( A \) represents the surface area of the walls of the cylinder \( 4\pi al \) over which the source is applied to abide by Heelan’s presentation. The same geometric relationship governs the radial and vertical components and the spherical components for the axial source and the radial source (Eq. 3.26).

It can be seen from Table 3.1 and Appendix B and Appendix C that these results exactly agree with Heelan’s and Brekhovskikh’s although the notation is slightly different (Heelan, 1952, 1953a,b; Brekhovskikh, 1960, 1980).

### 3.3 Calculations for the fluid-filled borehole: P-SV

The method of stationary phase has been used to calculate far field radiation pattern formulas from an empty borehole in the last section. The procedure is identical for a fluid-filled borehole except that the algebra and the physics is more complex.

The numerators of the Cramer’s rule solution for an axial source of length \( 2l \) in a fluid-filled borehole are from Table F.2
\[ A = G(\omega) \frac{2\sin(\lambda z)}{k_z} (2f \mu k_z [m_1 H_0^{(2)}(m_1 a) - \frac{H_1^{(2)}(m_1 a)}{a}]) J_1(fa) \quad (3.28) \]
\[ C = -G(\omega) \frac{2 \sin(\ell x)}{k_z} \left( f\left( -\rho \omega^2 + 2 \mu k_z^2 \right) H_0^{(2)}(l_1 a) + \frac{2 \mu l_1}{a} H_1^{(2)}(l_1 a) \right) J_1(f a) \] (3.29)

and denominator Eq. F.8

\[ f\left( -\rho \omega^2 + 2 \mu k_z^2 \right)^2 H_0^{(2)}(l_1 a) H_1^{(2)}(m_1 a) - \frac{2 \mu l_1 \rho \omega^2}{a} H_1^{(2)}(l_1 a) H_1^{(2)}(m_1 a) \] (3.30)

\[ + 4 \mu^2 k_z^2 l_1 m_1 H_1^{(2)}(l_1 a) H_0^{(2)}(m_1 a) J_1(f a) + \rho \mu \omega^2 l_1 H_1^{(2)}(l_1 a) H_1^{(2)}(m_1 a) J_0(f a) \]

Discarding terms in \( H_0^{(2)} \) and applying the expansions for small argument yields for the denominator

\[ \left[ f^2 + \frac{\rho \mu \omega^2}{\mu} \right] \frac{4 \mu \rho \omega^2}{\pi^2 m_1 a^2} \] (3.31)

where additional small argument expansions besides those presented in Eq. 3.5 have been utilized as follows

\[ J_0(z) \rightarrow 1 \] (3.32)

\[ J_1(z) \rightarrow \frac{z}{2} \]

It is immediately recognized that there is a pole in Eq. 3.31 which is a solution of the equation

\[ f^2 + \frac{\rho \mu \omega^2}{\mu} = 0 \] (3.33)

Remembering that \( f = \sqrt{\frac{\rho_f}{\mu} - k_z^2} \) this can be factored into

\[ \left( k_z + \omega \sqrt{\frac{\rho_f}{\mu} + \frac{1}{\alpha_f^2}} \right) \left( -k_z + \omega \sqrt{\frac{\rho_f}{\mu} + \frac{1}{\alpha_f^2}} \right) \] (3.34)

and the two solutions are

\[ k_z = \pm \omega \sqrt{\frac{1}{\alpha_f^2} + \frac{\rho_f}{\mu}} \] (3.35)

Each of these two solutions is a first order pole. As pointed out by Lee and Balch (1982), the denominator of Eq. 3.35 is the zero frequency tube wave equation of Biot
\[ \frac{1}{C_T} = \sqrt{\frac{1}{\alpha_f^2} + \frac{\rho_f}{\mu}} \]  

(3.36)

From Eq. 3.36 it is apparent that \( C_T < \alpha_f \) and that \( C_T \) does not change with frequency. Therefore, there is a linear relationship between \( \omega \) and the pole location.

Applying the small argument simplifications to the numerators of \( A \) and \( C \) for an axial source and dividing through by the denominator yields

\[
A = G(\omega) \frac{2\alpha l k_x (f^2 + \frac{\rho_f \omega^2}{\mu})}{\rho \omega^2 (f^2 + \frac{\rho_f \omega^2}{\mu})} \\
C = G(\omega) \frac{2\alpha m_1 (f^2 + \frac{\rho_f \omega^2}{\mu})}{\rho \omega^2 (f^2 + \frac{\rho_f \omega^2}{\mu})}
\]

(3.37)

\( A \) and \( C \) for the axial source have the tube wave pole in both the numerator and denominator so cancellation occurs. The absence of the tube wave pole for axial sources was also noticed by Winbow (1989). Thus \( A \) and \( C \) can be rewritten as

\[
A = G(\omega) \frac{2\alpha l k_x}{\rho \omega^2} \\
C = G(\omega) \frac{2\alpha m_1}{\rho \omega^2}
\]

(3.38)

which are equivalent to the expressions for an empty borehole and thus the integrals equal those for an empty borehole (Table 3.1). This is consistent with the observations made in Chapter 2 that the fluid potential from an axial source would be vanishingly small and that the presence of a fluid in the borehole had little effect on the radiation emanating from axial sources.

However, for radial and volume point sources, the tube wave singularity does not vanish. For a radial source after simplification the coefficient functions \( A \) and \( C \) are

\[
A = G(\omega) \frac{-ia^2 lf^2 (-\rho \omega^2 + 2\mu k_z^2)}{2 \rho \omega^2 (-k_z + \frac{\omega}{C_T})(k_z + \frac{\omega}{C_T})} \\
C = G(\omega) \frac{a^2 l k_z f^2 m_1}{\rho \omega^2 (-k_z + \frac{\omega}{C_T})(k_z + \frac{\omega}{C_T})}
\]

(3.39)

103
First consider integrals involving the coefficient functions \( A \) for the \( P \) waves. Such integrals will have saddle points at \( k_z = \frac{w \cos \phi}{\alpha} \). Since \( \cos \phi < 1 \) and \( \frac{w \cos \phi}{\alpha} < \frac{w}{C_T} \), the tube wave pole will never be crossed. There is no branch cut associated with \( f \), the fluid wavenumber by convention (Paillet and Cheng, 1986). The convention is easily understood here by virtue of the fact that since \( f \) is only present in the form \( f^2 \) it is single valued. A branch cut associated with \( l_1 \) is not crossed because the branch point \( \frac{w}{\alpha} > \frac{w \cos \phi}{\alpha} \). The steepest descent path and the branch cut for \( P \) waves are also presented in Fig. 3-2 and since no singularities are crossed the integral can be evaluated with routine application of the method of stationary phase as in the last section. The resulting radiation pattern formulas are presented in Table 3.1.

For \( S \) wave radiation from an empty borehole (Fig. 3-3) the integral procedure is fundamentally similar to that presented in Fig. 3-2. Again there are no branch cuts nor any poles crossed in the method of stationary phase analysis of the integral.

Conversely, the integrals involving \( S \) wave radiation from a fluid-filled borehole are fundamentally different as can be seen in Fig. 3-4. First, the saddle point will be at \( \frac{w \cos \phi}{\beta} \) and if \( \beta > C_T \), then \( \frac{w \cos \phi}{\beta} < \frac{w}{C_T} \) and the tube wave pole will never be crossed. Again the integrals are evaluated as in Fig. 3-2 using the method of stationary phase and the results are presented in Table 3.1. However, if \( \beta < C_T \), which is often the case with soft, low velocity sediments and especially the case for cased boreholes surrounded by low-velocity sediments, the tube wave pole will be crossed for some co-latitude \( \phi \). In fact, \( \frac{w \cos \phi}{\beta} \) will be less than \( \frac{w}{C_T} \) for some values of \( \phi \) and greater than \( \frac{w}{C_T} \) for others. The value of \( \phi \) where \( \frac{\omega \cos \phi}{\beta} = \frac{\omega}{C_T} \) is the complementary Mach angle \( \phi_c \) explained in the last chapter. Thus the values of \( z \) and \( R \) are critical to the discussion that follows.

The evaluation of an integral where the stationary point coincides with pole or is in the neighborhood of a pole is a difficult task. The fundamental assumption of the method of stationary phase is that the integrand be slowly varying in the neighborhood of the stationary point - an assumption clearly violated with the presence of
the pole. The evaluation of integrals of large parameter with singularities near stationary points is almost a branch of applied mathematics in and of itself. For branch cuts near stationary points, important work has been accomplished by Brekhovskikh (1960, 1980), Bleistein (1966) and others. For the case under consideration here, a pole near the stationary point, a solution has been derived by Felsen and Marcuvitz (1959, 1973), Bleistein (1966) and Bleistein and Handelsman (1976). The steps needed to evaluate radiation integrals by this method is presented in Appendix G and there it is seen that the end result is a calculation of the stationary phase solution, the residue and a function including a complex complementary error function. Although the complex complementary error function can be evaluated via standard packages (i.e. IMSL, Gautschi (1970)), these complicated integrals are no less numerical than the numerical solution presented in the earlier chapter. Moreover, this method of solution fails to yield an easily interpretable description of radiation pattern which is the desired result and is only valid for an uncased borehole surrounded by an infinite half space whereas most boreholes surrounded by low velocity sediments would be cased. Therefore, although for some instances the full solution presented in Appendix G might be valuable, the recommendation for the quantitative prediction of shear wave radiation into low-velocity sediments is to use the numerical algorithm developed in the last chapter. However, this does not preclude the extraction of important qualitative information from a simplification of the integrals by isolating the singularities.

The first strategy is to treat the tube wave pole in the manner that the Rayleigh pole is treated in seismology and evaluate it as a residue contribution. In so doing, the basic geometric decay and phase governing the Mach waves will be produced.
3.3.1 Calculation of residues: Mach waves

Recall the residue theorem (e.g. Mathews and Walker, 1970) states that an integral around a pole(s) can be evaluated as the sum of the residues times $2\pi i$ or symbolically

$$\int_C F(z)dz = 2\pi i \sum \text{residues} \quad (3.40)$$

For the single pole in this case at $k_z = \frac{\omega}{C_T}$, the evaluation of the integral around the tube wave pole by the residue theorem yields for a volume point source

$$U_{SV} = -\int_{-\infty}^{\infty} \frac{G(\omega)V_0\rho_f}{4\rho C_T} \left( \frac{1}{\beta^2} - \frac{1}{C_T^2} \right) \frac{1}{\beta} \sqrt{\frac{2}{\pi r}} e^{i\omega(t-\frac{x}{c_T} - \frac{\sqrt{M^2-1}r}{c_T})} e^{i\frac{\pi}{6}} d\omega \quad (3.41)$$

$$U_{zSV} = \int_{-\infty}^{\infty} \frac{G(\omega)V_0\rho_f}{4\rho} \left( \frac{1}{\beta^2} - \frac{1}{C_T^2} \right) \frac{1}{\beta} \sqrt{\frac{2}{\pi r}} e^{i\omega(t-\frac{x}{c_T} - \frac{\sqrt{M^2-1}r}{c_T})} e^{i\frac{\pi}{6}} d\omega$$

and for a radial source

$$U_{SV} = -\int_{-\infty}^{\infty} \frac{G(\omega)\Delta i\omega}{4\rho C_T} \left[ \frac{1}{\alpha_f^2} - \frac{1}{C_T^2} \right] \frac{1}{\beta^2} \sqrt{\frac{2}{\pi r}} e^{i\omega(t-\frac{x}{c_T} - \frac{\sqrt{M^2-1}r}{c_T})} e^{i\frac{\pi}{6}} d\omega \quad (3.42)$$

$$U_{zSV} = \int_{-\infty}^{\infty} \frac{G(\omega)\Delta i\omega}{4\rho} \left[ \frac{1}{\alpha_f^2} - \frac{1}{C_T^2} \right] \frac{1}{\beta^2} \sqrt{\frac{2}{\pi r}} e^{i\omega(t-\frac{x}{c_T} - \frac{\sqrt{M^2-1}r}{c_T})} e^{i\frac{\pi}{6}} d\omega$$

where $M$ is the Mach number, $M = \frac{C_f}{\beta}$, and $\Delta$ is the volume of the cylindrical cavity of length $2l$ over which the radial source is applied.

It can be seen in Eq. 3.41 and Eq. 3.42 that the expressions for radial and volume point sources are nearly identical. Recognizing that $G(\omega)$ is a stress wavelet for a radial source and $G(\omega)$ is a unitless wavelet for a volume point source, the expressions are identical except for the factor $\Delta \left( \frac{1}{C_T^2} - \frac{1}{\alpha_f^2} \right)$ in the radial source expression and the factor $V_0\rho_f$ in the point source. Thus these expressions governing Mach wave propagation for a radial and volume point source are essentially identical.

From Eq. 3.41 and Eq. 3.42 it can be seen that the phase delays necessary for Mach wave generation

$$e^{i\omega(t-\frac{x}{c_T} - \frac{\sqrt{M^2-1}r}{c_T})} \quad (3.43)$$

are present which is exactly equivalent to the delay found by geometric arguments (Eq. 2.54). Therefore, as demonstrated in Chapter 2, the traveltime will consist of
the traveltime up the borehole as a tube wave and the traveltime as a Mach wave. Additionally, from Eq. 3.41 and Eq. 3.42, it can be seen that the geometric decay is $\frac{1}{\sqrt{r}}$. de Bruin and Huizer (1989) had predicted a decay of conical wavefront of $\frac{1}{r}$ so this is a new result. The implication of this geometric decay is that the Mach waves decay like head waves and travel in the $r$ direction and that the Mach waves do not decay geometrically in the $z$ direction. This results in the large relative amplitude of Mach waves in comparison to $P$ waves for large values of $z$ as seen in data and will be shown in Chapter 4. The tube wave will exhibit attenuation in a real borehole of course but to first order, the geometric decay is independent of $z$.

3.4 Table of results for P-SV case

A table for far field radiation pattern expressions is presented for reference purposes below
<table>
<thead>
<tr>
<th>Source Type</th>
<th>Radiation Pattern Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Empty Borehole</td>
<td></td>
</tr>
<tr>
<td>Radial Source (2l)</td>
<td>( F_1(\phi) = G'(t - \frac{R}{\alpha})\left[\Delta(1 - \frac{2\beta^2 \cos^2 \phi}{\alpha^2})\right] )</td>
</tr>
<tr>
<td></td>
<td>( F_2(\phi) = G'(t - \frac{R}{\beta})\left[\frac{\Delta \sin 2\phi}{4\pi \mu \beta R}\right] )</td>
</tr>
<tr>
<td>Axial Source (2l)</td>
<td>( G_1(\phi) = -G(t - \frac{R}{\alpha})\left[\frac{A\beta^2 \cos \phi}{4\pi \mu \alpha^2 R}\right] )</td>
</tr>
<tr>
<td></td>
<td>( G_2(\phi) = G(t - \frac{R}{\beta})\left[\frac{A \sin \phi}{4\pi \mu R}\right] ) )</td>
</tr>
<tr>
<td>Fluid Filled</td>
<td></td>
</tr>
<tr>
<td>Radial Source (2l)</td>
<td>( F_{1f}(\phi) = G'(t - \frac{R}{\alpha})\left[\frac{\left(\frac{1}{\alpha^2} - \frac{\cos^2 \phi}{\alpha^2}\right)}{(\frac{\mu L}{\mu} + \frac{1}{\alpha^2} - \frac{\cos^2 \phi}{\alpha^2})} \Delta(1 - \frac{2\beta^2 \cos^2 \phi}{\alpha^2})\right] )</td>
</tr>
<tr>
<td></td>
<td>( F_{2f}(\beta &gt; C_T) = G'(t - \frac{R}{\beta})\left[\frac{\left(\frac{1}{\alpha^2} - \frac{\cos^2 \phi}{\beta^2}\right)}{(\frac{\mu L}{\mu} + \frac{1}{\alpha^2} - \frac{\cos^2 \phi}{\beta^2})} \Delta \sin 2\phi\right] )</td>
</tr>
<tr>
<td></td>
<td>( F_{2f}(\beta &lt; C_T)^* = \text{Numeric calculation required}^{**} )</td>
</tr>
<tr>
<td>Axial Source (2l)</td>
<td>Same as Empty Borehole</td>
</tr>
<tr>
<td>Point Source</td>
<td>( V_1(\phi) = G'(t - \frac{R}{\alpha})\left[\frac{1}{(\frac{\mu L}{\mu} + \frac{1}{\alpha^2} - \frac{\cos^2 \phi}{\alpha^2})} V_0\rho_f(1 - \frac{2\beta^2 \cos^2 \phi}{\alpha^2})\right] )</td>
</tr>
<tr>
<td></td>
<td>( V_2(\beta &gt; C_T) = G'(t - \frac{R}{\beta})\left[\frac{1}{(\frac{\mu L}{\mu} + \frac{1}{\alpha^2} - \frac{\cos^2 \phi}{\beta^2})} V_0\rho_f \sin 2\phi\right] )</td>
</tr>
<tr>
<td></td>
<td>( V_2(\beta &lt; C_T)^* = \text{Numeric calculation required}^{**} )</td>
</tr>
</tbody>
</table>

Table 3.1: Radiation Pattern Formulas for P-SV. *Inside the Mach cone. **radiation pattern prediction will be valid for large co-latitudes, i.e. near the horizontal axis (z=0). Further simplification of some of the denominators could be achieved by recognizing \( \frac{1}{\alpha^2} = \frac{\mu L}{\mu} + \frac{1}{\alpha^2} \).
3.5 Calculations for fluid-filled and empty boreholes: SH

Radiation from torsional sources is independent of the presence of the fluid in the borehole. The following integral describes radiation from a borehole, either empty or fluid-filled, into an infinite half-space from a torsional source of length $2l$

$$U_\theta = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{1}{\mu(mR)^2} \frac{-G(\omega)H_1^{(2)}(mr)}{2H_1^{(2)}(ma) - \frac{2}{a}} 2\sin\frac{lk_z e^{-ik_z z} e^{i\omega t}}{k_z} dk_z d\omega \tag{3.44}$$

which can be simplified by expanding in terms of $a$, the borehole radius, and $2l$. Further simplification through applying the expansion for large argument of the Hankel function yields

$$-\frac{G(\omega)ima^2l}{4\mu} \sqrt{\frac{2}{\pi mr}} e^{-im\tau} e^{-ik_z z} e^{i\omega t} \tag{3.45}$$

Applying the stationary phase approximation produces for the far field representation of azimuthal displacement (also Appendix C; Heelan, 1953a; Brekhovskikh, 1960, 1980)

$$U_\theta = G'(t - \frac{R}{\beta}) \frac{1}{R} \frac{1}{2\mu\beta} \sin \frac{\phi a^2l}{R} = G'(t - \frac{R}{\beta}) \frac{1}{4\pi\mu\beta} \frac{1}{R} \frac{1}{2\mu\beta} \sin \phi \Delta \tag{3.46}$$

3.6 Comparison of radiation pattern formulas

Radiation pattern formulas are presented graphically in this section and a comparison is made between the well known formulas for point forces and force couples in infinite media and the radiation patterns in this section so that the effects of the borehole can be isolated.

To provide a range of parameters from which to test, the radiation pattern formulas are evaluated using density and velocity parameters from four different lithology types as presented in Table 2.3.
3.6.1 Radiation from sources in infinite media and radiation from torsional and axial sources in boreholes

In this section, Heelan's results for the empty borehole displayed in Table 3.1 are compared to established results for radiation from axial and torsional sources embedded in infinite media. It will be shown that the radiation pattern formulas for torsional and axial sources are functionally identical to the expressions in infinite media.

Attenuation is not considered in this analysis because it would be expected to exhibit a uniform effect with azimuth and thus would not alter the results.

Axial sources

White (Eq. 6-5, 1983) gives the following formula for radiation from a vertically directed point force in an infinite elastic medium

\[ U_R = \frac{\cos \phi \beta^2}{4\pi \mu \alpha^2 R} g(t - \frac{R}{\alpha}) \]  

\[ U_\phi = -\frac{\sin \phi}{4\pi \mu R} g(t - \frac{R}{\beta}) \]  

which is a well known result. \( g(t - \frac{R}{\alpha}) \) is the wavelet describing the force applied as a function of time. By comparing Eq. 3.48 with the results for an axial source \((G_1, G_2)\) from Table 3.1 reproduced below

\[ U_R = G_1(\phi) = -\frac{A \cos \phi \beta^2}{4\pi \mu \alpha^2 R} G(t - \frac{R}{\alpha}) \]  

\[ U_\phi = G_2(\phi) = \frac{A \sin \phi}{4\pi \mu R} G(t - \frac{R}{\beta}) \]

it is immediately seen that the results are almost identical barring a sign change and realizing that \( G(t) \) is stress applied as a function of time. A sign change between the two sets of formulas is due to Heelan's definition of co-latitude \((\phi)\) (e.g. Fig. 3-5). In fact Eq. 3.48 and Eq. 3.49 are identical if

\[ g(t - \frac{R}{\alpha}) = AG(t - \frac{R}{\alpha}) \]  

110
The far field behavior for an axial source or a point source in an infinite medium is graphically displayed in Fig. 3-6. It can be seen by mentally rotating the patterns displayed in Fig. 3-6 about the vertical axis that the S wave radiation is torus shaped and the P wave radiation consists of two spheres. In Fig. 3-6 and subsequent figures the radiation patterns are displayed in relative scale so that the effect of Poisson ratio can be seen. In terms of absolute amplitudes, the radiation pattern predictions would indicate a greater amplitude for the lower velocity materials because of inverse velocity squared and inverse velocity cubed dependences. From the formula presented in Table 3.1 and Fig. 3-6 the relative P to S excitation is controlled by the factor \( \frac{\beta^2}{\alpha^2} \). Looking at the relative amplitudes of the radiation patterns in Fig. 3-6 it can be seen that the low Poisson ratio of the Berea Sandstone produces a relatively larger P wave amplitude compared to the S wave.

The equivalence of the axial source with the point source representation (Fig. 3-6, also White, 1965) and the fact explained in Chapter 2 that axial sources produce a vanishingly small tube wave leads one to propose that radiation from axial sources can be modelled with infinite media representations to a high degree of accuracy. In other words, it is not required to consider the effect of the borehole.

**Torsional sources**

Torsional sources also exhibit the same equivalence with infinite media representations. The azimuthal displacement for an \( SH \) source, a torque-inducing couple, in an infinite medium (White, Eq. 5.10, 1965) is

\[
U_\theta = -g'(t - \frac{R}{\beta}) \left[ \frac{1}{R} \frac{2h \sin \phi}{4\pi \mu \beta} \right]
\]

(3.50)

where \( 2h \) is the separation between point forces and \( g' \) is the time derivative of the force applied as a function of time instead of stress as with \( G' \). Eq. 3.46 from the \( SH \) analysis section is reproduced below

\[
U_\theta = G'(t - \frac{R}{\beta}) \left[ \frac{1}{R} \frac{\sin \phi \Delta}{4\pi \mu \beta} \right]
\]

(3.51)

111
and is identical to Eq. 3.50 if

\[ 2\eta g'(t - \frac{R}{\beta}) = G'(t - \frac{R}{\beta})\Delta \]  

(3.52)

The torsional radiation pattern is displayed in Fig. 3-7 and is torus-shaped in three dimensions. Also plotted on Fig. 3-7 is White's (1965) equivalent representation for the source in an infinite medium. Comparing Fig. 3-7 and Fig. 3-6 might lead one to conclude that the S wave radiation from the axial and torsional sources is identical, however this is false. The radial shear wave component for the axial source radiation pattern for S waves is multiplied by the cosine of the co-latitude angle \( \phi \) and along the radial axis, the cosine of \( \phi \) equals zero. Therefore, the shear wave is vertically polarized along the radial axis. Conversely, none of the torsional wave radiation is along the vertical. Thus, the shear wave radiation from torsional and axial sources is complementary but not identical.

Finally, due to the equivalence of the borehole and infinite media representations, torsional sources like axial sources can be well represented by infinite media formulas. However, the same equivalence is not seen for radial and volume point sources but there are still some important parallels.

### 3.6.2 Radiation from a radial source in an empty borehole

For a radial source of finite length, \( F_1 \) and \( F_2 \) are from Table 3.1

\[ F_1(\phi) = G'(t - \frac{R}{\alpha}) \frac{\Delta}{4\pi \mu c}(1 - \frac{2\beta^2 \cos^2 \phi}{\alpha^2}) \]

(3.53)

\[ F_2(\phi) = G'(t - \frac{R}{\beta}) \frac{\Delta}{4\pi \mu \beta} \sin 2\phi \]

(3.54)

Examining Fig. 3-8, it can be seen that \( F_1 \) is characterized in three dimensions as being tomato-shaped and in two dimensions by a peanut shape - the dimpling of the peanut shape along the borehole axis. This dimpling is due to reduced amplitude along this axis. \( F_2 \) is governed by the classic four-leaved rose radiation pattern.
The shear wave radiation pattern is equivalent to an infinite medium formulation for double force without moment (White, 1983). A double force without moment consists of two point forces separated by a distance 2h which act in opposing vertical directions and is reproduced in Fig. 3-8. The radiated displacement in shear is (Eq. 6.6, White, 1983)

\[ U_\phi = -\frac{h \sin 2\phi}{4\pi \mu \beta R} g'(t - \frac{R}{\beta}) \]  (3.55)

where \( g \) is a force wavelet and is seen to be identical to Eq. 3.54 if \(-hg = \Delta G\). The implication is that the shear wave radiation from a radial source is analogous to stretching the borehole along the borehole axis.

The P wave radiation pattern is similar to the double force without moment result as follows. The double force without moment P wave radiation is given by

\[ U_R = \frac{2h \cos^2 \phi}{4\pi \rho \alpha^2 \beta} g'(t - \frac{R}{\alpha}) \]  (3.56)

For comparison, if Eq. 3.53 is rewritten in the following manner

\[ F_1(\phi) = G'(t - \frac{R}{\alpha}) \Delta \frac{1}{4\pi \rho \beta^2 \alpha R} - G'(t - \frac{R}{\alpha}) \frac{2\Delta \cos^2 \phi}{4\pi \rho \alpha^3 R} \]  (3.57)

The latter half of Eq. 3.57 is seen to be equivalent to Eq. 3.56 if \( \Delta G = -gh \) just as before. The first half of Eq. 3.57 is just a spherical function \( \frac{1}{R} \) times a constant. Therefore, the radial source radiation pattern consists of a sphere perturbed by a double force without moment, a stretching along the axis of the borehole. For col-latitudes approaching 90 degrees, the horizontal axis, the cosine approaches zero so the latter half of Eq. 3.57 in turn approaches zero making the motion predominantly spherical.

For lower velocity sediments, Poisson’s ratio increases and \( \beta \) diminishes faster than \( \alpha \) causing the spherical portion to become more predominant. This effect is observed in the P wave radiation patterns presented in Fig. 3-8 for Pierre Shale which has the most prominently spherical radiation pattern.
3.6.3 Radiation from a radial source in a fluid-filled borehole

Radiation from a radial source in a fluid-filled borehole is fundamentally similar to that for an empty borehole, especially when high velocity sediments surround the borehole (Lee and Balch, 1982). The mathematical differences from Table 3.1 between the radiation pattern in an empty borehole versus a fluid-filled borehole for the $P$ wave is

$$\frac{\left(\frac{1}{\alpha_f^2} - \frac{\cos^2 \phi}{\alpha^2}\right)}{\frac{\rho L}{\mu} + \frac{1}{\alpha_f^2} - \frac{\cos^2 \phi}{\alpha^2}} = \frac{\left(\frac{1}{\alpha_f^2} - \frac{\cos^2 \phi}{\alpha^2}\right)}{\frac{1}{\alpha_f^2} - \frac{\cos^2 \phi}{\alpha^2}}$$

(3.58)

and for the $S$ wave is

$$\frac{\left(\frac{1}{\alpha_f^2} - \frac{\cos^2 \phi}{\beta^2}\right)}{\frac{\rho L}{\mu} + \frac{1}{\alpha_f^2} - \frac{\cos^2 \phi}{\alpha^2}} = \frac{\left(\frac{1}{\alpha_f^2} - \frac{\cos^2 \phi}{\beta^2}\right)}{\frac{1}{\alpha_f^2} - \frac{\cos^2 \phi}{\alpha^2}}$$

(3.59)

The interesting new feature of this radiation is that Eq. 3.59 will equal zero when $\cos \phi = \frac{\beta}{\alpha_f}$ which led Lee and Balch to predict the existence of lobes in the shear wave radiation pattern (See Fig. 1-2). The existence of these nodes is confirmed numerically later. However, the upper lobe is diminished in amplitude in the numerical solution because for small co-latitudes the path of steepest descent approaches the tube wave pole which degrades the analytic approximation. The prediction for shear wave lobes when $\beta < C_T$ is still valid but they will be dominated by the Mach wave.

Radiation patterns from radial sources in fluid-filled boreholes is shown in Fig. 3-9. In comparing Fig. 3-8 to Fig. 3-9 it is seen that the radiation patterns are identical for the Berea Sandstone and Solenhofen Limestone. However, the Lee-Balch sediment radiation pattern exhibits the shear wave lobes and the Pierre Shale radiation pattern is not valid inside the Mach cone. As shown in Fig. 3-9, very small lobes are predicted for the shear wave radiation pattern but would be unlikely to be observed in practice because of their extremely small amplitudes.
3.6.4 Radiation from a point source in a fluid-filled borehole

The radiation patterns for a volume point source in a fluid-filled borehole (Fig. 3-10) are nearly identical to those from a radial source (Lee and Balch, 1982; Winbow, 1989). The primary difference is the absence of shear wave lobes for the Lee-Balch sediment or Pierre Shale in comparison to Fig. 3-9.

The differences from Table 3.1 between radiation from a radial source and a point source for the $P$ wave are the absence of the $(\frac{1}{\sigma_j} - \frac{\cos^2 \phi}{\sigma^2})$ term in the numerator and substitution of $V_0 \rho_f$ for the volume $\Delta$. Similarly for the $S$ wave, the absence of the $(\frac{1}{\sigma_j} - \frac{\cos^2 \phi}{\sigma^2})$ term is the only difference. The $P$ wave radiation patterns are very similar to those for a radial source. (Fig. 3-9)

3.7 Comparison between numerical and analytical results

Once the radiation pattern formulas have been determined as in Tables 3.1 and 3.2, these formulas can be used to calculate synthetic seismograms to compare with those from Chapter 2. For instance, let $U(R, \phi)$ equal a displacement component and $F(R, \phi)$ be the radiation pattern result from the tables not including the wavelets $G(t - \frac{R}{v})$ (Eq. 3.24). $G(t)$ is calculated as the inverse fourier transform of the Ricker wavelet $G(\omega)$ such that $U$ equals

$$U(R, \phi) = \frac{F(R, \phi)}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{-i\omega t} e^{i\omega \frac{R}{v}} d\omega$$  \hspace{1cm} (3.60)

where $v$ is either the $P$ or $S$ wave velocity. If the derivative $G'(t - \frac{R}{v})$ is calculated than the following expression is used

$$U(R, \phi) = \frac{F(R, \phi)}{2\pi} \int_{-\infty}^{\infty} i\omega G(\omega) e^{-i\omega t} e^{i\omega \frac{R}{v}} d\omega$$  \hspace{1cm} (3.61)

The same model was used for each of the comparisons shown below and consists of a .2m diameter uncased borehole with a source borehole to receiver array distance
of 50m. The distance of 50m was arrived at by reasoning that except in unusual circumstances, 50m would likely be the minimum separation for a pair of boreholes to be used for a crosshole tomography experiment. Because most boreholes are fluid-filled, examples in this chapter will be restricted to fluid-filled boreholes. The volume of the point source will be 1600 cubic centimeters as in Chapter 3. The axial, radial, and torsional sources will have activation lengths of .5m and will exert a maximum stress of 1 bar. The array height will be 50m which gives a maximum source-receiver latitude \( \phi \) of 45 degrees which meets the maximum latitude likely to be seen in most crosshole tomography experiments. The interreceiver distance is 2m. The source wavelet is a Ricker wavelet of center frequency 500 Hz. No attenuation was considered in these comparisons.

### 3.7.1 Torsional and axial sources

The first type of source considered is radiation from a torsional source which as was shown earlier closely resembles the infinite medium approximation. The lithology surrounding the borehole is Berea Sandstone though the results apply equally well to lower velocity sediments. The comparison between the numerical and analytical prediction of displacement is made in Fig. 3-11 where it can be seen that the displacements predicted from the Thomson-Haskell algorithm and the radiation pattern formula (Eq. 3.46) agree to within 3% which is clearly acceptable. The seismograms are each normalized to the peak amplitude with the numerical prediction plotted as a solid line and the analytic as a dashed line. Since the two lines effectively plot on top of each other it can be seen that the waveforms are identical.

The comparison for axial sources is presented in Fig. 3-12. Like Fig. 3-11 the agreement is quite good between the radiation pattern (Table 3.1) and Thomson-Haskell prediction. The difference in magnitudes is 5% and again the waveforms qualitative agreement is excellent since again the analytic and numeric solutions plot on top of each other.
3.7.2 Radial source

In general radial sources and volume point sources are qualitatively similar so most of the discussion for these types of sources is relegated to the next section. The one difference is the presence of shear wave lobes in the radiation pattern for certain velocities as predicted by Lee and Balch (1982). That hypothesis is tested here. The velocity requirement is that the shear wave velocity is less than the fluid velocity but greater than the tube wave velocity. The location of the lobes is predicted for values of source-receiver co-latitude \( \phi \) of

\[
\phi = \cos^{-1}\left( \frac{\beta^2}{\alpha^2} \right)
\]  

(3.62)

as shown in the last section. The model is slightly different for this test. The borehole is surrounded by the Lee-Balch sediment (see Table 2.3) which is a Poisson solid that has a \( P \) wave velocity 1.4 times the fluid velocity. To model these lobes a greater azimuthal coverage was required so the receiver array is now only 25m away from the borehole and the 50m receiver array now has an interreceiver spacing of 1m. Inserting the parameters from Table 2.3 into Eq. 3.62 the prediction is for a lobe at co-latitude of 49 degrees or latitude of 41 degrees. The 50m array height with 25m source-receiver array spacing has a latitude of 63 degrees which is greater than 41 degrees, so the aperture is sufficient to see these lobes.

Fig. 3-13 shows the radial component seismograms from the numerical prediction (solid line) and the analytic prediction (dashed line) and shows excellent agreement up through the first lobe and the null. Moreover, the agreement in \( P \) is excellent throughout. A second shear lobe with reversed polarity is seen in the numerical modelling but the amplitude is not nearly as great as that predicted by the analytic expression. The reason for the breakdown in agreement is the proximity to the tube wave pole which begins adversely affecting the analytic prediction. The analytic prediction via the method of stationary phase requires the function to be regular near the saddle point and as the saddle point approaches the pole this assumption
becomes increasingly invalid. Nonetheless, the presence of the two lobes is confirmed but the enlarged amplitude in the analytic prediction is an artifact of the proximity of the pole.

It is unlikely that the second lobe would be seen in practice because of the high angle (latitude) between source and receiver which would be required to measure it. For instance in this case a latitude of 41 degrees was required in order to view the null.

3.7.3 Volume point source

Fig. 3-14 is a comparison of the radial components of seismograms calculated using the numerical and analytic predictions for a borehole surrounded by Berea Sandstone. The source is a volume point source and there is excellent agreement, both quantitative ($\sim 1\%$) and qualitative, between the two seismograms.

Continuing with the same model, Fig. 3-15 shows the same comparison as Fig. 3-14 but for a borehole surrounded by Pierre Shale. For the source receiver geometry outlined above, the propagation is outside the Mach Cone so there are no direct Mach wave effects. The waveform agreement seen in Fig. 3-15 is not nearly as good as that seen in Fig. 3-14 and the quantitative agreement in magnitudes shows a 40% difference. The quantitative agreement for $P$ waves is excellent.

Fig. 3-16 is a subset of Fig. 3-15 which truncates the receiver array to only 20m in height at a 1m interreceiver spacing. Since mathematically these particular receivers are farther away from the tube wave pole the agreement is much better. Instead of 40% predicted amplitude mismatch, the mismatch is just over 1%.

Therefore, sufficiently far away from the tube wave pole the agreement in radiation is excellent. But the approximation gets progressively worse as the tube wave pole is approached.

One final question might be answered with this kind of comparison and that question is how close to the borehole the analytic approximations are valid. Already
it is known that for shear wave radiation into low velocity sediments the analytic results are inadequate while they are adequate for high velocity sediments. Therefore a model was run comparing analytic and numerical results with a source borehole to receiver array spacing of 10m. This comparison is shown in Fig. 3-17 and seen to be quite adequate. Agreement within 10% allows the statement that even at 10m separation we are in the far field. Further evidence for this close proximity of the far field is provided by viewing Fig. 2-22 from Chapter 2 which shows very little influence of the tube wave beyond 5m.

3.8 Conclusions

The solution for radiation from an empty borehole was examined in this chapter and in Appendices B and C, and Heelan’s results were found to be completely correct both directly and indirectly. Lee and Balch’s (1982) extensions of Heelan’s results for calculating radiation from a fluid-filled borehole have also been verified and found to have limits in predicting shear wave radiation into low velocity sediments due to the presence of a tube wave pole. The radiation pattern predictions for $P$ wave propagation are valid for both empty and fluid-filled boreholes because the singularities in the integrand are far away from the stationary points (saddle points).

The tube wave pole in low velocity sediments is responsible for the generation of Mach waves. By evaluating the residue of the tube wave pole, the necessary phase delays for Mach wave propagation were demonstrated along with a geometric decay of $\frac{1}{\sqrt{r}}$ when inside the Mach cone. Because of the presence of the Mach wave, there is no classic radiation pattern for the shear wave in low velocity sediments. Instead the radiation is dependent upon both coordinates $r$ and $z$ and exhibits nonuniform geometric decay. Thus for low velocity sediments it is required to evaluate the integral numerically which was done in the last chapter or to use an integration strategy such as that of Felsen and Marcuvitz (1959, 1973) and Bleistein and Handelsman
(1976) which can evaluate an integral when the saddle point and a first order pole are coincident. Because of the complexity of the resulting integrand in terms of complex complementary error functions (See Appendix G) it is recommended that the numerical solution of the last chapter be utilized.

It was also found in this chapter that the the tube wave pole vanishes for axial source propagation and thus axial source propagation is independent of the presence of the fluid in a borehole - an important result also noted by Winbow (1989). Thus the axial source formulation is identical to that for an empty borehole and in fact identical to that for a vertical point force in infinite elastic media. The same equivalence with infinite medium formulations was also noted for torsional sources. This equivalence has important ramifications because it implies that for examining radiation from axial and torsional sources the borehole effects are minimal.

It was shown that radial and point sources are fundamentally alike except that the radial source radiation pattern will exhibit nodes in the shear wave radiation pattern when low velocity sediments surround the borehole. For high velocity sediments surrounding the borehole, the radial and point source shear wave radiation patterns are identical to those from two oppositely directed point forces along the borehole axis. In the case of a low velocity sediment surrounding a borehole, the tube wave radiates shear waves into the formation, the Mach wave concept.

Excellent agreement between the numerical and analytical predictions for high velocity sediments indicate the utility of the analytical predictions and the proximity of far field radiation to the borehole. The disagreement for shear wave radiation into low velocity formations is evidence for the utility of the numerical results.
Heelan and Brekhovskikh
Geometric Relation

\[ -z = R \cos \phi \]

\[ r = R \sin \phi \]

Figure 3-1: Geometry in the far field from Heelan and Brekhovskikh. The relationships between the distance \( R \), the radial distance \( r \), the vertical distance \( z \), and the co-latitude \( \phi \) are very important in the stationary phase analysis.
Figure 3-2: Graph of $k_z$ planes, saddle points, poles, branch cuts and paths of steepest descent used in the method of stationary phase for $P$ waves. There is a saddle point at $k_{z0} = \frac{\omega}{\alpha R}$ in both cases. Since $\frac{\omega}{R} < 1$ (Fig. 3-1), this saddle point is always less than the branch point $\frac{\omega}{\alpha}$. In the fluid-filled borehole case (bottom) $\frac{\omega}{c_T} > \frac{\omega}{\alpha}$ since $\alpha > c_T$ by definition. Hence neither the tube pole not the branch cut are crossed for these integrals.
Figure 3-3: Graph of $k_z$ plane, saddle point, branch cut and path of steepest descent used in the method of stationary phase for $S$ waves in an empty borehole. This graph of the $k_z$ plane is functionally similar to that for the $P$ wave in an empty borehole graph in Fig. 3-2. The same conclusions is reached, for the evaluation of the integral for an $S$ wave radiating from an empty borehole, no poles or branch cuts are crossed. There is no branch point associated with $\frac{\omega}{\alpha}$, abiding by a convention in acoustic well logging (e.g. Paillet and Cheng, 1986).
Figure 3-4: Graph of $k_z$ planes, saddle points, poles, branch cuts and paths of steepest descent used in the method of stationary phase for $S$ waves radiating from a fluid-filled borehole. Saddle point at $k_{z0} = \frac{\omega}{\beta R}$. In a fluid-filled borehole, there is a tube wave pole but if $\beta$ is greater than the tube wave velocity $C_T$ (top) then neither the pole nor the branch cut is crossed by the steepest descent path. However, as $\beta$ approaches $C_T$ the analytic prediction will get progressively worse. If $\beta < C_T$ (bottom) then the tube wave pole is crossed for some value of $\frac{\omega}{\beta R}$, at the complementary Mach angle, requiring special treatment. Again the branch cut will not be crossed in the integration. There is no branch point associated with $\frac{\omega}{\alpha_f}$ abiding by a convention in acoustic well logging (e.g. Paillet and Cheng, 1986).
Coordinate Transformations for Radiation Patterns

Figure 3-5: Picture shows the relationship between Heelan's definition of radial $U_R$ and tangential $U_\phi$ displacement and the radial $U_r$ and vertical $U_z$ components of displacement for $P$ and $S$ waves. Measurement of the angle $\phi$ from the bottom or south pole is a different convention than that often encountered when working in spherical coordinates. This convention is responsible for the difference in sign between White's formulas and the empty borehole formulas displayed in the text.
Figure 3-6: $P$ and $SV$ ($G_1, G_2$ from Table 3.1) radiation patterns for a vertically directed point force in an infinite medium or an axial source in a borehole. Velocities and densities may be found in Table 2.3. Both the absolute scale showing the greater amplitude for soft sediments and the relative scale showing the greater relative amplitude for $P$ waves as Poisson ratio decreases are shown. For the absolute scale, the $P$ wave is magnified by a factor of 4.
Figure 3-7: $SH$ radiation pattern for a torsional source (Eq. 3.51) of finite length applied to the wall of a borehole or a force couple producing a torque in an infinite medium (Eq. 5.10, White, 1965).
Figure 3-8: $P$ and $SV$ ($F_1, F_2$ from Table 3.1) radiation patterns for a radial source of finite length applied to the wall of an empty borehole. Velocities and densities may be found in Table 2.3.
Figure 3.9: $P$ and $SV$ ($F_1, F_2$ from Table 3.1) radiation patterns for a radial source of finite length applied to the wall of a fluid-filled borehole. Velocities and densities may be found in Table 2.3. Radiation patterns for Berea Sandstone and Solenhofen Limestone are nearly identical to those presented in Fig. 3.8. Presence of shear wave lobes is exhibited by Lee and Balch sediment. *SV radiation pattern for Pierre Shale is only valid for values of co-latitude $\phi$ less than $\phi_c = \cos^{-1} \frac{\beta}{C_p}$, the complementary Mach angle, and similarly for the other three quadrants.
Figure 3-10: $P$ and $SV$ ($F_1, F_2$ from Table 3.1) radiation patterns for a volume point source applied to the wall of a fluid-filled borehole. Velocities and densities may be found in Table 2.3. Radiation patterns for Berea Sandstone and Solenhofen Limestone are nearly identical to those presented in Fig. 3-8. *SV radiation pattern for Pierre Shale is only valid for values of co-latitude $\phi$ less than $\phi_c = \cos^{-1} \frac{\rho}{c_T}$, the complementary Mach angle, and similarly for the other three quadrants.
Figure 3-11: A comparison of seismograms for radiation from a torsional source. The seismograms were calculated using the Thomson-Haskell algorithm (solid line, .1296e-7 m displacement) and radiation pattern formula (dashed line, .1336e-7 m displacement) for a borehole surrounded by Berea Sandstone. Azimuthal component of displacement. 1 bar torsional source of .5m length. Source to vertical receiver array distance is 50m with receivers equally spaced between 0 and 50m in height at 2m intervals. 500 Hz Ricker Wavelet. No time shift applied to data. The fact that the seismograms effectively plot on top of each other plus the peak amplitudes are in close agreement bear out the usefulness of the analytic approximation.
Figure 3-12: A comparison of seismograms for radiation from an axial source. The seismograms were calculated using the Thomson-Haskell algorithm (solid line, $8915e-7m$ peak displacement) and radiation pattern formulas (dashed lines, $8521e-7$ peak displacement) for a fluid-filled borehole surrounded by Berea Sandstone. Radial component of displacement. 1 bar axial source of $.5m$ length. Source to vertical receiver array distance is $50m$ with receivers equally spaced between 0 and $50m$ in height at $2m$ intervals. $500$ Hz Ricker Wavelet. No time shift applied to data. The fact that the seismograms effectively plot on top of each other plus the peak amplitudes are in close agreement bear out the usefulness of the analytic approximation.
Figure 3-13: A comparison of seismograms for radiation from a radial source. The seismograms were calculated using the Thomson-Haskell algorithm (solid line, 0.982e-7 m peak displacement) and radiation patterns (dashed line, 1.01e-7 m peak displacement) for a fluid-filled borehole surrounded by Lee-Balch sediment (See Table 2.3). Radial component of displacement. 1 bar radial source of 0.5 m length. Source to vertical receiver array distance is now 25 m with receivers equally spaced between 0 and 50 m in height at 1 m intervals. 500 Hz Ricker Wavelet. No time shift applied to data. Notice appearance of shear wave lobes as predicted by Lee and Balch (1982). However, upper lobe is less well developed in the numeric rather than the analytic prediction indicating upper lobe in analytic prediction is an artifact due to the proximity of the tube wave pole.
Figure 3-14: A comparison seismograms for radiation from a point source. The seismograms were calculated using the Thomson-Haskell algorithm (solid line, .170e-04m peak displacement) and radiation patterns (dashed line, .169e-04m peak displacement) for a fluid-filled borehole surrounded by Berea Sandstone. Radial component of displacement. 1600 cc volume point source. Source to vertical receiver array distance is 50m with receivers equally spaced between 0 and 50m in height at 2m intervals. 500 Hz Ricker Wavelet. No time shift applied to data.
Lithology: Pierre Shale
Purpose: Point Source - Numeric and Analytic Prediction
Component: Radial
Gain: 1.500
Maximum Amplitude: .303, .501 e-03
Clipping Factor: None

Figure 3-15: A comparison of seismograms calculated using the Thomson-Haskell algorithm (dashed line, .303e-3m displacement) and radiation patterns (solid line .501e-3m displacement) for a fluid-filled borehole surrounded by Pierre Shale.Radial component of displacement. 1600 cc volume point source. Source to vertical receiver array distance is 50m with receivers equally spaced between 0 and 50m in height at 2m intervals. 500 Hz Ricker Wavelet. No time shift applied to data. Outside Mach cone so no direct observation of Mach waves but Mach cone adversely influences the magnitude prediction.
Figure 3-16: A comparison of seismograms calculated using the Thomson-Haskell algorithm (dashed line, .151 e-3m displacement) and radiation pattern (solid line, .150e-3m displacement) for a fluid-filled borehole surrounded by Pierre Shale. Radial component of displacement. 1600 cc volume point source. Source to vertical receiver array distance is 50m with receivers equally spaced between 0 and 20m in height at 1m intervals. 500 Hz Ricker Wavelet. No time shift applied to data. Outside Mach cone so no direct observation of Mach waves. Much better agreement in amplitude seen between the numeric and analytic solution in comparison to Fig. 3-15 because these receivers are further away from the Mach cone, closer to the $z = 0$ plane.
Lithology: Berea Sandstone
Purpose: Point Source - Numeric and Analytic Predictions
Component: Radial
Gain: 1.500
Maximum Amplitude: .365,.378e-03
Clipping Factor: None

Figure 3-17: A comparison of seismograms calculated using the Thomson-Haskell algorithm (solid line, .3645e-3 m displacement) and radiation patterns (dashed line, .3377e-3 m displacement) for a fluid-filled borehole surrounded by Berea Sandstone. Only 10m separation between source and vertical receiver array yet qualitative agreement is excellent between the analytic approximation and the numeric results. However, the amplitude difference is 11%. Radial component of displacement. 1600 cc volume point source. Receivers equally spaced now between 0 and 20m in height at 1m intervals. 1000 Hz Ricker Wavelet. No time shift applied to data.
Chapter 4

Modelling of Field Data Sets

4.1 Introduction

In this chapter, experimental data will be modelled using the numerical and analytical techniques developed in Chapters 2 and 3. There is excellent agreement between theoretical predictions of radiation and radiation data recorded in high velocity sediments as shown by Fehler and Pearson's (1984) data (Fig. 4-1). Moreover, the theoretical results presented in Chapter 3 and the numerical results presented in Chapter 2 show that radiation into high velocity sediments is well understood. Conversely, the theoretical results predict that radiation data taken in low velocity sediments, will be of the most interest because of the generation of Mach waves. Therefore, two experiments published in papers by White and Sengbush (1963) and de Bruin and Huizer (1989) which have been done in low velocity sediments have been assembled from the literature and will be examined here. A third data set from the experiment described by Winterstein and Paulsson (1990) allowing comparison of axial source radiation and experimental data is also discussed in this chapter.
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Table 4.1: Station Locations for White and Sengbush (1963) experiment.

4.2 White and Sengbush (1963) experiment

In the summer of 1953, a set of crosshole and surface field experiments was performed in Limon, Colorado culminating in two important papers, the first one by McDonal et al. (1958) which investigated attenuation and the second one by White and Sengbush (1963) entitled “Shear waves from explosive sources” which tested radiation pattern predictions of Heelan (1953a). To perform the radiation pattern experiment, it was desirable to have a homogeneous background lithology. The requirement was fulfilled at the site due to the presence of the Pierre Shale which ranges in depth from 100 to 4000 ft. below the surface.

The field program used a large variety of dynamite sources and recording geometries. However, only the most pertinent experiment will be examined here and a diagram of this experiment is displayed in Fig. 4-2. Referring to the figure, the dynamite booster charges were fired at 600 ft depth in borehole 24 and the six receiver boreholes were laid out in a semi-circular fashion in a vertical plane including the source borehole. Table 4.1 gives the coordinates of each receiver borehole. Both the source and the receiver boreholes were fluid-filled and the receivers were three component geophones which measured particle velocity for each component.

The pivotal figure from White and Sengbush’s paper was a comparison of experimentally determined S wave to P wave amplitude ratios versus Heelan’s theoretical
predictions plotted on a polar grid. This figure is reproduced in Fig. 4-3. Because the $P$ wave radiation pattern in these low velocity sediments is nearly spherical as shown in Chapter 3, the amplitude ratio closely resembles the $S$ wave radiation pattern. As can be seen the comparison between the theoretical predictions and the data was not good because the experimentally determined $S$ to $P$ wave amplitude ratio has a strong vertical orientation. White and Sengbush cited this as evidence that tube waves, which are not accounted for in Heelan’s predictions for radiation from an empty borehole, are strongly influencing the propagation of shear waves.

In fairness to Heelan, the boreholes used by White and Sengbush were fluid-filled and the Heelan theory was for empty boreholes. More appropriate predictions would be those of Lee and Balch (1982) but as pointed out in Chapter 3, the Lee and Balch (1982) radiation patterns do not yield results inside the Mach cone and near the boundary of the Mach cone the Lee and Balch prediction is not good. The Mach number for this experiment is 1.14 so that the complementary Mach angle is 28.7 degrees (Fig. 2-16). In other words, receivers at co-latitudes less than 28.7 degrees in Fig. 4-3 are inside the Mach cone so the shear wave arrivals at these receivers are actually Mach waves. This explains why the radiation pattern in Fig. 4-3 seemingly jumps at about 30 degrees co-latitude.

To model the amplitude effects of the Mach wave, numerical calculations were performed using the Thomson-Haskell algorithm developed in Chapter 2. The parameters for the model included a 20cm diameter borehole, a $Q$ value of 10 for $S$ waves and 32 for $P$ waves (McDonal et. al., 1958; pg. 250, Toksöz and Johnston, 1981), a $P$ wave velocity of 2150 m/sec and an $S$ wave velocity of 860 m/sec (White and Sengbush, 1963).

The $S$ to $P$ ratio of this numerical prediction is displayed in Fig. 4-4 versus the data of White and Sengbush. The agreement shown in Fig. 4-4 is very good and a substantial improvement over Heelan’s prediction (Fig. 4-3). One note is that Sta. 26 at a co-latitude of 90 degrees was not precisely located (White and Sengbush, 1963).
which explains the non-zero shear wave amplitude at this angle.

The difference between the Heelan prediction and the White and Sengbush data is due to the fact that the geometric decay of the Mach wave as shown in Chapter 3 is \( \frac{1}{\sqrt{r}} \) where \( r \) is the distance away from the borehole and this predominates in the vertical direction over the geometric decay of the \( P \) waves which is the inverse of the total distance (\( \frac{1}{R} \)).

Synthetic seismograms calculated using these velocity parameters are displayed in Fig. 4-5 as a function of azimuth and support the large shear wave amplitude in a vertical direction. A comparison of seismograms which were digitized from White and Sengbush's paper and these synthetics is made on a relative basis in Fig. 4-6. Some notes about this figure are that the two horizontal components were rotated into one radial component and the data seismograms were not displayed for Station 26 because it was improperly located and Station 31 because of excessive noise. The data and synthetic seismograms were aligned on the first arrival of the \( P \) wave since no absolute time scale was available. Therefore all of the misfit in time is seen with the \( S \) or Mach wave arrival.

4.3 de Bruin and Huizer (1989) experiment

In a recent paper (de Bruin and Huizer, 1989), an analysis was made of a rather unique experiment which provides a good test for theoretical predictions. For the experiment, radiation from point sources located in a borehole was measured on a receiver array only 20m away. The unique aspect of the experiment was that the vertical array of receivers had been buried for many years with sand so coupling was predicted to be ideal (de Bruin and Huizer, 1989). Because of the long term burial there should be no receiver borehole tube waves because there is effectively no receiver borehole.

The length of the receiver array was 120m with a 2.5m spacing between receivers
and the source was fired at 70m depth in the source borehole (Fig. 4-7, Fig. 5 from de Bruin and Huizer (1989)). The receivers were vertical component geophones only. Additional parameters of interest for modelling purposes are that the source borehole was fluid-filled, 10 inches in diameter, and cased with steel. $Q$ values were assumed to be the same as those for Pierre Shale cited above because of the similarity in velocities between the two sites.

The source was 12.5 g of dynamite and the velocities of the surrounding media included $P$ wave velocities of 1800 m/sec, $S$ wave velocities of 600 m/sec and tube wave velocities of 1200 m/sec. Calculation of the tube wave velocity using the formula from Marzetta and Schoenberg (1985, also Eq. 2.49, this thesis) gives the same value to two significant figures assuming a fluid velocity of 1450 m/sec.

The crosshole seismic data is shown in Fig. 4-7 and the secondary arrivals are recognized as dominant on this figure. It can be seen that there are multiple crisscrossing arrivals offset in time which could not be due to a single shot of dynamite. de Bruin and Huizer (1989) attributed the strong secondary arrivals to a tube wave travelling up and down the borehole radiating shear conical waves, the Mach waves. The multiple arrivals noticed in this experiment provide one of the strongest examples of the existence of Mach waves in field data.

Because the source borehole was cased, the tube wave velocity was much greater than the shear wave velocity. In fact the complementary Mach angle $\phi_c = \cos^{-1} \frac{\beta}{c_T}$ was determined to be 60 degrees and the Mach number for this experiment was 2. Thus at a distance of 20m from the borehole, it was predicted the Mach cone would exist 10m above the source depth (60m and above) and 10m below the source depth (80m and below). It is difficult to see this in Fig. 4-7 but an expanded view provided in Fig. 4-8 makes this identification of the Mach cone easier. The identification is hampered by the unknown scaling factor applied to the de Bruin and Huizer data but close inspection allows delineation of the front of the Mach cone. The snapshots presented in Fig. 2-21 provide more complete evidence.
For modelling purposes only the first arrivals were modelled. When comparing the synthetics with the data only the arrivals below the source-receiver axis are modelled because of the symmetry of the model about a horizontal axis. A 100 Hz Ricker wavelet was used and particle velocities were computed.

A qualitative comparison with the de Bruin and Huizer data set is presented in Fig. 4-9. Good general agreement is seen and especially well duplicated is the near linear moveout in the Mach cone corresponding to the tube wave velocity as discussed in Chapter 2. The hyperbolic moveout of the $P$ wave in both the synthetic and the data is easily recognizable.

To reinforce some of the conclusions made in Chapter 2, a model was run without casing and is displayed as the top half of Fig. 4-10 along with the model with casing in the bottom half from Fig. 4-9 for comparison. These vertical components show that the effect of the addition of casing, thus increasing the tube wave velocity, expands the Mach cone to include a greater portion of the array. Therefore, the Mach cone is difficult to see in the model without casing because most of the seismogram is dominated by the spherical wave front. The model with casing has the much more clearly discernible Mach cone. This example reinforces the observation that the Mach waves will be a much more important phenomenon when casing is added to the borehole environment.

## 4.4 Chevron experiment

The new data set examined here was provided by Chevron Oil Field Research Co. and was taken in the Coyote Field adjacent to Chevron’s research site in La Habra, California as part of a comprehensive research program. The geology at the Coyote Field is a mostly homogeneous, late tertiary marine shale located in the Los Angeles basin (Winterstein and Paulsson, 1990).

The purpose of the experiment was to investigate methods of measuring anisotropy
and to test Chevron's downhole axial vibrator source (Paulsson, 1988). The research program included a crosshole tomography experiment, a vertical seismic profiling experiment and a thorough logging program. A recent paper (Winterstein and Paulsson, 1990) extensively examined the complete data set but only one shot gather from the crosshole experiment will be examined here. The particular shot gather displayed in this thesis was not examined in the Winterstein and Paulsson (1990) paper.

The shot gather of interest is at a 200 ft (61m) source depth. The radiation was recorded with a three component geophone march up the borehole from 750 to 10 ft (228.6-3.1m) below the surface in 10 ft (3.05m) increments. A cross section diagram outlining the experiment is presented in Fig. 4-11. It can be seen from this diagram that the source borehole elevation is 10 ft (3.1m) above that of the receiver borehole and that the distance from the source to the receiver borehole is 75m.

The activation length of the axial source (Paulsson, 1988) was approximately 8 inches (20cm). Power was provided through hydraulics and the source was a frequency controlled vibrator. This allowed sweep across different frequencies and the sweep used was 10-640 Hz. For the purpose of modelling it was assumed that the pressure exerted on casing in the source borehole was 1 bar.

The receiver was a three component clamped geophone initially described by Wuenschel (1976). The vertical and \( H1 \) components were very satisfactory although the \( H2 \) component suffered from ringiness due to improper coupling at crosshole frequencies. This downhole geophone was originally designed for VSP frequencies and works well in that lower range (Turpening, personal communication).

The source borehole was cased with 7 in (17.8cm) casing that had the common designation of 23 pounds per foot density and thus an inside diameter of 6.37 inches (16.2cm) and an outside diameter of 7 in (17.8 cm) (Austin, 1983). These casing parameters were used for modelling purposes. The receiver borehole was cased with 13.375 in (34.0cm) casing but the receiver borehole effects are not modelled in this thesis.
Data conditioning consisted of first deconvolving the data which was performed at Chevron prior to receipt. A 350 Hz low pass anti-aliasing filter was applied but there still was substantial ringing in the H2 component. The ringing was removed by aligning the traces and applying a deconvolution filter that preserved the amplitude of the first break. Finally, H1 and the specially processed H2 component were rotated into a radial component and a transverse component, the latter one not being considered here. This processing was accomplished at our laboratory by fellow student, Chengbin Peng, as part of another study. No processing other than low pass filtering was applied to the vertical component. The vertical component shot gather is displayed in Fig. 4-13. Fig. 4-14 shows the rotated radial component shot gather before the special processing.

The source borehole was pumped dry prior to the experiment and thus no source generated borehole tube waves were to be expected. Additionally, the receiver borehole was also dry thus preventing source or receiver borehole tube wave contamination. Since axial sources do not generate tube waves as shown in Chapter 3 this was not necessary on a theoretical basis. However, since the source was a prototype the dry borehole was an operational requirement.

### 4.4.1 Well logs

Well log information proved to be of use in interpretation of the data. A large number of well logs were run in the open hole in the source and receiver boreholes prior to complete casing of these two wells and include

1. cement bond logs
2. borehole compensated sonic logs
3. compensated neutron density logs
4. caliper logs
5. gamma ray logs

6. grain density logs

7. resistivity logs

The logs range from a total depth of 800 ft to 140 ft below the surface in the receiver borehole and from a total depth of 950 ft to 50 ft below the surface in the source borehole.

Space considerations prohibit the display of each log; only the sonic log for each hole is displayed in Fig. 4-12 which is taken from the Winterstein and Paulsson (1990) paper. However some of the log properties will be addressed here. The resistivity logs for instance show almost no deviation between the deep and shallow measurements which indicates that there are no invasion effects perceptible. This conclusion is collaborated by the sonic logs which show excellent agreement between the short and long spacing measurements (Fig. 4-12). Winterstein and Paulsson (1990) pointed out that the boreholes are very straight with less than a degree of deviation. The caliper logs show that the source borehole prior to casing is in gauge and only slightly larger than the drill bit size which is desirable.

The sonic logs show a shallow $P$ wave velocity gradient (Winterstein and Paulsson, 1990) of

$$v_p = 1696 + 1.775z \text{ m/sec}$$

where $z$ is the depth below the surface which is valid from 61 to 183m (200-600 ft). The $S$ wave velocity gradient was not determined from logging information.

Thus logging measurements confirm that the site chosen was very homogeneous and ideal for the modelling presented in this thesis.

4.4.2 Modelling of data and discussion

The near surface is complicated because it is above the water table. Since the model requires a homogeneous medium and since the model is symmetric with respect to
the source location it was decided to only model the receivers below the source level at 190 ft (58m) down to the bottom of the borehole at 750 ft (229m). The processed radial component is displayed in Fig. 4-17 and it can be seen that the ringingness that plagued this component has been removed by comparison of this figure with Fig. 4-13.

Very good qualitative agreement is seen by comparing the vertical component of the numerical model (Fig. 4-16) with the data (Fig. 4-15). The numerical simulation of the radial component is presented in Fig. 4-18 and qualitatively shows good agreement with the data presented in Fig. 4-17, especially for the P wave radiation. However, the fit for the radial component is not as good as for the vertical component. The P wave prediction is good but the S wave prediction failed to account for the large amplitude shear wave opposite to the source. It is likely that the reason for the large amplitude shear wave opposite to the source is due to the rotation of the shear wave particle motion from a purely vertical direction. A large shear wave gradient of

\[ \beta = 362 + 1.8z \text{ m/s} \]  

(Winterstein and Paulsson, 1990) determined by interval velocity analysis plus the noticed anisotropy (Winterstein and Paulsson, 1990) would cause this bending of the wavefield. The numerical algorithm cannot model these inhomogeneities in the velocity field but nonetheless the prediction of the decay and relative amplitudes are very reasonable. Quantitative comparison was not possible because of lack of knowledge of the gain applied to the receivers. Since an axial source can be mimicked by a vertical point force in an infinite medium there might be some future success in modelling anisotropy and shear wave gradients with a simpler technique.

### 4.5 Conclusions

Experimental radiation from volume point sources in a fluid-filled borehole and from an axial source in a dry borehole have been modelled in this chapter. The qualitative agreement between the synthetic seismograms for both point and axial sources and the
experimental data is excellent and points out the utility of the Thomson-Haskell algorithm as implemented in Chapter 2. But more importantly, the modelling confirms the suspicions from the experimental data that in fluid-filled boreholes surrounded by low velocity sediments the generation of Mach waves is an important component of the full wavefield, especially for cased boreholes where the Mach cone extends further in the formation. The Chevron experiment was done in an empty borehole so nothing can be stated about tube wave propagation but no tube waves would be expected for an axial source.
FIG. 1. Amplitudes of $P$ and $S$ waves due to small explosions in one borehole as measured in a neighboring borehole. Amplitudes have been corrected for geometrical spreading ($1/R$). Solid lines represent angular dependence of radiation pattern predicted by equations (6) and (7). $P$-wave amplitudes are indicated by a $p$, $S$-wave amplitudes are indicated by an $s$. $\phi$ is the angle between the raypath and the borehole axis.

Figure 4-1: Fig. 1 from Fehler and Pearson (1984) showing agreement between experimental data taken in granite and the rose petal shape $S$ wave radiation pattern and peanut shaped $P$ wave radiation pattern (both in two dimensions). Although Fehler and Pearson (1984) used a moment tensor representation for the source, geometrically, the results are equivalent to Heelan's (1953a).
Figure 4-2: Geometry of the White and Sengbush experiment (Fig. 1, White and Sengbush, 1963). Six receiver boreholes were drilled in a semicircular arc in a vertical plane including the source borehole (24). The shotpoint was at 600 ft depth in the source boreholes. The coordinates for the receiver boreholes and their co-latitudes are displayed in Table 6.1. It should be noted that both source and receiver boreholes were fluid-filled but not cased and the receivers were three component receivers.
Figure 4-3: Fig. 6 from White and Sengbush (1963). Results of experiment to test Heelan’s radiation pattern prediction. Y axis is S wave to P wave amplitude ratio and the experimental data shows a strong vertical directivity of S wave amplitudes that Heelan’s theory cannot predict, a very poor fit. Heelan’s theory was for dry boreholes and the White and Sengbush experiment was performed in fluid-filled boreholes so the difference was surmised to be due to the presence of radiation from tube waves in the fluid-filled case.
Figure 4.4: Polar plot displays ratio of \( S \) over \( P \) wave amplitude in Pierre Shale versus co-latitude of station. Data points are from Fig. 6 (White and Sengbush, 1963; also see Fig. 4-3). Theoretical prediction is from Thomson-Haskell technique as developed in Chapter 2. Model parameters are described in text. When compared to Fig. 4-3 it can be seen with the numerical solution the vertical skewness has been reproduced and much better agreement is seen.
Figure 4-5: Plot of synthetic seismograms with White and Sengbush parameters. Vertical component is on top and radial on bottom for each pair. Station number and co-latitude is plotted on each station. Time for each seismogram is from 0 to 150 msec.
Figure 4-6: Comparison of seismograms digitized from White and Sengbush’s (1963) paper (dashed) and calculated using the algorithm from Chapter 2 (solid). Seismograms are aligned on P-wave arrival so misfit in time is biased toward the S wave. Qualitative agreement is excellent. Station 31 is not included because of excessive noise and Station 26 is improperly located (White and Sengbush, 1963). Time for each seismogram is from 0 to 150 msec.
Figure 4-7: Crosshole data from de Bruin and Huizer (Fig. 5, 1989) including original caption. Source borehole and receiver borehole are placed 20m apart. Dynamite source. 10 inch cased borehole. Delta of vertical array is 2.5m. Vertical array was filled with sand for many years such that coupling is theoretically ideal (de Bruin and Huizer, 1989). Strong secondary reverberations are noticed for long periods. Geophones are vertical component only.
Fig. 7. Response in the vertical array to a shot with 12.5g dynamite. (the discarded trace just above the source level was a dead trace).

Figure 4-8: Subset of Fig. 4-7 consisting of period from 0 to 300 msec to be used for modelling. Fig. 7 of de Bruin and Huizer (1989). Portion of vertical array from 70-120 m is modelled in Fig. 4-9 although actual location of 70m is two traces above that labelled. Geophones are vertical component only.
Figure 4-9: Numerical model of the data (bottom) and actual data (top) from de Bruin and Huizer (1989). Delta of vertical array is 2.5m. 200 Hz Ricker wavelet used for modelling. Notice source wavelet in data is ringier than Ricker wavelet but the qualitative agreement is very good with the primary arrivals presented in Fig. 4-8. Linear moveout of the radiated tube wave due to the propagation of Mach waves is duplicated. Scaling factor of de Bruin and Huizer data is unknown.
Figure 4-10: Second numerical model of the data from de Bruin and Huizer (1989). Uncased borehole (top) and bottom half of Fig. 4-9, the cased borehole model, is duplicated here in the bottom. Notice change in moveout of secondary arrival from Fig. 4-9. For the uncased model, the moveout is at a much shallower slope due to the reduction in tube wave velocity and shrinking of the Mach Cone. The presence of the Mach cone is much more prevalent in the bottom half of the figure representing the cased borehole.
Figure 4-11: Cross section diagram of the crosshole shot gather provided by Chevron. The source borehole was located 75m away from the receiver borehole. The topographic profile of the surface was essentially flat although there is a 10ft (3.05m) drop between the source and receiver borehole. The source was fired at 200 ft below the surface which corresponds to 190 ft in the receiver borehole. The three component clamped geophone was moved from 750 to 10 ft below the surface for a total of 75 stations.
Figure 4-12: Borehole compensated sonic logs from the Chevron test site. Fig. 5 from Winterstein and Paulsson (1990). Notice general uniformity and gradient with depth for $P$ wave velocity.
Figure 4-13: Plot of the vertical component of the shot gather. Data is remarkably clean for crosshole data and shews clearly discernible $P$ and $S$ waves. No further processing was done to this vertical component beyond low pass filtering and that which was performed by Chevron, i.e. deconvolution, before receipt of the data. Reflection is believed to be from a low velocity zone in the higher section seen in Fig. 4-12 at about 80 ft in depth.
Figure 4-14: Plot of the radial component of the shot gather prior to processing. Ringiness is due to a poor H2 component on the receiver. Further processing was done as explained in the text to remove this ringing and the result is presented in Fig. 4-17.
Figure 4-15: Subset of the vertical component data shown in Fig. 4-13. This portion of the data set was used for comparison with model results presented in the next figure.
Figure 4-16: Synthetic seismograms calculated from the Thomson-Haskell algorithm presented in Chapter 2 for an axial source in an empty borehole with the 75m source-receiver borehole separation. Vertical component to compare with Fig. 4-15 showing good agreement. In particular the gradual extinction of the S wave with depth is duplicated fairly well.
Figure 4-17: Subset of the radial component data shown in Fig. 4-14 after processing. The $P$ wave is much more refined after removal of the ringing. It can be seen that the $P$ wave amplitude is decreasing in this radial component compared to Fig. 4-15 with depth because the $P$ wave is becoming more vertical. This portion of the data set was modelled using the algorithm's results and the model prediction is presented in the next figure.
Figure 4-18: Synthetic seismograms calculated from the Thomson-Haskell technique presented in Chapter 2 for an axial source in an empty borehole with the 75m source-receiver borehole separation. Radial component to compare with Fig. 4-17 showing good agreement in $P$. Agreement with $S$ waves is not as good near the 190 ft level and is most likely due to rotation of the shear wave particle motion from a purely vertical plane due to a shear wave velocity gradient and anisotropy.
Chapter 5

Conclusions

The overall objective of this thesis was to enhance the description of radiation from downhole seismic sources. This was first done by developing the Thomson-Haskell algorithm for the calculation of numerical results in the near and far field and by extending the far field analysis to contend with Mach wave propagation in low velocity sediments. Comparison of the numerical and analytical results where possible showed good agreement. The contribution of each individual chapter to this enhanced description is described below.

In Chapter 1, the current understanding of radiation from downhole seismic sources was presented along with a review of the work which has led us to this juncture. Finally, Chapter 1 provides a comprehensive overview of the thesis.

Chapter 2 was a cornerstone chapter in this thesis in which the Thomson-Haskell algorithm was developed for the numerical calculation of radiation outside a borehole. The integral expressions for the calculation of the radiation from axial, radial, torsional sources in empty boreholes and these same sources as well as volume point sources in fluid-filled boreholes was presented. Numerical integration was accomplished using the discrete wavenumber technique and was tested by comparison with a boundary integral technique developed by Bouchon and Schmitt (1989a,b).

The fundamental observation of the numerical modelling was the differences in
radiation due to the presence of low velocity or high velocity material surrounding the borehole. For instance, in low velocity sediments where the shear wave velocity is less than the tube wave velocity, the tube wave velocity travels in a super shear condition. This results in the generation of conical wavefronts or Mach waves which were well modelled with the algorithm. The hybrid traveltime of these Mach waves has contributions due to propagation as a tube wave and propagation in the medium at shear wave velocity. Taking one quadrant, Mach waves exist from a co-latitude of \( \phi_c = \cos^{-1} \frac{\theta}{c_T} \) to 90 degrees, \( \phi_c \) being the complementary Mach angle. The Mach waves also propagate as plane waves with normal vector in the \( \phi_c \) direction. In an uncased borehole, direct observation of Mach waves in the surrounding medium is unlikely unless an experiment was explicitly designed to measure them. However, outside the borehole the Mach waves will be essentially planar in nature and it is likely that reflections of Mach waves off interfaces may be observed beyond the first shear wave arrival. The presence of steel casing increases the tube wave velocity without changing the shear wave velocity so the Mach cone is expanded and direct or indirect observation of Mach waves becomes more likely. \( P \) wave propagation is not affected by the presence of Mach waves and only slightly affected by the presence of the borehole or the presence of fluid in the borehole.

In Chapter 3, the analysis of the far field radiation was undertaken and the analytic approximations necessary to carry out the integrations by the method of stationary phase were presented. Analytical results were exclusively limited to the borehole surrounded by an infinite half space case, the one layer case. The results of Heelan (1953a) for radiation from an empty borehole, which have come under some criticism, were shown to be correct with more direct proof relegated to Appendices B and C. Secondly, the results of Lee and Balch for radiation from a fluid-filled borehole were analyzed and the shortcomings identified for the prediction of radiation from low velocity sediments due to the presence of Mach waves. Extensions of the Lee and Balch treatment by calculation of the residue of the tube wave pole to isolate the
effect of the Mach waves proved to be successful. A geometric decay of \( \frac{1}{\sqrt{r}} \) was found and a hybrid travel time to a receiver \((r, z)\) inside the Mach cone of

\[
t = \frac{z}{C_T} + \frac{r\sqrt{M^2 - 1}}{C_T}
\]  

(5.1)

where \( C_T \) is the tube wave velocity and \( M \) equals the Mach number. This hybrid traveltime calculated through the use of the residue theorem was identical to that calculated in Chapter 2 based on geometrical arguments alone.

For an uncased borehole, it was possible to compare the numerical results with the analytic predictions. It was shown that when low velocity sediments surround a fluid-filled borehole, the predictions for shear wave radiation near the Mach cone was not very good but away from the Mach cone near the \( z = 0 \) plane the predictions were very good. The prediction of radiation for \( P \) waves was good for all instances. Comparison of numerical and analytical results when high velocity sediments surrounded the borehole showed excellent agreement. Combining the results of Chapters 2 and 3 allows that statement that the far field is as close as 10m when high velocity sediments surround a borehole. Similarly, when low velocity sediments surround the borehole, the far field is as close as 10m for \( P \) wave radiation only. This 10m prediction is based on the vanishing influence of tube waves for this case. The similarity of far field representations of downhole seismic sources in boreholes and sources in infinite media was also explored in Chapter 4.

In Chapter 4, the numerical results were compared to data sets published by White and Sengbush (1963) and de Bruin and Huizer (1989) which were taken in low velocity sediments with point sources. It was shown that the data in these experiments exhibit the influence of Mach waves especially for the de Bruin and Huizer (1989) experiment in which the boreholes were cased. The Mach wave hypothesis helped fully explain the anomalous results of White and Sengbush. An additional data set provided by Chevron Oil Field Research Co. demonstrating radiation from an axial source also fit the algorithm's predictions fairly well although the present formulation of the algorithm cannot account for shear wave velocity gradients or anisotropy.
5.1 Practical implications

There are a number of practical implications for the results of this thesis many of which are borne out by previous work and experiments. A broadly stated goal for downhole seismic source design is to increase the amount of coherent energy radiating into the formation. The results of this thesis show that increasing the amount of radiation can be done in many fashions.

- For high velocity sediments surrounding the borehole, there will be no generation of Mach waves but there will still be diffraction effects due to energetic tube waves impinging on bed boundaries, fractures, borehole constrictions, etc. Therefore, these discontinuities could be modelled independently of the borehole.

- Similarly, for the $P$ wave train exclusively, even in low velocity sediments, there are only small effects due to the presence of the borehole and the presence of the fluid in the borehole. The far field can be considered as close as 10m away.

- For low velocity formations having shear wave velocity less than the tube wave velocity, the shear wave field emanated from the borehole will consist of a spherical wavefront tangent to a Mach cone. The Mach wave will travel as essentially a plane wave with normal vector having a direction given by $\cos \phi = \frac{\rho}{\sigma r}$ which is the inverse of the Mach number. Consequently the moveout across a vertical array if it is in the conical wavefront will provide a measure of the tube wave velocity and not the shear wave velocity. Also the radiation will be more energetic in a vertical direction as noticed by White and Sengbush (1963) and de Bruin and Huizer (1989). The geometric decay is $\frac{1}{\sqrt{r}}$ where $r$ is the radial distance away from the borehole.

- The effect of introducing casing is manifold. In all cases the tube wave velocity is increased, thus increasing the efficiency of the borehole waveguide and
reducing the amplitude of radiation into the formation. However, because the introduction of casing might allow the use of more energetic sources this may very well be a moot issue. The introduction of casing in a low velocity environment will make the direct observation of Mach waves more likely which one should be aware of.

- Axial and torsional sources are theoretically not affected by the presence of the borehole or a presence of a fluid in the borehole. In fact the radiation pattern for an axial or torsional source is well approximated by formulations for the vertical point force and torque-inducing seismic couple in infinite media (e.g. White, 1983) provided the waveforms are properly substituted for by stress waveforms.

- Radiation from radial and volume point sources are substantially alike as has also been noticed by Winbow (1989) and Lee and Balch (1982). One exception is that in the domain $\alpha_f > \beta > C_T$ shear wave lobes are noticed in the radiation patterns for radial sources and in numerical results. These lobes have not been seen experimentally. Experimental verification will have to await future work and requires a) development of radial sources and b) sufficient aperture to measure the shear wave lobes.

- Radiation from radial sources and volume sources consists of two parts. One is a spherical pattern and one is equivalent to a stretching along the borehole axis. Thus future modelling might be able to utilize this fact to superpose these solutions in an infinite medium to simulate the borehole effects.

5.2 Future Work

It has been emphasized in this thesis that shear wave radiation into low velocity sediments will be affected by radiation from the tube wave in the source borehole due to the generation of Mach waves. These Mach waves add a complexity to the resulting
wavefield but also represent a potential opportunity. In a cased borehole especially, one is given in effect an energetic, repeatable line source generating plane waves, the Mach waves, that will reflect off of interfaces. This repeatability and geometric coverage might be exploited in crosshole experiments. On a more technical basis, the algorithm needs to be extended to include velocity gradients and transverse isotropy.

The introduction of the Thomson-Haskell method for numerical calculation, the extension of the far field analysis and the analogy found between Mach waves and supershear tube wave propagation has enhanced our understanding of radiation from downhole seismic sources. Nonetheless, it is only one small step towards the final goal of unravelling the full waveform, the full spectrum of information available in crosshole tomography and reverse VSP data.
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179


180


Appendix A

Wave Equation Relationships in Cylindrical Coordinates: Unification of Wave Propagation Across Different Symmetry Systems

A.1 Introduction

When studying wave propagation in cylindrical coordinates, it is very important to address the fundamental decisions of which symmetry system to use, which eigenfunction to use, etc. For instance, in a cylindrical coordinate system \((r, \theta, z)\) there are three different symmetries commonly assumed: no symmetry, symmetry in \(\theta\), and symmetry in \(z\). Considering symmetry in \(\theta\) (axisymmetry) there are three further subdivisions for a total of 5 symmetry strategies. Compounding the different choices of symmetry strategies available is the use of different time dependencies, the use of different definitions for potentials and the usage of different types of special func-
tions, i.e. Bessel, Hankel or modified Bessel, for the eigenfunctions. This appendix describes the interrelationship between displacement potential, wave equation, and stress-strain relations for these different symmetry systems.

The second half of this appendix will demonstrate how the scalar wave equations are separable for each symmetry strategy in terms of the appropriate Bessel and exponential eigenfunctions using the method of separation of variables. Combined these two appendices provide a thorough review of the mathematics of wave propagation in a cylindrical medium.

A.1.1 Types of symmetry assumptions

The first type of symmetry is the degenerate assumption of no symmetry and can be considered the parent representation. The displacement field is represented by three displacement potentials $\phi, \psi$, and $\chi$ corresponding to a longitudinal $\phi$ and two transverse waves $\psi, \chi$. One of the transverse waves is polarized vertically to the longitudinal motion ($SV$) and the other horizontally ($SH$) and there is complete coupling between $P-SV$ and $SH$ motion for this case. When axisymmetry is assumed the $P-SV$ and $SH$ problems may be solved independently.

The second representation assumes axisymmetry so the $P-SV$ and $SH$ problems can be solved independently. There are two kinds of waves governed by shear wave velocity $SV$ ($\psi$ potential) and compressional wave velocity $P$ ($\phi$ potential), in other words the $P-SV$ case. $\psi$ is transformed to simplify the algebra and boundary conditions and this representation is very frequently found in the geophysics literature. This representation was used extensively in Chapters 2 and 3.

For the third representation, axisymmetry is assumed and $P-SV$ motion except the potential $\psi$ is not transformed. Developments assuming this strategy may be found in a few important references in the geophysics literature and also that of more general physics and mechanics. This representation was used by Heelan (1952, 1953a, b) and Brekhovskikh (1960, 1980) and therefore in Appendices A and C of this
thesis which analyze their work.

The fourth representation also assumes axisymmetry where one calculates the solution for torsional \((SH)\) motion and utilizes the displacement potential \(\chi\) exclusively. In Chapter 2, this representation was used for the torsional source problem.

The fifth and final representation is commonly encountered with line source problems where symmetry in the \(z\) axis is assumed and the displacement potentials \(\phi\) and \(\chi\) are used. Although this representation was not used in this thesis it’s important to be aware of it when comparing these results to results presented in that system.

The existence of these five representations although fundamentally related and useful can be a source of considerable confusion when working with elastic wave propagation in cylindrical coordinates. In this thesis, the axisymmetric representations - the second through fourth representations - were used along with developments in all five systems but the distinctions will be carefully noted.

### A.2 Displacement potential and wave equation relationships

Displacement fields are calculated using the Laplacian operators. These operators in cylindrical coordinates are (e.g. Schey, 1973) given by the following equations. One note is that an arbitrary vector field is given the symbol \(\mathbf{K}(r, \theta, z)\) and an arbitrary scalar field is given by \(A(r, \theta, z)\)

Divergence: Operates on vector field \(\mathbf{K}\) and yields a scalar field

\[
\nabla \cdot \mathbf{K} = \frac{1}{r} \frac{\partial (rK_r)}{\partial r} + \frac{1}{r} \frac{\partial K_\theta}{\partial \theta} + \frac{\partial K_z}{\partial z} \tag{A.1}
\]

Gradient: Operates on a scalar field \(A\) and yields a vector field

\[
\nabla A = \hat{e}_r \frac{\partial A}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial A}{\partial \theta} + \hat{e}_z \frac{\partial A}{\partial z} \tag{A.2}
\]
Curl: operates on vector field $\vec{K}$ and yields a vector field

$$\nabla \times \vec{K} = \hat{e}_r \left( \frac{1}{r} \frac{\partial K_z}{\partial \theta} - \frac{\partial K_\theta}{\partial z} \right) + \hat{e}_\theta \left( \frac{1}{r} \frac{\partial K_z}{\partial r} - \frac{\partial K_z}{\partial r} \right) + \hat{e}_z \left( \frac{1}{r} \frac{\partial (r K_\theta)}{\partial r} - \frac{1}{r} \frac{\partial K_r}{\partial \theta} \right) \quad (A.3)$$

Laplacian: operates on scalar field $A$ and yields a scalar field

$$\nabla^2 A = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial A}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 A}{\partial \theta^2} + \frac{\partial^2 A}{\partial z^2} \quad (A.4)$$

### A.2.1 No symmetry assumptions

An arbitrary displacement field $\vec{U}$ can be represented in terms of the scalar displacement potentials $\phi, \psi, \chi$. In the literature this strategy has been adopted by many authors including White and Zechman (1968) for studying flexural waves (waves that are functions of $\theta$ to the first order only), White and Tongtaow (1981) for describing wave motion inside a borehole propagating through a transversely isotropic medium and Greenfield (1978), Lee (1986) for describing radiation emanating from a borehole, and Schoenberg (1986) for radiation incident on a borehole. The displacement field and the displacement potentials are related by the Helmholtz representation or theorem (e.g. pg. 548, Malvern, 1969; Box 6.5, Aki and Richards, 1980) as follows assuming $\vec{U}$ is continuously differentiable.

$$\vec{U} = \nabla \phi + \nabla \times \vec{K} \quad (A.5)$$

This is the sum of an irrotational scalar field represented by the displacement potential $\phi$ and a solenoidal (equivoluminal) vector field, $\vec{K}$. Potentials are a mathematical convenience and the reader should be reminded that there are in fact many different types of potentials. For instance, in seismology body-force potentials (e.g. Aki and Richards, 1980) are spoken of whereas in underwater and general acoustics velocity or pressure potentials (Wood, 1949; Brekhovskikh, 1960, 1980; Morse and Ingard, 1968) are commonly used.

The "units" of displacement potentials as used here are meters squared for $\phi, \chi$ and meters cubed for $\psi$. The reason for the difference in units is the difference in the order
of the vector differential operators which are required to produce the displacements. For $\phi, \chi$ the differential operators are first order and for $\psi$ they are second order.

$\mathbf{K}$ can be further decomposed into two scalar potentials $\psi$ and $\chi$ (pg. 1762-1767, Morse and Feshbach, 1953; pg. 58-62, Miklowitz, 1978) representing respectively $SV$ and $SH$ waves both travelling at the shear wave velocity. The decomposition is

$$\mathbf{K} = \chi \hat{e}_z + \nabla \times (\psi \hat{e}_z) \quad (A.6)$$

or writing in terms of coordinates where $\hat{e}_z$ is the unit vector in the $z$ direction

$$\mathbf{K} = \left( \frac{1}{r} \frac{\partial \psi}{\partial \theta}, -\frac{\partial \psi}{\partial r}, \chi \right) \quad (A.7)$$

which yields for the curl of $\mathbf{K}$

$$\nabla \times \mathbf{K} = \hat{e}_r \left( \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta \partial z} + \frac{1}{r} \frac{\partial \chi}{\partial \theta} \right) + \hat{e}_\theta \left( \frac{1}{r} \frac{\partial \psi}{\partial \theta} - \frac{\partial \chi}{\partial r} \right) +$$

$$\hat{e}_z \left( -\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial \psi}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right) \quad (A.8)$$

The vector field $\mathbf{K}$ is most commonly considered as divergence free and many elementary statements of the Helmholtz theorem tacitly assume this to be true. But to quote Morse and Feshbach (Vol. II, pg. 1763, 1953)

"We usually assume that the vector field $\mathbf{K}$ has zero divergence. A few exceptions will be made, particularly in the case of the vector Laplace equation."

where the notation $\mathbf{K}$ has been substituted in the above quote. Wave propagation in cylindrical coordinates with no symmetry assumptions often proves to be one of those exceptions outlined by Morse and Feshbach. For instance, it can be seen that $\mathbf{K}$ is comprised of a scalar function multiplied by the unit vector $\hat{e}_z$ and the curl of a vector function (Eq. A.6). Taking the divergence of the curl term yields zero by a well known vector identity. However taking the divergence of $\chi \hat{e}_z$ yields $\frac{\partial \chi}{\partial z}$ which

187
is not necessarily zero. The requirement that the divergence equal zero is sometimes referred to as a gauge condition and the gauge condition is not satisfied for the most general case in cylindrical coordinates. Therefore, in general $P-SV$ and $SH$ motion are completely coupled if no symmetry is assumed (e.g. Pilant, 1979).

Adding $\nabla \phi$ and the cross product $\nabla \times \vec{K}$ (Eq. A.8) using the Helmholtz representation (Eq. A.5), the displacement potential relations (pg. 216, Miklowitz, 1978) are

$$U_r = \frac{\partial \phi}{\partial r} + \frac{\partial^2 \psi}{\partial r \partial z} + \frac{1}{r} \frac{\partial \chi}{\partial \theta}$$

$$U_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} + \frac{1}{r} \frac{\partial^2 \psi}{\partial \theta \partial z} - \frac{\partial \chi}{r \partial \theta}$$

$$U_z = \frac{\partial \phi}{\partial z} - \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial \psi}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2}$$

For infinitesimal strain, the strains in cylindrical coordinates can be calculated from the formulas for curvilinear coordinate systems (e.g. Love, 1927; Morse and Feshbach, 1953; Ben-Menahem and Singh, 1981). The curvilinear coefficients are $h_r = 1, h_\theta = r, h_z = 1$. In terms of displacement these strains are (pg. 56, Love, 1927; pg. 217, Miklowitz, 1978; pg. 163, White, 1983)

$$\varepsilon_r = \frac{\partial U_r}{\partial r}, \quad \varepsilon_\theta = \frac{1}{r} \left( \frac{\partial U_\theta}{\partial \theta} + U_r \right), \quad \varepsilon_z = \frac{\partial U_z}{\partial z}$$

$$\varepsilon_{r\theta} = \frac{1}{r} \frac{\partial U_r}{\partial \theta} + \frac{U_\theta}{r} + \frac{1}{r} \frac{\partial U_\theta}{\partial r}, \quad \varepsilon_{r\theta} = \frac{1}{r} \frac{\partial U_z}{\partial \theta} + \frac{\partial U_\theta}{\partial z}, \quad \varepsilon_{rz} = \frac{\partial U_r}{\partial z} + \frac{\partial U_z}{\partial r}$$

The stresses in terms of strains are calculated using Hooke's law

$$p_r = \lambda \Delta + 2\mu \varepsilon_r, \quad p_\theta = \lambda \Delta + 2\mu \varepsilon_\theta, \quad p_z = \lambda \Delta + 2\mu \varepsilon_z$$

$$p_{r\theta} = \mu \varepsilon_{r\theta}, \quad p_{rz} = \mu \varepsilon_{rz}, \quad p_{\theta z} = \mu \varepsilon_{\theta z}$$

where $\Delta$ is a dilatation defined by

$$\Delta = \nabla^2 \phi = \varepsilon_r + \varepsilon_\theta + \varepsilon_z$$
In most problems of interest, boundary conditions which are functions of the radial and tangential stress \( p_r \) and \( p_{rz} \) arise. Therefore these two terms are written out below

\[
\begin{align*}
    p_r &= \lambda \nabla^2 \phi + 2\mu \frac{\partial}{\partial r} \left( \frac{\partial \phi}{\partial r} + \frac{\partial^2 \psi}{\partial r \partial z} + \frac{1}{r} \frac{\partial \chi}{\partial \theta} \right) \\
    p_{rz} &= \mu \left( \frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial^3 \psi}{\partial z^2 \partial r} + \frac{1}{r} \frac{\partial^2 \chi}{\partial z \partial \theta} + \frac{\partial^2 \phi}{\partial r \partial z} - \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial \psi}{\partial r} \right) \right) - \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right) \right) \\
    p_{rz} &= \mu \left( 2 \frac{\partial^2 \phi}{\partial r \partial z} + \frac{1}{r} \frac{\partial^2 \chi}{\partial z \partial \theta} - \frac{\partial}{\partial r} \left( \nabla^2 \psi - 2 \frac{\partial^2 \psi}{\partial z^2} \right) \right)
\end{align*}
\]

where \( p_{rz} \) has been further simplified.

With the preceding development the equations of motion in an isotropic homogeneous medium (Harkrider, 1964, Aki and Richards, 1980)

\[
(\lambda + 2\mu) \nabla(\nabla \cdot \vec{U}) + \mu \nabla \times (\nabla \times \vec{U}) = \rho \frac{\partial^2 \vec{U}}{\partial t^2}
\]

are now satisfied implicitly by

\[
\nabla^2 \phi = \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2}
\]

\[
\nabla^2 \psi = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2}
\]

\[
\nabla^2 \chi = \frac{1}{\beta^2} \frac{\partial^2 \chi}{\partial t^2}
\]

where \( \alpha^2 = \frac{\lambda + 2\mu}{\rho} \) and \( \beta^2 = \frac{\mu}{\rho} \) and \( \nabla^2 \) is the Laplacian operator (Eq. A.4). Lamé's theorem (pg. 58-62, Miklowitz, 1978; Aki and Richards, 1980) allows the definition of these wave equations from the initial Helmholtz scalar separation. Lamé's theorem for the most general case will not be proved here for the sake of brevity. However, the summation of the radial and vertical forces that leads to the solution of Lamé's theorem will be presented later for the axisymmetric case where \( \psi \) is transformed.

In the development just completed, scalar potentials and the scalar Helmholtz equations were satisfied allowing separation into three scalar displacement potentials.
$\phi, \psi,$ and $\chi$. These three potentials can be equated to vectors through vector differential operators to demonstrate the solution of the vector Helmholtz equation. A set of these basis vectors are the vector $\mathbf{L}, \mathbf{M}, \mathbf{N}$ which are known as Hansen vectors (Hansen, 1935; (Ben-Menahem and Singh, 1981). Such a treatment has been accomplished by Ben-Menahem and Singh (1968a,b, 1981), Pao and Mow (1971), Pilant (1979) and others. Advantages of using Hansen vector theory besides demonstrating the solution of the vector Helmholtz equation is the generalization of the relationships developed to any of the six coordinate systems in which vector separability of the Helmholtz equation is allowed and the relationship of Hansen vectors to boundary conditions and important integrals of mathematical physics. The Hansen vector theory and its applicability to cylindrical coordinates and the five different symmetry strategies addressed in this appendix is presented in Appendix E.

A.2.2 Axisymmetry – transformation of $\psi$

Axisymmetry implies that there is no variation allowed in properties or sources with respect to azimuth ($\theta$). This strategy is the most frequently used in the literature for modelling wave propagation inside a borehole (e.g. Biot, 1952; White, 1965) and is analogous to the case of two dimensional propagation in Cartesian media (e.g. Ewing et al., 1957; Ben-Menahem and Singh, 1981). With axisymmetry, discontinuities are allowed in normal and tangential stresses and displacements only precluding the description of torsional sources which can be addressed independently.

$U_\theta$ and derivatives with respect to $\theta$ in the displacement potential relations (Eq. A.9) vanish. $\vec{K}$ is tangent to a circle perpendicular to the $z$ axis and has no radial or vertical component $K_r, K_z = 0, \chi = 0$ such that Eq. A.7 is rewritten

$$\vec{K} = -\varepsilon_\theta \frac{\partial \psi}{\partial r} \quad (A.16)$$

The gauge condition $\nabla \cdot \vec{K} = 0$ is automatically satisfied since $\frac{\partial \chi}{\partial z} = 0$ identically.

The quantity $-\frac{\partial \psi}{\partial r}$ of Eq. A.16 is taken as the $\theta$ component of a transformed
displacement potential \( \psi \) such that the new \( \psi \) equals \(-\frac{\partial \psi}{\partial r}\). The \( \psi \) symbolism is maintained. Although the preceding definition is how the transformation is explained this is not quite true in practice. The derivative of the eigenfunction is taken but the derivative of the argument of the eigenfunction following a chain rule argument is discarded. By discarding the derivative of the argument the dimensionality of \( \phi, \chi \) and the new \( \psi \) are all now meters squared.

Because of the widespread use of this transformation for axisymmetric media, many authors immediately use this formulation without describing its origins. The reason this transformation is used is that first it helps simplify the algebra which can be daunting at times. Secondly, this transformation helps maintain the analogy between axisymmetric propagation in Cartesian media \((x, y, z)\) and axisymmetric propagation governed by a cylindrical coordinate system. For instance, in Cartesian media when propagation is independent of one component, \( y \) for instance, then \( \vec{\psi}(\psi_x, \psi_y, \psi_z) \) has only a \( \psi_y \) component. Therefore, to simplify the algebra \( \frac{\partial \psi_y}{\partial x} \) is abbreviated as \( \psi \), similarly for cylindrical coordinates where \( \psi_\theta = -\frac{\partial \psi}{\partial r} \) is abbreviated as \( \psi \) (pg. 45, Pilant, 1979). Another strategy provided by Ewing et al. (pg. 10, 1957) was to equate the transformed \( \psi \) to an intermediate potential \( W \) and use \( \psi \) for subsequent derivations.

Rewriting the curl of \( \vec{K} \) (Eq. A.8) and simplifying in terms of the transformed potential \( \psi \) yields

\[
\nabla \times \vec{K} = -\hat{e}_r \left( \frac{\partial \psi}{\partial z} \right) + \hat{e}_z \left( \frac{1}{r} \frac{\partial}{\partial r} (r \psi) \right)
\]

(A.17)

Combining \( \nabla \phi \) (Eq. A.2) and using the Helmholtz representation (Eq. A.5) gives the displacement potential relations below

\[
U_r = \frac{\partial \phi}{\partial r} - \frac{\partial \psi}{\partial z}
\]

(A.18)

\[
U_z = \frac{\partial \phi}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (r \psi)
\]

The strain displacement relations (Eq. A.10) apply but with terms set to zero because
of the independence with respect to $\theta$. These simplified strain displacement relations are

$$
\varepsilon_r = \frac{\partial U_r}{\partial r}, \quad \varepsilon_\theta = \frac{U_r}{r}, \quad \varepsilon_z = \frac{\partial U_z}{\partial z}
$$  \tag{A.19}

Similarly the stresses simplify using Hooke's law

$$
p_r = \lambda \Delta + 2\mu \varepsilon_r, \quad p_\theta = \lambda \Delta + 2\mu \varepsilon_\theta, \quad p_z = \lambda \Delta + 2\mu \varepsilon_z
$$  \tag{A.20}

where $\Delta$ is a dilatation (Eq. A.12).

### Lamé's Theorem

A brief outline of Lamé's theorem for this axisymmetric case follows. First the equations of motion are developed after White (1983). Referring to Fig. A-1, stresses and resulting forces are determined over a cylindrical element and are summed in both the radial and vertical direction. The forces are derived by summing the oppositely directed stresses multiplied by the area of application and dividing by the volume of the cylindrical element. Areas of application for each stress are indicated by a hashed pattern in Fig. A-2. Because of axisymmetry there are no net $\theta$ forces but $\theta$ components do contribute to the radial force (White, 1983). Elongation in the $r$ and $z$ direction indicated by a dashed outline contributes an extra term to the sum over $p_{r\theta}$ in the radial direction and $p_z$ in the vertical. The sum of these forces is equal to mass times acceleration yielding (pg. 162-164, White, 1983)

$$
\frac{\partial p_r}{\partial r} + \frac{p_r - p_\theta}{r} + \frac{\partial p_{r\theta}}{\partial z} = \rho \frac{\partial^2 U_r}{\partial t^2}
$$  \tag{A.21}

$$
\frac{\partial p_{r\theta}}{\partial r} + \frac{p_{r\theta}}{r} + \frac{\partial p_z}{\partial z} = \rho \frac{\partial^2 U_z}{\partial t^2}
$$  \tag{A.22}

192
Proceeding with Lamé’s theorem, the potential representations are calculated for Eq. A.21 and Eq. A.22. By examining the displacement potential relations one can see that there are no interdependencies, i.e. terms in \( \phi \) are independent of terms of \( \psi \) so each equation can be solved in term of \( \phi \) and then \( \psi \).

Substituting the following stress potential relations into Eq. A.21

\[
\frac{\partial p_r}{\partial r} = (\lambda + 2\mu) \frac{\partial^3 \phi}{\partial r^3} + \lambda \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \phi}{\partial r} \right) + \lambda \frac{\partial^3 \phi}{\partial r \partial z^2} \quad (A.23)
\]

\[
\frac{r_r - p_\theta}{r} = 2\mu \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \phi}{\partial r} \right)
\]

\[
\frac{\partial p_{rz}}{\partial z} = 2\mu \frac{\partial^3 \phi}{\partial r \partial z^2}
\]

and then summing

\[
(\lambda + 2\mu) \left( \frac{\partial^3 \phi}{\partial r^3} + \frac{\partial^3 \phi}{\partial r \partial z^2} + \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \phi}{\partial r} \right) \right) = \rho \frac{\partial^3 \phi}{\partial r \partial t^2} \quad (A.24)
\]

and factoring out a \( \frac{\partial}{\partial r} \), the wave equation independent of \( \theta \) is derived

\[
\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial^2 \phi}{\partial z^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} = \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2} \quad (A.25)
\]

Performing the same procedure for the transformed \( \psi \) yields

\[
\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} - \frac{\psi}{r^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2} \quad (A.26)
\]

These new equations of motion are still wave equations as in Eq. A.15 but because \( \psi \) has been transformed the left-hand side cannot be considered the Laplacian for the shear potential. Using these wave equations and carrying out some algebra, \( p_r, p_{rz} \) can be rewritten

\[
p_r = \rho \frac{\partial^2 \phi}{\partial t^2} - 2\mu \left( \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial r \partial z} \right) \quad (A.27)
\]

\[
p_{rz} = \rho \frac{\partial^2 \psi}{\partial t^2} - 2\mu \left( \frac{\partial^2 \psi}{\partial z^2} - \frac{\partial^2 \phi}{\partial r \partial z} \right)
\]
A quick exercise will show that the Laplacian from Eq. A.26 can be recovered by substituting in the original transformation, \( \psi = -\frac{\partial \psi}{\partial r} \). In doing so, the equation
\[
- \frac{\partial}{\partial r} \frac{\partial^2 \psi}{\partial r^2} - \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) - \frac{\partial}{\partial r} \left( \frac{\partial^2 \psi}{\partial z^2} \right) = -\frac{\partial}{\partial r} \left( \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2} \right)
\] (A.28)
is obtained. Dividing out the common factor of \( \frac{\partial}{\partial r} \) from both sides yields the wave equation
\[
\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2}
\] (A.29)
with the left-hand side recognized as a Laplacian operating on \( \psi \).

### A.2.3 Axisymmetry — no transformation of \( \psi \)

The development in this symmetry system is identical to the previous axisymmetric development except \( \psi \) is not transformed. This symmetry strategy is used more often in applied physics and mechanics literature than geophysics literature. However, some very important references in the geophysics literature use this strategy Heelan (1952, 1953a,b), Brekhovskikh (1960, 1980) after Heelan, and Ewing et al. (1957). Again the \( SH \) problem can be considered independently. \( U_\theta, \chi \), and derivatives with respect to \( \theta \) are set to zero (Miklowitz, 1978). Rewriting Eq. A.16.

\[
\vec{K} = \hat{\epsilon}_\theta \left( -\frac{\partial \psi}{\partial r} \right)
\] (A.30)

\[
\nabla \times \vec{K} = \hat{\epsilon}_r \left( \frac{\partial^2 \psi}{\partial r \partial z} \right) - \hat{\epsilon}_z \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) \right)
\]

As with the previous axisymmetric treatment, the gauge condition \( \nabla \cdot \vec{K} = 0 \) is now automatically satisfied since \( \frac{\partial \chi}{\partial z} = 0 \) identically for axisymmetry.

Summing \( \nabla \times \vec{K} \) and adding it to \( \nabla \phi \) using the Helmholtz representation as before, the displacement potential relations are

\[
U_r = \frac{\partial \phi}{\partial r} + \frac{\partial^2 \psi}{\partial r \partial z}
\] (A.31)

\[
U_z = \frac{\partial \phi}{\partial z} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right)
\]
The strain-displacement relations (Eq. A.19) and Hooke’s law relationships (Eq. A.20) are equivalent to those for the previous strategy so will not be repeated although it should be recognized that the displacement elements $U_r, U_z$ are slightly reformulated which will affect the boundary conditions, the wave equations and the form of the governing eigenfunctions.

The Hooke’s law relationships (Eq. A.20) remain unchanged from the previous strategy and thus are not repeated here. Because $\psi$ has not been recorded here, the simplified wave equation relationships have been maintained for $\phi, \psi$ (Eq. A.15) with the Laplacian operator.

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2}
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2}
\]

### A.2.4 Axisymmetry – torsional motion

A second type of axisymmetry is to examine the torsional source problem independently (pg. 217-220, Miklowitz, 1978; White, 1983).

$\vec{K}$ and the curl of $\vec{K}$ can be rewritten

\[
\vec{K} = \hat{e}_z(\chi)
\]

\[
\nabla \times \vec{K} = -\hat{e}_\theta \frac{\partial \chi}{\partial r}
\]

The gauge condition $\nabla \cdot \vec{K} = 0$ is not necessarily satisfied here since $\nabla \cdot \vec{K} = \frac{\partial \chi}{\partial z}$. But $SH$ motion can be solved for independently because $\phi, \psi$ equal zero identically. By utilizing the Helmholtz representation, azimuthal displacement and stresses are obtained

\[
U_\theta = -\frac{\partial \chi}{\partial r}
\]

\[
\epsilon_{r\theta} = -\frac{U_\theta}{r} + \frac{\partial U_\theta}{\partial r}, \quad \epsilon_{\theta z} = \frac{\partial U_\theta}{\partial z}
\]

\[
p_{r\theta} = \mu \epsilon_{r\theta}, \quad p_{\theta z} = \mu \epsilon_{\theta z}
\]

195
with the remaining stress and strain elements equalling zero.

The shear displacement potential $\chi$ satisfies the wave equation (Eq. A.15)

$$\frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r} \frac{\partial \chi}{\partial r} + \frac{\partial^2 \chi}{\partial z^2} = \frac{1}{\beta^2} \frac{\partial^2 \chi}{\partial t^2}$$

(A.36)

### A.2.5 No Axisymmetry – symmetry in $z$

This strategy has been adopted by Sezawa (1927), Viktorov (1958), Siggins and Stokes (1987), and Siggins (1989) to calculate wave propagation emanating from line sources parallel to the $z$ axis. Because of the symmetry in $z$, derivatives with respect to $z$ must vanish which implies $\psi$ equals zero by taking the curl of $\vec{K}$. One point to note is that Viktorov, Siggins and Stokes, and Siggins use the potential $\psi$ for $\chi$ but the $\chi$ symbolism will be used here in agreement with previous definitions.

$\vec{K}$ and the curl of $\vec{K}$ are rewritten

$$\vec{K} = \hat{e}_r(\chi)$$

$$\nabla \times \vec{K} = \hat{e}_r \left( \frac{1}{r} \frac{\partial \chi}{\partial \theta} \right) - \hat{e}_\theta \left( \frac{\partial \chi}{\partial r} \right)$$

(A.37)

The gauge condition is not necessarily satisfied here since $\nabla \cdot \vec{K} = \frac{\partial \psi}{\partial z}$. Therefore $P$ and $SH$ motions are coupled through the boundary conditions. The displacement potential relations using the Helmholtz representation (Eq. A.5) are

$$U_r = \frac{\partial \phi}{\partial r} + \frac{1}{r} \frac{\partial \chi}{\partial \theta}$$

$$U_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} - \frac{\partial \chi}{\partial r}$$

(A.38)

and the symmetry between $U_r$ and $U_\theta$ is readily apparent. Applying the strain-displacement relations (Eq. A.10) yields

$$\varepsilon_r = \frac{\partial U_r}{\partial r} \quad \varepsilon_\theta = \frac{1}{r} \left( \frac{\partial U_\theta}{\partial \theta} + U_r \right) \quad \varepsilon_{r\theta} = \frac{1}{r} \frac{\partial U_r}{\partial \theta} + r \frac{\partial}{\partial r} \left( \frac{U_\theta}{r} \right)$$

(A.39)

$\phi$ and $\chi$ satisfy the wave equations (Eq. A.15) with dependence on $z$ removed

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2}$$

(A.40)
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \chi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} = \frac{1}{\beta^2} \frac{\partial^2 \chi}{\partial t^2}
\]

which completes the development of the wave equation relationships for the five different symmetry systems.

### A.3 Transforming the resultant wave equations using separation of variables

#### A.3.1 Separation of variables - no symmetry assumptions

When no symmetry assumptions were made (Section A.2.1) scalar wave equations (Eq. A.15) of the form

\[
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2} \tag{A.41}
\]

\[
\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2}
\]

\[
\frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r} \frac{\partial \chi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} + \frac{\partial^2 \chi}{\partial z^2} = \frac{1}{\beta^2} \frac{\partial^2 \chi}{\partial t^2}
\]

were obtained where the left-hand side is the Laplacian (Eq. A.4) of each potential.

It is well known that scalar wave equations in cylindrical coordinates are separable with Bessel and exponential eigenfunctions (e.g., Ben-Menahem and Singh, 1981). Proposed solutions are of the form

\[
\phi(r, \theta, z, t) = R(r)\Theta(\theta)Z(z)T(t) \tag{A.42}
\]

and similarly for \( \psi, \chi \).

The general procedure for separation of variables is presented in texts on differential equations. The exercise will be performed here for the potential \( \phi \) only - the procedure for \( \psi, \chi \) being identical. Substituting this proposed form for \( \phi \) (Eq. A.42)
into the first equation of Eq. A.41 and dividing by $\phi$ produces five quotients which each contain only one dependent variable. The quotient $\frac{R''}{R}$ for instance is dependent on $r$ and not on $z, \theta, t$ etc. For brevity's sake, the functional dependencies have been omitted in the transformation of the wave equations presented below

$$\alpha^2 \left[ \frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{\Theta''}{\Theta} + \frac{Z''}{Z} \right] = \frac{T''}{T}$$  \hspace{1cm} (A.43)

By setting the right-hand side of this equation equal to $-\omega^2$, which is a separation constant (e.g. Mathews and Walker, 1970; Ben-Menahem and Singh, 1981), an ordinary differential equation with exponential solution is obtained

$$\frac{T''}{T} = -\omega^2$$  \hspace{1cm} (A.44)
$$T(t) = e^{\pm i\omega t}$$

where the sign of $\omega$ is either plus or minus. The $e^{\pm i\omega t}$ is commonly referred to as the time dependence. Both signs are commonly used in the geophysical literature.

The next step will be to calculate radial wavenumbers which are square root functions, the difference in phase of the square root will be a factor of $i$ if the time dependence is negative instead of positive. This difference in phase allows the substitution of modified Bessel functions for Bessel functions and vice versa since the modified Bessel Functions are fundamentally Bessel functions of imaginary argument.

Rewriting Eq. A.43 after some rearrangement yields

$$\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{\Theta''}{\Theta} + \frac{\omega^2}{\alpha^2} = -\frac{Z''}{Z}$$  \hspace{1cm} (A.45)

Again setting the right-hand side equal to a separation constant, $k_z^2$, gives

$$\frac{Z''}{Z} = -k_z^2$$  \hspace{1cm} (A.46)
$$Z(z) = e^{\pm ik_z z}$$

The convention concerning the signs of the exponentials in the transform is developed as follows. In cartesian coordinates, well known solutions for the wave equations
in one dimension are of the form

\[ F(t - \frac{x}{v}) + G(t + \frac{x}{v}) \]  \hspace{1cm} (A.47)

where \( F \) represents a wave travelling in the \( x \) direction and \( G \) represents a wave travelling in the \(-x\) direction with velocity \( v \) where a proper combination of \( F \) and \( G \) equals a standing wave. In cylindrical coordinates solutions are of the form

\[ \frac{F(t - \frac{R}{v})}{R} + \frac{G(t + \frac{R}{v})}{R} \]  \hspace{1cm} (A.48)

Particular solutions to Eq. A.48 are the following

\[ \frac{e^{i\omega(t-\frac{R}{v})}}{R}, \frac{e^{i\omega(t+\frac{R}{v})}}{R} \]  \hspace{1cm} (A.49)

\[ \frac{e^{-i\omega(t+\frac{R}{v})}}{R}, \frac{e^{-i\omega(t-\frac{R}{v})}}{R} \]

An example follows for the choice of signs for the \( K_0 \) eigenfunctions. Using a relationship from Gradshteyn and Ryzhik (Eq. 6.677.5, 1980) allows the writing of the Hertzian oscillator

\[ \frac{e^{-i\omega \frac{R}{v}}}{R} = \frac{1}{\pi} \int_{-\infty}^{\infty} K_0(\sqrt{k_z^2 - \frac{\omega^2}{v^2} R'}) e^{ik_z z} dk_z \]  \hspace{1cm} (A.50)

where \( R \) is the distance to a point and \( R' \) is the distance projected into the \( r\theta \) plane. Eq. A.50 uses the positive exponential, \( e^{ik_z z} \). Therefore a mapping of the particular solutions to the wavenumber sign conventions for \( K_0 \) eigenfunctions can be accomplished as follows

\[ \frac{e^{i\omega(t-\frac{R}{v})}}{R} \rightarrow e^{i\omega t} e^{ik_z z} \]  \hspace{1cm} (A.51)

\[ \frac{e^{-i\omega(t+\frac{R}{v})}}{R} \rightarrow e^{-i\omega t} e^{ik_z z} \]

In the far field literature, (i.e. Heelan (1952, 1953a,b), Brekhovskikh (1960,1980), Lee and Balch, (1982)), the first and the last representations are used so that functions of \( f(t - \frac{R}{v}) \), in other words, causal functions are yielded. In the near field literature either
particular solution is desired because the results after Fourier transformation are presented in real time. Therefore, in the near field literature the latter representation is used.

Rearranging Eq. A.45 and multiplying both sides by $r^2$

$$r^2 \left[ \frac{R''}{R} + \frac{1}{r} \frac{R'}{R} - k_z^2 + \frac{\omega^2}{\alpha^2} \right] = -\frac{\Theta''}{\Theta}$$ (A.52)

The left-hand side is again independent of $\theta$ so setting the right-hand side equal to the separation constant $n^2$ gives

$$\frac{\Theta''}{\Theta} = -n^2$$

$$\Theta(\theta) = e^{in\theta}$$ (A.53)

The sign of $n$ is usually positive by convention. By Euler's formula, the term $e^{in\theta}$ can be decomposed into a sum of sines and cosines. For the most general solution of the potential, an infinite number of these terms will be summed and for the terms to remain single valued in the variable $\theta$ they must replicate with a period of $2\pi$. For this to happen, $n$ must be an integer (Lee, 1986).

Dividing both sides by $r^2$ of Eq. A.52 and multiplying through by $R$

$$R'' + \frac{1}{r} R' + \left( -k_z^2 + \frac{\omega^2}{\alpha^2} - \frac{n^2}{r^2} \right) R = 0$$ (A.54)

This is a Bessel's equation of order $n$ (Bowman, 1958). Solutions are Bessel Functions of the first, $J_{\pm n}(lr)$, second, $Y_n(lr)$ and third kind, $H_n^{(1)}(lr)$, $H_n^{(2)}(lr)$ (Abramowitz and Stegun, 1964, Eq. 9.1.1) of order $n$ in which Bessel functions of the third kind are known as Hankel functions. $l$ is a radial wavenumber equal to $\sqrt{\frac{\omega^2}{\alpha^2} - k_z^2}$.

The modified Bessel Functions $I_n(z)$, $K_n(z)$ are fundamentally Bessel functions of imaginary argument. Therefore $I_n(\text{i}lr)$, $K_n(\text{i}lr)$ are also solutions. By putting $l' = il$ where $l' = \sqrt{k_z^2 - \frac{\omega^2}{\alpha^2}}$, the solutions can be written $I_n(l'r)$ and $K_n(l'r)$. For the potentials $\psi, \chi$ solutions are of the same general form but with new radial wavenumber $m$ and $m'$ which equal $\sqrt{\frac{\omega^2}{\beta^2} - k_z^2}$ and $\sqrt{k_z^2 - \frac{\omega^2}{\beta^2}}$ respectively.
Although the Bessel functions and modified Bessel functions each solve Bessel’s equation of order \( n \), only in combinations do they span the whole space of possible solutions. General solutions for \( R_n \) are any of the following

\[
R_n = A_nJ_n(lr) + B_nY_n(lr) \quad (A.55)
\]
\[
R_n = A_nH_n^{(1)}(lr) + B_nH_n^{(2)}(lr) \quad (A.56)
\]
\[
R_n = A_nK_n(l'r) + B_nI_n(l'r) \quad (A.57)
\]

where \( A_n, B_n \) are coefficients and functions of \( k, \omega, \) and \( \theta \). The superscript on \( n \) designates the type of Bessel function used in constructing the general solution. The first two general solutions (Eq. A.55, Eq. A.56) can easily be seen to be equivalent by recalling a basic property of Bessel functions (Abramowitz and Stegun, 1964)

\[
H_n^{(1)}(z) = J_n(z) + iY_n(z) \quad (A.58)
\]
\[
H_n^{(2)}(z) = J_n(z) - iY_n(z)
\]

which allows the rewriting of the Hankel functions in terms of Bessel functions.

All solutions of \( \phi \) are superposed which can be accomplished by integrating over all \( k_z \), all of \( \omega \), and integer values of \( n \). Hence solutions for \( \phi \) are any of the following

\[
\phi = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} (A_nJ_n(lr) + B_nY_n(lr))e^{ik_zz}e^{-i\omega t}e^{i\theta}dk_z d\omega \quad (A.59)
\]
\[
\phi = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} (A_nH_n^{(1)}(lr) + B_nH_n^{(2)}(lr))e^{ik_zz}e^{-i\omega t}e^{i\theta}dk_z d\omega
\]
\[
\phi = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} (A_nK_n(l'r) + B_nI_n(l'r))e^{ik_zz}e^{-i\omega t}e^{i\theta}dk_z d\omega
\]

\( \psi \) and \( \chi \) are similarly represented but with wavenumbers \( m, m' \) instead of \( l, l' \), for instance

\[
\psi = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} (C_nK_n(m'r) + D_nI_n(m'r))e^{ik_zz}e^{-i\omega t}e^{i\theta}dk_z d\omega \quad (A.60)
\]
\[
\chi = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} (E_nK_n(m'r) + F_nI_n(m'r))e^{ik_zz}e^{-i\omega t}e^{i\theta}dk_z d\omega
\]

201
where for the sake of brevity only those solutions in terms of the modified Bessel functions are reproduced for \( \psi \) and \( \chi \). This pattern of designating coefficients \( A, B \) for the irrotational potential \( \phi \), \( C, D \) for the solenoidal potential \( \psi \), and \( E, F \) for the solenoidal potential \( \chi \) will be adhered to whenever possible throughout this thesis.

The choice of the factor \( \frac{1}{(2\pi)^2} \) in Eq. A.59 and Eq. A.60 is an important issue. Fundamentally, there is latitude in the choice of normalizing factor in front of Fourier transform (e.g. Bracewell, 1978), whether it be \( \frac{1}{(2\pi)^2} \) or \( \frac{1}{2\pi} \) or 1 is irrelevant. The choice of these constants is irrelevant because they will divide out in subsequent analysis. However, these constants must be carried through the analysis and the use of different conventions by different authors must be recognized.

### A.3.2 Use of Sommerfeld radiation conditions to reduce the dimensionality of the general integrals

The Sommerfeld radiation condition states that solutions are not allowed to diverge in an outward direction. This means that the governing asymptotic behavior which is exponential cannot be real and positive for outgoing waves. The asymptotic behavior of solutions of Bessel’s equations of order \( n \) as \( z \) goes to infinity is the following to the first term with some subsequent simplification. (Abramowitz and Stegun, 1964)

\[
J_n(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right) \quad (A.61)
\]

\[
Y_n(z) \sim \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right) \quad (A.62)
\]

\[
H_n^{(1)}(z) \sim \frac{\pi}{2z} e^{i\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)} \quad (A.63)
\]

\[
H_n^{(2)}(z) \sim \frac{\pi}{2z} e^{-i\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)} \quad (A.64)
\]

\[
I_n(z) \sim \frac{1}{\sqrt{2\pi z}} e^z \quad (A.65)
\]

\[
K_n(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \quad (A.66)
\]

The asymptotic behavior is governed by the real part of the exponential function,
the imaginary part only contributing to oscillation. The asymptotic behavior of $J_n$ (Eq. A.61) and $Y_n$ (Eq. A.62) are very similar and by referring to Fig. A-3 it can be seen that they sinusoidally decay exponentially to infinity. Additionally, $J_n$ is finite at 0 and $Y_n$ goes to $-\infty$ as $z$ goes to zero. Both $J_n, Y_n$ diverge if the imaginary part of $z$ goes to infinity.

The argument of the Bessel functions is $lr$ a function of wavenumber times radius - with $r$ being real and $l = \sqrt{\omega^2 - k_z^2}$. This argument diverges to imaginary infinity when $k_z^2 >> \frac{\omega^2}{c^2}$. What is physically desirable is to have a pair of wave functions that have opposite asymptotic behavior so that one solution will decay to infinity and one which will remain finite at the origin. It has been shown that $J_n, Y_n$ do not satisfy this criteria, but the Hankel functions and modified Bessel Functions do, however. For this reason, they are found much more frequently as general solutions in wave propagation literature.

The advantage of the oppositely asymptotically behaving wave duality is that the coefficient on the term that asymptotically increases can be set to zero in the outermost layer. This duality saves considerable computational and algebraic manipulation.

The asymptotic behavior of the modified Bessel functions $K_n, I_n$ is seen for real argument in Fig. A-4. $K_n$ decays to zero at infinity whereas $I_n$ is finite at the origin. It is difficult to plot Hankel functions for real arguments $H_n^{(1)}, H_n^{(2)}$ (e.g. Abramowitz and Stegun, 1964) as was done with $J_n, Y_n$ and $K_n, I_n$ but the behavior can be seen by substituting an $iz$ for $z$ in the asymptotic relations (Eq. A.61). In so doing, it is seen that the $H_n^{(2)}$ resembles $K_n$ and governs the behavior in the outermost layer.

The relations just developed are valid off the axis where $r > 0$. This condition is met for media surrounding a cylinder which is the usual case treated. Inside the cylinder it is required to avoid the singularities in the Bessel and modified Bessel functions at $r = 0$. For instance when Hankel functions $H_n^{(1)}, H_n^{(2)}$ are used for the general solution, there is a potential singularity at zero since $Y_n(r)$, a component of both Hankel functions (Eq. A.59) approaches negative infinity as $r$ approaches...
0. Therefore, inside the borehole, it is common to set the coefficient on the Hankel functions \( H_n^{(1)} \) and \( H_n^{(2)} \) equivalent using Eq. A.59 to yield the Bessel function \( J_n \), which has finite values throughout as an incoming wave and \( H_n^{(2)} \) as an outgoing wave. (Tsang and Rader (1979a), Cheng and Toksöz (1981), Lee and Balch (1982)).

### A.3.3 Separation of variables - transformation of \( \psi \)

When addressing the separation of variables method with axisymmetry and the transformed potential \( \psi \) considered in (Section A.2.2) important simplifications immediately arise. These simplifications arise because the dependence on \( \theta \) is removed and therefore the infinite sum of Bessel functions over \( n \) is removed in the integrals of Eq. A.59. Complications arise because of the introduction of a new form of wave equation in terms of \( \psi \). These wave equations are

\[
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2} \tag{A.67}
\]

\[
\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} - \frac{\psi}{r^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2}
\]

Proposed solutions are of the form

\[
\phi(r, z, t) = R_\phi(r)Z(z)T(t) \tag{A.68}
\]

\[
\psi(r, z, t) = R_\psi(r)Z(z)T(t)
\]

As can be seen by examining Eq. A.67, the two solutions for \( R_\phi, R_\psi \) will not be equivalent. The solution for \( \phi \) is

\[
R_\phi''(r) + \frac{1}{r} R_\phi'(r) + \left(-\frac{k^2}{\alpha^2} + \frac{\omega^2}{\alpha^2}\right) R = 0 \tag{A.69}
\]

and for \( \psi \) is

\[
R_\psi''(r) + \frac{1}{r} R_\psi'(r) + \left(-\frac{k^2}{\beta^2} + \frac{\omega^2}{\beta^2} - \frac{1}{r^2}\right) R = 0 \tag{A.70}
\]

204
The equation for $R_\phi$ is seen to be a Bessel’s equation of order 0 and similarly the solution for $R_\psi$ a Bessel’s equation of order one. Solutions for $R_\phi, R_\psi$ can be represented in the following forms using Bessel, modified Bessel and Hankel Functions.

\[ R_\phi(r) = AJ_0(lr) + BY_0(lr) \quad \quad (A.71) \]
\[ R_\phi(r) = AH_0^{(1)}(lr) + BH_0^{(2)}(lr) \]
\[ R_\phi(r) = AK_0(l'r) + BI_0(l'r) \]
\[ R_\phi(r) = CJ_1(lr) + DY_1(lr) \]
\[ R_\psi(r) = CH_1^{(1)}(mr) + DH_1^{(2)}(mr) \]
\[ R_\psi(r) = CK_1(m'r) + DI_1(m'r) \]

Again utilizing the Sommerfeld radiation condition, it is easy to recognize $K_0, K_1, H_0^{(2)}, H_1^{(2)}$ as outgoing waves and $I_0, I_1, H_0^{(1)}, H_1^{(1)}$ as incoming waves. The use of the modified Bessel functions $K$ and $I$ for axisymmetry with the transformed $\psi$ is made by Biot (1952), White (1983), Cheng et al. (1982) and many others and is the most common for describing wave propagation inside an axisymmetric borehole. The use of Hankel functions is made by White (1965). Both systems are used by Cheng and Toksöz (1981) and Tsang and Rader (1979) use both Hankel functions and Bessel functions.

Writing the resulting integrals which are recognized as a double Fourier transform in $k_z$ and $\omega$ results in

\[ \phi = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} (AK_0(l'r) + BI_0(l'r))e^{i k_z z} e^{-i \omega t} dk_z d\omega \quad \quad (A.72) \]
\[ \phi = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} (AH_0^{(1)}(lr) + BH_0^{(2)}(lr))e^{i k_z z} e^{-i \omega t} dk_z d\omega \]
\[ \psi = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} (CK_1(l'r) + DI_1(l'r))e^{i k_z z} e^{-i \omega t} dk_z d\omega \]
\[ \psi = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} (CH_1^{(1)}(lr) + DH_1^{(2)}(lr))e^{i k_z z} e^{-i \omega t} dk_z d\omega \]
A.3.4 Separation of variables, axisymmetric media, no reCoding of ψ

Again, because of the axisymmetry, dependence on θ is removed. From Section A.2.3, the wave equations

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2}$$  \hspace{1cm} (A.73)

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2}$$

and proposed solutions are of the form

$$\{ \phi, \psi \} = R_{\phi, \psi}(r)Z(z)T(t)$$  \hspace{1cm} (A.74)

where a Bessel’s equation of order zero is yielded

$$R''_{\phi, \psi} + \frac{1}{r} R'_{\phi, \psi} + \left( -k_z^2 + \frac{\omega^2}{\{\alpha, \beta\}^2} \right) R_{\phi, \psi} = 0$$  \hspace{1cm} (A.75)

Rewriting the integrals (Eq. A.59, Eq. A.60) and restricting the eigenfunctions to be modified Bessel and Hankel functions as before yields

$$\phi = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} (AK_0(l'r) + BI_0(l'r)) e^{ik_z^2} e^{-i\omega t} dk_z \, d\omega$$  \hspace{1cm} (A.76)

$$\psi = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} (CH_0^{(1)}(mr) + DI_0(m'r)) e^{ik_z^2} e^{-i\omega t} dk_z \, d\omega$$

which are again recognized as double Fourier transforms and the superposition of incoming and outgoing waves.
A.3.5 Separation of variables, axisymmetry, torsional motion

From Section A.2.4, wave equations of the form

\[ \frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r} \frac{\partial \chi}{\partial r} + \frac{\partial^2 \chi}{\partial z^2} = \frac{1}{\beta^2} \frac{\partial^2 \chi}{\partial t^2} \]  \hspace{1cm} (A.77)

were generated and solutions of the form

\[ \chi = R_\chi(r)Z(z)T(t) \]  \hspace{1cm} (A.78)

are desired where \( R_\chi \) solves the following Bessel’s equation of order zero.

\[ R''_\chi + \frac{1}{r} R'_\chi + \left( -k_z^2 + \frac{\omega^2}{\beta^2} \right) R_\chi = 0 \]  \hspace{1cm} (A.79)

Rewriting the integrals (Eq. A.59) as superpositions of incoming and outgoing waves yields

\[ \chi = \frac{1}{(2\pi)^2} \int \int_{-\infty}^{\infty} (EK_0(m'r) + FI_0(m'r))e^{ik_z z}e^{-i\omega t}dk_z d\omega \]  \hspace{1cm} (A.80)

\[ \chi = \frac{1}{(2\pi)^2} \int \int_{-\infty}^{\infty} (EH_0^{(1)}(mr) + FH_0^{(2)}(mr))e^{ik_z z}e^{-i\omega t}dk_z d\omega \]

A.3.6 Separation of variables, symmetry in z

For the case of symmetry along the z axis as presented in Viktorov (1958), Siggins and Stokes (1987), Siggins (1989) as presented in Section A.2.5, wave equations of the form

\[ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2} \]  \hspace{1cm} (A.81)

\[ \frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r} \frac{\partial \chi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \chi}{\partial \theta^2} = \frac{1}{\beta^2} \frac{\partial^2 \chi}{\partial t^2} \]

where proposed solutions were of the form

\[ \{ \phi, \chi \} = R_{\phi,\chi}(r)Z(z)T(t) \]  \hspace{1cm} (A.82)
$R_{\phi,x}$ solves the following Bessel's equation of order $n$.

$$R''_{\phi,x} + \frac{1}{r} R'_{\phi,x} + \left(-k_z^2 + \frac{\omega^2}{\{\alpha, \beta\}^2} - \frac{n^2}{r^2}\right) R_{\phi,x} = 0 \quad (A.83)$$

Rewriting the integrals (Eq. A.59) as superpositions of incoming and outgoing waves

$$\phi = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} (\text{AK}_n(l'r) + \text{BI}_n(l'r)) e^{-i\omega t} e^{i\theta n} d\omega$$

$$\phi = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} (\text{AH}_n^{(1)}(l') + \text{BH}_n^{(2)}(l')) e^{-i\omega t} e^{i\theta n} d\omega$$

$$\chi = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} (\text{EK}_n(m'r) + \text{FI}_n(m'r)) e^{-i\omega t} e^{i\theta n} d\omega$$

$$\chi = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} (\text{EH}_n^{(1)}(mr) + \text{FH}_n^{(2)}(mr)) e^{-i\omega t} e^{i\theta n} d\omega$$

In formulating the propagation independent of $z$, one of the integrals over minus infinity to plus infinity has been eliminated at the expense of requiring an infinite summation over $\theta$ or devising a strategy to truncate the sum over $\theta$ at small values of $n$.

**A.4 Conclusions**

The description of the interrelationships between the wave equation, displacement potential, sign conventions, eigenfunction conventions and separation of variable results for the different symmetry systems was addressed in this appendix. The precedent of literature dictates that at times any one of these conventions may be used. Thus this appendix provides background useful in understanding the text of the thesis.
Cylindrical Element

Figure A-1: Elementary cylindrical element. Elongation of the chord in the radial direction is shown.
Figure A-2: This figure shows the stresses and forces for axisymmetric wave propagation over an elementary cylindrical element. Stresses are indicated by arrows. Resulting forces indicate below each element. Dashed outline on uppermost left and lowermost right elements corresponds to elongation leading to an extra force term.
Figure A-3: Zeroth and first order Bessel Functions of the first and second kind for real argument. Computed from a program in Press et al. (1986). The behavior of both $J_n$ and $Y_n$ mimic a sinusoidal decaying exponential.
Figure A-4: After Abramowitz and Stegun (1964). Behavior of modified Bessel functions of orders zero and one for real argument. The exponentially decreasing nature of $K_{0,1}$ and the exponentially increasing nature of $I_{0,1}$ is clearly seen.
Appendix B

Mathematics and physics of radiation from empty boreholes

The purpose of this appendix is to verify and explain the mathematics and physics of Heelan (1952, 1953a) used in describing radiation from empty boreholes. Because there has been extensive criticism of Heelan’s work (Jordan, 1962; Abo-Zena, 1977) this explanation will go into great detail but will not exclusively rely on the approach developed by Heelan. For instance, the mathematics and physics of a parallel treatment of the radiation problem will be developed - the parallel treatment having been initiated by Brekhovskikh (1960,1980) and completed in Appendix C. This parallel treatment is necessary because the contour Heelan used in performing contour integration is unknown and consequently Heelan’s contour analysis could not be verified. However, by extending Brekhovskikh’s parallel treatment in this appendix which is a much more well known type of contour analysis, the analytic approximations culminate in integration by the method of stationary phase. The integration is presented in Appendix C and is similar to the treatment presented in Chapter 3 except use is made of the Weyl integral instead of the Sommerfeld integral.
B.1 Heelan’s Analysis

In Heelan’s model, a stress is applied to a finite length \(2l\) of an empty borehole which is surrounded by an infinite elastic medium (See Fig. 2-2). The stress is axisymmetrically applied to the boundary between the borehole and the infinite medium and centered on the origin. Since the empty borehole represents a free surface there are no displacement boundary conditions but instead boundary conditions are functions of normal, azimuthal, and tangential stress.

With the axisymmetric radial, axial and torsional sources described by Heelan, the \(P - SV\) and \(SH\) problems can be solved independently. For the \(SH\) case, the boundary condition is the vanishing of azimuthal stress and for \(P-SV\) the vanishing of tangential and normal stress.

Because axisymmetry was assumed, there is no dependence on the \(\theta\) component and no summation over \(\theta\). Therefore, Heelan writes the displacement potentials \(\phi, \psi, \chi\) with integral transforms of the form

\[
\phi = \text{Re} \int_0^\infty e^{ikVt} dk \int_C f_0(\sigma, k) H_0^{(1)}(\sigma r) e^{\sigma \sqrt{\sigma^2 - k^2}} d\sigma \\
\psi = \text{Re} \int_0^\infty e^{ikVt} dk \int_C g_0(\sigma, k) H_0^{(1)}(\sigma r) e^{\sigma \sqrt{\sigma^2 - h^2}} d\sigma \\
\chi = \text{Re} \int_0^\infty e^{ikVt} dk \int_C n_0(\sigma, k) H_0^{(1)}(\sigma r) e^{\sigma \sqrt{\sigma^2 - h^2}} d\sigma
\]

The origin of these displacement potentials will be described subsequently. In Heelan’s notation, \(V\) and \(v\) are the compressional and shear wave velocities, \(k, h\) are wavenumbers and \(kV = hv\). \(\sigma\) is an axial wavenumber, \(\sqrt{\sigma^2 - k^2}\) and \(\sqrt{\sigma^2 - h^2}\) are the radial wavenumbers, \(e^{ikVt}\) is the positive time dependence, and \(r\) is the radius of investigation. It is useful to recognize that \(kV, hv\) has the dimension of frequency.

In calculating the \(P-SV\) problem Heelan used displacement potentials \(\phi\) and \(\psi\). But unlike the most common treatments for axisymmetric problems (Biot, 1952; White, 1965, 1983) Heelan does not transform \(\psi\) and thus Heelan’s work uses symmetry strategy 3 of Appendix A of this thesis. Likewise Heelan’s analysis for the \(SH\) case and the use of the potential \(\chi\) corresponds to symmetry strategy 4 of Appendix A.
A departure from conventions in describing source radiation for borehole geophysics purposes was that the integrals of Eq. B.1 represent a Hankel function of axial wavenumber, $\sigma$, multiplied by $r$ instead of radial wavenumber multiplied by $r$, the common convention. This was due to a reversal of the final two steps in the separation of variables procedure Heelan used. However, in performing the separation of variables procedure in this manner, Heelan’s results are governed by the Sommerfeld integral and thus a superposition of cylindrical waves which will be shown later.

### B.1.1 Separation of variables procedure

In Appendix A, it was shown that a common method to separate variables for this type of problem is to first separate in $t$ yielding a time function $e^{iwt}$ ($e^{ikvt}$), then in $z$ yielding a depth function $e^{ikz}$ ($e^{i\sigma z}$) and finally in $r$ yielding a modified Bessel or Hankel function of radial wavenumber times radius $H_0^{(1)}(lr)$, $H_0^{(1)}(mr)$, where $l$ and $m$ equal $\sqrt{k_z^2 - \omega^2/\alpha^2}$, $\sqrt{k_z^2 - \omega^2/\beta^2}$ respectively or in Heelan’s notation $\sqrt{\sigma^2 - k_z^2}$, $\sqrt{\sigma^2 - \gamma^2}$.

The separation of variables procedure used by Heelan (Heelan did not specifically address this issue in either the thesis or papers) reversed the final two steps and put the radial wavenumber under the exponent of $z$. To wit, having separated the variables over frequency Eq. 2.46 can be rewritten

$$\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} + k^2 + \frac{Z''}{Z} = 0$$  \hspace{1cm} (B.2)

ignoring $\theta$ because of axisymmetry. In the separation of variables procedure described in Appendix A the term $\frac{Z''}{Z}$ was brought over to the right-hand side and set equal to the axial wavenumber $k_z^2$ ($\sigma^2$) but to achieve Heelan’s solutions, this separation of variables is rearranged in a different manner. Both the $Z$ term and the wavenumber term are brought over to the right-hand side yielding

$$\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} = -k^2 - \frac{Z''}{Z} = \sigma^2$$  \hspace{1cm} (B.3)

which solutions in $R$ are Bessel functions, modified Bessel functions or Hankel functions of order zero. Heelan uses the radiation condition to limit the solutions to
outgoing waves governed by the Hankel function $H_0^{(1)}(\sigma r)$.

Solving for $Z$ yields the function

$$Z = e^{z\sqrt{\sigma^2 - k^2}}$$ (B.4)

An identical procedure is followed for the integrals dependent on the wavenumber $h$ instead of $k$. Thus it can be seen that the equations and the boundary conditions that Heelan solves are comparable to the developments in Appendix A - the only difference is a rearrangement of the separation of variables procedure.

### B.1.2 Development of the displacement potential relations

Using the separation of variables procedure results in the general solution by superposing all potential solutions and thus an integration over all of the values of the parameters $\sigma$ and $\omega$. The displacement potential $\phi$ becomes

$$\phi = \int \int_{-\infty}^{\infty} A(\sigma,k)H_0^{(1)}(\sigma r)e^{z\sqrt{\sigma^2 - k^2}}e^{ikvt}d\sigma dk$$ (B.5)

but complex values of the axial wavenumber $\sigma$ must be allowed so Heelan from conception evaluates Eq. B.5 as the following contour integral

$$\phi = \int_{-\infty}^{\infty} \int_{C} A(\sigma,k)H_0^{(1)}(\sigma r)e^{z\sqrt{\sigma^2 - k^2}}e^{ikvt}d\sigma dk$$ (B.6)

It is common to consider Eq. B.6 as a double Fourier transform with a factor of $(2\pi)^{-2}$ in front. This factor is not introduced in this development although it was argued in Appendix A that this factor is in fact irrelevant if maintained throughout the analysis. Heelan specified that $\phi, \psi, \chi$ are real and thus Heelan used a one sided Fourier transform operator $\text{Re} \int_0^{\infty} e^{ikvt}dk$ that has also been used by other geophysicists (e.g. Brekhovskikh, 1960, 1980; Sommerfeld, 1949; Gilbert, 1964; Pilant, 1979; Ben-Menahem and Singh, 1981).

Since Heelan’s use of the one sided Fourier transform operator has been the source of some criticism (Abo-Zena, 1977) it is briefly discussed here. The validity of this
shorthand notation is easily seen by use of the full Fourier integral theorem although issues of integrability, differentiability and continuity (e.g. Wiener, 1933) will not be addressed here.

The Fourier integral theorem (Sommerfeld, 1949; Bracewell, 1978) can be written in the following equivalent forms

\[ f(x) = \frac{1}{\pi} \int_{\omega=0}^\infty \int_{-\infty}^\infty f(u) \cos \omega(x - u) \, du \, d\omega \]  \hspace{1cm} (B.7)

\[ f(x) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{i\omega(x-u)} \, du \, d\omega \]

\[ f(x) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} e^{i\omega x} \int_{-\infty}^{\infty} f(u) e^{-i\omega u} \, du \, d\omega \]

First consider the integral

\[ \frac{1}{\pi} \int_{\omega=0}^\infty \int_{-\infty}^\infty f(u) e^{i\omega(x-u)} \, du \, d\omega = \]  \hspace{1cm} (B.8)

\[ = \frac{1}{\pi} \int_{\omega=0}^\infty \int_{-\infty}^\infty f(u) \cos \omega(x - u) \, du \, d\omega + \frac{i}{\pi} \int_{\omega=0}^\infty \int_{-\infty}^\infty f(u) \sin \omega(x - u) \, du \, d\omega \]

For \( f(u) \) defined real, the two integrals on the right-hand side will be real and imaginary respectively. Taking the real part of the right-hand side of Eq. B.8 leaves the left integral

\[ \frac{1}{\pi} \int_{\omega=0}^\infty \int_{-\infty}^\infty f(u) \cos \omega(x - u) \, du \, d\omega \]  \hspace{1cm} (B.9)

which is just equal to Eq. B.7. Rewriting the real part of Eq. B.8 as

\[ \text{Re} \frac{1}{\pi} \int_{\omega=0}^\infty F(\omega) e^{i\omega x} \, d\omega \]  \hspace{1cm} (B.10)

where \( F(\omega) \) is defined to be

\[ F(\omega) = \int_{-\infty}^{\infty} f(u) e^{-i\omega u} \, du \]  \hspace{1cm} (B.11)

Thus for \( f(x) \) is real, the following representation is valid

\[ \text{Re} \frac{1}{\pi} \int_{\omega=0}^\infty F(\omega) e^{i\omega x} \, d\omega \]  \hspace{1cm} (B.12)

The \( \frac{1}{\pi} \) normalization factor can be ignored as argued previously.

The preceding has shown that Heelan's use of this operator was valid.

Thus using Eq. B.6 as a template for radial, axial and torsional sources Heelan wrote the displacement potentials using Eq. B.1 (Heelan, 1953a, Eq. 5).
B.1.3 Relationship to the Sommerfeld integral

Heelan’s integral transforms can be related to classic physical results such as the Sommerfeld integral by addressing a parenthetical remark made by Heelan. Heelan states in a footnote to the first paper (1953a) and the thesis (1952) that if \( f_0 = \frac{-\sigma}{2\sqrt{\sigma^2 - k^2}} \) then the contour \( C \) can be deformed onto the real axis and the resulting integral is the classic Hertzian oscillator \( \frac{e^{ikR}}{R} \), where \( R = \sqrt{r^2 + z^2} \). The resulting integral upon doing this substitution is the Sommerfeld integral as shown below.

Assuming Heelan’s supposition is correct, the unknown contour \( C \) can be deformed onto the real axis after the substitution of \( f_0 = \frac{-\sigma}{2\sqrt{\sigma^2 - k^2}} \) yielding the integral

\[
\phi = \text{Re} \int_0^\infty \int_{-\infty}^\infty \frac{-\sigma}{2\sqrt{\sigma^2 - k^2}} H_0^{(1)}(\sigma r) e^{z\sqrt{\sigma^2 - k^2}} e^{ikvt} d\sigma dk \tag{B.13}
\]

Temporarily setting aside the integral with respect to \( k \) and recalling the following integral (Gradshteyn and Ryzhik, 1980, 4th edition, Eq. 6.616.3; Erdélyi et al., 1953, Eq. II-7.14.53)

\[
\int_{-\infty}^\infty e^{itz} H_0^{(1)}(r\sqrt{\alpha^2 - t^2}) dt = -2i e^{i\alpha \sqrt{r^2 + x^2}} \tag{B.14}
\]

with arguments restricted to the domain

\[0 < \text{arg} \sqrt{\alpha^2 - t^2} < \pi, 0 < \text{arg} \alpha < \pi \quad r \text{ and } x \text{ are real}\]

and putting Eq. B.14 into Heelan’s notation produces

\[
\int_{-\infty}^\infty e^{itz} H_0^{(1)}(r\sqrt{k^2 - t^2}) d\sigma = -2i e^{i\sqrt{r^2 + x^2}} \tag{B.15}
\]

Now substituting \( it = \sqrt{\sigma^2 - k^2} \) and simplifying

\[
\frac{1}{2} \int_{-\infty}^\infty \frac{-\sigma}{\sqrt{\sigma^2 - k^2}} H_0^{(1)}(\sigma r) e^{z\sqrt{\sigma^2 - k^2}} d\sigma = \frac{e^{ikR}}{R} \tag{B.16}
\]

It can be seen that Eq. B.16 and Eq. B.13 are in fact equivalent. The range limitations on Eq. B.14 are satisfied by assuming \( k \) has a positive imaginary component which requires attenuation.

218
The purpose of this section is to point out that the integral resulting from Heelan's supposition (Eq. B.16) is in fact a Sommerfeld integral. Aki and Richards (1980, Eq. 6.7) identify the Sommerfeld integral as the following

$$\frac{e^{i k R}}{R} = \int_0^\infty \frac{\sigma}{\sqrt{\sigma^2 - k^2}} J_0(\sigma r) e^{i z \sqrt{\sigma^2 - k^2}} d\sigma$$  \hspace{1cm} (B.17)

which Aki and Richards later rewrite (Aki and Richards, 1980, Eq. 6.15)

$$\frac{i}{2} \int_{-\infty}^{\infty} \frac{\sigma}{\sqrt{\sigma^2 - k^2}} H_0^{(1)}(\sigma r) e^{i z \sqrt{\sigma^2 - k^2}} d\sigma$$  \hspace{1cm} (B.18)

both equations having been rewritten in Heelan's notation. A difference in phase $i$ between Eq. B.16 and Eq. B.18 is attributed to Aki and Richards different definition of radial wavenumber.

The integrand of Eq. B.18 is identified as a cylindrical wave governed by axial and radial wavenumbers. Thus Heelan's results represent the decomposition of a spherical wave into cylindrical waves abiding by Sommerfeld integral theory.

### B.1.4 Boundary conditions for radial and axial sources

As mentioned in Chapter 2, boundary conditions at the empty cavity are vanishing of normal stress and tangential stress for the $P$-$SV$ case and vanishing of azimuthal stress for the $SH$ case. In an unusual manner Heelan equated the potential $\psi$ to the "negative" of the potential $\psi$ discussed in Appendix A. Thus the stresses below are reversed in sign for the potential $\psi$ presented in Appendix A (Section A.2.3). The stresses are

$$p_r = \lambda \nabla^2 \phi + 2 \mu \frac{\partial}{\partial r} \left( \frac{\partial \phi}{\partial r} - \frac{\partial^2 \psi}{\partial r \partial z} \right)$$ \hspace{1cm} (B.19)

$$p_{rz} = \mu \frac{\partial}{\partial r} \left( 2 \frac{\partial \phi}{\partial z} + \nabla^2 \psi - 2 \frac{\partial^2 \psi}{\partial z^2} \right)$$

$$p_{r\theta} = \mu \left( \frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r} \frac{\partial \chi}{\partial r} \right)$$

and similarly the displacements are given by

$$U_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 \psi}{\partial r \partial z}$$  \hspace{1cm} (B.20)
\[ U_\theta = \frac{\partial \chi}{\partial r} \]
\[ U_z = \frac{\partial \phi}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) \]

Heelan mathematically represents radial, axial and torsional sources as stresses of constant magnitude \( P, Q, S \) multiplied by a time varying amplitude factor \( G(t) \) in turn multiplied by a "boxcar" function in depth \( F(z) \). The boxcar function is defined for a cylindrical length \( l \) centered at zero as \( F(z) = 0 \quad |z| > l, \quad F(z) = 1 \quad |z| \leq l \).

From Chapter 2, the system of equations can be written for the \( P-SV \) case

\[
\begin{bmatrix}
  D_{31} & D_{33} \\
  D_{41} & D_{43}
\end{bmatrix}
\begin{bmatrix}
  f_o(\sigma, k) \\
  g_o(\sigma, k)
\end{bmatrix}
+ \begin{bmatrix}
  PG(t)F(z) \\
  QG(t)F(z)
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\] (B.21)

and for the \( SH \) case

\[
\begin{bmatrix}
  D_{11}
\end{bmatrix}
\begin{bmatrix}
  n_o(\sigma, k)
\end{bmatrix}
+ \begin{bmatrix}
  SG(t)F(z)
\end{bmatrix} = \begin{bmatrix}
  0
\end{bmatrix}
\] (B.22)

where the \( D_{3i} \) represent the normal stress boundary condition and the the \( D_{4i} \) represent the tangential stress boundary condition. Also since only one layer is considered in Heelan's far field analysis the \( D \) symbolism is used for the matrix elements instead of \( G \).

This is a slight departure from Heelan's notation. Heelan uses the notation \( P(t), Q(t) \) and \( S(t) \) and specifies only in a parenthetical remark that they are proportional to each other. What this proportionality means is that \( P(t) \) and \( Q(t) \) have the same time function but different magnitudes. \( S(t) \) for most cases of interest would also have the same time function but because the \( P-SV \) and \( SH \) problems can be solved independently this is not a requirement. The common time function \( G(t) \) is used to clarify this important relationship.

Heelan makes use of the Fourier transform of the boxcar function which is commonly referred to as the "sinc" function

\[ F(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin l\tau}{\tau} e^{i\tau z} d\tau \]
and transforms it to

\[ F(z) = \frac{i}{\pi} \int_C \frac{\sigma \sinh(l\sqrt{\sigma^2 - k^2})}{\sigma^2 - k^2} e^{z\sqrt{\sigma^2 - k^2}} d\sigma \]  

(B.23)

using the mapping \( \sqrt{\sigma^2 - k^2} = i\tau, -\pi \leq \arg \tau \leq 0 \).

Since the \( P-SV \) radiation problem can be solved independently it is treated separately from the \( SH \) as is done here.

### B.1.5 P-SV case

Assume the most general case where both radial and axial sources of magnitude \( P \) and \( Q \) respectively are applied. If it was desired to study radial sources exclusively, \( Q \) would be set equal to zero and vice versa. The transformed values of \( PG(t)F(z) \) and \( QG(t)F(z) \) are the following from Eq. B.23

\[ PG(t)F(z) \rightarrow P(\sigma,k) = PG(k) \frac{i\sigma \sinh(l\sqrt{\sigma^2 - k^2})}{\pi(\sigma^2 - k^2)} e^{z\sqrt{\sigma^2 - k^2}} \]  

(B.24)

\[ QG(t)F(z) \rightarrow Q(\sigma,k) = QG(k) \frac{i\sigma \sinh(l\sqrt{\sigma^2 - k^2})}{\pi(\sigma^2 - k^2)} e^{z\sqrt{\sigma^2 - k^2}} \]  

(B.25)

having brought the boundary conditions under the integral.

From Eq. B.21 Cramer’s rule solutions are

\[ f_o(\sigma,k) = \frac{Q(\sigma,k)D_{33} - P(\sigma,k)D_{43}}{D_{31}D_{43} - D_{33}D_{41}} \]  

(B.26)

\[ g_o(\sigma,k) = \frac{-Q(\sigma,k)D_{31} + P(\sigma,k)D_{41}}{D_{31}D_{43} - D_{33}D_{41}} \]

The normal and tangential stresses are calculated using Eq. B.19 where the following relationships for \( \psi \) and \( \phi \) have been used

\[ \nabla^2 \phi = -k^2 \phi \quad \nabla^2 \psi = -h^2 \psi \]  

(B.27)

and the stresses are as follows

\[ D_{31} = \left( -\lambda k^2 - 2\mu \sigma^2 \right) H_0^{(1)}(\sigma a) + 2\mu \sigma \frac{H_1^{(1)}(\sigma a)}{a} \right) e^{z\sqrt{\sigma^2 - k^2}} \]  

(B.28)

221
\[ D_{33} = 2\mu\sigma^2\sqrt{\sigma^2 - k^2} \left[ H_0^{(1)}(\sigma a) - \frac{H_1^{(1)}(\sigma a)}{\sigma a} \right] e^{\sqrt{\sigma^2 - k^2}} \]
\[ D_{41} = -2\mu\sigma\sqrt{\sigma^2 - k^2} H_1^{(1)}(\sigma a) e^{\sqrt{\sigma^2 - k^2}} \]
\[ D_{43} = \mu\sigma(h^2 + 2\sigma^2 - 2k^2) H_1^{(1)}(\sigma a) e^{\sqrt{\sigma^2 - k^2}} \]

An algebraic difficulty arises because in Eq. B.28 there are two potentials with two different exponential factors, \( e^{\sqrt{\sigma^2 - k^2}} \) and \( e^{\sqrt{\sigma^2 - k^2}} \).

To ease this algebraic burden, Heelan implements a very complex transformation. First a factor \( \varrho^2 \) equal to \((\sigma^2 + h^2 - k^2)\) is substituted into the \( \psi \) potential under the radical, although this crucial interim step was not defined in Heelan’s papers and only very loosely defined in Heelan’s thesis. In the final solution, for the \( f_\varrho(\sigma, k) \) potential, all \( \varrho \) terms cancel so there is no reverse transformation. However the solution for \( g_\varrho(\varrho, k) \) is calculated and in order to transform it back to \( g_\varrho(\sigma, k) \) \( \sigma^2 \) is set equal to \((\sigma^2 - h^2 + k^2)\).

Rewriting the potentials and temporarily setting aside the integral signs
\[ \phi = f_\varrho(\sigma, k) H_0^{(1)}(\sigma r) e^{\sqrt{\sigma^2 - k^2}} \quad \text{(B.29)} \]
\[ \psi = g_\varrho(\varrho, k) H_0^{(1)}(\varrho r) e^{\sqrt{\sigma^2 - k^2}} \]
and the change in \( \psi \) from Eq. B.1 is evident. So now rewriting the Eq. B.28 in the following form produces
\[ D_{31} = \left( (-\lambda k^2 - 2\mu\sigma^2) H_0^{(1)}(\sigma a) + 2\mu\sigma \frac{H_1^{(1)}(\sigma a)}{a} \right) e^{\sqrt{\sigma^2 - k^2}} \quad \text{(B.30)} \]
\[ D_{33} = 2\mu\varrho^2\sqrt{\sigma^2 - k^2} \left[ H_0^{(1)}(\varrho a) - \frac{H_1^{(1)}(\varrho a)}{\varrho a} \right] e^{\sqrt{\sigma^2 - k^2}} \]
\[ D_{41} = -2\mu\sigma\sqrt{\sigma^2 - k^2} H_1^{(1)}(\sigma a) e^{\sqrt{\sigma^2 - k^2}} \]
\[ D_{43} = \mu\varrho(h^2 + 2\sigma^2 - 2k^2) H_1^{(1)}(\varrho a) e^{\sqrt{\sigma^2 - k^2}} \]

The denominator for the Cramer’s rule solution is \( D_{31}D_{43} - D_{41}D_{33} \) which after some simplification equals
\[ D_{31}D_{43} - D_{41}D_{33} = \]
\[ 4\sigma \mu^2 g^2 (\sigma^2 - k^2) H_1^{(1)}(\rho a) H_0^{(1)}(\rho a) \] (B.31)

\[-\mu \rho (h^2 + 2\sigma^2 - 2k^2)(\lambda k^2 + 2\mu \sigma^2) H_1^{(1)}(\rho a) H_0^{(1)}(\sigma a) \] (B.32)

\[ + 2\sigma \mu^2 h^2 \rho \frac{H_1^{(1)}(\rho a) H_1^{(1)}(\sigma a)}{a} \] (B.33)

where the exponential factors have been discarded because of cancellation throughout the system of equations.

Heelan uses the expression “only predominant terms are kept in the expansion”. What this means is that in the limit as \( z \) goes to zero, the ratio \( \frac{H_0^{(1)}(z)}{H_1^{(1)}(z)} \) also goes to zero. A similar argument was used in Chapter 3 for the modified Bessel functions \( K_{\sigma} \) and \( K_{1} \) and also by Abo-Zena (1977). Thus factors in the denominator and numerator can be ignored which have \( H_0^{(1)} \) terms (Eq. B.31, Eq. B.32). The only term that survives in the denominator is therefore Eq. B.33. Isolating Eq. B.33 gives for the denominator

\[ 2\sigma \mu^2 h^2 \rho \frac{H_1^{(1)}(\rho a) H_1^{(1)}(\sigma a)}{a} \] (B.34)

The numerator for the calculation of \( f_0(\sigma, k) \) \((Q(\sigma, k) D_{33} - P(\sigma, k) D_{43})\) leaving off terms in \( H_0^{(1)} \) is

\[-Q(\sigma, k)2g^2 \mu \sqrt{\sigma^2 - k^2} \frac{H_1^{(1)}(\rho a)}{\rho a} - P(\sigma, k) \mu \rho (h^2 + 2\sigma^2 - 2k^2) H_1^{(1)}(\rho a) \]

and the numerator for \( g_0(\rho, k) \) \((P(\sigma, k) D_{41} - Q(\sigma, k) D_{31})\) is similarly

\[-P(\sigma, k)2\sigma \mu \sqrt{\sigma^2 - k^2} H_1^{(1)}(\sigma a) - Q(\sigma, k)(2\mu \sigma H_1^{(1)}(\sigma a)) \] (B.35)

Evaluate \( f_0 \) first. Substitute the values of \( P(\sigma, k), Q(\sigma, k) \) into Eq. B.35 to obtain

\[-\frac{QG(k)i\sigma \sinh(l\sqrt{\sigma^2 - k^2})}{\pi(\sigma^2 - k^2)} 2\mu \rho \sqrt{\sigma^2 - k^2} \left[ \frac{H_1^{(1)}(\rho a)}{a} \right] \] (B.36)

\[-\frac{PG(k)i\sigma \sinh(l\sqrt{\sigma^2 - k^2})}{\pi(\sigma^2 - k^2)} \mu \rho (h^2 + 2\sigma^2 - 2k^2) H_1^{(1)}(\rho a) \]

Dividing through by the denominator (Eq. B.34) yields

\[-QG(k)i\sinh(l\sqrt{\sigma^2 - k^2}) \frac{1}{\pi h^2 \mu (\sigma^2 - k^2)} \left[ \frac{1}{H_1^{(1)}(\sigma a)} \right] \] (B.37)

\[-PG(k)i\sinh(l\sqrt{\sigma^2 - k^2}) \frac{2\sigma^2 - 2k^2}{h^2} \left[ \frac{a}{H_1^{(1)}(\sigma a)} \right] \]
Substituting in the following expansion for small arguments

\[ \sinh x \simeq x \]

\[ H_1^{(1)}(z) \simeq \frac{2}{i\pi z} \]  \hspace{1cm} (B.38)

finally yields (Heelan, 1953a, Eq. 8a)

\[ f_0(\sigma, k) = \frac{PG(k)\Delta\sigma}{8\pi\mu\sqrt{\sigma^2 - k^2}} \left( 1 + \frac{2\sigma^2}{h^2} - \frac{2k^2}{h^2} \right) + \frac{QG(k)\Lambda\sigma}{8\pi\mu h^2} \] \hspace{1cm} (B.39)

where \( \Delta \) equals the volume of the source \( 2\pi a^2l \) and \( \Lambda \) equals the surface area of the finite length cylinder \( 4\pi al \), where the factor of two arises due to the cavity length being \( 2l \).

Following the same procedures for \( g_0(\varphi, k) \) yields

\[ -\frac{PG(k)ia\sigma \sinh(\sqrt{\sigma^2 - k^2})}{\pi\sqrt{\sigma^2 - k^2} \varrho H_1^{(1)}(\varrho a)} - \frac{QG(k)ia\sigma \sinh(\sqrt{\sigma^2 - k^2})}{\pi\mu h^2(\sigma^2 - k^2) \varrho H_1^{(1)}(\varrho a)} \] \hspace{1cm} (B.40)

Applying the expansions for small arguments yields \( g_0(\varphi, k) \)

\[ \frac{PG(k)\pi a^2 l\sigma}{2\pi\mu h^2} + \frac{QG(k)\pi a l\sigma}{2\pi\mu h^2 \sqrt{\sigma^2 - k^2}} \] \hspace{1cm} (B.41)

Which can be transformed to \( g_0(\sigma, k) \) by replacing \( \sqrt{\sigma^2 - k^2} \) with \( \sqrt{\sigma^2 - h^2} \)

\[ \frac{PG(k)\pi a^2 l\sigma}{2\pi\mu h^2} + \frac{QG(k)al\sigma\pi}{2\pi\mu h^2 \sqrt{\sigma^2 - h^2}} \] \hspace{1cm} (B.42)

and finally yielding (Heelan, 1953a, Eq. 8b) analogous to Eq. B.39

\[ g_0(\sigma, k) = \frac{PG(k)\Delta\sigma}{4\pi\mu h^2} + \frac{QG(k)\Lambda\sigma}{8\pi\mu h^2 \sqrt{\sigma^2 - h^2}} \] \hspace{1cm} (B.43)

### B.1.6 SH case

The calculation of \( n_o \) is particularly simple because the \( SH \) problem can be solved independently. Adding the transformed expression for the torsional source to the azimuthal stress and requiring the sum to vanish yields

\[ \frac{SG(k)ia\sigma \sinh(\sqrt{\sigma^2 - h^2})}{\pi(\sigma^2 - h^2)} e^{x\sqrt{\sigma^2 - h^2}} + n_o\mu \sigma^2 H_2^{(1)}(\sigma a)e^{x\sqrt{\sigma^2 - h^2}} = 0 \] \hspace{1cm} (B.44)
For small arguments, the substitution for sinh is made and the additional substitution

\[ H_2^{(1)}(z) \simeq \frac{4}{i\pi z^2} \quad (B.45) \]

to yield (Eq. 8c, Heelan, 1953a)

\[ n_o(\sigma, k) = \frac{SG(k)\Delta\sigma}{8\pi\mu\sqrt{\sigma^2 - h^2}} \quad (B.46) \]

with the previous definition of \( \Delta \) as volume of the finite cylindrical length applying.

Thus Heelan's algebra to this point has been proven completely sound. However, the criticism goes beyond the algebraic results.

### B.2 Criticisms of Heelan's results

Any discussion of Heelan's results would not be complete without addressing the criticism Heelan's work received. Jordan (1962), Hazebroek (1966), and Abo-Zena (1977) have criticized Heelan's work but Abo-Zena's criticism is the most comprehensive and compelling.

Abo-Zena's work (1977) also examined radiation from empty boreholes but used a different form of integral transform. Although as pointed out by White (1983), Abo-Zena's far field results are equivalent to Heelan's results, Abo-Zena chose to devote the appendix of the 1977 paper to criticism of Heelan's results.

The real story is slightly more complex. Heelan's results are equivalent only if a \( \frac{1}{\mu} \) correction is made to Eq. 54 of Abo-Zena's paper (1977). Simple dimensionality analysis shows that Abo-Zena's results are missing a factor with the units of shear modulus and that this \( \frac{1}{\mu} \) factor was left off in the development of Eq. 30 to Eq. 54 of Abo-Zena's paper. But Abo-Zena's results are in fact an extension of Heelan's in that a spatially nonuniform but still axisymmetric source can be used with Abo-Zena's use of the Laplace transform. However, this added capability will not be of use here.

Abo-Zena has four principal criticisms. The first criticism is shared with Hazebroek and is primarily an issue of semantics. Abo-Zena (1977) and Hazebroek (1966)
point out that Heelan’s work is not for a cylinder of finite length since contributions from the ends of the cylinder are not considered. Thus Heelan’s results are not for an isolated cavity but instead for a stress applied over a finite length of an infinite cavity which is true.

The second objection Abo-Zena has is due to a typographical error in Heelan’s first paper that is not present in Heelan’s thesis (1952). On page 687, Heelan describes the functions \( f_o(\sigma, k), g_o(\sigma, k), n_o(\sigma, k) \) with \( k \) being a wavenumber. On page 688 the lower case \( k \) is mistakenly converted to a lower case \( r \) in the typesetting of the heading - the heading displaying \( f_o(\sigma, r), g_o(\sigma, r), n_o(\sigma, r) \) but the thesis displaying the correct \( f_o(\sigma, k) \), etc. (pg. 19, Heelan, 1952). Abo-Zena mistakenly interprets this heading as requiring Heelan to evaluate \( f_o, g_o, n_o \) as \( r \) goes to infinity thus yielding a point source approximation. In fact, the dependence is not on \( r \) as a variable but \( r \) at the borehole wall (\( r = a \)) which is a parameter. The far field expansion is done in the parameter \( a \) using an expansion for small arguments. A well known example of an expansion for small arguments is the approximation \( \sin a = a \). Heelan does not evaluate the functionals in terms of \( r \) thus due to a misprint this particular criticism is unjustified.

A third criticism of Heelan’s paper by Abo-Zena (1977) was related to Heelan’s use of the one-sided Fourier transform. Abo-Zena recognized Heelan’s operator as a Fourier transform but stated that the boundary conditions could not be equated under the integrand. It was shown earlier through the use of the Fourier integral theorem that Heelan’s use of the one-sided Fourier transform is just a short-hand notation for the more conventional two-sided Fourier transform operator. Therefore, Heelan’s boundary conditions can be brought under the integral without penalty.

A final criticism of Abo-Zena concerns Heelan’s contour integral analysis although Abo-Zena does not explicitly show or describe the contour. The issue of the unspecified contour is the principal reason for this examination of Heelan’s work. Abo-Zena states that the contour \( C \) from Heelan’s papers cannot be equivalent for each integral and thus this is a second reason why boundary conditions cannot be placed under
the integral sign. However, Abo-Zena does not prove this assertion but it is justified since Heelan's contour was not specified. It is our opinion that Heelan did in fact use the same contour but without a precise definition of what Heelan's contour was Heelan's integration nor the final results can be verified. The saddle points of Heelan's integrals are the same saddle points used for the integrals in Chapter 3.

Because the nature of Heelan's contour is unresolved an indirect proof of the validity of Heelan's results is required. For this reason a parallel development using the Weyl integral (Stratton, 1941; Sommerfeld, 1949; Brekhovskikh, 1960, 1980) as suggested by Brekhovskikh (1960, 1980) and extended here will unambiguously demonstrate the correctness of Heelan's results. This indirect proof is provided in Appendix C.
Appendix C

Brekhover'skikh's analysis and notation

In Appendix C, it was shown that although the work of Heelan for the empty borehole received extensive criticism, the algebra was irrefutable. The chief criticism of Heelan's work was due to Abo-Zena (1977) and was principally concerned with Heelan's omission of the outlining of the complex contour used for the analysis. A parallel treatment initiated by Brekhover'skikh allows verification of Heelan's treatment and also the treatment for the empty borehole presented in Chapter 3.

Abo-Zena's principal objection could be restated as follows "since the contour Heelan used is unknown the contour integration is suspect". However, Brekhover'skikh (1960, 1980) also addressed Heelan's problem of radiation from an empty borehole but used the well known Weyl integral and contours (Stratton, 1941; Brekhover'skikh, 1960, 1980) to expand the potentials in terms of plane waves instead of cylindrical waves.

There are many reasons why an extensive analysis of Brekhover'skikh's work is required. Brekhover'skikh's first edition (1960) was written before the aforementioned criticisms were published so is very sparse in details nor did the second edition (1980) improve in this respect. Secondly, the treatment of Brekhover'skikh did not include axial
sources which is a very important omission which is rectified here. Thirdly, it is not intuitively obvious that Brekhovskikh’s treatment in terms of the Weyl integral will yield the same results as Heelan’s. Finally, there are minor misprints and algebraic errors in Brekhovskikh’s work which must be addressed.

Because the two editions of Brekhovskikh’s book (1960, 1980) have slightly different section numbers it is wise to note that the following analysis will exclusively reference the second edition (1980).

C.1 Development of the displacement potential relations

The most fundamental description is the plane wave decomposition of a scalar field, once the development of this is accomplished, the decomposition of the potential field will follow matter of factly. It will be seen that the Hertzian oscillator, $\frac{e^{ikR}}{R}$ or equivalently, $\frac{e^{imR}}{R}$, in Brekhovskikh’s treatment is introduced from conception.

Consider a spherical wave radiated at the origin and its decomposition into plane waves. Brekhovskikh (Eq. 26.15, 1980) also Ben-Menahem and Singh (1981) introduce the following form for the Hertzian oscillator

$$\frac{e^{ikr}}{r} = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i(k_x x + k_y y)}}{\sqrt{k^2 - k_x^2 - k_y^2}} dk_x dk_y$$

(C.1)

where $r = \sqrt{x^2 + y^2}$ and $k$ is a constant. Eq. C.1 is a decomposition of the Hertzian oscillator into plane waves in the $x$-$y$ plane where $z = 0$ exclusively. It is necessary to downward and upward continue Eq. C.1 to include all of $z$. To perform this continuation, a third component is introduced to the exponential, $ik_z z$, where $k_z = \sqrt{k^2 - k_x^2 - k_y^2}$ for $z \geq 0$ and $-k_z = \sqrt{k^2 - k_x^2 - k_y^2}$ for $z \leq 0$ and $k = \frac{\omega}{c}$. Thus the integrals (Eq. 26.17, Brekhovskikh, 1980; Eq. 6.4, Aki and Richards, 1980)

$$z \geq 0 \quad \frac{e^{ikR}}{R} = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i(k_x x + k_y y + k_z z)}}{k_z} dk_x dk_y$$

(C.2)
\[ z \leq 0 \quad \frac{e^{ikR}}{R} = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i(k_x x + k_y y - k_z z)}}{k_z} dk_x dk_y \]

are produced where \( R \) now equals \( \sqrt{x^2 + y^2 + z^2} \). Eq. C.2 represents the expansion of a spherical wave into plane waves (Brehkovskikh, 1980) with the wavenumber vector \( \vec{k} \) of the plane waves having components \( k_x, k_y, k_z \). A difference in phase \( i \) between Aki and Richards and Brehkovskikh's treatment is attributable to Brehkovskikh's \( k_z \) equalling Aki and Richards' \( ik_z \).

The wavenumber vector \( \vec{k} \) can also be expressed in terms of polar angles \( \phi, \theta \) and Stratton terms this expansion the Weyl solution or Weyl integral (Sec. 9.29, Stratton, 1941). However, \( \vec{k} \) can in addition be expressed in terms of variables \( k, \theta \) which leads to the Sommerfeld integral as developed by Aki and Richards (1980) among others. A good discussion of the distinction between Weyl and Sommerfeld integrals may be found in Box 6.1 of Aki and Richards (1980).

Both representations, the Sommerfeld and the Weyl, require transformations of the coordinate system of the integral (Eq. C.2) which necessitates the use of a Jacobian. For a general integral in coordinates \( (k_x, k_y) \) to be transformed to coordinates \( (a, b) \) the formulation is

\[
\iint f(k_x, k_y) \, dk_x \, dk_y = \iint f(k_x(a, b), k_y(a, b)) \left| \frac{\partial(k_x, k_y)}{\partial(a, b)} \right| \, da \, db \quad (C.3)
\]

where the term \( \left| \frac{\partial(k_x, k_y)}{\partial(a, b)} \right| \) is the absolute value of the Jacobian of the transformation.

Before proceeding to the transformation in terms of the Weyl integral which is of fundamental importance to this thesis, the transformation in terms of the Sommerfeld integral will be performed first. \( k_z \) and \( k_y \) are set equal to

\[ k_x = k_r \cos \phi \quad k_y = k_r \sin \phi \quad (C.4) \]

and thus

\[ k_z = \sqrt{k^2 - k_x^2 - k_y^2} = \sqrt{k^2 - k_r^2} \quad (C.5) \]

and the magnitude of the determinant of the Jacobian of the transformation now equals \( k_r \). The transformed integrals which are precursors of the Sommerfeld integral
(Eq. 6.6, Aki and Richards, 1980) are rewritten to this notation as

\[
z \geq 0 \quad \frac{e^{ikR}}{R} = \frac{1}{2\pi} \int_0^\infty dk_r \int_0^{2\pi} \frac{k_r}{k_z} e^{i(k_r r \cos \phi + k_z z)} d\phi
\]

\[
z \leq 0 \quad \frac{e^{ikR}}{R} = \frac{1}{2\pi} \int_0^\infty dk_r \int_0^{2\pi} \frac{k_r}{k_z} e^{i(k_r r \cos \phi - k_z z)} d\phi
\]

Applying a suitably transformed version of Poisson’s integral (Eq. 2.3.2, Watson, 1944)

\[
J_0(k_r r) = \frac{1}{2\pi} \int_0^{2\pi} e^{ik_r r \cos \theta} d\theta
\]

to Eq. C.6 yields the Sommerfeld integral below

\[
z \geq 0 \quad \frac{e^{ikR}}{R} = \int_0^\infty \frac{k_z}{k_z} J_0(k_r r) e^{ik_z z} dk_r
\]

\[
z \leq 0 \quad \frac{e^{ikR}}{R} = \int_0^\infty \frac{k_r}{k_z} J_0(k_r r) e^{-ik_z z} dk_r
\]

### C.1.1 Weyl integral formulation

Having completed the integration in terms of the Sommerfeld integral, a superposition in terms of cylindrical waves, the transformations for the Weyl integral, a superposition in plane waves, are now introduced. \(k_x\) and \(k_y\) are set equal to

\[
k_x = k \sin \theta \cos \phi \quad k_y = k \sin \theta \sin \phi
\]

and thus

\[
k_z = \sqrt{k^2 - k_x^2 - k_y^2} = k \cos \theta
\]

and the absolute value of the Jacobian (Eq. C.3) for the transformation equals \(k^2 \sin \theta\).

The transformation is not complete without considering the properties of the new contour. For instance, in order to reconstruct the spherical source, the simple Hertzian oscillator, or for that matter any source which is not an infinite distance away from the target requires the superposition of both homogeneous and inhomogeneous plane waves (Stratton, 1941). Homogeneous waves are waves for which planes of constant phase have constant amplitude whereas for inhomogeneous plane waves planes of
constant phase have variable amplitudes. To include inhomogeneous plane waves complex values are allowed for the angle \( \theta \) and thus the range of \( \theta \) is from zero to \( \frac{\pi}{2} - i \infty \). As will be discussed later, in general the contour is deformable among these limits (Stratton, 1941). This leads to the following form for the integral (Eq. 26.19, Brekhovskikh, 1980)

\[
\frac{z}{R} \geq 0 \quad e^{ikR} = \frac{ik}{2\pi} \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} \frac{e^{i(k_x z + k_y y + k_z z)}}{k_z} \sin \theta d\phi \\
\frac{z}{R} \leq 0 \quad e^{ikR} = \frac{ik}{2\pi} \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} \frac{e^{i(k_x z + k_y y - k_z z)}}{k_z} \sin \theta d\phi
\]

(C.11)

For readers following this analysis with Brekhovskikh’s book(s) it would be a good point to state that there is a misprint present in the second edition of Brekhovskikh’s book (1980) but not in the first (Brekhovskikh, 1960) and it is not carried through in the analysis. This misprint is that the term \( dk_x dk_y dk_z \) should read \( \frac{dk_x dk_y}{k_z} \) just prior to Eq. 26.19 (Brekhovskikh, 1980).

The requirement is to approximate any scalar field \( \vartheta \) such as a potential in terms of this integral superposition of plane waves (Eq. C.11). To accomplish this \( \varepsilon \) function \( V(\theta) \) is introduced to multiply times the integrand to achieve the scalar field \( \vartheta \). Brekhovskikh treats the function \( V(\theta) \) as a reflection coefficient function but it can be thought of in more general terms. Upon completing the the transform, the following is produced

\[
\frac{z}{R} \geq 0 \quad \vartheta = \frac{ik}{2\pi} \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} V(\theta) e^{i(k_x z + k_y y + k_z z)} \sin \theta d\phi \\
\frac{z}{R} \leq 0 \quad \vartheta = \frac{ik}{2\pi} \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} V(\theta) e^{i(k_x z + k_y y - k_z z)} \sin \theta d\phi
\]

(C.12)

Expanding in terms of the full expansions for \( k_x, k_y, k_z \) (Eq. C.9) produces (Brekhovskikh, 1980, Eq. 26.24)

\[
\frac{z}{R} \geq 0 \quad \vartheta = \frac{ik}{2\pi} \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} V(\theta) e^{i(k(\cos \phi + y \sin \phi) \sin \theta + z \cos \theta)} \sin \theta d\phi \\
\frac{z}{R} \leq 0 \quad \vartheta = \frac{ik}{2\pi} \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} V(\theta) e^{i(k(\cos \phi + y \sin \phi) \sin \theta - z \cos \theta)} \sin \theta d\phi
\]

(C.13)
and now performing the substitution \( x = r \cos \phi_1, y = r \sin \phi_1 \) for the parameters \( x, y \) yields

\[
\begin{align*}
  z \geq 0 \quad \vartheta &= \frac{ik}{2\pi} \int_{0}^{\frac{\pi}{2} - i\infty} d\theta \int_{0}^{2\pi} V(\theta) e^{ikr \sin \theta \cos(\phi - \phi_1) + ikr \cos \theta \sin \theta} d\phi \\
  z \leq 0 \quad \vartheta &= \frac{ik}{2\pi} \int_{0}^{\frac{\pi}{2} - i\infty} d\theta \int_{0}^{2\pi} V(\theta) e^{ikr \sin \theta \cos(\phi - \phi_1) - ikr \cos \theta \sin \theta} d\phi
\end{align*}
\]  

(C.14)

where a double angle formula from trigonometry has been used. Finally, use is made of the Poisson integral transformation (Eq. C.7) to obtain

\[
\begin{align*}
  \vartheta &= ik \int_{0}^{\frac{\pi}{2} - i\infty} V(\theta) J_0(kr \sin \theta) \sin \theta e^{ikz \cos \theta} d\theta \\
  \vartheta &= ik \int_{0}^{\frac{\pi}{2} - i\infty} V(\theta) J_0(kr \sin \theta) \sin \theta e^{-ikz \cos \theta} d\theta
\end{align*}
\]  

(C.15)

As an aside, if \( V(\theta) \) equals 1 Eq. C.15 is another expression for a Hertzian oscillator. Eq. C.15 is still not in its final form because the preferred solution is in terms of Hankel or modified Bessel functions which may be more easily thought of as incoming or in this case outgoing waves (e.g. Appendix A).

A Bessel function identity (Abramowitz and Stegun, 1964, Eqs. 9.1.3-4) is used

\[
J_0(z) = \frac{1}{2} [H_0^{(1)}(z) + H_0^{(2)}(z)]
\]  

(C.16)

Upon substituting this value for \( J_0 \), splitting the integral into two integrals, using the formula \( H_0^{(2)}(e^{-\pi iz}) = -H_0^{(1)}(z) \) (Abramowitz and Stegun, 1964, Eq. 9.1.6), and then reversing the sign of integration the following integrals are obtained

\[
\begin{align*}
  z \geq 0 \quad \vartheta &= ik \int_{-\frac{\pi}{2} - i\infty}^{\frac{\pi}{2} - i\infty} V(\theta) H_0^{(1)}(kr \sin \theta) \sin \theta e^{ikz \cos \theta} d\theta \\
  z \leq 0 \quad \vartheta &= ik \int_{-\frac{\pi}{2} - i\infty}^{\frac{\pi}{2} - i\infty} V(\theta) H_0^{(1)}(kr \sin \theta) \sin \theta e^{-ikz \cos \theta} d\theta
\end{align*}
\]  

(C.17)

Brehkovskikh refers to the contour from \( -\frac{\pi}{2} + i\infty \) to \( \frac{\pi}{2} - i\infty \) as "\( \Gamma_1 \)", a notation that shall be used subsequently here. The contour is shown diagrammatically in Fig. C-1.
C.2 Properties of the $\Gamma_1$ Contour

The contour $\Gamma_1$ is a member of a family of contours described by Sommerfeld (e.g. Sommerfeld, 1949) best known for the evaluation of Hankel functions with an excellent discussion of the properties of these contours is provided by Stratton (pg. 367-368, 1941). A basic property of the family is a radiation condition that requires the contributions from the end points to vanish and this requirement is satisfied by the cosine function at $\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

The dominant behavior of the integrals described thus far is exponential of the form $e^{i\varrho \cos \theta}$ where $\varrho$ is some constant or a function that is constant under the particular variable of integration such as $kz$. Defining $\theta$ in terms of real and imaginary parts, $\theta = \theta' + i\theta''$ allows the following decomposition

$$i\varrho \cos \theta = \varrho \sin \theta' \sinh \theta'' + i\varrho \cos \theta' \cosh \theta''$$  \hspace{1cm} (C.18)

Terms of the form $e^{-\varrho \infty}$ will vanish which require $(\theta', \theta'')$ pairs as follows

$$(\theta', \theta'') = (\frac{-\pi}{2}, \infty)$$ \hspace{1cm} (C.19)

$$(\theta', \theta'') = (\frac{\pi}{2}, -\infty)$$

Thus the vanishing endpoint contributions coupled with the desire to span all values of imaginary wavenumber dictate that the path begin at $(\frac{-\pi}{2}, \infty)$ and terminate at $(\frac{\pi}{2}, \infty)$.

However, the preceding is too limiting. In fact any pair of an infinite choice of endpoint pairs beginning and terminating at opposite imaginary infinities separated by $\pi$ can be chosen but the choice of the principal pair (Eq. C.19) is usually sufficient. For example if $\theta$ is replaced by $\theta - \theta_0$ the same relationship would hold but shifted to the right by an amount $\theta_0$. Within these endpoint limits, and of course strongly dependent on the singularities of the integrand, the contour is deformable throughout although deformation through a saddle point often produces a crossing of the real axis at 45 degrees.
In the method of stationary phase approximation (i.e. Brekhovskikh, 1980), the endpoints and the location of the saddlepoint are shifted to the right by an amount \( \theta_0 \). The particular member of the family provided by \( \Gamma_1 \) can be broken down into segments consisting of an integral over inhomogeneous waves, a segment integrating over the real \( \theta \) axis consisting of homogeneous waves and another segment integrating over inhomogeneous waves. Again Fig. C-1 can be referenced.

### C.3 Displacement potential relations continued

Eq. C.17 represents Eqn. 26.27 from Brekhovskikh (1980). The background for Brekhovskikh’s treatment is now complete so Heelan’s integrals from Appendix B are now revisited. Brekhovskikh uses Eq. C.17 for the scalar fields \( \phi, \psi, \chi \) and writes the Heelan integrals for \( z \leq 0 \) as

\[
\phi = \text{Re} \int_0^\infty e^{-ikVt} \int_{\Gamma_1} f_\alpha(\theta, k)H_0^{(1)}(kr \sin \theta)e^{-ikz \cos \theta \sin \theta} d\theta dk \\
\psi = \text{Re} \int_0^\infty e^{-ikVt} \int_{\Gamma_1} g_\alpha(\theta, k)H_0^{(1)}(kr \sin \theta)e^{-ikz \cos \gamma \sin \theta} d\theta dk \\
\chi = \text{Re} \int_0^\infty e^{-ikVt} \int_{\Gamma_1} n_\alpha(\theta, k)H_0^{(1)}(kr \sin \theta)e^{-ikz \cos \gamma \sin \theta} d\theta dk
\]

Note Brekhovskikh uses a negative time dependence \( e^{-ikVt} \) and Brekhovskikh’s integrals have been rewritten to conform with Heelan’s notation. Additionally, \( \kappa \) is a wavenumber. Brekhovskikh’s use of the negative time dependence will be maintained. Brekhovskikh as Heelan did specifies that \( kV = \kappa \nu \) and also that \( k \sin \theta = \kappa \sin \gamma \).

The boundary conditions are set up with the same mathematical model as before (Eq. B.21) although Brekhovskikh did not consider \( Q \), axial stress.

The Fourier transform of the boxcar function is the sinc function and can be written

\[
F(z) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{\sin(\ell t)}{t} e^{izt} dt
\]

It is assumed that Brekhovskikh makes the following change of variable, \( t = -k \cos \theta \),
in transforming the boxcar function to

\[ F(z) = \frac{1}{\pi} \int_{\Gamma_1} \frac{\sin(\theta)}{\cos \theta} e^{-ikz \cos \theta} \sin \theta d\theta \quad (C.22) \]

### C.4 P-SV case

In terms of Brekhovskikh’s notation the elements of the D matrix (Eq. B.28) are

\[ D_{31} = \left( -\lambda k^2 H_0^{(1)}(ka \sin \theta) - 2\mu k^2 \sin^2 \theta \left[ H_0^{(1)}(ka \sin \theta) - \frac{H_1^{(1)}(ka \sin \theta)}{ka \sin \theta} \right] \right) e^{-ikz \cos \theta} \]

\[ D_{33} = -2i \mu k^2 \sin^2 \theta \kappa \cos \gamma \left[ H_0^{(1)}(ka \sin \theta) - \frac{H_1^{(1)}(ka \sin \theta)}{ka \sin \theta} \right] e^{-ikz \cos \gamma} \quad (C.23) \]

\[ D_{41} = 2i \mu k^2 \sin \theta \cos \theta H_1^{(1)}(ka \sin \theta) e^{-ikz \cos \theta} \]

\[ D_{43} = -\left( 2\mu k \sin \theta \kappa \cos \gamma - \mu k^3 \sin \theta \frac{V^2}{v^2} \right) H_1^{(1)}(ka \sin \theta) e^{-ikz \cos \gamma} \]

where sign changes from the Appendix B results are due to the exponential factors changing from \( e^{\sqrt{\sigma^2 - k^2}} \) to \( e^{-ikz \cos \theta} \) affecting the signs of the derivatives. Additionally, a \( \sin \theta \) term that cancels has been left off both sides of these equations.

As with Heelan’s work a substitution of a term \( \varrho \) into \( \psi \) such that \( \kappa \cos \gamma \rightarrow k \cos \theta \) is made

\[ \phi = f_\varrho(k, \theta) H_0^{(1)}(kr \sin \theta) e^{-ikz \cos \theta} \quad (C.24) \]

\[ \psi = g_\varrho(\varrho, k, \theta) H_0^{(1)}(\varrho r) e^{-ikz \cos \theta} \]

but this \( \varrho \) factor will divide out in subsequent analysis. This provides a transformed Eq. C.23.

\[ D_{31} = \left( -\lambda k^2 H_0^{(1)}(ka \sin \theta) - 2\mu k^2 \sin^2 \theta \left[ H_0^{(1)}(ka \sin \theta) - \frac{H_1^{(1)}(ka \sin \theta)}{ka \sin \theta} \right] \right) e^{-ikz \cos \theta} \]

\[ D_{33} = -2i \mu k^2 \cos \theta \left[ H_0^{(1)}(\varrho a) - \frac{H_1^{(1)}(\varrho a)}{\varrho a} \right] e^{-ikz \cos \theta} \quad (C.25) \]

\[ D_{41} = 2i \mu k^2 \sin \theta \cos \theta H_1^{(1)}(ka \sin \theta) e^{-ikz \cos \theta} \]

\[ D_{43} = -\left( 2\mu \varrho k^2 \cos^2 \theta - \varrho \mu \varrho k^2 \frac{V^2}{v^2} \right) H_1^{(1)}(\varrho a) e^{-ikz \cos \theta} \]
The $e^{-ikz \cos \theta}$ terms will be discarded henceforth. Rewriting the above equation only including the terms with the Hankel function of the first order yields for the matrix $D$

$$D_{31} = 2\mu k \sin \theta \frac{H_1^{(1)}(ka \sin \theta)}{a}$$  \hspace{1cm} (C.26)$$
$$D_{33} = 2i\mu k \cos \theta \frac{H_1^{(1)}(qa)}{a}$$
$$D_{41} = 2i\mu k^2 \sin \theta \cos \theta H_1^{(1)}(ka \sin \theta)$$
$$D_{43} = -\left(2\mu \rho^2 \cos^2 \theta - \mu k^2 \frac{V^2}{v^2}\right)H_1^{(1)}(qa)$$

The denominator $D_{31}D_{43} - D_{41}D_{33}$ is after some simplification

$$\frac{2\mu^2 k^3 \sin \theta V^2}{v^2 a} \rho H_1^{(1)}(qa)H_1^{(1)}(ka \sin \theta) \hspace{1cm} (C.27)$$

It is again found that for $f_o$ the transformation in $\rho$ can be neglected and the numerator for the solution of $f_o$ ($Q(k, \theta)D_{33} - P(k, \theta)D_{43}$) is

$$\frac{PG(k) \sin(ik \cos \theta)}{\pi \cos \theta} \left[2\mu k^2 \cos^2 \theta - \mu k^2 \frac{V^2}{v^2} \rho\right]H_1^{(1)}(qa)$$
$$+ \frac{QG(k) \sin(ik \cos \theta)}{\pi \cos \theta} \left(\frac{2i\mu k \cos \theta \rho}{a}\right)H_1^{(1)}(qa) \hspace{1cm} (C.28)$$

and the numerator for the solution of $g_o$ is ($P(k, \theta)D_{41} - Q(k, \theta)D_{31}$)

$$\frac{PG(k) \sin(ik \cos \theta)}{\pi \cos \theta} \left(2i\mu k^2 \sin \theta \cos \theta\right)H_1^{(1)}(ka \sin \theta)$$
$$- QG(k) \frac{\sin(ik \cos \theta)}{\pi \cos \theta} \left(\frac{2\mu k \sin \theta}{a}\right)H_1^{(1)}(ka \sin \theta) \hspace{1cm} (C.29)$$

where again terms in $H_0$ have been neglected. As with the results in Heelan, solving for $f_o$ by dividing through with the denominator Eq. C.27 gives

$$\frac{PG(k) \sin(ik \cos \theta)}{\pi \cos \theta} \left(\frac{2V^2}{v^2} \cos^2 \theta - 1\right) \frac{a}{2\mu k \sin \theta H_1^{(1)}(ka \sin \theta)}$$
$$+ QG(k) \frac{\sin(ik \cos \theta)}{\pi \cos \theta} \left(\frac{iv^2 \cos \theta}{\mu k^2 \sin \theta H_1^{(1)}(ka \sin \theta)}\right) \hspace{1cm} (C.30)$$

Using the following expansions for small arguments

$$\sin x \simeq x \hspace{1cm} (C.31)$$
$$H_1^{(1)}(z) \simeq \frac{2}{i\pi z} $$

237
which then yields for \( f_o \)

\[
f_o(k, \theta) = +iPG(k) \left( \frac{2v^2}{V^2} \cos^2 \theta - 1 \right) \left( \frac{k\Delta}{8\pi \mu} \right) - QG(k) \frac{A \cos \theta}{8\pi \mu} \frac{v^2}{V^2} \tag{C.32}
\]

where \( \Delta, A \) have been previously defined as volume and length of the empty cavity of length \( 2l \).

For \( g_o(k, \theta) \) divide through by the denominator to obtain

\[
\frac{PG(k) \sin(\alpha k \cos \theta)}{\pi \cos \theta} - \frac{ia v^2 \cos \theta}{\mu k V^2 \phi H_1^{(1)}(\phi a)} \quad \frac{QG(k) \sin(\alpha k \cos \theta)}{\pi \cos \theta} - \frac{v^2}{\mu V^2 k^2 \phi H_1^{(1)}(\phi a)} \tag{C.33}
\]

Performing the same expansions for small arguments yields

\[
- PG(k) \frac{\pi a^2 l v^2 \cos \theta}{2\pi \mu V^2} - QG(k) \frac{i\pi a l v^2}{2\pi \mu V^2 k} \tag{C.34}
\]

Terms in \( \phi \) have dropped out leaving

\[
- PG(k) \frac{\Delta v^2 \cos \theta}{4\pi \mu V^2} - QG(k) \frac{i A v^2 \cos \theta}{8\pi \mu V^2 k \cos \theta} \tag{C.35}
\]

Now making the inverse transformation for \( \phi, k \cos \theta = \kappa \cos \gamma, g_o \) can be written

\[
g_o(k, \theta) = -PG(k) \frac{\Delta v^2 \cos \theta}{4\pi \mu V^2} - QG(k) \frac{i A v^2 \cos \theta}{8\pi \mu V^2 \kappa \cos \gamma} \tag{C.36}
\]

The terms in \( PG(k) \) were given by Brekhovskikh (Eq. 33.6, 1980) and the terms in \( QG(k) \) are presented here for the first time. There is a sign change in the results for \( g_o \) with the \( P \) term from Brekhovskikh’s work. The sign discrepancy was corrected in further integration so it is concluded that the sign error was an error in Eq. 33.6 of Brekhovskikh.

### C.4.1 Solution by the method of stationary phase

There are many excellent books which have sections on evaluation of integrals by the method of stationary phase including Brekhovskikh (1960, 1980), Mathews and Walker (1970) and numerous others.
It is necessary to evaluate the far field displacements using Eq. B.20. The integrals are thus

\[
U_r = -\text{Re} \int_0^\infty e^{-ikvt} \int_{\Gamma_1} f_o(k, \theta) k \sin \theta H_1^{(1)}(kr \sin \theta) e^{-ikz \cos \theta} \sin \theta \, d\theta \, dk
\]

\[
-\text{Re} \int_0^\infty e^{-ikvt} \int_{\Gamma_1} g_o(k, \theta) i k \kappa \sin \theta \cos \gamma H_1^{(1)}(kr \sin \theta) e^{-ikz \cos \gamma} \sin \theta \, d\theta \, dk
\]

\[
U_z = -\text{Re} \int_0^\infty e^{-ikvt} \int_{\Gamma_1} i f_o(k, \theta) k \cos \theta H_0^{(1)}(kr \sin \theta) e^{-ikz \cos \theta} \sin \theta \, d\theta \, dk
\]

\[
-\text{Re} \int_0^\infty e^{-ikvt} \int_{\Gamma_1} g_o(k, \theta) k^2 \sin^2 \theta H_0^{(1)}(kr \sin \theta) e^{-ikz \cos \gamma} \sin \theta \, d\theta \, dk
\]

Expanding the first integral of Eq. C.37 with \( f_o \) yields

\[
\text{Re} \int_0^\infty -k \left( \frac{iP\Delta}{8\pi \mu} k G(k) \left( \frac{2v^2}{V^2} \cos^2 \theta - 1 \right) - Q G(k) \frac{A v^2 \cos \theta}{8\pi \mu V^2} \right) dk d\theta
\]

\[
H_1^{(1)}(kr \sin \theta) e^{-ikz \cos \theta} \sin^2 \theta e^{-ikvt} \, dkd\theta
\]

which is simplified by applying the following expansions for large argument (Abramowitz and Stegun, 1964)

\[
H_1^{(1)}(z) \to \sqrt{\frac{2}{\pi z}} e^{iz} e^{-i\pi \frac{z}{4}}
\]

and for the integrals for \( U_z \)

\[
H_0^{(1)}(z) \to \sqrt{\frac{2}{\pi z}} e^{iz} e^{-i\pi \frac{z}{4}}
\]

Applying Eq. C.40 and introducing the identities \( r = R \sin \theta_0, \ z = -R \cos \theta_0 \) (Fig. 3-1) transforms Eq. C.38 to

\[
\text{Re} \int_0^\infty \int_{\Gamma_1} e^{-ikvt} \left( \frac{iP\Delta}{4\pi \mu} k G(k) \left( \frac{2v^2}{V^2} \cos^2 \theta - 1 \right) \right)
\]

\[
- Q G(k) \frac{A v^2 \cos \theta}{4\pi \mu V^2} \sin^2 \theta e^{ikR \cos(\theta - \theta_0)} \sqrt{\frac{2}{\pi R \sin \theta \sin \theta_0}} e^{-i\frac{z^2}{4}} \, dk \, d\theta
\]

The stationary point is \( \theta = \theta_0 \). Applying the method of stationary phase approximation as in Chapter 3 produces an integral in \( k \) only

\[
\text{Re} \int_0^\infty e^{-ikvt} \left( \frac{iP\Delta}{4\pi \mu} k G(k) \left( \frac{2v^2}{V^2} \cos^2 \theta_0 - 1 \right) \right)
\]

\[
- Q G(k) \frac{A v^2 \cos \theta_0}{4\pi \mu V^2} \sin \theta_0 e^{ikR} \right) \frac{1}{R} \, dk
\]

239
For carrying out the integration with respect to $k$, use is made of the following operator

$$-ikV Re \int_{-\infty}^{\infty} G(k)e^{-ikV(t-\frac{R}{V})}dk = \frac{d}{dt} Re \int_{-\infty}^{\infty} G(k)e^{-ikV(t-\frac{R}{V})}dk = \frac{d}{dt} G(t-\frac{R}{V}) \quad (C.44)$$

which when applied to Eq. C.43 yields for the resulting integral

$$-\frac{PA}{4\pi \mu V} \left( \frac{2v^2}{V^2} \cos^2 \theta_0 - 1 \right) \sin \theta_0 \frac{1}{R} \frac{d}{dt} G(t - \frac{R}{V}) - Q \frac{Av^2 \cos \theta_0}{4\pi \mu V^2} \sin \theta_0 \frac{1}{R} G(t - \frac{R}{V}) \quad (C.45)$$

Heelan's Eq. 13 for comparison is

$$\frac{PA}{4\pi \mu V} \left( 1 - \frac{2v^2}{V^2} \cos^2 \theta_0 \right) \sin \theta_0 \frac{1}{R} \frac{d}{dt} G(t - \frac{R}{V}) - Q \frac{Av^2 \cos \theta_0}{4\pi \mu V^2} \sin \theta_0 \frac{1}{R} G(t - \frac{R}{V}) \quad (C.46)$$

and the equivalence is readily seen.

Although the procedure is cumbersome, the integration by the method of stationary phase substantially simplified the resulting integrands. Except for a factor of two that was divided into the development there is no change in the boundary condition equations in the integrand. This is a remarkably simple result from integral expressions as complicated as those in Eq. C.37.

Proceeding with the same integration procedure with the other integrands results equivalent to Heelan's Eq. 13 are obtained (Table 4.1)

$$U_r = -\frac{PA}{4\pi \mu V} \left( 2 \frac{v^2}{V^2} \cos^2 \theta - 1 \right) \sin \theta \frac{1}{R} \frac{d}{dt} G(t - \frac{R}{V}) - Q \frac{Av^2}{4\pi \mu V^2} \cos \theta \sin \theta \frac{1}{R} G(t - \frac{R}{V})$$

$$+ \frac{PA}{4\pi \mu v} \sin 2\theta \cos \theta \frac{1}{R} \frac{d}{dt} G(t - \frac{R}{v}) + Q \frac{A}{4\pi \mu} \sin \theta \cos \theta \frac{1}{R} G(t - \frac{R}{v}) \quad (C.47)$$

$$U_z = \frac{PA}{4\pi \mu V} \left( \frac{2v^2}{V^2} \cos^2 \theta - 1 \right) \cos \theta \frac{1}{R} \frac{d}{dt} G(t - \frac{R}{V}) + Q \frac{Av^2}{4\pi \mu V^2} \cos^2 \theta \frac{1}{R} G(t - \frac{R}{V})$$

$$+ \frac{PA}{4\pi \mu v} \sin 2\theta \sin \theta \frac{1}{R} \frac{d}{dt} G(t - \frac{R}{v}) + Q \frac{A}{4\pi \mu} \sin^2 \theta \frac{1}{R} G(t - \frac{R}{v})$$

One notational note is that the subscript on $\theta_0$ has been dropped and that the symbol $\theta$ has been used for the polar angle instead of $\phi$.

In performing the integration by stationary phase analysis with the integrals including $g_\alpha$ in Eq. C.37 special treatment was required. Namely, a change in variable
from $\theta$ to $\gamma$ is required using the relations

$$k \sin \theta = \kappa \sin \gamma \quad k \cos \theta d\theta = \kappa \cos \gamma d\gamma \quad H_0^{(1)}(kr \sin \theta) = H_0^{(1)}(\kappa r \sin \gamma) \quad (C.48)$$

and for reference these integrals in $\gamma$ are

$$\begin{align*}
Re \int_0^\infty e^{-ikVt} \int_{\Gamma_1} \left( \frac{P\Delta ikV}{4\pi \mu} G(k)i\kappa \cos^2 \gamma \sin^2 \gamma - \frac{Q\Delta}{8\pi \mu} G(k)\kappa \sin^2 \gamma \cos \gamma \right) d\gamma & = e^{-\kappa \pi} \sqrt{\frac{2}{\pi \kappa R \sin \gamma \sin \gamma_0}} e^{i \kappa R \cos (\gamma - \gamma_0)} d\gamma dk \\
\text{and} \\
\int_0^\infty e^{-ikVt} \int_{\Gamma_1} \left( \frac{P\Delta}{4\pi \mu} G(k)\kappa^2 \sin^3 \gamma \cos \gamma + \frac{i Q\Delta}{8\pi \mu} G(k)\kappa \sin^3 \gamma \right) d\gamma & = e^{-\kappa \pi} \sqrt{\frac{2}{\pi \kappa R \sin \gamma \sin \gamma_0}} e^{i \kappa R \cos (\gamma - \gamma_0)} d\gamma dk 
\end{align*} \quad (C.49)$$

with use made of the relationship $r = R \sin \gamma, z = -R \cos \gamma$, completely analogous to the relationship for $\theta$ and the stationary point is likewise $\gamma = \gamma_0$. Another example utilizing the transformation in $\gamma$ is provided by the $SH$ case shown below.

### C.5 SH case

At the borehole wall, there is only one boundary condition, the vanishing of azimuthal stress. Equating this boundary condition to the source, an azimuthal stress discontinuity, yields

$$p_{r\theta} = \mu \left( \frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r} \frac{\partial \chi}{\partial r} \right) = -S \quad (C.51)$$

Substituting the values for $\chi$ and $S$ into Eq. C.51 delivers

$$SG(k) \frac{\sin(lk \cos \theta)}{\pi \cos \theta} e^{-ikz \cos \theta} + n_0 \mu k^2 \sin^2 \theta H_2^{(1)}(ka \sin \theta) e^{-ikz \cos \gamma} = 0 \quad (C.52)$$

where the derivative operator, $H_1^{(1)'}(z) = -H_2^{(1)}(z) + \frac{H_1^{(1)}(z)}{z}$, has been used along with the following expansion for small argument

$$H_2^{(1)} \to -\frac{4}{i\pi z^2} \quad (C.53)$$
An additional expansion is that $e^x$ can be expanded in the Taylor series and for small arguments only the first term $z$ is kept. Thus $n_0$ can be written as in Brekhovskikh (1960, 1980, Eq. 33.6)

$$n_0(k, \theta) = SG(k) \left( \frac{\cos \theta}{\cos \gamma} \right) \left( \frac{vk\Delta}{8\pi i\mu V} \right)$$  \hspace{1cm} (C.54)

where $\Delta$ is the volume of the cylinder of length $2l$. The number 8 in the numerator of Eq. C.54 is given as the number 4 in Brekhovskikh’s book but it has been shown that this must be a misprint. The number 8 abides by Heelan’s development bearing in mind the different algebraic formulation. The number is self correcting in subsequent steepest descent analysis leading to the belief that this error is only typographical in nature.

Proceeding with the formulation of the integral

$$U_\theta = \frac{\partial \chi}{\partial r} = -Re \int_0^\infty e^{-ikVt} dk \int_{\Gamma_1} SG(k) \frac{\Delta \cos \theta \nu k}{8\pi \mu i V \cos \gamma} k \sin \theta H_1^{(1)}(kr \sin \theta) e^{-i\kappa \cos \gamma z} d\theta$$  \hspace{1cm} (C.55)

Making the transformations $k \sin \theta = \kappa \sin \gamma, \ d\theta = \frac{\kappa \cos \theta}{\kappa \cos \gamma} d\gamma$ and the substitution for large argument (Eq. C.40)

$$-Re \int_0^\infty kV e^{-ikVt} dk \int_{\Gamma_1} SG(k) \frac{\Delta}{8\pi \mu i V} \sqrt{\frac{2}{\pi \kappa R \sin \gamma \sin \gamma_0}} e^{-i\frac{\pi}{4} \gamma} e^{i\kappa \sin \gamma_0} e^{-i\kappa \cos \gamma z} d\gamma$$  \hspace{1cm} (C.56)

Substituting for $r$ and $z$ ($r = R \sin \gamma_0, z = -R \cos \gamma_0$) as for $\theta_0$ yields

$$-Re \int_0^\infty kV e^{-ikVt} dk \int_{\Gamma_1} SG(k) \frac{\Delta}{8\pi \mu i V} e^{-i\frac{3\pi}{4}} \sqrt{\frac{2}{\pi \kappa R \sin \gamma \sin \gamma_0}} e^{i\kappa R \cos(\gamma - \gamma_0) \kappa \sin \gamma_0} \gamma d\gamma$$  \hspace{1cm} (C.57)

and the integration using the method of stationary phase with the stationary point at $\gamma = \gamma_0$ yields for the calculation of far field $SH$ radiation

$$-Re \int_0^\infty ikV G(k) e^{-ikVt} dk \frac{S\Delta \sin \gamma_0}{4\pi \mu V} e^{i\kappa R}$$  \hspace{1cm} (C.58)

and after simplification

$$U_\theta = \frac{d}{dt} G(t - \frac{R}{v}) \frac{S\Delta \sin \gamma_0}{4\pi \mu V}$$  \hspace{1cm} (C.59)

equivalent to Heelan’s result (Eq.13, Heelan, 1953).
C.6 Conclusions

The importance of this appendix was to substantiate the analysis of Heelan utilizing the mechanism of the Weyl integral as proffered by Brekhovskikh and corrected and expanded upon here. While doing so, the relationship between a scalar field and its plane wave decomposition, the Weyl integral, or its cylindrical wave decomposition, the Sommerfeld integral, was developed. This analysis is equivalent to that presented in Chapter 3 but the Chapter 3 analysis is simpler and treats axial and radial sources separately. However, the results of Heelan and Brekhovskikh have precedence and needed to be examined, corrected and expanded as was done here.
Figure C-1: Contour $\Gamma_1$ used by Brekhovskikh (1980, 2nd Edition, Fig. 28.1) for the Weyl integral. A member of a family of contours described by Stratton (1941, Fig. 67, contour $C_1$). Includes all homogeneous and inhomogeneous plane waves and is conditionally deformable at will as long as it terminates at the endpoints $(\frac{\pi}{2}, \infty)$ and $0, \frac{\pi}{2}$. Often advantageous to cross the origin to achieve a saddle point.
Brekhovskikh Steepest Descent Path

Figure C-2: Deformed steepest descent path \( \Gamma \) incorporating stationary phase approximation used by Brekhovskikh (1980, 2nd Edition, Fig. 28.1) along with original contour \( \Gamma_1 \).
Appendix D

Modifications of the Bouchon Schmitt boundary integral technique

In Chapter 2, the problem of axisymmetric sources applied to radially layered media was developed but a requirement was that the layered media consisted of cylindrical shells. However, it is possible to have an irregular cross section along the length of the borehole and still maintain axisymmetry. For example, a regular constriction in borehole radius along the length of the borehole maintains axisymmetry.

Wave propagation in an irregular borehole is a topic that has been addressed by Smith and Schuster (1985), and Bouchon and Schmitt (1989a,b) through use of the boundary integral technique. The difference between the work of Bouchon and Schmitt and Schuster and Smith is the means of evaluating the boundary integrals. Bouchon and Schmitt used the discrete wavenumber method whereas Schuster and Smith used quadrature techniques. Results of this research demonstrated that tube waves exhibit nearly complete reflections off borehole irregularities.

Here, the boundary integral technique of Bouchon and Schmitt (1989a,b) will be slightly modified to study wave propagation outside a borehole as a means of testing
the algorithm for radially layered media presented in Chapter 2. This can be done because a regular borehole is a degenerate form of an irregular borehole. The Bouchon and Schmitt technique can also be used for the investigation of radiation outside an irregular borehole which demonstrates the radiation of P and S waves into a formation from tube wave reflections inside a formation. However, the computational burden of the present formulation is such that these calculations have only been carried out for a very short distance so are of limited utility.

The basic strategy behind the boundary integral technique is as follows. N pairs of ring sources are placed inside and outside the borehole adjacent to each other and the N pairs are equally spaced along the length L of the borehole. The number of pairs, N, is chosen such that there is at least three points per the minimum wavelength of interest. For example, if the minimum wavelength was 9 cm, the maximum distance between pairs of ring sources would be 3cm.

The boundary conditions of continuity of normal displacement and normal stress and vanishing of tangential stress must be satisfied for each pair of ring sources. The coefficient to solve for in the case of each interior ring source is the magnitude of the pressure, Q, and the coefficients to solve for in the case of each ring source outside the borehole are the magnitudes of the horizontal, \( F_x \), and vertical, \( F_z \), force components. Thus the problem is set up with 3N coefficients and 3N boundary conditions for a completely determined system. The \( 3N \times 3N \) system is then inverted using a standard Gauss-Jordan elimination with partial pivoting routine as done in Bouchon and Schmitt (1989a,b). Computation times are such that the use of supercomputers is often advantageous for this algorithm. The integrals are calculated using the discrete wavenumber method and summed over \( 2M + 1 \) wavenumbers with \( 2M + 1 \) equalling N thus requiring \( N \) to be odd.

Consider one of the \( N \) points, \( j \) for instance, then the expression for the normal displacement inside the borehole at point \( j \) is the superposition of the Green's Function response from each of the \( N \) interior ring sources. This superposition is as follows
(Bouchon and Schmitt, 1989a,b)

\[ U_{r,j} = -\frac{4\pi}{L} \sum_{i=0}^{N} Q_i \sum_{m=-M}^{M} f_m G_{\alpha f,i} e^{ikz_m(z_j-z_i)} \]  

where \( G_{\alpha f,i} \) is defined in Eq. D.4.

The radial and vertical displacement for an exterior ring source \( j \) outside the borehole are the superpositions of the Green’s function for the horizontal and vertical force and are given by (Bouchon and Schmitt, 1989a,b)

\[ U_{r,j} = \sum_{i=1}^{i=N} \sum_{m=-M}^{M} \left( k_{zm}^2 G_{\beta,2} - \frac{\partial}{\partial r} \left( \frac{1}{r} \right) G_{\alpha,2} \right) e^{ikz_m(z_j-z_i)} \]  

\[ + \sum_{i=1}^{i=N} \sum_{m=-M}^{M} k_{zm} \left( -m_m G_{\beta,1} + l_m G_{\alpha,1} \right) e^{ikz_m(z_j-z_i)} \]  

\[ U_{z,j} = \sum_{i=1}^{i=N} \sum_{m=-M}^{M} k_{zm} \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \left( G_{\beta,2} - G_{\alpha,2} \right) e^{ikz_m(z_j-z_i)} \]  

\[ + \sum_{i=1}^{i=N} \sum_{m=-M}^{M} \left[ -m_m^2 G_{\beta} + k_{zm}^2 G_{\alpha} \right] e^{ikz_m(z_j-z_i)} \]  

where \( \varepsilon_n \) is the Neumann factor. The stresses can be derived from the displacements (Eq. D.3 as in Bouchon and Schmitt (1989a,b)) and the \( G \) functions are provided below

\[ G_{\alpha f}(r,r_i) = \begin{cases} 
I_0(f_m r) K_0(f_m r_i) & \text{for } r \leq r_i \\
I_0(f_m r_i) K_0(f_m r) & \text{for } r \geq r_i 
\end{cases} \]  

\[ G_{\alpha f,i}(r,r_i) = \begin{cases} 
-I_1(f_m r) K_0(f_m r_i) & \text{for } r \leq r_i \\
I_0(f_m r_i) K_1(f_m r) & \text{for } r \geq r_i 
\end{cases} \]  

\[ G_\alpha(r,r_i) = \begin{cases} 
I_0(l_m r) K_0(l_m r_i) & \text{for } r \leq r_i \\
I_0(l_m r_i) K_0(l_m r) & \text{for } r \geq r_i 
\end{cases} \]  

\[ G_\beta(r,r_i) = \begin{cases} 
I_0(m_m r) K_0(m_m r_i) & \text{for } r \leq r_i \\
I_0(m_m r_i) K_0(m_m r) & \text{for } r \geq r_i 
\end{cases} \]  

\[ G_{\alpha,1}(r,r_i) = \begin{cases} 
-I_1(l_m r) K_0(l_m r_i) & \text{for } r \leq r_i \\
I_0(l_m r_i) K_1(l_m r) & \text{for } r \geq r_i 
\end{cases} \]
\[ G_{\beta,1}(r, r_i) = \begin{cases} -I_1(m_mr)K_0(m_mr_i) & \text{for } r \leq r_i \\ I_0(m_mr_i)K_1(m_mr) & \text{for } r \geq r_i \end{cases} \]

\[ G_{\alpha,2}(r, r_i) = \begin{cases} I_1(l_mr)K_1(l_mr_i) & \text{for } r \leq r_i \\ I_0(l_mr_i)K_1(l_mr) & \text{for } r \geq r_i \end{cases} \]

\[ G_{\beta,2}(r, r_i) = \begin{cases} I_1(m_mr)K_1(m_mr_i) & \text{for } r \leq r_i \\ I_0(m_mr_i)K_1(m_mr) & \text{for } r \geq r_i \end{cases} \]

The procedure for calculation of displacements outside the borehole is straightforward. The displacements and stresses are calculated for each interior and exterior ring source \( j \) using the above equations. The \( 3N \times 3N \) system is then inverted to yield the coefficients \( Q_i, F_{ri}, \) and \( F_{zi}. \) Once these coefficients are determined, the far field displacement is calculated with the following summations

\[ U_r = \frac{1}{\rho \omega^2 L} \sum_{i=0}^{N} F_{ri} \sum_{m=0}^{M} \varepsilon_n \left( k_{zm}^2 I_1(m_mr_i)K_1(m_mr) - l_m^2 I_1(l_mr_i)K_1(l_mr) \right) \cos(k_{zm}(z - z_i)) \]

\[ -\frac{1}{\rho \omega^2 L} \sum_{i=0}^{N} F_{zi} \sum_{m=0}^{M} \varepsilon_n k_{zm} \left( -m_m I_0(m_mr_i)K_1(m_mr) + l_m I_0(l_mr_i)K_1(l_mr) \right) \sin(k_{zm}(z - z_i)) \]

\[ U_z = -\frac{1}{\rho \omega^2 L} \sum_{i=0}^{N} F_{ri} \sum_{m=0}^{M} \varepsilon_n k_{zm} \left( -m_m I_1(m_mr_i)K_0(m_mr) + l_m I_1(l_mr_i)K_0(l_mr) \right) \sin(k_{zm}(z - z_i)) + \frac{1}{\rho \omega^2 L} \sum_{i=0}^{N} F_{zi} \sum_{m=0}^{M} \varepsilon_n \left( -m_m^2 I_0(m_mr_i)K_0(m_mr) + k_{zm}^2 I_0(l_mr_i)K_0(l_mr) \right) \cos(k_{zm}(z - z_i)) \]

The difference between calculation at an exterior ring source and at a distance \( r \) away from the borehole is that it can be assumed that the radius \( r \) is greater than any radius \( r_i \) on the borehole wall so the \( G \) functions in Eq. (D.4) can be evaluated explicitly and the partial derivative operators can then be expanded. Additionally, since the distance to the point of investigation is fixed, there is only need to calculate one set of modified Bessel functions \( K_1(mr) \) and \( K_0(mr) \). This saves considerable
computation time.

The source contribution to the displacements and stresses of the interior sources has not been specified in the above treatment but is a Hertzian oscillator calculated as

\[
U_{r_{f,j}} = \frac{-V_0}{2\pi L} \sum_{m=-M}^{M} f_m K_1(f_m r_j)e^{ik smz_j},
\]

\[
U_{z_{f,j}} = \frac{V_0i}{2\pi L} \sum_{m=-M}^{M} k_{zm} K_0(f_m r_j)e^{ik smz_j},
\]

where \( V_0 \) is the volume of the source similar to the presentation in Chapter 2. One note is that the sign convention on \( V_0 \) used by Bouchon and Schmitt is a sonic well logging convention and opposite to the one used in this thesis. This convention was corrected in the figures comparing the two.
Appendix E

Hansen Vector Theory and Cylindrical Coordinates

In Appendix A, the development of the scalar wave equations, the use of the Helmholtz representation, the choice of eigenfunctions, and the key issue of separability was outlined for 5 commonly used symmetry assumptions when working with cylindrical coordinates.

The approach was conventional in terms of decomposition of the wavefields using the Helmholtz representation directly as is often done in physics (Morse and Feshbach, 1953; Ewing et al., 1957). The end result was the scalar potentials $\phi, \psi$, and $\chi$ representing longitudinal and two transverse waves respectively.

Another but similar approach has been demonstrated by Hansen (1935), Ben-Menahem and Singh (1968a,b, 1981), Pao and Mow (1971), Eringen and Suhubi (1975), Pilant (1979) and others which leads to a direct solution of the more rigorous vector Helmholtz equation and demonstrates vector separability. This approach is to use the mechanism of Hansen vectors (Ben-Menahem and Singh, 1981) which are essentially vector differential operators (div, grad, curl)applied to the scalar potentials $\phi, \psi$, and $\chi$ which produce vectors $\mathbf{L}, \mathbf{M},$ and $\mathbf{N}$.

The reason for the introduction of Hansen vectors is that theoretical develope-
ments utilizing Hansen vectors can be used in unmodified form across different coordinate systems and have great utility in describing boundary conditions. It can be shown directly using Hansen vector theory that 11 coordinate systems are separable satisfying the scalar Helmholtz equation but only six coordinate systems are separable satisfying the vector Helmholtz equation (Ben-Menahem and Singh, 1968a, 1981) and cylindrical coordinates are one of these types.

The treatment here is largely derived from Ben-Menahem and Singh (1981) and that book should be consulted for a more thorough treatment. However, the explicit expressions for cylindrical coordinate systems are derived from Ben-Menahem and Singh's formulas and a slightly different notation is used to conform with the developments of this thesis. As with the conventional treatment, the goal of Hansen vector theory is to establish three scalar potentials that individually satisfy the scalar Helmholtz wave equation and three vectors that collectively satisfy the vector Helmholtz wave equation.

In Appendix A, the displacement vector \( \vec{U} \) for cylindrical coordinates was decomposed into the following form (Eqs. 2.6,2.7)

\[
\vec{U} = \nabla \phi + \nabla \times (\chi \hat{e}_z) + \nabla \times \nabla \times (\psi \hat{e}_z)
\]  

(E.1)

Instead the decomposition can be written in terms of an equivalent vector decomposition with the Hansen vectors

\[
\begin{align*}
\mathbf{M} &= \nabla \times (\hat{e}_z \chi) = (\nabla \chi) \times \hat{e}_z \\
\mathbf{N} &= \frac{1}{k_c} \nabla \times \nabla \times (\hat{e}_z \psi) \\
\mathbf{L} &= \frac{1}{k_c} \nabla \phi
\end{align*}
\]  

(E.3)

where the factor \( \frac{1}{k_c} \) is a factor to keep the dimensions equivalent across the vectors and is set equal to 1 for the cases considered here. There is a difference in sign between the potential \( \psi \) presented in Appendix A and the \( \psi \) presented here but is irrelevant due to symmetry considerations.
The displacement vector \( \vec{U} \) can be written in terms of the Hansen vectors as follows

\[
\vec{U} = \mathbf{L} + \mathbf{M} + \mathbf{N} \quad (E.4)
\]

Does the Hansen vector formalism provide a separable system of vector coordinates that satisfy the vector Helmholtz equations? Yes, it can be shown that the following vector Helmholtz equations can be satisfied

\[
\nabla^2 \mathbf{M} + \mathbf{M} = 0 \quad (E.5)
\]
\[
\nabla^2 \mathbf{N} + \mathbf{N} = 0
\]
\[
\nabla^2 \mathbf{L} + \mathbf{L} = 0
\]

as demonstrated below.

The cylindrical coordinate system has unit orthogonal vectors \( \hat{e}_r, \hat{e}_\theta, \hat{e}_z \). It can be seen that \( \pm \hat{e}_z \) is normal to planes \( z = z_0 \). If on the other hand \( \theta \) or \( r \) were to be held constant the surfaces generated would not be planes but instead a half plane and a cylinder so it would be impossible to choose an orthogonal basis vector to one of these two surfaces. Since the goal is to construct a solution of orthogonal vectors a full plane is necessary as provided by the vector \( \hat{e}_z \) which is normal to the plane \( z = z_0 \).

The vector \( \mathbf{M} = \nabla \times (\hat{e}_z \psi) \) where

\[
\nabla^2 \psi = -\psi \quad (E.6)
\]

is a vector that is tangential to planes of \( z = z_0 \). Applying the scalar operator \((\nabla^2 + 1)\) to the vector \( \mathbf{M} \) which commutes with the curl vector operator (Ben-Menahem and Singh, 1981) yields

\[
\nabla^2 \mathbf{M} + \mathbf{M} = \nabla \times [\psi \nabla^2 \hat{e}_z + 2(\nabla \psi \cdot \nabla \hat{e}_z)] \quad (E.7)
\]

where the vector identities A.35 and A.44 from Ben-Menahem and Singh (1981) were used to obtain intermediate results prior to simplification.

253
A sufficient condition that Eq. E.7 equals zero is that the two terms in brackets \( \psi \nabla^2 \hat{e}_z \) and \( (\nabla \psi \cdot \nabla \hat{e}_z) \) vanish. For cylindrical coordinates this condition is met only by \( \hat{e}_z \) (Ben-Menahem and Singh, 1981). Therefore, the Hansen vectors solve the vector Helmholtz equation Eq. E.5.

The first symmetry assumption presented in Section A.2.1 of Appendix A was the degenerate assumption of no symmetry where the potentials \( \phi, \psi, \chi \) were completely coupled. The Hansen vectors can then be written

\[
\mathbf{L} = \hat{e}_r \frac{\partial \phi}{\partial r} + \hat{e}_\theta \frac{\partial \phi}{\partial \theta} + \hat{e}_z \frac{\partial \phi}{\partial z} \\
\mathbf{N} = -\hat{e}_r \frac{\partial^2 \psi}{\partial r \partial z} - \hat{e}_\theta \left( \frac{1}{r} \frac{\partial \psi}{\partial \theta} \right) - \hat{e}_z \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \\
\mathbf{M} = \frac{1}{r} \frac{\partial \chi}{\partial \theta} \hat{e}_r - \frac{\partial \chi}{\partial r} \hat{e}_\theta
\]

If axisymmetry is assumed, the displacement components can be written for the \( P-SV \) case from Section A.2.3 of Appendix A (Eq. A.32) and for the \( SH \) problem (Eq. A.34)

\[
U_r = \frac{\partial \phi}{\partial r} + \frac{\partial^2 \psi}{\partial r \partial z} \\
U_z = \frac{\partial \phi}{\partial z} - \frac{1}{r} \frac{\partial \psi}{\partial r} (r \frac{\partial \psi}{\partial r})
\]

and the Hansen vectors can be written

\[
\mathbf{L} = \frac{\partial \phi}{\partial r} \hat{e}_r + \frac{\partial \phi}{\partial z} \hat{e}_z
\\
\mathbf{M} = -\frac{\partial \chi}{\partial r} \hat{e}_\theta
\\
\mathbf{N} = \frac{\partial^2 \psi}{\partial r \partial z} \hat{e}_r - \left( \frac{\partial^2 \psi}{\partial r^2} + \frac{\partial \psi}{\partial r} (\frac{\partial \psi}{r}) \right) \hat{e}_z
\]

For the second strategy of Appendix A where the potential \( \psi \) is transformed to \(-\frac{\partial \psi}{\partial r}\) the displacement components can be written

\[
U_r = \frac{\partial \phi}{\partial r} + \frac{\partial \psi}{\partial r} \\
U_z = \frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial r} \frac{\psi}{r}
\]
so the Hansen vectors are

\[ L = \frac{\partial \phi}{\partial r} \hat{e}_r + \frac{\partial \phi}{\partial z} \hat{e}_z \]  
\[ M = 0 \]  
\[ N = \frac{\partial \psi}{\partial z} \hat{e}_r - \left( \frac{\partial \psi}{\partial r} + \frac{\psi}{r} \right) \hat{e}_z \]  

(E.13)

For the last strategy of Appendix A where symmetry was assumed in the z axis, the displacement potential equations (Eq. A.40) are

\[ U_r = \frac{\partial \phi}{\partial r} + \frac{1}{r} \frac{\partial \chi}{\partial \theta} \]  
\[ U_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} - \frac{\partial \chi}{\partial r} \]  

(E.14)

where the Hansen vectors are the following

\[ L = \frac{\partial \phi}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{e}_\theta \]  
\[ N = \frac{1}{r} \frac{\partial \chi}{\partial \theta} \hat{e}_r - \frac{\partial \chi}{\partial r} \hat{e}_\theta \]  
\[ M = 0 \]  

(E.15)
Appendix F

Algebra for Hankel functions

The elements of the D matrix were presented in Chapter 2 where modified Bessel functions were used as eigenfunctions. This appendix provides the analogous algebra for the case where Hankel functions are used as the eigenfunctions. The displacement potentials are

\[
\begin{align*}
\phi_f &= \frac{1}{(2\pi)^2} \int \int_{-\infty}^{\infty} B_f J_0(fa)e^{-ik_z r} e^{i\omega t} dk_z d\omega \\
\phi_i &= \frac{1}{(2\pi)^2} \int \int_{-\infty}^{\infty} (A_i H_0^{(2)}(l_ir_i) + B_i H_0^{(1)}(l_ir_i))e^{-ik_z r} e^{i\omega t} dk_z d\omega \\
\psi_i &= \frac{1}{(2\pi)^2} \int \int_{-\infty}^{\infty} (C_i H_1^{(2)}(m_ir_i) + D_i H_1^{(1)}(m_ir_i))e^{-ik_z r} e^{i\omega t} dk_z d\omega
\end{align*}
\]

where \( a \) is equal to \( r_1 \) and is the radius of the borehole. The \( \phi_f \) potential, an incoming wave, is in terms of \( J_0 \) because of the presence of a logarithmic singularity for the Hankel function \( H_0^{(1)} \) at zero.

The source term can be written (Lee and Balch, 1982) as an outgoing wave

\[
S = \frac{-V_0 e^{-i\omega R}}{4\pi R} = \frac{iV_0}{8\pi} \int_{-\infty}^{\infty} H_0^{(2)}(fR)e^{-ik_z r} dk_z
\]  \hspace{1cm} (F.1)

hence analogous to Eq. 2.28

\[
S_f = \frac{iG(\omega)V_0}{4}
\]  \hspace{1cm} (F.2)

The elements of the D matrix (cf. Eq. 2.8) are then given by

\[
D_{11} = -l_i H_1^{(2)}(l_ir_i)
\]  \hspace{1cm} (F.3)
\[ D_{12} = -l_i H_1^{(1)}(l_i r_i) \]
\[ D_{13} = i k_z H_1^{(2)}(m_i r_i) \]
\[ D_{14} = i k_z H_1^{(1)}(m_i r_i) \]
\[ D_{21} = -i k_z H_0^{(2)}(l_i r_i) \]
\[ D_{22} = -i k_z H_0^{(1)}(l_i r_i) \]
\[ D_{23} = m_i H_0^{(2)}(m_i r_i) \]
\[ D_{24} = m_i H_0^{(1)}(m_i r_i) \]
\[ D_{31} = (-\rho_i \omega^2 + 2 \mu_i k_z^2) H_0^{(2)}(l_i r_i) + \frac{2 \mu_i l_i}{r_i} H_1^{(2)}(l_i r_i) \]
\[ D_{32} = (-\rho_i \omega^2 + 2 \mu_i k_z^2) H_0^{(1)}(l_i r_i) + \frac{2 \mu_i l_i}{r_i} H_1^{(1)}(l_i r_i) \]
\[ D_{33} = 2 \mu_i k_z [m_i H_0^{(2)}(m_i r_i) - \frac{H_1^{(2)}(m_i r_i)}{r_i}] \]
\[ D_{34} = 2 \mu_i k_z [m_i H_0^{(1)}(m_i r_i) - \frac{H_1^{(1)}(m_i r_i)}{r_i}] \]
\[ D_{41} = 2 \mu_i k_z l_i H_1^{(2)}(l_i r_i) \]
\[ D_{42} = 2 \mu_i k_z l_i H_1^{(1)}(l_i r_i) \]
\[ D_{43} = (-\rho_i \omega^2 + 2 \mu_i k_z^2) H_1^{(2)}(m_i r_i) \]
\[ D_{44} = (-\rho_i \omega^2 + 2 \mu_i k_z^2) H_1^{(1)}(m_i r_i) \]

and the elements of the \( D_f \) matrix are given by

\[
D_{f11} = -f H_1^{(2)}(fa) \\
D_{f12} = -f J_1(fa) \\
D_{f31} = -\rho f \omega^2 H_0^{(2)}(fa) \\
D_{f32} = -\rho f \omega^2 J_0(fa)
\] (F.4)

The simplification provided by Schmitt and Bouchon (1985) for calculating the inverse matrix \( D \) with modified Bessel functions as eigenfunctions Eq. 2.13 has a completely analogous counterpart for Hankel functions. The Wronskian governing
Hankel functions is

\[ H_1^{(1)}(z)H_0^{(2)}(z) - H_0^{(1)}(z)H_1^{(2)}(z) = \frac{-4i}{\pi z} \]  

(F.5)

and the inverse matrix is thus

\[ D_i^{-1} = \frac{\pi r_i}{4i\omega^2\rho_i} \begin{vmatrix} D_{32} & -D_{42} & -D_{12} & D_{22} \\ -D_{31} & D_{41} & D_{11} & -D_{21} \\ -D_{34} & D_{44} & D_{14} & -D_{24} \\ D_{33} & -D_{43} & -D_{13} & D_{23} \end{vmatrix} \]  

(F.6)

F.1 Particular models: P-SV

For an empty borehole the Cramer’s rule denominator is

\[ (-\rho^2 + 2\mu k_z^2)^2 H_0^{(2)}(l_1 a)H_1^{(2)}(m_1 a) - \frac{2\mu l_1 \rho \omega^2}{a} H_1^{(2)}(l_1 a)H_1^{(2)}(m_1 a) \]

\[ + 4\mu^2 k_z^2 l_1 m_1 H_0^{(2)}(m_1 a)H_1^{(2)}(l_1 a) \]  

(F.7)

with the Cramer’s rule numerators being given by

<table>
<thead>
<tr>
<th>Source Type</th>
<th>Cramer’s Rule Numerator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radial Source (2l)</td>
<td>[ A = -G(\omega)(-\rho^2 + 2\mu k_z^2)H_1^{(2)}(m_1 a) \frac{2\sin lk_z}{k_z} ]</td>
</tr>
<tr>
<td></td>
<td>[ C = G(\omega)2\mu ik_z l_1 H_1^{(2)}(l_1 a) \frac{2\sin lk_z}{k_z} ]</td>
</tr>
<tr>
<td>Axial Source (2l)</td>
<td>[ A = G(\omega)2\mu ik_z [m_1 H_0^{(2)}(m_1 a) - \frac{H_1^{(2)}(m_1 a)}{a} \frac{2\sin lk_z}{k_z} ]</td>
</tr>
<tr>
<td></td>
<td>[ C = -G(\omega)[(-\rho^2 + 2\mu k_z^2)H_0^{(2)}(l_1 a) + \frac{2\mu l_1}{a} H_1^{(2)}(l_1 a)] \frac{2\sin lk_z}{k_z} ]</td>
</tr>
</tbody>
</table>

Table F.1: Cramer’s rule numerators for an empty borehole (Hankel Functions).
Similarly for a fluid-filled borehole the denominator is

\[ f\left[ (-\rho \omega^2 + 2\mu k_z^2) H_0^{(2)}(l_1 a) H_1^{(2)}(m_1 a) - \frac{2\mu_1 l_1 \rho_1 \omega^2}{a} H_1^{(2)}(l_1 a) H_1^{(2)}(m_1 a) \right] \]  

(F.8)

\[ + 4\mu^2 k_z^2 l_1 m_1 H_1^{(2)}(l_1 a) H_0^{(2)}(m_1 a) J_1(fa) - \rho \rho_1 \omega^4 l_1 H_1^{(2)}(l_1 a) H_1^{(2)}(m_1 a) J_0(fa) \]

and the solutions for the numerators are given by the table below.

One note is that a Wronskian relationship has been used similar to the one for modified Bessel functions to simplify the numerators for the volume point source.

\[ J_0(z) H_1^{(2)}(z) - J_1(z) H_0^{(2)}(z) = \frac{2i}{\pi z} \]  

(F.9)

Although this Wronskian relationship is not explicitly listed in any of the standard reference works on Bessel functions (Watson, 1944; Abramowitz and Stegun, 1964) it can be derived by rearrangement and making use of the relationship

\[ H_n^{(2)}(z) = J_n(z) - i Y_n(z) \]  

(F.10)
### Cramer's Rule Numerator for Fluid-Filled Borehole (Hankel Functions)

<table>
<thead>
<tr>
<th>Source Type</th>
<th>Numerator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radial Source (2l)</td>
<td>[ B_f = -G(\omega) \frac{2 \sin l k_z}{k_z} \rho \omega^2 l_1 H_1^{(2)}(l_1 a) H_1^{(3)}(m_1 a) ]</td>
</tr>
<tr>
<td></td>
<td>[ A = -G(\omega) \frac{2 \sin l k_z}{k_z} f(-\rho \omega^2 + 2 \mu k_z^2) H_1^{(2)}(m_1 a) J_1(f a) ]</td>
</tr>
<tr>
<td></td>
<td>[ C = G(\omega) \frac{2 \sin l k_z}{k_z} 2 \mu ik_z f l_1 H_1^{(2)}(l_1 a) J_1(f a) ]</td>
</tr>
<tr>
<td>Axial Source (2l)</td>
<td>[ B_f = G(\omega) \frac{2 \sin l k_z}{k_z} (2 \mu ik_z l_1 m_1 H_0^{(2)}(m_1 a) H_1^{(2)}(l_1 a) ]</td>
</tr>
<tr>
<td></td>
<td>[ + i k_z (-\rho \omega^2 + 2 \mu k_z^2) H_0^{(2)}(l_1 a) H_1^{(2)}(m_1 a)) ]</td>
</tr>
<tr>
<td></td>
<td>[ A = G(\omega) \frac{2 \sin l k_z}{k_z} (2 f \mu ik_z [m_1 H_0^{(2)}(m_1 a) - \frac{H_1^{(2)}(m_1 a)}{a}]) J_1(f a) ]</td>
</tr>
<tr>
<td></td>
<td>[ - \rho f \omega^2 i k_z J_0(f a) H_1^{(2)}(m_1 a)) ]</td>
</tr>
<tr>
<td></td>
<td>[ C = -G(\omega) \frac{2 \sin l k_z}{k_z} (f(-\rho \omega^2 + 2 \mu k_z^2) H_0^{(2)}(l_1 a) ]</td>
</tr>
<tr>
<td></td>
<td>[ + \frac{2 \mu l_1}{a} H_1^{(2)}(l_1 a)) J_1(f a) + \rho f \omega^2 l_1 J_0(f a) H_1^{(2)}(l_1 a)) ]</td>
</tr>
<tr>
<td>Volume point source</td>
<td>[ B_f = G(\omega) \frac{i V_0}{4} (-f([-\rho \omega^2 + 2 \mu k_z^2] H_0^{(2)}(l_1 a) H_1^{(2)}(m_1 a) ]</td>
</tr>
<tr>
<td></td>
<td>[ - \frac{2 \mu l_1 \rho \omega^2}{a} H_1^{(2)}(l_1 a) H_1^{(2)}(m_1 a) + 4 \mu^2 k_z^2 l_1 m_1 H_1^{(2)}(l_1 a) H_0^{(2)}(m_1 a) ]</td>
</tr>
<tr>
<td></td>
<td>[ H_1^{(2)}(f a) + \rho \rho f \omega^4 l_1 H_1^{(2)}(l_1 a) H_1^{(2)}(m_1 a) H_0^{(2)}(f a) ]</td>
</tr>
<tr>
<td></td>
<td>[ A = G(\omega) \frac{i V_0}{4} \frac{2 i \rho f \omega^2}{\pi a} (-\rho \omega^2 + 2 \mu k_z^2) H_1^{(2)}(m_1 a) ]</td>
</tr>
<tr>
<td></td>
<td>[ C = G(\omega) \frac{i V_0}{4} \frac{4 \rho f \mu ik_z \omega^2 l_1}{\pi a} H_1^{(2)}(l_1 a) ]</td>
</tr>
</tbody>
</table>

Table F.2: Cramer's rule numerators for a fluid-filled borehole (Hankel Functions).
F.2 Particular models: SH

For $SH$ waves the potential $\chi$ is given by

$$\chi = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} (E_i H_0^{(2)}(mr) + F_i H_0^{(1)}(mr)) e^{-ikz} e^{i\omega t} dk_d\omega \quad (F.11)$$

The elements of the $D$ matrix are

$$D_{11} = m_i H_1^{(2)}(m_ir_i) \quad (F.12)$$
$$D_{12} = m_i H_1^{(1)}(m_ir_i)$$
$$D_{21} = \mu_i (m_i^2 H_0^{(2)}(m_ir_i) - \frac{2m_i H_1^{(2)}(m_ir_i)}{r_i})$$
$$D_{22} = \mu_i (m_i^2 H_0^{(1)}(m_ir_i) - \frac{2m_i H_1^{(1)}(m_ir_i)}{r_i})$$

and the inverse of the $D_i$ matrix is

$$D_i^{-1} = \frac{\pi r_i}{4i\mu m^2} \begin{vmatrix} D_{22} & -D_{12} \\ -D_{21} & D_{11} \end{vmatrix} \quad (F.13)$$

with the particular solution of a torsional source applied to the wall of a borehole surrounded by an infinite half space given by

$$E = \frac{-G(\omega)}{\mu (m^2 H_0^{(2)}(ma) - \frac{2m H_1^{(2)}(ma)}{a})} \frac{2\sin lk_z}{k_z} \quad (F.14)$$
Appendix G

Uniform Asymptotic Expansion of the Shear Wave Integrals

Integrals of the form
\[ \int_{-\infty}^{\infty} F(k_z) e^{Rj(k_z)} dk_z \]  
(similar to Eq. 3.15) where \( R \) is a large parameter are usually evaluated by a straightforward application of the method of stationary phase. However, if there is a singularity near the saddle point such as a pole, the traditional method of stationary phase cannot be utilized because the integral will no longer be regular near the saddle point. To evaluate this type of integral, a uniform asymptotic expansion may be used. The first application of this type of analysis was accomplished by Pauli (1938) in solving a special case of Sommerfeld’s wedge diffraction problem. The special case was determining the diffracted field when the area of interest is in the shadow of the wedge. By being in the the shadow zone, a pole was introduced in the vicinity of the steepest descent path so the integral had to be transformed into an asymptotic series expansion, now known as an uniform asymptotic expansion.

Uniform asymptotic expansion is a very powerful technique for evaluating integrals with saddle points close to critical points. Critical points in complex integration refer to poles, branch cuts and points which have a major influence on the resulting integral
These asymptotic expansions are called "uniform" because the error progressively decreases as more terms in the expansion are kept. The use of asymptotic expansions for solving integration problems has been accomplished by Friedman (1959) who solved the problem of having two nearby stationary points and Felsen and Marcuvitz (1959, 1973) and Felsen (1963) who treated the case for stationary points near poles of order \( n \) where \( n \) is an integer or \( n = \pm \frac{1}{2} \). Bleistein (1966) generalized the uniform asymptotic expansion integration technique to include poles of any order, branch points and endpoints of integration near stationary points. The general theory of critical points near saddle points is well treated in the books by Felsen and Marcuvitz (1973) and Bleistein and Handelsman (1976).

For the integrals in Chapter 3, the endpoints are at \( \pm \infty \) and are well behaved at these limits. Therefore, these integrals nicely fit the criteria of Bleistein and Handelsman (1976) for the appropriateness of uniform asymptotic expansion techniques. This Appendix will closely follow the work of Felsen and Marcuvitz for the case of a saddle point near a simple pole (pg. 399-402, Felsen and Marcuvitz, 1973) as originally developed by Van der Waerden (1951). A more general treatment may also be found in Bleistein and Handelsman (1976) in their solution of the Klein-Gordon equation and for other integrals with steepest descent paths near other types of critical points.

The integrals under consideration from Chapter 3, Eq. G.1, have a pole at \( k_{zp} = \frac{\omega}{c_T} \) and a saddle point at \( k_{s0} = -\frac{\omega}{\beta} \cos \phi \) (remember \( \cos \phi = \frac{\omega}{c_T} \)). The saddle point will be coincident with the pole at \( k_z = \frac{\omega}{c_T} \) for some value of \( \phi \) if \( C_T > \beta \). In Chapters 2 and 3 this angle \( \phi \) was identified as the complementary Mach angle \( \phi_c \). Even if \( C_T < \beta \), the pole will have an influence as \( \phi \) approaches \( \frac{\pi}{2} \). Thus a uniform asymptotic expansion can help evaluate the integral in both cases. The integrals for the \( P \) waves are usually far enough away from the tube wave pole to exhibit almost no influence due to this pole.

As typical examples of these functions, the radial component for a shear wave from a radial source would have the following definitions for the functions \( f \) and \( F \)
(Eq. 3.39)

\[ F(k_z) = \frac{a^2 l G(\omega) i k_z f^2 \sqrt{m}}{2 \rho \omega^2 (k_z + \frac{\omega}{c_T})(k_z - \frac{\omega}{c_T})} \sqrt{\frac{2}{\pi r}} e^{i z z} \]  

\[ f(k_z) = i (k_z \cos \phi - m \sin \phi) \]

\[ f'(k_z) = i (\cos \phi + \frac{k_z}{m} \sin \phi) \]

\[ f''(k_z) = i \frac{(m^2 \sin \phi + k_z^2 \sin \phi)}{m^3} \]

The saddle point is defined by the equation

\[ f'(k_{z0}) = 0 \quad k_{z0} = -\frac{\omega}{\beta} \cos \phi \quad f''(k_{z0}) = \frac{i \beta}{\omega \sin^2 \phi} \]  

and since \( f''(k_{z0}) \neq 0 \), the saddle point is a simple saddle point (order 1) meeting a requirement.

The solution to the integral is given by (Eq. 4.4.2, Felsen and Marcuvitz, 1973)

\[ I \sim e^{R f(k_z)} \{ \pm i 2 A \sqrt{\pi e^{-R b^2}} Q(\mp i b \sqrt{R}) + \sqrt{\frac{\pi}{R}} (h F(k_{z0}) + \frac{A}{b}) \} \]

where

\[ b = \sqrt{f(k_{z0}) - f(k_{zp})} \]  

\[ h = \sqrt{-2 \over f''(k_{z0})} \quad \text{arg} \ h = \frac{\pi}{4} \]

\( Q \) is the complex complementary error function multiplied by \( \frac{\sqrt{\pi}}{2} \) given by

\[ Q(z) = \int_{z}^{\infty} e^{-y^2} dy \]

and \( A \) is

\[ A = \lim_{k_z \to k_{zp}} [(k_z - k_{zp}) F(k_z)] \]

The ± sign and ∓ correspond to the sign and anti-sign of \( Im \ b \). The error function in essence corrects the integral for the influence of the pole on the steepest descent path.
For the cases here

\[ k_{z_0} = \frac{-\omega \cos \phi}{\beta} \]  

(G.9)

\[ k_{z_p} = \frac{\omega}{C_T} \]

\[ f(k_{z_0}) = -\frac{i\omega}{\beta} \]

\[ f''(k_{z_0}) = \frac{i\beta}{\omega \sin^2 \phi} \]

\[ f(k_{z_p}) = i\omega \left( \frac{\cos \phi}{C_T} - \frac{\sin \phi \sqrt{M^2 - 1}}{C_T} \right) \]

\[ b = \sqrt{-i\omega \left( \frac{1}{\beta} + \frac{\cos \phi}{C_T} - \frac{\sin \phi \sqrt{M^2 - 1}}{C_T} \right) \}

\[ b^2 = -i\omega \left( \frac{1}{\beta} + \frac{\cos \phi}{C_T} - \frac{\sin \phi \sqrt{M^2 - 1}}{C_T} \right) \]

\[ h = e^{i\frac{\pi}{4}} \sin \phi \sqrt{\frac{2\omega}{\beta}} \]

\[ A = \frac{a^2 i\omega \cos^2 \phi G(\omega)}{2\mu \left( \frac{-\cos \phi}{\beta} + \frac{1}{C_T} \right)} \left( \frac{1}{\alpha^2} - \frac{\cos^2 \phi}{\beta^2} \right) \left( \frac{\omega^2}{\beta^2} - \frac{\omega^2 \cos^2 \phi}{\beta^2} \right) \frac{1}{4} \sqrt{\frac{2}{\pi \tau}} e^{\frac{i\pi}{4}} \]

\[ F(k_{z_0}) = \frac{G(\omega) a^2 i\omega \cos^2 \phi G(\omega)}{2\mu \left( \frac{-\cos^2 \phi}{\beta^2} - \frac{1}{C_T^2} \right)} \left( \frac{1}{\alpha^2} - \frac{\cos^2 \phi}{\beta^2} \right) \left( \frac{\omega^2}{\beta^2} - \frac{\omega^2 \cos^2 \phi}{\beta^2} \right) \frac{1}{4} \sqrt{\frac{2}{\pi \tau}} e^{\frac{i\pi}{4}} \]

Hence the asymptotic representation of the integral equals

\[ e^{-\frac{i\omega R}{\beta}} \left\{ \pm i2\sqrt{\pi} e^{i\omega R \left( \frac{1}{\beta} + \frac{\cos \phi}{C_T} - \frac{\sin \phi \sqrt{M^2 - 1}}{C_T} \right)} \left( \frac{1}{\alpha^2} - \frac{\cos^2 \phi}{\beta^2} \right) \left( \frac{\omega^2}{\beta^2} - \frac{\omega^2 \cos^2 \phi}{\beta^2} \right) \frac{1}{4} \sqrt{\frac{2}{\pi \tau}} e^{\frac{i\pi}{4}} \right\} \]

\[ \cdot \bar{Q}(\mp i\sqrt{R} \sqrt{-i\omega \left( \frac{1}{\beta} + \frac{\cos \phi}{C_T} - \frac{\sin \phi \sqrt{M^2 - 1}}{C_T} \right)}) \]

\[ + \sqrt{\frac{\pi}{R}} \left( e^{i\frac{\pi}{4}} \sin \phi \sqrt{\frac{2\omega G(\omega) a^2 i\omega \cos^2 \phi G(\omega)}{2\mu \left( \frac{-\cos \phi}{\beta^2} - \frac{1}{C_T^2} \right)}} \right) \]

\[ + \sqrt{\frac{\pi}{R}} \left( e^{i\frac{\pi}{4}} \sin \phi \sqrt{\frac{2\omega G(\omega) a^2 i\omega \cos^2 \phi G(\omega)}{2\mu \left( \frac{-\cos \phi}{\beta^2} + \frac{1}{C_T} \right)}} \right) \]

\[ + \sqrt{\frac{\pi}{R}} \left( e^{i\frac{\pi}{4}} \sin \phi \sqrt{\frac{2\omega G(\omega) a^2 i\omega \cos^2 \phi G(\omega)}{2\mu \left( \frac{-\cos \phi}{\beta^2} + \frac{1}{C_T} \right)}} \right) \]

(G.10)

which can be simplified remembering that \( z = -R \cos \phi, r = R \sin \phi \) and it turns out
that the \( \sqrt{\frac{\pi}{R}} h F(k_0) \) term of Eq. G.4 (3rd eqn of Eq. G.10) equals the method of stationary phase solution from Chapter 3, Table 3.1.

\[
I \sim -e^{-i\omega(\frac{\pi}{2\rho} + \frac{\sqrt{M^2-1}}{C_T})}\left(\frac{\pm a^2 \omega \cos^2 \phi G(\omega)}{\mu(-\cos \phi + \frac{1}{C_T})} \left(\frac{1}{\alpha_f^2} - \frac{\cos^2 \phi}{\beta^2} - \frac{\omega^2 \cos^2 \phi}{\beta^2}\right) \frac{1}{r} \sqrt{\frac{2}{r}} e^{-i\frac{\pi}{4}}\right)
\cdot Q(\mp i \sqrt{-\omega R(\frac{1}{\beta} + \frac{\cos \phi}{C_T} + \frac{1}{C_T})^{-1}})
\cdot \frac{e^{i\omega R}}{R} G(\omega) a^2 \omega \cos \phi \sin 2\phi (\frac{1}{\alpha_f^2} - \frac{\cos^2 \phi}{\beta^2})
\frac{e^{-i\omega R}}{R} G(\omega) a^2 \omega \cos \phi \sin 2\phi (\frac{1}{\alpha_f^2} - \frac{\cos^2 \phi}{\beta^2})
\frac{2\mu(-\cos \phi + \frac{1}{C_T})}{2\mu(-\cos \phi + \frac{1}{C_T})\sqrt{-\omega R(\frac{1}{\beta} + \frac{\cos \phi}{C_T} + \frac{1}{C_T})^{-1}}} e^{i\frac{\pi}{4}}\frac{1}{r} e^{i\frac{\pi}{4}}
\]

(G.11)

and reintroducing the integral over \( \omega \)

\[
U_r \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \left\{ -e^{-i\omega(\frac{\pi}{2\rho} + \frac{\sqrt{M^2-1}}{C_T})}\left(\frac{\pm a^2 \omega G(\omega)}{\mu(-\cos \phi + \frac{1}{C_T})} \left(\frac{1}{\alpha_f^2} - \frac{1}{C_T} \right) \left(\frac{\cos \phi}{\beta^2} - \frac{\omega^2 \cos^2 \phi}{\beta^2}\right) \frac{1}{r} \sqrt{\frac{2}{r}} e^{-i\frac{\pi}{4}}\right)
\cdot Q(\mp i \sqrt{-\omega R(\frac{1}{\beta} + \frac{\cos \phi}{C_T} + \frac{1}{C_T})^{-1}})
\cdot \frac{e^{i\omega R}}{R} G(\omega) a^2 \omega \cos \phi \sin 2\phi (\frac{1}{\alpha_f^2} - \frac{\cos^2 \phi}{\beta^2})
\frac{e^{-i\omega R}}{R} G(\omega) a^2 \omega \cos \phi \sin 2\phi (\frac{1}{\alpha_f^2} - \frac{\cos^2 \phi}{\beta^2})
\frac{2\mu(-\cos \phi + \frac{1}{C_T})}{2\mu(-\cos \phi + \frac{1}{C_T})\sqrt{-\omega R(\frac{1}{\beta} + \frac{\cos \phi}{C_T} + \frac{1}{C_T})^{-1}}} e^{i\frac{\pi}{4}}\frac{1}{r} e^{i\frac{\pi}{4}}\right\} d\omega
\]

(G.12)

Similarly for the calculation of \( U_z \),

\[
F(k_z) = \frac{a^2 \omega G(\omega) k_z f^2 \frac{m^3}{2}}{2\rho \omega^2 (k_z + \frac{\omega}{C_T})(k_z - \frac{\omega}{C_T})} \sqrt{\frac{2}{\pi r}} e^{i\frac{\pi}{4}}
\]

(G.13)

\( f(k_z) \) remains the same (Eq. G.2) and

\[
U_z \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \left\{ -e^{-i\omega(\frac{\pi}{2\rho} + \frac{\sqrt{M^2-1}}{C_T})}\left(\frac{\pm a^2 \omega G(\omega)}{2\rho} \left(\frac{1}{\alpha_f^2} - \frac{1}{C_T^2} \right) \left(\frac{\omega^2}{\beta^2} - \frac{\omega^2 \cos^2 \phi}{\beta^2}\right) \frac{1}{r} \sqrt{\frac{2}{r}} e^{-i\frac{\pi}{4}}\right)\right\} d\omega
\]

(G.14)
\[
\begin{align*}
Q & \left( \pi i \sqrt{-i \omega R \left( \frac{1}{\beta} + \frac{\cos \phi}{C_T} - \frac{\sin \phi \sqrt{M^2 - 1}}{C_T} \right)} \right) \\
+ e^{-\frac{\pi R}{B}} G(\omega) a^2 i \omega \cos \phi \sin 2\phi \left( \frac{1}{\alpha^2} - \frac{\cos^2 \phi}{\beta^2} \right) \\
+ \frac{2 \mu (\frac{1}{C_T^2} - \frac{\cos^2 \phi}{\beta^2})}{R} \\
- e^{-\frac{\pi R}{B}} a^2 i \omega \left( \frac{1}{\alpha^2} - \frac{\omega^2}{\beta^2} \right) \frac{\omega^2}{\beta^2} \sqrt{2} e^{i \pi r} \\
& \frac{4 \rho \sqrt{-i \omega R \left( \frac{1}{\beta} + \frac{\cos \phi}{C_T} - \frac{\sin \phi \sqrt{M^2 - 1}}{C_T} \right)}}{d\omega} \\
\end{align*}
\]

For the radial component of a volume point source

\[
F(k_z) = \frac{V_0 \rho f G(\omega) i k_z \sqrt{m}}{4 \pi \rho (k_z + \frac{w}{C_T}) (k_z - \frac{w}{C_T})} \sqrt{\frac{2}{\pi r}} e^{i \frac{m}{r}} \tag{G.15}
\]

again \( f(k_z) \) remains the same (Eq. G.2) so

\[
U_{zvol} \sim \int_{-\infty}^{\infty} e^{i \omega t} \left\{ -e^{-i \omega (\frac{1}{\beta} + \frac{\sqrt{M^2 - 1}}{C_T})} \left( \frac{1}{\omega} V_0 \rho f G(\omega) \frac{\omega^2}{\beta^2} \right) \frac{1}{\sqrt{r}} e^{i \frac{m}{r}} \right\} \tag{G.16}
\]

\[
\cdot Q(\pi i \sqrt{-i \omega R \left( \frac{1}{\beta} - \frac{z}{C_T} - \frac{r \sqrt{M^2 - 1}}{C_T} \right)} ) \\
+ e^{-\frac{\pi R}{B}} G(\omega) V_0 \rho f i \omega \cos \phi \sin 2\phi \\
& \frac{2 \mu (\frac{1}{C_T^2} - \frac{\cos^2 \phi}{\beta^2})}{R} \\
+ e^{-\frac{\pi R}{B}} a^2 \sqrt{2} e^{i \pi r} \\
& \frac{8 \pi \rho C_T \sqrt{-i \omega R \left( \frac{1}{\beta} + \frac{\cos \phi}{C_T} - \frac{\sin \phi \sqrt{M^2 - 1}}{C_T} \right)}}{d\omega}
\]

and similarly for the vertical

\[
F(k_z) = \frac{V_0 \rho f G(\omega) k_z \sqrt{m}}{4 \pi \rho (k_z + \frac{w}{C_T}) (k_z - \frac{w}{C_T})} \sqrt{\frac{2}{\pi r}} e^{i \frac{m}{r}} \tag{G.17}
\]

and the integral equals

\[
U_{zvol} \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i \omega t} \left\{ e^{-i \omega (\frac{1}{\beta} + \frac{\sqrt{M^2 - 1}}{C_T})} \left( \frac{1}{\omega} V_0 \rho f G(\omega) \frac{\omega^2}{\beta^2} \right) \frac{1}{\sqrt{r}} e^{i \frac{m}{r}} \right\} \tag{G.18}
\]

\[
\cdot Q(\pi i \sqrt{-i \omega R \left( \frac{1}{\beta} + \frac{\cos \phi}{C_T} - \frac{\sin \phi \sqrt{M^2 - 1}}{C_T} \right)} ) \\
+ e^{-\frac{\pi R}{B}} G(\omega) V_0 \rho f i \omega \sin \phi \sin 2\phi \\
& \frac{2 \mu (\frac{1}{C_T^2} - \frac{\cos^2 \phi}{\beta^2})}{R} \\
+ e^{-\frac{\pi R}{B}} a^2 \sqrt{2} e^{i \pi r} \\
& \frac{8 \pi \rho C_T \sqrt{-i \omega R \left( \frac{1}{\beta} + \frac{\cos \phi}{C_T} - \frac{\sin \phi \sqrt{M^2 - 1}}{C_T} \right)}}{d\omega}
\]
\[-e^{-\frac{\omega R}{\beta}} \frac{V_0 \rho f G(\omega) i(\omega^2 - \frac{\omega^2}{\beta^2})^\frac{3}{2} \sqrt{\frac{2}{r}} e^{iT}}{8\pi \rho \sqrt{-\omega R \left(\frac{1}{\beta^2} + \frac{\cos \phi}{C_T} - \frac{\sin \phi \sqrt{M^2 - 1}}{C_T}\right)}}}\} d\omega\]

It can be seen in Eq. G.12, Eq. G.14, Eq. G.16, and Eq. G.18 that the Mach wave time delay

\[- \frac{R \cos \phi}{C_T} + \frac{R \sin \phi \sqrt{M^2 - 1}}{C_T} = \frac{z}{C_T} + \frac{r \sqrt{M^2 - 1}}{C_T}\]  \hspace{1cm} (G.19)

is present throughout. The geometric decay is governed both by $\frac{1}{\sqrt{r}}$ and $\frac{1}{\sqrt{R}}$. 

268