MODELING, DESIGN AND CONTROL OF
FLEXIBLE MANIPULATOR ARMS

by

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FLEXIBLE MANIPULATOR ARMS

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ABSTRACT

A tool for analyzing the interaction between the control system and the flexible components of manipulator arms has been developed. Distributed and lumped parameter models of the various arm components were combined via transfer matrices and numerical techniques were used to derive frequency domain information on the complete arm model. This modeling technique was used in a design context for a specific arm under construction. In this use it indicated the sizing required of the structural members and suggested that more general conclusions on structural requirements for adequate rigidity could be reached. Additional study indicated that for the assumed common form of control a rule of thumb limiting the arm bandwidth to one-half the lowest locked actuator natural frequency was quite accurate for the link, joint combinations explored. This rule replaces more conservative estimates of rigidity requirements.

The requirements for strength (stress limitations) and rigidity were expressed nondimensionally in terms of the pertinent arm performance specifications. Analysis of single link arms based on these requirements indicated the relative significance of strength and stiffness constraints for regions of a three dimensional space of arm specifications. The results indicate practical application for improved manipulator control schemes.

Thesis Advisor: Professor Daniel E. Whitney
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CHAPTER 1

INTRODUCTION

Subject Background and Thesis Goals

This thesis concerns itself with the analysis of flexibility in the design of manipulator arms. Manipulators, known alternatively as teleoperators, have as their purpose the physical manipulation of the environment. Manipulators in the broad sense include cranes, booms and derricks as well as hot lab and industrial manipulators pictured in Figure 1.1. They are capable of and intended for arbitrary motions within a finite number of degrees of freedom. This is distinct from linkages which are constrained mechanically to achieve a specified, desired motion. The manipulator arm motion is determined external to it by, for example, human operator or computer generated input signals to its actuators. The ability of a manipulator to accomplish its specified functions depends to a large extent on adequate fidelity in its response to the range of input signals. Thus flexibility, while inevitable, is undesirable insofar as it degrades that fidelity.

Another source of decreased fidelity is the limited torque and speed of the actuators and their attenuated response to high frequency signals. This is aggravated by a desire for rigidity, which must ultimately come from increased material in the flexible
Figure 1.1 Example: Manipulators

a) Hot Lab Manipulator

b) Industrial Manipulator
components. This in turn increases mass and thus the torque required to accurately produce the commanded motion.

The arm control system stands at a unique juncture between these two limitations. It provides the signals to the actuators for moving the arm and for regulating its position under disturbances. Thus it partially determines the excitation given to the flexible arm components and responds to eliminate those excitations. The way in which it performs those functions determines the extent to which arm flexibility deteriorates performance.

The effect of the interaction between the arm flexibility and the arm control system has been poorly accounted for in manipulator design. Attempts have been made to account for the flexibility by modeling it as a point compliance located in the joint or along the beam. One notable attempt [1] to control a flexible one link manipulator model, feeding back some of the flexible state variables has been made. No previous general study of arm devices as distributed systems has been found. In addition no design tool oriented towards analyzing the characteristics of such systems was available. It is the goal of this thesis to respond to both of those needs.

**Summary and Reader's Guide**

The reader might consider this thesis in three sections corresponding to the three areas of contribution listed below.

Chapter 2 discusses the arm model and how it can be used to obtain
useful information on the interaction between the flexible arm dynamics and the arm control system. Chapter 3 then uses the model concept to explore the limits imposed by beam flexure for a particular control scheme commonly found in servomechanisms, for a variety of related arm configurations. A rule of thumb is proposed relating the arm flexibility to the maximum speed of the arm system with this control.

The third section consisting of Chapters 4 and 5 uses the rule of thumb developed in Chapter 3 as a criterion to compare the limitations of flexibility with the limitations of stress. It poses the question, "Will a practical arm designed on the basis of strength ever be too flexible, and if so, when?" The answer to that question requires that strength requirements be expressed in terms of the parameters of the arm and the task it is to perform. This is found in Chapter 4. Chapter 5 uses the results of Chapters 3 and 4 to indicate where in a nondimensional design space one would expect flexibility to be most important.

Chapter 6 presents some indications of the effects of the assumptions that were made in the course of the preceding chapters, and offers conclusions and suggestions for future work.

Each chapter (except Chapter 4) uses results of the previous chapters. If these results are accepted out of hand, each chapter should be understandable when read alone.
Areas of Contribution

The contributions of this thesis are chiefly in the area of design of manipulator arms and their control systems:

(1) The thesis has produced a tool oriented to the design of manipulators which enables ready analysis of the flexible dynamics for most practical arm structure and control system designs. Two methods used which are novel to this tool include: the use of two dimensional searches to find the eigenvalues of the mixed distributed-lumped system, and the use of the Fast Fourier Transform to obtain the impulse response from the frequency response. This tool is implemented on the digital computer to allow the designer to obtain frequency response, eigenvalues, eigenfunctions and impulse time responses for a linear, mixed distributed-lumped parameter arm model. (Chapter 2)

(2) This tool has been used to study the flexural motion of the frequently reoccurring combinations of one and two beam-like links with one or two rotary joints regulated by feedback of each joint's position and velocity to itself. A simple rule of thumb for the limitations of this type of control has been developed. It should provide a good approximation for the flexural rigidity required to build an arm with dominant system eigenvalues of a given magnitude and adequate damping. (Chapter 3)

(3) The stress limitations of arm beams are compared to the limitations imposed by rigidity to show that beam flexure is the limiting constraint over a practically important part of a
nondimensional design space. It shows furthermore that more sophisticated control schemes utilizing information on the flexible state variables have the potential for extending the capabilities of manipulator arms beyond those achievable with simple joint rate and position feedback currently in use. (Chapters 4 and 5)

Manipulators--Their Problems and Promise

Man has long concerned himself with extending his capabilities through the use of machines. He has contrived through automation to relieve himself of hazardous and unpleasant tasks by designing unfeeling mechanical surrogates. Mechanical arms provide a more general solution to this design problem than previous mechanisms. Under direct human control manipulators physically remove man from his task. When the manipulator is under computer control man can be mentally separated as well.

In task automation, intelligence in an electric control signal replaces the intelligence provided by cams, gears and linkages in the more common variety of automation. Tasks that have not been automated remain so because of at least two reasons. The size of the task may be too small to justify the investment in highly specialized, though very efficient machines. In other cases the variability of the task overwhelms the intelligence that can be designed into cams, gears and linkages. In both these cases the computer controlled manipulator offers promise of a breakthrough. This should strike a responsive note in a country which supports a
high standard of living yet tries to remain competitive internationally in the manufacture of goods which are relatively labor intensive.

"Hot" lab manipulators designed to remove an operator from a hazardous radioactive environment provided much of the incentive for early development of these devices. Space and undersea exploration present other hazardous environments in which manipulators can and have replaced man's physical presence. A stricter definition of hazardous by current Federal regulations (OSHA) opens new domains for manipulators. These concerns, coupled with a desire to enable the ultimate use of coal and undersea oil, make manipulators even more attractive.

Survey of Pertinent Literature

The literature pertinent to this thesis falls into three areas. First, in the application area of manipulators there is found minimal discussion of the problems of flexible components. Second, authors in applied mechanics have analyzed problems of vibrating beams and systems of beams in great detail. Finally, authors in the area of controls have discussed the control of the vibration of single distributed beams.

The first work in the area of manipulators which considers flexible members is a study by Mirro [1] of a single beam in flexure with payload. The beam was pinned at one end and its rotation controlled by state variable feedback. The state variables considered included the moment at three cross sections of the lumped flexible beam model. In [2] Book reported a systematic procedure for
obtaining compliance matrices for spatial arms subject to end point loadings. It was indicated how this procedure could be used to obtain vibration and control information when the number of lumped masses resulted in a state vector of manageable dimension. The effects of distributed flexibility in manipulator design are also reported by Whitney, Lynch and Book in [3].

The control of a single distributed beam modeled by the Bernoulli-Euler equations has been treated by a number of authors. Komkov [4,5] has developed the conditions for optimality for control of transverse vibration of beams from Pontryagin's principle, assuming a criterion which minimized the beam energy after a fixed time interval.

Time domain responses of controlled, distributed beams are usually treated by approximations of some type. If one formulates the open loop solution to the beam equations in modal coordinates a truncation which retains the lower modes is logical. The exact number of modes which must be retained to obtain an accurate response of the closed loop system is difficult to justify. This procedure is used by Vande Vegte [6] and by Maki and Vande Vegte [7] to obtain optimal and constrained optimal feedback of the modal state variables for single beams subject to various simple boundary conditions. Vande Vegte derived from a quadratic index an optimal control applied to the beam endpoint which fed back all the modal state variables. This was compared to the constrained optimal control which effectively varied the termination impedance of the beam. He concluded for the
single beam case that the improvement in performance of the optimal over the constrained optimal was unlikely to justify the increase in complexity. Koehne [8] considered a slightly different performance criterion, which weighted only the modes included in the model. The modal truncation can be performed after the feedback loops are imposed as described by Berkman and Karnopp in [9]. This method is subject to truncations and approximations of other sorts. Optimization of the response based on this formulation has not been found in the literature.

One essential of the time domain methods in the literature is that the system be described by a partial differential equation and simple boundary conditions. This precludes the direct extension of those works to two beams.

Another approach to the analysis of distributed one dimensional systems well summarized by Paynter [10], Pestel and Leckie [11] and Brown [12], is essentially a frequency domain method. The transfer matrix approach conveyed by Pestel was relied on heavily in this thesis.

The application of these techniques to the control of bending vibrations was considered by Vaughan [13] who obtained results for a single beam by analog simulation. Both Vaughan and Vande Vegte [14], who clarified and extended Vaughan's work, considered effects of the termination of a Bernoulli-Euler Beam on wave reflection in the beam.

One of the benefits of the transfer matrix frequency domain approach is that it is an input-output relation between the variables
at each end of the beam. Thus the boundary conditions may be the simple alternatives of pinned, clamped, sliding, or free; or they may be an additional transfer matrix. This alternative is dealt with extensively in Pestel, but not in the context of control. No previous reference to control applied between two distributed beams using this or any other method has been found.

A Note of Social Concern

The inventions of mankind have not always produced the greatest good for the greatest numbers. The capability of a general purpose mechanical arm to isolate man's body from hostile or unpleasant surroundings is its greatest attribute. When that arm is coordinated with a digital computer, this isolation can be extended to his mind and thus to his conscience. The isolation of conscience from the activities which do violence to the human race can only lessen the incentive to find an acceptable alternative for violence. This can only lessen the number of people who are forced to scrutinize any justification for violence. The burden of conscience must rest with the master of such a device and with its designer, who may be the reader of this thesis.
CHAPTER 2

MODELING OF MANIPULATOR ARMS WITH DISTRIBUTED FLEXIBLE COMPONENTS

One of the goals of this study was an improved understanding of the interaction of the various design components of manipulator arms so that arm design in the future would be based on more reliable and more logical procedures and tools. Discussed in this chapter is an arm modeling procedure that has been established and implemented on the digital computer as a significant step toward that goal. This implementation is design oriented, allowing for easy alteration in the arm configuration and parameters. It focuses on the flexibility of the various structural components and the interaction of that flexibility with the joint servo control to affect dynamic performance. The model implementation is based on frequency domain techniques and yields to the designer such information as natural frequencies, complex eigenvalues, mode shapes, the system frequency response in the form of Bode diagrams or polar plots, and via its inverse Fourier transform the time impulse response. This model has been verified with several experimental cases by comparing natural frequencies.
2.1 Arm Model

Since the arm model, its implementation and verification are an important product of this research, its rationale, features, verification and limitations will be discussed in some detail.

2.1.1 Modeling Rationale

Out of a number of possible models a selection has been made which results in features and limitations. The rationale for the selection will be discussed in the design context in which this study was carried out.

Small Motions about a Reference Position. The most pervasive decision in the modeling process was perhaps to separate gross motions from small motions. This enables one to describe the arm as a linear system. The nonlinear equations of motion for a rigid arm are quite complex and lengthy and the general flexible case seems beyond present capabilities. A linear model also allows for better comprehension of the issues of design. Aside from convenience one must discuss the validity and pertinence of a small motion model. The validity of the model to describe small motions is supported by the experiments discussed later. The pertinence of a small motion model to the design question is claimed on the basis of the following:

(1) The frequency of structural vibrations is such that as the arm is normally operating many cycles will occur before the configuration of the arm has significantly changed. Thus the interaction of the control system and structural flexibility is well
described by a small motion model.

(2) The first order effect of the gross motion on the structural arm vibrations (e.g. Inertial effects) can be represented by a disturbance loading, which is adequately described by the small motion model.

(3) In most manipulator tasks the most stringent position accuracy requirements, therefore the most need for consideration of structure/servo interaction, occurs when the arm configuration is not rapidly changing.

**Mixed Distributed Lumped Model.** Observing manipulator arms of various designs one will find for general purpose arms one or more slender members which generally support the forces both transmitted to the payload and generated by the body forces of gravity and acceleration on the arm itself. Although these members are frequently not smooth enough to be called beams the contention here is that they can be modeled as such. Their compliance is most significant when supporting bending or torsional loads. If a significant portion of the mass of the arm is distributed throughout the length of this member, the distributed nature must be recognized in the modeling. The most straightforward approach is to describe the arm using partial differential equations. Alternatives include lumping the mass and flexibility of the beam into separate elements or truncation which retains the lower modes of the distributed elements.
A beam is considered to be distributed in one dimension only, which is reasonable for most arm models. It is conceivable that distributed effects in two or three dimensions would be important, but a model valid for design purposes can be constructed in most cases without this complexity.

Other elements which are appended to the beam element such as joints, actuators, and some payloads seem to contribute essentially mass or essentially compliance and are more conveniently considered as lumped elements. The control action itself is concentrated at the joints of the arm and thus is well considered as lumped. The lumped parameter dynamics are described by ordinary differential equations and in this case the equations considered will be linear.

**Why Not Lumped Parameter Approximations?** One alternative is to model the arm entirely as a collection of massless springs and rigid masses. This method is intuitively appealing and practical in some cases. Such a procedure was used by the author in [2].

Arms were described by a product of 4 x 4 matrices which performed a coordinate transformation between coordinate systems fixed to each end of the various components of the arm. When constant forces or moments are applied to an arm, the steady state deflection that occurs can be described by 6 x 6 compliance matrices relating the 6 components of rotation and translation to the 6 components of moment and force. To obtain this matrix for an arm with several beams connected at arbitrary angles in three dimensional space, a
sizeable bookkeeping problem arises since one must describe first the force and moment vector on the beam or other arm component, then the effect of its deformation on the end point position. This can all be done very efficiently using the transformation matrices. Differentiating the product of these matrices with respect to end point forces, one finds via the chain rule a sum of matrix products where one matrix in each product is the differential of an original transformation matrix. Upon summing these products, one finds the transformation of coordinates between the arm endpoint before and after the application of a force or moment. This transformation contains the elements of the compliance matrix mentioned above. For a more detailed explanation of this technique the reader is referred to [2].

The arm compliance matrix can be inverted to obtain a spring matrix. When combined with a 6 x 6 inertia matrix describing a lumped mass, this technique results in a second order system with 6 equations, or 12 equations in state variable form. Control at the arm joints is described by a control matrix obtainable from transformation matrices as well. When the mass of the arm--payload system can be concentrated into few lumps, this approach is quite feasible. This would be the case, for example, when zero gravity exists and a very light arm can be attached to a very large payload. It would be possible to accurately model the motion of a six degree of freedom arm with 6 x 6 spring, control, and inertia matrices (assuming the manipulator was mounted on something much larger than even the payload). When
the arm is as massive or larger than the payload, which is usually the case, additional lumped masses must be added, creating additional state variables. Exactly where one would best locate the lumped masses of a complex system can only be determined by comparing its response to experimental measurements or a more accurate model. This is especially cumbersome in the design context where the system is being changed with each iteration of the design, and the accuracy of the model is never certain. It was this problem that lead the author to confront the distributed nature of the arm with the model presented in this thesis. Approximations must still be made, but the designer has the option of using distributed beam elements in the model as well as lumped springs and masses if they seem more appropriate.

The advantage of the lumped approximation over the distributed model used here is the ease with which one can move between the time and frequency domains. This enables one to take advantage of the tools of time domain analysis as well as frequency domain techniques.

Frequency Domain Model. Frequency domain techniques, while still a workhorse in many practical applications are not currently the vogue of controls engineers. State space models, with optimal control determined in the time domain are more popular and yield promise in many areas including manipulators. In the flexible arm design context they lack the versatility and ease of model construction and alteration that results with the present model.
The lack of versatility is especially acute with the mixed parameter problem at hand. Beam vibration control studies in the time domain have been published [5, 6, 8], for single uniform beams. A manipulator model based on a single distributed beam described in terms of its lower modes of vibration with control optimization in the time domain has been studied by Mirro [1]. When connecting two or more distributed beams the problem of boundary conditions between the systems of partial differential equations restricts time domain models severely.

Frequency domain techniques have long been available for beams but the technique is not frequently used in the context of control of distributed systems. The boundary condition problem is very conveniently solved utilizing the sequential (one dimensional) nature of beams and of arms in general. The numerical techniques are relatively straightforward. The lack of a mathematically convenient optimality criteria is not in itself a severe restriction since there is no concise definition of optimality for general purpose arms anyway. The frequency domain techniques will be discussed in more detail in a later section.

2.1.2 Mathematical Basis for the Arm Model

The elements we consider an arm to be modeled by (e.g. beams, masses, springs) are described most fundamentally by their differential equations (ordinary or partial). Using the general form of the steady state solution (particular solution) of these equations the
transfer matrices can be derived. This is a process of replacing the arbitrary constants of the assumed solution with the system boundary conditions, which remain unspecified at this point. The resulting equations can be placed in a convenient matrix form. When arranged to describe the variables at one point in the element by multiplying the vector of variables at another point of the element, the matrix is termed a transfer matrix.

This section will discuss transfer matrices and the ways in which they can be manipulated to obtain useful information. Transfer matrices for many of the elements used in this thesis are developed in Pestel [11] and that development will not be duplicated here. Figure 2.3 displays some of the transfer matrices pertinent to this thesis. Appendix D develops other transfer matrices which are not specifically given by Pestel.

**Transfer Matrix Approach.** The transfer matrix approach provides a versatile method of describing the interaction between the linear components of a system when that interaction occurs at no more than two stations of the component. For beams these two stations correspond physically to the two ends of the beam. For pure rotary springs these stations correspond to the ends of the springs. For rigid body inertias these stations correspond to the points of attachment. When three or more components interact at a single station it is still possible to use the transfer matrix approach if this interaction is well defined. The transfer matrix method is well explained
by Pestel and Leckie [11] and only the essentials will be discussed here.

The interaction between two components is described by means of a vector of state variables. At the station where two components are joined the value of their state variables is identical. A transfer matrix is used to describe the relation between the state variables at the two stations of each component. If the component is a static component (does not involve differentials with respect to time) such as an ideal spring, the transfer matrix is a function only of the component parameters. For dynamic components such as an ideal mass, the transfer matrix is also a function of the time derivatives of the state variables. For linear components (described by linear differential equations) it is convenient to deal with the Fourier transform of these equations which yields the steady state amplitude and phase (or complex amplitude) of the state variables under a pure sinusoidal excitation of frequency \( \omega \). The transfer functions for these components are functions of \( \omega \).

The state vector \( \mathbf{z} \) that is used in the arm models consists of four variables displayed in Figure 2.1 for a beam with flexure in the \( x-z \) plane. These four state variables are sufficient to describe most arm vibrations of interest. Arm flexure in two planes can be described if these motions are decoupled. In addition torsional compliance of beams can be accounted for when vibrations out of the plane of two beams is studied. The sufficient conditions for decoupling the motion are discussed later. In general the neutral
Figure 2.1 State Vector for Arm Models

\[ z_1 = \begin{bmatrix} -W \\ \psi \\ M \\ V \end{bmatrix} = \begin{bmatrix} \text{-displacement} \\ \text{angle} \\ \text{moment} \\ \text{shear force} \end{bmatrix}_{\text{at station 1}} \]

\[ z_0 = B z_1 \]

B = beam transfer matrix

Figure 2.2 Transfer Matrix Representation of an Arm

\[ \begin{bmatrix} -W \\ \psi \\ M \\ V \end{bmatrix}_0 = R_1 B_2 R_3 A_4 C_5 B_6 R_7 \begin{bmatrix} -W \\ \psi \\ M \\ V \end{bmatrix}_7 \]
**Figure 2.3 a Distributed Beam Transfer Matrix (Timoshenko model)**

\[
\mathbf{B} = \begin{bmatrix}
    c_0 - \sigma c_2 & l(c_1 - (\sigma + \tau)c_2) & ac_2 & \frac{a}{\beta_1}(\sigma c_1 + (\beta_4 + \sigma^2)c_3) \\
    \frac{\beta_1}{a}c_3 & c_0 - \tau c_2 & \frac{a}{l}(c_1 - \tau c_2) & ac_2 \\
    \frac{\beta_1}{a}c_2 & \frac{l}{a}[-\tau c_1 + (\beta_4 + \tau^2)c_3] & c_0 - \tau c_2 & l(c_1 - (\sigma + \tau)c_2) \\
    \frac{\beta_1}{a}(c_1 - \sigma c_3) & \frac{\beta_1}{a}c_2 & \frac{\beta_1}{l}c_3 & c_0 - \sigma c_2
\end{bmatrix}
\]

where

\[
a = \frac{\mu}{EJ} \\
\beta_1 = \frac{\mu a^3}{EJ} \\
\sigma = \frac{\mu a^3}{GA_s} \\
\tau = -\frac{\mu}{EJ} (\mu^2 a^3) \\
\lambda_1 = + \sqrt{\beta_1 + \frac{1}{2}(\sigma - \tau)^2 \pm \frac{1}{2}(\sigma + \tau)} \\
A = \frac{1}{\lambda_1^2 + \lambda_2^2}
\]

- \( GA_s \) = Shear stiffness
- \( EJ \) = Bending stiffness
- \( \mu \) = Mass per unit length
- \( i \) = Radius of gyration of cross sectional area perpendicular to the beam’s neutral axis
- \( l \) = length of the beam
- \( \omega \) = vibration circular frequency
Figure 2.3b Angle Transfer Matrix

\[
A = \begin{bmatrix}
1/\cos \phi & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
m_s \omega^2 \sin \phi \tan \phi & 0 & 0 & \cos \phi
\end{bmatrix}
\]

where:

\[z_i = A z_{i+1}\]

\[m_s = \sum_{j=i}^{n} m_j\]

\[m_j = \text{mass of element } j\]

\[n = \text{first joint (angle) element greater than } i, \text{ or the total number of elements, whichever is smaller}\]

\[\omega = \text{vibration circular frequency}\]

Figure 2.3c Rigid Mass Transfer Matrix

\[
R = \begin{bmatrix}
1 & l & 0 & 0 \\
0 & 1 & 0 & 0 \\
m_\omega^2(l-h) & I_g \omega^2 + m\omega^2(l-h) & 1 & l \\
m\omega^2 & m\omega^2(l-h) & 0 & 1
\end{bmatrix}
\]

\[z_i = R z_{i+1}\]

\[I_g = \text{mass moment of inertia, axis perpendicular to the plane of vibration, through the center of gravity, c.g.}\]

\[m = \text{mass of the body}\]

\[\omega = \text{vibration circular frequency}\]
Figure 2.3d Controlled Rotary Joint Transfer Matrix

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1/k(j\omega) & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Pinned connection between two elements. Relative angle is related to the moment which affects each of the two elements. \(k(j\omega)\) is essentially the transfer function relating moment to angle. In the simplest case \(k=\text{constant}\), a rotary spring determines the relation between angle and moment, \(k(j\omega)\) can account for feedback of angular velocity and other derivatives or integrals of the variable \(\Psi\), for filter dynamics included for compensation, and for servo motor and transmission dynamics.
axis of the undeformed arm must lie in a plane for these four state variables to sufficiently describe the arm. Two link designs which predominate arms built today (except for short wrist segments) automatically qualify. Additional arm links or beam like supports modeled as part of the arm may of course be arranged in nonplanar configurations. There is no conceptual difficulty in extending the state vector to the complete three dimensional case. The twelve state variables then needed (flexure in two planes, twisting, and compression) lengthen numerical computation disproportionately to the information obtained. State vectors with between 4 and 12 variables may give added information in some specific cases. Four state variables are all that will be considered here.

For decoupled motion it is sufficient for small motions that a planar arm have its joints either in or perpendicular to the plane of the arm, which is usually the case. For a typical arm two sets of data describe the small motions of the arm. One includes only flexure. The other includes distributed flexure and a lumped torsional compliance for out of plane motion. For arms that can be configured so as not to meet the requirements for decoupled motion an extended state vector or a configuration which does meet the requirements can be studied to obtain design information.

Given the transfer matrix $B$ for a component and the state vector at one of its stations $z_i$, the state vector $z_{i-1}$ at the other station is given by the matrix multiplication.

$$z_{i-1} = B \cdot z_i$$
It is thus a simple matter when components are connected serially (two components per station except for the end components) to find an overall transfer matrix by multiplication of the individual matrices to eliminate the intermediate state vectors. This is demonstrated for an arm model in Figure 2.2.

**Numerical Operations with Transfer Matrices.** The product of matrices such as appears in Figure 2.2 essentially is the implementation of the model of the arm which consists of beams, lumped masses, controlled joints and angles, joined end on end. The implementation of the model provides ways of getting useful information from that model. One possibility is to express analytically the elements of each component matrix, multiply the matrices and obtain a single matrix each element of which is a sum of products of the original matrix. While this is in fact done for simple cases in Appendix A, it is not recommended for more complex cases unless the same configuration is to be used many times. The alternative is to evaluate each term before the matrix product is taken, then multiply the numerical values. This is a procedure which can be carried out in a straightforward fashion by digital computer.

The advantage to numerical evaluation of this nature is that the complex functions need not be manipulated avoiding large amounts of designer time and potential for mistakes.

The disadvantages are of three types:
(1) More computer time is required to evaluate expressions which might be simplified using trigonometric and hyperbolic identities. The simplification is not apt to be great unless there are identical components or at least many identical parameters. Additional computer savings may be observed when some of the transfer matrices have many zero or unity elements. Straightforward multiplication of these takes as long for the computer as do non-zero or non-unity elements. It may not be difficult for the designer to combine several simple elements into one matrix analytically if this combination is to be used frequently.

(2) Numerical errors may become significant. The larger number of calculations may cause roundoff errors to become significant especially in some cases (evaluation of determinants) which require taking the difference of two large, nearly equal numbers. This difficulty has been encountered only in rare and unusual cases and been solved by using extended precision in those cases. It is also possible to get a numerical overflow in the product of the transfer matrices.

(3) For simple cases the analytical expressions resulting from the matrix product may give the designer insight into the problem that the numerical results obscure.

**Boundary Conditions and Forcing Functions.** The transfer matrix, whether it describes a single component or a group of them, expresses the relation between the state variables at its two stations. In order
for the transfer relation to be valid between state vectors, at mostour of the eight state variables may be arbitrarily established.
In fact for physical systems only two of the state variables at
each station may be determined and more precisely for these state
vectors only one of the associated variables of displacement or
force and angle or moment may be arbitrarily specified. This speci-
fication may be as a simple boundary condition, as a forcing function,
or as a linear combination of variables which may implicitly include
the other variables of the same state vector. This last case is in
essence what one does when he appends another component by multi-
plying another transfer matrix. In this case one merely transfers
the specification to another station in the extended system.

For simple boundary conditions one prescribes two non-associated
variables of the arm. Figure 2.4 displays the physically possible
combinations of zero state variables. The non-trivial solution of
this case can result in solving for the natural frequency of complex
eigenvalues (for damped systems). Additionally one can solve for
the eigenfunctions of the system (the mode shape at the eigenvalue).
The imposition of a forcing function yields the steady state forced
response of the system, assuming the forcing function is a sinusoid
of frequency \( \omega \). These techniques will be discussed in the following
sections.

\textbf{Natural Frequencies and Eigenvalues.} If disturbed from the equilibrium
position and then allowed to move freely after \( t = 0 \) (without
Figure 2.4 Physically Possible Boundary Conditions

Clamped $W = \psi = 0$

Pinned $W = M = 0$

Sliding $\psi = V = 0$

Free $M = V = 0$

Figure 2.5 Frequency Determinant

$$U = B_1 R_2 A_3 C_4 B_5 R_6$$

$$z_0 = \begin{bmatrix} -W \\ \psi \\ V \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{bmatrix} \begin{bmatrix} -W \\ \psi \\ V \end{bmatrix}$$

For a nontrivial solution:

$$u_{11} u_{32} - u_{12} u_{31} = 0$$
disturbance or outside input) the state variables of a linear system will be described over time by a function of the form

\[(2.1) \quad z_{ik}^* = e_1 e^{j\omega_1 t} + e_2 e^{j\omega_2 t} + e_3 e^{j\omega_3 t} + \ldots + e_k e^{j\omega_k t}\]

\[z_{ik}^* = the \ i^{th} \ element \ of \ z_k^*\]

For a lumped system there will be a finite number of these terms while a distributed system may theoretically have a countably infinite number.

For undamped systems the \(\omega_i\) appear as the frequencies of vibration and assume real values. For damped systems some of the \(\omega_i\) will have complex or pure imaginary values and it is more conventional to deal with the eigenvalues \(s_i = j\omega_i\). Complex or pure imaginary values of \(s\) (complex or pure real values of \(\omega\)) will always occur in pairs with the same real (imaginary) part and an imaginary (real) part with the same absolute value and opposite sign. When this is the case we will deal explicitly only with the value of \(s\) (\(\omega\)) with the positive imaginary (real) part.

The values of \(\omega_i\) for an arm system model depend only on the parameters of, and the boundary conditions on, the system. They are independent of initial conditions, independent of which state variable

*The assumption here is that the \(\omega_i\) are distinct. For physical systems this is always true if one cares to look at the values with enough accuracy. The more general case \(\omega_i = \omega_k\) does not restrict the results presented.*
is observed, and independent of the point in the system at which it
is being observed. The values $a_1$ depend on all of these quantities.

The time function (Eq. 2.1) which describes a state variable
of the system must be consistent with the differential equations of
the individual components, the general solution of which is the same
form as Equation 2.1, also involving terms $e^{j\omega_1 t}$. This general
solution is the source of the transfer matrix for the component.
For an isolated single component the rest of the system acts as a
complex boundary condition which must be considered when determining
the values of the $\omega$ for that component. The state variables between
two components assume values over time that must conform with the
general solution of two sets of differential equations. This can only
occur if the values of $\omega$ are the same in both solutions and in fact
the same as the general solution for the entire system.

The transfer matrix technique allows one to simultaneously
consider all the components and the boundary conditions on the system
and thus determine the $\omega_1$ of interest. Multiplying transfer matrices
eliminates the intermediate state variables at the interface
between components and expresses state variables at one end of an
arm directly in terms of the other end. Imposing two boundary
conditions at each end restricts the values $\omega_1$ can assume for a
nontrivial solution of the remaining state variables. These $\omega_1$ are
the same $\omega_1$ appearing in Eq. 2.1. The restriction is developed in
Fig. 2.5 for specific boundary condition on a specific arm model.
In general for a system represented by a matrix product $U$ such that
\[
\begin{bmatrix}
  z_{10} \\
  z_{20} \\
  z_{30} \\
  z_{40}
\end{bmatrix} = U \begin{bmatrix}
  z_{1n} \\
  z_{2n} \\
  z_{3n} \\
  z_{4n}
\end{bmatrix} = U \mathbf{z}_n =
\begin{bmatrix}
  u_{11} & u_{12} & u_{13} & u_{14} \\
  u_{21} & u_{22} & u_{23} & u_{24} \\
  u_{31} & u_{32} & u_{33} & u_{34} \\
  u_{41} & u_{42} & u_{43} & u_{44}
\end{bmatrix}
\]

With the boundary conditions

\[ z_{i0} = z_{j0} = 0 \quad \text{at station } 0 \]

and \[ z_{kn} \neq 0, \ z_{n} \neq 0 \text{ at station } n \]

(implying the remaining two variables at station n are zero)

requires for a nontrivial solution that the frequency determinant

\[
(2.2) \quad d = \begin{vmatrix}
  u_{ik} & u_{i} \omega \\
  u_{jk} & u_{j} \omega 
\end{vmatrix} = u_{ik} u_{j} \omega - u_{jk} u_{i} \omega = 0 \quad \omega = \omega_i
\]

The elements of \( U \) and thus the terms of the frequency determinant are generally complex functions of \( \omega \). This being the case one must numerically search for values of \( \omega \) where \( d = 0 \). When dealing with systems with no damping one can restrict the search to real values of \( \omega \) or values of the eigenvalue \( s \). In general however, one must search over the complex plane for values of \( s = s_i \), where both the real and imaginary parts of \( d \) are zero. In order to use conventional
search routines one can search for minimum values of 
\[ \|d\| = \sqrt{(\text{Im } d)^2 + (\text{Re } d)^2}, \]
then check to see if \( d = 0 + j0 \) for the values of \( s \) returned. This topic will be discussed in more
detail in Section 2.2.4.

For single beams and variable impedance terminations Vande
Vegte found solutions to complex eigenvalues and displayed them as
root loci. He reversed the procedure that was used in this thesis.
The real and imaginary parts of the eigenvalues were specified, as
well as some of the termination impedances. The remainder of
impedances were then solved for by requiring the frequency determinant
to be zero. This procedure is acceptable for analysis but might be
cumbersome for design, since it assumes that an eigenvalue can exist
at a given position, and that that eigenvalue is the one desired
(i.e. that it corresponds to the mode of interest and not some higher
mode). This would be difficult to determine for complex arms. In
the analysis of Chapter V a procedure similar to this is used for
analyzing a single beam and mass.

**Modal Shapes.** Associated with each eigenvalue is an arm shape
called the modal shape which describes the relative amplitude of all
points of the arm when vibrating at that frequency. Looking at the
problem from another perspective, there is an arm shape which when
it constitutes the initial condition will result in arm vibration
described by a single eigenvalue, to the exclusion of all other
system eigenvalues.
The transfer matrix method can be employed to find the mode shape after the eigenvalue has been found. Refer again to the equation

\[ z_0 = U \ z_n. \]

Two of the state variables at each end are specified by the boundary conditions, and the remaining four state variables can be solved for in terms of each other. By normalizing one of the state variables to be equal to one the remaining three are specified.

More specifically consider the boundary conditions resulting in Equation 2.2. The homogeneous equations which precede the frequency determinant are

\[
\begin{bmatrix}
  u_{ik} & u_{il} \\
  u_{jk} & u_{jl}
\end{bmatrix}
\begin{bmatrix}
  z_{in} \\
  z_{jn}
\end{bmatrix}
= 0
\]

If \( z_{in} \) is required to equal one, the solution for \( z_{jn} \) is

\[ z_{jn} = \frac{-u_{ik}}{u_{il}} = \frac{-u_{ik}}{u_{jl}} \]

Selecting the appropriate 2 x 2 submatrix from \( U \) will enable one to solve for the unspecified state variables at station 0.

In order to visualize the modal shape values of the state variables at intermediate points are helpful. For this one must refer to the transfer matrices of the separate components. For lumped components the knowledge of the state variable at either end
is usually adequate for visualization. For distributed beams, however, the trigonometric and hyperbolic functions describing the shape within the component are far from obvious. For plotting these functions one can essentially divide the component into smaller components thus creating additional intermediate state vectors (for purposes of plotting only, not for finding eigenvalues). When plotted versus distance along the axis of the arm the state variables indicate the shape, angle, moment and shear amplitudes along the arm.

Steady State Frequency Response. In the previous section two state variables at each end of the arm were specified to be equal to zero, thus establishing simple boundary conditions which enabled one to search for the eigenvalues of the arm system. Another way to specify the system is to impose sinusoidal forcing functions of frequency $\Omega$ and arbitrary but constant amplitude on from one to four of the state variables (subject to the same constraints on combinations of state variables as for boundary conditions indicated in Figure 2.4). The procedure here is actually more straightforward than for finding eigenvalues.

Consider a linear system with at least a small amount of damping (present in all physical systems) forced by a single sinusoid of constant frequency for some appropriately long time. The response of that system at any point will be described completely by a complex amplitude times a sinusoid of the same frequency $\Omega$. Alternatively the response could be described by a real amplitude and a phase angle.
This complex amplitude as a function of $\Omega$ is termed the steady state frequency response.

Consider once again an arm model with describing transfer matrix $U$. 

Now

$$
\begin{bmatrix}
-w \\
\psi \\
M \\
V_0
\end{bmatrix}
= U
\begin{bmatrix}
-w \\
\psi \\
M \\
V_n
\end{bmatrix}
$$

Assume the rows and columns of $U$ are rearranged to form $\bar{U}$ such that the first two state variables of the rearranged state vectors $z_0$ and $z_n$ are forced with a sinusoid of arbitrary but constant complex amplitude. Then

$$z_0 = \bar{U} z_n$$

Let us partition this matrix expression such that

$$z_0 =
\begin{bmatrix}
f_0 \\
r_0
\end{bmatrix}
= 
\begin{bmatrix}
\bar{U}_{11} & \bar{U}_{12} \\
\bar{U}_{21} & \bar{U}_{22}
\end{bmatrix}
\begin{bmatrix}
f_n \\
r_n
\end{bmatrix}
= \bar{U} z_n$$

$f_0$ and $f_n$ contain the forced state variables and $r_0$ and $r_n$ contain the remaining state variables, termed the response variables.

(2.3) $$f_0 = \bar{U}_{11} f_n + \bar{U}_{12} r_n$$
\begin{equation}
(2.4) \quad r_0 = \tilde{u}_{21} \frac{f}{n} + \tilde{u}_{22} r_n
\end{equation}

solving explicitly for $r_0$ and $r_n$

from (2.3) $r_n = \tilde{u}_{12}^{-1} f_0 - \tilde{u}_{12}^{-1} \tilde{u}_{11} \frac{f}{n}$

from (2.4) $r_0 = \tilde{u}_{21} \frac{f}{n} + \tilde{u}_{22} (\tilde{u}_{12}^{-1} f_0 - \tilde{u}_{12}^{-1} \tilde{u}_{11} \frac{f}{n})$

\begin{equation}
(2.5) \quad \begin{bmatrix}
  r_0 \\
  r_n
\end{bmatrix} = \begin{bmatrix}
  \tilde{u}_{22} & \tilde{u}_{12}^{-1} \\
  \tilde{u}_{12}^{-1} & -\tilde{u}_{12}^{-1} \tilde{u}_{11}
\end{bmatrix} \begin{bmatrix}
  f_0 \\
  \frac{f}{n}
\end{bmatrix}
\end{equation}

Assuming $\tilde{u}_{12}^{-1}$ exists.

Equation (2.5) expresses the four response state variables in terms of the four forced state variables. The value of $\omega$ at which the transfer matrices will be evaluated will be the forcing frequency $\Omega$. The forcing vectors $f_0$ and $f_n$ will contain the (possibly complex and distinct) forcing amplitudes. Complex amplitudes can be used to represent a phase shift between the various forced state variables. $r_0$ and $r_n$ will contain the amplitudes of responses.

In practice it is seldom informative to force more than one state variable simultaneously. Then it is possible to simplify equation (2.5) assuming there is only one non zero element in $f_0$ and $f_n$. 
Once again it is usually preferable to numerically evaluate the individual component transfer matrices prior to multiplication, enabling straightforward implementation on the digital computer.

**Impulse Time Response.** It is well known that the steady state frequency response is equivalent to the Fourier transform of the impulse time response for linear systems. This has long been used to obtain the power spectral density from time records. The inverse transform of the complex frequency response (which contains magnitude and phase information) can thus be used to obtain the impulse response. This enables one to visualize the system time response to an impulse disturbance. Thus if one desires the impulse response of a given state variable, displacement \( w(t) \) for example, to an impulse disturbance at another state variable, shear force \( V \) at the same station for example, one first uses Equation (2.5) to solve for the frequency response of that variable, in our example \( w(j\omega) \). One then uses the inverse Fourier transform given by

\[
(2.6) \quad z_{hi}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} z_{hi}(j\omega)e^{j\omega t} \, d\omega
\]

\( h = \text{response state variable index} \)
\( i = \text{response station index} \)

to determine the time impulse response for example \( w(t) \).

In practice we desire to evaluate this response digitally using discrete samples of both the time and frequency responses. This introduces a great number of subtle consequences which will not be
discussed in this section. Suffice it to say that the inverse transform can be performed economically with a slight change in the very efficient FFT (Fast Fourier Transform) algorithm [15]. The use of the algorithm in the inverse fashion is not well documented and will be discussed in Section 2.1.4.

2.1.3 Experimental Verification of Arm Model

To get some indication of the accuracy of the model two sets of experiments have been performed. The first of these is summarized in Figure 2.6 and Table 2.1. The second is summarized in Figures 2.7 and 2.8 and Table 2.2. In both cases, the predicted natural frequencies of the undamped systems are compared to the resonant frequencies observed when forcing the system with a sinusoidal torque. In the first set of experiments, the frequency was determined via stroboscope. The second experiment was conducted by Octavio Maizza-Neto, also in the Department of Mechanical Engineering at MIT. It was performed using automatic frequency sweeping and measurement of the amplitude of the endpoint of the system via accelerometer. Figure 2.8 plots the acceleration amplitude measured versus frequency for this case.

The motor rotor inertia was the only value which could not be measured. Table 2.1 is based on values of that parameter taken from a motor catalogue. By using 30% lower values the maximum error was reduced to 3.7% for the first three natural frequencies. Thus it is possible that the errors in Table 2.1, although not excessive
### TABLE 2.1

RESULTS OF EXPERIMENT IN FIGURE 2.6

<table>
<thead>
<tr>
<th>In or out of Plane</th>
<th>Payload (slugs) (kg)</th>
<th>$l_1$ (ft) (m)</th>
<th>$l_2$ (ft) (m)</th>
<th>$\phi$ (Degrees)</th>
<th>Predicted $\omega_1$</th>
<th>Measured $\omega_1$</th>
<th>Predicted $\omega_2$</th>
<th>Measured $\omega_2$</th>
<th>Predicted $\omega_3$</th>
<th>Measured $\omega_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>in</td>
<td>$3.44 \times 10^{-3}$ (.0502)</td>
<td>.921 (.281)</td>
<td>.885 (.270)</td>
<td>45</td>
<td>163</td>
<td>162</td>
<td>499</td>
<td>567</td>
<td>----</td>
<td>----</td>
</tr>
<tr>
<td>out</td>
<td>$3.44 \times 10^{-3}$ (.0502)</td>
<td>.921 (.281)</td>
<td>.885 (.270)</td>
<td>45</td>
<td>157</td>
<td>161</td>
<td>549</td>
<td>565</td>
<td>1045</td>
<td>1025</td>
</tr>
<tr>
<td>in</td>
<td>$1.12 \times 10^{-3}$ (.0163)</td>
<td>.921 (.281)</td>
<td>.888 (.271)</td>
<td>45</td>
<td>233</td>
<td>232</td>
<td>556</td>
<td>605</td>
<td>1357</td>
<td>1350</td>
</tr>
<tr>
<td>in</td>
<td>0</td>
<td>.921 (.281)</td>
<td>.946 (.288)</td>
<td>45</td>
<td>263</td>
<td>264</td>
<td>585</td>
<td>635</td>
<td>1477</td>
<td>1464</td>
</tr>
<tr>
<td>in</td>
<td>$3.44 \times 10^{-3}$ (.0502)</td>
<td>.921 (.281)</td>
<td>.889 (.271)</td>
<td>0</td>
<td>201</td>
<td>211</td>
<td>520</td>
<td>583</td>
<td>1081</td>
<td>1100</td>
</tr>
<tr>
<td>in</td>
<td>$1.12 \times 10^{-3}$ (.0163)</td>
<td>.921 (.281)</td>
<td>.889 (.271)</td>
<td>0</td>
<td>260</td>
<td>268</td>
<td>596</td>
<td>643</td>
<td>1359</td>
<td>1377</td>
</tr>
<tr>
<td>in</td>
<td>0</td>
<td>.921 (.281)</td>
<td>.949 (.271)</td>
<td>0</td>
<td>284</td>
<td>292</td>
<td>631</td>
<td>667</td>
<td>1466</td>
<td>1481</td>
</tr>
</tbody>
</table>

Average % error 4.1%
Figure 2.6 Experimental Verification - Clamped Elbow Joint

Torque motor
rotor inertia = $3.98 \times 10^{-4} \text{ft-lb} \cdot \text{sec}^2$
($5.75 \times 10^{-4} \text{nt-m} \cdot \text{sec}^2$)
rods-dia. = .25 in. (.00635 m)
material - carbon steel
joint mass = .00117 slugs (.0171 kg)

payloads, aluminum
dia. = 0.75 in (.0191 m)
lengths { .307 ft (.0936 m)
{ .060 ft (.0183 m)

Figure 2.7 Experimental Verification - Pinned Elbow Joint

Units: $J$: ft.-lb. \cdot \text{sec}^2 (\text{nt-m-sec}^2)$
m: slugs (kg)
$\mu$: slugs/ft. (kg/m)
$E$: lb$_f$/ft. (nt/m$^2$)
r$_{2}$: ft. (m)
lengths: ft. (m)
Figure 2.8 Amplitude of Acceleration vs. Frequency

Acceleration amplitude

77.24  80.76
145.0  136.6

Model predicted natural frequencies (Hz)

278
477
795
1063

244.6  401.6  688.3  859.2  1366
417.0  893.1

1591

measured resonances (Hz)
TABLE 2.2

COMPARISON OF PREDICTED AND OBSERVED (FIG. 2.7)
NATURAL FREQUENCIES—EXPERIMENTAL CASE

<table>
<thead>
<tr>
<th>Model (Hz)</th>
<th>Experimental (Hz)</th>
<th>% Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>77.2</td>
<td>80.76</td>
<td>4.3%</td>
</tr>
<tr>
<td>145.0</td>
<td>136.6</td>
<td>6.0%</td>
</tr>
<tr>
<td>278</td>
<td>244.6</td>
<td>13.8%</td>
</tr>
<tr>
<td>477</td>
<td>401.6</td>
<td>18.8%</td>
</tr>
<tr>
<td></td>
<td>417.0</td>
<td>14.4%</td>
</tr>
<tr>
<td>795</td>
<td>683.3</td>
<td>15.6%</td>
</tr>
<tr>
<td>1063</td>
<td>859.2</td>
<td>23.8%</td>
</tr>
<tr>
<td></td>
<td>893.1</td>
<td>19.0%</td>
</tr>
<tr>
<td>1366</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1591</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(averaging 4%) do not do the model justice.

Predictions of the first three natural frequencies appear good in both cases, with accuracy decaying after that. Figure 2.8 displays acceleration amplitude.

The decay of the amplitude peaks indicate that the first two modes will dominate the response. This decay will be even greater for displacement amplitude. To obtain the relative magnitude of these peaks one must divide by the frequency squared. Table 2.3 shows the relative amplitude of the first six modes for acceleration and displacement.
TABLE 2.3

RELATIVE AMPLITUDE OF FIRST SIX RESONANCES

<table>
<thead>
<tr>
<th>Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative Acceleration</td>
<td>1</td>
<td>.868</td>
<td>.162</td>
<td>.226</td>
<td>.166</td>
<td>.170</td>
</tr>
<tr>
<td>Relative Displacement</td>
<td>1</td>
<td>.304</td>
<td>.0177</td>
<td>.00914</td>
<td>.00229</td>
<td>.00150</td>
</tr>
</tbody>
</table>

The quality of the data seems good in both cases. The multiple peaks at some resonances in Figure 2.8 are probably due to the constantly changing frequency which does not allow the beam to reach steady state, and with the mixing of two nearly equal frequencies from the torque motor and the beams produces a beat like phenomenon.

For the higher modes a decay in accuracy is not surprising. Since the total motion involved in these vibrations is very small, nonlinear effects such as backlash from the nonzero tolerance at the joint become more significant. Because of the small amplitude it is not felt that this significantly affects the model for practical use.

2.1.4 Inverse Fourier Transform via Digital Methods

The calculations of the small motion response of the arm system have been carried out in the frequency domain. This information is related to the time domain response via an almost symmetrical pair of transformations—the Fourier transform and its inverse. It is at best awkward to perform these transformations via analog hardware,
and in view of the digital nature of the calculations to this point
digital calculations are preferred. This introduces distortion due
to sampling of continuous phenomena.

The digital Fourier transform (DFT) and its inverse have exact
properties in themselves and can be used to approximate their con-
tinuous counterparts. This can be done most efficiently using the
algorithm known as the fast Fourier transform or FFT to compute the
DFT. Only by understanding the DFT can the sampled approximation
yield information on the continuous case with the least distortion
and greatest efficiency.

Fourier Series for Continuous Functions. For a periodic time function
it is well known that there exists a representation in terms of a
weighted sum of sines and cosines, or in terms of complex exponentials
which are harmonics of the fundamental frequency $\Omega$. This is called
the Fourier series:

\begin{equation}
(2.7a) \quad f(t) = \frac{1}{T} \sum_{i = -\infty}^{\infty} F_i \exp (j \Omega t) \quad i = \ldots -2, -1, 0, 1, 2, \ldots
\end{equation}

\begin{equation}
\text{Where } j = \sqrt{-1}
\end{equation}

$\Omega = \text{the fundamental frequency in rad/sec}$

\begin{equation}
(2.7b) \quad F_i = \frac{T/2}{-T/2} \int f(t) \exp (-j i \Omega t) \, dt
\end{equation}

\begin{equation}
\text{Where } T = 2\pi/\Omega = \text{Period of one cycle in sec.}
\end{equation}

The $F_i$ are the Fourier coefficients.
It is usually possible to truncate the series at a finite number of harmonics, which depends on the accuracy desired.

For aperiodic functions it seems intuitive that $T$ goes to infinity and $\Omega = 2\pi/T$ goes to zero. In fact in the limit

\[(2.8a) \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) \exp(j\omega t) \, d\omega\]

\[(2.8b) \quad F(j\omega) = \int_{-\infty}^{\infty} f(t) \exp(-j\omega t) \, dt\]

$F(j\omega)$ is the Fourier integral and is equivalent to the Fourier transform of the time signal.

The above are true for any time signals regardless of their origin. Certain additional relations will hold when the time function is the response of a linear system. When $f(t)$ is the response of a linear system to an impulse input to one variable at a point in the system at $t = 0$, $F(j\omega)$ is equivalent to the steady state frequency response resulting from a unity amplitude sinusoid forcing the same variable. This is in fact what is obtained from the frequency domain arm model.

**Discrete Approximations.** Starting in either the frequency or time domain, one can at least formally transform to the other domain via Equations (2.8). In practice however one must approximate the transform by replacing the integration with summation and replacing the differential $d\omega$ with the increment $\Delta\omega$ resulting in
(2.9) \[ f(t) = \hat{f}(t) = \frac{2}{\pi} \sum_{i=-n}^{n} F(ji \Delta \omega) \exp (ji \Delta \omega t) \Delta \omega \]

or \[ \hat{f}(t) = \frac{\Delta \omega}{2\pi} \sum_{i=-n}^{n} F(ji \Delta \omega) \exp (ji \Delta \omega t) \]

By comparing this with Equation (2.7) one realizes that this approximation must be periodic and of period \(2\pi/\Delta \omega\), indicating that distortion of some sort has occurred.

Certain steps must be taken to assure that the distortions resulting from the discrete approximations do not render the approximate time response \(\hat{f}(t)\) useless. For \(\hat{f}(t)\) to result from a stable physical system the response for \(t < 0\) (before any input) must be zero. Noting in Equation (2.8b) the two sided nature of the Fourier integral, and recalling the periodic nature of \(\hat{f}(t)\) one can correctly conclude that the results of Equation (2.9) must be at least approximately zero for half of each period. This requires sufficient time for the impulse response to settle out, placing a lower constraint on the period \(2\pi/\Delta \omega\) (or upper constraint on \(\Delta \omega\)).

It is not readily apparent in the frequency domain what an adequately small value of \(\Delta \omega\) is. If one knows the location of the system eigenvalues he can estimate the settling time of each exponential. For complex conjugate root pairs for example
\[ t_s \approx \frac{4\zeta}{\omega_n} \]

Where \( \zeta \) = damping ratio of the root pair
\( \omega_n \) = distance from origin of the pair
\( t_s \) = time to point where response remains within 2% of steadying state

One may make \( \Delta \omega \) smaller by either increasing the number of samples or by limiting the upper value of the frequency response considered. By lowering the upper value of frequency \( \Omega_f \) that will be considered one removes the components of the time response with frequency greater than \( \Omega_f \). By looking at the frequency response for the system one can estimate \( \Omega_f \). Resonant peaks that are not down at least 20 db. from the magnitude at \( \omega = 0 \) should be included if feasible. Thus for given settling times and given frequency \( \Omega_f \) one arrives at a minimum sampling interval in the frequency domain analogous to the Nyquist sampling interval in the time domain. Just as small time intervals between samples are required to evaluate the frequency response for large frequency, small frequency intervals are required to evaluate the time response for long times.

2.2 Numerical Implementation

The arm modeling method has been implemented on a digital computer. This section will describe briefly the program and how the user may interact with it. Also described are numerical difficulties that have arisen in use and how they have been handled. A listing of
the computer programs is found in reference [24].

2.2.1 Computer Capabilities

Although it was originally intended to include all the programs in one package for interactive use by the designer, this goal has been abandoned for the present time due to the modest core storage of the machine used. The Interdata Model 70, with 40K 16 bit words of core storage at the M.I.T. Joint Civil-Mechanical Engineering Computer Facility was used. This is a mini computer handling interactive graphics and console input which is very useful in a design situation. An Interdata Model 80 with 32K 16 bit words of storage was used for some of the more extensive eigenvalue searches due to a lower price structure and a faster CPU. The program as it presently exists is divided into several compatible packages. Table 2.4 classifies the programs in seven more or less related categories. These could at some point be combined into an integral program with complete communication between all the program parts. Categories I, II, and III utilize descriptions of the arm and subroutines in VI and VII Categories IV and V use output from II and III to obtain additional information.

2.2.2 Nature of User Input

This section will briefly describe the nature of the user input to the system to indicate the designer effort required (not to indicate how to use the programs).
TABLE 2.4
PROGRAM CATEGORY OUTLINE FOR
NUMERICAL IMPLEMENTATION

I. Natural frequency calculation (no damping)
   A. Single Precision
   B. Double Precision

II. Eigenvalue search (two dimensional search, systems with
damping)
   A. Card and console input only
   B. Interactive graphics implementation

III. Frequency response calculation
   A. Logarithmic frequency scale
      1. Bode plot
      2. Polar (Nyquist) plot
      3. Modified polar plot
   B. Linear frequency scale (equal increment in frequency for
      FFT input)

IV. Mode shape calculation and display

V. Fast Fourier transform

VI. Component transfer matrix calculation
   A. Distributed Beam
      1. Bernoulli-Euler model
      2. Timoshenko model (includes shear and rotary inertia)
   B. Rigid Body
      1. General
      2. Uniform cross section
   C. Angle in the arm shape
      1. In the plane of vibration
      2. Perpendicular to the plane of vibration
   D. Controlled Rotary Joint
      1. Transfer function control
      2. With flexible shaft and gear reduction
   E. Parallel elements (combines certain elements in parallel
      by clamping them at each end).
TABLE 2.4 (Continued)

F. Discontinuity in one state variable (with its associated variable equal to zero e.g. pinned connection between beams or pinned connection to ground)

VII. Search Routines

A. Pattern search—2 dimensional
B. IBM SSP root finding algorithm

Arm Description. The description of the arm to be modeled is in terms of the arm elements or components selected from Category VI of Table 2.4. Each element requires one and sometimes more data cards giving its parameters and are arranged in the order of occurrence on the arm. In certain cases there are restrictions as to what combinations of elements can be used. For instance angles in and out of the plane of vibration would result in a nonplanar arm which cannot be handled by four state variables. Other restrictions include (1) arm discontinuity can be used only with natural frequency or eigenvalue (I and II) calculations are presently implemented. (2) Parallel elements cannot be used with mode shape calculations. (3) Flexible shafts incorporated into the controlled joint must connect to a pinned, clamped, or sliding end of the arm.

Calculation Description. Presently the description of the desired calculation for natural frequencies and frequency response requires one card for describing arm boundary conditions, calculation type,
number of increments, and extreme frequencies considered. In addition the forcing variable must be specified for frequency response calculations. A number of selections of output alternatives and extended calculations are available by data switch and console input at run time.

For eigenvalue calculations additional starting points for the two dimensional search can be read in from card or console in one implementation, or input graphically via joystick and crosshairs in another implementation.

Mode shape calculations require the input of the eigenvalues for the system for the mode for which the shapes are to be calculated, boundary conditions, and the number of points at which the shape is to be calculated for each beam element.

To calculate the inverse Fourier transform to obtain the time impulse response, the frequency response must first be calculated with equal spacing between calculations (Table 2.4, IIB). The results are stored on disc to be used by the FFT algorithm. The specification of the total number of points to be used and the frequency range they cover is input by card.

2.2.3 Nature of Program Output

The program output is of the nature described in Table 2.4. In cases where there are arrays of data (such as the values of frequency response) this data can either be plotted or plotted and printed to allow more precise comparisons. In this case there is also a selection
of which endpoint state variables are to be plotted. The three types of frequency plots are the Bode diagram, Polar plot of magnitude vs. phase, and the modified polar plot which can be used in stability analysis for certain nonlinear arm elements. The graphics assisted eigenvalue search program also yields a plot of the roots as they are found.

2.2.4 Numerical Difficulties

The arms modeled to date have resulted in few numerical difficulties. The difficulties encountered and ways of dealing with them are described below.

False Roots in Eigenvalue Search. For a damped arm system the frequency determinant is a complex number. Evaluated at the system eigenvalue both real and imaginary parts should be zero. Since conventional root finding algorithms deal only with real values, alternative methods must be found. At present the procedure is to minimize the complex modulus via a pattern search program developed by Prof. D.E. Whitney. The global minimums must be zero, but the search routine may return false eigenvalues corresponding to local minimums. These can usually be detected by simply looking at the modulus value. When the modulus value is near zero there is a possibility that the nonzero determinant is due either to the finite word size of the computer or to the fact that a false root has been returned. To resolve this ambiguity one can look at the real and imaginary parts of the determinant. If an actual system root has
been returned, and it is a single distinct root, the real and imaginary parts must both change signs in the neighborhood of the root. In normal operation where the change in roots with design changes is being observed the user confidence in root positions is high, and only when unexpected root locations are returned is the doubt sufficient to check the sign change. The experience has been that this check seldom fails to confirm a root whose determinant value was reasonably low (say less than 0.1). What is considered "reasonably low" varies with the arm and the position of the root.

**Convergence to the "Wrong" Eigenvalue.** Since the eigenvalue search bases its actions on the shape of the determinant modulus over the complex plane, it will converge to different roots depending on where the search is begun. Thus one can repeatedly "find" the same root and not find a desired root. Under these circumstances graphical display and input becomes very helpful, allowing the user to quickly modify the starting point of the search based on the displayed results of the previous search. Search routines specifically designed to take advantage of the particular problem of complex root finding might do much better and cut down on the user interaction required.

A more critical numerical problem arises when it is practically impossible to make the routine coverage to a root that exists. This has been observed for extreme values of arm servo control parameters. It corresponds to shapes of the determinant that change very rapidly
in the region of the eigenvalues. It seems to be aggravated by two
roots which are very close. Fortunately when this happens it also
seems that the root changes very slowly with parameter changes so
that roots determined with different parameter values can be used.

Numerical Overflow. In only one case has the problem of numerical
overflow been encountered. The problem was solved for that case by
using extended computer word size, which may not be practical in all
cases for all computers.

2.3 Use of Model in a Design Context

The model described above was first used in a real manipulator
design context. The example model output given here was part of the
evaluation of the arm structure for a manipulator under design at
the Charles Stark Draper Laboratory. The study was conducted at an
early point in the design, and before the generalizing results of
the following chapter were obtained. Thus if the same study were
conducted now, it could proceed more directly to a solution.

2.3.1 Example of Program Use

The arm was to be a high performance arm slightly more than
one meter in length and capable of carrying a payload of ten kilo-
grams. An estimate was made of the weight of the hydraulic actuators
and control valves which would supply the torque required. One of
the design goals was to build an arm capable of responding to input
frequencies of up to 10 hz. The question that the modeling exercise
was to answer was, "What structure will provide the rigidity necessary to build a 10 hz arm?" The transient response desired was one with minimal overshoot. For the second order system which approximated the control of each joint minimal overshoot implied a damping ratio near one. Thus the first approach was to establish feedback gains which would yield dominant eigenvalues with a damping ratio of one and a magnitude of 10 hz for a rigid inertia equivalent to that of the structure, actuators and payload.

The model of the arm is indicated schematically in Figure 2.9. Nine elements are used in this model of the arm; four distributed beams, one controlled joint, and four rigid masses. A five element model was also used for some studies. That model was obtained by increasing the density of the outer link to account for the mass of the 4 kg and 8 kg masses.

Figure 2.10a shows the log magnitude and phase versus frequency of the frequency response of the five element model for a structure outer radius of 0.04 m. The joint feedback gains would achieve 10 hz. eigenvalues with damping ratios of 1.0 for an equivalent rigid inertia. The real and imaginary parts of this frequency response are displayed in Figure 2.10b, arranged for inverse Fourier transformation via the Fast Fourier Transform Algorithm. This (inverse) transformation yields the time impulse response displayed in Figure 2.10c.

As would be expected from the peak in the magnitude plot Figure 2.10a, and verified by the impulse response of Figure 2.10c,
Figure 2.9 Example: Nine Element Arm Model

Inner radius = 0.035 m  
Inner radius = 0.038 m

Material: Aluminium, Density = 2705 kg/m³, E=7.1x10¹⁰ nt/m²
G=2.65x10¹⁰ nt/m²
Figure 2.10a  Bode Diagram, Force Loading at Endpoint
Joint feedback gains for equivalent inertia rigid
system eigenvalues of 10Hz. and 1.0 damping ratio.

Figure 2.10b  Samples of Complex Frequency Response. Arranged for FFT.
Figure 2.10c  Impulse Response via Inverse Fourier Transform of b.

Figure 2.11  Impulse Response with Increased Velocity Feedback. System of Figure 2.10 but with equivalent rigid inertia damping ratio of 1.2.
the response is certainly not one of minimum overshoot as desired, or expected from a rigid analysis. An attempt to remedy this by increasing the velocity feedback gains as might be done for a rigid arm (to a damping ratio of 1.2 for an equivalent rigid inertia) results in the impulse response of Figure 2.11. This response is slightly more oscillatory than before the velocity gain was increased.

A better understanding of the nature of these occurrences may be obtained from Figure 2.12. Figure 2.12a gives a close up of the imaginary plane near the origin, and displays the complex roots responsible for the oscillatory behavior. This root, which is due to the flexibility of the structure, has a maximum damping with variations in velocity feedback which occurs at values of the velocity gain which are lower, not higher, than the gains first used.

Figure 2.12 also displays the locus of the dominant closed loop eigenvalues as the outer radius is varied for the nine element model, with feedback gains that would produce second order roots of magnitude 10 Hz. and damping ratio 1.0 in an equivalent rigid inertia. As can be seen, the oscillatory conjugate root pair that dominates for small radii moves to the left, eventually leaving a real root to dominate. That real root approaches the -10 Hz. expected for a rigid system as the arm radius becomes larger. Roots of a higher, lightly damped mode displayed in Figure 2.12b move away from the origin and remain lightly damped.

An acceptable outer radius for the arm might in fact be 0.05 m., and the impulse response for that case is shown in Figure 2.13.
Figure 2.12 Root Locus for Arm of Figure 2.9.

- 5 element model varying rigid system damping.
- 9 element model varying outer radius.

\[ \text{X} \] 0.05 \( \Delta \) 0.045 1.2
\[ \Delta \] 0.055 0.04

a) Detail view near origin.

b) View of some higher modes.

Figure 2.13 Impulse Response of Nine Element Model.
Outer radius = 0.05m.
It displays the third order nature one might expect from the root locus plots of Figure 2.12.

2.3.2 Conclusions from the Design Exercise

Based on a design experience such as the one just described, one might conclude several things.

1) The model is capable of yielding design information for reasonably complicated arm models sufficient to evaluate the structural rigidity.

2) Flexibility limits the validity of rigid arm design procedures.

3) Flexibility can result in effects counter to intuition based on rigid arm systems, i.e. increased oscillation with increased velocity feedback.

4) More efficient design effort might result if the general nature of the interaction between the joint control and the flexible components existed.

5) Damping inherent in the material, provided for by complex modulii, had no noticeable effect for reasonable values when velocity feedback was present.

The following chapter will attempt to develop a more general understanding of the joint control – structure interaction as suggested in 4) above.

2.4 Computer Programs

The computer programs referred to in this thesis will be listed and explained in [24].
CHAPTER 3

CHARACTERIZATION OF THE ARM SERVO STRUCTURE INTERACTION

The previous chapter presented a procedure for modeling an arm which included the flexibility of the arm components, and the control system which regulated the arm joints. In this chapter results are presented which were obtained by using that modeling procedure. These results are interpreted to indicate the limitations which the flexibility of a given arm design will impose on the feedback gains of the type of joint control assumed. The arm performance, in particular the speed of response and settling time, depend on these gains. Thus limitations on the gains are limitations on the arm performance.

3.1 Two Equal Link Examples

The most typical of general purpose arm configurations seems to be one with two long beam like segments of approximately equal length with rotary joints. An additional short segment incorporating the end effector is also typical but will not be considered in this study.

The beam segments in this study are considered to be identical in all properties but have different boundary conditions. They are joined by a servo controlled rotary joint. This "elbow" joint is
assumed to operate on the position and velocity error feedback
of the joint, analogous to a rotary spring and dashpot. One end of
the two link arm has free boundary conditions and the other end is
clamped to ground in the first case, and in the second case is
connected to ground via a rotary "shoulder" joint with the same
control scheme as the elbow joint.

The modeling process discussed in Chapter 2 was used to obtain
the dynamic characteristics of the two cases. The display of the
dominant eigenvalues is relied on heavily to convey the changes in
system behavior due to changes in joint servo control parameters.
The frequency determinant was analytically simplified for this study
as described in Appendix A.

3.1.1 Two Link, One Joint Case

First let us consider the two identical link, one joint case
illustrated in Figure 3.1. This case includes distributed beam
dynamics on either side of a controlled joint. It proves to be very
informative because after nondimensionalization the results for the
complete range of independent parameters can be displayed on one plot
of the root loci.

Nondimensionalization of Parameters. Table 3.1 displays the important
physical parameters for this problem and a set of convenient non-
dimensional groupings. They are nondimensional frequency (complex)
$\bar{\omega}$ which is multiplied by $\sqrt{-1}$ to obtain the eigenvalue, the non-
dimensional servo frequency $\bar{\omega}_s$ and damping ratio $\zeta$. The servo
Figure 3.1 One Joint, Two Link Arm Example

\[ \omega_c = \text{1st structural frequency (joints clamped)} \]

controlled joint \((\omega_s \& \zeta)\)

\[ \frac{l}{2} \quad l \]
frequency would result from a rigid inertia equivalent to that of the outer link with only the joint position feedback.

As observed in Figure 3.2 and in Figure 3.3 in more detail this simple case has root loci which fall into three characteristic groups as follows:

1. \( \tilde{\omega} \ll .386 \)
2. \( \tilde{\omega} \approx .386 \)
3. \( \tilde{\omega} \gg .386 \)

Appearing in Figure 3.4 are the higher order roots that are not dominant.

**Low Servo Bandwidth** (Figures 3.2a and 3.3a). For case 1 with \( \tilde{\omega}_s \) much less than .386 rad/sec and \( \zeta \) not much greater than 1.0 the behavior is essentially that of a rigid arm. For \( \zeta < 1 \) the complex conjugate roots \( r_1 \) and \( r_2 \) near the origin indicate oscillatory behavior. For \( \zeta \) just greater than one the real roots indicate overdamped behavior. There will theoretically also be vibrations of an infinite number of modes of the distributed beams. Their effect will be less important in most cases because their time constants are several times higher and their amplitudes several times lower than the lower, dominant modes. As \( \zeta \) is increased past one, one of these real roots, \( r_1 \), moves toward the origin and \( r_2 \) moves away from the origin, eventually meeting a third root, \( r_3 \), moving toward the origin. Further increases in \( \zeta \) cause \( r_2 \) and \( r_3 \) to become complex. As \( \zeta \) gets very large these roots move in an arc and approach the imaginary axis. The value they approach is the clamped elbow natural frequency of the arm, which is
Figure 3.2 Three Characteristic Root Loci
Each plot indicates the position of the three dominant roots of the arm of Figure 9 for different servo bandwidth. Arrows indicate direction of root movement for increasing values of damping ratio.

a. Rigid Region (ωₙ < 0.386)
Characteristic second order system for reasonable values of damping. Figure 3.1la for detail.

b. Transition Region (ωₙ = 0.386)
A triple root occurs at the critical value of ωₙ = 0.386 for ζ = 0.915.
Figure 3.1lb for detail.

c. Floppy Region (ωₙ >> 0.386)
It is no longer possible to bring the dominant complex roots near the real axis. Figure 3.1lc for detail.
Figure 3.3a: Detail Root Loci of Dominant Roots Varying $\tau$, $\omega / \omega_c = 0.167$
Figure 3.3b  Detail Root Loci of Dominant Roots
Varying $\zeta$, $\omega_s/\omega_c=0.383$
Figure 3.3c: Detail Root Loci of Dominant Roots
Varying $\xi$, $\omega_n/\omega_C=0.50$
Figure 3.3d Summary of Complex Root Loci

- \( \omega_k / \omega_c \) constant
- \( \zeta \) constant
Figure 3.4 Higher Order Root Loci Varying $\zeta$.

Higher Order Roots
- $\omega_s/\omega_c = 0.167$
- $\omega_s/\omega_c = 0.383$
- $\omega_s/\omega_c = 0.500$

Values of $\zeta$ appearing for each case are:
0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7,
0.8, 0.9, 1.0, 2.0, 3.0, 4.0, 5.0.
the value one on the vertical axis of the nondimensional plot of Figure 3.3.

**TABLE 3.1**

NONDIMENSIONAL AND PHYSICAL VARIABLES FOR THE TWO FLEXIBLE LINK, ONE JOINT CASE

<table>
<thead>
<tr>
<th>Physical Variables</th>
<th>Nomenclature</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Joint position feedback gain (angular)</td>
<td>k</td>
<td>LF</td>
</tr>
<tr>
<td>Joint velocity feedback gain (angular)</td>
<td>c</td>
<td>LFT</td>
</tr>
<tr>
<td>Frequency (complex)</td>
<td>( \omega )</td>
<td>( T^{-1} )</td>
</tr>
<tr>
<td>Mass density/unit length</td>
<td>( \mu )</td>
<td>( FT^{-2}L^{-2} )</td>
</tr>
<tr>
<td>Total arm length</td>
<td>( \ell )</td>
<td>L</td>
</tr>
<tr>
<td>Stiffness product</td>
<td>EI</td>
<td>FL²</td>
</tr>
</tbody>
</table>

(E = Young's modulus and I = cross sectional area moment of inertia)

From these variables we construct for convenience:

First natural frequency of a cantilevered beam of length \( \ell \)

\[
\omega_c = 3.52 \sqrt{\frac{EI}{\mu \ell^4}}
\]

Servo frequency for a rigid arm structure

\[
\omega_s = \sqrt{\frac{24k}{\mu \ell^3}}
\]

If one replaces EI and \( \mu \) as independent dimensional parameters with
\( \omega_c \) and \( \omega_s \), one can form the following nondimensional groups:

<table>
<thead>
<tr>
<th>Nondimensional Variable</th>
<th>Nomenclature</th>
<th>Grouping</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>( \bar{\omega} )</td>
<td>( \omega/\omega_c )</td>
</tr>
<tr>
<td>Servo frequency</td>
<td>( \bar{\omega}_s )</td>
<td>( \omega_s/\omega_c )</td>
</tr>
<tr>
<td>Servo damping ratio</td>
<td>( \zeta )</td>
<td>( \frac{12c}{\mu k^3 \omega_s} )</td>
</tr>
</tbody>
</table>

**Critical Servo Bandwidth** (Figures 3.2b and 3.3b). As \( \bar{\omega}_s \) is increased, the point at which \( r_1 \) and \( r_2 \) join the real axis, and the point at which \( r_2 \) and \( r_3 \) leave the imaginary axis move closer together. For a critical value of \( \bar{\omega}_s \) found numerically to be about 0.386 these two points meet and there exists a triple root for \( \zeta \approx 0.915 \) in the region of \((-0.67 + j0.0)\). In this region the position of the roots is very sensitive to slight variations in \( \bar{\omega}_s \) and \( \zeta \). This critical point seems to be the farthest from the origin that the nearest root of the arm can be located, and thus the fastest overdamped response of these dominant roots obtainable with simple joint position and velocity feedback.

**High Servo Bandwidth** (Figures 3.2c and 3.3c). For \( \bar{\omega}_s \gg 0.386 \) the root locus plot looks quite different than for \( \bar{\omega}_s \ll 0.386 \). A single root \( r_3 \) remains on the real axis moving toward the origin as \( \zeta \) increases. The two complex conjugate roots \( r_1 \) and \( r_2 \) remain imaginary for all
values of $\zeta$. They move from a value on the imaginary axis for $\zeta = 0$, through an almost semicircular path to a point whose value is the clamped elbow natural frequency. As $\bar{\omega}_s$ becomes larger the semi-circles become smaller and the real root becomes more negative for the same value of damping ratio $\zeta$. In this region the spring is too stiff to allow motion which would dissipate the vibrational energy of the beam. This is essentially a poor match between the mechanical impedances of the structure and joint for the purposes of dissipating energy.

**Summary for Two Link, One Joint Case.** Figure 3.3d displays contours of constant position feedback gain (equivalent to constant $\bar{\omega}_s$) for a wide range of values of $\bar{\omega}_s$. Also indicated in dashed lines are the contours of constant $\zeta$. This plot shows the transition from the rigid to the floppy extreme cases for the lowest complex conjugate pair.

Conclusions one might support with a graph of this type include:

1. Through careful design one may be able to achieve dominant system eigenvalues on the order of one-half or more of the lowest clamped joint natural frequency using this simple control. For example see $\omega_s/\omega_c = 5$, $\zeta = 0.8$.

2. Noticeable deviations from rigid behavior become apparent when the eigenvalues of the rigid dynamics are of a magnitude of about one-fourth the lowest clamped joint natural frequency. For example see $\omega_s/\omega_c = 0.33$. Excitation of the higher modes by periodic excitations may require that flexibility be considered for any arm.
(3) Consideration of arm flexibility becomes crucial when a rigid design procedure produces dominant eigenvalues the magnitude of which is greater than one-third the lowest clamped joint natural frequency. A rigid analysis of the behavior would indicate that increased velocity feedback would result in increased damping on the dominant complex root pair. The opposite result occurs in the actual flexible system and the effect is to decrease damping for the complex conjugate pole pair. See for example \( \omega_s/\omega_c = 0.5, \zeta = 0.8 \).

The first conclusion above is the most important for use in the remainder of the thesis. It will be supported with other examples in the remainder of this chapter.

3.1.2 Two Link, Two Joint Case

Having considered the two link one joint case the next logical step in complexity is two links and two joints. This configuration more resembles an actual manipulator although when the additional joint is locked or if the joint drive is self locking and not being driven the previous result is more realistic. The step is made with considerable increase in the difficulty with which one can thoroughly understand the case. With the simplifying assumptions of identical links and feedback constrained to two variables per joint one must deal with four independent parameters in the flexible case after nondimensionalizing the problem and establishing the nominal joint angles about which vibration is being studied. While this is not so great as to prohibit meaningful exploration of the case, it makes compact and intuitive display impossible.
Limiting Case—Rigid Links. In order to establish bounds for the flexible link behavior we will first consider the limiting case of rigid links. In one nominal position this case can be described by three nondimensional parameters as shown in Table 3.2. As chosen here these correspond to nondimensional gains in the feedback of the velocity and position of the proximal or shoulder joint and the velocity of the distal or elbow joint. The problem is nondimensionalized with respect to the total arm length, the elbow position feedback gain and the undamped natural frequency with the shoulder joint clamped.

Lagrange's equations were used to develop the frequency function for the rigid link case and that development is sketched briefly in Appendix B. For the constrained feedback the rigid motion is very close to that of a double pendulum for small vibrations.

Nondimensionalization of Two Link, Two Joint Example. By nondimensionalizing the two equal link, two joint case we can reduce a problem of eight parameters to one of five parameters. For root locus plots one of these parameters is the location of the roots which will be a function of the other four parameters. The choice of nondimensional groupings will depend on convenience and the physical significance to the problem at hand. In the case of two links and one joint the choice was influenced by the prior understanding of a simple second order system which was the limiting case as the links became very rigid relative to the joint. When two joints are present the rigid counterpart is a fourth order system which is generally not nearly so intuitively comforting.
TABLE 3.2
NONDIMENSIONAL AND PHYSICAL VARIABLES OF THE TWO RIGID LINK, TWO JOINT PROBLEM

<table>
<thead>
<tr>
<th>Physical Variables</th>
<th>Nomenclature</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distal Joint position feedback gain (angular)</td>
<td>$k_2$</td>
<td>FL</td>
</tr>
<tr>
<td>Distal Joint velocity feedback gain (angular)</td>
<td>$c_2$</td>
<td>FLT</td>
</tr>
<tr>
<td>Proximal Joint Position feedback gain (angular)</td>
<td>$k_1$</td>
<td>FL</td>
</tr>
<tr>
<td>Proximal Joint velocity feedback gain (angular)</td>
<td>$c_1$</td>
<td>FLT</td>
</tr>
<tr>
<td>Mass Density/unit length</td>
<td>$\mu$</td>
<td>FT$^2$L$^{-2}$</td>
</tr>
<tr>
<td>Total Arm length</td>
<td>$\ell$</td>
<td>L</td>
</tr>
<tr>
<td>Frequency</td>
<td>$\omega$</td>
<td>T$^{-1}$</td>
</tr>
</tbody>
</table>

Construct the following variable equivalent to the natural frequency of the arm with the proximal joint clamped, and with simple position feedback of gain $k_2$ to the distal joint with dimensions T$^{-1}$.

$$p = \sqrt{\frac{24k_2}{\mu \ell^3}}$$

<table>
<thead>
<tr>
<th>Nondimensional Variables</th>
<th>Grouping of Physical Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\omega}$</td>
<td>$\omega/p$</td>
</tr>
<tr>
<td>$\bar{k}$</td>
<td>$k_1/k_2$</td>
</tr>
<tr>
<td>$\bar{c}_1$</td>
<td>$c_1p/k_2$</td>
</tr>
<tr>
<td>$\bar{c}_2$</td>
<td>$c_2p/k_2$</td>
</tr>
</tbody>
</table>
By retaining nondimensionalization compatible with the rigid analysis, however, we can draw on our previous analysis and readily see when there is a significant departure from the rigid behavior. Thus we chose the nondimensional groupings of Table 3.3 which enables us to see in what range of flexibility the rigid analysis is valid. Notice we can move from the rigid analysis whose parameters appear in Table 3.2 to the flexible analysis by additionally specifying one value, the ratio of the clamped joint, cantilevered frequency to the rigid (clamped shoulder) frequency, \( \frac{\omega_c}{\bar{p}} = \bar{\omega}_c \).

Table 3.4 shows alternative parameters that will be used for improving on the rigid analysis when the links are significantly flexible.

**Performance Criteria.** When dealing with the flexible or rigid case one needs to develop some criteria for good dynamic performance. While some criteria are obvious such as stability, others depend on the specific application of the arm and in the practical design case are almost always the result of a combination of subjective designer opinion and constraints of weight, power, or previous design decisions. The quadratic cost function of linear optimal control is a popular way to specify performance but is more the result of mathematical convenience than a correct estimation of the cost. For an indication of how different criteria can result in different systems refer to Figure 3.5. It displays eigenvalues of optimal fourth order systems by three criteria: Integral of Time times Absolute Error (ITAE),
### TABLE 3.3

NONDIMENSIONAL AND PHYSICAL VARIABLES OF THE TWO FLEXIBLE LINK, TWO JOINT CASE. (FOR DEVIATION FROM RIGID ANALYSIS.)

<table>
<thead>
<tr>
<th>Physical Variables</th>
<th>Nomenclature</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distal joint position feedback gain (angular)</td>
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<tr>
<td>Distal joint velocity feedback gain (angular)</td>
<td>$c_2$</td>
<td>LFT</td>
</tr>
<tr>
<td>Proximal joint position feedback gain (angular)</td>
<td>$k_1$</td>
<td>LF</td>
</tr>
<tr>
<td>Proximal joint velocity feedback gain (angular)</td>
<td>$c_1$</td>
<td>LFT</td>
</tr>
<tr>
<td>Frequency (complex roots)</td>
<td>$\omega$</td>
<td>$T^{-1}$</td>
</tr>
<tr>
<td>Mass density/unit length</td>
<td>$\mu$</td>
<td>$FT^2L^{-2}$</td>
</tr>
<tr>
<td>Stiffness product ($E =$ Young's modulus, $I =$ cross section moment of inertia)</td>
<td>$EI$</td>
<td>$FL^2$</td>
</tr>
<tr>
<td>Total arm length</td>
<td>$\ell$</td>
<td>L</td>
</tr>
</tbody>
</table>

From these physical variables we construct for convenience:

First natural frequency of a cantilevered beam:

$$\omega_c = 3.52 \sqrt{\frac{EI}{\mu \ell^4}} \equiv T^{-1}$$

Rigid natural frequency with proximal joint clamped:

$$p = \sqrt{\frac{3k_2}{\mu \ell^3}} \equiv T^{-1}$$

We will use the variables $\omega_c$ and $p$ instead of $EI$ and $\mu$ and non-dimensionalize with respect to $p$, $k_2$ and $\ell$ to obtain the following groupings:

| $\frac{\omega}{p}$ | $\frac{k_1}{k_2}$ | $\frac{c_1}{p/k_2}$ | $\frac{c_2}{p/k_2}$ | $\frac{\omega_c}{p}$ |
TABLE 3.4
NONDIMENSIONAL AND PHYSICAL VARIABLES OF TWO FLEXIBLE
LINK, TWO JOINT CASE. (CONSTANT LINK PARAMETERS.)

<table>
<thead>
<tr>
<th>Physical Variables</th>
<th>Nomenclature</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distal joint position feedback gain (angular)</td>
<td>$k_2$</td>
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</tr>
<tr>
<td>Distal joint velocity feedback gain (angular)</td>
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<td>LFT</td>
</tr>
<tr>
<td>Proximal joint position feedback gain (angular)</td>
<td>$k_1$</td>
<td>LF</td>
</tr>
<tr>
<td>Proximal joint velocity feedback gain (angular)</td>
<td>$c_1$</td>
<td>LFT</td>
</tr>
<tr>
<td>Frequency (complex roots)</td>
<td>$\omega$</td>
<td>$T^{-1}$</td>
</tr>
<tr>
<td>Mass density/unit length</td>
<td>$\mu$</td>
<td>$FT^2L^{-2}$</td>
</tr>
<tr>
<td>Stiffness product ($E = $ Young's modulus, $I =$ cross section moment of inertia)</td>
<td>$EI$</td>
<td>$FL^2$</td>
</tr>
<tr>
<td>Total arm length</td>
<td>$\ell$</td>
<td>$L$</td>
</tr>
</tbody>
</table>

From these physical variables we construct for convenience:

First natural frequency of a cantilevered beam:

$$\omega_c = 3.52 \sqrt{\frac{(EI/\ell^4)\ell}{\mu}} T^{-1}$$

Cantilever beam endpoint rotational spring constant: $\alpha = EI/\ell \equiv LF$

Use $\omega_c$ and $\alpha$ instead of $\mu$ and $EI$. Nondimensionalize with respect to $\omega$, $\omega_c$, and $\alpha$ to obtain:

$$\frac{\omega}{\omega_c}, \frac{k_1}{\alpha}, \frac{k_2}{\alpha}, \frac{c_1 \omega_c}{\alpha}, \frac{c_2 \omega_c}{\alpha}$$
Figure 3.5 System Roots for Optimal Fourth Order Systems by Various Criteria

Criteria:
- △ Solution time
- □ ITAE
- ○ Townsend (quadratic)

Figure 3.6 Root Nomenclature Indicated on Four Undamped Roots

\( r_2^* = \text{Complex Conjugate of } r_2 \)

\[ r_3 = r_4 \]

\[ r_1 = r_2^* \]
Solution Time criterion, and a quadratic cost function developed by Townsend [16] for manipulator arms. Townsend's control gains resulted in all overdamped roots and a rather sluggish system. The other two criteria, ITAE and solution time, result in four underdamped roots and therefore a much faster response, but with some overshoot.

In an effort to make this analysis independent of the criterion selected, the roots of the arm in a broad range of configurations which might be termed acceptable will be discussed.

The transient performance of systems with various root pole placements and desirable placements for poles is extensively discussed by Graham and Lathrop [17].

Feedback Control Configuration and Its Limitations. As suggested previously the control configuration which is being considered in this section is not complete state variable feedback, even in the rigid case. This would involve feedback between the joints (inter-joint feedback) as well as within the joint (intra-joint feedback). Four additional feedback gains would be involved, two velocity and two position gains. The additional complexity would enable complete freedom in the rigid case of placing the eigenvalues of the system wherever desired, and in some arms the improved performance may well be worth the added complexity. Such interjoint feedback is extremely rare in arms produced today, even experimental arms. In fact many arms do not have velocity feedback and are dependent on friction in drive trains which is highly variable from unit to unit, and cannot
be readily controlled.

Thus while we can achieve many acceptable pole placements it would be accidental if we could achieve the optimal, even if that optimum were validly and precisely defined. For example we cannot match the eigenvalues dictated by the standard forms of the ITAE or solution time criteria or in general duplicate the eigenvalues of a quadratic cost optimal system. More complete feedback would give one more degree of freedom in placing system eigenvalues arbitrarily, freedom not available in the present case. The feedback gains determined from the matrix Riccatti equation for a quadratic cost criterion for example assumes all measured state variables can be fed back to any control variable.

The justification for this simplification then is that it more adequately describes the practical problem. This constraint in feedback is not entirely a simplification. Design techniques for complete state variable feedback are more readily available than for restricted feedback.

Discussion of the Root Loci—Two Link, Two Joint Rigid Link. Figure 3.6 displays the location of the complex conjugate root pair for the case in question with no velocity feedback or damping. We will reference the roots as \( r_1, r_2, r_3, \) and \( r_4 \) where \( r_1 = r_2^* \) and \( r_3 = r_4^* \). The lower indexes refer to roots nearer the origin. The relative position of the undamped roots is determined by the ratio of position feedback at the two joints \( k_1/k_2 \). This relation is displayed in Figure 3.7.
Figure 3.7a Undamped Natural Frequencies for a Rigid Two Joint Arm. Nondimensionalized with respect to a one joint (elbow) arm with same feedback gains. Plotted versus ratio of position feedback gains.
Figure 3.7b  First Undamped Natural Frequency for a Rigid Two Joint Arm.  
(Expanded Scale)
In the nondimensional case considered the lower root asymptotically approaches 1 as the inboard "spring" $k_1$, is made stiffer, $k_2$ remaining fixed at 1. The position of these roots is moved to the left as the velocity is fed back with increasing negative gain. The series of Figures 3.8a through 3.8d shows these variations in detail.

For any value of $k_1/k_2$, the roots can be brought to the negative real axis by increasing $c_1$ and $c_2$ appropriately. The difficulty arises in achieving sufficient damping for the pair $r_1$ and $r_2$ without greatly overdamping roots $r_3$ and $r_4$. For relatively low values of $k_1/k_2$ (see Figure 3.8a) it is impossible to achieve a damping of the order of 0.7 on $r_1$ and $r_2$ without bringing the roots $r_3$ and $r_4$ down to become two real roots, one of approximately the same magnitude as $r_1$ and $r_2$. For values of $k_1/k_2 = 2$ a reasonable response can be obtained of a third order nature for $c_1 = 6.5$, $c_2 = 0.1$. As noticed in Figure 3.8, $r_1$ and $r_2$ are much nearer the origin but this can be compensated for by increasing $k_2$ when dimensional values are computed, thus shrinking the scale of the plot.

As $k_1/k_2$ is increased to 5 and 10, higher values of $c_2$ are required to damp the lower modes sufficiently. The high values of both $c_1$ and $c_2$ in turn overdamps the higher modes, bringing a slow real root near the origin. However, as $k_1/k_2$ continues to increase to 50 (see Figure 3.8d) the lower mode becomes essentially independent of $c_1$, and its damping can be controlled by $c_2$ at will. This approaches the one joint case. For $k_1/k_2 = 50$, $c_1 = 2$, $c_2 = 1.7$ the two complex conjugate pairs will both have damping greater than 0.7.
Figure 3.8a Root Loci of Nondimensional Rigid
Two Joint Arm: $k_1 = 2$

$k_1 = 2$
values of $c_i$ are 0, 5, & 10
Figure 3.8b Root Loci of Nondimensional Rigid Two Joint Arm: \( k_1=5 \)

Values of \( c_1=0, 5, 10, 15 \)

- \( \Delta c_2=1 \)
- \( \nabla c_2=1 \)
- \( \square c_2=2 \)

\[ \text{Re}(j\omega/p) \]
\[ \text{Im}(j\omega/p) \]
Figure 3.8c Root Loci of Nondimensional Rigid Two Joint Arm: $k_1 = 10$

- $k_1 = 10$
- $c_1$ varies from 0 in steps of 5
  - $\Delta c_2 = 1$
  - $\nabla c_2 = 1.0$
  - $\square c_2 = 1.5$
  - $\circ c_2 = 2.0$

$\Re(j \omega/p)$ vs. $\Im(j \omega/p)$
Figure 3.8d Root Loci of Non-dimensional Rigid Two Joint Arms: $k_1 = 50$

$c_1$ varies from 0 in steps of 10.

$\triangle c_1 = 1.0$

$\blacklozenge c_1 = 1.5$

$\square c_1 = 1.7$

$c_1 = 0.0$
Thus there are many parameter values where performance might be satisfactory. The one preferred would depend on the application at hand. The area with \( k_1/k_2 \) large seems preferrable in terms of performance but the higher gains would require larger shoulder motors if the operation is to remain within the linear range. This is feasible however. It may be difficult to achieve the low values of damping desired at the joints especially when large gear reductions are involved.

When \( k_1/k_2 \) is small (the two gains are more nearly equal), the lower pair of roots remains much nearer the origin. (Figure 3.8a). To have a response of comparable speed in a system with this control it is necessary to have a higher \( k_2 \), which has the effect of shrinking the nondimensional scales of Figure 3.8. This may not be desireable since a larger value of \( k_1 \) requires more torque if the response is to remain linear. This requires more massive motors and drives at the outer joint, or compliant transmission shafts which aggravate the vibration problems.

**Shift from Rigid Roots for Flexible Links.** To demonstrate the change of the roots as the links become more and more flexible, examples were chosen from the rigid case which had reasonable root locations. Appearing in Figure 3.9 are the following cases, nondimensionalized as in Table 3.3.

\[
\begin{align*}
\tilde{k}_1 &= 2 \\
\tilde{k}_1 &= 10 \\
\tilde{k}_1 &= 50
\end{align*}
\begin{align*}
\tilde{c}_1 &= 5 \\
\tilde{c}_1 &= 15 \\
\tilde{c}_1 &= 10
\end{align*}
\begin{align*}
\tilde{c}_2 &= .5 \\
\tilde{c}_2 &= 2 \\
\tilde{c}_2 &= 1.7
\end{align*}
\]
Figure 3.9 Variation in Roots with $\omega_c$. Nondimensionalization
by Table 3.3

$\Delta k_1 = 2, c_1 = 5, c_2 = 5$
$\triangledown k_1 = 10, c_1 = 15, c_2 = 2$
$\square k_1 = 50, c_1 = 10, c_2 = 1.7$

remains fixed while additional root moves in
For each case the stiffness parameter of the arm $\omega_c$ is varied from one to 10 and the root position is plotted. This is not presented as evidence of the limits of flexibility for reasonable performance, but as evidence of the limits of a rigid design procedure. As $\omega_c$ decreases the complex conjugate root pair, $r_1$ and $r_2$ become less damped in all cases, moving from the position of the rigid root. The real root near the origin in these cases moves nearer the origin indicating a slower response. For $k_1 = 10$ it is observed that the rigid root near the origin does not move but an additional real root moves in causing similar effects of slower response and increased phase shift.

Several things can be observed qualitatively from these three examples. Higher values of position feedback $k_1$ result in the complex conjugate pair being more sensitive to $\omega_c$. Initially one might think that this favors lower $k_1$ for more flexible links. This is not conclusive since to get a comparable speed from the two systems $k_1/k_2 = 10$ and $k_1/k_2 = 50$ one would have to use a larger value of $p = \sqrt{3k_2/\mu l^3}$ when moving to dimensional parameters. If one used $p = 2$ for the case $k_1/k_2 = 10$, $\omega_c = \omega_c p$ would also double. The present nondimensionalization scheme indicates limits of the rigid analysis but is not well adapted to improving on that analysis. For this reason a slightly different scheme will now be used which fixes the links while allowing us to vary all the parameters of the joint control system.
Improvement on Control with Constant Flexible Link Parameters. In order to demonstrate how one might improve on the servo control parameters using the flexible model we will take an example from Figure 3.9. First we will convert from the nondimensional parameters of Table 3.3 which allowed ease in comparison to the rigid case, to the nondimensional parameters of Table 3.4, which holds the structural parameters constant and displays directly what we can accomplish with the control. The case chosen for demonstration is:

<table>
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<tr>
<th>Variable Structural Parameters</th>
<th>Constant Structural Parameters</th>
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<tr>
<td>(Table 3.3 and Figure 3.9)</td>
<td>(Table 3.4 and Figure 3.10)</td>
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<tr>
<td>( \bar{k}_1 = 50 )</td>
<td>( \bar{k}_1 = 6.5 )</td>
</tr>
<tr>
<td>( \bar{\omega}_c = 2 )</td>
<td>( \bar{k}_2 = 0.13 )</td>
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First the velocity feedback gain \( \bar{c}_2 \) to the outer joint is varied until a maximum damping on the complex root pair is found as shown in Figure 3.10. Then \( \bar{c}_1 \), the velocity feedback to the inner joint is varied until the slow real root moves in, slowing the response. If the maximum damping ratio shown in the figure of 0.58 is not adequate for the application at hand, other values of \( k_1 \) and \( k_2 \) could be tried or the slow real root with slow response would have to be tolerated.
Figure 3.10  Adjustment of Flexible Roots. Nondimensionalization by Table 3.4

\[ k_1 = 6.5 \quad k_2 = 0.13 \]

\( \Delta \circ c_1 = 0.2 \quad c_2 = 0.15, 0.30, 0.40, 0.50 \quad \square c_2 = 0.3, 0.4, 0.5 \)

\( \diamond \circ c_1 = 15 \quad c_2 = 0.40 \)

\[ \text{RE}(j\omega/\omega_c) \]
3.2 One Link, One Joint, with Payload Mass

The final distributed combination that will be presented here is the one link, one joint with payload example. The nondimensionalization for this example is given in Table 3.5. The locus of the dominant value \( \bar{\omega} \) for this system are displayed in Figure 3.11 a, b, c, and d for values of \( \bar{m} \) of 0.1, 0.2, 1.0 and 10.0. As will be observed from these figures the maximum dominant eigenvalue that can be obtained with damping ratio of order 0.7 has approximately one half the magnitude of the clamped joint frequency, which is the point of convergence of the root loci for large values of \( \omega_s \).

3.3 Simple Lumped Model

It is informative to investigate the simplest model which might have a response at least qualitatively like the mixed distributed-lumped model response. The simplest lumped parameter model investigated is shown in the figure below. It consists of a spring \( k_s \) and dashpot \( c \) in parallel which represent the servo parameters. These elements are connected at one end to ground and to the spring \( k_b \) at the other end. The spring with constant \( k_b \) represents the structure compliance and is connected to the mass \( m \). Although a system with simple linear displacement is shown, results would be identical for a rotational system but harder to draw. The spring dashpot system has a total complex "spring constant" of

\[
k_T = \frac{1}{\frac{1}{k_s + sc + k_b}}
\]
where $s = j\omega$ is the Laplace variable evaluated on the imaginary axis. Writing the equations of motion as

$$m\ddot{x} + k_l x = 0$$

one obtains the characteristic equation which is

$$s^3 + \frac{k_s + k_b}{c} s^2 + \frac{k_2}{m} s + \frac{k_1 k_2}{mc} = 0$$

By defining the following nondimensional variables

$$Z = s/\sqrt{k_s/m}$$

$$k = k_b/k_s$$

and $$\zeta = c/(2m k_s/m)$$

one obtains a nondimensional form of the characteristic equation

$$Z^3 + \frac{1 + k}{2\zeta} Z^2 + kZ + \frac{k}{2\zeta} = 0$$

the roots of which are a function of only two variables, $k$ and $\zeta$. These roots are plotted in the figure below for variations in $k$ and $\zeta$.

Qualitatively these greatly resemble the root loci for the distributed parameter systems explored previously. For velocity feedback increasing from zero the complex conjugate roots initially become more negative in their real parts. For $k_b \gg k_s$ it is possible to achieve critical damping for appropriate values of velocity feedback (damping). For still higher values of $\zeta$ the real root moving away from the origin meets another moving toward the origin at which point a complex conjugate pair is formed once again.
For lesser values of $k_b$ critical damping cannot be achieved. Eventually $k_b$ becomes so weak that an acceptable value of damping on the dominant pole pair cannot be achieved. This corresponds to a minimum allowable stiffness of the structure. The rule of thumb developed in the distributed case was based on the ratio of the natural frequency $\omega_s$ when the structure was essentially rigid to the natural frequency $\omega_c$ when the structure was clamped to ground. For this lumped model

$$\frac{\omega_s}{\omega_c} = \sqrt{\frac{k_s}{m}} \sqrt{\frac{k_b}{m}} = \sqrt{\frac{k_s}{k_b}}$$

A damping ratio of 0.7 can still be achieved with $k = k_b/k_s = 5.0$ which corresponds to $\omega_s/\omega_c = 0.447$. Maximum damping begins to be inadequate for $k = 3.33$ or $\omega_s/\omega_c = 0.548$ for which the maximum damping ratio is 0.547. This corresponds well with the distributed model and represents the extreme case when the arm mass is negligible compared to the payload mass.
TABLE 3.5

PHYSICAL AND NONDIMENSIONAL VARIABLES, ONE LINK, ONE JOINT WITH PAYLOAD MASS EXAMPLE

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<tr>
<th>Physical Variables</th>
<th>Nomenclature</th>
<th>Dimensions</th>
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<td>Joint Position Feedback Gain</td>
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<td>LF</td>
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<td>Joint Velocity Feedback Gain</td>
<td>c</td>
<td>LFT</td>
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<tr>
<td>Arm Length</td>
<td>l</td>
<td>L</td>
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<tr>
<td>Frequency (complex)</td>
<td>ω</td>
<td>T⁻¹</td>
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<tr>
<td>Payload Mass</td>
<td>m_p</td>
<td>L⁻¹ FT²</td>
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<tr>
<td>Stiffness Product</td>
<td>EI</td>
<td>FL²</td>
</tr>
<tr>
<td>Arm Density/Unit Length</td>
<td>μ</td>
<td>FT² L⁻²</td>
</tr>
</tbody>
</table>

Forming the first clamped joint natural frequency when \( m = 0 \)

\[
ω_c = 3.52 \sqrt[4]{\frac{EI}{μl^4}}
\]

the servo frequency for a rigid arm structure

\[
ω_s = \sqrt{\frac{k}{\frac{3}{μl^3} + m l^2}}
\]

and the damping ratio for a rigid arm structure

\[
ζ = \frac{c}{2 \sqrt{k \left(\frac{3}{μl^3} + m l^2\right)}}
\]

The arm system can be described in terms of the following nondimensional groups:

\[
\tilde{ω} = \frac{ω}{ω_c}
\]

\[
\tilde{ω}_s = \frac{ω_s}{ω_c}
\]

\[
\tilde{ζ} = \frac{ζ}{μ l}
\]

\[
m = \frac{m_p}{μ l}
\]
Figure 3.11a Root Loci for One Link, One Joint case.
\[ m = 0.0 \quad \omega_g = 0.33, 0.40, 0.50. \]
\[ w = 0.3333 \]

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\[ w = 0.3333 \]

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\[ w = 0.4000 \]

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\[ w = 0.4000 \]
Figure 3.11a (continued) Data Points

\[ \omega_p = 0.4000 \]

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\[ \omega_p = 0.5000 \]

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\[ \omega_p = 0.5000 \]

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Figure 3.11 b Root Loci for One Link, One Joint Case:
\( m = 0.2 \quad \omega_s = 0.25, 0.33, 0.50. \)
**Figure 3.1lb (continued) Data Points**

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### Figure 3.11b (continued) Data Points

**$\bar{\omega}_g = 0.3333$**

$X$ = symbol

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**$\bar{\omega}_g = 0.5000$**

$\Lambda^g$ = symbol

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Figure 3.1lc Root Loci for One Link, One Joint Case.
$m = 100, \omega_n = 0.1667, 0.25, 0.3333.$
### Figure 3.11c (continued) Data Points

\[ \omega_g = 0.1667 \]
\[ \triangledown = \text{symbol} \]

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\[ \omega_3 = 0.1667 \]
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Figure 3.11d Root Loci for One Link, One Joint case.
\( \bar{m} = 10.0 \)  \( \bar{\omega}_s = 0.05, 0.075, 0.09. \)
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Figure 3.12 Simple Lumped Model

Figure 3.13 Root Loci for Simple Lumped Model

- $k_b/k_s = 0.1 \left( \omega_s/\omega_n = 0.316 \right)$
- $= 0.2 \left( = 0.447 \right)$
- $= 0.3 \left( = 0.547 \right)$
- $\zeta$ indicated
CHAPTER 4

ARM STRENGTH

This chapter will discuss the strength considerations in the design of the arm structure. The specifications of the arm payload, its task, and its environment will be related to the nominal sizing of the arm cross section based on stress limitations. This relation will then be used in the following chapter to compare strength limitations to limitations resulting from flexibility.

Of primary importance in designing an arm structure is that it not fail under the specified applications. In a specific design this is a complicated requirement which may incorporate many considerations. The stresses arising in the material most directly affects the dynamic performance of the arm. Dynamic requirements may be expressed directly in terms of forces and moments necessary to give the required acceleration to a payload of given mass and shape. These forces and moments must be supported by the arm structure, and the larger they are the larger the cross section of the arm must be. The structure mass in turn creates additional demands on the arm actuators which must accelerate the structure along with the payload. The structure design also directly affects the flexible dynamics of the arm, since the stiffness depends on the structure cross section, but through different relations than arm strength. Thus the cross
section of the arm may be constrained by either strength or stiffness in different designs.

The assumption will be that the arm structure consists of hollow cylindrical beams. In most arms it is hard to find something that actually looks like a beam since the structure consists largely of actuator housings and various multifunctional components. To paraphrase a popular line however "the model is the medium" by which one can relate these components to their task as part of the structure. For a general design study and scaling laws the beam contains the essence of structural components.

This discussion is divided into two main parts. First the application of the theory of strength of materials and the elementary theory of elasticity as applied to beams of the assumed cross section is discussed. Then the results are applied to the application at hand--designing manipulator arms.

4.1 Beam Stress

The following is a brief discussion of strength of materials as applied to beams of the assumed cross section. The basic assumptions include a cylindrical beam stressed in the elastic range. The deformations that occur are assumed to be small for the purpose of determining the stress. All beam configurations considered are statically determinate thus deflections need not be calculated to determine the stresses. The beam is assumed to be at equilibrium under the applied and inertial loads. Local effects such as stress
concentrations and end effects are ignored which strictly speaking requires for validity that the beam be smooth and slender. With the above assumptions in mind we will proceed.

Consider the beam in Figure 4.1 under equilibrium with the applied moments \( M \), forces \( F \) and force distributions \( g(x) \), and with an axis named \( x \) coincident with the neutral axis of the beam. At any cross section perpendicular to \( x \) the beam will be supporting force and moment vectors which we will resolve into components. The force and moment vectors will be represented in terms of an \( x \) component \( F_x \) (compression or tension) and \( M_x \) (torsion) and two more components in mutually perpendicular directions. By properly choosing the \( z \) axis the moment component \( M_y \) will be zero. Having thus established the \( z \) direction one can express the components \( F_y \) and \( F_z \). The assumption will be made that the component \( F_z \) will be zero. This is justified because the largest moments will arise from forces on the beam, and when these forces are coplanar the resulting moment vector will be perpendicular to the force vector at every cross section. Thus the resultant forces on any face will appear as in Figure 4.2, \( M_z \) is termed the bending moment and \( F_y \) the shear stress.

Theory of Failure. In order to determine when a beam is adequately designed one must determine when it will fail. For engineering materials a common criterion is the maximum principal normal or shear stresses compared to limiting values of those stresses for the material used. More elaborate criteria such as provided by dislocation
Figure 4.1  Beam at Equilibrium under Applied and Inertial Loads

Figure 4.2  Force and Moment Components and Coordinate Systems
theory or even consideration of the stress concentration factors is impractical without a detailed design. Since the loading on general purpose arms tends to be in arbitrary directions, and since as will be seen it results chiefly in flexure of the beam, the flexural endurance limit for the rather ductile materials usually used in arm design is most appropriate. If the arm is used in an application which does not result in complete stress reversals (for example for significant gravity loading at constant orientation to a beam) the Soderberg diagram [18] can be used to modify the flexural endurance limit found in tables of material properties.

**Stress Distributions in the Beam.** With the choice of coordinates and the assumption that $F_z = 0$ it is not overly difficult to find the stress distributions arising in the beam given forces and moments $F_x, F_y, M_x, \text{ and } M_z$. These are sketched in Figure 4.3 and given by the equations below. First the notation used in this section will be established.

- $r_1$ inner radius of cross section
- $r_2$ outer radius of cross section
- $I = \pi(r_2^4 - r_1^4)/4$ area moment of inertia about a diameter.
- $J = \pi(r_2^4 - r_1^4)/2$ polar moment of inertia of cross section about its center
- $A = \pi(r_2^2 - r_1^2)$ cross sectional area
- $c$ distance of a point from the neutral plane of bending
Figure 4.3 Stress Distributions for Isolated Loading Components

Figure 4.4 Shear Form Factor Determination for Assumed Cross Section
$r$  distance of a point from the $x$ axis

$\gamma$  angle between a radial line through a point and the $x$ axis

For pure axial loads $F_x$

\[(4.1a) \quad \sigma_x = \frac{F_x}{A} \quad \text{(constant over the entire cross section)}\]

for pure bending loads $M_z$

\[(4.1b) \quad \sigma_x = \frac{M_z c}{I} \quad \text{(linearly varying with distance from the neutral axis)}\]

For pure torsional loads $M_x$

\[(4.1c) \quad \tau_{xz} = \frac{M_x r \cos \gamma}{J} \quad \tau_{xy} = \frac{M_x r \sin \gamma}{J} \quad \text{(tangential, varying linearly with radius)}\]

For pure shear loads $F_y$

\[(4.1d) \quad \tau_{xy} = \frac{F_y A' \bar{y}}{J} \quad \text{(constant for given value of } y, \text{ symmetric about the neutral plane)}\]

where (see Figure 4.4)

- $A'$  area for $y > c$
- $\bar{y}$  distance from the neutral axis to the centroid of $A'$
- $b$  width of cross section at $c$
For the assumed cross sectional shape:

\[ A' = r_2^2 \left( \frac{\pi}{2} - \sin^{-1} \frac{c}{r_2} \right) - c \sqrt{r_2^2 - c^2}, \quad c > r_1 \]

\[ A' = r_2^2 \left( \frac{\pi}{2} - \sin^{-1} \frac{c}{r_2} \right) - r_1^2 \left( \frac{\pi}{2} - \sin^{-1} \frac{c}{r_1} \right) - c \left[ \sqrt{r_2^2 - c^2} - \sqrt{r_1^2 - c^2} \right], \quad c < r_1 \]

\[ y = \frac{A_2 OC_2 - A_1 OC_1}{A_2 - A_1} \]

\[ OC_1 = 0 \text{ for } c > r_1 \]

\[ OC_1 = \frac{4r_1(r_1^2 - c^2)^{3/2}/r_1^3}{3(2 \cos^{-1} \left( \frac{c}{r_1} \right) - 2c \sqrt{r_1^2 - c^2}/r_1^2)} \]

for \( OC_2 \) substitute \( r_2 \) for \( r_1 \) in the expression for \( OC_1 \).

Under the appropriate assumptions which are approximately true (such as that plane sections remain plane) one can superimpose the stress distributions which result from the isolated loadings to obtain the stress distributions which result from the combined loadings. It will be convenient to describe the location of a point in terms of the polar coordinates of Figure 4.2 but to retain the Cartesian coordinates for describing the stress at that point. The components
of stress are:

\[ (4.3) \quad \sigma_x = \frac{F_x}{A} + M_z c/I \]

\[ \tau_{xy} = \frac{F A' y}{Ib} + \frac{M r}{J} \sin \gamma \]

\[ \tau_{xz} = \frac{M r}{J} \cos \gamma \]

\[ \sigma_y = \sigma_z = \tau_{yz} = 0 \]

In general at a point the stress tensor will have three unique shear stresses and three unique normal stresses. The orientation of the coordinate system x-y-z was selected for the ease it afforded in determining the stress distributions. It does not however yield the maximum stresses for use in the failure criteria. There does exist a coordinate system where shear stresses are zero and normal stresses maximum. The values of those stresses are given by the roots \( s_1, s_2, \) and \( s_3 \) of the equation

\[ (4.4) \quad s^3 - (\sigma_x + \sigma_y + \sigma_z) s^2 + (\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x - \tau_{yz}^2 - \tau_{xz}^2 - \tau_{xy}^2) s - (\sigma_x \sigma_y \sigma_z + 2\tau_{yz} \tau_{xz} \tau_{xy} - \sigma_x \tau_{yz}^2 - \sigma_y \tau_{xz}^2 - \sigma_z \tau_{xy}^2) = 0 \]

which is simply the characteristic equation of the stress tensor [19].

The maximum shear stress is the maximum of

\[ (4.5) \quad \tau_1 = \pm (s_1 - s_2) \quad \tau_2 = \pm (s_1 - s_3) \quad \tau_3 = \pm (s_2 - s_3) \]
If one substitutes the zero quantities of Equation (4.3) into Equation (4.4) the characteristic equation degenerates to second order which can be solved rather simply:

\( s_{1,2} = \frac{-\sigma_x \pm \sqrt{\sigma_x^2 + 4(\tau_{xz}^2 + \tau_{xy}^2)}}{2}, \quad s_3 = 0 \)

**Maximum Principal Stress.** The cross section must be increased until the maximum principal stress is below the maximum allowable for the material. At a given cross section the maximum could theoretically occur at any point. One could conceivably differentiate (4.6) with respect to the space variables and set that expression equal to zero, but the complicated expressions relating the force \( F_y \) to shear stress are not conducive to that approach. One observes that none of the stress components has a maximum at any point other than the outer radius except for the shear stress resulting from \( F_y \). By selecting various values of loading and computing the principal stresses one can empirically learn where the maximums will occur.

Figure 4.5 shows schematically where the maximums occur for the 15 combinations of the 4 types of loadings. No variation in the radius at which the maximum occurs was noticed for variation of the ratio of inner to outer radius from 0 to 0.99 and a wide range of load combinations. In all cases the maximum was observed to occur on the outer radius, but for different values of the angle \( \gamma \).
Figure 4.5 Observed Maximum Principal Stresses for Combined Loading
4.2 Arm Structure Synthesis

The stress arising in an arm structure will be determined by
the application of the arm, more specifically the mass and inertia
of the payload, the accelerations the arm is required to impart to
the payload, and the gravity of the environment. Given these
quantities it should be possible to work back through a known arm
configuration from the payload to a fixed point, calculating the
required cross section parameters on the way. This will be done in
this section and certain simplifying assumptions made to facilitate
analysis. By comparing the magnitudes of stress terms further
simplifications can be justified which lead to manageable expressions
for use in the following chapter.

Arm Stress as Determined by Arm Specifications. If it is required
that the arm be capable of some minimum acceleration to the payload
mass \( m_p \), certain worst cases arise which determine the torque required
of the motors. The arm structure must be capable of supporting the
torques generated by the motors at all configurations the arm will
assume. Solving for the exact worst case insofar as the structure
is concerned is complicated so we will conservatively assume certain
most severe conditions coexist—highest acceleration and most
vulnerable configuration which will result in a conservative design.

The acceleration of gravity \( g \) must be counteracted in the
worst case and is included in the acceleration \( a_p \) of the payload.
In order to move the center of gravity (c.g.) of the payload, which
is located a distance \( e \) from the end of the arm, with acceleration \( a_p \) the force \( m_a \) and moment \( m_a e \) must be provided by the beam (see Fig. 4.6). In the worst case these must be provided transverse to the beam. Notice that \( a_p \) used for designing the beam is generally greater than the specified arm requirements for accelerating the payload. Consider a specification as follows: For the configuration in Figure 4.7 provide specified acceleration \( a_s \) in all directions. This would require actuators capable of providing \( a_s \) in three perpendicular directions. The acceleration between those directions will be greater however by a factor of \( \sqrt{3} = 1.73 \). Acceleration transverse to the beam will always be less than \( 1.73a_s \) but will in some cases be at least \( 1.414a_s \), plus an axial component, thus \( a_p = 1.73a_s \) is not overly conservative. This would create principle stresses of

\[
(4.7) \quad s_p = \frac{-M_z c}{2I} \pm \frac{1}{2} \sqrt{\left(\frac{M_z c}{I}\right)^2 + 4\left(\frac{A'y}{Ib} F_y\right)^2}
\]

Studies discussed previously show that maximum stress occurs at the outer radius at some point at an angle \( \gamma \) between the \( y \) and \( z \) axes. Thus

\[
(4.8) \quad 2s_p = \frac{-M_z r_2}{I} \cos \gamma \pm \sqrt{\left(\frac{M_z r_2}{I} \cos \gamma\right)^2 + 4\left(\frac{\sin^3 \gamma}{3I} (r_2^2 - r_1^2) F_y\right)^2}
\]

At \( x = 0, M_z = m_a e, F_y = m_a \) (see Figure 4.6). For \( x = x_1 > 0 \) the acceleration of the beam for \( x < x_1 \) must also be provided for. If the beam is rotating about a length \( \ell \) and the
Figure 4.6 Payload Acceleration and Resulting Forces and Moments

Figure 4.7 Example Configuration for Specification of Acceleration
acceleration has a gravity or other constant component \( g \) the acceleration at \( x \) is

\[
a(x) = \frac{L}{L + e} (a_p - g) + g
\]

Thus, assuming slender beams, the force and moment at each cross section varies as follows

\[
\frac{d}{dx} F_y = \mu(x) a(x) \quad \text{and} \quad \frac{d}{dx} M_z = F_y
\]

One can express \( \mu(x) \) in terms of \( F_y (x) \) and \( M_z (x) \) by requiring the maximum principal stress not be exceeded. These two first order differential equations can be easily integrated numerically to give the lightest design but at each step one must search for the value of \( \gamma \) at which the principal stress is maximum and for the minimum radius which will support that stress. When arriving at a discontinuity such as an actuator mass, the appropriate discontinuity in force is assumed. Furthermore since the actuators are most sensibly located inboard of or at the joint they power, the actuator size can be determined from the moment calculated for it to accelerate, and the determined mass added to the mass that must be accelerated by the inboard actuators and supported by the inboard structure.

Alternatively a constant \( \mu(x) \) can be assumed for a segment of the beam and the value required at the inboard end of the segment calculated. In this case for the first segment:
\[
F_y(x) = \int_0^{x_1} \mu(x) a(x) \, dx
\]

or finally:

\[(4.9) \quad F_y(x_1) = a_p m_p + \mu_1 \left[ \frac{\ell (a_p - g)}{\ell + e} + g \right] x_1 - \frac{\mu_1}{2(\ell + e)} (a_p - g) x_1^2 \]

\[(4.10) \quad M_z(x_1) = a_p m_e + a_p m x_1 + \frac{\mu_1}{2} \left[ \frac{\ell (a_p - g)}{\ell + e} + g \right] x_1^2 - \frac{\mu_1 (a_p - g)}{6(\ell + e)} x_1^3 \]

This avoids the need for numerical integration but still requires two searches for maximum stress and minimum radius.

**Insignificant Stress Terms.** It is desirable to simplify Equation (4.8) for \( s_p \) if possible by ignoring the effects of \( F_y \) at the cross section at the inboard end of a constant cross section segment. By comparing expressions (4.9) and (4.10) for \( M_z \) and \( F_y \) one can see that \( M_z > F_y x/2 \). Thus if we substitute the \( M_z = F_y x/2 \) into expression (4.8) for \( s_p \) we will find an upper bound on the error in ignoring \( F_y \) at a cross section. The original \( F_y \) term will be kept in square brackets for comparison.
\[ 2s_p = \frac{-F_y x r_2}{2I} \cos \gamma \pm \sqrt{\left( \frac{F_y x r_2}{2I} \cos \gamma \right)^2 + \left( \frac{4 \left( \frac{\sin^2 \gamma (r_2^2 - r_1^2)}{3I} F_y \right)^2}{9} \right)} \]

Considering the maximum effect of both of the terms under the radical, i.e. assuming \( r_1 = 0 \), and that both \( \cos \gamma \) and \( \sin \gamma = 1 \),

\[ 2s_p = \frac{-F_y x r_2}{2I} \left[ 1 \pm \sqrt{1 + \frac{4}{x^2} \left[ \frac{4r_2^2}{9} \right]} \right] \]

Thus for example when \( x/r = 5 \), an error in the term under the radical sign of 0.0711 can be expected. Using the Binomial Series to express the square root one observes the form

\[ \beta \left[ 1 + (1 + \Delta)^{1/2} \right] = \beta \left[ 2 + \Delta/2 \right] = 2\beta \left[ 1 + \Delta/4 \right] \]

Thus it is observed that only one-fourth that error occurs in the maximum principal, stress is which a percentage error of 1.7%. Since the beams have already been presumed slender the following approximation will be assumed valid:

\[ s_{p, \text{max}} = \frac{M r_2}{I} \]

when \( \mu = \text{constant over a beam segment and} \)

\[ (4.12) \quad M_z = a_{p, \text{p}} + a_{p, \text{p}} x_1 + \frac{\mu_1}{2} \left[ \frac{\lambda(a_p - g)}{(\lambda + e)} + g \right] x^2 - \frac{\mu_1(a_p - g)}{6(\lambda + e)} x^3 \]
\[ \mu_1 = \rho \pi (r_2^2 - r_1^2) \]
\[ I = \frac{\pi}{4} (r_2^4 - r_1^4) \]

The variables in the above equations can be categorized as follows: Specifications of the desired arm: \( m_p, e, g, a_p, l \)
Constants of the material used: \( s_{p,\text{max}}, \rho \)
Results of the calculations (11) and (12): \( M_z \) (supplied by the actuator), \( \mu_1, r_2, r_1 \)

Notice that \( r_1 \) and \( r_2 \) are not completely specified by the above analysis. For example the ratio \( r_1/r_2 \) can be fixed and then the value \( r_2 \) determined from strength. Considering only the structural weight, a lighter structure can theoretically always be obtained by making \( r_1/r_2 \) larger creating a more efficient cross section. In practice buckling may constrain the minimum wall thickness of beams. Additionally for arms specifically, a large radius on the arm restricts the motion the arm is capable of when joint angles approach 180°.

**Choice of Limiting Stress.** The limiting stress varies with the application and the material used. It is not the intent here to perform a detailed failure analysis, but only to indicate the nature of the interaction of strength requirements with rigidity requirements. The limiting stress varies not only in its numerical value but also in its nature. For arms with long life requirements and stress which will be varying significantly, endurance stresses are appropriate to
prevent fatigue failure. For ferrous materials there appears to be a limiting stress after which the life is essentially infinite [20]. Nonferrous metals do not display such a limit. If loads are essentially static the yield stress (either shear or tensile) is the appropriate number for comparison. Because one of the principal stresses is identically zero under the assumed loading, and because another is very small since the contribution of shear force is negligible at the inboard end of a constant cross section segment, the maximum shear stress is numerically equal to the maximum principal stress. The quantity $M_zc/I$ thus becomes our only criterion of the stress the application or material might demand. The value that will be generally assumed is specific examples to follow is the flexural endurance limit.

**Design Procedure.** The above procedure can be used to design the outer link in a straightforward manner. Torque requirements at the inboard end of the link specify actuator torques and thus size. The actuator weight is added to the payload and outer link to effectively obtain a payload mass and distance $e_2$ for designing the next link, and so on for other links. Since the only effects of the outer beam on the inner beam are the forces and moments it imposes on it:

$$e_2 = \frac{M_z(x_1)}{F_y(x_1) + m_a p \frac{(l - x_1)}{(l + e_1)}}$$
CHAPTER 5

STRENGTH AND DYNAMIC STIFFNESS AS DESIGN CONSTRAINTS

Two interacting constraints to any mechanical design are the strength and the stiffness of its components. Both are complex constraints whose limits are established by the materials used and the application. Both have their dynamic and static aspects. Some design parameters affect them both but through different relations, and some design parameters affect only one or the other.

Strength is seemingly a more compelling requirement since inadequate provisions for strength can result in catastrophic failures if a component fractures. This will occur if the ultimate strength of the arm is exceeded, of course, but much more insidious mechanisms of failure will be more demanding in the arm design. Yielding of the material, for example, must be avoided under the operating conditions and, depending on the design life of the arm, the endurance of the material to cyclic loadings must be allowed for. Thus an expendable space manipulator which must function for a few thousand cycles would have different limitations than an industrial manipulator which must function for millions of cycles. This is one of the dynamic aspects of strength. Another is the fact that inertial loads may account for much of the total stress arising in the arm.
Thus strength limits the acceleration one can apply to the payload. The nature of this limitation was discussed in Chapter 4.

The dynamic aspects of stiffness manifest themselves in the form of vibrations, with characteristic frequencies and amplitudes. The case of deflection under constant loads or static deflection is also important in the arm design, especially if the feedback of arm position information does not account for this. The static stiffness at the arm endpoint can be described by a six by six compliance matrix relating all components of the force and moment vectors at the endpoint to the components of the displacement and incremental rotation vectors. A systematic method for finding these compliances is described by the author in another report [2]. (These compliances can be used to obtain dynamic information by lumped parameter methods which is also treated in [2].) The static aspects of stiffness and the characterization of their effect on manipulator performance will be largely ignored in this treatment.

One important aspect of the dynamic flexibility of an arm is the limitation it imposes on the speed at which small motions can be made and the arm brought to rest. In Chapters 2 and 3 this effect was characterized as a maximum magnitude of the dominant eigenvalues of the arm system with reasonable damping. The type of control assumed was joint torque proportional to the difference between the joint angle and angular velocity and a reference angle and angular velocity. With this simple control the dominant system eigenvalues could be placed at a distance from the origin of one-half the clamped
joint natural frequency and still achieve a damping ratio of the order of 0.7.

Arm strength and stiffness are qualitatively affected in the same way by a number of arm parameters including payload mass, arm material density, length, and cross section parameters. When one of these parameters is varied to make the arm stronger, a stiffer arm results. Other parameters directly affecting only strength include the maximum allowable stress for the arm material, inertial loading from accelerating the arm and its payload, and the gravity or other constant body forces present in its environment. Stiffness as characterized by the first natural frequency of the arm is additionally affected by the value of Young's modulus for the material chosen.

For none of these parameters is a direct tradeoff involved between stiffness and strength. One might ask therefore why is such extensive consideration required. The reasons are several. First, it is important to understand for what types of arms flexibility will be important. Estimation of the capabilities of an arm design based on strength requirements may be incorrect for a manipulator with excessive flexibility. For other arms the flexibility resulting from such a design might be insignificant. Secondly, it is desired to determine on a theoretical basis if and when flexibility should be considered in the design of the control system. Certain techniques for analyzing the interaction between control and flexibility were developed in the first part of this thesis. These can be applied at
some cost in design time and effort. This section will use the analysis itself to indicate if the analysis was needed. Finally, it might be possible to develop improved control schemes which recognize and act on the flexible motions of an arm. This section might indicate whether such schemes have a practical application.

5.1 Grounds for Comparison of Strength to Stiffness

If the strength and stiffness limitations are to be compared, a common basis for comparison must be developed. This comparison has been alluded to and will be developed below.

Arm strength is the ability to withstand loadings which create stresses in the arm. These loadings arise from attempts to move and stop the arm and to maintain a position in a gravity or other force field. Limitations on these loadings limit the capability of the arm to perform its specified functions by limiting the speed with which it can move. This speed limitation can yield a basis for comparison to stiffness limitations.

The stiffness is the tendency of the arm to resist load-induced deflections which for dynamic loading may take the form of vibrations. For a distributed system such as a manipulator arm these may take place at an infinite number of frequencies. The values of these frequencies depend on the boundary conditions as discussed in Chapter 2 and these boundary conditions include any feedback gains which are part of a servo control. The eigenvalues ($\sqrt{-1}$ times the complex frequency) of the complete arm–servo–payload system lower-limit the time in which a given move can be made using "servo control."
"Servo control" is defined here as control of motion achieved by adjusting the reference position of some or all of the state variables of a linear regulator, which determines the torque applied to the joint of an arm by summing the products of constant gains times the error between each state variable and its reference.

Servo control, or some type of feedback control which acts to null errors, is essential in the face of operating uncertainties including disturbances exogenous to the arm system (wind gusts, collision by foreign bodies), noise in the arm system (component parameter variations, electrical noise in control signals, noise in the motion command), and uncertainty of the task to be performed (unknown payload masses or unexpected variations thereof, dimensional variations in the task descriptions, etc.).

The limitations of servo control can be severe for certain types of manipulator arms. Consider such a control system designed to track "reasonably" a time varying reference signal with a given bandwidth. For a rigid arm structure this control system must have a lowest undamped natural frequency somewhat higher than the frequency of the highest component of the reference signal one desires to track. These are the servo frequencies \( \omega_s \) discussed in Chapter 3. The servo frequencies required for tracking may be higher than \( \omega_c/2 \) (\( \omega_c \) = the clamped joint natural frequency) for the actual flexible system which would result in less than the desired damping. Therefore, gains which would result in good tracking performance for a rigid arm model of a given length and mass would be too high to maintain good
regulation of the actual flexible arm when fast settling time for the arm vibrations is needed. Whether the tracking for the flexible arm is satisfactory or not depends on whether one really cares about settling the vibrations of the arm when it is performing large scale motions. The postulation that for many tasks one does not care will lead us to distinguish (as has been done before for human motions) two types of motion and two types of control appropriate to them. These are termed gross motions and fine motions.

5.1.1 Gross Motion--Fine Motion Distinction

In this section justifications from a number of sources will be given of a brief qualitative nature for the proposed distinction between gross and fine motions. Gross motions are defined to be motions whose primary purpose is a large change in the configuration of the arm. Large could be defined with respect to the deflection of the arm (structure and servo controlled joints) which demanded the maximum torque the actuators were capable of delivering. This deflection is the maximum position error that remains within the linear range of the servo system. In other cases it may be appropriate to define large with respect to the average error resulting from preprogrammed open loop torque commands directly to the actuators, where this error includes all the uncertainties of the arm, environment and task previously discussed. These definitions are chosen because the fine motion servo control would be used within this range to damp out vibrations and correct errors resulting from the
gross motion command.

The most intuitive rationale for the distinction between gross and fine motions is the human arm. It has a complicated control system but observations of its performance have been successfully modeled by a system with two modes of control—an open loop bang-bang mode for gross motions and a continuous closed-loop mode for fine motion [21]. Studies of human performance in industrial tasks have acknowledged this distinction in the "therblig" system for categorizing the components of tasks, where one of the subdivisions considered is the transport subtask, as distinct from fine motion tasks. Since many of the things that manipulators are projected to do have previously been done by humans, or will be done under direct human command, it is logical to consider manipulator design in the context of human motion characteristics. Note that not all human motions fit this categorization well, such as the ballistic moves of an athlete throwing a ball accurately with fast, large arm motions. That this motion is atypical is evidenced by the high salaries of athletes and the fact that assembly line workers do not throw parts together.

If this anthropomorphic justification leaves one uneasy, he may resort to the results of optimal preview control. Information about the future is discounted heavily as that future becomes more distant. If the target is a given system state the deviations in the optimal state trajectory due to random noise have little effect on the control when the state is far from the target state. When near
the target, these deviations become the dominant factor.

The above descriptions are admittedly general and imprecise. Putting them on a quantitative basis is an appropriate thesis in itself, and is not necessary for the development that follows. What is necessary is that the two modes of motion be acknowledged, that gross motion speed be related to arm strength, that fine motion speed be related to arm stiffness, and that some concept of a balance between these two speeds be established. The following two sections are dedicated to these goals.

5.1.2 The Limiting Gross Motion Speed—Bang-Bang Control

The minimum time with which a rotary inertia can be moved between two points and stopped by a torque source with limited output results from applying the maximum torque, first in one direction, then in the other. Since gross motions are large with respect to vibration amplitudes by definition, we will assume that this type of control gives a characteristic time for the gross motion speed an arm is capable of achieving. If gross motion control is of the servo variety there will be a minimum servo frequency which will be required to track the triangular velocity command accurately. Let us not constrain the fine motion control to have the same gains if this is the case, since to do so might limit performance in certain cases.

If we further assume constant gravity in the direction of motion, the torque history for a round trip in time $\tau$ will be as described in
Figure 5.1. Integrating this torque twice yields the position of the inertia as a function of time as also indicated in Figure 6.1 and its accompanying equations. The result is a fundamental frequency $2\pi/\tau$ which is a function of the acceleration produced by the actuator, the distance moved, and the acceleration of gravity. For rotational motions the distance can be expressed as the product of the distance to the center of rotation and the angle $\theta$ of rotation. The quantity $2\pi\sqrt{\theta/\tau}$ is a normalized frequency which describes the gross motion time to move a fraction $\theta/2\pi$ of the workspace of the arm whose gross motion we represent. Gravity which varies as a function (say, cosine) of position could be used, but since this would depend on the two particular positions selected, it would add two more impediments to generalizing the results obtained.

5.1.3 Gross Motion and Fine Motion Speeds for Good Performance

As stated in the first part of this Chapter, strength and stiffness are not direct tradeoffs. The gross motion speed and the fine motion speed with the assumed servo control can be traded off against each other for the following reason. To accelerate the payload, a minimum cross sectional area moment of inertia is required to keep structural members within stress limits. Increasing the cross section above the minimum would increase $\omega_c$ and thus the achievable (with appropriate gains) fine motion speed. The increased arm inertia however would lower gross motion speeds for the same actuators. It also reduces the range of motions which can be classified as "fine" by our previous definition, because the higher feedback gains require
Figure 5.1 Bang-Bang Control Speeds Constant Gravity

for zero velocity at $\tau/2$

$$( a_p - g ) \tau_1 = ( a_p + g ) ( \tau/2 - \tau_1 )$$

switching point $\tau_1$

$$\tau_1 = \frac{(a_p + g)\tau}{4a_p}$$

velocity at switching point

$$v(\tau_1) = \frac{(a_p - g)(a_p + g)\tau}{4a_p}$$

distance traveled

$$d = \frac{\tau v(\tau_1)}{2} + v(\tau_1)(\tau/2 - \tau_1)/2$$

or

$$d = \frac{\tau^2}{16a_p} \left( a_p^2 - g^2 \right) = \ell \theta$$

$$\theta = \frac{1}{\tau^2} \frac{a_p^2 - g^2}{\ell 16a_p}$$
higher torques for the same errors from the reference position. Within the fine motion range, however, increasing the cross sectional radius allows increases in the fine motion speed by appropriately increasing the gain.

Because of the tradeoff between gross and fine motion speeds the best performance will not occur at extreme values of either. Just where the optimum balance between the two occurs is highly task dependent. How much of the task involves fine motion and how much involves gross motion? Suffice it to characterize the balance as a ratio between the gross motion and fine motion frequencies that can be carried as a description of the arm specifications, along with the payload mass, acceleration, and arm length.

What this section will attempt to develop is an indication of the structure and control imposed limitations on gross motions and fine motions. In order to do this we have already assumed a type of fine motion control, i.e. simple servo control. Let us initially assume that this same control system is used to track a time varying position and velocity command to achieve gross motions.

Based on the torque limitations of the actuators as developed in section 5.1.2, the bang-bang control frequency bounds any gross motion frequency (for a single joint) including that achieved by servo tracking. As discussed in Chapter 3, $\omega_c$, the clamped joint natural frequency, bounds the fine motion frequency which is represented by $\omega_s$. In fact, for a damping ratio of the order of 0.7 we must have $\omega_s < \omega_c/2$. We will then define $\eta$ as the ratio:
\[
\eta = \frac{2\pi \sqrt{\theta/\tau^2}}{\omega_c}
\]

One may find that by using the fine motion servo control to achieve gross motions, he restricts some alternatives that might otherwise be open to him. Consider the case with zero gravity where the bang-bang control results in a symmetrical triangular profile over time for the velocity of the arm. In order to track reasonably a periodic triangular velocity command with a fundamental frequency \( \Omega \) using servo control, one would have to have a frequency response for that servo that was flat to a frequency of say 10\( \Omega \). This is necessary to pass the lower harmonics of the triangular wave without attenuation, and to avoid the significant phase shift that usually accompanies the attenuation of the magnitude of the frequency response. Since \( \omega_s \) represents the break frequency for our assumed servo control, this would imply \( \omega_s = 10\Omega \). But \( \Omega \) is simply the gross motion frequency, and by our rule of thumb \( \omega_s \leq \omega_c / 2 \). Therefore, to track a gross motion command with the fine motion servo we will require values of \( \eta \approx 0.05 \). Whether this is constraining or not depends on whether the design of the arm based on strength would have a value \( \eta < 0.05 \) anyway. Values \( \omega_c > \omega_s > \omega_c / 2 \) can also be used if damping is not important in the task, but damping deteriorates rapidly for \( \omega_s > \omega_c / 2 \).

5.1.4 Real World Actuators

The preceding is based on an admittedly simplified view of actuators, since real actuators have nonlinear dynamic characteristics
of their own which can cause instabilities at some values of gain and extreme attenuation and phase shift at high frequencies. The only constraint assumed here is a limitation on the maximum torque. This assumption is probably the best for electric torque motors if the rotor inertia is considered as separate from the actuator. Hydraulic valves and actuators are highly nonlinear and have a modest frequency response when operated open loop. This response can be improved with feedback compensation of the actuator itself, making it appear more like a pure torque source. This enhances the validity of the simplifying assumptions to be made. Pressure feedback has been used at the C.S. Draper Lab and other places to significantly increase the range of frequencies over which hydraulic motors can be used effectively as manipulator actuators [22]. These constraints are largely independent of the issue at hand with at least one notable exception. This exception is the addition of torque amplification devices such as gear boxes to high speed actuators (e.g. servo motors) which have the following effect: (1) They add weight, but so does a more powerful actuator, and this effect can be accounted for if one has available a reasonable relation between the torque and the weight for the given device. (2) They add Coulomb friction which if too large prevents or severely restricts back driving the actuator. Thus the joint control system cannot be used to damp vibrations of the arm structure. (3) They tend to result in gross motions that are limited by the maximum speed of the actuator rather than the maximum torque. Finally, they add (4) compliance and (5) backlash.
The second effect is especially severe for highly flexible manipulators, as can be seen from the results of Chapter 3. It might be desirable to overcome the static friction by dither, or more sophisticated control, to redesign the arm actuator, or to stiffen the structure and accept the penalties in gross motion times.

The third effect would add at least one more parameter depicting the speed limitation and change the nature of the gross motion time expression to reflect a constant speed over a large part of the gross motion cycle. These effects could be included, particularly for a well defined actuator type, but will not be included here.

The compliance of speed reduction devices is very severe when the structure of the arm is very rigid. When most of the compliance is in the reduction, one would do a different study since attempts to stiffen the beams reaches a point of diminishing returns.

The fourth effect, backlash, can result in undesirable instability in the arm response or a limit cycle behavior.

5.1.5 Example of Strength and Stiffness Calculations

To clarify the above design issues consider the example design calculation below. The formulae useful for this purpose are as follows. Nomenclature not given with the equation is found in Table 5.1. The moment required to accelerate a one link arm and its payload from Equation (4.11) with $e = 0$:

\begin{equation}
M_z = a \frac{m}{p} l + \frac{u k^2}{2} \left( \frac{2a}{3} + \frac{g}{3} \right) \quad \text{Where} \quad m = \rho \pi r_2^2 (1 - k_2^2)
\end{equation}
The relation between the moment and the stress at a cross section, equation (4.12):

\[
(5.2) \quad s = 4 \frac{M_z}{\pi r_2^2 (1 - k_r^4)}
\]

The relation between the parameters of the arm and its clamped joint frequency as approximated by the Equation (Appendix A):

\[
(5.3) \quad \omega_c = \sqrt{\frac{3E \pi r_2^4 (1 - k_r^4)}{4(M_p + 0.23 \rho \pi l r_2^2 (1 - k_r^2)}}}
\]

and the gross motion frequency for moving a rotational inertia through an angle \( \theta \) and back in time \( \tau \) (Figure 5.1):

\[
(5.4) \quad 2\pi \sqrt{\frac{\theta}{\pi^2}} = \frac{\pi}{2} \sqrt{\frac{a_p^2 - g^2}{a_p}}
\]

Consider an arm which is to be 10 feet long and is to accelerate its payload in Earth's gravity with 1.58 g's. One would also like to specify the payload mass, and perhaps a minimum value for \( \omega_c \) in order to assure adequate damping is achievable for the servo control system which has a specified bandwidth \( \omega_s \). The equation solution is easier if one poses values for the other parameters and checks the resulting mass and frequency. The following is one iteration of such a solution procedure.

First choosing a material, aluminum 2014-T4, T451, we find the following material properties:
Fatigue Strength - $20 \times 10^3 \text{ lb}_f/\text{in.}^2$
Shear Strength - $38 \times 10^3 \text{ lb}_f/\text{in.}^2$
Yield Strength - $42 \times 10^3 \text{ lb}_f/\text{in.}^2$
Young's Modulus - $1 \times 10^7 \text{ lb}_f/\text{in.}^2 = E$
Density - $5.371 \text{ slugs/ft}^3 = \rho$

Assuming one is designing for infinite life, the stress should not exceed the fatigue strength of $20 \times 10^3 \text{ lb}_f/\text{in.}^2$. We will use this value in the following calculations although in practice a designer might choose a design factor $> 1$ by which to divide this stress to obtain a permissible operating stress $s_p$.

Assuming an actuator capable of generating, for example $100 \text{ ft. lb}_f$, one can calculate a minimum radius capable of withstanding the resulting stress via Equation (2). Assuming $k_r = 0.9$ the calculation is as follows:

$$r_2 = \left[ \frac{4M_z}{s_p \pi (1 - k_r^4)} \right]^{1/3} = 0.0504 \text{ ft. (0.606 in.)}$$

Solving for the payload from equation (5.1):

$$m_p = \frac{M_z}{a_p} - \frac{\mu l_p^2}{6a_p^3} (2a_p + g) = .161 \text{ slugs (5.17 lb}_m)$$

Checking the locked joint natural frequency (equation 5.3):

$$\omega_c = 1.016 \text{ rad. sec. (.1617 hz)}$$
Assuming \( \omega_s = \omega_c / 2 \), which is our rule of thumb for the limits of servo control with reasonable damping, one finds that a single cycle of the fine motion would require:

\[
\frac{2\pi}{\omega_s} = 12.37 \text{ seconds}
\]

The time required to reduce a disturbance deflection or vibration set up in a gross motion to a small fraction of its initial value using servo control is proportional to this time.

For a bang-bang gross motion control the time \( \tau \) is found from Equation (5.4) to be:

\[
\tau = 0.573 \sqrt{\theta}
\]

Thus while it may be possible to move the joint 45° and return in a time:

\[
\tau = 0.508 \text{ sec.},
\]

this motion would be followed by a period of fine motion settling which would be measured in multiples of:

\[
\frac{2\pi}{\omega_s} = 12.37 \text{ sec.}
\]

The ratio of the two limiting frequencies \( 2\pi \sqrt{\theta}/\tau \) and \( \omega_c \) is termed \( \eta \):

\[
\eta = \frac{2\pi \sqrt{\theta}}{\tau \omega_c}
\]
By using the same actuator and increasing the radius of the arm it is possible to change the values \( \eta \) assumes. The clamped joint frequency would increase (allowing \( \omega_s \) to increase) and the gross motion frequency would decrease and one could no longer meet the 1.58 g's acceleration specified. The exact value of \( \eta \) preferred would depend on the task at hand. Some tasks require precise fine motion control and accurate adjustment of position as the task proceeds (such as assembly tasks) and their speed will be more constrained by the settling time and thus \( \omega_c \), requiring designs with small \( \eta \) (probably \( \eta \approx 0.1 \)). Other tasks require fast gross motions and for a large part of the task the settling time is irrelevant (such as picking up large sand castings from the sand, shaking them clean, and dumping them in a container). Here, larger \( \eta \) would be acceptable.

Since the actuator deforms the beam while powering it through the gross motions, there is interaction between flexibility and gross motions. In the example above this is sizeable, because the beam is so flexible. (\( \eta \) is large) The maximum deflection the assumed actuator can create in the beam with a force resistance at the payload is:

\[
\delta = \frac{Mz^2}{3EI} = 1.33 \text{ ft.}
\]

and is 17% of the total end point travel for a 45° move. One would expect a large part of time to make a single 45° move with such a system to be the result of this deflection. This can be interpreted to mean either that the bang-bang speed is not a good estimate of gross
motion speed for such a system, or that motions of this size are not really gross motions, but in a transition region between gross and fine motions. Whatever the interpretation, if one is forced to operate such a system because of weight, space or other limitations, the control we are assuming is inadequate. Open loop torque commands contoured to produce a "deadbeat" response (essentially all vibrations cancelled as a part of the command) are needed for gross motions. For fine motions and regulation of the position under uncertain disturbances a more sophisticated feedback control is needed, measuring or reconstructing the additional state variables of the flexible system. The analysis here will indicate when such sophistication might be valuable.

One can conclude from the above example that for some (at least one) single link arms, flexibility is a more demanding requirement than arm strength. Is this always true or seldom true? The remainder of this chapter will pose that question in a general way for a broad range of manipulator parameters covering orders of magnitude of lengths, payload masses and accelerations, actuator torques, environments and material properties. For each point in a multi-dimensional design space a different manipulator exists with different properties. Meaningful simplifications of that design space will be displayed to aid in understanding the constraints that exist in different parts of that space.
5.1.6 Logic of Arm Design and Scaling Laws

Having made the arguments above one realizes that the ultimate goal of this soul searching process is the structuring of the arm design problem. Restated, the goal is a logical procedure for taking specifications of what is desired, calling on the technology available, and specifying the free parameters to create a design which fulfills the original specifications, if possible. In most cases this is a very fuzzy process with little structure as might be represented by Figure 5.2. By asking the right questions one can begin to specify what makes one design better than another. If precise answers can be obtained to enough of these questions, and they can be related to the free parameters, the design will be completely specified. It is unlikely that anything but the simplest design task will be specified by solving a few simultaneous equations, but when the chance presents itself to constrain some of the design parameters and relate them directly to the specifications, even with reasonable simplifying assumptions, the opportunity should not be passed up, because it rationalizes the design process. With the developments to this point we might suggest the structure of Figure 5.3 as useful for the small but important arm design subtask discussed here: the structure-control system interaction. It does not include actuator mass effects, power transmission weight or compliance but these could be added.

The desire here is not to design a specific case but to indicate the significance of flexibility for the range of specification
Figure 5.2 Unstructured Design Process

Task & Arm Specifications

Designer Experience Judgement

Technical Feasibility

Strength Considerations

Stiffness Considerations

Control Considerations

Figure 5.3 Possible Design Logic for the Structure Control System Interaction (Symbols listed in Table 5.1)

Fine Motion Control

Design Comparison

Gross motion

Fine motion

Strength Relations

Clamped Joint Frequency

Gross Motion Control

Feasible Materials

\[ \omega_c = \frac{2\pi}{\tau \omega_c} \]

\[ \frac{\theta}{\tau^2} \]
parameters. By plotting the contours of a surface of constant $\eta$ for variations in the other specification parameters one describes a constraint on the ratio of gross motion to fine motion speeds. (Some parameters will, of course, have to remain fixed to allow one to plot the contours.) If a value of $\eta$ results which is lower than demanded by the task one can look on the difference as a margin of safety. Alternatively one can increase $\eta$, perhaps by using a material with higher strength which would result in a lighter more flexible design. This may not be economical since the higher strength may have a higher cost than the extra weight.

These plots are termed scaling laws, since they scale the parameters in terms of each other. That is they indicate what an equivalent change is, of two parameters in terms of their effect on a third, all other parameters held constant.

5.2 Solving for the Simultaneous Occurrence of Strength and Flexibility Constraints

In Chapter 4, equations (4.11) and (4.12) related the arm specification parameters (length, payload, acceleration and gravity) to the arm strength parameters (cross section and materials). These same parameters directly (as discussed in Chapters 2 and 3) or indirectly affect the flexible natural frequencies of the beam. In this section we will nondimensionalize the parameters of both problems in such a manner as to best portray the effect of the basic arm specifications on arm design. Using a ratio of gross motion to fine motion frequencies as proposed earlier in this chapter the dynamic and
strength requirements will be combined to show which of the two, strength or flexibility, constrain the design of the arm. This will be done by solving for the points where both constraints occur simultaneously.

5.2.1 Exploration of the Parameter Space

Table 5.1 displays the thirteen physical parameters that determine the arm as proposed. This is too many to explore effectively and to derive any benefit from studying the general problem.

The number of parameters will be reduced by nondimensionalization, by assuming constraining relations between parameters, and by fixing some parameters to reasonable values for the cases of interest.

**Nondimensionalization.** One of the most valuable tools for the reduction of parameters is nondimensionalization. It must be used with care to preserve the usefulness of results. Table 5.1 displays the groupings which seem to be most valuable to the task at hand. Variables were nondimensionalized with respect to two materials properties and the acceleration of gravity. The density of the arm material and Young's modulus are important parameters but are fairly constant for a given material type such as aluminum or steel. For this reason they were chosen instead of the material strength (maximum allowable stress) which varies widely with the treatment the material has received and the application of the arm. Similarly for arms in a given environment the gravity will be essentially constant. These considerations and the fact that one is constrained to choose variables which exhibit
### Table 5.1

Nondimensionalization of Strength and Stiffness Parameters \( (\Lambda = \frac{F}{\rho g} \equiv L) \)

<table>
<thead>
<tr>
<th>Description</th>
<th>Physical Nomenclature</th>
<th>Dimensions</th>
<th>N.D. Grouping</th>
<th>N.D. Nomenclature</th>
<th>Substitution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Payload Mass</td>
<td>( m_p )</td>
<td>( FT^2L^{-1} )</td>
<td>( m_p/\rho\Lambda )</td>
<td>( \bar{m} )</td>
<td>( \rho \Lambda )</td>
</tr>
<tr>
<td>2. Distance from payload attachment to its center of gravity.</td>
<td>( e )</td>
<td>( L )</td>
<td>( e/\Lambda )</td>
<td>( \bar{e} )</td>
<td>( e\Lambda )</td>
</tr>
<tr>
<td>3. Arm length</td>
<td>( \ell )</td>
<td>( L )</td>
<td>( \ell/\Lambda )</td>
<td>( \bar{\ell} )</td>
<td>( \ell\Lambda )</td>
</tr>
<tr>
<td>4. Payload acceleration</td>
<td>( a_p )</td>
<td>( LT^{-2} )</td>
<td>( a_p/g )</td>
<td>( \bar{a} )</td>
<td>( a g )</td>
</tr>
<tr>
<td>5. Acceleration of gravity</td>
<td>( g )</td>
<td>( LT^{-2} )</td>
<td>( g/g )</td>
<td>( \bar{1} )</td>
<td>( g )</td>
</tr>
<tr>
<td>6. Maximum arm stress</td>
<td>( s_p )</td>
<td>( FL^{-2} )</td>
<td>( s_p/E )</td>
<td>( \bar{s} )</td>
<td>( s E )</td>
</tr>
<tr>
<td>7. Arm material mass density per unit volume</td>
<td>( \rho )</td>
<td>( FL^2L^{-4} )</td>
<td>( \rho/\rho )</td>
<td>( 1 )</td>
<td>( \rho )</td>
</tr>
<tr>
<td>8. Outer cross sectional radius</td>
<td>( r_2 )</td>
<td>( L )</td>
<td>( r_2/\Lambda )</td>
<td>( \bar{r} )</td>
<td>( r\Lambda )</td>
</tr>
<tr>
<td>9. Inner cross sectional radius</td>
<td>( r_1 )</td>
<td>( L )</td>
<td>( r_1/r_2 )</td>
<td>( k_r )</td>
<td>( \Lambda r_k r )</td>
</tr>
<tr>
<td>10. Actuator Torque on Arm</td>
<td>( M_z )</td>
<td>( FL )</td>
<td>( M_z/EA^3 )</td>
<td>( \bar{M} )</td>
<td>( M E \Lambda^3 )</td>
</tr>
<tr>
<td>11. Shear Force in the Arm</td>
<td>( F_y )</td>
<td>( F )</td>
<td>( F/EA^2 )</td>
<td>( \bar{F} )</td>
<td>( F E \Lambda^2 )</td>
</tr>
<tr>
<td>12. Young's Modulus of Elasticity</td>
<td>( E )</td>
<td>( FL^{-2} )</td>
<td>( E/E )</td>
<td>( 1 )</td>
<td>( E )</td>
</tr>
<tr>
<td>13. Frequency of Vibration</td>
<td>( \omega_c )</td>
<td>( T^{-1} )</td>
<td>( \omega_c/g/\Lambda )</td>
<td>( \bar{\omega}_c )</td>
<td>( \omega_c g/\Lambda )</td>
</tr>
</tbody>
</table>
all the three primary dimensions in a linearly independent fashion, resulted in the particular choice. One disadvantage to the choice is the fact that the important case of gravity equal to zero cannot be considered except in the limit. Another slight inconvenience is the magnitude of the nondimensional numbers resulting, which is very small. Nevertheless it is felt that this combination will convey the maximum information in this particular case.

Further Reduction in Parameters. Nondimensionalization leaves one with ten out of the original 13 parameters to deal with. By displaying contours on a plane and several sets of contours on different planes we can effectively display four variables. Thus something must be done with the other six variables. If one assumes an "optimum" or at least a desirable ratio between gross motion and servo controlled fine motion bandwidth, one can assume a constraining relation between the nondimensional acceleration $\tilde{a}$ and the cantilevered first natural frequency $\tilde{\omega}_c$ and one more variable can be eliminated. $\tilde{F}_y$ at the end of the arm segment will not be significant if one assumes rotary actuators and only concerns himself with a single link. Assuming a geometric factor which expresses the inner radius as a constant times the outer radius eliminates one more variable. By assuming the lightest possible design for a given maximum allowable stress one can eliminate the moment $\tilde{M}_z$ by expressing it in terms of the stress and the outer radius. By further assuming the distance to the center of gravity of the payload from the end of the arm is negligible compared
to the length of the arm one has five variables and two equations, one resulting from the strength requirements and endpoint acceleration (4.11) and one resulting from the frequency determinant of a clamped-free beam with a mass on one end. The arm radius can theoretically be eliminated from these simultaneous equations leaving one with a single relation between the remaining four primary variables. These are the nondimensional arm length $\bar{l}$, payload mass $\bar{m}$, payload acceleration $\bar{a}$ and the ratio of the maximum allowable stress to Young's modulus $\bar{s}$. The first two variables will be plotted for contours of constant acceleration, and these plots will be displayed for several values of stress/modulus.

This verbal description and rationalization is now followed with the actual manipulation of the equations and a description of how the outer cross sectional radius can be eliminated from the final relation using the two nonlinear simultaneous equations.

5.2.2 Development of Parameter Relationships--Single Link Case

For this development all symbols not defined as they are used are described in Table 5.1. From Equation 4.11 relating moment at a given cross section to arm specifications and design parameters, assuming a single link of length $\ell$, and Equation 4.12 relating maximum stress at a cross section to the cross sectional parameters one finds that:

$$m_p = \frac{s}{p} \frac{\pi r^3 (1 - k^4)}{4a_p \ell} - \frac{\rho \pi r^2 (1 - k^2)}{2} \left[ 2 + \frac{1}{3} \frac{g}{a_p} \right]$$
m_p is the payload mass that will produce a stress s_p in an arm with the given dimension (k_r and r_2), material (\rho) in gravity, g, with payload acceleration a_p.

As developed in Appendix A the frequency relation for a uniform cantilevered beam of length \ell with a point mass at its end is:

\begin{equation}
(5.6a) \quad \ell^3 \frac{m}{p} \frac{c}{\omega^2} + 1 + \frac{3 \ell^3}{\omega^2} \frac{c}{\omega^2} \left[ \cos \beta \sinh \beta - \cosh \beta \sin \beta \right] = 0
\end{equation}

\begin{equation}
(5.6b) \quad \beta^4 = \frac{\mu \ell^3 \omega c}{EI} = \frac{4 \rho \ell^4 \omega c}{E \ell^2 (1 + k_r^2)} \quad \text{for the assumed cross section. The gross motion frequency } 2\pi \sqrt{\frac{\theta}{\ell^2}} \text{ for bang-bang control against constant gravity which was developed earlier in this chapter requires}
\end{equation}

\begin{equation}
(5.7) \quad \frac{\theta}{\ell^2} = \frac{1}{\ell} \frac{a_p^2 - g^2}{16a_p}
\end{equation}

By substituting into equation 5.5 the nondimensional variables as indicated in Table 5.1, factoring and dividing through by common terms one obtains:

\begin{equation}
(5.8) \quad \bar{m}_s = \frac{g \pi r}{4a} \left( 1 - k_r^4 \right) - \frac{\pi}{6} \frac{r^2}{(1 - k_r^2) \bar{m}} \left[ 2 + \frac{1}{a} \right]
\end{equation}

Where the subscript s on \bar{m} indicates it results from the strength relation. \bar{m}_s is sketched in Figure 4 as a function of \bar{r} with the remaining parameters assuming fixed values. The assumption of a ratio between gross motion and fine motion frequencies, and the rule of
Figure 5.4a Simultaneous Solution of the Strength and Stiffness Equations.

Figure 5.4b Strength and Stiffness Curves with No Intersection with Physical Meaning.

Figure 5.4c The Limiting Case for Intersection between $\bar{m}_f$ and $\bar{m}_s$. 
thumb of Chapter 3 that the limiting value for servo controlled fine motion frequencies is one-half the first clamped joint natural frequency lead to the definition of \( \eta \) as:

\[
\eta = \frac{2\sqrt{\frac{\theta}{\omega_c^2}}}{\omega_c} = \frac{1}{2} \sqrt{\frac{1}{\frac{1}{2} a_p^2 - \frac{g^2}{16 a_p^2}}}
\]

Nondimensionalization yields:

\[
\eta = \frac{\pi}{2} \sqrt{\frac{1}{\frac{1}{2} \frac{a^2 - 1}{a}}} \frac{(\bar{a}^2 - 1)}{a}
\]

Solving for \( \bar{\omega}_c \):

\[
\bar{\omega}_c = \frac{\pi}{2\eta} \sqrt{\frac{(a^2 - 1)}{\bar{a}^2}}
\]

Substituting this into (5.6) and nondimensionalizing and solving for \( \bar{m} \) yields:

\[
\bar{m}_f = \frac{-\cos \beta \cosh \beta - 1}{\cos \beta \sinh \beta - \cosh \beta \sin \beta} \left[ \frac{\bar{r}^4 (1 - k_r^4) \beta^3}{\pi^2 \bar{\alpha}^3 (a^2 - 1)} - \frac{\eta^2}{a} \right]
\]

where the subscript \( f \) on \( \bar{m} \) indicates its origin as the frequency relation. \( \bar{m}_f \) is the nondimensional payload mass which will result in a frequency \( \bar{\omega}_c \) for the dimensions \( (\bar{r}, k_r, \bar{\alpha}) \) and material \( (E, \rho) \) parameters. \( (E \text{ and } \rho \text{ appear implicitly in the nondimensional groupings.}) \)

\( \bar{\omega}_c \) has been expressed in terms of the parameters \( a_p \) and \( \eta \). This expression for \( \bar{m}_f \) is also sketched in Figure 5.4 versus \( \bar{r} \) for assumed
values of the other parameters.

Theoretically Equations (5.8) and (5.10) can be solved simultaneously to eliminate one variable from the two expressions. This would yield the family of balanced designs which were just strong enough to meet the stress requirements (nondimensional stress equal to $\bar{s}$) and were simultaneously just stiff enough to achieve the minimum value of $\eta$. Unfortunately $\bar{r}$, the variable we would like to eliminate, is involved in both equations in such a way as to make it quite difficult to express it explicitly in terms of the other parameters. With $\bar{m}$, on the other hand, this is easily done for either equation. In order to plot the desired parameters as functions of each other the following procedure is proposed. First $\bar{m}$ is expressed explicitly from both relations. Values of the independent variables are selected and a numerical search varying $\bar{r}$ is conducted to find the point where the two relations yield the same value of $\bar{m}$, which is plotted as the dependent variable with $\bar{r}$ and intermediate parameter along the curve.

The details of this solution procedure are indicated in Appendix C along with simplifications that are valid for some parameter ranges. Some discussion of the nature of that solution is informative, however. Figure 5.4a indicates the solution we are seeking where $\bar{m}_f = \bar{m}_s$ at an intersection of the two appropriate curves. It will be noticed that $\bar{m}_f$ has an infinite number of branches corresponding to the infinite number of natural frequencies of the distributed beam. We are interested only in the rightmost curve which corresponds to the lowest natural frequency. Furthermore, for the
intersection to have physical meaning it must occur for positive mass. For certain values of the assumed parameters there will exist no intersection of the two curves with physical meaning as indicated in Figure 5.4b. Since \( \bar{m}_f \) increases as the fourth power of \( \bar{r} \) while \( \bar{m}_s \) increases as the third power of \( \bar{r} \) in the region of interest, the limiting case for an intersection with physical meaning will occur when \((\bar{r})_{m_f} = 0 = (\bar{r})_{m_s} = 0\). This limiting case is indicated in Figure 5.4c. In these cases \( \bar{m}_f > \bar{m}_s \) for all values of \( \bar{r} \) for which \( \bar{m}_s > 0 \). This means that an arm designed for strength to carry any mass with the parameter values assumed in Figure 5.4c will always be stiff enough. The relationship between the assumed parameters that results in a picture such as Figure 5.4c is developed in Appendix C and is presented in Equation (C.6) which is repeated here:

\[
\bar{r} = \frac{\pi^2}{5.508} \frac{s^2(1 + k^2)(a^2 - 1)}{\eta^2 a(2a + 1)^2}
\]  

Equation (5.11) is partially a product of the solution procedure; but it does indicate that it is possible to have designs that, unlike our earlier example, have adequate rigidity when designed for strength. Other cases when this will be true are indicated in Figure 5.4a when \( \bar{r} \) is greater than \( \bar{r}_2 \), where the intersection of \( \bar{m}_s \) and \( \bar{m}_f \) occurs. Equation (5.11) and Figure 5.4 illustrate one thing which causes us to pursue the topic further. Arm designs are potentially limited by either strength or stiffness, depending on the parameters of the arm. To find out which one occurs in a particular case we must actually
solve for the intersections as indicated in Figure 5.4a. The following section discusses the results of that solution.

5.2.3 Display and Interpretation of Results

Figure 5.5 displays the results of the simultaneous solution of Equations (5.8) and (5.10) for reasonable values of the parameters. Figure 5.6 displays some of the results in a three dimensional sketch. This section will discuss and interpret these results with the aid of the approximate relations developed in Appendix C. Because of the frequent use of the word nondimensional it will be abbreviated by n.d.

Figures 5.5 and 5.6 are visualizations of the design space for the strength and stiffness characteristics of our simple one link manipulator. To interpret these plots correctly one must keep in mind what is held constant, the assumptions made, and the relations used in obtaining them, including the nondimensionalizations described in Table 5.1.

All the plots are for a constant value of nondimensional stress $\bar{s}$. This is the maximum stress arising in the beam and in this case is numerically equal to 0.002, corresponding to the fatigue strength of the alluminum alloy of the Example problem. Any point in the plot of Figure 5.6 will have a maximum n.d. stress of 0.002. This is a result of using the strength relation to obtain the plots. By using the frequency relation as well we can determine the value of $\eta$ which will result from a design made on the basis of strength. Assuming constant $\bar{a}$ and constant $k_r$ in addition to constant $\bar{s}$, the
Values of $\tilde{a}$:
1 - 1.58
2 - 2.51
3 - 3.98
4 - 6.31
5 - 10.0

Figure 5.5a Flexible Manipulator Design Space. Contours of constant $\tilde{a}$ for the surface $\eta = 0.05$. ($\tilde{s} = 0.002$, $k_F = 0.9$)
Figure 5.5a Flexible Manipulator Design Space. Contours of constant \( \bar{a} \) for the surface \( \eta = 0.05 \). (\( s = 0.002 \), \( k_r = 0.9 \))

Values of \( \bar{a} \):
- 1 - 1.58
- 2 - 2.51
- 3 - 3.98
- 4 - 6.31
- 5 - 10.0
Values of $\bar{a}$:
1 - 1.58
2 - 2.51
3 - 3.98
4 - 6.31
5 - 10.0

Figure 5.5b Flexible Manipulator Design Space. Contours of constant $\bar{a}$ for the surface $\eta = 0.1$. ($\varepsilon = 0.002$, $k_r = 0.9$)
Figure 5.5c Flexible Manipulator Design Space: Contours of Constant $a$ for the surface $n = 0.2$, $a = 0.002$, $k_e = 0.9$.
Values of $a$:
1 = 1.58
2 = 2.51
3 = 3.96
4 = 6.31
5 = 10.0

Figure 5.5c: Flexible Manipulator Design Space. Contours of constant $a$ for the surface $\eta = 0.4$ ($k = 0.002, k_F = 0.9$).
Figure 5.5e Flexible Manipulator Design Space. Moment required for arm specifications with $\eta = 0.1$. ($s = 0.002$, $k_r = 0.9$)

Values of $a$:
1 - 1.58
2 - 2.51
3 - 3.98
4 - 6.31
5 - 10.0
Figure 5.5 Flexible Manipulator Design Space. Radius required for arm specifications with \( \eta = 0.1 \), \( g = 0.002 \), \( k_x = 0.9 \).
Values of $\bar{a}$:
1 - 1.58
2 - 2.51
3 - 3.98
4 - 6.31
5 - 10.0

Figure 5.5h  Flexible Manipulator Design Space. Cantilevered natural frequency resulting from arm specifications with $\eta = 0.1$. ($s = 0.002$, $k_r = 0.9$)
Figure 5.6: Flexible Manipulator Design Space. Surfaces of constant $\eta$. Actuator moment ($M$), and position feedback gain ($k$). $\eta$ achieves $\omega_0 = \omega_c/2$. Assumed $s = 0.002$, $k_r = 0.9$. 

$M = 10^{-25}$, $R = 10^{-21}$, $\eta = 10^{-22}$, $\eta = 10^{-8}$, $\eta = 10^{-4}$.
Figure 5.6 Flexible Manipulator Design Space. Surfaces of constant $\eta$. Actuator moment ($\bar{M}$), and position feedback gain ($K$). $K$ achieves $\omega_s = \omega_c / 2$. Assumed $s = 0.002$, $k_r = 0.9$. 
locus of the design points with a given value of \( \eta \) will be a line.

If one changes \( \tilde{a} \) he will get a different line. Several values of \( \tilde{a} \) are plotted in Figure 5.5a for \( \eta = 0.05 \) and \( k_R = 0.9 \). These lines can be visualized as contours of constant \( \tilde{a} \) on a surface of constant \( \eta \) when one is looking down the \( \tilde{a} \) axis of a three dimensional plot.

Such a three dimensional plot is sketched in Figure 5.6. In addition to the value of \( \eta = 0.05 \), values of \( \eta = 0.1, 0.2, \) and 0.4 have been used to obtain the contours of Figures 5.5b, c, and d and in turn the additional surfaces of constant \( \eta \) sketched in Figure 5.6.

The variables on the axes of Figures 5.5 and 5.6 are the primary specifications of the arm, its length, \( \tilde{l} \), payload mass, \( \tilde{m} \), and payload acceleration, \( \tilde{a} \), all nondimensionalized as described in Table 5.1 with respect to the material density, Young's modulus, and the acceleration of gravity. One can pick arbitrary values of these specifications and design an arm with a maximum stress \( \tilde{s} = 0.002 \). All points in Figure 5.6 represent that family of designs. As a result of these arbitrary choices, other parameters are determined. The gross motion frequency \( 2\pi \sqrt{\theta / \tau} \) is determined by \( \tilde{a} \) and \( \tilde{l} \) alone. The n.d. radius \( \tilde{r} \) is determined by \( \tilde{m}, \tilde{l}, \tilde{a}, \tilde{s}, \) and \( k_R \). The n.d. clamped joint frequency, \( \tilde{\omega}_c \), depends in turn on \( \tilde{r} \) and so on for other variables including the n.d. actuator moment \( \tilde{M} \) and the n.d. position gain \( \tilde{K} \) necessary to achieve \( \omega_s = \omega_c / 2 \). These are plotted in Figure 5.5 as well. Figures 5.5a, b, c, and d for example show contours of constant \( \tilde{a} \) on the surfaces of constant \( \eta \). These surfaces are sketched in Figure 5.6 and the contours indicated. This is the surface of
primary interest in the vein of this thesis, since it indicates the flexibility of an arm designed on the basis of strength. We could have plotted in Figure 5.5 contours of constant $\bar{a}$ for surfaces of constant $\bar{M}$ for example. Instead it was chosen to plot the value of these secondary parameters along the contour of constant $\bar{a}$ for each $\eta$ chosen. Surfaces of constant $\bar{M}$ can be obtained graphically from Figures 5.5 and have been sketched in Figure 5.6.

While the n.d. plot is very compact it might be desirable to return to the physical variables to give the reader a feeling for the plots. Consider an example where:

$$m_p = 3.11 \cdot 10^{-21} \text{ slugs}$$

$$\ell = 0.833 \cdot 10^7 \text{ ft.}$$

This corresponds to the same material and gravity used in the example of section 5.1.5. Thus the length axis runs from .833 ft. to 83.3 ft. in the figures. The length axis in Figure 5.6 cuts across a line of constant mass equal to 3.11 slugs (100 lb$_m$). The acceleration scale runs from 1.58 to 10 g's. (1g = 32.17 ft./sec$^2$). Thus a 8.33 ft. arm designed for a maximum n.d. stress $\bar{s} = .002$ which is to have an acceleration of 1.58 g's and a payload mass of 100 lb$_m$, will fall on the midpoint of the length axis in Figure 5.6. We see that this falls between the surfaces of $\eta = 0.2$ and $\eta = 0.4$. If this is adequate for the task and control systems available the design is complete and strength is in fact the constraining design requirement. If not, another iteration must be made. To achieve lower $\eta$ one has
several alternatives which are made clear by the approximate relations in Equation (C.12) which is repeated here:

\[
\frac{m}{p} = 13.55 \frac{-k^2 (1 - k_r^4) (a^2 - 1)^3}{\eta^6 a^7}
\]

Making explicit the nondimensional groupings:

\[
\frac{\rho g \frac{2}{3} m}{E^3 p} = \frac{\rho g}{E^6} \frac{s^4 k^2 (a_p^2 - g^2)^3 (1 - k_r^4)}{\eta^6 a_p^7}
\]

and solving for \(\eta\):

\[
\eta = \frac{s^{2/3}}{E^{1/2}} (1 - k_r^4)^{1/6} \left[ \frac{13.55 k^2 (a_p^2 - g^2)^3}{g^{3/2} \frac{a_p^7}{m}} \right]^{1/6}
\]

Thus if \(\eta\) is too high one can either change materials to increase \(E\), operate at a lower stress, or change the geometric constant \(k_r\). A change in operating stress would be displayed on a new series of plots and would entail designs with a larger radius, more structure weight, and actuator torque all of which increase cost. Increasing \(E\) will decrease \(\eta\) but materials with a higher modulus such as steel tend to weigh more entailing the same costs. Reducing \((1 - k_r^4)\) (thinner walls) is a good tactic since it can theoretically reduce \(\eta\) to any value desired. Unfortunately limits posed by the maximum outer diameter of the arm and the buckling strength of a thin wall prevent this from going on forever. This effect can be achieved in large
booms by using a complicated structure instead of a tube which moves the load bearing material as far from the neutral axis as possible and yet maintains members which have size adequate to withstand local impacts without buckling. This is observed for example in construction derricks.

The other thing the designer could do is use a better control system which could produce adequate damping at a higher value of $\eta$. If a control system which allowed an increase in $\eta$ by a factor of two could be built it would be "worth" a material which had a modulus that was a factor of four larger. As discussed earlier, using a single servo controller both to track gross motion velocity commands and to effect fine motions, locks the designer into a value of $\eta = 0.05$. As can be seen from our example and the large region surrounding it, this results in practical cases in which designs are constrained by flexibility. They must operate at lower than maximum allowable stress in order to provide for the rigidity required to make $\eta = 0.05$. The metal which must be added in order to do this will require more actuator torque to maintain the payload acceleration and gross motion time specified.

It is possible to improve on the above situation if the designer will allow the complexity of two modes of control in the arm control system. Consider servo control for the fine motions and either open loop torque commands or a servo control (with higher gains than the fine motion servo uses) for the gross motion control. Now the limitation depends on what is an adequate balance for the fine
and gross motion speeds of the task. Recall the examples of the assembly line tasks compared to picking up hot sand castings. The limitation can be expressed as:

\[
\frac{\text{gross motion frequency}}{\text{fine motion frequency}} \propto \frac{2\pi \theta/\tau}{\omega_s} \quad \frac{2\pi \theta/\tau}{\omega_c/2} = 2\eta
\]

Further improvement might be obtained by improved fine motion control. One might, for example, feed back the flexible state variables of the arm and thus improve on the oft stated rule of thumb for the dominant frequency with servo control which is:

\[
\omega_s \leq \omega_c/2
\]

The practical benefit of this control would depend on the task at hand.

5.2.4 Design Synthesis Perspective

From a pure design synthesis viewpoint the plots can be interpreted as follows. A proposed material, say aluminum, has almost constant values of \(E\) and \(\rho\) but widely variable strength depending on treatment and alloy materials added. The values of \(E\), \(\rho\), and \(g\) could be used to nondimensionalize the specifications. For charts with the appropriate value of \(\eta\) one could enter and find the value \(\bar{s}\) for which the intersection of the specified mass and length fell on the specified acceleration contour. The appropriate nondimensional section radius \(\bar{r}\) for the design could then be read
from a corresponding set of plots since it is uniquely determined by the problem as stated and was in fact solved for when $\tilde{m}_s$ and $\tilde{m}_f$ were equated. Likewise the moment required to accelerate the arm is specified. The entire design is specified with only two arbitrary choices, $E$ and $\rho$. It is an optimal design in the respect that it is the only design which will perform exactly as specified with the material chosen. It is more practical to look at this design as the nominal design, deviations from which can be considered as margins of safety or reasons for concern. The results could only be used as a first rough cut but which with some engineering analysis might aid in determining what is feasible.

For design, other arrangements of the variables may be more useful in some cases. For example, if payload mass is well defined, it need not be displayed explicitly as done here. For manipulators operating only in zero g it would be required to redefine the non-dimensional variables. This would be a rather straightforward variation in the procedure used here.

5.2.5 Effect of the Simplifying Assumptions

One might consider how a plot such as Figures 5.5 and 5.6 would be altered by assumptions more realistic for manipulator arms. For example, such arms frequently have much non structural weight, such as actuators, transmission lines, and instrumentation. To the extent that we can account for this mass with an increase in structure density and an increase in payload, we can expect that Figure 5.5 and 5.6
would not change.

The assumption of a point payload mass is unrealistic for the short length and large mass regions of Figures 5.5 and 5.6, as is the assumption of slender beams in those same regions.

Actuators of a family whose weight--torque characteristics could be included in the analysis with some complexity, but has not been done here.

The effect of tapering on both \( \omega_c \) and \( \theta/\tau \) will be discussed in the following chapter along with the effects of additional mass located at the arm midpoint.

5.3 Conclusions

From this chapter one can conclude that there are regions when flexibility constrains manipulator design, and that this region can be characterized in terms of the specifications of the arm, the task it is to perform, its environment, and the materials used in its construction. This characterization has been displayed for a reasonable range of parameter values in nondimensional form for a simple one link arm. This characterization indicates that improved control of the flexible motion would be very valuable in tasks which require accurate control of small motions, and consistent with lower arm weight, torques, and gross motion speed.
CHAPTER 6

COMPLICATIONS, CONCLUSIONS AND RECOMMENDATIONS

This final chapter will discuss in addition to the traditional conclusions and recommendations, the effect of certain complications which were not included in the previous studies.

6.1 Discussion of Complicating Effects

The studies in the previous chapters, especially Chapters 3, 4, and 5, dealt with the limiting effects of beam flexure on dynamic arm performance. Certain assumptions, made in order to hold down the number of variables to allow a more general treatment of the problem, might not be found in a real arm design. These assumptions will be treated in an isolated fashion in this section, observing the effects of the additional variable without the complications of many of the variables treated in the previous sections.

6.1.1 Effects of Tapering on Flexural Frequencies and Gross Motion Speeds

One of the more obvious improvements on the arm designs assumed in the previous sections would result from tapering the beam to add additional metal to the inboard end of the arm where its added rigidity is most crucial, and the added weight has little effect. This not only aids rigidity but decreases the torque requirements for a given gross motion speed. Previous beam segments were assumed uniform. Stress
restrictions will limit the amount of tapering one can allow. Larger payloads will allow less tapering, and in general make it less advantageous to taper.

Figure 6.1 displays the nomenclature of the tapered beam, and Table 6.1 presents the nondimensional and physical parameters of significance. Based on the analysis of Conway et al. [23] the first cantilevered frequency is given for several taper ratios and the comparison of the frequency to the frequency of a uniform beam is given in Table 6.1. Also appearing in Table 6.1 is the gross motion frequency of the tapered beam, compared to the gross motion frequency for a uniform beam, and the ratio of gross motion frequencies to the first cantilevered frequency.

It will be observed from Table 6.1 that cantilevered frequency is increased more than the bang-bang gross motion frequency, thus lowering the ratio $2\pi\sqrt{\frac{E}{\rho}}/\omega_c^*$. This extends the range of servo controlled fine motions by up to 25%. It is expected that this extension would be less when a payload is present and the strength at the distal end of the arm is provided for.

6.1.2 Stepped Beams, and Stepped Beams with Payload

Stepped beams offer some of the advantages of tapered beams and are easier to construct. They can be easily analyzed using the modeling procedures of this thesis. Two uniform sections of the stepped beam will be assumed. For Bernoulli Euler beam models one can isolate two parameters which might vary between the uniform segments. One is
TABLE 6.1
TAPERED BEAM FREQUENCIES

Based on the nomenclature of Figure 6.1 and the development in Conway et al. [23], the first cantilevered natural frequency for a tapered beam is given as:
\[ \omega_t = \nu(1 - k_t)^2 \omega_c / 3.52 \]
where \( \omega = \) first cantilevered natural frequency of uniform beam of length \( \ell \) and radius \( r_a \):
\[ k_t = \frac{r_b}{r_a} \]
\[ \nu = \text{a factor which depends on the value of } k_t \text{ alone, and has been tabulated for several values in } [23]. \]

The moment of inertia about the large end of a tapered beam is found to be:
\[ I_t = (0.1 + 0.3k_t + 0.6k_t^2)I_c \]
where \( I_c = \) the moment of inertia of a uniform beam of length \( \ell \) and radius \( r_a \).

uniform: \[ \eta_c = (\text{gross motion frequency})/(1st \text{ cantilevered frequency}) \]

tapered: \[ \eta_t = (\text{gross motion frequency})/(1st \text{ cantilevered frequency}) \]
\[ \frac{\eta_t}{\eta_c} = \sqrt{\frac{I_c}{I_t}} / (\frac{\omega_t}{\omega_c}) \]

Tabulating the results for several values of \( k_t \):

<table>
<thead>
<tr>
<th>( k_t )</th>
<th>0.5</th>
<th>0.333</th>
<th>0.25</th>
<th>0.10</th>
<th>0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nu )</td>
<td>18.5</td>
<td>11.9</td>
<td>10.4</td>
<td>8.89</td>
<td>8.72</td>
</tr>
<tr>
<td>( \omega_t/\omega_c )</td>
<td>1.314</td>
<td>1.503</td>
<td>1.662</td>
<td>2.046</td>
<td>2.477</td>
</tr>
<tr>
<td>( \sqrt{I_c/I_t} )</td>
<td>1.581</td>
<td>1.936</td>
<td>2.169</td>
<td>2.712</td>
<td>3.162</td>
</tr>
<tr>
<td>( \eta_t/\eta_c )</td>
<td>0.8312</td>
<td>0.7764</td>
<td>0.7663</td>
<td>0.7547</td>
<td>0.7837</td>
</tr>
</tbody>
</table>
Figure 6.1 Tapered Beam Nomenclature.
the stiffness product EI and the other is the density per unit
length, \( \mu \). The material chosen affects the modulus \( E \) and the density
\( \mu \). The cross section geometry affects both the cross section area
moment of inertia \( I \) and the density. Density can also be increased
by non structural weight such as wires and instrumentation.

Table 6.2 shows the nondimensionalization used on this problem.
Figures 6.2a and b display the variation of the cantilevered natural
frequency as the two parameters \( \mu \) and EI of the distal section are
varied independently. In Figure 6.2a one will observe a large
variation of the natural frequency with density of the outer link,
but EI has little effect. When payload mass is present the effect
of EI is much larger as displayed in Figure 6.2b. Also, constructed
in these figures are lines which indicate the changes in both EI and
\( \mu \) when the radius of the beam is changed.

6.1.3 Joint and Payload Mass

Figure 6.3 indicates contours of constant cantilevered natural
frequency for variations in the payload mass \( m_p \) and mass located at
the beam midpoint \( m_j \), expressed as a fraction of the beam weight. The
small variations with \( m_j \) indicate that the results of the previous work
will be distorted only slightly. It also indicates the value of
locating inboard as far as possible, when that mass does not add
compliance. If undamped compliance is added further study is required
to determine the net effect.
### TABLE 6.2

**STEP BEAM NONDIMENSIONALIZATION**

<table>
<thead>
<tr>
<th>Physical Variable</th>
<th>Nomenclature</th>
<th>Dimension</th>
</tr>
</thead>
</table>

**Inboard Link:**

- Density/unit length  \( \mu_1 \)
- Stiffness product EI \( \frac{E}{I_1} \)
- Length  \( \ell \)

**Outboard Link:**

- Density/unit length  \( \mu_2 \)
- Stiffness product EI \( \frac{E}{I_2} \)
- Length  \( \ell \)
- Frequency  \( \omega \)
- Payload Mass  \( m \)

If one constructs the first cantilevered frequency of a beam of length 2\( \ell \), with \( \mu_1 \), and \( EI_1 \):

\[
\omega_c = 3.52 \sqrt{\frac{EI_1}{\mu(2\ell)^4}}
\]

he may construct the following nondimensional groupings:

\[
\begin{align*}
\overline{EI} &= \frac{E_{I_2}}{E_{I_1}} \\
\overline{\mu} &= \frac{\mu_2}{\mu_1} \\
\overline{\omega} &= \frac{\omega}{\omega_c} \\
\overline{m} &= \frac{m}{(2\mu_1 \ell)}
\end{align*}
\]
Figure 6.2  Stepped Beam Cantilevered Frequencies.

a) No payload mass ($\bar{m} = 0.0$)

- $E_2/E_1 = 1.0$
- $E_2/E_1 = 0.2$
- Varying $r_2$ (distal link)

b) $\bar{m} = 0.1$

- $E_2/E_1 = 1.0$
- $E_2/E_1 = 0.9$
- $E_2/E_1 = 0.8$
- Varying $r_2$ (distal link)
Figure 6.3 Sensitivity of Cantilevered Frequency to Joint Mass ($m_j$) and Payload Mass ($m_p$).
6.2 Recommendations for Future Work

The recommendations for future work fall into three categories: experimental, analytical and numerical. Experimental studies of the following types are recommended:

(1) Measurements on actual manipulator arms with both locked joints and controlled joints should be made. The measurements could be compared to the model results to indicate the number and types of model elements necessary to achieve reasonable correspondence between the two.

(2) Controlled experiments on a two degree of freedom, planar arm should be performed. The construction should allow easy variation of the feedback control scheme, and the parameters within that scheme. Instrumentation of the arm flexure would allow measurements and control based on those measurements.

Additional analytical and numerical studies should cover the following aspects:

(3) Studies of arm control with the links oriented at different angles with respect to each other should be performed. This would indicate if the same rule of thumb that resulted from the study in Chapter 3 for two links aligned with each other could be applied to these cases. This is a straightforward extension using the modeling procedures of the thesis. The additional variable would make a general analysis difficult, but specific cases should at least be explored.
(4) Additional control schemes should be explored. The model can be extended to account for feedback from each joint to the actuators of the other joints. This control scheme is used in few of the arms in operation today, but promises considerable improvements over independent control of the joints. Measurements made at points in the arm other than the joints can also be modeled with reasonable extensions of the model. An exhaustive general study of these complications would likely be impossible, but specific cases could be analyzed. This need was indicated by the results of Chapter 5.

(5) Systematic methods for improving the control system parameters are greatly needed to facilitate control system design using this model. This will be especially true if the additional complexity of recommendation (4) is present. Possible approaches include sensitivity analysis of eigenvalues of the closed loop system to control gains.

(6) An improved method of finding system eigenvalues is needed. The present search technique makes use of few of the properties of functions of complex variables to facilitate the search process. Especially valuable would be a procedure for indicating how many eigenvalues existed in a given region of the complex plane, and then finding them. Possible approaches include contour mapping and observation of the phase and magnitude relations and variation of
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the frequency function for neighboring points.

(7) Torsional compliance has been modeled and included in the experimental results but not in the control interaction studies. Torsion would result from control applied to joints whose axis was not perpendicular to the plane of the arm.

Some of the recommended extensions will appear in the contract report to be written at the completion of the contract period ending August 31, 1974 [24].

6.3 Summary of Conclusions

The results of this thesis support the conclusions summarized in this section. This summary is elaborated in Chapters 2, 3 and 5.

(1) The transfer matrix approach is a valid, useful, and convenient way to model control system and structure interactions of manipulators in the linear range of motions. It has the adaptability required for use in a design context when combined with the numerical procedures of this thesis. It also has the efficiency for extensive analysis of particular configurations of components when combined with analytical simplifications of the matrix expressions.

(2) For simple servo control based on independent feedback of joint velocity and position of one and two joint manipulator arms, flexibility limits the magnitude of the dominant system eigenvalues. For the cases explored this limitation is the first natural frequency of the system with the joints
clamped. To achieve a damping ratio of the order of 0.7 a factor of two difference between the dominant eigenvalue magnitude and the clamped joint natural frequency must exist. This factor seems reasonably constant with a variety of beam and mass configurations.

(3) For arms with the common control scheme assumed, flexibility is the most constraining factor in the design of the arm structure for a wide variety of practical performance specifications. More sophisticated control schemes are of practical interest in these cases if they deal more effectively with the flexible arm components.
APPENDIX A

Analytical derivation of Frequency Functions

CASE 1: One Joint, Two Links

\[ \text{pinned joint with spring and dashpot} \]

The relevant state variables at each point in the system

\(-w = \text{deflection perpendicular to the x axis}\)

\(\psi = \text{angle or slope of neutral axis with x axis}\)

\(M = \text{moment vector out of plane of vibration}\)

\(V = \text{shear in the direction of } w\)

The relations between the state variables at each end of the two identical beam segments is given by the transfer matrix for a Bernoulli Euler beam.

\[ \mathbf{z}_{\text{left}} = \mathbf{B} \mathbf{z}_{\text{right}} \]

\[
\begin{bmatrix}
-w \\
\psi \\
M \\
V
\end{bmatrix}
= 
\begin{bmatrix}
c_0 & lc_1 & ac_2 & acl_3 \\
\beta c_3 / \lambda & c_0 & ac_1 / \lambda & ac_2 \\
\beta c_2 / a & \beta ^4 c_3 / a & c_0 & lc_1 \\
\beta ^4 c_1 / \lambda & \beta ^4 c_2 / a & \beta ^4 c_3 / \lambda & c_0
\end{bmatrix}
\begin{bmatrix}
-w \\
\psi \\
M \\
V
\end{bmatrix}
\]

Where

\[ c_0 = (\cosh \beta + \cos \beta )/2 \]
\[ \beta ^4 = \omega ^2 \mu /EI \]
\[ c_1 = (\sinh \beta + \sin \beta )/2 \]
\[ a = \lambda ^2 /EI \]
\[ c_2 = (\cosh \beta - \cos \beta )/2\beta ^2 \]
\[ c_3 = (\sinh \beta - \sin \beta )/2\beta ^3 \]
and \( \mu = \) density/unit length
\( \omega = \) circular frequency of vibration
\( E = \) Young's Modulus
\( I = \) cross sectional area moment of inertia

Across the joint \( z_{\text{left}} = K z_{\text{right}} \)

\[
\begin{bmatrix}
-w \\
\psi \\
M \\
V
\end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1/k & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-w \\
\psi \\
M \\
V
\end{bmatrix}
\]

left \hspace{1cm} \text{right}

\( k = k_R + j \omega c \)

\( k_R = \) spring constant
\( c = \) damping coefficient
\( j = \sqrt{-1} \)

Let \( z_0 = \) the state vector at the left end of the arm
\( z_1 = \) the state vector at the right end of the arm

Then \( z_0 = B K B z_1 = U z_1 \)

if \( B \) describes the identical beam segments and \( K \) describes the spring loaded joint

Boundary Conditions require

\[
\begin{bmatrix}
-w_0=0 \\
\psi_0=0 \\
M_0 \\
V_0
\end{bmatrix}
\]

\[
\begin{bmatrix}
-w_1 \\
\psi_1 \\
M_1=0 \\
V_1=0
\end{bmatrix}
\]
Thus the equation
\[ z_0 = U z_1 \]
Contains a homogeneous set of equations
\[ -u_{11} w_1 + u_{12} \psi_1 = 0 \]
\[ -u_{21} w_1 + u_{22} \psi_2 = 0 \]
For this always to be true the determinant of the submatrix of \( U \) must \( \neq 0 \) i.e.
\[ \begin{vmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{vmatrix} = 0 \]
All that is required is a great deal of algebraic manipulation and this requirement can be expressed in terms of the elements of the matrices \( A \) and \( K \) as follows:

\[ -\sinh^2 \beta (1 - 2 \cos^2 \beta) + \cos^2 \beta + \frac{\beta l}{2ka} [\cos^2 \beta \sinh \beta \cosh \beta \]
\[ - \cos \beta \sin \beta \cosh^2 \beta ] = 0 \]

As \( k_r \to \infty \) the expression
\[ -\sinh^2 \beta (1 - 2 \cos^2 \beta) + \cos^2 \beta = 0 \]
dominates. This is equivalent to expressions for single beams of length \( 2l \). As \( k_r \to 0, c \to 0 \)
\[ \cos^2 \beta \sinh \beta \cosh \beta - \cos \beta \sin \beta \cosh^2 \beta = 0 \]
dominates and is equivalent to pinned beams without springs or dashpots.

Finally as EI becomes very large \( \beta \to 0, a \to 0 \) and via a series expansion of the trigonometric and hyperbolic function the frequency condition can be shown to converge to
\[ \omega^2 = \frac{3k}{\mu l^3} = 0 \]
Which is true for a simple second order system with inertia $\mu \lambda / 3$.

CASE 2: Two Joints, Two Links

Now $U = K_1 B K_2 B$

which is equal to the transfer matrix in the one joint example premultiplied by $K_1$.

Let us designate this as $U = K_1 D$
The frequency determinant requires evaluation of additional terms.

$$U = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1/k_1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
d_{11} & d_{12} & \cdots \\
d_{21} & d_{22} & \cdots \\
d_{31} & d_{32} & \cdots \\
d_{41} & d_{42} & \cdots
\end{bmatrix}$$

The frequency determinant is

$$\begin{vmatrix}
u_{11} & u_{12} \\
u_{21} & u_{22}
\end{vmatrix} = \begin{vmatrix}
d_{11} & d_{12} \\
d_{21} + \frac{d_{31}}{k_1} & d_{22} + \frac{d_{32}}{k_1}
\end{vmatrix}$$

$$= \frac{d_{11} d_{22} - d_{12} d_{21}}{k_1} + \frac{d_{11} d_{32}}{k_1} - \frac{d_{12} d_{31}}{k_1} = 0$$

determinant for the one joint case

requires only that $d_{31}$ and $d_{32}$ be evaluated since the detailed procedure for the one joint case would give $d_{11}$ and $d_{12}$ as
$$d_{11} = \cos^2\beta + \sinh^2\beta + \frac{\beta \ell}{4k_a} (\sinh \beta + \sin \beta)(\cosh \beta - \cos \beta)$$

$$d_{12} = \frac{\ell}{\beta} (\cosh \beta \sinh \beta + \cos \beta \sin \beta) + \frac{\ell^2}{4k_a} (\sinh^2 \beta - \sin^2 \beta)$$

The values of $d_{31}$ and $d_{32}$ are found by multiplication and simplification to be:

$$d_{31} = \frac{\beta^3 \ell}{4ak_2} (\cosh \beta - \cos \beta)(\sinh \beta - \sin \beta) + \frac{\beta^2}{2a}(\cosh^2 \beta - \cos^2 \beta$$

$$+ \sinh^2 \beta + \sin^2 \beta)$$

$$d_{32} = \frac{\beta^2 \ell^2}{4a^2k_2} (\sinh \beta - \sin \beta)^2 + \frac{\beta \ell}{a}(\cosh \beta \sinh \beta - \cos \beta \sin \beta)$$

These expressions, or values obtained from them, can then be fed into the frequency determinant.
Case 3: One Joint, One Link with Payload

The point mass transfer matrix:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\frac{m\omega^2}{\beta^2} & 0 & 0 & 1 \\
\end{bmatrix}
\]

is pre-multiplied by the beam transfer matrix stated in case 1.

The result is pre-multiplied by the joint transfer matrix. If the determinant of the 2x2 upper left hand submatrix of the last result is taken upon simplification, one obtains:

\[
1 + \cos \beta \cosh \beta + (\cos \beta \sinh \beta - \sin \beta \cosh \beta) \left( \frac{\beta^2}{ka} + \frac{m\omega^2al}{\beta^3} \right)
\]

\[- \frac{2k^2m\omega^2}{k\beta^2} (\sin \beta \sinh \beta) = 0\]

As \( k \) becomes a very large real number the result simplifies to the clamped beam and payload case:

\[
1 + \cos \beta \cosh \beta + (\cos \beta \sinh \beta - \sin \beta \cosh \beta) \frac{m\omega^2al}{\beta^3} = 0
\]

In this case the first natural frequency is approximated by the equation:

\[
\omega = \frac{3EI}{k^3(m + 0.23\mu l)}
\]
APPENDIX B

Rigid Body Analysis of Two-Link Two-Joint Case

Two Link Two Joint Arm

The kinetic energy of the system for small angles $\psi_1$ and $\psi_2$

$$T = \frac{1}{2} \left[ (I_1 + m_1 h_1^2 + m_2 \dot{\theta}_1^2) \dot{\psi}_1^2 + (I_2 + h_2^2 m_2) \dot{\psi}_2^2 + 2 h_2 l_1 m_2 \dot{\psi}_1 \dot{\psi}_2 \right]$$

The potential energy (with no gravity) with rotary springs of constant $k_1$ and $k_2$ at the joints

$$V = \frac{1}{2} \left[ k_1 \psi_1^2 + k_2 (\psi_2 - \psi_1)^2 \right]$$

Lagrange's equations require for $i = 1,2$ that

$$\frac{d}{dt} \left[ \frac{\partial (T - V)}{\partial \psi_i} \right] - \frac{\partial (T - V)}{\partial \dot{\psi}_i} = 0$$

$I_1$ = mass moment of Inertia about c.g. of link 1

$I_2$ = mass moment of Inertia about c.g. of link 2

$m_1, m_2$ = mass of links 1 and 2
For the homogeneous solution to linear differential equations

\[ \ddot{\psi}_1 = -\omega^2 \psi_1 \]

where \( \omega \) = frequency of vibration in radians/sec.

Computing the appropriate derivatives and arranging the results in matrix form yields:

\[
\begin{bmatrix}
  k_1 + k_2 - (I_1 + m_1 h_1^2 + m_2 l_1^2) \omega^2 & -k_2 h_2 \rho_1 m_2 \omega^2 \\
  -k_2 - h_2 \rho_1 m_2 \omega^2 & k_2 - (I_2 + h_2^2 m_2) \omega^2
\end{bmatrix}
\begin{bmatrix}
  \psi_1 \\
  \psi_2
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]

(B1)

The determinant of the 4 x 4 matrix must be zero, which gives a condition for determining the natural frequencies \( \omega \).

**Equal Links**

If the two links are identical (B1) simplifies to

\[
\begin{bmatrix}
  k_1 + k_2 - 4m l^2 \omega^2 / 3 & -k_2 - l^2 m \omega^2 / 2 \\
  -k_2 - l^2 m \omega^2 / 2 & k_2 - ml^2 \omega^2 / 3
\end{bmatrix}
\begin{bmatrix}
  \psi_1 \\
  \psi_2
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]

(B2)

where \( \lambda = l_1 = l_2 \), \( m = m_1 = m_2 \) \( I_1 = I_2 = m l^2 / 3 \)

which expands and simplifies to

\[
(\lambda^3) k_1 k_2 - \frac{m}{3} \lambda^2 \omega^2 (k_1 + 8k_2) + \frac{7}{36} m^2 \lambda^4 \omega^4 = 0
\]

(B3)

Equation (B3) will be used to obtain natural frequencies in both the damped and undamped cases.
Undamped Natural Frequencies

If we make the substitution of variables into equation (B3)

\[ q^2 = \frac{m \ell^2 \omega^2}{3k_2} \]

and divide by \( k_2^2 \) we obtain

(B4) \[ \frac{k_1}{k_2} - \frac{(k_1 + 8) q^2 + \frac{7}{4}}{k_2^2} q^4 = 0 \]

The variable \( q \) corresponds to the natural frequency with the inboard joint clamped. The solution of (B4) yields two roots which are a nondimensionalized form of the natural frequencies of the original problem.

Damped Natural Frequencies (complex)

The damped case can be obtained directly from equation (B3) by a change in notation.

\[ k_1 \rightarrow k_1 + j \omega c_1 \]
\[ k_2 \rightarrow k_2 + j \omega c_2 \]

and

\[ s = j \omega \quad I = m \frac{\ell^2}{3} \]

The characteristic equation for determining the values of \( s \) is

\[ 7s^4/4 + s^3I(c_1 + 8c_2) + s^2[I(k_1 + 8k_2) - c_1c_2] + s(k_1c_2 + k_2c_1) + k_1k_2 = 0 \]
APPENDIX C

STRENGTH AND STIFFNESS CONSTRAINTS

Details of the Simultaneous Solution of the Strength and Stiffness (Frequency) Equations

The strength relation in equation (5.8) is a cubic in \( \tilde{r} \) with two roots at \( \tilde{r} = 0 \) and one root at \( \tilde{r}_s \) (s for strength) where:

\[
\tilde{r}_s = \frac{\pi \left( 1-k_r^2 \right) \bar{I} \left( 2 + 1/\bar{a} \right) 4 \bar{a} \bar{I}}{6 \bar{S} \bar{I} \left( 1-k_r^2 \right)}
\]

or upon simplifying:

\[
(C.1) \quad \tilde{r}_s = \frac{2 \bar{I}^2 \left( 2\bar{a} + 1 \right)}{3 \bar{S} \left( 1+k_r^2 \right)}
\]

The frequency relation, equation (5.10), has an infinite number of branches. Its zeros are the zeros of the numerator and it has asymptotes at the zeros of the denominator. The numerator is the frequency function for a clamped-free beam which is known to have zeros at:

\[ \beta_{1,n}^2 = 3.52, 22.0, 61.7, 121, 200, \ldots \]

The denominator is the frequency function for a clamped-pinned beam which has zeros at:

\[ \beta_{1,d}^2 = 15.4, 50.0, 104, 178, 272, \ldots \]

The desired branch is that corresponding to the zero of the numerator with \( \beta_{1,n}^2 = 3.52 \). Notice that from the definition of \( \beta \) in Equation (5.6b):

\[
(C.2) \quad \bar{r} = \frac{\pi}{\beta^2 \eta} \sqrt{\frac{\bar{I}^3 \left( \bar{a}^2 - 1 \right)}{\left( 1+k_r^2 \right) \bar{a}}}
\]
Thus the values of $\bar{r}_{f}$ at which the poles and zeros of (5.10) occur are inversely proportional to $\beta_{1}^{2}$. The zero for the branch we are interested in occurs at $\bar{r}_{f}$ (f for frequency):

\begin{equation}
\bar{r}_{f} = \frac{\pi}{3.52 \eta} \sqrt{\frac{\beta_{3}^{2} (\bar{a}^{2} - 1)}{(1 + k_{r}^{2}) \bar{a}}}
\end{equation}

and there are an infinite number of branches for $\bar{r} < \bar{r}_{f}$ as sketched in Figure 5.4 in solid lines. Also sketched is an approximation to the frequency function which becomes more exact for large $\bar{m}$ as long as the slender beam assumption remains valid. This is given in Appendix A as:

\begin{equation}
\omega_{c} \approx \sqrt{\frac{3 \pi r_{2}^{4} (1 - k_{r}^{4})}{4 \ell^{3} (m_{p} + 0.23 \rho \ell \pi r_{2}^{2} (1 - k_{r}^{2}))}}
\end{equation}

Solving for $m_{f}$:

\begin{equation}
m_{f} \approx \frac{3 \pi r_{2}^{4} (1 - k_{r}^{4})}{4 \ell^{3} \omega_{c}^{2}} - 0.23 \rho \ell \pi r_{2}^{2} (1 - k_{r}^{2})
\end{equation}

Substituting for $\omega_{c}$ from Equation (5.5):

\begin{equation}
m \approx \frac{3 \eta^{2} \pi r_{2}^{4} (1 - k_{r}^{4}) a_{p}}{\eta \ell^{2} (a_{p}^{2} - q^{2}) - 0.23 \rho \ell \pi r_{2}^{2} (1 - k_{r}^{2})}
\end{equation}

Nondimensionalization of (C.4) yields:

\begin{equation}
\bar{m}_{f} \approx \frac{3 \eta^{2} \bar{r}^{4} (1 - k_{r}^{4}) \bar{a}}{\eta \ell^{2} (\bar{a}^{2} - 1) - 0.23 \ell \pi \bar{r}_{2}^{2} (1 - k_{r}^{2})}
\end{equation}

By comparing the approximation (C.5) (which is essentially exact when $m_{b} \gg m_{p}$ and differs from the actual frequency by 3%) with $\bar{m}_{f} = 0$ of the mass from the frequency requirement to the mass from the strength requirement (C.3) one observes that for large $\bar{r}$, (C.5)
varies as the fourth power of \( r \), while (5.8) varies as the third power of \( r \). Thus, if at some value of \( r = \bar{r}_1 \), \( \bar{m}_f \) is less than \( \bar{m}_s \), the two curves will intersect at \( \bar{r}_2 > \bar{r}_1 \) as shown in Figure 5.4. This will happen if the curve for \( \bar{m}_f \) crosses the axis at a larger \( r \) than the curve for \( \bar{m}_s \), that is if \( \bar{r}_f > \bar{r}_s \). The desired intersection of the two curves in Figure 5.4 corresponds to one point in the volume sketched in Figure 5.6, i.e. that point with the mass \( \bar{m}(= \bar{m}_f = \bar{m}_s) \), acceleration \( \bar{a} \), length \( \bar{L} \), and frequency ratio \( \eta \) for which Figure 5.4 was drawn. Furthermore for every point in Figure 5.6 there is an intersection of two curves such as appears in Figure 5.4. While we have not shown conclusively that there will be no solution if \( \bar{r}_f < \bar{r}_s \), the physical nature of the problem and the fact that two solutions instead of one would exist, leads us to conclude that in this case no solution exists (at least with physical meaning). The limiting case when \( \bar{r}_f = \bar{r}_s \) requires:

\[
(C.6) \quad \bar{L} \bigg|_{\bar{r}_f = \bar{r}_s} = \bar{L} = \frac{\eta^2 \bar{s}^2 (1 + k_r^2)(\bar{a}^2 - 1)}{5.508 \eta^2 \bar{a} (2 \bar{a} + 1)^2}
\]

and no solution of the equation \( \bar{m}_f = \bar{m}_s \) will exist for \( \bar{L} > \bar{L}_0 \). This means the arm, when designed for strength, will always have a clamped natural frequency higher than required by specification of \( \eta \), regardless of the value of \( \bar{m} \). Thus, if plotted on a linear scale, a surface of constant \( \eta \) would intersect the plane \( \bar{m} = 0 \). Making explicit some of the nondimensional groupings to express the physical length at which this occurs, we find:

\[
l = \frac{E \bar{L}}{\rho g} = \frac{E \eta^2}{\rho g 5.508} \frac{\bar{s}^2 (1 + k_r^2)(\bar{a}^2 - 1)}{\eta^2 \bar{a} (2 \bar{a} + 1)^2}
\]
For relatively low values of $\eta$ this point will tend to occur at lengths beyond practical interest. As $\eta$ is increased or as $\rho$ is increased (such as to represent nonstructural weight) this length becomes of practical interest. It will appear as an asymptote on a log log plot.

**Simplifications Based on Order of Magnitude**

Repeating here the nondimensional strength relation Equation (5.8):

\[
(C.7) \quad \bar{m}_s = \frac{\bar{s} \pi \bar{r}^3 (1 - k_r^4)}{4 \bar{a} \bar{l}} - \frac{\pi \bar{r}^2 (1 - k_r^2) \bar{l}}{6 (2 + 1/\bar{a})}
\]

and the approximation of the cantilevered frequency, Equation (C.5):

\[
(C.8) \quad \bar{m} = \frac{3 \eta^2 \bar{r}^4 (1 - k_r^4) \bar{a}}{\pi^2 \bar{l}^2 (\bar{a}^2 - 1)} - 0.23 \bar{r} \pi \bar{r}^2 (1 - k_r^2)
\]

To understand the relative magnitude of these terms consider some realistic numbers as an example.

**Example:** Material: Aluminum 2014-T4, T451

- Fatigue Strength $- 20 \times 10^3$ lb$_f$/in$^2 = s_p$
- Shear Strength $- 38 \times 10^3$ lb$_f$/in$^2$
- Yield Strength $- 42 \times 10^3$ lb$_f$/in$^2$
- Young's Modulus $- 1 \times 10^7$ lb$_f$/in$^2 = E$
- Density $- 5.371$ slugs/ft$^3 = \rho$

**Example Environment** - Earth, gravity $= 32.17$ ft/sec$^2 = g$

Expressing the parameters in nondimensional form:

- $\bar{s} = s_p / E = 0.002$
- $\bar{k} = \rho g / E = 1.2 \bar{l} \times 10^{-7}$, $\bar{l}$ = arm length in ft.
\[ \bar{m} = m_p \rho^2 g^3 / E^3 = 0.3216 \times 10^{-21}, \bar{m}_p = \text{payload mass in slugs} \]

\[ \bar{a} = a_p / g = 0.0311 \bar{a}_p, \bar{a}_p = \text{payload acceleration in ft/sec}^2 \]

Assuming \( k_r = 0.9 \), and substituting into the equation for \( \bar{m}_f \) and \( \bar{m}_s \)

\[ \bar{m}_s = 4.502 \frac{\bar{r}^3}{a} 	imes 10^3 - 0.1194 \times 10^{-7} \bar{r}^2 \bar{r}(2 + 1/a) \]

\[ \bar{m}_f = \eta^2 \frac{0.0726 \times 10^{14} \bar{a} \bar{a}}{(a^2 - 1) \bar{r}^2} - 0.1647 \times 10^{-7} \bar{r}^2 \]

If one will set \( \bar{m}_s = \bar{m}_f \), assume \( \bar{a} = 2 \), divide through by \( \bar{r}^2 \) and solve the resulting quadratic equation for \( \bar{r} \) the result is:

\[ \bar{r} = \ell 23.24 \times 10^{-10} \left( 1 \pm \sqrt{1 - 0.177 \ell \eta^2} \right) \]

which for \( \eta^2 \ell \) less than 0.2 is will approximated with error of less than 1% by:

\[ \bar{r} = 0.465 \times 10^{-8} \ell / \eta^2 \]

or \( \bar{r} = 0.03875 \ell / \eta^2 \)

(and is also proportional to \( (\bar{a} - 1) \bar{a} \) for values of \( \bar{a} \) other than 2.)

Substituting this back to solve for \( \bar{m}_f \) one obtains:

\[ \bar{m} = \frac{\ell^2 (a^2 - 1)^3}{\eta^6 \bar{a}} \left( 2.166 \times 10^{-7} \left[ 1 - \frac{\ell a^3 \eta}{(a^2 - 1)} \right] 1.69 \times 10^3 \right. \]

For \( \eta = 0.1, \bar{a} = 2 \) and \( \ell < 0.22 \times 10^{-6} (\ell < 185 \text{ ft.}) \).

one can neglect the second term in brackets with less than 0.1% error and

\[ \bar{m}_f \approx \frac{2.166 \times 10^{-7} \ell^2 (a^2 - 1)^3}{\eta^6 \bar{a}} \]

This agrees with the exact solution presented in Figure 5.5 displaying a second-power dependence of \( \bar{m} \) and \( \ell \). This is readily
observed from the slope of the log log plots in the regions where the approximation holds. It also illustrates how to describe the contours readily without the laborious search used to obtain Figure 5.5.

The approximations described are reasonably good in the range of lengths of interest for small values of $\eta$. In general the equation

$$\bar{m}_f - \bar{m}_s \approx r$$

takes the form:

$$Ar^2 + Br + C = 0$$

The approximation $r = -B/A$ is good when $|4AC| \ll B^2$, that is when:

$$4(1-k_r^2) \pi \left( \frac{2}{3} + \frac{1}{6\bar{a}} - 0.23 \right) \left[ \frac{3\eta^2 (1-k_r^4) \bar{a}}{r^2 (\bar{a}^2 - 1)} \right] \ll \left( \frac{\bar{s} \pi (1-k_r^4) / 4 \bar{a}}{r^2} \right)^2$$

or

$$0.192 \left( 0.437 + 1/(6\bar{a}) \right) / (1 + k_r^2) \ll \frac{\bar{s}^2}{(\bar{a}^3 \bar{f} \eta^2)}$$

Assuming $1 + k_r^2 = 1$, $\bar{a} = 1$, and $\bar{s} = .001$, all "conservative" assumptions, this will simplify to:

$$\bar{n} \ll 0.1384 \times 10^6 / \eta^2$$

If there is a 10 to 1 difference in the quantities the approximation will have an error of 2.5%. Solving for the radius at which

$$\bar{m}_s = \bar{m}_f = \bar{m}$$

$$\bar{r} = \frac{\bar{S} \pi^2 \bar{f} (\bar{a}^2 - 1)}{12 \eta^2 \bar{a}^2}$$

Substituting into (C.8) to find $\bar{m} = \bar{m}_f$:

$$\bar{m} = \frac{\bar{S}^2 (\bar{a}^2 - 1)^2 \pi^2 \bar{f}^2 (1-k_r^2)}{(144 \eta^2 \bar{a}^4)} \left[ \frac{0.6459 (1+k_r^2)(\bar{a}^2 - 1) \bar{S}^2}{\eta^2 \bar{a}^3} - 0.23 \bar{f} \right]$$

where the last term inside the brackets can be ignored in most instances with small error, giving the approximation:
(C.13) \[
\bar{m} = 13.55 \frac{I \left( 1 - k_r^4 \right) (\bar{a}^2 - 1)^3 \bar{s}^4}{\eta^b \bar{a}^7}
\]

which shows the basic variation of the allowable payload mass with all the parameters of interest. The traits of this relation appear in Figure 5.5.
D.1 Rigid Mass

Using the state variables first displayed in Figure 2.1, with additional subscripts \( R \) and \( L \) for right and left.

\[ m = \text{mass of element.} \]

\[ I_g = \text{mass moment of inertia about the center of gravity.} \]

A summation of forces yields:

(D.1.1) \[ V_R + V_i - V_L = 0 \]

where the d'Lambert force

\[ V_i = m \omega^2 w_i = m \omega^2 (w_L - \psi h) \]

from (D.1.1) \[ V_R = -m \omega^2 w_L + m \omega^2 h \psi + V_L \]

The displacement of the right end can be expressed:

\[ \psi_R = \psi_L - \ell \psi \]

Summation of Moments about the center of gravity:

D.1.2 \[ M_i + V_L (\ell - h) - M_i + V_R h - M_R = 0 \]

The d'Lambert moment

\[ M_i = I_g \omega^2 \psi_i = I_g \omega^2 \psi_z \]

with D.1.2 \[ M_R = -m \omega^2 h w_L + [-I_g \omega^2 + m \omega^2 h^2] \psi_L + M_L + \ell V_L \]

In Matrix form:

\[
\begin{bmatrix}
-w \\
\psi \\
M \\
V
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
m \omega^2 h & [I_g \omega^2 + m \omega^2 h^2] & 1 & \ell \\
m \omega^2 & m \omega^2 h & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-w \\
\psi \\
M \\
V
\end{bmatrix}
\]
D.2  In Plane Bending of Jointed Beam

The full state vector including compression and bending deflection and angle.

\[
\mathbf{z}' = \begin{bmatrix} u \\ w \\ \psi \\ M \\ V \\ N \end{bmatrix}
\]

- compressive displacement
- bending displacement
- bending angle
- bending moment
- bending force in plane of paper, perpendicular beam axis
- Compressive force in plane of paper, parallel beam axis

Assume the beam is infinitely stiff to axial forces and we desire to reduce the state vector to a \(4 \times 1\) vector \(\mathbf{z}\) where:

\[
\mathbf{z} = \begin{bmatrix} w \\ \psi \\ M \\ V \end{bmatrix}
\]

The abbreviations \(c\phi = \cos \phi\) and \(s\phi = \sin \phi\) will be used. Using the full state vector the state at station 1 can be written as follows:
\[
\begin{bmatrix}
    u \\
    w \\
    \psi \\
    M \\
    V \\
    N
\end{bmatrix} =
\begin{bmatrix}
    c\phi & s\phi \\
    s\phi & c\phi \\
    1 \\
    1 \\
    c\phi & s\phi \\
    s\phi & c\phi
\end{bmatrix}
\begin{bmatrix}
    1 & 0 & 0 & 0 & 0 \\
    0 & 0 & -w \\
    0 & B & 0 & \psi \\
    0 & 0 & 0 & M \\
    0 & 0 & 0 & V \\
    -\omega^2 u_2 + N_z n
\end{bmatrix}
\begin{bmatrix}
    u \\
    N
\end{bmatrix}
\]

Where \( B \) is the 4 x 4 matrix of the beam in bending alone.

\[
\begin{bmatrix}
    c\phi & s\phi \\
    s\phi & c\phi
\end{bmatrix}
\begin{bmatrix}
    u_z \\
    B z_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
    u_1 \\
    c\phi
\end{bmatrix}
\begin{bmatrix}
    1 \\
    1 \\
    1 \\
    N_1
\end{bmatrix}
B z_2
\begin{bmatrix}
    0 \\
    s\phi u_2 \\
    0 \\
    -s\phi(-\omega^2 u_2 + N_2) \\
    0
\end{bmatrix}
\]

From the Figure above:

\[
u_2 = w_2 \tan \phi, \quad N_2 = 0
\]

\[
\begin{bmatrix}
    1/cos\phi \\
    c\phi + s\phi \tan \phi \\
    s\phi (\omega^2 \tan \phi)
\end{bmatrix}
\begin{bmatrix}
    0 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1
\end{bmatrix}
B z_2
\]

\[
\begin{bmatrix}
    1/cos\phi \\
    c\phi + s\phi \tan \phi \\
    s\phi (\omega^2 \tan \phi)
\end{bmatrix}
\begin{bmatrix}
    0 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1
\end{bmatrix}
B z_2
\]
Assuming no torsional moments in the outer beam:

\[ M_1 = M_2 \cos \phi \]

In matrix form:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1/\cos \phi & \frac{\alpha \sin^2 \phi}{\cos \phi} & 0 \\
0 & 0 & \cos \phi & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix}
\]
D.3 Vibrations out of the Plane of a Jointed Arm

Vibrations out of the plane of the arm must include torsion of inboard elements.

If one assumes the polar moment of inertia of the inboard beam about its axis to be insignificant compared to the inertia of the second beam, the twisting of the first beam acts as a pure rotational compliance $\alpha$. Angle of twist of inboard beam $= \gamma_1 = -\alpha M_2 \sin \theta$, of outboard beam $= \gamma_2$. Angle of bending of inboard beam $= \psi_1$, of outboard beam $= \psi_2$. The angle of twist for each beam is perpendicular to its angle of bending and for small values they may be treated as vectors. Thus a transformation of coordinates yields:

$$
\psi_2 = \psi_1 \cos \phi + \gamma_1 \sin \phi
$$

Substituting for $\gamma_1$ and solving for $\psi_1$

$$
\psi_1 = \psi_2 / \cos \phi + (\alpha \sin^2 \phi / \cos \phi) M_2
$$
D.4 Parallel Elements

Given two parallel elements whose state vectors $p$ and $q$ with elements $p_j$ and $q_j$ are described by $4 \times 4$ transfer matrices $B$ and $C$ as follows:

\begin{align*}
(D.4.1) \quad p_i &= Bp_{i+1}, \quad q_i = Cq_{i+1} \\
\text{and whose junctions are defined by:} \quad p_1 &= q_1 = r_1; \quad p_2 = q_2 = r_2; \\
r_3 &= p_3 + q_3; \quad r_4 = p_4 + q_4 \ \text{at both stations $i$ and $i+1$.} \end{align*}

In matrix form:

\begin{align*}
(D.4.2) \quad r_i &= p_i + \begin{bmatrix} 2^0 & 2^0 \\ 2^0 & 2^I \end{bmatrix} q_i \quad \text{and} \\
r_{i+1} &= p_{i+1} + \begin{bmatrix} 2^0 & 2^0 \\ 2^0 & 2^I \end{bmatrix} q_{i+1}
\end{align*}

where $2^0 = 2 \times 2$ zero matrix and $2^I = 2 \times 2$ identity matrix.

Combining (D.4.1) and (D.4.2):

\begin{align*}
(D.4.3) \quad r_i &= Bp_{i+1} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} Cq_{i+1}
\end{align*}

We desire the form $r_i = Dr_{i+1}$. Thus we must express $p_{i+1}$ and $q_{i+1}$ in terms of $r_{i+1}$. First express $q_{i+1}$ in terms of $p_{i+1}$:

\[ \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}_i = \begin{bmatrix} B_{11} & B_{12} \end{bmatrix} p_{i+1} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_i = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix} q_{i+1} \]

since

\[
\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}_i, \quad B_{11} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}_i + B_{12} \begin{bmatrix} p_3 \\ p_4 \end{bmatrix}_i+1 = C_{11} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}_i+1 + C_{12} \begin{bmatrix} q_3 \\ q_4 \end{bmatrix}_i+1
\]
Solving for \( q_3 \) and \( q_4 \):

\[
\begin{bmatrix}
q_3 \\
q_4
\end{bmatrix}_{i+1} = \begin{bmatrix}
C_{12}^{-1} (B_{11} - C_{11}) & C_{12}^{-1} B_{12}
\end{bmatrix} p_{i+1}
\]

Plugging this into (D.4.3):

\[
x_{i+1} = \left\{ \mathbf{B} + \begin{bmatrix} 0 & 0 \\
0 & I
\end{bmatrix} C \begin{bmatrix}
\begin{array}{c|c}
2I & 20 \\
\hline
C_{12}^{-1} (B_{11} - C_{11}) & C_{12}^{-1} B_{12}
\end{array}
\end{bmatrix} \right\} p_{i+1}
\]

Thus from (D.4.2):

\[
p_{i+1} + \begin{bmatrix} 0 & 0 \\
0 & I
\end{bmatrix} q_{i+1} = p_{i+1} + \begin{bmatrix} 0 & 0 \\
0 & I
\end{bmatrix} \begin{bmatrix}
0 & 20 \\
\hline
C_{12}^{-1} (B_{11} - C_{11}) & C_{12}^{-1} B_{12}
\end{bmatrix} p_{i+1} = r_{i+1}
\]

or solving for \( p_{i+1} \):

\[
(D.4.4) \quad p_{i+1} = \begin{bmatrix}
\begin{array}{c|c}
2I & 20 \\
\hline
C_{12}^{-1} (B_{11} - C_{11}) & C_{12}^{-1} B_{12}
\end{array}
\end{bmatrix}^{-1} r_{i+1}
\]

Again from (D.4.2) and (D.4.4):

\[
x_{i} = \left\{ \mathbf{B} \begin{bmatrix} I & 0 \\
C_{12}^{-1} (B_{11} - C_{11}) & C_{12}^{-1} B_{12}
\end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\
C_{21} & C_{22}
\end{bmatrix} \right\} x_{i+1}
\]
BIBLIOGRAPHY


Wayne John Book was born November 28, 1946 to Edwin and Leona Moeller Book, in San Angelo, Texas. Early years were spent on the family farm near Miles, Texas, with education beginning in the Miles Rural School, where he comprised the top five percent of his Senior Class. Athletics and agriculture as well as science and mathematics were chief interests in these years. In 1965 he entered the University of Texas at Austin and emerged four years later with the degree of Bachelor of Science in Mechanical Engineering with honors. Graduate studies began at the Massachusetts Institute of Technology in 1969 in the Department of Mechanical Engineering. Research Assistantships in Civil Engineering, then Mechanical Engineering, an N.S.F. traineeship, part-time Teaching Assistantship, loans, summer jobs and consulting provided sustenance during the next five years. This period was highlighted by his marriage to Judith Howard Rand on April 28, 1973. Lesser events included the Masters Degree (June 1971) and Mechanical Engineers Degree (January 1972) and his election as captain of the Rugby Club.