# Algebraic Treatment of The Whitney Conditions

by


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**Abstract:** In this work, we give an algebraic treatment of the Whitney conditions. We extend the theory of Whitney stratification to algebraic schemes over an algebraically closed field of characteristic zero. Precisely, we prove the following theorem.

**Theorem.** Let $X$ be an algebraic scheme over an algebraically closed field of characteristic zero. Let $Y$ be a smooth closed subscheme of $X$. Assume that $X$ admits a proper closed imbedding into a smooth scheme $M$ and that $X - Y$ contains a smooth, open dense subscheme of $X$. Let $\kappa : C(X) \to X$ be the conormal scheme of $X$ in $M$. Denote by $D_Y$ the exceptional divisor of the blow-up of $C(X)$ along $\kappa^{-1}(Y)$ and let $\xi_Y : D_Y \to Y$ be the induced morphism. Let $m_x(P_k(X,x))$ denote the multiplicity of the local polar variety of codimension $k$ of $X$ at $x$. Then, the following statements are equivalent.

($i$) The pair $(X^{\mathrm{sm}}, Y)$ satisfies the Whitney conditions.

($ii$) For every integer $k$ for $0 \le k \le \dim X - 1$, $m_x(P_k(X,x))$ is independent of the point $x$ of $Y$.

($iii$) The morphism $\xi_Y : D_Y \to Y$ is equidimensional.

($iv$) The ideal sheaf $\mathcal{J}$, which define the intersection $C(X) \cap C(Y)$ in $C(X)$, is

integral in $\mathcal{O}_{C(X)}$ over the ideal $\mathcal{I}$ which define $\kappa^{-1}(Y)$.

($v$) Let $D_\alpha$ be an irreducible component of $(D_Y)_{\mathrm{red}}$ and let $V_\alpha$ its image in $\mathsf{P}(C_Y X)$. Then,

$$D_\alpha = C(V_\alpha / Y, \mathsf{P}(C_Y M)/Y)$$

The above theorem was first proved by D. T. Lê and B. Teissier in the context of complex analytic spaces but their proof uses trancendental methods in an essential way.

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Para Roxana, Claudio y Gabriela
con mucho amor

Trabajadores de mi patria:
Tengo fé en Chile y su destino.
Superaran otros hombres este momento
gris y amargo en que la traición pretende imponerse.
Sigan ustedes sabiendo que más temprano que tarde,
de nuevo se abriran las anchas alamedas
por donde pase el hombre libre,
para construir una sociedad mejor.
Viva Chile, Viva el Pueblo. Vivan los Trabajadores.
*Salvador Allende*
(The last democratical President of Chile)

# Contents

# Chapter 1

# Introduction

In his attempt at resolving singularities, Oscar Zariski was led to pose the following problem. For each complex analytic variety $X \subset \mathbb{C}^n$, find a natural stratification (i.e., a decomposition) of $X$ into smooth closed subvarieties with the following property: if $H \subset \mathbb{C}^n$ is a smooth hypersurface which avoids the "exceptional points" (i.e., the zero-dimensional strata) and cuts all the positive dimensional strata transversally, then, some suitable process of resolving the singularities of $X \cap H$ will propagate along the strata to resolve the singularities of $X$ outside the exceptional points. Hence, by induction on the dimension, one could resolve almost all the singularities of $X$. The remaining step would be to transform the exceptional points into nonexceptional ones. In short, he was asking for a *equisingular* stratification of $X$ (i.e., a stratification such that the singularities in each strata are alike in some strong sense). Precisely, he asked the following question: "If $Y$ is a subvariety of $X$ and if $y$ is a closed point of $Y$, what is to be meant by saying that $X$ is *equisingular at $y$ along $Y$*? We want a definition, preferably algebraic, which will meet satisfactorily a series of stringent tests, whether algebraic, algebro-geometric, or (in the complex domain) topological in nature. By this we mean that whatever property is used in the definition of equisingularity, that property should be proved to be equivalent to each of a series of other properties which we intuitively associate with the concept of equisingularity, and which, together, cover just about everything that one could possibly expect from a correct definition of that concept. In the definition of equisingularity we must not ask for too much nor for too little. For instance, it would be too much to require that $x$ and $y$ have an analytically isomorphic neighborhood, where $x$ is a generic point of $Y$. It would be too little to ask only for equimultiplicity, i.e., to ask only that $x$ and $y$ have the same multiplicity for $X$" ( [38, introduction]). In particular, "a convincing theory of equisingularity should agrees with what one would expect from equisingularity when tested in examples against the behavior of $X$ under a monoidal transformation centered at $Y$."

**Branch Loci**

A satisfactory theory of equisingularity exists only in the case where $X$ is a hypersurface and its singular locus is of codimension 1. Assume that $X$ is imbbeded in an affine space $\mathbb{A}^n$. Let $\pi : X \rightarrow \mathbb{A}^{n-1}$ be the projection defined by $\pi(x_1, ..., x_n) = (x_1, ..., x_{n-1})$ and let $B_\pi$ be the branch locus of $\pi$. $X$ is equisingular at $y$ along $Y$ if $B_\pi$ has a simple point at $\pi(y)$. The underlying reason why this case lends itself to a complete treatment is the fact that the concept of equivalent singularities of algebraic plane curves is well established. In fact, $X$ is equisingular at $y$ along $Y$ if and only if there exists $Y$-transversal sections $\Gamma_x$ and $\Gamma_y$ of $X$ at $x$ and $y$ respectively (i.e., $\Gamma_x$ is a surface of $\mathbb{A}^n$ having at $x$ a simple point and such that $T_x\Gamma_x \bigoplus T_xY = T_x\mathbb{A}^n$), such that the curves $C_x := X \cap \Gamma_x$ and $C_y := X \cap \Gamma_y$ are equivalents (in the sense of their behavior under a monoidal transformation, as defined in [39], Definitions 2,3 or 4). In fact, O. Zariski proved that this definition of equisingularity (in the codimension 1 case) satisfies all the desired properties that one expects. Most importantly, he proved that this definition behaves nicely with respect to a monoidal transformation [40, VII, page 314]. He also showed that if $X$ is equisingular at 0 along $Y$, then $X$ is equimultiple along $Y$ [41, 4.5], and moreover that $X$, as an imbedded variety in $\mathbb{A}^n$, is topologically equivalent, locally at 0, to the direct product $C_0 \times Y$.

In view of his sucess of giving a complete treatment of the concept of equisingularity in the case that $X$ is an imbedded hypersurface of $\mathbb{A}^n$ whose singular locus $Y$ has codimension 1 in $X$, by the existence of an "equisingular projection" $\pi : X \to \mathbb{A}^{n-1}$, O. Zariski proposed the following general algebraic definition of equisingularity (by induction on the codimension): If $X$ is an hypersurface of $\mathbb{A}^n$ and $Y$ is a closed nonsingular subspace of $X$ then, $X$ is equisingular at the point 0 of $Y$ along $Y$ if there exists a projection $\pi : \mathbb{A}^n \to \mathbb{A}^{n-1}$ such that $\mathrm{Ker}(d\pi) \cap X = 0$ (i.e., $\pi$ maps $Y$ isomorphically to $Y_\pi \subset \mathbb{A}^{n-1}$ and $\pi^{-1}(0) \not\subset X$) and, the branch locus $B_\pi$ of $\pi|X$, which is a hypersurface of $\mathbb{A}^{n-1}$, is equisingular at 0 along $Y_\pi$ [42, Definition 3, page 489]. With this definition of equisingularity, certain statements which are not obvious either become straightforward consequences or can be proved inductively without too much difficulty. For example, the equisingularity of $X$ at 0 along $Y$ implies the equimultiplicity of $X$ at 0 along $Y$. Also, that equisingularity is "generically" satisfied (i.e., the set of points $y \in Y$ such that $X$ is equisingular at $y$ along $Y$ is the complement of a proper analytic subvariety of $Y$). More evidence that this is a good definition of equisingularity is provided by the following theorem of A. N. Varchenko: if $X$ is equisingular at 0 along $Y$, then $X$ is topologically isomorphic to $X_0 \times Y$, where $X_0$ is a suitable $Y$-transversal section of $X$ at 0 [35, Theorem 1]. Nevertheless, a weakness of the definition was revealed in the following example due to J. Briançon and J. P. Speder: $X$ is the hypersurface of $\mathbb{C}^4$ defined by

$$X : z^6 + y^3 + tx^4y + x^6 = 0.$$

$X$ is equisingular at the origin along the singular locus $Y : x = y = z = 0$, but it does not have a "stable behavior along $Y$ under blowing-up $Y$" [43, page 14]. Therefore, Zariski's definition of equisingularity can not be accepted in the general case.

We deliberately chose the point of view of equisingular projections, because they are related to polar varieties (in fact, they define the polar variety of codimension 1 in $X$ ), and polar varieties play a key role in the numerical characterization of a new type of equisingularity, namely, Whitney equisingularity (or differential equisingularity). Whitney equisingularity is the theme of this work.

### Whitney Equisingularity

In [37], H. Whitney introduced the following two conditions of regularity on a triple $(X, Y, y)$, where $X \subset \mathbb{C}^n$ is a complex analytic space, $Y \subset X$, a non-singular subspace of $X$ and $y \in Y$ with $X^0 := X - Y$ non-singular.

Condition (a): If for a sequence of points $(x_i)$ in $X^0$ converging to $y$, the sequence $\{T_{x_i}X^0\}$ has a limit $T$ in a Grassmannian, then $T \supset T_yY$.

Condition (b): If for each sequence of points $(x_i)$ in $X^0$ converging to $y$ and each sequence of points $(y_i)$ in $Y$ converging to $y$, the sequence $\{\overline{x_iy_i}\}$ of secants has a limit $\ell$ and the sequence $\{T_{x_i}X^0\}$ has a limit $T$, then $\ell \subseteq T$.

Whitney's aim was to find sufficient geometric conditions on a pair $(X, Y)$ to guarantee that $X$ is topologically locally trivial along $Y$. He gave an example to show that condition (a) alone is not sufficient for this purpose. Then he proposed condition (b) and he conjectured this:

*If a triple of complex analytic spaces $(X, Y, y)$ satisfies both conditions (a) and (b), then there exists a $Y$-transversal section $X_y$ of $X$ at $y$ such that $(\mathbb{C}^n, X, y)$ is topologically isomorphic to $(\mathbb{C}^r \times Y, X_y \times Y, y)$.*

This conjecture is now a theorem, known as the Thom-Mather (isotopy) theorem. Whitney also showed that any complex analytic space $X$ has a "regular" stratification (or *Whitney stratification* as it is now known): There exists an analytic stratification $X = \cup_\alpha X_\alpha$, such that if $X_\alpha \subseteq \overline{X_\beta}$, then $(X_\beta, X_\alpha)$ satisfies both conditions $(a)$ and $(b)$. The Thom-Mather theorem indicates the importance of the Whitney conditions: they can be viewed as an approximation to the ideal concept of equisingularity.

In [14, 5.1], H. Hironaka introduced a modified version of the Whitney conditions, of *intrinsic trancendental nature*, in the case when $X \subset \mathbb{C}^n$ is a complex analytic space and $Y \subset X$ is a nonsingular closed subspace of $X$. Namely, $X$ is said to satisfy the strict Whitney condition $(a)$ (with exponent $e$) at the point $0 \in Y$ if there exists an open neighborhood $U$ of 0 in $X$ and a positive real number $C$ such that, for every point $x \in X^{sm} \cap U$, we have

$$\mathrm{dist}(T_0Y, T_xX) \leq C\mathrm{dist}(x, Y)^e.$$

Hironaka proved that if $(X, Y, 0)$ satisfies the Whitney conditions $(a)$ and $(b)$, then it satisfies the strict Whitney condition $(a)$ with no precise exponent $e$ [14, 5.2]. The strict Whitney condition $(a)$ with exponent 1 was called Condition $(w)$ by J. L. Verdier [36, 1.4] and he proved that the condition $(w)$ implies both conditions $(a)$ and $(b)$ of Whitney [36, 1.5].

The Whitney conditions caught the attention of O. Zariski and he proved that in the case when $X$ is a hypersurface in $\mathbb{C}^n$ and $Y$ has codimension 1 in $X$, then $X$ is equisingular at $y$ along $Y$ (in Zariski's sense) if and only if $X$ satisfies both conditions $(a)$ and $(b)$ at $y$ along $Y$ [40, 10.3]. In 1970, H. Hironaka proved that if the pair $(X, Y)$ satisfies the Whitney conditions at the point $y$ of $Y$, then $X$ is equimultiple along $Y$ in a neighborhood of $y$ in $Y$, or equivalently, $X$ is pseudo-normally flat along $Y$ in a neighborhood of $y$ in $Y$, i.e., the projective cone of $X$ along $Y$ is equidimensional over a neighborhood of $y$ in $Y$ [14].

In the above example of Briançon and Speder, $X$ satisfies the Whitney conditions along its singular locus $Y$ but, as we pointed out, it is not "stable under blowing-up $Y$." This example says, therefore, that the Whitney conditions are not acceptable as a definition of equisingularity. Nevertheless, as the work of B. Teissier et al indicates, they are important per se: they are related with the theory of Tangency and Duality of projective varieties, as we will describe below.

**The Work of Teissier and Others**

In the early 1970's, B. Teissier began working on an algebraic theory of the Whitney conditions. His starting point was the case where $X \subset \mathbb{C}^n$ is a hypersurface. In 1973, he found numerical conditions whose "constancy" characterizes the Whitney conditions. He introduced the sequence of Milnor's numbers associated with a family of hypersurfaces parameterized by $Y$, considering the following setup:

$$\begin{array}{ccc} X & \subset & Y \times \mathbb{C}^n \\ r \downarrow & & \swarrow p_1 \\ Y & & \end{array}$$

where $r$ is a retraction which admits a section $\sigma$, such that $\sigma(Y) = Y \times \{0\}$ and $r^{-1}(y) =: X_y$ is an hypersurface of $\mathbb{C}^n$ with isolated singularities. Let

$$\mu^*(X_y, y) := \{\mu^{(i)}(X_y, y)\}_{i=0}^n$$

where

$$\mu^*(X_y, y) := \inf\{\mu(X_y \cap H, y); H \text{ is an } (i+1)\text{-plane of } Y \times \mathbb{C}^n \text{ containing } Y\}$$

and where $\mu$ denotes the Milnor number of the hypersurface. Teissier, Briançon and Speder proved the following theorem.

**Theorem 1.0.1** ([*31*]). *Let $F(y, z_1, ..., z_n) = 0$ be the equation defining $X$ and let $f_y(z) := F(y, z)$ be the function which defines the fiber $X_y$ of $X$. If dim$Y = 1$, then the following statements are equivalent:*

*(i) $(X^0, Y)$ is Whitney regular along $Y$.*

*(ii) $\mu^*(X_y, y)$ is independent of $y$ in $Y$.*

*(iii)* $\frac{\partial F}{\partial y} \in \overline{\mathbf{m}_n(\frac{\partial F}{\partial z_1}, ..., \frac{\partial F}{\partial z_n})}$

*(iv) $e(\mathbf{m}_n(J(f_y)))$ is independent of $y$ in $Y$.*

In the above theorem $J(f_y)$ denotes the Jacobian ideal of $f_y$, and $e(\mathbf{m}_n(J(f_y)))$ denotes the multiplicity of the ideal $\mathbf{m}_n(J(f_y))$ in $\mathcal{O}_{X_y,y}$. Condition $(iii)$ of the above theorem was introduced by B. Teissier [32] as condition *c-cosecant*, and it is related with the idealistic Bertini theorem [32]. The above theorem gives a good converse of the Thom-Mather theorem in the case where $X$ is a hypersurface with smooth singular locus: The Whitney conditions hold if $X$ and *all* of its sections are topologically locally trivial along its singular locus.

In 1981, B. Teissier generalized the above theorem by replacing the $\mu^*$- sequence with the following one

$$M_y^*X := \{m_y(X,y), m_y(P_1(X,y)), ..., m_y(P_{d-1}(X,y))\}$$

where $d$ is the dimension of $X$ and $m_y(P_k(X,y))$ denotes the multiplicity at the point $y$ of $Y$ of the $k^{\text{th}}$ local polar variety $P_k(X,y)$. Teissier, inspired by the work of Hironaka [13] and the results of Canuto and Speder [3], considered the following normal-conormal diagram

$$\begin{array}{ccc} \mathbf{Bl}_{\kappa^{-1}(Y)}C(X) & \xrightarrow{\hat{e}} & C(X) \\ {\scriptstyle\kappa'}\downarrow & & \downarrow{\scriptstyle\kappa} \\ \mathbf{Bl}_Y X & \xrightarrow{e} & X \end{array}$$

where $t = \dim Y$, where $C(X)$ is the projectivized conormal space of $X$ in $\mathbf{C}^n$, where $Y$ is non-singular, and where $\mathrm{Bl}_A B$ denotes the blow-up of $B$ along $A$. Set $\xi := \kappa \circ \hat{e}$. Teissier proved the following theorem:

**Theorem 1.0.2** ([33]). *Let $X \subset \mathbf{C}^n$ be an integral complex analytic space of dimension $d$, and let $Y \subset X$ be a non-singular subspace of $X$ such that $X^0 := X - Y$ contains a smooth open dense subscheme. Then the following conditions are equivalent:*

*(i) The pair $(X^{\mathrm{sm}}, Y)$ satisfies the Whitney conditions at the point $0$ of $Y$.*

*(ii) The pair $(X^{\mathrm{sm}}, Y)$ satisfies the condition $(w)$ at the point $0$ of $Y$.*

*(iii) The sequence $M_y^*X$ is independent of $y$ in a neighborhood $U$ of $0$ in $Y$.*

*(iv) $\dim \xi_Y^{-1}(0) = n - t - 2$ where $\xi_Y : D_Y \to Y$ is the restriction of $\xi$ to the exceptional divisor $D_Y$ of $\mathrm{Bl}_{\kappa^{-1}(Y)}C(X)$.*

In [25], R. MacPherson introduced, in a purely obstructional way, the local Euler obstruction which plays an important role in his affirmative response to a conjecture of Deligne and Grothendieck on the existence of Chern classes for singular complex algebraic varieties (see [25], [16], [22]). In [22], D. T. Lê and B. Teissier proved a formula for the multiplicity of the local polar varieties, and, with the aid of Gonzales-Sprinberg's purely algebraic interpretation of the local Euler obstruction, they showed that the local Euler obstruction is an alternate sum of the multiplicity of the local polar variety. Theorem 1.0.2 says that the multiplicity of the local polar varieties are constant along the strata of a Whitney stratification (i.e., are constructible functions). Therefore, in view of the existence of Whitney stratifications for complex analytic spaces [37], Lê and Teissier showed that the local Euler obstruction is a constructible function. The constructibility of the local Euler obstruction was first proved by R. MacPherson in [25] and he asked for an algebraic treatment of this matter (see [16]).

In 1988, D. T. Lê and B. Teissier improved the above theorem. They related the Whitney conditions with the theory of projective duality and, therefore, they were able to give an explicit method to describe the set of limits of tangent hyperplanes of $X$ at the point 0 of $Y$ in the case that $X$ satisfies the Whitney conditions at 0 along $Y$. Precisely, they proved the following theorem.

**Theorem 1.0.3** ([*21*]) *Under the situation of Theorem* 1.0.2, *the following conditions are equivalent:*

*(i) The pair* $(X^{\mathrm{sm}}, Y)$ *satisfies the Whitney conditions at the point* 0 *of* $Y$.

*(ii) At every point of* $\kappa^{-1}(0)$, *the ideal* $\mathcal{I}$ *which defines the intersection* $C(X) \cap C(Y)$ *is integral in* $\mathcal{O}_{C(X)}$ *over the ideal* $\mathcal{J}$ *which defines* $\kappa^{-1}(Y)$ *in* $C(X)$.

*(iii) If* $(D_Y)_{red} = \cup D_\alpha$ *is the decomposition of the exceptional divisor* $D_Y$ *of* $\mathrm{Bl}_{\kappa^{-1}(Y)} C(X)$ *in its irreducible components and if* $V_\alpha := \kappa'(D_\alpha) \subset Y \times \mathbf{P}^{n-t-1}$ *then, for each* $\alpha$, *the component* $D_\alpha$ *is equal to the relative conormal space of* $V_\alpha$ *in* $Y \times \mathbf{P}^{n-t-1}$ *and all the fibers of the morphism* $D_\alpha \to Y$ *have the same dimension in a neighborhood of* 0 *in* $Y$.

In [5], T. Gaffney generalized Theorem 1.0.1 for 1-parameter families of complete intersection, with isolated singularities (CIIS). He replaced the $\mu^*$-invariants of $X_y$ by the polar multiplicities $m_y(P_k(X_y, y))$ and the integral closure of an ideal by the integral closure of a submodule of a free module. He also replaced the multiplicity of an ideal for the Buchsbaum-Rim multiplicity of a module [2]. The restriction on $X$ to be a CIIS in T. Gaffney's result appears only in the use of the Buschsbaum-Rim multiplicity (see [6]).

In [11, 5.1], J.P. Henry and M. Merle, inspired by the work of Zariski in the codimension 1 case and the equivalence $(i) \Leftrightarrow (iii)$ of Theorem 1.0.2, gave a complete numerical characterization for Zariski's equisingularity in the case where $X \subset \mathbb{C}^n$ is a complex analytic space of dimension no larger than 3. They require the constancy of the following sequences:

(1) The polar multiplicities of $X$.

(2) The polar multiplicities of $\mathcal{F}(X)$ and $\mathcal{D}(X)$ where $\mathcal{F}(X)$ denotes the *cusp* variety of $X$ and $\mathcal{D}(X)$ denotes the *double fold* variety of $X$ [11, 3.1.1;4.1.1].

(3) The polar multiplicities of the singular locus $S(X)$ of $X$.

(4) The polar multiplicities of the singular locus of $S(X)$.

Although the statements of Theorem 1.0.2 and Theorem 1.0.3, other than 1.0.2$(ii)$, are algebro-geometric, the proof of Lê and Teissier use topological methods in an essential way, as we will explain below.

**Some Comments on the Proof of Theorem 1.0.2:**

$(i) \Leftrightarrow (ii)$ : This equivalence follows from the work of Hironaka and Verdier, as discussed above.

$(i) \Rightarrow (iii)$ : The key case of this proof is the case that $Y$ is 1-dimensional. In this case, the key step is to show that the polar curve $P_{d-1}(X, 0)$ is empty. In order to do so, Teissier proved that if $p : \mathbb{C}^n \to \mathbb{C}^2$ is a generic projection (i.e., such that define a generic polar curve of $X$ at 0), then the Kernel of $p$ does not contain any limit of secants $\{\overline{x_i y_i}\}$ with $x_i \in P_{d-1}(X, 0)$ and $y_i \in Y$ ( [33, Lemme-clé. V 1.2.2]; [9]). The proof of this result is transcendental in nature.

$(iii) \Rightarrow (iv)$ : The first step is to prove that there exists a generic hyperplane section $H$ of $X$ containing $Y$ which is transversal to the limits of tangent hyperplanes of $X$ at 0 and, such that,

$$m_x P_k(X, x) = m_x P_k(X \cap H, x).$$

The next step is to proceed by induction on the dimension of $X$. This part of the proof is algebraic and was nicely simplified by J. P. Henry and M. Merle [10]

$(iv) \Rightarrow (ii)$ : Teissier proved that the function

$$\psi(x) := \frac{\operatorname{dist}(T_x X, Y)}{\operatorname{dist}(x, Y)},$$

is bounded in a neighborhood $U$ of 0 in $X$. Therefore, the pair $(X, Y)$ satisfies the Condition $(w)$ at the point $0 \in Y$.

**Some Comments on the Proof of Theorem 1.0.3:**

$(i) \Leftrightarrow (ii)$ : A straightforward computation shows that the triple $(X, Y, 0)$ satisfies $(ii)$ if and only if it satisfies condition $(w)$ [21, 1.3.8]. Then, use the fact that condition $(w)$ is equivalent to the Whitney conditions.

$(ii) \Rightarrow (iii)$ : This part of the proof is incorrect. Lê and Teissier claim that the cone $C_{C(X) \cap C(Y)} C(X)$ (of the projectivized conormal of $X$ in $\mathbb{C}^n$ along its intersection with the projectivized conormal of $Y$) is Lagrangian in $\mathbb{C}^n \times \check{\mathbb{P}}^{n-1}$, by using the principle of "Lagrangian specialization" [21, 1.2.6] and a projectivized version of [20, 4.4.2] and [30, Appendix 4]. But, the canonical isomorphism

$$\Theta : \operatorname{Def}(T^* M, T^*_Y M) \longrightarrow T^*(\operatorname{Def}(M, Y)/\mathbb{A}^1)$$

constructed in [20, 4.4.2] is not $\mathbb{G}_m \times \mathbb{G}_m$-equivariant; hence, it does not induce a morphism from $\operatorname{Def}(\mathbb{P}(T^* M), \mathbb{P}(T^*_Y M))$ to $\mathbb{P}(T^*(\operatorname{Def}(M, Y)/\mathbb{A}^1))$. Otherwise the proof is correct in spirit.

**This Work**

In this work, we give a complete algebraic treatment of the Whitney conditions: we extend the theory of Whitney stratifications to the category of algebraic schemes (i.e., when $X$ is integral, of finite type, and separate over the ground field) over an algebraically closed field of characteristic zero. The restriction on the characteristic is due to the use of the algebro-geometric version of Sard's lemma.

In Chapter $II$, Section 1, we define the local polar variety of $X$ at $x$ via the Nash blow-up of $X$ as defined in [33, IV,1]. An important role in the definition of the local polar varieties is played by Kleiman's transversality lemma [17] (see Proposition 2.1.3 ).

In Chapter $II$, Section 2, we define the conormal scheme of a subscheme of a smooth variety as in [18, 3]. We also describe the local polar variety of $X$ at $x$ via the conormal scheme of $X$ as in [33, IV,4] (see Proposition 2.2.2).

In Chapter $II$, Section 3, we give a formula for the multiplicity of the local polar variety (see Lemma 2.3.5). The proof of this formula follows the same path as in the proof of [16, Lemma 2]. A similar description for the multiplicities of the polar varieties was given by Lê and Teissier [22, 5.1.1].

In Chapter $II$, Section 4, we give a new proof of Lê and Teissier's theorem (see Theorem 2.4.7) that the multiplicities of the local polar varieties of $X$ are constant along a nonsingular closed subscheme $Y$ of $X$ if and only if the exceptional divisor $D_Y$ of $\mathrm{Bl}_{\kappa^{-1}(Y)}C(X)$, the blow-up of $C(X)$ along $\kappa^{-1}(Y)$, is equidimensional over $Y$. Our proof is as follows: First, we show the existence of "generic local hyperplane sections" which satisfies strong local transversality conditions: they are transversal to every limit of tangent hyperplanes to $X$ at points of $Y$ (see Lemma 2.4.3). We use those transversality conditions to prove that the multiplicity of the local polar variety and the equidimensionality of the exceptional divisor $D_Y$ of $\mathrm{Bl}_{\kappa^{-1}(Y)}C(X)$ over $Y$ behave nicely with respect to hyperplane sections (see Lemma 2.4.4 and Lemma 2.4.6). The proof of Lemmma 2.4.4 does not use the Bertini-idealistic theorem as was done by Lê and Teissier (see [22]). In fact, we only need the emptyness of some polar varieties to guarantee the existence of generic hyperplane sections $H$ satisfying the following transversality conditions:

$$C(X, M) \cap C(H, M) = \emptyset.$$

Altogether, this allows us to reduce the proof of Theorem 2.4.7 to the case where $Y$ is 1-dimensional. In this case, if we assume that $D_Y$ is equidimensional over $Y$, then, with the aid of the intersection theory [4], we are able to show that our formula for the multiplicity of the local polar varieties "specializes," i.e.,

$$m_x(P_k(X, x)) = \int (\alpha)_x$$

for some 1-cycle $\alpha \in A_1(D_Y)$. Therefore, by conservation of numbers [4, 10.2], we have that the multiplicities of the local polar varieties are constant along $Y$. The converse follows from [10, Theorem 1].

In Chapter $III$, Section 1, we describe the theory of Lagrangian schemes as developed by S. Kleiman [18]. The only new result in this section is Lemma 3.1.5, which states that the natural projection from the absolute projective cotangent scheme to the relative projective cotangent scheme maps absolute Lagrangian subschemes into relative ones.

In Chapter $III$, Section 2, we describe the construction of the deformation to the normal cone as in [4, 5]. We also state the behavior of the deformation to the normal cone with the formation of the cotangent bundle as in [20, 4.4.2] and [30, Appendix 4].

In Chapter $III$, Section 3, we describe the theory of integral dependence of ideal sheaves as developed by M. Lejeune-Jalabert and B. Teissier [23]. We also prove a result concerning with "specialization of integral dependence" (see Proposition 2.3.7), which is slightly different from the one proved by B. Teissier in [34, Appendix]: we drop the condition of finiteness of the scheme defined by the ideal sheaf $\mathcal{I}$ over the base scheme.

In Chapter $III$, Section 4, we give a set-theoretical definition of the Whitney conditions as follows (see Definition 2.4.5):

Condition (a): The pair $(X^{\mathrm{sm}}, Y)$ satisfies the Whitney condition $(a)$ if

$$(D_Y)_{\mathrm{red}} \subset \mathsf{P}(\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2) \times_Y \mathsf{P}((\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2)^{\vee})$$

Condition (b): The pair $(X^{\mathrm{sm}}, Y)$ satisfies the Whitney condition $(b)$ if

$$(D_Y)_{\mathrm{red}} \subset I_Y := \text{ incidence correspondence of } \mathsf{P}(\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2) \times_Y \mathsf{P}((\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2)^{\vee})$$

Our definition of the Whitney conditions agrees with the original one given by Whitney in the complex domain (see [37]): Our definition of condition $(a)$ is equivalent to the following one. Set-theoretically,

$$\kappa^{-1}(Y) \subset C(Y, M)$$

Now, $\kappa^{-1}(Y)$ is, set-theoretically, the set of all limits, at points of $Y$, of "hyperplanes of $M$" which are tangent to $X^{\mathrm{sm}}$ and, $C(Y, M)$ parametrizes the "hyperplanes of $M$" which contain the tangent space of $Y$. Thus, the condition $\kappa^{-1}(Y) \subset C(Y, M)$ is equivalent to the original definition of condition $(a)$.

Now, set-theoretically, we have that the schemes $D_Y$ and $\mathsf{P}(C_Y X) \times_Y \kappa^{-1}(Y)$ have the same irreducible components. A point of $\mathsf{P}(C_Y X) \times_Y \kappa^{-1}(Y)$ is of the form $(y, \ell, H)$, where $y \in Y$, $\ell \in (\mathsf{P}(C_Y X))_y$ and $H \in \kappa^{-1}(y)$. Thus, $\ell$ is a limit, at the point $y$ of $Y$, of secant from point of $X$ to points of $Y$, and $H$ is a limit, at the point $y$ of $Y$, of tangent hyperplanes at smooth point of $X$. Our definition of condition $(b)$ says that $\ell \in H$, which is the original definition of the Whitney condition $(b)$.

With the above definition in mind, we give a new proof of the following theorem of Lê and Teissier.

**Theorem 1.0.4** ([*21*, 2.1.1]). *Let $X$ be a $d$-dimensional integral scheme of finite type and separate over an algebraically closed ground field $K$ of characteristic zero.*

*Let $Y$ be a smooth integral proper closed subscheme of $X$ of dimension $t$ such that, $X^0 := X - Y$ contains a smooth open dense subscheme. Assume that $X$ admits a proper imbedding in a smooth ambient scheme $M$. Then, the following conditions are equivalent.*

*(i) The pair $(X^{\mathrm{sm}}, Y)$ satisfies the Whitney conditions (a) and (b).*

*(ii) The sequence $M_x^*(X)$ is independent of the point $x$ of $Y$.*

*(iii) The morphism $\xi_Y : D_Y \to Y$ is equidimensional.*

*(iv) $\mathcal{I}_{C(Y,M)\cap C(X,M)}\mathcal{O}_{C(X,M)} \subseteq \overline{\mathcal{I}_{\kappa^{-1}(Y)}\mathcal{O}_{C(X,M)}}$.*

*(v) Let $(D_Y)_{red}$ denote the scheme $D_Y$ with its reduced induced structure. Let $(D_Y)_{red} = \cup D_\alpha$ be its decomposition in its irreducible component, and let $V_\alpha$ denote the image of $D_\alpha$ in $\mathrm{P}\left(\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2\right)$ via $\kappa'$, equipped with the induced reduced closed subscheme structure. Then, for each $\alpha$ we have that*

$$D_\alpha = C\left(V_\alpha/Y; \mathrm{P}(\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2)/Y\right).$$

Our proof is in the following way:

$(v) \Rightarrow (i)$ : This follows from the fact that the relative conormal scheme always lies in the incidence correspondence.

$(iii) \Rightarrow (iv)$ : We prove first, by induction on the dimension of $Y$, that a condition like $(iv)$ is generically satisfied on the fibers (see Lemma 3.4.4). Then, since $D_Y$ is equidimensional over $Y$, we have, by "specialization of integral dependence" (see Proposition 3.3.7), that $(iv)$ holds (globally).

$(iv) \Rightarrow (v)$ : By using the Hamiltonian isomorphism and Lemma 3.1.5, we prove that the natural projection

$$\alpha : \{\mathrm{P}(T_{\mathrm{P}(T_Y^*M)}\mathrm{P}(T^*M)) - \mathrm{P}(T_{\mathrm{P}(T_Y^*M)}q^{-1}(Y))\} \to \mathrm{P}(T_Y^*M) \times_Y \mathrm{P}(T_YM),$$

maps $\mathrm{P}(C_{C(X)\cap C(Y)}C(X))$ into a $Y$-Lagrangian subscheme of $I_Y := \mathrm{P}(T^*(\mathrm{P}(T_Y^*M)/Y))$ and that the image of $\alpha$ is exactly $I_Y$. Condition $(iv)$ says that $\alpha$ induces a finite morphism from $\mathbf{P}(C_{C(X)\cap C(Y)}C(X))$ to $D_Y$. Therefore, $D_Y$ is $Y$-Lagrangian in $I_Y$.

$(i) \Rightarrow (v)$ : From the above discussion and the assumption that $D_Y$ is entirely contained in the incidence-correspondence, we have that $\alpha$ induces a dominant morphism from $\mathbf{P}(C_{C(X)\cap C(Y)}C(X))$ to $D_Y$ and, therefore, $D_Y$ is $Y$-Lagrangian in $I_Y$.

$(v) \Rightarrow (iii)$ : Notice that in $(v)$ we are not requiring that $D_Y$ to be equidimensional over $Y$ as in Theorem 1.0.3 $(iii)$. Therefore this proof is not immediate. First we reduce to the case when $Y$ is 1-dimensional (by using the fact that the multiplicities of the polar varieties behave nicely with respect to hyperplane sections and that Condition $(iii)$ is equivalent to the equimultiplicity of the polar varieties). Then, since $D_\alpha$ is $Y$-Lagrangian, we have that $D_\alpha$ contains an open dense subscheme which is smooth over $Y$ (therefore equidimensional over $Y$) and, since $D_\alpha$ is irreducible and $Y$ is 1-dimensional, we have, by dimension counting, that $D_\alpha$ is (globally) equidimensional over $Y$.

# Chapter 2

# Multiplicities of Polar Varieties

## 2.1 Polar Varieties and Nash Blow-Up

([33]; [22])

**Setup 2.1.1** *Let $X$ be a $d$-dimensional integral scheme of finite type and separate over an algebraically closed ground field $K$. We assume that $X$ admits a proper imbedding in a smooth ambient scheme $M$ of dimension $m$, and that the smooth locus $X^{\mathrm{sm}}$ is dense in $X$.*

We recall the following definition.

**Definition 2.1.2** *Let $S$ be a ground scheme, and $\mathcal{E}$ be a quasicoherent $\mathcal{O}_S$-module. We define the $S$-scheme $\mathsf{P}(\mathcal{E})$ by*

$$\mathsf{P}(\mathcal{E}) := \mathrm{Proj}\,(\mathrm{Sym}\,\mathcal{E})$$

*and, in general, we define the $S$-scheme $\mathrm{Gr}^n(\mathcal{E})$ as the $S$-scheme which represents the Grassmann functor of locally free quotients of rank $n$ of $\mathcal{E}$. In particular we have that*

$$\mathrm{Gr}^1(\mathcal{E}) = \mathsf{P}(\mathcal{E}).$$

Consider the $X$-scheme $\mathrm{Gr}^d(\Omega^1_X)$. Let $\hat{X}$ be the unique component of $\mathrm{Gr}^d(\Omega^1_X)$ dominating $X$, and let $\upsilon : \hat{X} \to X$ be the induced morphism. By definition of $\mathrm{Gr}^d(\Omega^1_X)$ we have a locally free quotient $\tilde{\Omega}$ of $\upsilon^*\Omega^1_X$; its dual is called the *Nash Tangent Bundle* and it is denoted by $TX$.

Under the situation of the Setup 2.1.1, there is a more geometrical construction of the bundle $TX$. The Nash blow-up $\hat{X}$ may be identified with a closed subscheme of $\mathrm{Gr}^{m-d}(\iota^*\Omega^{1\vee}_M)$ where $\iota : X \to M$ is the closed imbedding of $X$ in $M$. Now

$$\mathrm{Gr}^{m-d}(\iota^*\Omega^{1\vee}_M) = X \times_M \mathrm{Gr}^{m-d}(\Omega^{1\vee}_M)$$

and, over $\mathrm{Gr}^{m-d}(\Omega^{1\vee}_M)$ we have, by definition, an exact sequence of locally free sheaves

$$0 \to L \to p^*\Omega^{1\vee}_M \to S \to 0$$

where $p : \mathrm{Gr}^{m-d}(\Omega^{1\vee}_M) \to M$ is the structural morphism. The restriction $L \mid \hat{X}$ coincides with $TX$.

We will define next the local polar variety of $X$ at a closed point $x$ of $X$ via the Nash blow-up $\hat{X}$ of $X$ ([33, $IV$, 1]). Let $V$ be an open neighborhood of $x$ in $M$, on which we have an étale morphism $g : V \to \mathbb{A}^m$ with $g(x) = 0$ ([1, $VII$, 5.8]). Let $U := V \cap X$. We have, by ([1, $IV$, 1, 5.2]), that

$$\Omega^{1\vee}_M \mid = g^*\Omega^{1\vee}_{\mathbb{A}^m}.$$

Consider the Nash blow-up diagram of $X$

$$\begin{array}{ccc} \hat{X} & \subset & X \times_M \mathrm{Gr}^{m-d}(\Omega^{1\vee}_M) \\ \upsilon \downarrow & & \swarrow p_1 \\ X & & \end{array}$$

We have that

$$\upsilon^{-1}(U) \subset U \times_M \mathrm{Gr}^{m-d}(\Omega^{1\vee}_M) = X \times_M \mathrm{Gr}^{m-d}(\Omega^{1\vee}_M \mid V) = U \times \mathrm{Gr}^{m-d}(\check{\mathbb{A}}^m) = U \times \mathrm{Gr}^{m-d}(\mathbb{A}^m).$$

Therefore we have the following diagram

$$\begin{tikzcd} & \mathrm{Gr}^{m-d}(\mathbb{A}^m) & \\ v^{-1}(U) \arrow[ur, "\gamma_U"] \arrow[dr, "v_U"'] & \subset & U \times \mathrm{Gr}^{m-d}(\mathbb{A}^m) \arrow[ul, "p_2"'] \arrow[dl, "p_1"] \\ & U & \end{tikzcd}$$

where $v_U := v \mid U$ and where $\gamma_U$ is induced by the projection $p_2$. Let

$$\mathcal{D} : 0 \subset D_{m-1} \subset \ldots \subset D_{d-k+1} \subset \ldots \subset D_1 \subset \mathbb{A}^m,$$

be a complete flag in $\mathbb{A}^m$ with $\operatorname{codim}(D_i, \mathbb{A}^m) = i$. Let

$$\sigma_k(\mathcal{D}) := \{T \in \mathrm{Gr}^{m-d}(\mathbb{A}^m); \dim(T \cap D_{d-k+1}) \geq k\}$$

be the Schubert variety associated to $\mathcal{D}$. The scheme structure is given as in [4, 14]. This is a subvariety of $\mathrm{Gr}^{m-d}(\mathbb{A}^m)$ of codimension $k$.

**Proposition 2.1.3** ([*33, IV, 2*]). *For each integer $k$ for $0 \leq k \leq d$ there exists an open dense set $W_k$ of $\mathrm{Gr}^{d-k+1}(\mathbb{A}^m)$ such that, for each $D_{d-k+1}$ in $W_k$, we have*

*(i) Set $U^{\mathrm{sm}} := U \cap X^{\mathrm{sm}}$. Then, $\gamma_U^{-1}(\sigma_k(D_{d-k+1})) \cap v^{-1}(U^{\mathrm{sm}})$ is schematically dense in $\gamma_U^{-1}(\sigma_k(D_{d-k+1}))$, and the former scheme is either empty or has codimension $k$ in $v^{-1}(U)$*

*(ii) $v^{-1}(x) \cap \gamma_U^{-1}(\sigma_k(D_{d-k+1}))$ is either empty or has dimension equal to $\dim v^{-1}(x) - k$.*

**Proof** Let $\Gamma := \mathrm{GL}\,(m, K)$. The action of $\Gamma$ over $\mathbb{A}^m$ induces an action on $\mathrm{Gr}^{m-d}(\mathbb{A}^m)$ and on $\mathrm{Gr}^{d-k+1}(\mathbb{A}^m)$. Both actions are related by the equality

$$\mu \cdot \sigma_k(D_{d-k+1}) = \sigma_k(\mu^{-1} \cdot D_{d-k+1})$$

with $\mu \in \Gamma$.

Clearly $\upsilon^{-1}(U^{\mathrm{sm}}) \cap \gamma_U^{-1}(\sigma_k(D_{d-k+1}))$ is either empty or schematically dense in $\gamma_U^{-1}(\sigma_k(D_{d-k+1}))$. To compute the dimension of $\gamma_U^{-1}(\sigma_k(D_{d-k+1}))$, we observe that

$$\upsilon^{-1}(U) \cap (U \times \sigma_k(D_{d-k+1})) = \gamma_U^{-1}(\sigma_k(D_{d-k+1}))$$

where the intersection is taken in $U \times \mathrm{Gr}^{m-d}(\mathbb{A}^m)$.

By Kleiman's transversality lemma ( [17]), there exists an open dense subgroup $\Gamma^0$ of $\Gamma$ such that, if $\mu \in \Gamma^0$ then we have that

$$\begin{aligned}
&\dim(\upsilon^{-1}(U) \cap \mu \cdot (U \times \sigma_k(D_{d-k+1})))\\
&= \dim(\upsilon^{-1}(U) \cap (U \times \sigma_k(\mu^{-1} \cdot D_{d-k+1})))\\
&= \dim \upsilon^{-1}(U) - \operatorname{codim}(U \times \sigma_k(\mu^{-1} \cdot D_{d-k+1}); U \times \mathrm{Gr}^{m-d}(\mathbb{A}^m))\\
&= d - k.
\end{aligned}$$

Therefore, $(i)$ follows from the fact that $\Gamma$ acts transitively on $\mathrm{Gr}^{m-d}(\mathbb{A}^m)$.

The proof of $(ii)$ is analogous. □

**Definition 2.1.4** *We define the* local polar variety of $X$ at $x$ *with respect to* $U$, *denoted by* $P_k(X, U, x, D_{d-k+1})$, *by*

$$P_k(X, U, x, D_{d-k+1}) := \upsilon_U(\gamma_U^{-1}(\sigma_k(D_{d-k+1})))$$

*for* $0 \leq k \leq d-1$ *and where* $D_{d-k+1}$ *belongs to the open dense set* $W_k$ *of Proposition* 2.1.3. *The scheme structure is the reduced induced structure as a closed subset of* $U$.

**Remark 2.1.5** *It follows from Proposition* 2.1.3 *that*

$(i)$ $P_k(X, U, x, D_{d-k+1})$ *is a reduced closed subscheme of* $U$ *of pure codimension* $k$ *or empty.*

$(ii)$ $\gamma_U^{-1}(\sigma_k(D_{d-k+1}))$ *coincides set-theoretically with the strict transform of* $P_k(X, U, x, D_{d-k+1})$ *via* $\upsilon_U$.

## 2.2 Polar Varieties and Conormal Scheme

([33]; [12])

**Setup 2.2.1** *Use the notation and hypothesis of* (2.1.1).

Let $\iota : X \to M$ be the closed imbedding of $X$ in $M$. Over $X^{\mathrm{sm}}$ there is a surjection

$$\partial i \mid X^{\mathrm{sm}} : \iota^* \Omega^1_M \mid X^{\mathrm{sm}} \to \Omega^1_X \mid X^{sm}.$$

The *(projective) conormal scheme* of $X$ in $M$, $C(X, M)$, is defined as

$$C(X,M) := \text{closure of } \mathsf{P}((\mathrm{kernel}\partial i|X^{\mathrm{sm}})^\vee) \ \text{ in } X \times_M \mathsf{P}(\Omega^{1\vee}_M).$$

There is a proper morphism $\kappa_X : C(X,M) \to X$ which is induced by the first projection. If no confusion arise, we will denote $C(X,M)$ by $C(X)$ and $\kappa_X$ by $\kappa$.

Let $x$ be a (closed) point of $X$. As before, we consider an open neighborhood $V$ of $x$ in $M$ over which we have an *é*tale morphism $g : V \to \mathsf{A}^m$. Therefore, we have that

$$\Omega^1_M \mid V = g^* \Omega^1_{\mathsf{A}^m}.$$

Now, consider the following diagram

$$\begin{tikzcd}
C(X) \arrow[r, phantom, "\subset"] \arrow[dr, "\kappa"'] & X \times_M \mathsf{P}(\Omega^{1\vee}_M) \arrow[d, "p_1"] \\
 & X
\end{tikzcd}$$

Let $U := V \cap X$. We have that

$$\kappa_U^{-1} \subset U \times_M \mathsf{P}(\Omega^{1\vee}_M) = \mathsf{P}(\Omega^{1\vee}_M \mid U) = U \times \check{\mathsf{P}}^{m-1}.$$

Therefore we have the following diagram

$$\begin{tikzcd}
 & \check{\mathsf{P}}^{m-1} & \\
\kappa^{-1}(U) \arrow[ur, "\lambda_U"] \arrow[rr, phantom, "\subset"] \arrow[dr, "\kappa_U"'] & & U \times \check{\mathsf{P}}^{m-1} \arrow[ul, "p_2"'] \arrow[dl, "p_1"] \\
 & U &
\end{tikzcd}$$

where $\lambda_U$ is induced by $p_2$ and where $\kappa_U$ is induced by $p_1$.

**Proposition 2.2.2** ([*33, IV,* 4.1.1]). *For each integer $k$, $0 \leq k \leq d$, there exists an open dense set $V_k$ of $\mathrm{Gr}^{k+m-d-1}(\check{\mathbb{P}}^{m-1})$, the Grassmannian of projective subspaces of dimension $d-k$ in $\check{\mathbb{P}}^{m-1}$, such that for every $L^{d-k}$ in $V_k$, we have that*

*(i) Set $U^{\mathrm{sm}} := X^{sm} \cap U$. Then, $\lambda_U^{-1}(L^{d-k}) \cap \kappa^{-1}(U^{\mathrm{sm}})$ is schematically dense in $\lambda_U^{-1}(L^{d-k})$, and the former scheme is either empty or has pure codimension $k+m-d-1$ in $\kappa^{-1}(U)$.*

*(ii) $\kappa^{-1}(x) \cap \lambda_U^{-1}(L^{d-k})$ is either empty or has dimension equal to* $\dim \kappa^{-1}(x) - (k+m-d-1)$.

*(iii) The intersection $D_{d-k+1}$ in $\mathbb{A}^m$ of all hyperplanes of $\mathbb{A}^m$ represented by the points of $L^{d-k}$ is a linear subspace of $\mathbb{A}^m$ of codimension $d-k+1$ and $D_{d-k+1}$ belongs to $W_k$, where $W_k$ is the open dense set of Proposition* 2.1.3, *and we have that*

$$(\kappa_U(\lambda_U^{-1}(L^{d-k})))_{\mathrm{red}} = P_k(X, U, x, D_{d-k+1})$$

*where $(\ )_{\mathrm{red}}$ means the reduced induced structure.*

**Proof** The proof of $(i)$ and $(ii)$ are analogous to the proof of 2.1.3.

For the proof of $(iii)$ see the discussion of [33, IV,page 433]. □

**Remark 2.2.3** *It follows from the above proposition that $\lambda_U^{-1}(L^{d-k})$ coincides set-theoretically with the strict transform of $P_k(X, U, x, D_{d-k+1})$ via $\kappa_U$.*

## 2.3 Multiplicities of Polar Varieties

([22]; [28]; [16]; [11])

**Setup 2.3.1** *Use the notation and hypothesis of* (2.1.1) *and denote the algebraic multiplicity of the local polar variety of $X$ at $x$ by* $m_x(P_k(X, U, x, D_{d-k+1}))$

**Proposition 2.3.2** ([*22*, 5.1.1]) *The multiplicity of the local polar variety of $X$ at $x$ is given by*

$$m_x(P_k(X,U,x,D_{d-k+1})) = (-1)^k \int c_k(TX) \cap \{s(v^{-1}(x),\hat{X})\}_k$$

*where $s(v^{-1}(x),\hat{X})$ denotes the Segre class of $v^{-1}(x)$ in $\hat{X}$.* □

In particular we have that the multiplicity of the local polar variety $P_k(X,U,x,D_{d-k+1})$ is independent of the choice of the open neighborhood $U$ of $x$ in $X$ and of the choice of the linear subspace $D_{d-k+1}$ of $\mathbb{A}^m$. We will denote this multiplicity by $m_x(P_k(X,x))$.

In 1981, Gonzales-Sprinberg gave a purely algebraic interpretation of the local Euler obstruction introduced by R. MacPherson (see [7]; [25]). In ([22]), D. T. Lê and B. Teissier used Gonzales-Sprinberg's result and Proposition 2.3.2 to obtain the following description of the local Euler obstruction.

**Corollary 2.3.3** ([*22*, 5.1.2]).

$$\mathrm{Eu}_x X = \sum_{k=0}^{d-1} (-1)^k m_x(P_k(X,x))$$

*where $\mathrm{Eu}_x X$ denote the local Euler obstruction of $X$ at $x$ as defined in* ([*4*, 4.2.9]). □

The formula of Proposition 2.3.2 was reproved by V.Navarro-Aznar ([28]) in the context of algebraic schemes (i.e. when $X$ is integral of finite type and separate over an algebraically closed field of arbitrary characteristic). In this section, with the aide of intersection theory, we will give an analogous formula for the multiplicity of the local polar variety, but instead of considering the Nash blow-up we will consider the conormal scheme.

**Lemma 2.3.4** *Let $S$ be a scheme and let $\mathcal{E}$ be a locally free sheaf of rank $e+1$ over $S$. Let*

$$0 \to T \to \pi^*\mathcal{E} \to \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \to 0$$

*be the tautological sequence of locally free sheaves over* $\mathbf{P}(\mathcal{E})$ *where* $\pi : \mathbf{P}(\mathcal{E}) \to S$ *is the structural morphism. Let* $\alpha \in A_*(S)$*. Then,*

$$\pi_*(c_j(T) \cap \pi^*\alpha) = \begin{cases} (-1)^e\alpha & \textit{if } j = e \\ 0 & \textit{otherwise} \end{cases}$$

**Proof** By linearity we can assume that $\alpha = [V]$, with $V$ an irreducible $k$- dimensional subscheme of $S$. Then we have that

$$c_j(K) \cap \pi^*[V] \in A_{k+e-j}(\mathbf{P}(\mathcal{E}))$$

and $\pi_*(c_j(K) \cap \pi^*[V])$ is supported in $V$.

Therefore, by counting dimension, we have that

$$\pi_*(c_j(K) \cap \pi^*[V]) = \begin{cases} N \cdot [V] & \text{if } j = e \\ 0 & \text{otherwise} \end{cases}$$

Since the degree $N$ can be computed on the fibers of $\pi$, we may assume that $\mathcal{E}$ is trivial. In this case, by Whitney sum, we have that

$$c_j(K) = (-1)^j c_1(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1))^j$$

Therefore, $N = (-1)^e$. □

Over $C(X)$ we have a tautological locally free sheaf of rank $m-1$, constructed as follows. Let

$$0 \to T \to q^*\Omega_M^{1\vee} \to \mathcal{O}_{\mathbf{P}(\Omega_M^{1\vee})}(1) \to 0$$

be the tautological sequence of locally free sheaves over $\mathbf{P}(\Omega_M^{1\vee})$, where $q : \mathbf{P}(\Omega_M^{1\vee}) \to M$ is the structural morphism.

The restriction of $T$ to $C(X)$ is a locally free sheaf of rank $m-1$, which we denote by taut.

**Lemma 2.3.5** *([16, lemma 2]; [11, 2.3.1])*

$$m_x(P_k(X,x)) = (-1)^{k+m-d-1} \int c_{k+m-d-1}(\mathrm{taut}\,) \cap \{s(\kappa^{-1}(x), C(X))\}_{k+m-d-1}$$

**Proof** Over $\hat{X}$ we have an exact sequence of locally free sheaves

$$0 \to TX \to p^*\Omega_M^{1\vee} \mid \hat{X} \to \mathcal{Q} \to 0$$

where $p : \mathrm{Gr}^{m-d}(\Omega_M^{1\vee}) \to M$ is the structural morphism. Since

$$\mathbf{P}(\mathcal{Q}) \subset \mathbf{P}(p^*\Omega_M^{1\vee} \mid \hat{X}) = \hat{X} \times_X \mathbf{P}(\iota^*\Omega_M^{1\vee})$$

we have a natural morphism

$$\mu : \mathbf{P}(\mathcal{Q}) \to \mathbf{P}(\iota^*\Omega_M^{1\vee}).$$

Since $\mathbf{P}(\mathcal{Q})$ is contained in the incidence correspondence of $\mathrm{Gr}^{m-d}(\iota^*\Omega_M^{1\vee})$ and $\mathbf{P}(\iota^*\Omega_M^{1\vee})$ ( [19, 2.1.1]), we have that $\mu$ maps $\mathbf{P}(\mathcal{Q})$ onto $C(X)$.

Let $\pi$ denote the structural morphism from $\mathbf{P}(\mathcal{Q})$ to $\hat{X}$. We have a commutative diagram

$$\begin{array}{ccc} \mathbf{P}(\mathcal{Q}) & \xrightarrow{\mu} & C(X) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\kappa} \\ \hat{X} & \xrightarrow{\upsilon} & X \end{array}$$

in which $\mu$ and $\upsilon$ are birational, $\kappa$ is proper and $\pi$ is proper and flat morphisms (see [30, 1.2]). Since $\mathbf{P}(\mathcal{Q})$ is contained in the incidence correspondence of $\mathrm{Gr}^{m-d}(\iota^*\Omega_M^{1\vee})$ and $\mathbf{P}(\iota^*\Omega_M^{1\vee})$ we have that $\pi^*(TX)$ is a subsheaf of $\mu^*(\mathrm{taut})$. Therefore, we have an exact sequence of locally free sheaves over $\mathbf{P}(\mathcal{Q})$

$$0 \to \pi^*TX \to \mu^*(\mathrm{taut}\,) \to \mu^*(\mathrm{taut}\,)/\pi^*TX \to 0.$$

On $\mathbf{P}(\mathcal{Q})$ we have a commutative diagram of locally free sheaves

$$\begin{tikzcd}
& \pi^*(TX) \arrow[r, "\sim"] \arrow[d, tail] & \pi^*(TX) \arrow[d, tail] & \\
& \mu^*(\mathrm{taut}) \arrow[r, tail] \arrow[d, two heads] & \mu^*(q^*\Omega^{1\vee}_M|C(X)) \arrow[r, two heads] \arrow[d, two heads] & \mu^*(\mathcal{O}_{\mathbf{P}(\Omega^{1\vee}_M)}(1)|C(X)) \arrow[d, "\wr"] \\
& \mu^*(\mathrm{taut})/\pi^*(TX) \arrow[r, tail] & \mathcal{Q} \arrow[r, two heads] & \mu^*(\mathcal{O}_{\mathbf{P}(\Omega^{1\vee}_M)}(1)|C(X))
\end{tikzcd}$$

and the sequence

$$0 \to \mu^*(\mathrm{taut})/\pi^*(TX) \to \mathcal{Q} \to \mu^*(\mathcal{O}_{\mathbf{P}(\Omega^{1\vee}_M)} \mid C(X)) \to 0 \qquad (2.3.5.1)$$

is the tautological sequence over $\mathbf{P}(\mathcal{Q})$.

Since $\pi$ is flat of relative dimension $m-d-1$, we have by ([4, 4.2]) that

$$\pi^*(\{s(v^{-1}(x), \hat{X})\}_k) = \{s(\pi^{-1}v^{-1}(x), \mathbf{P}(\mathcal{Q}))\}_{k+m-d-1} \qquad (2.3.5.2)$$

By Lemma 2.3.4, applied to the sequence (2.3.5.1), we have that

$$\pi_*(c_j(\mu^*(\mathrm{taut}\,)/\pi^*(TX)) \cap \pi^*\alpha) = \begin{cases} (-1)^{m-d-1}\alpha & \text{if } j = m-d-1 \\ 0 & \text{otherwise} \end{cases} \qquad (2.3.5.3)$$

for $\alpha \in A_*(\hat{X})$.

Therefore we have that

$$\int c_{k+m-d-1}(\mathrm{taut}) \cap \{s(\kappa^{-1}(x), C(X))\}_{k+m-d-1}$$

$$= \int c_{k+m-d-1}(\mu^*(\mathrm{taut})) \cap \{s(\pi^{-1}v^{-1}(x), \mathbf{P}(\mathcal{Q}))\}_{k+m-d-1} \qquad ([4, 4.2(a)])$$

$$= \int c_{k+m-d-1}(\mu^*(\mathrm{taut})) \cap \pi^*\{s(v^{-1}(x), \hat{X})\}_k \qquad (2.3.5.2)$$

$$= \int \sum_{j=o}^{k+m-d-1} c_{k+m-d-1-j}(TX) \cap \pi_*(c_j(\mu^*(\mathrm{taut})/\pi^*(TX)) \cap \pi^*\{s(v^{-1}(x), \hat{X})\}_k) \qquad ([4, 3.2(e)])$$

$$= (-1)^{m-d-1} \int c_k(TX) \cap \{s(v^{-1}(x), \hat{X})\}_k \qquad (2.3.5.3)$$

$$= (-1)^{k+m-d-1} m_x(P_k(X, x)) \qquad (2.3.2)$$

## 2.4 The Main Theorem

([33])

**Setup 2.4.1** *Use the notation and hypotheses of* (2.1.1) *and, let $Y$ be a smooth integral proper closed subscheme of $X$ of dimension $t$ such that, $X^0 := X - Y$ contains a smooth open dense subscheme. From* ([*1*, $VII$, 5.8]) *we have that for each closed point $x$ in $Y$ there exists an open neighborhood $V$ of $x$ in $M$ and an étale morphism $g : V \to \mathbb{A}^m$ such that $V \cap Y = g^{-1}(\mathbb{A}^t)$ and $g(x) = 0$. Set $U := V \cap X$. In particular we have that $Y$ is regularly imbedded in $M$ of pure codimension $m - t$* ([*1*, 5.13]).

**Remark 2.4.2** ([*27*]). *Let $A$ be a commutative ring (with unity) and let $I$ be an ideal of $A$. Assume that $I$ is generated by a regular sequence in $A$. Then, the natural map from the Symmetric algebra to the Rees algebra*

$$\mathrm{Sym}_A(I) := \bigoplus_{d\geq 0} S^d(I) \longrightarrow \bigoplus_{d\geq 0} I^d$$

*is an isomorphism.*

This remark tell us that, since $Y$ is regularly imbedding in $M$, the blow-up of $M$ along $Y$ is isomorphic to the projective bundle associated to $\mathcal{I}_Y^M$ where $\mathcal{I}_Y^M$ denotes, from now on, the ideal sheaf of $Y$ in $M$. i.e.,

$$\mathrm{Bl}_Y M := \mathrm{Proj}\,(\bigoplus_{d\geq 0}(\mathcal{I}_Y^M)^d) \xrightarrow{\sim} \mathbf{P}(\mathcal{I}_Y^M) := \mathrm{Proj}\,(\mathrm{Sym}\,\mathcal{I}_Y^M)$$

Furthermore, the exceptional divisor $E_Y M$ of $\mathrm{Bl}_Y M$ is $Y$-isomorphic to $\mathbf{P}(\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2)$ and, under this isomorphism the normal sheaf of $E_Y M$ in $\mathrm{Bl}_Y M$ correspond to $\mathcal{O}_{\mathbf{P}(\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2)}(-1)$.

We will consider the following diagram

$$\begin{tikzcd}
X \times_M \mathbf{P}(\mathcal{I}_Y^M) \times_M \mathbf{P}(\Omega_M^{1\vee}) \arrow[d] & \mathrm{Bl}_{\kappa^{-1}(Y)} C(X) \arrow[l] \arrow[r, "\hat{e}"] \arrow[d, "\kappa'"'] & C(X) \arrow[r] \arrow[d, "\kappa"] & X \times_M \mathbf{P}(\Omega_M^{1\vee}) \arrow[dl] \\
X \times_M \mathbf{P}(\mathcal{I}_Y^M) & \mathrm{Bl}_Y X \arrow[l] \arrow[r, "e"] & X &
\end{tikzcd} \quad (2.4.2.1)$$

Denote by $\xi$ the composition $\kappa \circ \hat{e}$. Let $D_Y$ denote the exceptional divisor of $\mathrm{Bl}_{\kappa^{-1}(Y)} C(X)$, let $\xi_Y : D_Y \to Y$ be the restriction of $\xi$ to $D_Y$ and, let $\sigma : Y \to M$ denote the closed imbedding of $Y$ in $M$. Then, by Remark 2.4.2 and the universal property of blowing-up, we have the following diagram.

$$\begin{tikzcd}
\mathsf{P}(\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2) \times_Y \mathsf{P}(\sigma^*\Omega_M^{1\vee}) \arrow[d] & D_Y \arrow[l] \arrow[r, "\hat{e}"] \arrow[d, "\kappa'"'] & \kappa^{-1}(Y) \arrow[r] \arrow[d, "\kappa"] & \mathsf{P}(\sigma^*\Omega_M^{1\vee}) \arrow[dl] \\
\mathsf{P}(\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2) & e^{-1}(Y) \arrow[l] \arrow[r, "e"] & Y &
\end{tikzcd} \tag{2.4.2.2}$$

The next lemma guarantee the existence of "generic local hyperplane sections" of $M$ satisfying certain strong transversality conditions with respect to the schemes $X$ and $Y$ in a neighborhood of a point $x$ of $Y$. Those conditions will allow us to describe the behavior of the local polar varieties and of the exceptional divisor $D_Y$ of $\mathrm{Bl}_{\kappa^{-1}(Y)} C(X)$ under those generic hyperplanes sections (see Lemma 2.4.4 and Lemma 2.4.6). Altogether will play a key role in the sequel.

**Lemma 2.4.3** *Consider the diagram and notations of* 2.2.1. *There exists an open dense subset $W$ of $\check{\mathsf{P}}^{m-1}$ such that for every $H$ in $W$ we have that*

*(i)* $\lambda_U^{-1}(H) = \emptyset$.

*(ii)* $C(H, \mathsf{A}^m) \cap C(\mathsf{A}^t, \mathsf{A}^m) = \emptyset$, *where the intersection is considered in* $\mathsf{A}^m \times \check{\mathsf{P}}^{m-1}$

**Proof** Let $V_d$ be the open dense subset of $\check{\mathsf{P}}^{m-1}$ as in Proposition 2.2.2. Then, by (2.2.2), we have that for every $H$ in $V_d$, $\lambda_U^{-1}(H) = \emptyset$.

Let $\check{\mathsf{P}}^{m-t-1}$ be the subspace of $\check{\mathsf{P}}^{m-1}$ parametrizing the hyperplanes of $\mathsf{A}^m$ containing $\mathsf{A}^t$. Let

$$W := V_d \cap (\check{\mathsf{P}}^{m-1} - \check{\mathsf{P}}^{m-t-1}).$$

Since $C(\mathsf{A}^t, \mathsf{A}^m) = \mathsf{A}^t \times \check{\mathsf{P}}^{m-t-1}$, we have that for every $H$ in $W$ both conditions $(i)$ and $(ii)$ are satisfied. $\square$

**Lemma 2.4.4** ([*33*, 5.4.3], [*28*, 4.2], [*22*, 4.1.6, 4.1.9]). *(transversality of polar varieties) For each integer $k$ with $0 \leq k \leq d-1$, there exists an open dense subset of the space*

$$I_k := \{(D_{d-k+1}, H);\ \ D_{d-k+1} \subseteq H\} \subset \mathrm{Gr}^{d-k+1}(\mathbb{A}^m) \times \check{\mathbb{P}}^{m-1}$$

*such that, for every $(D_{d-k+1}, H)$ in that open subset we have that*

*(i)* $(P_k(X, U, x, D_{d-k+1})\widehat{\cap} H')_{red} = P_k(X \cap H', U \cap H', x, D_{d-k+1})$

*(ii)* $m_x(P_k(X, x)) = m_x(P_k(X \cap H', x)) = m_x(P_k(X, U, x, D_{d-k+1})\widehat{\cap} H')$

*where $H' := g^{-1}(H)$ and where $(P_k(X, U, x, D_{d-k+1})\widehat{\cap} H')$ denotes the closure of $P_k(X, U, x, D_{d-k+1}) \cap U^{\mathrm{sm}} \cap H'$ in $U$.*

**Proof** Let $p_1 : I_k \to \mathrm{Gr}^{d-k+1}(\mathbb{A}^m)$ and $p_2 : I_k \to \check{\mathbb{P}}^{m-1}$ be the natural projections. It follows from ([19, 2.1.1]) that $I_k$ is a Grassmannian bundle over $\check{\mathbb{P}}^{m-1}$.

Let $W_k$ be the open dense subset of $\mathrm{Gr}^{d-k+1}(\mathbb{A}^m)$ of Proposition 2.1.3 and, let $W$ be the open dense subset of $\check{\mathbb{P}}^{m-1}$ of Lemma 2.4.3. For each $H$ in $W$ let $W_k(H)$ denote the open dense subset of $p_2^{-1}(H)$ as in Proposition 2.1.3. Let

$$W' := p_1^{-1}(W_k) \cap \Big( \bigcup_{H \in W} W_k(H)\Big).$$

Since $I_k$ is a bundle over $\check{\mathbb{P}}^{m-1}$ we can construct an open dense subset $\tilde{W}$ of $I_k$, contained in $W'$, such that for every $(D_{d-k+1}, H)$ in $\tilde{W}$ we have that

$(a)$ $P_k(X, U, x, D_{d-k+1})$ and $P_k(X \cap H', U \cap H', x, D_{d-k+1})$ define a (generic) polar variety, where $H' := g^{-1}(H)$.

$(b)$ $H$ satisfies the conditions of Lemma 2.4.3.

Since the conditions $(i)$ and $(ii)$ are local, we may assume that we have a global étale morphism $g : M \to \mathbb{A}^m$ such that $g(x) = 0$ and such that we have a fiber square diagram

$$\begin{array}{ccc} Y & \xrightarrow{\ \sigma\ } & M \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbb{A}^t & \xrightarrow{\ \iota\ } & \mathbb{A}^m \end{array}$$

By Lemma 2.4.3 we have that

$$\begin{aligned} C(H',M)\cap C(X,M) &= \emptyset \qquad (2.4.4.1)\\ C(H',M)\cap C(Y,M) &= \emptyset. \end{aligned}$$

From the exact sequence of locally free sheaves of $\mathcal{O}_{H'}$-modules

$$0\to \mathcal{I}^M_{H'}/(\mathcal{I}^M_{H'})^2\to \Omega^1_M\otimes\mathcal{O}_{H'}\to\Omega^1_{H'}\to 0$$

we have an $H'$-morphism

$$p:(H'\times_M \mathbf{P}(\Omega^{1\vee}_M)-C(H',M))\longrightarrow \mathbf{P}(\Omega^{1\vee}_{H'}),$$

which induces an isomorphism

$$q^o:(X^{\mathrm{sm}}\cap H')\times_X C(X,M)\xrightarrow{\sim} C(X\cap H',H')\times_{X\cap H'}(X^{\mathrm{sm}}\cap H').$$

From (2.4.4.1), we have that the morphism $q^o$ can be extended to the closure. i.e., we have an $X\cap H'$-morphism

$$q: X\widetilde{\cap}H'\to C(X\cap H',H')$$

where

$$X\widetilde{\cap}H':=\ \text{closure of}\ C(X,M)\times_X(X^{\mathrm{sm}}\cap H')\ \text{ in }\ C(X,M).$$

We consider the following diagram

$$\begin{tikzcd}
\check{\mathbf{P}}^{m-1} & \check{\mathbf{P}}^{m-1}-\{H\} \arrow[l] \arrow[r, "p"] & \check{\mathbf{P}}^{m-2} \\
C(X,M) \arrow[u, "\lambda_X"] \arrow[d, "\kappa_X"'] & X\widetilde{\cap}H' \arrow[l] \arrow[r, "q"] \arrow[u] \arrow[d] & C(X\cap H',H') \arrow[u, "\lambda_{X\cap H'}"'] \arrow[d, "\kappa_{X\cap H'}"] \\
X & X\cap H' \arrow[l] \arrow[r, "\sim"] & X\cap H'
\end{tikzcd}$$

Since $D_{d-k+1}$ is contained in $H$, we have that

$$L^{d-k}=\ \text{closure of}\ p^{-1}(L^{d-k-1})\ \text{ in }\ \check{\mathbf{P}}^{m-1}$$

where $L^{d-k}$ is the subspace of $\check{\mathbb{P}}^{m-1}$ parametrizing the hyperplanes of $\mathbb{A}^m$ containing $D_{d-k+1}$ and, $L^{d-k-1}$ is the subspace of $\check{\mathbb{P}}^{m-2}$ parametrizing the hyperplanes of $H$ containing $D_{d-k-1}$. Therefore we have that

$$\lambda_X(L^{d-k}) \cap (X \widetilde{\cap} H') = q^{-1}(\lambda_{X \cap H'}^{-1}(L^{d-k-1})).$$

It follows from Proposition 2.2.2 that

$$\kappa_X(\lambda_X^{-1}(L^{d-k}) \cap (X \widetilde{\cap} H')) = P_k(X, x, D_{d-k-1}) \widehat{\cap} H'$$

and

$$\kappa_X(q^{-1}(\lambda_{X \cap H'}(L^{d-k-1}))) = P_k(X \cap H', x, D_{d-k-1}).$$

This prove $(i)$. The second equality of $(ii)$ follows from $(i)$ and the first equality follows from [28, Theorem 4.2]. □

**Lemma 2.4.5** *Let $N$ be a smooth scheme and let $Z$ be a pure dimensional closed subscheme of $N$. Let $Z'$ be a pure dimensional scheme and let $p : Z' \to Z$ be a proper morphism. Let $F_k = \{x' \in Z';\ \ \dim_{x'} p^{-1}(p(x')) \geq k\}$ and denote by $B_k$ its image under the morphism $p$. Assume that $B_k$ is proper in $Z$ and that $p^{-1}(B_k)$ is proper in $Z'$. Let $H$ be a non-singular closed subscheme of $N$ such that $H$ is dimensionwise transverse to $Z$ and to every irreducible component of $B_k$ for every $k$. Then, the inverse image of $H \cap Z$ via $p$ coincides, set-theoretically, with the closure of $p^{-1}((Z - B_k) \cap H)$ in $Z'$.*

**Proof** It is enough to prove that for every point $x$ of $p^{-1}(H \cap Z)$, $\dim_x p^{-1}(H \cap B_k)$ is strictly smaller than the minimum of the dimension of the non-imbedded irreducible components of $p^{-1}(H \cap Z)$ through $x$. Because, in this case, $p^{-1}(H \cap B_k)$ is nonwhere dense in $p^{-1}(H \cap Z)$ and, therefore, $p^{-1}(H \cap (Z - B_k))$ is dense in $p^{-1}(H \cap Z)$.

Fix an integer $k$. If $j \geq k$ then $p^{-1}(B_j) \subseteq p^{-1}(B_k)$. Therefore we have that

$$j + \dim B_j \leq \dim F_j \leq \dim p^{-1}(B_j) \leq \dim p^{-1}(B_k)$$

So,

$$\dim B_j \leq \dim p^{-1}(B_k) - j$$

Now, since $H$ is transverse to every irreducible component of $B_j$, we have that

$$\dim(H \cap B_j) = \dim B_j - h \leq \dim p^{-1}(B_k) - j - h$$

where $h$ is the codimension of $H$ in $N$. Since

$$p^{-1}(H \cap B_k) = \bigcup_{j \geq k} p^{-1}(H \cap (B_j - B_{j+1}))$$

we have that

$$\dim_x p^{-1}(H \cap B_k) \leq \dim_x p^{-1}(B_k) - h$$

Let $(p^{-1}(H \cap Z))_\alpha$ be a non-imbedded irreducible component of $p^{-1}(H \cap Z)$ through $x$. Then we have that

$$\dim_x (p^{-1}(H \cap Z))_\alpha \geq \dim_x Z' - h$$

Therefore we have that

$$\begin{aligned} \dim_x p^{-1}(H \cap B_k) &\leq \dim_x p^{-1}(B_k) - h \\ &< \dim_x Z' - h \\ &\leq \dim_x (p^{-1}(H \cap Z))_\alpha \end{aligned}$$

$\square$

**Lemma 2.4.6** *Let $H$ be a smooth closed subscheme $M$ of pure codimension $j$, which satisfies the following transversality conditions,*

$$\begin{aligned} C(H, M) \cap C(X, M) &= \emptyset \\ C(H, M) \cap C(Y, M) &= \emptyset \end{aligned}$$

*Assume that the morphism $\xi_Y : D_Y \to Y$ is equidimensional. Then, the morphism $\xi_{Y \cap H} : D_{Y \cap H} \to Y \cap H$ is also equidimensional, where $D_Y$ and $D_{Y \cap H}$ are as in (2.4.2).*

**Proof** We will use the notation of the proof of (2.4.4). Consider the following commutative diagram

$$\begin{tikzcd} \mathrm{Bl}_{\kappa_X^{-1}(Y)}C(X,M) \arrow[d, "\hat{e}_X"'] & \widehat{X\cap H} \arrow[l] \arrow[r, "\hat{q}"] \arrow[d] & \mathrm{Bl}_{\kappa_{X\cap H}^{-1}(Y\cap H)}C(X\cap H,H) \arrow[d, "\hat{e}_{X\cap H}"] \\ C(X,M) \arrow[d, "\kappa_X"'] & \widetilde{X\cap H} \arrow[l] \arrow[r, "q"] \arrow[d] & C(X\cap H,H) \arrow[d, "\kappa_{X\cap H}"] \\ X & X\cap H \arrow[l] \arrow[r, "\sim"] & X\cap H \end{tikzcd}$$

where $\widehat{X\cap H}$ is the strict transform of $X\cap H$ via the morphism $\xi := \kappa_X \circ \hat{e}_X$, which is equal to the strict transform of $\widetilde{X\cap H}$ via $\hat{e}_{X\cap H}$, and $\hat{q}$ is the morphism constructed by the universal property of the blow-up.

Since the morphism $\xi_Y : D_Y \to Y$ is equidimensional and $C(H,M)\cap C(Y,M) = \emptyset$ we have, by Lemma 2.4.5 applied to the morphism $\xi : \mathrm{Bl}_{\kappa^{-1}(Y)}C(X,M) \to X$, that

$$\widehat{X\cap H} = \mathrm{Bl}_{\kappa^{-1}(Y)}C(X,M) \times_X (X\cap H).$$

Since the morphism $q$ is induced by the projection

$$p : (H \times_M \mathsf{P}(\Omega_M^{1\vee}) - C(H,M)) \longrightarrow \mathsf{P}(\Omega_H^{1\vee})$$

we have, by a general fact of central projections, that the morphism $q$ is finite. Since the morphism $\hat{q}$ is induced by the projection

$$\widetilde{X\cap H} \times_{C(X\cap H,H)} \mathrm{Bl}_{\kappa^{-1}(Y\cap H)}C(X\cap H,H) \longrightarrow \mathrm{Bl}_{\kappa^{-1}(Y\cap H)}C(X\cap H,H),$$

we have, from the finiteness of $q$, that the morphism $\hat{q}$ is also finite.

Therefore, we have a finite $Y\cap H$-morphism

$$\tilde{q} : D_Y \times_Y (Y\cap H) \longrightarrow D_{Y\cap H}.$$

The conclusion of the lemma follows from the finiteness of the morphism $\tilde{q}$. □

For each integer $k$ for $0 \leq k \leq d-1$, let $m_x(P_k(X,x))$ denote the multiplicity of a generic local polar variety of codimension $k$ of $X$ at the point $x$, as in Lemma 2.3.5. We will denote the sequence of this multiplicities by $M_x^*(X)$. i.e.,

$$M_x^*(X) := \{m_x(P_0(X,x)); m_x(P_1(X,x)); \ldots ; m_x(P_{d-1}(X,x))\}$$

**Theorem 2.4.7** ([*33*, V, 1.2]). *The following conditions are equivalents.*

*(i) The morphism $\xi_Y : D_Y \to Y$ is equidimensional*

*(ii) The sequence $M_x^*(X)$ is independent of the point $x$ in $Y$.*

**Proof** Assume $(i)$. In order to prove that $(ii)$ holds we will reduce to the case when $Y$ has dimension 1, by showing that there exists a codimension $t-1$ nonsingular subscheme $G'$ of $M$ satisfying the transversality conditions of Lemma 2.4.6. We will show the existence of $G'$ by proving that the polar variety of codimension bigger than $d-t$ is empty. Without lost of generality we may assume that there is a global étale morphism $g : M \to \mathbb{A}^m$ satisfying the conditions of Setup (2.4.1). By [4, 4.2.(a)], we have that

$$\{s(\kappa^{-1}(x), C(X))\}_{k+m-d-1} = \hat{e}_*\{s(D_Y(x), \mathrm{Bl}_{\kappa^{-1}(Y)}C(X))\}_{k+m-d-1}$$

Therefore, by Lemma 2.3.4 and the projection formula, we have that

$$m_x(P_k(X,x)) = (-1)^{k+m-d-1} \int c_{k+m-d-1}(\hat{e}^*(\mathrm{taut})) \cap \{s(D_Y(x), \mathrm{Bl}_{\kappa^{-1}(Y)}C(X))\}_{k+m-d-1} \quad (2.4.7.1).$$

Since the morphism $\xi_Y : D_Y \to Y$ is equidimensional, we have that

$$\dim D_Y(x) = m - 2 - t. \qquad (2.4.7.2)$$

Therefore, since $s(D_Y(x), \mathrm{Bl}_{\kappa^{-1}(Y)}C(X))$ is supported in $D_Y(x)$, we have by (2.4.7.1) and (2.4.7.2) that

$$m_x(P_k(X,x)) = 0 \text{ for } k \geq d - t \qquad (2.4.7.3)$$

or, equivalently,

$$P_k(X,x)) = \emptyset \text{ for } k \geq d-t$$

Therefore, by the same reasoning as in the proof of Lemma 2.4.3 and Lemma 2.4.4, we can find a general linear subspace $G$ of $\mathbb{A}^m$ of codimension $t-1$ such that

$$\begin{aligned} m_x(P_k(X,x)) &= m_x(P_k(X \cap G', x)) \text{ for } 0 \leq k \leq 0 \\ C(G', M) \cap C(X, M) &= \emptyset \\ C(G', M) \cap C(Y, M) &= \emptyset \end{aligned} \tag{2.4.7.4}$$

where $G' := g^{-1}(G)$.

Therefore, by Lemma 2.4.6, we have that the morphism

$$\xi_{Y \cap G'} : D_{Y \cap G'} \longrightarrow Y \cap G' \tag{2.4.7.5}$$

is equidimensional.

Therefore, by (2.4.7.3), (2.4.7.4) and (2.4.7.5), we may assume that $Y$ is 1-dimensional. In this case, consider the following diagram.

$$\begin{array}{ccccc} D(x) & \xrightarrow{j} & D & \xrightarrow{\alpha} & \overline{\mathrm{Bl}_{\kappa^{-1}(Y)} C(X)} \\ \downarrow & & \downarrow & & \downarrow \eta \\ D_Y(x) & \longrightarrow & D_Y & \longrightarrow & \mathrm{Bl}_{\kappa^{-1}(Y)} C(X) \\ \downarrow & & \downarrow & & \downarrow \xi \\ \{x\} & \xrightarrow{\iota} & Y & \longrightarrow & X \end{array}$$

where $\eta : \overline{\mathrm{Bl}_{\kappa^{-1}(Y)} C(X)} \longrightarrow \mathrm{Bl}_{\kappa^{-1}(Y)} C(X)$ is the normalization of $\mathrm{Bl}_{\kappa^{-1}(Y)} C(X)$ and $D$ is the inverse image of $D_Y$ via $\eta$.

Since $\xi_Y : D_Y \to Y$ is equidimensional and $\eta$ is a finite morphism, we have that no irreducible component of $D$ is contained in $D(x)$. Therefore, since $\overline{\mathrm{Bl}_{\kappa^{-1}(Y)} C(X)}$ is normal, we have that $\iota$, $j$ and $\alpha$ are regular imbeddings of codimension 1. Therefore, we have an exact sequence of vector bundles on $D(x)$

$$0 \to N_{D(x)} D \to N_{D(x)} \overline{\mathrm{Bl}_{\kappa^{-1}(Y)} C(X)} \to j^* N_D \overline{\mathrm{Bl}_{\kappa^{-1}(Y)} C(X)} \to 0.$$

Since $N_{D(x)}D$ is the pullback of $N_{\{x\}}Y$, which is a trivial line bundle, we have, by Whitney sum, that

$$c(N_{D(x)}\overline{\mathrm{Bl}_{\kappa^{-1}(Y)}C(X)}) = c(j^* N_D \overline{\mathrm{Bl}_{\kappa^{-1}(Y)}C(X)}) \qquad (2.4.7.6)$$

where $c$ denotes the total Chern class. Therefore we have that

(2.4.7.7) :

$$\{s(D(x), \overline{\mathrm{Bl}_{\kappa^{-1}(Y)}C(X)})\}_k$$

$$= \{c(N_{D(x)}\overline{\mathrm{Bl}_{\kappa^{-1}(Y)}C(X)})^{-1} \cap [D(x)]\}_k \qquad ([4, 4.1(a)])$$

$$= \{c(j^* N_D \overline{\mathrm{Bl}_{\kappa^{-1}(Y)}C(X)})^{-1} \cap [D(x)]\}_k \qquad (2.4.7.6)$$

$$= (-1)^{m-3-k} c_1(j^* N_D \overline{\mathrm{Bl}_{\kappa^{-1}(Y)}C(X)})^{m-3-k} \cap [D(x)] \qquad \text{(dimension counting)}$$

$$= ((-1)^{m-3-k} c_1(N_D \overline{\mathrm{Bl}_{\kappa^{-1}(Y)}C(X)})^{m-3-k} \cap [D])_x \qquad ([4, 10.1])$$

$$= (\{c(N_D \overline{\mathrm{Bl}_{\kappa^{-1}(Y)}C(X)})^{-1} \cap [D]\}_{k+1})_x \qquad \text{(dimension counting)}$$

$$= (\{s(D, \overline{\mathrm{Bl}_{\kappa^{-1}(Y)}C(X)})\}_{k+1})_x \qquad ([4, 4.1(a)])$$

Therefore we have that

$$(-1)^{k+m-d-1} m_x(P_k(X, x))$$

$$= \int c_{k+m-d-1}(\hat{e}^*(\mathrm{taut})) \cap \{s(D_Y(x), \mathrm{Bl}_{\kappa^{-1}(Y)}C(X))\}_{k+m-d-1} \qquad (2.4.7.1)$$

$$= \int c_{k+m-d-1}(\eta^* \hat{e}^*(\mathrm{taut})) \cap \{s(D(x), \overline{\mathrm{Bl}_{\kappa^{-1}(Y)}C(X)})\}_{k+m-d-1} \qquad ([4, 4.2(a)])$$

$$= \int c_{k+m-d-1}(\eta^* \hat{e}^*(\mathrm{taut})) \cap (\{s(D, \overline{\mathrm{Bl}_{\kappa^{-1}(Y)}C(X)})\}_{k+m-d})_x \qquad (2.4.7.7)$$

$$= \int (c_{k+m-d-1}(\eta^* \hat{e}^*(\mathrm{taut})) \cap \{s(D, \overline{\mathrm{Bl}_{\kappa^{-1}(Y)}C(X)})\}_{k+m-d})_x \qquad ([4, 10.1])$$

Since $\xi \circ \eta : D \to Y$ is a proper morphism we have, by conservation of number ( [4, 10.2]) applied to the 1-cycle $\alpha$ in $A_1(D)$, where

$$\alpha = c_{k+m-d-1}(\eta^* \hat{e}^* \mathrm{taut}) \cap \{s(D, \overline{\mathrm{Bl}_{\kappa^{-1}(Y)} C(X)})\}_{k+m-d},$$

that $m_x(P_k(X,x))$ is independent of the point $x$ in $Y$. In other words $(ii)$ holds.

Conversely, let $x$ be a closed point of $Y$, and let $Z$ be an irreducible component of $D_Y(x)$. We have to show that

$$\dim Z \leq m - 2 - t$$

Since this is a local condition, we may assume that we have a commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "h"'] & M \arrow[d, "g"] \\ \mathbb{A}^t \arrow[r] & \mathbb{A}^m \end{tikzcd}$$

with $g$ and $h$ étale morphisms such that $h(x) = g(x) = 0$. Under this assumption we have the following diagram

$$\begin{tikzcd} X \times \mathbf{P}^{m-t-1} \times \check{\mathbf{P}}^{m-1} \arrow[d] & \mathrm{Bl}_{\kappa^{-1}(Y)} C(X) \arrow[l] \arrow[r, "\hat{e}"] \arrow[d, "\kappa'"'] & C(X) \arrow[r] \arrow[d, "\kappa"] & X \times \check{\mathbf{P}}^{m-1} \arrow[dl] \\ X \times \mathbf{P}^{m-t-1} & \mathrm{Bl}_Y X \arrow[l] \arrow[r, "e"] & X & \end{tikzcd}$$

We also have that

$$m - 1 - d \leq \dim \kappa^{-1}(x) \leq \dim \kappa^{-1}(Y) \leq m - 2.$$

The first inequality is due to the theorem of dimension of the fibers of a morphism and, the last inequality is due to the assumption that $Y$ is properly contained in $X$. Therefore we have that

$$\dim(\hat{e}(Z)) = k + m - d - 1$$

for some integer $k$ with $0 \leq k \leq d - 1$.

By Kleiman's transversality lemma ([17]), we can choose $L^{d-k}$ in the open dense set $V_k$ of Proposition 2.2.2 such that

$$\dim(\hat{e}(Z) \cap \lambda^{-1}(L^{d-k})) = 0$$

where $\lambda : C(X) \to \check{\mathbf{P}}^{m-1}$ is induced by the projection. Also, we can choose $L^{d-k}$ such that the above intersection lies in the open dense subscheme of $\hat{e}(Z)$ over which the morphism $\hat{e} : Z \to \hat{e}(Z)$ is equidimensional. Therefore we have that

$$\dim \hat{e}^{-1}(\hat{e}(Z) \cap \lambda^{-1}(L^{d-k})) = \dim Z - (k + m - d - 1) \qquad (2.4.7.8).$$

By Kleiman's transversality lemma we can choose $L^{d-k}$ in such a way that

$$\hat{e}^{-1}(\lambda^{-1}(L^{d-k})) = \overline{P_k(X, x, D_{d-k+1})} \qquad (2.4.7.9),$$

where $D_{d-k+1}$ is as in Proposition 2.2.2 and, $\overline{P_k(X, x, D_{d-k+1})}$ is the strict transform of $P_k(X, x, D_{d-k+1})$ via the morphism $\xi$ (see Remark 2.2.3 or [12, 3.2.1]). Therefore we have that

$$\hat{e}^{-1}(\hat{e}(Z) \cap \lambda^{-1}(L^{d-k+1})) = Z \cap \overline{P_k(X, x, D_{d-k+1})} \qquad (2.4.7.10).$$

Since $Z \subseteq D_Y(x)$, we have that

$$\kappa'(Z \cap \overline{P_k(X, x, D_{d-k+1})}) \subseteq (\mathbf{P}(C_Y P_k(X, x, D_{d-k+1})))_x \qquad (2.4.7.11).$$

Since the sequence $M_x^*(X)$ is independent of the point $x$ of $Y$, we have (see [24, Theorem 4.(b)]) that the morphism

$$e : \mathbf{P}(C_Y P_k(X, x, D_{d-k+1})) \longrightarrow P_k(X, x, D_{d-k+1})$$

is equidimensional. Therefore, if $P_k(X, x, D_{d-k+1})$ is not empty, we have that

$$\dim(\mathbf{P}(C_Y P_k(X, x, D_{d-k+1})))_x = d - k - t - 1. \qquad (2.4.7.12)$$

Therefore we have that

$$
\begin{aligned}
&\dim Z \\
&= \dim \hat{e}^{-1}(\hat{e}(Z) \cap \lambda^{-1}(L^{d-k})) + (k+m-d-1) && (2.4.7.8) \\
&= \dim(Z \cap \overline{P_k(X,x,D_{d-k+1})}) + (k+m-d-1) && (2.4.7.10) \\
&= \dim \kappa'(Z \cap \overline{P_k(X,x,D_{d-k+1})}) + (k+m-d-1) \\
&\leq \dim(\mathsf{P}(C_Y P_k(X,x,D_{d-k+1})))_x + (k+m-d-1) && (2.4.7.11) \\
&= m-2-t && (2.4.7.12)
\end{aligned}
$$

$\square$

# Chapter 3

# Whitney Conditions and Projective Duality

## 3.1 Lagrangian Theory

([18]; [29])

**Setup 3.1.1** *Fix a base scheme $S$ over an algebraically closed field $K$, and a smooth ambient scheme $Z$ of constant relative dimension $N \geq 1$. Form the projectivization of the relative cotangent bundle*

$$I := \mathsf{P}(\Omega^{1\vee}_{Z/S})$$

*and denote its structure map by*

$$p : I \to Z$$

*The bundle $I$ carries two canonical maps, whose composition*

$$\omega_S : \mathcal{O}_I(-1) \to p^*\Omega^1_{Z/S} \to \Omega^1_{I/S}$$

*is called the $S$-*contact form *(see[18, 3.1]).*

**Definition 3.1.2** *Let $g : C \to I$ be an $S$-map. For any map of $\mathcal{O}_I$-modules*

$$\eta_S : L \to \Omega^1_{I/S},$$

*Let $\eta_S|C$ stand for the composition $\partial g \circ g^*\eta_S$*

$$\eta_S|C : g^*L \to g^*\Omega^1_{I/S} \to \Omega^1_{C/S}$$

*Then $C$ or $C/I$ will be said to satisfy the (twisted first order partial differential) equation $\eta_S = 0$ if $\eta_S|C$ vanishes on a $S$-smooth, dense open subscheme $C^0$ of $C$. If $C$ satisfies the equation $\omega_S = 0$ and if $C^0/S$ has pure relative dimension $N-1$, then $C$ will be called $S$-Lagrangian.*

**Remark 3.1.3** *Analogously, we can define the concept of (absolute) Lagrangian as follows:*

*Let $J := \mathbf{P}(\Omega^{1\vee}_Z)$ and $q : J \to Z$ its structural morphism. The bundle $J$ carries two canonical morphisms, whose composition*

$$\omega : \mathcal{O}_J(-1) \to q^*\Omega^1_Z \to \Omega^1_J$$

*is called the (absolute) contact form.*

*Let $g : C \to J$ be a morphism. For any map of $\mathcal{O}_J$-modules*

$$\eta : L \to \Omega^1_J$$

*let $\eta|C$ stand for the composition $\partial g \circ g^*\eta$*

$$\eta|C : g^*L \to g^*\Omega^1_J \to \Omega^1_C.$$

*Then $C$ will be said to satisfy the equation $\eta = 0$ if the morphism $\eta|C$ vanishes on a smooth, dense open subscheme $C^0$ of $C$. If $C$ satisfies the equation $\omega = 0$ and if $C^0$ has pure dimension equal to $\dim Z - 1$, then, $C$ will be called (absolute) Lagrangian.*

**Example 3.1.4** *([18, Proposition 3.2]). Let $f : V \to Z$ be an $S$-map. Let $V^0$ denote the largest open subscheme of $V$ on which $V/S$ is smooth and on which $f$ is an immersion; the latter condition means that on $V^0$ the Jacobian map*

$$\partial f : f^*\Omega^1_{Z/S} \to \Omega^1_{V/S}$$

*is surjective. Assume that $V^0$ is scheme-theoretically dense in $V$. Then the scheme $C(V/S, Z/S)$ defined by*

$$C(V/S, Z/S) := \text{ closure of } \mathbf{P}((ker(\partial f|V^0))^\vee) \text{ in } V \times_Z I$$

*is $S$-Lagrangian.*

**Lemma 3.1.5** *In the Setup of (3.1.1), assume that the ground field $K$ has characteristic zero. Denote by $h : Z \to S$ the structural morphism. From the exact sequence of $\mathcal{O}_Z$-modules*

$$0 \to h^*\Omega^1_S \to \Omega^1_Z \to \Omega^1_{Z/S} \to 0$$

*there is a natural projection*

$$\varphi : [\mathbf{P}(\Omega^{1\vee}_Z) - \mathbf{P}(h^*\Omega^{1\vee}_S)] \longrightarrow \mathbf{P}(\Omega^{1\vee}_{Z/S}).$$

*Let $J := \mathbf{P}(\Omega^{1\vee}_Z)$ and let $\omega$ denotes its contact form (as in Remark 3.1.3). If $g : C \to J$ satisfies the equation $\omega = 0$ (in the sense of Remark 3.1.3), and the image of $C$ in $J$ does not meet the center of the projection $\varphi$, then $\varphi \circ g : C \to I$ is well defined, and satisfies the equation $\omega_S = 0$, where $I$ and $\omega_S$ are as in (3.1.1).*

**Proof** If $g : C \to J$ satisfies the equation $\omega = 0$ then there exists a smooth open dense subscheme $C^0$ of $C$ over which $\omega|C$ vanishes, where $\omega|C$ is the composition

$$\omega|C : g^*\mathcal{O}_J(-1) \to g^*\Omega^1_J \to \Omega^1_C.$$

By generic smoothness ([8, III,10.7]), we may assume that $C^0$ is $S$-smooth.

Set $U := J - \mathbf{P}(h^*\Omega_S^{1\vee})$. Then, by [8, II,5.12], we have that

$$\varphi^*\mathcal{O}_I(-1) = \mathcal{O}_J(-1)|_U \qquad (3.1.5.1).$$

Since the image of $C$ under $g$ does not meet $\mathbf{P}(h^*\Omega_S^{1\vee})$, we have a well defined morphism

$$\varphi \circ g : J \longrightarrow I$$

and, by (3.1.5.1), we have that

$$g^*\varphi^*\mathcal{O}_I(-1) = g^*\mathcal{O}_J(-1) \qquad (3.1.5.2).$$

Now, $\omega|C$ induces a section, which we also denote by $\omega|C$, of the $\mathcal{O}_C$-module $g^*\mathcal{O}_J(1) \otimes \Omega_C^1$, i.e.,

$$\omega|C \in H^0(C, g^*\mathcal{O}_J(1) \otimes \Omega_C^1).$$

Analogously,

$$\omega_S|C \in H^0(C, g^*\varphi^*\mathcal{O}_I(1) \otimes \Omega_{C/S}^1).$$

From the exact sequence of $\mathcal{O}_C$-modules

$$\Omega_C^1 \to \Omega_{C/S}^1 \to 0$$

and from (3.1.5.2), we have a map

$$\alpha_* : H^0(C, g^*\mathcal{O}_J(1) \otimes \Omega_C^1) \longrightarrow H^0(C, g^*\varphi^*\mathcal{O}_I(1) \otimes \Omega_{C/S}^1)$$

which sends $\omega|C$ to $\omega_S|C$.

Since $\omega|C$ vanishes on $C^0$, $\omega_S|C$ vanishes too. □

**Definition 3.1.6** *We define the relative affine cotangent scheme $T^*(Z/S)$ of $Z$ over $S$ by*

$$T^*(Z/S) := \mathrm{Spec}\,(\mathrm{Sym}\,\Omega_{Z/S}^{1\vee})$$

*Let $q : T^*(Z/S) \to Z$ be the structural morphism. Set $T = T^*(Z/S)$. The bundle $T$ carries two canonical maps, whose composition*

$$\alpha_S : \mathcal{O}_T \to q^*\Omega^1_{Z/S} \to \Omega^1_{T/S}$$

*is called the* affine $S$-contact form *on $T^*(Z/S)$*

**Definition 3.1.7** *Let $g : C \to T$ be an $S$-map. For any map of $\mathcal{O}_T$-modules*

$$\eta^p_S : L \to \Omega^p_{T/S}$$

*let $\eta^p_S|C$ stand for the composition $(\wedge^p \partial g) \circ g^*\eta^p_S$*

$$\eta^p_S|C : g^*L \to g^*\Omega^p_{T/S} \to \Omega^p_{C/S}$$

*Then $C$ or $C/T$ will be said to satisfy the (twisted p-order partial differential) equation $\eta^p_S = 0$ if $\eta^p_S|C$ vanishes on a $S$-smooth, dense open subscheme $C^0$ of $C$. If $C$ satisfies the (2-order partial differential) equation $d\alpha_S = 0$ and if $C^0$ has pure relative dimension $N$, then $C$ will be called* affine $S$-Lagrangian *in $T$.*

**Remark 3.1.8** *Analoguosly we can define the concept of (absolute)* affine Lagrangian *scheme as follows:*

*Let $T^*(Z) := \mathrm{Spec}(\mathrm{Sym}\,\Omega^{1\vee}_Z)$ be the absolute cotangent scheme of $Z$. Let $r : T^*(Z) \to Z$ be the structural morphism. Set $R = T^*(Z)$. The bundle $R$ carries two canonical maps, whose composition*

$$\alpha : \mathcal{O}_R \to r^*\Omega^1_Z \to \Omega^1_R$$

*is called the (absolute) affine contact form on $T^*(Z)$.*

*Let $g : C \to R$ be a morphism. For any map of $\mathcal{O}_R$-modules*

$$\eta^p : L \to \Omega^p_R$$

*let* $\eta^p|C$ *stand for the composition* $(\wedge^p \partial g) \circ g^* \eta^p$

$$\eta^p|C : g^*L \to g^*\Omega^p_R \to \Omega^p_C$$

*Then* $C$ *or* $C/R$ *will be said to satisfy the (twisted p-order partial differential) equation* $\eta^p = 0$ *if* $\eta^p|C$ *vanishes on a smooth, dense open subscheme* $C^0$ *of* $C$. *If* $C$ *satisfies the (2-order partial differential) equation* $d\alpha = 0$ *and if* $C^0$ *has pure dimension equal to* $\dim Z$, *then* $C$ *will be called (absolute)*affine Lagrangian *in* $R$.

**Example 3.1.9** *Let* $f : V \to Z$ *and* $V^0$ *as in Example* 3.1.4. *Then the scheme* $T^*_{(V/S)}(Z/S)$ *defined by*

$$T^*_{(V/S)}(Z/S) := \text{ closure of } \operatorname{Spec}\left((ker(\partial f|V^0))^\vee\right) \text{ in } V \times_Z T$$

*is an affine* $S$*-Lagrangian subscheme of* $T$.

**Notation 3.1.10** *Sometimes we will denote the projective cotangent bundle* $\mathbf{P}(\Omega^{1\,\vee}_{Z/S})$ *by* $\mathbf{P}(T^*(Z/S))$ *and, the projective conormal scheme* $C(V/S, Z/S)$ *by* $\mathbf{P}\left(T^*_{(V/S)}(Z/S)\right)$.

Let $\sigma : Z \to T^*(Z/S)$ be the zero-section imbedding of $Z$ in $T^*(Z/S)$. Let $\dot{T}^*(Z/S) := T^*(Z/S) - \sigma(Z)$. Set $\dot{T} = \dot{T}^*(Z/S)$. Let

$$\pi : \dot{T} \longrightarrow I$$

be the natural projection, where $I = \mathbf{P}(\Omega^{1\,\vee}_{Z/S})$ as in the Setup 3.1.1. Let $g : C \to T$ be an $S$-morphism, let $\dot{C} := g^{-1}(\dot{T})$ and let $\dot{g} : \dot{C} \to \dot{T}$ be the induced morphism.

**Lemma 3.1.11** *([18, 3.6];[29, Proposition 10.1]). Assume that the ground field* $K$ *has characteristic zero. Assume that* $g : C \to T$ *is a closed imbedding and that the image of* $C$ *is conic in* $T$ *(i.e., stable under the action of* $\mathbb{G}_m$ *on* $T$*). Denote by* $\mathbf{P}(C)$ *the image of* $\dot{C}$ *in* $I$. *Let* $V$ *the image of* $C$ *in* $Z$, *equipped with the induced reduced* *closed subscheme structure. Then, the following statements are equivalent.*

*(i) $C$ is an affine $S$-Lagrangian subscheme of $T$.*

*(ii) $C = T^*_{(V/S)}(Z/S)$*

*(iii) $\mathsf{P}(C) = C(V/S, Z/S)$*

*(iv) $\mathsf{P}(C)$ is a (projective) $S$-Lagrangian subscheme of $I$.*

**Proof** The equivalence $(i) \Leftrightarrow (ii)$ follows from [29, Proposition 10.1]. The equivalence $(ii) \Leftrightarrow (iii)$ follows from the fact that $C$ is conic. Finally, the equivalence $(iii) \Leftrightarrow (iv)$ follows from [18, Corollary 3.6]. □

## 3.2 Deformation to the Normal Cone

([4, 5.1])

**Setup 3.2.1** *Let $Y$ be a closed subscheme of a scheme $M$. Let $\mathcal{I}$ be the ideal sheaf of $Y$ in $M$. We define the normal cone of $Y$ in $M$, $T_Y M$, by*

$$T_Y M := \operatorname{Spec}\left(\oplus_{n\geq 0}\mathcal{I}^n/\mathcal{I}^{n+1}\right).$$

We will construct a scheme $\operatorname{Def}(M, Y)$, together with a closed imbedding of $Y \times \mathsf{A}^1$ in $\operatorname{Def}(M, Y)$, and a flat morphism

$$\rho : \operatorname{Def}(M, Y) \longrightarrow \mathsf{A}^1$$

such that the following diagram

$$\begin{array}{ccc} Y \times \mathsf{A}^1 & \subset & \operatorname{Def}(M, Y) \\ p_2 \searrow & & \swarrow \rho \\ & \mathsf{A}^1 & \end{array}$$

commutes, and such that

(*i*) over $\mathbb{A}^1 - \{0\}$, $\rho^{-1}(\mathbb{A}^1 - \{0\}) = M \times (\mathbb{A}^1 - \{0\})$ and the imbedding is the trivial imbedding

$$Y \times (\mathbb{A}^1 - \{0\}) \subset M \times (\mathbb{A}^1 - \{0\})$$

(*ii*) over $\{0\}$, the divisor $\rho^{-1}(0)$ is equal to $T_Y M$.

The construction of the scheme $\mathrm{Def}\,(M, Y)$ goes as follows:

Let $\mathrm{Bl}_{Y\times\{0\}}M \times \mathbb{A}^1$ be the blow-up of $M \times \mathbb{A}^1$ along $Y \times \{0\}$. From the sequence of imbeddings

$$Y \times \{0\} \subset M \times \{0\} \subset M \times \mathbb{A}^1$$

the blow-up $\tilde{M}$ of $M$ along $Y$ is imbedded as a closed subscheme of $\mathrm{Bl}_{Y\times\{0\}}M \times \mathbb{A}^1$. Let

$$\mathrm{Def}\,(M, Y) := \mathrm{Bl}_{Y\times\{0\}}M \times \mathbb{A}^1 - \tilde{M}$$

From the sequence of imbeddings

$$Y \times \{0\} \subset Y \times \mathbb{A}^1 \subset M \times \mathbb{A}^1$$

the blow-up of $Y \times \mathbb{A}^1$ along $Y \times \{0\}$ is imbedded as a closed subscheme of $\mathrm{Bl}_{Y\times\{0\}}M \times \mathbb{A}^1$. Since $Y \times \{0\}$ is imbedded as a Cartier divisor in $Y \times \mathbb{A}^1$, we have that

$$\mathrm{Bl}_{Y\times\{0\}}Y \times \mathbb{A}^1 = Y \times \mathbb{A}^1$$

The exceptional divisor of $\mathrm{Bl}_{Y\times\{0\}}M \times \mathbb{A}^1$ is

$$\mathbf{P}(T_Y M \oplus \mathbf{1}) := \mathrm{Proj}\left(\bigoplus_{n\geq 0}\left(\bigoplus_{j=0}^{n}(\mathcal{I}^j/\mathcal{I}^{j+1})\cdot t^{n-j}\right)\right)$$

The imbedding of $Y \times \{0\}$ in $\mathbf{P}(T_Y M \oplus \mathbf{1})$ is the zero-section imbedding of $Y$ in $T_Y M$ followed by the canonical open imbedding of $T_Y M$ in $\mathbf{P}(T_Y M \otimes \mathbf{1})$.

The divisors $\mathsf{P}(T_Y M \oplus \mathbf{1})$ and $\tilde{M}$ intersects in the scheme $\mathsf{P}(T_Y M)$. Therefore, the imbedding of $Y \times \mathbb{A}^1$ in the blow-up of $M \times \mathbb{A}^1$ along $Y \times \{0\}$ do not intersect $\tilde{M}$.

In other words, we have an imbedding

$$Y \times \mathbb{A}^1 \subset \mathrm{Def}(M, Y).$$

The morphism $\rho : \mathrm{Def}(M, Y) \to \mathbb{A}^1$ is the restriction of the composition

$$\mathrm{Bl}_{Y \times \{0\}} M \times \mathbb{A}^1 \to M \times \mathbb{A}^1 \xrightarrow{p_2} \mathbb{A}^1$$

to $\mathrm{Def}(M, Y)$.

**Proposition 3.2.2** ([*20*, *Proposition*4.4.2]; [*30*, *Appendix*4]). *Let $Y$ and $X$ be two closed subschemes of a smooth scheme $M$. There exists canonical isomorphisms over $\mathbb{A}^1$.*

$$\begin{array}{lll} (1) & \theta: & \mathrm{Def}(T^*M; T^*_Y M) \longrightarrow T^*\left(\mathrm{Def}(M, Y)/\mathbb{A}^1\right) \\ (2) & \psi: & \mathrm{Def}(T^*_X M; T^*_X M \cap T^*_Y M) \longrightarrow T^*_{\left(\mathrm{Def}(X, X\cap Y)/\mathbb{A}^1\right)}\left(\mathrm{Def}(M, Y)/\mathbb{A}^1\right) \end{array}$$

□

## 3.3 Integral Dependence over Ideal Sheaves

([23])

**Setup 3.3.1** *Let $Z$ be a noetherian scheme and let $\mathcal{I}$ be a coherent $\mathcal{O}_Z$-ideal sheaf which define a closed subscheme $W$ of $Z$.*

**Definition 3.3.2** *Let $I$ be an ideal of a noetherian ring $A$. An element $h$ of $A$ is said to be integral over $I$ if it satisfies an integral dependence relation of the form*

$$h^k + a_1 h^{k-1} + \cdots + a_k = 0, \quad \textit{with} \quad a_i \in I^i \qquad \textit{for} \quad 1 \le i \le k.$$

*The integral closure $\overline{I}$ of $I$ is the ideal*

$$\overline{I} := \{x \in A;\ \ x \text{ is integral over } I\}$$

**Remark 3.3.3** *If $A$ is a normal ring (i.e., reduced and integrally closed in its field of fraction), and $I$ is an invertible ideal of $A$ (i.e., generated by a non-zero divisor of $A$), then $I = \overline{I}$.*

**Definition 3.3.4** *Let $Z$ and $\mathcal{I}$ as in* (3.3.1), *we define the integral closure $\overline{\mathcal{I}}$ of the ideal sheaf $\mathcal{I}$ as the sheaf associated to the presheaf*

$$U \to \overline{\mathcal{I}(\mathcal{U})}.$$

*$\overline{\mathcal{I}}$ is a quasi-coherent $\mathcal{O}_Z$-ideal sheaf.*

**Lemma 3.3.5** *([23, Proposition 4.18]). Let $\overline{Z}$ be a normal scheme and let $\mathcal{I}$ be an invertible $\mathcal{O}_{\overline{Z}}$-ideal sheaf. Let $D$ be the Cartier divisor of $\overline{Z}$ defined by the ideal sheaf $\mathcal{I}\mathcal{O}_{\overline{Z}}$ and let $D = \bigcup_{\alpha \in A} D_\alpha$ be the decomposition of $D$ in its irreducible components. Let $h \in H^0(\overline{Z}, \mathcal{O}_{\overline{Z}})$ then, the following conditions are equivalent.*

*(i) $h\mathcal{O}_{\overline{Z},x} \subseteq \mathcal{I}\mathcal{O}_{\overline{Z},x}$ for every point $x$ of $\overline{Z}$.*

*(ii) For every $\alpha \in A$, there exists a point $x_\alpha \in D_\alpha$ such that $h\mathcal{O}_{\overline{Z},x_\alpha} \subseteq \mathcal{I}\mathcal{O}_{\overline{Z},x_\alpha}$.*

**Proof** Assume $(i)$. Let $\varphi \in K(\overline{Z})$ be the rational function of $\overline{Z}$ which generate the invertible ideal $\mathcal{I}^{-1} \cdot h$. Consider the polar loci of $\varphi$. i.e., the closed subscheme $P_\varphi$ of $\overline{Z}$ defined by the coherent $\mathcal{O}_{\overline{Z}}$-ideal sheaf $\mathcal{P}_\varphi$, where $\mathcal{P}_\varphi$ is defined by

$$\Gamma(U, \mathcal{P}_\varphi) := \{h \in \Gamma(U, \mathcal{O}_{\overline{Z}});\ \ h \cdot (\varphi|U) \in \Gamma(U, \mathcal{O}_{\overline{Z}})\}.$$

Clearly $P_\varphi \subset D$ and, $h\mathcal{O}_{\overline{Z},x} \subset \mathcal{I}\mathcal{O}_{\overline{Z},x}$ if and only if $x \notin P_\varphi$. Since $\overline{Z}$ is normal, we have that $P_\varphi$ is either empty or has codimension 1 in $\overline{Z}$. By hypothesis we have that for each $\alpha \in A$, $x_\alpha \notin P_\varphi$ and, since $P_\varphi \subseteq D$, we have that $P_\varphi = \emptyset$ (otherwise every irreducible component of $P_\varphi$ would be an irreducible component of $D$). Therefore $(i)$ holds. $\square$

**Lemma 3.3.6** ([*23, Theorem2.1*]). *Let $h \in H^0(Z, \mathcal{O}_Z)$ and let $\varphi : \widetilde{Z'} \to Z$ be the normalization of the blow-up $Z'$ of $Z$ along $W$. The following conditions are equivalents.*

*$(i)$ $h \in H^0(Z, \overline{\mathcal{I}})$*

*$(ii)$ $h\mathcal{O}_{\widetilde{Z'}} \subset \mathcal{I}\mathcal{O}_{\widetilde{Z'}}$*

**Proof** Assume $(i)$. Then, pulling back the integral dependence relation of $h$ in $Z$ via $\varphi$ we have that

$$h\mathcal{O}_{\widetilde{Z'}} \subset \overline{\mathcal{I}\mathcal{O}_{\widetilde{Z'}}}$$

But, since $\widetilde{Z'}$ is normal, we have that

$$\overline{\mathcal{I}\mathcal{O}_{\widetilde{Z'}}} = \mathcal{I}\mathcal{O}_{\widetilde{Z'}}$$

Thus, $(ii)$ holds.

Conversely, $\widetilde{Z'}$ is $Z$-isomorphic to Proj $(\oplus_{n\geq 0}\mathcal{J}^n)$ for some coherent ideal sheaf $\mathcal{J}$ of $\mathcal{O}_Z$. Since condition $(i)$ is a local condition, we may assume that $\mathcal{J}$ is generated by global sections $g_1, ..., g_m$ which are not zero-divisor in $\mathcal{O}_{\widetilde{Z'}}$. Under this assumptions, we have that $\widetilde{Z'}$ can be cover by a finite number of open subschemes $V_i$ such that

$$\widetilde{Z'}|V_i = \operatorname{Spec}\mathcal{O}_X[\frac{g_1}{g_i}, ..., \frac{\overset{\wedge}{g_i}}{g_i}, ..., \frac{g_m}{g_i}]$$

(here we use the usual convention that the element under $\wedge$ is ommited). From $(ii)$, we have that $h$ can be expressed as a polynomial in $\frac{g_k}{g_i}$ with $k \neq i$ (with coefficient in $\mathcal{I}$). Therefore, by eliminating denominators, we can find an integer $n$ sufficiently big such that

$$h\mathcal{J}^n \subset \mathcal{I}\mathcal{J}^n$$

Thus, $(i)$ holds from [26, Theorem 2.1]. $\square$

**Proposition 3.3.7** *([34, Appendix]). (Specialization of integral dependence). Let $Z$ be a pure dimensional noetherian scheme and let $W$ be a closed subscheme of $Z$ of pure dimension $s$. Let $p : Z' \to Z$ be a proper morphism, where $Z'$ is a pure dimensional noetherian scheme. Let $W'$ be the inverse image of $W$ in $Z'$, and let $\mathcal{I}$ be the ideal sheaf of $W'$ in $Z'$. Let $\overline{Z'}$ be the normalization of the blow-up of $Z'$ along $W'$ and denoted by $\overline{\pi}$ the morphism from $\overline{Z'}$ to $Z'$. Denote by $D$ the exceptional divisor of $\overline{Z'}$ and assume that the morphism $p \circ \overline{\pi} : D \to W$ is equidimensional. For an element $h$ of $H^0(Z', \mathcal{O}_{Z'})$, the following conditions are equivalents.*

*(i)* $h \in H^0(Z', \overline{\mathcal{I}})$.

*(ii) There exists an open dense subscheme $U$ of $W$ such that for every point $y$ of $U$, we have that*

$$h\mathcal{O}_{Z'(y)} \subseteq \overline{\mathcal{I}\mathcal{O}_{Z'(y)}}.$$

*(iii) There exists an open dense subscheme $V$ of $W$ such that for every point $y'$ of $p^{-1}(V)$, we have that*

$$h\mathcal{O}_{Z',y'} \subseteq \overline{\mathcal{I}\mathcal{O}_{Z',y'}}.$$

**Proof** Assume $(i)$. Since an integral dependence relation pulls back to an integral dependence relation, we have that $(ii)$ holds. By definition of the integral closure of an ideal sheaf, we have that $(iii)$ holds trivially.

Next, let $D = \cup D_\alpha$ be the decomposition of $D$ in its irreducible components. Since $\overline{Z'}$ is normal, therefore nonsingular in codimension 1, we can find for each $D_\alpha$ a non-empty open subscheme $U_\alpha$ of $D_\alpha$ such that for every point $x$ of $U_\alpha$ we have that

$(a)$ $\overline{Z'}$ is nonsingular in $x$.

$(b)$ $p \circ \overline{\pi} : D_\alpha \to W$ is smooth at $x$ ([8, Corollary 10.7]).

$(c)$ The strict transform of the subscheme of $Z'$ defined by the ideal sheaf $h\mathcal{O}_{Z'}$, is empty in a neighborhood of $x$.

$(d)$ $W$ is nonsingular at $y := p \circ \overline{\pi}(x)$.

Let $(t_1, \ldots, t_s)$ be a local coordinate system for $W$ about $y$ (i.e., $(t_1, \ldots, t_s)$ generate the maximal ideal of $\mathcal{O}_{W,y}$). From (a) and (b), we have that the $s$-sections

$$t'_j := (p \circ \overline{\pi})^* t_j, \qquad 1 \leq j \leq s$$

give rise to a local coordinate system

$$(t'_1, \ldots, t'_s, u, b_1, \ldots, b_{d-1})$$

for $\overline{Z'}$ about $x$ such that,

$(e)$ $\mathcal{I}\mathcal{O}_{\overline{Z'},x} = u^{a_\alpha}\mathcal{O}_{\overline{Z'},x}$

(f)h$\mathcal{O}_{\overline{Z'},x} = Au^{b_\alpha}\mathcal{O}_{\overline{Z'},x}$, with $A$ inversible in $\mathcal{O}_{\overline{Z'},x}$ (this follows from (c): the strict transform of $h\mathcal{O}_{Z'}$ via $\overline{\pi}$ is defined by $A\mathcal{O}_{\overline{Z'}}$ in a neighborhood of $x$ in $\overline{Z'}$).

Assume $(iii)$. We can shrink the open dense subscheme $U_\alpha$ of $D_\alpha$ such that the point $y = p \circ \overline{\pi}(x)$ lies in $U$. Since an integral dependence relation pulls back to an integral dependence relation, we have that

$$h\mathcal{O}_{\overline{Z'},x} \subseteq \overline{\mathcal{I}\mathcal{O}_{\overline{Z'},x}}$$

Since $\mathcal{O}_{\overline{Z'},x}$ is normal and $\mathcal{I}\mathcal{O}_{\overline{Z'},x}$ is invertible, we have, by Remark 3.3.3, that

$$\overline{\mathcal{I}\mathcal{O}_{\overline{Z'},x}} = \mathcal{I}\mathcal{O}_{\overline{Z'},x}$$

Therefore, we have that

$$Au^{b_\alpha}\mathcal{O}_{\overline{Z'},x} = h\mathcal{O}_{\overline{Z'},x} \subseteq \overline{\mathcal{I}\mathcal{O}_{\overline{Z'},x}} = \mathcal{I}\mathcal{O}_{\overline{Z'},x} = u^{a_\alpha}\mathcal{O}_{\overline{Z'},x}$$

and, since $A$ is invertible in $\mathcal{O}_{\overline{Z'},x}$, we have that $b_\alpha \geq a_\alpha$. Therefore $(i)$ holds in view of Lemma 3.3.5.

Assume $(ii)$. We can shrink the open dense subscheme $U_\alpha$ of $D_\alpha$ such that the point $y = p \circ \overline{\pi}(x)$ lies in $V$. Since an integral dependence relation pulls back to an integral dependence relation, we have that

$$h\mathcal{O}_{\overline{Z'(y)}} \subseteq \overline{\mathcal{I}\mathcal{O}_{\overline{Z'(y)}}} \qquad (3.3.7.1)$$

Since,

$$\mathcal{O}_{\overline{Z'(y)},x} \cong \mathcal{O}_{\overline{Z'},x}/(t'_1, \ldots, t'_s, u^{a_\alpha})\mathcal{O}_{\overline{Z'},x},$$

we have that $\mathcal{O}_{\overline{Z'(y)},x}$ is regular, hence normal. Now,

$$\mathcal{I}\mathcal{O}_{\overline{Z'(y)},x} \cong u^{a_\alpha}\mathcal{O}_{\overline{Z'},x}/(t'_1, \ldots, t'_s, u^{a_\alpha})\mathcal{O}_{\overline{Z'},x} = 0 \qquad (3.3.7.2),$$

and

$$h\mathcal{O}_{\overline{Z'(y)},x} \cong Au^{b_\alpha}\mathcal{O}_{\overline{Z'},x}/(t'_1, \ldots, t'_s, u^{a_\alpha})\mathcal{O}_{\overline{Z'},x} \qquad (3.3.7.3).$$

Therefore, by (3.3.7.1), we have that

$$h\mathcal{O}_{\overline{Z'(y)},x} \subseteq \overline{\mathcal{I}\mathcal{O}_{\overline{Z'(y)},x}} = \mathcal{I}\mathcal{O}_{\overline{Z'(y)},x} = 0.$$

Since $A$ is invertible in $\mathcal{O}_{\overline{Z'},x}$, we have by (3.3.7.3) that $b_\alpha \geq a_\alpha$. Therefore $(i)$ holds by $(e)$, $(f)$ and Lemma 3.3.5. □

## 3.4 The Main Results

**Setup 3.4.1** *Use the notation and hypothesis of* (2.4.1) *and, recall the diagrams of* (2.4.2.1) *and* (2.4.2.2).

**Lemma 3.4.2** ([*20*, 4.6.13]). *Let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module. Let*

$$\mathsf{V}(\mathcal{E}) := \operatorname{Spec}(\operatorname{Sym}\mathcal{E}),$$

*and,*

$$\mathsf{V}(\mathcal{E}^\vee) := \operatorname{Spec}(\operatorname{Sym}\mathcal{E}^\vee).$$

*Let $\sigma : Y \to \mathsf{V}(\mathcal{E})$ and $\check{\sigma}: Y \to \mathsf{V}(\mathcal{E}^\vee)$ be the zero-section imbeddings. There exists a canonical $Y$-isomorphism*

$$F : T^*\mathsf{V}(\mathcal{E}) \longrightarrow T^*\mathsf{V}(\mathcal{E}^\vee).$$

*The isomorphism $F$ satisfies the following properties.*

*(i) $F$ identifies $T^*_{\sigma(Y)}\mathsf{V}(\mathcal{E})$ (resp. $T^*_{\check{\sigma}(Y)}\mathsf{V}(\mathcal{E}^\vee)$) to the zero-section of $T^*\mathsf{V}(\mathcal{E}^\vee)$ (resp. $T^*\mathsf{V}(\mathcal{E})$).*

*(ii) $F$ identifies $T^*\mathsf{V}(\mathcal{E}) \times_{\mathsf{V}(\mathcal{E})} \sigma(Y)$ (resp. $T^*\mathsf{V}(\mathcal{E}^\vee) \times_{\mathsf{V}(\mathcal{E}^\vee)} \check{\sigma}(Y)$) with $\mathsf{V}(\mathcal{E}^\vee) \times_Y T^*Y$ (resp. $\mathsf{V}(\mathcal{E}) \times_Y T^*Y$).*

*(iii) If $\alpha_{\mathcal{E}}$ (resp. $\alpha_{\mathcal{E}^\vee}$) denotes the affine contact form on $T^*\mathsf{V}(\mathcal{E})$ (resp. $T^*\mathsf{V}(\mathcal{E}^\vee)$) as in Definition 3.1.6 then,*

$$F^*d\alpha_{\mathcal{E}^\vee} = d\alpha_{\mathcal{E}} \qquad\qquad [15, 5.1.3]$$

□

Let

$$L : T_{T^*_Y M}T^*M \longrightarrow T^*(T_Y M)$$

be the isomorphism induced by the isomorphism $\theta$ of Proposition 3.2.2.

**Lemma 3.4.3** ([20, 4.7]). *There exists a canonical isomorphism of vector bundles over $T^*M$*

$$H : T^*(T^*M) \longrightarrow T(T^*M),$$

*where*

$$T(T^*M) := \mathrm{Spec}\,(\mathrm{Sym}\,\Omega^1_{T^*M}).$$

*The isomorphism $H$ induces an isomorphism of vector bundles over $T^*_Y M$, which we also denote by $H$,*

$$H : T^*(T^*_Y M) \xrightarrow{\sim} T_{T^*_Y M}T^*M.$$

*Furthermore, we have a commutative diagram of isomorphisms*

$$\begin{array}{ccc} T^*(T_Y^*M) & \xrightarrow{-H} & T_{T_Y^*M}T^*M \\ F\searrow & & \swarrow L \\ & T^*(T_YM) & \end{array} \qquad (3.4.3.1)$$

*where $F$ is the isomorphism of Lemma 3.4.2 associated to the locally free $\mathcal{O}_Y$-module $\left(\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2\right)^\vee$.* □

**Lemma 3.4.4** *Let $U$ be an open dense subscheme of $Y$ over which the morphism $\xi_Y : D_Y \to Y$ is equidimensional. Then, for every point $y$ of $U$, we have that*

$$\mathcal{I}_{C(Y,M)\cap C(X,M)}\mathcal{O}_{C(X,M)(y)} \subseteq \overline{\mathcal{I}_{\kappa_X^{-1}(Y)}\mathcal{O}_{C(X,M)(y)}} \qquad (*).$$

**Proof** By induction on the dimension of $Y$.

If $\dim Y = 0$, then $C(Y,M)\cap C(X,M) = \kappa_X^{-1}(Y)$. Therefore, $(*)$ holds trivially.

In general, since $(*)$ is local, we may assume that there is a (global) étale morphism $g : M \to \mathbb{A}^m$ satisfying the conditions of Remark 2.4.1 for a fixed point $y$ of $U$ (i.e., with $g(y) = 0$).

Let $H'$ be a (generic) hyperplane of $\mathbb{A}^m$ satisfying the conditions of Lemma 2.4.3, and let $H := g^{-1}(H')$. i.e., we have that

$$\begin{aligned} C(H,M)\cap C(X,M) &= \emptyset \\ C(H,M)\cap C(Y,M) &= \emptyset \end{aligned}$$

Choose $H'$ in such a way that $H$ satisfies also the conditions of Lemma 2.4.5 for the morphism

$$\xi : \mathrm{Bl}_Y C(X,M) \longrightarrow X.$$

Since $\xi$ is equidimensional over $X^0 := X - Y$, we have that $H$ also satisfies the conditions of Lemma 2.4.5 for the morphism

$$\xi_Y : D_Y \longrightarrow Y.$$

Consider the notations and the diagram (2.4.6.1). We have that

$$X \widehat{\cap} H = \mathrm{Bl}_Y C(X, H) \times_X (X \cap H).$$

Analogously, we have a finite morphism

$$\tilde{q} : C(Y, M) \times_Y (Y \cap H) \longrightarrow C(Y \cap H, H).$$

By Lemma 2.4.6, we have that the morphism

$$\xi_{Y \cap H} : D_{Y \cap H} \longrightarrow (Y \cap H)$$

is equidimensional over $U \cap H$. Therefore, by induction hypothesis, we have that

$$\mathcal{I}_{C(Y \cap H, H) \cap C(X \cap H, H)} \mathcal{O}_{C(X \cap H, H))(y)} \subseteq \overline{\mathcal{I}_{\kappa^{-1}_{X \cap H}(Y \cap H)} \mathcal{O}_{(C(X \cap H, H))(y)}}$$

Pulling back this integral dependence relation via the morphism $q$, we obtain an integral dependence relation

$$\mathcal{I}_{(\widetilde{X \cap H}) \cap (C(Y.M) \times_Y (Y \cap H))} \mathcal{O}_{(\widetilde{X \cap H})(y)} \subseteq \overline{\mathcal{I}_{\kappa_X^{-1}(Y \cap H)} \mathcal{O}_{(\widetilde{X \cap H})(y)}} \qquad (3.4.4.1)$$

By Lemma 2.4.5, we have that

$$(X \widetilde{\cap} H) = \kappa_X^{-1}(X \cap H)$$

Therefore, we have that

$$\mathcal{I}_{(\widetilde{X \cap H}) \cap (C(Y.M) \times_Y (Y \cap H))} \mathcal{O}_{(\widetilde{X \cap H})(y)} = \mathcal{I}_{C(X,M) \cap C(Y,M)} \mathcal{O}_{C(X,M)(y)}$$

and,

$$\mathcal{I}_{\kappa_X^{-1}(Y \cap H)} \mathcal{O}_{(\widetilde{X \cap H})(y)} = \mathcal{I}_{\kappa_X^{-1}(Y)} \mathcal{O}_{C(X,M)(y)}.$$

Therefore, from (3.4.4.1), we have that $(*)$ holds. $\square$

**Definition 3.4.5** *([37]).*

*(i) We say that the pair $(X^{\mathrm{sm}}, Y)$ satisfies the Whitney condition $(a)$ if*

$$(D_Y)_{red} \subset \mathsf{P}\left(\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2\right) \times_Y \mathsf{P}\left((\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2)^\vee\right)$$

*(ii) We say that the pair $(X^{\mathrm{sm}}, Y)$ satisfies the Whitney condition $(b)$ if $(D_Y)_{red} \subset I_Y :=$ incidence correspondence of $\mathsf{P}\left(\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2\right)$ and $\mathsf{P}\left((\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2)^\vee\right)$.*

**Lemma 3.4.6** ([*18*, 4.2]).

*(i) There is a canonical $\mathsf{P}\left(\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2\right)$-isomorphism (resp. $\mathsf{P}\left((\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2)^\vee\right)$-isomorphism ) from $I_Y$ to the projectivization of the relative cotangent bundle of $\mathsf{P}\left(\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2\right)$ (resp. $\mathsf{P}\left((\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2)^\vee\right)$).*

*(ii) Let $\omega$ and $\check{\omega}$ denote the contact form of $I_Y/\mathsf{P}\left(\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2\right)$ and $I_Y/\mathsf{P}\left((\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2)^\vee\right)$ respectively (see 3.1.1). Then,*

$$\omega + \check{\omega} = 0.$$

□

**Lemma 3.4.7** *Assume that the pair $(X^{\mathrm{sm}}, Y)$ satisfies the Whitney conditions $(a)$ and $(b)$. If $H$ is a nonsingular closed subscheme of $M$ of codimension $j$ which satisfies the following conditions*

(1) *The intersection of $H$ with $Y$ is smooth of dimension $t - j$.*

(2) $C(H, M) \cap C(X, M) = \emptyset$

(3) $C(H, M) \cap C(Y, M) = \emptyset$.

*Then, the pair $((X \cap H)^{\mathrm{sm}}, Y \cap H)$ also satisfies the Whitney conditions $(a)$ and $(b)$.*

**Proof** Consider the following diagram

$$\begin{array}{ccc} Y \cap H & \xrightarrow{\iota'} & H \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\iota} & M \end{array}$$

Since $\iota$ and $\iota'$ are regular imbeddings of the same codimension, we have that

$$C(Y \cap H, H) = C(Y, M) \times_Y (Y \cap H)$$

Therefore, we have that

$$\mathbf{P}\left(\Omega^{1\,\vee}_{C(Y\cap H,H)/Y\cap H}\right) = (Y \cap H) \times_Y \mathbf{P}\left(\Omega^{1\,\vee}_{C(Y,M)/Y}\right) \qquad (3.4.6.1)$$

Consider the diagram (2.4.6.1). Since the morphism

$$\hat{q} : X\widehat{\cap}H \longrightarrow \mathrm{Bl}_{\kappa^{-1}_{X\cap H}(Y\cap H)} C(X \cap H, H)$$

is finite, and since $X\widehat{\cap}H$ is the strict transform of $X \cap H$ via the morphism $\xi := \kappa_X \circ \hat{e}_X$, which is equal to the strict transform of $X\widetilde{\cap}H$ via the morphism $\hat{e}_{X\cap H}$, we have a finite morphism

$$\tilde{q} : D_Y \cap (X\widehat{\cap}H) \longrightarrow D_{Y\cap H}$$

which is induced by $\hat{q}$. Since

$$D_Y \cap (X\widehat{\cap}H) \subseteq D_Y \times_Y (Y \cap H),$$

we have, by (3.4.6.1), that $D_{Y\cap H}$ is contained in the incidence correspondence

$$I_{Y\cap H} := \mathbf{P}\left(\Omega^{1\,\vee}_{C(Y\cap H,H)/Y\cap H}\right).$$

In other words, the pair $((X \cap H)^{\mathrm{sm}}, Y \cap H)$ satisfies also the Whitney conditions $(a)$ and $(b)$. $\square$

Let $(D_Y)_{\mathrm{red}}$ denote the scheme $D_Y$ with its reduced induced structure. Let $(D_Y)_{\mathrm{red}} = \cup D_\alpha$ be its decomposition in its irreducible component, and let $V_\alpha$ denote the image of $D_\alpha$ in $\mathbf{P}\left(\mathcal{I}^M_Y/(\mathcal{I}^M_Y)^2\right)$ via $\kappa'$, equipped with the induced reduced closed subscheme structure.

**Theorem 3.4.8** ([*21*, 2.1.1]). *Let $X$ be a d-dimensional integral scheme of finite type and separate over an algebraically closed ground field $K$ of characteristic zero. Let $Y$ be a smooth integral proper closed subscheme of $X$ of dimension $t$ such that, $X^0 := X - Y$ contains a smooth open dense subscheme. Assume that $X$ admits a proper imbedding in a smooth ambient scheme $M$. Then, the following conditions are equivalent.*

*(i) The pair $(X^{\mathrm{sm}}, Y)$ satisfies the Whitney conditions $(a)$ and $(b)$.*

*(ii) The sequence $M_x^*(X)$ is independent of the point $x$ of $Y$.*

*(iii) The morphism $\xi_Y : D_Y \to Y$ is equidimensional.*

*(iv) $\mathcal{I}_{C(Y,M)\cap C(X,M)}\mathcal{O}_{C(X,M)} \subseteq \overline{\mathcal{I}_{\kappa^{-1}(Y)}\mathcal{O}_{C(X,M)}}$.*

*(v) For each $\alpha$ we have that*

$$D_\alpha = C\left(V_\alpha/Y; \mathbf{P}(\mathcal{I}_Y^M/(\mathcal{I}_Y^M)^2)/Y\right).$$

**Proof** The equivalence between $(ii)$ and $(iii)$ was proved in Theorem 2.4.7.

Assume $(v)$. Then, by Lemma 3.4.6, we have that $(i)$ holds.

Assume $(iii)$. Then, by Lemma 3.4.4 and Proposition 3.3.7, we have that $(iv)$ holds.

Next, consider the following commutative diagram of isomorphisms (see Lemma 3.4.3).

$$\begin{array}{ccc} T^*(T_Y^*M) & \xrightarrow{-H} & T_{T_Y^*M}T^*M \\ & F \searrow \qquad \swarrow L & \\ & T^*(T_YM) & \end{array} \qquad (3.4.8.1)$$

We are going to prove first that the image of $T_{T_X^*M\cap T_Y^*M}T_X^*M$ in $T^*(T_Y^*M)$ via $(-H)^{-1}$ is an affine Lagrangian subscheme of $T^*(T_Y^*M)$.

By Proposition 3.2.2, we have that the isomorphism

$$L : T_{T_Y^*M}T^*M \longrightarrow T^*(T_YM)$$

maps $T_{T^*_X M\cap T^*_Y M}T^*_X M$ isomorphicaly into $T^*_{T_Y X}T_Y M$. i.e., we have a commutative diagram

$$\begin{array}{ccc} T_{T^*_X M\cap T^*_Y M}T^*_X M & \xrightarrow{\sim} & T^*_{T_Y X}T_Y M \\ \downarrow & & \downarrow \\ T_{T^*_Y M}T^*M & \xrightarrow{L} & T^*(T_Y M) \end{array} \qquad (3.4.8.2)$$

On the other hand, from Lemma 3.4.2, we have that the isomorphism $F$ preserves the differential of the affine contact forms. i.e.,

$$F^*(d\alpha_{T_Y M}) = d\alpha_{T^*_Y M}$$

where $\alpha_{T_Y M}$ (resp. $\alpha_{T^*_Y M}$) is the affine contact form on $T^*(T_Y M)$ (resp. $T^*(T^*_Y M)$). Thus, the isomorphism $F^{-1}$ maps affine Lagrangian subschemes of $T^*(T_Y M)$ into affine Lagrangian subschemes of $T^*(T^*_Y M)$ (see Remark 3.1.8).

Therefore, since $T^*_{T_Y X}T_Y M$ is an affine Lagrangian subscheme of $T^*(T_Y M)$, we have that the image of $T^*_{T_Y X}T_Y M$ in $T^*(T^*_Y M)$ via $F^{-1}$ is an affine Lagrangian subscheme of $T^*(T^*_Y M)$. i.e., it satisfies the equation

$$d\alpha_{T^*_Y M} = 0.$$

Now, from (3.4.8.1) and (3.4.8.2), we have that

$$F^{-1}(T^*_{T_Y X}T_Y M) = (-H)^{-1}(T_{T^*_X M\cap T^*_Y M}T^*_X M)$$

Therefore,

$$(-H)^{-1}(T_{T^*_X M\cap T^*_Y M}T^*_X M) \text{ is Lagrangian in } T^*(T^*_Y M) \qquad (3.4.8.3)$$

Since,

$$H : T^*(T^*_Y M) \xrightarrow{\sim} T_{T^*_Y M}T^*M$$

is an isomorphism of vector bundles over $T^*_Y M$, we have that $H$ induces an isomorphism, which we also denote by $H$, of projective bundles over $\mathsf{P}(T^*_Y M)$. i.e.,

$$H : \mathsf{P}\left(T^*(\mathsf{P}(T^*_Y M))\right) \xrightarrow{\sim} \mathsf{P}\left(T_{\mathsf{P}(T^*_Y M)}\mathsf{P}(T^*M)\right).$$

Since $(-H)^{-1}(T_{T_X^* M \cap T_Y^* M} T_X^* M)$ is a conic subscheme of $T^*(T_Y^* M)$, we have, from (3.4.8.3) and Lemma 3.1.11, that the image of $\mathbf{P}\left(T_{\mathbf{P}(T_X^* M) \cap \mathbf{P}(T_Y^* M)} \mathbf{P}(T_X^* M)\right)$ via $H^{-1}$ is a (projective) Lagrangian subscheme of $\mathbf{P}\left(T^*(\mathbf{P}(T_Y^* M))\right)$ in the sense of Remark 3.1.3.

Let $\pi : \mathbf{P}(T_Y^* M) \to Y$ be the structural morphism. From the exact sequence of locally free sheaves

$$0 \to \pi^* \Omega_Y^1 \to \Omega^1_{\mathbf{P}(T_Y^* M)} \to \Omega^1_{\mathbf{P}(T_Y^* M)/Y} \to 0$$

we get a morphism

$$\beta : [\mathbf{P}(T^*(\mathbf{P}(T_Y^* M))) - \mathbf{P}(T_Y^* M) \times_Y \mathbf{P}(T^* Y)] \longrightarrow \mathbf{P}(T^*(\mathbf{P}(T_Y^* M))/Y).$$

From Lemma 3.4.6 $(i)$, we have that

$$\mathbf{P}(T^*(\mathbf{P}(T_Y^* M))/Y) = I_Y$$

where $I_Y$ is as in Definition 3.4.5. Therefore, we have that the morphism $\beta$ induces a morphism, which we also denote by $\beta$,

$$\beta : [\mathbf{P}(T^*(\mathbf{P}(T_Y^* M))) - \mathbf{P}(T_Y^* M) \times_Y \mathbf{P}(T^* Y)] \longrightarrow I_Y \qquad (3.4.8.4)$$

Since $Y$ and $M$ are smooth, we have a sequence of regular imbeddings

$$\mathbf{P}(T_Y^* M) \xrightarrow{\jmath} q^{-1}(Y) \xrightarrow{\iota} \mathbf{P}(T^* M)$$

where, $q : \mathbf{P}(T^* M) \to M$ is the structural morphism.

Therefore, we have a sequence of locally free sheaves

$$0 \to \jmath^* \mathcal{I}_{q^{-1}(Y)}^{\mathbf{P}(T^*M)} / (\mathcal{I}_{q^{-1}(Y)}^{\mathbf{P}(T^*M)})^2 \to \mathcal{I}_{\mathbf{P}(T_Y^*M)}^{\mathbf{P}(T^*M)} / (\mathcal{I}_{\mathbf{P}(T_Y^*M)}^{\mathbf{P}(T^*M)})^2 \to \mathcal{I}_{\mathbf{P}(T_Y^*M)}^{q^{-1}(Y)} / (\mathcal{I}_{\mathbf{P}(T_Y^*M)}^{q^{-1}(Y)})^2 \to 0$$

and, from this sequence, we get a morphism

$$\alpha : \left[ \mathsf{P}\left(T_{\mathsf{P}(T_Y^*M)}\mathsf{P}(T^*M)\right) - \mathsf{P}\left(T^{q^{-1}(Y)}_{\mathsf{P}(T_Y^*M)}\right)\right] \longrightarrow \mathsf{P}\left(j^*\mathcal{I}^{\mathsf{P}(T^*M)}_{q^{-1}(Y)}/(\mathcal{I}^{\mathsf{P}(T^*M)}_{q^{-1}(Y)})^2\right)$$

Since,

$$\begin{aligned}\mathsf{P}\left(j^*\mathcal{I}^{\mathsf{P}(T^*M)}_{q^{-1}(Y)}/(\mathcal{I}^{\mathsf{P}(T^*M)}_{q^{-1}(Y)})^2\right) &= \mathsf{P}(T_Y^*M) \times_{q^{-1}(Y)} \mathsf{P}\left(T_{q^{-1}(Y)}\mathsf{P}(T^*M)\right) \\ &= \mathsf{P}(T_Y^*M) \times_Y \mathsf{P}(T_YM)\end{aligned}$$

We have that the morphism $\alpha$ induces a morphism, which we also denote by $\alpha$,

$$\alpha : \left[ \mathsf{P}\left(T_{\mathsf{P}(T_Y^*M)}\mathsf{P}(T^*M)\right) - \mathsf{P}\left(T^{q^{-1}(Y)}_{\mathsf{P}(T_Y^*M)}\right)\right] \longrightarrow \mathsf{P}(T_Y^*M) \times_Y \mathsf{P}(T_YM) \qquad (3.4.8.5)$$

Now,

$$I_Y \subset \mathsf{P}(T_Y^*M) \times_Y \mathsf{P}(T_YM)$$

and we are going to prove that the image of $\alpha$ is $I_Y$.

From Lemma 3.4.2 $(ii)$, we have that the isomorphism

$$F : T^*(T_Y^*M) \longrightarrow T^*(T_YM)$$

identifies $T_Y^*M \times_Y T^*Y$ with $T^*(T_YM) \times_{T_YM} \sigma(Y)$, where $\sigma : Y \to T_YM$ is the zero-section imbedding.

On the other hand, the isomorphism

$$L^{-1} : T^*(T_YM) \longrightarrow T_{T_Y^*M}T^*M$$

identifies $T^*(T_YM) \times_{T_YM} \sigma(Y)$ with $T_{T_Y^*M}p^{-1}(Y)$ where, $p : T^*M \to M$ is the structural morphism.

Therefore, we have that the isomorphism

$$H : \mathsf{P}\left(T^*(\mathsf{P}(T_Y^*M))\right) \xrightarrow{\sim} \mathsf{P}\left(T_{\mathsf{P}(T_Y^*M)}\mathsf{P}(T^*M)\right).$$

identifies $\mathsf{P}(T_Y^*M) \times_Y \mathsf{P}(T^*Y)$ with $\mathsf{P}\left(T_{\mathsf{P}(T_Y^*M)}q^{-1}(Y)\right)$. Therefore, by (3.4.8.4) and (3.4.8.5), we have a commutative diagram

$$\begin{array}{ccc} \left[\mathbf{P}\left(T_{\mathbf{P}(T_Y^*M)}\mathbf{P}(T^*M)\right) - \mathbf{P}\left(T^{q^{-1}(Y)}_{\mathbf{P}(T_Y^*M)}\right)\right] & \xrightarrow{\alpha} & \mathbf{P}(T_Y^*M) \times_Y \mathbf{P}(T_Y M) \\ {\scriptstyle H^{-1}}\downarrow & & \uparrow \\ [\mathbf{P}(T^*(\mathbf{P}(T_Y^*M))) - \mathbf{P}(T_Y^*M) \times_Y \mathbf{P}(T^*Y)] & \xrightarrow{\beta} & I_Y \end{array} \qquad (3.4.8.6)$$

Thus, the image of $\alpha$ is the incidence correspondence $I_Y$.

Since the image of $\mathbf{P}\left(T_{\mathbf{P}(T_Y^*M)\cap\mathbf{P}(T_X^*M)}\mathbf{P}(T_X^*M)\right)$ via $H^{-1}$ is a Lagrangian subscheme of $\mathbf{P}\left(T^*(\mathbf{P}(T_Y^*M))\right)$, we have, by Lemma 3.1.5, that every irreducible component of $H^{-1}\left[\mathbf{P}\left(T_{\mathbf{P}(T_Y^*M)\cap\mathbf{P}(T_X^*M)}\mathbf{P}(T_X^*M)\right)\right]$ which is not contained in $\mathbf{P}(T_Y^*M) \times_Y \mathbf{P}(T^*Y)$, is mapped via $\beta$ into a $Y$-Lagrangian subscheme of $I_Y$.

Assume $(iv)$. Then, we have a (globally defined) dominat, finite morphism

$$\tilde{\alpha} : \mathbf{P}\left(T_{\mathbf{P}(T_Y^*M)\cap\mathbf{P}(T_X^*M)}\mathbf{P}(T_X^*M)\right) \longrightarrow \mathbf{P}\left(T_{\kappa^{-1}(Y)}\mathbf{P}(T_X^*M)\right)$$

which is induced by $\alpha$. In particular, we have that $H^{-1}\left[\mathbf{P}\left(T_{\mathbf{P}(T_Y^*M)\cap\mathbf{P}(T_X^*M)}\mathbf{P}(T_X^*M)\right)\right]$ does not meet the center of the projection $\beta$ and, it is mapped dominantly into

$$D_Y = \mathbf{P}\left(T_{\kappa^{-1}(Y)}\mathbf{P}(T_X^*M)\right)$$

Therefore, $D_Y$ is $Y$-Lagrangian in $\mathbf{P}\left(T^*((\mathbf{P}T_Y^*M)/Y)\right)$. Hence, $(v)$ holds (see [18, 3.6]).

Assume $(i)$. Consider the following commutative diagram of imbeddings

$$\begin{array}{ccccc} \mathbf{P}(T_X^*M) \cap \mathbf{P}(T_Y^*M) & \xrightarrow{j'} & \kappa^{-1}(Y) & \xrightarrow{\imath'} & \mathbf{P}(T_X^*M) \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ \mathbf{P}(T_Y^*M) & \xrightarrow{\jmath} & q^{-1}(Y) & \xrightarrow{\iota} & \mathbf{P}(T^*M) \end{array}$$

Then, we have that

$$\begin{aligned}
\mathbf{P}\left(T_{\mathbf{P}(T_Y^*M)\cap \mathbf{P}(T_X^*M)}\mathbf{P}(T_X^*M)\right) &\subset \mathbf{P}\left(f^*T_{\mathbf{P}(T_Y^*M)}\mathbf{P}(T^*M)\right)\\
&= [\mathbf{P}(T_X^*M)\cap \mathbf{P}(T_Y^*M)] \times_{\mathbf{P}(T_Y^*M)} \mathbf{P}\left(T_{\mathbf{P}(T_Y^*M)}\mathbf{P}(T^*M)\right)
\end{aligned}$$

and,

$$\begin{aligned}
\mathbf{P}\left(j'^*T_{\kappa^{-1}(Y)}\mathbf{P}(T_X^*M)\right) &\subset \mathbf{P}\left(j'^*g^*T_{q^{-1}(Y)}\mathbf{P}(T^*M)\right)\\
&= \mathbf{P}\left(f^*j^*T_{q^{-1}(Y)}\mathbf{P}(T^*M)\right)
\end{aligned}$$

Therefore, the morphism $\alpha$ of (3.4.8.5) induces a morphism

$$\overline{\alpha}:\left[\mathbf{P}\left(T_{\mathbf{P}(T_Y^*M)\cap \mathbf{P}(T_X^*M)}\mathbf{P}(T^*M)\right)-\mathbf{P}\left(f^*T_{\mathbf{P}(T_Y^*M)}q^{-1}(Y)\right)\right]\longrightarrow \mathbf{P}\left(T_{\kappa^{-1}(Y)}\mathbf{P}(T_X^*M)\right)$$

The image of $\overline{\alpha}$ is equal to the image of

$$T:=\left[\mathbf{P}\left(T_{\mathbf{P}(T_Y^*M)}\mathbf{P}(T^*M)\right)-\mathbf{P}\left(T_{\mathbf{P}(T_Y^*M)}q^{-1}(Y)\right)\right]$$

via $\alpha$, intersected with $\mathbf{P}\left(T_{\kappa^{-1}(Y)}\mathbf{P}(T_X^*M)\right)$.

But, from (3.4.8.6), we know that the image of $T$ via $\alpha$ is exactly the incidence correspondence

$$I_Y = \mathbf{P}\left(T^*((\mathbf{P}T_Y^*M)/Y)\right)$$

Therefore, since $D_Y = \mathbf{P}\left(T_{\kappa^{-1}(Y)}\mathbf{P}(T_X^*M)\right)$ is contained in $I_Y$, we have that the morphism $\overline{\alpha}$ is dominant. Therefore, from the discussion bellow (3.4.8.6), we have that $D_Y$ is $Y$-Lagrangian. i.e., $(v)$ holds.

Assume $(v)$. We are going to prove that $(iii)$ holds. But, first, we will reduce to the case when $Y$ is 1-dimensional by using the equivalence $(iii) \Leftrightarrow (ii)$ and $(v) \Leftrightarrow (i)$ of this theorem.

Since condition $(v)$ holds, we have that

$$\kappa^{-1}(Y)\subset \mathbf{P}(T_Y^*M)$$

set-theoretically. In particular, we have that

$$\dim \kappa^{-1}(x) \leq m-1-t$$

for every point $x$ of $Y$. Therefore, by Proposition 2.2.2, we have that

$$P_k(X,x) = \emptyset \text{ for } k > d-t \qquad (3.4.8.7)$$

Therefore, by the same reasoning as in the proof of Lemma 2.4.3 and Lemma 2.4.4 and by shrinking $M$ if necessary, we can find a (generic) non singular closed subscheme $H$ of $M$ of codimension $t-1$ such that

$(a)$ $H$ passes through a (specific) point $x$ of $Y$.

$(b)$ The intersection of $Y$ and $H$ is smooth.

$(c)$ $C(X,M) \cap C(H,M) = \emptyset$

$(d)$ $C(Y,M) \cap C(H,M) = \emptyset$

$(e)$ $m_x(P_k(X,x)) = m_x(P_k(X \cap H, x))$ for $0 \leq k \leq d-t$.

Therefore, by Lemma 3.4.7, we have that the pair $((X \cap H)^{\text{sm}}, Y \cap H)$ satisfies the Whitney conditions $(a)$ and $(b)$ (here we are using the equivalence $(v) \Leftrightarrow (i)$ of this Theorem, which had been already proved) . Thus, in order to prove that condition $(iii)$ holds (which is equivalent to the equimultiplicity of the local polar varieties in view of Theorem 2.4.7), we may assume, by (3.4.8.7) and $(e)$, that $Y$ has dimension 1.

In this case, since every irreducible component $D_\alpha$ of $(D_Y)_{\text{red}}$ is $Y$-Lagrangian, we have that for every $\alpha$ there exists an open dense subscheme $D_\alpha^0$ of $D_\alpha$ over which the morphism

$$\xi_Y : D_\alpha \longrightarrow Y$$

is smooth (of relative dimension $m-3$). In particular, we have that for every $\alpha$ the morphism

$$\xi_Y : D_\alpha \longrightarrow Y$$

is dominant. Since $Y$ has dimension 1 and $D_\alpha$ is irreducible, we have that

$$\xi_Y : D_\alpha \longrightarrow Y$$

is equidimensional. In other words, $(iii)$ holds. $\square$

# Bibliography


[1] Altman, A., and Kleiman, S., "Introduction to Grothendieck duality theory," *Springer Lect. Notes in Math.* **146**.

[2] Buchsbaum, D.A., and Rim, D.S., "A generalized Kozul complex II. Depth and multiplicity," *Trans. Amer. Math. Soc.* **111** (1964) 197-224.

[3] Briançon, J., and Speder, J.P., "Les conditions de Whitney impliquent $\mu^*$ constant," *Ann. Inst. Fourier*, **26** (1976) 153-163.

[4] Fulton, W., "Intersection Theory," *Ergebnisse der Math. Springer*, **2** (1984).

[5] Gaffney, T., "Multiplicities and Equisingularity of ICIS germs," *Manuscript.*

[6] Gaffney, T., "integral Closure of Modules and Whitney Equisingularity," *Preprint*

[7] Gonzales-Sprinberg, G., "L'obtruction locale d'Euler et le théorème de MacPherson," *Asterisque* **82-83** (1981).

[8] Hartshorne, R., "Algebraic Geometry," *Springer-Verlag GTM*, **52**

[9] Henry, J.P., and Merle, M., "Limites D'Espaces Tangents et Transversalité de Variétés Polaires," *Springer Lecture Notes*, **961** (1981) 189-199.

[10] Henry, J.P., and Merle, M., "Limites de Normales, Conditions de Whitney et éclatement d'Hironaka," *Proc. A.M.S. Symp. in Pure Math.*, **40** (1983) 575-584.

[11] Henry, J.P., and Merle, M., "Fronces et Doubles Plis," *to appear.*

[12] Henry, J.P., and Merle, M., and Sabbah, C., "Sur la condition de Thom stricte pour un morphisme analytique complexe," *Annales E.N.S.*, **17**, 2, (1984).

[13] Hironaka, H., "Equivalences and deformations of isolated singularities," Woods Hole Conference on Algebraic Geometry, *Amer. Math. Soc.* (1964).

[14] Hironaka, H., "Normal Cones in Analytic Whitney Stratifications," *Publ. Math. I.H.E.S.*, **36** (1970).

[15] Kashiwara, M., and Schapira, P., "Microlocal Study of Sheaves," *Asterisque*, **128** (1985).

[16] Kennedy, G., "MacPherson's Chern Classes of Singular Algebraic Varieties" *Preprint.*

[17] Kleiman, S. L., "On the Transversality of a General Translate," *Comp. Math.*, **28** (1974).

[18] Kleiman, S. L., "About the Conormal Scheme" *Springer Lecture Notes*, **1092** (1983).

[19] Kleiman, S. L., "Geometry on Grassmannians and applications to splitting bundles and smoothing cycles," *Publ. Math. I.H.E.S.*, **36** (1969) 281-298.

[20] Laumon, G., "Transformations Canoniques et Spécilisation pour les $\mathcal{D}$-modules filtrés," *Astérisque*, **130** (1985).

[21] Lê, D.T., and Teissier, B., "Limites d'espaces tangents en géométrie analytique," *Comment. Math. Helvetica*, **63** (1988) 540-578.

[22] Lê, D.T., and Teissier, B., "Varietés Polaires Locales et Classes de Chern des Varietés Singulières," *Ann. Math.*, **114** (1981) 457-491

[23] Lejeune-Jalabert, M., and Teissier, B., "Dépendence Intégrale sur les Idéaux et Équisingularité," *Séminaire Ecole Polytechnique* (1974).

[24] Lipman, J., "Reduction, blowing-up and multiplicities," *Proc. Conf. Transcendental Methods in Commutative Algebra.*

[25] MacPherson, R., "Chern Classes for Singular Algebraic Varieties," *Ann. Math.*, **100** (1974).

[26] Matsumura, H., "Commutative Ring Theory," *Cambridge Studies in Adv. Math.*, **8** (1986).

[27] Micali, A., "Sur les algèbres universelles," *Inst. Fourier, Grenoble*, **14**, 2, (1964).

[28] Navarro, V., "Sur les multiplités de Schubert locales des faisceaux algébriques cohérents, *Compositio Math.*, **48** (1983).

[29] Pham, F., "Singularités des Systèmes Différentiels de Gauss-Manin," *Progress in Math.*, **2**.

[30] Sabbah, C., "Quelques Remarques sur la géométrie des Espaces Conormaux," *Astérisque*, **131** (1985).

[31] Teissier, B., "Cycles évanescents, sections planes et conditions de Whitney," *Asterisque* **7-8** (1973).

[32] Teissier, B., "The Hunting of Invariants in the Geometry of Discriminants," *Proc. Nordic Summer School "Real and Complex Singularities"*, **Oslo 1976**.

[33] Teissier, B., "Variétés polaires II: Multiplicité polaires, sections planes, et conditions de Whitney," Actes de la conférence de géometrie algébrique à la Ràbida. *Springer Lecture Notes*, **961** (1981) 314-491.

[34] Teissier, B., "Cycles Evanescents et Résolution Simultanée, I et II," *Springer Lec. Notes*, **777** (1980).

[35] Varchenko, A. N., "The Relation between Topological and Algebro-Geometric Equisingularities According to Zariski," *Funct. Anal. Appl.*, **7** (1973) 87-90.

[36] Verdier, J. L., "Stratifications de Whitney et Théorème de Bertini-Sard," *Inventiones Math.*, **36** (1976) 295-312.

[37] Whitney, H., "Tangents to an analytic variety," *Ann. Math.*, **81** (1964) 496-549.

[38] Zariski, O., "Equisingular Points on Algebraic Varieties," *Seminari dell'Istituto Nazionale di Alta Matematica'*, (1962-1963).

[39] Zariski, O., "Studies in equisingulariy I. Equivalent Singularities of Plane Algebroid Curves," *Amer. J. Math.*, **87** (1965) 507-536.

[40] Zariski, O., "Contribution to the problem of equisingularity," *Centro Internazionale Matematico Estivo (C.I.M.E.), Questions on Algebraic varieties. III Ciclo, Varena*, (1969) 261-343.

[41] Zariski, O., "Studies in equisingulariy II. Equisingularity in co-dimension 1 (and characteristic zero)," *Amer. J. Math.*, **87** (1965) 972-1006.

[42] Zariski, O., "Some Open Questions in the Theory of Singularities," *Bull. Amer. Math.Soc.*, **77** (1971) 481-491.

[43] Zariski, O., "The Elusive Concept of Equisingularity and Related Questions," *Johns Hopkins Centennial Lectures (Supplement to the Amer. J. Math.*, (1977) 9-22.