ESSAYS ON ASSETS AND CONTINGENT COMMODITIES

by

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ABSTRACT

ESSAYS ON ASSETS AND CONTINGENT COMMODITIES

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Submitted to the Department of Economics on August 18, 1969 in partial fulfillment of the requirement for the degree of Doctor of Philosophy.

Formally, the last four chapters of this thesis are specializations of a general problem—that of choosing optimal lifetime consumption and portfolio plans for an individual whose lifetime is of uncertain length—which is described in Chapter I. Each chapter is also of independent interest.

Chapter II is an examination of the two-period portfolio model. The well-known model with one risky and one safe asset is summarized and extended. Conditions under which the two assets are gross substitutes are indicated. The properties of models in which there are many risky assets are then examined. It is shown that some results of the model with one risky asset do not extend to the model with many risk assets except in special cases. A Slutsky-Hicks equation for assets is derived. The special case of concentrated distributions is presented and strong results are derived for this case.

The asset model of Chapter II is related to the contingent commodity model in Chapter III. It is shown that, where there are fewer assets than states of nature, the asset model is equivalent to a contingent commodity model in which there are linear constraints on the choice of contingent commodities.

In Chapter IV the multi-period model where the date of death is certain is presented. Consumption functions for three classes of utility functions are presented. It is proved that these are the only classes of utility function which generate linear consumption functions.

The lifetime model with uncertainty over the date of death and with an insurance asset is analyzed in Chapter V. It is shown that life insurance is more likely to be purchased by individuals owning human capital than by individuals owning non-human capital. Time profiles of consumption and the purchases of insurance are derived from computer simulations. It is shown that an individual with a concave utility function is quite likely to reject the purchase of fair insurance.

Thesis Supervisor: Franklin M. Fisher
Title: Professor of Economics
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Many people have helped me during the writing of this thesis. Professors Duncan Foley, Franklin Fisher and Paul Samuelson constituted my thesis committee, and each was freely available for advice and discussion. Franklin Fisher took over as thesis supervisor when Duncan Foley went on leave.

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My wife Rhoda drew the diagrams and graphs, prepared the bibliography and helped in the tedious task of proofreading. I owe her thanks for far more.

Ellyce Anapolsky and Risa Goldberg typed the first draft. Mrs. Eileen Smith typed the final version of a very difficult manuscript with speed and accuracy.

Financial support from the Woodrow Wilson Foundation is recorded with thanks.

Miguel Sidrauski was to have been my thesis supervisor; his death was a deep loss, both personally and professionally. This thesis is dedicated to his memory.
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CHAPTER I

INTRODUCTION

I. Summary

The four essays which follow this chapter relate to the theory of the demand for assets. The problems investigated in the dissertation are described in the summary of the essays presented below.

In Chapter II—and in the remaining chapters—consumption and portfolio choice decisions are made simultaneously. The basic purpose of Chapter II is to study the properties of demand functions for assets in a two period model in a way analogous to the analysis of the demand for goods. Assets differ from goods in not providing utility directly. Most work in this area to date has either concentrated on the effects of changes in wealth on asset demands or used a particular utility function—the quadratic. An equation for assets analogous to the Slutsky-Hicks equation for goods is derived; wealth and substitution effects are distinguished. The two-asset and many-asset models are studied in detail.

The asset model of Chapter II is related to the well-known contingent commodity model in Chapter III. The contingent commodity model is one in which utility functions are additive and strong statements

---

1 Tobin [37] points out that the separation of these decisions has become entrenched in macroeconomic models. In [37] Tobin makes an attempt to move toward an integration of the two decisions.
on demand functions for contingent commodities can be made. It is shown that the asset model is equivalent to a contingent commodity model in which there are linear constraints on the choice of contingent commodities so long as there are fewer assets than states of nature.

In Chapter IV we take up the consumption and portfolio choice plans of an individual living a lifetime of known length. The qualitative properties of the demand functions do not change from period to period for three particular classes of utility functions. It is shown that these are the only utility functions for which demand functions are linear and explicit consumption functions for these utility functions are presented. A number of previously known results for these utility functions are related to the linearity properties to which they give rise.

In the final chapter a lifetime model in which the date of death is uncertain is developed. This allows the introduction of life insurance and is of interest both in making the lifetime model more realistic and as a way of handling terminal conditions. The model developed should be more amenable to empirical testing than other portfolio models because the mortality function is readily available. The introduction of life insurance represents a contribution toward bringing the study of an important group of financial intermediaries—insurance companies—within the scope of monetary economics.

Formally, each chapter is a special case of the general problem which we outline below. The general technique for analyzing the demand for assets in a discrete-time model has been set out by P. A. Samuelson [28];

\[1\]

\[1\]It is of interest that the portfolio problem was given as an exercise in Bellman [3], p. 47, No. 28, in 1957.
Samuelson explains the rather fine assumptions made--particularly with regard to the time at which the outcomes of events become known--at some length. Robert C. Merton [20] has studied the continuous time case where the return on risky assets is generated by a Wiener process. He provides explicit solutions for the paths of consumption and portfolio composition for utility functions exhibiting constant relative and constant absolute risk aversion. Samuelson gives an explicit solution for the path of consumption for utility functions of constant relative risk aversion (constant elasticity of marginal utility) and shows that portfolio composition in this case is constant where distributions of returns on risky assets are time-independent and not serially correlated.

In the remainder of this chapter we set out the general problem and indicate what specializations of this problem are made in subsequent chapters.

II. The Maximand

There is a maximum number of periods, T, for which the individual can live. He knows that if he is alive at T, he will be dead at T+1.\(^1\) Let \(d_t\) be the probability that the individual dies at the beginning of period \(t\), or the end of period \(t-1\); \(t=2, \ldots, T+1\). This distribution is known to the individual.

The maximand is expected utility of consumption over the individual's lifetime--as seen at the beginning of the lifetime--plus the

\(^1\)This assumption is commented upon at length in Chapter V.
expected utility of bequests. The utility function for consumption is assumed to be additive through time and in canonical form for the use of the von Neumann-Morgenstern expected utility theorem,\(^1\) and each utility function—that of consumption and that of bequests—is assumed strictly concave in each period with positive marginal utility of consumption and bequests respectively. For convenience we also assume all utility functions thrice differentiable.

The individual may live a life of 1 year, 2 years, \(..., T\) years. If he dies at the end of the first year his total utility is

\[
(1) \quad U_1(C_1) + V_2(G_2)
\]

where \(U_1(C_1)\) is the first period utility of consumption function and \(V_2(G_2)\) is the utility of bequests (denoted \(C_2\)) left in the second period. The probability that his utility is given by (1) is \(\bar{p}_2\).

Similarly, if the individual lives two years his utility is given by

\[
(2) \quad U_1(C_1) + U_2(C_2) + V_3(G_3)
\]

where the notation is obvious. The probability that he lives only two years is \(\bar{p}_3\).

Writing out the relevant expressions for utility for lifetimes of 3, \(..., T\) years, and weighting each expression by the probability with which it may be relevant we obtain for the expected value of the maximand, \(U\),\(^2\)

\[\text{---}\]

\(^1\)See Pollak [24].

\(^2\)Yaari [39], examining an insurance problem in continuous time, names the analogous continuous-time maximand a "Fisher utility function."
\begin{align}
E_1[U] &= E_1 \left[ \sum_{t=1}^{T} \left( \bar{\pi}_t^a U_t(C_t) + \bar{\pi}_{t+1}^d V_{t+1}(C_{t+1}) \right) \right] \\
\text{where } \bar{\pi}_t^a = \sum_{t+1}^{T+1} \bar{\pi}_z^d \text{ is the probability of being alive in period } t.
\end{align}

We use the expectation operator in (3) to indicate that in the problem to be analyzed none of the variables in the utility functions of (3) except \(C_t\) is determinate at \(t=1\). The subscripted 1 on the expectation operator indicates that the expectation is evaluated in period 1.

III. Assets

In each period, \(t\), the individual may purchase or sell one-period bonds, which bear a one-period interest rate of \(r_t\). We define \((1+r_t) = R_t\). The individual is assumed to know, with certainty, all one-period interest rates over his lifetime; this is a strong requirement which could be dispensed with by making bonds an uncertain asset. The quantity of bonds bought in period \(t\) is denoted \(B_t\).

In addition, there are \((m-1)\) risky assets—equities—each having a random one-period rate of return \(X_{i}^t - 1\); \(i = 1, \ldots, m-1\). Limited liability ensures \(X_{i}^t - 1 \geq -1\); no more can be lost on a stock than is invested in it. It is useful to note that

\begin{align}
X_{i}^t &= \frac{p_{i}^t}{p_{i}^{t-1}}
\end{align}

where \(p_{i}^t\) is the price of the \(i\)'th stock in period \(t\) per unit of the consumption good. \(p_{i}^{t-1}\) is to be understood as including any dividends.
In this chapter we shall not work directly with the price of a given stock but rather with the quantity that can be purchased with one unit of the consumption good. Had our primary concern been with market equilibrium, then it would have been preferable to work with prices.\footnote{Stone \cite{34} develops a theory of asset pricing.}

The one-period rate of return on each stock is assumed by the investor to be a function of the state of nature, $\theta$, which will obtain in the next period. Thus

\begin{equation}
X_i = X_i(\theta) \quad i = 1, \ldots, m-1.
\end{equation}

$\theta$ may either take only a finite number of values or may be a continuous random variable. The distributions may be subjective to the particular investor, but he is assumed to attach some distribution to the rate of return on each asset. This assumption is important and unrealistic; it ignores costs of obtaining information which may explain many asset-market phenomena. For instance, in unconstrained portfolio problems with more assets than states of nature, the investor in our model will be shown to take a position in every asset; this does not accord with casual observation.

We shall in general assume that the distributions of the $X_i(\theta)$ are independent of time. From (4) we see that an independence assumption on the $X_i(\theta)$ distributions necessarily implies Markovian distributions for $P^t_i(\theta)$. Suppose for example that

\begin{equation}
\text{Prob} \left[ X_i = \frac{1}{2} \right] = \frac{1}{2} = \text{Prob} \left[ X_i = 2 \right]
\end{equation}
i.e., the stock price is as likely to halve as to double each period. Then where \( P_1 \) = 1, the possible courses of prices through \( t=1, 2, 3, 4 \) is illustrated by Figure 1 where each vertex in any given time period is equiprobable. This assumption is equivalent to that of unitary elasticity of price expectations.\(^1\) A belief that "what goes up must come down" is not consistent with the independence assumption on \( X_1(\theta) \) but would be consistent with the Markovian assumption.

\[
\begin{align*}
&1 \\
&\frac{\frac{1}{2}}{1} \\
&\frac{\frac{1}{4}}{1} \\
&\frac{\frac{1}{8}}{1} \\
&\frac{\frac{1}{2}}{2} \\
&\frac{\frac{1}{2}}{2} \\
&\frac{\frac{1}{2}}{2} \\
&\frac{8}{1} \\
&\frac{t=1}{t=2} \\
&\frac{t=3}{t=4}
\end{align*}
\]

Figure 1

The value (in consumption goods) of each asset bought in period \( t \) is denoted \( A_t \). We do not impose restrictions on short sales of assets in this dissertation.\(^2\)

---

\(^1\)Hicks [17], pp. 204-205.

\(^2\)Myers [22] introduces non-negativity constraints explicitly in the context of a two-period model.
The life insurance asset we introduce is single period term insurance. (Multi-period term insurance is discussed in Chapter V, Section VI.) The amount of premiums paid in any period is \( I_t \) and the value of payments received in the following period by the beneficiaries in the event of the policy-holder's death is \( Q_t I_t \). \( 1/Q_t \) is accordingly the premium rate. If the policy-holder lives no payments are made and he has no coverage for the next period. Since bonds pay off whether the individual is dead or alive and insurance only if he is dead, we require \( Q_t > R_t \).

IV. Budget Constraints

The individual's initial wealth, \( W_1 \), is given. This wealth may be consumed and/or invested, where investments include the cost of insurance premiums. Thus the first period budget constraint is just

\[
W_1 = \sum_{i=1}^{m-1} \left( A_i + B_i + C_i + I_i \right)
\]

Note that we are assuming that the individual receives no labor income.\(^1\)

In Chapters IV and V we allow labor income in some sections and indicate how the budget constraints are modified. In general, though, all income is derived from asset holdings.

\(^1\)We cannot in general introduce income by taking \( W_t \) to include the present value of all income: firstly, asset demands are different functions of wealth and income (shown in Chapter V); secondly, the discount rate has to be modified to take account of the probability of dying and it is shown in Chapter V that the discount factor depends on the cost of insurance.
Period 2 wealth is

$$W_2 = \sum_{i=1}^{m-1} A_i^2 X_i^2(\theta) + B_1 R_1$$

If the individual dies at the end of period 1 his bequests are

$$G_2 = W_2 + Q_1 I_1$$

In general the constraints are

$$W_t = \sum_{i=1}^{m-1} A_i^t + B_t + I_t + C_t$$

where

$$W_t = \sum_{i=1}^{m-1} A_i^{t-1} X_i^t(\theta) + B_{t-1} R_{t-1}$$

and

$$G_t = W_t + Q_{t-1} I_{t-1}$$

V. Method of Solution

For the finite horizon problem we start at the end; in the last period in which the individual is alive he has only to choose his consumption and the composition of his portfolio. No insurance can be bought in this period since there is no uncertainty of survival—death is certain. Define

$$J_1[W_T] = \max \left\{ \begin{array}{c} C_T \\ A_i^T \\ \end{array} \right\} \sum_{i=1}^{m-1} V_T \left( \sum_{i=1}^{m-1} A_i^T (X_i^T(\theta) - R_T) + (W_T - C_T) R_T \right)$$
where we have substituted in from the budget constraint for \( B_T \).

Finding \( J_T[W_T] \) is simply the two period portfolio problem with no bequests, slightly recast, for if there were no bequests and the individual were alive at \( T+1 \) he would consume all his wealth. It is this problem that we take up in Chapters II and III.

The first order conditions for an unconstrained interior maximum are

\[
(14) \quad 0 = U_T'(C_T) - R_T E_T[V_{T+1}'(G_{T+1})] \\
(15) \quad 0 = E_T[V_{T+1}'(X_{T+1}^i(\theta) - R_T)] \quad i = 1, \ldots, m-1
\]

(For the definition of \( G_{T+1} \) refer to either (11) and (12) or to (13).)

It will be shown in Chapter II that if an interior solution exists, it is a maximum. In the absence of explicit constraints on \( C \) and \( G \) or conditions on the utility functions there is no guarantee that \( C_T > 0 \) or \( G_{T+1} \geq 0 \). We shall assume that a solution exists in which \( C_T > 0 \) and \( G_{T+1} > 0 \).

The solution for \( J_T[W_T] \) is then used in the previous period's optimizing problem:

\[
(16) \quad J_2[W_{T-1}] = \max \quad U_{T-1}(C_{T-1}) + \Pi^a_T E_{T-1}[J_T[W_T]] + \Pi^m_T E_{T-1}[V_T(G_T)] \\
\begin{align*}
C_{T-1} \\
A_{T-1} \\
I_{T-1}
\end{align*}
\]

\(^{1}\)E is subscripted by \( T \) to indicate that the distributions are conditional on \( \theta^T \) if they are Markovian.
where $\Pi_T^d$ denotes the conditional probability of death in period $T$, conditional on having survived to period $T-1$; $\Pi_T^a$ is the conditional probability of living to period $T$.

First order conditions are:

\begin{align*}
(17) \quad 0 &= U_{T-1}(C_{T-1}) - R_{T-1} \Pi_T^a E_{T-1}[J_1'(W_{T})] - R_{T-1} \Pi_T^d E_{T-1}[V_T'(G_T)] \\
(18) \quad 0 &= \Pi_T^a E_{T-1}[J_1'(W_{T})(X^T_1(\theta) - R_{T-1})] + \Pi_T^d E_{T-1}[V_T'(G_T)(X^T_1(\theta) - R_{T-1})] \\
&\quad i = 1, \ldots, m-1 \\
(19) \quad 0 &= -R_{T-1} \Pi_T^a E_{T-1}[J_1'(W_{T})] + (Q_{T-1} - R_{T-1}) \Pi_T^d E_{T-1}[V_T'(G_T)]
\end{align*}

The interpretation of (17)-(19) is straightforward. (17) simply states that the marginal utility of current consumption is equal to the weighted (by probabilities) sum of expected marginal utilities of consumption in period $T$ if alive and expected marginal utility of bequests if dead. The $R$ factor indicates the rate at which marginal utilities in the two periods can be substituted. (18) is more easily interpreted as

\begin{align*}
(18') \quad 0 &= U_{T-1}(C_{T-1}) - \Pi_T^a E_{T-1}[J_1'(W_{T})X^T_1(\theta)] - \Pi_T^d E_{T-1}[V_T'(G_T)X^T_1(\theta)]
\end{align*}

This is similar to (17) except that the marginal rate of substitution is not now the certain $R_{T-1}$ but the stochastic rate $X^T_1$. Instead of having a certain rate of substitution of consumption between the two periods as in (17) we now in effect are working with the expected
rates of substitution.\textsuperscript{1} From (19) we see that the expected gain in marginal utility from buying insurance must be equal to the loss of marginal utility which would be entailed by borrowing to pay the premiums.

These results are analogous to those that would be obtained under certainty if assets were treated as goods and expected marginal utilities as actual marginal utilities.

The first order conditions for earlier periods are analogous to those of (17)-(19) with $t$ appearing for $T-1$ and $t+1$ for $T$. It can be shown that the probability weighting on the utility and bequest functions for each period which result from continued backward maximizing to obtain $J_t[W_t]$, are those given by (3), as they have to be.

In Chapter IV we specialize the multiperiod problem just outlined by assuming $\pi^d_t = 0$, $t = 2, \ldots, T$, $\pi^d_T = 1$ --- that is, that there is no uncertainty of survival. In Chapter V we examine the full problem but work with a particular class of utility functions.

VI. [Further] Special Assumptions

We shall assume henceforth

\begin{equation}
U_t(C_t) = \frac{U(C_t)}{(1+\lambda)^{t-1}} \quad t = 1, \ldots, T.
\end{equation}

\textsuperscript{1}We also have the "util-prob" results of Samuelson and Merton [29], i.e.,

\begin{equation}
\pi^d_t E_{T-1}[J'_1(W_t)(X_i(\theta) - X_j(\theta))] + \pi^d_t E_{T-1}[V'_T(C_T)(X_i(\theta) - X_j(\theta))] = 0
\end{equation}
(ii) \( V_t(G_t) = \hat{b}(t)V(G_t) \) \( t = 2, \ldots, T+1 \)

Earlier assumptions repeated here are

\[ U'(C_t) > 0 \quad U''(C_t) < 0 \quad V'(G_t) > 0 \quad V''(G_t) < 0 \]

and \( U'''(C_t) \) and \( V'''(G_t) \) exist.

In general the \( \hat{b}_t \) function (smoothed) would probably look something like Figure 2; the sharp early rise occurs as family obligations grow.
CHAPTER II

TWO-PERIOD PORTFOLIO MODELS

I. Introduction and Review

We shall be treating the two-period problem solely from the viewpoint of individual equilibrium and not considering conditions for market equilibrium. From the viewpoint of the individual, the two-period problem is one in which he has an initial endowment of wealth, part of which is to be consumed immediately; the remainder is invested in a bond and/or in risky assets. Then one period later the returns on the risky assets are known and the investor consumes all his assets as of that time. Alternatively one may think of the two-period model as a model of the decisions facing someone in the last period of his life--this is the problem described in Chapter I, (13), where we are to find \( J_1(W_t) \).

There is by now an extensive literature on the two-period model. The pioneering contributions are those of Markowitz [19] and Tobin [36] which deal with portfolio decisions which can be made on the basis of the first and second moments of the probability distributions of returns on risky assets--usually referred to as mean-variance

\[1\] We have chosen to work with a model including one safe asset. This choice is not without content: many of the qualitative properties of the solution depend heavily on the existence of the safe asset. This is particularly true in multi-period models where knowledge of future interest rates enables the investor to behave myopically in some cases. See Mossin [21].
Later work on the two-period portfolio problem seems to have developed largely out of Arrow's Helsinki lectures [2]. In these lectures Arrow introduced measures of risk aversion which characterize the individual's utility function. These measures—which will be used repeatedly in this chapter—are absolute risk aversion, \( \gamma(C) \)

\[
\gamma(C) = - \frac{U''(C)}{U'(C)}
\]

and relative risk aversion, \( \beta(C) \)

\[
\beta(C) = - \frac{U''(C)C}{U'(C)}
\]

\( \beta(C) \) is also the elasticity of the marginal utility of \( C \). It is to be understood that \( C \) is the argument of the relevant utility function: in our case utility is attached to consumption but in Arrow's lectures it is attached to wealth. These measures are invariant under positive affine transformations of the utility function.

The measures were introduced independently by Pratt [25] who also gave utility functions for which they are constant. These are: constant absolute risk aversion utility functions (CARA is the designation we shall use subsequently),

\[
U(C) = -\frac{e^{-\gamma C}}{\gamma}, \quad \gamma > 0
\]

\[^1\text{Though Pratt's important article [25] which overlaps some of Arrow's work was published before the Arrow lectures.}\]
and constant relative risk aversion utility functions (CRRA)

\[ U(C) = \frac{C^{1-\beta}}{1-\beta} , \quad \beta > 0 \]

The behavioral interpretation of the measures of risk aversion given by Arrow [2], p. 34, is useful and makes for an intuitive understanding of some subsequent results. The index of absolute risk aversion measures the favorable odds which an individual demands to accept a given small bet. If \( \gamma \) is constant he demands the same odds at all levels of wealth; if \( \gamma \) is decreasing he demands less favorable odds as he becomes wealthier. Similarly, the index of relative risk aversion measures the odds which an individual demands to accept a bet which is a given small proportion of his wealth; for \( \beta \) constant he demands the same odds at all levels of wealth.

For reasons having to do with observation of betting behavior and the need for utility to be bounded to prove the expected utility theorem, Arrow believes that absolute risk aversion is decreasing and relative risk aversion increasing. On balance he believes the index of relative risk aversion hovers about unity, being greater than unity for small values of wealth and less than unity for large values of

\[ 1 \text{The logarithmic utility function belongs to the CRRA family: it is the function obtained in the limit by letting } \beta \to 1. \]

Evaluating \( \lim_{\beta \to 1} \frac{C^{1-\beta} - 1}{1-\beta} \) by L'Hospital's rule, we have

\[ \lim_{\beta \to 1} \frac{C^{1-\beta} - 1}{1-\beta} = \frac{-\log C}{-1} = \log C \]
of wealth. This ensures boundedness below and above respectively.\(^1\)

Arrow then proves a number of results in a portfolio model with one safe asset (called "money") and one risky asset—there is no consumption. These are: the risky asset is held in positive amounts if and only if its mean return exceeds the safe rate of return; holdings of the risky asset increase with wealth if and only if absolute risk aversion is decreasing; the elasticity of demand for money is at least one for an individual whose utility function shows increasing relative risk aversion; a uniform increase in the rate of return on the risky asset across all states of nature decreases the demand for the safe asset for utility functions of decreasing absolute risk aversion.\(^2\)

Diamond's paper \([6]\) examines the two-period model with utility being a function (not necessarily additive) of consumption in the two periods, and with one safe and one risky asset. Diamond analyzes changes in risk where the distribution of returns on the risky asset is "squeezed in" uniformly about the safe rate of return: as a simple discrete example consider the distribution in which \(X_1 = 0.5\) and \(X_2 = 2.0\), each with probability \(\frac{1}{2}\), and \(R = 1.1\). Then a new distribution which Diamond would analyze would be \(\tilde{X}_1 = 0.7\) and \(\tilde{X}_2 = 1.7\), each with

\(^1\)It would be most convenient if it turns out that utility functions are logarithmic, since the consumption decision is then independent of financial variables. See Samuelson \([28]\) for a comment on Arrow's insistence on boundedness.

\(^2\)Diamond \([6]\), p. 11, shows that this last result does not go through to a model where there is also a consumption decision.
probability $\frac{1}{2}$. This fulfills his requirement since

$$\frac{X_1 - R}{X_1 - R} = \frac{2}{3} = \frac{X_2 - R}{X_2 - R}$$

We shall see in Chapter III why this is a particularly interesting case.

Sandmo's [30] model is the same as that of Diamond except that his utility function is additive. Sandmo finds that the Arrow result on the wealth effect for the risky asset goes through to a model which includes consumption. Using Arrow's hypothesis that absolute risk aversion is decreasing and relative risk aversion increasing he shows that both assets and consumption are normal goods if both are held in positive amounts. He does not consider the case of borrowing, partly since he identifies his safe asset with money and not bonds. We shall show below that bonds may not be normal goods if borrowing is allowed, even under Sandmo's restrictions on the utility function. He also examines the effects on consumption and portfolio choice of a parallel shift in the density function of the rate of return on the risky asset, i.e., when the rate of return on the risky asset increases uniformly across all states of nature.¹

We have stated that both Arrow and Sandmo identify the safe asset with money. In addition, they claim backing for their result that the wealth elasticity of the demand for the safe asset is greater than one by referring to empirical studies of the demand for money. This seems

¹There are errors in some proofs in this paper. We show this in Appendix 1 to this chapter, where we also comment on some other features of Sandmo's paper.
highly implausible. Money is dominated in portfolio models by bonds, and explanations of the demand for money should not be sought primarily in this framework.

Other work in this area is that of Dreze and Modigliani [9] and Stone [34]. Stone is chiefly concerned with market equilibrium but his thesis is of interest both for its summary of the literature and for its introduction of a new measure of risk. His measure of risk reflects properties of both the utility function and the probability distributions of returns on risky assets: it is

$$\phi = E[\tilde{W} - U(\tilde{W})]$$

where $\tilde{W}$ is the stochastic level of wealth obtained by choosing some portfolio. Stone normalizes utility functions as to origin and scale and shows that equilibrium for the individual requires the ratio of the differences between expected return and marginal risk entailed by buying an extra unit of a security to be proportional to their prices. This is a very attractive result but we find it more useful to concentrate on measures of risk which relate solely to the utility function.

Since so much is known about the one-risky, one-safe asset model with or without consumption, we shall not develop it fully. We shall, in Sections III-V, examine the multiasset model at some length. Before doing so we shall comment in Section II on some fresh results which can be obtained in the two asset model. In Section VI we examine the case of concentrated distributions.
II. The One-Risky Asset Model

Formally we now take up equations (14) and (15) of Chapter I and specialize by assuming \( m=2 \): there is only one risky asset. We normalize the weighting of the utility of bequests function by setting \( \hat{\delta}(t+1)=1 \). The time subscripts on equations I (14)-(15) will be dropped; this should not cause any confusion.

The first order conditions are now:

(5) \[ 0 = U'(C) - R \mathbb{E} [V'(G)] \]

(6) \[ 0 = \mathbb{E}[V'(G)(X-R)] \]

As a reminder \( G = (W-C)R + A(X-R) \), and \( X \) is the return on the risky asset. Henceforth we shall write \( Z \equiv (X-R) \).

Sufficient second order conditions for a maximum are

(7) \[
\begin{align*}
J_{CC} &< 0 \\
J_{AA} &< 0 \\
J_{CA} &> 0 \\
J_{AC} &> 0
\end{align*}
\]

where \( J \) is the maximand defined in equation (13), Chapter I.

The Hessian of the system is given by

(8) \[
[H] = \begin{bmatrix}
U''(C) + R^2 \mathbb{E}[V''(G)] & - R \mathbb{E}[V''Z] \\
- R \mathbb{E}[V''Z] & \mathbb{E}[V''Z^2]
\end{bmatrix}
\]

We show in Section III that the conditions (7) are satisfied.

The questions we take up in this section—utilizing (5), (6) and (7)—are I. Does Arrow's result that the risky asset is held only if
its mean return exceeds the safe interest rate go through to this model including consumption?

2. Are there any similar conditions on bond holdings?

3. It is frequently taken in the literature that assets are gross substitutes.\(^1\) We know that in models of behavior under certainty substitution properties are least ambiguous when there are only two goods. Are bonds and equity (which is what we call the risky asset from now on) gross substitutes in general, or for particular types of utility function?

To discuss this we introduce a shift parameter, \(h\), in the density function of \(X\). Let

\[
(9) \quad X' = X + h
\]

so that an increase in \(h\) (from zero) results in an increase in the mean return on equities but changes no other moments. To discuss gross substitution we shall consider the effects of increases in \(R\) and in \(h\) on the demands for the two assets.\(^2\)

Another shift parameter we introduce is \(t\) -- note \(t\) has nothing to do with time and is simply a shift parameter -- where

\[
(10) \quad X'' = X + t (\bar{X} - X)
\]

\(^1\)For instance, Tobin and Brainard [38], Brainard [5], and Foley and Sidrauski [13].

\(^2\)Our \(h\) is the same as Sandmo's \(\gamma\), \textit{op. cit.}. We show in Chapter III that an increase in \(h\) has similar effects on asset demands to a multiplicative increase in the rate of return across all states of nature--except for multiplication by a positive constant. This is shown there to be a consequence of the existence of the safe asset.
so that an increase in \( t \) (from zero) decreases the variance of \( X \). A decrease in \( t \) will also affect other moments.

\[
E[(X' - \overline{X})^k] = E[(X - \overline{X})^k(1-t)^k], \text{ so that an increase in } t \text{ reduces the absolute value of all non-zero moments but the first.}
\]

4. Given our measure of dispersion -- viz., \( t \) -- what effect does an increase in \( t \) have on consumption?

1. The risky asset is held if and only if its mean return is greater than the return on bonds.

Proof: Consider equation (6)

\[
(6) \quad 0 = E[V'((W-C)R + A(X-R))(X-R)]
\]

From (6) it follows that

\[
(11) \quad E[V'(G)(X-\overline{X})] \geq 0 \text{ as } \overline{X} \lessgtr R \text{ since } E[V'(G)] > 0
\]

But \( E[V'(G)(X-\overline{X})] = \text{covariance} [V'(G)X] \).

Since \( V'(G) \) is monotone decreasing in its argument it will be decreasing in \( X \) depending on whether \( A \lessgtr 0 \). Thus

\[
(12) \quad [V'(G)X] \geq 0 \text{ as } A \lessgtr 0
\]

From (11) and (12):

\[
(13) \quad A \lessgtr 0 \text{ as } \overline{X} \lessgtr R
\]

---

\(^1\) The operation described in (10) where \( t \) is free to vary defines a family of what Rothschild and Stiglitz [26] call "mean preserving spreads."
2. Bond Holdings

The question now arises: is there a similar condition which ensures that bond holdings will be positive? One obvious condition is $R > \bar{X}$, but we shall be assuming $\bar{X} > R$. Actually, there seems to be no particular reason to require an interior solution in which bond holdings are non-negative, though those who identify the safe asset with money generally require this.

It is not a consequence of an implicit assumption that bankruptcy in the second period is to be avoided that bond holdings should be positive. All that non-bankruptcy requires is that

$$AX_{\text{min}} + BR > 0$$

where $X_{\text{min}}$ is the minimum return on the risky asset which occurs with non-zero probability. Since $X = 0$ is the case where the return on a stock is $-100\%$, the condition $B > 0$ is unlikely to emerge from non-bankruptcy conditions.\(^1\)

\(^1\)It is as well here to point out another condition we have been implicitly assuming, and which will be formalized as the "no-arbitrage condition" in Chapter III. Designating $X_{\text{max}}$ the maximum return on the risky asset which occurs with non-zero probability, we require

$$X_{\text{min}} < R < X_{\text{max}}$$

For if $X_{\text{min}} > R$ then the individual would borrow as much as he could, and if $X_{\text{max}} < R$ the individual would short the stock as much as he could. The distribution of $X$ is subjective so that (15) might not hold for given $R$ and price of equity (which we are now taking to be unity). If we were considering market equilibrium it is clear that we would require
We show in Appendix 2 to this chapter that sufficient conditions to ensure non-negativity of consumption and bequests are

\begin{equation}
\lim_{C \to 0^+} U'(C) = \infty
\end{equation}

and

\begin{equation}
\lim_{G \to 0^+} V'(G) = \infty
\end{equation}

These conditions (16) and (17) are helpful, but not greatly so. They do, for instance, apply to utility functions of constant relative risk aversion. They do not, however, apply for CARA utility functions. But note that condition (14) should not be expected to guarantee non-negativity of bond holdings.

3. Gross Substitution

In this section we shall be using equations (5) and (6) for various comparative static exercises. We want to obtain the derivatives $\frac{\partial A}{\partial h}, \frac{\partial A}{\partial R}, \frac{\partial B}{\partial h}, \frac{\partial B}{\partial R}$ We shall also comment on wealth effects for bonds.

(For the definition of $h$ refer to equation (9) of this chapter). We

\[
\max_i \frac{X_i^i}{\min_i} < R < \min_i \frac{X_i^i}{\max_i} \quad i=1, \ldots, S
\]

where there are $S$ individuals. Again, since probabilities are subjective we might well have $\max_i X_i^i_{\min} > \min_i X_i^i_{\max}$. One might then require a constrained maximum in which limits on both borrowing and shorting are imposed. (There are doubtless other reasons for the existence of such constraints in practice, such as lack of confidence in the subjective distributions of others by lenders.)
shall denote the determinant of the Hessian of the system (5) and (6) by $|H|$. 

$$
(18) \quad \frac{\partial A}{\partial h} = - \frac{\partial W}{\partial h} \bigg|_{J=J} \frac{\partial A}{\partial W} \frac{E[V'(G)][R^2E[V''(G)] + U''(C)]}{|H|}
$$

The second term is the substitution term which is always positive; the first term is the wealth effect which is positive for decreasing absolute risk aversion. (Note $- \frac{\partial W}{\partial h} |_{J=J}$ is the compensating variation in wealth which holds utility constant when $h$ increases.) Thus this expression is positive for decreasing or constant absolute risk aversion so long as $A$ is positive (i.e., $\bar{X} > R$).

$$
(19) \quad \frac{\partial A}{\partial R} = B \frac{\partial A}{\partial W} + \frac{E[V'(G)][U''(C) + RE[V''(G)X]]}{|H|}
$$

The second term is an unambiguously negative substitution effect and the first term is a wealth effect which is negative for $B < 0$ and decreasing absolute risk aversion.

From (18) and (19) we conclude that for an individual who is borrowing and who has decreasing absolute risk aversion,

$$
\frac{\partial A}{\partial h} > 0 \quad \text{and} \quad \frac{\partial A}{\partial R} < 0
$$

We continue now to bond holdings. To discuss $\frac{\partial B}{\partial h}$ and $\frac{\partial B}{\partial R}$ we have first to consider wealth effects on the demand for bonds.
\[
\frac{3B}{3W} = \frac{u''(C) E[v''(G)zX]}{|H|}
\]

To obtain a clearer idea of the properties of this expression we define

\[
K(X;A,B) = -\frac{v''(G)X}{v'(G)}
\]

It may be shown that \( \frac{3B}{3W} \geq 0 \) as \( K'(X;A,B) \geq 0 \). \( K(X;A,B) \) is a weighted combination of the measures of relative and absolute risk aversion:

\[
(21) \quad BR_Y + AK(X;A,B) = B
\]

Now if \( A \) is positive and \( B \) positive, then \( \frac{3B}{3W} \) is positive for constant or increasing relative risk aversion. Figure 1 illustrates what we may deduce from (21) for \( A > 0 \) and \( B > 0 \).

Where \( B < 0 \), then for constant absolute risk aversion \( \frac{3B}{3W} > 0 \) and for constant relative risk aversion \( \frac{3B}{3W} < 0 \). This case is summarized in Figure 2.

On the hypothesis of decreasing absolute and increasing relative risk aversion, then, both assets are normal if both are held. But for an individual who is borrowing, an increase in wealth leads to more borrowing if relative risk aversion is constant and less borrowing if absolute risk aversion is constant.

---

1B is an endogenous and not an exogenous variable; it would be more satisfactory to know in terms of the exogenous variable when the cases \( B < 0 \) and \( B > 0 \) are applicable. There is some interest in Figures 1 and 2 though since we can always observe whether \( B \geq 0 \) for a particular individual.
\[ \frac{\partial B}{\partial R} = B [3B - 1] - \frac{E[V'(G)]U''(C) + E[V''(G)X^2]}{|H|} \]

The sum of the last term and \(-B/R\) is the wealth-compensated substitution.
effect: it is definitely positive only for an individual who is borrowing. We shall see in Chapter III where we define "transformation terms" why the expression \(-B/R\) appears here. But (from Figure 2) for an individual who is borrowing, \(\frac{B}{R} \frac{\partial B}{\partial W}\) will definitely be positive only for utility functions of diminishing, constant or only slightly increasing relative risk aversion. In other cases \(\frac{\partial B}{\partial R}\) is ambiguous.

\[
(23) \quad \frac{\partial B}{\partial h} = A \left[ \frac{\partial B}{\partial h} - 1 \right] + \frac{E[V'(G)][U''(C) + R E[V''(G)X]]}{|H|} \]

The substitution term is now always negative (for \(A > 0\)); thus where \(\frac{\partial B}{\partial W} < 0\), \(\frac{\partial B}{\partial h}\) is negative. The wealth effect will definitely be negative \(\frac{\partial B}{\partial h}\) when \(B < 0\) (the individual is borrowing) and relative risk aversion is slightly increasing, constant or diminishing (See Figure 2).

Summarizing this discussion on gross substitution we do have a strong proposition: if relative risk aversion is constant or only slightly increasing, and the individual is borrowing, the two assets are gross substitutes.

\[
(24) \quad \frac{\partial A}{\partial h} > 0, \quad \frac{\partial A}{\partial R} < 0, \quad \frac{\partial B}{\partial R} > 0, \quad \frac{\partial B}{\partial h} > 0 \quad (\text{for } B < 0). \]

It is interesting that it is precisely this type of utility function that Arrow believes most likely. Notice, though, that we cannot show gross substitution for an individual who is not borrowing even for these types of utility function.
4. The effects of a decrease in variance on consumption.

We want now to derive \( \frac{\partial C}{\partial t} \). (For a definition of \( t \), see equation (10) of this chapter.)

\[
\frac{\partial C}{\partial t} = - \frac{\partial W}{\partial t} \bigg|_{J=J} \frac{\partial C}{\partial W} \left[ \frac{R}{H} E[V''(G)G] \right] E[V'(X-\bar{X})]
\]

The wealth effect of a reduction in variance (more strictly, an increase in \( t \)) is always positive; the second term is zero for constant absolute risk aversion and negative for decreasing absolute risk aversion (so long as \( \bar{X} > R \)). Thus for utility functions with the property of decreasing absolute risk aversion wealth and substitution effects of a change in \( t \) work in opposite directions. But for increasing absolute risk aversion (a property of the quadratic) \( \frac{\partial C}{\partial t} > 0 \).

5. Summary of comparative statics for the two-period two-asset model.

This summary is presented in Table I.

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1 Recall that an increase in \( t \) affects all non-zero moments other than the first.
### TABLE I

QUALITATIVE PROPERTIES OF SOLUTION TO TWO-PERIOD TWO-ASSET MODEL

(Assuming $x > R$, which implies $A > 0$)

<table>
<thead>
<tr>
<th>Utility Function</th>
<th>$\frac{\partial C}{\partial W}$</th>
<th>$\frac{\partial C}{\partial r}$</th>
<th>$\frac{\partial C}{\partial h}$</th>
<th>$\frac{\partial C}{\partial t}$</th>
<th>$\frac{\partial A}{\partial W}$</th>
<th>$\frac{\partial A}{\partial R}$</th>
<th>$\frac{\partial A}{\partial h}$</th>
<th>$\frac{\partial A}{\partial t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Increasing</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$&lt; 0$ for $B &gt; 0$</td>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
<td>$&lt; 0$</td>
</tr>
<tr>
<td>Absolute Risk</td>
<td>$0 &lt; \frac{\partial C}{\partial W} &lt; 1$</td>
<td>$\geq 0$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
<td></td>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
</tr>
<tr>
<td>Aversion</td>
<td>$\geq 0$</td>
<td></td>
<td></td>
<td></td>
<td>$&lt; 0$ for $B &gt; 0$</td>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
<td>$&lt; 0$</td>
</tr>
<tr>
<td>Constant</td>
<td>$\frac{R}{1+R}$</td>
<td></td>
<td></td>
<td></td>
<td>$&lt; 0$</td>
<td>$&lt; 0$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
</tr>
<tr>
<td>Absolute Risk</td>
<td>$\frac{BR}{\gamma R(1+R)}$</td>
<td>$\frac{A \gamma}{\gamma (1+R)}$</td>
<td>$\frac{A \gamma (X-R)}{\gamma (1+R)}$</td>
<td>$0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Aversion</td>
<td>$&lt; 0$ for $B &gt; 0$</td>
<td>$&gt; 0$</td>
<td>$&lt; 0$</td>
<td>$&gt; 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Decreasing</td>
<td>$0 &lt; \frac{\partial C}{\partial W} &lt; 0$</td>
<td>$\leq 0$</td>
<td>$\leq 0$</td>
<td>$\leq 0$</td>
<td>$&lt; 0$ for $B &lt; 0$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
</tr>
<tr>
<td>Absolute Risk</td>
<td>$0 &lt; \frac{\partial C}{\partial W} &lt; 0$</td>
<td>$&lt; 0$</td>
<td>$&lt; 0$</td>
<td>$&lt; 0$</td>
<td>$&lt; 0$ for $B &lt; 0$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
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<tr>
<td>Aversion</td>
<td>$&lt; 0$ for $B &lt; 0$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td>$\frac{C}{W}$</td>
<td>$- \frac{B(1-B)}{R B W C}$</td>
<td>$- \frac{A(1-B)}{R B W C}$</td>
<td>$\frac{A(1-B)(X-R)}{R B (W/C)}$</td>
<td>$\frac{A}{W}$</td>
<td>$&lt; 0$ for $B &lt; 0$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
</tr>
<tr>
<td>Relative Risk</td>
<td>$\frac{A}{W}$</td>
<td>$&lt; 0$ for $B &lt; 0$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Aversion</td>
<td>$&lt; 0$ for $B &gt; 0$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
<td>$&gt; 0$</td>
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<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Contd. on P. 30A
<table>
<thead>
<tr>
<th>( \frac{\partial B}{\partial W} )</th>
<th>( \frac{\partial B}{\partial r} )</th>
<th>( \frac{\partial B}{\partial h} )</th>
<th>( \frac{\partial B}{\partial t} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>&gt; 0 for ( B &gt; 0 )</td>
<td>&gt; 0</td>
<td>( \gtrless ) 0</td>
<td>&lt; 0 for ( B &gt; 0 )</td>
</tr>
<tr>
<td>( \frac{1}{1+R} )</td>
<td>&gt; 0 for ( B &lt; 0 )</td>
<td>( \gtrless ) 0</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>&gt; 0 for ( B &gt; 0 )</td>
<td>( \gtrless ) 0</td>
<td>( \gtrless ) 0</td>
<td>( \gtrless ) 0</td>
</tr>
<tr>
<td>( \frac{B}{W} )</td>
<td>&gt; 0 for ( B &lt; 0 )</td>
<td>&lt; 0 for ( B &lt; 0 )</td>
<td>( \gtrless ) 0</td>
</tr>
</tbody>
</table>

Contd. from P. 30
III. The Many-Risky Asset Model.

In this section we restate the maximand and first order conditions, prove that second orders conditions are satisfied, and show that the Arrow result in the single risky asset model that a risky asset is held in positive amounts if and only if its mean return exceeds the interest rate does not go through to this model.¹

1. The maximand is as given in equation (13) of Chapter I:

\[ J(W) = \text{Max}_{C,A} \left\{ U(C) + E\left[V \left( \sum_{i=1}^{m-1} A_i (X_i-R) + (W-C)R \right) \right] \right\} \]

and first order conditions are

(25) \[ 0 = U'(C) - RE[V'(G)] \]

(26) \[ 0 = E[V'(G)(z_i)] \quad i = 1, \ldots, m-1 \]

Once again, \( z_i \equiv X_i - R \).

Second order conditions require that the Hessian

\[ [H] = \begin{bmatrix} R^2 E[V''(G)] + U''(C) & - R E[V''(G) z_i] \\ - R E[V''(G) z_j] & E[V''(G) z_i z_j] \end{bmatrix} \quad i, j = 1, \ldots, m-1 \]

¹For reasons which will become clear in Chapter III we assume that there are at least as many states of nature as assets.
be negative definite. This is always the case.

Proof: Define new util-prob distributions \([29]\) such that

\[
[H] = \begin{bmatrix}
R^2 E[V''(G)] + U''(C) & -E[V''(G)] E[RZ_i] \\
- E[V''(G)] E[RZ_j] & E[V''(G)] E[Z_i Z_j]
\end{bmatrix}
\]

The \(\tilde{E}\) indicates that the expectation is now over the util-prob distribution. Then

\[
|H| = E[V''(G)]^m \begin{vmatrix}
R^2 + U''(C) & -\tilde{E}[RZ_i] \\
\frac{E[V''(G)]}{E[V''(G)]} & -\tilde{E}[RZ_j]
\end{vmatrix}
\]

Aside from the \(\frac{U''(C)}{E[V''(G)]}\) term the determinant in \((29)\) is the determinant of a variance-covariance matrix and is positive definite. The \(\frac{U''(C)}{E[V''(G)]}\) term clearly does not affect the positivity of \(|H|\). Thus \(E[V''(G)]\)

\(|H|\), which is the determinant of an \(m \times m\) matrix, is of sign \((-1)^m\).

One can show by the same technique that its principal minors alternate in sign.

2. It is not now true that a risky asset is held if and only if its mean return exceeds the interest rate; whether an asset is held depends also on the covariance of its returns with returns on other assets.

Consider, for example, the quadratic utility function \(U(C) = C - \frac{bC^2}{2}\). and assume that the same utility function holds in both periods with no
discounting of the future. The solution for asset holdings and consumption is:

\[
\begin{bmatrix}
\text{cov}(X_i, X_j) + \frac{Z_i Z_j}{1 + R^2} & -RZ_i \\
-RZ_j & \frac{1}{1+R^2}
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
A_1 \\
\vdots \\
A_{m-1}
\end{bmatrix} \\
C
\end{bmatrix} = \begin{bmatrix}
\frac{Z_i}{1 - b} \\
\frac{1}{b} - R(1 - RW)
\end{bmatrix}
\]

where \( Z_i = X_i - R \) and \( \bar{Z}_i = \bar{X}_i - R \).

For a numerical example, fix the parameter values at

\[
\bar{X}_1 = 3, \quad \bar{X}_2 = 1.5, \quad R = 2, \quad b = .001
\]

\[
\sigma_1^2 = 4 \quad \sigma_2^2 = 9.75, \quad \sigma_{12} = -4.5
\]

The solution is

\[
A_1 = 225 - .15W
\]

\[
A_2 = 75 - .05W
\]

\[
C = .75W - 125
\]

\[
B = .45W - 175.
\]

There is now a composite asset consisting of 3/4 unit of \( A_1 \) and 1/4 unit of \( A_2 \) which is held in the portfolio in positive amounts at all levels of wealth for which the marginal utility of consumption is positive. The

\[1\]The form of (30) illustrates nicely the observation that consumption (of period 1) can be thought of as a particular type of asset which gives a definite period 1 return.
second risky asset, \( A_2 \), is held (because of its negative covariance with \( A_1 \)) even though its mean return is less than that on bonds. The mean return on the composite asset exceeds \( R \).

Now we solve the problem again except that we make the covariance of returns positive: \( \sigma_{12} = 4.5 \). The solution then is:

\[
\begin{align*}
A_1 &= \frac{1}{285} \left[ 36,000 - 24W \right] \\
A_2 &= \frac{1}{285} \left[ 13W - 19,500 \right] \\
C &= \frac{1}{285} \left[ 75W - 3,500 \right] \\
B &= \frac{1}{285} \left[ 221W - 13,000 \right]
\end{align*}
\]

(32)

Again the risky assets are held in the portfolio in a fixed ratio:

\[
\frac{A_1}{A_2} = \frac{-24}{13}
\]

It may be shown that in this case in ranges of wealth for which marginal utility is positive in both periods, holdings of \( A_1 \) are positive and holdings of \( A_2 \) are negative.

The property that the risky assets are held in fixed ratios in the portfolios in these two examples is not an accident. We prove in Appendix 3 to this chapter that the ratios of holdings of risky assets are independent of the level of wealth for this utility function. \(^1\)

\(^1\) It is known that this is true when there is no consumption decision but the simple proof used in that case—that the vector on the right hand side of (30) is proportional to \([(1/b) - RW] \) -- breaks down here for we now have an extra \((1/b)\) term included.
IV. Wealth and Substitution Effects on the Demand for Consumption.

The results of this section are summarized by the statement that the number of assets has little effect on the properties of the demand function for consumption. In this section we consider the derivatives $\frac{\partial C}{\partial W}$, $\frac{\partial C}{\partial h_j}$, $\frac{\partial C}{\partial r_j}$, and $\frac{\partial C}{\partial R}$, paying particular attention to CRRA and CARA utility functions. We also give some consideration to the appropriate analogy in the case of uncertainty to a change in the interest rate in the certainty model.

1. It is easy to show that $0 < \frac{\partial C}{\partial W} < 1$. For the constant relative risk aversion family,

$$\frac{\partial C}{\partial W} = \frac{C}{W}$$

and for CARA utility functions:

$$\frac{\partial C}{\partial W} = \frac{R}{1+R}$$

2. $\frac{\partial C}{\partial h_j} = \frac{A_j}{R} \frac{\partial C}{\partial W} - \frac{E[V'(G)]}{|H_j,m|} |H_j,m|$

where $|H_j,m|$ is the cofactor of the $j$, $m$'th element of $[H]$. 

$$= - \left. \frac{\partial W}{\partial h_j} \right|_{J=\bar{J}} \frac{\partial C}{\partial W} - \frac{E[V'(G)]}{|H_j,m|} |H_j,m|$$
This is made up of a wealth effect which is positive (negative) for an individual who is long (short) in the asset whose rate of return has risen and a substitution term of ambiguous sign.

(i) Constant Relative Risk Aversion:

\[
\frac{\partial C}{\partial h_j} = \frac{A_j C}{RW} \frac{\beta-1}{\beta} < 0 \text{ as } \beta > 1 \text{ for } A_j > 0.
\]

(ii) Constant Absolute Risk Aversion:

\[
\frac{\partial C}{\partial h_j} = \frac{A_j}{(1+R)}
\]

Examining (36) we notice that changes in any \( h_j \) do not affect consumption—and therefore saving—when the utility function is logarithmic.

More interestingly, the direction in which consumption changes when any \( h_j \) changes depends only on whether the utility function is bounded for constant relative risk aversion utility functions.

3. The effect of an increase in \( t_j \) (i.e., a "squeezing-in" of the distribution of returns on any given asset—for a definition of \( t_j \) see equation (10) of this chapter and subscript the relevant variables there by \( j \)) on the demand for consumption is a simple multiple of the effect of an increase in \( h_j \).

\[
\frac{\partial C}{\partial t_j} = \bar{z}_j \frac{\partial C}{\partial h_j}
\]
This is an interesting result for it indicates that increases in mean which leave other moments unchanged and proportionate decreases in the absolute value of all other moments which leave the mean unchanged affect consumption symmetrically. We now show that changes in $h_j$ and $t_j$ where $t_j = -\bar{z}_j h_j$ leave the individual's welfare unchanged.

\begin{equation}
(39) \quad J[W] = U(G) + E[V(\sum_{i \neq j} A_i \bar{z}_i + A_j (X_j + h_j + t_j (\bar{X}_j - X_j) - R) + (W-C)R)]
\end{equation}

(39) is a simple statement about how changes in $h_j$ and $t_j$ affect welfare; we now differentiate (39) totally with respect to $h_j$ and $t_j$ to find the ratio in which these two variables must change to keep utility constant:

\begin{equation}
(40) \quad \frac{\partial h_j}{\partial t_j} \bigg|_{J=j} = -\frac{E[V'(G)\bar{X}_j - X_j]}{E[V'(G)]} \quad \text{and, using (26)}
\end{equation}

\begin{equation}
= -\frac{(\bar{X}_j - R) E[V'(G)]}{E[V'(G)]}
\end{equation}

This relationship is explored further in Section V of this chapter and in Chapter III, Section VI.

4.

\begin{equation}
(41) \quad \frac{\partial C}{\partial R} = \frac{B}{K} \frac{\partial C}{\partial W} + \frac{E[V'(G)]}{|H|} \begin{bmatrix} h_{m,m-1} \vdots \vdots \vdots \\
\end{bmatrix} \begin{bmatrix} 1 \\
1 \\
\end{bmatrix}
\end{equation}

where $H_{m,m-1}$ is the mx(m-1) submatrix of $[H]$. This is made up of a
wealth effect and a substitution term of ambiguous sign.

(i) Constant Relative Risk Aversion:

\[ \frac{\partial C}{\partial R} = \frac{BC [\beta - 1]}{RW \beta} \geq 0 \text{ as } \beta \geq 1 \text{ for } R > 0 \]

(ii) Constant Absolute Risk Aversion:

\[ \frac{\partial C}{\partial R} = \frac{(W-C)R \gamma - 1}{\gamma R(1+R)} \]

5. We have been able to make no very clear statements about the effects of financial variables on the demand for consumption—even the substitution effect is in general ambiguous. Yet we do know that under certainty the effects of an increase in the safe interest rate on the demand for consumption can at least be broken down into a positive wealth effect and a negative substitution effect. Is there an analogy in the uncertainty model?

The closest analogy to the intertemporal wealth and substitution effects of a change in the safe rate of interest under certainty is the wealth and substitution effects of a uniform increase in the rate of return on all assets across all states of nature. Let this increase be \( \epsilon \):

\[ \frac{\partial C}{\partial \epsilon} = \frac{W-C}{R} \frac{\partial C}{\partial W} + \mathbb{E}[V'(G)] \frac{|H_m|}{|H|} \]

\[ = -\frac{\partial W}{\partial \epsilon} |J = J| \frac{\partial C}{\partial W} + \mathbb{E}[V'(G)] \frac{|H_m|}{|H|} \]
where $|H_{m,m'}|$ is the cofactor of the $m,m'$th element of $[H]$. The first term is the positive wealth effect and the second term a negative substitution effect.

There are three reasons for regarding this as the "pure" intertemporal substitution effect. First, it is easy to show that the two period certainty case of a change in the interest rate can be written formally as a special case of (44). Second, the mathematical reason for negativeness of the substitution term in standard consumer theory applies here also.\footnote{This could as well relate to an intertemporal problem where all goods are dated.} Third, in economic terms, an increase in $\varepsilon$ represents a change in the rate at which period two consumption can be substituted for period one consumption for any given portfolio behavior.

The substitution effect of an increase in $\varepsilon$ is closely related to the effects of changes in two other parameters one might want to introduce. Suppose first that the utility valuation placed on period 1 consumption changed multiplicatively, i.e.,:

$$U(C) = \mu \tilde{U}(C)$$

and then consider a decrease in $\mu$. The effect of this on period 1 consumption is equal to $R$ times the substitution effect of an increase in $\varepsilon$. Similarly if we were to introduce a discount factor $(1+\epsilon)$ into the second period utility function, the effect of a decrease in the discount factor on consumption would be $\frac{R}{(1+\epsilon)^2}$ times the substitution effect of an increase in $\varepsilon$.\footnote{This could as well relate to an intertemporal problem where all goods are dated.}
(i) Constant Relative Risk Aversion:

\[
\frac{\partial C}{\partial \varepsilon} = \frac{C(W-C)}{RW} \left[ \frac{\beta-1}{\beta} \right] \geq 0 \text{ as } \beta > 1
\]

(ii) Constant Absolute Risk Aversion:

\[
\frac{\partial C}{\partial \varepsilon} = \frac{(W-C)R\gamma-1}{R\gamma(1+R)} = \frac{\partial C}{\partial R}
\]

V. The Demand for Assets

In this section we show first that the Arrow result in the two-asset model that the demand for the risky asset is an increasing function of wealth if absolute risk aversion is decreasing does not go through to this model and present some special cases for which the result does hold. We then examine own- and cross-effects of changes in financial variables on the demand for risky assets.

1. It can be shown that for the CRRA family, \( A_j \frac{\partial A_j}{\partial W} > 0 \), for the CARA family \( A_j \frac{\partial A_j}{\partial W} = 0 \), and for the quadratic family, \( A_j \frac{\partial A_j}{\partial W} < 0 \). It is then reasonable to conjecture that \( A_j \frac{\partial A_j}{\partial W} \) decreases as absolute risk aversion is decreasing, constant, or increasing. This is not, however, true for general utility functions with many risky assets, as we now show.

We shall assume that there are three assets and three states of nature. We are constrained not to work with the three specific utility
functions mentioned above. Instead we use piecewise quadratic utility functions and let the marginal utility function be piecewise linear. In particular, let the utility functions $U(C)$ and $V(G)$ be

$$U(C) = C - \frac{.001}{2} C^2$$

$$V(G) = \begin{cases} 
G - .001G^2 & 0 \leq G \leq 20 \\
.4 + G - .002G^2 & 20 \leq G \leq 40 \\
3.6 + G - .004G^2 & 40 \leq G \leq 125
\end{cases}$$

Absolute risk aversion is increasing in this example.\(^1\)

Initial wealth, $W_0 = 165.98$.

Let $[X]$ be the matrix of returns on assets across states of nature: $X_{ij}$ is the return on the $j$'th asset in the $i$'th state of nature.

$$[X] = \begin{bmatrix} 
3.27 & 1.628 & 1.1 \\
.54 & 1.06 & 1.1 \\
.445 & .682 & 1.1 
\end{bmatrix}$$

and $\pi_1 = .176$, $\pi_2 = .536$, $\pi_3 = .288$, where $\pi_i$ is the probability of state of nature $i$.

Solving for consumption and asset holdings:

---

\(^1\)This approach was suggested by Michael Rothschild; Joseph Stiglitz suggested a counterexample of this type to the conjecture that the 2-asset results went through here. The utility function we are using is not three times differentiable everywhere but is three times differentiable at the points we shall be considering. It may be recalled that our assumption in Chapter II was that utility functions to be used were thrice differentiable. This is not a serious complication since our utility functions could be smoothed to ensure three times differentiability.
(49) \[ C = 137.60, \quad A_1 = 4, \quad A_2 = -56, \quad 3 = 80.38. \]

It can then be shown that

\[
\frac{\partial A_1}{\partial W} \frac{\partial A_2}{\partial W} = \frac{21}{394.3} > 0 \quad \text{but since } A_1 > 0 \text{ and } A_2 < 0
\]

it is clear that we cannot have both \( A_1 \frac{\partial A_1}{\partial W} > 0 \) and \( A_2 \frac{\partial A_2}{\partial W} > 0. \)

There is however a local theorem for utility functions which are of almost constant absolute risk aversion. Suppose specifically that

\[
\frac{-V''(G)}{V'(G)} = \gamma + \mu G.
\]

Then, for \( \mu \) small, \( -\mu A_j \frac{\partial A_j}{\partial W} > 0. \)

Proof:

\[
\frac{\partial A_j}{\partial W} = \frac{U''(C)}{|H|} E[V''(G)Z_i Z_j] \quad \text{RE} \left[ \frac{E[V''(G)Z_i]}{V'(G)} \right]
\]

where the determinant in the numerator is from an \((m-1) \times (m-1)\) matrix.

\[
= -\frac{U''(C)}{|H|} E[(\gamma + \mu (C_i Z_1 + (W-C)R)) V'(G)Z_i Z_j] \quad \text{RE}[(\gamma + \mu G)V'(G)Z_i]
\]

where we have simply substituted in the definition of \( \gamma + \mu G. \) Adding \( R A_k \) of each of the first \( m-2 \) columns (where \( k \) is the column index) to the \( m-1 \) st, and neglecting terms in \( \mu^2 \), we obtain

\[
\frac{\partial A_j}{\partial W} = (-1)^{m-2} \frac{U''(C) \quad RA_j}{|H|} E[(\gamma + \mu G) V'(G)Z_i Z_j] \quad \text{RE}[(\gamma + \mu G) V'(G)Z_i Z_{m-1}]
\]

1 The presentation of the counterexample has been cryptic. The example was obtained by using the fact—shown in Chapter III—that where there are as many assets as states of nature, a price system for contingent commodities can be defined. In Appendix 4 to this chapter we show the intermediate steps between (48), (49) and (50); the reader will, however, have to understand parts of Chapter III to work the example for himself.
(53) can be rewritten as the sum of determinants, each of which but one will have two columns multiplied by $u$; once more neglecting terms in $u^2$ we have

$$\frac{\partial A_j}{\partial w} = \frac{(-1)^{m-2} U''(C) R A_j u}{|H|} E[V'(G)Z_1 Z_j] \cdots E[V'(G)Z_1 Z_{m-1}]$$

$|H|$ is of sign $(-1)^{m}$, the determinant in the numerator is positive because it can be transformed into the determinant of a variance-covariance matrix, so that sign $\frac{\partial A_j}{\partial w} = \text{sign} \ U''(C)uA_j = \text{sign} - uA_j$.

Now it is clear from (51) that absolute risk aversion is increasing (decreasing) as $u$ is positive (negative). We have thus shown that the Arrow result goes through for utility functions which are of almost constant absolute risk aversion.

Summarizing this section, the simplicity of the result on wealth effects for the one-safe one-risky asset model is lost in general, though for some utility functions the two-asset result goes through. We proved that the simple result also goes through when absolute risk aversion is nearly constant. We shall see in Chapter III why these complications occur.

2. The effect of an increase in the own rate of return on holdings of a risky asset can be broken down into wealth and substitution terms; the substitution effect is always positive.

$$\frac{\partial A_j}{\partial h_j} = \frac{A_j}{R} \frac{\partial A_j}{\partial w} - \frac{E[V'(G)]}{|H|} H_{m-1,m-1}$$
Where the wealth effect is positive or zero, an increase in the own rate of return increases the demand for that asset. This is exactly analogous to the Slutsky-Hicks equation and leaves us with the same ambiguity.

(i) Constant Relative Risk Aversion:

\[
\frac{\partial A_j}{\partial h_j} > 0 \quad \text{for} \quad A_j > 0
\]

For an individual with constant relative risk aversion who is shorting the \( j \)’th asset, wealth and substitution effects of a change in its rate of return work in opposite directions. It is always true, though, that \( \frac{\partial w_j}{\partial h_j} > 0 \) where \( w_j \) is the proportion of the portfolio held in the \( n \)’th asset.

(ii) Constant Absolute Risk Aversion:

\[
\frac{\partial A_j}{\partial h_j} > 0.
\]

There are no wealth effects in this case, and so no ambiguity.

The effect on holdings of asset \( j \) of a change in the rate of return on another asset, \( k \), are not restricted in sign, even for our special utility functions. This is because the cross-substitution effect is of ambiguous sign.

Instead of concentrating on the effects of an increase in \( h_j \) to examine the effects of an increase in mean, it might be desirable to think of a multiplicative increase in returns; e.g., define \( X' = k_i X_{i1} \)
and then consider changes in \( k_1 \) from unity. The interest in this lies in the fact that if we were explicitly to introduce the prices of assets - instead of fixing each price at unity - and assume stationary expectations for the next period's prices, then an increase in \( k_1 \) would be equivalent to a fall in the price of the \( i \)'th asset. One finds that

\[
(56) \quad \frac{\partial A_i}{\partial k_j} = R \frac{\partial A_i}{\partial h_j} - A_j
\]

and

\[
(57) \quad \frac{\partial A_i}{\partial k_i} = R \frac{\partial A_i}{\partial h_i} \quad i=1, \ldots, j=1, \ldots, m-1
\]

Now (56) is interesting: the substitution term in (55) is positive, but terms other than the wealth effect in (56) need not be positive, for there is an additional term, viz., \(-A_j\). We name the term \(-A_j\) in (56) a "transformation term" and explain its origin in Chapter III, Section VI.

Despite the presence of the \(-A_j\) term in (56), it is clear that there is no difficulty in translating rate of return effects into price effects by use of (56) and (57). Notice, though, that the expectations behavior assumed in calculating \( \frac{\partial A_i}{\partial h_i} \) differs from that assumed in the derivation of \( \frac{\partial A_i}{\partial k_i} \).

The effects of an increase in the interest rate on demands for risky assets are ambiguous: the sign of the substitution term may be either positive or negative. This is true also for the special utility
functions we have been considering.

It remains only to consider the effects of increases in \( t_j \) — i.e., a "squeezing in" of the distribution of \( X_j \) about its mean (see equation (10)) — on the demands for \( A_j \) and \( A_i \). The following may readily be shown:

\[
(58) \quad \frac{\partial A_j}{\partial t_j} = A_j + \bar{z}_j \frac{\partial A_j}{\partial h_j}
\]

and

\[
(59) \quad \frac{\partial A_i}{\partial t_j} = \bar{z}_j \frac{\partial A_i}{\partial h_j}
\]

These results have the very interesting implication that if the rate of return on an asset increases at the same time as its variance (and the absolute values of other moments) is increased, and the ratio of the change in \( t_j \) to \( h_j \) is \( \frac{h_j}{t_j} = \bar{z}_j \), then holdings of all risky assets but that for which the rate of return and variance (i.e., \( t \)) have increased, will not change at all, but holdings of the asset itself fall. We know that the consumption decision is unaffected, so that bond holdings must rise. Similarly a decrease in mean and decrease in variance of returns on asset \( j \) in the appropriate ratio increases holdings of that asset and reduces bond holdings, leaving holdings of all other assets and consumption unchanged.
3. We conclude this section with some comments on the general many asset problem. The many risky asset case is seen to be more complicated than the one-risky asset case in that the signs of comparative-static derivatives are more frequently ambiguous. This is the case for comparative-static exercises involving asset holdings, but not to the same extent for exercises involving consumption decisions.

We have found no particular utility function which gives demand functions with properties—such as gross substitution—which are frequently assumed in the literature. Nor, when there are many risky assets, are wealth effects dependent solely on the curvature of the utility function. There are certain special utility functions for which definite results on wealth effects and the effects of changes in own rates of return are available.

The indefiniteness of results in the n-risky asset case is not particularly surprising since specification of the properties of assets has been minimal.

We have shown the close relationship between theories of behavior under certainty where goods are objects of choice and this theory of behavior under uncertainty where assets are objects of choice. The results here are no weaker than in classical price theory: the Slutsky-Hicks equation is no more informative than equation (55) of this chapter.
VI. The Special Case of Concentrated Distributions.

1. It is not in general easy to solve for asset holdings and consumption in terms of the parameters of the probability distribution even when a distribution and a utility function are specified—unless the utility function happens to be quadratic.\(^1\) In the special case where the distributions of returns on risky assets are concentrated in such a way that moments of higher than some given order can be neglected, we can study the effects of changes in given moments and of wealth on portfolio decisions—this is what we now propose to do.\(^2\)

In what follows the moments of the distributions will be taken about the safe interest rate. It is a consequence of the requirement that no arbitrage possibilities exist that the interest rate be contained within the range of the distribution of returns on each asset. By describing a distribution as concentrated we mean that moments about the interest rate of order higher than 3 of that distribution can

---

\(^1\)Solutions can easily be found for the constant relative and constant absolute risk aversion utility functions if the distribution is uniform. The normal is ruled out except in approximations of the sort we shall now use since the range of the normal is \(-\infty\) to \(+\infty\) and limited liability makes negative values inapplicable. A solution for some utility functions and for a single asset for the lognormal distribution is obtainable by use of Aitchison and Brown's [1] three-parameter lognormal family (P. 14). I was unable to solve the single risky asset problem for the gamma distribution and a constant relative risk aversion utility function. See Feldstein [12] for comments on distributions.

\(^2\)I am grateful to Professor Samuelson for suggesting this problem. I have had most useful discussions with him and Robert Merton on this section. Dreze and Modigliani [9] examine a similar case—of what they call infinitesimal risks—to study the effects of uncertainty on the consumption decision.
be neglected. It will be obvious how to modify our results if, say, the third order moment cannot be neglected.

2. We begin with the one safe-one risky asset model. The maximand is

\[ U(C) + E[V(W-C)R + A(X-R))] \]

Now expanding \( V((W-C)R + A(X-R)) \) in Taylor series about the point where \( X=R \):

\[ V((W-C)R + A(X-R)) = V((W-C)R) + A V'(W-C)R)(X-R) + \frac{A}{2} V''((W-C)R)(X-R)^2 \]

\[ + O(A^3(X-R)^3) \]

so that the maximand becomes (neglecting higher-order terms)

\[ U(C) + V((W-C)R) + A \left[ (X-R)V'(W-C)R) + \frac{A}{2} E[(X-R)^2] V''(W-C)R) \right] \]

\[ = U(C) + V((W-C)R) + A V'(W-C)R) \left[ (X-R) - \frac{A}{2} E[(X-R)^2] \right] \gamma (W-C)R) \]

where \( \gamma = -\frac{V''}{V'} \) is the Arrow-Pratt measure of absolute risk aversion. We shall assume that derivatives of \( V(.) \) up to the fourth exist.\(^1\)

---

\(^1\)One can assume that the distributions with which we are working are obtained by some sort of limiting "squeezing" process on arbitrary distributions. There is then a question of what sort of limiting process is used. For some discussion of a similar problem, see Merton [20], Section 10. We shall discuss two types of limiting process: (i) in which the variance and distance of the mean from the safe rate are of the same order, say \( O(h) \) (this is the type of process considered by Merton in [20]); (ii) in which the variance is \( O(h^2) \) and the distance of the mean from the safe rate is \( O(h) \).
Solving first for holdings of the risky asset: differentiate (62) with respect to $A$ to obtain

\[(63) \quad A = \frac{(X-R) T((W-C)R)}{E[(X-R)^2]} \]

where $T = \frac{1}{Y}$ is the risk tolerance function.

Note that $C$ in (63) is an endogenous variable so that some care should be taken in using (63). It is clear from (63) that $A > 0$ requires $X > R$; also that for $T(.)$ constant, (the CARA family), holdings of the risky asset will be greater the greater is $T$. Further, if we can show $\frac{\partial C}{\partial W} < 1$ (and we show this below), then $A$ is an increasing (decreasing) function of wealth if $T(.)$ is an increasing (decreasing) function of its argument.

Now, using (62) to obtain the first order condition for consumption:

\[(64) \quad 0 = U'(C) - RV'((W-C)R) - A(X-R) RV''((W-C)R) - \frac{RA^2 E[X-R]^2}{2} V''' \]

With some rearrangement, involving the use of (63) and the derivative of $T(.)$, we get

\[(65) \quad 0 = U'(C) - RV'((W-C)R) \left[ \frac{2\sigma_x^2 + (X-R)^2 (1+T'((W-C)R))}{2(\sigma_x^2 + (X-R)^2)} \right] \]

where $T'(.) = \frac{V'(.) V'''(.)}{(V''(.)^2} - 1$ is the derivative of the risk tolerance function. For an interior solution to exist it is necessary that the coefficient of $-RV'(.)$ be positive. This coefficient will
certainly be positive for increasing risk tolerance (decreasing absolute risk aversion) but need not be for decreasing risk tolerance.

3. Using (63) and (65) we now present solutions for asset holding and consumption decisions for three particular utility functions. Then in Section 4 we discuss the results (63) and (65) for general utility functions.

For the examples we use the same utility functions in both periods (no discounting): (a) constant relative risk aversion, (b) constant absolute risk aversion, (c) quadratic.

(a) For the CRRA utility function

\[ T((W-C)R) = \frac{(W-C)R}{\beta} \quad \Rightarrow \quad T'(.) = \frac{1}{\beta} \]

Thus

\[ C^{-\beta} = R((W-C)R)^{-\beta} \frac{[2\sigma_x^2 + (\bar{X}-R)^2 \frac{(1+\beta)}{\beta}]}{2(\sigma_x^2 + (\bar{X}-R)^2)} \]

(66) \[ C = \frac{Rk}{1+Rk} \bar{W} \quad \text{where} \quad \hat{k} = \left[ R[2\sigma_x^2 + (\bar{X}-R)^2 \frac{(1+\beta)}{\beta}] \right]^{-\frac{1}{\beta}} \frac{2(\sigma_x^2 + (\bar{X}-R)^2)}{2(\sigma_x^2 + (\bar{X}-R)^2)} \]

Note that for \( \beta=1 \) (logarithmic utility function) \( \hat{k} = R^{-1} \)

and \( C = \frac{\bar{W}}{2} \).

Solving for \( A \) for this utility function:

\[ A = \frac{\bar{X} - R}{E[(X-R)^2]} \quad \frac{(W-C)R}{\beta} \]
\[(67) \quad \therefore \quad \frac{A}{(W-C)} = \frac{R(X-R)}{\beta(\sigma_x^2 + (X-R)^2)} \] where \( \frac{A}{W-C} \) is the portfolio share for the risky asset.

Now assume

(i) \((X-R)\) is of order \(h\) and \(\sigma_x^2\) is \(O(h)\) and let \(h \to 0\).

Then from \((66)\)

\[(68) \quad \frac{C}{W} = \frac{1}{\beta} \frac{\beta}{1+RR} \frac{-1}{\beta} \] which is exactly the consumption decision which would be made if there were no risk in the problem.

From \((67)\)

\[(69) \quad \frac{A}{(W-C)} = \frac{R(X-R)}{\beta\sigma_x^2} \] which is an expression similar to Merton's [20] (29) for the continuous time case. \((69)\) is a very attractive and plausible result: the properties of the demand function for assets are so obvious as to require no comment.

(ii) For \((X-R)\) of \(O(h)\) and \(\sigma_x^2\) of \(O(h^2)\) the results are a good deal more disturbing since then the consumption decision takes uncertainty into account, while examining

\[(70) \quad \frac{A}{W-C} = \frac{R}{\beta(X-R)} \]
we see that the asset decision essentially does not.

(b) For the CARA utility function: \( T'(\cdot) = 0 \) so that

\[
(71) \quad e^{-\gamma C} = R \frac{2\sigma_x^2 + (\bar{X}-R)^2}{2(\sigma_x^2 + (\bar{X}-R)^2)} e^{-\gamma R(W-C)} = R^\gamma e^{-\gamma R(W-C)}
\]

Taking logarithms of both sides

\[-\gamma C = \log [R\xi] - \gamma (W-C)R\]

\[C(1+R) = WR - \log [R\xi] \gamma \frac{1}{1+R}\]

(72) \[C = \frac{R}{1+R} W - \log [R\xi] \gamma \frac{1}{1+R}\]

and, solving for \( A \):

(73) \[A = \frac{(\bar{X}-R)}{\gamma (\sigma_x^2 + (\bar{X}-R)^2)}\]

(c) For a quadratic utility function, \( U(C) = C - bC^2 \) :

(74) \[C = \frac{1-Rq}{b[1+R^2 q]} + \frac{R^2 q}{1+R^2 q} W \text{ where } q = \frac{\sigma_x^2}{\sigma_x^2 + (\bar{X}-R)^2}\]

and

(75) \[A = \frac{\bar{X}-R}{\sigma_x^2 + (\bar{X}-R)^2} \left[ \frac{1-b((W-C)R)}{b} \right] = \frac{(\bar{X}-R)(1+R)}{b[\sigma_x^2(1+R^2) + (\bar{X}-R)^2]} - \frac{R(\bar{X}-R)}{[\sigma_x^2(1+R^2) + (\bar{X}-R)^2]}\]

---

1This feature of our results does not depend on any particular utility function; it is puzzling and may reflect an error in some of the earlier development. We simply point it out here and defer further comment to part 4 of this section.
The reader may satisfy himself that these are the exact solutions for the quadratic utility function obtained in equation (30) of this chapter.

4. In general, from (62) one can show that holdings of the risky asset are an increasing (decreasing) function of wealth if and only if risk tolerance is an increasing (decreasing) function of wealth.

Inspection of (62) reveals that a proof of this requires that we show

\[ \frac{\partial C}{\partial W} < 1, \]

which we now do. The proof is as follows: totally differentiate (65) to obtain

\[ (76) \quad 1 - \frac{\partial C}{\partial W} = \frac{U''}{U'' + R^2V'' \left( \frac{2\sigma_x^2 + (\bar{X}-R)^2}{2\sigma_x^2 + (\bar{X}-R)^2} + \frac{R^2V''(\bar{X}-R)^2T''}{2(\sigma_x^2 + (\bar{X}-R)^2)} \right)} > 0 \]

The denominator is negative as a second order condition for a maximum. Thus an increase in wealth increases holdings of the risky asset if the risk-tolerance function is increasing in its arguments.

There is one difficulty with the present approximation: it is not possible to show, as we can when we are not approximating, that

\[ 0 < \frac{\partial C}{\partial W}. \]

For \( \frac{\partial C}{\partial W} > 0 \) the sum of the second and third elements of the denominator of the expression in (76) must be negative, and without specification on the curvature of the risk tolerance function, this is not guaranteed. A sufficient condition for \( \frac{\partial C}{\partial W} > 0 \) is

\[ T'' - T' < 0: \]

a risk tolerance function with \( T'' < 0 \) and \( T' > 0 \) clearly satisfies this. But notice that \( T'' \) involves \( V^{****} \), and we do not
usually presume to much knowledge—*a priori* or otherwise—of this derivative. For the three examples, however, we did have $0 < \frac{\partial C}{\partial W} < 1$.

Notice from (65) that if $\bar{X} - R$ is $O(h)$ and $\sigma_x^2$ is $O(h^2)$, then the consumption decision is exactly that made under certainty, since in the certainty model with a safe asset, the first order condition for a maximum is

$$U'(C) = R V'((W-C)R)$$

This is very attractive. Also, one obtains from (63) that

$$A = \frac{R(\bar{X} - R)}{\sigma_x^2} T((W-C)R)$$
as $h \to 0$

which is likewise attractive.

But the result alluded to in the footnote above when $(\bar{X} - R)$ is $O(h)$ and $\sigma_x^2$ is $O(h^2)$ is puzzling and unsatisfactory. Nonetheless, in this case, where the consumption decision is affected by uncertainty, one can show, from (65), that

$$(77) \quad \frac{\partial C}{\partial \sigma_x^2} - T'(.) - 1 \quad \text{where} \quad \sim \quad \text{means "of the same sign as."}$$

For decreasing risk tolerance an increase in variance always causes a reduction in consumption. A family of utility functions for which $T'(.)$ is constant is the constant relative risk aversion family for

---

1 Sandmo [30] shows this for a quadratic utility function and seems to believe (P. 119) that the result is a consequence of risk aversion. The result $\frac{\partial C}{\partial \sigma_x} < 0$ clearly does not hold in general for $V' < 0$. $rac{\partial C}{\partial \sigma_x}$
which $T'(.) = \frac{1}{\beta}$. Thus, for this family, $\frac{\partial C}{\partial \sigma_x^2} > 0$ as $1 < \beta$.  

For CARA utility functions $T' = 0$, so that $\frac{\partial C}{\partial \sigma_x^2} < 0$.

By dint of some manipulation one can also show

\[(78) \quad \frac{\partial C}{\partial \bar{x}} = 1 - T'(.) \]

5. We continue now to the multi-asset problem where we obtain a very striking result. Essentially repeating the steps (61) and (62) one obtains as a maximand:

\[(79) \quad U(C) + V((W-C)R) + V'((W-C)R) \left[ \sum A_i (\bar{X} - R)^i \right] + \frac{\gamma(\beta)}{2} \sum_{i,j} A_i A_j \Sigma \Sigma E[(X^i - R)(X^j - R)] \]

First order conditions for risky assets are

\[0 = (\bar{X}^i - R) + \gamma A^i E[(X^i - R)^2] + \gamma(\beta) \sum_{j \neq i} A_j E[(X^i - R)(X^j - R)] \quad i = 1, \ldots, m-1. \]

In matrix notation

\[(80) \quad [\Omega] [A] = T((W-C)R) [\bar{X} - R] \]

where $\Omega$ is the $(m-1) \times (m-1)$ matrix $[E[(X^i - R)(X^j - R)]]$, $A$ is the $(m-1) \times 1$ vector of risky assets, $T$ is the (scalar) value of the risk aversion function, and $(\bar{X} - R)$ is the $(m-1) \times 1$ vector $[\bar{X}^i - R]$.

Thus

\[(81) \quad [A] = T((W-C)R) [\Omega]^{-1} [\bar{X} - R] \]

\[\text{Merton [20] obtains the same result—cf. the equation between his (54) and (55).}\]
The striking result is that holdings of all risky assets are now proportional to the risk-tolerance function, the argument of which can once more be shown to be an increasing function of wealth. Thus for this case of concentrated distributions—as for certain utility functions—ratios of risky assets are independent of the level of wealth.

This last result provides an interesting case where mean-variance analysis of the Markowitz-Tobin type is strictly theoretically justifiable. Whether "blue-chip" stocks have sufficiently concentrated distributions as to make their distributions conform to the requirements of this section is an empirical question.
Appendix 1: The Sandmo Article.

The errors in Sandmo's proofs [30] are on P. 114 where he shows that the share of money in the portfolio is an increasing function of wealth, and in Appendix B. The error is essentially the same one, so we discuss the proof in Appendix B. X and Y are random variables, a and m are holdings of the risky asset and money respectively.

Consider the expression (we use our own numbering)

\[ K = E[W''(Y)X^2] + E[W''(Y)X] \]

We now quote.

"Now multiply K by (a+m) and add and subtract, on the right hand side, the expression aXE[W''(Y)X]. After some rearrangement we then obtain

\[ K(a+m) = mE[W''(Y)X^2] + (a+m+aX) E[W''(Y)X] \]

Since a+m+aX = Y, we can write

\[ K(a+m) = mE[W''(Y)X^2] + E[W''(Y)XY]. \]

To obtain (1.2) from (1.1) by the operation he describes, Sandmo apparently assumes

\[ aE[W''(Y)X^2] = aX E[W''(Y)X]. \]

Note that X is stochastic. (1.4) is justifiable if there exists some value of X, say \( \tilde{X} \), to guarantee the equality. But then (1.2) should be written with \((a+m+\tilde{X})\) instead of \((a+m+aX)\). He could not then pass to (1.3). Anyway, it is true that actual Y (second period wealth) = a+m+aX but this depends on the outcome of X, i.e., Y is stochastic.
Thus even in its original form, the deduction (13) from (1.2) is improper.

Two further comments. On p. 115 it is stated that an individual can have a proportional consumption function and a wealth elasticity of demand for money greater than one. In the type of model he works with the only proportional consumption function arises from the CRRA family in which all wealth elasticities are unity. (This is proved in Chapter IV).

Second, Sandmo works with a quadratic utility function to study the effects of risk on portfolio and consumption decisions. He shows, for instance that $\frac{\partial C}{\partial x} < 0$ and seems to believe that this is a general result. It is not: see Section IV of this chapter.
Appendix 2: Non-negativity of Consumption and Bequests.

We shall show that conditions (16) and (17) of this chapter ensure non-negativity of consumption and bequests. We want to consider three possible solutions in which either C and/or G is zero.

(i) \( C = 0 \) but bequests are positive across all states of nature occurring with finite probability (this qualification is to be understood whenever "\( G > 0 \)" is used henceforth)

(ii) \( G = 0 \) but C is positive

(iii) \( C = 0 \) and \( G = 0 \)

(i) The fact that \( C=0 \) and \( G>0 \) implies

\[ U'(C) > E[V'(G)] \]

Consider now increasing C to y and reducing bond holding by y, thus satisfying the budget constraint. By the mean value theorem the gain in utility from increasing consumption is

\[ yU'(\tilde{C}) \quad \text{where} \quad \tilde{C} \in (0,y) \]

and the loss in utility from reducing bequests is

\[ yRE[V'(\tilde{G})] \quad \text{where} \quad \tilde{G} \in (G,G-y) \]

But because \( \lim_{C \to 0^+} U'(C) = \infty \), there always exists some y such that

\[ U'(y) > E \left[ V'(G-yR) \right] \]

Since \( U''(C) < 0 \) and \( V''(G) < 0 \) this implies that the gain in utility from increasing consumption outweighs the loss in expected utility from
reducing bequests uniformly across all states of nature. Thus the original strategy was not optimal.

If $C$ were not initially zero but actually negative then the process just described could be repeated until $C$ became positive. If at any stage $G$ reached zero, then we would have to turn to the proof given in (iii) below.

(ii) Suppose $G = 0$ in some but not all states of nature. This implies positive savings. Consider then reducing holdings of the risky asset and increasing bond holdings. By a proof essentially similar to that just given this increases total utility.

Similarly, if $G$ were zero in all states of nature, all wealth would be consumed in period 1. But now consider reducing consumption by $y$ and increasing bond holdings to $y$. This increases total utility.

(iii) We have now to distinguish situations in which either

$$\lim_{C \to 0^+} U(C) = -\infty$$

or

$$\lim_{C \to 0^+} V(C) = -\infty$$

from those in which neither holds.

If either $C$ or $G$ is zero when either of these is the case, then it is obvious that the plan is not optimal for an alternative plan--viz., $C = W/2$, $B = W/2$ -- which ensures positive consumption and bequests,
is superior. So to complete the proof we need only confine ourselves to utility functions for which \( U(0) \) and \( V(0) \) are finite.

If \( C=0 \) we confine ourselves first to the portfolio, the total period one value of which is \( W \). The fact that there exist situations where \( G=0 \) implies \( A > 0 \). Holding \( C \) at 0 we can now rearrange the portfolio to ensure \( G > 0 \) in all states of nature by increasing bond holdings. Having done this we can go on to repeat the proof of (i) to make \( C > 0 \).
Appendix 3: Proof that Ratios of Risky Assets are Independent of Wealth for the Quadratic Utility Function.

Rewrite (30) as

\[
\begin{bmatrix}
A_1 \\
\vdots \\
A_{m-1} \\
\end{bmatrix} = \left( \frac{1}{b} - RW \right) \Omega^{-1} \begin{bmatrix}
\bar{z}_1 \\
\vdots \\
\bar{z}_{m-1} \\
\end{bmatrix} + \begin{bmatrix}
\Omega^{-1} \\
\vdots \\
\Omega^{-1} \\
\end{bmatrix} \begin{bmatrix}
0 \\
\vdots \\
\frac{1}{b} \\
\end{bmatrix}
\]

where $\Omega$ is the matrix on the left hand side of (30). Denote elements of $\Omega^{-1}$ by $W_{i,j}$. Then

\[
(3.2) \quad A_i = \left( \frac{1}{b} - RW \right) \sum_{j=1}^{m} W_{ij} \bar{z}_j + \frac{W_{im}}{b} \quad \text{where } \bar{z}_m \equiv 0
\]

Thus if $\frac{A_i}{A_k}$ is to be constant we require

\[
(3.3) \quad \sum_{j=1}^{m} W_{ij} \frac{\bar{z}_j}{\bar{z}_m} = \frac{W_{im}}{W_{km}}
\]

We may multiply numerator and denominator of the left hand side of (3.3) by $-R$; but then going back to (30), we see that, except for division by $|\Omega|$, each of the numerator and denominator is simply the sum of the product of the cofactor of each element of one column of a matrix by the corresponding element of another column (the last), except for the (1) factor in the $\text{(m,m)}$ th element of $[\Omega]$. We know that
(3.4) \[-R \sum_{j=1}^{m} W_{ij} \overline{Z}_j + W_{i,m} = 0\]

since this is the expansion of the cofactors of one column by elements of another column. Thus the left-hand side of (3.3) is just

\[
\frac{W_{im}}{W_{km}}
\]

whence \(A_i/A_k\) is constant.
Appendix 4: Example of Wealth Effects for Risky Assets

Find the inverse of \([X]\) in equation (48):

\[
(X)^{-1} = \begin{bmatrix}
.4 & -1 & .6 \\
-.1 & 3 & -2.9 \\
-.1 & -1.454 & 2.5
\end{bmatrix}
\]  

(4.1)

Obtain the price vector of contingent commodities by summing the columns of \([X]^{-1}\).

\[
P_1 = .2 \\
P_2 = .546 \\
P_3 = .2
\]  

(4.2)

Let \(G_i\) be purchases of the \(i\)'th contingent commodity. Then the solution to the contingent commodity problem is given by

\[
G_1 = 10 \\
G_2 = 30 \\
G_3 = 50
\]

\(C = 137.60\)

This implies that asset holdings are

\[
A_1 = 4 \\
A_2 = -56 \\
B = 80.38
\]

where we use the transformation

\([G] = [X][A]\)
\( G \) is the vector of contingent commodities, \( A \) the asset vector.

Next we go on to obtain the vector \( \frac{\partial G}{\partial W} \). It is fairly straightforward to show that \( \frac{\partial G_i}{\partial W} = -k \frac{\partial V_i}{\partial W} \) where \( k \) is positive and the same for each \( i \) at the optimum; \( k \) varies as the optimum position varies.

Calculating

\[
\begin{bmatrix}
\frac{\partial G_1}{\partial W} \\
\frac{\partial G_2}{\partial W} \\
\frac{\partial G_3}{\partial W}
\end{bmatrix} = k \begin{bmatrix} 490 \\
220 \\
75
\end{bmatrix}
\]

From (4.3) we may readily obtain \( \frac{\partial A}{\partial W} \) by use of the relationship

\[
(4.4) \quad \begin{bmatrix}
\frac{\partial A_1}{\partial W} \\
\frac{\partial A_2}{\partial W} \\
\frac{\partial B}{\partial W}
\end{bmatrix} = [X]^{-1} \begin{bmatrix}
\frac{\partial G}{\partial W}
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
\frac{\partial A_1}{\partial W} \\
\frac{\partial A_2}{\partial W} \\
\frac{\partial B}{\partial W}
\end{bmatrix} = k \begin{bmatrix} 21 \\
394.3 \\
-219
\end{bmatrix}
\]
CHAPTER III

ASSETS AND CONTINGENT COMMODITIES

I. Introduction

In this chapter we shall examine the relationship between the two-period asset model of Chapter II and a contingent commodity model. A contingent commodity is a commodity deliverable if a particular state of nature occurs in period 2, but not otherwise. It is paid for in period 1.

In Section II we discuss the relationship between the contingent commodity and asset models with the aid of diagrams. Section II is included chiefly for heuristic purposes, and may be avoided if the reader so wishes; we believe, though, that this section makes the remainder of the chapter more understandable.

In Section III the contingent commodity and asset models are defined and relationships between them clarified. In Sections IV-VI we undertake comparative static exercises of the sort done in Chapter II. We believe that the new perspective of the contingent commodity model throws much light on our earlier results.
II. Diagrammatic Discussion.

1. In Chapter II, Section II we dealt with only two assets and since there was uncertainty about the state of nature which would prevail in period 2, there were at least two states of nature. Thus the number of assets did not exceed the number of states of nature. In the remainder of Chapter II we assumed that there were at least as many states of nature as assets.

Where the number of assets is equal to the number of states of nature, then, under certain assumptions, the assets span the states of nature and there is a system of contingent commodities which is equivalent to the set of assets.

This is demonstrated graphically in Figure 1 where there are only two states of nature. Let $G_1$ and $G_2$ (the contingent commodities) be bequests or consumption in states of nature 1 and 2 respectively. We can define period 2 indifference curves on which

$$E[V(G)] = \Pi_1 V(G_1) + \Pi_2 V(G_2) = \text{constant}.$$ 

These have the slope

$$(1) \quad \left. \frac{3G_1}{3G_2} \right|_{V=V} = -\frac{\Pi_2 V'(G_2)}{\Pi_1 V'(G_1)}$$

so that on the 45° line their slope is $-\frac{\Pi_2}{\Pi_1}$. Now suppose one unit of wealth is to be invested in period 1. If this is invested in asset 1, then the contingent consumptions possible are shown by the point $(X_1^1, X_1^2)$. Similarly, investment in asset 2 makes contingent consumptions
\((x_1^1, x_2^1)\) possible. Any linear combination of these patterns of consumption is feasible: the line drawn through the two points is the budget line. By calculating its intercepts with the \(G_1\) and \(G_2\) axes we can find the prices of the contingent commodities \(G_1\) and \(G_2\). To the left of 1 the individual is short in asset 1 and to the right of 2 he is short in 2.\(^1\) The constraints \(G_1 > 0\) and \(G_2 > 0\) clearly do not imply the constraints \(A_1 > 0\) and \(A_2 > 0\).

\(^1\)A diagram like Figure 1 can be used to demonstrate why the risky asset of Chapter II, Section II is held only if its mean return exceeds \(R\). Where there is a bond, its returns lie on the 45° line. The slope of the indifference curve on the 45° line is \(-\Pi_2/\Pi_1\); then calculate the slope of the budget line and note that \(\Pi_1/\Pi_2\); whether a tangency occurs to the left (or to the right) of the 45° line depends on whether \(X > R\) (or \(X < R\)).
2. What if there are more assets than states of nature? In Figures 2 and 3 we have 3 assets and 2 states of nature. The vectors of returns are now linearly dependent. In the figures we indicate the patterns of contingent consumptions made possible by investing one unit of wealth in period 1 by the shaded area. If one of the assets were a bond then one of the vertices of each figure would lie on the 45° line. The numbers 1, 2, 3 relate to the returns on assets 1, 2 and 3 respectively. Since the assets span the states of nature it
is obvious that no more than 2 assets need ever be held at one time.\(^1\)

There is no obvious rule in terms solely of the patterns of returns indicating which assets are actually bought by the investor. This clearly depends on the utility function, for tangency could occur on any of the straight line segments of the frontier, or even on all three in Figure 2 and on both segments in Figure 3.\(^2\) It would then be a matter of indifference which assets were chosen. If one asset is strongly dominated by another\(^3\) then the frontier has a positive slope and any amount of period 2 consumption in either state of nature is feasible because of the possibility of arbitrage.

Where there are more assets, \((n+1)\) say, than states of nature, \(n\), the formal first order conditions constitute a set of dependent equations and cannot be solved to give optimal levels of \(n+1\) assets. A complete procedure would be to solve \(n\) sets of \(n\) equations and compare the values of the maximand for each set of \(n\) assets.

---

\(^1\)One should note that on the frontier in Figure 2 the individual is shorting asset 2 and holding asset 1 to the left of point 1, holding positive amounts of assets 1 and 3 between 1 and 3, and holding 3 and shorting 2 to the right of 3. Thus 2 is never held in positive amounts in the portfolio. A linear combination of assets 1 and 3 would always dominate 2 were it to be held in positive amounts.

On the frontier of Figure 3 assets 1 and 3 are never held in positive amounts and asset 2 is always held. To the left of 2 the individual is short in 3 and to the right of 2 he is short in 1.

\(^2\)Figures 2 and 3 are similar to diagrams used in discussing the non-substitution theorem of linear programming where, in that case, \(G_1\) and \(G_2\) are the commodities produced. This is, though, a case of joint production and it will not be true that one set of assets is used for any consumption levels. It is clear that for homothetic indifference maps (constant relative risk aversion--this equivalence is proved by Stiglitz [32]) the same assets in the same proportions would be chosen at all levels of wealth.

\(^3\)That is, has higher returns in both states of nature.
From now on, in discussing assets, we shall assume that there are at least as many states of nature as assets. Whether this is actually the case is a complex problem which turns on one's beliefs about subjective probability distributions. It might be that states of nature are regarded as continuous in some range. It is also frequently stated that there are "too few" contingent commodity markets, though it is not clear that those who make this statement have considered equities as linear combinations of contingent commodities.

Although we shall not take up the case of more assets than states of nature—which we may term the "efficient contingent commodity problem"—it may be of some interest. It would be an explanation—among many others—of the fact that individuals do not usually hold all assets. To assure the continued existence of many assets one would require probability distributions and/or utility functions to differ among individuals. This is entirely reasonable.

We shall distinguish the case where there are at least as many states of nature as assets as the "asset problem."¹

3. We now take up the case of fewer assets than states of nature. We shall assume that there are two assets and three states of nature. In Figure 4, C₁, C₂ and C₃ are the contingent commodities. The rays OX₁ and OX₂ indicate combinations of contingent commodities which are

¹Stiglitz [33] has exploited the properties of the equivalence of asset and contingent commodity problems where there are as many assets as states of nature.
obtained by investing in assets 1 and 2 respectively. Of course, the investor is not confined to purchasing only one asset or the other.

Any combination of contingent commodities which lies in the cone $OX_1X_2$ is purchasable by the investor if he has sufficient wealth. In the absence of restrictions on short sales, the attainable set of contingent
commodities is not $OX_1 X_2$ but $OH_1 H_2$ where $OH_1$ lies in the $G_1 G_2$ plane and $OH_2$ in the $G_1 G_3$ plane (i.e., $G_3$ is zero on the ray $OH_1$ and $G_2$ is zero on the ray $OH_2$). What we have shown so far is that the pattern of returns on assets—i.e., the rays $OX_1$ and $OX_2$—defines an attainable set of contingent commodities which the investor can purchase.

An attainable set may instead be described by placing linear restrictions on the investor's choice of contingent commodities. We can impose the constraint that the investor choose only from the cone $OH_1 H_2$. It should be clear that such a constraint does not define a unique pattern of returns on contingent commodities: any two non-dependent rays through the origin in the plane $OH_1 H_2$ serve as a basis for the cone and represent a pattern of returns on assets. A set of constraints on the investor's choice of contingent commodities defines a set of patterns of returns, but not a unique pattern of returns.

The attainable set describes the contingent commodities which may be purchased in the absence of a budget constraint. The plane $ABC$ indicates the budget plane for a given price system for contingent commodities and a given allocation of wealth $\hat{W}$ to second period consumption. The points D and E on $OX_1$ and $OX_2$ are the combinations of contingent commodities purchasable by investing $\hat{W}$ in assets 1 and 2 respectively. (We have fixed the prices of all assets at unity). F and K are the intersections of DE with $OH_2$ and $OH_1$ respectively. The triangle OFK describes the budget set, the set of contingent commodities which the individual can buy, given his savings decision.
We want next to consider the relationships between the pattern of returns and the price system for contingent commodities which describes the budget plane, such as ABC.

Suppose first that the points D and E are given—that we know what quantities of contingent commodities can be obtained by investing \( \hat{w} \) in assets 2 and 1 respectively. The points D and E define the triangle OFK. The budget set OFK is consistent with many price systems for contingent commodities; many planes which intersect the G axes at positive G may be rotated about the FK axis. The budget set OFK is not, though, consistent with any price system: for instance a set of price planes can be generated by tilting ABC on the AB axis, and only one of these planes—ABC—would be consistent with the budget set OFK. Thus a pattern of returns in an asset problem—the points D and E—imposes some constraints on the price system of an associated contingent commodity problem.

Next assume that a price system for contingent commodities—ABC, say—is given, as well as the constraint that the investor can choose only from the OH1H2 cone. This defines a unique budget set. Such a budget set is consistent with many possible patterns of returns on assets—any non-dependent rays through the origin which intersect the FK line describe an admissible pattern of returns. The requirement that the budget set of a contingent commodity problem be the same as that of an associated asset problem places restraints on the pattern of returns from assets; we will investigate these restrictions carefully in Section II below.
At this stage we want only to restate what we have shown above: the requirement that the attainable sets for the two problems be the same generates a set of relationships between the two problems and these relationships are then constrained by requiring budget sets to be the same.

Indifference surfaces in $G_1, G_2, G_3$ space exist. They are loci of constant expected utility, on which $\Pi_1 V(G_1) + \Pi_2 V(G_2) + \Pi_3 V(G_3) = \text{constant}$, where $\Pi_i$ is the probability of state of nature $i$ occurring. For $\Pi_1 = \Pi_2 = \Pi_3$ the indifference surfaces are symmetric about a generalized $45^\circ$ line through the origin (henceforth called the $45^\circ$ ray). If one asset were a bond, returns from it in terms of contingent commodities would lie on the $45^\circ$ ray. So long as the attainable set is unaltered we can do comparative statics by considering indifference curves which lie in the $OH_1H_2$ plane.

The attainable set is preserved if the rate of return on one of the assets increases multiplicatively; the budget set, though, is changed by such an increase. The position of the "budget line", $FK$, within the attainable set will pivot on $D$ as the rate of return on asset 1 moves from $E$ to $E'$. An additive increase in the rate of return on any risky asset shifts both the attainable set and the budget set unless there is a bond, in which case the attainable set is unaltered. This is shown in Section III below. Notice that if the pattern of returns on one of the assets changes in such a way as to maintain the budget set, the optimum in terms of contingent commodities is in no way disturbed. Thus if the return pattern were to change from $E$ to $E''$ on one asset,
the equilibrium in terms of assets but not in terms of contingent commodities would be disturbed.

III. Asset and Constrained Contingent Commodity Models.

1. An asset problem is characterized by its returns matrix, $X$, which is $nxm$; $X_{ij}$ is the return on the j'th asset in the i'th state of nature. There are n states of nature and m assets, $m \leq n$.

$A$ is the $mxl$ vector of assets.

The $X$ matrix is semi-positive with some positive element in each column; the most that one can lose on a particular asset is all the money invested in it, and each asset must pay something in some state of nature. In addition, we impose the requirement that it be of rank $m$ so that there are essentially $m$ distinct assets.\(^1\)

In addition we impose a "no-arbitrage" requirement on $X$: this is the requirement that no weighted combination of the columns of $X$, with the sum of the weights being zero, be weakly greater than zero.

\[ \sum_{j=1}^{m} \mu_j X_j \leq 0 \quad \text{for all } \mu \text{ such that } \sum_{j=1}^{m} \mu_j = 0. \]  

We place no restriction on the sign of $\mu_j$ since short sales are allowed.

\(^1\)This will ensure that there is a positive element in each row of $X$ so that there is no state of nature in which no consumption is possible. This is a strong assumption.
The asset problem is to find

\[ J[W] = \max \{ U(C) + E[V(\sum_{j=1}^{m} A_j X_j)] \} \]
\[
\begin{align*}
\{ & C \} \\
\{ & A_j \} \\
\end{align*}
\]

subject to \( W = C + \sum_{j=1}^{m} A_j \)

We are not in this chapter distinguishing the bond by calling it \( B \); where we wish to include a bond in the problem we denote it \( A_m \).

2. The constrained contingent commodity problem is to find

\[ J[W] = \max \{ U(C) + \sum_{i=1}^{n} \Pi_i V(G_i) \} \]
\[
\begin{align*}
\{ & C \} \\
\{ & G_i \} \\
\end{align*}
\]

subject to \( \sum_{i=1}^{n} \Pi_i G_i + C = W \)

and

\[ \hat{\alpha} G = 0 \] where \( \hat{\alpha} \) is \((n-m)xn\).

We denote \((n-m)\) by \( \tau \) henceforth.

A constrained contingent commodity problem is characterized by its constraint matrix, \( \hat{\alpha} \), defined in (5), and its price vector \( P \), which is \( l \times n \). \( G \) is the \( n \times l \) vector of contingent commodities. Since \( G \succ 0 \) (where \( \succ \) means that some elements of \( G \) are positive) and \( \hat{\alpha} G = 0 \), \( \hat{\alpha} \) cannot be positive.
3. **Definition 1A:** The attainable set for an asset problem, \( S_A \), is defined by

\[
(6) \quad S_A = \{ G | G = XA, A \in \mathbb{R}^m, G \in \mathbb{R}^n, G \geq 0 \}
\]

**Definition 1B:** The budget set for an asset problem is defined by \([1 \ldots 1] A = \hat{W}\) where \([1 \ldots 1]\) is \(1 \times m\).

Denote the \([1 \ldots 1]\) vector by \(e\) henceforth.

**Definition 2A:** The attainable set for a constrained contingent commodity (CCC) problem, \( S_G \), is defined by

\[
(7) \quad S_G = \{ G | \hat{\alpha} G = 0, G \in \mathbb{R}^n, G \geq 0 \}
\]

**Definition 2B:** The budget set for a contingent commodity problem is defined by \(PG = \hat{W}\).

4. Throughout our primary concern will be with the asset-contingent commodity relationship. We take up first the case \(m=n\); then \(\hat{\alpha}\) does not exist and there is, as we now show, a unique mapping from \(X\) to \(P\) but not from \(P\) to \(X\).

**Theorem III.1:** Where there are as many assets as states of nature, the \(X\) matrix of returns on assets defines a unique price system, \(P, P > 0\), for contingent commodities.

**Proof:**

1. \( G = XA \)

and \( P \cdot G = \hat{W} = eA \)

so that
\( (8') \quad P X A = e A \)

\( (8') \) is a relationship which holds at all levels of wealth so that \( A \) can take on any values. Accordingly we require

\( (8) \quad P X = e \)

This is a set of \( n \) linear equations in \( n \) unknowns (the \( P_i \)) and has a unique solution. Geometrically, given \( n \) points which define an \( n \)-dimensional plane, the intersections of that plane with the axes are unique.

(ii) In order for \( P \) to be acceptable as a price vector, \( P \) must be positive. This can be shown to be true from the "no-arbitrage" condition (2) and Farkas's Lemma, \(^1\) as follows.

"Exactly one of the following alternatives holds: Either the equation

\[ P X = e \]

has a non-negative solution or the inequalities

\[ X U \geq 0, \quad e U < 0 \]

have a solution."

Suppose there existed a vector \( U \), the sum of the elements of which was negative and for which \( X U > 0 \). Then it would be true, since \( X \) is semipositive, that any element of \( U \) could be increased until \( e U = 0 \) and we would then have \( X U \geq 0 \). But this would violate the no-arbitrage

\(^1\) See Dorfman, Samuelson and Solow [8], p. 504; also Gale [14], p. 44. We state Gale's version but change the notation.
condition (2). Hence the solution to equation (8) is non-negative.

It is clear from the form of (8) that P is semipositive; actually it is positive. For suppose that Pn were zero so that

\[ [P_{n-1} : 0][\hat{x}] = e \]

Now consider the set of equations

\[
\begin{bmatrix}
X_{n-1} \\
\vdots \\
1 \cdots 1
\end{bmatrix} \begin{bmatrix}
u_1 \\
\vdots \\
u_n
\end{bmatrix} = \begin{bmatrix}
b_1 \\
b_{n-1} \\
b_n
\end{bmatrix} \quad b_1, \ldots, b_{n-1} \geq 0 \\
b_n < 0
\]

or \([\hat{x}] U = b,\)

where \(X_{n-1}\) consists of the first \(n-1\) rows of \(X\) and the \(b\) vector is arbitrary aside from the specifications in (9). If \(\hat{x}\) is non-singular there exists a \(U\) vector which satisfies (9). Also since \(X_{n-1}\) is non-negative, some element of \(U\), say \(u_j\), must be positive. Then we can create a new matrix

\[
\begin{bmatrix}
\frac{X_{n-1}}{0 \ldots 0 X_{n-1} 0 \ldots 0}
\end{bmatrix} = \tilde{X}
\]

such that \([P_{n-1} : 0] \tilde{x} = e \quad , \quad [P_{n-1} : 0] \geq 0 \]

and \(\tilde{X}U \geq 0\), with \(eU < 0\), which contradicts Farkas's Lemma.

Thus \(P_n\) could not have been zero; we know that \(P\) is non-negative, so \(P > 0\).

What if \(\hat{x}\) is singular? It is clear since \(X\) is nonsingular that \(\hat{x}\) is of rank at least \(n-1\); not all the cofactors of the last row of \(\hat{x}\) can
can be zero. Say the cofactor of the k'th element of the last row of X is nonzero; then the matrix X with 1+ε for arbitrarily small ε replacing 1 in the k'th place in the last row will be nonsingular. But for small enough ε it will still be true that εJ < 0. Hence the contradiction again.

Thus, for m=n, given a semipositive X matrix satisfying the no arbitrage condition there is a unique price vector P > 0 associated with X. Q.E.D.

Suppose, working in the opposite direction, that P is given. Then (8) is a set of n equations in n² unknowns and does not have a unique solution. One simple solution would be a diagonal matrix with 1/P in the relevant positions: this would really be a contingent commodity model since each asset pays off only in one state of nature.

4. In the case m=n all the restrictions on the P and X matrices come from the requirement that budget sets for the two problems be the same. For m < n the equivalence of attainable sets imposes some restrictions. As previously, we will study the asset-contingent commodity relation first.

Theorem III.2. The returns matrix X, of an asset problem, defines a unique attainable set for a CCC problem, where there are fewer assets than contingent commodities.
Proof: We have $G = XA$. Solving for the first $m$ contingent commodities in terms of assets,

$$G_m = XA$$

where $X$ is now written as

$$
\begin{bmatrix}
X_m \\
X_t
\end{bmatrix}
$$

$X_m$ is $m \times m$ and is chosen to be non-singular, (this is always possible by renumbering assets since the rank of $X$ is $m$) and $G_m$ is $m \times l$.

Thus

$$A = X_m^{-1}G_m$$

and solving for $G_{m+1}, \ldots, G_n$ in terms of the $G_m$ vector:

$$G_t = XA$$

$$= X_m X_m^{-1} G_m , \quad \text{or}$$

$$[-X_m X_m^{-1} : I] G = 0$$

where $[-X_m X_m^{-1} : I]$ is $\tau \times n$

or

$$[\alpha : I] G = 0.$$  

Of course, by elementary row operations on $[\alpha : I]$ which is a constraint matrix $\hat{\alpha}$, many different constraint matrices can be obtained. Any such constraint matrix defines the same attainable set so that we lose no generality by usually writing $\hat{\alpha}$ as $[\alpha : I]$. Notice that $[\alpha : I] X = [0]$ where $[0]$ is an $\tau \times m$ matrix; since $X$ is semipositive and contains some positive element in each row and each column $\alpha$ must contain some negative elements. Q.E.D.
Next we impose the requirement that budget sets be the same, in order to study the constraints on the P vector imposed by the X matrix. Once again equation (8) is required to hold; this is now a set of m linear equations in n unknowns and does not have a unique solution. By Farkas's Lemma we could again show that there exists a non-negative P and by a variant of the previous proof that there exists a price vector with m positive elements and \( \tau \) zero elements. Since (8) determines m prices in terms of the remaining \( \tau \) we can specify any \( \tau \) prices; in particular let \( P_{m+1}, \ldots, P_n = 0 \). It need not then be true that \( P_1, \ldots, P_m > 0 \) since the no-arbitrage condition applies to X and not to every subset of its rows.

It does not matter that the P vector could include negative numbers, since all that concerns us is the intersection of the budget plane and the attainable set. In terms of Figure 4, we can fix any one intersection point with an axis (A, B, or C)—say we choose A—but then the points A, D and E have to lie in the same plane; the intersection of this plane with the other axes determines the remaining prices. By specifying some prices to be zero we have required that the price plane never intersect the axes for those goods, which requirement, together with points like D and E, defines a plane just as well as specifying particular intersection points with those axes.

Thus given a returns matrix X from an asset problem a unique attainable set is defined: the attainable set may be described by many constraint matrices \( A \) but each can be obtained from \( [\alpha : I] \) by elementary row operations. An X matrix defines a unique \( [\alpha : I] \)
matrix. The returns matrix determines \( m \) prices in terms of the remaining \( r \) but does not define a unique price vector unless \( n = m \) -- there are as many assets as states of nature.

5. The constrained contingent commodity-asset relation is more flexible; this can be seen geometrically from Figure 1 where the plane \( OH_1H_2 \) can be spanned by many return vectors. We are now given a matrix \( \hat{a} \) and a vector \( P \). Our initial requirement is that attainable sets be the same.

From \( \hat{a}G = 0 \) and \( G = XA \) we have

\[
\hat{a}XA = 0. \tag{14}
\]

Now this cannot restrain \( A \) in any way since it does not involve the constraints \( G \geq 0 \), and our only other constraint on \( A \) is that \( A \in \mathbb{E}^m \). Accordingly we require

\[
\hat{a}X = [0] \quad \text{where} \ [0] \text{ is } \mathbb{R}^m, \text{ or that each column of } X \text{ lie in the null space of } \hat{a}. \tag{15}
\]

In order for \( X \) to be an acceptable returns matrix it has to be semipositive. If \( a \) were nonpositive then we could simply let \( X = \begin{bmatrix} I & \frac{1}{-a} \end{bmatrix} \).

For budget sets to be the same we require equation (8) to hold so that the inner product of the price vector and each column of \( X \) will be unity. This can be viewed simply as a normalization on the column sums of \( X \); given any \( X \) matrix which satisfies (15) and

---

\(^1\)Working from \( X \) to \( \hat{a} \) above we found that (15) holds.
semipositivitv, we can calculate $P X_j$ where $X_j$ is a column of $X$ and then divide each element of the $j'$th column by $P X_j$ to obtain a new returns matrix which satisfies both (15) and (8). The remaining question is, of course, whether this matrix also satisfies (2).
This is not taken any further but may be of interest.

IV. The Solution to the Constrained Contingent Commodity (CCC) Problem.

To find the solution to the CCC problem, the maximand of which is

$$ U(C) + \sum_{i=1}^{m} \Pi_{i} V(G_i), $$

subject to the constraints

$$ P G = \bar{W} $$

and

$$ \lambda G = 0 $$

we set up the Lagrangean, $L$

$$ L = U(C) + \sum_{i=1}^{n} \Pi_{i} V(G_i) + \mu \left( \sum_{i=1}^{n} P_i G_i + \bar{C} - \bar{W} \right) + \lambda [\alpha : I] G $$

where $\lambda$ is an $1 \times t$ vector of multipliers. We assume that the $[\alpha : I]$ matrix is obtained from a given return matrix $[X]$ and that $P$ is the price vector with its last $t$ components zero corresponding to $X$.

First order conditions for a solution to the original problem are$^{1}$

$$ 0 = U'(C) + \mu $$

$$ 0 = \Pi_{i} V'(G_i) + \mu P_i + \sum_{j=1}^{t} \lambda_j \alpha_{ji} \quad i=1, \ldots, n $$

$^{1}$For an actual computational procedure for solving a CCC problem with non-negativity constraints see Edwardsen [10].
\[
0 = \nabla \mathbf{P}^\top \mathbf{G} + \mathbf{C} - \mathbf{W}
\]
\[
0 = [\alpha : \mathbf{I}] \mathbf{G}
\]

Regrettably we have now to present the Hessian of the system (17):

(18) \[
[\mathbf{H}] =
\begin{bmatrix}
\mathbf{U}'' & \mathbf{P}_1 \mathbf{V}_1'' & 0 & 1 & 0 \\
0 & \mathbf{P}_n \mathbf{V}_n'' & 0 & \mathbf{P}_1 & \mathbf{T} \\
1 & 1 & \mathbf{P}_1 & \ldots & \mathbf{P}_m & 0 & \ldots & 0 \\
\tau & 0 & \alpha & \mathbf{I} & 0 & 0 & 0
\end{bmatrix}
\]

The notations down the left hand side of the matrix indicate the number of rows in each block. Sufficient conditions for a maximum are (17) and the requirement that \(|\mathbf{H}|\) be of sign \((-1)^{n+1}\) with the bordered principal minors up to that obtained by striking out the first \(m\) rows and columns alternating in sign. Before proceeding to comparative statics, we need a lemma.

**Lemma III.1:** The determinant of any matrix of the form

(19) \[
\mathbf{K} = \begin{bmatrix}
\beta & \alpha^\top \\
\alpha & 0
\end{bmatrix}
\]

where \(\beta\) is a negative \(n \times n\) diagonal matrix, and \(\alpha\) is any \(\tau \times n\) matrix, \(\tau < n\), is of the sign \((-1)^n\).
Proof: Premultiply K by E, defined below:

\[
E K = \begin{bmatrix}
I_n & 0 \\
-\alpha \beta^{-1} & I_r
\end{bmatrix}
\begin{bmatrix}
\beta & \alpha^T \\
\alpha & 0
\end{bmatrix}
= \begin{bmatrix}
\beta & \alpha^T \\
0 & \alpha \beta^{-1} \alpha^T
\end{bmatrix}
\]

|E K| = |E| |K| = |K|, so

(20) \[|K| = \prod_{i=1}^{n} \beta_i, \quad |\alpha \beta^{-1} \alpha^T| = \prod_{i=1}^{n} \beta_i, \quad |\alpha (-\beta^{-1}) \alpha^T|\]

\((-\beta^{-1})\) is a positive diagonal matrix; define \(\gamma\) as the positive diagonal matrix with \(\sqrt{-\beta_i}\) on the diagonal, and then \(\alpha (-\beta^{-1}) \alpha^T = \alpha \gamma \gamma^T = (\alpha \gamma)(\alpha \gamma)^T\)

which is positive definite.\(^1\)

Since \(\beta_i < 0\) for all \(i\), \(|K|\) is of the sign \((-1)^n\).

The lemma makes it clear that \(|H|\) satisfies sufficient second order conditions for a maximum since its relevant bordered principal minors will alternate in sign with the determinant of \(|H|\) itself of sign \((-1)^{n+1}\).

---

\(^1\)This follows simply from the definition of positive definiteness: consider the quadratic form \(X'AA'^TX = (X'A)(A'^TX) = Y'Y\) which is a sum of squares. Definiteness rather than semi-definiteness requires \(r < n\) which is equivalent to requiring fewer constraints than variables. We assume this for the lemma and assume it for the problem to make sense.
V. Comparative Statics and Wealth Effects.

In this section we first examine the basic equation used in deriving the comparative statics of an asset problem from a contingent commodity problem, and then go on to discuss wealth effects for consumption and assets respectively.

1. Equation (21) which follows is used repeatedly in the rest of this chapter and careful attention should be paid to it, though there is no difficulty in deriving the equation.

From \( G = XA \), we have

\[
\frac{\partial G}{\partial \xi} = \frac{\partial X}{\partial \xi} A + X \frac{\partial A}{\partial \xi}, \quad \text{where} \ \xi \ \text{is any parameter,}
\]

so that

\[
(21) \quad \frac{\partial A}{\partial \xi} = [X_m^{-1}]^{-1} [\frac{\partial G}{\partial \xi} - \frac{\partial X}{\partial \xi} A].
\]

\([X_m]^{-1}\) is a non-singular matrix consisting of the first \( m \) rows of \( X \), where the assets have been renumbered to assure non-singularity of \( X \). In Section VI below we shall indicate why the terms \(-[X_m^{-1}]^{-1} \frac{\partial X}{\partial \xi} A\) are called "transformation terms."

Notice also that given any parameter change in an asset problem we have to find the effects of this change on both the price vector and the constraint vector \((P \text{ and } \hat{\alpha})\) of a CCC problem to obtain \( \frac{\partial G}{\partial \xi} \).

2. The marginal propensity to consume was found always to lie between zero and one in earlier chapters. Making use of the lemma it
is quite easy to show that the same result holds here.¹

3. The interesting question about wealth effects for assets is whether the ambiguity of these effects arises from the presence of the constraints in the contingent commodity problem or from the transformation (21) or both. The constraints may be excluded as the only cause since we showed that the ambiguity exists for some utility functions even when there are as many assets as states of nature in our example of Chapter II, Section V.

Wealth effects for contingent commodities may be either positive or negative: that is, the presence of the constraints above can reverse the normally positive direction of wealth effects. To show this we present an example, using our standby of a quadratic utility function.²

The example may also be of use in understanding Section III of this chapter.

Example: Let U(C) = C−.0005C² for the first and second periods so that there is no discounting. Let the matrix of returns, X, be given by

\[
X = \begin{bmatrix}
4 & 2 \\
1 & 2 \\
3 & 2
\end{bmatrix}
\]

¹ Use Cramer's Rule: \( \frac{3C}{\hat{A}} = \frac{\hat{H}}{\hat{A}} \) where \( \hat{H} \) is the same as \( |H| \) except that the \( U'' \) term in \( \frac{\hat{A}}{|H|} \) is replaced by zero in \( |H| \). Then expand down the first column, and next across the first row.

² Borch [4] points out that indifference curves from quadratic utility functions are circular in mean-variance space; we have taken care in all our examples to confine ourselves to regions of positive marginal utility.
where the second asset is a bond. The probabilities of states of nature 1, 2, and 3 are 0.5, 0.1, and 0.4 respectively. From the returns matrix we derive the constraint matrix in the form \([a : I]\); it is \([-\frac{2}{3} \quad -\frac{1}{3} \quad 1]\). Two of the price vectors which are admissible are \([0 \quad \frac{1}{4} \quad \frac{1}{4}]\) and \([\frac{1}{5} \quad \frac{1}{3} \quad 0]\). We actually worked with the first but used the second to check the results; for example, the price-weighted sum of constants in demand functions should be zero, and the price-weighted sum of marginal propensities to consume out of wealth should be unity. The reader may, if he wishes, take some other acceptable price vector, such as \([-\frac{1}{2} \quad 0 \quad 1]\) and check these properties.

The resultant demand functions are:

\[
\begin{align*}
(22) & \quad C = 153 + 0.565 \ W \\
& \quad G_1 = 1050 - 0.036 \ W \\
& \quad G_2 = -988 + 1.325 \ W \\
& \quad G_3 = 372 + 0.417 \ W
\end{align*}
\]

The range of wealth in which we have an interior solution and positive marginal utility is restricted, but exists. It may be confirmed that for \(W=1450\) these requirements are met: at \(W=1450\)

\[
\begin{align*}
(23) & \quad C = 973 \\
& \quad G_1 = 998 \\
& \quad G_2 = 932 \\
& \quad G_3 = 977
\end{align*}
\]

At this point wealth effects for the first contingent commodity are negative.
It may be interesting briefly to obtain the asset solution from (22):

we know that

\[
\begin{bmatrix}
C \\
G
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
0 & X
\end{bmatrix}
\begin{bmatrix}
C \\
A
\end{bmatrix}
\]

and we can solve for A by taking the inverse of any two by two non-singular submatrix of X. This gives

(24) \[A_1 = 678 - .453 \, W\]
\[A_2 = -831 + .888 \, W\]

It should be noted that at \(W=1450\), \(A_1 \frac{\partial A_1}{\partial W} < 0\) which is in accordance with the results for risky assets of Chapter II for quadratic utility functions.

In general, using (21)

(25) \[\frac{\partial A}{\partial W} = X_{m}^{-1} \frac{\partial G_{m}}{\partial W}\]

where \(X_{m}^{-1}\) is the inverse of some particular \(m \times m\) submatrix of X of rank \(m\) and \(\frac{\partial G_{m}}{\partial W}\) is an \(m \times 1\) vector excluding the same elements of the full \(\frac{\partial G}{\partial W}\) vector as the rows that are excluded from the X matrix in selecting \(X_{m}\). Since elements of both \(X_{m}^{-1}\) and \(\frac{\partial G_{m}}{\partial W}\) may be negative, and \(X_{m}^{-1}\) and \(\frac{\partial G_{m}}{\partial W}\) are not obviously related, one would in general expect that \(\frac{\partial A}{\partial W}\) could consist of elements of any sign.

---

1. There are some discrepancies in the third significant figure for coefficients in some demand functions depending on which 2x2 submatrix of X is used; this is due to rounding error in the original demand functions.

2. "Obviously" is required since the X matrix determines both the constraint matrix and price vector.
We do know, though, that the sign pattern of $\frac{dA}{dW}$ is restricted in particular cases: with two assets $A \frac{dA}{dW} > 0$ as the utility function shows decreasing, constant or increasing absolute risk aversion (A is now the risky asset); for three particular utility functions the same result for risky assets goes through when there are more than two assets.

One can prove the two-asset proposition from the contingent commodity viewpoint where there are only two states of nature, but the proof throws no light on the reason for this result. Indeed, it is hard to see how a better explanation than that of Arrow [2] in describing better behavior can be given. Similarly, one can prove the linearity of Engel curves for those three utility functions which carry the two-asset result through to many asset models, but without any great gain in insight.

It may be of some interest to indicate the implications of decreasing absolute and decreasing relative risk aversion for the indifference map.

![Figure 5](image-url)
Where there are only two commodities we know that $G_1 > G_2$ or $G_1 < G_2$ or $G_1 = G_2$ at all levels of wealth; the Engel curve never crosses the 45° line. (Proof: with one safe and one risky asset, two states of nature, the no-arbitrage condition, and the mean on the risky asset greater than $R$, the risky asset is always held in positive amounts. The no-arbitrage condition means that the return on the risky asset in one state of nature—say the first—is greater than $R$, and the return in the second state less than $R$; thus $G_1 > G_2$ if the risky asset is held. If the mean on the risky asset is the same as that on the bond, the risky asset is not held and $G_1 = G_2$.) Assuming $G_1 > G_2$, the Engel curve for utility functions of decreasing absolute risk aversion has a slope greater than unity; for decreasing relative risk aversion the Engel curve has elasticity greater than unity.

VI. Price and Rate of Return Effects.

In this section we shall undertake some of the same sorts of exercises as were done in Chapter II. In Section 1 we explain how the result expressed in Chapter II, equation (44), can be derived in the CCC model. This equation expressed what we called the closest analogue in the uncertainty model to the effects of a change in the interest rate in the certainty model. We back this up in Section 1 by use of the composite commodity theorem.

In Section 2 the Slutsky-Hicks equation for assets of Chapter II, equation (55), is derived from the CCC model.

In Section 3 we introduce the "transformation term" by giving
an example in which it appears.

In Section 4 we discuss the Slutsky-Hicks equation for risky assets when there is no bond.

Finally, in Section 5, we show that the strong result of equations (58) and (59) of Chapter II in which it was shown that changes in mean (increases in \( h_1 \)) and variance (decreases in \( t_1 \) -- recalling that this affects other moments) increase the demand for \( A_1 \), decrease the demand for bonds, and leave all other demands unchanged, is a simple transformation effect.

1. In Chapter II, Section IV we suggested that the appropriate analogue of a change in the interest rate in the certainty model occurs in the uncertainty model when the rate of return on all assets in all states of nature rises uniformly. The analysis of this section will make it clear that this depends on the existence of a safe asset and suggests that the more appropriate analogue is a multiplicative increase in the rate of return on all assets. It also explains a somewhat puzzling result which was not given in Chapter II, but is easily derived: viz., the effects of an additive increase in the rate of return on all assets on consumption are the same except for a multiplicative constant as those of a multiplicative increase.

(i) Consider first an additive increase in the rate of return on all assets; instead of the matrix \( X \) we now have the matrix \( X + \varepsilon \) where \( \varepsilon_{ij} = \varepsilon \) for all \( i, j \). In the presence of a bond this leaves
the \([\alpha : I]\) matrix unchanged: since \([\alpha : I]X = [0]\) and one column
of \(X\) consists of the constants, the sum of the elements of each row of
\([\alpha : I]\) must be zero so that adding a constant to each element of \(X\)
does not affect the \([\alpha : I]X = [0]\) equation.

(ii) Next consider the \(P^2 = e\) equation. We can as well
write the \(X\) matrix as \([X_{m-1} : R]\) since the bond is the \(m'\)th asset.
Let \(P\) be the original price vector: we require

\[(26)\quad P^2 = 1 \quad j = 1, \ldots, m-1\]

\[(27)\quad P^2 = 1\quad \text{or}\quad \sum_{i=1}^{m} P^2 = \frac{1}{R}\]

Now let the rate of return on all assets change by \(\varepsilon\); denote by \(\hat{P}\)
the new price vector. We will show that

\[(28)\quad \hat{P} = \frac{R}{R+\varepsilon} P\quad \text{is a new admissible price vector, where}\quad \frac{R}{R+\varepsilon}
\]
is a scalar.

We now require

\[(29)\quad \hat{P}(X_j + \varepsilon) = 1\quad \text{or}\quad \hat{P}X_j + \varepsilon \sum_{i=1}^{m} \hat{P} = 1, \quad j = 1, \ldots, m-1\]

and

\[(30)\quad \hat{P}(R+\varepsilon) = 1\quad \text{or}\quad \sum_{i=1}^{m} \hat{P} = \frac{1}{R+\varepsilon}\]

It is easy to confirm that \(\hat{P}\) given by (28) satisfies (29) and (30).

Thus a simple proportionate reduction (in absolute value) of
all prices gives a new admissible price vector.
(iii) Previously the budget constraint was

$$C + P \cdot G = W;$$

it is now

$$C + \frac{R}{R+C} P \cdot G = W.$$ 

Second-period consumption can accordingly be treated as a composite commodity and it is seen that the increase in the return on all assets is equivalent to an increase in the rate of interest where future consumption is a single good.

This development, though, depends heavily on the existence of a bond. It is thus clear that the more general analogue would be a uniform multiplicative increase in the payoffs from each asset across all states of nature; it is then easy to confirm that the $[\omega^T I]$ matrix is unchanged and relative prices in the $P$ vector are unchanged whether or not there is a safe asset.

2. In the appendix to this chapter we show the derivation of the vector $\frac{3A}{3h}$ which is needed to explain the Slutsky equation for assets. It is shown there that

\[ \text{For the rest of this chapter we shall be writing } X \text{ for what until now has been } X_m. \text{ This is to reduce the number of subscripts, and should cause no confusion. Thus in what follows } X^{-1} \text{ is the element in the } m, i' \text{th place of the inverse of our new } \bar{X} = \text{what we have been calling } X_m. \]
\begin{equation}
\begin{bmatrix}
\frac{\partial A}{\partial h_1} \\
\frac{\partial G_1}{\partial p_j}
\end{bmatrix} = \left[ X^{-1} \right] \begin{bmatrix}
\frac{\partial X_{m1}}{\partial h_1} \\
\vdots \\
\frac{\partial X_{mn}}{\partial h_1}
\end{bmatrix} = \begin{bmatrix}
0 \\
\vdots \\
0 \\
A_{1/R}
\end{bmatrix}
\end{equation}

(31) is an interesting expression; it should be clear that we will obtain something like regular wealth and substitution effects for the first \( m - 1 \) assets from the first term, but for the \( m \)th asset—the bond—there is an additional term. This is not to say that the additional term for bonds arises because there is a bond—rather the additional term for risky assets is zero because there is a bond.\(^1\)

We want to examine three elements of \( \frac{\partial A}{\partial h_1} \): the first, representing the own-effects of an increase in \( h_1 \); the second, representing cross-effects; and the last, being the effect on bond holdings of a change in \( h_1 \).

\begin{equation}
\frac{\partial A_1}{\partial h_1} = \sum_{k=1}^{m} \frac{X_{1k}^{-1}}{X_{1k}} \frac{\partial X_{1k}}{\partial h_1} = \sum_{j=1}^{m} \frac{\partial A_j}{\partial p_j} \frac{\partial X_{m1}}{\partial h_1}
\end{equation}

\begin{equation}
\frac{\partial A_2}{\partial h_1} = \sum_{k=1}^{m} \frac{X_{2k}^{-1}}{X_{2k}} \frac{\partial X_{2k}}{\partial h_1} = \sum_{j=1}^{m} \frac{\partial A_j}{\partial p_j} \frac{\partial X_{m1}}{\partial h_1}
\end{equation}

\(^1\) We do not prove this here. It would be a useful exercise, though, for the reader to derive \( \frac{\partial A}{\partial h_1} \) where there is no bond and examine the vector in that expression analogous to \( \mathbf{h}_1 \).
(34) \[ \frac{\partial A_m}{\partial h_1} = \sum_{k=1}^{m} X_{mk}^{-1} \sum_{j=1}^{m} \frac{\partial G_k}{\partial p_j} \frac{\partial X_{mj}^{-1}}{\partial h_1} = A_1 - \frac{A_1}{R} \]

As in standard consumer theory

(35) \[ \frac{\partial G_k}{\partial p_j} = -\mu H^{-1} \sum_{j=1}^{k+1} G_j \frac{\partial G_k}{\partial W} \]

where \( H^{-1}_{j+1,k+1} \) is an element of the inverse of \( H \).

The own-substitution effect is as usual negative (\( \mu \) is negative as may be seen from equation (17).) Wealth effects, we have shown above, are ambiguous.

Taking up (32): change the order of summation and use (35):

(36)

\[ \frac{\partial A_1}{\partial h_1} = -\sum_{j=1}^{m} \frac{\partial X_{mj}^{-1}}{\partial h_1} \sum_{k=1}^{m} X_{lk}^{-1} G_k \frac{\partial G_j}{\partial W} - \mu \sum_{j=1}^{m} \frac{\partial X_{mj}^{-1}}{\partial h_1} \sum_{k=1}^{m} X_{lk}^{-1} H_{j+1,k+1} \]

Before proceeding we need a lemma:

**Lemma III.2.** \[ \frac{\partial X_{mj}^{-1}}{\partial h_1} = -\frac{X_{lj}^{-1}}{R} \]

**Proof:** For the proof we show that \[ \frac{\partial X_{ml}^{-1}}{\partial h_1} = -\frac{X_{lj}^{-1}}{R} \]

---

\( ^1 \) The peculiar numbering arises from our treating consumption as the first good.
Taking the inverse by the adjoint method we have

\[(37) \quad -\frac{X_{11}^{-1}}{R} = -\frac{1}{R\Delta} |X_{11}|\]

where $|X_{11}|$ is the cofactor of the 1,1'th element of $X$.

Now expand $|X_{11}|$ down its last column, which consists of constants equal to $R$. Write the cofactors of these elements as $|X_{jm}|_{11}$, so

\[(38) \quad -\frac{X_{11}^{-1}}{R} = -\frac{1}{\Delta} \sum_{j=2}^{m} |X_{jm}|_{11}\]

Now $X_{ml}^{-1} = \frac{1}{\Delta} |X_{lm}|$ and expanding $|X_{lm}|$ down its first column,

\[(39) \quad X_{ml}^{-1} = \frac{1}{\Delta} \sum_{j=2}^{m} (X_{lj} + h_l) |X_{lj}|_{1m}\]

so that\(^1\)

\[(40) \quad \frac{\partial X_{ml}^{-1}}{\partial h_l} = \frac{1}{\Delta} \sum_{j=2}^{m} |X_{lj}|_{1m}\]

The absolute values of the determinants in each of the sums in (38) and (40) is the same but their signs are opposite. Consider, for example, the determinant obtained by striking out the first two rows and first and last columns of $X$: in equation (38) it appears as $|X_{2m}|_{11}$ with sign $(-1)^m$, and in (40) it appears as $|X_{12}|_{1m}$ with sign $(-1)^{m+1}$.

\(^1\)We have already shown that $\frac{\partial \Delta}{\partial h_l} = 0$. 
Thus, equating left hand sides of (38) and (40):

\[ \frac{\partial X_{ml}}{\partial h_1} = - \frac{X_{ll}^{-1}}{R} \]

and the same proof applies for all other elements of the vectors to show

\[ \frac{\partial X_{mj}}{\partial h_1} = - \frac{X_{lj}^{-1}}{R} \]

Q.E.D.

Using the lemma and (36):

\[ \frac{\partial A_1}{\partial h_1} = \sum_{j=1}^{m} \frac{X_{lj}^{-1}}{R} \frac{\partial g_j}{\partial W} \sum_{k=1}^{m} X_{lk}^{-1} G_k + \frac{\mu}{R} \sum_{j=1}^{m} X_{lj}^{-1} \sum_{k=1}^{m} X_{lk}^{-1} H_{j+1,k+1}^{-1} \]

Rearranging (42) we arrive at

\[ \frac{\partial A_1}{\partial h_1} = \frac{A}{R} \frac{\partial A}{\partial W} + \frac{\mu}{R} (X_{lj}^{-1}) [\hat{H}_{ll}^{-1}] (X_{lj}^{-1})' \]

The first term is the wealth effect; the second is a quadratic form where \( \hat{H}_{ll}^{-1} \) is the \( m \times m \) submatrix of the inverse of \( H \) obtained by striking out the first row and column and last \( 2 \times 2 \) rows and columns of \( H^{-1} \). \( X_{lj}^{-1} \) is the first row of \( X^{-1} \).

\( \hat{H}_{ll}^{-1} \) is at least negative semi-definite\(^1\) so that we know the substitution term cannot be negative (recall \( \mu < 0 \)). The rank of \( \hat{H}_{ll}^{-1} \) is at most \( m \) and can actually be shown to be \( m \).\(^2\)

\(^1\)Samuelson [27], p. 379.

\(^2\)A proof which is exactly analogous to that of Hicks [17], pp. 310-311, on the rank of the substitution matrix can be constructed.
Thus the substitution term is positive.

This is the derivation of the Slutsky-Hicks equation for assets from the underlying contingent commodity framework. Of course the standard substitution term is negative and ours is positive, but this arises from the fact that an increase in $h_1$ is analogous to a decrease in the price of an asset. The existence of the safe asset seems to play a special role here, and it should be interesting to consider the case where there is no such asset to see whether positivity of the substitution term is guaranteed there. This is done in part 5 of this section.

We shall now expand (33) and (34) briefly:

\[
\frac{\partial A_2}{\partial h_1} = \frac{A_1}{R} \frac{\partial A_2}{\partial W} + \frac{\mu}{R} (X_1^{-1}) \left[ \hat{H}_{11}^{-1} \right] (X_2^{-1})',
\]

The ambiguity of the cross substitution term arises from the fact that the second term is no longer a definite quadratic form (nor is it a quadratic form).

For bonds:

\[
\frac{\partial A_m}{\partial h_1} = \frac{A_1}{R} \frac{\partial A_m}{\partial W} + \frac{\mu}{R} (X_1^{-1}) \left[ \hat{H}_{11}^{-1} \right] (X_m^{-1})' - \frac{A_1}{R}
\]

Now it may be shown that $\frac{A_1}{R}$ is the compensating variation in wealth so that $\frac{A_1}{R} \frac{\partial A_m}{\partial W}$ is the wealth effect and the third term is definitely not. Nor does it appear to have the form of a substitution term, as the second terms in (43), (44) and (45) do. What is it?

It arose from a change in the transformation $G=XA$; even were the same
equilibrium in terms of contingent commodities chosen after the increase in \( h_1 \) as before, the demand for assets would be different. For want of a better word we call it a "transformation term". The reason there is no transformation term for the risky assets is that there is a bond; this will be shown below.

3. To provide an heuristic justification of the usage "transformation term", consider increasing the return on the first asset across all states of nature so as to leave both the constraint matrix and price vector unchanged. In particular, let

\[
X_1' = X_1 + K(X_1 - R)
\]

where \( X_1 \) is the first column of \( X \).

Notice particularly that variations in \( K \) produce exactly the type of change in distribution with which Diamond worked, as stated in Chapter II, Section I.

It is easy to see that an increase in \( K \) affects neither \( \hat{\alpha} \) nor \( P \): the equilibrium in terms of contingent commodities is unaltered by a change in \( K \), so that, from (21)

\[
0 = \frac{\partial X}{\partial K} A + X \frac{\partial A}{\partial K}
\]

or\(^1\)

---

\(^1\) The essential steps in obtaining the second expression in (48) from the first are to note that \( \frac{\partial X}{\partial K} \) has zero's everywhere but in the first column and then to remember that \( X^{-1} \) may be written in terms of cofactors of elements of \( X \).
\[
(48) \quad \frac{\partial A}{\partial K} = -X^{-1} \frac{\partial X}{\partial K} A = \begin{bmatrix} -A_1 \\ 0 \\ \vdots \\ 0 \\ A_1 \end{bmatrix}
\]

This example indicates that we may obtain terms in comparative static expressions which relate neither to wealth nor to traditional substitution effects but which arise from the repackaging of a set of assets which in no way alters the individual's (contingent commodity) consumption possibilities. It is these terms that we name "transformation terms." The types of change in distribution with which Diamond [6] was concerned were exactly those which amount to a repackaging of assets without affecting consumption possibilities.

4. It is time to rid ourselves of a bond, and we do this thoroughly: assume now that there is no linear combination of assets which yields the same outcome in each state of nature. Graphically, the 45° ray does not belong to the attainable set. We consider first a multiplicative increase in the returns on the first asset in all states of nature: let \( X \) be written \( \begin{bmatrix} k_1 X_1 & X_2 & \ldots & X_m \end{bmatrix} \) and consider an increase in \( k_1 \), from one. Then by exactly the same type of manipulation as before:

\[
(49) \quad \frac{\partial A_1}{\partial k_1} = A_1 \frac{\partial A_1}{\partial W} + \mu(X_1^{-1})(\hat{u}_{11})^{-1}(X_1^{-1})' - A_1
\]

---

1Some circumspection is in order here: the traditional substitution term is, after all, the "remainder" after the wealth effect has been separated. Our sentence above is based on the forms of the substitution terms in (43)-(45) and on the example just given.
where the second term is a positive substitution term and the third is a transformation term.

\[
(50) \quad \frac{\partial A_2}{\partial x_1} = A_1 \frac{\partial A_2}{\partial \overline{w}} + \mu \left( x_2^{-1} \right) [\hat{\lambda}_{11}]^{-1} \left( x_1^{-1} \right),
\]

In (50) there are only wealth and substitution terms.\(^1\)

It is not possible to obtain any simple expression for the comparative statics of an additive increase in the return on one asset when there is no bond. Both the constraint matrix and the price vector will change: the effects of both these changes provide the \(\frac{\partial G}{\partial h_1}\) vector.

5. We carry out only one more exercise in this vein. We showed in Chapter II that, in the presence of a bond, a uniform increase in the return on an asset (\(h_1\) say) coupled with a shift of its distribution towards its mean in the form:

\[
(52) \quad x_1' = x_1 + t_1(x_1 - \overline{x})
\]

where \(\frac{h_1}{t_1} = (\overline{x} - \overline{x})\)

\(^1\)Notice that if we had bonds in the problem, then we would be able to show exactly the results on the differences between the effects of additive and multiplicative increases in rates of return expressed in equations (56) and (57) of Chapter II. It is also clear that the term \(-A_j\) in (57) of Chapter II is a transformation term.
will leave the holdings of all risky assets but the first unchanged and will affect the demand for bonds. This is a pure transformation effect. For consider the new column

\[
\hat{x}_1 = x_1 + h_1 + t_1 (x_1 - \bar{x}_1)
\]

\[
= x_1 + t_1 (\bar{x}_1 - R) + t_1 (x_1 - \bar{x}_1)
\]

\[
= x_1 + t_1 (x_1 - R)
\]

This is exactly the change described in obtaining equation (48).¹

VII. Conclusion

This chapter illustrates the close connection between asset and contingent commodity models. Certain results in the asset problem become quite transparent when viewed from the contingent commodity framework.

It has probably been noted that we have avoided much discussion of the effects of changes in the attainable set in the last section. These effects are a good deal more complicated than the effects of changes we have considered and have been excluded for that reason.

¹The result of Chapter II was obtained first and I did not actually guess the nature of the result until first writing the above paragraph; this seems to me to indicate that the contingent commodity approach can be of help in understanding asset problems.
It should also be noted that we have confined ourselves to finite distributions. Even within this restricted group of distributions, some interest attaches to further work on the constrained contingent commodity-asset problem relationship.
Appendix: Derivation of the Vector $\frac{\partial A}{\partial h_1}$

In this appendix we shall obtain the expression for $\frac{\partial A}{\partial h_1}$ is equation (31) of the text that is used in the derivation of the Slutsky-Hicks equation of Section VI, part 2.

It will be recalled from Chapter II, (9), that $h_1$ represents a uniform additive increase in the return on the first asset. It is clear by the same arguments used in Section VI, part 1 of this chapter, that the constraint matrix is unaffected by an increase in $h_1$.

Any effects of a change in $h_1$ on asset demands must accordingly come from its effects on the price vector. Suppose that we have set the last $\tau$ prices at 0 and are working with the price vector obtained by solving

$$PX_m = e$$

(1.1)

Now let

$$\hat{X}_m = [X_1 + h, X_2, \ldots, X_{m-1}, R]$$

(1.2)

where each element represents a column of $[\hat{X}_m]$.

Then it is easy to see that

$$[\hat{X}_m]^{-1} = \begin{bmatrix}
X_{m-1}^{-1} \\
\vdots \\
\hat{X}_m^{-1}
\end{bmatrix}$$

(1.3)

The partitioning of $\hat{X}_m^{-1}$ is by rows, and there should be no confusing
the matrix itself with its last row. In order to see that it is only
the last row of the inverse of $X_m^{-1}$ which is affected by the increased
payoff one need only consider taking the inverse by the adjoint method
and remembering that adding a multiple of one column to another column
leaves the value of the determinant unaltered.

The new price vector $\hat{p}$ is obtained as

$$\hat{p} = e^T [H_{m-1}]^{-1}$$

so will generally be different in the first $m$ elements from the
original price vector.

The $m \times 1$ vector $\left[ \frac{\partial G}{\partial h_i} \right]$ is then obtained from

$$\left[ \frac{\partial G}{\partial h_i} \right] = \left[ \frac{\partial X}{\partial h_i} \right] \left[ \frac{\partial P}{\partial h_i} \right]$$

$$\left[ \frac{\partial G}{\partial p_i} \right]$$ is an $m \times m$ matrix in which $i$ is the row index.

$$\frac{\partial p_j}{\partial h_i}$$ is a column vector given by

$$\left[ \frac{\partial X}{\partial h_i} \right]^{-1} e_i$$

so that, combining (1.4) and (1.5)

$$\left[ \frac{\partial G}{\partial h_i} \right] = \left[ \frac{\partial G}{\partial p_i} \right] \left[ \frac{\partial X}{\partial h_i} \right]^{-1} e_i$$
Using (21)

\[
(1.7) \quad \frac{\partial A}{\partial h_1} = X_m^{-1} \left[ \begin{bmatrix} \frac{\partial G_i}{\partial x_i} & \frac{\partial X_m^{-1}}{\partial h_1} \end{bmatrix}^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{\partial X_m^{-1}}{\partial h_1} \end{bmatrix} A \right]
\]

Now \( \frac{\partial X_m^{-1}}{\partial h_1} \) has zeros everywhere but in the first column, which is a column of 1's.

We now take aside the final term in (1.7).

\[
(1.8) \quad X_m^{-1} \left[ \begin{bmatrix} \frac{\partial X_m^{-1}}{\partial h_1} \end{bmatrix}^T A = \begin{bmatrix} A_1 & X_{1i}^{-1} \\ \vdots & \vdots \\ A_m & X_{mi}^{-1} \end{bmatrix} \right]
\]

The sum of the elements of each of the first \( m-1 \) rows of the inverse of \( X_m \) is zero (multiplying that sum by \( R \) involves the multiplication of the cofactors of one column of a determinant by the elements of another column) and the sum of the elements of the last row of \( X_m^{-1} \) is \( \frac{1}{R} \) (since \( R \sum_{i=1}^m X_{mi}^{-1} \) is the value of the determinant of \( X \)).

Thus, from (1.7), using (1.8), we obtain

\[
(1.9) \quad \frac{\partial A}{\partial h_1} = \left[ X_m^{-1} \right] \left[ \begin{bmatrix} \frac{\partial G_i}{\partial x_i} \\ \vdots \\ \frac{\partial X_m^{-1}}{\partial h_1} \end{bmatrix} \begin{bmatrix} \frac{\partial X_m^{-1}}{\partial h_1} \end{bmatrix} \right] - \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} A_1/R \\ \vdots \\ \vdots \end{bmatrix}
\]

since \( \frac{\partial X_m^{-1}}{\partial h_1} \) has zeros everywhere but in its last row.

(1.9) is equation (31) of the text of this chapter.
CHAPTER IV

MULTI-PERIOD MODEL: NO UNCERTAINTY OF DEATH

I. Introduction

We shall in this chapter examine the full solution to the general problem outlined in Chapter I, but specialize by assuming that the lifetime is of known length. Accordingly no life insurance assets are available. This is the case examined by Samuelson [28] and solved by him for the paths of consumption and asset holding through time for utility functions of constant relative risk aversion.\(^1\)

In Section II the concavity of the derived utility function in each period is discussed.

In Section III we define the ACRRA class of utility functions and discuss their properties. We then derive consumption functions and the properties of asset demand functions for these utility functions, and comment on the properties of the demand functions. We also discuss the phenomenon of "businessman's risk" in the light of these results.

In Section IV the consumption and asset demand functions for

\(^1\)Since writing this chapter I have benefitted from reading work by Hakansson [16] and Leland [18] in this same area. See also Fama [11].
CARA (constant absolute risk aversion) utility functions are stated and interpreted. We solve the problem for the case of quadratic utility functions in \( V \) and discuss the possibility of induced changes in the utility function through time, pointing out the effects of such changes on the demand functions.

In Section VI the solution to the problem for general utility functions is considered and the crucial role of linearity of demand functions in Sections III-V clarified. In Section VII we show that there are no utility functions other than those of Sections III-V which give rise to linear demand functions, and relate a number of previously known results on these utility functions to our results.

II. Concavity of Derived Utility Functions

1. It will be recalled from Chapter I, equation (16), that the maximand for the second-last decision period is

\[
J_2(W_{T-1}) = \max \left\{ \frac{E_{T-1}}{1+\lambda} \left[ J_1(W_T) \right] \right\}
\]

\[
\begin{cases}
C_{T-1} \\
A_{i,T-1}
\end{cases}
\]

where

\[
W_T = \sum_{i=1}^{m-1} A_i T-1 (X_i^T - R_{T-1}) + (W_{T-1} - C_{T-1}) R_{T-1}.
\]

We will now show that \( J_1(W_T) \) is a concave function. The definition of \( J_1(W_T) \) (from Chapter I, equation (13)) is
\[(2) \quad J_1[W_T] = \max \left\{ U_T(C_T) + E[V_{T+1}(G_{T+1})] \right\}
\]

\[
\begin{align*}
&= \max \left\{ U_T(C_T) + E \left[ \sum_{i=1}^{m-1} A_i^T (X_i^{T+1} - R_i) + (W_T - C_T)R_T \right] \right\} \\
&= \max \left\{ U_T(C_T) + E \left[ \sum_{i=1}^{m-1} A_i^T (X_i^{T+1} - R_i) + (W_T - C_T)R_T \right] \right\}
\end{align*}
\]

where \[G_{T+1} = \sum_{i=1}^{m-1} A_i^T (X_i^{T+1} - R_i) + (W_T - C_T)R_T.\]

It is a well-known theorem in dynamic programming under certainty that the function corresponding to \(J_1[W_T]\) is concave if both \(U_T(C_T)\) and the function corresponding to \(E[V_{T+1}(G_{T+1})]\) are concave.¹

Since \(V_{T+1}(G_{T+1})\) is concave and \(E\) is a linear operator, it follows that \(E[V_{T+1}(G_{T+1})]\) is a concave function so that \(J_1[W_T]\) is concave.

Then by the same proof as outlined above, \(J_2[W_{T-1}]\) is concave and similarly \(J_{T-t+1}[W_t]\) is concave for all \(t\).

It follows that the form of \(J_2[W_{T-1}]\) in (1) (or more generally, the form of \(J_{T-t+1}[W_t]\)) is the same as that of \(J_1[W_T]\) in the one-period maximizing problem of Chapter II: instead of the concave function \(V_{T+1}(W_{T+1})\) appearing inside the expectation operator we have the concave derived utility function \(\frac{J_1(W_T)}{1+2}\). Thus any properties of the solution of the two-period problem which depended only upon the concavity of the \(V_{T+1}(G_{T+1})\) function are preserved for the solution of the three or \(T\) period problem.

The more important of these properties are that first order conditions give a maximum; the marginal propensity to consume lies between zero and one; wealth and substitution effects can be distinguished; in the one-safe, one-risky asset case, the risky asset

¹Bellman [3], p. 20, Theorem 5.
is held if and only if its mean exceeds \( R_t \).

In some sense, then, particularly with regard to the qualitative properties of asset demand functions, there is little difference between 2-period and \( T \)-period models. The similarity of results of \( T \)-period and 2-period models is most marked where consumption in each period is a linear function of wealth.\(^1\)

2. There is a simple alternative proof of the concavity of \( J_{1}(W_T) \) which is of some economic interest. Substituting the definition of \( C_{T+1} \) into (2) above, differentiating with respect to \( W_T \), and making use of first order conditions for a maximum\(^2\) (which hold for all values of \( W_T \) since \( J_{1}(W_T) \) is a maximum), we find

\[
(3) \quad J_{1}'(W_T) = U_T'(C_T^*(W_T))
\]

where \( C_T^* \) is the optimal level of consumption.

Then\(^3\)

\[
(4) \quad J_{1}''(W_T) = U_T''(C_T^*(W_T)) \frac{\partial C_T^*}{\partial W_T}
\]

---

\(^1\) This is shown in Section VI below.

\(^2\) These are equations (14) and (15) of Chapter I.

\(^3\) It may not be obvious that (4) is a legitimate deduction from (3) which is, after all, usually referred to as an envelope condition. The deduction is valid because (2) is a relationship which holds along a path for all values of \( W_T \) and not only at one point, and \( \frac{\partial C_T^*}{\partial W_T} \) is taken where \( C_T^* \) must remain optimal as \( W_T \) varies.
We have shown in Chapter IV that

\[ 0 < \frac{\partial C_T}{\partial W_T} < 1 \]

so that \( J''(W_T) < 0 \) and \( J(W_T) \) is concave. This concludes the proof of concavity.

3. Now combining (4) and (3)

\[ -\frac{J''(W_T)}{J'(W_T)} = -\frac{U''(C_T)}{U'(C_T)} \frac{1}{\frac{\partial C_T}{\partial W_T}} \]

On the assumption \( 0 < C < W \), (6) indicates that where the absolute rate aversion of \( U(C) \) is constant or increasing in \( C \), the absolute risk aversion of \( J[W] \) is greater than that of \( U(C) \) at all points at which their arguments are equal. But the stronger statement—that

\[ -\frac{J''(\xi)}{J'(\xi)} > -\frac{U''(\xi)}{U'(\xi)} \]

cannot be made. It is easy to confirm that

\[ \frac{J''(\xi)}{J'(\xi)} = \frac{U''(\xi)}{U'(\xi)} \]

for CRRA utility functions, \( -\frac{J''(\xi)}{J'(\xi)} = -\frac{U''(\xi)}{U'(\xi)} \).

III. Consumption and Portfolio Decisions for ACRRRA Utility Functions

The ACRRRA (asymptotically constant relative risk aversion) utility functions are of the form

\[ U(C_T) = \frac{(C_T + \alpha)^{1-\beta}}{1-\beta} , \quad \beta > 0 , \quad \alpha \geq 0 \]

These are clearly a simple extension of the CRRA family, for which \( \alpha = 0 \).
Deriving the measures of absolute and relative risk aversion:

\[
\begin{align*}
(8) \quad - \frac{U''(C_t)}{U'(C_t)} &= \frac{\beta}{C_t + \alpha} \\
(9) \quad - \frac{U''(C_t)C_t}{U'(C_t)} &= \frac{\beta C_t}{C_t + \alpha}
\end{align*}
\]

From (8) it is seen that absolute risk aversion is a diminishing function of consumption and

\[
\begin{align*}
(10) \quad d \left( \frac{U''(C_t)C_t}{U'(C_t)} \right) &= \frac{\alpha \beta}{(C_t + \alpha)^2} \quad \forall \quad \alpha > 0 \quad \text{as} \quad \alpha \to 0
\end{align*}
\]

that is, for \( \alpha \) positive, relative risk aversion is increasing, and for \( \alpha \) negative, relative risk aversion is decreasing. The reason for the ACRRA designation lies in the property that

\[
\lim_{C_t \to \infty} \frac{-U''(C_t)C_t}{U'(C_t)} = \beta
\]

What is the interpretation of the \( \alpha \) term in the utility function? For \( \alpha \) negative it is reasonable to think of \( \alpha \) as the subsistence level of consumption. We shall assume that initial wealth is sufficiently large so that, were it invested in bonds, the subsistence level of consumption would be attainable in each period. In the absence of this assumption an interior maximum cannot exist.

There is no obvious interpretation of positive \( \alpha \) though it will be seen below that there is an explanation of the type of behavior
implied by a positive \( \alpha \).

2. The derivation of the demand functions for assets and consumption for ACRRA utility functions is shown in the appendix to this chapter. The demand functions are:

\[
C_t = k_t(W_t + \frac{\alpha}{R^{T+1-t}} \sum_{i=0}^{T+1-t} R^i) - \alpha = k_t(W_t + \alpha g_t) - \alpha
\]

\[
A_i = W_i (1 - k_t) (W_t + \alpha g_t)
\]

\[
B_t = (1 - \sum_{i=1}^{m-1} W_i) (1 - k_t) (W_t + \alpha g_t) - \alpha g_{t-1}
\]

where

\[
k_t = \sum_{i=0}^{T+1-t} a_i \quad \text{and} \quad a = \left( \frac{R^*}{1+\xi} \right)^{-1} \beta
\]

\( R^* \), defined in (1.10) of the appendix, is the expected utility return on the portfolio.

It is quite easy to show that \( k_t < k_{t+1} \) and \( g_t > g_{t+1} \).

The explanation of these results is easier when \( \alpha \) is negative--the subsistence consumption case--than when \( \alpha \) is positive. Assume first that \( \alpha \) is negative: calculating the amount of current wealth \( \hat{W}_t \), which would have to be laid aside if it were to be invested at the safe interest rate to yield an annuity equal to subsistence consumption, from the present to the end of the individual's lifetime:

\[\text{Professor Samuelson suggested that annuity income might be involved here.}\]
\[ \hat{W}_t = \alpha g_t \]

The consumption function indicates that consumption in excess of subsistence consumption is proportional, not to wealth, but to the amount of wealth which remains after taking care of subsistence consumption by purchasing an annuity which will provide this consumption with certainty. Similarly, holdings of risky assets are proportional to wealth reduced by the same amount.

We would expect the decision to provide for future subsistence consumption to show up in the demand function for bonds, (13). The demand for bonds is higher by the amount of \(-\alpha (g_{t-1})\) than it would be were it simply proportional to wealth reduced by \(\hat{W}_t\).¹ This excess is exactly the amount needed to provide a certain income of \(\alpha\) for all future periods.

Confirmation of this interpretation is seen by considering the case where initial wealth is just sufficient to ensure subsistence consumption in each period but no more; i.e., \(W_1 = \alpha g_1\). Then \(C_1 = -\alpha > 0, A_1^1 = 0\) for all \(i = 1, \ldots, m-1,\) and \(B_1 = -\alpha (g_1 - 1)\). In the next period \(W_2 = \alpha g_2\) and the same solution holds till the end of the individual's lifetime.

Thus when an individual begins life with just enough wealth to live at the subsistence level for the rest of his life, he consumes just enough to subsist in each period and rigorously avoids risk.

¹The -1 factor is accounted for by the fact that the annuity income we have calculated would pay \(\alpha\) in the present period as well as all future periods.
by holding all wealth in the form of bonds. If he has any wealth above that subsistence amount he begins to buy risky assets and to increase his consumption—he may or may not increase holdings of bonds.¹

It is clear from (12) that ratios of risky assets are fixed in this problem. The individual picks on a portfolio of risky assets and as his wealth varies shifts the composition of his portfolio between bonds and the composite risky asset. Note that it is only in the case \( \alpha = 0 \) that all ratios of asset holdings (including bonds) are constant at all levels of wealth. It is probably superfluous to comment that all demand functions are linear in wealth.

The explanation of behavior for positive \( \alpha \) is somewhat different. Suppose the individual receives income of \( \alpha \) in consumption goods each period and that \( C_t \) is interpreted as purchases of (excess demand for) consumption goods. Then in (1) \( C_t + \alpha \) is his actual consumption and it is proportional to his total wealth, where this is defined as current disposable wealth plus the discounted (at the safe rate) future stream of incomes. The demand for risky assets is directly proportional to total wealth and the demand for bonds is adjusted to take account of the fact that the individual is effectively holding bonds by having an assured income stream of \( \alpha \) each period. This is behavior characteristic of an individual with constant relative risk aversion. The

¹Duncan Foley points out that this discussion is exactly analogous to results in ordinary consumption theory about "necessary" bundles in the linear expenditure system. For such a discussion, and some comments on previous discussions of the topic, see Pollak [23].
interesting part of this explanation lies in the fact that we did not
arbitrarily prescribe that future income be discounted at the safe
rate and added to current disposable wealth, i.e., \( W_t \), to describe
the wealth relevant for the individual's consumption and portfolio
decisions but that we were led to this definition by an examination
of maximizing behavior.\(^1\)

Whether this explanation is found plausible or not, it should be
interesting to pay attention to the case \( \alpha > 0 \) as indicative of behavior
associated with utility functions of decreasing relative risk aversion.

3. This model allows us to consider the rather vague phenomenon of
"businessman's risk", examined in [28]: this is the suggestion that cer-
tain assets are too risky for "widows and orphans" but quite suitable for
businessmen. We shall try to obtain a more precise idea of what is meant
by "businessman's risk" by considering two of the arguments in [28].

First there is the suggestion that the businessman being more
wealthy faces less danger of falling below a subsistence level of
income. As a consequence of this suggestion we shall use as our
first criterion for the existence of "businessman's risk" the following:\(^2\)
businessman's risk exists if

\[
(B.1) \quad \frac{A_t}{B_t}\text{ is an increasing function of wealth.}
\]

\(^1\)It is by no means obvious before completing the analysis that
this type of answer will emerge. One may tell a similar story for
negative \( \alpha \)--the individual having an obligation to meet each period--
but we shall stay with the subsistence consumption description.

\(^2\)We shall assume that there is only one risky asset in the
following development.
Using (12) and (13) we obtain

\[
\frac{A_t}{B_t} = \frac{w(1-k_t)(W_t + ag_t)}{(1-w)(1-k_t)(W_t + ag_t) - a(g_t - 1)}
\]

\[
= \frac{1}{1-w - \frac{a(g_t - 1)}{w(1-k_t)(W_t + ag_t)}}
\]

For \(0 < w < 1\) (i.e., where the portfolio is diversified)

\[
\frac{\partial A_t}{\partial W_t} = \frac{\partial B_t}{\partial W_t} = -a
\]

where " means "of the same sign as".

Using definition (3.1) businessman's risk exists when subsistence income is explicitly taken into account for then \(-a > 0\) and

\[
\frac{\partial A_t}{\partial W_t} > 0.
\]

Second the businessman can look forward to a high salary in the future. This leads to our second criterion: businessman's risk may be said to exist when

\[
(B.2) \quad \frac{A_t}{B_t} \text{ is an increasing function of } a.
\]

Businessman's risk exists on criterion B.2 for

\[
\frac{\partial A_t}{\partial a} > 0
\]
Criteria B.1 and B.2 seem quite adequate descriptions of the idea behind the notion of businessman's risk and the phenomenon is seen to exist using these definitions. In [28], however, Samuelson goes on to suggest that the idea may refer to the length of time till the end of the horizon, the idea being that an individual who purchased more risky assets early in his life than late in life would provide an example of businessman's risk. Instead of pursuing the question of businessman's risk further by considering more definitions, we shall simply consider some questions relating to the effects of age on portfolio decisions in this model.

The first question is whether individuals with the same levels of wealth but of different ages buy differing amounts of the risky asset. We have first to define what is meant by "the same level of wealth." This is a difficult question. An individual with wealth $W$ at time $t+1$ is apparently better off than one with wealth $W$ at time $t$ since he can enjoy higher consumption for the remainder of his life than the younger person if both invest only in bonds. On the other hand, the younger person has more time in which to accumulate wealth. For the sake of definiteness we shall say that individuals are equally wealthy when they can earn the same income stream in each period for the rest of their lives by investing in bonds; that is, when the annuity income purchasable by $W$ is the same. We then want to compare an individual at time $t$ with wealth equal to $Yg_t$ with one at time $t+1$ with wealth equal to $Yg_{t+1}$, where $Y$ is their annuity income.
Now computing the difference in holdings of risky assets for two individuals identical in all respects but age:

\[(18) \quad A_{t+1} - A_t = w(Y+\alpha)[(1 - k_{t+1}) g_{t+1} - (1 - k_t) g_t] < 0\]

The inequality (18) holds since \(g_t > g_{t+1}\), and \((1 - k_{t+1}) < (1 - k_t)\).

An older person with the same amount of wealth (as defined above) as a younger person always holds fewer risky assets.\(^1\)

A more interesting question, but one which is more difficult to answer, is whether the portfolio composition shifts towards bonds as the individual ages: i.e., is it the case that

\[(19) \quad \frac{B_{t+1}}{A_{t+1}} > \frac{B_t}{A_t}.\]

It turns out that

\[(20) \quad \frac{B_{t+1}}{A_{t+1}} - \frac{B_t}{A_t} = \alpha \left[ \frac{g_t - 1}{(1 - k_t)(W_t + ag_t)} - \frac{g_{t+1} - 1}{(1 - k_{t+1})(W_{t+1} + ag_{t+1})} \right].\]

Using the same definition of wealth as above,

\[(21) \quad \frac{B_{t+1}}{A_{t+1}} - \frac{B_t}{A_t} > 0 \quad \text{as} \quad \alpha \left[ \frac{g_t - 1}{(1 - k_t) g_t} - \frac{g_{t+1} - 1}{(1 - k_{t+1}) g_{t+1}} \right] > 0\]

There are two effects at work here: the older the individual, the less bonds he needs to hold for annuity purposes, but, at the same time, the less does he save.

The sign of the bracketed expression in (21) is the same as

\(\text{\footnotesize{\(1\)Mossin [21] refers to "time effects" in portfolio problems and suggests that these are positive or negative as} a \text{\small{is} positive or negative. His "time effects" would be defined as} A_{t+1} < A_t \text{\small{for the same level of} W. His result is due to his assuming no consumption, for then} A_t = w(W + ag_t).}}\)
that of \((R-a)(1-aR)\);\(^1\) the sign of \((R-a)(1-aR)\) for different values of \(R\) and \(a\) is indicated in Figure 1. Recall that \(a = (R^*/l+\ell)^{-\frac{1}{B}}\)

![Figure 1.](image)

where \(R^*\) is the expected utility return on the portfolio.

There is very little that can be said about the sign of \((R-a)(1-aR)\) in general: using the definition of \(a\) one can show that

For \((1-\beta) > 0\), if \(R \geq (l+\ell)\), \(R > a\) and \(a\) is an increasing function of \(R\)

For \((1-\beta) < 0\), if \(R \leq (l+\ell)\), \(R < a\) and \(a\) is a decreasing function of \(R\)

Thus for high values of \(R\) and unbounded utility \((\beta < 1)\) one would expect \((R-a)(1-aR) < 0\) and for low values of \(R\) and bounded utility \((\beta > 1)\) again one would expect \((R-a)(1-aR) < 0\). These inequalities are not very useful but they do indicate that whether the portfolio

\(^1\)I am grateful to Ran Mosenson for providing a proof of this conjecture.
shifts towards bonds through time depends on the boundedness of the utility function as well as the value of $R$.

Most important, though, is to note that whether $\alpha(R-a)(1-aR) > 0$ and the portfolio composition shifts towards bonds as the individual ages, depends on the sign of $\alpha$. Portfolio composition stays constant through time for $\alpha = 0$.

4. The qualitative properties of the demand functions (11)-(13) with respect to changes in financial variables are readily derivable by the techniques used in Chapter II.¹

(22) $k_t = k_t(R, h_1, \ldots, h_{m-1}, t_1, \ldots, t_{m-1})$

$(\beta-1)B \frac{\partial k_t}{\partial R} > 0; (\beta-1)A_i \frac{\partial k_t}{\partial h_i} > 0; (\beta-1)A_i \frac{\partial k_t}{\partial t_i} > 0.$

(23) $w_i^t = w_i^t(R, h_1, \ldots, h_{m-1}, t_1, \ldots, t_{m-1})$

$\frac{\partial w_i^t}{\partial R} \geq 0; \frac{\partial w_i^t}{\partial h_i} \geq 0; \frac{\partial w_i^t}{\partial h_j} \geq 0; \frac{\partial w_i^t}{\partial t_i} \geq 0; \frac{\partial w_i^t}{\partial t_j} \geq 0$

These properties of the demand functions do not change through time; nor does the magnitude of the effects of parameter changes on $w_i^t$ vary.

¹We apologize for the clash of notation in (22): $t$ as a subscript or superscript represents time; $t_i$ inside the parentheses is a variable which affects the spread of the distribution of $X_i$. 
IV. Constant Absolute Risk Aversion (CARA) Utility Functions

1. The demand functions generated by CARA utility functions are in some ways similar to those obtained in Section III above.

We shall comment on the consumption function first and then examine asset demand functions. The consumption function is

\[ C_t = k_t W_t + m_t \]

where
\[
k_t = \frac{\Pi_{i=t}^{T} \frac{R_i}{R_i}}{1 + \sum_{i=t}^{T} \frac{R_i}{R_i}}
\]
\[
m_t = -\sum_{i=t}^{T} \log \left( \frac{\frac{R_i}{R_i}}{1 + \sum_{j=i+1}^{T} \frac{R_j}{R_j}} \right)
\]
\[
\gamma(1 + \sum_{i=t}^{T} \frac{R_i}{R_i})
\]

and
\[
a_t = \min \left\{ e^{-\gamma k_{t+1} \sum_{j=1}^{m-1} A_t A_j z_{t+1}} \right\}
\]

\( \gamma \) is the measure of risk aversion attached to the utility function.

If the interest rate is expected to remain unchanged throughout the individual's lifetime, then

\[ C_t = \frac{x}{R} W_t + m_t \]

\[
= \frac{W_t}{x_t} + m_t
\]
where \( \frac{W_t}{g_t} \) is the annuity income obtained by investing wealth at the safe rate. (The \( \frac{W_t}{g_t} \) formulation does not depend on a constant interest rate).

Before continuing to an examination of the \( m_t \) term, we should emphasize the striking feature of the consumption function (25). The propensity to consume out of wealth depends only on the safe interest rate; the effects of uncertainty on consumption are all taken care of in the constant term, \( m_t \).

The \( m_t \) term represents the demand for consumption goods resulting from expected earnings on holdings of risky assets. Since \( a_t \) is a minimum, \( a_t < 1 \) (setting \( A_i^t = 0, i=1, \ldots, m-1 \) gives \( a_t = 1 \) and the minimum cannot exceed this). Thus for \( t > r_t \) for all periods, \( m_t \) will be positive. The higher the discount rate the greater is \( m_t \); the more lucrative are risky assets the higher is \( m_t \). The higher the interest rate the more likely is \( m_t \) to be negative.

From the fact that \( k_{t+1} \) and \( A_i^t \) enter the expression for \( a_t \) multiplicatively one can show that, where the interest rate is constant, the value of \( a_t \) is independent of time;

\[
(26) \quad a_t = \min \left\{ E[e^{-\gamma k_{t+1}(L_i^{A_i^t \{t+1\}})}] = a \right\} \quad \forall \ t=1, \ldots, T.
\]

\( \{A_i^t\} \)

Assuming that the interest rate is not expected to change and that it is not equal to zero we may write
\[ m_t = \log \left( \frac{Ra}{1 + \gamma} \right) \frac{\left( \sum_{i=1}^{T-t} R^i (R^{T+2-t} - 1) \right)}{\gamma (R^{T+1-t} - 1)} \]

The coefficient of \( \log \left( \frac{Ra}{1 + \gamma} \right) \) is always negative; its behavior through time depends on both \( t \) and \( R \); early in the individual's life when we would expect \( R > \frac{T+1-t}{T-t} \) (this is a sufficient condition) the value of the coefficient rises as the individual ages. Thus for \( Ra > 1 + \gamma \), early in life the value of the constant in the consumption function is negative and rising through time; for \( Ra < 1 + \gamma \) the value of the term in the consumption function which is not related to wealth is positive and falling through time.

Demands for risky assets are independent of the level of wealth.

\[ A_i^t = A_i^t (R, h_1, \ldots, h_{m-1}, t_1, \ldots, t_{m-1}) ; \]
\[ \frac{\partial A_i^t}{\partial R} < 0; \frac{\partial A_i^t}{\partial h_i} > 0; \frac{\partial A_i^t}{\partial t_i} > 0; \frac{\partial A_i^t}{\partial t_j} > 0 \]

\[ B_t = W_t \left( \frac{g_t - 1}{g_t} \right) - \sum_{i=1}^{m-1} A_i^t - m_t . \]

The behavior of this individual is decidedly peculiar. He fixes on a portfolio of risky assets independently of the level of his wealth and borrows in order to hold that portfolio if his wealth is so small as to require that; there is an element in his consumption demand which is independent of the level of wealth and for a sufficiently high discount rate or sufficiently high expected earnings on his portfolio, this too is positive. He will borrow to undertake this consumption if his wealth is so small as to require this.
The marginal utility of consumption at zero is not infinite here so that there is a possibility of negative consumption in some period as the mathematical solution or negative bequests in the absence of direct constraints. The former is hard to define in economic terms though the latter is not absurd.

Marginally, this individual deals with uncertainty by avoiding it. Any increases in wealth are divided between bond-holdings and consumption, with bond-holdings being arranged through time to provide a constant annual consumption stream till the end of the horizon.

2. There are interesting effects of the passage of time on demands for risky assets. First, it is easy to show in the two-period model that

$$\frac{\gamma}{A_1} \frac{\partial A_1}{\partial \gamma} = -1$$

where $\gamma$ is the measure of absolute risk aversion of the second period utility function. Now the derived utility functions, going forward in time from the first decision period, are of increasing absolute risk aversion; recall the envelope condition

$$J'_{T+1-t} [W_t] = U'(C_t)$$
$$J''_{T+1-t} [W_t] = U''(C_t) \frac{\partial C_t}{\partial W_t} = U''(C_t)k_t.$$

The absolute risk aversion of the derived utility function is

$$-\frac{J''_{T+1-t}[W_t]}{J_{T+1-t}[W_t]} = \gamma k_t \quad \text{and} \quad k_t \text{ is an increasing function of time.}$$
Thus as the individual becomes older he buys less of each type of risky asset though the composition of his portfolio of risky assets does not change through time. The ratio of holdings of risky assets from period to period is given by

\[
\frac{A_i^t}{A_i^{t-1}} = \frac{k_t}{k_{t+1}} = \frac{R_{T+2-t} - R}{R_{T+2-t-1}}
\]

where we assume the same interest rate from period to period.

V. Quadratic Utility Functions

Despite its advantages as a pedagogic device and the simplicity of the results it provides, the quadratic utility function is not to be taken seriously as a description of preferences, for it implies negative marginal utility at high levels of consumption.

For what it is worth, we present the general result for consumption and asset-holdings decisions for the quadratic utility function. To simplify we assume there is only one risky asset.

\[
(30) \quad \begin{bmatrix}
E[z^2] & -Rz \\
-Rz & \hat{\alpha}_{t+1} + R^2
\end{bmatrix}
\begin{bmatrix}
A_t \\
C_t
\end{bmatrix} = \begin{bmatrix}
\frac{z}{b} \left[ \frac{R_{T+1-t-1} - RW_t}{bR_{T-t(R-1)}} \right] \\
\hat{\alpha}_{t+1} + R \left[ \frac{R_{T+1-t-1} - RW_t}{bR_{T-t(R-1)}} \right]
\end{bmatrix}
\]

where \( \hat{\alpha}_t = \sum_{i=0}^{T+1-t} \left[ \frac{1+i}{2} \frac{E[z^2]}{R^2} \sigma_z^2 \right]^i \), \( \sigma_z^2 = E[(z - \bar{z})^2] = \sigma_x^2 \).
Solving for $C_t$ and $A_t$

$$C_t = \frac{\hat{\sigma}^2 \sigma_{Z_t}^2 + \hat{a}_{t+1}(1+\zeta) E[\hat{z}^2] - \frac{R^2 \sigma_{Z_t}^2}{b} [R^{T+1-t-1}]}{b R^{T-t} (R-1)}$$

$$A_t = \frac{\hat{a}_{t+1}(1+\zeta) \hat{\sigma}^2}{R^2 \sigma_{Z_t}^2 + \hat{a}_{t+1}(1+\zeta) E[\hat{z}^2]}$$

2. Mossin [21] in examining Tobin's work in which a utility function quadratic in the portfolio rate of return is used, \(V(R) = R-\beta R^2\) where $R$ is the portfolio rate of return) points out that Tobin's formulation is consistent with a utility function quadratic in wealth\(^1\) if $\beta$ is taken to be a function of wealth. It may be interesting to reverse this reasoning and apply it to our problem.

It is possible that the parameters of the utility function are dependent on the level of wealth. The quadratic utility function implies that there are levels of wealth at which any extra wealth will simply be given away, and this is generally found implausible. It is not unreasonable to argue, though, that a poor man could believe he would give away wealth when rich, a rich man could believe he would give away wealth when super-rich, etc.

The individual now believes in each period, $t$, that his utility function at time $t$ and for all future periods will be

\(^1\)There is no consumption decision in either Tobin's or Mossin's model.
(33) \[ U(C_{t+t}) = C_{t+t} - \frac{bC^2_{t+t}}{2W_t} \quad t = 0, \ldots, T+1-t \]

Of course, his actual utility function next period will be different if his wealth changes, but he believes that (33) reflects his preferences till the end of the horizon. We discuss this at greater length below.

To make clear what this implies about attitudes to risk we calculate the measure of absolute risk aversion associated with the utility function (33):

\[
\frac{-U''(C_{t+t})}{U'(C_{t+t})} = \frac{b}{W_t - bC_{t+t}}
\]

The higher the current level of wealth, the less averse to risk is the individual in his decisions this period and the less averse to risk does he believe he will be in all future periods.

There are a number of points which require comment here. First, it is not obvious that the relevant wealth level is \( W_t \); and not, say, \( W_{t-1} \); one might want to introduce a one-period lag in the belief that it may take a little time for the learning process in the utility function to work itself out. One would be at liberty to do this, or to introduce a distributed lag on past levels of wealth or a "previous peak income" ratchet or anything of the sort.

Second, and more important, we should distinguish two assumptions about the use of the utility function (33). We shall be assuming that the individual acts as if he is unaware of future changes in his utility function: in this period he believes that from now to the end
of the horizon he will have a utility function with the parameter on the $C^2$ term equal to $-\frac{b}{2w_t}$; next period if his wealth changes he redoes all his calculations till the end of his horizon on the belief that that utility function will be applicable forever. This is a process which causes regrets. Alternatively, we could assume that the individual knows what his utility function each period will be and takes the dependence of its parameters on wealth into account and suffers no regrets.\footnote{Making this assumption for the quadratic utility function leads to a linear derived utility function; we have not therefore analyzed this case in any detail. The "no-regrets" procedure anyway strains credulity as a utility function more than the regrets procedure.}

3. We now solve the quadratic problem using the utility function (33). The solution is

\[
\frac{C_t}{W_t} = \frac{\frac{1}{b} \hat{\alpha}_{t+1}(1+\lambda) E[\bar{z}^2] - \frac{R \sigma_{\bar{z}}^2 (R^T-t-1)}{R^T-t(R-1)}}{\frac{R^2 \sigma_{\bar{z}}^2 + \hat{\alpha}_{t+1}(1+\lambda) E[\bar{z}^2]}}
\]

\[
\frac{A_t}{W_t} = \frac{\hat{\alpha}_{t+1}(1+\lambda) \frac{R^{T+2-t} - 1}{(R-1)R^{T-t}b} - R}{\frac{R^2 \sigma_{\bar{z}}^2 + \hat{\alpha}_{t+1}(1+\lambda) E[\bar{z}^2]}}
\]

The most interesting change in behavior indicated by the difference between (31) and (32), and (34) and (35) is that within each period this individual now looks as if he has constant relative risk aversion in that he consumes a given fraction of his wealth. In a cross-section
study within a period for individuals of the same age and same horizon an investigator might conclude that individuals with the utility function (33) had constant relative risk aversion and differed in their subjective probability distributions. In a time series study these individuals could be distinguished from those with constant relative risk aversion for their portfolio composition would change systematically with age and wealth. The other difference is that all decisions here are based on the first two moments of the distribution of returns on the risky asset: this formulation thus saves the simplicity of the quadratic utility function while avoiding one of its more objectionable implications—that holdings of the risky asset are a decreasing function of wealth.

It is quite easy to modify this analysis by making a function of both past and present levels of wealth. In so doing a variety of consumption functions estimated by empirical investigators can be obtained. It should be remembered that all our discussion so far has been based on the assumption that the individual plans as if his utility function will not change systematically with wealth, even though it does so each period.

VI. General Utility Functions

1. Each of the three classes of utility function investigated above has the property that consumption is linear in wealth in each period; this implies that the derived utility function $J_i[\bar{W}_{T+1-i}]$ $i = 1, \ldots, T$ is similar in form to the actual utility function $U_t(C_t)$. 

The similarity arises from the envelope condition

\[
J_i'(W_{T+1-i}) = U'(C_{T+1-i}(W_{T+1-i}))
\]

for then

\[
J_i''(W_{T+1-i}) = U''(C_{T+1-i}(\cdot)) \frac{3C_{T+1-i}}{3W_{T+1-i}}
\]

and

\[
J_i'''(W_{T+1-i}) = U'''(C_{T+1-i}(\cdot)) \frac{(3C_{T+1-i})^2}{(3W_{T+1-i})} + U''(\cdot) \frac{3^2 C_{T-1-i}}{3W_{T+1-i}}
\]

etc. In those cases where \( \frac{3^2 C_{T-1}}{3W_{T}} \) is zero, it is quite easy to obtain the derived utility function from the actual utility function for each period. Suppose \( C_t = a_t + b_t W_t \); then

\[
J'(W_t) = U'(a_t + b_t W_t)
\]

and \( J(W_t) = c_t + \frac{1}{b} U(a_t + b_t W_t) \). It is worth pointing out that the constant \( c_t \) in the derived utility function is not arbitrary but is given by solving

\[
c_t + \frac{1}{b} U(a_t + b_t W_t) = U(a_t + b_t W_t) + \frac{1}{1+\epsilon} E[U(W_{t+1})] \text{ for } c_t.
\]

We are at liberty to determine \( U(C_t) \) up to an affine transformation, but each definition of \( U(C_t) \) defines a unique derived utility function.\(^1\)

\(^1\)This is of no particular significance in finite horizon problems but should be borne in mind in the infinite horizon case; a discrepancy between the solution I had obtained for the infinite horizon case with a quadratic utility function and the limit as \( t \rightarrow \infty \) of the finite horizon solution was due to my having arbitrarily set the constant term for \( J_{\infty}(W) \) at zero in solving the infinite horizon problem.
Where consumption is linear in wealth, comparative static exercises of the type undertaken in Chapters III and IV for the two-period problem carry over simply to multiperiod problems. In any period in the multi-period problem the individual is finding

\[(39) \quad J_{t+1-t}(w_t) = \max \quad U(C_t) + \frac{E}{1+\lambda} [J_{t-t}(w_{t+1})] \]

\[\{C_t\} \quad \{A_t\} \]

In the two period problem he is finding

\[J[w_t] = \max \quad U(C_1) + \frac{E}{1+\lambda} [U(C_2)] \]

\[\{C_1\} \quad \{A_1\} \]

\[= \max \quad U(C_1) + \frac{E}{1+\lambda} [U(w_2)] \]

\[\{C_1\} \quad \{A_1\} \]

The comparative static results in the two-period case depend on the second derivatives of the two utility functions in (40); in any period of the multiperiod problem they will depend on the second derivatives of the \(U(C_t)\) and \(J_{t-t}(w_{t+1})\) functions. But the second derivative of the \(J_{t-t}(w_{t+1})\) is simply a constant less than unity multiplied by the second derivative of the \(U(w_{t+1})\) function and qualitative results will not differ from results for these utility functions in the two period case.

In general, though, there is little definite that can be said about the properties of the solution to the multiperiod problem. One
can easily show that \( \frac{\partial C_t}{\partial W_t} \) is always less than one and an increasing function of time at a given level of wealth. Other than this, however, we can only repeat our comment of Section II of this chapter: properties of the solution which depend only on concavity in the two-period problem are preserved for the multiperiod problem.

VIII. **Linearity**

We have shown above that the ACRRA, CARA and quadratic utility functions provide demand functions for consumption and assets which are linear in wealth. A number of results on these utility functions are known:

1. They are the only additive utility functions which have linear Engel curves, either locally or in the entire consumption space.\(^1\)

2. They are the only additive utility functions for which "portfolio separation theorems" apply. In these cases the individual's portfolio behavior may be described as first choosing one composite risky asset and then choosing the optimum amounts of this and the safe asset to hold as wealth varies.\(^2\)

---

\(^1\)Pollak [23] and Stiglitz [32]. "Locally" needs careful interpretation, which is provided by Pollak: the Engel curves may intersect the axes or radiate from some point within the consumption space (and not be defined to the south-west of that point) but are otherwise everywhere linear.

\(^2\)Stiglitz [33].
(3) "Partial myopia" in portfolio choice occurs with these utility functions.\(^1\) "Partial myopia" exists when, in each period, an individual can make portfolio decisions as if it is the last decision period, after which wealth will have to be invested at the safe rate till the end of the horizon. Total myopia occurs when the individual always behaves as if the next period were the last--this, in Mossin's model, holds only for constant relative risk aversion utility functions, and for no others.

It may help to understand Mossin's results if it is pointed out that there is no consumption in his model; nonetheless, the individual is assumed to attach utility to wealth at different dates. There is only a portfolio decision to be made in each period so that the total myopia result, for instance, arises from the fixed portfolio proportions in each period in that case. Similar definitions of myopia to Mossin's, based on portfolio behavior, could be given which would take his results through to our more general model. This would be based on the recurrence of various annuity type formulae in the demand functions of Chapter V.\(^2\)

---

\(^1\)Mossin [21].

\(^2\)There is an error in Mossin's proof [21], pp. 225-226 of necessity of these three utility functions for partial myopia; he claims these are the only utility functions for which such a property holds. In the course of his proof he wished to show that the condition

\[ E[U'(1+r)A + a(X-r)] = b[U'(1+r)A] \]

where \(b\) is independent of \(A\), can be satisfied only by the three utility functions in question. \(A\) is initial wealth and \(a\) the proportion of the portfolio in the risky asset; \(a\) is the choice parameter. \(X\) is the return on the risky asset. We now quote:
(4) We have shown that each of these utility functions provides demand functions for consumption and assets which are linear in wealth in each period. We now show that they are the only utility functions for which this property holds.

One proof of this is as follows: in the contingent commodity model these are the only utility functions which give linear Engel curves (this is easily obtained from Pollak’s proof). If there were any other utility functions which had linear Engel curves in the asset problem they would also have to provide linear Engel curves when the number of assets was equal to the number of states of nature—i.e., in the contingent commodity problem. But there are no other such utility functions.

"To demonstrate necessity we observe that [this condition] can be satisfied only if there exists, for each value of X, a factor h(X) such that

\[ U'((1+r)A + a(X-r)) = h(X) U'((1+r)A) \]  

(Equation (33) appears with Y as the argument of the utility function on the left hand side and A* as the argument on the right hand side; we have replaced these by their definitions.) Then b is to be taken as E[h(X)]. For each outcome of yield X, the value of U'(Y) is proportional to U'(A*) by a factor which is independent of A; if this were not the case, the weighted sum b of such factors could not be independent of A, either."

The statement after (33) is not correct. We should write (33) as

\[ U'((1+r)A + a(X-r)) = h(X,A) U'((1+r)A) \]  

and then

\[ E[h(X,A)] = b \]

Suppose \( h(X,A) = (X-X)A+b \) where \( X \) is the mean of the distribution of \( X \); then clearly \( E[h(X,A)] = b \) but the factor \( h(X,A) \) is not independent of \( A \). (I am grateful to Professor Franklin Fisher for this example.)

There are a number of other restrictions on the \( h(X,A) \) function implied by (33)' but these are not sufficient to ensure that \( h(X,A)=E(X) \). I have not, however, found a counterexample to the theorem itself, and suspect that it is true.
Alternatively, we proceed as follows:

(i) These are the only utility functions with the property

\[
\frac{U'(C)}{U''(C)} = \alpha + \beta C
\]

If linearity of Engel curves implies (41) then it implies that the three utility functions we have are the only ones providing linear Engel curves.

(ii) Let \( C_o = a(R,F(X)) + b(R,F(X)) W_o \)

\( A_o = e(R,F(X)) + d(R,F(X)) W_o \)

The first order conditions for a maximum in the two period problem are

\[
U'(C_o) = \frac{R}{1+\varepsilon} E[U'(W_o - C_o)R + A_o Z]
\]

\[
0 = E[U'(W_o - C_o)R + A_o Z]
\]

Linearity of \( C_o \) and \( A_o \) in wealth implies that the conditions

\[
U'(a+bW_o) = \frac{R}{1+\varepsilon} E[U'(W_o(1-b)-a)R + (c+dW_o)Z]
\]

\[
0 = E[U'(W_o(1-b)-a)R + (c+dW_o)Z]
\]

hold identically for all values of wealth.

From (42) we obtain

\[1\] See Mossin [21].
(47) \[ C_0 = \frac{\partial a}{\partial R} + \frac{\partial b}{\partial R} W_0 \]

Calculating \( \frac{\partial C_0}{\partial R} \) from the first order conditions (or from Chapter II)

\[
\frac{\partial C_0}{\partial R} = \frac{W_0 - C_0 - A_0}{R} \frac{\partial C_0}{\partial W_0} + \frac{E[U'(C_1)]}{R} \frac{\partial C_0}{\partial W_0} \left\{ \frac{E[U''(C_1)Z^2]}{1 + \xi} + \frac{E[U''(C_1)Z^2]}{1 + \xi} \right\} \]

Now differentiate (45) and (46) with respect to \( W_0 \) to obtain relationships between \( U''(C_0) \), \( E[U''(C_1)Z] \) and \( E[U''(C_1)Z^2] \) which hold identically in wealth.

Using these relationships and substituting for

\[
\frac{W_0 - C_0 - A_0}{R} \frac{\partial C_0}{\partial W_0} \text{ in (48)}
\]

(49) \[ \frac{\partial C_0}{\partial R} = \frac{b}{R} [W_0 (1 - (b+d)) - (a+c)] + \frac{1}{1 + \xi} \frac{E[U'(C_1)]}{U''(C_0)} \frac{[1 - (b+d)]}{U''(C_0)} \]

Using (49) and (47)

(50) \[ \frac{\partial a}{\partial R} (R, F(x)) + \frac{\partial b}{\partial R} (R, F(x)) W_0 = \frac{b}{R} [W_0 (1 - (b+d)) - (a+c)] + \frac{U'(C_0)[1 - (b+d)]}{RU''(C_0)} \]

or \[ \alpha + \beta W_0 = \frac{U'(C_0)}{U''(C_0)} \] (for \( 1 \neq (b+d) \))

whence

(51) \[ \alpha + \beta C_0 = \frac{U'(C_0)}{U''(C_0)} \] which is condition (41)
We have now to take up the condition $1 + b+d$. This means that the marginal propensity to hold bonds out of wealth is zero or

\[
B_o = g(R,F(X)) + h(R,F(X)) W_o
\]

where, at the existing value of $R$ and for the given distribution of $X$, $h(R,F(X)) = 0$.

Going back to Chapter II

\[
\frac{\partial B_o}{\partial R} = \frac{B_o}{R} \left[ \frac{\partial B_o}{\partial W_o} - 1 \right] - \frac{\mathbb{E}[U'(C_1)]}{1+\xi} \frac{\mathbb{E}[U''(C_o)] + \mathbb{E}_x [U''(C_1)(X)^2]}{1+\xi} \left[ \frac{2 \mathbb{E}[U''(C_1)] + U''(C_o)}{1+\xi} \right] \left[ \frac{\mathbb{E}[U''(C_1)^2]}{1+\xi} \right] - \frac{\mathbb{E}[U''(C_1)^2]}{1+\xi} \right]^{2}.
\]

On simplification, one obtains

\[
\frac{\partial g(R,F(X))}{\partial R} + \frac{\partial h(R,F(X))}{\partial R} W_o = -\frac{B}{R} + U'(C_o)
\]

or

\[
U'(C_o) = \hat{\alpha} + \hat{\beta} W_o = \hat{\alpha} + \hat{\beta} C_o
\]

so that

\[
\frac{U'(C_o)}{U''(C_o)} = \frac{\hat{\alpha} + \hat{\beta} C_o}{\hat{\beta}} = \alpha + \beta C.
\]

Notice that this proof uses linearity of all demand functions and not just the demand function for consumption goods.
Appendix: Derivation of Demand Functions for Consumption and Assets.

We shall derive the paths of consumption and asset holdings for ACERRA functions at some length to indicate the methods by which a solution can be obtained; the same methods were used to obtain the solutions for CARA and quadratics.

Since insurance is not explicitly included in this problem we shall assume that there are no bequests and that the T+1'st period is the last in which the individual is alive. His utility function for any period $t$ is just $^1$

$$U_t(C_t) = \frac{U(C_t)}{(1+\delta)^{t-1}} \quad t = 1, \ldots, T+1$$

In the last decision period the individual has to find

(1.1)

$$J_1[W_T] = \max \left\{ \begin{array}{c} (C_T + \alpha)^{1-\beta} \frac{(C_T + \alpha)^{1-\beta}}{(1-\beta)} + \frac{E}{(1+\delta)(1-\beta)} \left[ (W_T - C_T)R_T + \sum_{i=1}^{m-1} \left( X_i^{T+1} - R_T \right)^{\beta} \right] \end{array} \right\}$$

First order conditions for a maximum are

(1.2) \quad 0 = E[(W_T + \alpha)^{-\beta} Z_i^{T+1}] \quad i = 1, \ldots, m - 1

where $Z_i^{T+1} = X_i^{T+1} - R_T$

---

$^1$Alternatively, the individual may live T years and have a bequest function of the same form and with the same weighting as his utility-of-consumption function.
(1.3) \[ 0 = (C_T + \alpha)^{-\beta} - \frac{R}{1+\lambda} E[|W_{T+1} + \alpha|^{-\beta}] \]

Next we derive expressions for \( \frac{\partial C_T}{\partial W_T} \) and \( \frac{\partial A_i^T}{\partial W_T} \) in order to integrate back to demand functions. These derivatives are obtained by manipulations of the sort used in Chapter II, and are

(1.4) \[ \frac{\partial C_T}{\partial W_T} = \frac{(C_T + \alpha)}{W_T + \frac{\alpha}{R_T} (1+R_T)} \]

so that

(1.5) \[ C_T + \alpha = k_T(.) (W_T + \frac{\alpha}{R_T} (1+R_T)) \]

where \( k_T(.) \) is a constant of integration.

We use functional notation on \( k_T(.) \) since we are interested in its dependence on the parameters of the problem.

Then

(1.6) \[ \frac{\partial A_i^T}{\partial W_T} = \frac{A_i^T}{W_T + \frac{\alpha}{R_T} (1+R_T)} \]

so that

(1.7) \[ A_i^T = \xi_i^T(.) (W_T + \frac{\alpha}{R_T} (1+R_T)) \]

Substituting (1.5) and (1.7) back into (1.2) and (1.3), we find (1.2) and (1.3) homogeneous in \( (W_T + \frac{\alpha}{R_T} (1+R_T)) \), and eliminating this term obtain

(1.8) \[ 0 = k_T^{-\beta} - \frac{R_T}{1+\lambda} E[|(1-k_T)R_T + \sum_{i=1}^{m-1} \xi_i^T A_i^T + 1|^{-\beta}] \]

and
\[ 0 = E\left\{ (1-k_{i,T})R_{T} + \sum_{i=1}^{m-1} \xi_{i,T} z_{i,T+1}^{-\beta} z_{i,T}^{-\beta} \right\} \quad i = 1, \ldots, m-1 \]

Now solve formally for \( k_{T} \): define \( w_{T}^{*} = \xi_{T}^{T}/(1-k_{T}) \), and then

\[ k_{T}^{-\beta} = \frac{R_{T}}{1+\xi} (1-k_{T})^{-\beta} E\{ [R_{T} \sum_{i=1}^{m-1} w_{i,T} z_{i,T+1}^{-\beta}] \} \]

\[ = \frac{R_{T}^{*}}{1+\xi} (1-k_{T})^{-\beta} \]

where \( R_{T}^{*} = R_{T} E\{ [R_{T} \sum_{i=1}^{m-1} w_{i,T} z_{i,T+1}^{-\beta}] \} \) is the expected marginal utility return on the portfolio.

Finally

\[ k_{T} = \frac{(R_{T}^{*}/1+\xi)^{-\frac{1}{\beta}}}{(1+R_{T}^{*}/1+\xi)^{-\frac{1}{\beta}}} \frac{a_{1}}{1+a_{1}} \]

In the last decision period, then, demand functions for consumption and risky assets are as given in (1.5) and (1.7), with \( k_{T} \) of (1.5) being given in (1.11) and with \( \xi_{T}^{T} \) of (1.7) being obtained by solving (1.8) and (1.9).

Next solve explicitly for \( J_{1}[W_{T}] \):

\[ J_{1}[W_{T}] = \frac{k_{T}^{-\beta}}{1-\beta} \left( \frac{\alpha(1+R_{T})}{R_{T}} + W_{T} \right)^{1-\beta} \]

---

\(^{1}\)In this note we compare our results for the ACRRRA utility function with Samuelson's [28] for the CRRA function. Our \( k_{T} \) and \( w_{T} \) are exactly the same as his would be if his model were extended to many assets. The differences are: his \( k_{T} \) is the ratio \( C_{T}/W_{T} \) as well as the marginal propensity to consume out of wealth, whereas our \( k_{T} \) is only the marginal propensity to consume out of wealth (his consumption function is not degree one homogeneous in wealth); his \( w_{T} \) are portfolio proportions but ours are not, again due to non-homogeneity of the demand functions for risky assets in wealth. The similarities are more important: the \( k_{T} \) and \( w_{T} \) obtained in the Samuelson model are equal to those obtained here.
It is now simply a matter of retracing the steps from (1.1) to (1.11) to obtain the demand functions for the second last period. As expected the first order conditions are now homogeneous in \( [W_{T-1} + \frac{\alpha}{R_T R_{T-1}} (1+R_T + R_T R_{T-1})] \) and the equations for \( k_{T-1} \) and \( \xi_i^{T-1} \) are obtained by solving the expressions analogous to (1.8) and (1.9).

The following can then be shown

\begin{equation}
(1.13) \quad k_{T-1} = \frac{k_{T-1} a_{i_1}}{1+k_{T-1}} = \frac{a_2}{1+a_2}
\end{equation}

and

\begin{equation}
(1.14) \quad w_i^T = \xi_i^{T-1} / 1-k_T = \xi_i^{T-1} / 1-k_{T-1} = w_i^{T-1}
\end{equation}

Without going through more steps the general recursion relations can be obtained and solved, to give the demand functions

\begin{equation}
(1.15) \quad C_t + \alpha = k_t (W_t + \frac{\alpha}{R^{T+1-t}} \sum_{i=0}^{T+1-t} R^i) = k_t (W_t + \alpha g_t)
\end{equation}

\begin{equation}
(1.16) \quad \bar{A}_i = w_i (1-k_t) (W_t + \alpha g_t)
\end{equation}

\begin{equation}
(1.17) \quad B_t = \sum_{i=1}^{m-1} w_i (1-k_t) (W_t + \alpha g_t) - \alpha (g_t - 1)
\end{equation}

where

\begin{equation}
(1.18) \quad k_t = \frac{a_{T+1-t}}{T+1-t} \quad a = \frac{R^*}{1+\delta} - \frac{1}{\delta} \quad \text{where } R^* \text{ is defined in (1.10)}
\end{equation}

\( g_t \) is implicitly defined in (1.15).
CHAPTER V

THE LIFETIME MODEL WITH INSURANCE

I. Introduction

In this chapter we take up the problem described in Chapter I: that of finding the lifetime savings—consumption plan of an individual who is uncertain of the date of his death and who has the opportunity to buy (or sell) life insurance on his own life.

There is interest in a model which involves life insurance and an uncertain lifetime for several reasons. First, life insurance companies constitute one of the largest groups of financial intermediaries\(^1\) and this approach should furnish additional understanding of the demand for their product—in a general way, of course, there is no difficulty in understanding why such a demand exists. Second, decisions on life insurance are major portfolio decisions; since the probability distributions for the returns on life insurance policies are readily available from mortality tables this is an area in which the expected utility approach to portfolio decisions is amenable to empirical testing. Third, from a theoretical viewpoint, the introduction of an uncertain lifetime is a more sensible method of handling terminal conditions than the assumption that the lifetime is of known length. Also from the theoretical viewpoint, the use of two different

\(^1\)See Goldsmith [15], p. 98.
utility functions is a particular case of a problem alluded to in the literature\(^1\) as the "umbrella and rain problem" in which the utility function depends on the state of nature. Fifth, an insurance contract more closely resembles a contingent commodity than do equity and bonds and so the properties of the demand functions for insurance we obtain may be more definite than those for bonds and equity.

Finally, one can distinguish two classes of risk which confront the consumer. One class consists of risks which face the society as well as the individual—such as uncertainty over the rate of growth of the economy, the rate of inflation, etc. The other consists of risks which face the individual but not the society—the risk of dying is one of these, in the sense that the average mortality rates for the economy as a whole are quite predictable. The distinction is not perfect; it is closely related to the distinction between non-insurable and insurable risks. Financial institutions such as banks and mutual funds, which have been extensively analyzed, deal largely with risks which face society as well as the individual;\(^2\) insurance companies deal with the second type of risks—those facing only individuals—and such institutions have received less attention in monetary economics than have other types of financial intermediaries.

\(^1\)E.g., Diamond [7].

\(^2\)This is not a suggestion that the primary purpose of a bank is to deal with such risks.
II. The Model Recapitulated

We shall briefly outline the model of Chapter I again in order to familiarize the reader with it and to discuss some special features.

There is a maximum number of periods, T, for which the individual can live. If he is alive at T he will be dead at T+1. The probability that he dies at the beginning of any period t is \( \pi^d_t \), \( t=2, \ldots, T+1 \). The conditional probability of his dying at the beginning of period t, given that he was alive at t-1, is denoted \( \pi^d_t \). If he is alive in any period he obtains utility from consumption, denoted \( C_t \), and if he dies at the beginning of a period, utility attaches to the bequests he leaves, \( C_t \). He maximizes expected utility:

\[
E_1[\sum_{t=1}^{T} U(t) + \sum_{t=1}^{T} V(t) + \sum_{t=1}^{T} \pi^d_t C_t + \pi^d_{t+1} V_{t+1}(C_{t+1})] = E_1[\sum_{t=1}^{T} U(t) + \sum_{t=1}^{T} V(t) + \pi^d_t C_t + \pi^d_{t+1} V_{t+1}(C_{t+1})]
\]

In Sections III-VII of this chapter we take up special cases of the problem, in which the menu of assets, the sources of wealth and the form of the bequest function vary. In all sections we confine ourselves, for definiteness of results, to utility functions of constant relative risk aversion. In Sections 3-6 the elasticity of the bequest function and the utility-of-consumption function are the same. In the main, that is, we shall be assuming that the utility-of-consumption and the bequest function differ only by a multiplicative constant. This is a strong assumption which is made for tractability; in Section VII we allow the elasticity of the bequest function to differ from that of
the utility-of-consumption function and illustrate the difficulties which arise in that case.

In Sections III-VI we shall have

\[ U_t(C_t) = \frac{C_t^{1-\beta}}{1-\beta} \frac{1}{(1+i)^{t-1}} \]

(2)

\[ V_t(G_t) = \hat{b}_t \frac{G_t^{1-\beta}}{1-\beta} \]

The \( \hat{b}_t \) function is called the "weighting function for bequests"; and its form has been discussed in Chapter I. The discount factor in the utility of consumption function requires comment: it is sometimes argued that the discount factor is accounted for in part by the uncertainty of living—clearly our \( i \) cannot be taken to represent this uncertainty since we have taken explicit account of the uncertainty of living. The discount factor should instead be taken as a measure of the defectiveness of the imagination or of impatience (which may amount to the same thing); any reader who objects to this interpretation is free to set \( i=0 \).

The assumption of a maximum lifetime requires amplification. If the decision period is taken as one year, then the assumption that there is a last period is not wholly satisfactory; after all, there is no reason to assume that an individual who has survived 150 years cannot survive another year. One cannot, though, treat this as a stationary infinite horizon problem since stationarity would require the conditional probability of death in each period to be \( \frac{1}{2} \). A

\[ ^1 \text{This is the only probability which ensures that the probability of dying over the entire horizon is unity.} \]
possible approach would be to assume that after a certain date the conditional probability of death is large and constant and to solve the infinite horizon problem for these dates; then use the derived utility function from that stationary period in earlier periods when the probability of death distribution is non-stationary. Nonetheless, we shall be assuming a maximal lifetime in the knowledge that the approximation is not serious: the 1958 Standard Ordinary Mortality Table gives the number of deaths per 1000 at the age of 99 as 1000.¹

Except in Section VI we shall be considering single-period term insurance. The amount of premiums paid in any period is \( I_t \); if the individual dies at the end of the period \( t \) his heirs receive \( Q_t I_t \), but if he lives no payments are made. The purchase of insurance in period \( t \) entails no obligation to purchase insurance in the following periods. \( R_t \) is the return on bonds and for obvious reasons we assume \( Q_t > R_t \), \( t < T \). The individual cannot buy insurance in the last period of his life because death is then certain; equivalently one can say that the bond is an insurance asset for an individual in period \( T \). 

We shall sometimes talk of "fair insurance": insurance is fair when \( \Pi_{t+1}^d Q_t = R_t \). The expected profits of the insurance company

¹See [31], p. 583. This may be a residual figure since 99 is the last age for which there is an entry; the conditional probability of death at 99 is probably close to unity because \( \Pi_{97}^d = .448 \) and \( \Pi_{98}^d = .668 \).
are zero when insurance is fair for then it can invest one dollar of premiums in bonds in period \( t \) and receive \( R_t \) in period \( t+1 \) while expected payments in period \( t+1 \) are \( \Pi_{t+1}^d Q_t \). We shall also talk of "loading" in which case \( p \Pi_{t+1}^d Q_t = R_t \cdot p > 1; p - 1 \) is then the expected profit rate of the insurance company. Fair insurance is the "no loading" case of \( p = 1 \).

In Section III we analyze the simplest possible model in which the only assets are single period insurance and single period bonds. The individual begins life with an initial stock of wealth and receives no income. The analytical results obtained for the time path of the purchase of insurance are ambiguous so that a set of simulations was run. These will be described in greater detail in Section III. An extra asset-equity-is introduced in Section IV, and it is shown that the results are modified very little; we revert thereafter to the bonds-insurance model.

In Section V we introduce labor income which is received only if the individual is alive. It is shown that this makes the purchase of insurance more likely. Another set of simulations was run for this case.\(^2\)

\(^1\)Expected profits per dollar of premiums received are \( R_t - \Pi_{t+1}^d Q_t = R_t (1 - \frac{1}{p}) \), and discounting at \( R_t \) gives the profit rate of \( \frac{p - 1}{p} \) on insurance sales by the company.

\(^2\)The model of Section V is closely related to that of Yaari [39]. Yaari works in continuous time and does not specify any particular utility functions. He is particularly concerned with the existence of a solution, something we assume. He does not use an insurance asset but works with "actuarial notes" which pay off in the event of the individual's living, but not of his death. Our analysis could easily be recast in terms of such notes; our approach is slightly more natural when one is thinking of insurance.
In Section VI the individual is allowed to purchase two-period term insurance as well as single period insurance. No income is received. Finally, in Section VII we allow the elasticity of the bequest function to differ from that of the utility-of-consumption function.

III. Bonds and Single-Period Term Insurance

Employing the usual techniques to derive the last period consumption function—recalling that no insurance is bought in this period—we obtain

\[
C_T = \frac{R_T(R_T \hat{b}_{T+1})^{-\frac{1}{\beta}}}{1+R_T(R_T \hat{b}_{T+1})^{-\frac{1}{\beta}}} \hat{\nu}_T W_T = k_T W_T = \frac{\hat{\nu}_T}{\beta} W_T
\]

and

\[
J_1[W_T] = \hat{\gamma}_T W_T^{1-\beta} ; \hat{\gamma}_T > 1.
\]

In the second last period we find

\[
J_2[W_{T-1}] = \max \frac{C_{T-1}^{1-\beta}}{1-\beta} + \frac{\hat{\gamma}_T}{1+\beta} \frac{(W_{T-1} - C_{T-1})^{1-\beta}}{1-\beta} \left[ 1 - \frac{W_{T-1}^I}{R_{T-1}} \right]^{1-\beta}
\]

\[
+ \frac{\hat{\gamma}_T}{1+\beta} \frac{(W_{T-1} - C_{T-1})^{1-\beta}}{1-\beta} \left[ R_{T-1} + (Q_{T-1} - R_{T-1} W_{T-1}^I) \right]^{1-\beta}
\]

The variable $W_{T-1}^I$ denotes the proportion of the portfolio which
consists of insurance: the value of the portfolio is $W_{T-1} - C_{T-1}$. This implies that we are using the convention that the value of insurance bought is the amount of premiums paid and not the face value of the insurance policy. The variable $w_t^B = 1 - w_t^I$ is the proportion of the portfolio which consists of bonds.

We shall also write

$$b_t \left[ \frac{\beta}{1+\delta} \right] = b_t \quad \text{and} \quad \gamma_t \left[ \frac{\beta}{1+\delta} \right] = \gamma_t$$

The first order conditions for a maximum are

\begin{align*}
C_{T-1}^{-\beta} &= (W_{T-1} - C_{T-1})^{-\beta} \left[ \gamma_T \left( (1-w_t^I)R \right)^{1-\beta} + b_T [R + (Q-R)w_t^I]^{1-\beta} \right] \\
0 &= - R\gamma_T \left( (1-w_t^I)R \right)^{-\beta} + (Q-R)b_T [R + (Q-R)w_t^I]^{-\beta}
\end{align*}

Using (6), (5) may be simplified to

$$C^{-\beta} = (W-C)^{-\beta} \left[ \frac{R\gamma_TQ}{Q-R} \right] \left( (1-w_t^I)R \right)^{-\beta}$$

whence

\begin{align*}
C_{T-1} &= \frac{k_{T-1}}{1+k_{T-1}} W_{T-1} \\
\text{where} \\
k_{T-1} &= R(1-w_t^I) \left[ \frac{R\gamma_TQ}{Q-R} \right] - \frac{1}{\beta} \\
&= \frac{Q_{T-1}}{b_T \left[ Q_{T-1} \right]} + \left( \gamma_{T-1} Q_{T-1} \right) \frac{1}{\beta} \left( \frac{Q_{T-1} - R_T}{R_{T-1}} \right)^{\beta-1}
\end{align*}

1We shall occasionally drop time subscripts. The occasions are: first, when it should cause no confusion in the context of the chapter to do so; second, when the variable concerned—usually $R_t$—could without much loss of generality be taken as constant throughout the individual's lifetime; third, when some function—such as $w^I$—is not a function of time.
We may write the consumption function as

\[ C_{T-1} = k_{T-1}(Q_{T-1}, R_{T-1}, \hat{y}_T, \hat{b}_T, \pi_{T}^{d})W_{T-1}. \]

\[ (1-\beta)w_{I} \frac{\partial k_{T-1}}{\partial Q_{T-1}} < 0; \quad (1-\beta)w_{B} \frac{\partial k_{T-1}}{\partial R_{T-1}} < 0; \quad \frac{\partial k_{T-1}}{\partial \hat{y}_T} < 0; \quad \frac{\partial k_{T-1}}{\partial \hat{b}_T} < 0; \]

\[ \frac{\partial k_{T-1}}{\partial \pi_{T}^{d}} = 0; \quad \frac{R_{T-1}}{\pi_{T}^{d}} \frac{\partial C_{T-1}}{\partial R_{T-1}} = \frac{Q_{T-1}}{\pi_{T}^{d}} \frac{\partial C_{T-1}}{\partial Q_{T-1}}. \]

Clearly \( k_{T-1} \) is the propensity to consume in period \( T-1 \). The properties of the consumption function are obtained from (7).

We want first to consider whether the propensity to consume of period \( T-1 \) is necessarily smaller than that of period \( T \), as it is in the models of Chapter IV. Since a decrease in \( b_T \), the weighting on the bequest function, increases consumption of period \( T-1 \) it is clear that we can increase the propensity to consume of period \( T-1 \) without reducing that of period \( T \). We now show that we can in fact have situations in which \( k_{T-1} > k_T \).

To simplify matters we assume \( R_T = R = R_{T-1} \) and that, for the sake of the argument, insurance is fair. Using equations (3) and (7) we find that \( k_T > k_{T-1} \) requires

\[ \left( \frac{Q_{T-1} \pi_T^{d} \hat{b}_T}{Q_{T-1} \pi_T^{d}} \right)^{\frac{1}{\beta}} \left[ 1 - (1-\pi_T^{d}) \left( \frac{1}{Q_{T-1} \pi_T^{d}} \right)^{\frac{\beta-1}{\beta}} \left( \frac{1}{1+\pi_T^{d}} \right)^{\frac{1}{\beta}} \right] < \]

\[ \left( \frac{Q_{T-1} \pi_T^{d} \hat{b}_T}{Q_{T-1} \pi_T^{d}} \right)^{\frac{1}{\beta}} + (1-\pi_T^{d}) \left( \frac{1}{Q_{T-1} \pi_T^{d}} \right)^{\frac{1}{\beta}} \left( \frac{1}{1+\pi_T^{d}} \right)^{\frac{1}{\beta}} \]
The left hand side of (9) can be positive if \((l+\ell)\) is very large; then since \(\hat{b}_{T+1}\) is arbitrary, the entire left hand side can be made larger than the right hand side, thus violating the inequality \(k_T > k_{T-1}\). This is readily explicable: \((l+\ell)\) large implies that the derived utility function in the second-last period receives little weight, thus tending to increase the current propensity to consume.

Then the high \(\hat{b}_{T+1}\) means that if the individual survives to the last period, he worries greatly about bequests, and so reduces his consumption.

The fact that we can have \(k_T < k_{T-1}\) is an interesting result for it implies that an observation that the propensity to consume falls during some periods of an individual's life does not reflect a failure of the life-cycle consumption model. In particular, this is likely to happen when we expect the \(\hat{b}_T\) function to be increasing through time—as family obligations grow, for example.

The most interesting property of the consumption function relates to the effects on consumption of an increase in the conditional probability of death. The effect is of ambiguous sign, but the sign depends on the various factors in the way one would expect, as we now show.

\[
\frac{\partial k_{T-1}}{\partial \hat{n}_d} = \left(\frac{\hat{n}_R}{(1+\ell)\hat{b}_T}\right) [n_T^{-1}]^{1-\beta} - [n_T^{-d} (Q_{T-1} - R)]^{1-\beta}
\]

where " means "of the same sign as". The more does the "eat, drink and be merry ......." philosophy—in the form of a low \(\hat{b}\)—prevail, the more likely is it that an increase in the probability of death increases current consumption. Similarly the greater the weight.
on the derived utility function, the more likely is an increase in
the probability of death to increase current consumption.

There is an equivalent expression for the sign of \( \frac{\partial k_{T-1}}{\partial n_d^d} \)
which is of interest. If insurance is actuarially fair, or is loaded,
and it is actually purchased, then an increase in the probability of
death reduces consumption.

To prove this, use Cramer's rule and the first order conditions:
one can show that

\[
(10) \quad \frac{\partial k_{T-1}}{\partial n_d^d} = (Q^d - R) - w^I (Q - R).
\]

For fair or loaded insurance, \( Q^d - R < 0 \)

When the probability of death rises there are two effects on
the consumption decision: the weighting on the bequest function rises
and that on the derived utility function falls. The first effect
tends to increase consumption and the second to lower it. The portfolio
decision of equation (6) (or, when there are many assets, the set of
portfolio equations) in effect brings the two utility functions into
a desired balance, and it is due to this balance that the result occurs.

We now go on to consider the demand for insurance. Solve (6)
for \( w^I \):

\[
(11) \quad w^I = \frac{R(1-a)}{R+a(Q-R)} \quad \text{where} \quad a = \frac{(Q-R)}{R} \cdot \frac{b}{\gamma} - \frac{1}{\beta}
\]

so that the condition \( 0 < w^I \) requires, given \( Q > R \), that
(1-a) > 0 or

\( 0 < \hat{w}^I \text{ requires } \hat{R}\left(\frac{\hat{\gamma}^1}{1+\hat{x}} + \hat{b}^d\right) < Q\hat{b}^d \)

This is strictly analogous to the result that a risky asset is bought only if its mean return exceeds that on the safe asset so long as one takes account of the two different utility functions. The right-hand side \((Q\hat{b})^d\) is the bequest function weighted expected return on insurance and \((\frac{\hat{R}^1}{1+\hat{x}}\gamma + (\hat{R}^d)\hat{b})\) is the total utility function weighted return on bonds. This result is not an artifice of there being only two assets, but goes through for any number of assets, so long as one of them is safe, and so long as the bequest function and derived utility function differ only by a multiplicative constant.

Rewriting (12), the condition for the purchase of insurance, when insurance is actuarially fair, we obtain

\( 0 < \hat{w}^I \text{ requires } \hat{b} > \frac{\hat{\gamma}}{1+\hat{x}} \)

Recalling that \(\hat{\gamma} > 1\), no insurance will be bought unless the weighting on the bequest function exceeds the inverse of the discount factor. This is not hard to understand in terms of the model: a bond pays off whether one is dead or alive but an insurance asset pays off only if one is dead. If the expectation on the insurance asset is the same as that on the bond then by the earlier result one would not buy the insurance asset—except that now it is quite possible that one is more concerned about some states of nature than others—and a high \(\hat{b}\).
indicates such concern for states of nature in which one is dead. ¹

The insurance demand function has most of the properties one would expect of it.

\[ I_{T-1} = w_{T-1} (Q_{T-1}, R_{T-1}, \hat{\gamma}_T, \hat{b}_T, \hat{b}_{\Pi}^d) (1-k_{T-1})w_{T-1} \]

\[
\frac{\partial w}{\partial Q_{T-1}} > 0; \quad \frac{\partial w}{\partial R_{T-1}} < 0; \quad \frac{\partial w}{\partial \hat{\gamma}_T} < 0; \quad \frac{\partial w}{\partial \hat{b}_T} > 0; \quad \frac{\partial w}{\partial \hat{b}_{\Pi}^d} > 0.
\]

Also,

\[
\frac{\partial w}{\partial \hat{\gamma}_T} = -\frac{\hat{b}_T}{\hat{\gamma}_T} \frac{\partial w}{\partial \hat{\gamma}_T} \quad \text{and} \quad \frac{\partial w}{\partial \hat{b}_T} = -\frac{\pi_{\Pi}^d}{\hat{\gamma}_T} \frac{\partial w}{\partial \hat{b}_T}.
\]

The one surprising property of this demand function is that an increase in the insurance rate of return, \( Q_{T-1} \), has ambiguous effects on the purchases of insurance. The reason for the ambiguity becomes clear when we state (from (11))

\[ \frac{\partial (Qw)}{\partial Q} = w + Q \frac{\partial w}{\partial Q} > 0 \quad \text{or} \quad \frac{Q \frac{\partial w}{\partial Q}}{w} > -1. \]

Further, it can be shown—as we do below—that \( \frac{\partial (Qw)}{\partial Q} > 0 \), so that when the insurance rate of return rises the individual rearranges

¹The use of different multiplicative constants on the utility functions which apply in different states of nature is the simplest possible way of taking care of the "umbrella and rain" problem—in which the utility function depends on the state of nature. Our result that the individual may reject fair insurance or accept heavily loaded insurance depending on the weighting on his bequest function might help in understanding the paradox of the simultaneous purchase of insurance and gambling without appealing to utility functions convex in some range. One would have to be careful in defining the payoffs from gambling.
his portfolio so that insurance payments in the event of his death will be higher: the face value of his insurance policy rises. This result is proved in terms of the contingent commodity model, and it is incidentally seen that the ambiguity of $3\gamma_1 \partial / \partial Q$ arises from the presence of a transformation term.

Denoting by $G_1$ the contingent commodity receivable if alive, $G_2$ that receivable if dead, and by $B$ the amount of bonds:

$$\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} 0 & R \\ Q & R \end{bmatrix} \begin{bmatrix} I \\ B \end{bmatrix}$$

whence

$$\begin{bmatrix} I \\ B \end{bmatrix} = \begin{bmatrix} -1/Q & 1 \\ 1/R & 0 \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$$

and

$$\frac{3\gamma_1}{3Q} = -\frac{I}{Q} - \frac{1}{Q} \left( \frac{3G_1}{3Q} - \frac{3G_2}{3Q} \right)$$

The prices of the two contingent commodities are

$$p_1 = \frac{Q-R}{QR} \quad p_2 = \frac{1}{Q}$$

so that

$$\frac{3p_1}{3Q} = \frac{1}{Q^2} \quad \text{and} \quad \frac{3p_2}{3Q} = -\frac{1}{Q^2}$$

Thus an increase in $Q$ results in an increase in the price of the first contingent commodity and a fall in the price of the second. With two goods and additive utility functions it is easy to show that

$$\frac{3G_1}{3Q} - \frac{3G_2}{3Q} < 0.$$
The second term in (16) is accordingly positive.

The first term in (16) is a transformation term and for \( I > 0 \) its presence results in the ambiguity of the sign of (16). From (16) it is trivial to show that \( \frac{\partial (QI)}{\partial Q} > 0 \). From (16) one can also derive the equivalent statement that the elasticity of demand for insurance exceeds \(-1\). This completes the discussion of the term \( \frac{\partial I}{\partial Q} \).

We have not presented the signs of the derivatives \( \frac{\partial I_T}{\partial \xi} \), where \( \xi \) are the arguments of the demand function other than \( Q \), though they are easily calculable. If the reader is interested he may combine (8), (11) and (13) to do this.

The bond demand function is given by

\[
B_{T-1} = w^B(Q, R, \gamma, \hat{b}, \hat{n}) (1-k) W_{T-1}
\]

\[
\frac{\partial w^B}{\partial R} > 0; \quad \frac{\partial w^B}{\partial Q} > 0; \quad \frac{\partial w^B}{\partial \gamma} > 0; \quad \frac{\partial w^B}{\partial \hat{b}} < 0; \quad \frac{\partial w^B}{\partial \hat{n}} < 0.
\]

The ambiguity of \( \frac{\partial w^B}{\partial R} \) is once again due to a transformation term since

\[
\frac{\partial B}{\partial R} = -\frac{G_1}{R^2} + \frac{1}{R} \frac{\partial G_1}{\partial R}
\]

and the \( -\frac{G_1}{R^2} \) term introduces the ambiguity. By the same type of proof as before one can show that the interest rate elasticity of the demand for bonds exceeds minus one. Equivalently, when the interest rate increases the portfolio is rearranged in such a way that the income received next period from holding bonds rises.
We have spent a considerable amount of time exploring the properties of the consumption and asset demand functions for the period T-1; this is not because of an inherent interest in the second last period of life but because these functions are time independent. The next step is accordingly to solve the recursion relations for \( k_t \); all the other arguments of the demand functions are exogenous.

For earlier periods

\[
C_t = \left( \frac{k_t}{1+k_t} \right) W_t = k_t \hat{W}_t
\]

where

\[
\hat{k}_t = \left( \frac{R_t Q_t}{(1+\beta)(Q_t - R_t)} \right)^{-\frac{1}{\beta}} (1-W_t^I) R_t
\]

and \( \hat{\gamma}_t = \left( \frac{1+\hat{k}_t}{\hat{k}_t} \right)^{\beta} = k_t^{-\beta} > 1 \).

Denote \( \left\{ \frac{k_t}{k_t}, \frac{1}{\hat{\gamma}_t} \right\} \equiv R_t^* \), and then

\[
k_t = \hat{\gamma}_t^{-\beta} = \frac{T_t^* \prod_{i=t}^T R_i}{1 + \sum_{j=t}^T \prod_{j=t}^T R_j^*}
\]

We are not yet done since (19) is only an implicit solution for \( \hat{\gamma}_t \). Solving for \( k_t \) in terms of exogenous variables:

\[
k_t = \hat{\gamma}_t^{-\beta} = \frac{1}{T} \left[ \prod_{t}^{T-1} \eta_i \right] + \sum_{j=t}^{T-1} \eta_j \prod_{i=t}^{T-1} \xi_i
\]

and \( \prod_{i=t}^{T-1} \xi_i \equiv 1 \)
where $k_t$ is given in equations (3) and

$$
\xi_t = \left( \frac{\frac{1}{\Pi_{t+1}}}{\Pi_{t+1}} \right)^{\frac{1}{\beta}} \left( \frac{Q_t - R_t}{Q_t R_t} \right)^{\frac{\beta - 1}{\beta}}, \quad \eta_t = 1 + \left( \phi_t \Pi_{t+1} \right)^{\frac{1}{\beta}} \frac{1}{Q_t^{\frac{1}{\beta}}}
$$

Equation (20) can be used to answer questions on the comparative dynamics of the consumption path; for instance what happens to the propensity to consume in period $t$ when the weight on the final period bequest function rises?

$$
\frac{\partial k_t}{\partial b_{t+1}} = \sum_{i=t}^{T-1} \xi_i \frac{1}{k_t^2} \frac{\partial k_t}{\partial b_{t+1}} < 0.
$$

Similarly, an increase in any future $b_{t+1}$ will reduce $k_t$; an increase in any future $R$ will reduce $k_t$ for $1-\beta > 0$, as will an increase in any future $Q$ for $1-\beta > 0$. The effect on $k_t$ of any given change in one of these variables is smaller the further away in time the change is. Aside from the $\Pi_{t+1}$ variable, then, the qualitative properties of the consumption function with respect to future changes in variables are the same as those with respect to changes in concurrent variables.

The reason one has to be more chary about changes in $\Pi_{t+1}$ is not only that $\Pi_{t+1} + \Pi_{t+1} = 1$, but also that the sum of the probabilities of death for the entire lifetime is unity. Now $\Pi_{t+1}$ is not the probability of death at the end of period $t$ but the conditional probability of death. Given an increase in $\Pi_{t+1}$ it is possible to construct a new probability of death function such that the conditional
probability of death in all future periods is unchanged. Thus one can define a very simple (from the viewpoint of this problem) change in the mortality function which is quite amenable to analysis. In particular, for such changes our earlier result that \( \frac{\partial k}{\partial n_{t+1}} < 0 \) for loaded insurance and an individual who actually buys insurance goes through. The analysis of less restrictive changes in the mortality function is a good deal more complex.

Turning now to the insurance demand function for periods up to \( T-1 \):

\[
W_t^I = \frac{R_t (1 - \alpha_t)}{R_t + \alpha_t (Q_t - R_t)}
\]

and the condition for the purchase of insurance, given \( Q_t > R_t \), is that \( 1 > \alpha_t \). Then the discussion of the purchase of insurance above after equation (12) goes through here. If insurance is loaded, then the purchase decision depends only on the load factor:

\[
W_t^I \geq 0 \text{ as } \hat{b}_{t+1} \geq p\left[ \frac{\hat{n}_{t+1}^{-1}}{1 + \hat{b}_{t+1} \hat{n}_{t+1}^d} \right] + \hat{b}_{t+1} \hat{n}_{t+1}^d
\]

Thus even if one were to find an individual for whom \( \hat{n}_{t+1}^d = \hat{n}_t^d \) and \( \hat{b}_{t+1} = \hat{b}_t \), it is by no means certain that he will buy insurance.

---

1 If \( \hat{n}_t \) denotes a conditional probability and \( \hat{n}_t \) the actual probability of an event, and \( \hat{n}_t \) is some new probability of event \( t \), then if for all \( \hat{n}_{t+1}^j \), \( j=1, \ldots, T-t \), we have

\[
\hat{n}_{t+j} = \hat{n}_{t+j} \begin{bmatrix} (1 - E \hat{n}_i^t) \\ \frac{t}{1 - E \hat{n}_i^t} \end{bmatrix}
\]

\( \hat{n}_{t+j} \) for \( j=1, \ldots, T-t \), is unchanged.
in both periods \( t \) and \( t+1 \), since \( \hat{\gamma}_{t+1} \) is a function of time.

The properties of the \( w^I \) function are time independent so long as \( w^I_t \) is defined as a function of \( \hat{\gamma}_{t+1} \). But \( \hat{\gamma}_{t+1} \) is of course a function of the time paths of \( \hat{b} \), \( R \), \( \Pi^d \) and \( Q \). It is clear from (22) that the properties of the \( w^I \) function are the same as those set out in (13)—remembering that the appropriate changes in \( \Pi^d_{t+j} \) are those which leave all future \( \Pi^d_{t+j} \), \( j \geq 2 \) unchanged.

The meaning of the derivative \( \frac{\partial w^I_t}{\partial \Pi^d_{t+1}} > 0 \) should be made absolutely clear; this refers to the effects on insurance purchases of an increase in the probability of death which is not accompanied by a change in the premium. Thus we might think of it as referring to a situation in which the individual perceives an increase in his current probability of death (and decrease in future probabilities) but the insurance company does not—a case of intimations of mortality. More usually one would expect an increase in \( \Pi^d_{t+1} \) to lead to an increase in the premium. Frequently in practice an increase in \( \Pi^d_{t+1} \) due to fresh medical opinion leads to a refusal by the company to sell insurance.

The next question we want to ask is about the effects on insurance purchases of an increase in the probability of death which is accompanied by increase in the premium. Assuming \( pQ_t \Pi^d_{t+1} = R_t \), we obtain

\[
(23) \quad w^I_t = \frac{1}{1 + \frac{1}{\Pi^d_{t+1}} \left[ \frac{(1-p\Pi^d_{t+1})\hat{b}_{t+1}}{\left(\frac{\Pi^d_{t+1}}{1+\xi}k_{t+1}\right)} - \frac{1}{\beta} - \left(1 - p\Pi^d_{t+1}\right)\hat{b}_{t+1} - \frac{1}{\beta} \right]} \]

and, from (23),

\[
(24) \quad \frac{\partial W_t^I}{\partial \Pi_{t+1}^d} = R_p \left[ \frac{p(1-p)}{1-\Pi_{t+1}^d} \right] \frac{1}{\beta} - 1
\]

\[
+ \frac{p \Pi_{t+1}^d \hat{b}_{t+1} (1+\xi) k_{t+1} \Pi_{t+1}^d}{\beta} \frac{1}{(1-p)} \frac{1}{\beta} - 1
\]

The first term is of the same sign as \(\alpha - 1\) and so is negative if insurance is not being bought; the second term is negative if loading is positive. Thus an individual who is selling loaded insurance will increase his sales when the probability of death rises and the premium rises with it.\(^1\)

Because of the complexity of the effects of changes in other parameters on \(k_{t+1}\), little can be predicted about the course of insurance purchases through time without actually calculating the time profiles. Accordingly, a set of simulations of this model was run.\(^2\)

There were 32 runs in all, but the properties of all series were essentially similar so that the result of only one is reproduced here—in Graph 1. Before commenting on Graph 1 we describe the data used.

---

\(^1\)The reader may work out the similar— but weaker— statements which can be made about an individual who is buying insurance.

\(^2\)The simulations were run on the Sloan School of Management's IBM 1130. I would like to thank John Hill and Lovell Jarvis for assistance in performing the simulations.
Two different time series for \( \hat{b}_t \) were used, and these are given in Tables A1 and A2 in the appendix to this chapter. Also given in Table A3 is the \( n_t^d \) series. This consists of the last 50 entries in the Standard Ordinary Mortality Table,\(^1\) representing the conditional probabilities of death from ages 49 through 99. We confined ourselves to these ages since we hoped to weight the simulations in the direction of making the individual buy insurance.\(^2\) It would be a simple matter to extend the results back to age 20, but the major point is adequately made by the present results.

In Table I we show the alternative values for various parameters used in the different runs:

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( p )</th>
<th>( R )</th>
<th>( 1/1+t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1.05</td>
<td>1.10</td>
<td>0.92</td>
</tr>
<tr>
<td>2.0</td>
<td>1.20</td>
<td>1.05</td>
<td>0.84</td>
</tr>
</tbody>
</table>

There are sixteen combinations of these parameters, and there were two series for \( \hat{b}_t \), making 32 combinations.

\(^1\)Op. cit.

\(^2\)As it turned out, this hope was abortive. We shall comment on this below.
Graph 1: Lifetime Consumption and Insurance Decisions for an Individual Who Receives No Income.
The most outstanding feature of Graph I (for which the parameter values are those given in the first row of Table I and the $\hat{b}_t$ series is that of Table AI) is simply that the individual never buys insurance but always tries to sell it on his own life. (If we had put non-negativity constraints on $w^I$ he would never have bought insurance). In addition he tries to sell more insurance as he gets older. Reverting to (24) we recall that this expression is unambiguously negative for an individual who is selling insurance and for $p > 1$; thus it is the increasing probability of death which increases insurance sales through time since the other major factor which is changing through time---$\hat{\gamma}_t$---is tending to reduce insurance sales. This pattern occurred in all 32 simulations.

Two points require comment. First, we are working with the mortality function for the last 50 years of life: we stated that this was done to try to bias the results towards making the individual buy insurance. But we have now seen that the increase in the $\Pi_t$ series through time leads to increased sales of insurance. Might it be the case that extending the simulations back 30 years would reveal the purchase of insurance? The answer is "no", so long as we confine ourselves to orders of magnitude for $\hat{b}_t$ similar to those used in the later periods. Recall that $\hat{\gamma}_t$ is inversely related to $k_t$; further, $k_t$ is almost always an increasing function of time in the simulations. Then since $k_1$ in our simulations is about 0.1, a very high $\hat{b}$ would be required for the purchase of insurance in earlier periods (see equation (12)).
Second, when $p > 1$ and an individual is selling insurance, he is making an expected profit from so doing. One might want to redo the simulations with $p < 1$ and see whether the individual still sells insurance. If, for $p < 1$, he attempts to buy insurance, we could conclude that in practice he is unlikely either to buy or to sell insurance.

The other noteworthy feature of Figure 1 is that the propensity to consume is monotonically increasing. This tendency was prevalent almost everywhere but in one instance—the run in which $\beta=0.5$, $R=1.05$, $1/1+i=0.84$, $p=1.05$ and for the second $b$ series—the propensity to consume did drop once, from year 6 to year 7.

The comparative dynamic patterns expected between the different runs were observed—as for instance that an increase in $p$ led to more insurance sales, an increase in $\lambda$ to more consumption early in life.

The simulations prove nothing since the parameter values are arbitrary. This is particularly true for the $b_t$ series; it could be argued that these figures should be much larger and should not be of the same order of magnitude as the discount factor. Nonetheless, I conclude that an individual who has no labor income is unlikely to buy insurance. This result is not as surprising as it might appear at first when one considers that one of the major reasons for the purchase of life insurance is probably the fact that the death of the head of the family involves the loss of a major source of income.
IV. Bonds, Term Insurance and Risky Assets

The introduction of a risky asset (or many risky assets) alters the analysis of insurance and consumption demand functions very little. We shall attempt to keep the discussion brief; the same notation will be used for the weighting on the derived utility function \( \hat{\gamma}_T \) and the propensity to consume \( k_t \); other notation not so far used in this chapter is as in Chapter IV.

The derived utility function for the second-last period is the same as that of Chapter IV, equation (1.12) with \( \alpha \) of that chapter = 0;

\[
J_1[W_T] = \left( \frac{k_t^{1+\beta}}{1+\beta} \right)^{\frac{1}{1-\beta}} \frac{W_T^{1-\beta}}{1-\beta} = \frac{\hat{\gamma}_T W_T^{1-\beta}}{1-\beta}
\]

and
\[
\hat{k}_T^{\beta} = b_{T+1} E[(w_T^A X + w_T^B R)^{1-\beta}]
\]

In any earlier period, first order conditions for a maximum are

\[
0 = c_t^{\beta} - (w_t^I - c_t)^{\beta} \{ \hat{\gamma}_{t+1} E[(w_t^A Z + (1-w_t^I)R)^{1-\beta}] + b_{t+1} E[(w_t^A Z + R + (Q_t - R)w_t^I)^{1-\beta}] \}
\]

\[
0 = \hat{\gamma}_{t+1} E[(w_t^A Z + (1-w_t^I)R)^{-\beta} Z] + b_{t+1} E[(w_t^A Z + R + (Q_t - R)w_t^I)^{-\beta} Z]
\]

\[
0 = -\hat{\gamma}_{t+1} E[(w_t^A Z + (1-w_t^I)R)^{-\beta}] + (Q_t - R)b_{t+1} E[(w_t^A Z + R + (Q_t - R)w_t^I)^{-\beta}]
\]

Equations (27) and (29) are analogous to (6) and (7). Equation (28) is the additional equation which is the first order condition for equity; its interpretation is as for Chapter I, equation (19). From (29) one derives the identical condition for the purchase of insurance.
as (12). Incidentally, in this simplified two risky-asset model (insurance is a risky asset), equity is held only if its mean return exceeds the interest rate.¹

The qualitative properties of the consumption function are the same as those of (8) including the elasticity condition with respect to changes in R and Q, except for a slight change in \( \frac{\partial k_t}{\partial t+1} \) which is of the same sign as

\[
(30) \quad \omega_t^A E[(\omega_t^A Z + (1-\omega_t^I)R)^{-\beta}] + \frac{R}{Q_t-R} E[(\omega_t^A Z + (Q_t-R)\omega_t^I + R)^{-\beta}]
\]

\[
[(\Pi_{t+1}^d Q_t-R) - \omega_t^I (Q_t-R)]
\]

This has an affinity with condition (10) when there is no risky asset in that the second term of (30) is of the same sign as \( \frac{\partial k_t}{\partial t+1} \) in equation (10): the first term is negative for \( \omega_t^A > 0 \) and \( \omega_t^I > 0 \).² We may then say that if the individual is sufficiently worried about bequests to buy insurance when it is loaded, then an increase in the probability of death reduces his consumption, so long as \( \omega_t^A > 0 \) (i.e., \( \overline{X} > R \)).

The demand functions for insurance and bonds are likewise qualitatively the same as (13) and (17), respectively, for the risky asset

¹The proof is the same as that used in Chapter II.

²The proof of this depends on \( U'' < 0 \) and condition (28).
(31) \( A_{T-1} = w_{T-1}^A (Q_{T-1}, R_{T-1}, \gamma_T, b_T, \pi^d_T, f(z_T))(1-k_{T-1})W_{T-1} \)

\[
\begin{align*}
\frac{\partial w_{T-1}^A}{\partial Q_{T-1}} &> 0; \quad \frac{\partial w_{T-1}^A}{\partial R_{T-1}} < 0; \quad \frac{\partial w_{T-1}^A}{\partial b_T} < 0; \quad \frac{\partial w_{T-1}^A}{\partial \gamma_T} > 0; \quad \frac{\partial w_{T-1}^A}{\partial \pi^d_T} < 0
\end{align*}
\]

Somewhat surprisingly, the bond-equity ratio is not independent of \( w_{T-1}^I \).

Further, since \( \gamma_T, b_T \) and \( \pi^d_{T+1} \) will be changing through time, portfolio shifts between non-insurance assets can be expected through time.\(^2\)

This is in contrast to the model of Chapter IV.

Solving formally for the propensity to consume for earlier periods:

(32) \( C_t = \frac{\hat{k}_t W_t}{1 + \hat{k}_t} = k_t \sqrt{w_t} \)

where \( \hat{k}_t = \left[ \frac{R_t \gamma_{t+1} Q_t}{Q_t - R_t} \right]^{\frac{1}{\beta}} \)

and \( \gamma_t = \left( \frac{1 + \hat{k}_t}{\hat{k}_t} \right)^\beta = k_t^{-\beta} > 1 \)

Denote \( R_t^* = \frac{\gamma_{t+1} \hat{k}_t \gamma_t}{\hat{k}_t}^{\frac{1}{\beta}} \) and then

\[\text{\footnotesize \( To \ see \ this, \ set \ \frac{w_{T-1}^B}{w_{T-1}^A} = \lambda_{T-1} \) \ and \ solve \ the \ problem \ anew; \lambda \ is \ not \ independent \ of \ \gamma_T, b_T, \pi^d_T, Q_{T-1} \) and \( R_{T-1} \).

\[\text{\footnotesize \( I \ have \ not \ been \ able \ to \ calculate \ comparative \ elasticities.} \]
(33) \[ k_t = \frac{R^*_i}{\Pi R^*_j} \frac{T^*}{1 + \sum_{j=t}^{T} \Pi R^*_j} \]

with \[ R^*_T = \left[ \frac{1}{1 + \beta} \left[ E[w^A_T X + w^B_T R]^{1-\beta} \right] \right]^{-\frac{1}{\beta}} \]

\( k_t \) is not necessarily an increasing function of time.

The same statements as were made in section 3 on demand functions for assets for periods earlier than \( T-1 \) hold here. It is not, however, the case that if \( Q^d_t = R \), then \( \frac{\partial w^I_t}{\partial t} > 0 \); this derivative may be negative. Further, the invariance of \( w^I_t \) with respect to current financial variables if \( pQ^d_t = R \) is lost.

The ambiguities on the time profiles of the variables which exist in section 3 apply here as well.

V. Labor Income

The result of our simulations for the wealth-only model is not particularly surprising when one considers that in Sections III and IV we have had no assets which pay returns only when the individual is alive. There is in fact such an asset—human capital. One of the major reasons for the purchase of life insurance is probably the fact that the death of the head of the family involves the loss of a major source of income.

In this section we introduce labor income in a very simple way. We shall assume that if the individual is alive in any period \( t \), the
receives a definite income of \( Y_t \); if he dies at the beginning of the period his bequests include no income. There are a number of objections to this procedure: it might be appropriate to put the labor-leisure choice into the utility function; labor income is not certain—particularly not when one is drawing up lifetime plans in year 1; alternative income might well be available to one's family in the event of death. These—particularly the second—are reasonable objections but our method has the virtue of simplicity and leads to strong and sensible results.

The methods of deriving demand functions are exactly as previously, except that we shall not solve for portfolio proportions. For period \( T \) equations (3) and (4) apply. We shall use the same notation here as in Section III for some variables such as \( \gamma, k \) and \( \alpha \)—this should cause no confusion. Since (3) and (4) apply here, \( \gamma_T \) and \( k_T \) are the same as in Sections III and IV.

In the second last period,

\[
C_{T-1} = k_{T-1} W_{T-1} + \frac{V_{T-1} (Q_{T-1} - R_{T-1})}{\alpha_{T-1} (Q_{T-1} - R_{T-1}) + R_{T-1} (V_{T-1} Q_{T-1} + 1)} Y_T
\]

\[
= k_{T-1} W_{T-1} + \frac{V_{T-1} (Q_{T-1} - R_{T-1})}{\beta_{T-1}} Y_T
\]

where \( V_{T-1} = (Q_{T-1} b_T)^{-\frac{1}{2}} \) and \( \beta_{T-1} \) is defined implicitly above. In this period the marginal propensity to consume out of wealth is the same as that in the model without income; the marginal propensity to

\[w_{T-1} \text{ now includes principal and interest from bonds bought last period plus this period's labor income, } y_{T-1} \text{.} \]
consume out of next period's income is positive. (34) can be rewritten as
\[ C_{T-1} = k_{T-1} W_{T-1} + k_{T-1} \left( \frac{Q_{T-1} - R_{T-1}}{R_{T-1} Q_{T-1}} \right) Y_T \]

It will be recalled that when we worked with ACRRA utility functions in Chapter IV we could interpret the parameter \( \alpha \) there as future income and discovered that it could be discounted at the safe interest rate and treated as wealth. With fair insurance \( Q/R = 1/\Pi_T^d \), and then
\[ C_{T-1} = k_{T-1} \left[ W_{T-1} + \frac{\Pi_T Y_T}{R_{T-1}} \right] \]

so that next-period's income is weighted by the probability of receiving it and then discounted in reducing it to comparability with wealth.\(^1\)

The \( k_{T-1} \) function of (35) has the same properties as (8), noting that (8) is written with \( \partial k_{T-1}/\partial \Pi_T^d > 0 \). Whatever the sign of \( \partial k_{T-1}/\partial \Pi_T^d \), its value here is less than it is when no income is included—

it is more likely in a model where we include income than in the wealth model that current consumption falls when the probability of death rises.

The insurance demand function has the properties we expect of it:

\(^1\)Of course, this is proved here only for this utility function and for our particular definition of income: more properly income should be treated in a continuous time model such as that of Merton [20].
\[ I_{T-1} = \frac{1}{\hat{w}_{T-1}} \left[ (1-a_{T-1}) R_{T-1} + (1+V_{T-1} R_{T-1}) \frac{Y_T}{\hat{w}_{T-1}} \right] \]

\[ = \frac{I}{w_{T-1}} (1-k_{T-1}) w_{T-1} + \frac{1+V_{T-1} R_{T-1}}{\hat{w}_{T-1}} \frac{Y_T}{\hat{w}_{T-1}} \]

In particular, the demand for insurance is an increasing function of labor income next period. Our previous criterion for the purchase of insurance now becomes a criterion for the propensity to purchase insurance out of wealth to be positive: the earlier condition is now sufficient for the purchase of insurance. I have found no simple criterion for \( I > 0 \) here though one can obtain a complex criterion in terms of the parameters and the income-wealth ratio. Clearly insurance is more likely to be bought the higher is expected income and the greater is the weight on the bequest function.

Rewriting (36), we have

\[ I_{T-1} = \frac{I}{w_{T-1}} (1-k_{T-1}) w_{T-1} + \frac{I}{w_{T-1}} (0_{T-1}, R_{T-1}, \hat{y}_T, \hat{b}_T, \hat{\pi}_T)^T Y_T \]

with \( \frac{\partial y_{T-1}}{\partial Q_{T-1}} > 0; \frac{\partial y_{T-1}}{\partial R_{T-1}} > 0; \frac{\partial y_{T-1}}{\partial \hat{y}_T} < 0; \frac{\partial y_{T-1}}{\partial \hat{b}_T} > 0; \frac{\partial y_{T-1}}{\partial \hat{\pi}_T} > 0 \]

and \( w_{T-1}^I \) has the same properties as in (13).

\[ I \]

It seems from (36) that insurance purchases are more sensitive to expected income than to current wealth: specifically,

\[ \frac{\partial (Qy_I^I)}{\partial Q} > 0 \]

\[ \hat{y}_{T-1} \]

\[ \hat{b}_{T-1} \]

\[ \hat{\pi}_{T-1} \]

\[ \hat{b}_{T-1} \]

\[ \hat{\pi}_{T-1} \]
\[
\frac{\partial J_{T-1}}{\partial Y_T} - \frac{\partial J_{T-1}}{\partial W_{T-1}} = 1 + R_{T-1} (V_{T-1} + (\alpha_{T-1} - 1))
\]

This is not unambiguously positive but in some sense is "likely" to be—particularly since in the simulations we found without exception that \(\alpha_{T-1} > 1\).

The demand for bonds is an increasing function of current wealth and a decreasing function of expected income:

\[
(38) \quad B_{T-1} = \frac{1}{g_{T-1}} \left[ \alpha_{T-1} Q_{T-1} W_{T-1} - (1 + V_{T-1} Q_{T-1}) Y_T \right]
\]
\[= w_t^B (1 - k_{T-1}) W_{T-1} - y_t^B Y_T \]

The \(w_t^B\) function has the same properties here as in equation (17); the partial derivatives of the \(y_t^B\) function with respect to its arguments are of the same sign as those of the \(w_t^B\) function with respect to its arguments which are set out after equation (17).

The period two maximand is now \(J_2[W_{T-1}]\) but \(J_2[W_{T-1}, Y_T]\); by dint of some cancellation one obtains

\[
(39) \quad J_2[W_{T-1}, Y_T] = \frac{k_{T-1}}{1-\beta} \left[ W_{T-1} + \left( \frac{Q_{T-1} - R_{T-1}}{Q_{T-1} R_{T-1}} \right) Y_T \right]^{1-\beta}
\]

For \(Y_T = 0\) the derived utility function here is the same as it is in Section II—as it should be. Note that the envelope condition of Chapter IV, \(J'[W] = U'(C(W))\) holds here in the modified form

\[
\frac{\partial^2 J_2}{\partial W_{T-1}^2} = U'(C_{T-1}) \quad \text{and} \quad \frac{\partial^2 J_2}{\partial Y_T^2} = U'(C_{T-1}) \quad \frac{\partial C_{T-1}}{\partial Y_T}
\]
It should now be obvious how demand functions are modified for earlier periods: denoting \( \frac{Q_t + R_t}{m_t} \) by \( m_t \), demand functions are the same as (34), (36) and (38) with \( t \) appearing instead of \( T-1 \), except that \( Y_T \) in (34), (36) and (38) is replaced by

\[
Y_{t+1} = \sum_{j=2}^{T-t} \sum_{i=1}^{j-1} (Y_{T+j} - \frac{1}{m_{t+i}}); \tag{40}
\]

\( k_t \) is given by exactly the same recursion formula as in Section III and \( k_t = \gamma_t \).

Interpreting the above paragraph for the case of fair insurance (in which case \( m_t = \frac{1}{R_t} \)), the demand functions in earlier periods are similar to those of the second-last period but the income variable of (36), (38) and (40) is an annuity income discounted not at the safe rate, but at the higher rate \( \frac{1}{R_t} \).

Alternatively—and more clearly—to find the relevant income variable, discount each income at the safe rates back to the present, weight each discounted income by the probability of being alive to receive it, and sum—but only if insurance is fair. More generally the discount factor is related to \( m_t \).

A further set of simulations was run for the model of this section. This time only eight runs were made: only the \( \delta_t \) time series of Table A1 was used, and \( \frac{1}{1+\bar{x}} \) was fixed at 0.92. The individual was given no initial wealth; the income series used is presented in Table A4 of the appendix.
Graph 2: Net Worth and Consumption for an Individual Who Receives Income
Graph 3: Insurance and Bond Holdings for an Individual Who Receives Income
The results of a typical run—for the same parameter values as in Graph 1—are presented in Graphs 2 and 3. The individual now purchases insurance early in life and increases his purchases of insurance up to t=20 and then begins decreasing his purchases of insurance. At t=31 he begins selling insurance and sells it in increasing quantities in future years. The net worth position becomes negative fairly quickly but then by t=19 the net worth position is positive and continues increasing till near the end of the individual's life.

The question now arises of what institutional constraints prevent the sale by an individual of insurance on his own life. The most obvious difficulty from the viewpoint of an insurance company is that when an individual sells insurance on his life, the individual receives payments while he is alive and the insurance company has to collect its payments from his estate. In the event of a bankrupt estate the insurance company cannot cancel the policy, as it can when an individual defaults on premium payments. It appears that the purchase of annuities gives the individual a very similar lifetime pattern of insurance holdings to that of Graph 3. He never actually has negative insurance holdings, but his insurance coverage falls off in old age, and in old age he receives payments from the insurance company, as he would if he were selling life insurance. Of course, he has paid for these annuities earlier in life. But given the possible difficulties of collecting payments from an estate, this may be a second-best way of arranging lifetime insurance
holdings.\(^1\)

The conclusions I derive from the simulations of Sections III and \( V \) are that the major reason for the purchase of life insurance is that death involves the loss of human capital.

VI. Two-Period Term Insurance

In practice the individual buying life insurance is given the choice of term insurance for various periods and an assortment of savings plans. In this section, it is assumed that the individual lives a maximum of three years and can buy insurance each of his first two years; he is faced by a choice between two one-year policies and a two-year policy. This simplified model serves quite well to illustrate the point we wish to make.

The amount of two-year term insurance purchased is denoted by \( Z \), and the amount paid in the event of the individual's death at the end of either period 1 or 2 is \( m Z \). The remaining notation is as previously—including the terms \( a_t \) and \( V_t \) and \( g_t \) which were defined in Sections III and \( V \) respectively.

The consumption function in the last period—period 3 in this case—is as in Section 3. In the second period:

\[
(41) \quad C_2 = k_2 \left[ W_2 + \frac{m-Q_2}{Q_2} Z \right]
\]

and

\[
(42) \quad I_2 = w_2^I (1-k_2) W_2 - Z \left[ \frac{2(m-R)}{a_2(Q_2-R) + R(1+Q_2 W_2)} \right]
\]

\(^1\)By using a CRRA utility function we have ensured that the individual's estate is never bankrupt.
We do not comment on (41) and (42) since the major point of this section emerges when we examine first order conditions for an interior maximum for the first period.

These are

\[(43) \quad 0 = C_1^{-\beta} - RY_2 [B_2 R + (m-Q_2) z]^\beta - Rb_2 [B_2 R + Q_1 I_1 + m\bar{z}]^\beta\]

\[(44) \quad 0 = -RY_2 [B_2 R + \frac{m-Q_2}{Q_2} z]^\beta + (Q_1 - R) b_2 [B_2 R + Q_1 I_1 + m\bar{z}]^\beta\]

\[(45) \quad 0 = \left[\frac{m-Q_2}{Q_2} - R\right] \gamma_2 [B_2 R + \frac{m-Q_2}{Q_2} z]^\beta + (m-R) b_2 [B_2 R + Q_1 I_1 + m\bar{z}]^\beta\]

These are conditions for an interior maximum, and combining (44) and (45) we see that an interior maximum requires

\[(46) \quad (Q_1 - m) Q_2 = \frac{(Q_1 - R)}{R} (m-Q_2)\]

None of these are choice parameters for the individual and so there is no question of his being able to adjust his insurance purchases so that (46) is satisfied.

What if (46) is not satisfied? Then the individual will be attempting to buy as much of one sort of insurance as possible and sell as much of some other sort as he can. Accordingly we want to examine this condition carefully: it must have something to do with arbitrage possibilities.

First, we show that if insurance is fair in the sense that the insurance company makes no profit from the sale of either sort of insurance, (46) is satisfied. The present discounted value of
one unit of a two-year policy to the insurance company is

\[ \frac{R - m \Pi_2^\alpha}{R} + \frac{\Pi_2^\beta (R - m \Pi_3^d)}{R^2} \]

and setting this expression equal to zero and solving for \( m \):

\[ m = \frac{R^2 + R \Pi_2^\beta}{R \Pi_2^\alpha + \Pi_2^\beta \Pi_3^d} \]

(47)

It is a matter of computation to show that (46) is satisfied if \( m \) is

as given by (47) and \( \Pi_2^d Q_1 = R - \Pi_3^d Q_2 \) --remembering that

\[ \Pi_2^d = 1 - \Pi_2^\phi. \]

Second, note that if \( Q_1 = m \), then (46) requires \( m = Q_2 \). To see the arbitrage possibilities opened up if either \( Q_1 = m \) or

\( m = Q_2 \) and the other condition is not met, suppose \( m = Q_2 \) and

\( Q_1 > m \); then if the individual buys an arbitrarily large amount of

\( I_1 \), and sells \( z \) to the same value, his heirs receive large bequests

if he dies at the end of period 1. If he lives through period 2

he can cover himself against the large amount of two-period insurance

he has sold by buying \( I_2 \) to the same value; if he dies at the end of

period 2 his heirs' payments are covered by their receipts.

More generally, the condition (46) relates to the rank of the

following matrix:

\[
\begin{bmatrix}
B_1 & z & I_1 & I_2 \\
-1 & -1 & -1 & 0 \\
R & -1 & 0 & -1 \\
R & m & Q_1 & 0 \\
0 & m & 0 & Q_2 \\
\end{bmatrix}
\]
This is a returns matrix similar to that used in Chapter II: the assets are listed above the columns of \([X]\) and the states of nature to the left of the rows. If condition (46) is met the rank of \([X]\) is 3, as can be seen by calculating \(|X|\).

This condition on the rank of \(X\) ensures that the equation \([X][A] = 0\) where \(A\) as a vector of assets has a nontrivial solution. We now show what this has to do with arbitrage. The equations of \(XA=0\) are

\[
\begin{align*}
(i) & \quad B_1 + Z_2 + I_1 = 0 \\
(ii) & \quad RB_1 - Z - I_2 = 0 \\
(iii) & \quad RB_1 + mZ + 0_1 I_1 = 0 \\
(iv) & \quad mZ + 0_2 I_2 = 0
\end{align*}
\]

(49)

Suppose that (ii)-(iv) are solved: (iv) ensures that no net payments are made if the individual dies at the end of period 2; (iii) that no net payments are made if he dies at the end of period 1; (ii) that no net payments are made if he lives through period 2. Thus an asset vector which satisfies the last three equations ensures that the individual makes no net payments in all but one state of nature—viz., the first period. Unless (i) is also satisfied some net payments are made or received in the first period; choosing which assets to buy and which to sell the individual could always receive a net payment in the first period—and this payment could be made arbitrarily large since if \(A\) is a solution to (ii)-(iv), so is \(\lambda A\), for all \(\lambda\).
If the no-arbitrage condition (46) is met, then (43)-(45) constitute a set of dependent equations. It is a matter of indifference to the consumer what combination of single-period and two-period insurance is bought. In the absence of non-negativity constraints on the purchase of insurance the availability of multi-period insurance either presents arbitrage possibilities or makes no difference whatsoever to the individual buying insurance.

VII. The Bequest Function

Thus far we have assumed that the bequest function differs from the utility of consumption function only by a multiplicative constant. It can be argued that this is inappropriate: for instance, one may be more risk-averse on behalf of one's heirs than on one's own behalf. This difference in risk aversion would be reflected in differences in the two utility functions; confining ourselves to the constant relative risk aversion family, the index of relative risk aversion, \( \delta \), on the bequest function would be greater than the index, \( \beta \), on the utility of consumption function.

We provide no solutions for a full model in which the two elasticities differ; instead we examine a three period problem with bonds and insurance as an illustration of the difficulties which arise. In the last period

\[
J_1[W_T] = \frac{C_T^{1-\beta}}{1-\beta} + \frac{b_{T+1}^{1-\delta}}{1-\delta} [(W_T - C_T) R]^{1-\delta}
\]
\[
\frac{\delta - \delta}{\delta} + \frac{1}{\delta} \cdot \mathbb{R} = (bR) \cdot \frac{1}{\delta} \cdot \mathbb{R} \mathbb{W}
\]

The consumption function is not homogeneous in wealth; if \( \delta = 2\beta \)
then consumption is a quadratic function of wealth. More importantly,
since \( J_1'[W_T] = U'(C_T) \) the elasticity of marginal utility of the
derived utility function is neither \( \beta \) nor \( \delta \).

One period back, then, the individual is finding

\[
J_2[W_{T-1}] = \max_{C_{T-1}, I_{T-1}, B_{T-1}} \frac{C_{T-1}^{1-\beta}}{1-\beta} + \frac{\Pi^T_{T}}{1+\varepsilon} J_1[W_T] + \frac{\Pi^b_T}{1-\delta} [B_{T-1} R + Q_T + I_{T-1}]^{1-\delta}
\]

There are now three separate utility functions, each of a different
relative risk aversion, and no simple demand functions are obtainable
from (52).

VIII. Summary

A number of interesting conclusions emerge from this chapter.
In the model of Section III without labor income, the individual may
buy insurance which is heavily loaded against him if the weighting
on the bequest function is sufficiently large; on the other hand,
he may well reject fair or even favorable insurance if the weighting
on the bequest function is small. An increase in the probability of
death is more likely to increase current consumption the lower the
weighting attached to the bequest function. An increase in the
weighting on the bequest function unambiguously reduces current
consumption and increases insurance purchases. Bonds and insurance are not gross substitutes with respect to their rates of return. The demand for insurance as a proportion of the portfolio is invariant with respect to the interest rate provided that loading is constant. The elasticity of demand for bonds and insurance with regard to their respective rates of return exceed minus one.

The introduction of a risky asset does not change the condition for the purchase or sale of insurance, nor does it change the properties of the consumption and insurance demand functions significantly.

An individual who receives labor income is more likely to purchase insurance than an individual who lives off the proceeds of his wealth. If insurance is fair, then—in the consumption decision—future income is discounted at the safe rate and weighted by the probability of being alive to receive it in reducing it to comparability with wealth.

The simulations run indicate that an individual who lives off the proceeds of his wealth is unlikely ever to purchase life insurance. An individual who receives labor income is likely to purchase life insurance early in his life. In all simulations the individual tends to sell life insurance late in life: institutional reasons why companies do not engage in such transactions exist. The purchase of annuities generates consumption and bequest patterns similar to those of the simulations.

Multi-period term insurance either presents the individual with arbitrage possibilities, or makes no difference to his welfare.
Appendix: Data

TABLES A1 and A2: Weighting Functions for the Bequest Function

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$^1$Source [31], p. 583.
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BIBLIOGRAPHY


33. _______. Lecture on Portfolio Separation Theories and Mutual Funds at Harvard University, March, 1969.


BIOGRAPHICAL NOTE.

Stanley Fischer was born in Lusaka, Zambia in 1943. He received his schooling in Zambia, South Africa and Rhodesia. From 1962-1966 he attended the London School of Economics, obtaining the degrees of B.Sc.(Econ.) in 1965 and M.Sc.(Econ.) in 1966. He was enrolled in the Department of Economics at M.I.T. from September 1966 to August 1969; he was an instructor in economics in the spring semester of 1969.