TWO ESSAYS IN ECONOMIC THEORY

by

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ABSTRACT

Essay I: "Existence of Equilibrium, Optimality, and the Aggregate Excess Demand Correspondence in Economies with Externalities."

In Essay I we establish, under certain assumptions, the existence of a (non-cooperative) competitive equilibrium for a wide class of economies with externalities. The types of externalities covered in our existence theorem are the usual pure consumption (e.g., interdependent preferences) and pure production externalities, and also consumption-production (e.g., littering) and production-consumption (e.g., pollution) externalities. Most of the models of economies with externalities discussed in the literature fall within the class of economies covered by the theorem. Our proof of existence of equilibrium is similar in structure to the proofs of existence of equilibrium for economies without externalities found in the literature. The basic technique used in our proof is the use of Kakutani's Theorem to establish the existence of equilibrium for a compact economy.

We next analyze the relationship between equilibrium and optimality in economies with externalities, establishing the existence of a Pareto Optimum using fixed-point arguments. We demonstrate by use of a separating hyperplane argument that under certain assumptions a Pareto Optimum can always be supported by an equilibrium price vector and a system of lump sum and specific taxes and subsidies, where all participants face the same net price for "non-external" goods. By examples we demonstrate that a Pareto Optimum cannot generally be supported by a competitive equilibrium in a model where there are markets for externalities, because a competitive equilibrium for such models will usually not exist.

In the last part of Essay I, we analyze the structure of economies with externalities, showing the (jointly) feasible technology in such models will usually be non-convex, and establishing the existence and properties of the aggregate excess demand correspondence for economies with externalities.

Essay II: "The Diversification Problem in Portfolio Models."

In Essay II we attempt to give a partial answer to the question of how much diversification is optimal in choosing a portfolio of fixed size. The models we consider are the one period mean-variance criterion and the one period expected utility criterion portfolio choice models. We attempt to find general conditions on the joint probability distribution of the asset yields which are sufficient to ensure that an optimizing risk averse investor will hold a positive amount of each asset or will hold a positive amount of a particular asset. The main properties of the joint probability distribution we consider are non-positive (linear) correlation between the asset yields, and a type of non-positive (non-linear) correlation between the asset yields proposed by Samuelson, which we call S-Correlation.

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INTRODUCTION

In this essay we consider a class of economies with externalities, establishing under certain assumptions the existence of an "equilibrium" and a Pareto Optimum, and the relationship between the two. By "equilibrium" in this context we mean the classical (non-cooperative) competitive equilibrum concept used in defining the solution of the usual model without externalities. What this means is that all participants are price takers and no side payments are allowed. Of course this is not the only concept of equilibrium that has been considered for models with externalities. Coase[1*] and Buchanan[2] for example argue that side payments between participants will be a very likely result in a model with externalities. Charity of course is a good example of this. However actual examples of side payments arising out of externalities seem to be few compared to the many examples one can find in which there are no side payments. Casual empiricism would also seem to indicate that actual cases involving side payments are usually "good" externalities, e.g., charity, public health care. These externalities are good in the sense that making one person better off makes (most) other people better off. In the case of production externalities, the side payments in the case of "good" externalities are likely to mean the actual merger of firms. For example, in the classic apple grower-bee keeper illustration, it seems very likely that if the external

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*References are listed on p.93

benefits are significant that the apple grower will go into the bee keeping business or vice versa.

That one can find many examples of externalities where there are no side payments does not necessarily invalidate the Coase thesis. These might be explained by the costs of a change outweighing the value of the benefits of the change. To what extent we can accept the Coase thesis is not clear. This is complicated by the fact that there is no generally agreed upon concept of equilibrium in a model with side payments except perhaps the core, and economies with "bad" externalities do not in general have cores. Thus both the classical and the Coase analyses of economies with externalities each have serious defects. The classical analysis does not allow side payments, which certainly are present to some extent in the real world, while the Coase analysis does not provide an equilibrium concept which can be shown to exist for a class of models which approximate reality.

This essay unfortunately will not attempt a solution of these problems. Our much narrower aim is to establish that the classical analysis of externalities was not conducted in a vacuum, i.e., that an equilibrium in the classical sense can be shown to exist for the types of models of externalities the classical analysis considered. The problem of whether or not an equilibrium would exist in the various models with externalities seems to have been largely ignored. One reason for this is that much of the analysis was partial equilibrium analysis. The paper by McKenzie[3] is the only one in the existence literature which explicitly considers a model with externalities. The model he considered was one with only consumption externalities and assumed a stronger version of convexity of preferences than is desirable or necessary. This essay encompasses a much wider class of externalities.

That an equilibrium can be shown to exist for a class of economies with externalities is probably not surprising. However this lack of surprise should perhaps be tempered by the knowledge that the core does not exist for an interesting class of such economies. The intuition derived from the treatment of the existence of equilibrium problem for models without externalities would indicate that enough convexity and continuity assumptions for models with externalities would probably establish the existence of a (non-cooperative) equilibrium. But certain kinds of continuity and especially convexity assumptions in the context of externalities would be very objectionable. For example, the usual assumption that the aggregate production set for the economy be convex rules out the classic Marshallian example of industry increasing returns resulting from external economies between firms which individually exhibit non-increasing returns. Also, assuming in the case of consumption externalities that consumers' preferences are convex in the space of all consumers' consumptions would be objectionable, and would rule out many of the classical models of consumption externalities. The set of sufficient conditions which yield the existence theorem in this essay are satisfied by most of the models of externalities considered in the classical analysis of externalities. These sufficient conditions are probably not significantly more objectionable than the analogous conditions which are sufficient for existence in models without externalities.

This essay is made up of three parts. Part I describes the general model and presents an existence theorem. Part II establishes the existence of a Pareto Optimum for the model described in Part I and analyzes the relationship between equilibrium and optimality for the model. Part III analyzes the structure of the model described in Part I and defines and establishes the existence of the Aggregate Excess Demand Correspondence for the model.

PART I

THE MODEL AND THE EXISTENCE THEOREM

A: The Model

The class of externalities allowable within the framework of the model includes most of the types considered in the literature. These include:

a. Pure consumption externalities: a consumer's consumption may affect the preferences and/or the consumption possibilities of other consumers.

b. Pure production externalities: a producer's production activity may affect the production possibilities of other producers.

c. Production-consumption externalities: a producer's production activity may affect the preferences and/or consumption possibilities of consumers.

d. Consumption-production externalities: a consumer's consumption may affect the production possibilities of producers.

Externalities of type c) or d) have not had much emphasis in the literature, although those of type c) would seem to be an important real world phenomenon. We assume that all externalities are identifiable with the set of tradeable goods in the model.

As a basis for comparison we shall briefly review the general equilibrium model found in most of the conventional existence literature. In that model there are a fixed number of goods numbered $h = 1,\ldots,n$; a fixed number of consumers numbered $i = 1,\ldots,s$; and a fixed number of producers $e = 1,\ldots,t$.

A consumer, i, is a consumption possibilities set, $X^i \in R^n$, a set of preferences $(\geqq)_i$ defined on X^i, and an initial endowment $w^i \in R$. It is assumed that X^i is closed, bounded from below, and convex; that $(\geqq)_i$ is continuous and convex; and $\exists\, \bar{x}^i \in X^i$ such that $\bar{x}^i \ll w^i$.

A producer, e, is a net production possibilities set $Y^e \in R^n$. Y^e is assumed to be closed and convex, and $0 \in Y^e$. $y^e \in Y^e$ is a net input-output vector.

The strategy for the proof of existence is to show:

i) For X^i and Y^e compact, the excess demand correspondences of consumers and the supply correspondences of producers are upper semi-continuous (hereafter denoted u.s.c.) and convex-valued.

ii) For X^i and Y^e compact existence of equilibrium is established.

iii) It is shown under suitable assumptions that the set of feasible actions is compact.

iv) The original economy is "compactified" by bounding it by a set which contains the set of feasible actions.

v) It is shown that an equilibrium for the compactified economy is an equilibrium for the original economy.

The strategy of our existence proof will parallel i) - v). The model we consider has:

n goods, $h = 1,\ldots,n$

s consumers, $i = 1,\ldots,s$

t producers, $e = 1,\ldots,t$

An action for consumer i is the choise of a consumption vector $x^i \in R^n$. An action for producer e is the choice of an input and associated output vector $q^e = (q^{Ie}, q^{Oe}) \in R^{2n}$ where q^{Ie} is an input vector and q^{Oe} an associated output vector. We depart from the usual treatment of production by disaggregation into inputs and outputs rather than considering only net outputs. The reason for this is that intermediate goods would seem to be important in conveying externalities. For example, if a coal mine produces an externality (coal dust) which affects a nearby laundry, coal used as an intermediate good in the production of coal (e.g., burned) will also have an external effect. Of course the actions of consumer i are limited by his consumption possibilities and the actions of producer e are limited by his production possibilities, both of which we shall defined in the following sections. A set of proposed actions by the participants of the model is denoted $(z,y) \in R^{(s+2t)n}$ where $z = (x^1,\ldots,x^s)$ and $y = (q^1,\ldots,q^t)$.

A-1: Consumers

A consumer, i, has a consumption possibilities correspondence which for each vector of actions of other consumers and producers defines the consumption possibilities of i. This correspondence will be denoted $X^i(z,y)$ where $z = (x^1,\ldots,x^s)$, $y = (q^1,\ldots,q^t)$. Thus $X^i\colon R^{(s+2t)n} \to R^n$. $X^i(z,y)$ defines the set of consumptions for i given (z,y) which allow him to survive.

Assumptions about $X^i(z,y)$:

1) $X^i(z,y)$ does not depend on x^i, and is closed, convex, uniformly bounded from below, and continuous in (z,y), for all $(z,y) \in R^{(s+2t)n}$. ($X^i(z,y)$ is a correspondence, not a function, and thus by continuity we mean both upper and lower semi-continuous).

2) Each consumer, i, has an initial endowment function, $w^i(z,y)$, $w^i\colon R^{(s+2t)n} \to R^n$ which is continuous in (z,y), and $w^i(z,y) \geqq 0$, for all $(z,y) \in R^{(s+2t)n}$. $\exists\, \bar{w}^i$ such that $w^i(z,y) \leqq \bar{w}$, for all (z,y) and $\forall (z,y)\ \exists\, \bar{x}^i(z,y) \in X^i(z,y)$ such that $\bar{x}^i(z,y) \ll w^i(z,y)$.

Comments: That $X^i(z,y)$ is closed, convex and uniformly bounded from below, for all (z,y) is the obvious generalization of the assumptions made about the (fixed) consumption set in the usual existence problem. However, the assumption of continuity of $X^i(z,y)$ is a very strong assumption. Upper semi-continuity of $X^i(z,y)$ would be a more acceptable assumption, but unfortunately it is not strong enough. That $w^i(z,y)$ be uniformly bounded seems to be a reasonable assumption. As mentioned in our brief summary of the existence problem for economies without externalities, the usual assumption is that $\exists\, \bar{x}^i \in X^i$ such that $\bar{x}^i \ll w^i$. Our assumption in 2) is analogous. We could considerably weaken the assumption that $\bar{x}^i(z,y) \ll w^i(z,y)$ in a manner analogous to that of Debreu,[4] but only at the cost of a serious loss of clarity of exposition, and for this reason we will use the stronger assumption.

A consumer, i, has a preference relation $(\geqq)_i$ defined on $\{(z,y,p) \mid z \in R^{sn},\ y \in R^{2tn},\ p \in R^n\}$. Here $z = (x^1,\ldots,x^s)$ is a vector of all consumers' actions, $y = (q^1,\ldots,q^t)$ is a vector of all producers' actions, and $p = (p_1,\ldots,p_n)$ is a price vector. We allow consumers' preferences to be defined on prices also since it includes a model discussed in the literature and its inclusion causes no extra difficulties.

Assumptions about $(\geqq)_i$:

3) $(\geqq)_i$ is a complete continuous preorder on $R^{(s+2t+1)n}$. By continuous we mean the sets:

$$\{(z,y,p) \in R^{(s+2t+1)n} \mid (z,y,p)\ (\geqq)_i\ (\bar{z},\bar{y},\bar{p})\}$$

and

$$\{(z,y,p) \in R^{(s+2t+1)n} \mid (\bar{z},\bar{y},\bar{p})\ (\geqq)_i\ (z,y,p)\}$$

are closed in $R^{(s+2t+1)n}$, for all $(\bar{z},\bar{y},\bar{p}) \in R^{(s+2t+1)n}$.

4) Let $(z,y,p) = ((x^1,\ldots,x^{i-1}, x^i, x^{i+1},\ldots,x^s), y,p)$ and $(\bar{z},y,p) = ((x^1,\ldots,x^{i-1}, \bar{x}^i, x^{i+1},\ldots,x^s),y,p)$ be points in $R^{(s+2t+1)n}$ and $(z,y,p) \ (>)_i \ (\bar{z},y,p)$. Then $k(z,y,p) + (1-k)(\bar{z},y,p) \ (>)_i \ (\bar{z},y,p)$, for all $k \in (0,1)$.

Comments: The continuity of preferences we assume here is analogous to the continuity of preferences assumption made in the usual case. It rules out preferences such as lexicographic orderings. With externalities however continuity of preferences is probably a somewhat less reasonable assumption. For example, I may like my brother-in-law to have more, but I don't like him to have as much as I do. This sort of preference structure could be discontinuous. The convexity assumption in 4) is completely analogous to the usual convexity assumption. Notice it requires convexity of preferences only in a consumer's own consumption, given fixed actions of other participants.

Special Assumption I: $\exists$ non-empty compact convex sets: $Z_i \in R^n$, $i = 1,\ldots,s$; $Y_e \in R^{2n}$, $e = 1,\ldots,t$; such that $X^i(z,y) \in Z_i$, for all $(z,y) \in (\overset{s}{\underset{i=1}{X}} Z_i) \times (\overset{t}{\underset{e=1}{X}} Y_e)$. We will denote $\overset{s}{\underset{i=1}{X}} Z_i$ by Z and $\overset{t}{\underset{e=1}{X}} Y_e$ by Y.

Comment: The strategy here is to prove the existence theorem for a compact economy. Then we will show the original economy can be "compactified" in a manner which allows us to show that the equilibrium of the compactified economy is an equilibrium of the original economy.

Notation: Let $P = \{p \in R^n | \sum_{h=1} p_h = 1, 0 \leqq p_h \leqq 1, \forall h\}$. We will take P to be the set of possible price vectors. Let $Z \times Y \times P = W$. For

$$z \in Z, \ z = (x^1,\ldots,x^s),$$

let $z_i = x^i$, and

$$^iz = (x^1,\ldots,x^{i-1}, x^{i+1},\ldots,x^s) .$$

Definition: Let $B^i(z,y,p) = \{x^i \in X^i(z,y) | p \cdot x^i - w^i(z,y) \leqq v^i(z,y,p)\}$ where $v^i(z,y,p)$ is a continuous non-negative real-valued function on W. $B^i(z,y,p)$ is the budget correspondence for consumer i. Thus for each $(z,y,p) \in W$, $B^i(z,y,p)$ is the set of possible actions for consumer i which are both technically and economically feasible for consumer i. $v^i(z,y,p)$ will eventually be shown to be the share of profits function for consumer i.

Definition: The demand correspondence of consumer i, $f^i: W \to Z_i$ is defined

$$f^i(z,y,p) = \{\bar{x}^i \in B^i(z,y,p) | (\bar{z},y,p) \ (\geqq)_i \ (z',y,p),$$
$$\text{for all } \bar{z},\ z' \text{ such that } \bar{z}_i = \bar{x}^i,$$
$$^i\bar{z} = {}^iz' = {}^iz, \text{ and } z'_i \in B^i(z,y,p)\}.$$

Theorem I: Under assumptions 1)-4) and Special Assumption I $f^i(z,y,p)$ is upper semi-continuous and convex valued.

Proof: Under Special Assumption I, W is convex, and therefore connected. Thus since by 3) $(\geqq)_i$ is complete and continuous on W, by the theorem of Debreu[5] $\exists$ a continuous utility function U^i on W which represents $(\geqq)_i$.

Lemma 1: $B^i(z,y,p)$ is a continuous, convex-valued correspondence.

Proof: Since $v^i(z,y,p)$ is non-negative, by 2), $B^i(z,y,p)$ is non-empty, for all $(z,y,p) \in W$.

a) $B^i(z,y,p)$ is u.s.c.: Let $(z^n,y^n,p^n) \to (z,y,p)$, $x^i_n \in B^i(z^n,y^n,p^n)$, for all n and $x^i_n \to x^i$. We must show that $x^i \in B^i(z,y,p)$. By 1), $X^i(z,y)$ is continuous and thus $x^i \in X^i(z,y)$. Suppose $p \cdot x^i - w^i(z,y) > v^i(p,y)$. Since $(z^n,y^n,p^n) \to (z,y,p)$ and $w^i(z,y)$ and $v^i(z,y,p)$ are continuous, for large n we have $p^n \cdot x^i - w^i(z^n,y^n) > v^i(z^n,y^n,p^n) + \epsilon$ for some $\epsilon > 0$. But we also have $x^i_n \to x^i$ so that for large n we have $p^n \cdot x^i_n - w^i(z^n,y^n) > v^i(z^n,y^n,p^n)$. But this is a contradiction of $x^i_n \in B^i(z^n,y^n,p^n)$. Therefore $x^i \in B^i(z,y,p)$ and $B^i(z,y,p)$ is u.s.c.

b) $B^i(z,y,p)$ is l.s.c.: Let $x^i \in B^i(z,y,p)$ and $(z^n,y^n,p^n) \to (z,y,p)$, $(z^n,y^n,p^n) \in W$, for all n. We must show $\exists$ $\{x^i_n\}$ such that $x^i_n \in B^i(z^n,y^n,p^n)$, for all n and $x^i_n \to x^i$.

i) Suppose $p \cdot x^i - w^i(z,y) < v^i(z,y,p)$, and $x^i \in \text{int } X^i(z,y)$. In this case by continuity of $w^i(z,y)$, $v^i(z,y,p)$ and $X^i(z,y)$, for large n $x^i \in X^i(z^n,y^n)$ and $p^n \cdot x^i - w^i(z^n,y^n) < v^i(z^n,y^n,p^n)$. Thus $x^i \in B^i(z^n,y^n,p^n)$ for large n, and the necessary sequence is obvious.

ii) Suppose $p \cdot x^i - w^i(z,y) < v^i(z,y,p)$, but $x^i \notin \text{int } X^i(z,y)$. Again, by continuity of $w^i(z,y)$ and $v^i(z,y,p)$ we have $p^n \cdot x^i - w^i(z^n,y^n) < v^i(z^n,y^n,p^n) + \epsilon$ for some $\epsilon > 0$, for large n. Since $X^i(z,y)$ is continuous, $\exists$ a sequence $\{x_n^i\}$ such that $x_n^i \in X^i(z^n,y^n)$, for all n and $x_n^i \to x^i$. For large n then we have $p^n \cdot x_n^i - w^i(z^n,y^n) < v^i(z^n,y^n,p^n)$ and thus $x_n^i \in B^i(z^n,y^n,p^n)$ for large n. Again, the necessary sequence is now obvious.

iii) Suppose $p \cdot x^i - w^i(z,y) = v^i(z,y,p)$ and $x^i \in \text{int } X^i(z,y)$. By continuity of $X^i(z,y)$, $x^i \in X^i(z^n,y^n)$ for large n. By 2) and since $0 \notin P$ $\exists$ $\bar{x}^i \in X^i(z,y)$ such that $p \cdot \bar{x}^i - w^i(z,y) < 0$, and thus $p \cdot \bar{x}^i < p \cdot x^i$. Again, by continuity of $X^i(z,y)$, $\exists$ a sequence $\{\bar{x}_n^i\}$ such that $\bar{x}_n^i \in X^i(z^n,y^n)$ for all n and $\bar{x}_n^i \to \bar{x}^i$. By continuity of of $w^i(z,y)$ we will thus have $p^n \cdot \bar{x}_n^i - w^i(z^n,y^n) < 0$ for large n.

For large n let $x_n^i = x^i$ if $x^i \in B^i(z^n,y^n,p^n)$ or if $x^i \notin B^i(z^n,y^n,p^n)$ let x_n^i be the point in $[\bar{x}_n^i, x^i]$

such that $p^n \cdot x_n^i - w^i(z^n,y^n) = v^i(p^n,y^n)$. This is possible because $p^n \cdot \bar{x}_n^i - w^i(z^n,y^n) < 0$ for large n and $\{\bar{x}_n^i, x^i\} \in X^i(z^n,y^n)$ for large n, since $X^i(z^n,y^n)$ is convex. x_n^i is unique because either $x_n^i = x^i$ or $x^i \notin B^i(z^n,y^n,p^n)$ which means $p^n \cdot x^i - w^i(z^n,y^n) > v^i(z^n,y^n,p^n)$ and since for large n $p^n \cdot \bar{x}_n^i - w^i(z^n,y^n) < 0$, $\exists$ only one point $x_n^i \in \{\bar{x}_n^i, x^i\}$ such that $p^n \cdot x_n^i - w^i(z^n,y^n) = v^i(z^n,y^n,p^n)$. Clearly $x_n^i \in B^i(z^n,y^n,p^n)$ and since $\bar{x}_n^i \to \bar{x}^i$ and $p \cdot \bar{x}^i < p \cdot x^i$, we have $x_n^i \to x^i$, and the necessary sequence has been found.

iv) Suppose $p \cdot x^i - w^i(z,y) = v^i(z,y,p)$ and $x^i \notin \text{int } X^i(z,y)$. Since $X^i(z,y)$ is continuous $\exists$ a sequence $\{x_n^i\}$ such that $x_n^i \in X^i(z^n,y^n)$, for all n and $x_n^i \to x^i$. By 2) and since $0 \notin P$ $\exists\, \bar{x}^i \in X^i(z,y)$ such that $p \cdot \bar{x}^i - w^i(z,y) < 0$, and so $p \cdot \bar{x}^i < p \cdot x^i$. Again, by continuity of $X^i(z,y)$ $\exists$ a sequence $\{\bar{x}_n^i\}$ such that $\bar{x}_n^i \in X^i(z^n,y^n)$ for all n and $\bar{x}_n^i \to x^i$. By continuity of $w^i(z,y)$, we have $p^n \cdot \bar{x}_n^i - w^i(z^n,y^n) < 0$, for large n. For large n let $\tilde{x}_n^i = x_n^i$ if $x_n^i \in B^i(z^n,y^n,p^n)$ or if $x_n^i \notin B^i(z^n,y^n,p^n)$, let $\tilde{x}_n^i$ be the point in $(\bar{x}_n^i, x_n^i)$ such that $p^n \cdot \{\tilde{x}_n^i - w^i(z^n,y^n)\} = v^i(z^n,y^n,p^n)$. This is possible because $p^n \cdot \bar{x}_n^i - w^i(z^n,y^n) < 0$ for large n and $(\bar{x}_n^i, x_n^i) \subset X^i(z^n,y^n)$ since $X^i(z^n,y^n)$ is convex. $\tilde{x}_n^i$ is because either $\tilde{x}_n^i = x_n^i$ or $x_n^i \notin B^i(z^n,y^n,p^n)$, which means that $p^n \cdot x^i_n - w^i(z^n,y^n) > v^i(z^n,y^n,p^n)$ and since $p^n \cdot \bar{x}^i_n - w^i(z^n,y^n) < 0$ for large n, $\exists$ only one $\tilde{x}^i_n \in \{\bar{x}^i_n, x^i_n\}$ such that $p^n \cdot \tilde{x}^i_n \cdot w^i(z^n,y^n) = v^i(z^n,y^n,p^n)$. Clearly, $\tilde{x}^i_n \in B^i(z^n,y^n,p^n)$ and since $\bar{x}^i_n \to \bar{x}^i$, $x^i_n \to x^i$, and $p \cdot \bar{x}^i < p \cdot x^i$, we have $\tilde{x}^i_n \to x^i$, and the necessary sequence has been found.

We will now prove Theorem I.

That $f^i(z,y,p)$ is non-empty and compact follows from the compactness of $B^i(z,y,p)$ and the existence of a continuous utility function U^i on W which represents $(\geqq)_i$.

Let x^i, $\tilde{x}^i \in f^i(z,y,p)$. Then by the convexity of $B^i(z,y,p)$, $(x^i, \tilde{x}^i) \subset B^i(z,y,p)$. Let $\bar{z}$ be such that $\bar{z}_i \in (x^i, \tilde{x}^i)$ and ${}^i\bar{z} = {}^iz$. Then $\bar{z}_i$ is attainable for i at (z,y,p) and $(\bar{z},y,p)$ $(\geqq)_i$ (z',y,p) for z′ such that $z'_i = \tilde{x}^i$, ${}^iz' = {}^iz$, by 4). Therefore $\bar{z}_i \in f^i(z,y,p)$, for all $\bar{z}_i \in (x^i, \tilde{x}^i)$, and thus $f^i(z,y,p)$ is convex valued.

$f^i(z,y,p)$ is u.s.c.: Let $(z^n,y^n,p^n) \to (z,y,p)$, $x^i_n \in f^i(z^n,y^n,p^n)$, for all n and $x^i_n \to \bar{x}^i$. We must establish that $\bar{x}^i \in f^i(z,y,p)$. $\bar{x}^i \in B^i(z,y,p)$ because $B^i(z,y,p)$ is continuous. Suppose $\bar{x}^i \notin f^i(z,y,p)$. Then $\exists$ $\tilde{z} \in Z$ such that $\tilde{z}_i \in B^i(z,y,p)$, ${}^i\tilde{z} = {}^iz$, and $U^i(\tilde{z},y,p) > U^i(\bar{z},y,p)$, where $\bar{z}_i = \bar{x}^i$ and ${}^i\bar{z} = {}^iz$. Let $\bar{z}^n$ be such that $\bar{z}^n_i = x^i_n$ and ${}^i\bar{z}^n = {}^iz^n$. Then since $U^i(z,y,p)$ is continuous and $(\bar{z}^n,y^n,p^n) \to (\bar{z},y,p)$, we have $U^i(\tilde{z},y,p) > U^i(\bar{z}^n,y^n,p^n) + \epsilon$ for some $\epsilon > 0$, for large n.

Since $B^i(z,y,p)$ is l.s.c. $\exists$ a sequence $\{\tilde{x}^i_n\}$ such that $\tilde{x}^i_n \in B^i(z^n,y^n,p^n)$ for all n and $\tilde{x}^i_n \to \tilde{z}^i$. Define $\tilde{z}^n$ by $\tilde{z}^n_i = \tilde{x}^i_n$

and ${}^{i}\tilde{z}^{n} = {}^{i}z^{n}$. Then $\tilde{z}^{n} \to \tilde{z}$. But by continuity of $U^{i}(z,y,p)$, for large n, $U^{i}(\tilde{z}^{n},y^{n},p^{n}) > U^{i}(\bar{z}^{n},y^{n},p^{n})$. But this is a contradiction $x_{n}^{i} \in f^{i}(z^{n},y^{n},p^{n})$. Therefore $x^{i} \in f^{i}(z,y,p)$ and $f^{i}(z,y,p)$ is u.s.c.

We have now proved that for compact economies satisfying our assumptions, consumers' demand correspondences are upper semi-continuous. It should be noted that this demand correspondence is different from the demand correspondence defined in models without externalities because it is defined not only on prices, but also on the space of other participants actions. What interpretation to give to this demand correspondence in any dynamic visualization of the model is not clear, the problem being, given a price vector p, what decides what (z,y) is so that the consumer knows what $f^{i}(z,y,p)$ is? Many different conceptualizations are possible here. More will be said about this after we consider producers in the next section.

A-2: Producers

A producer, e, (e = 1,...,t) has a production possibilities correspondence Y^e: $R^{(s+2t)n} \to R^{2n}$ denoted $Y^e(z,y)$. A typical element of the range of $Y^e(z,y)$ is $q^e = (q^{Ie}, q^{Oe}) \in R^{2n}$. $q^e \in Y^e(z,y)$ means that if (z,y) represents the actions of all other participants then $q^e = (q^{Ie}, q^{Oe})$ is technologically feasible input-output combination for producer e. It is assumed that $Y^e(z,y)$ does not depend on $y_e = q^e$.

Notation: For $y = (q^1,\dots,q^t)$, let $y_e = q^e$,

$$^e y = (q^1,\dots,q^{e-1}, q^{e+1},\dots,q^t),$$

and

$$\hat{y}_e = q^{Oe} - q^{Ie} = \hat{q}^e$$

$\hat{y}_e$ $(= \hat{q}^e)$ is just the net output vector corresponding to the input-output combination $y_e = q^e = (q^{Ie}, q^{Oe})$. Similarly define $\hat{Y}^e(z,y) = \{\hat{q}^e | q^e \in Y^e(z,y)\}$. Then $\hat{Y}^e(z,y)$ is the net output correspondence of producer e.

Assumptions about $Y^e(z,y)$:

5) $\exists$ a closed convex set $Y^e \in R^{2n}$ such that $Y^e(z,y) \subset Y^e$, for all $(z,y) \in R^{(s+2t+1)n}$.

6) $Y^e(z,y)$ is a continuous convex-valued correspondence. (This implies $Y^e(z,y)$ is closed, for all (z,y)).

(7) $0 \in Y^e(z,y)$, for all (z,y).

Comments: That $Y^e(z,y)$ is closed and convex for all (z,y) is analogous to the usual assumption made about production sets in the no externalities case. This means that given fixed actions of other participants, producer e's production possibilities do not exhibit indivisibilities or increasing returns. This however is perfectly consistent with aggregate increasing returns resulting from externalities between producers. In this case, however, it is necessary to interpret a single producer as a firm rather than as an industry. Our assumption that $Y^e(z,y)$ be a continuous correspondence is very strong, but it is consistent with most treatments of production externalities found in the literature, where an individual producer is assumed to have a continuous (and usually differentiable) production function which has as its arguments the producer's inputs, and the actions of other producers.

Special Assumption II: For Y_e defined in Special Assumption I, $Y^e(z,y) \subset Y_e$, for all $(z,y) \in Z \times Y$. (Y^e is compact and convex by assumption.)

Definition: The supply correspondence of producer e, $r^e: W \to Y_e$ is defined

$$r^e(z,y,p) = \{\bar{q}^e \in Y^e(z,y) \,|\, p \cdot \hat{\bar{q}}^e \geqq p \cdot \hat{q}^e, \text{ for all } q^e \in Y^e(z,y)\}.$$

<u>Theorem II</u>: Under assumptions 5) and 6) and Special Assumptions I and II, $r^e(z,y,p)$ is an upper semi-continuous convex valued correspondence.

<u>Proof</u>: That $r^e(z,y,p)$ is non-empty follows from Special Assumption II which implies $Y^e(z,y)$ is compact. The convexity of $r^e(z,y,p)$ is a result of the convexity of $Y^e(z,y,p)$ by 6).

$r^e(z,y,p)$ is u.s.c.: Let $(z^n,y^n,p^n) \to (z,y,p)$, $q_n^e \in r^e(z^n,y^n,p^n)$, for all n and $q_n^e \to q^e$. We must show that $q^e \in r^e(z,y,p)$. By the continuity of $Y^e(z,y)$, $q^e \in Y^e(z,y)$. Suppose $q^e \notin r^e(z,y,p)$. Then $\exists$ $\bar{q}^e \in Y^e(z,y)$ such that $p\cdot\hat{\bar{q}}^e > p\cdot\hat{q}^e$.

Since $Y^e(z,y)$ is l.s.c., $\exists$ a sequence $\{\bar{q}_n^e\}$ such that $\bar{q}_n^e \in Y^e(z^n,y^n)$, for all n and $\bar{q}_n^e \to \bar{q}^e$. Since $q_n^e \to q^e$, we have $p^n\cdot\hat{q}_n^e \to p\cdot\hat{q}^e$, and since $p\cdot\hat{\bar{q}}^e > p\cdot\hat{q}^e$, for large n, we have $p^n\cdot\hat{\bar{q}}_n^e > p^n\cdot\hat{q}_n^e$. But this is a contradiction of $q_n^e \in r^e(z^n,y^n,p^n)$, for all n. Therefore $r^e(z,y,p)$ is u.s.c.

<u>Definition</u>: We define the profit function of producer e, $\pi^e(z,y,p)$ as:

$$\pi^e(z,y,p) = \max_{q^e \in Y^e(z,y)} p\cdot\hat{q}^e$$

By Special Assumption II, $\pi^e(z,y,p)$ is well defined. (Clearly, $\pi^e(z,y,p)$ is single-valued.)

The following obvious corollary is stated without proof.

Corollary: $\pi^e(z,y,p)$ is continuous.

Assumption: Each consumer, i, owns a proportion d_e^i of the e^{th} firm. $\sum_{i=1}^{s} d_e^i = 1$, $d_e^i \in [0,1]$, for all i,e.

Definition: Let $v^i(z,y,p)$ used in the definition of $B^i(z,y,p)$ be defined

$$v^i(z,y,p) = \sum_{e=1}^{t} d_e^i \cdot \pi^e(z,y,p).$$

By the Corollary, $v^i(z,y,p)$ is continuous and by 7) $v^i(z,y,p) \geqq 0$.

We have now completely described consumers, producers and their actions for a compact economy. We return now to the problem of how we interpret the correspondences $f^i(z,y,p)$ and $r^e(z,y,p)$. As mentioned at the end of the last section, the interpretation of these correspondences in a dynamic context is not clear. In a strictly static framework we might visualize Walras' auctioneer calling out not only prices but also a "reference" set of activities (z,y). Thus the auctioneer would call out a $(z,y,p) \in W$. On the basis of this, consumers would respond with $f^i(z,y,p)$ and producers with $r^e(z,y,p)$. In finding a solution (equilibrium) the auctioneer must find a point in W, (z^*,y^*,p^*) which has <u>two</u> properties:

i) $\Sigma z_i^* - \Sigma w^i(z^*,y^*) - \Sigma \hat{y}_e^* \leqq 0$,

and

ii) $z_i^* \in f^i(z^*,y^*,p^*)$ and $y_e^* \in r^e(z^*,y^*,p^*)$.

Thus the auctioneer must call out both an equilibrium price vector <u>and</u> an associated set of equilibrium actions. It is not at all clear what the <u>tatonnement</u> stability properties of an equilibrium would be in a model with externalities.

In a truly dynamic framework things are much more complicated. One dynamic characterization of the model might be: $z_i^t \in f^i(z^{t-1}, y^{t-1}, p^t)$ where (z^{t-1}, y^{t-1}) was the set of actions, actual or proposed at time t - 1 by the various participants. However, if (z^{t-1}, y^{t-1}) are the actual actions at time t - 1, this means (z^{t-1}, y^{t-1}) was technically feasible, which need not be the case. Thus the mechanics of a dynamic model with externalities are not obvious. This essay will not attempt to solve this problem, we will only demonstrate that a non-cooperative equilibrium exists. However, the dynamic properties of such a model are very important because many policy prescriptions come from the analysis of the static model with externalities, especially in the field of Public Finance. This analysis is even more suspect because of the increased likelihood of multiple equilibria in models with externalities. This will be discussed in Part III of this essay.

B: The Existence Theorem for Compact Economies

In this section assumptions 1)-7) and Special Assumptions I and II are assumed to hold. We will demonstrate that an equilibrium exists for economies satisfying these assumptions.

Definition: For $(z,y) = ((x^1,\dots,x^s), (q^1,\dots,q^t)) \in Z \times Y$ let $M: Z \times Y \to P$ be defined:

$$M(z,y) = \{p \in P \mid p\cdot[(\Sigma_i(x^i - w^i(z,y))) - \Sigma_e\hat{q}^e] \geqq \tilde{p}\cdot[(\Sigma_i(x^i - w^i(z,y))) - \Sigma_e\hat{q}^e],\ \forall\,\tilde{p} \in P\}.$$

The correspondence $M(z,y)$ will be used to demonstrate the existence of an equilibrium.

Lemma 2: $M(z,y)$ is an upper semi-comtinuous convex-valued correspondence.

Proof: $M(z,y)$ is non-empty and convex-valued because P is compact and convex.

$M(z,y)$ is u.s.c.: Let $(z^n,y^n) \to (z,y)$, $p^n \in M(z^n,y^n)$, $\forall n$ and $p^n \to p$. We must show that $p \in M(z,y)$. $p \in P$ because P is closed. Suppose $p \notin M(z,y)$. Then $\exists\,\bar{p} \in P$ such that $\bar{p}\cdot[\Sigma x^i - \Sigma w^i(z,y) - \Sigma \hat{q}^e] > p\cdot[\Sigma x^i - \Sigma w^i(z,y) - \Sigma \hat{q}^e]$. But since $(z^n,y^n) \to (z,y)$ and $p^n \to p$ we have for large n, $\bar{p}\cdot[\Sigma x_n^i - \Sigma w^i(z^n,y^n) - \Sigma \hat{q}_n^e] > p^n\cdot[\Sigma x_n^i - \Sigma w^i(z^n,y^n) - \Sigma \hat{q}_n^e]$. But this is a contradiction of $p^n \in M(z^n,y^n)$. Thus $M(z,y)$ is U.S.C.

Notation: Let $f(z,y,p) = \overset{s}{\underset{i=1}{X}} f^i(z,y,p)$

$$r(z,y,p) = \overset{t}{\underset{e=1}{X}} r^e(z,y,p)$$

Definition: Define Φ: $W \rightarrow W$ by $\Phi(z,y,p) = f(z,y,p) \times r(z,y,p) \times M(z,y,p)$. Notice that $(\bar{z},\bar{y},\bar{p}) \in \Phi(z,y,p)$ means: $\bar{z}_i \in f^i(z,y,p)$, $\forall i$, $\bar{y}_e \in r^e(z,y,p)$, $\forall e$ and $\bar{p}$ maximizes $p\cdot[\Sigma\, x_i - \Sigma\, w^i(z,y) - \Sigma\, \hat{q}_e]$ on P. The mapping $\Phi(z,y,p)$ is very similar to the mapping φ defined by Debreu.[6] Our strategy of proof will be quite similar also.

Theorem III. $\Phi(z,y,p)$ has a fixed point, (z^*,y^*,p^*), and if $p^*\cdot[\Sigma\, x^{*i} - \Sigma\, w^i(z^*,y^*) - \Sigma\, \hat{q}^{*e}] = 0$, (z^*,y^*,p^*) is an equilibrium.

Proof: Since $f(z,y,p)$, $r(z,y,p)$, and $M(z,y)$ are U.S.C. and convex valued, $\Phi(z,y,p)$ is also. Therefore, since W is compact and convex, by Kakutani's Theorem $\Phi(z,y,p)$ has a fixed point.

Thus $\exists\, (z^*,y^*,p^*)$ such that $(z^*,y^*,p^*) \in \Phi(z^*,y^*,p^*)$. This means, by the definition of $\Phi(z,y,p)$, that $z^* \in f(z^*,y^*,p^*)$, $y^* \in r^e(z^*,y^*,p^*)$, $p^* \in M(z^*,y^*)$.

i) $p^* \in M(z^*,y^*)$ means that

$$p^*\cdot[\Sigma\, x^{*i} - \Sigma\, w^i(z^*,y^*) - \Sigma\, \hat{q}^{*e}] \geqq p\cdot[\Sigma\, x^{*i} - \Sigma\, w^i(z^*,y^*) - \Sigma\, \hat{q}^{*e}],$$

$\forall\, p \in P$.

ii) $z^* \in f(z^*,y^*,p^*)$ means that $x^{*i} \in B^i(z^*,y^*,p^*)$, $\forall i$, and therefore $p^*\cdot[x^{*i} - w^i(z^*,y^*)] - v^i(z^*,y^*,p^*) \leqq 0$. But $v^i(z^*,y^*,p^*) = \sum_{e=1}^{t} d^i_e\, p^*\cdot\hat{q}^{*e}$. From $\sum_{i=1}^{s} d^i_e = 1$, we have $\sum_{i=1}^{s} \sum_{e=1}^{t} d^i_e\, p^*\cdot\hat{q}^{*e} = p^*\cdot \sum_{e=1}^{t} \hat{q}^{*e}$. Therefore $p^*\cdot[x^{*i} - w^i(z^*,y^*)] - v^i(z^*,y^*,p^*) \leqq 0$, $\forall i$ implies $p^*\cdot[\Sigma\, x^{*i} - \Sigma\, w^i(z^*,y^*) - \Sigma\, \hat{q}^{*e}] \leqq 0$. (This is just Walras' Law.)

Let $p^h \in R^h$ be a vector with a one in its h^{th} component and zeros elsewhere. Then $p^h \in P$, $h = 1,\ldots,n$. Since $p^h \in P$, $\forall h$, by i) and ii) we have

$$\Sigma x^{*i} - \Sigma w^i(z^*,y^*) - \Sigma \hat{q}^{*e} \leqq 0.$$

Therefore we have $x^{*i} \in f^i(z^*,y^*,p^*)$, $\forall i$ and $q^{*e} \in r^e(z^*,y^*,p^*)$, $\forall e$ and $[\Sigma x^{*i} - \Sigma w^i(z^*,y^*) - \Sigma \hat{q}^{*e}] \leqq 0$. For (z^*,y^*,p^*) to be an equilibrium it must be the case that if $[\Sigma x^{*i} - \Sigma w^i(z^*,y^*) - \Sigma \hat{q}^{*e}]_h < 0$ then $p^*_h = 0$.

Suppose $p^* \cdot [\Sigma x^{*i} - \Sigma w^i(z^*,y^*) - \Sigma \hat{q}^{*e}] = 0$. Then since $\Sigma p^*_h = 1$, $0 \leqq p_h \leqq 1$, $\forall h$ it cannot be the case that $[\Sigma x^{*i} - \Sigma w^i(z^*,y^*) - \Sigma \hat{q}^{*e}]_h < 0$ $\forall h$. Therefore $[\Sigma x^{*i} - \Sigma w^i(z^*,y^*) - \Sigma \hat{q}^{*e}]_k = 0$ for some k. Since $p^* \in M(z^*,y^*)$ and $p^k \in P$ it must be that if $[\Sigma x^{*i} - \Sigma w^i(z^*,y^*) - \Sigma \hat{q}^{*e}]_h < 0$ then $p^*_h = 0$.

The assumption that $p^* \cdot [\Sigma x^{*i} - \Sigma w^i(z^*,y^*) - \Sigma \hat{q}^{*e}] = 0$ need not in general be satisfied. However, when we construct the existence argument for the original non-compact economy a non-satiation assumption will allow us to infer that this condition holds.

We have now proved under certain assumptions that an equilibrium exists for a compact economy.

C: The "Attainable" Set

Definition: We will call the vector (z,y) resource feasible (r.f.) if $\Sigma x^i - \Sigma w^i(z,y) - \Sigma \hat{q}^e \leqq 0$.

We will call a vector (z,y) technologically feasible (t.f.) if $x^i \in X^i(z,y), \forall i$ and $y^e \in Y^e(z,y), \forall e$. For a vector (z,y) to be attainable for our model it must be both r.f. and t.f.

In this section we will prove under certain assumptions that the set of r.f. actions in our model is bounded. This is a stronger result than we need, since we need only show that the set of attainable actions is bounded. However, it will greatly simplify our arguments to do it this way. Our arguments in this section parallel those of Arrow and Debreu[7] for an economy without externalities.

Assumptions:

8) For each e, the Y^e as defined in 5) is such that $\hat{Y}^e$ is closed. ($\hat{Y}^e$ will be convex by (5) since Y^e is convex.)

9) Let $\hat{Y} = \sum_{e=1}^{t} \hat{Y}^e$. Then $\hat{Y} \cap \Omega^n = \{0\}$.

10) $\hat{Y} \cap (-\hat{Y}) \subset \{0\}$.

Comments: That $\hat{Y}^e$ is closed is analogous to the usual assumption that the (fixed) net output set of a producer is closed. 10) is the usual irreversibility of production assumption. 9) and 10) are probably much stronger than necessary because $\hat{Y}$ contains much more than the set of t.f. net output vectors of the economy.

By 1) the $X^i(z,y)$ are uniformly bounded from below. Let $\bar{x}$ be such that $\bar{x} < 0$ and $x^i > \bar{x}$, $\forall x^i \in X^i(z,y)$, $\forall i$. Let $Z_i = \{x^i \in R^n | x^i \geqq \bar{x}\}$. Then $X^i(z,y) \subset Z_i$, $\forall (z,y)$, $\forall i$. By 2), for each i $\exists \bar{w}^i$ such that $w^i(z,y) < \bar{w}^i$, $\forall (z,y)$. Since $w^i(z,y) \geqq 0$, $\forall (z,y)$, $\Sigma \bar{w}^i \geqq 0$ and $\Sigma \bar{w}^i > \Sigma w^i(z,y)$, $\forall (z,y)$. Let $\Sigma \bar{w}^i = \bar{w}$.

<u>Definition</u>

$$\tilde{X}^i = \{x^i \in Z_i | \exists x^j \in Z_j \text{ and } y^e \in Y^e \text{ such that } x^i + \sum_{j \neq i} x^j - \bar{w} - \Sigma \hat{y}^e \leqq 0\}$$

$$\hat{Y}'^e = \{\hat{y}^e \in \hat{Y}^e | \exists x^j \in Z_j \text{ and } y^k \in Y^k \text{ such that } \Sigma x^j - \bar{w} - \sum_{k \neq e} \hat{y}^k - \hat{y}^e \leqq 0\}.$$

Since $0 \in Y^e$, $\forall e$ and $\bar{w} > \bar{x}$, these sets are non-empty. ($\tilde{X}^i$ contains the set of r.f. actions for i and $\hat{Y}'^e$ contains the set of r.f. net outputs for e.)

<u>Lemma 3</u>: For economies satisfying 1), 2), 5), 7), 8), 9) $\hat{Y}'^e$ and $\tilde{X}^i$ are bounded, $\forall$ e,i.

<u>Proof</u>: <u>$\hat{Y}'^e$ is bounded</u>: Suppose $\hat{Y}'^e$ is not bounded. Then $\exists$ sequences $\{\hat{y}^k_n\}$, $\{x^i_n\}$ k = 1,...,t; i = 1,...,s such that

$$\lim_{n \to \infty} |\hat{y}^e_n| = \infty, \quad \sum_{k=1}^{t} \hat{y}^k_n \geqq \sum_{i=1}^{s} x^i_n - \bar{w}, \forall n.$$

Since $x^i_n \in Z_i$, $\forall n$, we have $x^i_n \geqq \bar{x}$, $\forall n$. Let $s\bar{x} = \bar{v}$, then

$$\sum_{k=1}^{t} \hat{y}^k_n \geqq \bar{v} - w, \forall n.$$

For each n let $b^n = \max_k |\hat{y}^k_n|$. For large n we must have $b^n \geqq 1$.

By 5) $\hat{Y}^e$ is closed, $\forall e$ and by 7), $0 \in \hat{Y}^e$, $\forall e$, thus $(1/b^n)\hat{y}_n^k +$ $(1 - 1/b^n)\cdot 0 = (1/b^n)\hat{y}_n^k \in \hat{Y}^k$ for large n, $\forall k$, by convexity of $\hat{Y}^e$. $|(1/b^n) \cdot \hat{y}_n^k| \leqq 1$, for large n by definition of b^n and so $\{(1/b^n) \cdot \hat{y}_n^k\}$ is contained in a compact set, $\forall k$. Therefore each sequence has a convergent subsequence: $\{(1/b^{n_p}) \cdot \hat{y}_{n_p}^k\}$, $(1/b^{n_p}) \cdot \hat{y}_{n_p}^k \to \hat{y}^k$, $\forall k$ and since $\hat{Y}^k$ is closed $\hat{y}^k \in \hat{Y}^k$, $\forall k$. Since $\sum_{k=1}^{t} (1/b^{n_p}) \cdot \hat{y}_{n_p}^k \geqq (1/b^{n_p}) \cdot (\bar{v} - \bar{w})$ and $b^{n_p} \to \infty$ we have $\sum_{k=1}^{t} \hat{y}^k \geqq 0$. By 9) we have then that $\sum_{k=1}^{t} \hat{y}^k = 0$. But then, for any k $\sum_{m \neq k} \hat{y}^m = -\hat{y}^k$. Since by 5) and 7), $0 \in \hat{Y}^k$, $\forall k$ $\sum_{m \neq k} \hat{y}^m \in \hat{Y}$ and thus $-\hat{y}^k \in \hat{Y}$. But by 10) this means $\hat{y}^k = 0$, $\forall k$. But this is not possible because $|\hat{y}_{n_p}^k| = b^{n_p}$, for some k, $\forall n_p$, and therefore for some k $|\hat{y}_{n_p}^k| = b^{n_p}$ for infinitely many n_p.

Thus $\hat{Y}'^e$ is bounded. $\tilde{X}^i$ is bounded: By definition of $\tilde{X}^i$, $\forall x^i \in \tilde{X}^i$ we have $\bar{x} \leqq x^i \leqq \Sigma \hat{y}^e - \sum_{j \neq i} x^j + \bar{w}$ for some $x^j \in Z_j$, $y^e \in Y^e$. But this means

$$x^i + \sum_{j \neq i} x^j - \bar{w} - \Sigma \hat{y}^e \leqq 0 \quad \text{for} \quad x^i \in Z_i,\ x^j \in Z_j,\ y^e \in Y^e.$$

But this means $\hat{y}^e$ is bounded, $\forall e$ because $\hat{y}^e \in \hat{Y}'^e$. Also, $x^j \in Z_j$, $\forall j$ implies $x_j \geqq \bar{x}$, $\forall j$. Therefore

$$\bar{x} \leqq x^i \leqq \Sigma \hat{y}^e - (s-1)\bar{x} + \bar{w}$$

and $\hat{y}^e$ is bounded, $\forall e$ and so $\tilde{X}^i$ is bounded.

Assumption

11) For any $k > 0$ let $B(k) = \{ u \in R^n |\ |u_h| \leqq k, \forall h\}$ and for any $j > 0$ let $C(j) = \{v \in R^{2n} |\ |v_h| \leqq j, \forall h\}$. Then for any $G \subset \hat{Y}^e$, $G \subset B(n)$, $\exists H \subset R^{2n}$ such that $H \subset Y^e$, $G \subset \hat{H}$, and $H \subset C(j)$ for some $j > 0$.

What this assumption guarantees is that bounded net output sets are achievable by bounded sets of input-output vectors. Therefore, since $\hat{Y}'^{e}$ is bounded $\exists$ a bounded set $\bar{Y}'^{e}$ such that $\hat{\bar{Y}}'^{e}$ contains $\hat{Y}'^{e}$.

D: The General Existence Theorem

In this section we finally demonstrate the existence of an equilibrium for the model constructed in section A. The strategy of our proof is to compactify the economy using the results of section C which show the set of attainable actions is compact. Using the results of section B, we shown an equilibrium exists for the compactified economy and then demonstrate that this equilibrium is also an equilibrium for the original economy.

Notation: We will denote the economy constructed in section A and satisfying assumptions 1)-11) by the vector:

$$E = \{(X^i(z,y),\ Z_i,\ w^i(z,y),\ (\geqq)_i),\ (Y^e(z,y),\ Y^e),\ (d^i_e)\}$$

where Z_i is as defined in section C.

Let K be a cube in R^n which contains each of the sets $\tilde{X}^i$, $\hat{Y}'^e$ (defined in section C) in its interior. Since $\forall\ (z,y)\ \exists\ \bar{x}^i(z,y)$ such that $\bar{x}^i(z,y) \ll w^i(z,y)$ and $0 \in \hat{Y}^e$, $\forall e$, by construction, $w^i(z,y) \in \text{int } \tilde{X}^i$, $\forall\ (z,y)$, and thus $w^i(z,y) \in \text{int } K$, $\forall (z,y)$. Since $0 \in \hat{Y}'^e$, $\forall e$, $0 \in K$.

Let $\tilde{Z}_i = Z_i \cap K, \forall i$ and $\tilde{X}_i(z,y) = X^i(z,y) \cap K, \forall i, \forall (z,y)$. Then $\tilde{Z}_i$ is non-empty convex and compact, $\tilde{X}^i(z,y)$ is non-empty convex and compact, and $\forall (z,y)\ \exists\ \bar{x}^i(z,y) \in \tilde{X}^i(z,y)$ such that $\bar{x}^i(z,y) \ll w^i(z,y)$. $\tilde{X}^i(z,y)$ is a continuous convex correspondence and $\tilde{X}^i(z,y) \subset \tilde{Z}_i$, $\forall (z,y)$.

By 11), since $\hat{Y}^e \cap K$ is a compact set contained in $\hat{Y}^e$, $\exists$ a bounded set $G^e \subset Y^e$ such that $\hat{Y}^e \cap K \subset \hat{G}^e$. Let $\tilde{Y}^e$ be such that $\tilde{Y}^e \supset G^e$, $\tilde{Y}^e \subset Y^e$, where $\tilde{Y}^e$ is convex and compact. Let $\tilde{Y}^e(z,y) = Y^e(z,y) \cap \tilde{Y}^e$.

Then $\tilde{\Psi}^e(z,y)$ is continuous, convex, compact and non-empty (by 7)) and $\tilde{Y}^e(z,y) \subset \tilde{Y}^e$, $\forall (z,y)$.

Now define a new economy, $\tilde{E}$, by

$$\tilde{E} = \{(\tilde{X}^i(z,y),\ \tilde{Z}_i,\ w^i(z,y),\ (\geqq)_i),\ (\tilde{Y}^e(z,y),\ \tilde{Y}^e,\ (d_e^i)\}\ .$$

Then $\tilde{E}$ satisfies assumptions 1)-7) and Special Assumptions I and II.

Assumption:

12) $X^i(z,y)$ has no satiation consumption, $\forall (z,y), \forall i$. (This means $\nexists\ \bar{x}^i \in X^i(z,y)$ such that $(\bar{z},y)\ (\geqq)_i\ (z',y)$, $\forall z'$ such that ${}^i\bar{z} = {}^iz' = {}^iz$, $\bar{z}_i = \bar{x}^i$, $z_i' \in X^i(z,y)$).

Theorem IV. The Economy E which satisfies assumptions 1)-12) has an equilibrium.

Proof: By assumptions 1)-11) and Theorem III the economy $\tilde{E}$ has an equilibrium if the fixed point of Φ, (z^*,y^*,p^*) has the property that $p^*\cdot[\Sigma\, x^{*i} - \Sigma\, w^i(z^*,y^*) - \Sigma\, \hat{q}^{*e}] = 0$. Since the fixed point has the property that $\Sigma\, x^{*i} - \Sigma\, w^i(z^*,y^*) - \Sigma\, \hat{q}^{*e} \leqq 0$, $x^{*i} \in \tilde{X}^i$, $\forall i$ and $q^{*e} \in \tilde{Y}$, $\forall e$. Therefore $x^{*i} \in \text{int } K$. Suppose $p^*\cdot[x^{*i} - w^i(z^*,y^*)] - v^i(z^*,y^*,p^*) < 0$. By 12) there is a point $\bar{x}^i \in X^i(z^*,y^*)$ such that $\bar{x}^i$ is preferred to x^{*i} at (z^*,y^*,p^*). But since $X^i(z,y)$ and $\tilde{X}^i(z,y)$ are convex, and $\tilde{X}^i(z,y) = X^i(z,y) \cap K$, $k\,\bar{x}^i + (1-k)\ x^{*i} \in \tilde{X}^i(z,y)$ for some $k \in (0,1)$ since $x^{*i} \in \text{int } K$, and by 4) $k\,\bar{x}^i + (1-k)\ x^{*i}$ is preferred to x^{*i} at (z^*,y^*,p^*), $\forall k \in (0,1)$. But $p^*\cdot[(k\,\bar{x}^i + (1-k)x^{*i}) - w^i(z^*,y^*)] - v^i(z^*,y^*,p^*) \leqq 0$ for some $k \in (0,1)$, and this is a contradiction of $x^{*i} \in \tilde{f}^i(z^*,y^*,p^*)$. Therefore by assumption 12), $p^*\cdot[x^{*i} - w^i(z^*,y^*)] - v^i(z^*,y^*,p^*) = 0$, $\forall i$.

By Walras Law, then $p^*\cdot[\Sigma\, x^{*i} - \Sigma\, w^i(z^*,y^*) - \Sigma\, \hat{q}^{*e}] = 0$, so the fixed point (z^*,y^*,p^*) is an equilibrium for $\tilde{E}$. Since (z^*,y^*,p^*) is an equilibrium for $\tilde{E}$:

i) $z^* = (x^{*1},\ldots,x^{*s})$ is such that $x^{*i} \in \tilde{X}^i(z^*,y^*)$, $\forall i$. By construction then $x^{*i} \in X^i(z^*,y^*)$, $\forall i$.

ii) $y^* = (q^{*1},\ldots,q^{*t})$ is such that $q^{*e} \in \tilde{Y}^e(z^*,y^*)$, $\forall e$. By construction then $q^{*e} \in Y^e(z^*,y^*)$, $\forall e$.

iii) $\Sigma\, x^{*i} - \Sigma\, w^i(z^*,y^*) - \Sigma\, \hat{q}^{*e} \leqq 0$, so (z^*,y^*) is attainable for E.

iv) y^{*e} is profit maximizing in $\tilde{Y}^e(z^*,y^*)$. We must show that y^{*e} is profit maximizing in $Y^e(z^*,y^*)$. As shown earlier, $y^{*e} \in \text{int } K$. Suppose $\exists\, \bar{y}^e \in Y^e(z^*,y^*)$ such that $p^*\cdot \hat{\bar{y}}^e > p^*\hat{y}^{*e}$. $\tilde{Y}^e(z^*,y^*) = Y^e(z^*,y^*) \cap K$ and $Y^e(z^*,y^*)$ convex implies $k\, \bar{y}^e + (1-k)y^{*e} \in Y^e(z^*,y^*) \cap K$ for some $k \in (0,1)$. But this is not possible because then $k\bar{y}^e + (1-k)y^{*e} \in \tilde{Y}^e(z^*,y^*)$ and $p^*\cdot[k\hat{\bar{y}}^e + (1-k)\hat{y}^{*e}] > p^*\hat{y}^{*e}$, which is a contradiction of y^{*e} being profit maximizing in $\tilde{Y}^e(z^*,y^*)$. Therefore y^{*e} is profit maximizing in $Y^e(z^*,y^*)$.

v) x^{*i} is optimal in $B^i(z^*,y^*,p^*) \cap \tilde{X}^i(z^*,y^*)$. We must show that x^{*i} is optimal in $B^i(z^*,y^*,p^*)$. As shown earlier, $x^{*i} \in \text{int } K$. Suppose $\exists\, \bar{x}^i \in B^i(z^*,y^*,p^*)$ such that $\bar{x}^i$ is preferred to x^{*i} at (z^*,y^*,p^*). Since $B^i(z^*,y^*,p^*)$ is convex, $k\, \bar{x}^i + (1-k)\, x^{*i} \in B^i(z^*,y^*,p^*)$, $\forall\, k \in (0,1)$ and $k\, \bar{x}^i + (1-k)\, x^{*i} \in K$ and thus $\in B^i(z^*,y^*,p^*) \cap K$ for some

$k \in (0,1)$. But $k\, \bar{x}^i + (1-k)\, x^{*i}$ is preferred to x^{*i} at (z^*,y^*,p^*), $\forall\, k \in (0,1)$. This is a contradiction of x^{*i} being optimal in $B^i(z^*,y^*,p^*) \cap \tilde{X}^i(z^*,y^*)$. Therefore x^{*i} is optimal in $B^i(z^*,y^*,p^*)$.

vi) Since (z^*,y^*,p^*) is an equilibrium for $\tilde{E}$, if $(\Sigma\, x^{*i} - \Sigma\, w^i(z^*,y^*) - \Sigma\, \hat{q}^{*e})_h < 0$ then $p_h = 0$.

Therefore (z^*,y^*,p^*) is an equilibrium for E.

The following is an example of a two consumer, two producer, three good economy with externalities.

Notation: x_1^A is consumption of good 1 by Mr. A.

x_2^B is consumption of good 2 by Mr. B., etc.

w_1^A is the endowment of good 1 of Mr. A, etc.

L^A is the endowment of labor of Mr. A, etc.

x_1^I is the output of good 1 by firm I.

x_1^{II} is the input of good 1 used by firm II.

x_2^{II} is the output of good 2 by firm II.

L_I is the input of labor used by firm I, etc.

Data:

Utility function of Mr. A: $U^A(x_1^A, x_2^A;\ x_1^B, x_2^B) = (x_1^A + x_1^B)^{\frac{1}{4}}\ (x_2^A)^{\frac{3}{4}}$

Endowment of Mr. A: $w_1^A = w_2^A = 1$

$$L^A = \begin{cases} 18 - x_1^I, & 0 \leqq x_1^I \leqq 16 \\ 2, & x_1^I \geqq 16 \end{cases}$$

Utility function of Mr. B: $U^B(x_1^B, x_2^B; x_1^A, x_2^A) = (x_1^B - x_1^{A^2} + 6x_1^A)\ x_2^B$

Endowment of Mr. B: $w_1^B = w_2^B = 1$

$$L^B = \begin{cases} 12 - x_1^I + x_2^{II}, & -8 \leqq x_2^{II} - x_1^I \leqq 10 \\ 20, & x_2^{II} - x_1^I \leqq -8 \\ 2, & x_2^{II} - x_1^I \geqq 10 \end{cases}$$

Technology of Firm I:

$$x_1^I = (1 + \tfrac{1}{6}\, x_2^{II})L_I$$

Technology of Firm II:

$$x_2^{II} = \begin{cases} \min\left(x_1^{II}, \dfrac{12L_{II}}{x_1^I}\right) & x_1^I \geqq 1 \,. \\ \min(x_1^{II}, 12L_{II}) \,, & 0 \leqq x_1^I \leqq 1 \,. \end{cases}$$

This model satisfies assumptions 1)-12). To see what is happening in this example, suppose that production of good 1 produces as a by-product noxious fumes, which have a deleterious effect on the health of consumers A and B, reducing the amount of labor they can sell. These fumes also have a deleterious effect on the laborers working for firm II. The production of good 2 by firm II produces as a by-product ozone, which negates to some extent the noxious fumes of firm I for Mr. B and laborers working for firm II. Mr. A does not live close enough to firm II to receive any benefits. Mr. A likes Mr. B's consumption of good 1 and Mr. B likes increases in Mr. A's consumption of good 1 up to a point (until $x_1^A = 3$), but dislikes increases in x_1^A beyond this level.

We will take $p_1 = 1$, $p_2 = p$, p_L = price of labor (L). The demand functions of Mr. A and Mr. B for good 1 are:

$$x_1^A = \begin{cases} \tfrac{1}{4}(1 + p + p_L(18 - x_1^I) - \tfrac{3}{4}\, x_1^B \\ \text{if this is non-negative and } 0 \leqq x_1^I \leqq 16 \,. \\ 0, \text{ if negative} \end{cases}$$

$$x_1^B = \begin{cases} \frac{1}{2}(1 + p + p_L(12 - x_1^I + x_2^{II})) - \dfrac{x_1^{A^2} - 6x_1^A}{2} \\ \text{if this is non-negative and } -8 \leqq x_2^{II} - x_1^I \leqq 10. \\ 0, \text{ if negative} \end{cases}$$

Given these demand functions and the production functions, you can easily show that an equilibrium for this model is:

$$p^* = 3 , \quad p_L^* = 2$$

$$x_1^{A*} = 0, \quad x_2^{A*} = 16/3; \quad x_1^{B*} = 8, \quad x_2^{B*} = 8/3$$

$$L^{A*} = 6; \quad L^{B*} = 6$$

$$x_1^{I*} = 12, \quad L_I^* = 6$$

$$x_2^{II*} = 6, \quad x_1^{II*} = 6, \quad L_{II}^* = 6$$

At this equilibrium Mr. A is exerting an external economy on Mr. B at his consumption level of x_1^A (i.e., $\partial U^{B*}/\partial x_1^A > 0$) and Mr. B is having a similar effect on Mr. A $\left(\frac{\partial U^{A*}}{\partial x_1^B} > 0\right)$. Firm I is exerting an external diseconomy on Mr. A, Mr. B, and Firm II through its production of x_1^I ($\partial L^{A*}/\partial x_1^I$, $\partial L^{B*}/\partial x_1^I$, $\partial x_2^{II*}/\partial x_1^I < 0$). Firm II is exerting an external economy on Mr. B and Firm I through its production of x_2^{II} ($\partial L^{B*}/\partial x_2^{II}$, $\partial x_1^{I*}/\partial x_2^{II} > 0$).

As is usually the case when externalities are present, this equilibrium is not a Pareto Optimum. For example keeping production levels fixed, Mr. B can benefit both himself and Mr. A by giving some of x_I^{B*} to Mr. A. This is because $[\partial U^{B*}/\partial x_1^A - \partial U^{B*}/\partial x_1^B] > 0$. Pareto Optimality in this example will be discussed in Part II of this essay.

We have now demonstrated the existence of an equilibrium for a class of economies with externalities. An examination of the models with externalities found in the literature would find that almost all of these models satisfy the assumptions of our theorem.

The following is an example of a two good, two person pure exchange economy for which an equilibrium does not exist. The notation will be the same as in the previous example.

<u>Data</u>: Utility Function of Mr. A: $U^A = (x_1^A - x_1^B)^{\frac{1}{4}} (x_2^A)^{\frac{3}{4}}$

Endowment of Mr. A: $w_1^A = 4 = w_2^A$

Utility Function of Mr. B:

$$U^B = \begin{cases} (x_1^B - x_1^{A^2} + 6x_1^A)x_2^B , & \text{if} \quad x_2^A \geqq \frac{5}{2} x_2^B \\ (x_1^B - x_1^{A^2} + 6x^A)(x^B)^{\frac{1}{2}} & \text{if} \quad x_2^A < \frac{5}{2} x_2^B \end{cases}$$

Endowment of Mr. B: $w_1^B = 4 = w_2^B$

In this example Mr. B's preferences are discontinuous (they do not satisfy assumption 3)). Mr. B's satisfaction derived from consuming good 2 is discontinuously reduced if Mr. A's consumption of good 2 falls below $\frac{5}{2} x_2^B$. This type of discontinuity of preferences is certainly not inconceivable in the presence of consumption externalities.

The demand functions for A and B are:

$$x_1^A = \begin{cases} \frac{1}{4}(4 + 4p - 3x_1^B) , & \text{if this is} \geqq 0 \\ 0, \text{ otherwise} & \end{cases}$$

$$x_1^B = \begin{cases} \frac{1}{2}(4+4p+x_1^{A^2} - 6x_1^A), & \text{if } x_2^A \geqq \frac{5}{2}\, x_2^B, \text{ and this is } \geqq 0 \\ \frac{2}{3}(4+4p+x_1^{A^2} - 6x_1^A), & \text{if } x_2^A < \frac{5}{2}\, x_2^B, \text{ and this is } \geqq 0 \\ 0, \text{ otherwise} \end{cases}$$

Again we take $p_1 = 1$, $p_2 = p$. An equilibrium would be the solution of either of the two following sets of equations:

I.

i) $$x_1^A = \begin{cases} \frac{1}{4}(4+4p-3x_1^B), & \text{if this is } \geqq 0 \\ 0, & \text{otherwise} \end{cases}$$

ii) $$x_1^B = \begin{cases} \frac{1}{2}(4+4p+x_1^{A^2} - 6x_1^A), & \text{if this is } \geqq 0 \\ 0, & \text{otherwise} \end{cases}$$

iii) $x_1^A + x_1^B = 8$

iv) $x_1^A + px_2^A = 4+4p = x_1^B + px_2^B$

v) $x_2^A \geqq \frac{5}{2}\, x_2^B$

II.

i′) $$x_1^A = \begin{cases} \frac{1}{4}(4+4p-3x_1^B), & \text{if this is } \geqq 0 \\ 0, & \text{otherwise} \end{cases}$$

ii′) $$x_1^B = \begin{cases} \frac{2}{3}(4+4p+x_1^{A^2} - 6x_1^A), & \text{if this is } \geqq 0 \\ 0, & \text{otherwise} \end{cases}$$

iii′) $x_1^A + x_1^B = 8$

iv′) $x_1^A + px_2^A = 4 + 4p = x_1^B + px_2^B$

v′) $x_2^A < \frac{5}{2} x_2^B$

The only solution of I. i)-iv) is $p^* = 3$; $x_1^{A*} = 0$, $x_2^{A*} = 16/3$; $x_1^{B*} = 8$, $x_2^{B*} = 8/3$. But this is inconsistent with I. v), so no solution of I exists.

The only solution of II. i′)-iv′) is $p^* = 2$; $x_1^{A*} = 0$, $x_2^{A*} = 6$; $x_1^{B*} = 8$, $x_2^{B*} = 2$. But this is inconsistent with II. v′), so no solution of II. exists.

Therefore this example has no equilibrium.

PART II

OPTIMUM

In Part II we will discuss (Pareto) optimality and the relationship between equilibrium and optimality in the model constructed in Part I. In the usual model without externalities if consumers' consumption sets and preferences are convex, an equilibrium where no consumption is a satiation consumption is a Pareto Optimum (Debreu[8]). Therefore the set of assumptions sufficient to guarantee existence of equilibrium plus a non-satiation assumption are sufficient to guarantee existence of a Pareto Optimum. (These assumptions are much stronger than necessary.) However with externalities, it is not true generally that an equilibrium is a Pareto Optimum. We will show though that as in the non-externalities case, the set of assumptions sufficient for existence of a Pareto Optimum is much weaker than the set sufficient for existence of an equilibrium. The main problem in establishing the existence of an optimum is to show the set of attainable actions is compact.

A: The Existence of a Pareto Optimum

Let $\tilde{Y}^e$ and $\tilde{Z}_i$ be as defined in Part I, section D, and let $\tilde{Y} = \mathop{X}_{e=1}^{t} \tilde{Y}_e$, $\tilde{Z} = \mathop{X}_{i=1}^{s} \tilde{Z}_i$. Consider the mapping Ψ: $\tilde{Z} \times \tilde{Y} \rightarrow \tilde{Z} \times \tilde{Y}$ where $\Psi(z,y) = [\mathop{X}_{i=1}^{s} (\tilde{X}^i(z,y))] \times [\mathop{X}_{e=1}^{t} \tilde{Y}^e(z,y)]$. Under the assumptions of Part I Ψ is a continuous convex-valued correspondence and $\tilde{Z} \times \tilde{Y}$ is compact and convex. Thus by Kakutani's theorem Ψ has a fixed point. (Notice that we only need $X^i(z,y)$ and $Y^e(z,y)$ to be <u>upper semi-continuous</u> and convex-valued to make this argument.) Let (z^*,y^*) be a fixed point of Ψ. Then $x^{*i} \in \tilde{X}^i(z^*,y^*)$, $\forall i$ and $q^{*e} \in \tilde{Y}^e(z^*,y^*)$, $\forall e$. Since $\tilde{X}^i(z,y) \subset X^i(z,y)$, $\forall i$ and $\tilde{Y}^e(z,y) \subset Y^e(z,y)$, $\forall e$, we see that a fixed point of Ψ is a vector of technologically feasible (t.f.) actions. Let F be the set of fixed points of Ψ. Then F is the set of t.f. actions in $\tilde{E}$.

<u>Lemma 4</u>: F is compact.

<u>Proof</u>: F is clearly bounded. ($\tilde{Z}$ and $\tilde{Y}$ are bounded.) Suppose F is not closed. Then there exists a sequence $\{(z^n,y^n)\}$ such that $(z^n,y^n) \rightarrow (z,y)$, $(z^n,y^n) \in \Psi(z^n,y^n)$, $\forall n$ but $(z,y) \notin \Psi(z,y)$. But this is a contradiction of Ψ upper semi-continuous. Thus F is closed. (F will not in general be convex.)

Now consider the mapping ξ: $\tilde{Z} \times \tilde{Y} \rightarrow R^n$ where $\xi(z,y) = z - \Sigma\, w^i(z,y) - \Sigma\, \hat{q}^e$. Since $w^i(z,y)$ is continuous, $\forall i$ $\xi(z,y)$ is a continuous function, and furthermore 0 is contained in the range of ξ because $w^i(z,y) \in \tilde{Z}_i$, $\forall i$ and $0 \in \tilde{Y}^e$, $\forall e$. Let $B = \{(z,y) \in \tilde{Z} \times \tilde{Y} | \xi(z,y) \leqq 0\}$. Then B is non-empty and compact since ξ is continuous. Let $A = B \cap F$. Then A is the set of attainable actions in E. A is compact since B and F are. We must show that A is non-empty.

Lemma 5: A is non-empty.

Proof: Consider the mapping α: $\tilde{Z} \to \tilde{Z}$ where $\alpha(z) = \sum_{i=1}^{s} w^i(z,0)$. By the continuity of the $w^i(z,)$, we have that $\alpha(z)$ is continuous and since $\tilde{Z}$ is compact and convex α has a fixed point. Let $\bar{z}$ be a fixed point of α. Then $(\bar{z},0) \in F$ because $0 \in \tilde{Y}^e(z,y)$, $\forall(z,y)$ and $w^i(z,y) \in \tilde{X}^i(z,y)$, $\forall(z,y)$. Also, $(\bar{z},0) \in B$ because by definition of α, $\sum_{i=1}^{s} w^i(\bar{z},0) = \bar{z}$. Thus $(\bar{z},0) \in A$. As mentioned earlier, A is the set of attainable actions for the economy E and as we have shown, A is non-empty and compact.

Theorem V: The economy E has a Pareto Optimum.

Proof: Under the assumptions on preferences, each consumer, i, has a continuous utility function $U^i(z,y)$ which represents his preferences $(\geq)_i$. (We drop the assumption that U^i depends on prices also because it is meaningless in this context.)

Consider the function β: $A \to R^s$ where $\beta(z,y) = \sum_{i=1}^{s} U^i(z,y)$, and the function γ: $R^s \to R$, where $\gamma(x)$ is any continuous strictly increasing function (increasing in each of its arguments).

Then the function $\gamma[\beta(z,y)]$ is a continuous real valued function on A, and since A is compact, this function attains its maximum. Clearly, a maximizer of this function in A is a Pareto Optimum for the economy E.

The assumptions we have used to guarantee the existence of an optimum are stronger than the corresponding assumptions used in the treatment of the problem for economies without externalities. For example, we needed to have the $\tilde{X}^i(z,y)$ to be convex, which is not necessary in the usual case. As indicated earlier, the assumption we used in showing Ψ has a fixed point and F is compact only requires that $\tilde{X}^i(z,y)$ and $\tilde{Y}^e(z,y)$ are convex and u.s.c.

B: Optimality and "Equilibrium"

As is well known, a competitive equilibrium with externalities is not in general a Pareto Optimum. This arises of course because individual optimization is not necessary or sufficient for joint optimization when the individual objective functions are interrelated. In some situations, mainly when there are only good externalities, this lack of optimality of equilibrium can be explained by the absence of sufficient markets. This is discussed in the paper by Shapley and Shubik,[9] which demonstrates that in a model with only "good" production externalities, under the usual assumptions the core will not be empty. In such a situation if there are markets for externalities a competitive equilibrium will exist in the model with the added markets and it will be in the core (and therefore a Pareto Optimum). To see how this would work, consider the following example.

Data:

Utility function of Mr. A: $U^A = (x_1^A + x_1^{B^2})x_2^A$

Endowment of Mr. A: $w_1^A = 2,\ w_2^A = 6$

Utility function of Mr. B: $U^B = x_1^B\ x_2^B$

Endowment of Mr. B: $w_1^A = 2 = w_1^B$

For the usual model where externalities are not traded, the demand functions for good 1 (with $p_1 = 1$, $p_2 = p$) are:

$$x_1^A = \begin{cases} 1 + 3p - x_1^{B^2}/2, & \text{if } \geqq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$x_1^B = 1 + p$$

The equilibrium for this model will be:

$$p^* = 1; \ x_1^{A*} = 2, \ x_2^{A*} = 6; \ x_1^{B*} = 2, \ x_2^{B*} = 2$$

Now consider a model where the externality (Mr. B's consumption of good 1) is tradable. In this case there is a market for Mr. B's consumption of good 1 -- i.e., Mr. B sells a new good (his consumption of good 1) to Mr. A. Let p^B be the price of this good.

The demand functions for this model come from the solution of the following problems. (Let ${}^A x_1^B$ be the demand by A for B's consumption of good 1.)

$$\max_{\{x_1^A,\ x_2^A,\ {}^A x_1^B\}} \ (x_1^A + {}^A x_1^{B^2}) x_2^A \quad \text{subject to} \quad x_1^A + p x_2^A + p^B \cdot {}^A x_1^B = 2 + 6p$$

$$\max_{\{x_1^B,\ x_2^B\}} \ x_1^B x_2^B \quad \text{subject to} \quad x_1^B + p x_2^B = 2 + 2p + p^B x_1^B$$

The demand functions for this model will be:

$$x_1^A = 0$$

$$x_2^A = \frac{2 + 6p}{3p}$$

$${}^A x_1^B = \frac{4 + 12p}{3p^B}$$

$$x_1^B = \frac{1 + p}{1 - p^B}$$

$$x_2^B = \frac{1 + p}{p}$$

The equilibrium for this model will be:

$$p^* = \tfrac{1}{3},\ p^{B*} = \tfrac{2}{3};\quad x_1^{A*} = 0,\ x_2^{A*} = 4,\ {}^{A}x_1^{B*} = 4;\quad x_1^{B*} = 4,\ x_2^{B*} = 4.$$

This allocation is a Pareto Optimum and it is in the core since it is an equilibrium supported by the original income distribution.

This Pareto Optimum can also be supported by a system of lump sum and specific taxes and subsidies. Let I^A and I^B be arbitrary income levels for A and B ($p_1 = 1$). Then the <u>original</u> demand functions can be written:

$$x_1^A = \begin{cases} I^A/2 - x^{B^2}/2\,, & \text{if } \geqq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$x_1^B = I^B/2$$

Let S_1^B denote a per unit subsidy on B's consumption of good 1 and let S_2 be a per unit subsidy on both A and B's consumption of good 2. Let T^A and T^B denote lump sum taxes on A and B. Let $\bar{I}^A$ and $\bar{I}^B$ denote the values of the endowments of A and B evaluated at the original equilibrium prices ($p^* = 1$). ($\bar{I}^A = 8$, $\bar{I}^B = 4$) Then in this tax and subsidy scheme, the demand functions can be written: (for $p_1^* = p_2^* = 1$)

$$x_1^A = \begin{cases} \dfrac{\bar{I}^A - T^A}{2} - x^{B^2}/2\,, & \text{if } \geqq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$x_2^A = \begin{cases} \dfrac{\dfrac{\bar{I}^A - T^A}{2} + x_1^{B^2}/2}{(1 - S_2)}, & \text{if } \dfrac{\bar{I}^A - T^A}{2} - x_1^{B^2}/2 \geqq 0 \\ \dfrac{\bar{I}^A - T^A}{(1 - S_2)}, & \text{otherwise} \end{cases}$$

$$x_1^B = \frac{\bar{I}^A - T^A}{2(1 - S_1^B)}$$

$$x_2^B = \frac{\bar{I}^B - T^B}{2(1 - S_2)}$$

The Pareto Optimum ($x_1^{A*} = 0$, $x_2^{A*} = 4$; $x_1^{B*} = 4$, $x_2^{B*} = 4$) can be supported by the original equilibrium prices ($p^* = 1$) and lump sum taxes and subsidies:

$$T^{A*} = 4; \quad T^{B*} = -4; \quad \text{with } S_1^{B*} = S_2^* = 0 .$$

Therefore this Pareto Optimal allocation can be supported either by a set of markets including a market for the externality, or by the original equilibrium prices and a system of lump sum and specific taxes and subsidies, where the net price paid by each participant for the "non-external" good is the same. Later in this section we will show under certain assumptions that a Pareto Optimum in a model with externalities can always be supported by a suitable system of lump sum and specific taxes and subsidies, where all participants face the same net price for non-external goods. It is not true however that a Pareto Optimum can always be supported by a set of markets including markets for externalities. To see this consider the following example:

Data:

$$\text{Utility function of Mr. A:} \quad U^A = \begin{cases} (x_1^A - x_1^{B^2})x_2^A , & \text{if } x_1^B \leqq 5 \\ (x_1^A - 25)x_2^A , & \text{if } x_1^B > 5 \end{cases}$$

$$\text{Endowment of Mr. A:} \quad w_1^A = 6, \; w_2^A = 2$$

$$\text{Utility function of Mr. B:} \quad U^B = x_1^B \, x_2^B$$

$$\text{Endowment of Mr. B:} \quad w_1^B = 2 = w_2^B$$

If externalities are not tradable the demand functions for good 1 will be:

$$x_1^A = \begin{cases} 3 + p + \dfrac{x_1^{B^2}}{2} , & \text{if this is } \leqq 6 + 2p \text{ and } x_1^B \leqq 5 \\ 6 + 2p, & \text{otherwise} \end{cases}$$

$$x_1^B = 1 + p$$

An equilibrium for this model will be

$$p^* = 1; \quad x_1^{A*} = 6, \; x_1^{B*} = 2; \quad x_1^{B*} = 2, \; x_2^{B*} = 2 \; .$$

This of course is not a Pareto Optimal allocation. For example, if we reduce B's consumption of good 1 to $\frac{3}{2}$ and increase his consumption of good 2 to 8/3 then he has the same utility as at the equilibrium. But this reallocation leaves A with 13/2 of good 1 and 4/3 of good 2 giving him a utility level of 17/3 which is bigger than his utility at the equilibrium.

Now consider adding a market in the externality. (Let ${}^A x_1^B$ be the demand of A for B's consumption of good 1.) The demand functions for this situation are the solutions to the following two problems:

$$\max_{\{x_1^A,\ x_2^A,\ {}^Ax_1^B\}} \begin{cases} (x_1^A - {}^Ax_1^{B^2})x_2^A\ , & \text{for } x_1^B \leqq 5 \\ \\ (x_1^A - 25)x_2^A\ , & \text{for } x_1^B > 5 \end{cases}$$

$$\text{subject to } x_1^A + px_2^A = 6 + 2p - p^B \cdot {}^Ax_1^B$$

$$\max_{\{x_1^B,\ x_2^B\}} \ x_1^B x_2^B \quad \text{subject to } x_1^B + px_2^B - p^B x_1^B = 2 + 2p$$

The demand function of B for x_1^B is:

$$x_1^B = \frac{1+p}{1-p^B} \qquad \text{if } p^B < 1$$

(If $p^B > 0$, A pays B for B's consumption of x_1^B; if $p^B < 0$, B pays A for B's consumption of x_1^B.)

The demand function of A for ${}^Ax_1^B$ is:

$${}^Ax_1^B = 0\ , \qquad \text{if } p^B \geqq 0$$

$${}^Ax_1^B = \infty\ , \qquad \text{if } p^B < 0$$

Obviously, there is no p^B which will clear the market (i.e. ${}^Ax_1^B = x_1^B$). Thus in this case a set of markets including a market for the externality does not have an equilibrium.

Now consider the allocation $x_1^{A*} = 7$, $x_2^{A*} = 10/7$; $x_1^{B*} = 1$, $x_2^{B*} = 18/7$. This is a Pareto Optimum. Writing the demand functions of A and B as we did in the previous example we have (for $p_1^* = p_2^* = 1$):

$$x_1^A = \begin{cases} \dfrac{(I^A - T^A)/2 + x_1^{B^2}/2}{(1 - S_1^A)}, & \text{if } \leqq I^A - T^A \text{ and } x_1^B \leqq 5 \\[2ex] \dfrac{I^A - T^A}{1 - S_1^A}, & \text{otherwise} \end{cases}$$

$$x_2^A = \begin{cases} \dfrac{(I^A - T^A)/2 - x_1^{B^2}/2}{(1 - S_2)}, & \text{if } \geqq 0 \\[2ex] 0, & \text{otherwise} \end{cases}$$

$$x_1^B = \frac{I^B - T^B}{2(1 - S_1^B)}$$

$$x_2^B \quad \frac{I^B - T^B}{2(1 - S_2)}$$

Starting at the original equilibrium we have $p^* = 1$, $\bar{I}^A = 8$, $\bar{I}^B = 4$. The original equilibrium prices ($p^* = 1$) and the following system of taxes and subsidies will support the Pareto Optimum allocation ($x_1^{A*} = 7$, $x_2^{A*} = 10/7$; $x_1^{B*} = 1$, $x_2^{B*} = 18/7$):

$$T^{A*} = 29/7; \quad T^{B*} = -8/7; \quad S_1^{A*} = 32/49; \quad S_1^{B*} = -11/7; \quad S_2^* = 0$$

(In this scheme B receives a lump sum subsidy and pays a specific tax on his consumption of good 1.)

$$\text{Total taxes} = 29/7 + 1 \cdot 11/7 = 40/7$$

$$\text{Total subsidies} = 8/7 + 7 \cdot 32/49 = 40/7$$

B.1: Graphical Depiction of Pareto Optimum

In a two consumer, two good, pure exchange economy we can give a graphical description of Pareto Optima in a manner analogous to the graphical analysis of such a model without externalities. We have two consumers A and B with utility functions $U^A(x_1^A, x_2^A; x_1^B, x_2^B)$, $U^B(x_1^B, x_2^B; x_1^A, x_2^A)$, and a total resource constraint, $x_1^A + x_1^B = \bar{x}_1$, $x_2^A + x_2^B = \bar{x}_2$. We will define the <u>resource feasible utility functions</u> of A and B, denoted $\bar{U}^A$ and $\bar{U}^B$, as

$$\bar{U}^A(x_1^A, x_2^A) = U^A(x_1^A, x_2^A; \bar{x}_1 - x_1^A, \bar{x}_2 - x_2^A)$$

$$\bar{U}^B(x_1^B, x_2^B) = U^B(x_1^B, x_2^B; \bar{x}_1 - x_1^B, \bar{x}_2 - x_2^B)$$

Under the usual assumptions these utility functions will have well defined indifference curves, but the indifference curves won't always be downward sloping or convex. Consider the previous example:

$$U^A = \begin{cases} (x_1^A - x^{B^2})x_2^A, & \text{if } x_1^B \leqq 5 \\ (x_1^A - 25)x_2^A, & \text{if } x_1^B > 5 \end{cases}$$

$$w_1^A = 6, \quad w_2^A = 2$$

$$U^B = x_1^B x_2^B$$

$$w_1^B = w_2^B = 2$$

In this case we have

$$\bar{U}^A = \begin{cases} [x_1^A - (8 - x_1^A)^2]x_2^A, & \text{if } 8 - x_1^A \leqq 5 \\ [x_1^A - 25]x_2^A, & \text{if } 8 - x_1^A > 5 \end{cases}$$

$$\bar{U}^B = x_1^B \, x_2^B$$

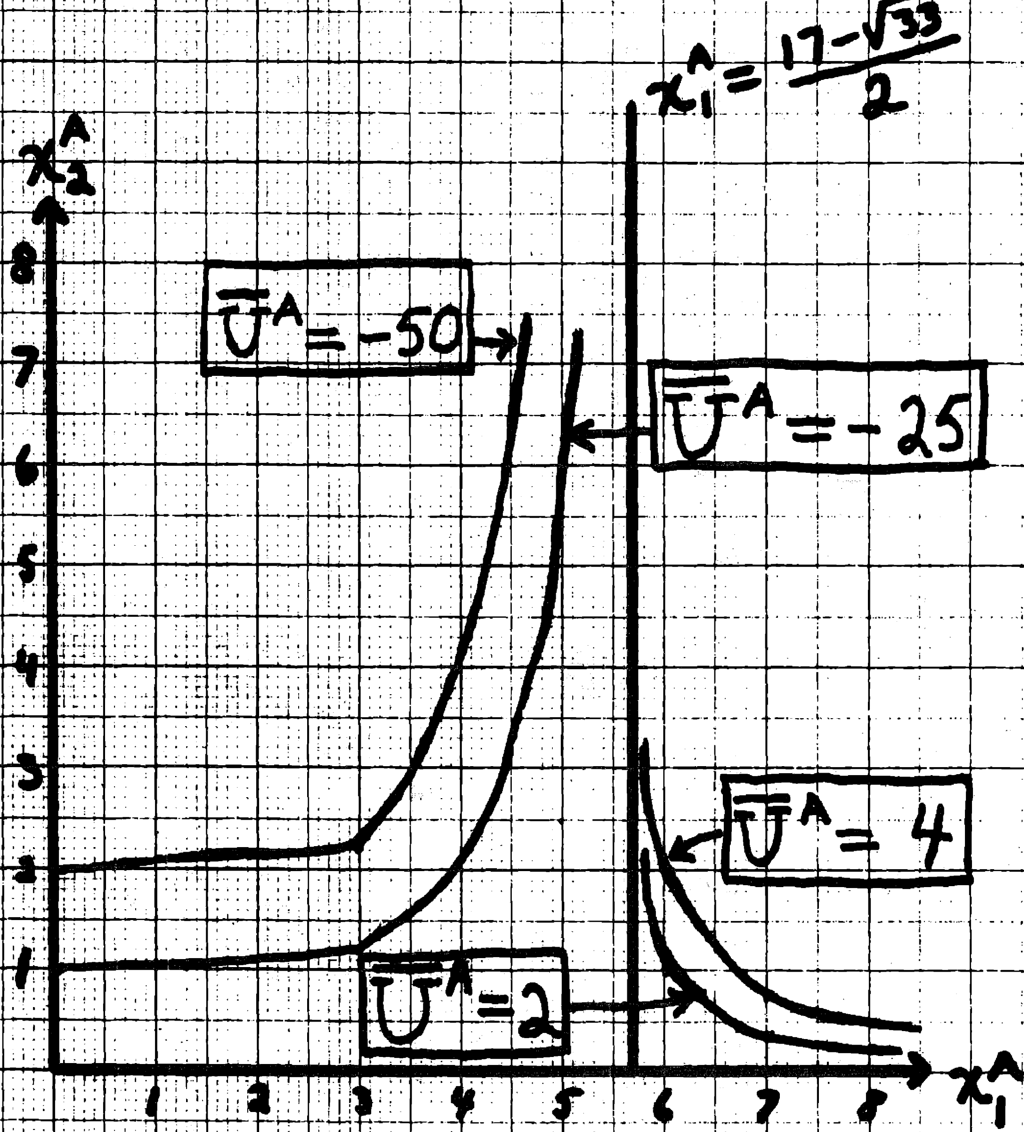
The locus in (x_1^A, x_2^A) space of the indifference curve $\bar{U}^A = -25$ is:

$$\underset{(x_2^A \geqq 0)}{x_2^A} = \begin{cases} -25[x_1^A - (8 - x_1^A)^2]^{-1}, & \text{if } 8 - x_1^A \leqq 5 \\ -25[x_1 - 25]^{-1}, & \text{if } 8 - x_1^A > 5 \end{cases}$$

Therefore

$$dx_2^A/dx_1^A\Big|_{\bar{U} = -25} = \begin{cases} 25[1 + 2(8 - x_1^A)][x_1^A - (8 - x_1^A)^2]^{-2} & \text{if } 8 - x_1^A < 5 \\ 25[x_1^A - 25]^{-2}, & \text{if } 8 - x_1^A > 5 \end{cases}$$

From this we see that the indifference curve corresponding to $\bar{U}^A = -25$ is positively sloped. The following diagram is a graph of the indifference curves of $\bar{U}^A$.



$\overline{U}^A = 0$ is the set of points:

$$\{x_2^A = 0\} \cup \{x_1^A = \frac{17-\sqrt{33}}{2}\}$$

If we graph the indifference curves of $\bar{U}^A$ and $\bar{U}^B$ in an Edgeworth box, the (<u>interior</u>) Pareto Optima will be points of tangency of these two sets of indifference curves. Consider again the Pareto Optimal point, $x_1^{A*} = 7$, $x_2^{A*} = 10/7$; $x_1^{B*} = 1$, $x_2^{B*} = 18/7$. For this allocation $\bar{U}^{A*} = U^{A*} = 60/7$; $\bar{U}^{B*} = U^{B*} = 18/7$. In the following Edgeworth box diagram we will depict this Pareto Optimum and how it is supported by the tax and subsidy system we derived earlier. Through the Pareto Optimal point we will draw the indifference curves:

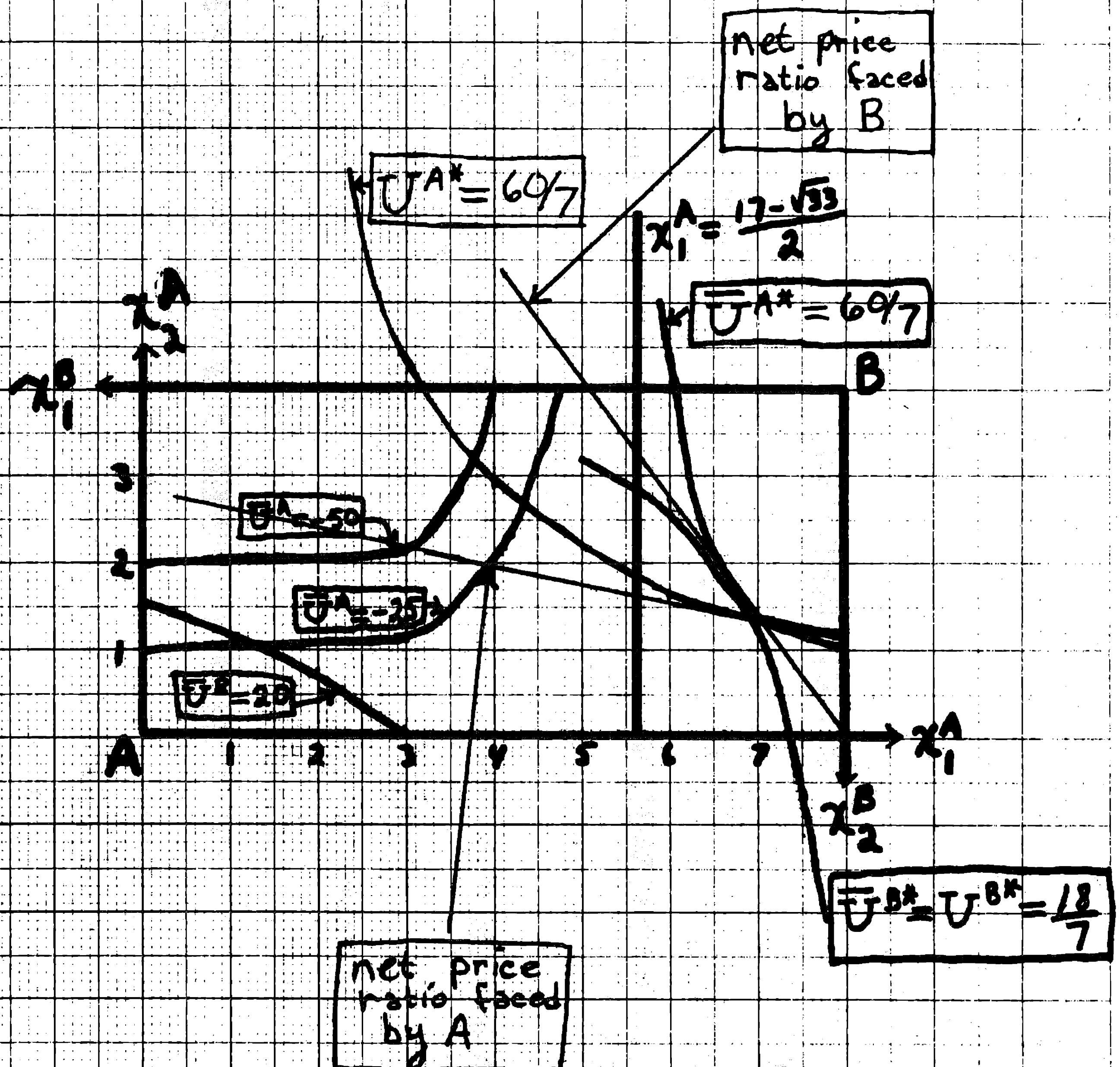
1) $\bar{U}^{A*} = 60/7$,

2) $\bar{U}^{B*} = 18/7$;

3) $U^A(x_1^A, x_2^A; 1, 18/7) = 60/7$, denoted $U^{A*} = 60/7$; and

4) $U^B(x_1^B, x_2^B; 7, 10/7) = 18/7$, denoted $U^{B*} = 18/7$.

Notice that the original endowment point is in the interior of the new budget sets of A and B. This is because of the lump sum transfers. The locus of Pareto Optima for $x_1^A < \frac{17 - \sqrt{33}}{2}$ is the line $x_2^A = 0$.


net price ratio faced by B
$U^{A*} = 60/7$
$x_1^A = \frac{17-\sqrt{33}}{2}$
$\bar{U}^{A*} = 60/7$
x_2^A
x_1^B
B
3
$\bar{U}^A = -50$
2
$\bar{U}^A = -25$
1
$\bar{U}^B = 20$
A
1 2 3 4 5 6 7
x_1^A
x_2^B
$\bar{U}^{B*} = U^{B*} = \frac{18}{7}$
net price ratio faced by A

To get a better understanding of why an equilibrium with added markets does not exist in this example, let us go back to our first example of this section:

$$U^A = (x_1^A + x_1^{B^2})x_2^A$$

$$w_1^A = 2, \quad w_2^A = 6$$

$$U^B = x_1^B \, x_2^B$$

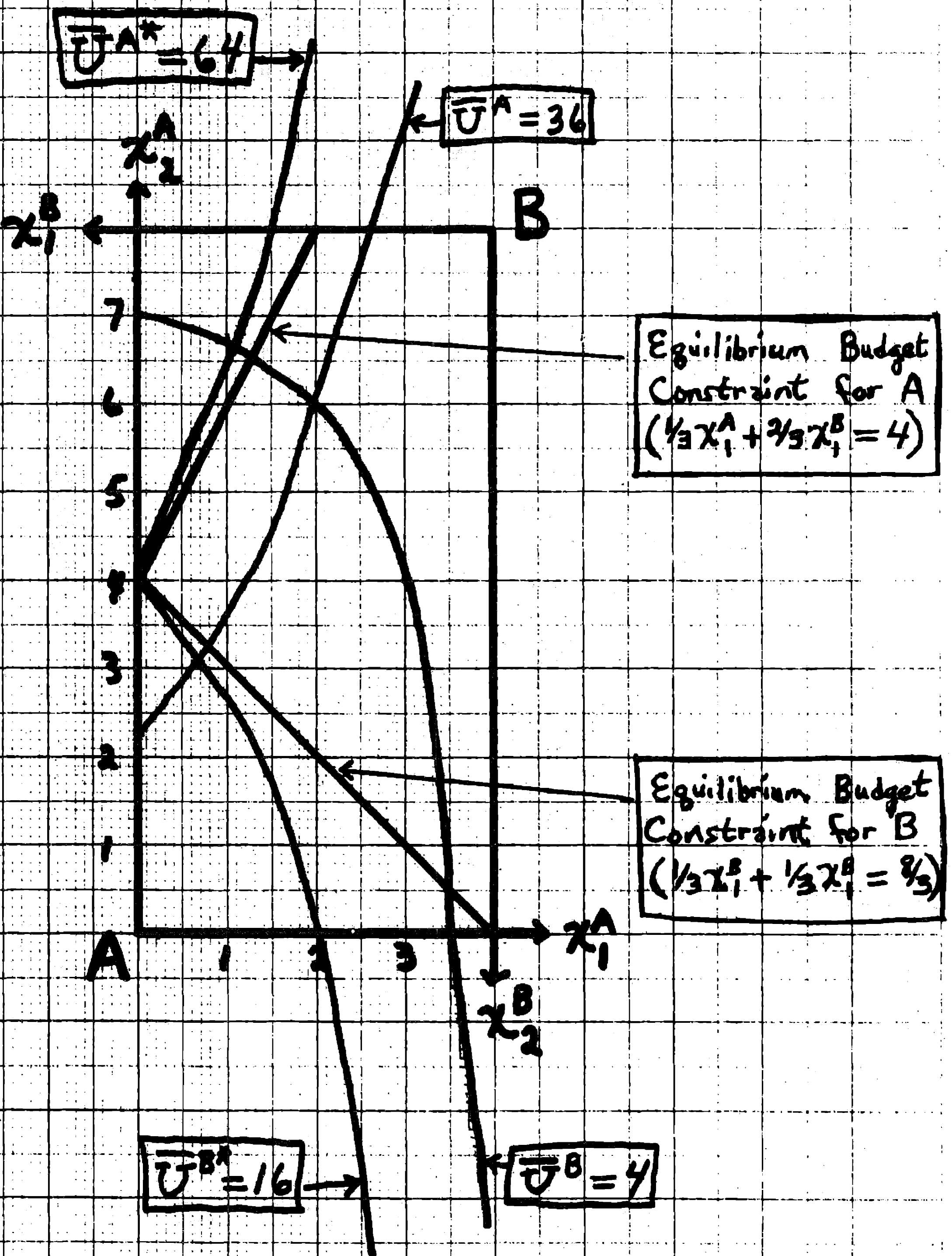
$$w_1^B = 2 = w_1^B$$

In this example we have:

$$\bar{U}^A = (x_1^A + (4 - x_1^A)^2)x_2^A$$

$$\bar{U}^B = x_1^B \, x_2^B$$

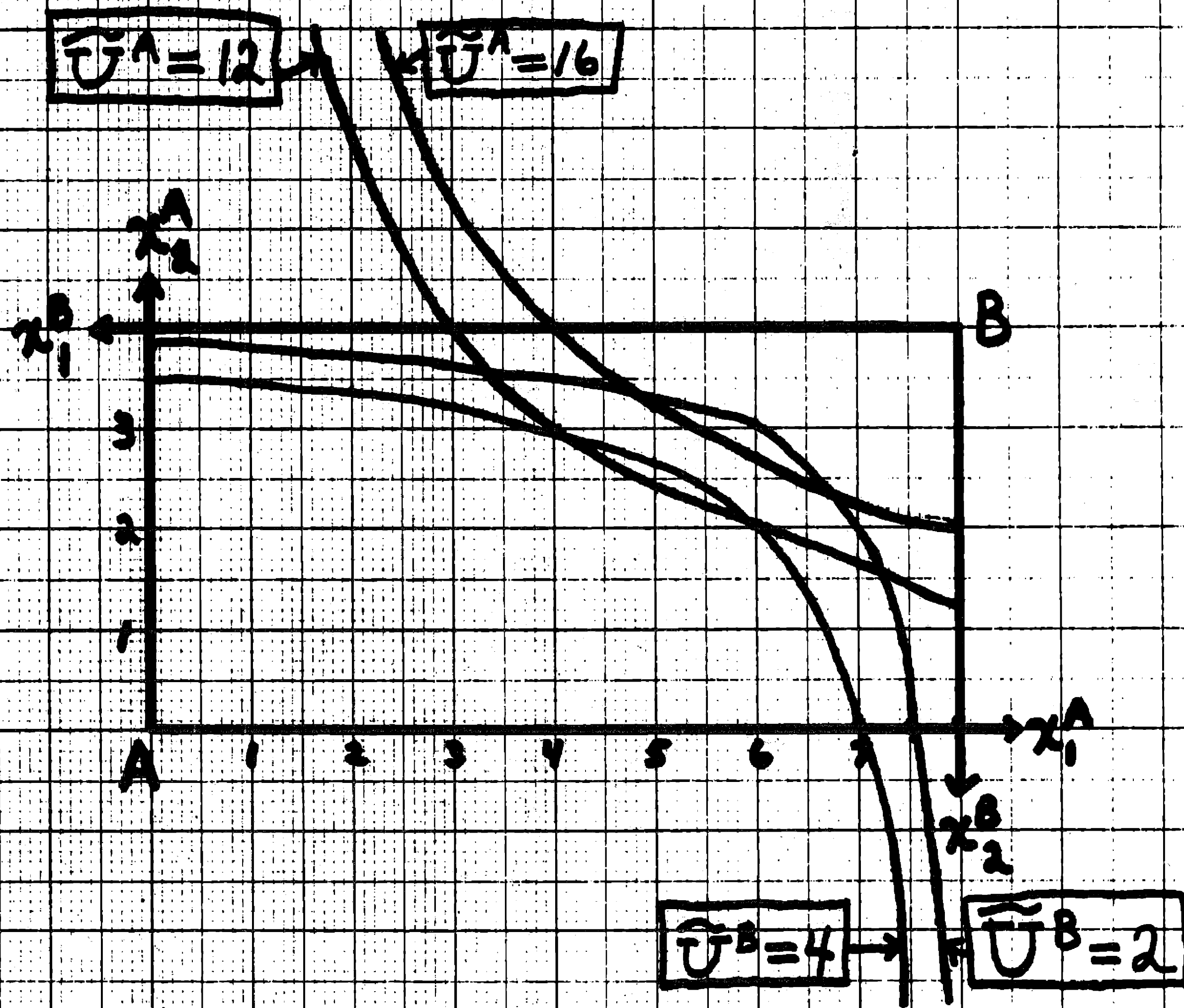
The $(\bar{U})$ indifference curves through the original endowment point are: $\bar{U}^A = 36$, $\bar{U}^B = 4$. In the following Edgeworth box diagram we will depict the equilibrium with a market for the externality ($p^* = \frac{1}{3}$, $p^{B*} = \frac{2}{3}$; $x_1^{A*} = 0$, $x_2^{A*} = 4$; $x_1^{B*} = 4$, $x_2^{B*} = 4$). The $(\bar{U})$ indifference curves through this point are: $\bar{U}^{A*} = 64$, $\bar{U}^{B*} = 16$. The locus of Pareto optima in the Edgeworth box will be the line $x_1^A = 0$.



$\overline{U}^{A*} = 64$
$\overline{U}^{A} = 36$
x_2^A
x_1^B
B
7
6
5
4
3
2
1
A
1
2
3
x_1^A
x_2^B
Equilibrium Budget Constraint for A ($1/3\, x_1^A + 2/3\, x_1^B = 4$)
Equilibrium Budget Constraint for B ($1/3\, x_1^B + 1/3\, x_1^B = 8/3$)
$\overline{U}^{B*} = 16$
$\overline{U}^{B} = 4$

Notice that the original endowment points are in the <u>interior</u> of A and B's budget sets. This is because A and B's endowments of good 1 are valued at 1 per unit which is above both the price of consuming good 1 for B and the price of ${}^{A}x_1^B$ for A. The total cost to both participants of B's consumption of x_1^B is 1 per unit.

Now let us return to the example in which an equilibrium with a market for the externality does not exist. Let $\tilde{U}^A = U^A(x_1^A, x_2^A; 0,0)$. As we have seen earlier, for $p^B \geqq 0$ A's demand for ${}^{A}x_1^B$ is zero. In the following Edgeworth box diagram we draw some indifference curves for $\tilde{U}^A$ and U^B. $\tilde{U}^A$ is the valid utility function to consider in a search for an equilibrium since we must have $p^B \geqq 0$ (otherwise A's demand for ${}^{A}x_1^B$ is infinite). However, we see from the indifference curves of B that B will never choose $x_1^B = 0$ in any budget situation. Thus no equilibrium can exist.



$\tilde{U}^A = 12$
$\tilde{U}^A = 16$
x_2^A
x_1^B
B
3
2
1
A
1
2
3
4
5
6
7
x_1^A
x_2^B
$\tilde{U}^B = 4$
$\tilde{U}^B = 2$

B.2: Pareto Optimum and Individual Efficiency

It has been pointed out by Murakami and Negishi[10] that in a model with pure production externalities a Pareto Optimal allocation could require some producer to operate at an individually inefficient production point (i.e., in the interior of his production set). In such a situation the Pareto Optimum cannot be supported by a system of taxes and subsidies in the usual sense because in such a situation an inefficient producer would have to face zero prices. In a model with only pure production externalities the possibility of this situation arising is small (a necessary condition is that the producer exerts an external economy through all his actually used inputs and an external diseconomy through all his actually produced outputs). As we shall see the problem becomes much more complicated in a general model with various kinds of externalities.

Our analysis will parallel that of Murakami and Negishi and use non-linear programming as the basic technique. To do this of course we will have to assume differentiability, etc. We will assume that the Pareto Optimum can be described as the saddle point of the appropriate Kuhn-Tucker Lagrangean.

$U^i(z,y)$ is the utility function of consumer i. We assume that the production correspondence of the e^{th} producer can be represented by the implicit equation $g^e(z,y) \leqq 0$.

A P.O. can be described as the maximizer of the following problem:

$$\max_{\{(z,y) \in \tilde{Z} \times \tilde{Y}\}} U^1(z,y) \quad \text{subject to}$$

1) $U^i(z,y) \geqq k_i \; , \quad i \neq 1$

2) $g^e(z,y) \leqq 0 \; , \quad \forall e$

3) $\Sigma\, x^i + \sum_e q^{Ie} - \Sigma\, w^i(z,y) - \sum_e q^{Oe} \leqq 0$

We will assume that

$$(z,y) \geqq 0 \; , \quad \forall (z,y) \in \tilde{Z} \times \tilde{Y} \; .$$

We formulate the Lagrangean:

$$L = U^1(z,y) - \sum_{i=2}^{s} \gamma_i (k_i - U^i(z,y)) - \sum_e \lambda_e g^e(z,y)$$
$$- \sum_{h=1}^{n} p_h \left[\sum_i x_h^i + \sum_e q_h^{Ie} - \sum_e q_h^{Oe} - \sum_i w_h^i(z,y) \right]$$

Denote $\partial U^i(z,y)/\partial q_j^{Ik}$, etc. by $U^i_{(Ikj)}$, and $\partial U^i/\partial x_j^m$, etc. by $U^i_{(mj)}$. Similarly, denote $\partial w_h^i/\partial q_j^{Ik}$ by ${}^{(Ikj)}w_h^i$ and $\partial w^i/\partial x_j^m$ by ${}^{(mj)}w_h^i$. The F.O.C.'s for the saddle point of L are:

i) $$L_{(Ikj)} = U^1_{(Ikj)} + \sum_{i=2}^{s} \gamma_i U^i_{(Ikj)} - \sum_e \lambda_e g^e_{(Ikj)} - p_j +$$
$$\sum_{h=1}^{n} p_h \left(\sum_i {}^{(Ikj)}w_h^i(z,y) \right) \leqq 0$$

and

$$q_j^{Ik} \cdot L_{(Ikj)} = 0 \; ; \quad k = 1,\ldots,t; \quad j = 1,\ldots,n \; .$$

ii) $$L_{(Okj)} = U^1_{(Okj)} + \sum_{i \neq 1} \gamma_i U^i_{(Okj)} - \sum_e \lambda_e g^e_{(Okj)} + p_j +$$
$$\sum_{h=1}^{n} p_h \left(\sum_i {}^{(Okj)}w_h^i(z,y) \right) \leqq 0$$

and

$$q^{Ok} \cdot L_{(Okj)} = 0 \ ; \quad k = 1,\ldots,t \ ; \quad j = 1,\ldots,n.$$

iii) $$L_{(mj)} = U^1_{(mj)} + \sum_{i\neq 1} \gamma_i U^i_{(mj)} - \sum \lambda_e g^e_{(mj)} - p_j + \sum_h p_h \left(\sum_i^{(mj)} w^i_h(z,y) \right) \leqq 0$$

and

$$x^m_j \cdot L_{(mj)} = 0, \quad m = 1,\ldots,s; \ j = 1,\ldots,n \ .$$

iv) $$-(U^i(z,y) - k_i) \leqq 0 \ , \quad \gamma_i \geqq 0 \ ,$$

and $$\gamma_i (k_i - U^i(z,y)) = 0 \ , \quad i = 2,\ldots,s.$$

v) $$g^e(z,y) \leqq 0 \ , \quad \lambda_e \geqq 0 \ , \quad \text{and} \quad \lambda_e g^e(z,y) = 0 \ , \quad e = 1,\ldots,t \ .$$

vi) $$\left[\sum_i x^i_h + \sum_e q^{Ie}_h - \sum_i w^i(z,y) - \sum_e q^{Oe} \right] \leqq 0 \ , \quad p_h \geqq 0 \ ,$$

$$p_h \cdot [\ \ ] = 0 \ , \quad h = 1,\ldots,n.$$

(These F.O.C.'s are evaluated of course at the P.O. point.)

We shall consider first the model discussed by Murakami and Negishi, i.e., U^i is not dependent on z except for z_i and is not dependent on y; g^e is not dependent on z, and w^i is a constant.

In this case our F.O.C.'s become

i') $$-\sum_e \lambda_e g^e_{(Ikj)} - p_j \leqq 0 \ , \quad q^{Ik}_j \left[-\sum_e \lambda_e g^e_{(Ikj)} - p_j \right] = 0$$

$$\text{ii}') \quad -\sum_e \lambda_e g^e_{(Okj)} + p_j \leqq 0 , \quad q^{Ok}_j[-\sum_e \lambda_e g^e_{(Okj)} + p_j] = 0$$

$$\text{iii}') \begin{cases} U^1_{(1j)} - p_j \leqq 0 , & x^1_j[U^1_{(1j)} - p_j] = 0 \\ \gamma_i U^i_{(ij)} - p_j \leqq 0 , & x^i_j[U^i_{(ij)} - p_j] = 0 , \quad i \neq 1 \end{cases}$$

If every good is desirable, $U^1_{(1j)} > 0$, $\forall j$ then $p_j > 0$, $\forall j$. Assume $p_j > 0$, $\forall j$. Suppose $g^k(z,y) < 0$, for some k -- then $\lambda_k = 0$. Then we have

$$\sum_{e \neq k} \lambda_e g^e_{(Ikj)} = -p_j < 0 , \quad \text{for} \quad q^{Ik}_j > 0$$

$$\sum_{e \neq k} \lambda_e g^e_{(Okj)} = p_j > 0 , \quad \text{for} \quad q^{Ok}_j > 0 .$$

Thus we see that for $g^{Ik}_j > 0$ it must be true that $\lambda_e > 0$ and $g^e_{(Ikj)} < 0$ for some e -- and for $q^{Ok}_j > 0$ it must be true that $\lambda_e > 0$ and $g^e_{(Okj)} > 0$ for some e. $g^e_{(Ikj)} < 0$ means that producer k exerts an external economy through his use of good j as an input on producer e. For example, suppose producer e produces Q using u and v as inputs, and producer k uses b as an input. Then producer e's production possibilities might be given by: $Q - a_1 u - a_2 v - a_3 b \leqq 0$. Similarly $g^e_{(Okj)} > 0$ means that producer k exerts an external diseconomy through his output of good j on producer e.

Thus we see that if all goods are desirable and if producer k operates inefficiently at the P.O., it is necessary that for every input he actually uses he exert through this input an external economy on some efficient producer, and for every output he actually produces he exert an external diseconomy through this output on some efficient producer.

Now we shall consider the case where there are both consumption and production externalities but no consumption-production externalities. Again we will assume w^i is a constant.

Our F.O.C.'s are

$$\text{i}'')\quad -\sum_e \lambda_e g^e_{(Ikj)} - p_j \leqq 0\ ; \qquad q_j^{Ik}[-\sum_e \lambda_e g^e_{(Ikj)} - p_j] = 0$$

$$\text{ii}'')\quad -\sum_e \lambda_e g^e_{(Okj)} + p_j \leqq 0\ ; \qquad q_j^{Ok}[-\sum_e \lambda_e g^e_{(Okj)} + p_j] = 0$$

$$\text{iii}'')\quad U^1_{(mj)} + \sum_{i\neq 1} \gamma_i U^i_{(mj)} - p_j \leqq 0\ ; \qquad x_j^m[U^1_{(mj)} + \sum_{i\neq 1} \gamma_i U^i_{(mj)}] = 0$$

We see that this case is analogous to the one just considered except now we cannot so easily assume that $p_j > 0,\ \forall j$. The necessary conditions in this case for a producer to operate inefficiently at the P.O. are:

a) An actually used input which is also a "desirable" good ($p_j > 0$) must exert an external economy on some efficient producer.

b) An actually produced output which is also a "desirable" good ($p_j > 0$) must exert an external diseconomy on some efficient producer.

c) An actually used input or produced output which is a "free" good ($p_j = 0$) must either exert no external effects on any efficient producer or if it exerts an external economy on some efficient producer it must exert an external diseconomy on some other efficient producer.

Of course there are no interesting necessary conditions for a good to be desirable or "free". A good will be "desirable" if $U^1_{(mj)} > 0$ and $U^i_{(mj)} \geqq 0$, $\forall i \neq 1$ for some m. (Of course if $U^i(z,y) = k_i$, $\forall i \neq 1$, then 1 and i are interchangeable here.) Obviously, however, this is not a necessary condition for "desirability". We shall consider now the general case, described by the F.O.C.'s given in i)-vi). Suppose, again, that producer k operates inefficiently. Then we have:

$$\hat{i})\quad \sum_{e \neq k} \lambda_e g^e_{(Ijk)} = -p_j + U^1_{(Ikj)} + \sum_{i \neq 1} \gamma_i U^i_{(Ikj)} + \sum_{h=1}^{n} p_h \Big(\sum_i^{(Ikj)} w^i_h \Big)$$

$$\text{for } q^{Ik}_j > 0 \;;$$

$$\hat{ii})\quad \sum_{e \neq k} \lambda_e g^e_{(Okj)} = p_j + U^1_{(Okj)} + \sum_{i \neq 1} \gamma_i U^i_{(Okj)} + \sum_h p_h \Big(\sum_i^{(Okj)} w^i_h \Big)$$

$$\text{for } q^{Ok}_j > 0 \;;$$

$$\hat{iii})\quad U^1_{(mj)} + \sum_{i \neq 1} \gamma_i U^i_{(mj)} - \sum_e \lambda_e g^e_{(mj)} - p_j + \sum_h p_h \Big[\sum_i^{(mj)} w^i_{(h)} \Big] = 0$$

$$\text{for } x^m_j > 0 \,.$$

For the case where producer k exerts no external effects on any consumer, we have a situation analogous to that just treated, except that whether a good is "desirable" or "free" is much more complicated.

We see also from $\hat{i}$), that since $p_j \geqq 0$ it cannot be the case that through an actually used input, (inefficient) producer k exerts only external diseconomies on <u>efficient</u> producers and consumers (i.e.,

$U^i_{(Ikj)} < 0$), while having a "non-augmenting" effect on consumers' endowments ($^{(Ijk)}w^i_h \leq 0$) of "desirable" goods. Similarly it cannot be the case that through an actually produced output inefficient producer k exerts only external economies on <u>efficient</u> producers and consumers ($U^i_{(Okj)} \geqq 0$), while having an augmenting effect on consumers' endowments ($^{(Ojk)}w^i_h \geq 0$) of "desirable" goods.

Another way of stating this is that if good j is "desirable" and/or [(inefficient) producer k exerts only diseconomies on consumers and has (only) a "non-augmenting" effect on endowments] through actually used input q^{Ik}_j, then he must exert an external economy on some <u>efficient</u> producer through q^{Ik}_j. Similarly, if good j is desirable and/or [inefficient producer k exerts only external economies on consumers and has (only) an augmenting effect on endowments] through actually produced output q^{Ok}_j, then he must exert an external diseconomy on some <u>efficient</u> producer through q^{Ok}_j.

Of course these conditions are very similar to the original Murakami-Negishi conditions. If we let

$$G^{Ikj}(z,y;\ \gamma,\lambda,p) = \sum_{e\neq k} \lambda_e g^e_{(Ikj)} - U^1_{(Ikj)} - \sum_{i\neq 1} \gamma_i U^i_{(Ikj)}$$

$$- \sum p_h \left(\sum_i {}^{(Ikj)} w^i_h \right)$$

and define $G^{Okj}(z,y;\ \gamma,\lambda,p)$ similarly, we can define producer k as producing a <u>net aggregate external economy</u> at the P.O. $(z,y;\ \gamma,\lambda,p)$ through actually used input q^{Ik}_j if $G^{Ikj}(z,y;\ \gamma,\lambda,p) < 0$; and producer k as producing a <u>net aggregate external diseconomy</u> at the $(z,y;\ \gamma,\lambda,p)$ through actually produced output q^{Ok}_j if $G^{Okj}_j(z,y;\ \gamma,\lambda,p) > 0$.

Using these definitions we see that a necessary condition for producer k to operate inefficiently at a P.O. is that he exert a net aggregate external economy through all actually used "desirable" inputs, and a net aggregate external diseconomy through all actually produced "desirable" outputs. Unfortunately this is not a very useful necessary condition except in the case that the producer affects all participants in the same way through a given good, because otherwise the direction of the net external effect depends on the values of the multipliers.

B.3: The Second Optimality Theorem of Welfare Economics for Economies With Externalities

The Second Optimality Theorem of Welfare Economies for economies without externalities states that under certain assumptions (convexity, continuity, and non-satiation) a Pareto Optimum can be supported by an "ordinary" price system (i.e., all participants face the same prices) with a lump sum redistribution of income, i.e., that a Pareto Optimum is an equilibrium for a suitable distribution of income. As is well known, this theorem is not generally true for a model with externalities. It might be instructive at this point to show show how the theorem breaks down in the presence of externalities.

Let (z,y) be a P.O. Following the notation of Debreu, let $X_i^{x_i}(z,y) = \{x_i' \in X^i(z,y) \mid (z',y) \ (\geqq)_i (z,y)$, for ${}^i z' = {}^i z$, and $z_i' = x_i'\}$. Then by our earlier assumptions, $X_i^{x_i}(z,y)$ is convex. Similarly $\hat{Y}^e(z,y)$ is convex, $\forall e$. Assume some consumer, $i*$, is not satiated at (z,y) and let $\dot{X}_{i*}^{x_i*} = \{x_i' \in X^i(z,y) \mid (z',y) \ (>)_i \ (z,y)$, for and $z_{i*}' = x_{i*}'\}$. Then the set $\dot{G} = \dot{X}_{i*}^{x_i*} + \sum_{i \neq i*} X_i^{x_i} - \sum_e \hat{Y}^e$ is convex. The usual proof at this point observes that in the case of no externalities the point $w = \sum_{i=1}^{s} w^i \notin \dot{G}$ because (z,y) is a P.O. However this is not the case in general in the presence of externalities. This is because $\dot{G}$ contains many (inconsistent or inconsistently ranked) net aggregate consumption vectors. Thus there are points, $(\bar{z}, \bar{y})$ such that $\bar{x}^{i*} \in \dot{X}_{i*}^{x_i*}$, $\bar{x}^i \in X_i^{x_i}$, $i \neq i*$ and $y^e \in Y^e(z,y)$ and $\sum \bar{x}^i - \sum \hat{y}^e = \sum w^i(z,y)$, but, $(\bar{z}, \bar{y}) \ {}_i(<)(z,y)$ for some i and/or $y^e \notin Y^e(\bar{z},\bar{y})$ for some e, or $w^i(\bar{z},\bar{y}) \neq w^i(z,y)$ for some i, or $\bar{x}^i \notin X^i(\bar{z},\bar{y})$ for some i.

As another approach one might define a new set $H(z,y) = \{\Sigma\, \bar{x}^i - \Sigma\, \hat{\bar{y}}^e \,|\, (\bar{z},\bar{y})\ (\geqq)_i\ (z,y),\ \forall i$ and $(\bar{z},\bar{y})\ (>)_i\ (z,y)$ for some i, where $\bar{z} = (\bar{x}^1,\ldots,\bar{x}^s)$, and $(\bar{z},\bar{y}) \in (\mathop{X}_{i=1}^{s} X^i(\bar{z},\bar{y})) \times (\mathop{X}_{e=1}^{t} Y^e(\bar{z},\bar{y}))\}$. Of course in this case by the assumption that (z,y) is a P.O. it must be that $\sum_i w^i(\tilde{z},\tilde{y}) \notin H(z,y),\ \forall(\tilde{z},\tilde{y})$. What goes wrong here is that the set H(z,y) will in almost all cases be non-convex, so that we cannot apply the usual separating hyperplane argument. Even in the unlikely case that H(z,y) is convex so that a set of prices could be found -- these prices would not in general support the P.O. in the usual sense because H(z,y) restricts consumers and producers to jointly consistent allocations. Thus at the set of prices found in this manner if consumers and producers are allowed to choose at those prices over the sets $X^i(z,y)$, $Y^e(z,y)$ -- <u>not</u> the restriction of those sets implied by H(z,y) -- producers and consumers will in general not choose the P.O. point. More will be said about the non-convexity of H(z,y) in Part III of this essay.

There are some cases in which a P.O. can be supported by a price system in the usual manner for certain types of externalities. For example, the paper by Sydney G. Winter, Jr.[11] shows that in a model with only consumption externalities where everyone's preferences "agree" in the sense that if $z(\geqq)_i\bar{z}$, where ${}^iz = {}^i\bar{z}$, then $z\ (\geqq)_{i'}\ \bar{z}$, $\forall i'$, the Second Theorem holds (under the usual assumptions). The reason for this is that for these special type of preferences and by transitivity it is easily shown that $\Sigma w^i \not\in \dot{G}$. Even in this special case, however, it is not true in general that a C.E. is a P.O. The reason for this is that gifts are not permitted in the usual C.E. model, and so the income distribution will generally be incorrect. An example of this situation is the first example of Part II of this essay. It is well known that in a model with externalities, under certain assumptions a Pareto Optimum can be supported by a redistribution of income and what I shall call a quasi-price system. A quasi-price system is a price system where different participants may face different prices for the same good. The usual interpretation of a quasi-price system is a price system with a set of specific taxes and subsidies (specific to both goods and participants). Under the usual convexity and continuity assumptions and the assumption that no consumer is satiated and all producers are producing at (individually) efficient production points at the Pareto Optimum, the demonstration of the existence of a quasi-price system which supports the P.O. is almost trivial. (We will say a quasi-price system supports a P.O. if all production points are profit maximizing and all consumption points minimize the cost of preferred or indifferent consumption points, at the set of quasi-prices). The reason for this is that if (z,y) is <u>any</u> (not necessarily Pareto Optimal) <u>attainable</u> allocation where each producer's production point is individually efficient, no consumer is satiated, and the usual convexity and continuity assumptions hold, (z,y) can be supported by a quasi-price system. To see this we need only note that by our non-satiation assumption, $x^i \notin \text{int } X_i^{x_i}(z,y), \forall i$ and by our efficiency assumption, $y^e \notin \text{int } Y^e(z,y), \forall e$. Since $X_i^{x_i}(z,y)$ and $Y^e(z,y)$ are convex, $\forall i,e$ by the Minkowski Theorem for each i there exists a (non-trivial) supporting hyperplane for $X^{x_i}(z,y)$ through x_i, and for each e there exists a (non-trivial) supporting hyperplane for $Y^e(z,y)$ through y^e. If we denote the hyperplane for consumer i by p^i ($p^i \in R^n$) and the hyperplane for producer e by $\bar{p}^e$ ($\bar{p}^e \in R^{2n}$) then $P = (\overset{s}{\underset{i=1}{X}} p_i) \times (\overset{t}{\underset{e=1}{X}} \bar{p}^e)$ is a quasi-price system supporting (z,y). Of course all we have done here is find a hyperplane for each consumer which supports his (non-thick) indifference curve which contains his consumption point at that point, and a hyperplane for each producer which supports his production set at his (efficient) production point.

A somewhat less trivial result holds in the case that (z,y) is a P.O. For this result we will need the following definition:

Definition: Good h will be said to be a non-external good if no consumer's consumption of good h has any external effect on any other consumer or producer (i.e., $X^i(z,y)$ and $w^i(z,y)$ are not affected by $x_h^{i'}$, $i' \neq i$, $(\geqq)_i$ is not affected by $x_h^{i'}$) and no producer's use or production of good h has any external effect on any other producer or consumer.

Theorem VI: Let (z,y) be a P.O. and suppose

i) $X^i(z,y)$ is convex, $\forall i$.

ii) $(\geqq)_i$ is continuous on $Z \times Y$, $\forall i$.

iii) $(\geqq)_i$ is convex in x^i, $\forall i$.

iv) $Y^e(z,y)$ is convex, $\forall e$, and $y^e \notin \text{int } Y^e(z,y)$, $\forall e$.

v) x^i is not a satiation consumption for i at (z,y), $\forall i$.

Then $\exists$ a quasi-price system supporting (z,y) which has the property that if good h is a non- external good, all particpants face the same price for good h.

Proof: First a comment on v): in the usual (non-externalities) case we need only assume one consumer is not satiated. With externalities however this is not enough. What we need v) for is to guarantee that $x_i \notin \text{int } X_i^{x_i}(z,y)$, $\forall i$, and this is not necessarily true if only one consumer is non-satiated at (z,y) since it is not necessarily Pareto superior to transfer goods from a satiated consumer to an unsatiated consumer when there are externalities.

Let any non-external goods be numbered h = 1,...,M. For $x^i \in X^i(z,y)$, $x^i = (x_1^i,\ldots,x_n^i)$, let ${}^M x^i = (x_{M+1},\ldots,x_n)$ and let $x_M^i = (x_1^i,\ldots,x_M^i)$. For $q^e \in Y^e(z,y)$, $q^e = ((q_1^{Ie},\ldots,q_n^{Ie}),(q_1^{Oe},\ldots,q_n^{Oe}))$, let ${}^M q^e = ((q_{M+1}^{Ie},\ldots,q_n^{Ie}),(q_{M+1}^{Oe},\ldots,q_n^{Oe}))$ and let $q_M^e = ((q_1^{Ie},\ldots,q_M^{Ie}),(q_1^{Oe},\ldots,q_M^{Oe}))$. Let

$$\dot{Q}(z,y) = \left\{ \left(\mathop{X}_{i=1}^{s} {}^M\bar{x}^i \right) \times \left(\mathop{X}_{e=1}^{t} {}^M_* \bar{q}^e \right) \times \left(\sum_{i=1}^{s} \bar{x}_M^i - \sum_{e=1}^{t} \hat{q}_M^e \right) \right| \text{ where}$$

$$_*\bar{q}^e = (\bar{q}^{Ie}, -\bar{q}^{Oe}),\ \bar{q}^e \in Y^e(z,y),\ \forall e \quad \text{and} \quad \bar{x}^i \in \dot{X}_i^{x_i}(z,y),\ \forall i \Big\}$$

$\dot{Q}(z,y)$ is just $(\overset{s}{\underset{i=1}{X}} \dot{X}^{x_i}_i(z,y)) \times (\overset{t}{\underset{e=1}{X}} Y^e(z,y)) = \dot{W}(z,y)$ with the non-external goods aggregated. By assumptions iii) and iv) $\dot{W}(z,y)$ is convex and therefore $\dot{Q}(z,y)$ is convex also, being the Cartesian product of sums of convex sets. Clearly $(z,y) \notin \dot{W}(z,y)$, but by ii), $(z,y) \in$ closure $W(z,y)$. Furthermore, $v = (\overset{s}{\underset{i=1}{X}} {}^M x^i) \times (\overset{t}{\underset{e=1}{X}} {}^M_* \bar{q}^e) \times (\sum_{i=1}^{s} x^i_M - \sum_{e=1}^{t} \hat{q}^e_M) \notin \dot{Q}(z,y)$. To see this, since (z,y) is a P.O., we have $\Sigma x^i - \Sigma \hat{q}^e = \Sigma w^i(z,y)$. If $v \in \dot{Q}(z,y)$, then there exists an allocation $(\tilde{z},\tilde{y})$ such that ${}^M\tilde{x}^i = {}^M x^i$, $\forall i$ and ${}^M\tilde{q}^e = {}^M q^e$, $\forall e$, where $\Sigma \tilde{x}^i - \Sigma \hat{\tilde{q}}^e = \Sigma w^i(z,y)$, and $\tilde{x}^i \in \dot{X}^{x_i}_i$, $\forall i$. But $(\tilde{z},\tilde{y})$ is a technologically feasible allocation since it only differs from (z,y) in non-external goods. Thus we have

$$(\tilde{z},\tilde{y}) (\succ)_i (z,y), \ \forall i, \ \tilde{x}^i \in X^i(z,y), \ \forall i, \ \tilde{q}^e \in Y^e(z,y), \ \forall e$$

and $\Sigma \tilde{x}^i - \Sigma \hat{\tilde{q}}^e = \Sigma w^i(\tilde{z},\tilde{y})$. But this is a contradiction of (z,y) being a P.O. Therefore, $v \notin \dot{Q}(z,y)$. Since $(z,y) \in$ closure $\dot{W}(z,y)$, $v \in$ closure $\dot{Q}(z,y)$.

By the Minkowski Theorem, there is a non-trival hyperplane through v which separates v from $\dot{Q}(z,y)$. We will denote this hyperplane by

$$P = \left({}^M p^1, \ldots, {}^M p^s; \ {}^M \bar{p}^1, \ldots, {}^M \bar{p}^t; \ \pi_1, \ldots, \pi_m\right)$$

P can be chosen (sign-wise) so that v minimizes $P \cdot \tilde{v}$ for $\tilde{v} \in$ closure $\dot{Q}(z,y)$. Now we need to show that x^i minimizes $(\pi, {}^M p^i) \cdot \tilde{x}^i$ for $\tilde{x}^i \in X^{x_i}_i(z,y)$ and y^e maximizes $(\pi, {}^M \bar{p}^{Ie}; \pi, {}^M \bar{p}^{Oe}) \cdot (\tilde{q}^{Ie}, \tilde{q}^{Oe})$ for $(\tilde{q}^{Ie}, \tilde{q}^{Oe}) \in Y^e(z,y)$. (Notice that $(-\tilde{q}^{Ie}, \tilde{q}^{Oe}) = -_*\tilde{q}^{Oe}$.) Suppose for example that $\exists \ \tilde{x}^1 \in X^{x_1}(z,y)$ such that $(\pi, {}^M p^1) \cdot \tilde{x}^1 < (\pi, {}^M p^1) \cdot x^1$. But then

$\tilde{v} = ({}^{M}\tilde{x}^{1}, {}^{M}x^{2},\ldots,{}^{M}x^{s};\ {}^{M}_{*}q^{1},\ldots,{}^{M}_{*}q^{t};\ \tilde{x}^{1}_{M} + \sum_{i=1} x^{i}_{M} - \sum \hat{q}^{e}_{M}) \in \text{clos. } \dot{Q}(z,y)$ and $P\cdot\tilde{v} < P\cdot v$ which is a contradiction. Therefore the quasi-price system

$$P = \left((\pi,{}^{M}p^{1}),\ldots,(\pi,{}^{M}p^{s});\ (\pi,{}^{M}\bar{p}^{O1};\ \pi,{}^{M}\bar{p}^{O1}),\ldots,(\pi,{}^{M}\bar{p}^{It};\pi,{}^{M}\bar{p}^{Ot})\right)$$

supports (z,y) and has the stated property. Starting at an initial equilibrium with prices $p^* = (p^*_1,\ldots,p^*_n)$ we can attain any Pareto Optimum which satisfies the assumptions of Theorem VI by a specific tax (and subsidy) system where for example the tax on the first peron's consumption of good M +1 is ${}^{M+1}p^{1} - p^*_{M+1}$ and a lump sum redistribution of income. Some observations should be made at this point about the policy implications of Theorem VI. First, lump sum redistributions of income will generally be necessary to attain any Pareto Optimum. This is important because it would appear that most policy measures actually enacted to "correct" externalities situations involve only specific taxes and subsidies. Our second observation, which is related to the first, is that "partial equilibrium" policy measures designed to "correct" externalities situations will not necessarily lead to Pareto superior allocations. Also, as we mentioned earlier in this essay, since the stability properties of equilibria in models with externalities are completely unknown, even if the correct policy measure is chosen in some situation, whether or not the measure will lead to the presumed result is uncertain.

PART III

THE STRUCTURE OF THE MODEL AND THE AGGREGATE EXCESS DEMAND CORRESPONDENCE

In this section we analyze the topological properties of the set of technologically feasible actions F (defined in Part II) and define what we shall call the aggregate excess demand correspondence for an economy with externalities. F is, in a sense, the only "interesting" set of actions. This is because if a vector of actions $(z,y) \notin F$, then $x^i \notin X^i(z,y)$ for some i, or $q^e \notin Y^e(z,y)$ for some e, i.e., (z,y) is not <u>technologically</u> possible. Of course in a dynamic framework actions not contained in F might well be of interest.

A: The Structure of the Model

F is the "compactified" disaggregated consumption-production set for an economy with externalities. F of course is different from the disaggregated consumption-production set of an economy without externalities. In such an economy this set is just a simple Cartesian production of the convex individual consumption and production sets. This is not true of F, where different agents' actions are connected by the joint feasibility constraint. F also is not generally convex.

To see this, recall that F is the set of fixed points of the mapping Ψ: $\tilde{Z} \times \tilde{Y} \rightarrow \tilde{Z} \times \tilde{Y}$, where $\Psi(z,y) = (\mathop{X}_{i=1}^{s} \tilde{X}^i(z,y)) \times (\mathop{X}_{e=1}^{t} \tilde{Y}^e(z,y))$. Define the "graph" of $\tilde{X}^i(z,y)$, G^i, by $G^i = \{(z,y) \in \tilde{Z} \times \tilde{Y} | z_i = x^i \in \tilde{X}^i(z,y)\}$ and the "graph" of $\tilde{Y}^e(z,y)$, G^e by $G^e = \{(z,y) \in \tilde{Z} \times \tilde{Y} | y_e = q^e \in \tilde{Y}^e(z,y)\}$.

The "graph" is analogous to the conventionally defined graph of a function. For example if f: $\mathbb{R} \rightarrow \mathbb{R}$, the graph of f, $G_f = \{(x,y) \in \mathbb{R}^2 | y = f(x)\}$. The only difference between our "graph" and the conventionally defined graph, is that the range of the correspondence is not in general located in the last coordinate for the "graph". Recall that $F = \{(z,y) | z_i = x^i \in \tilde{X}^i(z,y),\ \forall i \text{ and } y_e = q^e \in \tilde{Y}^e(z,y),\ \forall e\}$. Therefore, F is just the intersection of the "graphs" of the $\tilde{X}^i(z,y)$'s and $\tilde{Y}^e(z,y)$'s, i.e.,

$$F = \left(\bigcap_{i=1}^{s} G^i\right) \cap \left(\bigcap_{e=1}^{t} G^e\right) .$$

To see this, let $(z,y) \in (\bigcap_{i=1}^{s} G^i) \cap (\bigcap_{e=1}^{t} G^e)$. Then, by definition of G^i and G^e, we have $z_i = x^i \in \tilde{X}^i(z,y)$, $\forall i$ and $y_e = q^e \in \tilde{Y}^e(z,y)$, $\forall e$.

Since F is non-empty, this intersection of sets is non-empty. Now G^i is the "graph" of a continuous convex-valued correspondence. Therefore G^i will be compact. However, G^i will <u>not</u> in general be convex. To see this, consider for example the function $f: [0,1] \to [0,1]$, $f(x) = x^2$. f is certainly continuous and convex valued. But the graph of f, $G_f = \{(x,y) | y = x^2\}$ is certainly not convex. In fact, only if f is linear will G_f be convex! Therefore the G^i's and G^e's will not generally be convex. But since F is the intersection of the G^i's and G^e's, F will also generally not be convex. This will also be true of the set of attainable actions, $A = \{(z,y) \in F | \Sigma x^i(z,y) - \Sigma w^i(z,y) - \Sigma \hat{q}^e = 0\}$. Thus we have seen that the underlying technologically feasible set of actions of an economy with externalities will usually not be convex. One might ask whether this underlying t.f. set could in some cases arise from a non-convex economy without externalities (e.g., increasing returns). The answer to this is no, because although the t.f. set of actions of a non-convex economy will usually be non-convex, it will still be the simple Cartesian product of sets, which will not be the case in an economy with externalities.

B: The Aggregate Excess Demand Correspondence

Next, we shall establish the existence and properties of what we shall call the aggregate excess demand correspondence for our economy. In this exercise we invoke the assumptions of Part I.

Recall the compactified demand correspondence of consumer i is denoted $\tilde{f}^i(z,y,p)$ and the compactified supply correspondence of producer e is denoted $\tilde{r}^e(z,y,p)$. Consider the correspondence $K_p: \tilde{Z} \times \tilde{Y} \to \tilde{Z} \times \tilde{Y}$, where $K_p(z,y) = (\mathop{X}_{i=1}^{s} \tilde{f}^i(z,y,p)) \times (\mathop{X}_{e=1}^{t} \tilde{r}^e(z,y,p))$. Since the assumptions of Part I are in force, $\tilde{f}^i(z,y,p)$ is U.S.C. and convex-valued, $\forall$ i and $\tilde{r}^e(z,y,p)$ is U.S.C. and convex-valued, $\forall$ e. Therefore $K_p(z,y)$ will be U.S.C. and convex-valued, $\forall\, p \in P$, and by Kakutani's Theorem will have a fixed point. ($\tilde{Z} \times \tilde{Y}$ is compact and convex). Let d(p) be the set of fixed points of K_p. Then d(p) is the set of consistent demands and supplies at p. To see this, let (z,y) be a fixed point of K_p. This means that $(z,y) \in (\mathop{X}_{i=1}^{s} \tilde{f}^i(z,y,p)) \times (\mathop{X}_{e=1}^{t} \tilde{r}^e(z,y,p))$, i.e., $z_i = x^i \in \tilde{f}^i(z,y,p)$, $\forall$ i and $y_e = q^e \in \tilde{r}^e(z,y,p)$, $\forall$ e. d(p) therefore is the set of "stable" demands and supplies at p in the sense that no participant has any desire to change his action if d(p) is the set of actions at p.

d(p) is a correspondence, d(p): $P \to \tilde{Z} \times \tilde{Y}$. Let

$$G_{\tilde{f}^i}(p) = \{(z,y) \mid z_i = x^i \in \tilde{f}^i(z,y,p)\}$$

and

$$G_{\tilde{r}^e}(p) = \{(z,y) \mid y_e = q^e \in \tilde{r}^e(z,y,p)\}$$

$G_{\tilde{f}^i}(p)$ is just the "graph" of the correspondence $\tilde{f}^i(z,y,p)$, where p is kept fixed; for example if we define a new correspondence, $k^i_{\bar{p}}(z,y) = \tilde{f}^i(z,y,\bar{p})$, then $G_{\tilde{f}^i}(p)$ is just the "graph" of $k^i_{\bar{p}}(z,y)$. To determine the topological properties of d(p) it is useful to note that

$$d(p) = (\bigcap_i G_{\tilde{f}^i}(p)) \cap (\bigcap_e G_{\tilde{r}^e}(p)) \ ,$$

and we can therefore determine the properties of d(p) by determining the properties of $G_{\tilde{f}^i}(p)$ and $G_{\tilde{r}^e}(p)$

<u>Lemma</u>: d(p) is an u.s.c. correspondence.

<u>Proof</u>: First we must show $G_{\tilde{f}^i}(p)$ and $G_{\tilde{r}^e}(p)$ are u.s.c.: Let $p^n \to p$, p^n and $p \in P$, and $(z^n,y^n) \in G_{\tilde{f}^i}(p^n)$, $\forall n$, and $(z^n,y^n) \to (z,y)$. We must show that $(z,y) \in G_{\tilde{f}^i}(p)$. $(z^n,y^n) \in G_{\tilde{f}^i}(p^n)$, $\forall n$ implies $(z^n,y^n) \in \tilde{Z} \times \tilde{Y}$, $\forall n$ and $z^n_i = (x^i)^n \in \tilde{f}^i(z^n,y^n,p^n)$. But $\tilde{f}^i(z,y,p)$ is u.s.c., and therefore $(z^n,y^n,p^n) \to (z,y,p)$, and $z^n_i = (x^i)^n \in \tilde{f}^i(z^n,y^n,p^n)$, $\forall n$ implies that $\lim (x^i)^n = z_i = x^i \in \tilde{f}^i(z,y,p)$. Therefore $G_{\tilde{f}^i}(p)$ is u.s.c., and in the same manner we can show that $G_{\tilde{r}^e}(p)$ is u.s.c. Since d(p) is the (non-empty) intersection of u.s.c. correspondences, d(p) is u.s.c. As was shown for the set F, since d(p) is the intersection of the "graphs" of correspondences, d(p) will generally <u>not</u> be convex valued.

Let

$$D(p) = \{\Sigma x^i - \Sigma w^i(z,y) - \Sigma \hat{q}^e \mid (z,y) \in d(p)$$
$$z = (x^1,\ldots,x^s);\ y = (q^1,\ldots,q^t)\}$$

Then D(p) may properly be called the Aggregate Excess Demand Correspondence for the compactified economy $\tilde{E}$, because D(p) is the set of consistent aggregate excess demands at p. D(p) is completely analogous to the aggregate excess demand correspondence defined in an economy without externalities.

Since d(p) is u.s.c. but not generally convex D(p) will also have these properties. For the usual general equilibrium model without externalities treated in the existence literature the aggregate excess demand correspondence is u.s.c. and convex. It is interesting to note that the properties of D(p) are the same as the properties of the aggregate excess demand correspondence of a non-convex economy.

Consider the following example of a 2-person, 2-good pure exchange economy.

Notation: x_1^A is consumption of good 1 by Mr. A.

x_2^B is consumption of good 2 by Mr. B, etc.

Data:

Utility function of Mr. A: $U^A(x_1^A, x_2^A; x_1^B, x_2^B) = (x_1^A + x_1^B)^{\frac{1}{4}}(x_2^A)^{\frac{3}{4}}$

Endowment of Mr. A: $w_1^A = w_2^A = 4$

Utility function of Mr. B: $U^B(x_1^B, x_2^B; x_1^A, x_2^A) = (x_1^B - x_1^{A^2} + 6x_1^A)x_2^B$

Endowment of Mr. B: $w_1^B = w_2^B = 4$

Good 1 is numeraire so $p_1 = 1$, $p_2 = p$. For points of interior maximum

$(x_1^A, x_1^B > 0)$ the demand functions for good 1 are:

i) $x_1^A = 1 + p - \frac{3}{4} x_1^B \qquad (x_1^A, x_1^B \geqq 0)$

ii) $x_1^B = 2 + 2p + \frac{x_1^{A^2} - 6x_1^A}{2} \qquad (x_1^A, x_1^B \geqq 0)$

To get the set of consistent demands at p, we solve i) and ii) simultaneously. Substituting ii) into i) gives us:

$$x_1^A = 1 + p - \frac{3}{4}\left[2 + 2p + \frac{x_1^{A^2} - 6x_1^A}{2}\right].$$

Solving for x_1^A:

$$3x_1^A - 10x_1^A + 4p + 4 = 0 \ ,$$

and solving using the quadratic formula, we have:

iii) $x_1^A = \frac{10 \pm \sqrt{100 - 12(4p + 4)}}{6}$ (for points of <u>interior</u> maximum)

Thus, for example, for p = 1, we have the set of consistent demands

$$[x_1^A = 2 \ ; \quad x_1^B = 0]$$

or

$$[x_1^A = 4/3; \quad x_1^B = 8/9]$$

Denote the <u>excess</u> demands by z_1^A, etc. (Then $z_1^A = x_1^A - 4$, $z_2^B = x_2^B - 4$; etc.). Then the excess demand functions are:

iv) $z_1^A = p - 6 - \frac{3}{4} z_1^B$

v) $z_1^B = 2p - 6 + \frac{z_1^{A^2} - 2z_1^A}{2}$

These excess demand functions are valid only for points of <u>interior</u> maximum. For example, in iii) consider p = 3. For p = 3 iii) has no real solutions, i.e., there is no set of consistent demands given demand functions i) and ii). The problem here is that i) and ii) are not valid for p = 3 because the solutions of the consumer maximum problems are not interior solutions at p = 3.

The "global" demand functions are:

$$\text{i}') \quad x_1^A = \left\{ \begin{array}{ll} 1 + p - \frac{3}{4} x_1^B , & \text{if} \quad 1 + p - \frac{3}{4} x_1^B \geqq 0 \\ 0, \text{ otherwise} & \end{array} \right\} \quad x_1^B \geqq 0$$

$$\text{ii}') \quad x_1^B = \left\{ \begin{array}{ll} 2 + 2p + \dfrac{x_1^{A^2} - 6x_1^A}{2} , & \text{if} \quad 2+2p + \dfrac{x_1^{A^2} - 6x_1^A}{2} \geqq 0 \\ 0, \text{ otherwise} & \end{array} \right\} \quad x_1^A \geqq 0$$

This gives us global versions of iv) and v):

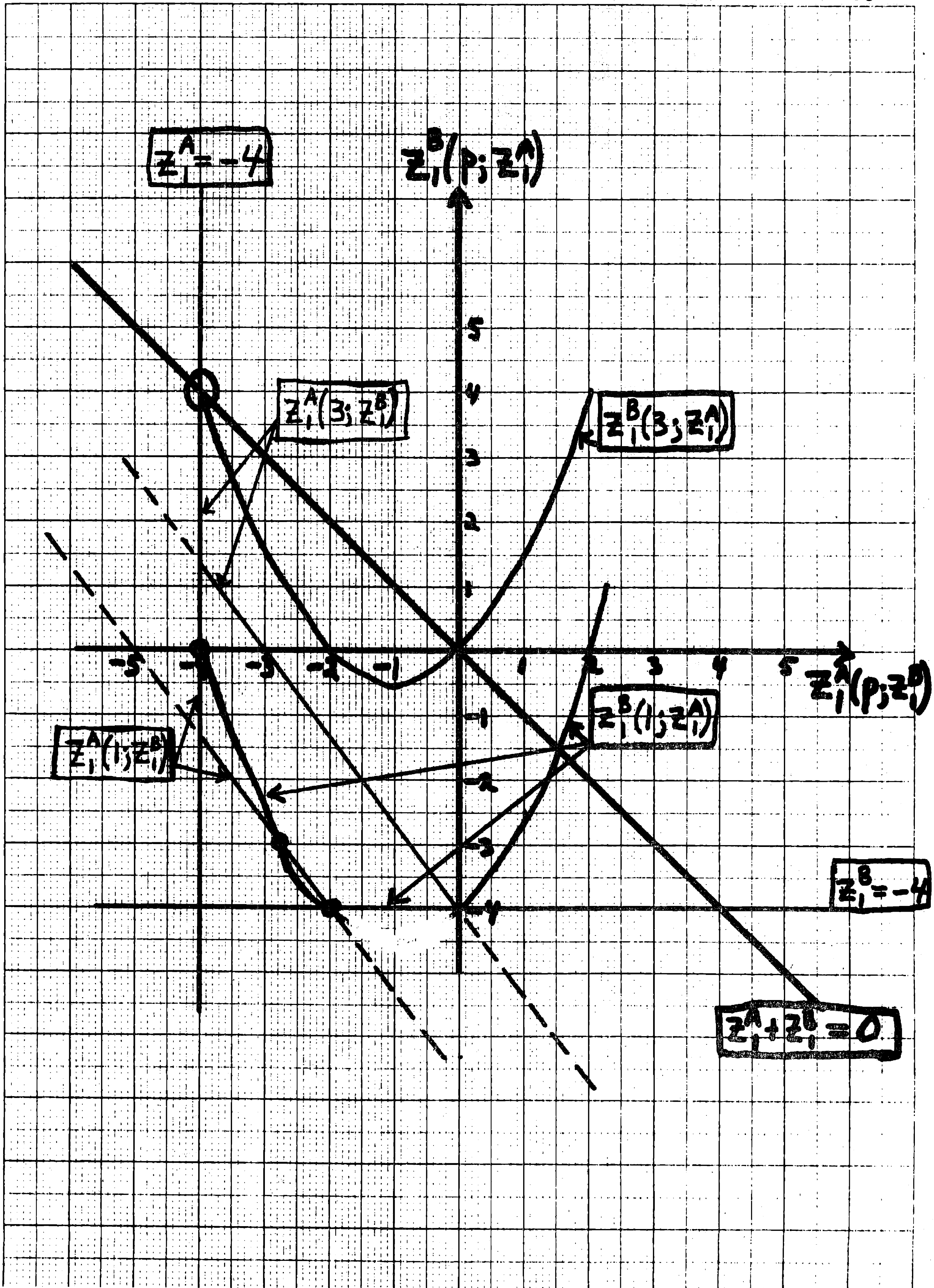
$$\text{iv}') \quad z_1^A = \left\{ \begin{array}{ll} p - 6 - \frac{3}{4} z_1^B , & \text{if} \quad p - 6 - \frac{3}{4} z_1^B \geqq -4 \\ -4, \text{ otherwise} & \end{array} \right\} \quad z_1^B \geqq -4$$

$$\text{v}') \quad z_1^B = \left\{ \begin{array}{ll} 2p - 6 + \dfrac{z_1^{A^2} + 2z_1^A}{2} , & \text{if} \quad 2p - 6 + \dfrac{z_1^{A^2} + 2z_1^A}{2} \\ -4, \text{ otherwise} & \end{array} \right\} \quad z_1^A \geqq -4$$

In the following diagram we depict excess demands of A and B in the following manner:

$z_1^B(1;\ z_1^A)$ for example is the excess demand of B for good 1 as a function of z_1 for p = 1. $z_1^B(3;\ z_1^A)$ is the excess demand of B for good 1 as a function of z_1^A for p = 3, etc., i.e., $z_1^B(3; z_1^A)$ is the graph of v′) for p = 3.

Points marked • are points of consistent excess demand for p = 1. They are the points: (-2,4), (-8/3, -28/9), (-4,0). The point marked 0 is the point of consistent excess demand for p = 3. The line $z_1^A + z_1^B = 0$ represents the locus of possible equilibria. If $[z^A(\bar{p}; z_1^B), z_1^B] \cap [z_1^A, z_1^B(\bar{p}; z_1^A)]$ contains a point on this line for some $\bar{p}$, then $\bar{p}$ is an equilibrium price. Thus, since the point marked ○ has this property, p = 3, is an equilibrium, since it is a point of consistent excess demands for good 1, and the sum of the excess demands for good 1 is zero. By Walras Law therefore the excess demand for good 2 will also be zero.



$Z_1^A = -4$
$Z_1^B(p; Z_1^A)$
5
4
3
2
1
-1
-2
-3
-4
-5
-4
-3
-2
-1
1
2
3
4
5
$Z_1^A(p; Z_1^B)$
$Z_1^A(3; Z_1^B)$
$Z_1^B(3; Z_1^A)$
$Z_1^B(1; Z_1^A)$
$Z_1^A(1; Z_1^B)$
$Z_1^B = -4$
$Z_1^A + Z_1^B = 0$

We see from the diagram that for this example the Aggregate Excess Demand Correspondence is clearly not convex-valued at p = 1.

$$\text{For } p = 1, \quad \begin{matrix} z_1^A + z_1^B = \\ z_2^A + z_2^B = \end{matrix} \begin{bmatrix} -6 \\ 6 \end{bmatrix} \text{ or } \begin{bmatrix} -52/9 \\ 52/9 \end{bmatrix} \text{ or } \begin{bmatrix} -4 \\ 4 \end{bmatrix}$$

Therefore $d(p)\Big|_{p=1} = \{(-6,6), (-52/9, 52/9), (-4, 4)\}$

For $0 < p < 13/12$, D(p) will take on three distinct values for each p. For $p = 13/12$, D(p) will take on two distinct values. For $p > 13/12$, D(p) will be single valued.

With externalities, the possibility of multiple equilibria is greater. Consider the following example.

Data:

Utility function of Mr. A: $(x_1^A + x_1^B)^a \, (x_1^A + x_2^B)^{1-a}$ where $a = 11/40$

Endowment of Mr. A: $w_1^A = 33/8, \quad w_2^A = 7/8$

Utility function of Mr. B: $(x_1^B - x_1^{A^2} + 6x_1^A)x_2^B$

Endowment of Mr. B: $w_1^B = 0, \quad w_2^B = 10$

The demand functions for good 1 for A and B are:

$$x_1^A = \begin{cases} (I^A + I^B) - x_1^B, & \text{if } \geqq 0 \text{ and } \leqq I^A \\ 0, & \text{if } < 0 \\ I^A, & \text{if } > I^A \end{cases}$$

$$x_1^B = \begin{cases} I^B/2 + \dfrac{x_1^A - 6x_1^A}{2}, & \text{if } \geqq 0 \text{ and } \leqq I^2 \\ 0, & \text{if } < 0 \\ I^B, & \text{if } > I^B \end{cases}$$

where

$$I^A = 33/8 + 7/8\ p; \quad I^B = 10\ p.$$

For p = 1, the set of consistent demands are:

I) $x_1^A = \frac{1}{2},\ x_2^A = 9/2;\ \ x_1^B = 29/8,\ x_2^B = 51/8$

II) $x_1^A = 7/2,\ x_2^A = 3/2;\ \ x_1^B = 5/8,\ x_2^B = 75/8$

Notice that allocations I) and II) are both equilibria at p = 1. Let U_I^A be the utility of Mr. A at allocation I, etc. Then we have:

$$U_I^B = (51/8)^2, \quad U_{II}^B = (75/8)^2;$$

$$U_I = U_{II} = (33/8)^a(87/8)^{1-a}\ .$$

Thus although I) and II) are both equilibria at p = 1, II) is Pareto-superior to I)! This of course could not occur in an economy without externalities.

FOOTNOTES

(Bracketed numbers refer to the listing in the Bibliography.)

1 [5]

2 [3]

3 [9]

4 [7]

5 [6], pp. 55-59.

6 [2], p. 83.

7 [6]

8 [6], pp. 94-95.

9 [14]

10 [11]

11 [17]

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ESSAY II: The Diversification Problem in Portfolio Models

Introduction

There is an old adage which states that "the prudent investor should diversify his holdings". In this essay we will attempt to analyze this adage in the context of a one period optimal asset choice model. The two main variants of this model are the mean-variance criterion model and the expected utility criterion model. The extent to which an investor with a mean-variance criterion should diversify can be quite easily analyzed, and the results are well known. However, as is well known, if the probability family is not restricted to Gaussian or if the mean-variance utility function is not consistent with maximizing the expected value of some quadratic utility function, the mean variance criterion is suspect since it leads to implausible choices. For this reason the expected utility criterion would appear to be more reasonable for a general model although of course the expected utility criterion also has some defects.

Unfortunately, the expected utility criterion model is much less robust in the type of general theorems it will allow. Most of the known diversification theorems in the expected utility model are found in the paper of Samuelson.[1] This essay presents a mathematical lemma useful for considering these sorts of problems and attempts to extend the Samuelson results.

A. The Mean-Variance Model

The mean-variance model is defined by the following assumptions.

An investor must choose from among n securities, $i = 1, \ldots, n$ in planning his portfolio for one period. The one period yields of at least some of the assets are uncertain. p_i^t is the price he must pay for one unit of the ith security at time t. p_i^{t+1} is the (possibly) unknown price of the security at time t + 1. It is assumed that the investor knows (or has a fixed estimate of) $E(p_i^{t+1}/p_i^t) = \mu_i$ and $E[(p_i^{t+1}/p_i^t - \mu_i)(p_j^{t+1}/p_j^t - \mu_j)] = \sigma_{ij}$. He has an initial wealth of W_0.

His portfolio optimization problem then becomes:

$$\max_{\{X_i\}} U[E\{\sum_i p_i^{t+1} X_i\}, \text{var } \{\Sigma\, p_i^{t+1} X_i\}]$$

subject to $\sum_i p_i^t X_i = W_0$, where X_i is the number of units bought of the ith security and U is a mean-variance utility function.

By defining new variables: $Z_i = p_i^{t+1}/p_i^t$, $\lambda_i = p_i^t X_i/W^0$ the problem can now be restated:

$$\max_{\{\lambda_i\}} U[E\{(\Sigma\lambda_i Z_i)W^0\}, \text{var } \{(\Sigma\lambda_i Z_i)W^0\}]$$

subject to $\Sigma\, \lambda_i = 1$.

Let $U_1 = \partial U/\partial[E\{(\Sigma\lambda_i Z_i)W^0\}]$ and $U_2 = \partial U/\partial[\text{var}\{(\Sigma\lambda_i Z_i)W^0\}]$. The usual assumptions about U require $U_1 > 0$, $U_2 < 0$. (If $U_2 > 0$ we call the investor a risk lover.) Of course, $U_2 < 0$ is consistent with the notion of a "prudent" investor.

Notice that if $\mu_i = \mu_j, \forall i, j$ and $U_2 < 0$ then the previous problem becomes:

$$\min_{\{\lambda_i\}} \text{var } [(\Sigma\lambda_i Z_i)W_0].$$

In the remaining analysis we will assume, without loss of generality, that $W_0 = 1$. We will also assume that no linear combination of the Z_i's has zero variance.

<u>Lemma 1</u>: Suppose $\sigma_{ij} \leqq 0, \forall i \neq j$. Then the problem:

$\min_{\{\lambda_i\}} \text{var } (\Sigma\lambda_i Z_i)$ subject to $\Sigma\lambda_i = 1$ has a solution $(\lambda_1, \ldots, \lambda_n)$ with the property $\lambda_i > 0, \forall i$.

<u>Proof</u>:

$$\text{var } (\Sigma\lambda_i Z_i) = E[\Sigma\lambda_i Z_i - \Sigma\lambda_i\mu_i]^2 = E[\Sigma\lambda_i(Z_i - \mu_i)] = \sum_i \sum_j \lambda_i\lambda_j\sigma_{ij}.$$

Suppose the solution $(\lambda_1, \ldots, \lambda_n)$ has the property that $\lambda_1, \ldots, \lambda_k < 0$; $\lambda_{k+1}, \ldots, \lambda_n \geqq 0$. Then

1) $$\sum_i \sum_j \lambda_i\lambda_j\sigma_{ij} = \sum_i \left(\sum_{j=1}^{k} \lambda_i\lambda_j\sigma_{ij} + \sum_{j=k+1}^{n} \lambda_i\lambda_j\sigma_{ij}\right)$$

$$= \sum_1^k \sum_1^k \lambda_i\lambda_j\sigma_{ij} + 2 \sum_1^k \sum_{k+1}^n \lambda_i\lambda_j\sigma_{ij} + \sum_{k+1}^n \sum_{k+1}^n \lambda_i\lambda_j\sigma_{ij}.$$

2) $$\sum_1^k \sum_1^k \lambda_i\lambda_j\sigma_{ij} = \text{var } \left(\sum_1^k \lambda_i Z_i\right) > 0.$$

3) Also, $\sum_1^k \sum_{k+1}^n \lambda_i\lambda_j\sigma_{ij} \geqq 0$ since $\sigma_{ij} \leqq 0$ and $\lambda_1, \ldots, \lambda_k < 0$,

$\lambda_{k+1}, \ldots, \lambda_n \geqq 0$.

Now consider $(\lambda_1^*, \ldots, \lambda_n^*) = \left[0, \ldots, 0, \frac{\lambda_{k+1}}{\sum_{k+1}^{n} \lambda_i}, \frac{\lambda_{k+2}}{\sum_{k+1}^{n} \lambda_i}, \ldots, \frac{\lambda_n}{\sum_{k+1}^{n} \lambda_i}\right]$

Since $\sum_{i=1}^{n} \lambda_i = 1$, $\sum_{k+1}^{n} \lambda_i > 1$. Let $\sum_{k+1}^{n} \lambda_i = b$

$$\text{var}\,(\sum_i \lambda_i^* Z_i) = \sum_{k+1}^{n} \sum_{k+1}^{n} (\lambda_i/b)(\lambda_j/b)\,\sigma_{ij}.$$

By 2) and 3) and since $b > 1$,

$\text{var}\,(\sum_i \lambda_i^* Z_i) < \text{var}\,(\sum_i \lambda_i Z_i)$. But this is a contradiction of the assumption that $(\lambda_1, \ldots, \lambda_n)$ is a solution. Therefore $\lambda_i \geqq 0, \forall i$.

Now suppose $\lambda_1 = 0$.

Consider $(\lambda_1', \ldots, \lambda_n') = (\varepsilon, (1 - \varepsilon)\, \lambda_2, \ldots, (1 - \varepsilon)\, \lambda_n)$, for $0 < \varepsilon < 1$.

Then $\sum \lambda_i' = 1$

$$\text{var}\,[\sum \lambda_i' Z_i] = \sum \sum \lambda_i' \lambda_j' \sigma_{ij} = \varepsilon^2 \sigma_{11} + 2 \sum_{2}^{n} \varepsilon(1 - \varepsilon)\, \lambda_j \sigma_{1j}$$
$$+ \sum_{2}^{n} \sum_{2}^{n} (1 - \varepsilon)^2\, \lambda_i \lambda_j \sigma_{ij}.$$

Therefore $d/d\varepsilon\, [\text{var}\,[\sum \lambda_i' Z_i]]\Big|_{\varepsilon=0} = 2 \sum_{2}^{n} \lambda_j \sigma_{1j} - 2 \sum_{2}^{n} \sum_{2}^{n} \lambda_i \lambda_j \sigma_{ij}$.

$\sum_{2}^{n} \lambda_j \sigma_{1j} \leqq 0$ since $\lambda_j \geqq 0$ and $\sigma_{1j} \leqq 0$.

$\sum_{2}^{n} \sum_{2}^{n} \lambda_i \lambda_j \sigma_{ij} = \text{var}\,(\sum_{2} \lambda_i Z_i) > 0$.

Therefore $d/d\varepsilon\, [\text{var}\,[\sum \lambda_i' Z_i]]\Big|_{\varepsilon=0} < 0$.

Since var $[\Sigma\lambda_i'Z_i]\Big|_{\varepsilon=0}$ = var $[\Sigma\lambda_i Z_i]$, this means that the variance can be decreased by having $\lambda_1 > 0$. Therefore $\lambda_i > 0, \forall i$.

We now have the following diversification theorem for the mean-variance model.

<u>Theorem 1</u>: If $U_2 < 0$, $\mu_i = \mu_j, \forall i, j$, $\sigma_{ij} \leq 0$, $i \neq j$ and no linear combination of the Z_i's has zero variance then the solution of the problem $\max_{\{\lambda_i\}} U[E\{\Sigma\lambda_i Z_i\}, \text{var}\{\Sigma\lambda_i Z_i\}]$ subject to $\Sigma\lambda_i = 1$ has the property that $\lambda_i > 0, \forall i$.

Of course, the practical usefulness of this theorem is very limited. Unfortunately most securities' yields are positively correlated. Empirical studies (e.g., Evans and Archer[2]), however, indicate that at least for large portfolio values, significant reduction of variance is achieved by considerable diversification.

As we mentioned in the introduction to this essay, the mean-variance criterion can lead to implausible choices. Consider the following example.

Let E = expected value of any prospect

V = variance of any prospect.

Consider the utility function

$U(E,V) = E/1+V$

This utility function has the properties $U_1 > 0$, $U_2 < 0$.

Now consider the following two prospects:

P1: 10 with probability .99

101 with probability .01

P2: 10 with probability .98

101 with probability .02

$E_1 = 11$, $V_1 = 81.99$; $E_2 = 12$, $V_2 = 162.34$.

For this utility function, P1 is preferred to P2, but this ranking is completely implausible.

The only family of mean-variance utility functions which will not lead to such implausibilities are of the form

$U(E, V) = aE + b(V + E^2)$.

The utility functions in this family are consistent with maximizing the expected value of a quadratic utility function. Mean-variance utility functions not in this family lead to implausible decisions because variance is not a good measure of risk.

In the previous example it is clear that although $V_2 > V_1$, P2 is less risky in some sense than P1. These implausibilities lead us to the expected utility criterion model.

B. The Expected Utility Model

The expected utility model choice problem can be written:

$$\max_{\{x_i\}} E\{U(\sum_i p_i^{t+1} x_i)\} \text{ subject to } \sum_i p_i^t x_i = W_0.$$

Again, by defining $Z_i = p_i^{t+1}/p_i^t$, $\lambda_i = \frac{p_i^t x_i}{W_0}$, this can be rewritten:

$$\max_{\{\lambda_i\}} E\{U((\Sigma\lambda_i Z_i)W_0)\} \text{ subject to } \Sigma\ \lambda_i = 1.$$

The usual assumptions about U are $U' > 0$, $U'' < 0$. (If $U'' > 0$, we call the investor a risk lover.) $U'' < 0$ is consistent with the notion of a "prudent" investor. In the expected utility model, Theorem 1 is no longer valid as the following example shows.

There are three states of nature possible, A, B and C, and each has a probability of 1/3 of occurrence. There are two assets, the random yields of which are denoted Z_1 and Z_2. The value of the asset yields in each of the states is given in the following table.

	A	B	C
Z_1	.5	1	1.5
Z_2	0	3	0

$$E\{Z_1\} = 1/3 \cdot .5 + 1/3 \cdot 1 + 1/3 \cdot 1.5 = 1$$

$$E\{Z_2\} = 1/3 \cdot 0 + 1/3 \cdot 3 + 1/3 \cdot 0 = 1$$

$$\sigma_{12} = 1/3(-1/2)(-1) + 0 + 1/3(1/2)(-1) = 0$$

Thus the asset yields have equal means and zero covariance. For a mean-variance criterion utility function, by Theorem 1 a positive amount of each would be held. Assume $W_0 = 1$. Let U be some utility function with $U' > 0$, $U'' < 0$.

$$E\{U(\lambda Z_1 + (1 - \lambda)Z_2)\} = 1/3\ U(.5\lambda) + 1/3\ U(\lambda + 3(1 - \lambda)) + 1/3\ U(1.5\lambda)$$

$$\frac{dE\{U\}}{d\lambda} = 1/6\ U'(.5\lambda) - 2/3\ U'(-2\lambda + 3) + 1/2\ U'(1.5\lambda) = F(\lambda).$$

The first order conditions for maximizing E{U}require that $F(\lambda) = 0$, for λ^* the maximizer of E{U}.

$$F(0) = 1/6\ U'(0) - 2/3\ U'(3) + 1/2\ U'(0) = 2/3\ U'(0) - 2/3\ U'(3)$$
$$F(1) = 1/6\ U'(.5) - 2/3\ U'(1) + 1/2\ U'(1.5)$$

Since $U'' < 0$, we must have $F(0) > 0$, and $F'(\lambda) < 0, \forall\lambda$. Therefore $\lambda^* > 0$. Furthermore it is clearly possible to find a concave U such that $F(1) > 0$ (e.g., for $U(x)=\log x$, $F(1) = 0$). For U such that $F(1) > 0$ the solution requires $\lambda^* > 1$ -- i.e. -- the investor "shorts" the second asset. Of course given the nature of the assets this is an entirely plausible solution. Thus Theorem 1 is not valid in the expected utility model.

It is not even true that if one prospect has a bigger mean and smaller variance than another prospect that the first is necessarily preferred to the second. Consider the following example.

$$P_1 = \begin{cases} 0, \text{ with probability } .8 \\ 100, \text{ with probability } .2 \end{cases}$$

$$P_2 = \begin{cases} 10, \text{ with probability } .998 \\ 1010, \text{ with probability } .002 \end{cases}$$

$E\{P_1\} = 20 > E\{P_2\} = 12$ and var $P_1 = 1600 <$ var P_2.

Consider the utility function $U(x) = P^{\frac{1}{2}}$.

$E\{U(P_1)\} = .8 \cdot 0 + .2 \cdot 10 = 2$

$E\{U(P_2)\{ = .998 \cdot \sqrt{10} + .002\sqrt{1010} > 2$

Thus P_2 is preferred to P_1 for this utility function.

As we have seen in the earlier example, in our search for a theorem analogous to Theorem 1 in the expected utility model, non-positive (Pearsonian) correlation between the asset yields will not be a strong enough assumption. Samuelson[3] showed that if the yields were distributed independently with equal means then a risk-averting expected utility maximizer would hold a positive amount of each asset. He then argued that since going from zero to negative correlation in the mean-variance model makes everything even better, going from independence to some stronger type of negative correlation would have the same result in the expected utility model. He proposed a stronger form of negative correlation which we will discuss next.

Let Z_1 and Z_2 be random variables with joint density function denoted $dP(z_1, z_2)$ and joint cumulative distribution function denoted $P(z_1, z_2)$.

Let $P(z_1|z_2) = \text{Prob}\{Z_1 \leq z_1|Z_2 = z_2\}$.

<u>Definition</u>: Z_1 will be said to be negatively S-correlated with Z_2 if $\partial P(z_1|z_2)/\partial z_2 \geq 0, \forall(z_1, z_2)$. Z_1 will be said to be positively S-correlated with Z_1 if $\partial P(z_1|z_2)/\partial z_1 \leq 0, \forall(z_1, z_2)$.

<u>Note</u>: The differentiability here is not necessary, -- the definition applies also for finite differences between z_2 points <u>with positive density</u>.

For example, consider the following density functions for the random variables Z_1 and Z_2:

i) There are three states of nature, A, B and C, each with a probability of occurrence of 1/3.

	A	B	C
Z_1	1	3	5
Z_2	7	4	0

$P(z_1|z_2)$: $P(1|0) = 0,\ P(1|4) = 0,\ P(1|7) = 1$

$P(3|0) = 0,\ P(3|4) = 1,\ P(3|7) = 1$

$P(5|0) = P(5|4) = P(5|7) = 1$

Therefore Z_1 is negatively S-correlated with Z_2.

ii) $dP(z_1, z_2) = z_1 + z_2,\ 0 \leq z_1 \leq 1$.

$$P(z_1|z_2) = \int_0^{z_1} dP(z_1|z_2) = [\int_0^{z_1} (Z_1 + z_2)dZ_1] \div [\int_0^{1} (Z_1 + z_2)dZ_1]$$

$$= (z_1^2/2 + z_1z_2)/(1/2 + z_2).$$

$$\frac{\partial P(z_1|z_2)}{\partial Z_2} = \frac{(1/2 + z_2) \cdot z_1 - (z_1^2/2 + z_1 z_2)}{(1/2 + z_2)^2} = \frac{1/2 - z_1^2/2}{(1/2 + z_2)^2}$$

which is ≥ 0 because $0 \leq z_1 \leq 1$.

Therefore Z_1 is negatively S-correlated with Z_2.

iii) $dP(z_1, z_2) = z_1 + 1/z_2$, $0 \leq z_1 \leq 1$, $1 \leq z_2 \leq b$

where b is such that $(b - 1)/2 + \log b = 1$.

$$P(z_1|z_2) = [\,{}_0\int^{z_1} (Z_1 + 1/z_2)dZ_1] \div [\,{}_0\int^{1} (Z_1 + 1/z_2)dZ_1]$$

$$= (z_1^2/2 + z_1/z_2)/(1/2 + 1/z_2).$$

$$\frac{\partial P(z_1|z_2)}{\partial Z_2} = \frac{(1/2 + 1/z_2) \cdot (-z_1/z_2^2) - (z_1^2/2 + z_1/z_2)(-1/z_2^2)}{(1/2 + 1/z_2)^2}$$

$$= \frac{1/2(-z_1 + z_1^2)}{z_2^2(1/2 + 1/z_2)^2}$$

which is ≤ 0 because $0 \leq z_1 \leq 1$.

Therefore Z_1 is positively S-correlated with Z_2.

iv) If Z_1 and Z_2 are distributed as bivariate normal then Z_1 is negatively S-correlated with z_2 if $\rho \leq 0$ and Z_1 is positively S-correlated with Z_2 if $\rho \geq 0$. To see this, recall for the bivariate normal

$$P(z_1|z_2) = {}_{-\infty}\int^{z_1} k \exp [\frac{-(Z_1 - b)^2}{2\sigma_1^2(1 - \rho^2)}]dZ_1$$

where $b = \mu_1 + \rho(\sigma_1/\sigma_2)[z_2 - \mu_2]$.

$$\text{Therefore } \partial P(z_1|z_2)/\partial z_2 = \frac{2k(\rho\sigma_1/\sigma_2)}{2\sigma_1^2(1 - \rho^2)} \int_{-\infty}^{z_1} (Z_1 - b) \exp [\ \ ] dZ_1$$

and the sign of this expression = - sign ρ because

$\int_{-\infty}^{z_1} (z_1 - b) \exp [\ \ ] dZ_1 < 0$ for all $z_1 < \infty$ because

$E\{Z_1|z_2\} = b$.

The definition of S-correlation is not a symmetric one -- i.e. -- z_1 may be negatively S-correlated with z_2 but z_2 may not be negatively S-correlated with z_1. (As we shall see in Lemma 5 it is not possible for z_1 to be strictly negatively S-correlated with z_2, while z_2 is positively S-correlated with z_1.) The following example shows that S-correlation is not a symmetric property.

There are two states of nature, A and B, with prob {A} = 9/10, prob {B} = 1/10. The density function for random variables Z_1 and Z_2 is:

State A

Z_1 = 5, with probability = 1

$Z_2 = \begin{cases} 3, \text{ with probability } = 1/2 \\ 4, \text{ with probability } = 1/2 \end{cases}$

State B

Z_1 = 10, with probability = 1

$Z_2 = \begin{cases} 1, \text{ with probability } = 1/2 \\ 3, \text{ with probability } = 1/6 \\ 4, \text{ with probability } = 1/3 \end{cases}$

From this density function we get the cumulative conditional distribution functions:

$P(Z_2 \leq 1|Z_1 = 5) = 0$

$P(Z_2 \leq 1|Z_1 = 10) = 1/2$

$P(Z_2 \leq 3|Z_1 = 5) = 1/2$

$P(Z_2 \leq 3|Z_1 = 10) = 2/3$

$P(Z_2 \leq 4| \quad) = 1$

$P(Z_1 \leq 5|z_2 = 1) = 0$

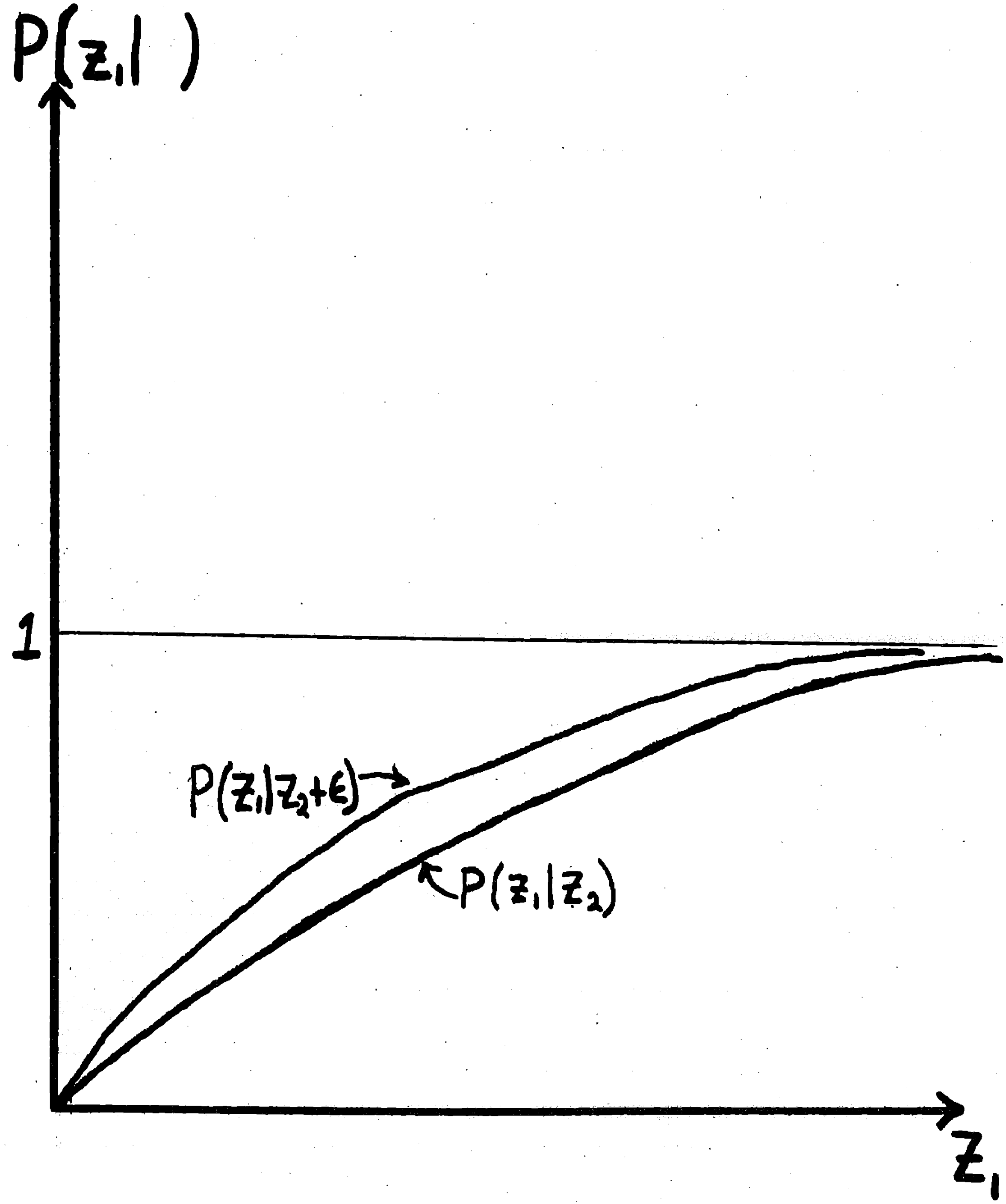
$P(Z_1 \leq 5|z_2 = 3) = 27/28$

$P(Z_1 \leq 5|z_2 = 4) = 27/29$

We see from the definition (using first differences between Z_1 points of positive density) that Z_2 is negatively S-correlated with Z_1, but since $P(Z_1 \leq 5|Z_2 = 4) - P(Z_1 \leq 5|Z_2 = 3) < 0$, Z_1 is not negatively S-correlated with Z_2.

Z_2 negatively S-correlated with Z_1 means that for $\varepsilon > 0$, $P(Z_2 \leq a|Z_1 = b + \varepsilon) - P(Z_2 \leq a|Z_1 = b) > 0$ (for b, ε such that b and $b + \varepsilon$ are points of positive density for Z_2). This means that an increase in the given value of Z_2 shifts the conditional cumulative distribution $P(z_1|z_2)$ to the left.

Thus negative S-correlation is plausible as a stronger version of negative correlation. The shifting of the conditional cumulative distribution function is depicted in the following diagram.


P(z₁|)
1
P(z₁|z₂+ε)
P(z₁|z₂)
Z₁

To determine the properties of negative S-correlation we need the following mathematical results.

Lemma B: Let Ψ, Φ, and v be Riemann-Stieltjes integrable with respect to P, where dP(Y) is a probability density function of a scalar random variable, Y, and v is a non-increasing (non-decreasing) function of Y on $[0, \infty)$ which has the property that $v(Y) > 0$ for $Y > 0$. Suppose:

a) $\exists\ \bar{Y} \in (0, \infty)$ such that $\Psi(Y) \leqq \Phi(Y)$ for all $Y \leqq \bar{Y}$ and $\Phi(Y) \leqq \Psi(Y)$ for all $Y > \bar{Y}$, and

b) $\int_0^\infty \Psi(Y)v(Y)dP(Y) = \int_0^\infty \Phi(Y)\Psi(Y)dP(Y).$

Then $\int_0^\infty \Psi(Y)dP(Y) \geqq (\leqq) \int_0^\infty \Phi(Y)dP(Y).$

Proof: Suppose v is non-increasing.

1) $\int_0^{\bar{Y}} [\Psi(Y) - \Phi(Y)]v(Y)dP(Y) \leqq 0$

and $\int_{\bar{Y}}^\infty [\Psi(Y) - \Phi(Y)]v(Y)dP(Y) \geqq 0$, because $v(Y) \geqq 0$.

2) $-\int_0^{\bar{Y}} [\Psi(Y) - \Phi(Y)]v(Y)dP(Y) = \int_{\bar{Y}}^\infty [\Psi(Y) - \Phi(Y)]v(Y)dP(Y),$

by b).

3) Let $\bar{v} = v(\bar{Y}) > 0$.

Then $v(Y) \geqq \bar{v}$, for $Y \leqq \bar{Y}$

$v(Y) \leqq \bar{v}$, for $Y \geqq \bar{Y}$,

since v is non-increasing.

4) Therefore, $-\int_0^{\bar{Y}} [\Psi(Y) - \Phi(Y)]\bar{v}dP(Y) \leqq \int_{\bar{Y}}^\infty [\Psi(Y) - \Phi(Y)]\bar{v}dP(Y)$

from 2) and 3).

Therefore $\int_0^\infty \Psi(Y)dP(Y) \geqq \int_0^\infty \Phi(Y)dP(Y)$.

The proof for v non-decreasing is analogous.

Corollary: Let Ψ, Φ, and dP be as in Lemma B and suppose $P(Y) > 0$ for some $Y < \bar{Y}$ and $P(Y') < 1$ for some $Y' > \bar{Y}$.

Suppose $v(Y)$ is strictly monotone-decreasing (monotone-increasing) and non-negative on $(0, \infty)$. Suppose:

a') there exists $\bar{Y} \varepsilon (0, \infty)$ such that $\Psi(Y) < \Phi(Y)$ for all $Y \varepsilon (0, \bar{Y})$ and $\Phi(Y) < \Psi(Y)$ for $Y \varepsilon (\bar{Y}, \infty)$, $\Psi(\bar{Y}) \leqq \Phi(\bar{Y})$ and

b') $\int_0^\infty \Psi(Y)v(Y)dP(Y) \geqq (\leqq) \int_0^\infty \Phi(Y)v(Y)dP(Y)$.

Then $\int_0^\infty \Psi(Y)dP(Y) > (<) \int_0^\infty \Phi(Y)dP(Y)$.

Proof: Suppose v is strictly monotone-decreasing.

1) $\int_0^{\bar{Y}} [\Psi(Y) - \Phi(Y)]v(Y)dP(Y) < 0$ and

$\int_{\bar{Y}}^\infty [\Psi(Y) - \Phi(Y)]v(Y)dP(Y) > 0$ by the property of P and $v \geqq 0$.

2) $-\int_0^{\bar{Y}} [\Psi(Y) - \Phi(Y)]v(Y)dP(Y) = \int_{\bar{Y}}^\infty [\Psi(Y) - \Phi(Y)]v(Y)dP(Y)$

by b').

3) Let $\bar{v} = v(\bar{Y}) > 0$.

Then $v(Y) > \bar{v}$, $Y < \bar{Y}$ and $v(Y) > \bar{v}$, $Y > \bar{Y}$.

4) Then $-\int_0^{\bar{Y}} [\Psi(Y) - \Phi(Y)]\bar{v}dP(Y) < \int_{\bar{Y}}^{\infty} [\Psi(Y) - \Phi(Y)]\bar{v}dP(Y)$.

5) Therefore, $\int_0^{\infty} \Psi(Y)dP(Y) > \int_0^{\infty} \Phi(Y)dP(Y)$.

The following diagram depicts Ψ, Φ, and v which have the stated property. By the Corollary, in diagrams a) and b) if

$\int \Psi(Y)v(Y)dP(Y) \geqq \int \Phi(Y)v(Y)dP(Y)$ then $\int \Psi(Y)dP(Y) > \int \Phi(Y)dP(Y)$ and in diagrams c) and d) if

$\int \Phi(Y)v(Y)dP(Y) \geqq \int \Psi(Y)v(Y)dP(Y)$ then $\int \Phi(Y)dP(Y) > \int \Psi(Y)dP(Y)$.

The following result is from the paper by Hanoch and Levy.[4]

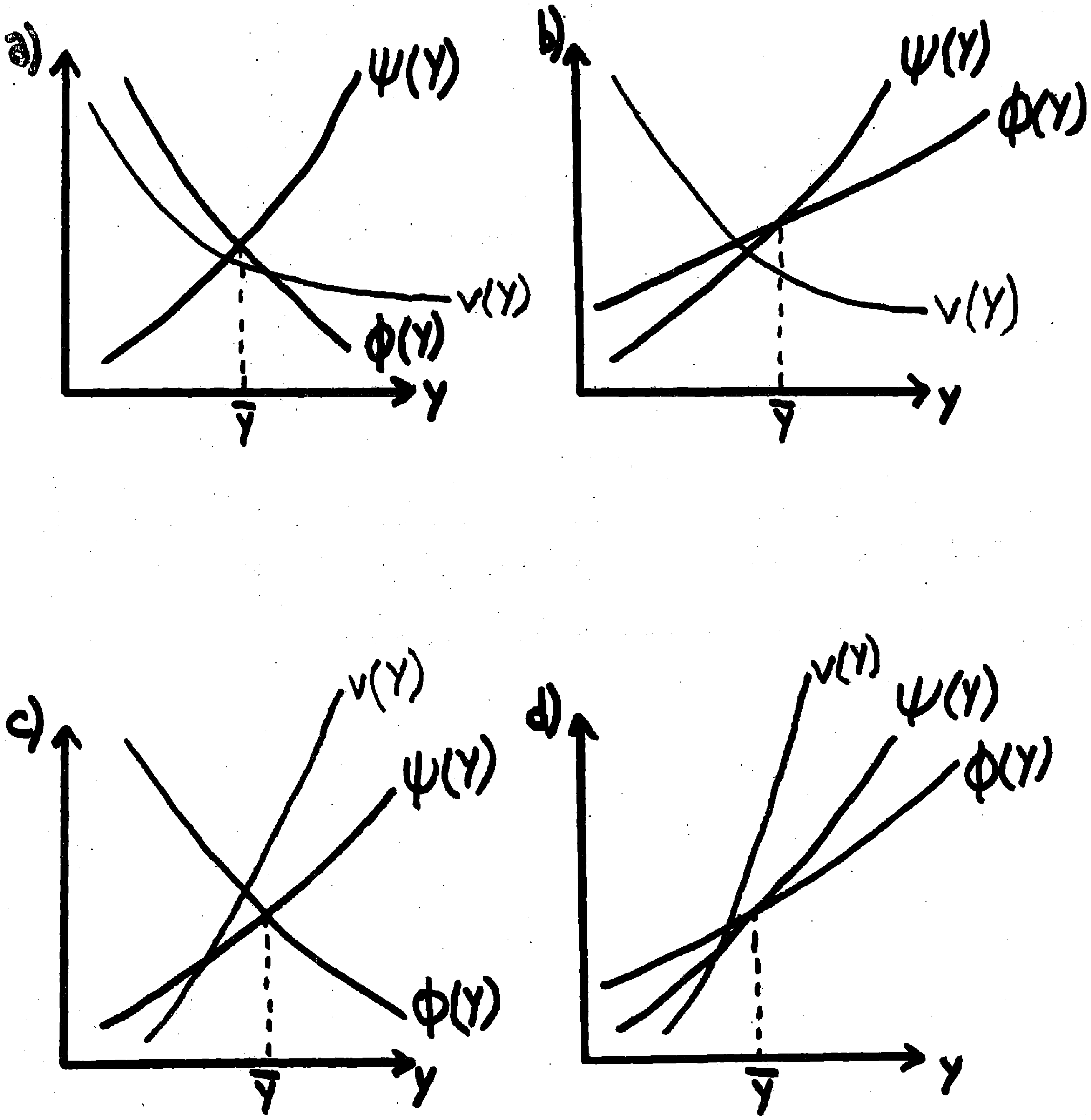
Lemma 3: Let F(x) and G(x) be cumulative distribution functions and let H(x) be a function which is Riemann-Stieltjes integrable with respect to F and G. Let $E_F\{f\} = \int f(x)dF(x)$, etc.

If $F(x) \leqq G(x)$, for all x and (<) holds for some x_0 then:

$E_F\{H(x)\} > (<) E_G\{H(x)\}$ if $H' > 0$ ($H' < 0$).

(Again -- the differentiability is not necessary here. The basic strategy of the proof is to integrate $H(x)[dF(x) - dG(x)]$ by parts.)

* For the rest of this essay we will assume $Z_i \geqq 0$, which, of course, is a reasonable assumption since $Z_i = p_i^{t+1}/p_i^t$.


a)
ψ(Y)
v(Y)
φ(Y)
Y
Ȳ
b)
ψ(Y)
φ(Y)
v(Y)
Y
Ȳ
c)
v(Y)
ψ(Y)
φ(Y)
Y
Ȳ
d)
v(Y)
ψ(Y)
φ(Y)
Y
Ȳ

We can now prove the following results.

Let $E\{Z_1|z_2\} = \int z_1 dP(z_1|z_2)$ -- i.e., the conditional expectation of Z_1 given $Z_2 = z_2$.

Lemma 4: If $\partial P(z_1|z_2)/\partial z_2 \geqq 0, \forall (z_1, z_2)$, then $\partial E\{Z_1|z_2\}/\partial z_2 \leqq 0$. (The differentiability is not necessary here.)

Proof: $E\{Z_1|z_2\} = \int z_1 dP(z_1|z_2)$.

Let $\varepsilon > 0$.

$E\{Z_1|z_2 + \varepsilon\} - E\{Z_1|z_2\} = \int z_1 dP(z_1|z_2 + \varepsilon) - \int z_1 dP(z_1|z_2)$.

By the assumptions of the Lemma,

$P(z_1|z_2) \leqq P(z_1|z_2 + \varepsilon)$ for $\varepsilon > 0$.

Furthermore, the function $U(z_1) = z_1$ is monotone-increasing.

Therefore by Lemma 3

$\int z_1 dP(z_1|z_2) \geqq \int z_1 dP(z_1|z_2 + \varepsilon)$,

so that $E\{Z_1|z_2 + \varepsilon\} - E\{Z_1|z_2\} \leqq 0$ for $\varepsilon > 0$

and the result is proved. (Notice if $\partial P(z_1|z_2)/\partial z_2 < 0$, then $\partial E\{z_1|z_2\}/\partial z_2 > 0$.)

Lemma 5: If Z_1 is negatively S-correlated with Z_2 (or vice versa) then $\sigma_{12} \leqq 0$.

Proof: $\sigma_{12} = \iint z_1 z_2 dP(z_1, z_2) - E\{Z_1\}E\{Z_2\}$.

$\iint z_1 z_2 dP(z_1, z_2) = \int z_1 E\{Z_2|z_1\} dP(z_1)$

where $dP(z_1)$ is the marginal density function of Z_1

$(dP(z_1) = \int_{(z_2)} dP(z_1, z_2))$.

Z_1 negatively S-correlated with Z_2 means

$\partial P(z_1|z_2)/\partial z_2 \geqq 0$.

Therefore, by Lemma 4, $\partial E\{Z_2|z_1\}/\partial z_1 \leqq 0$.

Now, using the notation of Lemma B, let $v(z_1) = z_1$, $\Psi(z_1) = E\{Z_2\}$,

$\Phi(z_1) = E\{Z_2|z_1\}$, and dP be $dP(z_1)$.

Then v is strictly increasing, Φ is non-increasing and Ψ is a constant.

$$\int E\{Z_2|z_1\}dP(z_1) = E\{Z_2\} = \int E\{Z_2\}dP(z_1).$$

Therefore by the contrapositive of the Corollary to Lemma B we must have

$$\int z_1 E\{Z_2|z_1\}dP(z_1) \leqq \int z_1 E\{Z_2\}dP(z_1) = E\{Z_1\}E\{Z_2\}.$$

Therefore $\iint z_1 z_2 dP(z_1, z_2) - E\{Z_1\}E\{Z_2\} \leqq 0$,

and so $\sigma_{12} \leqq 0$. (Notice that if $\partial P(z_1|z_2)/\partial z_2 > 0$, then $\sigma_{12} < 0$.)

Consider the following example:

There are three states of nature, A, B, and C, each with probability = 1/3.

The values of random variables Z_1 and Z_2 in each state are given in the following table:

	A	B	C
Z_1	1	5	3
Z_2	1	2	3

$\sigma_{12} = -1/3$

$P(z_1|z_2)$: $P(1|1) = 1$

$P(1|2) = 0$

$P(1|3) = 0$

Therefore Z_1 is not negatively S-correlated with Z_2, and so negative S-correlation is a stronger condition than negative (Pearsonian) correlation.

We state without proof the following obvious result:

Lemma 6: If $\partial P(z_1|z_2)/\partial z_2 = 0, \forall(z_1, z_2)$ then Z_1 and Z_2 are independent random variables.

We now have the apparatus necessary to derive some diversification theorems.

Theorem 2: If $U' > 0$, $U'' < 0$, $E\{Z_1\} = E\{Z_2\}$, and if Z_2 is negatively S-correlated with Z_1, or Z_1 and Z_2 are independent, then the solution of the problem

$$\max_{\{\lambda\}} E\{U(\lambda Z_1 + (1 - \lambda)Z_2)\}$$

has the property that $0 < \lambda < 1$.

Proof: Suppose $\lambda \geqq 1$.

The First Order Conditions (F.O.C.) for the problem require:

1) $\iint z_1 U'(\lambda z_1 + (1 - \lambda)z_2)dP(z_1, z_2) = \iint z_2 U'(\lambda z_1 + (1 - \lambda)z_2)dP(z_1, z_2)$

Iterating integrals we have:

2) $$\iint z_2 U'(\lambda z_1 + (1 - \lambda)z_2)dP(z_1, z_2) = \int \left[\int z_2 U'(\lambda z_1 + (1 - \lambda)z_2)dP(z_2|z_1) \right] dP(z_1)$$

where $dP(z_1)$ is the marginal density of Z_1.

Now using the notation of Lemma B, let $v(z_2) = U'(\lambda z_1 + (1 - \lambda)z_2)$, $\Psi(z_2) = z_2$, $\Phi(z_2) = E\{Z_2|z_1\}$, and $dP = dP(z_2|z_1)$.

Then we have $\Psi(z_2)$ is monotone-increasing, $\Phi(z_2)$ is constant, and $v(z_2)$ is non-decreasing ($v'(z_2) = \partial U'(\lambda z_1 + (1 - \lambda)z_2)/\partial z_2 = (1 - \lambda)U''(\lambda z_1 + (1 - \lambda)z_2)$, and $U'' < 0$, $1 - \lambda \leqq 0$ by assumption).

Furthermore,

$$\int \Psi(z_2)dP(z_2|z_1) = E\{Z_2|z_1\} = \int \Phi(z_2)dP(z_2|z_1).$$

Therefore by the contrapositive of the Corollary B we have:

3) $$\begin{aligned} &{}_{(z_2)}\int z_2 U'(\lambda z_1 + (1 - \lambda)z_2)dP(z_2|z_1) \\ &\geqq {}_{(z_2)}\int E\{Z_2|z_1\}U'(\lambda z_1 + (1 - \lambda)z_2)dP(z_2|z_1) \\ &= E\{Z_2|z_1\}E\{U'(\lambda z_1 + (1 - \lambda)Z_2)|z_1\} \end{aligned}$$

for <u>all</u> z_1.

Iterating the integrals in 1) and substituting 3) into 1) we have:

4) $\int z_1 E\{U'|z_1\} dP(z_1) \geqq \int E\{Z_2|z_1\} E\{U'|z_1\} dP(z_1)$.

Again, applying the notation of Lemma B, let $v(z_1) = E\{U'|z_1\}$,

$\Psi(z_1) = z_1$, $\Phi(z_1) = E\{Z_2|z_1\}$, and $dP = dP(z_1)$.

Clearly $\Psi(z_1)$ in monotone-increasing and by Lemma 4 $\Phi(z_1)$ is non-increasing.

5) $\partial/\partial z_1[E\{U'|z_1\}] = E\{\lambda U''|z_1\} + \int_{(z_2)} U' \; \partial/\partial z_1[dP(z_2|z_1)]$.

Since $\lambda \geqq 1$ and $U'' < 0$ by assumption, we have $E\{\lambda U''|z_1\} < 0$.

For $\varepsilon > 0$, consider

$\int U'(\lambda z_1 + (1 - \lambda)z_2) dP(z_2|z_1 + \varepsilon) - \int U'(\lambda z_1 + (1 - \lambda)z_2) dP(z_2|z_1)$

$\partial/\partial z_2[U'(\lambda z_1 + (1 - \lambda)z_2)] = (1 - \lambda)U'' \geqq 0$.

Therefore by Lemma 3

$\int U' dP(z_2|z_1 + \varepsilon) - \int U' dP(z_2|z_1) \leqq 0$,

since $\partial P(z_2|z_1)/\partial z_1 \geqq 0$.

6) Thus $\partial/\partial z_1[E\{U'|z_1\}] < 0$, so $v(z_1) = E\{U'|z_1\}$ is strictly monotone-decreasing.

7) Furthermore, $\int \Psi(z_1) dP(z_1) = E\{Z_1\}$ and $\int \Phi(z_1) dP(z_1) = E\{Z_2\}$, and by assumption, $E\{Z_1\} = E\{Z_2\}$.

Therefore by the Corollary to Lemma B and 6) and 7) we have:

8) $\int z_1 E\{U'|z_1\}dP(z_1) < \int E\{Z_2|z_1\}E\{U'|z_1\}dP(z_1).$

But this is a contradiction of 4). Therefore $\lambda < 1$. By a completely analogous argument we can also show $\lambda > 0$. Therefore $\lambda \varepsilon (0, 1)$.

<u>Corollary</u>: Let $(\lambda_1^*, \ldots, \lambda_{n-1}^*)$ be the maximizers of the problem:

a) $\max_{\{\lambda_i\}} E\{U(\sum_{i=1}^{n-1} \lambda_i Z_i)\}$ subject to $\sum_{i=1}^{n-1} \lambda_i = 1$, where $U' > 0$, $U'' < 0$.

Suppose $\sum_{i=1}^{n-1} \lambda_i^* Z_i$ is negatively S-correlated with Z_n or $\sum_{i=1}^{n-1} \lambda_i^* Z_i$ and Z_n are independent, and $E\{\sum_{i=1}^{n-1} \lambda_i^* Z_i\} \leqq E\{Z_n\}$. Then the solution of the problem:

b) $\max_{\{\lambda_i\}} E\{U(\sum_{i=1}^{n} \lambda_i Z_i)\}$ subject to $\sum_{i=1}^{n} \lambda_i = 1$ has the property that $\lambda_n \neq 0$.

<u>Proof</u>: Let $\sum_{i=1}^{n-1} \lambda_i^* Z_i = X$, and $Z_n = Y$.

Then by the assumptions of the Corollary and by Theorem 2 we have that the solution of the problem

1) $\max_{\{\lambda\}} E\{U(\lambda X + (1 - \lambda)Y)\}$

has the property that $\lambda < 1$. Therefore it cannot be optimal to have $\lambda_n = 0$ since the solution of 1) is admissable as a solution of b).

Of course Theorem 2 and its Corollary are of limited usefulness for the problem of actual portfolio management. Finding a security which satisfies the assumptions of the Corollary in any actual situation would probably be rare. This could, however, in some cases, be used to explain the yield of certain securities relative to the yield of the "market". By this we mean any security which the "market" is negatively S-correlated with would probably have a yield not significantly greater than the yield of the "market" if expectations of most participants are approximately the same.

Definition: Z_1 will be said to be strongly positively S-correlated with Z_2 if Z_1 is positively S-correlated with Z_2 and $E\{Z_1|z_2\}$ is a strictly convex function of z_2. (By Lemma 4, $E\{Z_1|z_2\}$ is a monotone-increasing function of z_2.)

Example: Let $Z_1 \sim U[0, 1]$ and let $Z_2 = 3/2\ Z_1^2$.

Then $P\{z_2|z_1\} = \begin{cases} 1, \text{ if } z_2 \leqq (3/2)\, z_1^2,\ 0 \leqq z_1 \leqq 1 \\ 0, \text{ otherwise} \end{cases}$

and $E\{Z_2|z_1\} = (3/2)\, z_1^2$. Clearly, Z_2 is positively S-correlated with Z_1 and $E\{Z_2|z_1\}$ is a strictly convex function of z_1. Therefore Z_2 is strongly positively S-correlated with Z_1.

The theory of warrant pricing presented in the paper by Samuelson and Merton[5] has as one result that the yield on a warrant is strongly positively S-correlated with its associated stock.

Theorem 3: If $U' > 0$, $U'' < 0$, $E\{Z_1\} = E\{Z_2\}$ and if Z_1 is strongly positively S-correlated with Z_2, then the solution of the problem

$$\max_{\{\lambda\}} E\{U(\lambda Z_1 + (1 - \lambda)Z_2)\}$$

has the property that $\lambda \notin [0, 1]$.

Proof: Suppose $0 \leqq \lambda \leqq 1$.

The First Order Conditions for the problem require:

$$1) \iint z_1 U'(\lambda z_1 + (1 - \lambda)z_2)dP(z_1, z_2) = \iint z_2 U'(\lambda z_1 + (1 - \lambda)z_2)dP(z_1, z_2).$$

Iterating integrals we have:

2) $\iint z_1 U'(\lambda z_1 + (1 - \lambda)z_2)dP(z_1, z_2)$

$$= \int [\int z_1 U'(\lambda z_1 + (1 - \lambda)z_2)dP(z_1|z_2)]dP(z_1).$$

Using the notation of Lemma B, let $v(z_1) = U'(\lambda z_1 + (1 - \lambda)z_2)$, $\Psi(z_1) = z_1$, $\Phi(z_1) = E\{Z_1|z_2\}$, and $dP = dP(z_1|z_2)$.

Then $\Psi(z_1)$ is strictly increasing, $\Phi(z_1)$ is constant, and $v(z_1)$ is non-increasing ($v'(z_1) = \lambda U''(\lambda z_1 + (1 - \lambda)z_2) \leqq 0$ since $\lambda \geqq 0$ and $U'' < 0$).

Furthermore, $\int \Psi(z_1)dP(z_1|z_2) = E\{Z_1|z_2\} = \int \Phi(z_1)dP(z_1|z_2)$.

Therefore by the Corollary to Lemma B we have

3) $\int z_1 U'(\lambda z_1 + (1 - \lambda)z_2)dP(z_1|z_2) \leqq E\{Z_1|z_2\}E\{U'|z_2\}$,

for <u>all</u> z_2.

Iterating the integrals in 1) and substituting 3) into 1) we have:

4) $E\{Z_1|z_2\}E\{U'|z_2\}dP(z_2) \geqq \int z_2 E\{U'|z_2\}dP(z_2)$.

Again, applying the notation of Lemma B, let $v(z_2) = E\{U'|z_2\}$, $\Psi(z_2) = E\{Z_1|z_2\}$, $\Phi(z_2) = z_2$, and $dP = dP(z_2)$. By assumption, $\Psi(z_2)$ is strictly convex and since $\int \Psi(z_2)dP(z_2) = E\{Z_1\} = \int \Phi(z_2)dP(z_2) = E\{Z_2\}$, $\exists\ \bar{z}_2$ such that $\Psi(z_2) < \Phi(z_2)$ for $z_2 < \bar{z}_2$ and $\Psi(z_2) > \Phi(z_2)$ for $z_2 > \bar{z}_2$.

5) $\partial/\partial z_2[E\{U'|z_2\}] = E\{(1 - \lambda)U''|z_2\} + \quad U'\ \partial/\partial z_2[dP(z_1|z_2)]$.

Since $(1 - \lambda) \geqq 0$ and $U'' < 0$, $E\{(1 - \lambda)U''\} \leqq 0$. Furthermore, by

Lemma 3, since Z_1 is positively S-correlated with Z_2 and $\lambda \geqq 0$, $\int U' \, \partial/\partial z_2[dP(z_1|z_2)] \leqq 0$. To see this, consider

$$\int U'dP(z_1|z_2 + \varepsilon) - \int U'dP(z_1|z_2), \text{ for } \varepsilon > 0.$$

Since Z_1 is positively S-correlated with Z_2, $P(z_1|z_2 + \varepsilon) \leqq P(z_1|z_2)$, $\forall(z_1, z_2)$. Also, $\partial/\partial z_1(U') = \lambda U'' \leqq 0$. Therefore, by Lemma 3

$$\int U'dP(z_1|z_2 + \varepsilon) - \int U'dP(z_1|z_2) \leqq 0.$$

6) Thus, $\partial/\partial z_2[E\{U'|z_2\}] \leqq 0$, so $v(z_2)$ is non-increasing.

Since $\int \Psi(z_2)dP(z_2) = E\{Z_1\} = E\{Z_2\} = \int \Phi(z_2)dP(z_2)$ by the Corollary to Lemma B.

8) $\int E\{Z_1|z_2\}E\{U'|z_2\}dP(z_2) \leqq \int z_2E\{U'|z_2\}dP(z_2)$.

Since it must be the case that either $\lambda > 0$ or $(1 - \lambda) > 0$, if $\lambda > 0$ then 4) holds with strict inequality which is a contradiction of 8), or if $(1 - \lambda) > 0$, then 8) holds with strict inequality which is a contradiction of 4).

Therefore $\lambda \notin [0, 1]$.

Thus we have provided conditions sufficient for a risk-averting expected utility maximizer to "short". This contradicts the notion that a prudent investor would never go "short".

We have seen that strengthening the concept of correlation from (Pearsonian) linear correlation to (Samuelsonian) non-linear S-correlation gives us some diversification theorems for the expected utility model. The results, however, are even weaker than the mean-variance model results since we do not have an n-asset theorem. What we are searching for in the n-asset case is a multi-dimensional analogue of the problem considered by Hanoch and Levy.[6] The problem they considered was the following:

Let $F(z)$ and $G(z)$ be cumulative distribution functions for random variables X and Y. Under what conditions on F and G will an expected utility maximizer always prefer X to Y? -- i.e., Under what conditions on F and G will we have $\int U(z)dF(z) > \int U(z)dG(z)$ for either [all possible functions U with $U' > 0$] or [for all possible functions U with $U' > 0$ and $U'' < 0$]? They show that a <u>necessary</u> <u>and</u> <u>sufficient</u> condition for $\int U(z)dF(z) > \int U(z)dG(z)$ for all U such that $U' > 0$, $U'' < 0$ is that

$$\int_{-\infty}^{t} [G(z) - F(z)]dz \geqq 0 \text{ and } G(z_0) \neq F(z_0) \text{ for some } z_0.$$

We can rephrase the n-asset diversification problem in the following manner:

Let $P(z_1,\ldots, z_n)$ be the joint cumulative distribution function for random variables $Z_1,\ldots, Z_n$. Let $\lambda = (\lambda_1,\ldots, \lambda_n)$, $\Sigma \lambda_i = 1$, and let Y_λ by the (one-dimensional) random variable, $Y_\lambda = \Sigma \lambda_i Z_i$. Denote the cumulative distribution function of Y_λ by $P_\lambda(y)$. Let

$I^+ = \{(\lambda_1,\ldots, \lambda_n) | \Sigma \lambda_i = 1, \lambda_i > 0, \forall i\}$ and let

$I^- = \{(\lambda_1,\ldots, \lambda_n) | \Sigma \lambda_i = 1$, and $\lambda_{i'} \leq 0$ for some $i'\}$. The n-asset diversification problem can now be restated: Under what conditions on $P(z_1, \ldots, z_n)$ will we have that for any $\lambda \in I^-$ and U such that $U' > 0$, $U'' < 0$, $\exists \lambda^* \in I^+$ such that $\int U(y)dP_{\lambda^*}(y) > \int U(y)dP_{\lambda}(y)$?

This, of course, is very similar to the Hanoch and Levy problem, but much more complicated. Our Theorem 2 is an answer to this problem for $n = 2$. For $n > 2$ it would be necessary to look at the properties of $dP_{\lambda}(y)$ which is extremely complicated if the Z_i's are not independent. Of course, by the Samuelson theorem, one condition on P which works is that the Z_i's are independent and have equal means. Whether or not there are other conditions (other than $P(z_1, \ldots, z_n)$ being a symmetric function) on P which will work remains an unsolved problem.

Footnotes

(Bracketed numbers refer to the listing in the Bibliography.)

1 [5].

2 [2].

3 [5].

4 [3].

5 [6].

6 [3].

7 [5].

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