COMPUTER MODELS OF THE BEAM-
PLASMA INTERACTION

by

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Submitted to the Department of Electrical Engineering on March 15, 1968 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

We present the results of computer experiments of the electron beam-plasma interaction at the electron plasma frequency. In our models a fresh beam is continuously injected, so that we do not assume spatially periodic boundary conditions as is often done. We compare the collected beam velocity distribution of our computer experiments with that of a laboratory experiment with an 8 kilovolt beam. The laboratory velocity distribution of the collected beam is relatively narrow, indicating a weak interaction. We find that plasma density gradients along the direction of beam flow are necessary to explain the narrow beam velocity spread. A uniform plasma model, with or without a reasonable temperature or even with a very high plasma collision frequency, produces a much wider beam velocity spread than actually observed. A homogeneous plasma is resonant everywhere at \( \omega_p \), so that beam bunching at \( \omega_p \) produces a large response in the plasma. Plasma density gradients along the direction of beam flow eliminate this resonance, and reduce the electric fields much more than plasma temperature, collisions, or finite beam diameter effects.

The computer results of the homogeneous beam-plasma models simulate the meniscus often seen in low voltage beam-plasma and hot cathode experiments, but not in high voltage experiments. The meniscus is a localized region of strong electric fields oscillating at \( \omega_p \) that greatly scatter the beam in velocity.

Thesis Supervisor: Abraham Bers
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>2</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>3</td>
</tr>
<tr>
<td>TABLE OF CONTENTS</td>
<td>5</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>10</td>
</tr>
<tr>
<td>CHAPTER</td>
<td></td>
</tr>
<tr>
<td>1 INTRODUCTION</td>
<td>15</td>
</tr>
<tr>
<td>1.1 Motivation</td>
<td>15</td>
</tr>
<tr>
<td>1.2 Historical Review</td>
<td>15</td>
</tr>
<tr>
<td>1.2.1 General Beam-Plasma Interaction</td>
<td>15</td>
</tr>
<tr>
<td>1.2.2 History of Low Voltage Beam-Plasma Interaction (the Meniscus)</td>
<td>19</td>
</tr>
<tr>
<td>1.2.3 History of High Power Beam-Plasma Interactions</td>
<td>20</td>
</tr>
<tr>
<td>1.2.4 History of Superparticle Models</td>
<td>21</td>
</tr>
<tr>
<td>1.3 Scope and Outline of Thesis</td>
<td>25</td>
</tr>
<tr>
<td>2 EXPERIMENTAL OBSERVATIONS</td>
<td>29</td>
</tr>
<tr>
<td>2.1 Experimental Apparatus</td>
<td>29</td>
</tr>
<tr>
<td>2.2 Velocity Analyzer Measurements</td>
<td>32</td>
</tr>
<tr>
<td>2.3 Axial Variations in Plasma Density</td>
<td>41</td>
</tr>
<tr>
<td>2.3.1 Introduction</td>
<td>41</td>
</tr>
<tr>
<td>2.3.2 Theoretical Axial Plasma Density Distribution in a Magnetic Mirror</td>
<td>42</td>
</tr>
<tr>
<td>2.3.3 Theoretical Axial Plasma Density Distribution in a Beam Generated Plasma with Short Mean Free Paths</td>
<td>44</td>
</tr>
<tr>
<td>2.3.4 Langmuir Probe Measurements</td>
<td>46</td>
</tr>
</tbody>
</table>
3. **A COMPUTER MODEL OF THE BEAM–PLASMA INTERACTION IN A UNIFORM PLASMA, WITH THE PLASMA REPRESENTED BY CHARGE SHEETS**

3.1 Introduction

3.2 Model with a Cold Plasma

3.3 Ordering Routine

3.4 Results of the Cold Plasma Model with $n_{po}/n_{bo} = 99$

3.4.1 Comparison of the Growth in Time with Linear Theory

3.5 Cold Plasma Sheet Model with Beam Density Equal to $1/19$ the Plasma Density

3.6 Heating by Longitudinal Waves in the Presence of Reflecting Walls

3.6.1 Approximate Theory of the Heating Rate of Wave–Mirror Heating

3.6.1.1 Velocity Gain

3.6.1.2 Probability Theory

3.6.2 Computer Experiments of Wave–Wall Heating

3.7 Warm Plasma Model

3.8 Summary of Chapter and Conclusions

4. **BEAM–PLASMA INTERACTION WITH A LOSSY PLASMA**

4.1 Introduction

4.2 Model Using a Linearized Plasma

4.3 Results of the Beam–Plasma Interaction with a Lossy Plasma

4.4 Conclusions
5 LINEARIZED ONE-DIMENSIONAL BEAM-PLASMA INTER-
ACTION WITH DENSITY GRADIENTS IN THE 
DIRECTION OF BEAM FLOW 146

5.1 Introduction 146

5.2 Linearized Calculations 147

5.3 Computer Experiments with a Linear Plasma 
Density Gradient along the Direction of 
Beam Flow 155

5.4 Conclusions 173

6 DISK BEAM-PLASMA INTERACTION IN A PLASMA WITH 
DENSITY GRADIENTS ALONG THE DIRECTION OF 
BEAM FLOW 176

6.1 Introduction 176

6.2 Comments on the Validity of Using Disks 177

6.3 Linear Theory of a Solid Beam Model Plasma 
Interaction with a Finite Diameter Beam 179

6.4 Model 184

6.5 Results of the Computer Model 192

6.6 Summary of the Results of this Chapter 206

7 CONCLUSIONS 207

7.1 Suggestions for Future Work 210

APPENDICES 211

A. Axial Density in a Magnetic Mirror 211

B. Program for a Warm Sheet Plasma 214

C. The General Wake of a Sheet Moving at an 
Arbitrary Positive Velocity Through a Cold 
Plasma with an Arbitrary Density Variation 214
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>D.</td>
<td>The Drag on One Sheet in an Unmodulated Sheet Beam of Infinite Extent</td>
<td>216</td>
</tr>
<tr>
<td>E.</td>
<td>One-Dimensional Linearized Beam-Plasma Interaction with a Linear Plasma Density Gradient and Collisional Loss</td>
<td>221</td>
</tr>
<tr>
<td>F.</td>
<td>Expected Values of Beam and Plasma Parameters for Values of $</td>
<td>\rho_b/\rho_{bo}</td>
</tr>
<tr>
<td>G.</td>
<td>Fields Generated by Charge Disk Moving at Constant Velocity Through a Uniform Cold Plasma</td>
<td>225</td>
</tr>
<tr>
<td>H.</td>
<td>Program for a Disk Beam and Plasma Density Gradients</td>
<td></td>
</tr>
<tr>
<td></td>
<td>BIBLIOGRAPHY</td>
<td>228</td>
</tr>
<tr>
<td></td>
<td>BIOGRAPHY</td>
<td>233</td>
</tr>
</tbody>
</table>
LIST OF COMMON SYMBOLS

a  acceleration
A  normalized acceleration
B  magnetic field
E  electric field
f  velocity distribution
g  density gradient (units of 1/lt^2)
k  wavenumber
L  distance between gun and collector
m  mass of electron
n_b  first order beam density
n_bo  zeroth order beam density
n_p  first order plasma density
n_po  zeroth order plasma density
q  electron charge
Q  sheet surface charge density
Q_tot  total charge in a tight beam bunch
r  disk radius
R  mirror ratio
t  time
v  velocity
v_b  first order beam velocity
v_d  disk velocity - a function of distance and time
v_p  plasma velocity
v_o  average injected beam velocity
v_T  thermal velocity
z  distance from gun
Z  normalized distance

\alpha \omega_r/v  permittivity of free space
collision frequency for momentum transfer
n_b  first order beam density
n_bo  zeroth order beam density
n_p  first order plasma density
n_po  zeroth order plasma density
\omega  peak plasma density of a sinusoidal plasma density variation
\phi  potential
\omega_n  radian frequency
natural oscillatory frequency of a lossy plasma
\omega_p  plasma frequency
\omega_pb  beam plasma frequency
\omega_po  maximum plasma frequency of a sinusoidal density variation
\omega_c  electron cyclotron frequency
LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Schematic of Experimental Apparatus.</td>
<td>30</td>
</tr>
<tr>
<td>2.2</td>
<td>Axial Magnetic Field Variation.</td>
<td>31</td>
</tr>
<tr>
<td>2.3</td>
<td>Beam Velocity Analyzer.</td>
<td>33</td>
</tr>
<tr>
<td>2.4a</td>
<td>Oscillograms of Velocity Analyzer Collector Current and Retarding Potential, p = 5.6x10^-4Torr.</td>
<td>34</td>
</tr>
<tr>
<td>2.4b</td>
<td>Same as Fig.2.4a, except p = 1.4x10^-3Torr.</td>
<td>35</td>
</tr>
<tr>
<td>2.5</td>
<td>Plot of Collector Current of Velocity Analyzer versus Retarding Voltage.</td>
<td>36</td>
</tr>
<tr>
<td>2.6</td>
<td>Collected Beam Velocity Distribution.</td>
<td>39</td>
</tr>
<tr>
<td>2.7</td>
<td>Analyzer Collector Current at low Retarding Voltages.</td>
<td>40</td>
</tr>
<tr>
<td>2.8</td>
<td>Theoretical Plasma Density Variation along the Axis, with a Parabolic Magnetic Mirror and very long Mean Free Paths, or with a Uniform Magnetic Field and a Diffusion Dominated Plasma.</td>
<td>45</td>
</tr>
<tr>
<td>2.9</td>
<td>Schematic of the Langmuir Probe.</td>
<td>47</td>
</tr>
<tr>
<td>2.10</td>
<td>Langmuir Probe Curves Taken in the Midplane at Distances from the Axis of 0.25 cm. and 0.5 cm.</td>
<td>48</td>
</tr>
<tr>
<td>2.11</td>
<td>Same Data as Fig.2.10, with the Ion Saturation Current Subtracted Out.</td>
<td>49</td>
</tr>
<tr>
<td>2.12</td>
<td>Ion Saturation Current versus Distance from gun at two Different Radii.</td>
<td>52</td>
</tr>
<tr>
<td>2.13</td>
<td>Radial Ion Saturation Current Variation at Various Distances from the Anode.</td>
<td>54</td>
</tr>
<tr>
<td>3.1</td>
<td>Schematic of the Acceleration (qE/m) versus Distance for the Sheet Plasma Model.</td>
<td>61</td>
</tr>
<tr>
<td>3.2</td>
<td>Snapshot of Beam Sheet Velocity and Plasma Sheet Acceleration versus Distance. nbo/npo = 1/99, t = 30/ωp.</td>
<td>67</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>3.3</td>
<td>Same as Fig. 3.2, except ( t = 70/\omega_p ).</td>
<td>69</td>
</tr>
<tr>
<td>3.4</td>
<td>Snapshot of Beam and Plasma Variables, ( n_{bo}/n_{po} = 1/99 ), ( t = 260/\omega_p ).</td>
<td>70</td>
</tr>
<tr>
<td>3.5</td>
<td>Same as Fig. 3.4, except ( t = 261.5/\omega_p ).</td>
<td>71</td>
</tr>
<tr>
<td>3.6</td>
<td>Same as Fig. 3.4, except ( t = 263/\omega_p ).</td>
<td>72</td>
</tr>
<tr>
<td>3.7</td>
<td>Same as Fig. 3.4, except ( t = 264.5/\omega_p ).</td>
<td>73</td>
</tr>
<tr>
<td>3.8</td>
<td>Same as Fig. 3.4, except ( t = 266/\omega_p ).</td>
<td>74</td>
</tr>
<tr>
<td>3.9</td>
<td>Beam Velocity Distribution Between 500 ( x (0.02v_o/\omega_p) ), (&lt;z&lt;700(0.02v_o/\omega_p) ), Averaged Over a Period Starting at ( t = 260/\omega_p ). Cold Plasma with ( n_{bo}/n_{po} = 1/99 ).</td>
<td>77</td>
</tr>
<tr>
<td>3.10</td>
<td>Same as Fig. 3.9, except ( 700(0.02v_o/\omega_p) ), (&lt;z&lt;900(0.02v_o/\omega_p) ).</td>
<td>78</td>
</tr>
<tr>
<td>3.11</td>
<td>Same as Fig. 3.9, except ( 800(0.02v_o/\omega_p) ), (&lt;z&lt;1000(0.02v_o/\omega_p) ).</td>
<td>79</td>
</tr>
<tr>
<td>3.12</td>
<td>Snapshot Comparing Beam Sheet Velocity of the Computer Experiment with that of the Asymptotic Linear Theory</td>
<td>87</td>
</tr>
<tr>
<td>3.13</td>
<td>Snapshot of Beam and Plasma Variables, ( n_{bo}/n_{po} = 1/19 ), ( t = 70/\omega_p ).</td>
<td>89</td>
</tr>
<tr>
<td>3.14</td>
<td>Plot Defining ( v_1^l ) and ( v_2^l ).</td>
<td>91</td>
</tr>
<tr>
<td>3.15</td>
<td>Same as Fig. 3.13, except ( t = 80/\omega_p ).</td>
<td>95</td>
</tr>
<tr>
<td>3.16</td>
<td>Same as Fig. 3.13, except ( t = 94/\omega_p ).</td>
<td>96</td>
</tr>
<tr>
<td>3.17</td>
<td>Same as Fig. 3.13, except ( t = 96/\omega_p ).</td>
<td>97</td>
</tr>
<tr>
<td>3.18</td>
<td>Same as Fig. 3.13, except ( t = 160/\omega_p ).</td>
<td>99</td>
</tr>
<tr>
<td>3.19</td>
<td>Same as Fig. 3.13, except ( t = 230/\omega_p ).</td>
<td>100</td>
</tr>
<tr>
<td>3.20</td>
<td>Plot of Potential Energy vs. Distance, in the Wave Frame, Defining the Potential Energies ( \beta ) and ( \delta ) at which Wall Collisions Occur.</td>
<td>103</td>
</tr>
</tbody>
</table>
3.21 Computer Experimental Results of Energy Gain vs. Number of Collisions for Hard Walls (Reflection at the same Phase), and for Reentry at a Random Phase.

3.22 Comparison of the Energy Gain of Electrons Reflected by a Constant Force Field in the Presence of a Wave that cuts off Sharply in Distance, with one that Decays Gradually.

3.23 Comparison of Computer Generated Maxwellian with True Maxwellian, for 1000 Sheets.

3.24 Dispersion Equation for real \( \omega \) for a Warm Plasma.

3.25 Snapshot of Beam and Plasma Variables. \( \frac{n_{bo}}{n_{po}} = 1/19, v_T = 0.075v_o, t = 112/\omega_p \).

3.26 Same as Fig.3.25, except \( t = 118/\omega_p \).

3.27 Same as Fig.3.25, except \( t = 220/\omega_p \).

3.28 Beam Velocity Distribution of Computer Experiment with Warm Plasma

4.1 Phasor to Determine the New Phase and Magnitude of the Field at a Cell Point when a Sheet Passes.

4.2 Corrections to the Accelerations and Positions of a Crossed Sheet.

4.3 Dispersion Relation for Real \( \omega \), for a Lossy Beam-Plasma Interaction.

4.4 Snapshot of Beam Velocity and Acceleration versus Distance.

5.1 Linear Solutions for Beam-Plasma Interaction with Linear Plasma Density Gradient.

5.2 Same as Fig.5.1, with the Two Other Modes.

5.3 Snapshot of Beam Sheet Velocity versus Distance, for Linear Plasma Density Gradient. \( t = 300/\omega_p \).
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.4</td>
<td>Snapshot of Acceleration versus Distance. $t = 300/\omega_{po}$.</td>
<td>159</td>
</tr>
<tr>
<td>5.5</td>
<td>Snapshot of Beam Velocity and Acceleration. $t = 400/\omega_{po}$.</td>
<td>162</td>
</tr>
<tr>
<td>5.6</td>
<td>Snapshot of Beam Velocity and Acceleration with a Negative Plasma Density Gradient.</td>
<td>163</td>
</tr>
<tr>
<td>5.7</td>
<td>Same as Figs. 5.3-5.6, but Plasma Density Gradient Doubled. $t = 360/\omega_{po}$.</td>
<td>166</td>
</tr>
<tr>
<td>5.8</td>
<td>Same as Fig. 5.7, except $t = 367/\omega_{po}$.</td>
<td>167</td>
</tr>
<tr>
<td>5.9</td>
<td>Same as Fig. 5.7 and 5.8, but Larger Modulation. $t = 400/\omega_{po}$.</td>
<td>169</td>
</tr>
<tr>
<td>5.10</td>
<td>Same as Fig. 5.7, but $n_{bo}/n = 0.002$, and a Sinusoidal Plasma Density Gradient Variation Instead of a Linear One. $t = 340/\omega_{po}$.</td>
<td>171</td>
</tr>
<tr>
<td>5.11</td>
<td>Same as Fig. 5.10, except $t = 400/\omega_{po}$.</td>
<td>174</td>
</tr>
<tr>
<td>6.1</td>
<td>Dispersion Relation of WKB Approximation to Beam-Plasma Interaction with Rigid Beam. Plasma with Linear Density Gradient.</td>
<td>183</td>
</tr>
<tr>
<td>6.2</td>
<td>Approximation to Nonoscillatory Force of Disks.</td>
<td>190</td>
</tr>
<tr>
<td>6.3</td>
<td>Comparison of WKB Theory with Computer Experiment.</td>
<td>193</td>
</tr>
<tr>
<td>6.4</td>
<td>Snapshot of Disk Velocity and Acceleration, in a Sinusoidal Plasma Density Variation.</td>
<td>194</td>
</tr>
<tr>
<td>6.5</td>
<td>Snapshot of Disk Velocity Distribution, Compared to that of the Laboratory.</td>
<td>196</td>
</tr>
<tr>
<td>6.6</td>
<td>Snapshot of Disk Velocity and Acceleration in a Uniform Plasma.</td>
<td>198</td>
</tr>
<tr>
<td>6.7</td>
<td>Same as Fig. 6.4, except the Beam is Sheets Instead of Disks. $t = 260/\omega_{po}$.</td>
<td>200</td>
</tr>
<tr>
<td>6.8</td>
<td>Same as Fig. 6.7, except $t = 320/\omega_{po}$.</td>
<td>202</td>
</tr>
<tr>
<td>6.9</td>
<td>Similar to Fig. 6.4, but Changes in Beam and Plasma Parameters.</td>
<td>203</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>D.1</td>
<td>Phasor Diagram to Determine the Total Wake Field of an Unmodulated Beam.</td>
<td>217</td>
</tr>
</tbody>
</table>
Chapter 1

INTRODUCTION

1.1 Motivation

The purpose of this thesis is to obtain a quantitative understanding of the steady-state beam-plasma interaction at the electron plasma frequency, when a fresh beam is continuously being injected. The small signal assumptions of linear theory break down in this case, because the dynamics of the beam, and sometimes of the plasma also, become nonlinear. Quasilinear theory also fails, because trapping, an inherently nonlinear process, is found to be quite evident. Hence we use a computer simulation of the interaction and model the beam with charge sheets or disks. In some cases the plasma is assumed to be linear, but in others the plasma too is modeled with sheets.

1.2 Historical Review

A great deal of work has been done on the beam-plasma interaction. We ignore in this review many articles which discuss the interaction when $\omega_p = \omega_c$, a situation which does not apply in our laboratory experiment, and which we have not investigated in our computer experiments. In this limited review we shall describe the early beam-plasma experiments and theories, those experiments in which a meniscus is seen, some high-power beam-plasma experiments, and other computer experiments.

1.2.1 General Beam-Plasma Interactions

The beam-plasma interaction has been of interest ever since Langmuir observed the scattering of primary electrons
in a hot cathode discharge, by what Tonks and Langmuir\(^2\) later deduced to be high frequency oscillations at \(\omega_p\).

In 1939, Merrill and Webb\(^3\) studied the scattering of primary electrons in an arc discharge, and found two regions only a few tenths of a millimeter wide in which the electrons were widely scattered in velocity. In these regions there were strong oscillations at the plasma frequency, and these oscillations were proposed as the scattering mechanism.

In 1948 Pierce\(^4\), in attempting to explain oscillations in microwave tubes, derived a theory for growth of oscillations at \(\omega_{pi}\), the ion plasma frequency. His model was that of a cold electron beam traversing cold stationary ions, with no transverse variations or magnetic field considered. Assuming an \(e^{i(\omega t - k z)}\) variation, and real \(\omega\), he predicted exponential growth of oscillations with increasing \(z\), with growth at all frequencies below \(\omega_{pi}\), and an infinite growth rate at \(\omega_{pi}\). Plasma electrons can be included in the analysis by merely substituting \(\omega_p^2 = \omega_{pi}^2 + \omega_{pe}^2\) for \(\omega_{pi}^2\).

Bohm and Gross\(^5\) clarified the physical processes of small amplitude waves in warm plasmas without a magnetic field. They found that the plasma frequency oscillation, which has zero group velocity in a cold plasma, goes into a forward wave with the introduction of temperature. They predicted that an electron beam sharply defined in velocity will lead to exponential growth with distance of these waves.
Looney and Brown\textsuperscript{6} were unable to excite plasma oscillations in any of their tubes constructed to satisfy the conditions of Bohm and Gross. They did, however, obtain oscillations by forming sheaths which allowed a return beam to flow, providing feedback.

Boyd, Field and Gould\textsuperscript{7} did find waves that grew exponentially with distance. In their experiment a premodulated beam was injected into a mercury arc discharge. The beam was modulated with a helix, and the modulation on the beam leaving the plasma was picked up by another helix. Maximum amplification occurred at the electron plasma frequency.

Sturrock\textsuperscript{8} has shown theoretically that the beam-plasma interaction of a finite diameter beam in an infinite plasma medium is convective; that is, that a pulse disturbance will propagate in the direction of the beam, growing as it does so, but the disturbance eventually returns to zero at the point of excitation. Using a finite beam theory, he calculates that Looney and Brown should have a growth rate of about \(20\text{db/cm}.\) Since their interaction length was only 1.5 cm., they presumably could not observe amplification of noise.

Self\textsuperscript{9} has criticized Sturrock's analysis because Sturrock predicted a maximum exponential spatial growth constant of \(\omega_p/2v_o\) (\(v_o\) is the zeroth order beam velocity), independent of beam density. Self found that a cold lossless, finite diameter beam in an infinite cold plasma with no applied magnetic field leads to an interaction which is actually an absolute instability;
that is the linear theory does not predict the formation of a steady state, but predicts continual growth in time. However, the interaction becomes convective (does reach a steady state for a constant excitation) with the inclusion of a very small amount of loss.

Allen and Kino\textsuperscript{10} obtained exponential spatial growth of oscillations in an apparatus similar to that of Boyd, Field, and Gould. The gain increased with frequency from $\omega_c$ up to $\omega_p$, then fell to zero again above $(\omega_c^2 + \omega_p^2)^{1/2}$. This agreed well with their theory, both qualitatively and quantitatively, except for the existence of secondary peaks in the gain curve above $\omega_p$.

Using a quasistatic analysis, Smullin and Chorney\textsuperscript{11,12} predicted the existence of both forward and backward slow waves in plasma loaded waveguides immersed in a magnetic field. For $\omega_p$ greater than $\omega_c$ (the electron cyclotron frequency), they predicted a backward wave near $\omega_p$. This wave would be expected to couple with the slow beam space charge wave to produce an absolute instability. For $\omega_c$ greater than $\omega_p$, a backward wave is predicted near $\omega_c$. This wave could be expected to couple with the slow beam cyclotron wave, again producing an absolute instability. Independently Trivelpiece and Gould\textsuperscript{13} made similar analyses and were able to observe the forward and backward waves experimentally.

Getty and Smullin\textsuperscript{14,15} obtained experimental confirmation of the interaction of an electron beam with the backward cyclotron wave of Smullin and Chorney, when $\omega_p = \omega_c$. 
Bers and Briggs\textsuperscript{16} have developed a stability criterion to
differentiate between absolute and convective instabilities.
They have applied the criterion to a large number of cases. In
particular, the cold beam, cold plasma interaction in one dimen-
sion is found to be convective, although in the absence of loss
or temperature an infinite growth rate in space is predicted.

Briggs\textsuperscript{17} has further investigated the linearized one-
dimensional cold beam-cold plasma interaction, and observed that
it is really a hybrid between a convective and absolute instability.
For a fixed excitation the response grows without limit in both
space and time.

Bers\textsuperscript{18} has derived a rigid beam model suitable for study
of a linearized beam-plasma interaction in finite geometries and
for finite temperature. This model allows a simplification of
the beam equation of motion by assuming the beam moves rigidly,
without shear.

Hsieh\textsuperscript{19} has done experiments on similar equipment as used
in this thesis. For $\omega_p \gg \omega_c$ he found broad band oscillations
near the electron and ion plasma frequencies. No radiation was
detected at the electron cyclotron frequency. The rf was radiated
in spikes, or short bursts on the order of 0.1 $\mu$s. duration.

1.2.2 History of Low Voltage Beam-Plasma Interaction
   (the Meniscus)

In low voltage gas discharges (voltage not more than a few
times the ionization potential of the gas), a meniscus-shaped
glow appears near the cathode. The meniscus has been intensively studied by Emeleus\textsuperscript{20,21} and his co-workers. They found, as did Merrill and Webb\textsuperscript{3}, that there were strong oscillations in the meniscus which scattered in velocity and in space the primary electrons emanating from the cathode fall region. Extensive bibliographies on this subject may be found in Crawford and Kino\textsuperscript{22} and Crawford and Self.\textsuperscript{23}

Cannara and Crawford\textsuperscript{24} have probed the meniscus region with a very small current electron beam. They concluded that a beam-plasma interaction was occurring, the intensity of which reached a maximum in the meniscus, and fell off rapidly beyond the meniscus.

At higher beam voltages (100v.-400v.), the meniscus becomes more diffuse\textsuperscript{23}. At 700v. Levitskii and Shashurin\textsuperscript{25} report a beam-plasma experiment in which the half of the glass drift tube near the gun had a uniform plasma density and only weak light output near the beam. Further downstream the plasma became much brighter and filled the tube. The plasma density approximately doubled in this region. The light was reported roughly axially uniform here, and not in the shape of a meniscus. Strong rf oscillations were detected in the downstream region, and were considered to provide the energy for secondary ionization by plasma electrons in this region.

1.2.3 History of High Power Beam-Plasma Interactions

Herrmann\textsuperscript{26} used a 4kv. beam of less than 80 ma. to generate a plasma in an interaction length of 15-50 cm., in
a uniform magnetic field. For constant magnetic field, he found that the collected beam velocity spread at first increased with plasma density to a very wide spread, but then decreased with further density increase. The oscillations began near the upper hybrid frequency \((\omega_p^2 + \omega_c^2)^{1/2}\), when \(\omega_p\) was about \(\omega_c/2\). Other authors 15,27 have reported an intense interaction when \(\omega_p = \omega_c\). Presumably the interaction when \(\omega_p \gg \omega_c\) is less intense in Herrmann's work than when \(\omega_p = \omega_c\). This would corroborate our relatively small beam velocity spread.

Other authors, for example Refs. 28 and 29, have measured the collected beam velocity distribution in beam-plasma systems, obtaining various degrees of spread. However, the narrow spreads are usually obtained in systems only a few wavelengths long, and the wide spreads in systems where \(\omega_p = \omega_c\).

Getty and Smullin,15 Bartsch,30 and Gatta31 have reported the presence of electrons with energies far exceeding those of the injected beam. In these experiments \(\omega_p\) is above \(\omega_c\) by a factor of three or more, but in these pulsed experiments it is not clear that the hot electrons were not entirely created during the plasma density buildup, when \(\omega_p = \omega_c\).

1.2.4 History of Superparticle Models

Superparticles are particles with charges much greater than that of an electron, but normally with the same charge to mass ratio as an electron (or ion). Examples are one-dimensional zero thickness sheets of disks, rods, spheres, or points. They
are used to model physical systems in computer experiments simulating systems such as traveling wave tubes, plasmas, galaxies, gases, etc. An extensive bibliography of super-particle models have been compiled by Birdsall.\textsuperscript{32}

Hartree\textsuperscript{33} was among the first to simulate electrons by sheets, doing so in an analysis of transients involved in charge injection into a vacuum tube diode.

Tien, Walker and Wolontis,\textsuperscript{34} obtained large signal solutions for traveling wave tubes. They used a disk model in a conduction drift tube (idealized helix). In this case the electric field of each disk varies with distance as $e^{-\alpha z}$.

Rowe\textsuperscript{35} has made extensive computer analyses of nonlinear effects in microwave tubes, including traveling wave tubes, klystrons, backward wave oscillators, and crossed field amplifiers.

Buneman\textsuperscript{36} used sheets to analyze the interaction of a beam of cold electrons drifting through cold ions. He used periodic boundary conditions, that is sheets that drifted out the right boundary re-entered from the left, and vice versa. He found that oscillations developed which randomized the electron velocities in a few tens of plasma oscillation periods. When a d.c. electric field was applied, "turbulence" generated by space charge oscillations resulted in a high resistivity. Buneman used both ion and electron sheets.
Dawson\textsuperscript{37} simulated a plasma by employing sheet electrons immersed in a uniform, stationary ion background. Between crossings, sheets oscillate sinusoidally at $\omega_p$, about an equilibrium plane. Hence their trajectories are known exactly, and no numerical integration routine need be used to calculate them. If two identical sheets cross, their equilibrium positions may merely be interchanged. Periodic boundary conditions were applied. He found experimental computer agreement with the theories describing Landau damping, Debye shielding, the thermalization of sheets, and the drag on a slow sheet.

Gould and Allen\textsuperscript{38} and Allen et al\textsuperscript{39} used a disk model of a beam drifting through a linearized plasma to analyze a beam-plasma amplifier. The plasma was not represented by sheets, but rather by the fluid equations. Each disk generated a wake in the plasma. Since the plasma was linear, the wakes were superposed. For a beam modulation frequency less than $\omega_p$, they found that the plasma created space charge forces that aided bunching, whereas a beam in free space tends to be debunched by beam space charge. They included longitudinal plasma density gradients by using sections of uniform plasma at different densities. They did not report modulating at a frequency corresponding to an $\omega_p$ of any of the uniform sections. By terminating the plasma before beam disk overtaking, they found that the beam could bunch at a total beam velocity spread of only forty per cent of the injected velocity, a condition which would allow good efficiency in a klystron.
Dunn and Ho$^{40}$ considered the beam-plasma interaction, using sheets for beam electrons, plasma electrons, and ions. Ion to electron mass ratios of fifty or less were used. The plasma was generated by beam impact ionization. They divided the drift region into ten or twenty cells, and allowed only two sheets of each species per cell. These two sheets had velocities of opposite sign. If two sheets of the same species going in the same direction were in the same cell simultaneously, they were combined into one sheet of combined mass and charge. Momentum was conserved in this process, but not energy. By observing the density in a cell as a function of time, they were able to detect both electron and ion plasma frequency oscillations.

Later, Dunn and Halsted$^{41}$ used a larger number of charge sheets and found that the ion plasma frequency oscillations vanished, indicating that the previous model was too coarse.

Dawson$^{42}$ has investigated the relaxation of a rectangular velocity distribution to a Maxwellian. He found that the relaxation time in his computer experiments increased as the number of sheets per Debye length increased. The thermalization was due to the wake generated by a sheet as it passes through the plasma, and the Landau damping of the wake by the other sheets.

Shanny, Dawson, and Greene$^{43}$ have included electron-ion collisions in their sheet models. They find that the total damping rate of plasma oscillations is a superposition of Landau damping and collisional damping.
1.3 Outline of Thesis

We investigate a number of models describing the beam-plasma interaction, with the goal of finding those parameters which most limit the interaction in our laboratory experiment.

Chapter 2 describes the experiment and the measured properties of the beam and plasma. We use an 8 kilovolt, 0.4 ampere, 0.25 cm. diameter beam, and find that the beam electrons are collected with velocities lying between 0.7 and 1.2 times the injected velocity. Attempts to duplicate this relatively narrow spread in a computer simulation of the interaction involves the bulk of the thesis.

In Chapter 3 we investigate the one dimensional beam-plasma interaction in a lossless, homogeneous plasma, first with a cold plasma, then with a warm one. Both the beam and plasma are represented by charge sheets, in a uniform, immobile ion background.

In Chapter 4 we introduce a large collisional loss into the plasma. We linearize the plasma, treating it analytically, but still model the beam with charge sheets.

In Chapter 5 we introduce plasma density gradients in a linearized plasma. The density gradients turn out to be the most important parameter limiting the interaction and give a narrow velocity spread in the emerging beam.

Chapter 6 introduces a finite radius for the beam by means of disks. In this chapter we approach quantitative agreement
with the laboratory collected beam velocity distribution.

Chapter 7 summarizes the thesis and the important conclusions.

We find that there are fundamental differences in the results of models employing a uniform plasma from those with plasma density gradients. The uniform models result in very strong electric fields that are, however, localized about the region where the beam dynamics become strongly nonlinear. These fields are strong enough to widely disperse the beam in velocity and, except for the very lossy model, to trap plasma electrons in a wave moving with a velocity comparable to the injected beam velocity.

If the plasma has density gradients along the direction of beam flow, the electric fields are much less intense and are more spread out spatially. With a finite diameter beam and plasma density gradients along the beam flow the collected beam velocity spread is comparable to the relatively narrow spread observed in our laboratory experiment, whereas the spread from the other models is too wide. This computer model is only twenty per cent as long as the experiment, but it does extend well beyond the point where the beam dynamics first become nonlinear.

We feel that the differences in the results of the uniform and nonuniform plasma models can explain differences in laboratory beam-plasma experiments. The uniform plasma models seem to
reproduce properties of the meniscus that is often seen in low power beam-plasma and hot cathode experiments.\textsuperscript{1-3,20-24}

The meniscus is a thin meniscus-shaped glowing region that appears near the cathode. It is a region with strong oscillations at the plasma frequency which scatter the beam in velocity. In hot cathode discharges, the primary electrons are scattered. Often a dark space appears on the anode side of the meniscus, which we also see reproduced in our computer experiments in the form of a low field region just beyond the meniscus. We cannot reproduce with our one-dimensional models certain three dimensional aspects of the meniscus, such as its slightly curved shaped or the diverging and converging beam rays that often appear on the anode side of it.

The meniscus does not seem to be observed in high voltage beam plasma experiments. We find in our computer experiments with a uniform plasma that the maximum plasma oscillatory kinetic energy of plasma electrons in the meniscus is limited to less than one-fourth of the kinetic energy of injected beam electrons; because if collisions or other losses do not limit the oscillation amplitude below this level, some plasma electrons are strongly accelerated to velocities comparable to the wave phase velocity, and are swept downstream away from the meniscus region. This process serves as a loss mechanism for the waves in the region of the meniscus, or strong field region, and limits the electric field amplitude. The remaining oscillating electrons
will then be unable to ionize if the beam voltage is less than about four times the ionization potential. If the beam voltage is higher, then ionization by plasma electrons is possible, and we expect the resulting plasma density gradients to wash out the meniscus.

The dielectric constant of a lossless, cold, uniform plasma with no magnetic field is:

$$\varepsilon = \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right),$$

(1.1)

which is negative if \(\omega < \omega_p\), the plasma frequency. If a beam is injected into the plasma and modulated at a frequency below \(\omega_p\), then an inspection of Gauss's law reveals that the electrons in a beam bunch will attract each other, rather than repel each other as in a vacuum. If \(\omega\) is very close to \(\omega_p\), the beam bunching will be strong. The plasma is resonant at \(\omega_p\), so this beam bunching is expected to produce a large response in the plasma, and indeed a violent beam-plasma interaction is found to occur in a uniform plasma.

If, however, the plasma density increases away from the gun, so will \(\omega_p\). Then a beam modulated at some \(\omega\) will be in a region of negative \(\varepsilon\) only after it passes the plane where \(\omega = \omega_p\). If at the point in space where the beam is strongly bunched \(\omega_p \neq \omega\), we expect the plasma response to be much less than in a uniform plasma. This reasoning explains the dramatic differences between our computer experiments with uniform and nonuniform plasmas.
Chapter 2

EXPERIMENTAL OBSERVATIONS

2.1 Experimental Apparatus

The experimental apparatus is similar to that used by Hsieh. It is shown in Figure 2.1. The beam is generated by a Varian Va-220 oxide coated cathode assembly, with a perveance \( I/V^{3/2} \), where \( I \) = current, \( V \) = voltage) of about \( 10^{-6} \). The cathode is mounted behind a cold rolled steel pole piece that shields the cathode from the magnetic field, and also serves as the accelerating anode. The anode opening is 3/16" in diameter, and the cathode is set 3/16" behind the anode to produce a nominal beam diameter of 0.1". The beam typically has an energy of 8 kilovolts, and a current of 0.4 amps. The glass drift region is 40 cm. long and 10 cm. in diameter. The magnetic field is generated by two Helmholtz coils, which produce the magnetic field shown in Figure 2.2, for a current in each magnet of 12.5 amps.

The vacuum is maintained by two CEC 4" water baffled diffusion pumps, one mounted in the center of the drift region, the other behind the gun. The base pressure ranges from \( 5 \times 10^{-7} \) Torr to \( 2 \times 10^{-6} \) Torr. The operating pressure is regulated by bleeding in dry hydrogen through a needle valve.

The pressure is measured by an RLE ionization gauge. It has been found that the measured pressure should be multiplied
Fig. 2.1 Schematic of experimental apparatus.
Fig. 2.2 Axial magnetic field. Magnet currents equal 12.5 amperes each.
by a factor of seven in \( H_2 \) to obtain the correct pressure. \(^{45}\)

Only corrected pressures are reported here.

The beam is pulsed with a pulse duration of up to 600\(\mu\)s. The repetition rate can be varied from 2 to 60 p.p.s.

2.2 Velocity Analyzer Measurements

The velocity analyzer is shown in Figure 2.3. The beam is sampled by a 1/64" hole in the back of the collector, and passes through two grids before reaching the analyzer collector. The first is biased +45v. to repel ions. The second is variably biased negatively to analyze the beam. The analyzer collector is biased at +45v. to inhibit secondary electrons from escaping. Varying this potential from +22.5 to +45 volts made no difference in the measured collected current, so apparently secondaries do not escape.

The collected current is passed through a 30 k\(\Omega\) resistor, and the resulting voltage is read on an oscilloscope. The connecting 10 foot RG58/U cable has a capacitance of about 300 pf. Since the beam acts as a current source, the resulting RC circuit (\(RC = 9\mu\)s.) acts as a low pass filter to facilitate measurements.

The oscillograms of collected analyzer current vs. retarding potential are given in Figure 2.4, and the plots of current vs. voltage read from these oscillograms are shown in Figure 2.5. The fact that the current does not reach zero for large voltages is probably caused by the presence of a hot electron component trapped by the magnetic mirrors. These hot electrons have been studied by Bartsch. \(^{30}\)
Fig. 2.3 Beam velocity analyzer.
Fig. 2.4a Oscillograms of velocity analyzer collector current and retarding potential. Capacitive collector current at start of trace caused by pulsing retarding potential. Current through 30 kΩ resistor. $p = 5.6 \times 10^{-4}$ Torr. Numbers to left of oscillograms give retarding potential normalized to beam potential. 100 µs./cm.
Fig. 2.4b Same as Fig. 2.4a, except $p = 1.4 \times 10^{-3}$. Because of a miscalibration of the two voltage dividers measuring beam voltage and retarding voltage, 6.8 kilovolts measured retarding voltage is actually 8 kilovolts.
Fig. 2.5 Analyzer collector current versus retarding voltage, normalized to the injected beam voltage.
The collected beam velocity distribution is obtained from the measured current vs. voltage characteristic of the analyzer. We know that

$$I \sim \int_{v}^{\infty} v' f_b(v') dv', \quad (2.1)$$

where $I$ is the analyzer collected current, $v$ is the lowest unrepelled velocity, and $f_b(v)$ is the collected beam velocity distribution. Differentiating Equation 2.1 with respect to $v$, we have

$$\frac{\partial I}{\partial v} \sim -vf_b(v). \quad (2.2)$$

The velocity is related to voltage by energy such that

$$\frac{mv^2}{2} = qV, \quad (2.3)$$

where $m$ and $q$ are the electron mass and charge, respectively, and $V$ is voltage. Hence

$$\frac{\partial}{\partial v} = \frac{\partial V}{\partial v} \frac{\partial}{\partial V}$$

or

$$\frac{1}{v} \frac{\partial}{\partial v} \sim \frac{\partial}{\partial V}. \quad (2.4)$$

Using Equation 2.4 in Equation 2.2 we find

$$f_b(v) \sim -\frac{\partial I}{\partial V}. \quad (2.5)$$
We graphically differentiate the I vs. V dependency, and arrive at the velocity distributions shown in Figure 2.6. The curves represent the velocity distribution 250μs, after the beam is turned on. The injected beam energy is 8 kilovolts, the injected current is 0.4 amps, and the magnetic field is that shown in Figure 2.2.

The collected current vs. retarding voltage at low voltages is shown in Figure 2.7. A plasma temperature can be obtained from this data since the beam and plasma velocities are well separated. We assume that the plasma distribution is a Maxwellian. After integrating over the transverse velocities,

$$ f_o = \frac{n_{p0}}{\sqrt{2\pi v_T}} e^{-mv^2/2qT_-}, \quad (2.6) $$

where $f_o$ is the plasma distribution function, $n_{p0}$ is the plasma density, $m$ the electron mass, $q$ the electron charge, $T_-$ the temperature in volts, and $v_T$ is the thermal velocity. If the grid has a retarding potential $V$, energy is conserved so that at the grid

$$ v_g^2 = v^2 - \frac{2qV}{m}, \quad (2.7) $$

where $v_g$ is the velocity at the grid. Solving Equation 2.7 for $v$, the corresponding velocity in the plasma, and inserting Equation 2.7 into Equation 2.6, we find that the distribution function $f_g$ at the grid is
$V_B = 8000$ VOLTS
$I_B = 0.4$ AMPS
$n_{po} = 6.9 \times 10^{12}$ cm$^{-3}$
$n_{bo} = 9.1 \times 10^9$ cm$^{-3}$

Beam radius = 0.125 cm.

$B_{\text{min}} = 430$ Gauss

Mirror Ratio = 3

System Length = 40 cm.

System Diameter = 10 cm.

Fig. 2.6 Collected beam velocity distribution.
Fig. 2.7 Analyzer collector current at low retarding voltages.
\[ f_g = e^{-V/T} f_o. \] (2.8)

\( f_g \) is a Maxwellian at the same temperature as \( f_o \), but with a reduced density. Hence the current through the grid scales as \( e^{-V/T} \). To obtain \( T \), we merely need to find the slope of the current vs. \( V \) on semilog paper, as given in Figure 2.7, and find the change in \( V \) necessary to change the current by a factor of \( e \). The ions are repelled by the first grid biased at 45\(v\); hence no ion saturation current need be subtracted.

We obtain a plasma temperature within the beam of 20.5 volts at \( p = 5.6 \times 10^{-4} \) Torr, and 12.5 volts at \( p = 1.4 \times 10^{-3} \) Torr. The data at \( p = 5.6 \times 10^{-4} \) Torr were taken on a different day than that shown in Figure 2.5. Because of a slight difference in beam alignment, the currents in the two figures at this pressure do not match.

2.3 Axial Variations in Plasma Density

2.3.1 Introduction

In our computer experiments we find that the introduction of plasma density gradients along the direction of beam flow is necessary to explain the relatively narrow spread in the collected beam velocity distribution seen in the laboratory. It is important to confirm that these gradients exist.

Our final computer model (Chapter 6) assumes a sinusoidal axial plasma density distribution. In section 2.3.4, we report Langmuir probe measurements which confirm this assumption. As
a check on these measurements, we examine the problem theoretically. Since our mean free paths are shorter than the system length, we need a diffusion theory in the presence of a magnetic field. The mean free path of one volt ions (the true ion temperature is unknown, but one volt seems reasonable) undergoing ion neutral collisions\(^46\) at \(1.4\times10^{-3}\) Torr \(H_2\) is about 3.5 cm., and of 12 volt electrons undergoing electron neutral collisions\(^47\) is about 24 cm., versus a system length of 40 cm. We should also include the effect of magnetic mirrors, but we have not been able to include mirrors in a diffusion theory. Hence in section 2.3.2, we present the expected axial density variation in a uniform magnetic field, from classical diffusion theory; and in section 2.3.3 we derive the expected density variation in a magnetic mirror, without collisions. Presumably the correct theory will be similar to a combination of the separate theories.

2.3.2 Theoretical Axial Plasma Density Distribution in a Beam-Generated Plasma with a Uniform Magnetic Field

Hopson\(^48\) has considered the axial plasma density distribution of a beam generated plasma in a uniform magnetic field, from classical diffusion theory. The mean free path of the electrons (24 cm vs. a system length of 40 cm.) is a little long to expect classical diffusion theory to be strictly valid, but the results do seem to agree with experiment. If the magnetic field is strong enough so that radial diffusion can be neglected and axial
diffusion is symmetric about the midplane, then ions and electrons must stream out axially together under ambipolar diffusion. The diffusion equation in the steady state is

\[ \nabla \cdot \vec{\Gamma} = n_{bo} \nu_{bi} + n_{po} \nu_{i}, \]  

(2.9)

where \( \vec{\Gamma} \) = particle current, \( n_{bo} \) = the zeroth order beam density, \( n_{po} \) = the plasma density, \( \nu_{bi} \) = the ionization frequency due to the beam, and \( \nu_{i} \) = the ionization frequency due to the plasma.

In ambipolar diffusion, the axial particle current is

\[ \Gamma_z = -D_a \frac{\partial n_{po}}{\partial z}. \]  

(2.10)

Combining Equations 2.9 and 2.10, we have

\[ \frac{\partial^2 n_{po}}{\partial z^2} + \frac{\nu_{i}}{D_a} n_{po} = \frac{n_{bo} \nu_{bi}}{D_a}. \]  

(2.11)

We assume the density at the end walls is zero, because the walls absorb the electrons and ions. The density cannot be exactly zero there, if \( \Gamma_z \) is to be finite, but the approximation should be good.

We consider the extremes of only ionization by the beam, and of only plasma ionization by the plasma electrons. If \( \nu_{bi} = 0 \) (no beam ionization), then the plasma temperature and density must adjust itself so that \( \nu_{i}/D_a = (\pi/L)^2 \), and

\[ n_{po} = n_{po} \text{sin}(\pi z/L), \]  

(2.12)
where \( n_{poo} \) is the midplane density.

If \( \nu_i = 0 \), (only ionization by the beam) then

\[
n_{po} = \frac{n_{bo} \nu_i}{D_a} [Lz-z^2]. \tag{2.13}
\]

Equations 2.13 and 2.13 are plotted in Figure 2.8. We note that the shape of the axial density distribution does not depend very strongly on whether primary or secondary ionization dominates, although the density is concentrated somewhat nearer the midplane if secondary ionization dominates.

2.3.3 Theoretical Axial Plasma Density Distribution 
in a Magnetic Mirror

The derivation of the variation of plasma density vs. axial distance in a magnetic mirror is given in Appendix A, and has been found in a different manner by Post.\(^49\) We ignore space charge electric fields and collisions. Neither can really be ignored, but the result should at least help our intuition in lieu of a diffusion theory in a nonuniform magnetic field. The result is

\[
n_{po} = n_{poo}[(1-B/B_{max})/(1-1/R)]^{1/2}, \tag{2.14}
\]

where \( R = B_{max}/B_{min} \), the mirror ratio. Equation 2.12 is also plotted in Figure 2.8, for a parabolic mirror and \( R = 3 \). This distribution rises and falls considerably faster at the ends than a sinusoid, and is more uniform in the middle. The mirrors tend to keep the plasma from the end walls, and are expected to produce a slightly greater concentration near the midplane than would a uniform field.
Fig. 2.8 Theoretical density distributions with a parabolic magnetic mirror or with a uniform magnetic field and a diffusion dominated plasma. The magnetic mirror ratio is 3.
2.3.4 Langmuir Probe Measurements

Plasma density measurements as a function of axial distance and radius were made with a Langmuir probe. The probe is illustrated in Figure 2.9. The probe tip is a sphere of tungsten. Aluminum oxide insulates the connecting wire up to the 5/16" O.D. glass tubing. The glass tubing slides axially through two Veeco sliding seals soldered together to insure axial alignment of the probe. The probe tip swings in an arc which lies in a plane perpendicular to the glass rod, and which passes through the beam center.

The V-I characteristics of measurements taken at two different radii in the midplane (z = 20 cm.) are shown in Figure 2.10. The measurements were taken at $p = 1.4 \times 10^{-3}$ Torr. At this pressure the discharge seems to be stable to the rotating instability observed by Hartenbaum\textsuperscript{50}. In this case the sphere diameter is 0.067 inches. The same data plotted on semilog paper are shown in Figure 2.11 (after subtracting the ion saturation current), from which we deduce a temperature of 4.6 volts at $r = 0.25$ cm., and of 3.9 volts at $r = 0.5$ cm. These results compare well with the 4 volts obtained by Parker\textsuperscript{51} in a system similar to ours. They are somewhat lower than the temperature of 12.5 volts obtained at this pressure from the velocity analyzer positioned behind the collector and centered on the beam.

There is a question about the extent to which the probe disturbs the plasma. The diamagnetic probe signal as picked up by
Fig. 2.9 Schematic of the Langmuir probe. The probe is free to slide axially or to rotate about its axis.
Fig. 2.10 Langmuir probe curves taken in the midplane at two different radii.

Probe tip diameter is 0.067 in.

-40 -20 0 20 40
VOLTS BIAS

0 40 80 120 160 200 240 280
PROBE CURRENT (mA)

r = 0.25 cm

r = 0.5 cm
Fig. 2.11 Same data as Fig. 2.10, with the ion saturation current subtracted out.
a small loop of wire is indeed perturbed if the Langmuir probe comes closer than 1 cm. from the axis, well outside the region we have measured, which is from 0.25 to 0.6 cm. from the axis. Bartisch $^{52}$ has found in a system similar to ours that the diamagnetic signal is perturbed by a Langmuir probe several inches from the beam. Bartisch $^{30}$ (in a beam-plasma discharge using a 10 kv., 6 amp beam has also found that almost all the diamagnetic signal is produced by a hot electron component ($T \approx 15$ kilovolts) which comprises less than one per cent of the density. The remaining plasma has a temperature of less than 10 volts. However, Parker $^{45}$, in another system similar to ours, removed the magnetic mirrors present in both Bartisch's case and ours, so that the hot electrons escaped. He found that neither the diamagnetic signal nor the plasma density, as measured by a microwave interferometer, were perturbed until the Langmuir probe touched the beam. Unfortunately, we did not monitor any other diagnostics to confirm that the cold plasma was not perturbed. Throughout this thesis we ignore the small percentage of very hot plasma electrons, and assume the beam interacts only with the relatively cold plasma electrons.

To measure the plasma density as a function of distance, we biased the probe at -22.5 volts and measured the ion saturation current. This voltage is more than sufficient to obtain ion saturation. Data were taken 300 $\mu$s after beam turn on, and read from an oscilloscope. The probe tip diameter here is 0.055 in.
The results at \( r = 0.25 \) and \( r = 0.5 \) cm. are shown in Figure 2.12. In these measurements it was not possible to align the probe accurately enough so that the probe tip could pass through the center of the beam for all distances from the gun. However, it was possible for the values shown. These values are the averages of the current measured at the symmetrical points on each side of the beam.

The density can be derived from the ion saturation current by a formula derived by Bohm, Burhop, and Massey\(^{53}\), which is

\[
I_+ = 0.4 \, q_n \, n_p A \left( \frac{qT_-}{M} \right)^{1/2}
\]

(2.15)

where \( A \) is the probe area, \( T_- \) the electron temperature, and \( M \) the ion mass. For a 0.055" spherical probe, and \( T_- \) equal to 4 volts

\[
n_{po} = 9.19 \times 10^{13} \, I_+,
\]

(2.16)

where \( I_+ \) is the ion saturation current. Equation 2.16 yields a peak density of \( 6.9 \times 10^{12} \text{cm}^{-3} \) at a radial distance of 0.25 cm. This corresponds to a plasma frequency of 23.5 GHz, which is about the highest radiated frequency observed by Hsieh.\(^{19}\) The values are compared to a sinusoid in Figure 2.12. The points taken at a radial distance of 0.25 cm. from the axis are a rather good fit to a sinusoid, those points taken further out at 0.5 cm. are a less good fit.

If we are to assume that the density on the axis also varies from the gun, the radial density decay constants should be
Fig. 2.12 Axial ion saturation current at two different radii. The peak density shown is $6.9 \times 10^{12} \text{cm}^{-3}$.

The probe tip diameter is 0.055 in.
independent of distance from the gun. Ion saturation current versus radial distance, at several different axial positions, is shown in Figure 2.13. The slopes at the different values of \( z \) are about the same with the exception of the data taken 10 cm. from the gun. The reason for this discrepancy is unknown. We have not plotted the density on axis because we do not know the forms of the density variations for small radii. This variation is probably different from that well outside the beam because of the larger temperature and hence much larger ionization frequencies inside the beam.

The exponential decay of the radial density is in rough agreement with results obtained from diffusion theory. We assume ambipolar diffusion. We have a glass drift tube so the radial ion and electron currents must be equal. If the axial currents are symmetric, they too will be ambipolar. Using a theory similar to that of Whitehouse and Wollman,\(^5\) we find for the radial and axial particle currents in a uniform magnetic field

\[
\Gamma_z = -D_a \frac{\partial n_{po}}{\partial z},
\]

\[
\Gamma_R = -b \frac{D_a}{a} \frac{\partial n_{po}}{\partial r},
\]

where \( D_a = \mu_+ \mu_- (T_+ + T_-) / (\mu_+ + \mu_-) \), \( b_a = 1 / (1 + \mu_+ \mu_- B^2) \), and \( \mu_+ \) and \( \mu_- \) are the ion and electron mobilities respectively. In a region of no ionization (well outside the beam), \( \nabla \cdot \Gamma = 0 \) in the steady state. The solution for \( n_{po} \) in the steady state is
Fig. 2.13 Radial ion saturation current variation at various distances from the anode. Lines are a least mean square fit, ignoring the points at 0.2 cm. The points shown are the average of two measurements taken symmetrically on each side of the beam.
\[ n_{po} = \sin(\pi z/L) K_0(\pi r/b_a^{1/2}L), \quad (2.19) \]

where \( K_0 \) is the modified Bessel function of zeroth order. At a pressure of \( 1.4 \times 10^{-3} \) Torr in \( H_2 \) and for our magnetic field, \( b_a \) varies from \( 9 \times 10^2 \) to \( 9 \times 10^3 \). \( K_0(x) \) can be approximated by \( (\pi/2x)^{1/2} \exp(-x) \). The resulting exponential decay constants predicted by Equation 2.16 vary with magnetic field from 0.13 cm to 0.4 cm., bracketing the observed decay constants of from 0.25 to 0.35 cm.

A summary of the experimental parameters is given in Table 2.1.

Table 2.1 SUMMARY OF EXPERIMENTAL PARAMETERS

| Beam Characteristics  |  
|-----------------------|---
| Beam voltage          | 8 kilovolts  |
| Injected beam velocity| 5.3 \times 10^9 \text{cm./sec.} |
| Beam current          | 0.4 \text{amps} |
| Nominal beam radius   | 0.05'' = 0.125 \text{cm.} |
| Beam density, assuming nominal beam radius | 9.1 \times 10^9 \text{cm}^{-3} |
| Collected beam velocity distribution | see Figure 2.6 |
| Imposed beam modulation | none |

| Plasma Characteristics  |  
|-------------------------|---
| Plasma density varies longitudinally as a sinusoid with peak density of | 6.9 \times 10^{12} \text{cm}^{-3}(r=0.25 \text{cm.}) |
| Plasma temperature at \( p=1.4 \times 10^{-3} \) Torr \( H_2 \) | 12 volts inside beam |
| Plasma density decays radially with decay constants of | 4.6 volts at midplane (r\approx0.25 \text{cm.}) |
|                         | 0.25-0.35 \text{cm.} |
Chapter 3
A COMPUTER MODEL OF THE BEAM-PLASMA INTERACTION
IN A UNIFORM PLASMA, WITH THE PLASMA REPRESENTED
BY CHARGE SHEETS

3.1 Introduction

In the rest of this thesis we will present a number of models which we have studied in attempts to explain the experimentally observed velocity distribution of the collected beam. The simplest model, the one-dimensional beam-plasma interaction in a uniform, cold, lossless plasma, is presented first. Later in this chapter we introduce a finite plasma temperature; in Chapter 4 a lossy cold plasma; in Chapter 5 density gradients in a cold, lossless plasma; and finally in Chapter 6 a finite diameter beam and plasma density gradients, the results of which agree qualitatively with the laboratory experiment.

Linear theory predicts that the one-dimensional beam-plasma interaction in a cold, lossless plasma should vary in the steady state as \( \exp j(\omega t-kz) \), with

\[
k = \frac{\omega}{v_o} + \frac{j\omega_{pb}/v_o}{(1-\omega^2/\omega_p^2)^{1/2}}, \tag{3.1}
\]

and \( \omega \) real. At \( \omega = \omega_p \), the spatial growth rate is infinite. (Actually, the cold, collisionless case is on the border line between a convective and an absolute instability, and the
concept of an infinite spatial growth rate only has meaning in the sense of, for example, a limit as the collision rate goes to zero.) However, the linear (small signal) assumptions are then invalid, and we need to include beam and plasma nonlinearities to find the correct behavior. We study these nonlinearities by a discrete model using charge sheets to represent the beam and plasma electrons.

In our model a fresh beam is continuously being injected into the plasma at the plane $z = 0$, and collected after passing through the plasma. It represents a continuously replenishable source of free energy. We shall find that in a homogeneous collisionless beam-plasma interaction, with or without plasma temperature, the wave amplitude becomes very large, being limited only by the acceleration of some plasma electrons to velocities on the order of the wave phase velocity, and subsequent randomization of these plasma electrons by the wave. The oscillations are strong and localized, and their properties are strikingly similar to those of the meniscus studied in laboratory experiments by many authors.\[1-3,20-24\] The meniscus is a thin glowing region that appears near the cathode in hot-cathode discharges and low voltage beam-plasma interactions, contains strong localized oscillations near $\omega_p$, and strongly scatters the beam in velocity.

3.2 Model with a Cold Plasma

The model to be used is similar to that of Dawson.\[37\] The beam and plasma electrons are represented by charge sheets.
A sheet is really a super-electron, consisting of many electrons moving together as a unit. It is a mathematical plane of zero thickness in the z direction and infinite extent in the x and y directions. Motion is restricted to the z direction. The ions are represented by a uniform immobile neutralizing background. There is no magnetic field.

At \( t = 0 \), the plasma sheets are uniformly spaced a distance \( 0.02v_o/\omega_p \) apart, with the first plasma sheet being \( 0.01v_o/\omega_p \) from the gun. A plasma sheet has a charge equal in magnitude to 0.99, the ion charge in an initial plasma intersheet spacing. The beam sheets neutralize the remaining ion charge. The last plasma sheet is \( 0.21v_o/\omega_p \) from the collector. There is a section of unneutralized ions near the collector, which is necessary to prevent plasma sheets from oscillating beyond the collector. The section of unneutralized ions at the collector causes a small reduction in the velocity of a collected beam sheet, such that a sheet passing at velocity \( v_o \) through the quiescent plasma is collected with velocity \( 0.98v_o \).

The first beam sheet is infinitesimally in front of the gun, the rest \( 0.2v_o/\omega_p \) apart, giving 10 plasma sheets per beam sheet. We have 314 beam and 314 plasma sheets in a distance of \( 2\pi v_o/\omega_p \) (a nominal wavelength). The system includes 1000 plasma sheets on the average about 100 beam sheets, and about three nominal wavelengths. Each beam sheet has 10/99 the surface charge density of plasma sheet, giving a ratio of total beam to plasma volume charge
density of 1/99. Other workers\textsuperscript{34,35,38} have found 16 to 32 beam sheets or disks per wavelength adequate. However, because of the large plasma to beam density ratio, we have 10 times as many plasma sheets as beam sheets, in an attempt to reduce the discreteness. We do not use even more plasma sheets because of computer costs.

To calculate the sheet trajectories we use the fact that an electron sheet in a uniform ion background performs sinusoidal oscillations about an equilibrium position. If sheet crossings occur, the equilibrium positions shift.

The equation of motion of a sheet (either beam or plasma) in the absence of crossings, is simply

$$\frac{d^2 z}{dt^2} = -\omega_p^2 (z-z_o)$$  \hspace{1cm} (3.2)

where $\omega_p^2 = q^2 n_o/m \varepsilon_o$, $q$ and $m$ are the electron charge and mass, respectively, $n_o$ is the ion density, $\varepsilon_o$ is the vacuum dielectric constant, and $z_o$ is the equilibrium position about which the sheet oscillates. The beam is neutralized, so that $n_o = n_{p0}^+ n_{b0}^-$.

At $t = 0$ the system is neutral and there is no electric field at $z = 0$. When a beam sheet is injected, we assume that the field due to a beam sheet appears only in front of it, so that the condition that $E(z=0) = 0$ is maintained. We discuss later another more important reason for having the field exist only in front of an injected beam sheet.
With zero electric field at the gun, the acceleration of any sheet is found by finding the total charge between the considered sheet and the gun, and adding a charge equal to one-half that of the charge of the considered sheet to find the average field seen by that sheet. The acceleration of a sheet is, from Gauss's law,

$$z_n = \frac{qF}{m} = \frac{q}{\varepsilon_0} (n_o |q| z + \sum_{i=1}^{n-1}(Q_i + Q_n/2).$$  \hspace{1cm} (3.3)

Here $Q_n$ is the surface charge density of the considered sheet, and the $Q_i$'s are negative. The beam and plasma sheets have different surface charge densities. A schematic of the fields is shown in Figure 3.1.

If we know the position, velocity, and acceleration of a sheet at time $t$, we can find these variables at any future time $t + \Delta t$, if no crossings occur involving that sheet. The new position and velocity are valid for any $\Delta t$.

$$z(t + \Delta t) = z(t) + \left[ \frac{a(t)}{\omega_p^2} \right] [1 - \cos \omega_p \Delta t]$$

$$+ \left[ \frac{v(t)}{\omega_p} \right] \sin \omega_p \Delta t,$$  \hspace{1cm} (3.4)

and

$$v(t + \Delta t) = v(t) + \left[ \frac{a(t)}{\omega_p} \right] \sin \omega_p t$$  \hspace{1cm} (3.5)

where $z$, $v$, and $a$ denote the position, velocity, and acceleration, respectively. For $\omega_p \Delta t \ll 1$. $z(t+\Delta t) \approx z(t) + \frac{1}{2} a(t) \Delta t^2 + v(t) \Delta t$. The equilibrium position is $z(t) + \frac{a(t)}{\omega_p^2}$. 
Fig. 3.1 Schematic of a snapshot of the acceleration $\left(\frac{qE}{m}\right)$ versus distance.
This model is exact if no sheet crossings occur. However, crossings do occur, even for a cold plasma, since the beam is moving through the plasma. We correct for crossings after each time-step $\Delta t = 0.05/\omega_p$, which allows a beam sheet moving at velocity $v_o$ to cross an average of two- and one-half plasma sheets per $\Delta t$. This choice of $\Delta t$ conserves energy within 0.02 per cent over any time interval of $30/\omega_p$. In the energy check, we compare the sum of the latest electric and kinetic energies with the sum of the electric and kinetic energies at the previous time ($30/\omega_p$ before) plus the difference in the injected and collected beam sheet kinetic energies. To calculate the injected energy, we include the field created throughout the interaction length by an injected beam sheet.

Because we know the trajectories of two crossing sheets exactly, we could in principle calculate the crossing time exactly. This would, however, involve solving a transcendental equation involving sines and cosines (see Equation 3.4). We use a simpler procedure. Let two sheets be denoted by subscripts 1 and 2, and let $z_1(t) < z_2(t)$, but $z_1(t+\Delta t) < z_2(t+\Delta t)$. During the interval between $t$ and $t + \Delta t$ the sheets cross. Our first estimate of the crossing time is

$$\Delta t_1 = \frac{[z_1(t)-z_2(t)]\Delta t}{[z_2(t+\Delta t)-z_2(t)-z_1(t+\Delta t)+z_1(t)]}$$

(3.6)
This is a straight line estimate from the average velocities. We then insert $\Delta t_1$ in place of $\Delta t$ into the exact trajectories (Equation 3.4). Replacing $z_1(t+\Delta t)$ and $z_2(t+\Delta t)$ by the exact $z_1(t+\Delta t_1)$ and $z_2(t+\Delta t_1)$ in Equation 3.6 now yields a better estimate of the actual crossing time $t + \Delta t_2$. The corrected (primed) trajectories at time $t + \Delta t$ are now

$$z'_{1,2}(t+\Delta t) = z_{1,2}(t+\Delta t) + \frac{A_{1,2}}{\omega_p} \left[ \frac{1}{2} \cos(\Delta t - \Delta t_2) \right]$$  \hspace{1cm} (3.7)

$$v'_{1,2}(t+\Delta t) = v_{1,2}(t+\Delta t) + \frac{A_{1,2}}{\omega_p} \sin(\Delta t - \Delta t_2)$$  \hspace{1cm} (3.8)

$$a'_{1,2}(t+\Delta t) = a_{1,2}(t+\Delta t) + A_{1,2} \cos(\Delta t - \Delta t_2)$$ \hspace{1cm} (3.9)

where $A_1 = qQ_2/m\varepsilon_0$ and $A_2 = -qQ_1/m\varepsilon_0$.

We do not explicitly need the equilibrium positions to calculate the sheet trajectories, but it is helpful to get a physical feeling for what happens to them when two sheets cross. At the equilibrium position a sheet feels no force. If sheet 1 of charge $Q_1$ moves to the right and crosses sheet 2 of charge $Q_2$, the equilibrium position of sheet 1 moves to the right a distance sufficient to uncover enough ion charge to cancel $Q_2$, or a distance $Q_2/n_0q$. Similarly, the equilibrium position of sheet 2 moves to the left a distance $Q_1/n_0q$.

Since the beam sheets are continuously moving through the plasma, the plasma sheets would acquire an average drift toward
the gun (to the left), if we did not insist that the field of a sheet appear only in front of it (to the right). We require that a sheet of charge $Q$ create a field $Q/\varepsilon_0$ in front of it, and have zero field behind it. This is tantamount to assuming that an ion sheet appears somewhere beyond the collector when a beam sheet appears at the gun. The appearance of a beam sheet at the gun causes a net shift to the right of the equilibrium positions of all the other sheets, but when that beam sheet crosses the other sheets, their equilibrium positions shift back, so on the average there is no net drift.

Suppose that instead of the above scheme, we let a beam sheet create a field behind it that, to prevent a net drift of the plasma sheets, vanishes on collection of the beam sheet. At the collector the beam sheets will be bunched, with $\omega_p$ being the most probable bunching frequency. Hence their disappearance would cause $\omega$ wavelength oscillations at $\omega_p$, which could have significant effects on the interaction, since the plasma is resonant at $\omega_p$, and the beam-plasma interaction is known to be very intense at this frequency. Our method prevents any spurious feedback of this sort.

We inject the beam sheets at intervals of $0.2\omega_p^{-1}$. At injection, there is a field created throughout the plasma of $Q_b/\varepsilon_0$, where $Q_b$ is the sheet surface charge density. Hence the electric field throughout the plasma changes stepwise in time at a frequency of $\omega = 10\pi\omega_p$. This frequency is well above $\omega_p$
and is not thought to be troublesome. At $z = 0$, we impose a two per cent velocity modulation on the beam ($v_b(z=0)/v_o = 0.02$) at $\omega_p$, from $t = 0$ to $t = 150/\omega_p$, and do not modulate thereafter. The oscillations continue to grow in the absence of modulation.

3.3 Ordering Routine

To find the acceleration of a sheet from Equation 3.3 it is necessary to order the sheets. The same basic ordering routine is used throughout this thesis. Each sheet is assigned a subscript $L$. If $z(L)$ is greater than $z(L+1)$, the sheets labeled $L$ and $L+1$ are interchanged. We then compare $z(L-1)$ with the new $z(L)$. If these two sheets are disordered, we interchange and compare $z(L-2)$ with the new $z(L-1)$, etc. We continue this process until two sheets are found in order, or the sheet nearest the gun is reached. We then jump up and compare $z(L+1)$ and $z(L+2)$. If they are disordered, we interchange and compare $z(L)$ with the new $z(L+1)$, etc. No case has been found in which the ordering routine has failed.

The routine is illustrated by an example. Consider the case (A - 2.1, B - 1.0, C - 5.2, D - 4.0, E - 3.0). Here A, B, C, D, E represent the labels assigned each sheet, and 2.1, 1.0, 5.2, 4.0, 3.0 represent the positions of each sheet. We want to obtain the situation A - 1.0, B - 2.1, C - 3.0, D - 4.0, E - 5.2. This will be obtained in the following sequence:
<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>1.0</td>
<td>5.2</td>
<td>4.0</td>
<td>3.0</td>
</tr>
<tr>
<td>1.0</td>
<td>2.1</td>
<td>5.2</td>
<td>4.0</td>
<td>3.0</td>
</tr>
<tr>
<td>1.0</td>
<td>2.1</td>
<td>4.0</td>
<td>5.2</td>
<td>3.0</td>
</tr>
<tr>
<td>1.0</td>
<td>2.1</td>
<td>4.0</td>
<td>3.0</td>
<td>5.2</td>
</tr>
<tr>
<td>1.0</td>
<td>2.1</td>
<td>3.0</td>
<td>4.0</td>
<td>5.2</td>
</tr>
</tbody>
</table>

We note that no two sheets will be interchanged at any step in the ordering process if they had not in fact crossed each other. If a sheet A has crossed three other sheets, then sheet A will be switched three times, each time with one of the crossed sheets. Switchings will normally be made in the order in which crossings occur. This fact allows trajectory corrections because of crossing to be made within the ordering routine. However, if sheet B crosses sheet C, and then sheet A crosses both B and C, the ordering routine will incorrectly assume A crossed both B and C before B crossed C. A small error will then be introduced.

This ordering routine seems as fast as any that could be used. It takes advantage of the fact that the sheets are likely to be only partially disordered, with crossings having occurred only with nearby sheets in each time-step. It can, however, handle any degree of disorder.

3.4 Results of the Cold Plasma Model with $n_{po}/n_{bo} = 99$

A snapshot of the beam sheet velocity and sheet acceleration at $t = 30/\omega_p$ is shown in Figure 3.2. Distance is normalized to $0.02v_o/\omega_p$ (the average plasma intersheet spacing), acceleration
$t = 30. / \omega_p$
$n_{bo} / n_{po} = 1/99$
2 per cent beam velocity modulation
Cold plasma

![Graph showing beam velocity and acceleration](image-url)

- Beam velocity
- Acceleration

Distance (units of $0.02v_o / \omega_p$)
to $0.02v_0\omega_p$, and velocity to $v_0$. The crosses represent the instantaneous velocity and position of each beam sheet. At this instant the beam is still laminar; that is, the beam velocity is a single-valued function of distance. The acceleration is plotted at the plasma sheet positions, and lines are drawn between these points. The discontinuities in the acceleration caused by the beam sheets are clearly seen in this graph. There are 10 plasma sheets for each beam sheet, and the acceleration of adjacent plasma sheets is almost the same, unless there is a beam sheet between them. In our units, a beam sheet causes a discontinuity in the acceleration of 0.1, a plasma sheet causes a discontinuity of 0.99.

In Figure 3.3 we see the snapshot of $t = 70/\omega_p$. We see that the beam has gone nonlaminar, that is the faster beam sheets have been able to overtake the slow ones. The acceleration ($qE/m$) is a maximum near the point that overtaking first occurs. At this time the maximum field is slightly over $7(0.02v_0\omega_p)$ and occurs near $z = 550(0.02v_0/\omega_p)$. The spatial growth rate is finite, and not infinite as predicted by Equation 3.1. The reason is that Equation 3.1 is a steady state solution, and in a finite time the beam cannot deliver to the oscillations the infinite energy required to satisfy Equation 3.1. The growth of the interaction in time is discussed more completely in section 3.4.1.

In Figures 3.4–3.8 are shown snapshots of the beam and plasma parameters vs. distance for times $1.5/\omega_p$ apart beginning
Fig. 2 - Scatter plot of beam sheet velocity and acceleration (cm/s^2).
Cold Uniform Plasma Sheets
One Dimensional Beam
T = 260 \omega_p

Fig. 3.4 Snapshots of beam and plasma variables. n_{po}/n_{bo} = 99.
Fig. 3.5 Snapshots of beam and plasma variables. \( \frac{n_{po}}{n_{bo}} = 99 \).
COLD UNIFORM PLASMA SHEETS
ONE DIMENSIONAL BEAM
T = 263/\omega_p

Fig. 3.6 Snapshots of beam and plasma variables, \( n_p/n_o = 99 \).
Fig. 3.7 Snapshots of beam and plasma variables. $n_{po}/n_{bo} = 99$. 
Fig. 3. Snapshots of beam and plasma variables. $n_{po}/n_{bo} = 99$. 

COLD UNIFORM PLASMA SHEETS
ONE DIMENSIONAL BEAM
$T = 266/\omega_p$
at \( t = \frac{260}{\omega_p} \). These figures are the result of programs run for over two hours on the IBM 7094. We have not reached the steady state, because the beam is found to be giving up a time average (averaged over the entire interaction length) of 9.7 per cent of its energy to the oscillations, and the fields are still growing. Further runs have not been made, since the fields are growing very slowly, and it did not seem worth the extra computer time.

At \( t = \frac{260}{\omega_p} \) the maximum field is almost three times that at \( t = \frac{70}{\omega_p} \), and the region of maximum field strength has moved upstream from about 550 to 300(0.02v_o/\omega_p). A traveling wave (by this we mean a wave with a finite phase velocity, but not necessarily a wave with constant shape or with a non-zero group velocity) has been excited, which has a phase velocity of about 0.8v_o near initial beam overtaking at \( t = \frac{70}{\omega_p} \), and about 0.68v_o at \( t = \frac{260}{\omega_p} \), judging from the wavelengths. Since the phase velocity is nearer to the beam velocity than to the induced plasma velocity, we expect the beam to get trapped first, and that is what happens. In this computer experiment the plasma does not get trapped for the time interval shown, but later in this chapter we will discuss some cases where it does. Near the gun the beam rides over the crests of the wave, since the beam is moving faster than the wave there. Further downstream the wave amplitude grows until the beam no longer can ride over the crest of the wave potential. Some beam sheets which have ridden
through the decelerating phase of the wave find themselves slowed to a velocity less than the wave velocity, and fall back into the accelerating phase again. Some beam sheets are actually reversed in direction before falling back into the accelerating phase (Figure 3.4). This process causes the beam sheets to seem to coil around themselves in the snapshot of beam velocity vs. distance.

The fields are large only over a distance of about one-and-one-half wavelengths. Beyond this region the beam sheets essentially drift downstream without much further interaction with the plasma. Bar graphs showing the beam velocity distribution averaged over the times corresponding to Figures 3.4 - 3.7 are shown in Figures 3.9 - 3.11, for different positions downstream. We note that the beam velocity spread downstream is much greater than that observed experimentally (see Chapter 2). The large localized fields that greatly spread the beam in velocity duplicate these aspects of the meniscus.1-3,20-24

Where overtaking first occurs, there is a large beam charge bunch that forms, as in a klystron. There are a large number of sheets within a short distance (near Z = 300 in Figure 3.7) occurring near the point of maximum deceleration. Hence energy is being extracted from the beam in this region in a very efficient manner. At Z = 400 in Figure 3.7 some of the beam electrons have fallen behind in phase so they are being accelerated. They extract energy from the oscillations and compete with the
Fig. 3.9 Beam velocity distribution between 500(0.03v_0/w_p) < z < 700(0.03v_0/w_p), averaged over a period starting at t = 0.
Fig. 3.10 Beam velocity distribution between $700(0.02v_o/w_p) < z < 900(0.02v_o/w_p)$, averaged over a period starting at $t = 260/w_p$. Cold plasma with $n_{po}/n_{bo} = 99$. 
oscillations for energy which had been deposited by the beam sheets at $z = 400$ a half period before (Figure 3.5).

Since the meniscus region is moving upstream, the beam on the average extracts energy from the oscillating fields on the downstream side of the meniscus, and gives up energy to the fields on the upstream side of the meniscus. The plasma itself has no means of energy transport, so any energy that the oscillations have must have been deposited directly by the beam.

Immediately downstream from the meniscus there is a region of very low intensity oscillations. This fact simulates another aspect of the meniscus seen experimentally, which is a dark space that appears just on the anode side of the meniscus. If light is generated by excitation of the neutral atoms by the plasma electrons, and not directly by beam impact (in these experiments the beam density is two to three orders of magnitude less than the plasma density), the low level oscillations are consistent with the dark space. We should point out again that we have not reached a steady state, but presumably a moderate amount of collisional loss could limit the interaction at the oscillation levels shown.

It is obvious that the interaction is quite nonlinear. The fields are strong enough to drive the plasma velocity to within forty per cent of the injected beam velocity. The plasma density reaches values almost five times the zeroth order plasma density (Figure 3.6).
Since the plasma is 99 times as dense as the beam, it is reasonable to think of the plasma as an oscillator, and the beam as a source driving the oscillator. In this case we might neglect the beam charge in Gauss's law, and write

\[ \frac{q}{m} \frac{\partial E}{\partial z} \approx \frac{q}{m_e \epsilon_o} \rho_p', \]

(3.10)

where \( \rho_p \) is the plasma electron density minus the zeroth order (unperturbed) plasma density. We expect the quantity \( q \rho_p \) to be proportional to the derivative of the acceleration (\( qE/m \)) vs. distance. This does seem to be the case. The peak in the plasma electron density in Figure 3.6 corresponds to a very steep gradient (of proper sign) in the acceleration in Figure 3.6. We note that at no time is there a steep negative gradient in the acceleration. This simply reflects the fact that the plasma electron density cannot go negative.

Even at large amplitudes, any individual plasma sheet oscillates sinusoidally in time at \( \omega_p \), with no higher time harmonics, if we ignore beam charge and no plasma sheet crossings occur (see Figure 3.2).

3.4.1 Comparison of the Growth in Time with Linear Theory

In the lossless case, the steady state prediction of linear theory is for an infinite spatial growth rate (see Equation 3.1). However, the small signal assumptions of linear theory are violated by this theory, and to compare our computer experiments with
linear theory we must look for a transient solution which
gives the buildup in time. Briggs\textsuperscript{17} has obtained an asymptotic
solution by the saddle-point method. However, he left out some
factors which we need for a quantitative comparison with our
computer experiments.

At $z = 0$ we impose a velocity modulation on the beam at
$\omega_p$, but no beam density modulation, starting at $t = 0$. That
is
\[ v_b(z=0) = v_{bac} \sin \omega_p t \ u_{-1}(t), \]  
(3.11)

where $v_b$ is the first order beam velocity, $v_{bac}$ is the
magnitude of the modulation and $u_{-1}$ is the unit step function.
We choose $v_{bac} = -10^{-4} v_o$, a value small enough to allow $v_b$
to become large relative to the modulation, so the asymptotic theory
can be valid, but to still be linear. We will show that a source
electric field of the form
\[ E_s(t, z) = E_0 \delta(z) t \ u_{-1}(t) \]  
(3.12)

will produce the required beam modulation. $E_0$ has units of
electric field times velocity. The transform of $E_s$ is given by
\[ E_s(\omega, k) = \int_0^\infty dt \int_{-\infty}^\infty dz \ e^{-j(\omega t - k z)} E_s(t, z) \]  
(3.13)
or
\[ E_s(\omega, k) = -E_0 / \omega^2. \]

The total electric field generated by $E_s$ in the beam-plasma system
is given by
\[ E(\omega,k) = \frac{\rho_b + \rho_p}{-jk\varepsilon_0} + E_s, \quad (3.14) \]

where we have used the transform of Gauss's Law, and have assumed that \( \rho_b \) and \( \rho_p \), the first order beam and plasma charge densities, respectively, are generated by the total electric field, but that \( E_s \) is generated by an external source. The plasma electron force equation is given by

\[ j\omega v_p = qE/m. \quad (3.15) \]

The beam electron force equation is given by

\[ j(\omega - kv) v_b = qE/m. \quad (3.16) \]

The plasma and beam conservation equations are, respectively,

\[ -jk(\rho_p v_p) = -j\omega \rho_p, \quad (3.17) \]

and

\[ -jk(\rho_b v_b + \rho_b v_o) = -j\omega \rho_b. \quad (3.18) \]

Combining Equations 3.14 - 3.18, we obtain for \( E(\omega,k) \),

\[ E(\omega,k) = \frac{E_s(\omega,k)}{1 - \left( \frac{\omega_p}{\omega} \right)^2 - \left( \frac{\omega_{pb}}{(\omega - kv_o)} \right)^2} \quad (3.19) \]
Combining Equations 3.13, 3.16, and 3.19, we obtain \( v_b(t,z) \):

\[
v_b(t,z) = -\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega \, dk \, e^{i(\omega t - kz)} \times \frac{(q/m) E_o}{j\omega^2 (\omega-kv_o) \{1 - \frac{\omega_p}{\omega^2} - \frac{\omega_pb}{(\omega-kv_o)^2}\}}.
\]  

Equation 3.20 can be rewritten to insure convergence of the inverse transform, and to see more easily the limits as \( \omega_p \to 0 \) or \( \omega_pb \to 0 \):

\[
v_b(t,z) = -\frac{1}{(2\pi)^2} \frac{qE_o}{jm} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega \, dk \, e^{i(\omega t - kz)} \left( \frac{1}{\omega^2 (\omega-kv_o)} + \frac{\omega_p^2}{\omega^2 (\omega-kv_o) (\omega^2-\omega_p^2)} + \frac{1}{(\omega-kv_o) v_o^2} \right) \times \frac{\omega_pb^2 \omega}{(\omega^2-\omega_p^2)^2 \{k-\frac{\omega}{v_o} + K(\omega)\} \{k-\frac{\omega}{v_o} - K(\omega)\}}.
\]  

(3.21)

where \( K(\omega) \equiv k_b (1-\omega_p^2/\omega^2)^{-1/2} \), and \( k_b \equiv \omega_pb / v_o \).

We perform the integration in the \( k \)-plane first. If \( \text{Im} \phi \to -\infty \), all the poles in the \( k \)-plane lie below the real \( k \) axis. Hence by the Bers-Briggs criterion, the contour of integration in the \( k \)-plane is above these poles. For \( z > 0 \) we close the contour below the real \( k \) axis. Some of the terms in Equation 3.21 can be integrated in the \( \omega \)-plane. By casuajly,
all the poles in the \( \omega \)-plane lie above the \( \omega \) contour. After a
canceling of terms, the result is

\[
v_b(t,z) = -\frac{qE_o}{mv_o^2 \pi} \int_{-\infty-j\sigma}^{\infty-j\sigma} d\omega \ e^{j\omega(t-z/v_o)} \frac{\cos K(\omega)z}{(\omega^2 - \omega_p^2)}.
\]  

(3.22)

At \( z = 0 \) the integral can be performed by contour integration.
The result is

\[
v_b(t,z=0) = \frac{qE_o}{mv_o \omega_p} \sin \omega_p(t) u_1(t),
\]

(3.23)

which agrees with Equation 3.11 with \( v_{bac} = -qE_o/mv_o \omega_p \).

For \( z > 0 \) the essential singularity at \( \omega = \omega_p \) prevents integration
of Equation 3.22 by Cauchy's theorem. However, the saddlepoint
method\(^5\) can be used. For an integral of the form \( \int_c e^f(t) \, dt \), we have

\[
\int_c e^f(t) \, dt = e^{f(t_0)} \frac{i^{\pi f''(t_0)}}{\sqrt{2\pi}}.
\]

(3.24)

The function \( \phi(\omega) = \omega \tau - K(\omega)z \), where \( \tau \equiv t-z/v_o \), has saddle points
at

\[
\omega_s = \frac{+\omega_p}{1 + (k_bz/\omega_p)^{2/3}} e^{-j2\pi/3} \left\{ \frac{k_bz}{\omega_p \tau} \right\}^{1/3}.
\]

(3.25)

We need

\[
K(\omega_s) = k_b (1 - \omega^2/\omega_s^2)^{-1/2} = k_b \frac{\omega_s}{\omega_p} e^{+j\pi/3} \left\{ \frac{k_bz}{\omega_p \tau} \right\}^{1/3},
\]

(3.26)

and

\[
K''(\omega_s) = \frac{3k_b \omega_s^3}{(\omega_s^2 - \omega_p^2)^{5/2}} = \frac{3k_b \omega_s^3}{\omega_p^5} \frac{k_bz^{5/3}}{\omega_p^5} e^{-j5\pi/3}.
\]

(3.27)
Hence Equation 3.22 is approximately, after adding the contributions at both saddle points,

\[ v_b(t, z) = -\frac{qE_o}{mv_o \sqrt{6\pi k_b z}} \frac{F^{1/6} e^{Re(j\omega_s - jK(\omega_s)z)\cos(\theta)u_{-1}(\tau)}}{\omega_p (1-F^2/3 + F^4/3)^{3/8}} \]

where \( F = (k_b z/\omega_p \tau) \), and

\[ \theta = Re\{\omega_s \tau - K(\omega_s)\} - 17\pi/12 + \frac{3}{4} \tan^{-1}\left\{\frac{3}{2} \frac{1/2 F^2/3}{(1 - \frac{1}{2} F^2/3)}\right\}. \]

(3.28)

For \( F = k_b z/\omega_p \tau \ll 1 \), Equation 3.28 reduces to

\[ v_b(t, z) = -\frac{qE_o}{mv_o} \frac{(3\sqrt{3}/4)(k_b z)^{2/3}(\omega_p \tau)^{1/3} e^{-\frac{3}{4}(k_b z)^{2/3}(\omega_p \tau)^{1/3}}}{\omega_p \sqrt{6\pi k_b z}} (k_b z/\omega_p \tau)^{1/6} \cos\{\omega_p \tau - \frac{3}{4}(k_b z)^{2/3}(\omega_p \tau)^{1/3} - 17\pi/12\} u_{-1}(\tau). \]

(3.30)

In Figure 3.12, Equation 3.28 is compared with the snapshot of the beam velocity at \( t = 150/\omega_p \). The beam is modulated at \( z = 0 \) such that \( v_b(z=0) = -10^{-4}v_o \sin(\omega_p t) u_{-1}(t) \). For small values of \( v_b/v_o \), the asymptotic theory is invalid. For \( v_b/v_o \) too large, the linear assumptions are invalid. There is a region in Figure 3.12 where \( 0.005 < |v_b/v_o| < 0.05 \) where the asymptotic theory and the computer experiment agree quite well, except for a phase shift of order \( \pi/30 \).
Fig. 3.12 Snapshot comparing the beam sheet velocity of the computer experiment with that of the asymptotic linear theory.
We should point out that what we had been calling \( \omega_p \) in other sections of this chapter is actually \( (\omega_{pe}^2 + \omega_{pb}^2)^{1/2} \). In the comparison with linear theory we modulated at the true \( \omega_{pe} \), whereas in the other parts of this chapter we modulated at \( (\omega_{pe}^2 + \omega_{pb}^2)^{1/2} \). The difference between the magnitudes of the first order beam velocity with the two modulations at \( t = 140/\omega_p \) is less than five per cent, although the response to \( \omega = \omega_{pe} \) is larger. In Chapters 4 - 6 we modulate at the true \( \omega_{pe} \).

3.5 Cold Plasma Sheet Model with Beam Density Equal to 1/19 the Plasma Density

Because of computer costs we did not let the model with beam density equal to 1/99 the plasma density run longer than \( t = 266/\omega_p \). In that case the cold plasma remains laminar, and the plasma velocity is a single valued function of distance. However, with a beam density to plasma density ratio of 1/19, some plasma sheets are accelerated to velocities equal to or greater than the wave phase velocity by \( t = 170/\omega_p \) (Figure 3.13). These plasma sheets cross other plasma sheets, so that the plasma becomes nonlaminar. The fast sheets are swept downstream, and extract energy from the waves. We feel that this loss mechanism probably would occur in the less dense beam case if we let the programs run longer.

At \( z = 0 \), we impose a velocity modulation of \( 0.02 v_o \) at \( \omega_p \) on the beam throughout the run. There is no beam density modulation at \( z = 0 \). In this model we collect beam sheets at the collector.
Fig. 3.13  Snapshot of beam and plasma variables. \( \frac{n_b}{n_p} = 1/19, \ t = 70/\omega_p. \)

Two per cent velocity modulation. imposed on beam at \( z = 0. \)

Velocities normalized to \( v_o, \) acceleration to \( 0.02v_o\omega_p. \)
after they pass through the plasma, but reflect plasma sheets at the collector and at the gun. Each sheet is still considered to be oscillating about an equilibrium position, but in this case a plasma sheet is allowed to oscillate beyond the collector. It is "reflected" by letting it oscillate back. Because of the large velocities obtained by the plasma sheets in this model, even by plasma sheets oscillating in a laminar fashion near the collector, setting the collector beyond the furthest penetration of any plasma sheet would have prevented many beam sheets from being collected. An alternative model (and one we will use in section 3.6 where we include plasma temperature) would be to specularly reflect the plasma sheets off a hard wall at the collector. We do not want to use a hard wall here, because an essential feature of plasma oscillations in a cold plasma is that the plasma velocity is a single valued function of distance, an attribute which would be destroyed at the wall.

We can construct a simple model to estimate the magnitude of the fields necessary to accelerate plasma electrons to the wave phase velocity. Early in the interaction the wave amplitudes are small, and the plasma sheets have much smaller velocities than the wave velocity $v_\phi$. Hence they merely ride over the crest of the wave as it goes by. We transform to a frame moving at $v_\phi$, and define an electric potential $\phi$ for the wave. Let the wave have a peak potential $\phi_0$, as shown in Figure 3.14.
Fig. 3.14 A particle at zero potential with velocity $v_1'$ in the wave frame has velocity $v_2'$ when $\phi = \phi_o$. 
Let a plasma sheet have an initial velocity \( v_1 \) in the lab frame, or \( v_1' = v_1 - v_\phi \) in the wave frame. A non-zero \( v_1 \) will let us consider the effects of plasma temperature later on. The sheet will have velocity \( v_1 \) whenever the sheet passes a plane such that \( \phi = 0 \). The velocity at the top of the potential hill will be

\[
v_2' = -(v_1'^2 - 2q\phi_o / m)^{1/2}
\]  
(3.31)

with the minus sign in front of the radical if \( v_2 < v_\phi \). If the sheet is moving at exactly the wave phase velocity, \( v_2' = 0 \) or

\[
\frac{q\phi_o}{m} = \frac{(v_1 - v_\phi)^2}{2}.
\]

(3.32)

The determination of \( \phi_o \) by Equation 3.32 is exact. However, we have only measured the electric field, and not \( \phi \). Since \( E = -\Delta\phi \), we must assume some spatial variation for \( \phi \) to obtain the approximate electric field required to accelerate the plasma sheet to the wave phase velocity. If we let \( \phi \sim R e^\nu j k z \), then

\[
\frac{qE_o}{m} = \frac{k(v_1 - v_\phi)^2}{2},
\]

(3.33)

where \( E_o \) is the peak electric field. If \( k = \omega_p / v_\phi \), then

\[
\frac{qE_o}{mv_\phi \omega_p} = \frac{(1 - v_1/v_\phi)^2}{2}.
\]

(3.34)

As a check on Equation 3.34, we note that if \( v_1 = v_\phi \), the field required to accelerate a plasma sheet to \( v_\phi \) vanishes, as expected.
If \( v_1 = 0 \) (cold plasma), \( qE/m = 0.5v_0^2 \). In our units, where \( qE/m \) is normalized to \( 0.02v_0^2 \), we expect \( A = 25v_0/v_\phi \), where \( A \) is the normalized acceleration. Here \( v_\phi \) is about \( 0.6v_0 \) in the peak field region, judging from the wavelength; hence \( A \) should be about 15. It is actually a little higher, being about 19. Our assumption of a sinusoidal field in space is not quite true, so our estimate is only approximate. The fields build up to about the same magnitude in the \( n_{po}/n_{bo} = 99 \) case as here, but the phase velocity is somewhat higher there (about \( 0.68v_0 \)), hence no plasma sheets are accelerated to \( v_\phi \) in that case.

A small difference in \( \phi_0 \) makes a big difference in the plasma velocity, when \( \phi_0 \) is near the potential required to accelerate a plasma sheet to \( v_\phi \) (the critical potential \( \phi_c \)); then the peak velocity in the lab frame of an initially cold sheet is only \( 0.9v_\phi \), as opposed to \( v_\phi \) if \( \phi_0 \) equals \( \phi_0 \). This explains why the plasma sheets with velocities approaching \( v_\phi \) seem to rise almost vertically above the other sheets in the snapshot of plasma velocity vs. distance. Small changes in the field amplitude with position near where \( \phi = \phi_c \) make striking differences in the plasma velocity.

The meniscus in this case is less well defined than in the \( n_{po}/n_{bo} = 99 \) case; that is the fields downstream from the plane of initial beam overtaking are larger relative to the meniscus. One possible reason for the difference is that the meniscus moves upstream so quickly that energy on the downstream side of the
meniscus cannot be as efficiently extracted as in the less dense beam case; transients can persist in this lossless model.

In Figure 3.15 we show the snapshots of the variables at \( t = 80/\omega_p \). We see that only 3 of the sheets accelerated to almost \( v_\phi \) at \( t = 70/\omega_p \) (Figure 3.13) were able to travel very far from their initial positions, and appear near \( Z = 800 \), where \( Z = z(0.02v_o/\omega_p) \). Another group of plasma sheets have been strongly accelerated around \( t = 76/\omega_p \), and appear near \( Z = 500 \). A third group is being strongly accelerated near \( Z = 250 \).

In Figure 3.16 are shown snapshots taken at \( t = 94/\omega_p \). The fields near the collector region. A in Figure 3.16 are starting to accelerate plasma sheets to velocities near \( v_\phi \). However, at \( t = 96/\omega_p \) (Figure 3.17), we see that these sheets were not able to travel far from their initial positions (region B in Figure 3.17). Apparently the beam is still bunched enough to give up energy to the oscillations downstream. Some downstream oscillations are also apparent in the \( n_{po}/n_{bo} = 99 \) case, but not to this extent.

At the regions where some plasma sheets have been accelerated to velocities near \( v_\phi \) and swept away by the wave, the remaining plasma sheets seem to have acquired a velocity distribution in addition to their oscillatory velocities. When two plasma sheets cross each other, they interchange equilibrium positions. Hence when a plasma sheet is strongly accelerated to a velocity near \( v_\phi \) by the wave and crosses a sheet downstream from it, the downstream sheet moves into the place vacated by the fast sheet (if the
Fig. 3.15  Snapshot of beam and plasma variables. $n_{bo}/n_{po} = 1/19$. $t = 80/\omega_p$. Two per cent velocity modulation imposed on beam at $z = 0$. Velocities normalized to $v_o$, acceleration to $0.02v_0/\omega_p$. For $t < 76/\omega$, these sheets were near $Z = 250$. For $t < 70/\omega$, these sheets were Rear $Z = 250$. 
Fig. 3.16  Snapshot of beam and plasma variables. \(\frac{n_{bo}}{n_{po}} = 1/19\). \(t = 94/\omega_p\). Two per cent velocity modulation imposed on beam at \(z = 0\). Velocities normalized to \(v_o\), acceleration to \(0.02v_0\omega_p\). In region A, plasma sheets are being accelerated to velocities near \(v_{\phi}\).
Fig. 3.17  Snapshot of beam and plasma variables. $n_{bo}/n_{po} = 1/19$, $t = 96/\omega$. $v_p(z=0) = 0.02v_o$.Velocities normalized to $v_o$, acceleration to $0.02v_0\omega$. In region B, most of the plasma sheets accelerated in region A of Fig. 3.16 did not obtain $v_\phi$, but fell back in phase relative to the wave.
downstream sheet is not also accelerated), but with an extra oscillatory velocity due to the crossing that the sheet immediately upstream from the fast sheet will not have. That is, the crossed sheet gets a kick when it is crossed, but the upstream sheets do not. Also, some plasma sheets are almost, but not quite, accelerated to \( v_\phi \), and fall back in phase. These sheets can acquire a sufficient velocity difference from that of their neighbors to cross some of them, again destroying the laminar flow.

By \( t = 160/\omega_p \) (Figure 3.18) the originally cold plasma sheets have acquired a velocity spread, presumably caused by the shifting of equilibrium plasma sheet positions described above. Many have been accelerated to velocities on the order of \( v_\phi \) and have been reflected by space charge at the ends. The plane of initial beam overtaking has moved upstream. We recall that the linear theory predicts a steady state with an infinitely fast spatial growth rate. The interaction is tending to this state in the small signal section near the gun; that is, these fields are still growing. The loss caused by the acceleration of sheets to velocities on the order of \( v_\phi \) does not affect the fields near the gun until plasma and beam sheets are reflected back to the gun region. These reflections become important by \( t = 230/\omega_p \) (Figure 3.19), where even the plasma at the gun has a velocity spread.

3.6 Heating by Longitudinal Waves in the Presence of Reflecting Walls

In Figure 3.19 we find that some plasma sheets have been
Fig. 3.18 Snapshot of beam and plasma variables. $n_{bo}/n_{po} = 1/19$. $t = 160/\omega_p$. $v_p(z=0) = 0.02v_o$. Velocities normalized to $v_o$, acceleration to $0.02v_o\omega_p$. 
Fig. 3.19 Snapshot of beam and plasma variables. $n_{b0}/n_{p0} = 1/19$, $t = 230/\omega_p$, $v_p(Z=0) = 0.02v_o$. Velocities normalized to $v_o$, acceleration to $0.02v_o\omega_p$, distance to $0.02v_o/\omega_p$. 
accelerated to velocities greater than $1.5v_o$. It is certainly possible that this velocity is attainable before the plasma sheet is reflected from the collector end. For example the plasma sheet near $Z = 900$ in Figure 3.15 has only had time to drift from the position where it was accelerated to a velocity of about $v_\phi$ near $t = 70/\omega_p$ at $Z = 250$, and it could not have suffered an end reflection. Nevertheless, for completeness we present a discussion of heating by reflections of various types of walls, in the presence of a longitudinal traveling wave. This heating could have had some effect on the plasma sheets which had wave speeds much higher than those in the main part of the plasma velocity distribution, although we have not attempted to separate this heating from the velocity gains due to the plasma sheets' being accelerated to $v_\phi$ and to higher velocities directly by the wave.

Another possible heating mechanism, which we have not investigated could be the interaction of plasma electrons with waves created by nonlinearities, if these waves have phase velocities in the range of some plasma electrons.

3.6.1 Approximate Theory of the Heating Rate of Wave-Mirror Heating

Consider a uniform, longitudinal wave with an electric field given by

$$E = E_o \cos(\omega t - kz) \hat{z}. \tag{3.35}$$
Let a low-energy electron (|v_1| \ll \omega/k) move parallel to this field and be contained within two hard walls, which specularly reflect the electron. The electron can gain much more energy from the wave if the walls are present than if they are not. This is seen qualitatively with reference to a simple physical model. An electron moving slowly relative to the wave's phase velocity experiences a force varying sinusoidally in time until it collides with a wall. Before the collision, it is alternately accelerated and decelerated, and has an approximate energy of \( \frac{m}{2} (v_1 + \frac{qE_0}{\omega m} \cos \omega t)^2 \), where \( v_1 \) is its net drift velocity. Suppose, however, that an electron suffers a perfectly elastic collision with a wall just as the electric field is changing from accelerating to decelerating. The magnitude of the electron velocity will continue to increase. This process can be repeated many times. Of course, the electron can also lose energy by this method, and a random walk will actually occur.

We shall assume that the phase of each wall collision is random with respect to the wave. For a long system, a random variation in the electron or phase velocity should randomize the phase of the collisions.

3.6.1.1 Velocity Gain

Consider a coordinate system moving with the wave, in which we can define a potential (Figure 3.20). As our reference point we shall consider the minimum of the potential energy for electrons. After an even number of collisions, we can again look at the energy
Fig. 3.20  A plot of potential energy vs. distance, in the wave frame. An electron moving in the direction of the wave suffers a reflection at point 2, at a potential $\delta$ above the potential minimum. Moving against the wave, it suffers a reflection at point 4, at a potential $\beta$ above the potential minimum.
of the electron at this point to see if it has gained or lost peak energy.

We shall denote velocities relative to the moving system by primes; those relative to the fixed system will not have primes.

Consider an electron with velocity \( v_1 \) in the fixed frame, at position 1 in Figure 3.20. In the moving frame its velocity is \( v'_1 = v_1 - v_\phi \), where \( v_\phi = \omega / k_\perp \). Its velocity at point 2, just before a collision at a potential \( \delta \) is

\[
v'_2 = -\sqrt{\frac{v'_1^2 - 2\delta}{m}}.
\]

(The velocity in the moving (primed) system is always negative for electrons moving slower than the wave's phase velocity.)

In the fixed system its velocity is

\[
v_2 = v'_2 + v_\phi.
\]

After the collision,

\[
v_3 = -v'_2 - v_\phi
\]

and

\[
v'_3 = -v'_2 - 2v_\phi.
\]

At point 4, just before another collision at potential \( \beta \),

\[
v'_4 = -\sqrt{\frac{v'_3^2 - 2(\beta-\delta)/m}{\beta}}
\]

(3.40)
After this collision

\[ v'_5 = -v'_4 - 2v_\phi. \]  \hspace{1cm} (3.41)

At point 6, on returning to the bottom of the potential well,

\[ v'_6 = -\sqrt{v'_5^2 + 2\beta/m}. \]  \hspace{1cm} (3.42)

In the fixed system,

\[ v_6 = v'_6 + v_\phi. \]  \hspace{1cm} (3.43)

In order to solve Equations 3.36 - 3.43, we must make some approximations. We shall assume that \( v_1^2, \delta^2/m^2v_\phi^2, \beta v_1/mv_\phi \), and \( \delta v_1/mv_\phi \) can all be neglected compared with \( v_\phi^2 \). This assumes that the initial velocity of the electron is small compared with the phase velocity of the wave, and that the potential energy of the wave is small compared with the kinetic energy of an electron moving at the wave's phase velocity. The last assumption implies also that the electron is not trapped by the wave. With these assumptions, the final velocity of the electron is

\[ v_6 \simeq v_1 + 2(\delta-\beta)v_\phi/mv_\phi^2. \]  \hspace{1cm} (3.44)

Here, \( v_6 \) is the velocity of an electron after suffering two abrupt, perfectly elastic collisions. The first collision occurs at a potential energy \( \delta \) above the potential minimum, the second at a potential energy \( \beta \) above the minimum. \( v_1 \) and \( v_6 \) are both
measured at the potential energy minimum. \( \delta \) and \( \beta \) will be treated
as independent random variables.

3.6.1.2 Probability Theory

Let us normalize the random variables by setting

\[
\delta = \eta v_1/2 + v_1/2 \quad (3.45)
\]

\[
\beta = \gamma v_1/2 + v_1/2, \quad (3.46)
\]

where \( v_1 \) is the peak-to-peak potential energy, and \( \eta \) and \( \gamma \) are
independent random variables such that \(-1 \leq \eta \leq 1; -1 \leq \gamma \leq 1\).

From Equations 3.44 - 3.46

\[
v_6 = v_1 + \frac{v_1}{mv_1} (\eta - \gamma) v \phi \quad (3.47)
\]

We assume that a collision at any phase of the wave is equally
probable. The probability density function of \( \eta \) is

\[
f_\eta(\eta_o) = \frac{1}{\pi \sqrt{1 - \eta_o^2}} \quad (3.48)
\]

The probability that \( \eta \) will have a value within the interval
\( d\eta_o \) at \( \eta_o \) is \( f_{\eta}(\eta_o) d\eta_o \) (\( \eta \) and \( \gamma \) have the same probability
distribution). The characteristic function\(^{36} \) of \( \eta \) is defined as

\[
\mathcal{M}_\eta(v) \equiv \int_{-1}^{1} f_{\eta}(\eta) e^{jv\eta} d\eta,
\]
or

\[
\mathcal{M}_\eta(v) = J_0(v), \quad (3.49)
\]

where \( J_0 \) is the zeroth order ordinary Bessel function of the
first kind.
Let

\[ Y = \sum_{i=1}^{N/2} \eta_i - \sum_{j=1}^{N/2} \gamma_j, \]

where \( \eta_i \) is the value of \( \eta \) at the \( i \)th collision, and \( N \) is the total number of collisions with both walls. The characteristic function of \( Y \) is

\[ \mathcal{M}_Y(v) = \left[ J_0(v) \right]^N, \quad (3.50) \]

and the variance of \( Y \) is

\[ \mathbb{E}(Y^2) = N/2. \]

Since the collisions are assumed to be independent random events, \( v_6 \) is proportional to \( Y \).

The expected value of \( v_6^2 \), after \( N \) collisions is,

\[ v_6^2 \approx v_1^2 + \frac{v_1^2}{(mv_2^2)^2} v_\phi^2 \frac{N}{2}. \quad (3.51) \]

Hence on the average an electron will gain energy by wall collisions.

We can do a similar analysis of a model in which the electrons re-entering the wave, after a collision with a wall, do so at a phase independent of the one they had just before the wall collision. At the first wall collision, let the electron leave the wave at potential \( \beta \), as before, but let it reenter the wave at a random potential \( \beta_1 \) independent of \( \beta \). Similarly, at the next wall collision let the electron leave the wave at a potential \( \delta \) and reenter at a random potential \( \delta_1 \) independent of \( \delta \). Equation 3.44 now becomes
\[ v_6 = v_1 + (\delta + \delta_1 - \beta - \beta_1) v_\phi / m v_\phi^2. \] (3.52)

If \( \delta = \delta_1 \) and \( \beta = \beta_1 \), Equation 3.44 is recovered. Equation 3.52 was twice as many independent random variables as Equation 3.44, but the coefficient multiplying these variables is only half as large. Following a similar derivation as that which led to Equation 3.51 we find

\[ v_6^2 = v_1^2 + \frac{v_1^2 v_\phi^2}{(m v_\phi^2)^2} \frac{N}{4}. \] (3.53)

The heating rate when the electron reenters the wave on a wall collision at a random phase relative to the entering phase is only half as large as that obtained (Figure 3.51) when these phases are the same.

3.6.2 Computer Experiments of Wave-Wall Heating

Computer experiments were run to check the theory of section 3.6.1, since a number of approximations are made there. Electrons are initially placed at \( z = \lambda / 2 \), where \( z = 2\pi / k \), with a velocity of \( 0.1\omega k \), in the presence of a uniform longitudinal wave of magnitude \( q E / m = 0.02(\omega^2 / k) \cos(\omega t - kz) \). The electrons are placed at the bottom of the wave potential. The electrons are reflected at random phase by allowing the end wall positions to vary randomly over a range of \(-\lambda < z < 0 \) and \( \lambda / 2 < z < \frac{3\lambda}{2} \). The trajectories are then calculated by the Milne integration routine started by the Runge-Kutta integration routine (see Chapter 4, section 4.2 for more detail on these routines). The electrons bounce back and
forth between the walls, and their velocities are measured, again at
the bottom of the wave potential, after every second collision, when
the velocities are positive. This process is carried out for a
number of electrons, and the square of their velocities is averaged.
The results are shown in Figure 3.21, and compared with the predic-
tion of linear theory (Equation 3.51). The computer experimental
kinetic energy gain is well above the predicted values. The most
probable reason is that our choice of $v_1 = 0.1 v_\phi$ is too large
compared to $v_\phi$ to justify keeping terms of order $v_1/v_\phi$ but dropping
the higher order term of $v_1^2/v_\phi^2$ as we did in section 3.6.1. The
error of about ten per cent per collision apparently grows. Never-
theless the electrons gain considerable energy from the reflections
in the presence of the wave, and non-linearities do not decrease
the heating below that predicted in section 3.6.1, but rather
seem to increase it.

We also ran a case such that not only is the phase random
at which an electron hits a wall, but the phase at which the
electron reenters the wave is again randomized. The analytic
theory predicts only half as much energy gain per reflection as
for specular reflection (compare Equations 3.51 and 3.52). The
computer experimental energy gain is well below that of the computer
results for specular reflections (Figure 3.21), but is above the
predictions of linear theory.

We feel that the ability of an electron to jump phase is
critical for it to be able to gain energy from a wall reflection.
Fig. 3.21 Computer experimental results of energy gain vs. number of collisions for hard walls (reflection at the same phase) and for reflection at random. The solid line represents the analytic theory for hard walls and the scatter plot represents the experimental data.
This hypothesis is strongly supported by computer experiments involving "soft" walls, or reflections in which the electron is reflected slowly, so that it cannot jump phase. In Figure 3.22 we see the results of a computer experiment in which the wave decays spatially, after passing a wall \((z = z_m)\), as \(qE/m = 0.02v_\phi \omega x e^{-k|z-z_m|} \cos(\omega t-kz) - 0.001v_\phi \omega\). In this case the electrons take about 30 periods to be reflected. Their furthest excursion past \(z_m\) is \(5/k\), so the wave is reduced in magnitude at that point by a factor of \(e^{-5}\). In this case there is no detectable energy gain, although there is a slight scatter in the values of \(v^2\) for the individual electrons. In the same figure we show the results of essentially truncating the wave field at the wall, but letting the electron be reflected by a constant force. In this case \(qE/m = 0.02v_\phi \omega e^{-1000k|z-z_m|} \cos(\omega t-kz) - 0.02v_\phi \omega\). In this model the electron can jump phase, and the results are similar to those obtained by allowing the electron to reenter the wave at a random phase.

In the results of section 3.5, with \(n_{p0}/n_{b0} = 19\), the fields near the collector are comparable to those elsewhere. The oscillating field is tied to the cold plasma sheets and vice versa. Hence the cold sheets cannot jump phase. However, hot plasma sheets which are reflected at the collector can leave the oscillations and jump phase on reentering the field, and can be heated. The transit time of an electron moving at \(v_o\) is \(20/\omega_p\), so by \(t = 230/\omega_p\) some hot
Fig. 3.22 Comparison of the energy gain of electrons reflected by a constant force field in the presence of a wave that cuts off sharply in distance, with one that decays gradually.
plasma sheets have made several transits. The reflections from the gun end should not cause much heating, since the fields there are small.

In the next section we will allow the plasma sheets to have temperature, and will reflect them with hard walls (specular reflection). The transit time of a sheet at the thermal velocity is \( 265/\omega_p \), longer than the time of the run. Hence the hard walls should not appreciably heat the temperature sheets, but could contribute to the heating of plasma sheets which have been accelerated to velocities on the order of \( v_\phi \), and whose transit time is much shorter. The hard walls do, however, allow us to study a uniform plasma, without end sheaths.

3.7 Warm Plasma Model

We now let the plasma have temperature. The primary motivation is to see if the interaction intensity is reduced enough to obtain the relatively narrow collected beam velocity distribution seen in the laboratory. This is not the case. We shall find, however, that the fields are reduced from the cold case, because plasma sheets in the tail of the distribution are more easily accelerated to the wave phase velocity, \( v_\phi \). The computer program for this model is given in Appendix B.

We assign velocities to each plasma sheet according to a Maxwellian velocity distribution. The Maxwellian is generated by using the random number generator RANNO. In the computer. Each
call of RANNO. gives an independent random number $x_i$ between 0 and 1 with a rectangular probability distribution. Each sheet is given a velocity

$$v = v_T(12/100)^{1/2} \sum_{i=1}^{100} (x_i - 0.5), \quad (3.54)$$

where $v_T$ is chosen to be $0.075v_o$. The velocity distribution generated is approximately

$$f(v) = \frac{1}{(2\pi)^{1/2} v_T} \exp(-v^2/2v_T^2). \quad (3.55)$$

The distribution generated for 1000 sheets is shown in Figure 3.23, where it is compared with a Maxwellian. We see that the fit is reasonably good.

We contain the plasma sheets by specular reflection off hard walls at each end, rather than by space charge, as in the cold plasma case. This containment scheme allows us to study a reasonably uniform plasma, and we do not have to worry about the tails of the Maxwellian being truncated by passing through sheaths at the ends, as would happen if we collected plasma sheets as well as beam sheets.

Reflections from hard walls create some programming problems. If two sheets cross near a wall, they may cross before or after a reflection, or both. A plasma sheet crossing a beam sheet may find that the beam sheet has been collected before crossing. These effects and others may produce unreasonable estimates of crossing times, but they all are eventually resolved.
Fig. 3.23 Comparison of computer generated Maxwellian with true Maxwellian, for 1000 sheets.
We have 7.5 sheets per Debye length, and an interaction length of about three wavelengths. We have 2000 plasma sheets and initially 100 beam sheets in the interaction region. The beam is 1/99 as dense as the plasma. The plasma intersheet spacing is half what it was in the cold case, and we normalize to it, which is $0.01v_o/\omega_p$. The acceleration is normalized to $0.01v_o\omega_p$.

We again impose a velocity modulation of $0.02v_o$ on the beam at $\omega_p$. The actual modulation in the laboratory would be by noise, but only signals with frequencies very near to $\omega_p$ are expected to be amplified. To see this, we can obtain the dispersion relation of a one dimensional beam going through a rectangular distribution plasma very simply. That is:

$$f_p(v) = \begin{cases} 
1/2v_r & |v| \leq v_r \\
0 & |v| > v_r 
\end{cases} \quad (3.56)$$

where $v_r = 3^{1/2}v_T$. This distribution ignores Landau damping, which should be negligible since $v_r$ is much less than the phase velocity. The dispersion relation is

$$1 - \frac{\omega_p^2}{\omega^2 - k^2v_r^2} - \frac{\omega_{pb}^2}{(\omega - kv_o)^2} = 0 \quad (3.57)$$

The roots to Equation 3.57 are plotted in Figure 3.24 for the parameters of this computer experiment, and for the actual experiment if $T_\perp = 12$ volts and the beam voltage is 8 kilovolts $V_T = 0.0373v_o$). The thermal velocity in the computer experiment
Fig. 3.24 Dispersion equations for real $\omega$ for thermal velocities shown. Only growing solutions shown, for square distribution plasma. $n_{bo}/n_{po} = .01$. $v_T$ is the mean square velocity of the plasma.
is about twice that of the laboratory, but the higher temperature should reduce the interaction level even more. The interaction is still not reduced nearly enough to allow the narrow laboratory collected beam velocity distribution (see Chapter 2). However, the larger temperature allows a wider bandwidth of amplification, so that we do not want to modulate over a wide band and perhaps introduce spurious interference effects that would not be present in the laboratory plasma with its lower temperature.

In Figure 3.25 are shown the results of the computer experiment, for \( v_T = 0.075v_o \), after a time of \( 112/\omega_p \). Figure 3.26 shows the results at a time of \( 118/\omega_p \). The circled plasma sheet has a velocity of about \( 0.35v_o \) in Figure 3.25, but in Figure 3.26, only \( 6/\omega_p \) later, this sheet has acquired a velocity of \( 0.8v_o \). The wave has grown large enough to accelerate a sheet in the tail of the plasma velocity distribution up to the wave phase velocity, and keep it in the accelerating phase of the wave for about a plasma period \( (2\pi/\omega_p) \).

The peak field where the acceleration of a plasma sheet to \( v_\phi \) normalized acceleration \( A \) is 21 or \( qE/mv_o \omega_p = 0.21 \). This field is less than in the cold case, which requires \( qE/mv_o \omega_p = 0.325 \). In the warm case, a sheet in the tail of the plasma velocity distribution needs to be accelerated less to obtain the same wave velocity than in the cold case, hence the maximum field is less. From Equation 3.34, if we let \( v_\perp = 2v_T \), to represent a sheet in the tail of the plasma velocity distribution, and use \( v_\phi = 0.7v_o \)
Fig. 3.25 Snapshot of beam and plasma variables. $n_b^0/n_p^0 = 1/99$. $t = 112/\omega_p$. Warm plasma with $v_T = 0.075v_o$. $v_b(z=0) = 0.02v_o$. Velocities normalized to $v_o$, acceleration to $0.01v_o\omega_p$.

The circled plasma sheet is located at $z = 8.9$.
Fig. 3.26 Snapshots of beam and plasma variables. The circled sheet has been accelerated to the wave phase velocity.
which we obtain from the wavelength, then the expected value of $A$ is 21.5, close to the observed value of $A = 21$. Our choice of $v_L = 2v_T$ in this case is somewhat arbitrary (see Figure 3.13), so the agreement is somewhat fortuituous.

The fields are largest in the region of initial beam overtaking and become much smaller on either side of this region, again simulating the meniscus. However, the region of large field amplitude extends further than in the cold plasma case (Figure 3.6). The reason is that in the smaller fields the beam breakup (i.e., overtaking and spreading in velocity) occur more gradually and over a larger region of space.

In Figure 3.27 we show the results at time $220/\omega_p$. A few more sheets have been accelerated to velocities on the order of $v_o$, some of which appear with negative velocities after reflecting off the hard walls at the collector. The point of initial beam overtaking (where the fields are maximum) has moved upstream, but the maximum field magnitude has not increased, presumably due to the energy drain caused by acceleration of some plasma sheets to velocities near $v_0$.

The beam at this time is giving up a time average of 12.6 per cent of its energy to the oscillations, over the entire interaction length shown. Some of this energy is needed to accelerate the fastest plasma sheets. However, some is also going into fields near the collector, which have a maximum field comparable to that at initial beam overtaking.
Fig. 3.27 Snapshots of beam and plasma variables.
In Figure 3.27 we compare the growth of the first order beam velocity with that predicted by linear theory, with the amplitude of the growing linear solution determined by the boundary conditions of two per cent velocity modulation \(v_b(z=0)/v_o = 0.02\) and no beam density modulation. There is also a decaying solution, with \(k\) conjugate to that of the growing solution of Figure 3.24, but not shown in that figure, which we need to match the boundary conditions. The amplitudes of the linear solutions \(|v_b|/v_o = 0.0146, (|qE|/m)/0.4v_0\omega_p = 57.6x0.0146\) are only approximate, since we have ignored two other decaying solutions to Equation 3.38 (which is fourth order in \(k\)). The discreteness noise evident in the acceleration plot of Figure 3.27 produces additional modulation, so it does not seem worthwhile to solve 4 rather involved complex algebraic equations for the exact solutions.

Noise such as is evident in the plots of acceleration has been studied by Dawson. He predicts that the distribution of displacements of the plasma sheets from their instantaneous equilibrium positions is

\[
P(x) = \exp\left[-(\omega_p^2x^2/2v_T^2)(1+2n_o\lambda_p)\right],
\]

(3.58)

where \(X\) is the displacement from equilibrium. The mean square displacement is \((v_T/\omega_p)/(1+2n_o\lambda_p)^{1/2}\), or in our units, 1.88 intersheet spacings. In our computer run, the plasma sheets start out uniformly spaced, but with random velocities. By \(t = 10/\omega_p\), the beam has lost a total amount of energy to the
oscillations of less than 1.5 per cent of the electric energy then present, but the plasma sheets are presumably at random positions relative to their equilibrium positions by this time. Ignoring the fields caused by the beam, which at this time is only very slightly bunched, and whose density is only 1/99 that of the plasma, the mean square field at the position of the plasma sheets corresponds to a mean square displacement from equilibrium of 1.89 intersheet spacings, compared to 1.88 predicted. The predicted noise value is equivalent to a normalized mean square acceleration of 1.88

The beam velocity distribution near the collector is compared with that of the laboratory in Figure 3.28. We see that the computer experiment has a much wider velocity spread than that of the laboratory, even though the interaction length of the computer experiment is only three per cent that of the laboratory. Since the beam is so widely spread in velocity, from Landau theory we do not expect the beam to be able to excite oscillations for positions much downstream from those shown here.

3.8 Summary of Chapter and Conclusions

We have found that modeling the beam-plasma interaction with a one-dimensional lossless plasma, with or without temperature, does not produce the narrow velocity distribution of the collected beam, observed in our experiment (Chapter 2), but a much wider one. However, a phenomenon; namely the meniscus, observed in these computer experiments is seen in many other beam plasma experiments at
much lower beam voltages than we use. We find that the computer runs duplicate the large, localized $\omega_p$ oscillations of the meniscus which scatter the beam widely in velocity, and also duplicate the dark space downstream from the meniscus. The fields build up until some plasma electrons are accelerated to the phase velocity of the waves. The introduction of plasma temperature seems only to reduce the field magnitude necessary to do this.
Fig. 3.28 Warm uniform plasma. Beam sheet velocity distribution of beam in last 15 per cent of interaction region, averaged over four times π/2ωp apart, beginning at t = 220/ωp. Compared with collected beam velocity distribution of laboratory experiment.
Chapter 4: Beam-Plasma Interaction with a Lossy Plasma

4.1 Introduction

We found in Chapter 3 that the beam plasma interaction in a uniform cold plasma is so intense that plasma electrons can acquire oscillatory velocities approaching that of the injected beam electrons. Even with the introduction of plasma temperature, the interaction is limited only by acceleration of the plasma electrons to the phase velocity of the wave. The beam velocity distribution is quite spread out, even in the three wave lengths considered.

It is of interest to consider the inclusion of loss into the plasma. In H₂, the electron collision frequency \(47 \nu \) at \( p = 1.4x10^{-3} \) Torr is \( 6.7x10^6 \) sec.\(^{-1} \) \( \omega_p \), however, is \( 10^{11} \) sec.\(^{-1} \). Hence \( \nu/\omega_p < 10^{-4} \). The collision frequency for scattering of the 8 kilovolt beam electrons through one radian \(57 \) is only \( 3.0x10^3 \) sec.\(^{-1} \), which is negligible compared to the plasma collision frequency, and it is ignored throughout this thesis.

It is impractical in terms of computer time to let the programs run for several collision times in this case. We did run a case of very large loss (\( \nu = 0.2\omega_p \)) which can give us insight into the problem. We find that the inclusion in the computer model of even so much loss results in a spread in beam velocity at the collector greatly exceeding that observed experimentally.
4.2 Model Using a Linearized Plasma

We have found it useful to employ a different model than that employed in Chapter 3. We linearize the plasma and treat it analytically, but still use sheets for the beam. This model allows a considerable saving in computer time and storage, so that it is practical to increase the number of wavelengths.

The presence of a large collision rate will allow us to reach a true steady state, independent of any transients induced in building up the interaction. We must, however, check that the fields are small enough to justify the assumption of plasma linearity. This requires that the $\nabla \cdot \nabla \nabla$ term be negligible compared to $\partial v/\partial t$ in the force equation, and that the first order plasma charge density, $\rho_p$, be small compared to the zeroth order plasma charge density, $\rho_{po}$.

The fields are generated by wakes created by each beam sheet as it moves through the plasma. Since the plasma is now assumed to be linear, the wakes due to different sheets can be superposed.

The field of a beam sheet moving through a lossy plasma can be found by using Fourier transforms. The plasma is represented by its equivalent dielectric constant,

$$\varepsilon(\omega, k) = \varepsilon_0 \{1 - \frac{\omega_p^2}{\omega(\omega - j\nu)}\}. \quad (4.1)$$

Note: The problem is one dimensional, and strictly speaking $B_0 = \infty$, so that the tensor nature of $\varepsilon$ can be ignored.
In Appendix C it is shown that the response due to a sheet of surface charge $Q$ passing through the plasma is

$$E(z,t) = \frac{Q}{\varepsilon_0} \left(1 + \frac{v^2}{4\omega_o^2}\right)^{1/2} \frac{1}{2} - \sqrt{t-t_1(z)} \right) e^{\frac{z}{2}}$$

$$x \cos \{\omega_o [t-t_1(z)] - \tan^{-1}(v/2\omega_o)\} u_{-1}[t-t_1(z)],$$

(4.2)

where

$$\omega_o^2 = \omega_p^2 \left(1 - \frac{v^2}{4\omega_p^2}\right),$$

(4.3)

and $t_1(z)$ is the time the sheet passes the plane $z$. We see that the oscillations are at a frequency slightly less than $\omega_p$, in a manner analogous to the way loss reduces the natural frequency of an LC circuit.

There is no field ahead of the sheet, and a decaying oscillatory field behind it. We note that the field depends only on the charge $Q$ of the sheet and the time $t_1$ that it passes the plane $z$. The fields are independent of sheet velocity or of any sheet velocity changes before or after $t_1$.

That there is no field ahead of the sheet can be understood in several ways. We have used Fourier transforms to derive Eq.4.2. This implies that a steady state has been reached, or that the sheet has been traveling for a very long time. If an unneutralized sheet suddenly appears in a plasma, there will instantaneously be fields of $\pm Q/2\varepsilon_0$ on each side.
of it, because the plasma electrons cannot respond instantaneously to shield the fields. The plasma far ahead of the sheet cannot know that the sheet is moving because the sheet field is independent of distance. Hence, the plasma at this plane experiences transient oscillations which eventually die out. The plasma electrons will end up being displaced a distance of \( +Q/2n_p q \) to produce a shielding field, which cancels the sheet field. When the sheet passes this plane, the sheet field and former shielding field are now of the same sign and add, instead of canceling, as they do before the sheet passes, to produce an instantaneous field of \( -Q/\varepsilon_0 \). The plasma oscillations occur until collisions damp them. The plasma electrons are then displaced a distance of \( -Q/2n_p q \) from their position when no sheet is present.

By using Fourier transforms we assume the beam sheets have been behind the gun for a long time. We could also have eliminated the transient field that is produced when a sheet suddenly appears at the gun (z=0) by requiring a motionless, neutralizing ion sheet to appear with it. Then a Gaussian pillbox taken about both sheets would enclose no net charge, and hence there will be no field except between the two sheets.

In any case, as the beam sheets move forward through the plasma, there will be a net backward displacement of the equilibrium positions of the plasma electrons. The reader will recall that in the sheet plasmas of Chapter 3, we place the neutralizing ion sheets behind the collector, to eliminate
the net displacement of the plasma sheets, but tolerate the infinite wavelength oscillations set up when the beam sheets are injected. That is, a field equal to $Q/\varepsilon_0$ appears between the gun and collector whenever a beam sheet is injected. The field then oscillates at $\omega_p$. We now choose to place the ion sheets behind the gun, eliminating the infinite wavelength oscillations; but we tolerate the net displacements. Hence plasma comes in from the collector when a beam sheet passes the collector, and passes out through the gun when a sheet is injected. There is an average plasma current to cancel the average beam current. We have decided to eliminate the spurious transient fields due to using large sheets, rather than worry about a current that has no noticeable effect in this model. There are no boundary conditions on the plasma, so the model is really that of a finite length beam in an infinitely long plasma. The collector is misnamed, since we do not annihilate the sheet when it passes the collector plane, but just ignore it. Its annihilation would produce transient fields. The effect of a sheet on the field at any plane depends only on the time the sheet passes that plane. Hence what the sheet does after it passes can have no effect, if we do not annihilate it.

Since the wake at a plane $z$ due to any sheet is a cosinusoidal oscillation in time, and since cosines at the same frequency can be combined into one cosinusoid, we need only keep track of the resultant magnitudes and phases. We
do so at cell points \(0.2v_o/\omega_o\) apart (about 31 per wavelength) and use a second order Lagrange polynomial to interpolate between cell points. We must be careful in interpolating if a sheet is within the three cell points between which we want to interpolate, since the sheet produces a field discontinuity. Hence we keep track of three separate sets of magnitudes and phases at each cell point. One set includes the fields of all the sheets that have passed the cell point, another set includes the fields of only those sheets that have passed the cell point one beyond the considered cell point, and the third set includes only those fields caused by sheets which have passed the second cell point beyond the considered cell point. The fields due to sheets within the three cell points are added on separately, using Eq.4.2.

The new magnitude formed when a sheet passes a cell point is found from a simple phasor diagram, as shown in Fig.4.1. Let the length \(A\) denote the magnitude of the field before the sheet passes, and \(\theta_A\) its phase; let \(B\) denote the field magnitude due to the sheet, and let \(\theta_B\) be its phase as determined by the time the sheet crosses the cell point. We wish to determine the resultant \(C\) and \(\theta_C\). \(C\) is, by the law of cosines,

\[
C = \sqrt{A^2 + B^2 + 2AB\cos(\theta_B - \theta_A)}
\]

(4.4)

and \(\theta_C\) is, by the law of sines,
Fig. 4.1  Phasor to determine the new phase and magnitude of the field at a cell point when a sheet passes. $A$ and $\theta_A$ determine the phasor of the field without the sheet, $B$ and $\theta_B$ determine the phasor of the sheet field, and $C$ and $\theta_C$ determine the resultant.
\[ \theta_C = \theta_A + \arcsin \left( \frac{B}{C} \sin(\theta_B - \theta_A) \right). \tag{4.5} \]

If \( B \cos(\theta_B - \theta_A) < -A \), we must take \( \pi \) minus the \( \arcsin \) to get the angle in the proper quadrant.

The same ordering method is used as in Chapter 3.

The Milne numerical integration routine\textsuperscript{58} is used to calculate the trajectories. It is

\[
\begin{align*}
  z(t+\Delta t) &= z(t)+z(t-2\Delta t)-z(t-3\Delta t)+ \frac{(\Delta t)^2}{4} [5a(t)+2a(t-\Delta t)+5a(t-2\Delta t)], \tag{4.6} \\
  x &= 5a(t)+2a(t-\Delta t)+5a(t-2\Delta t),
\end{align*}
\]

where \( a(t) \) is the acceleration at time \( t \). The velocity does not need to be calculated except for the output, when we find it by numerical differentiation. This routine is theoretically very accurate, the error\textsuperscript{58} being less than \( 17(\Delta t)^6/240 \) \( d^6z/dt^6 \). It is, however, not self starting in that we have to know the \( z \)'s at times less than \( t \) to find \( z(t+\Delta t) \). The Runge-Kutta\textsuperscript{58} routine is used to start the Milne method.

This is

\[
  z(t+\Delta t) = z(t)+v(t)\Delta t+ \frac{\Delta t}{6}(m_1+m_2+m_3),
\]

and

\[
  v(t+\Delta t) = v(t)+ \frac{1}{6} \left( m_1+2m_2+2m_3+m_4 \right), \tag{4.7}
\]

where
\[ m_1 = \Delta t \, a(t, x), \]
\[ m_2 = \Delta t \, a(t+\Delta t/2, x+v(t)\Delta t/2), \]
\[ m_3 = \Delta t \, a(t+\Delta t/2, x+v(t)\Delta t/2+\Delta t \cdot m_1/4), \]
\[ m_4 = \Delta t \, a(t+\Delta t, x+v(t)\Delta t+\Delta t \cdot m_2/2). \]  

(4.8)

The Milne routine is found to be slightly numerically unstable, that is small errors will grow.

We measure energy conservation as follows: The sum of the electric field energy density and the plasma kinetic energy density is

\[ \text{Energy density} = (1/2\varepsilon_0) \left| E_0^2 e^{-vt} \cos^2(\omega_0 t - \phi) \right. \]
\[ + \sin^2(\omega_0 t - \phi \tan^{-1} v/2\omega_0) \left. \right]. \]  

(4.9)

The Langevin force on the plasma electrons, \( F = -mv \cdot v_p \), produces a velocity dependent loss term so that loss is not constant over a cycle, but is greatest when \( v_p \) is greatest. Hence to measure the sum of the electric and plasma kinetic energy densities, we must know not only the present value of the magnitude of the electric field, \( E_0 e^{-vt/2} \), but also the phase. If \( v = 0 \), we need to know only \( E_0 \). To measure energy conservation over a time interval (typically of the order \( 20/\omega_p \)), we measure the electric and plasma kinetic energy from Eq. 4.9, given the present value of \( E_0 \), corrected for decay, and add it to the present beam kinetic energy. This sum, integrated over the total drift length, is compared with this value \( 20/\omega_p \) before, plus the difference in the injected and collected
beam kinetic energies, minus the energy dissipated over the time interval. The dissipation is step-wise integrated over the time interval, such that

\[
\text{total energy dissipated} = \sum_{t=t_{\text{min}}}^{t_{\text{max}}} \int_{z_{\text{collector}}} \frac{1}{2} \left| E_0(t,z) \right|^2 d\tau \times v \Delta t \sin^2 \left[ \omega_o t - \phi(t,z) + \tan^{-1} \frac{v}{2\omega_o} \right].
\] (4.10)

Energy in this program is conserved within five per cent of the average energy lost by the beam to the plasma. In the program we use later, with density gradients along the direction of beam flow, we use only the stable Runge-Kutta method and obtain energy conservation within 0.05 per cent of the total energy within any time interval of $50/\omega_p$.

When a crossing occurs, there are equal and opposite discontinuities in the accelerations of the crossed sheets. The Milne routine effectively fits a polynomial to the acceleration values known at discrete time steps, and integrates this polynomial twice over time to obtain the position. However, a discontinuity in the acceleration cannot be fitted well by the polynomial. We can remove the discontinuity in such a manner as to give the correct acceleration, position and velocity infinitesimally after the time of crossing, and make the acceleration and first derivative of the acceleration continuous, although not the second derivative.

Let the subscript "one" denote the sheet that crosses
that denoted by subscript "two". That is \( z_1(t+\Delta t) > z_2(t+\Delta t) \), but \( z_1(t) < z_2(t) \). On crossing, the acceleration attributable to one sheet, \( a_0 = qQ/m\varepsilon_0 \), is added to \( a_1(t), a_1(t-\Delta t), a_1(t-2\Delta t) \), and subtracted from \( a_2(t), a_2(t-\Delta t), a_2(t-2\Delta t) \). We wish to have the correct acceleration, position, and velocity for times greater than the crossing time. For this to be true, \( z_1,2(t), z_1,2(t-\Delta t), z_1,2(t-2\Delta t) \), and \( z_1,2(t-3\Delta t) \) must also be modified. In Fig. 4.2 we show how the corrections are made for the crossed sheet. The solid lines represent the old functions, which are correct for times less than the crossing time. The corrected functions, which are correct for times greater than the crossing time, are shown as dashed lines. We assume that the crossing sheet presents a constant force to the crossed sheet over the time \( \Delta t-\Delta t_1 \). This is a very good approximation if the sheets have nearly the same velocity. Note that if we reduce the acceleration of the crossed sheet for times less than \( t+\Delta t \), we must increase the previous velocities for the sheet to have the same velocity at the crossing time. This is reflected in the reduced \( z \)'s for times less than \( t+\Delta t_1 \).

The acceleration is a function of position and time. Hence its first derivative with respect to time depends on the actual velocity, but the first derivative of the velocity is in fact discontinuous. Hence the second derivative of the corrected acceleration is discontinuous. Mathematically, the error in the Milne routine is normally less than a factor
Fig. 4.2 Corrections to the accelerations and positions of a crossed sheet. The solid lines give the acceleration and position if the crossing is ignored. The dashed lines are the correct values after crossing but are incorrect before crossing. The crossing occurs at $t + \Delta t$. The corrections are necessary if the Milne routine is to integrate a continuous function.
proportional to \( \frac{d^6}{dt^6} \), so that this discontinuity in \( \frac{d^4z}{dt^4} \) reduces the accuracy of the Milne method. However, it is difficult to picture any great error being introduced, since even the discontinuity in field caused by a crossing is only three per cent of the maximum total field. It would be better (and simpler) to use the self-starting Runge-Kutta method throughout, and avoid the necessity of correcting the previous values. We do this in Chapters 5 and 6.

The time of crossing is estimated from the average velocities over \( \Delta t \),

\[
\Delta t_1 = \frac{z_2(t) - z_1(t)}{\frac{z_1(t+\Delta t) - z_1(t)}{\Delta t}} - \frac{z_2(t+\Delta t) - z_2(t)}{\Delta t}, \tag{4.11}
\]

where \( z_1, z_2(t+\Delta t) \) are calculated under the assumption of no crossing.

The beam is injected at \( z = 0 \) with a two per cent velocity modulation at \( \omega_0 \), but no density modulation. The dispersion equation from linearized theory is

\[
1 - \frac{\omega^2_p}{\omega(\omega-j\nu)} - \frac{\omega^2_{pb}}{(\omega-\kappa \nu_0)^2} = 0. \tag{4.12}
\]

The solution to Eq.4.12 in the steady state (real \( \omega \)) is plotted in Fig.4.3. In this figure \( \omega \) is normalized to \( \omega_p \). Here \( \omega_0 = (1-\nu^2/4\omega^2_p)^{1/2} = 0.995\omega_p \) and \( \nu = 0.2\omega_p \) as in the computer experiment. We modulate at \( \omega_0 \).

We ran the program in time intervals of typically \( 20/\omega_p \),
Fig. 4.3 Dispersion relation for real \( \omega \), of a one dimensional beam interacting with standing waves. \( \frac{k\nu_o}{\omega_p} \) vs. \( \frac{\omega}{\omega_p} \).
writing a binary pseudo-tape on disks at the end of the run, then reading it to resume the program. When a plot was desired, the binary tape was converted to a BCD tape, which was punched out on cards and read by a plotting program.

4.3 Results of the Beam-Plasma Interaction with a Lossy Plasma

A snapshot of the acceleration of a sheet and instantaneous sheet velocity versus distance is shown in Fig.4.4, where a steady state seems to have been reached. In this case \( t = 300.4/\omega_0, \omega_{pb}^2 = \omega_0^2/200, \nu = 0.2\omega_0 \). This large value of \( \nu \) is required to keep the plasma linear. From the linearized plasma force and conservation equations we can easily find from the magnitude of the acceleration that at initial beam overtaking, where the field is a maximum, that \( |v_p/v_\phi| = |\rho_p/\rho_{po}| < 0.05 \), where \( \rho_p \) and \( v_p \) are the first order plasma charge density and velocity respectively. Beyond overtaking \( (z > 150(0.2v_0/\omega_0)) \) waves with lower phase velocity form, and the condition of plasma linearity can be violated at a lower amplitude of electric field. \( |v_p/v_\phi| \) and \( |\rho_p/\rho_{po}| \) are still less than 0.1 throughout the drift region. The envelopes of beam velocity and acceleration are compared with linear theory. There is a fair comparison with linear theory up to the point of overtaking, except for a negative d.c. component present in both the acceleration and beam velocity that is not predicted by linear theory. In Appendix D we derive the drag on one sheet of an unmodulated sheet beam moving through
the plasma. The net acceleration on a sheet, counting that due to the sheet self field, is

\[
a = - \frac{\{qQ/m\varepsilon_0\} \{ -1 + e^{\nu t} + \nu/\omega_0 e^{\nu t}/2\sin(\omega_0 t) \}}{1 + e^{\nu t} - 2e^{\nu t} \cos(\omega_0 t)} (4.13)
\]

where \(\nu_0 t\) is the sheet separation. \(\Delta t\) is equal to the step time in the numerical integration formula. In the limit of a fluid beam (we let \(Q = \rho_0 v_0 \Delta t, \Delta t \to 0\),

\[
a = -(\omega_pb/\omega_p)^2 v_0 \nu, \quad (4.14)
\]

where \(\omega_pb^2 = \rho_0 q/m\varepsilon_0\). Normalizing distance to \(0.2v_0/\omega_0\), times to \(\omega_0^{-1}\), \(A\), the normalized acceleration, is \(-0.00495\), which is what Eq.4.13 equals to three significant figures. for \(\Delta t = 0.2\omega_0^{-1}\). From Fig.4.4 we see that the d.c. value of the acceleration is approximately as predicted. The d.c. value of the acceleration varies linearly as \(\nu\). Hence it should be negligible in the laboratory experiment, where \(\nu/\omega_p < 10^{-4}\). In the computer experiment, an unmodulated beam loses six per cent of its velocity in drifting from the gun to the collector. This is much less than that lost in the interaction (see Fig.4.4).

The fields are a maximum at the plane of initial beam overtaking. Here the beam charge density is a maximum. The phase is such as to maximally decelerate the beam at this plane, or to extract the maximum amount of energy from the beam. Beyond this plane two waves are evident. The first
Fig. 4.4 Collision frequency = $0.2\omega_p \cdot t = 300/\omega_o$. Exponentials are the envelopes of the growing solutions expected from linear theory for a 2 per cent velocity modulation of the beam, but no beam density modulation.
has a phase velocity comparable to that of the wave before overtaking occurs. The second wave, however, has a phase velocity equal to the velocity of clumps of beam charge that break off in velocity-distance space (Fig.4.4). The phasing is such that the clumps are maximally decelerated. This second wave widely spreads the beam in velocity space, yielding much wider spreads than observed in the laboratory (see Chapter 2).

The formation of the clumps is not completely understood. However, once formed, the clumps pass a plane at times $2\pi/\omega_o$ apart (a bunching frequency of $\omega_o$). Hence the wakes from each clump add coherently, their sum being limited only by the presence of loss, and their phase such as to maximumly decelerate the clump.

In the lossless cases described in Chapter 3, this clumping is not evident. Moreover, in those cases the slowest beam electrons fall back in wave phase, and are reaccelerated, so that the beam curls up on itself in the velocity-distance plot. There is some of this curling here, but the curling does not include those beam sheets which have been slowed the most, and the curling which is present is much slower, occurring over many more wavelengths. Part of the explanation of the difference between this case and Chapter 3 must lie in the fact that the fields at the plane of initial beam overtaking are only about ten per cent as large as those of Chapter 3. This fact produces a smaller beam velocity
spread at the plane of initial beam overtaking, which in turn means that the slowest beam sheets drift further before falling back into the overtaking phase. By this time the clump wave has formed, which traps the clumps in itself and uptraps them from the main wave, preventing curling of the sheets trapped in the slower wave.

4.4 Conclusions

The model described in this chapter is clearly not descriptive of the laboratory experiment. It does tell us that there is no hope of duplicating the narrow velocity spread observed in the lab with a uniform, lossy plasma, even if the fields are greatly reduced by collisions.

The lossy interaction is similar to the loss free case in that the field is a maximum at the plane of initial beam overtaking, so that a meniscus seems to form. However, the fields here are greatly reduced from the loss free case, and a slow wave is able to form in addition to a wave with a phase velocity near that of the wave before overtaking. This slow wave traps a clump of beam electrons with only a small velocity spread, and slows this clump to almost zero velocity.
Chapter 5: Linearized One-Dimensional Beam-Plasma Interaction with a Linear Density Gradient Along the Direction of Beam Flow

5.1 Introduction

In Chapters 3 and 4 we presented computer experiments of the beam-plasma interaction in uniform plasmas. In these cases we find that the fields are quite intense, unless an unrealistically large loss is introduced. Even in the very lossy case the beam is scattered in velocity far more than observed experimentally.

When the beam is modulated at $\omega_p$, the plasma is driven by the beam at its natural resonant frequency, and of course we expect a large response. In a uniform plasma, the frequency of the noise excited by the interaction is peaked around $\omega_p$.\(^{59}\)

The situation is different if the plasma is nonuniform along the direction of beam flow. The beam may be modulated at a frequency corresponding to a local $\omega_p$ near the gun, but as it drifts downstream and bunches, the $\omega_p$ changes. Hence the beam drives the plasma off resonance, and we expect a much smaller response.

In this chapter we first look at the linearized solutions to a one-dimensional beam-plasma interaction in a plasma with a linearly increasing plasma density gradient along the direction of beam flow. These solutions are obtainable in closed form, but are only valid for small signals, and are not
valid beyond initial beam overtaking. The linear solutions do give us insight into the effects of changing beam and plasma parameters as the point of initial beam overtaking is approached.

The linear theory is compared with computer experiments. We find that the gradient indeed significantly reduces the magnitude of the electric fields below the cold uniform case. However, the beam is still spread fairly widely in velocity, although the beam must drift several more wavelengths than in the uniform case to achieve this spread.

5.2 Linearized Calculations

We consider a one-dimensional beam interacting with a cold, collisionless, one dimensional plasma, whose density varies linearly with distance along the direction of beam flow. We assume a time variation of the form $e^{i\omega t}$, so that our results will be in terms of $\omega$ and $z$. The plasma frequency is:

$$\omega_p^2(z) = \omega^2 + gz. \quad (5.1)$$

At $z=0$, $\omega_p$ is equal to the modulation frequency $\omega$.

The linearized force equations for the plasma and beam are

$$jm\omega v_p = qE \quad (5.2)$$

$$jm\omega v_b + mv_o \frac{\delta v_b}{\delta z} = qE. \quad (5.3)$$
Here \( v_p \) is the first order plasma velocity, \( v_o \) is the zeroth order beam velocity, and \( v_b \) is the first order beam velocity. The electron charge is \( q \), and the electron mass is \( m \).

The charge conservation equations are

\[
\frac{\partial}{\partial z} \{ \rho_p (z) v_p \} = -j \omega \rho_p, \quad (5.4)
\]

and

\[
\frac{\partial}{\partial z} \{ \rho_b v_b + \rho_b v_o \} = -j \omega \rho_b, \quad (5.5)
\]

where \( \rho_p \), \( \rho_p \), \( \rho_b \), \( \rho_b \) are the zeroth and first order plasma and beam charge densities. Finally Poisson's equation is:

\[
\varepsilon_0 \frac{\partial E}{\partial z} = (\rho_p + \rho_b). \quad (5.6)
\]

Inserting Eqs. 5.4 and 5.5 into Eq. 5.6, we have

\[
\frac{\partial}{\partial z} \{ \rho_p v_p + \rho_b v_b + \rho_b v_o + j \omega \epsilon_0 E \} = 0. \quad (5.7)
\]

Integrating over \( z \), and setting the constant of integration equal to zero,

\[
\rho_p v_p + \rho_b v_b + \rho_b v_o + j \omega \epsilon_0 E = 0. \quad (5.8)
\]

Setting the constant of integration equal to zero means that we allow no spatially uniform electric fields, but allow only wave-type solutions.

Combining Eqs. 5.2-5.8, there results a differential equation in \( v_b \) only, which is
\[
\left[ \omega_p^2(z) - \omega^2 \right] v_o \frac{\partial^2 v_b}{\partial z^2} + \left\{ v_o \frac{\partial \omega_p^2(z)}{\partial z} + j2\omega \left[ \omega_p^2(z) - \omega^2 \right] \right\} \frac{\partial v_b}{\partial z} \\
+j \omega \frac{\partial \omega_p^2(z)}{\partial z} - \frac{\omega^2}{v_o} \left[ \omega_p^2(z) + \omega_{pb}^2 - \omega^2 \right] v_b = 0, \tag{5.9}
\]

where \( \omega_{pb} \) is the beam plasma frequency. Inserting Eq.5.1 into 5.9, we find that the solution of Eq.5.9 is

\[
v_b = e^{-j\omega z/v_o} \left\{ J_0 \left( \frac{2\omega_{pb}(-z)}{v_o g} \right)^{1/2} \right\}, \tag{5.10}
\]

where \( J_0 \) and \( Y_0 \) are the ordinary Bessel functions. For \( z < 0 \), there are two modes:

\[
v_{bI} = v_I e^{-j\omega z/v_o} J_0 \left[ C_1 (-z)^{1/2} \right], \tag{5.11}
\]

and

\[
v_{bII} = v_{II} e^{-j\omega z/v_o} Y_0 \left[ C_1 (-z)^{1/2} \right], \tag{5.12}
\]

where \( v_I \) and \( v_{II} \) are multiplicative constants. For \( z > 0 \),

\[
v_{bIII} = v_{III} e^{-j\omega z/v_o} I_0 \left[ C_1 z^{1/2} \right] \tag{5.13}
\]

and

\[
v_{bIV} = v_{IV} e^{-j\omega z/v_o} K_0 \left[ C_1 z^{1/2} \right], \tag{5.14}
\]

where \( C_1 = 2\omega_{pb}/v_o g^{1/2} \), \( I_0 \) and \( K_0 \) are the modified Bessel functions. These functions are shown in Figs.5.1 and 5.2. The electric fields corresponding to Eqs.5.11-5.14 are found from Eq.5.3. They are
Fig. 5.1

First-order beam velocities $v_{bI}$ and $v_{bII}$ vs. distance. Times are normalized to $\omega_p^2$, $\omega_p = 0.005 \omega_0$, $\omega_{bI} = 0.15 \omega_0$, and $g = 0.001 \omega_0$. There is little spatial growth of $v_b$, except in the region $z > 0$, where $v_b \approx (z)$. 
Fig. 5.2 First-order beam velocities $v_{b_{II}}$ and $-v_{b_{IV}}$ vs. distance. We note that the first-order beam velocity is large over a very small region near $z = 0$, where $\omega = \omega_p(z)$. 
\[ E_I = V_I C_2 e^{-jwz/v_o (z)^{1/2} J_1 [C_1 (z)^{1/2}]}, \]  
\[ E_{II} = V_{II} C_2 e^{-jwz/v_o (z)^{1/2} Y_1 [C_1 (z)^{1/2}]}, \]  
\[ E_{III} = V_{III} C_2 e^{-jwz/v_o z^{1/2} I_1 [C_1 z^{1/2}]}, \]  
\[ E_{IV} = -V_{IV} C_2 e^{-jwz/v_o z^{1/2} K_0 [C_1 z^{1/2}]}, \]

where \( C_2 = (m/q) (\omega_{p_b}/g)^{1/2} \).

The first order beam and plasma charge density and the first order plasma velocity of mode III are, from Eqs. 5.2, 5.4, 5.6, and 5.17,

\[ \rho_b = \frac{V_{III}}{V_o} \rho_{bo} e^{-jwz/v_o [C_1 z^{1/2}] - j\omega \nu_o (z)^{1/2} + j\omega v_o z^{1/2} \omega_{p_b}} \times I_1 [C_1 z^{1/2}], \]  
\[ \rho_p = j \frac{V_{III}}{V_o} \frac{\omega_{p_b}}{(g z)^{1/2}} \rho_{po} (z) e^{-jwz/v_o \nu_o (1 - \frac{j\omega v_o z^{1/2}}{2 \omega_{p_b}}) \nu_o} \times I_1 [C_1 z^{1/2}] + j\omega \nu_o \omega_{p_b} [C_1 z^{1/2}], \]  
and

\[ v_p = V_{III} \frac{\omega_{p_b} e^{-jwz/v_o}}{j(g z)^{1/2}} I_1 [C_1 z^{1/2}], \]

where \( V_{III} \) is the excitation velocity of mode III.

It is interesting to compare these results with those of a uniform plasma. In this case the solutions are of the
form $e^{-jkz}$, with $k$ given by

$$k = \frac{\omega}{v_o} + \frac{\omega_{pb}}{v_o (1 - \omega_p^2/\omega^2)^{1/2}}.$$  \hspace{1cm} (5.22)

For $\omega > \omega_p$, $k$ is real and the two solutions are purely oscillatory. For $\omega < \omega_p$, $k$ is complex, yielding a spatially growing and a spatially decaying solution. The growth rate is infinite if $\omega = \omega_p$.

In order to compare Eq.5.2 with Eqs.5.11-5.14 in the limit of a uniform plasma, we have

$$k_G = j \frac{b}{v_o} \frac{\partial b}{\partial z},$$ \hspace{1cm} (5.23)

and compare $k_G$ with $k$ in Eq.5.22.

Inserting Eq.5.13 into Eq.5.23, we have

$$k_G = \frac{\omega}{v_o} + j \frac{\omega_{pb}}{v_o (g^2)_{1/2}} \frac{I_1(C_1 z^{1/2})}{I_0(C_1 z^{1/2})}.$$ \hspace{1cm} (5.24)

From Eq.5.1, $g^2 = \omega^2_p - \omega^2$. As $g \to 0$, $z \to \infty$ if $\omega_p \neq \omega$. In the limit of large $z$, $I_1 \to I_0$, and the growing solution of Eq.5.22 is recovered. Similarly Eq.5.14 reduces to the decaying solution, and Eqs. 5.11 and 5.12 reduce to a linear combination of the purely oscillatory solutions.

Eq.5.12 blows up logarithmically at $z=0$, where $\omega = \omega_p(z)$. This fact may not be physically significant if a realistic amount of loss is included. At $z=0$, we find from Appendix E that
\[
\frac{v_{bII}}{v_{bI}} = \frac{2}{\pi} \ln \left( \frac{j \omega_p b \omega^2}{v_0 g} \right) \left( \frac{v}{\omega} \right)^{\frac{1}{2}},
\]

(5.25)

and

\[
\frac{E_{II}}{E_I} = \frac{2}{3} \frac{v_0 g}{\pi \omega \omega_p b \nu},
\]

(5.26)

for equal values of \(v_I\) and \(v_{II}\).

The logarithmic resonance is quickly washed out, \(v_{bII}/v_{bI}\) being about 2 for our parameters \((\omega_p = \omega, \omega_p^2/\omega^2 = 0.002,\)
\[\text{g} = \pi \omega_p^2/\text{L}, \nu/\omega = 10^{-4}, 2\pi \nu_0/\omega \text{L} = 0.002\]), but \(E_{II}/E_I\) is of order 15. In mode II the beam will pass through \(z=0\) without a resonance in beam velocity, but a large field will exist over a very narrow region. The resonance at \(z=0\) is possibly caused by the premodulated beam's driving the plasma at its resonance frequency.

For our purposes of explaining the narrow collected beam velocity spread observed experimentally, we feel we can ignore the \(Y_o\) solution. The reasons are:

1. The first order beam velocity is not strongly affected by it as compared to the \(I_o\) solution, if reasonable collisions are included;

2. The beam is unlikely to be modulated strongly at \(\omega_p\) until that \(\omega_p(z)\) is reached, making the excitation of either the \(Y_o\) or \(J_o\) solutions unlikely.

We have considered only a linearly increasing plasma density along the direction of beam flow. If we have a linearly decreasing plasma density, \((g\ \text{negative})\), the ordinary
and modified Bessel functions are interchanged in Eqs. 5.11-5.14.

Late in this research we found that Barston\textsuperscript{63} has worked out results similar to Eqs. 5.15 and 5.16.

5.3 Computer Experiments with a Linear Plasma Density Gradient along the Direction of Beam Flow

We present a computer experiment with the plasma density increasing linearly from the gun. The plasma is linearized, cold, lossless and one-dimensional. A cold, one-dimensional beam with 0.1 per cent velocity modulation, but no density modulation, is continuously injected into the plasma. The plasma is initially quiescent, and the programs are run until a steady state appears to be reached.

The beam sheets generate wakes as given in Appendix C. As in the lossy case described in Chapter 4, the wakes are superposed. An initial field (Appendix D, Eq. D.11) is set up in the plasma to simulate the steady-state field created by an infinite number of unmodulated beam sheets ahead of the first injected beam sheet. This field exactly cancels the drag caused by the self field of the first injected sheet of an unmodulated beam. The initial field combines with the wake of the first sheet such that the combined field exactly cancels the drag due to the self field of the second sheet, and so forth. Hence no sheet of an unmodulated beam sees any net force as it moves through a lossless plasma. The
only forces are caused by the effects of beam bunching.

The Runge-Kutta method is used to calculate the trajectories. The equations are,\(^5\) for \(\ddot{a}z(t)/\dot{a}t^2 \equiv a(t,z);

\[
  z(t+\Delta t) = z(t) + v(t)\Delta t + (\Delta t/6)(m_1 + m_2 + m_3)
\]

\[
  v(t+\Delta t) = v(t) + (1/6)(m_1 + 2m_2 + 3m_3 + m_4),
\]

\[
  m_1 = \Delta t \ a[t, z] \tag{5.27}
\]

\[
  m_2 = \Delta t \ a[t+\Delta t/2, z(t)+ (\Delta t/2)v(t)]
\]

\[
  m_3 = \Delta t \ a[t+\Delta t/2, z(t)+ (\Delta t/2)v(t)+0.25\Delta t m_1]
\]

\[
  m_4 = \Delta t \ a[t+\Delta t, z(t)+\Delta tv(t)+0.5\Delta tm_2].
\]

This method is more accurate and more convenient than the Milne method described in Chapter 4. It is more convenient because it is self starting; that is to calculate \(z(t+\Delta t)\), no points in the trajectory for times less than \(t\) are required. Hence when crossings occur no corrections need be made to previous points, such as those described in Chapter 4. In any case, the Runge-Kutta routine would be needed to start the Milne routine. The Milne routine is numerically unstable, allowing an energy error in the lossy case of about five per cent of the average energy transmitted from the beam to the plasma. The error with the Runge-Kutta method for the computer experiment described here, over any interval of \(50/\omega_p\), is less than 0.05 per cent of the total energy.
We impose a 0.1 per cent velocity modulation on the beam at a single frequency, corresponding to the plasma frequency at $z=0$ (the point of injection, or gun). The plasma density varies as

$$\omega_p^2(z) = \omega^2 + \omega_{po}^2 \frac{\pi z}{L}. \quad (5.28)$$

This density variation is a linear approximation to a sinusoidal density distribution. $\omega_{po}$ corresponds to the peak density of the sinusoid, and $L$ is the length of the plasma. $\omega_p^2(z=0) = 0.15 \omega_{po}^2$, so the sinusoid is actually truncated at $z=0$. In the laboratory the plasma density cannot be zero at the end, because if it were there could be no particle current at the end with less than an infinite diffusion velocity.

Distances are normalized to the average beam intersheet spacing, $0.4v_0/\omega_{po}$, times to $\omega_{po}^{-1}$, the acceleration ($qE/m$) to $0.4v_0\omega_{po}$. The density gradient is chosen so that $\pi/L=0.001$ $x (\omega_{po}/0.4v_0)$, and the beam density is such that $(\omega_{pb}/\omega_{po})^2=0.005$. The gradient is not as steep as in the laboratory plasma.

At $t=300/\omega_{po}$, snapshots of the beam sheet velocity and the acceleration of a test sheet are shown in Figs. 5.3 and 5.4 respectively. The results are compared with the linear theory (Eqs.5.13 and 5.17), and are seen to be in agreement in the small signal region. For the purpose of comparison with the linear theory we assume that the magnitude of mode III is equal to the total magnitude of the modulation, and mode IV is not excited at all. We were unable to excite mode
Fig. 5.3 Snapshot of the beam sheet velocity vs. distance in the computer experiment taken at $t = 300/\omega_{po}$. 

$V_T = 0.075v_o$

$\omega_p = 0.4\omega_{po}$

$V_{bIII}/V_{ob}$

$X$ = VELOCITY OF EACH BEAM SHEET

DISTANCE (UNITS OF $0.4v_b/\omega_{po}$)
Fig. 5.4 Snapshot of the acceleration of a test particle, taken at $t = \frac{300}{\omega_{po}}$. The small discontinuities in the field are related to the field discontinuity caused by a charge sheet. Agreement with the linearized theory is excellent for more...

$\frac{q E_{III}}{m}$
IV in the computer experiments.

The discontinuities in the acceleration plot are associated with sheet discreteness. The acceleration shown is drawn through points separated by on average intersheet spacing $(0.4v_0/\omega_p)$. As we move away from the gun, there are a series of points between which there is one sheet, and then suddenly, due to beam bunching, there are two points with either zero or two sheets between them. The discontinuity in the acceleration caused by one sheet is $0.005(0.4v_0\omega_p)$.

In Fig.5.3, the growth is compared to that predicted (Eq.3.24) in a uniform plasma with temperature corresponding to forty volts. The density of the warm plasma is chose to match the density at $z = 10(0.4v_0/\omega_p)$ in the nonuniform plasma, (close to the gun). The growth rate is less than what it would be if we had let $\omega_p = \omega_p$ or if we had let the temperature be a more realistic ten or twenty volts. Hence the density gradient reduces the growth rate in the linearized region well below that predicted for the uniform warm case, for the same beam density and excitation.

A steady state seems to be reached up to $Z = 150$ without the introduction of loss or temperature. The large beam bunch at initial beam overtaking now drives the plasma off resonance, since the modulation frequency $\omega$ is different from the $\omega_p$ at the plane of overtaking $Z=150$. This non-resonance is reflected in the fact that the overtaking bunch occurs at a zero of acceleration, and well away from the
acceleration minimum.

The peak field shown \((qE/m = 0.18(0.4v_o\omega_{p0}))\) drives the local plasma velocity at \(Z = 250\) to \(0.12v_o\) (derived from \(v_p = qE/jm\omega_p(z)\)). This is much less than the \(0.4v_o\) we found for the maximum velocity of the untrapped plasma sheets in the uniform case. It is large enough, however, to throw some suspicion on our assumption neglecting the \(v_p \cdot \nabla v_p\) term in the plasma force equation, which we used to derive the sheet wake in Appendix C. With a disk beam (Chapter 6), this nonlinearity will be no problem.

The fields are large enough to scatter the beam in velocity fairly widely. The interaction length shown is twenty per cent of the actual laboratory length, for the \(\omega_{p0}\) used in the computer experiment.

In Fig.5.5, we show the results at a later time, \(t = 400/\omega_{p0}\). A steady state now seems to be reached up to the plane \(Z = 300\). The beam is now more spread out in velocity, but the peak fields have not significantly increased from those at \(t=300/\omega_{p0}\).

Eq.5.10 is valid for negative \(g\) as well as positive \(g\), so that \(J_o\) and \(Y_o\) in Eqs.5.11-5.18 are interchanged with \(I_o\) and \(K_o\). In this case \(K_o\) is the growing solution for \(z<0\), so that if the beam is modulated at \(\omega_p(z=0)\) at a plane \(z<0\), initial beam overtaking is possible near \(z=0\), because the plasma is being driven strongly near its resonant frequency. Such a situation is shown in Fig.5.6. The plasma frequency
\[ \omega_p^2(z) = \omega^2 + 0.001z \]
\[ \omega = \omega_p(z=0) \]

0.1 per cent velocity modulation at \( z = 0 \)

\[ t = 400/\omega \]

\[ \omega_{pb} = 0.005 \omega^2 \]

**Fig. 5.5** Snapshot of beam and plasma variables.
Fig. 5.6 Interaction with a negative density gradient.
is given by \( \omega_p^2(z) = \omega_{po}^2 - 0.001(\omega_{po}^3/0.2v_o)z \). Distances are normalized to the intersheet spacing, \( 0.2v_o/\omega_{po} \). This intersheet spacing is half that of Fig.5.5, because the average frequency is higher. A velocity modulation of \( 0.001v_o \) is imposed at \( z=0 \), at a frequency of \( \omega = \omega_p(z=150(0.2v_o/\omega_{po})) \).

As in Fig.5.5, \( \omega_{pb}^2/\omega_{po}^2 = 0.005 \).

The peak field obtained in Fig.5.6 is over twice that obtained in Fig.5.5, such that \( qE/m = 0.18v_o \omega_{po} \), as compared to \( qE/m = 0.08v_o \omega_{po} \) in Fig.5.5. The linearized plasma approximation is not so good in this case, since \( v_p/v_o = 0.2 \). The maximum field occurs before the plane where \( \omega_p = \omega \), and at that plane (\( Z=150 \)), the fields are relatively small. There seems to be a meniscus around \( Z=110 \), with a "dark space" around \( Z=150 \), as we found in the uniform plasmas of Chapter 3. Again as in Chapter 3 the beam sheets near overtaking are at first strongly decelerated, then fall back into an accelerating phase and are re-accelerated, so as to curl around in the snapshot of velocity vs. distance.

We now return to the positive density gradient. The linear theory can help predict the effects of changing the ratio of beam density to plasma density or of changing the plasma density gradient. By manipulation of Eqs.5.13, 5.17, 5.19-5.21, we can find approximations for the envelopes of \( \rho_p, v_p, v_b \) and \( E \) in terms of \( \rho_b/\rho_{bo} \) near the initial beam overtaking for mode III. The details are given in Appendix E. These approximations are:
\[ |v_b/v_o| = \{\omega_{pb}/(gz)^{1/2}\} |\rho_b/\rho_{bo}| \]  
(5.29)

\[ \varepsilon_o |E|^2 /n_bo m v_o^2 = \{\omega^2 \omega_{pb}^2 / (gz)^2\} |\rho_b/\rho_{bo}|^2 \]  
(5.30)

\[ |v_p/v_o| = (\omega^2_{pb}/gz) |\rho_b/\rho_{bo}| \]  
(5.31)

\[ |\rho_p/\rho_{po}(z)| = (\omega^2_{pb}/gz) |\rho_b/\rho_{bo}|. \]  
(5.32)

From Eqs. 5.29 and 5.30 we expect lower amplitudes of 
\[ |v_b/v_o| \] and E near overtaking if the density gradient is increased; as also expect a lower spatial growth rate (Eq. 5.13). In Fig. 5.7 the density gradient is double that of Fig. 5.5, and corresponds better to the laboratory experiment. The plasma frequency is described by \[ \omega^2_p(z) = \omega^2 + 0.002 \ Z, \] where \( Z \) is normalized to \( 0.4v_o/\omega_{po} \). We increase the velocity modulation at \( z=0 \) from \( 0.001v_o \) to \( 0.004v_o \), but continue to modulate at \( \omega = \omega_p(z=0) = (0.15)^{1/2}\omega_{po} \). The increase in modulation amplitude allows us to better study the region beyond initial beam overtaking. The fields and growth rate are small than in Fig. 5.5, except for an apparent resonance near \( Z=225 \), where \( \omega_p = 2\omega \). The field is still increasing with time in this region. It seems that the plasma is responding to the second harmonic of the beam current. In Fig. 5.8 (\( t=367/\omega_{po} \)) a beam bunch near \( Z=200 \) forms and excites this wave. This bunch is seen forming near \( z=185 \) in Fig. 5.7. If \( \omega_p = 2\omega \), the wakes from bunches passing by every \( 2\pi/\omega \) seconds add coherently.

The resonance at \( \omega_p = 2\omega \) is not possible in Fig. 5.5, because \( \omega_p \) does not equal \( 2\omega \) until \( Z=450 \), which is not
Fig. 5.7 Snapshot of beam and plasma variables. $\omega_p(z=225) = 2\omega.$
Fig. 5.8 Snapshot of beam velocity and acceleration, \( \omega_p(z=225) = 2\omega \).

There seems to be a discrepancy, as \( \omega_p(z=225) \) should be 2 times \( \omega \).
included in the interaction region.

In Fig. 5.7 we increase the modulation amplitude to $0.02v_o$ in order to cause overtaking nearer the gun, and perhaps allow the beam enough extra drift length so that it is debunched before reaching the plane where $\omega_p = 2\omega$. The peak field is indeed smaller at the time of this snapshot ($t=400/\omega_p$, vs. $t=367/\omega_p$ in Fig. 5.8), but it is still growing near $Z=225$. We reduce the component of beam current at $2\omega$, but do not eliminate it.

An estimate of the magnitude of the fields obtainable in the plasma can be found from Eq. D.11 (Appendix D). This equation gives the steady-state fields excited by sheets passing a plane at intervals of $\Delta t$, and was originally derived to find the fields necessary to simulate those generated by an infinite number of unmodulated beam sheets ahead of the first injected beam sheet. This equation is

$$\frac{qE}{m} = \frac{qQ}{me_o \sin(\omega_p \Delta t/2)} \sin\{\omega_p(t-z/v_o) + \omega_p \Delta t/2\}, \quad (D.11)$$

where $Q$ is the sheet surface charge density. We wish to find the fields produced by clumps of beam sheets passing a plane at intervals of $2\pi/\omega$. If the bunch is fairly tight (all the sheets in the bunch pass the plane within a time interval short compared to the local $1/\omega_p$), we can lump the charge of the individual sheets into one big sheet, and replace $Q$ in Eq. D.11 by the total surface charge density of the sheets in the bunch, $Q_{\text{tot}}$. After the bunch passes, the fields
Fig. 5.9 Snapshots of beam velocity and acceleration vs. distance.
Same conditions as Fig. 5.8, except for larger modulation.
The resonance at the second harmonic is less well defined.
oscillate to their peak value of

$$|\frac{qE}{m}| = \frac{qQ_{tot}}{2me_0 \sin(\pi \omega_p/\omega)},$$

(5.33)

where we have replaced $\Delta t$ by $2\pi/\omega$. If $\omega_p = n_0 \omega$, where $n$ is an integer, the steady state field is infinite. This result is, of course, not surprising. A strongly bunched beam will have many higher harmonics of the modulation frequency. If the plasma is resonant at one of the harmonics, its response will be large. Note that the field amplitude off a resonance varies as $Q_{tot}$, which should vary directly as $n_0$. $Q_{tot}$ should vary inversely as $\omega$, since the fundamental wavelength, and hence the region from which the beam bunch could be formed, varies inversely as $\omega$.

Eqs. 5.29 and 5.30 predict that the first order beam velocity and electric field near overtaking are reduced if the plasma density is reduced. Eq. 5.33 predicts that beyond overtaking, the fields caused by a beam bunch should also be reduced, (if $\omega_p \neq n_0$). In Fig. 5.10 we show the results of a case similar to that of Fig. 5.8, except the beam density is reduced by a factor of 2.5, or $\omega_{pb}^2/\omega_{po}^2 = 0.002$. The plasma density is assumed to vary sinusoidally, such that $\omega_p^2 = \omega_{po}^2 \sin(0.15 + 0.002 Z)$, to better represent the laboratory conditions. The interaction length is about twenty per cent as long as a half period of the sinusoid. The difference between using a linear gradient and a sinusoid is probably negligible over this distance, except the sinusoid has a
Fig. 5.10 Snapshot of beam velocity and acceleration vs distance. $\omega_p = 2\omega$

at $Z = 246$. Compare with Fig. 5.8, where a higher beam density.
slightly smaller gradient for the largest values of Z shown. The plane where \( \omega_p = 2 \omega \) is further from the gun, occurring at \( Z=246 \). The beam velocity modulation is again \( 0.004 v_o \), at \( \omega = \omega_p(z=0) = (0.15)^{1/2} \omega_{bo} \). In this case the spatial growth rate of the interaction with a reduced beam density is quite reduced from that of Fig.5.8, as expected from Eq.5.13.

By assuming \( \rho_b/\rho_{bo} = 1 \), we can solve Eq.5.19 approximately for the plane of initial beam overtaking. We had thus expected this plane to occur near the plane where \( \omega_p = 2 \omega \), and for the second harmonic in the beam current to drive the plasma very strongly. We expected a result similar to that involving the negative density gradient (Fig.5.6), with large fields rapidly increasing in time near the plane of resonance. However, the fields are not large until beyond overtaking, and the second harmonic waves which do form (the wavelength of \( \lambda=20 \) in this region is half that near the gun indicating that a wave at frequency \( 2 \omega \) is dominant) seen to disrupt the beam bunching in the region \( Z<246 \). The second harmonic wave seems to be trying to prevent the buildup of fields. The explanation can be found from Gauss's law, which after the time transform is:

\[
\nabla \cdot \{ \varepsilon(\omega,z) \mathbf{E}(\omega,z) \} = q\rho_b(\omega,z),
\]

where \( \varepsilon(\omega,z) = \varepsilon_o(1-\omega_p^2/\omega^2) \). The plasma charge is included in \( \varepsilon \), and the beam charge is considered as free charge. If \( \varepsilon \) is positive, the fields are such as to debunch the beam and if \( \varepsilon \) is negative \( (\omega<\omega_p) \) the bunch attracts itself. Hence if the
sheets are in the region where $\omega_0(z) > 2\omega$, the beam density component at $\omega$ tends to bunch, that at $2\omega$ to debunch. As the beam goes nonlinear, the interaction at $\omega$ creates a beam density component at $2\omega$ which, however, tries to break itself up. Beyond the plane where $\omega_0 = 2\omega$ the beam bunches attract each other, but they can only drive the fields there off resonance.

There is still a small beam density (and hence current) component at $2\omega$ where $\omega_0 = 2\omega$, so after a long time a resonance is still expected at this plane. However in Fig.5.11, taken at $t=400/\omega_0$, the fields do not seem to have increased in magnitude from those at $t=340/\omega_0$ (Fig.5.10). The component of $\rho_b$ at $2\omega$ can be reduced if the beam is still linear at the plane where $\omega_0(z) = 2\omega$, or if the beam is nonlinear but is modulated over a wide band, so that the different frequencies can interfere with each other.

5.4 Conclusions

We have found that the introduction of an increasing plasma density gradient along the direction of beam flow can greatly reduce the electric field magnitude and beam velocity spread below that found in a uniform plasma. Reducing the beam density and holding everything else constant further reduces these variables in both the linear regime and beyond overtaking. This fact leads us into the next chapter, where we model the beam with disks. Using disks instead of sheets effectively reduces the beam density, and
\[
\omega_p^2 (z) = \sin_2 (\omega^2 + 0.002z) \omega_{p0}^2
\]
\[
\omega_p^2 = 0.15 \omega_{p0}^2
\]
\[
n_{bo}/n_{p0} = 0.002
\]

\[\times\] Beam sheet velocity

\[\text{---}\] Acceleration

\[v_b(z=0) = 0.004v_o\]
\[t = 400/\omega_{p0}\]

Fig. 5.11 Snapshot of beam sheet velocity and acceleration of the sheets. \(\omega_p(z = 246) = 2\omega.\)
allows us to approach the experimentally observed collected beam velocity distribution.
Chapter 6: Disk Beam-Plasma Interaction in a Plasma with Density Gradients Along the Direction of Beam Flow

6.1 Introduction

In Chapter 5, we found that a plasma density gradient along the direction of beam flow is an important factor in reducing the interaction, more so than temperature or collisions. We also found from the linear theory of Chapter 5 that a reduction in beam density relative to the plasma density reduces the beam velocity spread. Modeling the beam with disks to take account of the finite diameter of the beam effectively reduces the beam density, because a disk generates a smaller wake than that of a sheet of the same surface charge. In addition, a disk has purely repulsive fields on each side of it that tend to debunch the beam and reduce the strength of the interaction. This model should give a better simulation of beam bunching.

We modulate over a band of frequencies, instead of just one $\omega_p$ as we had been doing before. Waves at different frequencies will be expected to interfere with each other after the beam overtakes and goes nonlinear, and this interference should also reduce the interaction level. The real laboratory experiment is excited by wide band noise arising from the thermal fluctuations in the beam and the plasma.

In this chapter we first investigate the validity of using
disks. Then we derive a linear theory using a finite diameter rigid beam model valid for small signals, and compare it with a computer experiment. Finally we compare the results of a computer experiment with similar computer experiments using a uniform plasma, and with the same plasma (with density gradients) but with the disks replaced by sheets. The gradients are fundamental to achieving the reduction in interaction intensity needed for comparison with our laboratory measurements, reducing the fields by at least an order of magnitude. The finite beam diameter reduces the fields to one-third the strength due to beam sheets, and the beam velocity spread by a factor of two.

6.2 Comments on the Validity of Using Disks

The use of disks to model the beam-plasma interaction as we have done implies certain assumptions, which are: (i) no transverse beam motion; (ii) plasma transverse motion is unhindered; and (iii) no radial variation in $v_{bz}$, the first order beam velocity in the z direction. We are in effect saying, in assumptions (i) and (ii), that the beam sees an infinite magnetic field, whereas the plasma sees none. In any case no θ directed fields are allowed. To obtain a handle on the validity of these two assumptions, we examine the a.c. velocity in a uniform plasma (using Cartesian coordinates for simplicity):
\[ v = \frac{q}{jm} \begin{bmatrix}
\frac{\omega'}{\omega'} & \frac{\omega}{\omega'} & 0 \\
\frac{\omega'}{\omega'} & \frac{\omega}{\omega'} & 0 \\
0 & 0 & \frac{1}{\omega'}
\end{bmatrix} \begin{bmatrix}
E_x \\
E_y \\
E_z
\end{bmatrix} \]

(6.1)

where \( \omega' = \omega = \omega_p \) for the plasma, and \( \omega' = \omega_p - kv_0 \) for the beam.

In our experiment, the magnetic field varies over the mirror from 440 to 1600 Gauss, or \( \omega_c \) varies from \( 2\pi \times 1.2 \times 10^9 \) to \( 2\pi \times 4.5 \times 10^9 \); whereas the maximum \( \omega_p (\omega_{po}) \) of the sinusoidal density distribution is about \( 2\pi \times 23 \times 10^9 \). Hence for the plasma, \( \omega' = \omega_p >> \omega_c \) over most of the interaction length, and assumption (ii) is valid, as we can see from Eq. 6.1.

Assumption (i) requires \( |\omega_p - kv_0| << \omega_c \). To find \( k \), we anticipate the results of the next section and look at the results (Fig. 6.1) of a WKB approximation to the finite diameter beam-plasma interaction for the parameters of this computer experiment, with \( \omega = (0.15)^{\frac{1}{2}} \omega_{po} = 0.4 \omega_{po} \). \( R_e(k) \) is quite close to \( \omega/v_o \), so \( (\omega - kv_0) = -jIm(kv_0) \). \( Im(kv_0) \) varies with distance from the gun from about \( 0.2\omega \) to \( 0.036\omega \) and less.

Hence \( |\omega - kv_0| \) is less than \( \omega_c \), although only by a factor of 3 near the gun, and the assumption of the transverse beam motion being small compared to the longitudinal beam motion is a fairly good one, for comparable transverse and longitudinal electric fields. By symmetry, the transverse fields might be expected to be small compared to the longitudinal field,
making assumption (i) even better.

The effect of assumption (iii), that \( v_{bz} \) is independent of radius, is unknown. It implies that \( E_z \) is independent of radius over the disk.

6.3 Linear Theory of a Solid Beam Model of a Beam-Plasma Interaction with a Finite Diameter Beam

In this section we derive a linear theory with which we can compare the computer experiment. We assume a rigid beam\(^{18,64}\) consistent with the disk model, such that there is no radial beam motion and \( v_{bz} \) has no radial dependence. The plasma is cold and one dimensional, with a linear density gradient along the direction of beam flow. There is no magnetic field. The density gradient is described by

\[
\omega_p^2(z) = \omega^2 + gz. \tag{6.2}
\]

Within the quasistatic assumption, Poisson's equation is:

\[
\nabla \cdot (\kappa_{11} \nabla \phi) = -q n_b / \varepsilon_0, \tag{6.3}
\]

where \( n_b \) is the first order beam density, \( \phi \) is the electric potential, and

\[
\kappa_{11} = (1 - \omega_p^2(z) / \omega^2) \mathbf{I} = -(gz / \omega^2) \mathbf{I}, \tag{6.4}
\]

where \( \mathbf{I} \) is the unit tensor.

We perform a Bessel transform on \( \phi \) in the transverse direction; that is
\[ \phi_{k_T} = \int_0^\infty dr \, r \, J_0(k_T r) \phi_r, \quad (6.5) \]

and

\[ \phi_r = \int_0^\infty dk_T \, k_T J_0(k_T r) \phi_{k_T}. \quad (6.6) \]

These transforms assume an angularly symmetric mode. Combining Eqs. 6.3-5, we find

\[ \frac{g z}{2} \frac{2}{k_T} \phi_{k_T} - \frac{g}{\omega^2} \frac{\partial \phi_{k_T}}{\partial z} - \frac{g z}{\omega^2} \frac{\partial^2 \phi_{k_T}}{\partial z^2} = -q n_{b k_T} / \varepsilon_0. \quad (6.7) \]

The equation for conservation of beam electrons is

\[ n_b = -\mathbf{\nabla} \cdot \{ n_{b o}(r) \mathbf{\delta} \}, \quad (6.8) \]

where \( \mathbf{\delta} \) is the displacement of the beam from its equilibrium. By using the assumption of no radial beam motion, Eq. 6.8 becomes, after performing the transform,

\[ n_{b k_T} = -n_{b o k_T} \frac{\partial \mathbf{\delta}}{\partial z}. \quad (6.9) \]

Since there is no radial variation in \( \delta_z \), we average the field over the disk radius. The force equation is:

\[ \{ j \omega + v \mathbf{\omega} \cdot \frac{\partial}{\partial \mathbf{\omega}} \phi \}^2 n_{b o} \delta_z = -\frac{q}{m} \left\langle n_{b o} \frac{\partial \phi}{\partial z} \right\rangle, \quad (6.10) \]

where the average is defined as

\[ \langle \rangle \equiv \lim_{r_1 \to \infty} \frac{1}{2\pi r_1} \int_{r_1}^{r_2} \, dr \, r. \quad (6.11) \]
Hence

\[ \langle n_{bo} \phi \rangle = \lim_{r_1 \to \infty} \frac{2\pi}{\pi r_1^2} \int_0^{r_1} dr \int_0^\infty dk_T k_T n_{bok_T} J_0(k_T r) \]

\[ \times \frac{dk'}{\pi} k' \phi_{k'} J_0(k'r), \]

or

\[ \langle n_{bo} \phi \rangle = \lim_{r_1 \to \infty} \frac{1}{\pi r_1^2} \int_0^\infty dk_T k_T n_{bok_T} \phi_{k_T}. \quad (6.12) \]

To proceed further, we use the WKB approximation, that is we assume all variables vary as \( e^{-i \gamma_k(z)}dz \). This assumption allows us to replace \( \partial / \partial z \) by \(-jk\), if we assume \( k \) is slowly varying and ignore \( \partial k / \partial z \) to \( k^2 \). This assumption turns out to be not so good near the plane where \( \omega = \omega_p(z) \), but is good further downstream. We combine Eqs. 6.7, 6.9, 6.10, and 6.12 to obtain

\[ (\omega - k\nu_o)^2 = -\frac{q^2 k^2 \omega^2 2\pi}{m \epsilon_o} \frac{\int \frac{dk_T k_T n_{bok_T}^2}{2 g_0(z(k^2 + k_T^2) + jk)}}{\langle n_{bo} \rangle \pi r_1^2 g_0(z(k^2 + k_T^2) + jk)}. \quad (6.13) \]

Since \( n_{bo}(r) = n_{bo} \) for \( r \leq r_o \), and \( n_{bo}(r) = 0 \) for \( r > r_o \),

\[ n_{bok_T} = n_{bo} r_o J_1(k_T r_o) / k_T. \quad (6.14) \]

By conservation of charge,

\[ \lim_{r_1 \to \infty} \langle n_{bo} \rangle \pi r_1^2 = \pi r_o^2 n_{bo}. \quad (6.15) \]
Defining $\omega_{pb}^2 \equiv q^2 n_{bo}/m \epsilon_o$, and combining Eqs. 6.15 and 6.13, we have

$$
(\omega - k \nu_o)^2 = - \frac{\omega_{pb}^2 k^2 \omega^2}{g} \int_0^\infty \frac{dk_T J_1^2(k_T r_o)}{k_T[z(k^2+k_T^2)+jk]} .
$$

(6.16)

Carrying out the integral, this equation is:

$$
(\omega - k \nu_o)^2 = - \frac{\omega_{pb}^2 k^2 \omega^2}{g z(k^2+jk/z)} \{1-2I_1(r_o \sqrt{k^2+jk/z}) \}
\times K_1(r_o \sqrt{k^2+jk/z}) .
$$

(6.17)

Eq. 6.17 is solved for complex $k$ in terms of real $\omega$ for the parameters of the computer experiment (Fig. 6.1). In section 6.5 we will compare Eq. 6.17 with the results of the computer model. This gives us a check on the computer model.

To consider a uniform plasma, we replace $g z$ by $\omega_{pb}^2 - \omega^2$ in Eq. 6.17 and ignore the $jk/z$ terms, which arose from the density gradients. We find that

$$
(\omega - k \nu_o)^2 = \{\omega_{pb}^2/(1-\omega_{pb}^2)\}\{1-2I_1(kr_o)K_1(kr_o)\} .
$$

(6.18)

As $r_o \rightarrow \infty$, $I_\gamma(kr_o)K_\gamma(kr_o) \rightarrow i k r_o \rightarrow 0$, for any $\gamma$. We recover the one-dimensional limit, which is

$$
(\omega - k \nu_o)^2 = \omega_{pb}^2/(1-\omega_{pb}^2) .
$$

(6.19)

This result is in contrast to the result of authors 7,9 who do not assume a rigid beam but allow radial beam motion and who obtain
\[ \omega = (0.15)^{1/2} \omega_{po} \]
\[ r_o = 0.1875 \text{ cm.} = 0.075 \text{ in.} \]
\[ \omega_{pb} / \omega_{po} = 0.002 \]
\[ g = 0.002 \omega_{po}^3 / 0.4\nu_o \]
\[ \omega_p^2 = \omega^2 + 0.002\omega_{po}^3 z / 0.4\nu_o \]

Fig. 6.1 WKB approximation to rigid beam, plasma with linear density gradient interaction. See Eq. 6.17. Finite diameter beam of radius \( r_o \).
(ω-κν₀)² = \{ω₂²_p/(1-ω²_p/ω²)\}kν₀I₁(κν₀)K₀(κν₀). (6.20)

Since I₁(κν₀)K₀(κν₀) → 1 as \( r_0 \to \infty \), Eq. 6.20 does not go to the correct limit of Eq. 6.19, but is off by a factor of 2. The electric field of Eq. 6.20 is found to be generated entirely by surface charge at the beam edge. Hence on physical grounds we would not expect the correct limit, since the fields of the correct limit in Eq. 6.19 are generated entirely by volume charge. This point is puzzling, and deserves further study.

6.4 Model

The plasma is again linearized, as in Chapter 5, but we now use a sinusoidal density variation instead of a linear one. This variation corresponds better to the laboratory case. The plasma frequency varies as

\[ \omega_p^2 = \omega_p^2 \sin(0.15 + \pi z/L), \] (6.21)

where \( \omega_p^2 \) corresponds to the peak density. We normalize distance and time to \( 0.4v_o/\omega_p \) and \( \omega_p^{-1} \), respectively, as in Chapter 5. The peak plasma density assumed is \( 2.5 \times 10^{12} \text{ cm}^{-3} \) (\( \omega_p = 9 \times 10^{10} \text{ /sec.} \)), the beam density is \( 5 \times 10^9 \text{ cm}^{-3} \) (\( \omega_p^2/\omega_p = 0.002 \)) and the beam diameter is 0.15 inches. This beam diameter is larger than the nominal 0.1 inch for this gun, and allows for less than optimum focusing. The plasma density is smaller than that which has since been measured in the laboratory, which is \( 6.9 \times 10^{12} \text{ cm}^{-3} \). Parameters which are closer to the
laboratory ones are considered later, where we find little significant difference in the results. For the parameters used, the interaction length of the computer experiment is twenty per cent of the distance between zeroes of the sinusoidal plasma density distribution.

The fields generated by a beam disk moving at constant velocity through an infinite, uniform, linearized plasma with no magnetic field have been derived by Gould and Allen. This derivation is reproduced in Appendix G. The acceleration of a similar disk in front of the disk generating the field is

$$a(t, \Delta z) = \frac{q}{m} \frac{Q}{\varepsilon_0} \int_0^{\infty} J_1(x) x e^{-x \Delta z / r_0} \frac{dx}{x^2 + \alpha^2}, \quad (6.22)$$

where $q/m$ is the charge to mass ratio of an electron, $Q$ is the disk surface charge density, $\Delta z$ is the disk separation, and $\alpha = \omega_p r_0 / v_d$, where $r_0$ is the disk radius and $v_d$ is the disk speed. Later we will allow $v_d$ to vary, but here we hold it constant. The disks are injected with an average velocity $v_d = v_0$, and $\Delta z$ in this derivation at constant is $z - v_0 t$. If $\Delta z = 0$, the value of the integral is $I_1(\alpha)K_1(\alpha)$. The field producing the acceleration is nonoscillatory and repulsive. It is similar to the vacuum field of the disk, reduced due to shielding by the plasma, and it decays rapidly ahead of the disk.

The acceleration of a similar disk behind the disk is

$$a(t, \Delta z) = -\frac{q}{m} \frac{Q}{\varepsilon_0} \{1 - 2I_1(\alpha)K_1(\alpha)\} \cos \omega_p (t - z/v_d)$$

$$- \frac{q}{m} \frac{Q}{\varepsilon_0} \int_0^{\infty} J_1(x) x e^{-x \Delta z / r_0} \frac{dx}{x^2 + \alpha^2} \quad (6.23)$$
In addition to the mirror image of the nonoscillatory field there is an oscillatory field behind the disk, whose magnitude, for our numbers, is reduced about forty per cent from that of a sheet of the same surface charge density \( Q \). Hence the disk wake is reduced from that of a sheet, and in addition there are nonoscillatory repulsive forces that act to debunch the beam. The beam density reduction factor for the wake field in Eq.6.23 is similar to that of Eq.6.18, with \( k \) replaced by \( \omega_p/\nu_d \).

The difference in the fields generated in a plasma by a disk and by a sheet are related to the differences in their vacuum fields. The disk vacuum field decays with distance, but the sheet vacuum field is uniform, having magnitude \( Q/2\varepsilon_0 \) on each side. In equilibrium, cold plasma electrons will be repelled from the sheet a distance of \( Q/\rho_p \), cancelling the vacuum field everywhere. Hence there are two fields to the right of the sheet, the sheet vacuum field of \( +Q/2\varepsilon_0 \) and the shielding field of \( -Q/2\varepsilon_0 \). Let the sheet now move to the right. The sheet vacuum field remains constant at any point ahead of the sheet, so there will still be no net field in front. However, immediately after the sheet passes a point, the old shielding field and the new vacuum field add, producing a total field of \( -Q/\varepsilon_0 \). The plasma electrons, which are motionless before the passing, are first repelled from the sheet on sheet passing, and then oscillate at \( \omega_p \).
The situation with a disk is different. When the disk is far away, the disk vacuum field vanishes, and there is no shielding field from the plasma. As the disk approaches its vacuum field increases, and the plasma electrons are repelled from it. However, because the field is continually increasing, the inertia of the plasma electrons does not allow complete shielding, and there is a net field ahead of the disk. The shielding is more complete if the plasma density is higher, because more electrons can contribute to the shielding, or if the disk velocity is less, because the plasma has more time to respond. These facts are reflected in Eq. 6.22, which shows that the field ahead of a disk is a decreasing function of \( \omega_r \). 

At the instant the disk passes, the shielding field is less than it would be for a sheet, and this fact reduces the wake. After the disk passes, the fact that the disk vacuum field decreases rapidly means that the plasma electrons do not receive as much of a total kick away from the disk as they would from a sheet. This fact also reduces the oscillatory (wake) field.

As opposed to the sheet case, the plasma electrons have a velocity at the instant the disk passes. The nonoscillatory field behind the disk exactly cancels this velocity as the disk moves far away.

Although Eqs. 6.22 and 6.23 are derived for a uniform plasma and a uniform velocity disk, we use them in a plasma with a density gradient along the direction of beam flow,
and allow the disk velocity to change. The disk velocities turn out to change by less than ten per cent over a wavelength. The plasma density also changes slowly over a wavelength, there being over 100 wavelengths between zeroes of the sinusoidal plasma density. Since the gradient made no difference in the response of the plasma to a sheet, and we could exactly replace \( \omega_p \) of the uniform solution by \( \omega_p(z) \), we might expect this replacement in the wake of a disk to be good.

The assumed wake acceleration is

\[
a(t, z) = -\frac{q}{m} \frac{Q}{\varepsilon_0} \{1 - 2I_1[\alpha(z)]K_1[\alpha(z)]\} \cos(\omega_p(z)) \times [t - t_1(z)] u_{-1}[t - t_1(z)], \tag{6.24}
\]

where \( \alpha(z) = \omega_p(z)r_0/v_d(z) \), \( t_1(z) \) is the time at which the disk passes the plane \( z \), and \( u_{-1} \) is the unit step function. We keep track of the wake field the same way as in Chapters 4 and 5, summing the wakes from different disks at cell points \( 0.4v_o/p_0 \) (the average disk separation) apart.

To calculate the nonoscillatory force of, say, disk 1 on disk 2, we use Eq.6.22, substituting the \( \omega_p \) at the position of disk 2 and the present velocity of disk 1. These fields are not expressible in closed form, and are functions of \( \omega_p \), \( v_d \), and \( \Delta z \). We write an approximating function for Eq.6.22, which is

\[
a(t, z) = -\frac{q}{m} \frac{Q}{\varepsilon_0} I_1(\alpha)K_1(\alpha)/\{1 + 1.938y + 1.4886y^2 + \alpha(0.9989y + 0.6387y^3 + 0.2002y^4)\}, \tag{6.25}
\]
where \( y = z/r_o \). At \( z=0 \), Eq. 6.15 agrees exactly with Eq. 6.22. The coefficients of \( y, y^3 \), and \( y^4 \) which are multiplied by \( \alpha \) in Eq. 6.25 where chosen by a computer program to minimize the sum of the absolute magnitudes of the errors for \( \omega_p = \omega_{po} \) and \( \omega_p = \omega_{po}/2 \), with \( v_d = 5.3 \times 10^9 \text{cm./sec.} \) and \( r_o = 0.125 \text{cm.} \). We ended up using \( r_o = 0.1875 \text{cm.} \), and the comparison with the actual functions for this value of \( r_o \) is shown in Fig. 6.2 for several different values of \( \omega_p \). The approximation is not as good as if we had optimized using the correct \( r_o \), but it is still reasonable good. Since \( v_d \) always appears as \( \omega_p/v_d \), we have also allowed for changes in velocity. At \( z=0 \), \( \omega_p \) equals \((0.15)^{1/5}\omega_{po}\), the smallest \( \omega_p \) we allow in this computer experiment, and at \( z=300(0.4)\omega_{po}/\omega_{po} \), \( \omega_p = (0.75)^{1/5}\omega_{po} \), the largest. Changes in \( v_d \) will change \( \alpha \), but the force always lies between the curves labeled \( \omega_p = \omega_{po}/3 \) and \( \omega_p = 4\omega_{po}/3 \) in Fig. 6.2. Since the ordinate of Fig. 6.2 is \((qQ/m\epsilon_0)I_1(\alpha)K_1(\alpha)\), we can find the peak wake acceleration at different values of \( \omega_p \) from Fig. 6.2 and Eq. 6.24.

In calculating the total acceleration of a disk due to the nonoscillatory fields of the other disks, we include the fields from disks within 15.5 average disk separations on each side of the considered disk. Fields from disks further away are not included; their fields will be reduced by a factor of 30 or more from their peaks (See Fig. 6.2).

Just after the injection of just before the collection of a disk there will be fewer than 15 other disks on one side
Fig. 6.2 Comparison of the exact non-oscillatory acceleration due to a disk with the approximation to it.
or the other of the disk. In an effort to reduce end effects, nonoscillatory fields from imaginary disks behind the gun and collector are added on, so that all disks feel an average force of 15 disks on each side. The imaginary disks are placed as they would be in an unmodulated beam, i.e. if the imaginary disk is supposed to be the thirteenth disk from the considered disk, it is placed 13(0.4 v_o/\omega_{po}) away.

As the beam passes through the plasma it is velocity modulated by noise predominately at the local plasma frequencies. Because of the density gradient there is a band of plasma frequencies, so the modulation is wide band. We do not know the actual magnitude of this modulation, but simulate it by applying a modulation at z=0 of

$$\Delta v/v_o = \sum_{j=1}^{100} 1.5 \times 10^{-3}\left(\frac{\omega_{pj}}{\omega_{po}}\right)\cos\left(\frac{\omega_{pj}t}{\omega_{po}} - \theta_j\right), \quad (6.26)$$

where \(\omega_{pj}\) corresponds to the plasma frequency at \(z = j0.4v_o/\omega_{po}\). The disks are modulated over a frequency range of \((0.15)\frac{1}{2}\omega_{po}\) to \((0.35)\frac{1}{2}\omega_{po}\). These frequencies correspond to those within \(0<z<100(0.4v_o/\omega_{po})\). The phases of the hundred frequencies are assumed random with respect to each other. The beam remains linear over the distance corresponding to the plasma frequencies of the modulation (Fig.6.4). Hence we are justified in doing all the modulation at the origin for programming convenience, since the first order beam velocity at any \(\omega\) will not grow until the beam passes the plane where \(\omega = \omega_p\), but its envelope merely oscillates. For a sheet beam the
envelope varies as $J_0\{2\omega \omega \omega \omega (-z)^{1/2}v_o^{1/2}\}$. The total amplitude of the modulation is about $0.02v_o$.

The program used in this computer experiment is given in Appendix H.

6.5 Results of the Computer Model

As a check on the validity of the computer model we compare it with the linear theory. For this purpose only, the beam is modulated at the single frequency $\omega_p(z=0)$, which is $\omega = (0.15)^{1/2}\omega_p$. The beam diameter is $0.125\text{cm}$, $\omega_p/\omega_p$ is $0.1$, and $\omega_p^2 = \omega^2 + 0.001\omega_p^3z/(0.4v_o)$. After the envelope of $v_b$ has reached a steady state, we compare it in Fig.6.3 with the prediction of Eq.6.17, where we see that the agreement is good. The agreement gives us a basis for confidence in the validity of Eq.6.24 in the presence of a density gradient, and serves as a check on the programming. We were unable to do an energy check on this model, because of the complicated nature of the nonoscillatory fields. There is plasma kinetic energy associated with these fields, for which we have no formula. Also, on collection, disk bunching can carry electric energy out of the system, something the sheets in Chapter 5 can not do.

In Fig.6.4 is shown the snapshot of beam disk velocity and disk acceleration at $t = 260/\omega_p$, when a steady state seems to have been reached. We note that although overtaking occurs at $Z=140$ the beam is not spread very widely in velocity.
Fig. 6.3  Comparison of WKB linear theory envelope with computer experiment envelope. Nonuniform plasma with $\omega^2_{pb} = 0.01 \omega^2_{po}$. Envelope points taken at $t = 160.4/\omega_{p0}$, 160.8/\omega_{p0}, 164.4/\omega_{p0}$. 

$\omega_p(0) = (0.15)^{1/2} \omega_{po}$

$g = 0.001 \omega^3_{po}/0.4 v_o$

$r_o = 0.05$ in.
An averaged beam velocity distribution is shown in Fig. 6.5, and we see that the spread is comparable to the laboratory collected beam velocity distribution. It is true, however, that the computer interaction length is only twenty percent that of the laboratory, and the beam velocity spread would be expected to be larger, or at least not smaller, if the whole interaction length had been included.

The shapes of the computer and laboratory beam velocity distributions are different, with the laboratory one biased more to lower velocities. This may be expected, since the real plasma is lossy, whereas the computer plasma is not.

Our assumption of plasma linearity seems justified, at least to the neglect of second order terms in the plasma force and conservation equations. The peak oscillatory plasma velocity is about $0.02v_o$, and $\rho_p/\rho_{po}(z)$ is about 0.02.

There is no sign of a meniscus; indeed beyond initial beam overtaking the acceleration is approximately constant with distance. Again, this absence is due to the introduction of off-resonance effects by the plasma density gradient. The large beam bunch at initial overtaking cannot drive the plasma at the local $\omega_p$, because gain is only possible after the beam passes the plane where $\omega = \omega_p$.

To confirm that it is the density gradient that reduces the interaction, and not simply the finite diameter beam and/or the wide band modulation, we ran a case with a uniform plasma. The plasma frequency is equal to $\omega_{po}$ of the previous
Fig. 6.5 Disk velocity distribution in a section $20(0.4v_o/w_{po})$ long, adjacent to the collector, averaged over 7 snapshots taken $4/w_{po}$ apart, starting at $t = 260/w_{po}$. The laboratory experimental velocity distribution is also shown ($p = 1.4 \times 10^{-3}$ Torr). However, the computer interaction length is only 20% that of the laboratory.
example, but the disk beam is the same as before. The modulation is wide band and is given by Eq.6.26, except that all 100 frequencies are equally excited, the modulation extending from $0.5\omega_0$ to $1.5\omega_0$.

Because of the larger plasma frequency, and hence shorter wavelengths, we doubled the number of disks per unit length to continue having about thirty per wavelength. The distance and acceleration normalizations are hence changed to $0.2v_0/\omega_0$, the average interdisk spacing, and $0.2v_0\omega_0$, respectively.

The results are shown in Fig.6.6. This figure is a snapshot of the disk velocity and acceleration at $t = 300/\omega_0$. A steady state has not been reached, but the interaction is still growing. A narrow band signal has risen out of the noise, with a peak field ten times that reached with the density gradient, and with a much wider instantaneous velocity spread. Again the beam bunch at overtaking occurs in a decelerating phase of the wave. A meniscus is probably forming, although we have considered too short an interaction region to see all of it.

To separate the effect of using disks instead of sheets, we ran a case identical to the disk result of Fig.6.4, except that the disks are replaced by sheets. $\omega_{pb}/\omega_0$, $g$, and the modulation are identical in the two figures. The results at the same time ($t = 260/\omega_0$) as in Fig.6.4 are shown in Fig.6.7. The maximum fields are almost five times larger than in Fig.6.4, and the beam velocity spread is
Fig. 6.6 Disk beam in uniform plasma. Wideband excitation. 
\[ t = \frac{300}{\omega_p} \cdot \frac{\omega_\text{pb}^2}{\omega_p^2} = 0.002 \cdot r_0 = 0.075 \text{ in.} \]
Note the much larger fields and beam velocity spread than in the nonuniform plasma case (Fig. 6.4), even though the interaction length here is only 1/6 as long. The fields are still increasing in time.
\[
\sqrt{\frac{\omega}{p}} = \frac{\omega_{po}}{p} \sin(0.15 + 0.002z)
\]

\[
v_b(z = 0) = 0.0045 \sum_{j=0}^{100} \left( \frac{\omega_{p(j)}/p_{po}}{p_{po}} \sin(\omega_{p(j)}t - \phi_j) \right), \phi_j's \ random
\]

Fig. 6.7 Snapshot of sheet beam velocity and acceleration vs. distance.
almost twice as large. By $t = 320/\omega_p$, the fields have (Fig. 6.8) collapsed somewhat, being now only about three times as large as in Fig.6.4. These fields are larger compared to the disk model than expected from the disk wake reduction factor of $\{1-2I_0(\alpha)K_1(\alpha)\}$ given in Eq.6.24. This factor reduces the wake of a disk by forty per cent of less that of a sheet (see Fig.6.3, the ordinate of which yields $I_1(\alpha)K_1(\alpha)$). For a given number of sheets per bunch after overtaking, Eq.5.33 predicts that the fields should be directly proportional to the wake magnitude. Apparently the purely debunching nonoscillatory fields tend to break up the disk bunches and reduce these fields.

Fig.6.9 shows the results of increasing the plasma density to the value measured in the laboratory, $6.9 \times 10^{12} \text{cm}^{-3}$. The beam current is reduced to that actually collected, 0.4 amps, vs. 0.5 amps as used in Fig.6.4. However, the beam diameter is reduced to its nominal value of 0.125 cm., rather than 1.5 times this value. The reduction in $r_0$ assumes the worst possible case for our purposes (explaining the narrow velocity distribution of the collected beam). Allowing $r_0$ to be larger would reduce the disk surface charge density $Q$, which varies as $I/\pi r_0^2$, and this would reduce the fields (see Eq.6.23) which spread the beam in velocity. The various changes almost cancel each other. The parameters are:
Fig. 6.8 Snapshot of sheet beam velocity and acceleration vs. distance.
Fig. 6.9  Snapshot of disk velocity and acceleration vs. distance. Disk beam, sinusoidal plasma density variation along direction of beam flow.

\[ t = \frac{292}{\omega_{p_0}} \]
\[ \omega_p = \omega_{p_0} \sin(0.15 + 0.00113z) \]
\[ v_b(z=0) = 0.0045 \sum_{j=0}^{n_{p_0}} \{\omega_{p_0}(z+j)/\omega_{p_0}\sin(\omega_{p_0}(z+j)t-\phi_j), \phi_j \text{'s random} \]
\[ n_{b_0} = n_{p_0}(9.1/6900) \]
Fig. 6.4

\[ n_{po} = 2.5 \times 10^{12} \text{ cm}^{-3} \]
\[ n_{bo} = 5 \times 10^9 \text{ cm}^{-3} \]
\[ r_0 = 0.1875 \text{ cm.} \]
\[ \frac{\omega_{pb}}{\omega_{po}} = 0.002 \]

Fig. 6.9

\[ n_{po} = 6.9 \times 10^{12} \text{ cm}^{-3} \]
\[ n_{bo} = 9.1 \times 10^9 \text{ cm}^{-3} \]
\[ r_0 = 0.125 \text{ cm.} \]
\[ \frac{\omega_{pb}}{\omega_{po}} = 0.00132 \]

The beam density in Fig. 6.9 is increased, but the ratio of beam to plasma density is decreased.

Distance is again normalized to \(0.4v_o/\omega_{po}\), but since \(\omega_{po}\) is now increased, we must increase the normalized interaction length to obtain the same real interaction length. \(Z=300\) normalized in Fig. 6.4 is equivalent to \(Z=498\) in Fig. 6.9, so the actual interactions lengths of the two figures are almost equal. \(\omega_p^2\) is still given by \(\omega_p^2 = \omega_{po}^2 \sin\{0.15 + \pi z/L\}\), where \(L = 40 \text{ cm.}\), as before, but the normalized gradient is reduced from 0.002 to 0.00113. In order to modulate over about the same per cent bandwidth, we increase the summation in Eq. 6.26 to extend to \(\omega_p(Z=200)\). The peak applied modulation amplitude of the beam velocity is now about 0.03\(v_o\), in contrast to the 0.02\(v_o\) of Fig. 6.4.

There is only a slight difference in changing the parameters. The fact that the fields are comparable in the two figures is not too surprising. From Eq. 5.33, after including the wake field reduction factor in Eq. 6.24 which results from using disks instead of sheets, we expect the acceleration caused by the superposition of wake fields due to
beam bunches beyond overtaking to vary as $Q_{\text{tot}}\{1-2I_1[\alpha(z)]\} \times K_1[\alpha(z)]$, where $Q_{\text{tot}}$ is the beam charge in a bunch and $\alpha(z) = \omega_p(z)r_0/v_d$. Since these bunches can only form from beam charge within a distance on the order of a wavelength, we expect the acceleration to vary as $(n_{bo}/\omega_{po})\{1-2I_1(\alpha)K_1(\alpha)\}$. This factor is only 1.15 times as large in Fig.6.9 as in Fig.6.4, in spite of the fact that the beam density has almost doubled, and the wake reduction factor $\{1-2I_1[\alpha(z)]K_1[\alpha(z)]\}$ has increased slightly from 0.574 in Fig.6.4 to 0.605 in Fig.6.9. The peak fields are larger in Fig.6.9, but the mean square average field is somewhat less, being about 0.81 (unnormnalized) times that in Fig.6.4. The fields have probably not reached a steady state beyond $Z=300$ in Fig.6.9, and can yet grow.

We have assumed optimum beam focusing. If we had assumed a beam radius of 0.1875 cm. instead of the nominal 0.125 cm., the wake fields (see Eq.6.24) would be reduced by 47 per cent where $\omega_p(z) = 2\omega_{po}/3$, after allowing for the increase in $\{1-2I_1(\alpha)K_1(\alpha)\}$ to 0.722, and the reduction in beam density by a factor of $1/(1.5)^2$. The fields and beam velocity spread would be reduced, although the even greater reduction in the nonoscillatory fields would allow somewhat tighter bunches.

We do not know quantitatively the effect that modulating over a wide band has in reducing the fields at the second harmonic (where $\omega_p = 2\omega$) below those obtained when modulating at a single frequency. Presumably the harmonics of the different modulation frequencies can interfere with each other
when the beam goes nonlinear, and reduce the component of beam current at any single frequency, and hence the response of the plasma to this frequency component. If only a single frequency is applied, the magnitude of the beam current at $2\omega$ depends nonlinearly on the magnitude of the modulation at $\omega$, making such a comparison difficult.

6.6 Summary of Results in this Chapter

We have found that replacing the infinite beam sheets of Chapter 5 by finite diameter disks reduces the electric fields by about a factor of three and the beam velocity spread by about a factor of two. The disks allow us to at least approach agreement with the narrow beam velocity distribution observed in the laboratory. However, the interaction length we can model on the computer is only about twenty per cent that of the laboratory experiment.
Chapter 7: Conclusions

The principal goal of this thesis was to find those factors in the beam plasma interaction that limit the interaction intensity and result in the relatively narrow velocity distribution of the collected beam electrons that we observe experimentally. We find that the introduction of an increasing density gradient along the direction of beam flow is the most important factor in this reduction.

In Chapter 3 we studied the one-dimensional beam-plasma interaction in a lossless plasma. The interaction is so intense that it is only limited by the acceleration of some initially cold plasma sheets to velocities on the order of wave phase velocity. These sheets are swept away by the wave, and their acceleration acts as a loss mechanism for the wave. The introduction of plasma temperature reduces the field amplitudes needed to accelerate plasma sheets to high velocities, but does not reduce the fields nearly enough to allow other than a very widely spread beam velocity. These uniform, collisionless models show that neither beam nor plasma nonlinearities by themselves reduce the interaction level sufficiently to allow the narrow beam velocity we observe in the laboratory. However, these models do seem to duplicate several aspects of the meniscus often seen by others in low power beam-plasma experiments, where the beam is widely scattered in velocity, in a localized region of intense electric fields oscillating at \( \omega_p \). We feel that one of the essential differences between
these experiments and our laboratory experiment is that the meniscus is only seen at beam voltages not more than a few times the ionization potential of the gas. Hence the plasma electrons do not oscillate with peak kinetic energies exceeding the ionization potential. We find from the computer results that electrons oscillating to much over forty per cent of the beam velocity generally are accelerated on up to the wave phase velocity, and are carried away from the meniscus. If the beam voltage were higher, allowing a higher plasma kinetic energy, the resulting localized ionization would produce density gradients and wash out the meniscus.

In Chapter 4 we found that even introducing a collisional loss of $v = 0.2\omega_p$, a value of $v$ over three orders of magnitude greater than that actually existing, does not reduce the velocity spread. Indeed, the electric fields are greatly reduced. However, we find that frequencies very near to $\omega_p$ are still amplified the most, and beam bunches at a bunching frequency $\omega_p$ form beyond the overtaking plane. The wakes from these bunches add coherently, and are phased such as to maximally slow the bunches, spreading the beam velocity distribution.

In Chapter 5 we introduced plasma density gradients along the direction of beam flow, in a lossless cold plasma. With an increasing gradient there is no meniscus, and the electric fields are reduced by about a factor of four. However, the beam is still spread out in velocity, although only after
drifting over ten times the distance required to scatter the beam velocity in a uniform lossless plasma. For a plasma density that increases away from the gun, and for a beam modulated at a frequency $\omega$, amplification is only possible after the beam passes the point where $\omega = \omega_p$, and the spatial growth rate is thereafter finite. Hence in the region where the beam density is large, $\omega \neq \omega_p$, and the plasma response is greatly reduced from the uniform plasma case. There is, however, a possibility of a resonance at the region where $\omega_p(z) = 2\omega$, but it is not clear what the conditions are to strongly excite this resonance. Increasing the density gradient reduces the fields and velocity spread, as does reducing the beam density.

In Chapter 6 we introduced a finite diameter beam by means of disks. The disks reduce the fields from the results using beam sheets by about a factor of three and the beam velocity spread by a factor of two, such that the beam velocity spread is comparable to that seen in our laboratory experiment. We found that finite diameter beam in a uniform plasma produces a much more intense interaction than the same beam in the nonuniform plasma, with electric fields ten times as large, and a much wider beam velocity spread.

At least for $\omega_p > \omega_c$, we conclude that plasma density gradients along the direction of beam flow are much more important in limiting the beam plasma interaction than beam-plasma nonlinearities, temperature, collisions or finite
beam diameter.

7.1 Suggestions for Future Work

Our disk model assumes no radial beam motion and no radial variation in the z-directed beam motion. It would be interesting to determine the effects of removing these restrictions in a computer experiment by using, for example, a series of concentric rings to represent the beam.

A computer experiment to study the transient build up of the plasma density during the time interval when \( \omega_p = \omega_c \) could shed light on whether most of the hot electrons found in the beam-plasma interaction are produced at this time. This interaction produces strong transverse fields, so the sheets would have to be allowed to move transversely.
Appendix A: Axial Density in a Magnetic Mirror

On the midplane we assume that the plasma velocity distribution, \( f(v) \), is isotropic except for cut out loss cones. That is,

\[
f(v) = \begin{cases} 
  f_0(v), & \theta \geq \theta_c \\
  0, & \theta < \theta_c,
\end{cases}
\]

where \( \theta_c = \sin^{-1}\left(1/\sqrt{R}\right) \), the loss cone angle, and \( R \) is the mirror ratio. We ignore collisions and space charge fields.

We will use the conservation of magnetic moment, conservation of energy, and conservation of flux encircled by a cyclotron orbit to calculate the axial density variation. The angle \( \phi = \tan^{-1}(v_T/v_z) \) at any value of \( B \) can be found from the angle \( \theta = \phi \) at the midplane. From conservation of magnetic moment we find that

\[
\cos \phi = \sqrt{1-(B/B_{\text{min}})^2} \sin^2 \theta. \tag{A.1}
\]

As a particle moves into a region of higher \( B \), its \( v_z \) decreases. This fact gives an effective increase in density of \( \cos \theta/\cos \phi \).

There is a further increase in density due to the shrinking of the cyclotron orbit, since \( r_C^2B = \text{constant} \), where \( r_C \) is the cyclotron radius. This fact increases the density as \( B/B_{\text{min}} \).

Particles at the midplane with large values of \( \theta \) will not be able to travel far from the midplane. The density is, taking into account all of these effects,
\[ n = 4\pi \int_0^\infty dv \, v^2 f(v) \int_0^{\theta_e} d\theta \, \sin \theta (\cos \theta / \cos \phi) \]

\[ = 4\pi \int_0^\infty dv \, v^2 f(v) \int_{\sin^{-1}(1/\sqrt{\psi})}^{\sin^{-1}(1/\sqrt{R})} d\theta \, \frac{\sin \theta \cos \theta}{(1-\psi \sin^2 \theta)^{\frac{1}{2}}} \]

where \( \psi = B(z)/B_{\text{min}} \), and \( \theta_e = \sin^{-1}(1/\sqrt{\psi}) \). Normalizing to the density \( n_{\text{poo}} \) at the midplane, we find

\[ n = n_{\text{poo}} \left( \frac{1-\psi/R}{1-1/R} \right)^{\frac{1}{2}}. \] (A.2)

This result has been found in a different manner by Post.\(^{49}\)

Eq. A.2 is plotted in Fig. 2.8, for a parabolic mirror with mirror ratio \( R = 3 \).
Appendix B: Program for a Warm Sheet Plasma

This program is divided into three parts:

(1) the main program, ID9, which sets the initial conditions at t = 0, calculates the trajectories ignoring crossings, injects, modulates, and collects the beam sheets, and checks energy conservation;

(2) ΨD9 orders the sheets and corrects for crossings;

(3) RFLT9 reflects the plasma sheets at the ends.

See Chapter 3 for further details on this program.

To run the program on the MAC computer type

LOADGO ID9 ΨD9 RFLT9

Then type

TMAX = --, TAPEIN = --, TAPEΩT = --*

where TMAX is the maximum time of the run, in units of 1/ωp, TAPEIN is the input tape, TAPEΩT the output tape. If starting from t = 0, type ING = 0 instead of TAPEIN.

PRINT ID9 MAD 1 THRU 20
214 5.6

:R DISTANCES ARE NORMALIZED TO INTERSHEET SPACING
:R (0.01Vo/ωp), TIMES TO 1/ωp =
:R 1./SQR.((WPE.P.2+WBP.P.2).BEAM IS .01 AS DENSE AS PLASMA.
:R HOT PLASMA. 7.5 SHEETS/DEBYE LENGTH. VT = .075Vo
:R UNIFORM STATIONARY IONS. CONTINUOUSLY INJECTED BEAM.
PROGRAM COMMON
:1X1(2500),X2(2500),V1(2500),CHG(2500),A(2500)
:1,LMAX,LMIN,WDT,VO,XMAX,ΩB,ΩP,TER,SN,CN,ΩB2,GG,X8
INTEGER L,LMAX,LL,S,LMAX1,TAPEIN,TAPEΩT,GO,TET4
:1,LMIN,TEST1,TEST2,TEST5,TEST6,MMM,MNM,M,GG
:R IF TEST1 = 1, WE WRITE AN OUTPUT TAPE
TEST1 = 1
:R IF TEST4 = 1, WE PRINT OUTPUT DATA
TEST4 = 1
:R IF TEST5 = 1, WE CALCULATE ELECTRIC FIELD ENERGY CHANGE
:R DUE TO INJECTED SHEET, AT TIME OF INJECTION
TEST5 = 1
:R IF TEST6 = 1, WE PRINT ALL PLASMA SHEETS, O'E 1/20
TEST6 = 0
:R IF TEST2 = 1, WE PRINT ALL PLASMA SHEETS,
PRINT ID9 MAD 21 THRU 70
20 13.0
&R OTHERWISE 1 OUT OF 10.
&R MMM = NO. OF PRINTOUTS/TAPE
MMM = 1
TEST2 = 0
&R STEP TIME = .05/VP
WDT = .05
SN = SIN((WDT)
CN = COS((WDT)
VO = 100.
TMAX = 10.
&R QP IS PLASMA SHEET SURFACE CHARGE DENSITY, QB BEAM.
QP = .99
QB = .2
QB2 = 2.*QB
&R THERE ARE 2000 PLASMA SHEETS, 1 PER DISTANCE OF 1
&R (NORMALIZED).
XMAX = 2000.
LMAX = 2 18 4
LMIN = 8 4
M = 0
&R IF GO = 1, READ SHEET POSITIONS AND VELOCITIES FROM
&R TAPE, OTHERWISE GENERATE THESE.
GO = 1
PRINT COMMENT $ PRINT TMAX,TAPEIN,TAPEOT, (GO)$
R'A
W'R GO.E.1
READ BINARY TAPE TAPEIN,LMIN,LMAX,M,EEL,EVL,T,EVBL
READ BINARY TAPE TAPEIN,X1(LMIN)...X1(LMAX),V1(LMIN)...,
1 V1(LMAX),CHG(LMIN)...CHG(LMAX)
O'E
X1(LMIN) = 0.
V1(LMIN) = VO
CHG(LMIN) = QB
XX = .5
&R
&R SET INITIAL PLASMA MAXWELLIAN VELOCITY DISTRIBUTION
&R
B = 0
B = SETU.(6)
VT = .05*VO
SQ12 = SQRT.(12.)/SQRT.(100.)
T'H A7, FOR L = LMIN+1,1,L.G,LMAX
W'R ((L/21)*21.E.L
X1(L) = X1(L-21)+20.
V1(L) = VO
CHG(L) = QB
O'E
X1(L) = XX
XX = XX+1.
CHG(L) = QP
468+.550
PRINT ID9 MAD 71 THRU 115
' 2201.3
 Y = 0.
 T'H A9 4, FOR K = 1, 1, K.G.100
 Y = Y+RANNO.(K)
 V1(L) = VT*SQ12*(Y-50.)
 E' L
 :R
 :R CALCULATE INITIAL ENERGY
 :R ENER.
 EEL = EE
 T' H A6, FOR L = LMIN, 1, L.G.LMAX
 EVL = EVL+CHG(L)*V1(L).P.2
 W'R CHG(L).L.QB2, EVBL = EVBL+QB*V1(L).P.2
 E' L
 INTERNAL FUNCTION
 :R MEASURE ELECTRIC FIELD ENERGY
 E' D ENER.
 EE = 0.
 ACE = 0.
 W'R X1(LMIN).GE.0.
 X1(LMIN-1) = 0.
 O'E
 X1(LMIN-1) = X1(LMIN)
 E' L
 CHG(LMIN-1) = 0.
 T' H A2, FOR L = LMIN, 1, L.G.LMAX
 ACE = ACE+CHG(L-1)
 FIEL1 = ACE* X1(L-1)
 FIEL2 = ACE-(X1(L-1)+X1(L))/2.
 FIEL3 = ACE* X1(L)
 EE=EE+(X1(L)-X1(L-1))/6.*(FIEL1.P.2+4.*FIEL2.P.2+FIEL3.P.2)
 W'R L.E.LMAX
 FIEL1 = FIEL3+CHG(L)
 XX = XMAX* X1(L)
 FIEL2 = FIEL1-.5*XX
 FIEL3 = FIEL1-XX
 E' L
 F' N
 E' N
 :R
 :R CALCULATE TRAJECTORIES. USE SINUSOIDAL MOTION OF SHEETS
 :R IN UNIFORM ION BACKGROUND
 :R
 1.133+1.516
PRINT ID9 MAD 116 THRU 160

TH A20, FOR MNM = 1,1,MNM,G,MNM
WR MMM,NE.1,TMAX = TMAX+1.5
TH A3, FOR T = T,WDT,T,G,TMAX
M = M+1
WR X1(LMIN).L.0.
PT D43,X1(LMIN)
VS D43 = $H* X1(LMIN).L.0.,X1(LMIN) =*,E15.6*$
EL
CHG(LMIN-1) = 0.
X1(LMIN-1) = 0.
ACE = 0.
TH A4, FOR L = LMIN,1,L.G,LMAX
ACE = .5*(CHG(L-1)+CHG(L))*ACE
ACI = -X1(L)
A(L) = (ACE+ACI)
X2(L) = X1(L)
X1(L) = -A(L)*CN+V1(L)*CN+X1(L)+A(L)
V1(L) = A(L)*CN + V1(L)*CN
A(L) = A(L) - (X1(L)-X2(L))

GG = 0

:R ORDER SHEETS
ORD.
GG = 1
:R REFLECT SHEETS AT ENDS
TH A5, FOR L = LMIN,1,X1(L).G.0.
REFLEK.(L)
TH A10, FOR L = LMAX,-1,X1(L).L,XMAX
REFLEK.(L)
:R REORDER SHEETS AFTER REFLECTIONS
ORD.
:R
:R COLLECT BEAM
:R
PRINT COMMENT $ CBV$ 
LMAX1 = LMAX
TH A11, FOR L = LMAX,-1,L,L,LMAX-50,OR,X1(L).L,XMAX
WR CHG(L).L.QB2
:R FIND BEAM VELOCITY AT XMAX, FOR ENERGY CHECK
AA = A(L) + (X1(L)-XMAX)
V1C(L)=SQRT.(V1(L).P.2-2.*(AA*(X1(L)-XMAX)-(X1(L)-XMAX)).P.2/2.
:1 ))
PT D3,V1(L)/VO
1.716+.58 3
PRINT ID9 MAD 161 THRU 205
W 2119.6
ECOL = ECOL + V1(L).*P.*2*CHG(L)
LMAX1 = LMAX1-1
O'E
P'T D55,X1(L),CHG(L),V1(L)
V'S D55=$H* X1.G.XMAX.AND.CHG.NE.QB,X1,CHG,V1=*,3E15.8*$
E'L
V'S D3 = $S1,F15.8*$
A11
LMAX = LMAX1
:R
:R INJECT BEAM SHEET EVERY A DELTA T'S
:R
W'R (M/L)*4,E.M
W'R TEST5.E.1
:R MEASURE ELECTRIC FIELD ENERGY BEFORE INJECTION
ENER.
EE1 = EE
E'L
LMIN = LMIN-1
L = LMIN
X1(L) = 0.
X2(L) = X1(L)-VO*WDT
:R .C2 PER CENT VELOCITY MODULATE THE BEAM
V1(L) = VO*(1.-. C*SIN(T))
INN = INN+QB*V1(L).*P.*2
CHG(L) = QB
W'R TEST5.E.1
ENER.
:R MEASURE INCREASE IN ELECTRIC FIELD ENERGY DUE TO
:R SHEET INJECTION
INN = INN+(EE-EE1)
E'L
E'L
W'R ER.G..5,T'O GETOUT
:R
:R OUTPUT
:R
GETOUT CHG(LMIN-1) = 0.
W'R X1(LMIN).GE.0.
X1(LMIN-1) = 0.
O'E
X1(LMIN-1) = X1(LMIN)
E'L
ACE = 0.
T'H A15, FOR L = LMIN,1.L.G.LMAX
FOR 1.500+.600
PRINT ID9 MAD 206 THRU 252
2122.0
ACE = ACE + .5*(CHG(L-1)+CHG(L))
ACI = -X1(L)
A(L) = ACE+ACI
P'S T
W'R TEST T.E.1
W'R TEST2.E.0
PRINT COMMENT $  XB   VB   ABS
T'H A8, FOR L = LMIN,1,L.G.LMAX
W'R CHG(L).L, QB2.AND.TEST6.E.0 P'T D1,X1(L),V1(L)/VO,A(L)
V'S D1 = $S1,F13.6,2F15.8*$

PRINT COMMENT $  XP   VP   AS
T'H A9, FOR L = LMIN,1,L.G.LMAX
W'R CHG(L).G, QB2
W'R.ABS.V1(L)/VO,G..4.OR.TEST6.E.0 P'T D1,X1(L),V1(L)/VO,A(L)
W'R TEST6.E.0, L = L+20
E'L
Q'E
PRINT COMMENT $ CHG(L) X1 X2 V1/VO AS
T'H A19, FOR L = LMIN,1,L.G.LMAX
P'T D5, CHG(L), X1(L), X2(L), V1(L)/VO, A(L)
V'S D5 = $S1,5F16.8*$
E'L
E'L
:R
:R ENERGY CHECK
:R
EV = 0.
EVB = 0.
ENER.
T'H A18, FOR L = LMIN,1,L.G.LMAX
EV = EV+CHG(L)*V1(L).P.2
W'R CHG(L).L, QB2, EVB = EVB+QB*V1(L).P.2
EDIF=(EE+EV-EELEV-|NN+E|COL)/(EE+EV)
ENET = EDIF*(EE+EV)/(EVBL+|NN+E|BV-E|COL)
ENN = EVBL+|NN-EVBL-E|COL
ETOT = EE+EV
ETOTL = EELEV+|NN-E|COL
P'S ENET,ENN, EVB, EVBL, EDIF, ETOT, ETOTL, EE, EELEV, EV, EVL, |NN, E|COL
W'R TEST1.E.1
WRITE BINARY TAPE TAPEAT, LMIN, LMAX, M, EE, EV, T, EVB
V'S DTAP = $318, LE14.8*$
WRITE BINARY TAPE TAPEAT, X1(LMIN)...X1(LMAX), V1(LMIN)...:1
V1(LMAX), CHG(LMIN)...CHG(LMAX)
V'S DTAPI = $5E15.8*$
E'L
E'M
1.833+.500
EDL 0D9 MAD
W 2129.7
EXIT
P 3
S current line.

:R  JON DAVIS
EXTERNAL FUNCTION
END 0 ORD.

:R  ORDERS BEAM AND PLASMA SHEETS OF ID
PROGRAM COMMON
:1  X1(2500), X2(2500), V1(2500), CHG(2500), A(2500)
:1  LMAX, LMIN, WDT, VO, XMAX, QB, OP, T, ER, SN, CN, QB2, GG, X8
INTEGER L, LMAX, LL, S, LMAX1, TAPE1N, TAPE0T, GO
:1  INCR, LINC, LMIN, N, K, RR, GG, PR, PRR
T'H A5, FOR LL = LMIN, L, LL.G.LMAX-1
L = LL
S = L+1
W'R X1(L).G.X1(S)
RR = 0
R = 0
PR = 0
PRR = 0

:R  ESTIMATE CROSSING TIMES, FIRST APPROXIMATELY, THEN BY
:R  USING EXACT TRAJECTORIES
T2 = ((X2(S)-X2(L))*WDT)/(X1(L)-X2(L)-X1(S)+X2(S))
CS3 = COS.(T2-WDT)
SN3 = SIN.(T2-WDT)
X3 = -A(L)*CS3 + V1(L)*SN3 + X1(L) + A(L)
X4 = -A(S)*CS3 + V1(S)*SN3 + X1(S) + A(S)
T1 = (X2(S)-X2(L))*T2/(X3-X2(L)-X4+X2(S))

:R  ALLOW FOR CROSSINGS BEYOND ENDS. REFLECT PLASMA
:R  SHEETS ONLY.
W'R X4.G.XMAX OR. X3.L.0.
:R  CORRECT CROSSINGS DUE TO PREMATURE REFLECTIONS
W'R GG.E.1,T'0 A9
W'R X2(L).G.X2(S)
    PRINT COMMENT $ X4.G.XMAX OR. L.C,X2(L).G.X2(S)$
    T'0 A1C

E'L
W'R X4.G.XMAX
R = R+1
PRINT COMMENT $ X4.G.XMAX$

:R  REFLECT ONLY PLASMA SHEETS
W'R CHG(S).G.QB2
    REFLEK.(S)
O'E
    PRINT COMMENT $ CHG(S).L.QB2$
W'R CHG(L).G.QB2
PRINT COMMENT $ BUT CHG(L).G.QB2$ 
:R DO NOT NOW HAVE CROSSING 
REFLEK.(L) 
T'O A6 
O'E 
PRINT COMMENT $ AND CHG(L).L.QB2$ 
T1 = WDT 
T'O A8 
E'L 
E'L 
E'L 
W'R X3.L.G. 
RR = RR+1 
PRINT COMMENT $ X3.L.O.$ 
REFLEK.(L) 
E'L 
W'R R.G.1.OR.RR.G.1 
PRINT COMMENT $ LOOP IN ORDER$ 
ER = ER+1 
T'O A5 
E'L 
T'O A7 
E'L 
:R HERE WE CORRECT CROSSINGS DUE TO PREMATURE REFLECTIONS 
P'T D95,T1 
V'S D95 = $H* PLASMA UNREFLECTION,T1 =*F16.6*$ 
W'R X1(S).L.XMAX/2. 
X8 = 0. 
PR = PR+1 
REFLEK.(S) 
E'L 
W'R X1(L).G.XMAX/2. 
X8 = XMAX 
PRR = PRR+1 
REFLEK.(L) 
E'L 
W'R PR.G.1.OR.PRR.G.1 
PRINT COMMENT $ PR.G.1$ 
ER = ER+1 
T'O A5 
E'L 
T'O A7 
E'L 
W'R T1.G.1.2*WDT.OR.T1.L.-2*WDT 
:R ERROR CHECK 
ER = ER+1. 
PRINT COMMENT $ T1.G.WDT.OR.T1.L.$ 
P'S T1,T2,X3,X4,X1(L),X1(S),X2(L),X2(S),CHG(L),CHG(S) 
E'L
ORDER SHEETS AND CORRECT TRAJECTORIES. NOTE THAT WE WILL USE THE CORRECTED TRAJECTORIES TO CALCULATE FUTURE CROSSING TIMES

\[ \text{CN1} = \cos((WDT-T1)) \]
\[ \text{SN1} = \sin((WDT-T1)) \]
\[ \text{ALPHA} = X2(L) \]
\[ X2(L) = X2(S) \]
\[ X2(S) = \text{ALPHA} \]
\[ \text{ALPHA} = \text{A}(L) \]
\[ \text{A}(L) = \text{A}(S) - \text{CHG}(L) \]
\[ \text{A}(S) = \text{ALPHA} + \text{CHG}(S) \]
\[ \text{ALPHA} = X1(L) \]
\[ X1(L) = X1(S) - \text{CHG}(L) \times (1 - \text{CN1}) \]
\[ X1(S) = \text{ALPHA} \times \text{CHG}(S) \times (1 - \text{CN1}) \]
\[ \text{ALPHA} = V1(L) \]
\[ V1(L) = V1(S) - \text{CHG}(L) \times \text{SN1} \]
\[ V1(S) = \text{ALPHA} \times \text{CHG}(S) \times \text{SN1} \]
\[ \text{ALPHA} = \text{CHG}(L) \]
\[ \text{CHG}(L) = \text{CHG}(S) \]
\[ \text{CHG}(S) = \text{ALPHA} \]
\[ \text{W'R PR.E.1,REFLEK.}(L) \]
\[ \text{W'R PRR.E.1,REFLEK.}(S) \]
\[ L = L - 1 \]
\[ \text{W'RI} \text{ LL.LMIN,L} = \text{LMIN} \]
\[ \text{T} \text{O BETA} \]
\[ \text{E'L} \]

SHIFT MEMORY. SHEETS ARE ADDED TO THE TOP OF THE TABLE, COLLECTED FROM THE BOTTOM

\[ \text{W'RI} \text{ LMIN,L.5} \]
\[ \text{P'T D27} \]
\[ \text{V'S D27 =} / / / H* SHIFTED MEMORY* / / / *$ \]
\[ \text{INCR = 2490-LMAX} \]
\[ \text{W'RI} \text{ INCR.LE.0} \]
\[ \text{P'T DD10} \]
\[ \text{V'S DD10 =} H* \text{ INCR.LE.0**$} \]
\[ \text{ER = 20.} \]
\[ \text{E'L} \]
\[ \text{T'HA18, FOR L} = \text{LMAX,-1,L,L,LMIN} \]
\[ \text{LINC} = \text{L+INCR} \]
\[ \text{A(LINC)} = \text{A}(L) \]
\[ \text{V1(LINC)} = \text{V1}(L) \]
\[ \text{X1(LINC)} = \text{X1}(L) \]
\[ \text{X2(LINC)} = \text{X2}(L) \]
\[ \text{CHG(LINC)} = \text{CHG}(L) \]
\[ \text{LMIN} = \text{LMIN+INCR} \]
\[ \text{LMAX} = \text{LMAX+INCR} \]
\[ \text{E'L} \]
\[ \text{F'N} \]
\[ \text{E'N} \]
RFLT9 MAD

138.8

t

Current line.
:R JON DAVIS
:R SUBROUTINE TO REFLECT PLASMA SHEETS AT ENDS.
EXTERNAL FUNCTION (K)
E'O REFLEK.
J = K

PROGRAM COMMON
:1 X1(2500), X2(2500), V1(2500), CHG(2500), A(2500)
:1 LMAX, LMIN, WDT, V0, XMAX, Q8, QP, T, ER, SN, CN, QB2, GG, X8
INTEGER L, LMAX, LL, S, LMAX1, TAPEIN, TAPEOT, GO, TEST4, J, K
:1 LMIN, TEST1, TEST2, TEST5, M, GG
:R DO NOT REFLECT BEAM SHEETS
W'R CHG(J).G.QB2
PRINT COMMENT $ RFL$
W'R X1(J).L.0.,X8 = 0.
W'R X1(J).G.XMAX, X8 = XMAX
:R
:R ESTIMATE TIME OF REFLECTION, BY 2 STEP ITERATION.
:R
T2 = (X8 - X2(J)) WDT/(X1(J) - X2(J))
X3 = -A(J) COS(T2 - WDT) + V1(J) S1N(T2 - WDT) + X1(J) + A(J)
T1 = (X8 - X2(J)) T2/(X3 - X2(J))
W'R T1.G.1. 2*WDT.OR.T1.L. = 2*WDT
PRINT COMMENT $ T1.G.WDT IN REFLEK.$
P'S T1, X1(J), X2(J), V1(J), CHG(J), A(J), X8, X1(J+1), X2(J+1),
:1 X1(J-1), X2(J-1)
ER = ER+1
T'0 A1
E'L

:R
:R GIVEN T1, THE INSTANT OF REFLECTION, WE KNOW THE
:R X, V AND A AT THAT INSTANT AND CAN EXACTLY FIND THE
:R TRAJECTORY FROM THERE.
:R
AL = A(J)
VL = V1(J)
S1N2 = S1N(2* (T1- WDT))
COS2 = COS(2*(T1- WDT))
X1(J) = X1(J) + A(J) *(1. - COS2) + V1(J) * S1N2
V1(J) = -A(J) * S1N2 - V1(J) * COS2
A(J) = AL*COS2 - VL*S1N2
X2(J) = -A(J) * CN - V1(J) * SN + X1(J) + A(J)
E'L
E'N
Appendix C: The General Wake of a Sheet Moving at an Arbitrary Positive Velocity Through a Cold Plasma with an Arbitrary Density Variation

The beam sheet is represented by

$$\rho_b(z,t) = -Q \frac{\delta t_1(z)}{\delta z} \delta(t-t_1(z)),$$  \hspace{1cm} (C.1)

where \(Q\) is the sheet surface charge density, \(t_1\) is the time the sheet passes the plane \(z\), and \(\delta\) is the unit impulse. When Eq.C.1 is integrated over \(z\), the factor \(\delta t_1/\delta z\) allows the total charge per unit area to be \(Q\), as it must be.

We perform a Fourier transform in time, such that

$$\rho_b(z,\omega) = -Q \frac{\delta t_1(z)}{\delta z} e^{-j\omega t_1(z)}.$$  \hspace{1cm} (C.2)

The Fourier transform of the plasma force equation is, including a Langevin force term for collisions,

$$j\omega v_p = (qE/m) - vv_p.$$  \hspace{1cm} (C.3)

The plasma conservation and Poisson's equations are, respectively

$$\frac{\partial}{\partial z} \{\rho_p(z)v_p\} = -j\omega \rho_p,$$  \hspace{1cm} (C.4)

and

$$\varepsilon_0 \frac{\partial E}{\partial z} = \rho_p + \rho_b.$$  \hspace{1cm} (C.5)

Inserting Eqs.C.2, C.3, and C.4 into Eq.C.5 yields:
\[ \frac{\partial}{\partial z} \left[ \omega_p^2(z) + j\omega(j\omega + \nu) \right] E = + j\omega(j\omega + \nu) \frac{Q}{\varepsilon_0} \frac{1}{\omega} e^{-j\omega t_1}. \]  
(C.6)

We assume that the beam sheet will not excite any steady state fields that are independent of \( z \). Hence we can integrate Eq. C.6 over \( z \) and set the constant of integration equal to zero. This yields

\[ E(z, \omega) = \frac{(j\omega + \nu) Q}{(\omega - j\nu/2 - \omega_o)} \left( \frac{\varepsilon_0}{(\omega - j\nu/2 - \omega_o)} \right) e^{-j\omega t_1}, \]  
(C.7)

where \( \omega_o(z) = \omega_p(z) \left( 1 - \nu^2 / 4\omega_p^2(z) \right)^{1/2} \). Inverting the Fourier transform, we obtain

\[ E(z, t) = -\frac{Q}{\varepsilon_0} \left( 1 + \nu^2 / 4\omega_o^2 \right)^{1/2} \exp\left\{ -\left( \nu / 2 \right) t - t_1(z) \right\} \]  
\[ \times \cos\left\{ \omega_o \left[ t - t_1(z) \right] - \tan^{-1}\left( \nu / 2\omega_o \right) \right\} u_{-1}[t - t_1(z)]. \]  
(C.8)

If \( \nu = 0 \), Eq. C.8 simplifies to

\[ E(z, t) = -\frac{Q}{\varepsilon_0} \cos\{\omega_p(z)[t - t_1(z)]\} u_{-1}[t - t_1(z)]. \]  
(C.9)
Appendix D: The Drag on One Sheet in an Unmodulated Sheet Beam of Infinite Extent

As sheets pass through a plasma, each creates a wake as given by Eq.C.8,

\[
E = -\frac{Q}{\varepsilon_0} (1+\nu^2/4\omega_0^2)^{1/2} e^{-\nu[z-t_1(z)]/2} \cos(\omega_0[z-t_1(z)]) \\
-\tan^{-1}(\nu/2\omega_0) u_{-1}[z-t_1(z)],
\]

where \( \omega_0(z) = \omega_p(z)\{1-\nu^2/4\omega_p^2(z)\}^{1/2} \), \( \nu \) is the collision frequency, \( t_1(z) \) is the time the sheet passes the plane \( z \), and \( u_{-1} \) is the unit step function. There is a deceleration of each sheet due to its self field of \(-Q/2m\varepsilon_0\). However, each sheet also feels a force due to the sheets ahead of it. In this Appendix we find this force for an unmodulated sheet beam. It is needed to explain the d.c. value of the fields of the lossy plasma, Chapter 4, and to simulate an infinite beam ahead of the first injected sheets in Chapters 5 and 6.

Sheets of an unmodulated beam pass a plane \( z \) at times \( \Delta t \) apart. The steady state field presented to a sheet at \( z \) can be found from the phasor diagram of Fig.D.1. The phasor gives the phase \( \theta \) of a field at \( z \), such that the field is proportional to \( \cos(\omega_0 t+\theta) \). We let \( t=0 \) for convenience, and will add in the sign of \( -Q \) later. We let \( \hat{A} \) be the phasor describing the combined fields of the sheets ahead of the sheet, at the position of the considered sheet. \( \hat{B} \) describes the phasor of the field due to the considered sheet, and \( \hat{C} \) the resultant of
Fig. D.1 Phasor diagram at $t = 0$ to determine the total wake field of an unmodulated beam. $\vec{A}$ is the phasor of this field, $\vec{B}$ is the phasor of one sheet wake, and $\vec{C}$ is their resultant. To reach a steady state, the angle between $\vec{A}$ and $\vec{C}$ must be $\omega_o \Delta t$, and $|C| e^{-\nu \Delta t/2} = |A|$.
$\vec{A}$ and $\vec{B}$.  $\vec{C}$ must have an angle of $-\omega_0 \Delta t$ with respect to $\vec{A}$, so that a steady state is reached as the wake of each new sheet is added. The resultant will move clockwise discontinuously as the field of each sheet is added, with an average angular velocity of $\omega_0$.  $|C|e^{-\nu \Delta t/2}$ must equal $|A|$ for a steady state. Hence, from Fig. D.1,

$$
\theta_A - \theta_C = \omega_0 \Delta t, \quad \text{(D.1)}
$$

$$
|C| = Ae^{\nu/2\Delta t}, \quad \text{(D.2)}
$$

$$
\vec{B} = \frac{|Q|}{\varepsilon_0} (1 + \nu^2 / 4\omega_0^2)^{1/2} e^{-j \tan^{-1}(\nu / 2\omega_0)}. \quad \text{(D.3)}
$$

We know one side of a triangle, $|B|$, its opposite angle, $\omega_0 \Delta t$, and the ratio of the other two sides. This is sufficient to give $\vec{A}$. From the law of cosines

$$
|B|^2 = |A|^2 + |C|^2 - 2 |A| |C| \cos(\omega_0 \Delta t). \quad \text{(D.4)}
$$

Using Eq. D.2 and Eq. D.3, we find

$$
|A| = \frac{(|Q|/\varepsilon_0) (1 + \nu^2 / 4\omega_0^2)^{1/2}}{(1 + e^{\nu \Delta t/2} - 2e^{-\nu \Delta t/2} \cos(\omega_0 \Delta t))^ {1/2}}. \quad \text{(D.5)}
$$

Using the parallelogram in Fig. D.1 and the law of cosines again, we have

$$
|C|^2 = |A|^2 + |B|^2 - 2 |A| |B| \cos(\pi - \theta_A - \theta_B). \quad \text{(D.6)}
$$

From Eqs. D.2, D.3, D.5, and D.6 we find
\[ \theta_A = -\tan^{-1} \left( \frac{v}{2\omega_o} \right) - \cos^{-1} \left( \frac{e^{\nu\Delta t/2} \cos (\omega_o \Delta t) - 1}{(1 + e^{\nu\Delta t} - 2e^{\nu\Delta t/2} \cos \omega_o \Delta t)^{1/2}} \right). \]  

(D.7)

The acceleration of a sheet from all the sheets ahead of a considered sheet is \(-q/m \text{ sign}(Q) \text{ Re} |A| e^{j\theta_A} \), or

\[ \frac{qE}{m} = -\frac{qQ}{m\varepsilon_o} \frac{(1 + \nu^2/4\omega_o^2)^{1/2} \cos \theta_A}{(1 + e^{\nu\Delta t} - 2e^{\nu\Delta t/2} \cos \omega_o \Delta t)^{1/2}}. \]  

(D.8)

Eq.(D.8) does not include the sheet self field. \( \theta_A \) is in the first quadrant if \( \nu \) is large, in the second quadrant if \( \nu \) is small. That is, in a nearly lossless plasma the wakes from the sheets ahead will tend to cancel the drag on a sheet due to its self field, whereas if \( \nu \) is large, the drag on the considered sheet is increased. If \( \nu \) is large the wakes from sheets immediately ahead of the considered sheet dominate, and these tend to decelerate the considered sheet. Wakes from sheets which have passed by a time \( \pi/\omega_o \) before have decayed, and these are the ones which would have tended to accelerate the considered sheet.

The total acceleration of a sheet, including that due to its self field, is found by adding \(-qQ/2m\varepsilon_o\) to Eq.(D.8):

\[ \frac{qE}{m} = -\frac{qQ}{m\varepsilon_o} \left( \frac{1 + e^{\nu\Delta t} + (\nu/\omega_o) e^{\nu\Delta t/2} \sin \omega_o \Delta t}{1 + e^{\nu\Delta t} - 2e^{\nu\Delta t/2} \cos \omega_o \Delta t} \right). \]  

(D.9)

In the limit of a fluid beam (we let \( Q = \rho_b \nu_o \Delta t, \Delta t \to 0 \)), Eq.(D.9) is
\[ \frac{qE}{m} = -\frac{\omega_p}{\omega} v_0 \nu. \quad (D.10) \]

If \( \nu \) is zero, there is no net drag on the beam, but for
finite \( \nu \) the beam is slowed.

In the limit of \( \nu = 0 \), Eq.D.8 can be modified to move
with a sheet injected at \( t = 0, z = 0 \), with velocity \( v_0 \):

\[ \frac{qE}{m} = \frac{qQ}{2m\epsilon_0 \sin(\omega_p \Delta t/2)} \sin(\omega_p(t-z/v_0) + \omega_p \Delta t/2). \quad (D.11) \]

This is the field that the first sheet injected into the
plasma at \( z = 0, t = 0 \) would see to cancel its self field.

Eq.D.11 is used in Chapters 5 and 6 to prevent transients
that would otherwise be caused by the initial beam injections.
Appendix E: One-Dimensional Linearized Beam-Plasma Interaction with a Linear Plasma Density Gradient and Collisional Loss

Collisions are included in Eq. 5.2, such that

\[ j \omega v_p = qE - mv_v p. \]  \hspace{1cm} (E.1)

From Eqs. 5.1, E.1, 5.3-5.8, the resulting differential equation for the first order beam velocity is

\[ \omega_p^2(z) - \omega(z - j\nu) \frac{\partial^2 v_b}{\partial z^2} + \left( v_o \frac{\partial^2 v_b}{\partial z} + j2\omega \omega_p^2(z) - \omega(z - j\nu) \right) \frac{\partial v_b}{\partial z} = 0. \]  \hspace{1cm} (E.2)

The two solutions to this equation are, for \( z < 0 \),

\[ \begin{bmatrix} v_{bI} \\ v_{bII} \end{bmatrix} = \begin{bmatrix} v_I \\ v_{II} \end{bmatrix} e^{-j\omega z/v_o} \begin{bmatrix} J_0 \\ Y_0 \end{bmatrix} \begin{bmatrix} C_3(-z-j\nu \omega)^{1/2} \\ \gamma \end{bmatrix}, \]  \hspace{1cm} (E.3)

where \( v_I \) and \( v_{II} \) are the modulation amplitudes for the \( J_0 \) and \( Y_0 \) solutions, respectively, and \( C_3 = 2\omega_p \gamma (1-j\nu/\omega)^{1/2}/v_o \gamma^{1/2} \).

The electric fields are found from Eq. 5.3. They are

\[ \begin{bmatrix} E_I \\ E_{II} \end{bmatrix} = \begin{bmatrix} v_I \\ v_{II} \end{bmatrix} \begin{bmatrix} m \gamma \omega_p c_3 \gamma^{1/2} \\ \gamma \gamma \end{bmatrix} \begin{bmatrix} J_0 \\ Y_1 \end{bmatrix} \begin{bmatrix} C_3(-z-j\nu \omega)^{1/2} \\ \gamma \end{bmatrix}. \]  \hspace{1cm} (E.4)

For moderately large arguments, the magnitudes of \( J_y(x) \) and
\[ Y_\gamma(x) \text{ are } \sqrt{2/(\pi x)}, \text{ for any } \gamma \text{ and valid for complex } x. \text{ Hence if } v_I = v_{II}, \text{ the magnitudes of } v_{bI} \text{ and } v_{bII} \text{ will be almost equal for } z<<0, \text{ as will the magnitudes of } E_I \text{ and } E_{II}. \text{ We wish to compare the relative magnitudes when the argument of the Bessel functions is small.} \]

For small arguments \(^61\), \( J_0(x) = 1, Y_0(x) = (2/\pi)\ln(x/2), \)
\( J_1(x) = x/2, \) and \( Y_1(x) = -2/\pi x, \) valid for complex \( x. \) The magnitude of the arguments of the Bessel functions is a minimum at \( z=0. \) At \( z=0 \)
\[
\left| \frac{v_{bII}}{v_{bI}} \right| = \frac{2}{\pi} \ln \left( \frac{\omega \omega_{pb} (1-j\nu/\omega)^{1/2}}{\nu_{g}^{1/2} (-j\nu\omega)^{1/2}} \right), \tag{E.5}
\]

which for small values of \( \nu/\omega \) is approximately
\[
\frac{v_{bII}}{v_{bI}} = \frac{2}{\pi} \ln \left( \frac{\nu_{pb} \omega^2}{\nu_{g}^{1/2}} \left( \frac{\nu}{\omega} \right)^{1/2} \right). \tag{E.6}
\]

Similarly
\[
\left| \frac{E_{II}}{E_{I}} \right| = \frac{v^2 g^2}{\pi \omega^3 \omega_{pb} \nu}. \tag{E.7}
\]
Appendix F: Expected Values of Beam and Plasma Parameters for Values of $|\rho_b/\rho_{bo}|$, in a One-Dimensional Interaction in a Plasma with a Linear Density Gradient

Eq. 5.19 for the first order beam density is

$$\rho_b = \frac{v_{III}}{v_o} \rho_{bo} e^{-j\omega z/v_o} \left\{ -I_o(C_1 z^{\frac{1}{2}}) + \frac{j(gz)^{\frac{1}{2}}}{\omega_{pb}} \right\} x I_1(C_1 z^{\frac{1}{2}}),$$

(5.19)

where $C_1 = 2\omega_{pb}/v_o g^\frac{1}{2}$, and $\omega_{pb}^2 = \omega^2 + g z$. We assume the modulation is small enough so that the region where $\rho_b/\rho_{bo}$ approaches 1 allows the condition $(gz)^{\frac{1}{2}}/\omega_{pb} >> 1$ to be satisfied. In Fig. 5.3 $(gz)^{\frac{1}{2}}/\omega_{pb}$ is about 5 at $z = 125(0.4v_o/\omega_{po})$. We also assume that the argument of the $I_0$ and $I_1$ Bessel functions is large enough to justify the assumption $I_1 \approx I_o$. Since the argument is 7.5 at $z = 125(0.4v_o/\omega_{po})$, this approximation is correct within seven per cent. Eq. 5.19 then becomes

$$\frac{\rho_b}{\rho_{bo}} = \frac{v_{III}}{v_o} e^{-j\omega z/v_o} \frac{j(gz)^{\frac{1}{2}}}{\omega_{pb}} I_o(C_1 z^{\frac{1}{2}}).$$

(F.1)

But $v_b = v_{III} e^{-j\omega z/v_o} I_o(C_1 z^{\frac{1}{2}})$, hence

$$|v_b/v_o| = \left|\rho_b/\rho_{bo}\right| \frac{\omega_{pb}}{(gz)^{\frac{1}{2}}}. \quad \text{(F.2)}$$

Eq. 5.12 can be written

$$E_{III} = v_{bIII} \frac{m \omega_{pb}}{(gz)^{\frac{1}{2}}} e^{-j\omega z/v_o} \left\{ I_1(C_1 z^{\frac{1}{2}}) \right\}. \quad \text{(F.3)}$$
Inserting Eq. F.1 into Eq. F.3, with \( I_1 = I_o \), we obtain
\[
\frac{\varepsilon}{n_o m v_o^2} \omega_0^2 \left( \frac{\omega_p}{(gz)^2} \right)^2 |\rho_b/\rho_{bo}|^2. 
\]  

(F.4)

Combining Eqs. 5.2, F.1, and F.3, we have
\[
|v_p/v_o| = (\omega_p^2/gz)|\rho_b/\rho_{bo}|. 
\]  

(F.5)

In Eq. 5.20 near initial beam overtaking, we ignore \( \omega_p^2/(gz)^2 \approx 0.2 \) and \( \omega v_o/\omega_p^2 z = 0.04 \) to 1. With \( I_1 = I_o \), and Eq. F.1, we have
\[
|\rho_p/\rho_{po}(z)| = (\omega_p^2/gz)|\rho_b/\rho_{bo}|. 
\]  

(F.6)

Using Eq. F.1 to find the position \( z \) at which a particular (relatively large) value of \( |\rho_b/\rho_{bo}| \) is reached, we have
\[
(gz)^{1/2} I_o (C_1 z^{1/2}) = |\rho_b/\rho_{bo}| (v_o/v_{III}) \omega_p. 
\]  

(F.7)

For large arguments, \( I_o(z) = e^x/(2\pi x)^{1/2} \). Since \( C_1 = 2\omega_p v_o g^{1/2} \), we can rewrite Eq. F.7 as
\[
z^{1/2} \exp \left( 2\omega_p v_o g^{1/2} \right) = |\rho_b/\rho_{bo}| (v_o/v_{III}) \omega_p^{3/2} g^{1/4} 
\]
x \(
\frac{4\pi \omega}{v_o g^{1/2}} \). 

(F.8)
Appendix G: Fields Generated by a Charge Disk Moving at Constant Velocity Through a Uniform Cold Plasma

This appendix reproduces the results of Gould and Allen\(^{38}\).

We represent the plasma by its equivalent dielectric constant, after Fourier transforming in time and space,

\[
\varepsilon(\omega, \mathbf{k}) = \varepsilon_0 (1 - \omega_p^2 / \omega^2). \tag{G.1}
\]

We assume that the plasma is cold, homogeneous, isotropic, and lossless. The beam disks will be assumed to move at a constant velocity \(v_o\). Under the quasistatic assumption, Poisson's equation is

\[
k^2 \phi(\omega, \mathbf{k}) = \rho_b(\omega, \mathbf{k}) / \varepsilon(\omega, \mathbf{k}). \tag{G.2}
\]

The plasma space charge is included in the dielectric constant \(\varepsilon\), and \(\rho_b\) is the transform of the charge density represented by the beam disks. \(\rho_b(\omega, \mathbf{k})\) is:

\[
\rho_b(\omega, \mathbf{k}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j\omega t} e^{j\mathbf{k} \cdot \mathbf{r}} \rho_b(t, z, r) \, d\omega \, d\mathbf{k} \, d\mathbf{r}. \tag{G.3}
\]

where \(\rho_b(t, z, r)\) is

\[
\rho_b(t, z, r) = Qu_0(z - v_o t), \quad r \leq r_o, \tag{G.4}
\]

\[
= 0, \quad r > r_o.
\]

\(Q\) is the disk surface charge, \(u_o\) the unit impulse, \(v_o\) the diskspeed, and \(r_o\) the disk radius. We assume the disk passes the plane \(z=0\) at \(t=0\). Hence \(\rho_b(\omega, \mathbf{k})\) is
\[ \rho_b(\omega, \mathbf{K}) = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dz e^{-j\omega t} e^{jkwz} 2\pi \int_{0}^{\infty} d\theta e^{jk_T r (\cos \theta \cos \phi + \sin \theta \sin \phi)} a(t, z, r), \] 

(G.5)

where \( \theta \) is the space angle, \( \phi \) the transform angle, and we have scalar multiplied \( \mathbf{K} \) with \( \mathbf{r} \). We carry out the integrations, and find

\[ \rho_b(\omega, \mathbf{K}) = 2\pi q_d \left( \frac{2J_1(k_T r_0)}{k_T r_0} \right) u_0 (\omega-k_z v_0), \] 

(G.6)

where \( q_d = Q\pi r_0^2 \) is the total disk charge. We insert Eq.(G.6) into Eq.(G.2), and invert the transform to find \( \phi(t, z, r) \). The \( \omega \) integration is trivial:

\[ \phi(t, z, r) = \frac{q_d}{\varepsilon_0} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{2J_1(k_T r_0)}{k_T r_0} J_0(k_T r) e^{-jk_z(z-v_0 t)} k_T dk_T dk_z. \] 

(G.7)

We wish to find the acceleration of a similar disk at \( z \) from

\[ a(t, z) = -\frac{q}{m} \int_{0}^{\infty} dr \frac{\partial^2 \phi}{\partial z^2}. \] 

(G.8)

The acceleration is:

\[ a(t, z) = \frac{2}{m\varepsilon_0} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{2J_1(k_T r_0)}{k_T r_0} 2 jk_z e^{-jk_z(z-v_0 t)} \frac{\omega^2}{(k_z^2+k_T^2) \frac{\omega^2}{k_z v_0^2}} 
\]

\[ \times \frac{k_T dk_T dk_z}{(2\pi)^2}. \] 

(G.9)
We perform the $k_z$ integration first, and find poles at $k_z = \pm j k_T$ and $k_z = \pm \omega_p / v_o$. The contour of integration goes below the latter two poles, as we would find if we introduce loss into Eq.G.1. The resulting expressions for the acceleration of another disk at $z$ are, after a change of variables such that $x = k_T r_o$,

$$a(t,z) = \frac{q}{m} \frac{Q}{\varepsilon_0} \int_0^\infty \frac{J_1(x) x e^{-x (z-v_o t)/r_o}}{x^2 + \alpha^2} \, dx; \quad (z-v_o t > 0)$$

and

$$a(t,z) = - \frac{q}{m} \frac{Q}{\varepsilon_0} \{1-2I_1(\alpha)K_1(\alpha)\} \cos \omega_p (t-z/v_o)$$

$$- \frac{q}{m} \frac{Q}{\varepsilon_0} \int_0^\infty \frac{J_1^2(x) x e^{x (z-v_o t)/r_o}}{x^2 + \alpha^2} \, dx; \quad (z-v_o t < 0), \quad (G.10)$$

where $\alpha = \omega_p r_o / v_o$, and $q/m$ is the charge to mass ratio of an electron.
Appendix H: Program for a Disk Beam and Plasma Density Gradients

This program is divided into eight parts:

(1) the main program, GRADF1, sets the initial conditions at \( t = 0 \), reads the input tape and writes the output tape;

(2) RUNG1 injects the disks and calculates their trajectories;

(3) MAGSET combines the wake fields into one set of sinusoidal oscillations at cell points \( 0.4v_0/\omega_{p0} \) apart;

(4) ØRDER orders the disks, corrects for crossings, and collects the beam;

(5) RESULT prints the output and checks energy conservation;

(6) FØRC1 approximates the non-oscillatory fields of the disks;

(7) IK linearly interpolates the function \( I_1(y)K_1(y) \) from tables;

(8) ACSIN1 is basically the arcsin, with the proper quadrant for our purposes.

To run the program on the MAC computer, type

```
LOADGO GRADF1 RUNG1 MAGSET ØRDER RESULT FØRC1 IK ACSIN.
```

Then type

```
TMAX = --, ALPHA = --, BETA = --*
```

where \( TMAX \) is the maximum time in units of \( 1/\omega_{p0} \), \( ALPHA \) is the input tape, and \( BETA \) is the output tape. If starting from \( t = 0 \), type instead of \( ALPHA \), \( GØ = 0 \).
print gradf1 mad 1 thru 50

:R SEE CHAPTER 6 FOR COMPLETE DESCRIPTION.
:R JON DAVIS
:R DISK BEAM, INFINITE PLASMA INTERACTION. BEAM 0.002 AS
:R DENSE AS PLASMA, BEAM RADIUS IS 1.5*.05 INCHES.
:R SINUSOIDAL PLASMA DENSITY.
:R DISTANCE NORMALIZED TO 0.4VO/WPO, DELTA T (WDT) =
:R .04/WPO.

:R PROGRAM COMMON
:1 LMIN, ER, VO, T, X1(800), X2(800), X3(800), V, Z, N, LMAX, WDT, A,
:1 MAG1(500), MAG2(500), MAG3(500), PHI1(500), PHI2(500), PHI3(500),
:1 AB, SV, SV1, DAMP, PI2, AA, MAGGY, PHIT, THE, NUS,
:5 NU, SET, MOD, W1, ST, XS, WW(500), TEST, TEST3, U, T2, EE, EV
:6 , NN, NM, NNN, L, DONE, TMAX, NMAX, NUMB, VAV, ECOL, ORD, GRAD, IHN
:7, I1K1(70), W2I, E1(10), BR, NOM, COLL, FXUP, PHR(500), DELV
D'N ST(4), A(800), V(800),
:1 Z(505), SV(800), SV1(800), DONE(800)
:R WDT IS THE STEP TIME, NORMALIZED TO WPO
WDT = .4
:R TAPE 1161 CONTAINS I1K(Y)K1(Y) IN STEPS OF Y = 0.1
:R FROM 0 TO 6.
PI2 = 6.831853
:R U IS INCREASED BY 1 EACH STEP TIME. EVERY SET STEPS
:R WE PRINT RESULTS
READ BCD TAPE 1161, D8, I1K1(0) ... I1K1(59)
V'S D8 = $E15.8*$
U = 0
:R ER IS AN ERROR INDICATOR, USED TO
:R TRANSFER OUT OF A LOOP
ER = 0
:R LMAX IS THE LABEL OF THE DISK NEAREST THE COLLECTOR,
:R LMIN OF THE DISK NEAREST THE GUN. ORIGINALLY NO DISKS
:R ARE IN THE DRIFT REGION.
LMAX = 750
LMIN = 751
T2 = -WDT
:R T IS THE TIME, IN UNITS OF 1/WPO
T = 0.
:R IF GO.E.0, GENERATE INITIAL VALUES, OTHERWISE READ TAPE
GO = 1
:R N REFERS TO THE CELL POINTS WHERE PHASE AND MAGNITUDE
:R ARE KEPT. N GOES FROM 1 TO 300.
NMIN = 1
SET = 50000
:R IF MOD EQUALS 1, WE FULLY 2 PER CENT VELOCITY MODULATE
:R THE BEAM.
MOD = 1.
:R VO IS THE AVERAGE INJECTED BEAM VELOCITY, NORMALIZED
VO = 1./WDT
:R AB IS THE DISK SURFACE CHARGE AS A PERCENTAGE OF THE
2.700+1.533
print gradfl1 mad 51 thru 100

:R PEAK PLASMA CHARGE IN AN INTERDISK SPACING, ALIAS
:R THE BEAM TO PLASMA CHARGE DENSITY RATIO.
AB = 1./500.
:R TMAX IS THE MAXIMUM TIME OF THE RUN. IT IS NORMALLY
:R READ IN
TMAX = 30.
:R IF TEST.E.1, PRINT BEAM VELOCITY AND FIELDS
TEST = 1
:R IF TEST1.E.1, WE WRITE AN OUTPUT TAPE.
:R THESE ARE DISK PSUEDO TAPES.
TEST1 = 1
:R IF TEST3.E.1, PRINT LETTERS CBV (SHORT FOR COLLECTED
:R BEAM VELOCITY) JUST BEFORE PRINTING VELOCITY OF
:R COLLECTED DISKS.
TEST3 = 1
:R WE ALLOW FOR SECTIONS OF PLASMA TO BE ADDED AFTER
:R COLLECTOR. FOR THIS PURPOSE, EXPAND = 1, NMAX IS INCREASED
:R AND AN INITIAL FIELD IS IMPOSED TO TRY TO LESSEN
:R TRANSIENTS. PHC GIVES THE PHASE OF THIS FIELD.
NMAX = 300
PHC = 0.
:R BR IS THE NORMALIZED BEAM DISK RADIUS.
BR = 90.*.125*1.5/5.3*VO
:R NORM IS USED IN READING TAPE 1161 (SEE ROUTINE I1K).
NORM = 10.
:R THE E'S ARE COEFFICIENTS FOR THE APPROX. TO THE NON-
:R OSCILLATORY FIELDS OF THE DISKS (SEE ROUTINE FORC1).
E1(1) = 1.938/BR
E1(2) = 1.4886/BR.P.2
E1(3) = .9989/BR
E1(4) = .6387/BR.P.3
E1(5) = .2002/BR.P.4
T DD17
:R HERE WE TYPE IN MAXIMUM TIME AND TAPE NAMES.
V'S DD17 = $H* CHECK TMAX, TEST, TPES**$
:R ALPHA IS THE NAME OF THE PSUEDO TAPE TO BE READ
ALPHA = 1
:R BETA IS THE OUTPUT PSEUDO TAPE. ALPHA AND BETA ARE
:R NORMALLY READ IN AS DATA.
BETA = 1
PRINT COMMENT $ PRINT TAPES$
R'A
W'R GO.E.1
READ BINARY TAPE ALPHA, LMIN, LMAX, U,T, X1(LMIN)...X1(LMAX),
:1 X2(LMIN)...X2(LMAX), X3(LMIN)...X3(LMAX), SV(LMIN)...SV(LMAX)
:2 MAG1(1)...MAG1(NMAX), MAG2(1)...MAG2(NMAX), MAG3(1)...;
:3 MAG3(NMAX), PH11(1)...PH11(NMAX), PH12(1)...PH12(NMAX),
:4 PH13(1)...PH13(NMAX),
:3 V(LMIN)...V(LMAX), SV1(LMIN)...SV1(LMAX), T2, EE, EV, COLL
E'L

R 1.916+.766
print gradf1 mad 101 thru 150
W 2257.8

TMIN = T
INTEGER TEST1,COLL
:1 ER,L,G,H,K,N,R,P,Q,C,N,LMAX,LMIN,NMAX,INCR,1,S,UP,
:1 NNN,NN,LINC,U,STA,KM,LMAX1,NM,GGG,DRG,GO,TEST,GG,
:2 RR,R1,N1,G1,CR,CRO,LMAX2,SET,DONE,LL,CO,TEST3,ST,ORD
:1 ,ALPHA,BETA,NMIN,Z1,EXPAND
:R Z GIVES THE POSITIONS OF THE CELL POINTS
Z(1) = 0.
:R WW IS THE NORMALIZED WP AT EACH CELL POINT
WW(1) = SQRT(.15)
:R GRAD IS THE PLASMA DENSITY GRADIENT, NORMALIZED.
W21 = WW(1) .P 2
GRAD = .002
:R MAG1, PH11 ARE THE MAGNITUDE AND PHASE OF THE TOTAL WAKE
:R FIELDS AT EACH CELL POINT. MAG2, PH12 INCLUDE ONLY
:R WAKES FROM DISKS MORE THAN 1 CELL POINT AHEAD, MAG3
:R AND PH13 INCLUDE ONLY WAKES FROM DISKS MORE THAN 2
:R CELL POINTS AHEAD.
:R HERE WE INSURE CONSISTENCY OF THE END FIELDS.
Z = -10000.
MAG2(NMAX) = MAG1(NMAX)
MAG3(NMAX) = MAG1(NMAX)
MAG3(NMAX-1) = MAG2(NMAX-1)
PHI2(NMAX) = PH11(NMAX)
PHI3(NMAX) = PH11(NMAX)
:R PRINT IF WISH TO ADD PLASMA AT END.
PHI3(NMAX-1) = PHI2(NMAX-1)
PRINT COMMENT $ PRINT DATA (NMIN,NMAX,EXPAND)$

:R IF CHANGE NMIN,NMAX, BE SURE TO CHANGE NMAX IN GRADF1

R'A
:R SET UP POSITIONS AND WP'S OF CELL POINTS
T'H A1, FOR N = 2,1,N.G.NMAX
Z(N) = Z(N-1) + 1.
WW(N) = SQRT(SIN(W21*GRAD*Z(N)))
:R PHR IS A RANDOM PHASE THAT IS USED IN MODULATION.
RNSET = SETU.(2)
T'H A2, FOR K = 1,1,K.G.100
PHR(K) = 2.*3.1415927*RANNO.(K)
Z(NMAX+1) = 10000.
W1 = WW(1)
W' OR .GO.E.0.OR.EXPAND.E.1
T'H A40, FOR N = NMIN,1,N.G.NMAX
W' OR EXPAND.E.1,PHC = WW(N)*(T-Z(NMIN-1)/VO)
:R SET UP INITIAL CONDITIONS TO CANCEL DRAG OF DISK FROM
:R ITS WAKE FIELD, IF BEAM UNMULATED.
NUS = AB*(1.-2.*IK.(WW(N)/VO))
FACTOR = -3.1415927/2.*WDT*WW(N)/2. + WW(N)*Z(N)/VO + PHC
PHI1(N) = FACTOR

2.216+.666
PHI2(N) = FACTOR
PHI3(N) = FACTOR
FACTOR = NUS/(2.*SIN(WW(N)*WDT/2.))
MAG1(N) = FACTOR
MAG2(N) = FACTOR
MAG3(N) = FACTOR

:R MEASURE INITIAL ELECTRIC ENERGY
T'H A41, FOR N = NMIN+1,1,N.G,NMAX
EE = EE+.5*(MAG1(N).P.2+MAG1(N-1).P.2)/(AB*V0.P.2)
E'L
:R
:R MAIN LOOP OF PROGRAM
:R
T'H A0, FOR T = TMIN, WDT,T,G,TMAX
T'H A42, FOR L = LMIN,1,L,G,LMAX-1
Z1 = X1(L)
:R INSURE CONSISTENT MAGNITUDES.
MAG3(Z1+1) = MAG2(Z1+1)
MAG2(Z1+2) = MAG1(Z1+2)
E'L
W'R X1(LMAX).L,Z(NMAX-1)
MAG3(NMAX-2) = MAG2(NMAX-2)
MAG2(NMAX-1) = MAG1(NMAX-1)
E'L
:R RUNGE. CALCULATES TRAJECTORIES BY RUNGE-KUTTA METHOD.
RUNGE.
:R ORDER. REORDERS THE DISKS AND CORRECTS FOR CROSSINGS.
ORDER.
:R RESULT. PRINTER OUTPUT
RESULT.
:R ER IS THE ERROR INDICATOR.
W'R ER.GE.1
P'T D5,N,L,LMAX,LMIN,T
V'S D5 = $H* ER.GE.1, N =*,13,4H L =,13,7H LMAX =,
:1 13,7H LMIN =,13,4H T =,F10.2*$
T'O GETOUT
0
E'L
GETOUT
W'R TEST1.E.1
WRITE BINARY TAPE BETA,LMIN,LMAX,U,T,
:1 X1(LMIN)...X1(LMAX),X2(LMIN)...X2(LMAX),X3(LMIN)... 
:2 X3(LMAX),SV(LMIN)...SV(LMAX),MAG1(1)...MAG1(NMAX), 
:1 MAG2(1)...MAG2(NMAX),MAG3(1)...MAG3(NMAX),PHI1(1)... 
:1 PHI1(NMAX),PHI2(1)...PHI2(NMAX),PHI3(1)...PHI3(NMAX), 
:3 V(LMIN)...V(LMAX),SV1(LMIN)...SV1(LMAX),T2,EE,EV,COLL 
E'L
E'M
2.033+.383
PRINT RUNGL1 MAD 1 THRU 50

W 2325.8

:R CALCULATES TRAJECTORIES BY RUNGE-KUTTA METHOD
:R ALSO INJECTS DISKS AND MODULATES THEM AT ORIGIN.
EXTERNAL FUNCTION
E'0 RUNGE.
PROGRAM COMMON
:1 LMIN, ER, VO, T, X1(800), X2(800), X3(800), V, Z, N, LMAX, WDT, A,
:2 MAG1(500), MAG2(500), MAG3(500), PH11(500), PH12(500),
:3 PH13(500), AB, SV, SV1
:1 DAMP, P12, AA, MAGGY, PHIT, THE, NUS,
:5 NU, SET, MOD, V1, ST, XS, WW(500), TEST, TEST3, U, T2, EE, EV
:6, NN, NH, NNN, L, DONE, TMAX, NMAX, NUMB, VAV, ECOL, ORD, GRAD, INN,
:7 11K1(70), W21, E1(10), BR, NORM, COLL, FXUP, PHR(500), DELV
D'n A(800), V(800),
:1 SV(800), SV1(800),
:1 Z(505), MM(10), TR1(50), TR2(50), ST(4)
:2, DONE(800)
INTEGER
:1 ER, L, G, H, K, N, R, P, Q, C, M, LMAX, LMIN, NMAX, INCR, I, S, UP,
:1 NNN, NN, LINC, U, STA, KM, LMAX1, II, NM, GGG, GO, TEST, GG,
:2 RR, R1, UU, SET, ST, ORD, DONE
16 = 1./6.
:R INJECT A DISK
LMIN = LMIN-1
:R X1 IS THE DISK POSITION NOW, X2 THAT THE LAST STEP TIME,
:R X3 THE STEP TIME BEFORE THAT. V IS THE DISK VELOCITY.
X1(LMIN) = .000001
:R MODULATION BUILDS UP SLOWLY.
W' T.L.20.
MOD = T/20.
O'E
MOD = 1.
E'L
:R WE WIDE BAND MODULATE OVER THE WP'S OF Z BETWEEN 0 AND
:R 100, EACH FREQ. AT RANDOM PHASE.
DELV = 0.
TH A11, FOR N = 1,1,N,G.100
DELV = DELV+.0045*WW(N)*SIN.(WW(N)*T-PHR(N))
V(LMIN) = VO*(1.+MOD*DELV)
:R CALCULATE INPUT ENERGY
INN = INN+V(LMIN).*P.2
TH A14, FOR L = LMAX, -1, L.L.LMIN
N = X1(L)+2.
KM = 0
:R FORCE. CALCULATES THE NONOSCILLATORY FIELD OF A DISK.
RET = FORCE.(L)
:R FIND THE NUMBER OF DISKS IN THE 2 CELLS JUST AHEAD OF DISK
:R THE WAVES FROM THESE DISKS ARE NOT INCLUDED IN THE FIELDS
:R THE CELL POINTS AND ARE ADDED SEPARATELY.
TH A15A, FOR C = 1,1,C.G.30.OR.L+C.G.LMAX
W'R N.L.NMAX
R 1.416+.700
PRINT RUNGL MAD 51 THRU 100

2313.9

WR X1(L+C).LE.Z(N+1),KM = KM+1
OE
WR X1(L+C).LE.Z(N),KM = KM+1
15A
EL
NM = N+1
NN = N-1
NNN = N-2
:R USE LAGRANGIAN INTERPOLATION TO FIND THE TIME THAT
:R THE DISKS IMMEDIATELY AHEAD WERE AT X1 OF THE
:R CONSIDERED DISK.
T'HA17, FOR K = 1,1,K.G.KM
M = L + K
TR1(K) = WDT/((X2(M) - X3(M))*(X2(M) - X1(M))
TR2(K) = 2.*WDT/((X1(M) - X3(M))*(X1(M) - X2(M))
:R PERFORM RUNGE-KUTTA INTEGRATION.
T'HA19, FOR G = 1,1,G.G.4
AA = 0.
WR G.E.1
TINC = T
XINC = X1(L)
O'R G.E.2
TINC = T+.5*WDT
XINC = X1(L)+.5*WDT*V(L)
O'R G.E.3
XINC = XINC+.25*WDT*MM(1)
O'R G.E.4
TINC = T + WDT
XINC = X1(L)+WDT*V(L)+.5*WDT*MM(2)
EL
W2 = SQRT.(SIN.(W21+GRAD*XINC))
T'HA18, FOR K = 1,1,K.G.KM
M = L + K
NUS = AB*(1.-2.*IK.(W2/V(M)))
TF = T - WDT + TR1(K)*(XINC - X3(M))*(XINC - X1(M))
:1 + TR2(K)*(XINC - X3(M))*(XINC - X2(M))
AA = AA-NUS*COS.(W2*(TINC-TF))
NUS2 = AB/2.*(1.-2.*IK.(W2/V(L)))
WR N.L.NMAX
C1 = -(XINC - Z(NN))*(XINC - Z(NN))
C2 = .5*(XINC - Z(NN))*(XINC - Z(NN))
F3 = MAG3(NN)*COS.(WW(NN)*TINC-PHI3(NN))
F2 = MAG2(N)*COS.(WW(N)*TINC-PHI2(N))
F1 = MAG1(NN)*COS.(WW(NN)*TINC-PHI1(NN))
MN(G) = WDT*(-(F3+C1*(F2-F3)+C2*(F1-F3))+AA-NUS2+RET)
OE
F2 = MAG2(NN)*COS.(WW(NN)*TINC-PHI2(NN))
F1 = MAG1(N)*COS.(WW(N)*TINC-PHI1(NN))
MN(G) = WDT*(-(F2+(F1-F2)*(XINC-Z(NN)))+AA-NUS2+RET)
EL

2.400+.683
\texttt{print rung1 mad 101 thru 112}
\texttt{W 2329.3}
\texttt{X3(L) = X2(L)}
\texttt{X2(L) = X1(L)}
\texttt{X1(L) = X1(L) + WDT*V(L) + (WDT/6.)*((MM(1) + MM(2)) :1 + MM(3))}
\texttt{A(L) = (1./WDT)*MM(1)}
\texttt{V(L) = V(L) + 16*(MM(1) + 2.*MM(2) + 2.*MM(3)) :1 MM(4))}
\texttt{:R MAGSET. COMBINES THE WAKE FIELDS. IT CALCULATES MAG1,}
\texttt{:R PHI1,MAG2,ETC.}
\texttt{A14}
\texttt{MAGSET.}
\texttt{F'N}
\texttt{E'N}
\texttt{R 1.383+.350}
print magset mad 1 thru 50

R COMBINES WAKE FIELDS INTO 1 MAGNITUDE AND 1 PHASE, AT
R CELL POINTS 0.4VO/WP APART. MAIN PROGRAM IS GRADF1.
EXTERNAL FUNCTION
E'O MAGSET.
PROGRAM COMMON
:1 LMIN, ER, VO, T, X1(800), X2(800), X3(800), V, Z, N, LMAX, WDT, A,
:1 MAG1(500), MAG2(500), MAG3(500), PHI1(500), PHI2(500),
:1 PHI3(500), AB, SV, SV1, DAMP, P12, AA, MAGGY, PHIT, THE, NUS,
:5 NU, SET, MOD, W1, ST, XS, WM(500), TEST, TEST3, U, T2, EE, EV
:6 NN, NM, NNN, L, DONE, TMAX, NMAX, NUMB, VAV, ECOL, ORD, GRAD, INN
:7 TIKI(70), W21, E1(10), BR, NORM, COLL, FXUP, PHR(500), DELV
D'N A(800), AC(800), V(800),
:1 SV(800), SV1(800),
:1 Z(505), MM(10), ST(4)
:2 , DONE(800)
R SINCE MAGSET IS CALLED AFTER THE TRAJECTORIES HAVE BEEN
R ADVANCED, THE TIME IS T+WDT.
R T7 IS USED INSTEAD OF T.
T7 = T+WDT
INTEGER
:1 ER, LG, H, K, N, R, P, Q, C, M, LMAX, LMIN, NMAX, INCR, I, S, UP,
:1 NNN, NN, LINC, U, STA, KH, LMAX1, II, NM, GGG, GO, TEST, GG,
:2 RR, RI, UU, SET, ST, DONE, ORD, FXUP
R SEE ORDER. ROUTINE FOR USE OF FXUP. IT AVOIDS COUNTING
R A DISK WAKE TWICE.
WR FXUP.E.1, T'O FIX
R UP IS USED IF A DISK PASSES 2 CELL POINTS IN A STEP TIME
UP = 0
WR L.E., LMIN
R CALCULATE FIELDS AT ORIGON DUE TO DISK INJECTION.
T5 = T
B7 = MAG1(1)
R SEE RUNG1. FOR DELV. NUS IS DISK WAKE MAGNITUDE.
NUS = AB*(1.-2.*IK.(WW(1)/(VO*(1.+MOD*DELV))))
MAG1(1) = SQRT.(B7.P.2 + NUS.P.2 + 2.*NUS*
:1 B7*COS.(WW(1)*T5-PHI1(1)))
PHI1(1) = PHI1(1) + ACSIN.(B7,NUS,NUS/MAG1(1),
:1 (WW(1)*T5-PHI1(1)))
R SV1 IS THE TIME THE DISK PASSED THE LAST
R CELL POINT.
SV1(L) = T5
WR X1(L).GE.Z(N)
WR N.NE.2
P'T DD12,N,L,X1(L)
ER = 1
E'L
V'S DD12=$H* L=LMIN,N.NE.2*INR-K*,3HN = ,16,3HL = ,16,3HX =,
:1F12.6*$
TG = T5 + WDT*Z(N)/X1(L)
R TG IS THE TIME OF PASSING OF THE PRESENT CELL POINT
R 2.650+.783
print magset mad 51 thru 100
W 2334.5

:R SV1 IS THE TIME OF PASSING OF THE CELL POINT
:R JUST BEFORE, AND SV THE ONE BEFORE THAT.
SV(L) = T5
SV1(L) = TG
MAG2(1) = MAG1(1)
PH12(1) = PH11(1)
B7 = MAG1(2)
NUS = AB*(1.-2.*IK.(WW(2)/(V(L)-A(L)*T7-TG))))
MAG1(2) = SQRT.(B7.P.2 + NUS.P.2 + 2.*NUS*
:1 B7*COS.(WW(2)*T7-PH11(2)))
PH11(2) = PH11(2) + ACSIN.(B7,NUS,NUS/MAG1(2),
:1 (WW(2)*T7-PH11(2))
E'L
E'L

FIX

:R THE TIME OF CROSSING (TG) IS APPROXIMATED FROM THE POSI-
:R TIONS AT PREVIOUS STEP TIMES, AND A SECOND ORDEP INTER-
:R POLATION. SEE CHAPTER 4 FOR DETAILS OF COMBINING
:R WAKES.

W'R X1(L).GE.Z(N).AND.L.G.LMIN.AND.DONE(L).E.0
TR3 = WDT/((X2(L) - X3(L))*(X2(L) - X1(L)))
TR4 = 2.*WDT/((X1(L) - X3(L))*(X1(L) - X2(L)))
TG = T - WDT + TR3*(Z(N) - X3(L))*(Z(N) - X1(L))
:1 *TR4*(Z(N) - X3(L))*(Z(N) - X2(L))
W'RN.G.2
B7 = MAG3(NNN)
NUS = AB*(1.-2.*IK.(WW(NNN)/(V(L)-A(L)*T7-SV(L))))
MAG3(NNN) = SQRT.(B7.P.2 + NUS.P.2 + 2.*NUS*
:1 B7*COS.(PH13(NNN) - WW(NNN)*SV(L)))
PH13(NNN) = PH13(NNN) + ACSIN.(B7,NUS,NUS/MAG3(NNN),
:1 (WW(NNN)*SV(L) - PH13(NNN))
E'L
B7 = MAG2(NN)
NUS = AB*(1.-2.*IK.(WW(NN)/(V(L)-A(L)*T7-SV1(L))))
MAG2(NN) = SQRT.(B7.P.2 + NUS.P.2 + 2.*NUS*
:1 B7*COS.(PH12(NN) - WW(NN)*SV1(L)))
PH12(NN) = PH12(NN) + ACSIN.(B7,NUS,NUS/MAG2(NN),
:1 (WW(NN)*SV1(L) - PH12(NN))
SV(L) = SV1(L)
SV1(L) = TG
B7 = MAG1(N)
NUS = AB*(1.-2.*IK.(WW(N)/(V(L)-A(L)*T7-TG))))
MAG1(N) = SQRT.(B7.P.2 + NUS.P.2 + 2.*NUS*MAG1(N)
:1 *COS.(PH11(N) - WW(N)*TG))
PH11(N) = PH11(N) + ACSIN.(B7,NUS,NUS/MAG1(N),
:1 WW(N)*TG - PH11(N))
E'L
W'RX1(L).GE.Z(NM).AND.UP.E.0
:R A DISK MAY CROSS 2 CELL POINTS IN 1 STEP TIME (BUT, IT
:R TURNS OUT, NO MORE THAN 2).

R 2.166+.766
print magset mad 101 thru 116

w 2337.9

up = 1
n = n + 1
nm = n+1
nn = n-1
nnn = n-2
t'o a20
e'l
w'r up.e.1
n = n-1
nm = n+1
nn = n-1
nnn = n-2
e'l
up = 0
f'n
e'n
r 1.833+.300
print order mad 1 thru 50
V 2342.5

:R THIS ROUTINE REORDERS THE DISKS, CORRECTS FOR CROSSINGS,
:R COLLECTS THE DISKS, AND SHIFTS MEMORY.
EXTERNAL FUNCTION
E'O ORDER.
PROGRAM COMMON
:1 LMIN, ER, VO, T, X1(800), X2(800), X3(800), V, Z, N, LMAX, WDT, A,
:1 MAG1(500), MAG2(500), MAG3(500), PHI1(500), PHI2(500),
:2 PHI3(500), AB, SV, SV1
:1 DAMP, PI2, AA, MAGGY, PHIT, THE, NUS,
:5 NU, SET, MOD, W1, ST, XS, WU(500), TEST, TEST3, U, T2, EE
:6 ,EV, NN, NM, NNN, L, DONE, TMAX, NMAX, NUMB, VAV, ECOL, ORD, GRAD
:7 , INN, 1K1(70), W21, E1(10), BR, NORM, COLL, FXUP
D'N A(800), AC(800), V(800),
:1 SV(800), SV1(800),
:1 Z(505), NM(10), ST(4)
:2 , DONE(800)
INTEGER
:1 ER, L, G, H, K, N, R, P, Q, C, M, LMAX, LMIN, NMAX, INC, I, S, UP, FXUP,
:1 NNN, NN, LINC, U, STA, KM, LMAX1, I, NM, GGG, COLL, GO, TEST, GQ,
:2 L1, RR, R1, UU, SET, ST, N1, N2, CRO, CR, DONE, LL, TEST3, ORD, LCOL
:R
:R ORDER DISKS ( SEE CHAPTER 3 FOR DETAILS).
:R
AB2 = AB/2.
T'H A34, FOR L = LMIN,1,L.G.LMAX
:R DONE INSURES THAT A DISK ADDS TO THE FIELD AT A CELL
:R POINT ONLY ONCE.
A34
DONE(L) = 0
T'H A7, FOR LL = LMIN,1,LL.G.LMAX - 1
L = LL
BETA
S = L+1
W'R X1(L).G.X1(S)
CRO = X1(L)
CR = X1(S)
T1 =((X2(S) - X2(L))*WDT)/(X1(L) - X2(L) - X1(S) +
:1 X2(S))
W'R T1.G.WDT.OR.T1.L.0.
P'T DD7,T1
V'S DD7 = SH* T1.G.WDT,T1 =*,E12.4*S
ER = 1
T'O A30
E'L
:R CORRECT FOR CROSSINGS AS WELL AS REORDER
ALPHA = X1(L)
X1(L) = X1(S) - AB2*(WDT - T1).P.2
X1(S) = ALPHA + AB2*(WDT-T1).P.2
ALPHA = X2(L)
X2(L) = X2(S)
X2(S) = ALPHA
ALPHA = X3(L)

R 2.150+.900
X3(L) = X3(S)
X3(S) = ALPHA
ALPHA = DONE(L)
DONE(L) = DONE(S)
DONE(S) = ALPHA
W'R X1(L).L.CR
DONE(L) = 1
PRINT COMMENT $ UNCROSSED Z(N)$
E'L
ALPHA = A(L)
A(L) = A(S) - AB
A(S) = ALPHA + AB
ALPHA = SV(L)
SV(L) = SV(S)
:R IF ON REORDERING, A DISK ONLY NOW PASSES A CELL POINT,
:R WE CALL MAGSET. TO ADD IN ITS FIELDS
SV(S) = ALPHA
ALPHA = SV1(L)
SV1(L) = SV1(S)
SV1(S) = ALPHA
ALPHA = V(L)
V(L) = V(S) - AB*(WDT-T1)
V(S) = ALPHA + AB*(WDT-T1)
W'R X1(S).G.CRO+1.AND.DONE(S).E,0
L1 = L
L = S
N = CRO+2
NN = N-1
NNN = N-2
NM = N+1
PRINT COMMENT $ CROSSED Z(N)$
W'R N.LE.NMAX
FXUP = 1
MAGSET.
FXUP = 0
E'L
L = L1
E'L
L = L - 1
W'R L.L.LMIN,L = LMIN
T'O BETA
E'L
:R
:R   COLLECT BEAM
:R
W'R TEST3.E.1,P'T DD9
V'S DD9 = SH* CBV**$
LMAX1 = LMAX
LCOL = LMAX1-40
W'R LCOL.L.LMIN,LCOL=LMIN
1.383+.533
print order mad 101 thru 141
4 2347.7
Th A10, FOR L = LMAX1, -1, L.L.LCOL
W'R X1(L).G.Z(NMAX)
COLL = 2
LMAX = LMAX - 1
:R FIND ENERGY OF COLLECTED DISKS, AT Z(NMAX).
VVV = V(L) - A(L)*(T+WDT-SV1(L))
VAV = VAV+VVV
NUMB = NUMB + 1.
ECOL = ECOL + VVV*P.2
W'R TEST3.E.1,P'T D21, VVV/VO
V'S D21 = $1H ,F12.6*$
10 E'L
:R
:R SHIFT MEMORY
:R
W'R LMIN.L.5
P'T D27
V'S D27 = $H* SHIFTED MEMORY**$
INCR = 790 - LMAX
W'R INCR.LE.0
P'T DD10
V'S DD10 = $H* INCR.E.0 **S
ER = 1
SET = U+1
RESULT.
E'L
Th A11, FOR L = LMAX, -1, L.L.LMIN
LINC = L + INCR
A(LINC) = A(L)
X1(LINC) = X1(L)
X2(LINC) = X2(L)
X3(LINC) = X3(L)
SV(LINC) = SV(L)
SV1(LINC) = SV1(L)
V(LINC) = V(L)
1 DONE(LINC) = DONE(L)
LMIN = LMIN+INCR
LMAX = LMAX + INCR
E'L
F'N
E'N
1.900+.466
PRINT RESULT MAD 1 THRU 50
2353.0
"R THIS ROUTINE PRINTS THE OUTPUT AND ALSO CHECKS THE ENERGY.
"R THE ELECTRIC FIELD ENERGY IS FOUND BY INTEGRATING
"R ACCORDING TO SIMPSON'S RULE.
"R THE TOTAL ACCELERATION AND THAT DUE JUST TO THE NON-
"R OSCILLATORY FIELDS IS GIVEN.
EXTERNAL FUNCTION
E'O RESULT,
PROGRAM COMMON
:1 LMIN,ER,VO,T,X1(800),X2(800),X3(800),V,Z,N,LMAX,WDT,A,
:1 MAG1(500),MAG2(500),MAG3(500),PHI1(500),PHI2(500),PHI3(500),
:1 AB,SV,SV1,DAMP,P12,AA,MAGGY,PHIT,THE,NUS,
:5 NU,SET,MOD,V1,ST,XT,WT(500),TEST,TEST3,U,T2,EE
:6 ,EV,NN,NNM,NNN,L,DONE,TMAX,NMAX,NUMB,VAV,E COL,ORD,GRAD
:7 ,INN,IK1(70),W21,E1(10),BR,NORM,COLL,FXUP,PHR(500),DLEV
:N A(800),AC(800),V(800),
:1 SV(800),SV1(800),
:1 Z(505),NM(10),ST(4)
:2 ,DONE(800)
INTEGER
:1 ER,L,G,H,K,N,R,P,Q,C,N,LMAX,LMIN,NMAX,INCR,I,S,UP,
:1 NNN,NN,LLINC,STA,KN,LMAX1,11,NM,GGG,GO,TEST,GG,
:2 RR,R1,UU,SET,ST,N1,N2,DONE,ORD,LL,FXUP,COLL
"R
"R PLOT BEAM VELOCITY
"R
VBSQ = VO.P.2
U = U+1
W = (U/SET)*SET.E.U.OR.T.G.(TMAX=WDT/2.)
T1 = T+WDT
P'T D23,T1
V'S D23 = $4H T =,F18.8*$
P'T DD11
V'S DD11 = $H* X
V**$EV'L = EV
EV = 0.
T'H A47, FOR L = LMIN,1,L.G.LMAX
EV = EV + V(L).P.2
W'R TEST.E.1,P'T D28,X1(L),V(L)/VO
V'S D28 = $1H ,2F12.8*$
VAV = VAV/(NUMB*VO)
P'T D91,VAV,NUMB
V'S D91 = $7H VAV = ,F14.6,7HNUMB = ,F12.3*$
NUMB = 0.
VAV = 0.
EV = EV/VBSQ
"R
"R PLOT ACCELERATION
"R
P'T DD12
V'S DD12 = $H* Z
FIEL FORCE**$
print result mad 51 thru 100

* 2357.5
T'H A23, FOR N1 = 1,1,N1.G.NMAX.OR.(Z(N1-1).G.X1(LMAX)
  :1.AND.COLLE.E.0)
F = 0.
R = 0.
T'H A7, FOR L = LMIN,1,L.G.LMAX
Z1 = Z(N1)-X1(L)
W'R R.E.0.,NUM = 1K.(WW(N1)/VO)
R = 1
Z2 = .ABS.Z1
FF = NUM/(1.+E1(1)*Z2+E1(2)*Z2.P.2+WW(N1)*BR
  :1/V(L)*(E1(3)*Z2+E1(4)*Z2.P.3+E1(5)*Z2.P.4))
W'R Z1.L.0.,FF = -FF
F = F+FF

E'L
*:R DEM. GIVES THE DENOMINATOR OF THE APPROX. TO THE
:*R NON-OscILLATORY FORCE.
:R'N DEM.
Z5 = Z4
F'N 1.*E1(1)*Z5+E1(2)*Z5.P.2+WP3*BR*(E1(3)*Z5+E1(4)*Z5.P.3
  :1+E1(5)*Z5.P.4)
E'N
T'H A15, FOR K = N1,1,K.G.20
F = F+NUM/DEM.(1.*K,WW(N1)/VO)
T'H A16, FOR K = NMAX+1-N1,1,K.G.20
F = -NUM/DEM.(1.*K,WW(N1)/VO)+F
FIEL = -HIAG1(N1)*COS.(WW(N1)*T1-PK11(N1)) + F*AB
23 W'R TEST.E.1,P'T D27,Z(N1),FIEL,F*AB
V'S D27 = $1H ,F7.1,2F12.8*$S
*:R EE IS THE ELECTRIC ENERGY NOW. EEL THAT THE LAST TIME
:*R WE LOOKED. EV IS THE DISK KINETIC ENERGY NOW, EVL
:*R THE PREVIOUS. ECOL IS THE COLLECTED DISK K.E, INN IS
:*R THE INJECTED DISK K.E. ENN IS THE ENERGY LOST BY THE
:*R BEAM, ENET IS THE ENERGY ERROR DIVIDED BY ENN.
:*R ETOT IS THE TOTAL ENERGY, EDIF THE ENERGY ERROR /
:*R ETOT.
EEL = EE
ENER.
EE = (1./(AB*VBSQ))*POW
ECOL = ECOL/VBSQ
INN = INN/VBSQ
ENN = INN - ECOL -(EV-EVL)
ENET = (ENN - (EE-EEL))/ENN
T2 = T
P'T D11,ENET,EE,EV,ECOL,ENN,INN
V'S D11 = $7H ENET =E14.6,5H EE =E14.6,5H EV =E14.6,
  :17H ECOL =E14.6,6H ENN =E14.6,6H INN =E14.6*$S
ETOT = EE+EV
ETOTL = EVL+EEL+INN-ECOL
2.366+.666
481.4

EDIF = (ETOT-ETOTL)/ETOT
P'T D12,EEL,EVL,ETOT,ETOTL,EDIF
V'S D12 = $H* EEL =*,E14.6,H* EVL = *,E14.6,H* ETOT =*,E14.6,
:1 H* ETOTL =*,E14.6,H* EDIF =*,E14.6*$
P'T D71,LMIN,LMAX
V'S D71 = S7H LMIN =,18,7H LMAX =,18*$
ECOL = 0.
INN = 0.

E'1

INTERNAL FUNCTION
E'0 ENER.
:R ENER. FINDS THE ELECTRIC ENERGY OF THE FIELDS, BY A
:R SIMPSON'S RULE INTEGRATION.
L = LMIN-1
POW = 0.
K = 1
N = 1
W'R L.GE.LMAX,T'0 A55
LL = L+1
L = L+1
W'R X1(L).LE.Z(N+1).AND.L.LE.LMAX
K = 0
T'0 A57
E'1
L = L-1
M35 = MAG2(N)
P35 = PHI2(N)
M36 = MAG1(N+1)
P36 = PHI1(N+1)
X1 = Z(N+1)
T'H A58, FOR G = L,-1,G,L,LL
XX = X1(G)
W5 = SQRT.(SIN.(W21+GRAD*XX))
M37 = M35 + (M36-M35)*(XX-Z(N))
R = 0
RR = 0
W'R .ABS.(P35-P36).G.3.1415927
W'R P35.G.P36.AND.RR.E.0
P35 = P35-P12
R = 1
T'0 A61
E'1
W'R P35.L.P36.AND.R.E.0
P35 = P35+P12
RR = 1
T'0 A61
E'1
E'1
P37 = P35+(P36-P35)*(XX-Z(N))
NUS = AB*(1.-2.*IK.(W5/V(G)))
83+.450
print result mad 151 thru 179

M38 = SQRT.(M37.P.2+NUS.P.2+2.*NUS*M37*COS.(P37-W5*(
       :1 T+WDT)))
P38 = P37+ACOSIN.(M37,NUS,NUS/M38,W5*(T+WDT)-P37)
POW = POW + .5*(M36.P.2+M38.P.2)*(X1-XX)
X1 = XX
B7 = M35
NUS = AB*(1.-2.*IK.((WW(N)/(V(G)-A(G))*(T+WD-T-SV1(G)))))
M35 = SQRT.(M35.P.2+NUS.P.2+2.*NUS*M35*COS.(WW(N)*SV1(G)
       :1 -P35))
P35 = P35+ACOSIN.(B7,NUS,NUS/M35,WW(N)*SV1(G)-P35)
M36 = M38
P36 = P38

A58 W'R G.E.LL,POW = POW+.5*(MAG1(N).P.2+M38.P.2)*(1 X1-Z(N))
W'R ABS.(MAG1(N)-M35).G.1.E-4,P'S MAG1(N),MAG2(N),
       :1 MAC1(N+1),M35,PHI1(N),P35,PHI2(N),L,LL,PHI1(N+1)
       :2 X1(LL)
A55 N = N+1
W'R K.E.1
       POW = POW + .5*(MAG1(N).P.2+MAG1(N-1).P.2)
E'L
K = 1
W'R N,GE,NMAX,T'O A56
T'O A51
A56
F'N
E'N
F'N
E'N

R 2.400+.483
int for cl mad 1 thru 50

:R APPROXIMATES THE NON-OSSCILLATORY FIELD OF A DISK. SEE
:R CHAPTER 6 FOR DETAILS. DISKS WITHIN A DISTANCE OF
:R 15.5 (NORMALIZED) ON EACH SIDE ARE CONSIDERED.
:R PSUEUDO DISKS NEAR THE GUN AND COLLECTOR ARE ADDED
:R IN ATTEMPT TO ROUGHLY BALANCE NON-OSSCILLATORY FORCES.
EXTERNAL FUNCTION(R1)
E'O FORCE.
P\'N LMIN,ER,VO,T,X1(800),X2(800),X3(800),V,ZN,LMAX,WDT,A,
:1 MAG1(500),MAG2(500),MAG3(500),PHI1(500),PHI2(500),
:1 PHI3(500),AB,SV,
:1 SV1,DAMP,P12,AA,MAAGY,PHIT,THE,NUS,
:2 NU,SET,MOD,V1,ST,XS,WV(500),TEST,TEST3,U,T2,EE,EEV,
:2 NN,NM,Nnn,L,DONE,TMAX,NNMAX,NUMB,VAV,ECOL,ORD,GRD,LNN,
:2 T1K1(70),V21,E1(10),BR,NORM,COLL,FXUP
D\'N A(800),V(800),
:1 SV(800),SV1(800),
:2 Z(505),DONE(800),ST(4)
I'R R,GR,RI,SSI,COLL,LMAX,NNMAX,LMIN,K,FXUP
R = R1
WP2 = SQRT.(SIN.(V21+GRAD*X1(R)))
FF = 0.
Z1 = 0.
T'H A79, FOR G = R+1,1,Z1.G.15.5.OR.G.G.LMAX
WP1 = WP2/V(G)
Z1 = .ABS.(X2(G)-X1(R))
F = IK.(WP1)/(1.+E1(1)*Z1+E1(2)*Z1.P.2+WP1*BR*(E1(3)
:1 *Z1+E1(4)*Z1.P.3+E1(5)*Z1.P.4))
FF = FF-F
FF1 = 0.
Z1 = 0.
T'H A78, FOR G = R-1,-1,Z1.G.15.5.OR.G.G.LMIN
WP1 = WP2/V(G)
Z1 = .ABS.(X1(G)-X1(R))
F = IK.(WP1)/(1.+E1(1)*Z1+E1(2)*Z1.P.2+WP1*BR*(E1(3)*Z1
:1 +E1(4)*Z1.P.3+E1(5)*Z1.P.4))
FF1 = FF1+F
ANSW = 0.
W'R R-LMIN,L.15
WP1 = WP2/VO
NUM = IK.(WP1)
DELFU = 1.
T'H A3, FOR K = R-LMIN+1,1,K.G.15
ANSW = NUM/DEN.(X1(R)+DELFU,WP1) + ANSW
DELFU = DELFU+1.
E'L
INTERNAL FUNCTION (Z4,WP3)
E'O DEN.
Z5 = Z4
F'N 1.+E1(1)*Z5+E1(2)*Z5.P.2+1P3*BR*(E1(3)*Z5+E1(4)*Z5.P.3
:1 +E1(5)*Z5.P.4)
83+.616
E'N
W'RE COLL.E.0.OR.LMAX-R.L.15
W'R R.E.LMAX
XMAX = X1(LMAX)
O'E
XMAX = X2(LMAX)
E'L
WP1 = WP2/VO
NUM = 1K.(WP1)
DELFU = 1.
T'H A4, FOR K = LMAX-R+1,1,K.G.15
ANSW = -NUM/DEN.((XMAX+DELFU-X1(R),WP1) + ANSW
DELFU = DELFU+1.
E'L
F = (FF+FF1+ANSW)*AB
F'N F
E'N
1.500+.283
R READS THE FUNCTION 11(Y)K1(Y) FROM TABLES, R AND INTERPOLATES BETWEEN TABLE POINTS.
R WV IS WP(Z)/V(L)
EXTERNAL FUNCTION(WV)
E'O IK.
WV1 = BR*WV
I'R WPBI
P'N LMIN,ER,VO,T,X1(800),X2(800),X3(800),V,Z,N,LMAX,WPT,A,
:1 MAG1(500),MAG2(500),MAG3(500),PH11(500),PH12(500),
:2 PH13(500),AB,SV,
:1 SV1,DAMP,P12,AA,MAGY,PHIT,THE,NUS,
:1 NU,SET,MOD,W1,ST,XS,WV(500),TEST,TEST3,U,T2,EE,EV,
:2 NN,NM,NNN,L,DONE,TMAX,NMAX,NUMB,VAV,ECOL,ORD,GRAD,INN,
:2 T1K1(70),W21,E1(10),BR,NORM,COLL,FXUP
D'N A(800),V(800),
:1 SV(800),SV1(800),
:2 Z(505),DONE(800),ST(4)
R IF ARGUMENT GREATER THAN 5.7, APPROXIMATE FUNCTION.
W'R WV1.G.5.7
  NUMER = 1./2./WV1*(1.-3./8./WV1.P.2)
O'E
  WPB = NORM*WV1
  WPBI = WPB
  NUMER = T1K1(WPBI)*(WPB-WPBI)*(T1K1(WPBI+1)-T1K1(WPBI))
E'L
F'N NUMER
E'N
.433+.550
:R THIS ROUTINE IS BASICALLY THE ACSCIN, BUT THE
:R CORRECT QUADRANT IS INSURED. SEE CHAPTER 4. THERE
:R A = B4, B = B5, B/C = B1, B2 =
:R THETAB - THETAA
EXTERNAL FUNCTION (B4,B5,B1,B2)
E'O ACSVN.
B3 = B2
PI = 3.1415927
B10 = B1*SIN.(B3)
P'T D1,B10
V'S D1 = $14H B10.G.1.,B10=,E18.8*$
I'R B20
B20 = B10
B10 = 1.*B20
E'L
B6 = ASIN.(B10)
W'R -B5*COS.(B3).G.B4, B6 = PI - B6
F'N B6
E'N
1.066+.416
.TAPE. 1161 contains the function $I_1(y)K_1(y)$ for $y = 0$ to $y = 6$, in steps of 0.1. This tape is read by the function IK.

print .TAPE. 1161 1 thru 40
W 1732.1

.50000000E 00
.49330336E 00
.47998922E 00
.46357517E 00
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.42718673E 00
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.14174666E 00
.13801432E 00
.13446061E 00
.13107873E 00
.12785161E 00
.12477385E 00
R .766+.950
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62. Ibid., p.364.


BIOGRAPHY

The author was born January 12, 1941, in Jacksonville, Florida. He attended the public schools there, and graduated from Robert E. Lee High School in June, 1958. He then entered M.I.T., receiving the S.B. degree in Electrical Engineering in June, 1962. During his undergraduate years he held a National Merit Scholarship. During the summer of 1962 he worked for Sperry Microwave Electronics Laboratory in Clearwater, Florida.

Continuing at M.I.T., the author received the S.M. degree in Electrical Engineering in June, 1963. His Masters thesis was "The Forward Driven Varactor Frequency Doubler," supervised by Professor Robert Rafuse. The summer of 1963 was spent at Microwave Associates in Burlington, Massachusetts. From September, 1962 to June, 1966 he held a National Science Foundation Fellowship. From June, 1966 to January, 1968 he held a research assistantship in the Research Laboratory of Electronics.

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On June 15, 1963, the author was married to the former Heather Shilling of Ashland, Ohio. They had a son Stephen on September 16, 1967.