SEQUENTIAL MODELS IN ECONOMIC DYNAMICS

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ABSTRACT

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The first essay gives a model of money holding under employment uncertainty. The budget constraint is given as never running out of money in sequential trading. The demand for money and its dependence on the employment probability are analysed. The ergodic distribution of money balances is derived. These results are related to the theory of the consumption function and of market disequilibrium.

The second essay distinguishes two concepts of changing preferences in sequential decision making: One in which subsequent preferences are non-stationary, but coincide over the intersection of their domains, the other, in which they do not so coincide. A model of consumption behaviour investigates the second concept further with particular stress on problems of normative analysis.

The third essay formulates a model of computation costs which depend on the dimension of the commodity vectors that one compares. For a single decision an al-
Algorithm is given that approximates the optimal computation capacity. For repeated decision making the path that is actually pursued is more complex than any path the individual can compute, and no such algorithm is available.

The last essay gives sufficient conditions for the uniqueness of the optimal solution in the Ramsey problem with a fixed cost of making withdrawals from the bank.
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Asset Management with Trading Uncertainty

I. Introduction

It is our purpose in this paper to explore the connections between uncertainties inherent in the trading process and the holding of liquid assets, with the idea that the techniques we develop here are a step toward a unified theory of money, price formation and the trading process.

What we call "trading uncertainty" is uncertainty about an agent's immediate opportunities to buy and sell. It is inherent in the operation of real markets when information cannot be transmitted or processed costlessly. Trading uncertainty must be sharply distinguished from the "state-of-the-world" uncertainty which is treated in the theories of portfolio choice and market allocation of risk.¹/ State of the world uncertainty concerns events like changes in endowments, technology and tastes that are generally taken as exogenous from the point of view of the economic relations of production and exchange. The major theme of state of the world uncertainty is the ability of agents to reduce

¹/ This paper was written in joint authorship with Duncan K. Foley.

¹/ Cf. Arrow, Debreu, ch. 7.
private riskiness by exchanging goods in different states of the world, that is, by insurance or hedging, to evade some or all of the possible risk.

The central fact about trading uncertainty is that it cannot be traded away, since it inheres in the trading process itself. It could be eliminated only in a world where the transmission and processing of information was costless.

The picture of market equilibrium of conventional theory explicitly rules out trading uncertainty, since agents are supposed to be able to buy or sell any amount of any good at a known price. Trading uncertainty may enter in two guises depending on whether one views market disequilibrium in "Marshallian" or "Walrasian" terms. There may be several prices coexisting in the market at any moment, so that agents do not face a single known price. Or there may be only one price, but if it is not an equilibrium price agents cannot all buy or sell any amounts they want to. If excess supply or excess demand is rationed randomly in part, then agents will face trading uncertainty about their ability to buy or sell.

Costless and instantaneous transmission and processing of information would eliminate any divergence of price at a given moment, and would also presumably rule out trading at a disequilibrium price. This is the
case studied by conventional theory (even with the introduction of state of the world uncertainty). We believe it is a polar or asymptotic case to which real markets are only better or worse approximations.

The agents in real markets always bear some residual trading uncertainty. This uncertainty cannot be hedged or insured, since the insurance contract itself would be equivalent to the transaction it insures. Take, for instance, the case of an agent selling a house. Can the agent buy insurance as to the price he will get for the house? Clearly the issuer of such insurance is in precisely the position of having bought the house at the insured price. The problems of consummating and insuring the transaction are economically equivalent.

The major theme we want to develop in this paper is that there is an intimate connection between trading uncertainty and the holding of liquid assets. We show in one particular model that without trading uncertainty agents will not hold liquid assets asymptotically while with trading uncertainty they will hold almost always positive balances. This connection arises from stating the budget constraint in an intuitively appealing way: that the agent must not run out of stocks of purchasing power at any step of a sequential trading process.

This approach also offers an alternative and more tractable treatment of the idea advanced by Clower
(1965, 1967) and Leijonhufvud (1967, 1968) that Keynes' consumption multiplier represents a recognition that spending patterns are constrained not by potential income at current market prices, but by realized income, which will be smaller than potential income whenever trading takes place at nonequilibrium prices. In our treatment, the inability of an agent instantaneously to sell at going market prices reduces his current consumption, but not in a mechanical one-to-one fashion. In addition, the risk of inability to sell affects the agent's consumption spending even in periods when he himself is lucky enough to achieve a sale at the market price. In models of the kind we study here there appears to be a precise and subtle account of the distinction between "notional" and "effective" demand, and of the connection between "effective" demand and holdings of asset balances.

II. A Model of Trading Uncertainty

An economic agent exists in an infinite sequence of periods $t = 0, 1, 2, \ldots$ (We imagine these periods to be rather short in terms of calendar time, comparable to the time between transactions, on the order of hours or days). He buys a consumption good, $c$, at a known money price $p$, which is assumed to remain constant, and sells labour, $l$, at a known money wage $w$, assumed also to re-
main constant. There is, however, a probability \((1 - q)\) 
\((0 < q < 1)\) that the agent will find himself unable to sell 
labour at all in the period, or equivalently, that in 
that period the particular agent (not necessarily every 
agent) will face a zero wage. The agent also owns a 
non-interest bearing asset, \(m\), and cannot buy consump-
tion without paying \(m\) for it. Since \(m\) is the only asset 
in the model and it is used for transactions we call it 
"money", although much of what we say could apply to 
assets in general as a store of wealth.

The agent maximizes the expected stream of discon-
tered utility of consumption and labour. His decision 
variables are \(c_{1t}, c_{2t}\) and \(l_t\), where \(c_{1t}\) is his con-
sumption purchase when he is employed, \(c_{2t}\) his consump-
tion purchase when he is unemployed, and \(l_t\) his labour 
offering when he is employed. We can write this problem 
formally as:

\[
\text{Max } E\left( \sum_{t=0}^{\infty} \alpha^t u(c_t, l_t) \right) \quad (0 < \alpha < 1)
\]

subject to

\[
c_t = c_{1t} \geq 0
\]

\[
l_t = l_t \geq 0 \quad \text{with probability } q
\]

\[
m_{t+1} = m_t + w_l t + p c_{1t}
\]

\[
c_t = c_{2t} \geq 0 \quad \text{with probability } (1 - q)
\]

\[
l_t = 0
\]

\[
m_{t+1} = m_t - p c_{2t}
\]
m_t \geq 0 \text{ for all } t

m_0 \text{ given, } (0 \leq q \leq 1).

We make the following assumption on the utility function:

a) The instantaneous utility function \( u(c, l) \) is continuously differentiable, strictly concave and non-negative.

b) For all \( c, l: \) \( u_c > 0, \ u_l < 0 \).

Furthermore,

\[
\lim_{c \to 0} u_c(c, l) = \infty \text{ for all } l;
\]

\[
\lim_{l \to 0} u_l(c, l) = 0 \text{ for all } c;
\]

and there is some \( I \) such that

\[
\lim_{l \to I} u_l(c, l) = -\infty \text{ for all } c.
\]

Part b) ensures that the consumer never acquires an unlimited amount of money in any period.

We find it convenient to use the indirect utility functions \( v_1(\cdot), v_2(\cdot) \), defined as:

\[
v_1(e) = \max_{c, l \geq 0} u(c, l)
\text{ subject to } pc - wl \leq e
\]

\[
v_2(e) = \max_{c \geq 0} u(c, 0)
\text{ subject to } pc \leq e
\]

\( v_1(e) \) is the one period utility the agent can achieve by net dishoarding of \( e \) when he is employed; \( v_2(e) \) the
one period utility achieved by net dishoarding of e when he is unemployed. It is well known that under a), both $v_1(e)$ and $v_2(e)$ are strictly concave, nonnegative, increasing and continuously differentiable with $v_1(e)$ defined on $[wI, \infty)$, $v_2(e)$ defined on $[0, \infty)$. Furthermore,

$$\lim_{e \to 0} v'_2 = \infty$$

and as a consequence, for small e

$$v'_2(e) > v'_1(e).$$

The value of the optimal program under (1) depends, for given prices $p, w$ and probability of employment $q$, on initial money balances $m_0 = -m$ only. It is natural to describe the agent's choice in terms of $m_1$, money balances held over into the next period when he is employed, and $m_2$, money balances held over when he is not employed. At any moment, the agent balances the marginal utility gained from spending his money immediately and the marginal utility of holding money into the next period. If we define the value of the program for initial balances $m$ as

$$V(m) = \max B \sum_{t=0}^\infty \alpha^t u(c_t, l_t)$$

etc., as in (1), then the latter component is given by the value, discounted by one period, of the whole program, starting with end-of-period money balances. This line of reasoning leads to a dynamic programming formulation of the problem 2/; the function $V(.)$ must satis-
fy the functional equation:

\begin{align}
V(m) &= q \max_{m_1 \geq 0} \left( v_1(m - m_1) + \alpha V(m_1) \right) \\
&\quad + (1 - q) \max_{m_2 \geq 0} \left( v_2(m - m_2) + \alpha V(m_2) \right)
\end{align}

The total utility of money balances is given by taking the expectation from the employed and unemployed contingencies, where in each contingency utility is the sum of instantaneous utility from consuming and working in the current period and the discounted value of the program starting with \( m_1 \) (or \( m_2 \)) in the next period.

In the mathematical appendix, we prove the following:

**Proposition 1**: An optimal policy exists to problem (1). The value of problem (1), \( V(.) \) is a strictly concave, strictly increasing, differentiable function of initial monea balances \( m \) that satisfies equation (2).

III. Properties of the Optimal Policy

We continue to characterize the solution to the individual's problem in each period by \( m_1 \) and \( m_2 \), his end-of-period money balances. For given \( q \), these depend on initial balances \( m \) and satisfy the first order

---

\(^2/\) Cf. Levhari and Srinivasan.
conditions:

(3) \[ v_1^i(m-m_1) = \alpha V_m(m_1) + \mu_1 \]

(4) \[ v_2^i(m-m_2) = \alpha V_m(m_2) + \mu_2 \]

\[ \mu_1, \mu_2, \geq 0 \ ; \ \mu_1 m_1 - \mu_2 m_2 = 0 \]

**Proposition 2**: The policy correspondences \( m_1(m,q), m_2(m,q) \) are single-valued, continuous in \( m \) and satisfy for all \( q < 1 \):

a) \( m_2(m) > 0 \) for \( m > 0 \)

b) \( m_1(m) > 0 \) for \( m \geq 0 \)

c) \( 0 < m_1'(m) < 1 \) for \( m \geq 0 \) \( 3/ \)

d) \( 0 < m_2'(m) < 1 \) for \( m \geq 0 \) \( 3/ \)

e) For all \( q \), there exists \( m^+(q) \) such that

\[ m_1(m^+(q), q) = m^+(q). \]

**Proof**: The fact that \( m_1, m_2 \) are single-valued follows from strict concavity of \( v_1, v_2, V \), and the continuity of \( m_1, m_2 \) in \( m \) follows from the strict monotonicity and continuity of \( v_1^i, v_2^i, V_m \):

a) By the envelope theorem

(5) \[ V_m(m) = q v_1^i(m-m_1) + (1 - q) v_2^i(m-m_2) \]

As \( m \to 0, m - m_2 \to 0 \) and \( v_2^i(m - m_2) \to \infty \).

---

\( 3/ \) Where \( m_1', m_2' \) do not exist, c) and d) are to be interpreted as statements about the right and left hand derivatives.
Hence, for $q < 1$,  \[ \lim_{m \to 0} V_m(m) = \infty \]  But then, $m_2(m) = 0$, $m > 0$ implies  \[ v_2'(m-m_2) < \alpha V_m(m_2), \]  so that the first order condition (4) cannot be fulfilled. As long as there is a positive chance of being unemployed, it is worth holding some money over to avoid being caught without money or employment with an infinite marginal utility of consumption. The same argument establishes b), with an extension to the case $m = 0$, since $\lim_{m \to 0} v_1'(m) < \infty$. From results a) and b), it follows that for $q < 1$, $\kappa_1 = \kappa_2 = 0$ for all $m$.

c) Differentiating (3), we get:
\[ v_1''(m-m_1)(dm - dm_1) = \alpha V_{mm}(m_1) dm_1 \]
or:
\[ \frac{dm_1}{dm} = \frac{v_1''(m-m_1)}{v_1''(m-m_1) + \alpha V_{mm}(m_1)} \]
By strict concavity of $v_1$, $V$ we have $v_1'' < 0$, $V_{mm} < 0$ and hence,  \[ 0 < \frac{dm_1}{dm} < 1. \]
Exactly the same argument establishes d).

e) Suppose that there is a $\bar{q}$, such that for this probability $\bar{q}$ of employment, and all $m$: $m_1(m) > m$. From the first order conditions, we know that then, for all $m$:
\[ (6) \quad v_1'(0) < \alpha V_m(m) \]
From the envelope theorem,
(7) \[ V_m(m) = \alpha q V_m(m_1(m)) + \alpha (1 - q) V_m(m_2(m)) \]
\[ < \alpha V_m(m_2(m)), \]
since \( m_1(m) > m \geq m_2(m) \).

Writing \( \bar{m} = m_2(m) \), we have for all \( \bar{m} \):
\[ V_m(\bar{m}) > \alpha^{-1} V_m(m_2^{-1}(\bar{m})). \]

Applying this relation recursively, we have, for all \( m, n \):
\[ V_m(m) > \alpha^{-n} V_m(m_2^{-n}(m)). \]

But for all \( m, n, m_2^{-n}(m) \in [0, \infty) \), so that by (6),
\[ V_m(m_2^{-n}(m)) > \alpha^{-1} v_1'(0). \]

Substituting, we have for all \( m, n \):
\[ V_m(m) > \alpha^{-n-1} v_1'(0). \]

Hence, for all \( m \):
\[ V_m(m) = \infty \]

, in contradiction to proposition 1.

Q.E.D.

Thus, we have proved the following: For a positive probability of unemployment, the agent will never run down his money balances to zero (a), (b)); an increase in initial money balances will be used both to increase present net expenditure and to increase end-of-period holdings (c), (d)). Finally, if initial money balances are high enough, the agent will always have positive net expenditure.

Of these results, only e) carries over to the case where \( q = 1 \). Results a) and b) no longer hold, whereas
c) and d) hold only where the first order conditions (3) and (4) apply with \( \mu_1 = \mu_2 = 0 \); obviously \( \mu_1 > 0 \) or \( \mu_2 > 0 \) imply \( m_1' = 0 \), respectively \( m_2' = 0 \).

For \( q = 1 \), we have:

\[
V_m(m) = \alpha V_m(m_1(m)) + \mu_1.
\]

Suppose that \( m_1(m) \geq m \), for some \( m > 0 \). Since \( \alpha < 1 \) and \( V_m \) is decreasing in \( m \), we must have \( \mu_1 > 0 \), therefore, \( m_1(m) = 0 \), a contradiction. Hence, \( m_1(m) < m \), for all \( m > 0 \). By the monotonicity and nonnegativity of \( m_1(.) \), we must have \( m_1(0) = 0 \). But \( v_1'(0) < \alpha \) and hence, \( V_m(0) < \alpha \). Since \( \alpha < 1 \), we have for \( m = 0 \), \( q = 1 \),

\[
\mu_1 = (1 - \alpha) V_m(0) > 0.
\]

This implies that the nonnegativity constraint on \( m_1 \) is strictly binding. Therefore, there exists a whole interval \( I = [0, \epsilon] \), \( \epsilon > 0 \), such that \( m_1(m) = 0 \) for \( m \in I \).

IV. Dynamic Aspects of the Model

For initial money balances \( m \), one begins the next period with money balances \( m_1(m) \), if one is now employed, \( m_2(m) \), if not. We may thus regard money balances at any moment in time as a random variable the distribution of which depends only on the probability of employment \( q \) and last period's money balan-
ces. The sequence \( \left\{ m_t \right\}_{t=0}^{\infty} \) is a Markov process given by the following rule:

\[
m_{t+1} = \begin{cases} 
  m_1(m_t) & \text{with probability } q \\
  m_2(m_t) & \text{with probability } (1 - q).
\end{cases}
\]

This process has the following properties:

**Proposition 3:** When \( q = 1 \), the agent reaches \( m = 0 \) in finite time and remains there forever.

**Proof:** From the closing paragraph of the preceding section we know that for \( q = 1 \), \( m_{t+1} < m_t \) and that \( m_t \in [0, \varepsilon] \) implies \( m_{t+1} = 0 \in [0, \varepsilon] \).

Being monotone and bounded, the sequence \( m_t \) approaches a limit, say \( a \). Suppose \( a > 0 \). But we also have \( m_1(a) < a \). By continuity of \( m_1(.) \), there exists \( \delta > 0 \), such that \( m_1(a+\delta) < a \). But for \( T \) large enough, \( m_T < a + \delta \); therefore \( m_{T+1} < a \), contradictory to the assumption that \( a \) is a limit to the monotone sequence \( \left\{ m_t \right\}_{t=0}^{\infty} \). Hence, \( a = 0 \). But then, the interval \( [0, \varepsilon] \) is reached in finite time, from which the proposition follows. Q.E.D.

It should be noted that in the other extreme case of \( q = 0 \), we have \( \lim_{m \to 0} V_m(m) = \infty \) and therefore \( m = 0 \) implies \( \mu_2 = 0 \), so that the origin is only reached asymptotically.

The property that the probability distribution of money balances in the far future does not depend on
initial money balances carries over to the general case: There is an ergodic distribution of money balances which is approached as \( t \) becomes large.

**Proposition 4:** For all \( q \), there exists a cumulative distribution \( F^+(\cdot) \), defined on \([0, \infty)\), such that

\[
\lim_{t \to \infty} \text{Prob}(m_t \leq m) = F^+(m),
\]

regardless of initial \( m \). \( F^+(\cdot) \) is the unique distribution function that satisfies the functional equation

\[
(\delta) \quad F^+(m) = q F^+(m_1^{-1}(m)) + (1 - q) F^+(m_2^{-1}(m)).
\]

**Proof:** From proposition 2, we know that for every \( q \), there exists \( m^+ \), such that \( m > m^+ \) implies \( m_1(m) < m \) and \( m_2(m) < m \). All such \( m \) are therefore inessential. Once the agent reaches an \( m > m^+ \), he will never return to balances larger than \( m \).

Hence, for any \( q \), we only need concern ourselves with the interval \([0, m^+(q)]\) in studying the distribution of money balances for large \( t \).

Consider the space \( \mathcal{P} \) of probability distributions on \([0, m^+]\), with the metric

\[
\mathcal{F}(F, G) = \int_0^{m^+} |F - G|
\]

To see that this space is complete, consider any Cauchy sequence converging to a function \( F \), we must show that \( F \) is a probability distribution.

By Helly's Compactness Theorem (Tucker, p.83), such a sequence of probability distributions has a sub-
sequence, which converges pointwise to a distribution in $\mathcal{P}$, say $G$. Clearly, $G$ is a limit of the subsequence as a Cauchy sequence with the metric $\mathcal{G}$. But every subsequence of the original Cauchy sequence converges to the same limit $F$. Hence, $F = G$; therefore, $F \in \mathcal{P}$ and $(\mathcal{P}, \mathcal{G})$ is complete.

Next, consider the mapping $T: \mathcal{P} \rightarrow \mathcal{P}$ given by:

$$TF(m) = q F(m_1^{-1}(m)) + (1 - q) F(m_2^{-1}(m)).$$

If for any period, $F$ gives the distribution of money balances, then $TF$ gives the distribution of money balances in the subsequent period. We show that $T$ is a contraction mapping with respect to the metric $\mathcal{G}$. For any two functions $F, G$ in $\mathcal{P}$, we have

$$\mathcal{G}(TF, TG) = \int_0^{m^+} |TF - TG| \leq q \int_0^{m^+} |F(m_1^{-1}(m)) - G(m_1^{-1}(m))|$$

$$+ (1-q) \int_0^{m^+} |F(m_2^{-1}(m)) - G(m_2^{-1}(m))|$$

For $0 \leq m \leq m_1(0)$, $m_1^{-1}(m)$ does not exist; we may write $F(m_1^{-1}(m)) = G(m_1^{-1}(m)) = 0$. For $m_2(m^+) < m \leq m^+$, $m_2^{-1}(m)$ does not exist in $[0, m^+]$. However, if we imbed $\mathcal{P}$ in the space of probability distributions on $[0, \infty)$, $m_2^{-1}(m)$ for such $m$ exists in this larger interval, and, for $F, G$ in $\mathcal{P}$, we have $F(m_2^{-1}(m)) = G(m_2^{-1}(m)) = 1$, where $m_2(m^+) < m \leq m^+$.

Hence, we may write:
\[ j(TF, TG) \leq q \int_{m_1(0)}^{m_1^+} \left| F(m_1^{-1}(m)) - G(m_1^{-1}(m)) \right| m_1'(m) + (1 - q) \int_{m_2(m^+)}^{m^+} \left| F(m_2^{-1}(m)) - G(m_2^{-1}(m)) \right| m_2'(m) \]

\[ = q \int_{0}^{m^+} |F - G| m_1'(m) + (1 - q) \int_{0}^{m^+} |F - G| m_2'(m) \]

\[ \leq (1 - c) j(F, G), \]

where \( c > 0 \) is chosen so that \( 1 - c \geq \max(m_1'(m), m_2'(m)) \)

for all \( m \in [0, m^+] \), which is possible since \( m_1' \) and \( m_2' \) are smaller than one, and \( [0, m^+] \) is a compact interval.

Hence, \( T \) is a contraction mapping. By Banach's Fixed Point Theorem (Kolmogorov - Fomin, p.67), it has a unique fixed point and any sequence of functions \( \{ F_n \} \),

where \( F_n = T^n F_0 \) for some \( F_0 \), approaches that fixed point irrespective of the initial function \( F_0 \). The proposition immediately follows. Q.E.D.

**Corollary:** The expected value of money balances under the ergodic distribution is given by the equation

\[ E(m F^+) = q E(m_1(m) F^+) + (1 - q) E(m_2(m) F^+) . \]

**Proof:** By definition:

\[ E(m F^+) = \int_{0}^{m^+} m dF^+(m) \]

\[ = q \int_{0}^{m^+} m dF^+(m_1^{-1}(m)) + (1 - q) \int_{0}^{m^+} m dF^+(m_2^{-1}(m)) \]

\[ = q \int_{0}^{m^+} m_1(m) dF^+(m) + (1 - q) \int_{0}^{m^+} m_2(m) dF^+(m) \]
V. Economic Behaviour and the Probability of Being Employed

In this section, we first analyse some effects of changes in \( q \) on the ergodic distribution. In general, we will suppress the dependence of functions on \( q \), but when necessary, we write \( m_1(m; q) \), \( F^+(m; q) \) etc.

**Proposition 5:** a) If there is certainty about the prospects for employment, the stationary distribution is concentrated at zero money balances:

\[
F^+(0) = 1
\]

b) If there is uncertainty about the prospects for employment, that is, if \( 0 < q < 1 \), then money balances will almost never be run down to zero, that is:

\[
F^+(0) = 0
\]

**Proof:** a) The case \( q = 1 \) has been dealt with in proposition 3. In the case \( q = 0 \), we have:

\[
F^+(m) = F^+(m_2^{-1}(m)).
\]

Also, \( F^+(m^+(0)) = 1 \), and the sequence \( \left\{ m_2^n(m^+(0)) \right\} \) is strictly decreasing and goes to zero, where \( m_2^n \) is the \( n \)-fold composition of \( m_2 \). Thus, because of the monotonicity of the distribution function \( F^+ \), \( F^+(m) = 1 \) for all \( m > 0 \). By continuity to the right of \( F^+ \), this implies \( F^+(0) = 1 \).

b) By proposition 4, we have
\[ F^+(0) = q F^+(m_1^{-1}(0)) + (1-q) F^+(m_2^{-1}(0)). \]

For \( 0 < q < 1 \), \( m_1^{-1}(0) \) does not exist and \( m_2^{-1}(0) = 0 \).

Hence, \( F^+(0) = (1-q) F^+(0) \), so that \( F^+(0) = 0 \).

Q.E.D.

It should be noted that the discontinuity of \( F^+(0) \) in \( q \) at \( q = 1 \) and \( q = 0 \) does not arise from a discontinuity of the function \( F^+() \) itself at these points. In fact, as \( q \) approaches one or zero, the stationary distribution converges to the atomic distribution on \( m = 0 \).

This involves two things: First, probability mass becomes more and more concentrated close to the origin, i.e. high money holdings will become more and more improbable. Then, it also has to be shown that the limit of this process is the origin itself, that is to say that any positive holdings of money become improbable, if not excluded as \( q \) approaches either 0 or 1.

As \( q \) approaches one, this is shown, if the essential interval itself shrinks and if \( m^+ \) approaches the origin. In view of proposition 2, this is not trivial, since one has to show that \( m_1(0; q) \) is continuous at the point \( q = 1 \), even though \( V_m(0; q) \) is not. It is also not covered by the continuity proofs in the appendix, since we deal with the boundary of the region for which those proofs are valid.
At the other extreme, as \( q \) approaches zero, the situation is somewhat different, because here the essential interval will not vanish. Instead, we have to show directly that it becomes less and less probable for money balances to increase. For this reason, proposition 6 will be somewhat weaker for the case \( q \rightarrow 0 \) than for \( q \rightarrow 1 \).

**Proposition 6:** a) For any \( m > 0 \), there exists \( \delta > 0 \) such that \( |1-q| < \delta \) implies \( F^+(m; q) = 1 \).

b) For any \( m > 0, \varepsilon > 0, \) there exists \( \eta > 0 \) such that \( q < \eta \) implies \( F^+(m; q) > 1 - \varepsilon \).

**Proof:** a) We first show that for all \( m \),

\[
\lim_{q \rightarrow 1} m_1(m; q) \leq m,
\]

with strict inequality holding for \( m > 0 \). From equation (7), we have as \( q \rightarrow 1 \):\(^4/\)

\[
\lim_{q \rightarrow 1} V_m(m; q) = \alpha \lim_{q \rightarrow 1} q \cdot V_m(m_1(m); q)
\]

\[+ \alpha \lim_{q \rightarrow 1} (1-q) V_m(m_2(m); q)\]

The second term on the right hand side of this equation vanishes, unless \( V_m(m_2(m); q) \) were to grow out of bounds as \( q \) approaches 1. This in turn would require

---

\(^4/\) From proposition 2 a), b), we know that \( q < 1 \) implies \( m_1(m) > 0, m_2(m) > 0 \) for all \( m \), and hence

\[
\lim_{q \rightarrow 1} \mu_1 = \lim_{q \rightarrow 1} \mu_2 = 0.
\]
\[ \lim_{q \to 1} m_2(m; q) = 0. \]

But then, for \( m > 0, \) \( \alpha \lim_{q \to 1} V_m(0) > v_2'(m), \) in contradiction to the first order condition (4). Hence, we have, for \( m > 0, \)

\[ \lim_{q \to 1} V_m(m; q) = \alpha \lim_{q \to 1} V_m(m_1(m); q). \]

Since \( \alpha < 1 \) and \( V_m \) decreases in \( m, \) we must have

\[ \lim_{q \to 1} m_1(m) < m \text{ for all } m > 0. \]

From the monotonicity and nonnegativity of \( m_1(\cdot), \) it follows that \( \lim_{q \to 1} m_1(0) = 0. \)

The proposition follows immediately, because for \( q \) close enough to 1, \( m_1(m; q) < m, \) and \( m \) is an inessential state of the stochastic process so that \( F^+(m; q) = 1. \)

b) From equation (8), we have

\[ F^+(m) = q F^+(m_1^{-1}(m)) + (1-q) F^+(m_2^{-1}(m)) \]

\[ \geq (1-q) F^+(m_2^{-1}(m)). \]

Therefore, for all \( n, \)

\[ F^+(m) \geq (1-q)^n F^+(m_2^{-n}(m)), \]

where \( m_2^{-n}(\cdot) \) designates the \( n \)-fold inverse of the function \( m_2(\cdot). \)

Also, from proposition 2 e), we know that for all \( q \) there exists \( m^+(q) \) with \( m_1(m^+) = m^+ \) and, from proposition 4, \( F^+(m^+(q); q) = 1. \)

In the appendix, we show that \( m_1(\cdot) \) is continuous
in q; therefore \( m^+ \) is continuous in q and we can find \( \max m^+ \) as q varies over the compact interval \([0, 1]\).

Call this \( m^{++} \). Also, define a function \( f(\cdot) \), such that

\[
f(m) = \max_q m_2(m; q).
\]

Existence of this function follows from the continuity of the function \( m_2(\cdot) \) in q, which again is shown in the appendix. Clearly, \( f(\cdot) \) is continuous in m, with \( 0 < f' < 1 \). For every m, then, we can find n such that \( f^{-n}(m) \geq m^{++} \). By definition of the function f, we have, for all q, \( m_2^{-n}(m; q) \geq f^{-n}(m) \geq m^{++} \). Also, by definition of \( m^{++} \), \( F^+(m^{++}; q) = 1 \) for all q.

By monotonicity of \( F^+ \), we have for all q,

\[
F^+(m_2^{-n}(m)) = 1.
\]

Thus, for all \( m > 0 \), there exists n, such that for all q,

\[
F^+(m) \geq (1-q)^n.
\]

Clearly, we can choose q small enough so that

\[
F^+(m) \geq 1 - \varepsilon.
\]

Q.E.D.

We have thus proved that not only does total certainty on the prospects of employment lead to zero money holdings in the long run, but that furthermore in the proximity of the two certainty points, an increase in uncertainty leads to an increase in expected long run money holdings. It would be tempting to conclude that the maximum of expected long run money holdings
concurs with the point of maximum uncertainty in the statistical sense, where \( q = 1/2 \). However, this is in general not true. The point of maximum expected long run money holdings will generally depend on the utility function, notably its third derivatives.

We therefore propose a different interpretation of our results in terms of \textit{willingness} and \textit{ability} to hold and acquire money. At the one extreme of certain employment, the agent is always able to acquire more money if only he wants to. But given his time preference and the fact that money earns him no interest, he never wants to and even runs down whatever balances he starts out with. At the other extreme of certain unemployment, the agent is always willing to acquire more money, if only he could do so. But he never is able to. Again time preference and the fact that he has no income whatsoever induce him to run down his initial balances.

It is clear that what we call "ability" to acquire money varies monotonically with the probability of employment: the more likely the agent is to be employed, the more chances he has of earning money, from which to increase his holdings.

As the probability of employment varies, variations in the willingness to hold money can be analysed either through variations in the marginal utili-
lity of money or through variations in the end-of-period money balances \( m_1 \) and \( m_2 \). To show that these variations are monotone in \( q \), we formulate the following equivalent propositions:

**Proposition 7:** For all \( m \in (0, m^+] \), both employed and unemployed end-of-period money holdings decrease as the probability of employment \( q \) increases. The same holds for \( m_1(0; q) \).

**Proposition 8:** For all \( m \in (0, m^+] \) and all \( q_1, q_2 \), \( q_1 < q_2 \) implies \( V_m(m; q_1) > V_m(m; q_2) \).

**Proof:** Equivalence of the two propositions follows immediately from the first order conditions (3) and (4). In fact, where the derivatives of \( m_1 \) and \( m_2 \) with respect to \( q \) exist, we have, by total differentiation of (3) and (4):

\[
\frac{dm_1}{dq} = -\frac{\alpha V_{mq}(m_1; q)}{v_1'' + \alpha V_{mm}(m_1; q)}
\]

\[
\frac{dm_2}{dq} = -\frac{\alpha V_{mq}(m_2; q)}{v_2'' + \alpha V_{mm}(m_2; q)}
\]

Hence, it is sufficient to prove proposition (8). From lemma 5 in the appendix, both \( m_1 \) and \( m_2 \) are continuous in \( q \). Then \( V_m \) is continuous in \( q \). For \( m > 0 \), \( V_m \) is of bounded variation as \( q \) varies over the compact interval \([0, 1]\). Hence, \( V_m \) is almost everywhere
differentiable with respect to \( q \).

It is now sufficient to show that \( V_{mq} \) is negative wherever it exists. Differentiating equation (7) with respect to \( q \), we have:

\[
V_{mq}(m; q) = \alpha (V_m(m_1; q) - V_m(m_2; q)) \\
+ \alpha (q V_{mq}(m_1; q) + (1-q) V_{mq}(m_2; q)) \\
+ \alpha (q V_{mm}(m_1; q) \frac{dm_1}{da} + (1-q) V_{mm}(m_2; q) \frac{dm_2}{da})
\]

\[
= \alpha (V_m(m_1; q) - V_m(m_2; q)) \\
+ \alpha q V_{mq}(m_1; q) \frac{v''(m-m_1)}{v_1'' + \alpha V_{mm}(m_1; q)} \\
+ \alpha (1-q) V_{mq}(m_2; q) \frac{v''(m-m_2)}{v_2'' + \alpha V_{mm}(m_2; q)}
\]

For given \( q \), consider the supremum of \( V_{mq} \) over the interval \([0, m^+]\). Suppressing \( q \) in the arguments, we have:

\[
\sup_{[0,m]} V_{mq}(m) = \alpha (V_m(m_1(\tilde{m})) - V_m(m_2(\tilde{m}))) \\
+ \alpha (q V_{mq}(m_1(\tilde{m})) \frac{dm_1}{dm} + (1-q) V_{mq}(m_2(\tilde{m})) \frac{dm_2}{dm})
\]

\[
\leq \alpha (V_m(m_1(\tilde{m})) - V_m(m_2(\tilde{m}))) \\
+ \alpha (q \frac{dm_1}{dm} + (1-q) \frac{dm_2}{dm}) \sup_{[0,m]} V_{mq}(m)
\]

where \( \tilde{m} \) is the value for which \( V_{mq} \) achieves its supre-

---

5/ Kolmogorov - Fomin, p.331.
mom on \( [0, m^+] \). Rearranging terms, we have:

\[
(12) \sup_{[0, m^+]} V_{mq}(m) \leq \frac{\alpha}{1 - \alpha \left( \frac{dm_1}{dm} + \frac{dm_2}{dm} \right)} \left( V_m(\bar{m}_1) - V_m(\bar{m}_2) \right)
\]

But on the interval \( [0, m^+] \), we have \( m_1 \geq m \geq m_2 \), so that the right hand side of this inequality is negative. Hence, on the interval \( [0, m^+] \), \( V_{mq} \) is always negative.

Q.E.D.

It should be noted that the restriction to the essential interval is not always needed. For the use of the supremum above, any closed interval containing the essential interval will do. The use of \( m^+ \) was important only in that it ensured the negativity of the right hand side of (12). Any other condition, which ensures that \( m_1 \geq m_2 \), would serve the same purpose. For instance, if we knew that for all \( e \), \( v'_1(e) < v'_2(e) \), we could derive proposition 8 for all \( m \). Unfortunately, this does not appear to be true for all utility functions and depends on the complementarity of consumption and labour.

VI. The Consumption Function and Liquidity Preference

The consumption function and its corollary the multiplier are central puzzles in the problem of relating macroeconomic ideas to microeconomics. Why should current income constrain current spending in an economy
where agents command liquid assets? Both the life-cycle hypothesis of Modigliani and his associates and Friedman's permanent income hypothesis suggest that current income should affect spending only to the extent that it affects lifetime resources. Furthermore, to the extent that a change in current income is anticipated, it will not affect spending plans at all. On the other hand, Clower suggests that the consumption function has to be regarded as a short term liquidity phenomenon, the theoretical treatment of which requires a fundamental separation of earning and spending decisions.

The present model begins to illuminate this puzzle as well as the relation between consumption and the demand for liquid assets through its treatment of the budget constraint for sequential trading.

The preceding propositions establish two separate influences of unemployment on consumption, for given money balances m. First, the actually unemployed agent will consume $c_2(m)$ rather than $c_1(m)$. This effect corresponds in spirit to Clower's distinction between notional and effective demand. Notice that neither $c_1(m)$ nor $c_2(m)$ is really "notional" demand in the sense of the demand the agent would have if trading uncertainty did not exist at all and he could always sell as much labour as he wanted at the going rate. In that kind of world, $q = 1$, and money balances would in
the long run play no part. Notice, too that $c_2(m) < c_1(m)$ as long as consumption is not an inferior good; but this is not a result of any mechanical application of a "money constraint". There is only one constraint, that the agent never run out of money. Still, the fact that $c_2$ is smaller than $c_1$ does reflect the agent's response to his immediate failure to sell labour in the light of the necessity to use money as a means of payment, which is close in spirit to Clower's idea.

Second, since net spending during the period will increase with the probability of employment $q$, both in the employed and in the unemployed case, if we assume that neither consumption nor leisure are inferior goods, it follows that in all cases consumption will increase with the probability of employment. This change in spending occurs as the change in the estimated employment probability alters the agent's estimate of his total lifetime resources. It is thus akin to a change in expected permanent income.  

6/ The analytical relationship between $q$ and permanent income is somewhat cumbersome: Since there is no discounting in the model, which would make earlier earnings more important, we may evaluate expected permanent income from the ergodic distribution:

$$y_p = q w \int l(m;q) \, dF^+(m).$$
Both these effects on consumption work in the same direction. On the other hand, we have seen that the corresponding effects on money held over into the next period work in opposite directions. If widespread unemployment occurs without any change in expectations about individual probabilities of employment, then agents will dishoard to maintain their consumption (Proposition 2). If subjective estimates of the chance of unemployment increase, then agents will hoard to accumulate money (Proposition 7). For any short period of time, this provides an observable distinction between short term liquidity and long term behavioural aspects of consumption theory. Unfortunately, if the expectation of unemployment is linked to the actual experience, then over any longer period, this distinction disappears because the change of expected money holdings is indeterminate. It is probably this indeterminacy of the long run relationship between expected money holdings and expected consumption, which has diverted attention away from the interdependence of the consumption function and money demand (liquidity preference). However, the preceding analysis makes clear

Permanent income is monotone in q, if the indirect effects of q on l(m;q) and F^+(m;q) do not overcompensate for the direct effect on y_p. With constant labour supply I, we have y_p = q w I.
that it is not the difference between notional and effective demand that relates to liquidity preference, but rather the impact of different expectations on long run behaviour.

It may be argued that since our model has only one asset, it is a model of saving rather than of preference for liquid assets. However, the motive for saving is precautionary and therefore closely related to the motive for holding liquid assets. As in the life-cycle model, the consumer saves when he is employed for periods when he earns nothing. But whereas in the life-cycle model, the consumer knows the exact timing of periods in which he earns and periods in which he does not earn, in this model he knows nothing about the timing and only a probabilistic statement about the overall incidence of non-earning periods is possible. There is no return to holding the asset and, in the absence of uncertainty, asset holdings will be zero in the long run. In this sense, we think that the type of saving that occurs in this model may be considered as a case of liquidity preference.
VII. Different Employment Prospects for Employed and Unemployed Agents

It is quite unrealistic to assume that agents enter the job market each period anew with new uncertainty about their employment prospects. However, the foregoing model extends easily to the case where employment prospects differ according to whether one is presently employed or not. Let \( q_1 \) be the probability of being employed in the next period, if one is now employed, \( q_2 \), if one is out of work, with \( q_1 > q_2 \). Then, the state of an agent at the beginning of a period depends not only on his money balances \( m \), but also on whether or not not he was employed in the preceding period. The values of optimal programs are then given by a pair of functions \( V(m;1) \), \( V(m;2) \), satisfying the functional equations:

\[
(2^+a) \quad V(m; 1) = q_1 \max_{m_1} (v_1(m-m_1) + \alpha V(m_1; 1)) \\
\quad \quad \quad + (1-q_1) \max_{m_2} (v_2(m-m_2) + \alpha V(m_2; 2))
\]

\[
(2^+b) \quad V(m; 2) = q_2 \max_{m_1} (v_1(m-m_1) + \alpha V(m_1; 1)) \\
\quad \quad \quad + (1-q_2) \max_{m_2} (v_2(m-m_2) + \alpha V(m_2; 2))
\]

Since the convergence argument in the proof of proposition 1 in the appendix is uniform in \( q \), it can be shown along exactly the same lines that the functions
V(m; 1), V(m; 2) fulfilling the functional equations (2^+a) and (2^+b) exist and exhibit the same properties as the function V in proposition 1.

One notes that the maximizations in (2^+a) and (2^+b) are the same. The policy functions m_1 and m_2 are therefore independent of initial prospects. This is due to the fact that when the agent decides on net expenditure in the current period, he already knows with certainty whether he is presently employed or not. At that moment, the initial difference in employment prospects according to past employment is already a thing of the past and cannot affect the present decision on net expenditure.

The first order conditions for maximization are:

\[ (3^+a) \quad v_1'(m-m_1) = \alpha V_m(m_1; 1) + \mu_1 \]
\[ (4^+a) \quad v_2'(m-m_2) = \alpha V_m(m_2; 2) + \mu_2 \]

The envelope theorem gives the following expressions for the marginal utilities of money:

\[ (5^+a) \quad V_m(m; 1) = q_1 v_1'(m-m_1) + (1-q_1) v_2'(m-m_2) \]
\[ (7^+a) \quad = \alpha q_1 V_m(m_1; 1) + \alpha (1-q_1) V_m(m_2; 2) \]
\[ (5^+b) \quad V_m(m; 2) = q_2 v_1'(m-m_1) + (1-q_2) v_2'(m-m_2) \]
\[ (7^+b) \quad = \alpha q_2 V_m(m_1; 1) + (1-q_2) \alpha V_m(m_2; 2) \]

By investigation of the structure of these equations, we see immediately that all the results of section III carry over to the present case. Furthermore,
we notice that there exists no single probability $\bar{q}$, which, when expected independent of current employ-
ment, gives rise to the same set of consumption se-
quences as the differing probabilities $q_1$ and $q_2$. For suppose $\bar{q}$ existed. Then, by $(3^+)$ and $(4^+)$,
$V_m(m_1; 1) = V_m(m_1; \bar{q})$ and $V_m(m_2; 2) = V_m(m_2; \bar{q})$.
Substituting from (5) and $(5^+a)$ into the first of these
equations, we find that $q_1 = \bar{q}$, since, by assumption,
subsequent policies $m_1$, $m_2$ are the same under both
regimes. Similarly, substitution from (5) and $(5^+b)$
into the second equation gives $q_2 = \bar{q}$, a contradiction.
Hence, although the policy functions $m_1$, $m_2$ are inde-
dependent of whether one has begun the period with ini-
tial employment prospects $q_1$ or $q_2$, yet these policy
functions cannot be simulated by any single-probability
regime.

The results on the distribution of money balances
and its evolution over time carry over to the present
model as well. Note, however, that in this case, the
probability of being employed at time $t$ is not constant,
but depends on $t$ and on the initial probability of em-
ployment. In fact, we have:

$$q^t = q_1 q^{t-1} + q_2 (1 - q^{t-1}),$$

where the initial probability of employment $q^0$ is given
as either $q_1$ or $q_2$.

Employment probabilities thus evolve in an irredu-
cible, aperiodic Markov chain. It is well known (Feller, p. 393) that they approach a stationary probability \( q' \), given as

\[
(\delta^+ a) \quad q' = \frac{q_2}{1 - q_1 + q_2}.
\]

At the same time, for any distribution of money balances \( F_t \), the subsequent period's distribution is given as:

\[
T_t F_t(m) = q^t F_t(m_1^{-1}(m)) + (1 - q^t) F_t(m_2^{-1}(m))
\]

Examination of the first order conditions \((3^+)\) and \((4^+)\) shows that the slope of the policy functions \( m_1 \), \( m_2 \) is always strictly less than 1 (analogue of proposition 2). Since the argument in the proof of proposition 4 is independent of the probability \( q \), that proof carries over immediately to show that for all \( t \), the mapping \( T_t \) is a contraction mapping with the uniform contraction factor \( 1 - c \geq \max(m_1'(m), m_2'(m)) \), for all \( m \) in the essential interval on which \( m_1(m) \geq m \).

However, since the mappings \( T_t \) for subsequent distributions vary with \( t \), we have to introduce one further step, before we can apply Banach's Fixed Point Theorem.

For probabilities \( q \) and distribution functions \( F \) on the positive halfline, consider the space of ordered pairs \((q, F(.)\) ) with the metric
\[ \mathcal{S}^+(q, F; p, G) = \max(|q-p|, \mathcal{S}(F, G)), \]
where \[ \mathcal{S}(F, G) = \int |F - G| \]
as in section IV.

Now if present employment probability and money balance distribution are \( q, F \), then next period's employment probability and money balance distribution are given by the transformation

\[
T(q; F(.)) = \left( (qq_1 + (1-q)q_2); (qF(m_1^{-1}(.)) + (1-q)F(m_2^{-1}(.)) \right).
\]

This transformation is independent of the time at which it is applied. From the preceding discussion, it is clear that it is a contraction mapping. 7/

Hence, by Banach's Fixed Point Theorem, the mapping \( T \) has a unique fixed point. The first coordinate of this fixed point is given by \((8^+a)\). The stationary distribution of money balances obeys the functional equation

\[
(8^+b) \quad F^+(m) = q^i F^+(m_1^{-1}(m)) + (1-q^i) F^+(m_2^{-1}(m)).
\]

Finally, one notes that the results of section V apply without any change in the proofs.

---

7/ We remarked above that \( \mathcal{S}(T_2F, T_2G) \leq (1-c) \mathcal{S}(F, G) \), where \( T_2 \) for any \( q \) is the projection of \( T \) on the subspace of distributions on the positive halfline. Furthermore, \[ |qq_1 + (1-q)q_2 - pq_1 - (1-p)q_2| = |(q_1 - q_2)(q - p)| \leq (1-d)|q - p|, \] where \( 0 < d \leq 1 - \max(q_1, q_2) \)
VIII. Aggregate Demand and Market Information

We have seen that unemployment affects the demand for consumption both through a short run liquidity and through a long run behavioural effect. It has been suggested by Leijonhufvud and others that these effects impair the ability of markets to adjust from a disequilibrium position. Within the static context of our model, we want to make this notion more explicit by considering the behaviour of market aggregates.

Suppose that there are two commodities and two types of agents, a continuum of each. Each type supplies one good inelastically and buys the other good. Market conditions are characterized by four variables, namely, the prices of the two commodities, \( p_1 \) and \( p_2 \), and the probabilities that suppliers are able to sell the commodities at their given market price, \( q \) and \( s \) respectively.\(^8\)

\[
\mathfrak{s}(T(q; F(.)); T(p; G(.))) \leq (1-e)\mathfrak{s}((q,F);(p,G)),
\]

where \( 0 < e \leq \min(c,d) \).

\(^8\) We depart from the labour/consumption good setting, because the introduction of firms on the other side of the markets requires a more complicated treatment, which either analyses the payout of profits to stockholders or introduces a zero profit condi-
Furthermore, let money balances for agents of type one be distributed according to the function $F_1$. Then, if actual incidence of unemployment among agents of type one is independent of their money holdings and corresponds to their expectations, the total effective demand for good one is:

$$q \int_0^\infty x_1(m;q) dF_1(m) + (1-q) \int_0^\infty x_2(m;q) dF_1(m),$$

where $x_1(m;q)$, $x_2(m;q)$ are the demands for good one with money balances $m$ and employment prospects $q$, if the agent is, respectively is not able to sell good two.

Likewise, for type two the effective demand for good two will be:

$$s \int_0^\infty y_1(m;s) dF_2(m) + (1-s) \int_0^\infty y_2(m;q) dF_2(m),$$

where $y_1(m;s)$, $y_2(m;s)$ are the demands for good two with money balances $m$ and sales prospects $s$, if the agent is, respectively is not able to sell good one. $F_2$ is the distribution of money balances among agents of type two.

If these effective demands are actually realized,
then money balances of type one in the subsequent period are distributed as:

\[ F_{1}^{t+1}(m) = qF_{1}^{t}(m_{11}(m)) + (1-q)F_{1}^{t}(m_{12}(m)) \]

and likewise for type two, where the first subscript distinguishes the type of consumer and the second the contingency.

We see that this transformation is identical to the transformation T used in section IV to describe the distribution of money balances \( t+1 \) periods from the point of initial decision making. But where in that case, \( F \) was a distribution of money balances as expected for some future period, in the present context, it describes the actual distribution over different agents of one type.

If \( F_{1} \) happens to be the ergodic distribution for type one, then money balances for type one in subsequent periods are identically distributed. Hence, as a group, agents of type one are in this case neither accumulating nor decumulating money, although as individuals, their balances are constantly changing. In this case, the value of their spending on good one as a group will be equal to their achieved sales of good two:

\[ p_{1} \int_{0}^{m_{1}^{+}} (x_{1}(m;q) q + x_{2}(m;q)(1-q)) dF_{1}(m) = q p_{2} \bar{y} \]

Similarly, for type two:
\[ p_2 \int_0^{m_2^+} (s y_1(m; s) + (1-s)y_2(m; s)) \, dF_2^+(m) = sp_1 \bar{x} \]

If achieved sales by one type are to be equal to the effective demand by the other type, we get the pseudo-equilibrium condition:

\[ q p_2 \bar{y} = sp_1 \bar{x} \]

For such a pseudo-equilibrium, the total demand for money is given as:

\[ M = \int_0^{m_1^+} mdF_1^+(m) + \int_0^{m_2^+} mdF_2^+(m) \, . \]

For any choice of \( q, s \), there exist \( p_1, p_2 \) such that at these prices and sales prospects the market is in a pseudo-equilibrium and aggregate demand for any commodity is equal to expected achievable sales. Different choices of \( q, s \) result in different holdings of money balances.

There is nothing in the formalism of this model to suggest that the competitive equilibrium with \( q = s = 1 \) and zero money balances plays any special part. In fact, it is seen that any situation, in which \( q = s \), has a pseudo-equilibrium at the same relative price as the competitive equilibrium. This formalizes Leijonhufvud's notion that the reduction of effective demand by laid-off workers may be just large enough to justify the reduced production by firms and to prevent a return to equilibrium.
Of course, these considerations are subject to severe qualifications. The model says nothing about how prices are set and sales prospects are generated. Without an explicit dynamic structure, which analyses a process, in which both price formation and expectations of sales are endogenous, one should not draw any detailed conclusions from the above model. However, if we assume that the change of the money price in any market will be of the same sign as the effective excess demand, then the above model allows us to imagine situations, in which all the money prices decrease. A priori there is then no presumption that the relative price moves closer to its equilibrium value. In fact, one could imagine processes, in which relative prices remain constant, even though money prices change. In such a situation, the only adjustment towards equilibrium would be provided by a real balance effect, arising from the change in the demand for money as money prices decline.

9/ A more technical problem of the logic of market coordination is given by the fact that in the above model, both types make their decision on effective demand when they know their sales, which however depend on the other type's effective demand, which in turn depends on the other type's sales and hence, on their own effective demand, which is yet to be
Another phenomenon worth considering is the role of expectations in this model. We mentioned before that the sales prospects as arguments of the behavioural functions have to be interpreted as expectations. In the simple model above, such expectations are, in general, self-fulfilling in that a decreased estimate of future sales will decrease present demand, which reduces the sales of other market participants, hence, their effective demand and thus one's own future sales. This is similar to the phenomenon of self-fulfilling expectations in the theory of speculation, where the expectation of a price rise by a sufficiently large group of agents will in fact cause that price to rise. In the model above, all agents of one type have the same expectation about their sales prospects. It appears desirable to include in a dynamic model an account of how expectations are generated and to what extent they will be independent for different agents of a given type, or to what extent they will be homogeneous for agents of one type.

determined. This problem is easily solved in a dynamic model with inventory holding and decisions of end type based on last period's experience.
IX. Conclusion

The present model has certain formal similarities to recent work in stochastic growth theory. Iwai and Brock and Mirman analyse optimal growth of an economy in which production in each period depends not only on capital and labour inputs, but also on a random factor. They show that the distribution of per capita capital and output approaches a stationary distribution, which may be interpreted as the equivalent of the consumption turnpikes in optimal growth under certainty.

The present paper differs from their work in two respects. On the one hand, the proof of ergodicity of the distribution of money balances is simpler and does not make use of kernels, but uses a contraction mapping argument. More important is the fact that if we interpret our model in the context of price or production uncertainty, it is limited to two-point distributions, where one of the two points is the origin. Under the assumption that $\lim_{\alpha \to 0} u_c = \phi$, this class of distributions allows more specific propositions about optimal behaviour. In that sense, we believe that trading uncertainty, although it may be regarded formally as a special case of price or production uncertainty, merits a separate treatment, which we believe will throw new light on certain problems of disequilibrium adjustment.
The most important aspect of this line of reasoning lies, in our opinion, more in the imagery of its description of economic activity than in its mathematics. We can summarize the crucial element as follows.

First, there is a residual and unhedgeable uncertainty as to an agent's opportunities to buy and sell. Second, the agent confronts these opportunities in a sequence of decisions, not all at once at some initial trading session. Third, the agent sees his budget constraint not in terms of current expenditures equalling current income, nor as the present value of lifetime expenditures equalling the present value of lifetime income, but as never being able to incur negative money balances.

The conventional theory of consumer choice bears to this theory the same kind of relation that the theory of a "perfect" gas bears to that of actual gases. It is an asymptotic or polar case in which trading uncertainty is ignored. It can be derived from the present model by setting \( q = 1 \), that is, by eliminating

10/ The same can be said in a comparison of this model with the work of Merton, who introduces uncertainty about wage income into his model of consumption and portfolio selection and shows a general analogy between wage income uncertainty and asset return uncertainty.
trading uncertainty. Notice that the introduction of trading uncertainty implies certain qualitative changes in behaviour, such as the willingness to hold money, which do not disappear even when the randomness itself has been eliminated by aggregation.

The present work is clearly only a first step in applying the three principles we have just mentioned to problems of economic theory. It should be possible to include more than one asset by assuming that the alternative asset to money is not perfectly liquid in that it cannot always be sold at a known money price, just as in the present model, labour cannot always be sold. An attractive feature of the present model that we believe will carry over to more general models is that the distribution of assets arises in a natural way in the course of explaining aggregate phenomena. The relation between distributions of money and propositions reminiscent of Say's Law or Walras' Law seems particularly interesting.

At the present stage of development this line of research offers no complete solutions to the pressing problems of general equilibrium theory, particularly the need to explain the connected phenomena of price formation, money holding and unemployment of resources in a general equilibrium framework. The present work does touch on the last two of three topics from
the point of view of the individual agent. What is missing is any consideration of decentralized price formation in a sequential trading framework. It seems very desirable to us that solutions to these problems reflect one of the properties of the present model, that is that the conventional theory be derivable from the general theory as an asymptotic case when some cost goes to zero.
Appendix: Existence and Properties of the Optimal Policy

In this appendix, we prove proposition 1 by induction on the sequence of finite horizon problems with the same constraints and objective function.

Consider the sequence of functions:

\[ V^0(m) = q \, v_1(m) + (1-q) \, v_2(m) \]

\[ V^T(m) = q \, \max_{m_1^T} (v_1(m-m_1^T) + \alpha \, V^{T-1}(m_1^T)) + (1-q) \, \max_{m_2^T} (v_2(m-m_2^T) + \alpha \, V^{T-1}(m_2^T)) \]

By inspection, \( V^T \) is the value and \( m_1^T, m_2^T \) the policies of the T-period finite horizon problem of the same form as (1).

**Lemma 1:** Under assumptions a), b), for every T a unique \( V^T \) exists satisfying the above definition, which is strictly increasing, strictly concave and differentiable in m. The derivative satisfies:

\[ V^T_m(m) = q \, v_1^1(m-m_1^T(m)) + (1-q) \, v_2^1(m-m_2^T(m)). \]

**Proof:** \( V^0 \) clearly satisfies all the claims of the lemma by assumptions a), b). To make a proof by mathematical induction, assume that \( V^{T-1} \) satisfies all the claims.

\( \bar{V}^T(m) \) exists and is unique since it is the maximum value of a continuous function over a compact set.

It is strictly increasing because \( v_1, v_2 \) are
strictly increasing, so a strictly higher value can be achieved by spending more in the first period in each contingency, even holding the rest of the program constant.

$V^T(m)$ is strictly concave since it is the sum of maxima of strictly concave functions over convex sets.

The $m_1^T, m_2^T$ satisfy the first order conditions

$$v_1'(m-m_1^T) = \alpha V_m^{T-1}(m_1^T) + \mu_1; \mu_1 m_1^T = 0; \mu_1 \geq 0$$

$$v_2'(m-m_2^T) = \alpha V_m^{T-1}(m_2^T) + \mu_2; \mu_2 m_2^T = 0; \mu_2 \geq 0$$

Since $v_1, v_2$ and $V^{T-1}$ are everywhere differentiable and strictly concave in $m$ the $m_1^T, m_2^T$ functions are unique and continuous in $m$.

We can write:

$$V^T(m+h) - V^T(m) = q(v_1(m+h-m_1^T(m+h)) + \alpha V_m^{T-1}(m_1^T(m+h)) - v_1(m-m_1^T(m)) - \alpha V_m^{T-1}(m_1^T(m)))$$

$$+ (1-q)(\ldots).$$

The first summand gives:

$$v_1(m+h-m_1^T(m+h))-v_1(m-m_1^T(m))+\alpha V_m^{T-1}(m_1^T(m+h))-\alpha V_m^{T-1}(m_1^T)$$

$$= v_1'(\bar{e})(h-(m_1^T(m+h)-m_1^T(m)) + \alpha(m_1^T(m+h)-m_1^T(m))V_m^{T-1}(\bar{m})$$

$$= v_1'(\bar{e})h + (m_1^T(m+h)-m_1^T(m))(\alpha V_m^{T-1}(\bar{m}) - v_1'(\bar{e}))$$

by the mean value theorem, where $\bar{m}_1 \in [m_1^T(m), m_1^T(m+h)]$ and $\bar{e} \in [m-m_1^T(m), m+h-m_1^T(m+h)]$.

As $h$ approaches zero, $(m_1^T(m+h)-m_1^T(m))/h$ is bounded
between the right and left hand derivatives of \( m_1^T \); and by the first order conditions together with continuity of \( m_1^T, m_2^T, \alpha V^{T-1}(\bar{m}_1) - v_1^T(\bar{e}) \) approaches zero.

The same analysis is applied to the second summand. Combining the two results, we have:

\[
\lim_{h \to 0} \frac{V^T(m+h) - V^T(m)}{h} = qv_1'(m-m_1^T) + (1-q)v_2'(m-m_2^T)
\]

Q.E.D.

Now we consider the convergence of the \( V^T \) and \( V_m^T \)
functions as \( T \) becomes large.

**Lemma 2:** Let \( m_1^T(.) \), \( m_2^T(.) \) be the policy functions in the \( T \)-period problem. Then for all \( m \), \( m_1^T(m) \geq m_1^{T-1}(m) \) and \( m_2^T(m) \geq m_2^{T-1}(m) \).

**Proof:** The first order conditions and lemma 1 give, for all \( T \),

\[
v_1'(m-m_1^T) = \alpha (qv_1'(m_1^T-m_1^{T-1}(m_1^T)) + (1-q)v_2'(m_2^T-m_2^{T-1}(m_1^T))
\]

\[
v_2'(m-m_2^T) = \alpha (qv_1'(m_2^T-m_1^{T-1}(m_2^T)) + (1-q)v_2'(m_2^T-m_2^{T-1}(m_2^T))
\]

If \( m_1^T(\bar{m}) < m_1^{T-1}(\bar{m}) \) for some \( \bar{m} \), it means that the agent spends more in the employed contingency when he has \( T \) periods to face than when he has \( T-1 \) periods to face. The corresponding \( v_1'(m-m_1^T) < v_1'(m-m_1^{T-1}) \), and so in at least one of the second period contingencies his marginal utility must also be lower, meaning that in that contingency he spends more when he has \( T-1 \) periods to face than when he has \( T-2 \). Thus, there is at least
one chain of contingencies starting from \( \tilde{m} \), where the agent spends strictly more at each step when he faces \( T \) periods than when he faces \( T-1 \) periods. But this is impossible, because under the \( T-1 \) period program he ended the last period with zero money balances in every chain of contingencies, and a policy of spending strictly more at each step would imply negative money balances at the period \( T-1 \) in that chain, which is not permitted. Therefore, \( m^T_1(m) \geq m^{T-1}_1(m) \) and, by a parallel argument, \( m^T_2(m) \geq m^{T-1}_2(m) \) for all \( m \). Q.E.D.

**Lemma 3:** On any compact interval \([m, \tilde{m}]\), the functions \( m^T_1, m^T_2 \) converge uniformly to limiting functions \( m_1, m_2 \).

**Proof:** For all \( T \), \( m^T_1 \leq m + w\bar{l} \leq \bar{m} + w\bar{l} \)

\[
m^T_2 \leq m \leq \bar{m}
\]

Hence, the functions \( m^T_1, m^T_2 \) are uniformly bounded.

Furthermore, from the first order conditions, we have, for all \( T \),

\[
|m' - m''| < \epsilon \text{ implies } |m^T_1(m') - m^T_1(m'')| < \epsilon
\]

and

\[
|m^T_2(m') - m^T_2(m'')| < \epsilon
\]

Hence, the \( m^T_1 \) and \( m^T_2 \) are equicontinuous in \( m \). It follows by Arzela's Theorem (Kolmogorov-Fomin, p.102) and the monotonicity of the sequences shown in lemma 2 that the functions \( m^T_1, m^T_2 \) converge uniformly to limiting functions \( m_1, m_2 \). Q.E.D.

**Lemma 4:** The sequence of functions \( V^T(.) \) converges to
to a unique, differentiable function \( V(\cdot) \). The derivative \( V_m(\cdot) \) is the limit of the derivatives \( V_m^T \).

**Proof:** We appeal to the fact that if the derivatives of a sequence of functions are uniformly convergent on an interval, then the sequence converges uniformly to a limit function on the interval and the derivative of the limit function will exist and be equal to the limit of the derivatives of the original sequence (Apostol, p.402).

By lemma 1,

\[
V_m^T(m) = q v_1^i(m - m_1^T) + (1-q)v_2^i(m - m_2^T).
\]

Since \( v_1^i \), \( v_2^i \) are continuous and \( m_1^T \), \( m_2^T \) converge, it is clear that \( V_m^T \) converge to some value \( V_m \).

To see that this convergence is uniform on any compact interval excluding the origin, note that \( v_1^i \), \( v_2^i \) are uniformly continuous on any such interval. Then, since the \( m_1^T \), \( m_2^T \) converge uniformly, the \( V_m^T \) must converge uniformly as well. Q.E.D.

**Proposition 1:** An optimal policy exists to problem (1). The value of problem (1), \( V(\cdot) \) is a strictly concave, strictly increasing, differentiable function of \( m \) that satisfies equation (2).

**Proof:** The limit policy \((m_1, m_2)\) is clearly feasible for problem (1). Is there any better policy? Suppose that there were, and consider that the maximum utility achievable in the \( T^{th} \) period starting with money balances \( m \) is \( v_1(m + Tw \mid L) \), which is the amount the agent
would have if he worked the maximum amount and spent nothing in each period. The tail of any policy after T periods is thus worth less than 
\[ \alpha^T \sum_{t=0}^{\infty} \alpha^t v_1(m+(T+t)wI) \]
which goes to zero as T becomes large because \( v_1(.) \) is a concave function. Thus any policy that is strictly better for an infinite program than the limit policy would also yield more utility in the first T periods for some T than the \( m_1^T, m_2^T \) policy, which is a contradiction. Therefore, \((m_1, m_2)\) is the optimal policy and \( V(.) \) satisfies equation (2).

Since \( V(.) \) is the limit of a sequence of strictly concave, strictly increasing functions, it must be concave and non-decreasing. Since \( V(.) \) satisfies equation (2) by the principle of optimality, and \( v_1, v_2 \) are strictly increasing and strictly concave, it is clear that \( V(.) \) will also be strictly concave and strictly increasing.

Q.E.D.

**Lemma 5**: The functions \( m_1(m;q), m_2(m;q) \) are jointly continuous in \((m,q)\).

**Proof**: First we prove by induction that \( m_1^T, m_2^T, V^T \) are jointly continuous in \((m,q)\).

\[ m_1^0(m;q) = m_2^0(m;q) = 0, \text{ and} \]
\[ V^0(m;q) = q v_1(m) + (1-q) v_2(m), \]
so that \( m_1^0, m_2^0, V^0 \) are jointly continuous in \((m,q)\).

Assume \( m_1^{T-1}, m_2^{T-1}, V^{T-1} \) are jointly continuous in \((m,q)\). Then \( m_1^T \) is the solution to
\[
\max\left( v_1(m - m_1^T) + \alpha V^{T-1}(m_1^T) \right),
\]
and similarly for \( m_2^T \).

In lemma 6 below, we show that a unique maximizer is jointly continuous in parameters of the objective function whenever the objective function is jointly continuous in the parameters and the maximizing variable. \( m_1^T \) and \( m_2^T \) are unique by concavity of \( v_1, v_2, V^{T-1} \), so this lemma applies. Then, \( V^T \) is jointly continuous in \((m, q)\) by application of lemma 1.

Second, we show that \( V^T \) converges uniformly over any compact set to the limit function \( V \). In the proof of proposition 1, we show that

\[
|V^T(m; q) - V(m; q)| < \alpha^T \sum_{t=0}^{\infty} \alpha^t v_1(\bar{m} + (T+t)\bar{w})
\]

(where \( \bar{m} \) is an upper bound on \( m \)), which can be made as small as we like by choosing \( T \) large independent of \( m \) and \( q \). This establishes uniform convergence of the \( V^T \) sequence, and as a consequence, \( V^T \) is jointly continuous in \( m \) and \( q \).

Finally, note that \( m_1 \) and \( m_2 \) are unique maximizers of a jointly continuous function, so that lemma 6 establishes their joint continuity in \( m \) and \( q \). Q.E.D.

**Lemma 6:** Let \( X(\alpha) \) be the set of maximizers over a compact set \( S \) of a continuous function \( f(x; \alpha) \) defined over a set \( S \times T \). Then \( X(\alpha) \) is upper-semicontinuous. If each set of maximizers is a singleton set, the function \( X(\alpha) \) is continuous.
Proof: Consider a sequence $\{\alpha^i\} \rightarrow \alpha^+$ and a convergent subsequence $\{x^i\} \rightarrow x^+$ where $x^i \in X(\alpha^i)$. Suppose $x^+ \notin X(\alpha^+)$. Then there exists $\bar{x} \in S$ with $f(\bar{x}; \alpha^+) > f(x^+; \alpha^+)$. Then $f(\bar{x}; \alpha^+) - f(x^+; \alpha^+) > 2\varepsilon > 0$.

For $i > \max(I_1, I_2)$, we have:

$$|f(\bar{x}; \alpha^1) - f(x^i; \alpha^i)| < \varepsilon$$

and for $i > \max(I_1, I_2)$, we have:

$$|f(x^i; \alpha^1) - f(x^+; \alpha^+)| < \varepsilon$$

by continuity of the function $f(\cdot)$. Then for all $i > \max(I_1, I_2)$,

$$f(\bar{x}; \alpha^1) > f(x^i; \alpha^i).$$

But this contradicts $f(\bar{x}; \alpha^1) \leq f(x^i; \alpha^i)$, by construction of the sequence $x^i$. Thus no such $\bar{x}$ can exist and $x^+ \in X(\alpha^+)$. Q.E.D.
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Changing Preferences and Sequential Decisions

1. Introduction

Over the past fifteen years, a great deal of economic theory has been concerned with the extension of the neoclassical models of decision making and equilibrium to problems of intertemporal allocation. For the most part, this research was concerned with the nature of the optimal path of a set of variables under certain constraints, where optimality was defined with respect to a given, well-behaved ordering over the set of possible time paths.

While this approach solved the purely allocational problem of economic dynamics, it neglected the fact that in a dynamic context, decisions themselves are taken sequentially. The question is not one of choosing a single time path, but one of making a sequence of decisions, which results in some time path. As soon as one tries to take account of this fact, one faces several problems:

1. If one wants to leave some scope for later decisions, it cannot in general be possible for a single decision to fix the whole future time path of the variables. Instead, single decisions are made on the instantaneous values of the variables (present consumption) and a set of parameters constraining subsequent decisions (accumulation of capital). This raises the question how the agent evaluates the "utility" of those parameters.
ii: What relations are there between objective functions for decisions at different moments in time?

It may be argued that the dynamic programming formulation of the intertemporal allocation problem allows an interpretation of the neoclassical analysis in terms of a sequence of decisions. But not all models of sequential decision making are equivalent to models of single decisions on the whole optimal time path. In certain models of costly decision making, sequences of decisions may be easier to compute and therefore cheaper. In other models, the feasible set may be such that the precommitment of the whole future path of a variable that is involved in the neoclassical model is not possible. Again in other models, such a precommitment may be undesirable.

These problems become relevant, if the future path which is chosen if the first decision precommits all later behaviour differs from the one that is chosen in a sequence of decisions.

In that case, precommitment of the future path may be undesirable because of uncertainties which cannot be insured away by contingent contracts, so that the agent prefers the greater flexibility that is afforded by sequential decision making.

Both undesirability and unfeasibility of precommitment are important if the agent's preferences change over time. The later ordering will generally find a way to
void earlier precommitments, whereas the earlier ordering may want to keep down the costs of reversing a precommitment later on.

Beyond such intuitive notions, very little rigorous analysis of changing preferences has been attempted. Koopmans gives an axiomatic treatment of one approach to the problem which merges the phenomenon of changing preferences with that of uninsurable uncertainty: His agent does not yet know what his future preferences will be and therefore wants to be as little committed as possible so that he can accommodate any future change of preferences.

A different line has been pursued by Strotz and Pollak for the special case of the consumption-saving problem. A consumer at any time has preferences over the space of all present and future consumption, but expects that in the future, he will have a different ordering to govern any choices he makes then. In general, this expectation changes his present choice.

These two groups of models lead to an opposed view of the effects of changing preferences. In Koopmans' model, there is aversion to precommitment, whereas in the other models precommitment would be desired, if only it were feasible.

The present paper proposes to clarify the issue by distinguishing two different concepts of "changing preferences". In the first part, we give some basic concepts
and postulates that we believe will be useful in the further analysis of sequential decision making under changing preferences.

The second part analyses a particular model of systematically changing preferences. These occur, if the agent's computational capacities are limited, or in decentralized social units such as firms or families, in which different types of decisions are taken by different members of the unit. We lay particular stress on problems of normative analysis. We hope that the approach proposed here provides some insights into problems of social coordination, where neither a price system nor complete centralization are available.

I. Frameworks for Changing Preferences

1. Basic Structure and Concepts

We investigate the situation of a consumer who expects his preferences to change in the future. At any moment $t_0$ he has some preferences which are defined on the space of sequences of future consumptions $\left\{ c_t \right\}_{t=t_0}^\infty$, where $c_t$ is a vector that lists different commodities in different states of nature at time $t$. It will be convenient to single out those sets of states of nature which differ from each other by the preference ordering that the agent expects to have in them. Thus, if the agent foresees $T_t$ possible preference orderings for some future
time $t$, we write:

$$c_t = (c_{t1}, \ldots, c_{tT_t})$$

We assume that the number $T_t$ of anticipated orderings for $t$ is finite. At time $t_o$ he knows which ordering he has presently, so that he is not uncertain about the ordering under which he makes a decision.

Since we are not concerned with the axiomatics of well-behaved preferences, we assume:

**Assumption I:** For all $t_o$, $i_{t_o}$, the consumer's preference ordering at time $t_o$ in state of nature $i_{t_o}$ over the sequence of consumption vectors $\{c_t\}_{t=t_o}^\infty$ satisfies the following conditions:

i: It is a complete ordering;

ii: It is monotone in each of the $c_t$ and sensitive with respect to $c_{t_o}$;

iii: It is continuous with respect to whatever distance function is used;

iv: It has convex upper contour sets.\(^1\)

It is well known that this assumption implies:

**Assumption I\(^+\):** For all $t_o$, $i_{t_o}$ the ordering $\preceq_t$ can be represented by a continuous real-valued function $u_{t_o}^{i_{t_o}}$ on the space of sequences of consumption vectors $\{c_t\}_{t=t_o}^\infty$.

To simplify the notation, we write:

\(^1\) For a discussion of these assumptions, see Koopmans.
\[ t_o^c = \left\{ c_t \right\}_{t=t_o}^\alpha \]
\[ t_o^i^c = (c_{t_o}^i, t_o+1^c) \]
\[ t_o^f^c = (c_{t_o}^1, \ldots, c_{t_o}^{i-1}, c_{t_o}^{i+1}, \ldots, c_{t_o}^{T_{t_o}}, t_o+1^c) \]

We make no assumptions about the process that generates future preferences. He may expect subsequent preferences to be independently distributed, or he may assume that as he approaches the time \( t \), his experience will give him more information about his preferences at \( t \).

Since we use a state-preference approach, his probability judgments are part of his preferences and do not appear explicitly in the model. On the other hand, this means that if we make assumptions on the evolution of preference orderings, these impose constraints on the process that generates orderings.

The following concepts will play an important role in the discussion:

**Separability:** A utility function \( u_{t_o}^i \) is separable, iff there exist functions \( f_{t_o}^i, v_{t_o}^i \) such that for all \( t_o^c \)

\[ u_{t_o}^i(t_o^c) = f_{t_o}^i(c_{t_o}, v_{t_o}^i(t_o+1^c)). \]

**Stationarity:** A utility function \( u_{t_o}^i \) is stationary, iff there exists a function \( f_{t_o}^i \) such that for all \( t_o^c \)

\[ u_{t_o}^i(t_o^c) = f_{t_o}^i(c_{t_o}, u_{t_o}^i(t_o+1^c)). \]
Separability is the simplest property of an inter-temporal ordering. It means merely that the rate of substitution between consumptions at two dates is independent of the quantity of consumption at a third, preceding date. Stationarity is much stronger and requires that the rates of substitution between the second and third period, say, be the same as between the first and second period.

Also, since the function $u_{t_o i}$ is not sensitive to the consumption vectors $c_{t_o j}$, $j \neq i$, stationarity implies that preferences are sensitive to consumption in the $i$th state of nature only, not only in the first period, but ever thereafter. In most models, this means that the agent only expects to have the $i$th preference ordering all the time, so that stationarity is incompatible with uncertainty about future preferences.

Whereas separability and stationarity are properties pertaining to single preference orderings, the following concept relates to sets of orderings:

**Invariance:** a) Any two preference orderings represented by utility functions $u_{t_o i}$, $u_{t_1 j}$, with $t_o < t_1$, are invariant, iff there exists a function $f_{t_o ij}$ such that

$$u_{t_o i}(t_o^i c) = f_{t_o ij}(c_{t_o i}, c_{t_o +1}, \ldots, c_{t_1 -1}, t_1^j c, \ldots, u_{t_1 j}(t_1^j c)).$$

---

2/ Since the ordering at $t_o i$ is sensitive to $t_o^i c$ only, we omit the distinction between preferences $u_{t_o i}$.
b) A set of preference orderings is invariant, iff its elements are pairwise invariant.

The significance of this property is the following: If the consumer maximizes his preferences at \( t_0 \) subject to some constraint on the set of feasible paths, then the tail of the path, \( t_1^c \), that he would like to choose under the ordering at \( t_0 \) is the same as the path that he will actually choose at \( t_1 \) under his then ordering, provided the feasible set of sequences starting from \( t_1 \) is the same as under the original problem after consumption for the intervening periods has been fixed.

Invariance does not necessarily imply separability. It may happen that the change of preferences is endogenous and induced by consumption in the initial periods, \( t_0, \ldots, t_1 - 1 \). If we rule out this possibility, invariance implies separability, because then, under invariance, the ordering on the set of paths starting at \( t_1 \), which is induced by the ordering at \( t_0 \), is independent of earlier consumption.

If the agent foresees several orderings in the future, and if his present ordering is invariant with respect to more than one of them, then the present ordering cannot be stationary.

as defined on the space of sequences \( t_0^c \) and \( t_0^i^c \).
2. Preferences on Sets of Paths and Aversion to Precommitment

In general, present decisions do not precommit the whole future consumption path. Rather they determine present consumption and constrain the feasible set for a subsequent decision. Thus, they require a criterion for the comparison of future feasible sets.

Before we go into a detailed analysis of preferences on sets of consumption paths and their relationship to the preferences on consumption paths themselves that we discussed in the first section, we discuss some of the properties that Koopmans (1964) proposes for orderings of sets of consumption programs.

For the purpose of simplicity, we exclude the problem of intertemporal allocation and consider only those sets of consumption paths, which are t-uniform, that is to say, which have the same consumption up to and including time t. This procedure has the advantage that questions of separability play no role, since the beginnings of the paths that are compared are the same.

Koopmans argues that under uncertainty about future preferences one wants to leave as large a scope as possible for future choice in order not to rule out an alternative now which in the light of later preferences may be considered as better. Thus, any enlargement of a set of consumption paths should be considered an improvement. He
gives three basic formulations for this idea:

Assumption II.1: Let $A_t$, $A'_t$ be $t$-uniform and suppose that $A_t \supseteq A'_t$; then $A'_t$ is not preferred to $A_t$.

Assumption II.2: Let $A'_t$, $A''_t$ be $t$-uniform with $A'_t \cap A''_t = \emptyset$ and suppose that $A'_t$, $A''_t$ are indifferent. Then their union $A_t = A'_t \cup A''_t$ is preferred to either of them.

Assumption II.2*: Let $A'_t$, $A''_t$ be $t$-uniform, with $A'_t \cap A''_t = \emptyset$ and let there be $t' \geq t$ such that $A'_t$, $A''_t$ have subsets $A'_t'$, $A''_t'$, which are $t'$-uniform, with $A''_t \succ A'_t$; then if there exists a chain of eventualities such that consumptions for $A'_t'$, $A''_t'$ are chosen up to $t'$, the union $A'_t \cup A''_t$ is preferred to $A'_t$.

Assumption II.3: Let $A_t$, $A'_t$ be $t$-uniform with $A'_t \succ A_t$, then $A_t$ is preferred to $A'_t$.

Of these postulates, the first is the weakest and merely says that the enlargement of a set of programs can do no harm. The last one asserts that any proper enlargement will actually do good. Since this would include enlargement of a set by a program which is everywhere strictly dominated by a program already in the set, this postulate is rather implausible, given the monotonicity postulate I II.

The other two conditions try to find some middle ground, tying the assertion of strict preference in the conclusion to a qualification of the admissible enlargement.

II.2* merely requires that there be some eventuality under which the enlargement actually proves advantageous because a program is chosen which could not have been cho-
II.2+ excludes in particular those enlargements where the additional paths are monotonically worse than the members of the original set.

II.2 is more stringent and requires that the enlargement be as good as the original set itself.

These postulates formalize the notion that a consumer has a preference for the postponement of a decision, but for a further analysis, it is necessary to relate them to preferences on single consumption paths. It is useful to point out though that Koopmans' postulates are postulates about a single preference ordering and do not say anything about the relationship between different orderings at different moments of decision making.

3. Preferences on sets of Paths - General Principles

In ordinary consumer theory, it is the strength of the utility approach that it can be formulated in the choice theoretical setting of revealed preferences which makes extensive use of observable variables.

In the present context, the consumer does not choose a whole path under a single ordering. Therefore, the choice theoretical interpretation is no longer available, at least for orderings on consumption paths. Since choices are made on sets of paths, Koopmans formulates his axioms in terms of the choice between such sets and does not take re-
course to an underlying ordering of consumption paths.

This procedure merely sidesteps the issue. If underlying preferences are flexible and therefore not observable in a sequence of observations, then surely, one will not want to assume that preferences on sets of programs are inflexible and therefore observable. But then, for instance a condition like the antecedent of assumption II.2* is not observable, for it involves the ranking according to present preferences of sets of programs between which one only decides later, when one has changed one's preferences.

For lack of a better procedure, we drop any revealed preference interpretation and derive the ordering of sets of consumption paths from underlying orderings of consumption paths.

Such a derivation requires that we find functions, which assign to each set of paths a single consumption path, so that sets are ranked as these related paths are according to the underlying ordering.

A particular application of this procedure is well-known in consumer theory in the construction of the indirect utility function. There, each budget set is assigned its optimal element as evaluated by the direct utility function and budget sets are ranked according to their optimal elements.3/

3/ Koopmans seems to have had a similar principle in
This particular ranking is also used in dynamic programming to compare the values of future budget sets. In the present model, if the consumer anticipates a change of preferences, we would not, in general expect this ranking to be appropriate. The consumer will try to find his present optimal behaviour with respect to those paths that he will actually follow. Therefore, he wants to take account of the fact that in future, his changed preferences may follow a different path than he would choose under present preferences. This does not rule out the possibility of myopic behaviour, for we say nothing about the correctness of his anticipations. The consumer may refuse to acknowledge the future change in preferences, or he may anticipate the change incorrectly; we only argue that if he anticipates a certain change in his preferences, then he will take account of this change and its effect on his actual consumption in the future.

The structure of subsequent decisions is described in diagram 1:

![Diagram 1](image)

non-mind, when he proposed that 1-uniform sets be ranked as their best 1-uniform subsets.
At each moment, the consumer makes a decision $D$ so as to maximize a criterion function over some feasible set. The domain of his criterion function does not coincide with the space of decision variables. Each decision influences the subsequent decision's feasible set - with endogenous changes of taste also the subsequent criterion function - and therefore the subsequent decision. The domain of the criterion function includes the domain of subsequent criterion functions. Therefore, the anticipation of subsequent behaviour will have a feedback effect on the present decision. This line of thought leads to:

**Assumption III:** The consumer ranks feasible sets for subsequent decisions according to the points in them that he expects to be chosen in the subsequent decision. $+/$

This assumption includes the dynamic programming procedure as a special case if the choice from the future feasible set is the same under the current and under the anticipated future ordering. It is easy to see that this is the case if and only if any sequence of subsequent orderings is invariant. In this case, subsequent orderings will have the same indifference surfaces on the intersection of their domains.

4. **Nonstationary Invariant Preferences**

We use assumption III to analyse the notion that an enlargement of future feasible sets is always considered
This assumption appears to contain a vicious circle in that future choices depend on the ranking of feasible sets by future preferences. This circularity can be avoided by an inductive procedure: Behaviour in the T+1-horizon problem foresees behaviour in the T-horizon problem. Convergence is guaranteed in all cases in which the normal neoclassical problem corresponding to invariant preferences converges, for the path chosen under invariant preferences dominates any other that may be foreseen.

A more crucial problem is given by the fact that behaviour in the T-horizon problem need not be continuous and therefore optimal behaviour in the T+1-horizon problem need not exist. This is due to the fact that the T-1-horizon behaviour foreseen in the T-horizon problem need not have the curvature to give a unique solution to the T-horizon problem. But by lemma 6 in the appendix to the first paper in this thesis, T-horizon behaviour will be upper semicontinuous, if T-1-horizon behaviour is continuous. At the point where the T-horizon problem has two solutions, T-horizon behaviour is indifferent between two choices. If we introduce the convention that, of these, the one is taken, which according to preferences relevant for the T+1-horizon problem is preferable, the T+1-horizon problem becomes solvable and has an upper semicontinuous policy function.
advantageous.

From the remarks in section 2, it is obvious that assumption II.3 is incompatible with assumption III and the monotonicity postulate in I. For enlargement by an element, which is everywhere strictly dominated by some element in the original set, which is therefore not chosen under any ordering obeying II, cannot be an improvement.

For the weakest assumption II.1, we prove:

**Proposition 1:** Under assumptions I and III, assumption II.1 holds if and only if any sequence of preference orderings is invariant.

**Proof:** Sufficiency is obvious. To see necessity, consider the programs $t_{o+1,j}^c$, $j = 1, \ldots, T_{o+1}$ that the consumer will choose out of some feasible set at time $t_{o+1}$ under the orderings $t_{o+1,j}$ that he then has. If the orderings $t_{o+1,j}$, $j = 1, \ldots, T_{o+1}$ differ, these chosen programs will differ for some feasible set. Furthermore, unless present preferences and the $k^{th}$ ordering at time $t_{o+1}$ are invariant, the path $t_{o+1,k}^c$ will not be optimal under present preferences in the $k^{th}$ contingency at $t_{o+1}$. By the continuity of preference orderings, there exists a program $t_{o+1,k}^c$ in the neighbourhood of $t_{o+1,k}^c$ that is preferred by $t_{o+1,j}^c$, $j = 1, \ldots, T_{o+1}$, $j \neq k$, and which is preferred to $t_{o+1,k}^c$.
by the present preferences at $t_0$ for the contingency $k$ at $t_0+1$.

Now compare the two sets of programs:

$$A = \left\{ t_0+1, c_j^+, j = 1, \ldots, t_0+1; t_0+1, c_k^+ \right\}$$

$$A' = \left\{ t_0+1, c_j^+, j = 1, \ldots, t_0+1, j \neq k; t_0+1, c_k^+ \right\}$$

In the states of nature, $j = 1, \ldots, t_0+1, j \neq k$, the consumer at time $t_0+1$ will choose the same programs from both sets.

In state of nature $k$, the consumer at time $t_0+1$ will choose $t_0+1, c_k^+$ from $A$ and $t_0+1, c_k^+$ from $A'$. Hence, under assumption III, he will rank the two sets $A$, $A'$ like the vectors

$$(t_0+1, c_1^+, \ldots, t_0+1, c_{t_0+1}^+)$$

$$(t_0+1, c_1^+, \ldots, t_0+1, c_{t_0+1}^+, t_0+1, c_k^+, t_0+1, c_{t_0+1}^+)$$

By construction, the present preferences prefer $A'$ to $A$, even though it is contained in $A$, in contradiction to assumption II.1. Thus, subsequent preferences must be invariant.

Q.E.D.

Intuitively, this proof is very simple: Present preferences know which path they would like to follow if they could govern behaviour completely. Actual preferences will follow a different path, which is not optimal under pre-
sent preferences. Enlarging a feasible set is not harmless if it gives more scope to future preferences to move away from what present preferences would like. The exception to this pattern is the case where subsequent orderings are invariant, because then, the choice under the future ordering is the same that the present ordering would take.

Exactly the same argument can be used to show the necessity part of

**Proposition 2:** Under assumptions I and III, II.2 holds if and only if any sequence of preference orderings is invariant.

**Proof:** Since necessity follows from essentially the same argument as in proposition 1, we only show the sufficiency part. We note that II.2⁺ implies II.2. Therefore, it is enough to show that the invariance of subsequent orderings implies II.2⁺.

If the antecedent of II.2⁺ holds then in the sets A', A" given by the antecedent, there exist paths which coincide up to t' and which are - up to t' - chosen under some sequence of preference orderings. They have the further property that from t' onwards, the tail of those paths in A" is preferred to the tail of those in A' with the same beginning. Now suppose that the union of A' and A" is available, since all subsequent orderings are invariant, actual preferences at later moments will concure with the original judgment and in at least one contingency at t' select a continuation
from A'' rather than A'. The consumer is strictly better off if the union of A'_t, and A''_t, is available. Because of invariance, present preferences will consider themselves better off with the union of A' and A'' than merely with A'. The advantage of the enlargement here is that in any contingency, the consumer can take the best path from either set and unless one set dominates the other in all contingencies, he is better prepared for all eventualities if he can choose from both.

Thus, assumption II.2^+ is fulfilled and a fortiori also assumption II.2.

From propositions 1 and 2 we have a rather surprising corollary:

**Corollary:** Under assumptions I and III, the invariance of all sequences of preference orderings, assumption II.1, assumption II.2 and assumption II.2^+ are equivalent.

To understand the significance of these results, we write the utility function, which is invariant with respect to all subsequent utility functions^4/ as:

\[ u_{t_0}^i(\underline{c}) = f_{t_0}^i(c_{t_0}, u_{t_0+1,i}(t_{t_0+1,c}), \ldots \]

\[ \ldots, u_{t_0+1,T_{t_0+1}}(t_{t_0+1,T_{t_0+1}}(c)) \]

^4/ Note that propositions 1, 2 postulate invariance only for subsequent orderings, not for different orderings at the same time.
The utility function $u_{t_0}^{i}$ may be written as a quasi
utility function $f_{t_0}^{i}$ with the arguments present consump-
tion and futures utilities.

Thus, propositions 1 and 2 convert Koopmans' postu-
lates into a form which is tractable with the customary
tools of economic theory such as stochastic dynamic pro-
gramming.

Under invariance, present and future preferences are
the same over the intersection of their domains. In this
sense, Koopmans' "flexibility" of future preferences" is
a flexibility of present preferences as well. Preferences
do not change, but there is uncertainty about them.

Yet, if one changes the time perspective, one can for-
mulate the model in such a way that it makes sense to say
that preferences change.

If there is genuine uncertainty about future preferen-
ces and subsequent orderings are invariant, the agent as-
signs positive probability to several different orderings
in the future and wants to accommodate them now. Since
present preferences are - as of now - clearly and uniquely
given, they cannot be stationary. Thus, preferences in
Koopmans' model are non-stationary.

So far, we have referred to the dimensions of the do-
mains of orderings by their absolute situation in time. It
is also possible to refer to them by their position rela-
tive to the moment of decision making for which the orde-
ring is relevant. In this case, nonstationarity implies that subsequent orderings are no longer identical on equally denominated domains. If we let all orderings be defined on the space of sequences \( \{ c_{t-t_0} \}_{t=t_0}^\infty \) = \( \{ c_t \}_{t=0}^\infty \), then subsequent orderings appear to differ even if they are invariant.

This interpretation holds not only for matters of intertemporal allocation, but also for the allocation within one period. One is uncertain about tomorrow's rate of substitution between meat and fish tomorrow. Therefore, tomorrow's rate of substitution will not, in general, be equal to today's rate of substitution between meat and fish today, but it may be equal to today's rate of substitution between meat and fish tomorrow.

5. Non-Invariant Preferences

After the preceding discussion, the question raises itself what place there is in economic analysis for non-invariant preferences, i.e. those preferences, which may be considered to change if we denominate the dimensions of the commodity space by their absolute position in time.

The existence of nonstationary, invariant preferences was easy to understand. It is an everyday occurrence if an agent claims not to know yet his preferences at some future time, but wants to accommodate them whatever they
will be and therefore exhibits an aversion to precommit-
ment.

With noninvariant preferences, the enlargement of
the feasible set for a future decision is not always con-
sidered advantageous. There is no monotone relationship
between present and future utility. The consumer may ac-
tually want to precommit himself so as to prevent the
subsequent choice from deviating too far from what he con-
siders optimal now. There seems to be something "irratio-
nal" about such a conflict of subsequent orderings. What
justification is there to introduce them into economic
analysis?

The conflict between noninvariant orderings is not
the case of the old man saying "If only I had saved more
in my youth!". Since we exclude the past from the domain
of the preference orderings, this cannot occur. We do
not consider the intertemporal allocation between current
consumption and the future feasible set in this context. 5/
Rather we look at the case where marginal rates of substi-
tution differ for consumptions that lie in the future
with respect to both decisions. For instance, a 30-year
old man may wish to save much up to the age of 35 and then
raise his consumption, yet at the same time know that at
the age of 31, he will make 36 his new age for higher
consumption.

5/ But Strotz and Pollak show that the intertemporal allo-
One can make a rough distinction between historical and systematic changes of preferences in this sense, the former occurring once for all, the latter in a regular pattern. The practical occurrence of historical changes of preferences is quite obvious. A simple example would be given by a person who at one time vows never to drink coffee and five years later has become a regular coffee drinker. Unfortunately, this kind of change of taste is not very amenable to economic analysis, because it hardly ever is anticipated and cannot affect behaviour before it actually occurs. When it occurs it has to be taken as a positive fact.

Systematic changes of preferences occur most frequently as a decentralization phenomenon. This is obvious if the decision making unit itself consists of several agents.

In a family for instance, different members will in general have different preferences. Typically decisions are not made by one member alone, but rather different types of decisions are made by different members. This decentralization is not coordinated by a price system, but rather by the fact that decisions by any one member affect the feasible set of all other members with a feedback effect both on the future feasible set of the first
member and on those decision parameters of other members about which he may care.

Functional decentralization, which assigns different types of decisions to different preference orderings, is even more important in decentralizing firms. It is a well-known organizational phenomenon that each division of a firm has a tendency to overestimate its own importance for the total performance of the firm. Thus, one would expect attitudes to different types of risk to differ from division to division. The finance division of a firm may be more suspicious about the riskiness of a new investment than the production or research division with which the proposal originates.6/

Changing preferences as a decentralization phenomenon are not limited to decision units that consist of several individuals. They may occur with a single individual as well, simply because it is too bothersome to think of everything all the time.

An agent who does a sequence of transactions compares in each transaction the utility of his resources when used currently and when held over for future trans-

6/ One may not be able to distinguish between different perception of and different attitude to risk. Also, power rivalry between divisions may be the reason for different attitudes. The finance division may be averse to taking a risk in production while
actions. Suppose that he has a single invariant utility function. Then, in each transaction, the utility of holding money rather than spending it is given by an indirect utility function. Normally, this indirect utility function is computed from a consideration of all the other possible uses for his resources - even if these should be infinite.

In another paper in this thesis, we show that if there is a fixed cost to the dimension of the commodity space that he considers, he will only consider a finite number of alternative uses. For instance, in the solution of an infinite lifetime consumption problem, he may at each stage content himself with using a finite time horizon. In a more elementary model, a decision on the purchase of apples may think of pears and cherries as possible alternatives, but forget about bananas and peanuts. If we assume that preferences are separable between those commodities that are considered and those that are left out, it follows that the marginal utility of holding over resources is always smaller than if the whole space were considered.7/

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7/ Taking a high risk in financial investments, simply because the former reduces its influence relative to that of the production division.

7/ The separability assumption is not implausible, because one expects strongly complementary goods to
If the sequence of transactions traces out the whole space, the consumer will at each single transaction appear to be biased in favour of the immediate object of the transaction and against the holding over of resources. For instance the consumer, who considers a finite lifetime instead of the infinite lifetime that he actually goes through, appears to have a higher rate of time preference in each of his decisions. This is the same phenomenon as the self-overestimation of different divisions in a firm. It makes precise the intuitive notion that the concrete has, in general, a higher appeal to our taste than the abstract.

This kind of bias is very obvious in little children who sometimes turn with overenthusiasm from one activity to the next, without even completing the first. Their computational capacity is not yet enough developed to allow consistent planning over a longer period without constant changes of plans to give in to the latest whim.

A complete analysis of this phenomenon would start from the underlying consistent ordering and derive the sequence of apparent orderings by introducing explicitly the cost of making and computing a decision. But since neither the underlying ordering nor the decision making

be either both considered or both left out.
costs are easily observable we propose to treat this problem as a case of sequences of noninvariant preferences.

II. Systematic Changes of Preferences

1. General Principles

We analyse systematically changing preferences by means of a simple model of consumer behaviour. A model of systematic changes of preferences requires a distinction between different types of decisions so that one can assign different orderings to each type. Furthermore, we must introduce some kind of irreversibility so that later decisions cannot completely undo a former one, for otherwise the former decision would be irrelevant for any analysis. The simplest type of irreversibility is, of course, physical irreversibility. One need not go as far as to refer to the second law of thermodynamics; any actual act of consumption such as the eating of a cake has to be considered as irreversible.

A subtler and at the same time more general type of irreversibility is economic irreversibility, which appears whenever it is not economical to reverse an earlier decision. This is the case if the price at which the store-keeper takes a commodity back is far below the price at which it was purchased. If transactions are costly, purchases will in general be irreversible economically.
Models of transactions costs also provide for an easy distinction between different types of decisions for in inventory models, there is a clear separation of reorder decisions and decisions on the rate of consumption of the inventory.

The analysis of systematically changing preferences may proceed along positive or normative lines. Of these, the normative analysis is perhaps the more interesting. As a starting point, we take Koopmans' proposition that his model is a model of "rational" behaviour and has a normative significance, even if it does not describe actual behaviour. Since we have shown that his model is equivalent to the invariance of subsequent preferences, this would imply that decision making under noninvariant preferences is, in some sense irrational.

Such a statement is meaningless unless we define "rationality" in a model of changing preferences. If preferences are noninvariant because we deal with a decentralized firm, the underlying criterion function of the management or of stockholders can always be used for this purpose. In the case of the family, no such dictatorial ordering exists to provide an overriding criterion. Here, we have to satisfy ourselves with the Pareto ordering, which gives a partial ordering and is derived from the preferences of all the members of the family.
For changing preferences in a single individual, one applies the same principles. If the change in preferences is due to informational or computational costs, its normative analysis may use the underlying unconstrained ordering as a criterion. But since the latter is usually not revealed, it is operationally preferable to apply the Pareto ordering with respect to those preferences that are revealed and govern actual decisions.

This gives the following criterion of irrationality under changing preferences:

Given some exogenously determined feasible set, a set of preference orderings together with a set of rules assigning to each ordering some decision parameters in such a way that the set of orderings and the set of rules together determine a unique point of the feasible set, are called irrational if and only if there exists another set of orderings and rules assigning decision parameters to these orderings, which determine a new point of the feasible set, which, according to the first set of orderings, is Pareto superior to the point chosen under the first set of orderings and rules.

This definition deals both with the case of noninvariant preferences and the welfare theoretical problem of finding the socially optimal allocation. According to this criterion, the single preference ordering that selects a point from a feasible set is always ra-
tional. Similarly, for several orderings with disjoint domains, any set of rules that gives rise to a competitive allocation is rational. In normal welfare theory, the problems come when we introduce externalities and let the different orderings have non-disjoint domains. Precisely the same problems arise with noninvariant preferences.

In those cases in which the changing preferences themselves are a consequence of some information or computation cost, the term "irrational" is somewhat misleading. Then, we deal with a cost of decentralization, which has to be considered as part of a "super-optimization", which determines the optimal degree of decentralization by trading off this cost of decentralization against the higher informational and computational cost of centralized decision making.

2. A Simple Model

We use the consumer's optimum savings problem with a fixed cost of making withdrawals from the bank to discuss the above ideas in greater detail. It was already mentioned that this type of model allows an easy distinction between withdrawal and consumption decisions. To simplify the argument, we assume that there is no uncertainty about future behaviour. The consumer may be wrong in his forecast of future behaviour, i.e. he may be myopic, but he feels quite certain that
his expectation is correct.

In the fifth section of the first part of this paper, we argued that in a sequence of decisions, the preference ordering governing each single decision would have a bias in favour of the immediate object of the decision on hand. In this perspective, each decision is a decision between the holding of cash and an alternative use of resources. In this sense, we regard the withdrawal decision as a decision on saving, i.e. on how much to leave in the bank for later use instead of holding it in cash. The decision on the rate at which one consumes out of a given amount of cash is a consumption decision.

We now represent the above argument on a bias in favour of the specific object of the particular decision by assuming that the rate of time preference for withdrawal decisions is lower than for consumption decisions. While in the bank, the consumer has a higher preference for saving than when he sits down for dinner. This assumption is supported by the empirical observation that people in supermarkets regularly buy more than they intended to when they entered the store. 8/

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8/ This fact can of course be ascribed to the receipt of new information after they enter the store. But this cannot be distinguished from the variation in the reduction of the commodity space because of computational costs, which we discussed in I.5 above. The parti-
Now consider the situation of a consumer when he makes a withdrawal. After the withdrawal, he expects a certain consumption pattern which is governed by his preferences outside of the bank. According to this consumption pattern, he will run out of cash after a period of length $h$. Meanwhile, he will earn interest on whatever he leaves in the bank. After the period of length $h$, he comes back to the bank to make a new withdrawal. If he assumes that there is no change in the behaviour outside the bank, he faces the same problem as in the initial withdrawal with a new value of his bank balances.

If we write the value of the withdrawal problem for a bank balance $K$ and cash holdings $Y$ as $W(K,Y)$, we can use a standard dynamic programming formulation to write the withdrawal problem as:

$$(1) \ W(K, Y) = \max(\w(Z, h(Y+X)) + e^{-\delta h(Y+X)}W((K-X-a)e^{rh(Y+X)}, Y+X-Z)).$$

$X$ is the size of the withdrawal, $a$ the fixed cost of the withdrawal, $r$ the rate of interest paid by the bank and $\delta$ the rate of time preference relevant for withdrawal decisions. $w(Z, h(Y+X))$ is the utility he draws - according to his preferences in the bank - from consuming the amount $Z$ over the period $h(Y+X)$ according to the pattern given by his preferences while consuming. If his instantaneous utility function is $u(c)$, $w$ is given by:

$$w(Z, h(X+Y)) = \int_0^h e^{-\delta t}u(c'(t))dt,$$
Now consider the situation of a consumer when he makes a withdrawal. After the withdrawal, he expects a certain consumption pattern which is governed by his preferences outside of the bank. According to this consumption pattern, he will run out of cash after a period of length \( h \). Meanwhile, he will earn interest on whatever he leaves in the bank. After the period of length \( h \), he comes back to the bank to make a new withdrawal. If he assumes that there is no change in the behaviour outside the bank, he faces the same problem as in the initial withdrawal with a new value of his bank balances.

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W(K, Y) = \max_{X} (w(Z, h(Y+X)) + e^{-\delta h(Y+X)}w((K-X-a)e^{rh(Y+X)}), \quad Y+X-Z)).
\]

\( X \) is the size of the withdrawal, \( a \) the fixed cost of the withdrawal, \( r \) the rate of interest paid by the bank and \( \delta \) the rate of time preference relevant for withdrawal decisions. \( w(Z, h(Y+X)) \) is the utility he draws - according to his preferences in the bank - from consuming the amount \( Z \) over the period \( h(Y+X) \) according to the pattern given by his preferences while consuming. If his instantaneous utility function is \( u(c) \), \( w \) is given by:

\[
w(Z, h(X+Y)) = \int e^{-\delta t} u(c'(t)) dt, \text{ where } c'(t)
\]
maximizes $\int_0^t e^{-\gamma t} u(c(t)) dt$ subject to $\int_0^t c(t) dt = Z$.

$\gamma$ is the rate of time preference during consumption. 9/

Both the length of the withdrawal period and the amount $Z$ that is consumed over this period are determined by the preferences outside the bank. Simple efficiency considerations show that the consumer outside the bank will always use up all his cash before returning to the bank, that is: $Z = X + Y$. Thus, there is little loss of generality in setting $Y = 0$ and writing (1') $W(K) = \max_{X}(w(X, h(X)) + e^{-\gamma h(X)} W((K-X-a)e^{rh})).$

One can prove along exactly the same lines as the proof of proposition 1 in the fourth essay in this thesis that for every behavioural function $h(.)$ that is continuous, a function $W(.)$ exists which is unique and continuous and satisfies equation (1'). This function $W$ gives the value of the consumer's withdrawal problem for any bank balance $K$, if the consumer expects to behave as $h(X,K)$ once he has begun consuming.

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cular parametrization we use becomes plausible if we remember that the model of behaviour with time-preference is equivalent to a model without time preference with a Poisson distributed lifetime, where the rate of time preference corresponds to the inverse of expected lifetime. The lower rate of time preference in the bank represents a higher expected lifetime, which relates to our remarks on the reduction of the
A similar analysis is available for the problem of the consumer during consumption. It was already mentioned that he always uses up whatever cash he has on hand before returning to the bank. With a bank balance $K$ and cash balances $X$, he consumes the amount $X$ in an optimal manner over the period $h$, after which he expects a new withdrawal as given by the behavioural function $Y(.)$ that is determined by his preferences inside the bank and the capital $K^{r}h$ that he then has. If we write the value of this problem as $Z(K,X)$, we have:

$$
(2) \quad Z(K,X) = \max_{h} \left( z(X,h) + e^{-\frac{h}{h}}Z(K^{r}h - Y(K^{r}h) - a, Y(K^{r}h)) \right)
$$

Again, after the next withdrawal he faces the same problem as now with different values of capital in the bank and cash balances. The behavioural function $Y(.)$ which gives the next withdrawal is taken as a datum.

The utility of consumption in the current withdrawal period is given as $z(X,h) = \max \int_{t=0}^{t=h} e^{-s}u(c(t))dt$ subject to $\int_{t=0}^{t=h} c(t)dt = X$.

Again, the same proof as for proposition 4 in essay 4 applies to show that for every continuous behavioural function $Y(.)$, the function $Z(.)$ exists and is unique and continuous and fulfils equation (2). Furthermore, $Z$ gives the value of the optimal policy for the consumer.

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9/ The whole discussion draws heavily on the treatment
It should be noted that in both functional equations it is assumed that behaviour under the "other" preferences is stationary and does not change from step to step, i.e. the same functions h(.), Y(.) are applied at each step of the withdrawal-consumption sequence. It is possible to let these behavioural functions vary, for instance if one wants to represent the hope by the consumer inside the bank that as time goes on, he will become less of a spendthrift in his consumption. But this kind of analysis is much more complicated.

The same argument as in essay 4 on the single preference problem shows that optimal solutions to problems (1) and (2) are interior solutions. They satisfy the first order conditions:

\[(3) \quad w_x - e^{(r-\delta)}h_{W'} + (w_h - \delta e^{-\delta}h_W + r(K-a)e^{(r-\delta)}h_{W'}) \frac{dh}{dx} = 0\]

respectively:

\[(4) \quad z_h - e^{-\delta}h_z + rke^{(r-\delta)}h_{Z1} + rke^{(r-\delta)}h_{Z2} \frac{dy}{dk} = 0,\]

where, from the definition of the function z,

\[(5) \quad Z_2 = z_x - e^{(r-\delta)}h_{Z1} - e^{(r-\delta)}h_{Z2} \frac{dy}{dk}\]

of the problem for single preferences in essay 4.
The general character of these equations can be described as follows: The optimal withdrawal decision takes account of both direct and indirect effects of a change in the withdrawal size. The direct effect is the same that is considered in the single preference model. The indirect effect is the effect due to the ensuing change in the withdrawal period. It is given as the product of the marginal change of the withdrawal period with a change of the withdrawal size and the utility effect of a change in h. If we assume that the consumer outside the bank with his higher rate of time preference consumes faster than he would if his preferences in the bank determined all his behaviour, then the period h as expected in equation (3) is shorter than under the preferences in the bank. Therefore, the impact of a change in h on utility is positive.\(^{10/}\) If furthermore the withdrawal period is expected to increase with the size of the withdrawal, the last term in (3) is positive. Thus, the difference between the first two terms is negative. Hence, his withdrawal is larger than it would be, if his preferences in the bank determined the whole path.\(^ {11/}\)

---

\(^{10/}\) For the single preference behaviour, the sign of \(dh/dX\) is easily verified: In the notation of the paper on the single preference problem, we have
\[
dh/dX = -(w_{xh} - v_{kh}) / (w_{hh} + \gamma w_h + r(K-X-a)v_{kh}),
\]
which is positive.
Similarly, in equation (4), the sum of the first three terms is the direct effect of a change in the withdrawal period. The withdrawal period is longer than under the single preference decision, if the last term is positive. From (5), one sees that this is the case, if at all later stages, the withdrawal that is expected to be made is smaller than it would be, if the ordering outside the bank could determine withdrawals.\textsuperscript{12} Thus, if the consumer expects too small a withdrawal at later stages, he will react by consuming more slowly, so that the withdrawal period is longer.

\textsuperscript{11} These arguments are based on the assumption that the maximand is concave in $X$, $h$ at the point where the first order condition holds. This problem is discussed in detail in essay 4.

\textsuperscript{12} From (5) it follows that $(1 + e^{fr-g}Y/dK) \inf Z_2 > z_x - e^{fr-g}hZ_1$, which proves the statement of the text. If withdrawals are made to follow the single
Thus, we see that in the present setting, the consumer will always try to accommodate his alternative preferences. If he were myopic, he would behave as under the single preference ordering. While in the bank, he would make a smaller withdrawal than he wants to have when outside of the bank. While consuming, he consumes faster than his preferences inside the bank would like. As he becomes aware of the fact that his preferences are changing, he accommodates the alternative preferences by making a higher withdrawal when in the bank, by consuming more slowly when outside the bank. This result is somewhat unexpected. It rules out the possibility that the consumer reacts to his awareness of the changing behaviour by counteracting the alternative preferences, e.g. by withdrawing even less than if he were myopic in order to force the behaviour outside the bank into saving more. Apparently, the inefficiency created by having to go to the bank too often outweighs the advantage that might be gained by counteracting the effects of the other ordering. The preference for precommitment that was discussed in part I as a characteristic of non-invariant preferences takes into account the cost of the precommitment now and the possible undoing of it later.

The preceding argument rested on certain assumptions preference ordering, the right hand side is always positive.
about the expected behavioural functions. In so far, one expects that there exist expected behaviours $h(\cdot), Y(\cdot)$ for which these remarks do not hold, even though these behaviours probably have no simple interpretation.

Similarly, it is not, in general, possible to show that the policy functions $X(\cdot), h(\cdot)$ that solve (3) and (4) for any expected behavioural functions $\xi(\cdot), Y(\cdot)$ are continuous, because the uniqueness of the solution to (3) and (4) depends also on the curvature of the expected functions $h(\cdot), Y(\cdot)$. It is mainly for this reason that the game-theoretical question of whether there exist compatible policies has to be left open. Compatible policies would be functions $h^+, Y^+$ which have the property that when the consumer in the bank anticipates behaviour $h^+$, he will exhibit behaviour $Y^+$, and when the consumer outside the bank anticipates $Y^+$, he exhibits $h^+$. To show the existence of compatible policies, one would like to consider the mapping from pairs of expected behavioural functions to policy functions given by (3) and (4) and to show that it has continuity properties that guarantee the existence of a fixed point.

3. Normative Aspects of the Model

To consider normative aspects of the model, we look at the problem:
(6) \[ F(K, \alpha) = \max \int_0^\phi (e^{-\delta t} + \alpha e^{-\gamma t})u(c(t))dt \]
subject to the constraints on the withdrawal sequence under transactions costs.\(^{13/}\)

As \( \alpha \) varies on \([0, \infty)\), the set of paths that are solutions to (6) is the set of Pareto optimal paths with respect to the two orderings. A dynamic programming formulation of (6) is given by:

(7) \[ F(K, \alpha) = U(K) + \alpha V(K) \]
\[ = \max_{X, h} (f(X, h, \alpha) + e^{-\gamma h}U((K-X-a)e^{rh}) \]
\[ + \alpha e^{-\gamma h}V((K-X-a)e^{rh})) \]
where \( f(X, h, \alpha) \) is the value of the optimal program within the first withdrawal period:

\[ f(X, h, \alpha) = \max \int_0^h (e^{-\delta t} + \alpha e^{-\gamma t})u(c(t))dt \]
subject to
\[ X = \int c(t)dt. \]

We immediately note that the path chosen under changing preferences cannot be Pareto optimal in its allocation of consumption within withdrawal periods. For while the path under changing preferences depends for its allocation of consumption within withdrawal periods on the higher rate of time preference during consumption only, the optimality condition takes the other rate of time preference, the one in the bank, into account as well.

\(^{13/}\) For these constraints see equation (2) of essay 4 in this thesis.
At the path chosen under changing preferences, the preference ordering that the consumer has outside the bank is indifferent to small changes in the intraperiod allocation of consumption, while the ordering that he has in the bank is not so indifferent. The ordering in the bank would be willing to pay for a small reallocation of consumption in the withdrawal period, which could thus be beneficial to both orderings.\textsuperscript{14}\ Hence, the within period allocation of consumption is not Pareto optimal.

The underlying principle of this is quite general and extends to other models of noninvariant preferences: Whenever a set of decisions is organized in such a way that a single preference ordering uniquely determines the relation in which two commodities are consumed, and if the marginal rates of substitution are different for other orderings, the resulting allocation cannot be Pareto optimal. In the case of the decentralizing firm, the management ought to resign itself to the fact that if it gives up the day to day running of certain details to a subdivision it cannot rule on these details any more, but has to leave them to the taste of the division, even if this taste differs from the management's own. The management can only influence those aspects of the division's decision making that interact with decisions made outside the division.
Thus, there is always a cost to decentralization or noninvariant preferences if some aspect of decision making is completely under the control of one preference ordering at the expense of all others.

It is more difficult to determine whether the interaction of the different orderings in the decisions on the size of withdrawals and on the length of withdrawal periods is Pareto optimal. In view of the foregoing, we limit ourselves to paths whose within period allocation is uniquely determined by the preferences outside the bank.

There are two ways in which Pareto optimality can be investigated. One can consider the Pareto optimality of the path that is taken with respect to the two sequences of orderings, considering each type of ordering at a different moment as a new ordering. Or we can assume some initial moment and consider each sequence of invariant orderings as a single ordering. The first method appears to be more correct from a formal point of view. For subsequent orderings, even if they are invariant, are not identical as is evident by the fact that they have different domains. Also, it is more appropriate if one does not assume that the behavioural functions are stationary and the same at each step.

14/ The first order conditions for the within period allocation of consumption are:
Pareto optimality with respect to all the elements of the two sequences of orderings is a much stronger condition than with respect to the two "types" of orderings taken at some initial point. Hence, it should not surprise us to see that the game theoretic equilibrium, in which each ordering foresees correctly the behaviour under subsequent orderings cannot be Pareto optimal.\(^{15}\)

For each preference ordering chooses its decision variable in such a way that it is indifferent to small changes in it, because the direct and indirect effects balance each other off. But previous preference orderings are not indifferent to small changes in this variable and therefore would like to bribe the later orderings into choosing another value of their decision variable. However they are unable to do so.\(^{16}\)

The argument why the chosen path is not Pareto optimal with respect to the two sequences of orderings contains the reason why this result is not very interesting. For the reason why the earlier ordering cannot bribe the later ordering into changing its decision is that at the time when the later ordering takes its decision, the earlier ordering in the sense of this criterion is a thing of the past and never will recur. Thus, this criterion really considers a historical change of prefer-

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for preferences during consumption: \(e^{-\delta t}u'(c(t)) = \lambda\),

for Pareto optimality: \((e^{-\delta t} + \alpha e^{-\delta t})u'(c(t)) = \mu\).
ferences, which we noted before is not a good field for systematic analysis. Therefore, the other criterion that considers Pareto optimality only with respect to the two preference orderings from the same moment onwards does more justice to the special properties of systematically changing preferences. The procedure of identifying invariant preferences and considering them as one is clearly justified if the different types of orderings correspond to different decision making units such as divisions of a firm.

The first order conditions for Pareto optimality with respect to the two orderings are:

\[ u_x - \beta (r-\delta) h_U' + \alpha (v_x - \delta h_V') = 0 \]  
\[ u_h - r e^{-\delta h_U} + r(K-X-a) e^{(r-\delta) h_U'} + \alpha (v_h - \delta e^{s h_V} + r(K-X-a) e^{(r-\delta) h_V'}) = 0 \]

where \( \lambda, \mu \) are independent of \( t \) and the conditions hold for all \( t \leq h \). Unless \( \alpha = \phi \), the two conditions differ.

15/ A similar proposition is proved by Phelps and Pollak for the consumption-saving problem without transactions costs.

16/ The whole argument rests crucially on the fact that the first order conditions for the single preference problem (equations (6), (7) in essay 4) cannot be fulfilled simultaneously for different rates of time preference (see also footnote 17).
It is easy to see that a solution to (8) must have:

$$u_x - e^{(r-\delta)h_0} < 0$$

and

$$v_x - e^{(r-\gamma)h_0} > 0. \quad 17/$$

For any given consumption path with the property that consumption between any two withdrawals is governed by the rate of time preference outside the bank, $\gamma$, we evaluate this path at some rate of time preference $\delta$. Write the value of the path as

$$U(K; \delta) = u(X; h, \delta) + e^{-\delta h_0}u((X-X-a)e^{rh}),$$

where $u(X; h, \delta) = \int_0^X e^{-\delta t}u(c(t)) dt$ and $c'(t)$ maximizes $\int_0^X e^{-\delta t}u(c(t)) dt$ subject to $\int_0^X c(t) dt = X$.

Then,

$$u_x = v_x \int_c^t e^{(\delta-r)t} \frac{e^{\delta t}}{u''(c(t))} \frac{1}{\int_0^t e^{\delta s} \frac{1}{u''(c(t))} dt}$$

It follows that:

$$u_x \delta = -v_x \int_0^t e^{(\delta-r)t} \frac{e^{\delta t}}{u''(c(t))} \frac{1}{\int_0^t e^{\delta s} \frac{1}{u''(c(t))} dt}$$

$$= -hu_x + v_x \int_0^t e^{(\delta-r)t} \frac{e^{\delta t}}{u''(c(t))} \frac{1}{\int_0^t e^{\delta s} \frac{1}{u''(c(t))} dt} ds$$

where $v(X, h) = u(X, h, \gamma)$.

Hence, $u_x \delta < 0$, $u_x \delta + hu_x > 0$.
A similar analysis for (9) is not immediately available, because for a given consumption path, the derivative of \( u_h - \delta e^{-\delta h} U + r(K-X-a) e^{(r-\delta)h} U' \) with respect to \( \delta \) does not appear to be clearly signed.

Thus, we have to limit ourselves to saying that for a given length of the withdrawal period the Pareto optimal withdrawal is smaller than the preferences outside the bank would like it to be, but larger than required by those inside the bank.

If this path were to be duplicated for some \( \alpha \), by a path that is actually chosen and if furthermore, this path is a game theoretic equilibrium in that anticipations of behaviour are correct, then the valuation functions for the optimal path are the same as for the actually chosen path. That is to say, we can write:

\[
U(K) = W(K) \quad \text{and} \quad V(K) = Z(K-X^+-a, X^+),
\]

where \( X^+ \) is the optimal withdrawal if capital is \( K \). The first order conditions can then be written as:

\[
(3') \quad u_x - e^{(r-\delta)h} u' + (u_h - \delta e^{-\delta h} U + r(K-X-a) e^{(r-\delta)h} U')
\]
\[
\frac{dh}{dx} = 0
\]

\[
(4') \quad v_h - \delta e^{-\delta h} v + r(K-X-a) e^{(r-\delta)h} v' + r(K-X-a) e^{(r-\delta)h} \frac{dv}{dy} \frac{dy}{dk} = 0
\]

It follows that:

\[
\frac{\partial}{\partial \delta} (u_x - e^{(r-\delta)h} u') = u_x \delta + h u_x - e^{(r-\delta)h} u' - h(u_x - e^{(r-\delta)h} u')
\]
Together with the optimality conditions (8) and (9), these equations impose strict conditions on the derivatives dh/dX, dY/dK. If there exists behaviour that fulfils (8), (9), (3') and (4'), the withdrawal lies between those wanted - under single preference behaviour - by the two orderings, while the length of the withdrawal period is shorter than wanted when inside the bank, but longer than wanted outside the bank. I have not been able either to prove or to disprove the existence of such a path.

The problem with just the two preference orderings is more complicated than when we considered the Pareto optimality with respect to the two sequences of orderings; then, there was no optimality, because the earlier ordering had nothing it could trade to the later one. Now, when we have identified all invariant orderings, the earlier ordering has something to "trade", namely its own behaviour at a later decision.\[18/\]

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is positive at the point where \( u_x = e^{r-\delta} h_u' \). Hence, the term \( u_x - e^{(r-\delta)} h_u' \) is zero but once and is positive for higher rates of time preference, negative for lower rates of time preference.

\[18/\] This is similar to Samuelson's Consumption Loan Problem, in which each generation delivers to the older generation and when it has grown old, receives from the younger generation. Since one does not have the
The problem becomes even more difficult, if we analyse the normative aspects of behaviour under incorrect anticipations about the behaviour under the other preferences. In this case, the valuation functions used in the conditions for behaviour under the changing preferences and those in the conditions for optimality will differ, because the one refers to an anticipated path and the other to the actual path. The two differ because the anticipations are incorrect.

However, from the discussion of equations (3) and (4), one can derive that if the anticipations understate the deviation of the behaviour under the other preferences from the behaviour that current preferences would observe if they determined everything, then the path cannot be Pareto optimal. For we argued above that behaviour under (3) and (4) tries to accommodate the other preferences. Thus, if the anticipations have understated the named deviation, the consumer, if he becomes aware of his mistake will accommodate them even more. Thus, the other preference ordering will be better off, if he realizes his mistake, and so will, by definition, his current ordering. Therefore the path under mistaken anticipations cannot be optimal. A special case of this is myopic behaviour, in which the consumer always assumes that his current preferences govern
all his behaviour. From the preceding argument it follows that this can never be optimal.

4. Concluding Remarks

The present paper attempts to clarify the concept of changing preferences. It proposes a distinction between two different concepts, namely nonstationary invariant and noninvariant preferences. In the second part, it presents a model of sequential decision making with noninvariant preferences. It applies the Pareto ordering to analyse normative aspects of such behaviour. For a large class of cases, we find that behaviour under noninvariant preferences is not Pareto optimal, i.e. there exists another preference ordering - not necessarily stationary - with the property that if the different noninvariant orderings made a contract to leave all their decisions to this ordering, they would all be better off. In these cases, we can confirm Koopmans' proposition that there is something rational about invariant preferences.

This result is not quite conclusive because we have left open the existence of a simultaneous solution to equations (8), (9), (3'), (4'). But even if such a
solution should exist, it probably requires such a
degree of sophistication from the two preferences that
it is not very useful in problems in which the infor-
mational apparatus is limited. Thus, one can hardly
expect it to be important in models of decentralization.

Another question is to what extent the stationarity
of the behavioural anticipations will hold. Will the con-
sumer expect the other preferences to exhibit the same
behaviour at each step?

More fundamental is the problem to what extent the
change in preferences is really exogenous and to what
extent it can be willed. While at the bank, the consu-
mer may have the best will to force himself to eat less
once he starts eating. This intention may introduce some
myopic behaviour, but at the same time, it may in fact
reduce the gap between the two preference orderings.

More difficulties come in if we consider dynamics.
How does the consumer learn about the different beha-
viour under the other preferences? How is a stable pat-
tern of behaviour established? How can the Pareto opti-
mal solution - if it exists - be found?

Thus, the present paper shows that a theory of
sequential decision making under changing preferences
opens a wide field of analysis, but to reach a satis-
factory theory of such sequences of decisions, we need
a dynamic analysis of the interactions between different
decision making units.
with respect to sequences of orderings. This problem is resolved here by identifying invariant orderings.
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The Dimension of Commodity Space and a Model of Decision Making Costs

1. Introduction

Economic theory usually presupposes an understanding of the term "commodity". Introductory analysis may try to clarify the term by listing a few examples such as "apples, bread, cars, etc.", but little more is said.

The number of such commodities is taken to be given and to be exogenous to the economic process. One may find a reference to the effect that in principle, one should imagine a continuum of different commodities that could be produced and traded, but to this a note is immediately added that the limitation to a finite dimensional space has no severe effects. In part, this claim is born out by Bewley's recent extension of the main results of general equilibrium theory to infinitely dimensional commodity spaces. Yet his procedure is not entirely satisfactory. For although we want to consider commodity spaces that have an infinite dimension, potentially, the theory should be able to explain, why in reality, we only observe trading in a finite number of commodities as well as how this number is determined. Economic theory so far seems to have neglected to pose the reduction of the dimension of commodity space from its potential infinity to a finite number as an econo-
mic problem.

The present paper relates this question to the existence of fixed costs of decision making and trading. We assume that the computational apparatus of an agent is the more expensive, the higher is the dimension of the commodity bundles that it has to compare. In a very crude way, the consequences of this can be seen in the introductory listing of "commodities", where after "apples, bread, cars" the composite commodity "etc." is introduced. The listing economist finds it too bothersome for his purpose of explaining the term "commodity" to list more commodities and he names all those that he has not yet mentioned under the collective heading "etc.". Thus, we shall expect not only a reduction of the commodity space by the simple omission of commodities, but also by the combination of different commodities into a single composite commodity.

The second part of this paper goes on to discuss a particular model of decision making costs in an inter-temporal setting. In this, we come across a paradox, which precludes a complete analysis of the optimal path with the customary theoretical tools: Any attempt to explain all behaviour as optimizing behaviour remains essentially incomplete. This seems to be a fundamental problem in models with costly computing and making of decisions. In such models, optimizing behaviour
is still important, but the framework within which optimization takes place will have to be explained by another tool of analysis.

2. Fixed Costs and the Dimension of Commodity Space

We consider commodity spaces whose dimensionality has, potentially, the power of the continuum. This is the case, if commodities are distinguished by their colour, their shape or by different states of nature in which they occur, to name only a few examples.

There are two kinds of reduction of the dimensionality of this space. For instance, makers of cars pick a finite number of points out of the whole continuum of shapes and colours, in which cars could be produced. Points other than those chosen are not produced at all.

On the other hand, insurance companies do not drop certain states of nature from their program, but rather combine different states into a class of contingencies and offer contracts contingent upon any event in the class.

In both procedures, the number of degrees of freedom of any commodity bundle is reduced through the imposition of some a-priori constraints. This can usually be justified by some cost to the number of degrees of freedom of the commodity bundles that are compared. Such a cost can arise as an administrative or a transactions
cost. For instance, if each contract has to be written on a new sheet of paper or even, if each contract has to be mentioned in a separate line, agents will not concludede a continuum of contracts.

Similarly, fixed costs to the dimensionality of the commodity space may arise in the process of decision making itself, if computing machines are the more expensive - to buy or to run - the higher the number of degrees of freedom of the commodity bundles that they have to compare.

The a priori constraints will vary according to the nature of the problem. In the example of the carmaker, all constraints are zero-constraints: a set of variables is completely removed from consideration. In other cases, where sets of points are combined into a composite commodity, this new commodity may be a promise of delivery for all the points of the interval of which it is made up. The insurance company offers a contract, which is contingent on any one of a class of events. The amount delivered is the same for each point in the class. The commodity bundles have the form of step functions when seen in the original space of functions on a manifold. But other functional forms may appear as well. A consumer faced with the optimum savings problem may constrain himself to a space of piecewise exponential functions rather than step functions.

A more detailed investigation of the optimal functio-
nal form of the a priori constraints is needed. It would
be most desirable to have a concept of mathematical sim-
plicity that would be related to the number of eleme-
tary operations needed in comparing vectors under diffe-
rent constraints, so that one can relate the a priori
constraints in a more immediate way to the input in com-
putation capacity.

Intuitively, the use of step functions seems to be
the simplest procedure, but one can imagine that an
agent finds a cheap program for the comparison of ex-
ponential and piecewise exponential consumption paths,
which he uses all the more readily as he knows that in
general exponential functions play an important role in
growth theoretical problems.

Lacking the tools of a detailed analysis, we shall
work with the assumption that quantities are constant
over any interval, which is offered as a composite com-
modity, that is, all commodity bundles are step func-
tions when considered in the original space. Provisio-
nal though it is, this assumption is favoured by the
simplicity inherent in the concept of constancy on an
interval as well as by the special role that step
functions play in the derivation of the Lebesgue inte-
gral. We believe that all the results of this paper
continue to hold if one allows wider classes of a priori
constraints.
The question of whether or not certain groups of commodities are omitted completely, i.e. whether some constraints are zero-constraints, depends largely on the degree of substitutability between "adjacent" commodities. Two cars of similar colours and shapes are very good substitutes so that nobody minds if he has a zero consumption of the one, provided he can drive the other one. On the other hand, if the marginal utility of a commodity grows out of bounds as consumption of the commodity becomes small, the substitutability of neighbouring commodities is limited. This is the case if we deal with a continuum of states of nature and the marginal utility of consumption is infinite for zero consumption in all of them. Consumers are willing to pay a lot to avoid being stranded in any state of nature with zero consumption.

Another question concerns the measure of the sets on which the admissible functions are allowed to take on certain values. In the example of the carmaker, the set of points, on which car production is nonzero has the measure zero. In this case, Bewley's procedure of dealing with equivalence classes of functions which are equal except on a set of measure zero is no longer available, because all admissible functions are in this sense equivalent to the function with the constant value zero.¹/³

We summarize the main argument of this section in an
obvious but fundamental lemma which we think has to stand at the beginning of any analysis of infinite dimensional commodity spaces:

**Lemma:** Let a space of measurable functions on a manifold be given. Let an agent's objective function be defined on this space as the difference between the values of a continuous function defined on this space and a cost term which depends on the number of different positive values that an element of this space assumes. We assume that an increase in the number of different positive values raises the cost term by an amount which is bounded away from zero. Then, the function that is chosen from any constraint set to maximize the objective function takes on a finite number of values only: it is the sum of a step function and a function which is positive at a finite number of points.

The formulation of this lemma leaves the underlying space as well as the concept of continuity used deliberately vague. This is because we believe that a more precise formulation which clearly defines the function space and imposes a topology on it would add little to the stringency of the proof while limiting the general validity of the argument.

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1/ It may be argued that with limits to the precision of measurement and production, firms really offer small intervals with a guarantee that the actual
The proof is very simple. We use the fact that measurable functions can be defined as the limits of sequences of simple, i.e. step functions. One can find a step function that is arbitrarily close to the function that would be optimal in a given problem if there were no cost to the dimensionality of the commodity space. Since the objective function is continuous, the value of this step function falls short of the value of the optimum without the cost of dimensionality by an arbitrarily small amount.\footnote{If this amount is less than the cost of introducing an additional value into the range of admissible functions, then the improvement that can be reached through the admission of further values will always fall short of the cost of adding these values to the range of admissible functions. But then, the optimal function under consideration of the dimensionality cost cannot have more values than the step function considered. If the manifold that is the range of the delivery is in this interval, so that small deviations from the point explicitly offered are permitted.}

The critical step in this argument is the tacit assertion that the concepts of closeness used in the approximation of the unconstrained optimal function by a step function and in the use of the continuity of the objective function are the same. This is easily shown if the underlying space is a space of integrable
admissible functions is compact, this step function only takes on a finite number of values. If it is not compact, the objective function exhibits "eventual impatience", that is to say, it does not treat all regions of the manifold with equal importance. As one moves away from the center of the manifold, the effects of changes in the value of the function become eventually less and less important. Therefore, there exists a step function with a finite number of steps, which comes close enough to the value of the unconstrained optimal function. Thus, the number of values in the range of the optimal function with dimensionality costs is finite.

This lemma is neither surprising nor new. It has for a long time been known and used in inventory theory to analyse problems with fixed reorder costs. It says that if adjacent commodities are sufficiently similar, the agent takes a small loss in treating them alike rather than carrying the fixed cost of distinguishing between them.

On the other hand, it raises more questions than it settles. For it says nothing about the actual determination of those commodities that are eventually traded.

functions, we can impose the metric \( \| f - g \| = \int |f - g| \) and define continuity with respect to the metric topology. Then, Daniell's approach to integration assures us that as the unconstrained optimal func-
We have to distinguish between individual behaviour and market coordination. The problem of the individual is limited to the finding of an algorithm to compute the optimal number of steps and the optimal step function. The second part of this problem, namely finding the optimal step function for a given number of steps is difficult only because of the non-convexities involved. The first part of the problem, finding the optimal number of steps presents deeper logical difficulties which are discussed below.

The question of market coordination is completely in the dark. For how is one to assure that different individuals will want to combine the same classes of points into composite commodities? A priori, there is no presumption that the optimizing behaviour of different individuals will lead them to choose the same intervals for composite commodities. One agent may want to buy a contract contingent on 1 - 2 mm of rain the next day, while another wants to sell contracts contingent on .5 - 1.5 mm or 1.5 - 2.5 mm. Of course, one expects social coordina-

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4/ A similar problem is discussed in essay 4 in this the-
tion to take place through negotiations and adjustments of prices. In principle, this appears to be no more difficult than in general equilibrium theory with non-convexities and may be "reduced" to questions of who calls prices and serves as auctioneer. In the present context, the problem is somewhat more complicated because the number of intervals which have to be considered is of the power of the continuum. It would be very hard for any auctioneer to call a continuum of prices. And any social process, which serves the same functions as the auctioneer would presumably find the task equally hard. Similarly, agents would have to react to all these prices at the same time or else adjustment speeds may be pathological.

In the spirit of the whole argument of this section, we propose to treat this not only as a matter of the greater complexity of the price vector "called", but to give it a qualitative economic significance. Suppose for instance that the auctioneer is paid a wage that increases with the dimension of the price vector that he calls, or equivalently, that a disequilibrium adjustment process is the more costly the more commodities are traded or attempted to trade. In this case, it follows from

s sis, with piecewise exponential rather than step functions.

5/ If composite commodities are formed not only from intervals, but from any set of points, the number of
the lemma that it would be uneconomical to call prices for all the possible composite commodities. If we consider the problem of social coordination as a question of how many and which composite commodities are to appear in the adjustment process, then this problem has a cyclical logical structure similar to the one pointed out by Radner for individual optimization with computation costs: It requires an input in order to gain some information, before it is known whether the information gained is worth the cost of the input.

In principle, it does not seem possible to find solutions to such cyclical problems. Possibly, both the optimal computational capacity and the optimal social coordination in the present context cannot be found. The next section begins the analysis of a particular cyclical structure at the level of the individual decision maker.

3. The Optimal Computation Capacity in the Anticipatory Calculation

Consider a consumer who tries to solve the optimum savings problem under costly decision making. The costs of decision making depend linearly on the number of different positive values that consumption may have along any path that he considers, i.e. on the number of steps possible commodities is $2^c$, where $c$ is the power of
as his consumption path is a step function. An additional step in the consumption path costs him a constant amount $A$ in terms of utility.

Suppose furthermore that capital pays a fixed rate of interest $r$ and that the consumer has a well-behaved instantaneous utility function $u(c)$ and a rate of time preference $\delta$. Then, the value of the $n$-step-path problem is, for given $K$:

\[
V_n(K) = \max_{c,h} \left( u(c) \frac{1 - e^{-\delta h}}{c, h} + e^{-\delta h} V_{n-1} \left( (K - \frac{c}{r}) e^{rh} + \frac{c}{r} \right) \right)
\]

One might argue whether the spacing of steps should be subject to the consumer's initiative. His problem is much more complicated if he has to decide on the length of steps as well as on the consumption in different steps. The added complication is all the more serious as it not merely increases the number of decision parameters, but also introduces a non-convexity, since the convex-combination of two $n$-step paths with different points of discontinuity is not, in general, an $n$-step path. On the other hand, if we exclude the spacing of steps from the consumer's problem, we have to have a model of how they are determined. In particular, in order to be able to maintain the recursive
structure of (1), we have to make sure that the n discontinuity points of the n-step problem include the n-1 discontinuity points of the n-1-step problem as it is anticipated for the moment when the first step has finished. For this reason, we shall assume that the spacing of steps is a matter of individual decision making. We shall refer to the alternative problem as we go along, always under the assumption that the n-step problem has a recursive structure similar to that in (1·), so that although the spacing is fixed, it is arranged in such a way that the discontinuity points of the n-step path include those of the ensuing n-1-step path.

Now we are in a position to discuss Radner's problem that the consumer has no algorithm to find the optimal computational structure, if he has to incur the computational cost before he knows whether it is worth it. This is more a problem of the lumpiness of computational capacity than of the cyclical structure of the decision costs problem. If the consumer has to choose a computational structure beforehand, then surely he has no way of knowing the optimal n for his choice. But if he can buy computational capacity bit by bit and use whatever he has already bought while he thinks of buying more, then he can proceed gradually, buying computational capacity unit by unit. He can compute the optimal path for whatever capacity he has already bought and
compare it to the optimal path at the preceding stage. He can stop buying when the increase in computational capacity has shown a negative net pay-off for the first time, that is when the increase in the value of the optimal path due to the greater precision falls for the first time short of the cost of the additional unit of computational capacity. In this way, he can approximate the optimal computational structure up to one unit. The final computational capacity will satisfy the condition:

\[ (2) \quad V_{n-1}(K) - V_{n-2}(K) \geq A \geq V_n(K) - V_{n-1}(K) \]

If for any \( K \), the value of the optimal program is concave in the number of steps \( n \), condition (2) has a unique solution in \( n \). This solution has a smaller value than the number \( n-1 \), because the last step actually has a net negative effect. But on the other hand, under appropriate concavity assumptions, the number \( n-1 \) gives a global maximum of \( V_x - xA \), so that the number \( n \) that is chosen according to this algorithm is close to the optimal computation capacity. This would be ensured by:

**Concavity Assumption I:** For all \( n \) and \( K \),

\[ V_n(K) - 2V_{n-1}(K) + V_{n-2}(K) \leq 0 \ . \]

A somewhat weaker condition is:

**Concavity Assumption II:** For all \( n \) and \( K \),

\[ V_n(K) - 2V_{n-1}(K) + V_{n-2}(K) \leq 0 \text{ implies } V_{n+1}(K) - 2V_n(K) + V_{n-1}(K) \leq 0 \ . \]
Assumption I asserts that the value of the optimal path is everywhere concave in \( n \). Assumption II allows for the possibility that there are initially increasing returns to the expansion of computational capacity. It merely asserts that once returns begin to decrease, they go on doing so. Since the value of the optimal path is for all \( n \) bounded above by the value of the unconstrained optimum (the Ramsey path), provided the latter exists, it follows that returns to the expansion of computational capacity must be eventually decreasing, so that from some point onwards, under II, the value of the optimal path is concave in \( n \).

Thus, we have developed an algorithm, which under concavity assumptions I or II, determines a computational capacity with the properties that any higher computational capacity leads to a worse result and that it exceeds the optimal computational capacity by not more than 1.

---

6/ I have not found a proof that I or II holds in Ramsey problems, although I think it likely. Since the \( n \)-1-step policy is less than optimal in both the \( n \)-step and the \( n \)-2-step problems, we have the inequality:

\[
V_n - 2V_{n-1} + V_{n-2} \geq e^{-\delta n}(V_{n-1} - 2V_{n-2} + V_{n-3}),
\]

where the argument of the right hand side is that value of capital that is reached at time \( h \) on the
This algorithm is not available and Radner's paradox takes full hold, if the costs of decision making depend also on that part of decision making that determines the optimal capacity itself. In the above example, if the determination of the optimal $n$ according to our algorithm costs some amount $B$, then the consumer has no way of knowing beforehand, whether it is worth incurring this cost or not. To decide this, he would have to evaluate the difference in value between the optimal program under the computation capacity that satisfies (2) and the optimal program under a randomly picked capacity, but this would require that he already know the capacity that satisfies (2).

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n-step policy. Hence, for the expression in assumption I to be positive, the same expression would have to be positive one step earlier, at the earlier value of capital and a higher computation capacity. Going further backwards, there is a whole sequence of values of capital, representing a past history of the consumer, where the second difference of the optimal value with respect to $n$ is positive, and along this sequence, $n$ is increasing and growing out of bounds. This is implausible, since for any $K$, that second difference must eventually be negative to let the values $V_n$ converge towards the value of the
4. Anticipated and Realized Paths

The preceding analysis is incomplete because it only deals with a single decision. Consumption paths are compared as they appear at the moment of this decision. But in general, consumers will make sequences of decisions. From dynamic programming, we know that the approach outlined in the preceding section is not restrictive if subsequent decisions follow precisely the path that was anticipated in the first decision. In the present context there is no reason to assume that this will be the case. If at a later moment, e.g. at the end of the first step, the consumer has to make a new decision, part of what he originally treated as a problem has been removed from further consideration because it is past. In a finite dimensional space, a finite horizon problem, one might say that the problem has become less complex because its dimensionality has been reduced. With the infinite time horizon, this statement cannot be made. But one can argue the same point from the input side by looking at the computational capacity. One notes that in the new de-

Ramsey path. But as, along the sequence, the value of capital is constantly changing, this argument is not conclusive, however suggestive it may be.
cision at the end of the first step, an n-1-step computing machine is able to compute that path that, in the original decision, was to follow the first step. If the n-step machine that he used in the first decision is still available without additional cost, he will of course use it and choose a new n-step path instead of following the n-1-step tail of the original n-step path. Thus, the path that is actually chosen under repeated decision making with the same capacity for each decision differs from the path that is anticipated in any single decision along the sequence.

Thus, a consumer in his youth fails to distinguish between two consecutive days in his old age and assigns equal consumption to both, but when he has become old and no longer has to bother about consumption in the time between youth and old age, he finds it profitable to distinguish between the same two consecutive days and assign different consumptions to them, even though his computing capacity has not been increased.

To simplify the further discussion, we assume that preferences in subsequent decisions are invariant, that is to say that preferences for different decisions coincide on the intersection of their domains, where the dimensions of commodity space are labeled by their absolute position in time.7/

7/ The invariance property is discussed in detail in
It is easy to see that the realized path can never be worse than the anticipated path, if under subsequent decisions, the anticipated path is always available, but another path is preferred to it, because the feasible set is wider than was anticipated. If the anticipated and the realized paths differ, then the realized path is strictly better than the anticipated path.

If $n = 1$, the constant path is chosen and this decision is confirmed by all later decisions, so that the anticipated and realized paths coincide. (Diagram 1).

Also, in the limit as $n$ approaches $\infty$, the optimal anticipated path approaches the optimal unconstrained path (the Ramsey path). Since no path can overtake the Ramsey path and since in the unconstrained problem, anticipated and realized paths coincide, they must be equal in the limit as $n$ goes to $\infty$. (Diagram 2).

Diagram 1: $n = 1$  
Diagram 2: $n = \infty$

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8/ Since the Ramsey path is continuous, it can be represented by a countable number of parameters. In this
For finite \( n > 1 \), the problem becomes more complicated. We have to consider different constellations of the moments of revisions of decisions and of the anticipated discontinuity points. In the most appealing case the next decision takes place at the first discontinuity point of the anticipated path.\(^9\) At the moment of the new decision, the consumer can choose a new \( n \)-step path instead of the \( n-1 \)-step tail of the original path. The initial decision had been optimal in the intertemporal allocation between the first step and the subsequent \( n-1 \)-step path. The new decision substitutes an \( n \)-step for the \( n-1 \)-step path and is not bothered by the fact that this leads to a violation of marginal conditions between the first period which is already past and later periods. Under strict concavity, the realized path is strictly better than the anticipated path, since the latter could still have been chosen (Diagram 3).

\(^9\) If there is a current cost to decision making itself, the consumer will plan to have the next decision at this discontinuity point. At other points, his present anticipations do not allow him to foresee an immediate change in consumption so that a new decision
Diagram 3: \( n = 2, 1 \) revision

If the revision of the decision is planned for a moment before the first discontinuity point, the improvement is more subtle. For at the time, when the new decision is taken, both the old anticipated path and the newly chosen path foresee \( n \) steps. But whereas the first step of the old anticipated path is fixed both in its length and in its consumption level, being the end of a step, which had begun before, the first step of the new path is chosen freely. In this choice, the new decision will be able to take account of the fact that since the first decision, capital has been accumulated (if \( r > \xi \)), so that initial conditions have changed. It adjusts by increasing the level of initial consumption.

Thus, the early revision with the chance to adjust to changes in the parameters gives the realized path the possibility to distinguish better between consumption at

seems to remain without immediate effect. The next decision will be planned for a moment when it appears to have an immediate effect, that is for a discontinui-
different moments than the anticipated path. For continuous revisions of decisions, the realized path will even be a smooth function. Although it is not optimal with respect to the power of distinction that it reveals - the actual consumption sequence will obey no marginal condition related to the realized path - it is always better than the anticipated path on whose choice its first step is based.\footnote{ty point.}

The argument has to be qualified somewhat if the spacing of steps is not subject to choice, but given. If at the time, when the new decision is made, the discontinuity points for the new anticipated path are the same as those for the tail of the old one, it is still true that the tail of the old anticipated path is in the feasible set of the new decision. This is the case, if the new decision is made at the end of the first step and the problem has the recursive structure of (1). If there are earlier revisions of decisions, the assumption that the new discontinuity points are the same as those in the tail of the old anticipated path is very stringent. It may then be preferable to assume outright that the new anticipated path is better than the tail of the old one. Since this is the essence of a revision of a decision, to allow a correction
Analytically, we find that the realized path can be constructed from two policy functions \( c(\cdot), h(\cdot) \), giving the consumption level and the length of the period before the first revision of the decision for any capital \( K \). For while the anticipated path uses a different policy function at each step, first the \( n \)-step functions, then the \( n-1 \)-step functions etc., the realized path uses the same policy functions at each step.

For any two functions \( c(\cdot), h(\cdot) \), we can construct a sequence of values of capital at the moments of decision making:

\[
K_0 \text{ is given}
\]

\[
(3) \quad K_{i+1} = (K_i - \frac{c(K_i)}{r}) e^{r h(K_i)} + \frac{c(K_i)}{r}
\]

This gives rise to the consumption path:

\[
c(t) = c_1 = c(K_1) \text{ for all } t \text{ such that}
\]

\[
\sum_{j=0}^{i-1} h(K_j) \leq t < \sum_{j=0}^{i} h(K_j)
\]

The value of this path is given as:

\[
R(K_0) = \sum_{i=0}^{\infty} u(c_1) \frac{1 - e^{-\gamma h_1}}{\gamma} e^{-\gamma \frac{k'}{\theta} h_j}
\]

\[
(4) \quad = u(c_0) \frac{1 - e^{-\gamma h_0}}{\gamma} + e^{-\gamma h_0} R(K_1)
\]

It has to be stressed that the function \( h(\cdot) \) does

of the earlier decision, this assumption is rather natural.
not give the length of the first step of the anticipated path, but rather, the length of the period before the revision. Our formulation rests on the tacit assumption that the latter is not larger than the former. The beginning of the realized path corresponding to the n-step policy is the same as the beginning of the anticipated n-step path. Therefore, we can write:

\[
R_n(K_o) - V_n(K_o) = e^{-\gamma h_o}(R_n(K_1) - W_n(K_o, h_o)),
\]

where \(W_n(K_o, h_o)\) is the value of the tail of the anticipated path after \(h_o\) has elapsed. We have:

(5) \[
R_n(K_o) - V_n(K_o) = e^{-\gamma h_o}(R_n(K_1) - V_n(K_1) + V_n(K_1) - W_n(K_n, h_o))
\]

and:

\[
\inf_K (R_n(K) - V_n(K)) \geq \frac{1}{e^{\gamma h - 1}} (V_n(K_1) - W_n(K_o, h_o))
\]

\[
\sup_K (R_n(K) - V_n(K)) \leq \frac{1}{e^{\gamma h' - 1}} (V_n(K_1') - W_n(K_o', h_o'))
\]

where the 's and 's denote the specific values of the variables where the inf and sup occur.

We can now confirm the previous intuitive argument: For \(n = 1\) and \(n = \infty\), the right hand sides of the last two conditions are zero; the difference between the realized and the anticipated paths vanishes. For all other \(n\), the right hand sides are positive so that the realized path always overtakes the anticipated path. If \(h(.)\) happens
to coincide with the length of the first step of the anticipated path, equation (5) simplifies to:

\[
R_n(K_0) - V_n(K_0) = e^{-\mathcal{S}^{ho}(R_n(K_1) - V_n(K_1))} + V_n(K_1) - V_{n-1}(K_1).
\]

5. The Optimal Computation Capacity and the Realized Path

In order to evaluate the effect of an expansion of the computation capacity on the realized path, we have to compare the policy functions \(c_n(\cdot), c_{n-1}(\cdot)\) in the n-step respectively n-1-step problem. We note that the formal structure of problem (1) is the same as the one treated in essay 4 in this thesis. Therefore, it can be proved in the same way as is done there that the sequence of policy functions \(\left\{c_n(K)\right\}_{n=1}^{\infty}\) is everywhere strictly decreasing in \(n\). The n-step policy is everywhere strictly closer to the unconstrained optimal policy than the n-1-step policy. Therefore, the realized path under the n-step policy is closer to the Ramsey path than the realized path under the n-1-step policy. Since - apart from the computational constraints - the boundary of the set of feasible paths is linear and the n-step realized path lies on this boundary between the Ramsey path and the n-1-step realized path, it must be preferred to the latter since preferences are convex. Thus, we have:

\[
R_n - R_{n-1} \geq 0.
\]
Thus, the expansion of computational capacity also brings about an improvement in the realized paths. But the extent of this improvement is difficult to evaluate. Clearly, it must initially exceed the improvement in the anticipated paths, since we have $R_1 = V_1$ and $R_2 > V_2$. Similarly, for large $n$, the improvement in the realized paths must fall short of that in the anticipated paths, because for finite $n$, $R_n > V_n$, but $R_\infty = V_\infty$.

In general, the relation between the effects of an increase in $n$ on the realized path and on the anticipated path can be evaluated from equation (5) or equation (6). To simplify the exposition, we deal only with equation (6), that is to say with the case where the revision of decisions occurs at the end of the first step of the previously anticipated path. From (6) we have:

$$R_n - R_{n-1} - V_n + V_{n-1} = e^{-\gamma h_n}(R_n(k_1^n) - R_{n-1}(k_1^n))$$
$$- V_n(k_1^n) + V_{n-1}(k_1^n))$$
$$+ e^{-\gamma h_n}(V_n(k_1^n) - 2V_{n-1}(k_1^n))$$
$$+ V_{n-2}(k_1^n))$$
$$+ e^{-\gamma h_n}(R_{n-1}(k_1^n) - V_{n-2}(k_1^n))$$
$$+ e^{-\gamma h_{n-1}}(R_{n-1}(k_1^{n-1}) - V_{n-2}(k_1^{n-1}))$$

The first term on the right hand side is the same as the left hand side evaluated at the next value of capital $k_1^n$ and discounted back by $h_n$. The second term is not
greater than $V_{n+1}(K) - 2V_n(K) + V_{n-1}(K)$, because the n-step policy is less than optimal both in the n+1- and in the n-1-step problems.

Suppose that the number $n$ is the one determined by the algorithm described in section 3 above. Then, under the concavity assumption I or II, the term $V_{n+1} - 2V_n + V_{n-1}$ is negative and therefore, the second term in (7) is negative.

The difference between the last two terms is generally positive.\(^{11}\) This is a substitution effect that comes about because the use of the n-1-step policy instead of the n-step policy reduces the extent to which the value of the n-1-step problem falls short of that of the n-step problem ($V_n - V_{n-1} \leq e^{-\delta h_n}(V_{n-1}(K^n) - V_{n-2}(K^n))$). On the other hand, the same substitution increases the difference between the realized paths for n- and n-1-step policies. For we use the same argument that showed $R_n - R_{n-1} \geq 0$, to argue that

$$\frac{1 - e^{-\delta h_n}}{\delta} u(c_n) + R_{n-1}(K^n) e^{-\delta h_n} \geq \frac{1 - e^{-\delta h_{n-1}}}{\delta} u(c_{n-1}) + R_{n-1}(K^{n-1}) e^{-\delta h_{n-1}}$$

Even if from the next decision onwards, the agent always follows the n-1-step policy, yet he finds it profitable to follow the n-step policy in the first period.\(^{12}\)

---

\(^{11}\) If $h_n = h_{n-1}$, (h is fixed independently of n), this
This substitution effect works to increase the pay-off to the expansion of computational capacity beyond what it would be in the anticipated path. It is counteracted by the effect of the second term of (7), which depends on the decrease of the anticipated pay-off and which works to reduce the realized pay-off relative to the anticipated pay-off. This effect is caused by the fact that the realized path exceeds the anticipated path because the later decision has \( n \) rather than \( n-1 \) steps available. The excess of the realized path over the anticipated path depends on the difference in value, at the next step, between the \( n \)- and the \( n-1 \)-step problem. Thus, the improvement of the realized path with an increase in \( n \) will depend on the change in this differential over the anticipated path, which is exactly the second difference of the value of the anticipated path with \( n \).

The trade-off between these two effects is not clear cut and more specific propositions will depend on the properties of particular cases. For a large class of ca-

\[ \inf(R_n^i - V_n^i) \geq (V_n^i - V_{n-1}^i)/(e^{S^i h} - 1) \geq 0, \]

so that \( R_n - V_{n-1} \) is increasing in \( K \), from which the proposition follows.

The different effect of the substitution on the anti-
ses, the realized pay-off will fall short of the anticipated pay-off. This will be especially true for large n, where the policy functions lie close together and the substitution effect is small. If the substitution effect is small enough to be outweighed by the first term in (7), provided that term is negative, then, equation (7) leads to the inequality

$$R_n - R_{n-1} - V_n + V_{n-1} \leq 0.$$ 

The realized pay-off in this case is not only smaller than the anticipated pay-off, but it is smaller than the anticipated pay-off at the higher computation capacity. Since at the point chosen by condition (2), this latter difference is decreasing in n, this implies that the deviation of the realized pay-off from the anticipated pay-off exceeds the bounds permitted by the integer character of the problem. Where the anticipated pay-off fails for the first time short of the cost of expanding computation capacity, the realized pay-off has done so at least once already. The computing capacity that would fulfil a similar condition as (2) with respect to the realized paths is exceeded by at least one unit.

cipated and the realized paths is understood more clearly, if we consider that the realized n-step path has in fact an infinite sequence of steps, so that at any stage, the use of a policy which lies closer to the Ramsey path is advantageous.
More generally, the problem is the following: One wishes to determine the optimal computational capacity with respect to the realized rather than the anticipated path. If one assumes that the underlying structure is concave, one looks for \( n \) such that

\[(8a) \quad R_{n} - R_{n-1} \geq A > R_{n+1} - R_{n} \, .
\]

In the spirit of section 3 of this paper, one would be satisfied, if one could determine a capacity which exceeds the one given by (8a) by exactly one, that is if one could find \( n \) such that

\[(8b) \quad R_{n-1} - R_{n-2} \geq A > R_{n} - R_{n-1} \, .
\]

The realized paths take on, in general, an infinite number of values. Therefore, the consumer cannot compute the value of the realized path with his limited computational capacity.

The preceding discussion about the relation between the anticipated and the realized paths now showed that it serves the consumer little, if he uses the pay-off to the anticipated path as a proxy for the pay-off to the realized path, because the two will in general differ and not even the sign of the difference is clear. In many cases, the use of crite-

\[13/\] A very drastic example is the case of the logarithmic utility function with \( h \) fixed independently of \( n \). Consumption is \( rK(1-e^{-gn})/(1-e^{-rn}) \), independent
rion (2) to determine the computation capacity over-
estimates the need for it as given by the realized path.

This case seems to be somewhat paradoxical because
we can regard the difference between the realized and
the anticipated path as a difference between the single
and the repeated application of the same computational
structure. In the choice of the realized path, the
same input is applied more often. Thus, one would expect
its output to be greater. This is in fact what happens
on the way from the first to the second step. For higher
n however, the repeated application of the computing unit
itself introduces additional precision which reduces the
need for computational precision.

Going back to the example of the consumer in youth
and old age, it may occur that the consumer is not only
less precise with respect to what is far away and introdu-
ces more and more precision the closer it comes, but also,
that he is all the more prepared to be imprecise with re-
spect to consumption in his old age, because he knows
that when he will be old, he will have all the time he
needs to introduce the precision which he now foregoes.
It is the very knowledge that there are more decisions to
come which makes him unwilling to be too precise now.
6. Concluding Remarks

In this paper, we have first shown that if there is a cost to the dimensionality of the commodity space, then only a finite number of commodities will generally be traded in real markets. Then, we have formulated a model, in which the cost of the dimensionality of the commodity space appears as a computational cost and we have discussed Radner's problem that there is no algorithm to find the optimal computational input because this input has to be incurred before one knows whether it is worth it. If only a single decision is taken, one can give a procedure which approximates the optimal solution.

If there is a sequence of decisions all of which are taken under the same computational structure, this procedure is no longer applicable. There is no reassurance that it even approaches the optimal computational input. It is in principle not possible to find the optimal input, if repeated decisions are taken.

One suspects that the problem becomes even less solvable, if the computational structure, too, has to be decided upon anew each time. In this case, subsequent decisions have an even larger measure of freedom and become even more complicated to forecast. The central problem that repeated decision making gives individual behaviour a flexibility which the individual itself cannot anticipate still remains.
The proposition that the computation capacity cannot be determined by the individual's optimizing behaviour raises the question how it is determined. Presumably, individuals still intend to optimize, even if they have no way to find the optimum. Then how do they react when they find out from their past experience that they chose a less than optimal computation capacity? And how quickly do they find out about this? These questions suggest the necessity of a positive analysis of the processes which determine decision making structures and their evolution. This problem is less academic than it may appear at first sight. For it seems to be related to the very cyclical behaviour of firms' overhead expenses. In practice, one observes that firms will strongly cut down on their overhead when they are in a profit squeeze only to overexpand it in a boom. Since the administrative overhead serves a similar function as the computational input in our model of consumer behaviour, one suspects that this cycle represents a pattern of decision and error correction which is based on the fact that anticipations overestimate the need for additional overhead input. In this setting, the external conditions of the business cycle would merely set off the pattern by sharpening or blunting attention for particular cost factors.

Another set of questions is related to the compu-
tation technology. Is it possible to adjust for the imprecision of the computation under limited capacity and to find transformations which reduce the error? In our model for instance we noted that the limitation of computational capacity reduced the marginal utility of resources held into the future and led to too low a rate of saving. How well can the consumer do if he adjusts for this by multiplying the marginal utility of future capital by the uniform factor \((1+a)\), i.e. if he uses \(V'_{n-1}(1+a)\) instead of \(V'_{n-1}\) for his consumption decision? The introduction of a makes his problem more complicated by introducing another decision parameter. But perhaps if he uses the factor \((1+a)\) he can use a smaller number of steps in his consumption path. This line of reasoning leads on to another problem mentioned in this paper, that of the functional form of the a-priori constraints. Its further investigation will require a detailed knowledge of the technology of computation and will reduce the different functional forms to some basic input requirements. But it is not to be expected that such a deeper analysis of the problem of costly decision making will reduce the number of basic paradoxes that occur in this area.
Bibliography


A Note on Optimal Savings with Fixed Transactions Costs

1. Recent papers by Weitzman and Dixit, Mirrlees and Stern have discussed the problem of optimal growth with increasing returns to the scale of investment. They show that in such an economy, investment will be made at discrete moments in time, namely when the increased return from deferring investment in order to increase its size is at the margin equal to the utility cost of foregoing the benefits of new investment during the deferral period. A special case of these models is the consumer's savings problem if there are fixed costs of making transactions with the bank.

2. The papers referred to discuss problems of existence of an optimal solution as well as properties of this solution if it exists. They do not discuss the problem of the uniqueness of the optimal solution and the related question of the continuity of the resulting policy functions. The present paper looks at this latter question. In the course of the discussion, we come across a condition, which Dixit, Mirrlees and Stern have shown to be true asymptotically; namely, for the optimal solution to the consumer's saving problem to be unique, it is sufficient that the optimal length of the withdrawal period be always decreasing in the amount of capital that is available.
3. The present paper discusses the consumer's savings problem with a fixed cost of making withdrawals from the bank. The model presented here differs from that of Dixit, Mirrlees and Stern in that it introduces time discounting. Furthermore, it is assumed that the consumer's interest income is paid into his account. He withdraws whatever he needs for consumption purposes from the account and pays in cash for consumption goods. As opposed to the Dixit, Mirrlees, Stern model, where interest income was paid out in cash and savings invested at a fixed investment cost, the present discussion is more complicated, because the size of a withdrawal affects the consumer's saving not only through its direct effects on consumption, but also through its effect on the base, on which interest is paid. In a formal sense, the present approach has the advantage of allowing for compound interest.

4. A consumer has instantaneous utility of consumption \( u(c) \). He pays for his consumption in cash, which he withdraws from the bank at a fixed cost \( a \) per withdrawal. Any money in the bank earns interest \( r \). Let \( X_1 \) be the withdrawal at time \( t_1 \). This amount is consumed over a period \( h_1 \). If initial capital in the bank at time \( t_1 \) was \( K(t_1) \), capital in the bank after a period of length \( h_1 \) is \( (K(t_1) - X_1 - a) e^{rh_1} \).
If the consumer has an infinite time horizon, his problem can be formulated as follows:

(1) \[ \max \int_0^\infty e^{-\delta t} u(c(t)) \, dt \]

subject to:

(2) \[ K(t_{i+1}) = (K(t_i) - X_i - a) e^{rh_i} \geq 0 \]

\[ X_i = \int_{t_i}^{t_{i+1}} c(t) \, dt \geq 0 \]

\[ t_{i+1} = t_i + h_i \]

\[ h_i \geq 0 \]

for all \( i, i = 1, 2, \ldots \)

\[ K(0) = K, \text{ given} \]

\[ t_0 = 0 \]

In this formulation, the consumer, who is about to make a withdrawal, selects sequences of withdrawal sizes and of length of withdrawal periods to maximize the discounted integral of utility of the consumption path that is feasible under these two sequences. Implicitly, we already assume that it is always preferable to run down cash balances to zero before making a new withdrawal from the bank. This is easily understood, if one considers that any path which foresees a withdrawal while the consumer still has some cash is strictly overtaken by the path that first, without changing consumption, uses up the cash and earns more interest on the money that was to be with-
drawn as well as on the transactions cost.
We shall assume that the rate of interest \( r \) does not fail short of the rate of time preference \( \delta \). This assumption will have the effect that the consumer will want to accumulate capital and to increase consumption as time goes on.
Suppose that the consumer makes a withdrawal. Then, he consumes this amount over a period of length \( h \). Afterwards, he faces the same problem as initially, with a different value of his bank balance. Thus, we can transform problem (1) into the standard dynamic programming form:

\[
V(K) = \max_{X,h} \left( w(X, h) + e^{-\delta h} V((K - X - a)e^{rh}) \right),
\]

where:

\[
w(X, h) = \max_0^h e^{-\delta t} u(c(t)) dt
\]

subject to \( \int_0^h c(t) dt = X \).

\( V(K) \) is the value of the infinite horizon problem with initial bank balance \( K \), \( w(X, h) \) is the value of the optimal consumption path that uses the amount \( X \) over a period \( h \). Equation (3) makes use of the fact that any solution to problem (1) certainly must be optimal within the first withdrawal period as well as from the next withdrawal onwards.

5. We assume that the instantaneous utility function \( u(c) \) is twice continuously differentiable with
u'(c) > 0 and u''(c) < 0.

Furthermore, consider the family of twice differentiable functions \( F \), given as:

\[
F = \{ f(\cdot) : f'(x) + x f''(x) \leq 0, \forall x \} \tag{4}
\]

Any function \( f \) in \( F \) has the property that the function \( g(y) \) defined as:

\[
g(y) = f(e^y)
\]

is concave. For this reason, the family \( F \) is well known in optimal growth theory. If \( u \) is a member of \( F \), we know that there exists an optimal solution to the ordinary Ramsey problem with the instantaneous utility function \( u \) and income linear or concave in capital. With a fixed output-capital ratio \( r \), Ramsey's problem is written as:

\[
\text{Max} \int e^{-st} u(c(t)) dt
\]

subject to \( r K = c(t) + \dot{K}(t) \geq 0 \)

\( K(0) \) given.

If we let \( a = 0 \), any path with a strictly positive withdrawal period is overtaken by a path, which allows at least the same consumption at all times, but has a smaller withdrawal period, so that the optimal length of the withdrawal period is zero. Then, if we take limits in (2) as \( h_1 \) go to zero, we get the constraints of problem (5). Thus, the Ramsey problem (5) is equivalent to problem (1) for a zero with-
drawal cost.

**Proposition 1:** For all \( K > a \), an optimal solution to problem (1) exists if the instantaneous utility function is a member of \( F \). Its value for any initial capital \( K \) is given by the unique continuous function \( V(K) \) which satisfies the functional equation (3).

**Proof:** We prove proposition 1 by induction on the number of withdrawals, that is to say, we consider a sequence of problems, in which first only one more withdrawal from the bank is permitted, then two more, then three etc.

We define the sequence of functions:

\[
V_0(K) = w(K - a, \infty)
\]

\[
V_n(K) = \max_{X,h}(w(X,h) + e^{-\mathcal{S}h}V_{n-1}((K-X-a)e^{rh}))
\]

By inspection, the function \( V_n \) gives the value of the optimal program, if the consumer is restricted to making \( n \) further withdrawals (after the initial withdrawal).

We note that for all \( n \), \( V_n(K) \) is dominated by the value of an ordinary Ramsey problem with the same initial capital. This follows from the equivalence of Ramsey's problem to problem (1) for \( a = 0 \).

By induction, the functions \( V_n \) exist and are continuous, being the pointwise bounded maximum values of continuous functions over closed sets.
Now we show that for all \( K \), the sequence \( \{ V_n(K) \} \) is increasing in \( n \). It is always better to make \( n \) withdrawals than to be able to make only \( n-1 \) withdrawals. Since the \( n \)-withdrawal program can always follow the \( n-1 \) - withdrawal program up to the last withdrawal and then choose a 1-withdrawal program instead of the 0-withdrawal program, which withdraws the whole outstanding balance, it is sufficient to show that every 0-withdrawal program is dominated by a 1-withdrawal program.

This is certainly the case, if for every 0-withdrawal program, there exists a time \( s \), such that if the consumer follows the 0-withdrawal program up to \( s \), but leaves the consumption needs after \( s \) according to the 0-withdrawal program in the bank, these funds earn an interest income, which, by the time \( s \), exceeds the cost of making another withdrawal at time \( s \).

For any initial capital \( K \), let \( c_0(t) \) be the chosen consumption path under the 0-withdrawal problem. Let \( Y(t) \) be the resources needed to finance this path from time \( t \) onwards. Then we have:

\[
Y(s) = \int_s^\infty c_0(t) \, dt
\]

If the instantaneous utility function \( u \) is in the family \( F \), we have, by equation (6") in the appendix:

\[
Y(s) \geq -\int_s^{dc/dt} \, dt / \mathcal{G} = C(s) / \mathcal{G}
\]
From the same equation it follows by integration
that $c_9(s) \geq c_9(0) e^{-\mathcal{g} s}$, so that
$$Y(s) \geq c_9(0) e^{-\mathcal{g} s} / \mathcal{g}.$$  
If he leaves this amount in his account instead of withdrawing it immediately, then at time $s$, his account shows a balance of
$$K(s) = Y(s) e^{rs} = Y(s) + Y(s)( e^{rs} - 1)$$
$$\geq Y(s) + c_9(0) e^{-\mathcal{g} s} (e^{rs} - 1) / \mathcal{g}$$
$$\geq Y(s) + c_9(0)(e^{r-\mathcal{g}} s - 1) / \mathcal{g}$$
For $r \geq \mathcal{g}$, there clearly exists $s$, such that $K(s)$ is strictly greater than $Y(s) + a$, so that the 1-withdrawal policy is strictly better than the 0-withdrawal policy.\(^1\)

Being monotone and pointwise bounded, the sequence $\{V_n\}$ converges pointwise to a limiting function $V$. Clearly, $V$ satisfies the functional equation (3).
Suppose that there exists a program, which is feasible under (2) and gives a higher value than $V$; and call its value $W$.

\(^1\) For $r = \mathcal{g}$, the same result holds, if $u'^+cu''$ is bounded away from zero, so that $Y(s)$ decreases in $s$ at a rate strictly less than $r$. If $r = \mathcal{g}$ and $u'^+cu'' = 0$, $K(s)$ for large $s$ approaches $Y(s) + K - a$, so that for $K > 2a$, the 1-withdra-
Comparing \( W \) and the value of the \( n \)-withdrawal program \( V_n \), we know that the \( n \)-withdrawal program is certainly not worse than a policy which follows the first \( n \) steps of the program of value \( W \) and then withdraws the whole outstanding balance. Hence:

\[
W(K) - V_n(K) \leq e^{-\frac{\delta}{k}h_1(W(K(\sum_{i=0}^{n} h_i))) - V_o(K(\sum_{i=0}^{n} h_i)))}
\leq e^{-\frac{\delta}{k}h_1\max(|W(K(\sum_{i=0}^{n} h_i))|,|V_o(K(\sum_{i=0}^{n} h_i))|)}
\]

From our comparison of the Ramsey problem with problem (1), we know that \( W \) is always dominated by \( W^+ \), the value of the Ramsey problem (5) for the same initial capital. Furthermore, both \( W^+ \) and \( V_o \) are increasing in their argument and capital at time \( t_{n+1} = \sum_{i=0}^{n} h_i \) under the path with value \( W \) cannot exceed \( Ke^{rt_{n+1}} \), that is the amount, which one would have at time \( t_{n+1} \), if one made no withdrawals before \( t_{n+1} \). Thus, we have:

\[
W(K) - V_n(K) \leq e^{-\frac{\delta}{k}t_{n+1}\max(|W^+(Ke^{rt_{n+1}})|,|V_o(Ke^{rt_{n+1}})|)}
\]

It is easy to check that both \( W^+ \) and \( V_o \) are members of the family \( F \) if \( u \) is, so that the expression on the right hand side approaches zero for large \( n \), if \( t_{n+1} \) grows out of bounds. The latter must hold however, because the present value of future transactions costs payments along the path of value \( W, a_0 e^{-rt_1} \)
must not exceed initial capital and therefore must converge, so that the series \( \{ t_i \} \) diverges. Thus, the function \( W \) must be the limit of the sequence \( \{ V_n \} \) and therefore must be equal to the function \( V \). Thus, \( V(K) \) is in fact the value of the optimal program for problem (1).

The continuity of \( V \) follows immediately, since the constraint set (2) is closed and continuous in \( K \), while preferences are continuous on the space of all consumption paths.  

Q.E.D.

In this proof, tacit use has been made of the assumption that \( K > a \). If this condition is violated, none of the functions \( V_n \), \( n = 1,2, \ldots \) will exist. If \( K \leq a \), the agent cannot afford an immediate withdrawal, but waits until he can, that is until \( K > a \).

However, the set of moments, for which this inequality holds is not closed. Furthermore, from condition (4), we know that \( \lim_{c \to 0} u'(c) = \infty \). Hence, the agent wants to make the first withdrawal as soon as possible, provided he can afford it. Thus, he wants to make the first withdrawal as close as possible to the moment when \( K = a \), without in fact making the withdrawal at this moment itself. But then, there exists no optimal moment for the first withdrawal.

On the other hand, the assumption that \( K > a \), crucial though it is, is not very restrictive either.
For it is clear that a consumer will not use up all his cash, before he can afford to make another withdrawal, so that whenever he goes to the bank he has waited long enough so that he can afford to make a withdrawal.

6. We now discuss properties of the optimal policy for problem (3). First, we show that the optimal policy never occurs at the boundary of the set of feasible h, X.

Since an increase in capital allows at least increased initial consumption at a positive marginal utility, while the tail of the path may remain constant, the function V must be strictly increasing in K. Hence, V(K) > V(K - a). But V(K - a) is the value of the path that would be achieved with h = X = 0. Hence, this cannot be an optimal policy for problem (3).

Furthermore, since \( \lim_{c \to 0} u'(c) = \infty \), h > 0, X = 0 cannot be an optimal policy.

Finally, we refer to the arguments of the preceding section to propose that policies with h = \infty or X = K-a cannot be optimal.

Thus, the optimal solution to problem (3) occurs at the interior of the set of feasible h, X for all K. It obeys the following necessary conditions for an extremum:
(6) \[ w_n = \phi e^{-\psi} h \nu((K-X-a)e^{rh}) + r(K-X-a)e^{\phi-\gamma} h \nu', \]
\[ = 0 \]

(7) \[ w_x = e^{(r-\gamma)} h \nu'((K-X-a)e^{rh}) = 0 \]

Furthermore, we have from the envelope theorem:

(8) \[ V'(K) = e^{(r-\gamma)} h \nu'((K-X-a)e^{rh}) = w_x \]

Combining equations (7) and (8), we find that any two subsequent withdrawals must obey the condition:

\[ w_{x_i} - e^{(r-\gamma)} h_i w_{x_{i+1}} = 0 \]

This condition is the equivalent of the Euler equation in the ordinary Ramsey problem, which provides for an exponential decline of marginal utility over time. However, in the present model, this decline does not hold between any two pairs of consumption, because first order conditions are different within a withdrawal period and between withdrawal periods, and consumption within a withdrawal period is even declining as shown in the appendix. Therefore the modified Euler condition holds only between pairs of consumption, which are at an equal distance from the last withdrawal.

It is at this point that the formal advantage of the model with compound interest becomes evident. For clearly, the modified Euler condition holds not only between subsequent withdrawals, but between any two withdrawals along the optimal path that are separated
by a period of total length \( H \), we must have:

\[
W_{x_1} - e^{(r-g)H} W_{x_1+j} = 0,
\]

by repeated application of the modified Euler condition. This condition is independent of the number and spacing of withdrawals made in between, whereas the corresponding condition (19) in Dixit, Mirrlees and Stern makes the long run development of marginal utility depend on the whole pattern of withdrawals.

Substituting (7) into (6), we have:

\[
W_h + r(K-X-a) W_x - g e^{-gH} V((K-X-a)e^{rh}) = 0
\]

In the appendix we show that

\[
W_h = e^{-gH} (u(c(h)) - c(h) u'(c(h)))
\]

and

\[
W_x = e^{-gH} u'(c(h)).
\]

By substitution, we then find that

\[
(r(K-X-a) - c(h)) u'(c(h)) = g V((K-X-a)e^{rh}) - u(c(h))
\]

If we substitute for \( e^{-gH} V((K-X-a)e^{rh}) \) from equation (3), we find similarly for \( c(0) \):

\[
(r(K-X-a) - c(0)) u'(c(0)) = g V(K) - u(c(0))
\]

Equations (9) and (10) are the most important of the Keynes-Ramsey equations for our problem. In general, we may write:

\[
(r(K-X-a) - c(t)) u'(c(t)) = g \bar{V}(t) - u(c(t)),
\]

where \( \bar{V}(t) \) gives the value of the optimal program from time \( t \) onwards, evaluated at time \( t \).
The Keynes-Ramsey equation sets the value of saving at any moment, evaluated at the marginal utility of consumption at that moment, equal to the difference between that instantaneous utility that would be needed if the value of the rest of the optimal program were achieved by a constant consumption stream and the actual utility at that moment.

It is of some interest to note that the expression for saving in the Keynes-Ramsey equation does not use the actual interest income, which accrues to the account at that moment, \( r(K-X-a) e^{rt} \), but that this actual interest income is discounted back at the rate \( r \) to the last moment, at which the consumer could have "traded" interest income in the account for consumption, namely the moment of the last withdrawal, and then forward again to the moment \( t \) at the rate of interest paid on cash, which we have taken to be zero.\(^2\) Income thus is not measured as the actual interest income at that moment in the account, but as the interest income on the capital that the consumer would have, had he transformed the whole outstanding balance into cash at the last with-

\[^2\] If a rate of interest \( d, d < r \), is paid on cash, \( w_h \) is unchanged, but we have \( w_x = e^{(d-q)t} u'(c(h)) \), so that the left hand side of equation (9) becomes \( (r(K-X-a)e^{dh} - c(h)) u'(c(h)) \).
This is explained as follows: Let $c(0)$ be given. In the appendix, we investigate the behaviour of the consumption path within the withdrawal period and show that for given $c(0)$, consumption at all moments within the same withdrawal period is given as well. Then, let for given $c(0), X(t)$ be such as to allow the consumption path following $c(0)$ to be obeyed up to time $t$. If the withdrawal period were to last up to $t$, we would have for capital newly available for withdrawal at time $t$, $K(t) = (K-X(t)-a)e^{rt}$,

$$\frac{dK(t)}{dt} = (r(K-X(t)-a) - c(t)) e^{rt}.$$

Thus, we may regard the expression on the right hand side as the true saving at time $t$, as anticipated at the time of the last preceding withdrawal.

We immediately see that if initial consumption is monotone increasing in capital, then end-of-period saving must be positive, if $r \geq \xi$. By equation (7), consumption at the beginning of the next withdrawal period is higher than present initial consumption. Hence, capital at that moment must be higher than now. But then, there must be positive saving during the period, and, of course, saving is highest at the end of the withdrawal period.

On the other hand, even if there is positive overall
saving, it is not, in general, possible to show that there is also positive saving in the beginning of the withdrawal period. The consumer may dissave initially and make up for it at the end of the withdrawal period. This will certainly happen, if \( r = \gamma \). In this case, initial consumption is the same in all withdrawal periods, by equation (7). Since initial consumption is the highest within any withdrawal period, it is the highest ever reached in the whole program and almost all consumption in the program falls short of it. Therefore the steady-state consumption that provides the same utility as the optimal program is lower than initial consumption of a withdrawal period along the optimal path, i.e. the right hand side of equation (10) is negative. Since in this case, there is strictly positive dissaving at the beginning of the withdrawal period, a continuity argument can be used to show that for \( r \) close to \( \gamma \), but \( r > \gamma \), with strictly positive capital accumulation, there is still positive dissaving at the beginning of a withdrawal period.

The Keynes-Ramsey equation has the disadvantage that the term \( V(t) \) is not constant, if there is time preference, and that its use requires a knowledge of the subsequent path, which it is supposed to help find in the first place. However, one can combine the Keynes-
Ramsey equations for the end of the present withdrawal period, \( h \), and the beginning of the next, \( h^+ \), to obtain a single local condition for the determination of \( h \), which does not require knowledge of the whole path:

\[
(11) \quad (r(X+a) + c(h^+) - c(h)e^{rh}) \frac{d}{dh} u(c(h^+)) = u(c(h^+)) - u(c(h)),
\]

where \( c(h^+) \) is consumption at the beginning of the next withdrawal period, \( X \) the next withdrawal, which is foreseen for time \( h \).

Equation (11) allows an easy intuitive interpretation. For suppose that the next withdrawal is made a moment earlier. This leads to a cost \( r(X+a) \), because some interest is foregone. On the other hand, end-of-period consumption does not have to be withdrawn immediately, but remains in the account so that one has a net return \( c(h)(e^{rh} - 1) \).

Furthermore, we know that if we were to leave the consumption path unchanged, then consumption within the next withdrawal period would be allocated less than optimally. Therefore, we change consumption in the moment, by which the withdrawal is shifted, in such a way that the rest of the second withdrawal period is still optimal for intra-period allocation (see diagram 1). This has a monetary cost \( c(h^+) - c(h) \) and increases total utility by the
amount \( u(c(h^+)) - u(c(h)) \). Thus, condition (11) sets the gain in utility from the proposed marginal change equal to its net monetary cost, evaluated at the marginal utility of consumption at time \( h^+ \).

Diagram 1:

\[
\begin{align*}
\text{c} & \quad [c(h^+) - c(h)] \Delta k \\
\begin{array}{c}
\text{t} \\
\text{t_0} \\
\text{t_+h} \\
\end{array}
\end{align*}
\]

7. The problem of the uniqueness of the optimal solution to problem (3) is more difficult. To simplify the notation, we write:

\[ v(h, X, K) = w(X, h) + e^{-\int h v((K-X-a)e^{rh})} \]

The first order conditions (6) and (7) require that \( v_h = 0 \) and \( v_x = 0 \). From proposition 1, a solution to these equations exists, which corresponds to a maximum of \( v \) with respect to \( h \) and \( X \). If the function \( v \) is strictly concave in \( X \) and \( h \) at the point where the first order conditions hold, only one solution to equations (6) and (7) will exist. Because of the

\[ \text{3/ The proposed marginal change is not quite optimal in that it violates condition (7). Adjusting for this would decrease the monetary cost of the change by} \]

\[ \int_{t+ho}^{t+ho+h} \frac{dc}{dh} \, dt, \text{ while it decreases the utility} \]
convexity of the exponential function, this is not an obvious property. If we formulate the problem in terms of the set of feasible consumption paths, we have to note that the set of feasible paths is not convex, because the convex combination of two paths with different withdrawal points foresees withdrawals at the withdrawal points of both paths and hence requires a higher transactions cost than either of them.

Concavity of \( v \) requires:

\[
\begin{align*}
v_{hh} &= w_{hh} + g w_h - g v_h + r(K-X-a) v_{kh} < 0 \\
v_{xx} &= w_{xx} + v_{kk} < 0 \\
D &= v_{hh} v_{xx} - v_{xh}^2 > 0
\end{align*}
\]

We note that the derivatives of \( v \) obey the following conditions:

\begin{align*}
(12) \quad v_k &= e^{(r-g)h} v'((K-X-a)e^{rh}) \\
&= w_x, \text{ wherever (7) holds.} \\
(13) \quad v_{kk} &= e^{2(r-g)h} v''((K-X-a)e^{rh}) \\
(14) \quad v_{kh} &= (r-g) v_k + r(K-X-a) v_{kk} \\
&= e^{(r-g)h} ((r-g) v'((K-X-a)e^{rh}) \\
&+ r(K-X-a)e^{rh} v''((K-X-a)e^{rh})
\end{align*}

\[ \text{gain by } u'(c(h^*)) \int_{t_h}^{t_{h+k}} \frac{dc}{dh} \, dt, \text{ because initial consumption is lower. Thus, the interpretation of (11) also holds, if the change in the consumption path} \]
(15) \( v_{kx} = -v_{kk} \)

(16) \( v_{xh} = w_{xh} - v_{kh} \)

At the optimal point, \( v_h = 0 \), and the first of the concavity conditions becomes:

\( v_{hh} = w_{hh} + q w_h + r(K-X-a) v_{kh} \leq 0 \)

Expanding the determinant \( D \), we have:

\[
D = (w_{hh} + q w_h)(w_{xx} + v_{kk}) + r(K-X-a)v_{kh}w_{xx} \\
+ r(K-X-a)v_{kh}v_{kk} - w_{xh}^2 + 2w_{xh}v_{kh} - v_{kh}^2
\]

(17) \( D = -q w_x c(h)(w_{xx} + v_{kk}) + (r(K-X-a)-c(h))w_{xx}v_{kh} \\
- (r-q)v_{kh}v_{kk} - c(h)(v_{kh} - c(h)v_{kk}) w_{xx} \),

by repeated use of the properties of the function \( w \) as discussed in equations \((3')\) to \((14')\) in the appendix. Further transformations that use the results of the appendix as well as equations (12) to (16) at the point where the first order conditions (6) and (7) hold, give

(18) \( D = -q w_x c(h)v_{kk} - rw_x c(h)w_{xx} - (r-q)v_{kh}v_{kh} \\
+ (r(K-X-a)-c(h))(v_{kh} - c(h)v_{kk}) w_{xx} \).

By (17), \( D \) is positive, if \( v_{kh} \) is nonpositive and \( v_{kh} - c(h)v_{kk} \) is nonnegative. By (18), \( D \) is positive, if \( v_{kh} \) is nonpositive and \( v_{kh} - c(h)v_{kk} \) is nonpositive, provided both times that \( r(K-X-a) > c(h) \).

satisfies equation (7).
Combining the two conditions, we can say that the determinant $D$ is positive, if $r(K-X-a) > c(h)$ and

$\begin{align*}
  v_{kh} \leq 0.
\end{align*}$

Also, condition (19) guarantees that $v_{hh}$ and $v_{xx}$ are negative.

Condition (19) is the weakest condition for the strict concavity of $v$ at the optimal point that I could find. From equation (14), we see that it will certainly hold, if the function $V$ is a member of the family $F$ defined in (4), that is, if for all $K$,

$\begin{align*}
  V'(K) + KV''(K) \leq 0.
\end{align*}$

However, although the function $V$ lies between the functions $W^+$ and $V_0$, giving the values of the Ramsey problem and of the 0-withdrawal problem, both of which are in $F$, I have not been able to show that $V$ obeys condition (20).

Intermediate in strength between (20) and (19) is the condition

$\begin{align*}
  v_{kh} - c(h) v_{kk} \leq 0. \quad 4/
\end{align*}$

This condition has a simple intuitive interpretation:

If we compute the derivatives $dh/dK, dX/dK$, we find:

---

4/ That (21) implies (19) is evident. (21) is written as $(r - \sigma)(v_{k+} (K-X-a)v_{kk}) + (\sigma(K-X-a) - c(h))v_{kk}$, which, by (20), (12), (13) is negative, if

$\sigma(K-X-a) > c(h)$, which is shown below.
\[
\frac{dh}{dK} = - \frac{w_{xx}(v_{kh} - c(h)v_{kk})}{D}
\]

\[
\frac{dx}{dK} = - \frac{rv_{k}v_{kh}}{D} + c(0) \frac{dh}{dK}
\]

Thus, condition (21) implies that the length of the withdrawal period is non-increasing in K.

We are now in a position to prove

**Proposition 2:** The solution to equations (6) and (7) is unique, if the length of the optimal withdrawal period is non-increasing in K.

**Proof:** From equations (8), (22) and (23), we have:

\[
V''(K) = - \frac{w_{xx}w_{x}(g(c(h)v_{kk} + (r-g)v_{kh})}{D}
\]

Suppose that (19) holds. Then, the optimal solution is unique, if \( r(K-X-a) > c(h) \). From the argument in section 6, this is the case, if at the next withdrawal initial consumption is increasing in initial capital. But \( dc(0)/dK = V''(K)/u''(c) \) and, by assumption, (21) holds at the next withdrawal. But (21) implies (19), so that, by (24), \( V''(K) < 0 \), and at the next withdrawal, \( c(0) \) is increasing in K. Thus, it is sufficient to show that (19) holds everywhere.

This is not trivial and is not part of our assumption. For the assumption of our proposition lets (21) hold at those solutions of (6) and (7), which
give rise to a maximum of $v$, not everywhere.

For $v_{kh}$ to be negative everywhere, we require, by (14) that

$$(r - \varphi)V'((K-X-a)e^{rh}) + r(K-X-a)e^{rh}V''((K-X-a)e^{rh})$$

$$\leq 0$$

for all values of $K$, $X$, $h$.

This is equivalent to the requirement that

$$(r - \varphi)V'(K) + rKV''(K) \leq 0$$

for all values of $K$.

We can expand this condition by use of (24) and of our knowledge of the functions $v$ and $w$:

$$(r - \varphi)V'(K) + rKV''(K)$$

$$= (r - \varphi)V'(K) + r(K - X(K) - a) V''(K)$$

$$+ rX(K)V''(K) + raV''(K)$$

$$= (r - \varphi)V' + (ra + X + (r - \varphi)(X + \frac{w}{w_{xx}})) V''$$

$$+ (r - \varphi)V' \frac{w_x}{D} (\varphi c(h)v_{kk} + (r - \varphi)v_{kh})$$

$$- r(K-X-a)(\varphi c(h)v_{kk} + (r - \varphi)v_{kh}) \frac{w_{xx}w_x}{D}$$

$$= (r - \varphi)(X + c(h)/\varphi - c(0)/\varphi) V''$$

$$+ (ra + \varphi X + c(h)) V''$$

$$+ rc(h)V' \frac{dh}{dk}$$

By assumption, now both $dh/dk$ and $V''$ are nonpositive.

Furthermore, it is shown in the appendix that for $u$ in $F$, $X + c(h)/\varphi - c(0)/\varphi$ is nonnegative. There-
fore, the right hand side of (25) is nonpositive and hence, (19) must hold everywhere.\(^5\) Q.E.D.

An immediate consequence of the uniqueness of the optimal solution is the continuity of the policy functions \(h(K), X(K)\).

I have not been able to show that the antecedent of proposition 2 will hold in general. Before showing that it will hold for a particular class of functions, I want to discuss the significance of this condition.

It is easy to see that this condition must hold both for very small and for very large \(K\):

On the one hand, as \(K\) approaches \(a\), the withdrawal period needed merely to replete capital and allow the payment of the cost of another withdrawal grows out of bounds.

On the other hand, as \(K\) becomes large, \(h\) must go to zero. For otherwise, \(X\) grows out of bounds and if \(h\) remains constant at a positive value, there comes a point at which it is profitable to split the withdrawal period, because the extra interest earned on the amount needed for consumption in the second half of the period more than makes up

\(^5\) The use of differentials is not crucial to the argument even though it seems to beg the question
for the cost of an extra withdrawal. It is this behaviour in the limit, which has been described by Dixit, Mirrlees and Stern.

In general, for any two withdrawal periods of given total length \(6/\), and any given consumption path, the choice of withdrawal time trades off the benefit from earning increased interest on second period consumption as the first period is lengthened and the cost arising from the fact that consumption for the interim has to be withdrawn immediately and thus decreases the base on which interest is earned. As capital increases and the consumption path changes to accommodate this increase, this trade-off is shifted in favour of a lower withdrawal period. \(7/\)

by using continuity of the behavioural function. The argument can be refined by the use of \(\sup v_{kh}\).

\(6/\) In the two-period, fixed total time problem, the optimal path maximizes \(w(X,h) + e^{-\int_0^h w((K-X-a)e^{rh}, l-h)}, where l is given. First order conditions are (7), (11) and \(h\) is decreasing in \(K\), iff \(r(a+X_2) - c(h)e^{rh} + c(h^+) - r(c(h^+)- c(h))/3 \geq 0\), a condition which recurs in the proof of (21).

\(7/\) Conditions for the trade-off at the withdrawal time are:
However, since the withdrawal for the optimal path occurs at a discontinuity point of the consumption path, this trade-off imposes only inequality constraints, which cannot completely fix local behaviour. However suggestive the preceding remarks may be, they are in no way conclusive to show that the antecedent of proposition 2 must hold.

The notion that the withdrawal period decreases with initial capital has a certain intuitive appeal. For we noted above that the non-convexity of the feasible set arises as a wealth effect, because the convex combination of two paths requires a higher transactions cost and therefore a higher initial capital. As capital increases, this non-convexity loses in relative importance, on the one hand because consumption increases and therefore consideration of the interest foregone gains in weight, on the other hand, because for concave

\[ (A) \quad c(h^\uparrow)(e^{rh} - 1) \geq r(x_2 + a) \]
\[ (B) \quad c(h)(e^{rh} - 1) \leq r(x_2 + a) \]

For any \( h \), as the consumption path varies with an increase in initial capital, if \( u''' \geq 0 \), the inequality in (A) becomes more binding. The same would hold for (B), if the discontinuity point of the consumption path did not shift.
preferences the valuation of capital itself and hence of this wealth effect in utility terms decreases. This decrease of importance shows as a decrease in the length of the withdrawal period.

8. We now show that the antecedent of proposition 2 holds for the class of constant elasticity utility functions.

**Proposition 3:** For the class of constant elasticity utility functions:

\[(26)\quad u(c) = \frac{c^b}{b} \quad b < 0\]

\[u(c) = \ln c\]

the solution to equations (6) and (7) is unique.

**Proof:** For the constant elasticity functions, we have:

\[(27)\quad u(c) - cu'(c) = \frac{1-b}{b} \quad c^b = (1-b) u(c)\]

Now we use induction on the number of withdrawals. Suppose that (21) holds everywhere in the n-withdrawal problem. For given initial capital \(K_{n+1}\) in the \(n+1\)-withdrawal problem, let \(h_{n+1}, X_{n+1}\) be a solution to (6) and (7) and let \(K_n = (K_{n+1} - X_{n+1} - a)e^{rh_{n+1}}\) be capital at the beginning of the next withdrawal period. As we apply (25) together with the inductive hypothesis, we see that in the \(n+1\)-withdrawal problem, \(v_{kh}\) is negative and the solution to (6) and (7) is unique. Consider condition (21) in the \(n+1\)-withdrawal problem. We require
(r - q )V_n'(K_n) + rK_nV_n''(K_n) - c_{n+1}(h_{n+1})e^{rh_{n+1}}V_n''(K_n)

to be negative, by (13) and (14).

Applying (25) and the inductive hypothesis on (21) to V_n', we find that this expression is less than

(28) (r(a + X_n) - c_{n+1}(h_{n+1})e^{rh_{n+1}} + c_n(0; K_n)

- r(c_n(0; K_n) - c_n(h_n; K_n))/q )V_n''(K_n)

The end of the first withdrawal period, h_{n+1}, must satisfy equation (11). We may therefore write (28) as:

(29) \frac{u(c_n(0)) - u(c_{n+1}(h_{n+1}'))}{u'(c_n(0))}

- r(c_n(0; K_n) - c_n(h_n; K_n))/q )V_n''(K_n)

By (27), this is equal to

(30) \frac{1}{1 - b} \left( \frac{u(c_n(0)) - u(c_{n+1}(h_{n+1}'))}{u'(c_n(0))} + c_n(0)

- c_{n+1}(h_{n+1})e^{rh_{n+1}} - r(c_n(0) - c_n(h_n))/q )V_n''(K_n)

By renewed application of (11), this is equal to

(31) \frac{1}{1 - b} \left( r(a + rX_n) - r(c_n(0) - c_n(h_n))/q )V_n''(K_n)

= \left( \frac{ra}{1 - b} + r\left( \frac{X_n}{1 - b} + c_n(h_n) - c_n(0)/q \right) V_n''

In the appendix, it is shown that the last term is zero, so that (31) is equal to raV_n''/(1 - b), which is negative.

Thus, we have shown that if (21) holds everywhere
in the n-withdrawal problem, then, for constant elasticity utility functions it holds everywhere in the n+1-withdrawal problem. Furthermore, it certainly holds in the 0-withdrawal problem, where h is fixed at $\omega$.

Thus, for the constant elasticity utility functions, (21) holds for all n, and therefore also in the limit. Since (21) is a sufficient condition for the strict concavity of v, this completes the proof of proposition 3. Q.E.D.

We also have proved the following

**Corollary:** For isoelastic utility functions, the optimal length of the withdrawal period is decreasing in K.

9. The relationship between condition (21) and the uniqueness of the solution to (6) and (7) can be made clear in a rather immediate way, if we depart from the connection between (21) and the behaviour of h.

In equation (10), move the term $u(c(0))$ to the left hand side and differentiate totally to get, for given K:

\[
(32) \quad (r(K-X-a)-c(0))u''(c(0))dc(0) - ru'(c(0))dx = (r(K-X-a)-c(h) + r(c(0)-c(h))/\eta)u''(c(0))dc(0) - rc(h)u'(c(0)) dh = 0
\]
Thus, if we consider (10) as a condition on the pairs of \( h, c(0) \) that may be considered, we find that along this restriction, \( h \) is decreasing in \( c(0) \).

If we differentiate equation (7) totally, we get

\[
33 \ u''(c(0))dc(0) - ((r - \gamma) v_k + (r(K-X-a) - c(h)) v_{kk}) dh
= 0
\]

Thus, along the restriction given by (7), \( h \) is increasing in \( c(0) \), if and only if condition (21) holds. In that case, there can only be one common solution to (7) and (10). (Diagram 2).

**Diagram 2:**

![Diagram](image)

In section 6, we noted that for given \( c(0) \), capital available for withdrawal at time \( h \) changes with \( h \) as \((r(K-X-a) - c(h))e^{rh}\). On the other hand, capital needed to fulfil condition (7), increases in \( h \) as \(-(r - \gamma)V'/V''\). Thus, we may consider the term in condition (21) as an indicator of the rate at which the disequilibrium between capital available and capital needed at time \( h \) changes with \( h \), given present initial consumption \( c(0) \). If (21) holds, a lengthening of the withdrawal period will always re-
duce the difference between capital needed and capital available.

Since capital needed is for all h at least as large as initial capital while capital available at the time h=0, after the first withdrawal, is always less than initial capital, the smallest h that fulfills (7) for any c(0) must always fulfill (21), although this does not imply that the graph for (7) is monotone (Diagrams 3, 4).

**Diagram 3:**

We do notice however, that c(h⁺) changes at the rate

\[-(r - \varphi) u'/u'' \leq (r - \varphi)c(h⁺)\]

for \(u \in F\). Then, if \(c\) is linear or more than linear in \(K\), \(K(h)\) changes at a rate of less than \((r - \varphi)K(h)\)\(^8\). Then, (21) certainly holds if \(c(h) \leq \varphi(K-X-a)\).

Consider condition (10): Since \(V_n > V_{n-1}\), by (32) either \(c_n(0) < c_{n-1}(0)\) or \(h_n < h_{n-1}\). Suppose that for some \(n, K, c_n(0) > c_{n-1}(0)\), but \(c_{n-1}(0) < c_{n-2}(0)\) for all \(K\). Then, for any \(h,\) by the budget constraint, \(K_n(h) \leq K_{n-1}(h)\) and consequently, \(c_{n-1}(h^+) < c_{n-2}(h^+)\). But the marginal condition (7) would require \(c_{n-1}(h^+) > c_{n-2}(h^+)\). Thus, to fulfill (7), we must have \(h_n > h_{n-1}\) while (32) requires that
that $h_n < h_{n-1}$, a contradiction. Hence, we must have for all $n, K$, $c_n(0) < c_{n-1}(0)$.

Let $\tilde{k}$ be such that $c_n(0; K) = c_0(0; \tilde{k})$. From the preceding argument, $\tilde{k} < K$. Furthermore, $c_n(h_n; K) = c_0(h_n; \tilde{k}) = c_0(0; \tilde{k} - X_n)$. If $u \in F$, $c_0(0; \tilde{k} - X_n) \leq \zeta(\tilde{k} - X_n - a)$. Hence, for all $n$, $c_n(h_n) \leq \zeta(K - X_n - a)$, and clearly, this must hold in the limit as well.

Thus, (21) fails only, if at some point consumption is less than linear in initial capital.

10. The preceding notes investigated the problem of the uniqueness of an optimal solution to a class of non-convex problems. The model analysed is only one of a wide class of models with similar characteristics. The basic feature of all these models is the choice of a piecewise continuous function, in which the discontinuity points are chosen as part of the optimization problem. We have shown that the optimal solution is indeed unique for a particular class of instantaneous utility functions, and we have discussed the characteristics of the problem in general.

Our method of analysis extends immediately to the model of Dixit, Mirrlees and Stern. In that model, for any va-

\footnote{In this case, $V$ is a member of $F$ and the above argument completes the proof of footnote 4, that (4) implies (21).}
value of the state variable, consumption is unique because of the concavity of the instantaneous utility function.\textsuperscript{9/} Then, the length of the investment period is unique, if the valuation function \( V \) is concave, which again is the case, if the length of subsequent investment periods is decreasing in the state variable.

Similar problems turn up in other contexts as well. If an agent cannot or finds it too expensive to distinguish continuously between characteristics of a commodity, e.g. states of nature, colour, size etc., but wants to create categories of these characteristics, so that he does no longer distinguish between different points within a category, he reduces his commodity space from the space of measurable functions to the space of piecewise continuous functions - or even step functions - with given discontinuity points. In this context, the initial selection of the discontinuity points represents a problem, which is formally similar to the problem treated in this paper.

A typical case would be that of the insurance company creating risk classes or damage classes. It appears

\textsuperscript{9/} The Keynes-Ramsey equation is: \( u(c)+(Y-c)u'(c) = 0 \), which for given \( Y \) has a unique solution in \( c \). This uniqueness plays a similar role as the monotonicity of \( c(0) \) and \( h \) in our equation (10).
intuitively obvious that one should be able to apply the rationale behind condition (21) in this paper to argue that the distinction of risk classes will be the finer the higher the incidence of contracts in this general region of risks. However, in view of the preceding notes, one has to be cautious about such a conjecture. Although it will certainly be true in some overall sense, it is yet to be shown that it holds everywhere locally. Here again, one suspects that there is an intimate link between this conjecture and the uniqueness of the optimal division into risk classes.

11. The class of problems treated in this paper has assumed particular importance in inventory theory. Optimal policies in inventory problems with fixed reorder costs are generally given as a pair of numbers (S, s) for any value of the state variables, where the agent reorders to bring his inventory to S whenever his present inventory is at or below s. In this problem, too, uniqueness of the optimal solution has been a major point of discussion. Scarf introduced the concept of k-convex functions, i.e. functions that deviate from convexity by not more than a fixed amount k, to show that the critical value s was unique.

10/ A function is k-convex, iff for all $a > 0$ and all $x$, $k + f(a+x) - f(x) - af'(x) \geq 0$. 
It went unnoticed however that his argument does not imply the uniqueness of the optimal reorder value $S$. In our model, the uniqueness of $s$ corresponds to the fact that it is always optimal to run cash balances down to zero before making another withdrawal ($s=0$). It is to be hoped that proposition 2 and possible generalizations of proposition 3 throw some new light on the question of the uniqueness of $S$. 
Appendix: Properties of the function $w(X,h)$

The function $w$ is defined as follows:

$$(1') \quad w(X, h) = \max \int_0^h e^{-\int_0^t u(c(t)) dt} dt$$

subject to $X = \int_0^h c(t) dt$

The first order condition is:

$$(3') \quad u'(c(t)) = e^{\int_0^t w_x} c(t),$$

where $w_x$ is, of course, independent of $t$.

Taking partial derivatives with respect to $X$ we have:

$$(4') \quad u''dc(t) = e^{\int_0^t w_{xx}} dX;$$

similarly, partials with respect to $h$ give:

$$(5') \quad u''dc(t) = e^{\int_0^t w_{xh}} dh.$$  

Combining $(4')$ and $(2')$, we find that

$$(6') \quad w_{xx} = \frac{1}{h} \int_0^h e^{\int_0^t \frac{w_{xx}}{u''(c(t))} dt} dt$$

Also, from $(4')$, $(5')$ and $(2')$,

$$(7') \quad w_{xh} = -c(h) w_{xx}$$

Furthermore, if we take partials of $(3')$ with respect to $t$, we have:

$$(8') \quad u''dc(t) = g u' dt = g e^{\int_0^t w_x} dt.$$  

For the constant elasticity utility function, $(8')$ gives

$$(9') \quad dc(t) = -g c(t) dt / (1-b),$$

and condition $(4)$ implies

$$(8'') \quad dc(t) \geq -g c(t) dt$$
From (9') and (2') we find:

\[(10') \quad x = \frac{1 - b}{\xi} (c(0) - c(h)) .\]

Similarly, from (8'') and (2'), we have

\[(10'') \quad x \geq (c(0) - c(h))/\xi .\]

Substituting (8') in (6'), we have

\[(11') \quad w_{xx} = \frac{\xi w_x}{c(h) - c(0)} \leq -w_x/ X , \text{ by (10'')}, \text{ so that } w \text{ is a member of } F, \text{ if } u \text{ is.}\]

The preceding remarks hold without any change for $V_0$, since we have defined $V_0 = w(K-a, \infty)$.

Let $c^+(t)$ be the optimal path for (1'). Then

\[w(X,h) = \int_0^L e^{-\xi t} u(c^+(t)) dt\]

\[= - \left[ u(c^+(t))e^{-\xi t} \right]_0^L + \int_0^L e^{-\xi t} u(\cdot) \frac{de^+}{dt} dt\]

by integration by parts.

\[(12') \quad w(X,h) = (u(c(0)) - u(c(h)) e^{-\xi h} - w_x(c(0) - c(0))/\xi\]

Also, by (2'), (3'), (6') and (7'), we find that

\[(13') \quad \tilde{w}_h = e^{-\xi h} (u(c(h)) - c(h) u'(c(h))) \]

and hence,

\[(14') \quad \tilde{w}_{hh} = -\xi \tilde{w}_h - c(h) \tilde{w}_{xh} - \xi c(h) \tilde{w}_x .\]
Bibliography


Biographical Note

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