A THREE-DIMENSIONAL MATHEMATICAL MODEL OF THE HUMAN KNEE JOINT

by

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Submitted to the Department of Mechanical Engineering on April 2, 1990
in partial fulfillment of the requirements for the degree of
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Abstract

A mathematical model of the human knee joint is developed and applied to in vivo
kinematic and dynamic measurements of gait to estimate joint contact and muscle forces. The four degree of freedom knee model includes muscles, ligaments, and three-
dimensional articular surface geometry and is developed as part of a complete nine degree of freedom lower extremity model.

A method is developed for using a kinematic measurement system to generate a
gerometric model of the articular surfaces of the tibia and femur. A novel method is then
employed to estimate and remove soft tissue motion errors of in vivo knee kinematics by
including geometric constraints. The estimated knee kinematics are included in a dynamic
analysis to estimate the joint contact forces and muscle forces. Sensitivity to selected
modeling and processing parameters is discussed.

Difficulties in satisfying dynamic equilibrium for all degrees of freedom of the knee
model are presented in terms of the procedures for estimating knee joint kinematics without
regard to dynamics. A method is implemented to minimize these errors by iteratively
adjusting the estimated knee kinematics and muscle forces until convergence is obtained. A
further reduction in these errors is demonstrated by concurrently estimating kinematics and
dynamics.

A joint stiffness and stability analysis for human joints is derived including specific
requirements for muscles. This analysis is initially applied to a single degree of freedom
model of the human elbow joint to justify selection of the muscle stiffness model and also
to demonstrate that the method can successfully predict muscular co-contraction due to
stability constraints. The stability analysis is later applied to the knee as part of the muscle
and joint force estimation analysis.

For the experimental subject of this thesis, knee kinematics are estimated with an
accuracy between 2 and 6 mm and total forces in excess of 4.5 times body weight are
estimated for the joint during the stance phase of gait. Theoretical predictions of the
stability analysis applied to the elbow joint match experimental measurements fairly closely,
but further improvements in muscle models are required before the methods can be
successfully employed for the knee.

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I especially appreciate my parents, who have supported me despite not completely understanding why I spent so many years here at M.I.T. I recently heard one side of a telephone conversation of a student in our lab which sounded very familiar. "Hi, mom... I don't know mom, I hope 1991...". Through all of the trying circumstances my parents have encountered in their own lives, I have always been 100 percent confident that they would put their children first before any other priority. Their love means a lot to me and has given me a strong sense of self-esteem even when my thesis was not going as well as I would have liked.

Most importantly, I would like to thank my wife Caroline. Throughout my time as a doctoral student she has been my source of support and encouragement. She has worked very hard to help me to grow in the areas where I need to change, including forcing me to get organized and do the important things today, rather than working on interesting but tangential projects and putting off the essential work for later. I should have taken her advice some more. During these past few months she has picked up the slack and maintained our family while allowing me to concentrate on completing my thesis. Keeping our son Bobby in order while also taking care of our new baby daughter Elise is no easy task, especially in our small apartment. The years I have worked on my Ph.D. have certainly been more stressful for her than they have been for me, and I appreciate and love her very much.

Finally, my acknowledgements would be incomplete without mentioning that I became a Christian while I was a student here at M.I.T. I came to the Boston area very arrogant and prideful. Fortunately my pride was broken by the cross of Jesus Christ and not by M.I.T. A few verses from The Living Bible are very relevant to this.

I Corinthians 3:18

Stop fooling yourselves. If you count yourself above average in intelligence, as judged by the world's standards, you had better put this all aside and be a fool rather then let it hold you back from the true wisdom from above.

I Timothy 6:20,21

Keep out of foolish arguments with those who boast of their "knowledge" and thus prove their lack of it. Some of these people have missed the most important thing in life - they don't know God.

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NOMENCLATURE

\( a^s \)  
linear acceleration of center of mass of segment \( S \) relative to the GCS

\( F_{Lk} \)  
tensile force on the \( k \)th ligament

\( F_{Mk} \)  
tensile force on the \( k \)th muscle

\( F_{NET}^i \)  
net force vector of segment \( i \) acting on segment \( i-1 \), due to muscles, ligaments, and joint contact

GCS  
global coordinate system

\( H \)  
lower extremity inertia tensor

\( I^S \)  
inertia matrix for segment \( S \), relative to the GCS

\( J^S \)  
jacobian matrix of segment \( S \)

\( J^S_A \)  
angular velocity part of jacobian matrix for segment \( S \)

\( J^S_L \)  
linear velocity part of jacobian matrix for segment \( S \)

\( K \)  
total joint stiffness matrix

\( K_{Lk} \)  
stiffness of \( k \)th ligament

\( K_{Mk} \)  
stiffness of \( k \)th muscle

\( L \)  
Lagrangian

\( L_{Lk} \)  
length of \( k \)th ligament

\( L_{Mk} \)  
length of \( k \)th muscle

\( m^S \)  
mass of segment \( S \)

\( M_{NET}^i \)  
net moment vector of segment \( i \) acting on segment \( i-1 \), due to muscles, ligaments, and joint contact, calculated about origin of segment \( i \)

\( n^I_1, n^I_2 \)  
femur coordinates of unit normal vectors of knee contact points

\( n^T_1, n^T_2 \)  
tibia coordinates of unit normal vectors of knee contact points
\( q \)  

generalized coordinates for the lower extremity

\( q_{i/j} \)  

generalized coordinates for segment i relative to segment j

\( Q_i \)  

generalized force in direction of generalized coordinate i

\( r_i^f, r_i^t \)  

femur coordinates of locations of knee contact points

\( r_i^f, r_i^t \)  

tibia coordinates of locations of knee contact points

\( r_P^S \)  

coordinates of point P relative to segment S

\( R^S \)  

rotation matrix for segment S relative to the GCS

\( R_{i/j} \)  

rotation matrix for segment i relative to segment j

\( s \)  

dependent parametric and kinematic coordinates of the knee

\( T \)  

kinetic energy

\( u_{t1}, v_{t1}, u_{t2}, v_{t2} \)  

parametric coordinates for two contact points on tibia

\( u_{f1}, v_{f1}, u_{f2}, v_{f2} \)  

parametric coordinates for two contact points on femur

\( V \)  

potential energy

\( V_C^S \)  

linear velocity of center of mass of segment S

\( x^S \)  

position vector for segment S relative to the GCS

\( x_{i/j} \)  

position vector for segment i relative to segment j

\( x_P \)  

GCS coordinates of point P

\( \alpha^S \)  

angular acceleration of segment S

\( \theta_x, \theta_y, \theta_z \)  

Euler angles which define the rotation matrix

\( \omega^S \)  

angular velocity of segment S
Chapter 1

INTRODUCTION

1.1 Background

The knee is the largest joint in the human body. During the course of lifetime of a typical person, the knee joint withstands tens of millions of cycles with loads of several times body weight and usually functions without failure. The knee, however, is also the most frequently injured and surgically repaired joint. Existing methods for analyzing the mechanics of the knee joint are relatively simplistic compared to the complex orthopaedic surgical procedures which are performed ranging from arthroscopic surgery to total knee replacement. Although many of these surgical procedures produce very substantial improvements as measured by increased mobility and reduction of pain, the procedures have evolved based on empirical rather than analytical reasoning. Surgeons must rely primarily on intuition and experience rather than information based on fundamental mechanics because the mechanics of the injured knee joint, or even of the healthy knee joint, are not well understood. Difficulties in analyzing the knee arise from the interactions of the complex geometry, kinematics and numerous passive and active elements which control the motion of the joint. This research deals with the development of a model of the healthy human knee joint which may be used to investigate some of the underlying mechanical phenomena of the joint.

1.1.1 Review of Knee Anatomy

The skeletal knee is composed of three articulating bones: the femur, tibia, and patella. The femur, or thigh-bone, is the largest bone in the body. The distal portion of the femur at the knee joint terminates in two curved surfaces, the medial and lateral condyles (figure 1.1). Weight-bearing is supported by the femoral condyles resting on the two corresponding regions of the articular surface, the plateau of the tibia. The anterior surface of the distal femur is smooth and concave, providing a guide for the patella, or knee cap, to
slide up and down. All articulating surfaces are covered with a thin layer of hyaline cartilage. The patella acts as an attachment site for extensor muscles of the knee joint which have origins along the shaft of the femur or above the hip joint and insert into the tibia through the patella and the patellar ligament.

Many structures are involved in controlling the motion of the knee joint. The muscles are the only active structures, in that their force levels are subject to neural control as well as being functions of their lengths. More than a dozen muscles cross the knee joint, most of which also cross either the hip or ankle joints. Additionally, four major ligaments cross the joint and connect the femur to the tibia (figure 1.1): the anterior and posterior cruciate ligaments.
ligaments (ACL and PCL), which form a cross between their insertions; and the medial and lateral collateral ligaments (MCL and LCL), which are attached at the sides of the joint. Functionally (at least in the cadaver), the ACL limits backward sliding of the femur relative to the tibia and hyperextension of the knee, while the PCL limits forward sliding of the femur relative to the tibia, especially when the knee is flexed. The medial and lateral menisci are crescent-shaped elastic structures and run around the periphery of the articulating surfaces of the tibia, apparently serving to distribute the femoral condyle contact loads upon the tibial surfaces.

Relative motion of the femur with respect to the tibia in vivo has often been modeled as a simple hinge joint [45]. Its motion, however, is complex and depends on material properties of the ligaments, the activation levels of the knee muscles, and the externally applied loads. During normal function, the condyles of the femur appear to roll and slide on the tibia, primarily constrained by the articular surface geometry and forces in the ligaments and muscles. Even the motion of a cadaver knee is very complex; despite the fact that the muscle activation levels do not change, the axis of rotation changes significantly as the joint is flexed [19].

The purpose of this research is to examine motions and load-bearing of the tibia and femur. Hence, the kinematics of the patella will be omitted except as its motion may change the moment arms of knee muscles. Furthermore, knee kinematics will always refer to tibio-femoral motion without regard to the motion of the patella.

1.2 Previous Knee Studies

Knee research can be classified into two basic categories: studies of isolated elements of the knee and studies which analyze the entire knee as a system. Although the knee model in this thesis will require some estimates of material and geometric properties of isolated elements, the emphasis of the thesis is on the relationships between these elements and knee function, i.e. the system. For this reason, isolated elements of the knee will not be discussed here, but rather presented in later chapters where estimates of particular material properties are required. This literature review will be limited to studies relevant to more
The ultimate objective of most knee models is to estimate the forces on the knee joint (and possibly other joints as well) and the muscles which cross the joints of the lower extremity in vivo, especially during gait. Unfortunately, it has not yet been possible to measure the actual forces on a human knee joint in vivo (although both the force vector [54] and pressure distribution [25] have been measured in the human hip joint in vivo). Therefore, the accuracy of any particular solution can only be evaluated by comparing the muscle force estimates with measurements of muscle myoelectric activity (EMG). To further complicate matters, EMG has been shown to be directly related to muscle force only for some special cases of static postures. Hence, only the timing information of the EMG data can be usefully correlated with the muscle force estimates to check the results during dynamic activities such as gait.

1.2.1 Joint Force Estimates via Functional Grouping of Muscles

Early models of the human knee were greatly simplified in order to overcome two basic difficulties: (1) in vivo joint and muscle forces cannot be directly measured; and (2) the large number of muscles acting across the knee joint produce an underconstrained, indeterminate problem. Morrison [43] modeled the knee as composed of three muscle groups, four ligaments, and one joint contact force acting at an assumed joint center. His was primarily a sagittal plane analysis, although it did allow for a small joint force component in the medial-lateral direction. Typical maximum joint force magnitudes estimated by this method were approximately three times body weight (3.0 BW). Harrington [24] applied an analysis method similar to Morrison to estimate that the maximum force on the knee during gait was about 3.5 BW, and also concluded that the center of pressure was localized in the medial compartment through most of the stance phase. A limitations of these models is that no information on individual muscle forces can be estimated from the results. Since the timing patterns of EMG of actual muscles do not correspond to the muscle groups assumed, the results are not verifiable and may not be highly accurate.
1.2.2 Joint Force Estimates via Muscle Force Optimization

In order to solve the multiple redundancy problem without assuming only a few composite muscle groups, Crowinshield and Brand [17], Hardt [23], Patriarco et al [49], and Seirig and Arvikar [58], among others, solved for the muscle and joint forces using muscle force optimization methods. These methods, which are typically applied to gait data, select from among all of the feasible solutions to the joint equilibrium equations that set of muscle forces which minimize some objective function. Typical objective functions include weighted sums of squares or cubes of muscle forces or stresses and total chemical energy expended during a gait cycle. These methods have primarily been tested with kinematic data which assumes a pin-joint model of the knee. A more recent study has attempted to require equilibrium to be satisfied for both the flexion and abduction degrees of freedom at the knee (or maybe three degrees of freedom, check this reference), but, reportedly this produced errors in the joint force estimates and occasionally produced no feasible solutions [50]. Much work has been reported in applying different optimization criteria, but Patriarco et al [49] have shown this to be a much less significant factor in muscle and joint force predictions than the accuracy of the kinematic data or assumptions about muscle origins, insertions or lines of action. Nonetheless, optimization methods do estimate muscle forces which have reasonable time correlations with EMG measurements (with some limitations).

A major limitation of these methods is their inability to predict agonist and antagonist muscles as activated simultaneously, which has been observed to occur during many activities including gait. An agonist muscle is one which instantaneously produces a positive component in the direction of the required net moment about a specified axis, while an antagonist muscle produces a negative moment component in this direction. The combination of simultaneous agonist and antagonist activity, also called muscular co-contraction, requires more chemical energy to be expended than if just the minimum agonist activity produced the required net moment. Since co-contraction represents energy inefficiency or incremental joint force or muscle stress, it is not surprising that levels of co-contraction are consistently underestimated by muscle force optimization methods.
One might assume that muscle force optimization methods would never predict co-contraction to occur, but there are two cases when a minimum energy solution could allow co-contraction forces. If a muscle crosses more than one joint, then it may be an agonist muscle at one joint while an antagonist muscle at the other joint (for example, the rectus femoris muscle is a flexor of the hip and an extensor of the knee). Even if an antagonist muscle crosses only one joint, it may be required to be activated to offset moment components perpendicular to the net loading which are created by agonist muscles.

1.2.3 Simplified Mechanical Analyses of the Knee

More recently, some knee models have been developed which avoid optimization methods by changing some of the assumptions of the original knee models. Wongchaisuwat et al. [67] presented a static planar knee model which included a feedback law for the muscles and selected the muscle forces to stabilize the closed-loop system. The results of this analysis predicted unrealistic large interarticular friction forces (actual articular surfaces are almost perfect frictionless bearings). Johnson et al. [31] and Nissan [47] presented 3D models with simple kinematic assumptions. These models have been shown to be sensitive to kinematic modeling errors [47].

1.2.4 Measurement of 3D Knee Kinematics

A common assumption in the biomechanics literature is that the knee can be treated as a single degree of freedom joint, that is although its motion may include translations along and rotations about all three cartesian axes, the geometrical configuration can be completely specified by the flexion angle. Hence studies have been conducted to calculate how the knee would presumably move in vivo by measuring the motion of cadaver knees. For example, Duke et al. [19] measured the location of the axis of rotation of a cadaver knee as a function of flexion angle. These studies require the assumption that muscle forces cannot affect the knee kinematics other than by influencing the flexion angle.

Chao et al. [13] and Lafortune et al. [36] have attempted to measure the kinematics of
the knee in vivo, but the measurement techniques are at best able to provide accurate estimates of the joint angles. Although Lafortune's study involved using intracortical bone pins, the resulting translations were not reported with any confidence. Murphy [44], [45] has also measured in vivo knee kinematics using intracortical bone pins and has demonstrated dependence of the knee kinematics on muscle activity and external loading.

1.2.5 3D Knee Models with Passive Constraints

Many studies have investigated constraints on knee motion due to ligaments. Many researchers have measured in vitro kinematics of the knee with intact ligaments and externally applied loads [4], [9], [18], [27], [29], [33], [34], [40], [51], [57], [65]. These studies may provide some information on ligament function, but since muscle activities are completely ignored they cannot accurately represent how the knee functions in vivo. Furthermore, the complex geometry of the knee joint has not been included except for approximating the changing joint centers. Wismans et al [66] have presented a knee model which includes the geometry of the articulating surfaces of the knee and the ligamentous constraints. Although the model was not validated directly, it was able to match some predictions of knee response to loading in the literature.

1.2.6 3D Models with Passive and Active Constraints

Mikosz [42] developed a model which assumed a joint center located on the tibial plateau with an anterior-posterior component dependent on the flexion angle by an assumed functional relationship, and included 13 muscles plus ligaments and a joint capsule which were modeled using spring elements. This model required joint moments to be balanced by the muscle forces, while the shear forces were balanced by the ligaments, joint capsule and the articulating surfaces. Maximum joint forces were reported to be up to 7.0 BW during level walking with corresponding shear forces of nearly 2.0 BW. The correlations of the predicted muscle force patterns with EMG timing were somewhat reasonable, even predicting some co-contraction, but that is not surprising since the muscle forces were solved by "pre-conditioning" with EMG data rather than by using optimization methods.
Cheng [14] reported an iterative technique for estimating bony contact forces on the joint during one point of the gait cycle. The locations of the contact points on the medial and lateral tibial articulating surfaces were taken from a cadaver study by Ahmed and Burke [1] of the pressure distribution in the knee joint. An iterative technique was used to solve for the three force components of each of the two contact points by alternately using muscle optimization to estimate the muscle and ligament forces and then using equilibrium equations to solve for the joint reaction forces. The knee model included 13 muscles and 13 ligaments, and could be solved after assuming that each of the contact forces was parallel to the axis of the tibia and that equilibrium was satisfied exactly in only two of the six directions. The estimated joint contact force, which was assumed to correspond to the phase of the gait cycle at maximum knee joint loading, was calculated to be 4.3 BW, with 1.3 BW on the lateral condyle and 3.0 BW on the medial condyle.

Garg [21] presented a model of the knee which included 3 equivalent muscles and a model for complex single degree of freedom kinematics which depend only on the flexion angle. This model used average articular geometry from cadaver knees, and knee motion (i.e. flexion angle versus time) and external loading taken from the literature. Equilibrium was required to be satisfied with point contact assumed on each condyle. No limits were placed on ligament forces. The model was applied to gait data, with a maximum knee joint force calculated to be approximately 7.0 BW. During much of the gait cycle only a single equivalent muscle was assumed to be active. Because the joint contact force locations and directions were fixed in the calculations for the muscle and ligament forces, the ligament forces required to satisfy force and moment equilibrium in all six directions may have been above physiological limits.

1.3 Statement of Purpose

The purpose of this thesis is to develop a mathematical model of the human knee joint which can be used to accurately estimate muscle and joint forces in vivo. Several of the studies reviewed in the previous section have attempted to produce these same estimates, so it is important to differentiate exactly what makes this study unique and why it is being conducted. The fundamental difference between this and all previous in vivo knee joint
models is that this model will be applied to realistic knee kinematics and it will require all force and moment components to be accounted for by reasonable actions of muscles, ligaments, and joint contact forces.

1.3.1 A Knee Joint Model Including Articular Geometry

In differentiating the knee model of this thesis from previous in vivo models it is necessary to answer two questions. What exactly are "realistic" knee kinematics? What are "reasonable" actions of muscles, ligaments, and joint contact forces?

As mentioned previously, knee motion is characterized by the condyles of the femur rolling and sliding on the articulating surfaces of the tibia. Hence, the kinematics of the knee are strongly influenced by the geometry of the articulating surfaces of the tibia and femur. There may very well exist loading situations and knee motions which do not correspond to contact between both of the condyles of the femur and the tibia (perhaps during the swing phase of gait), but during periods of substantial load-bearing by the knee joint, condylar contact in both the medial and lateral compartments of the knee is a reasonable assumption. Given this assumption, "realistic" knee kinematics can be defined as those which correspond to contact between both condyles of the femur and the articulating surfaces of the tibia.

An empirical observation about healthy human articular joints is that they exhibit extremely low coefficients of friction. At each location on the surface, substantial stresses may occur only in the direction normal to the surface. Since the curvature of the articular surface of the tibia is not large (i.e. the slope of the surface does not change much over small distances on the surface), then even though the contact force actually corresponds to a pressure distribution, it will always be oriented approximately normal to the surface at the center of pressure. Therefore, knee joint contact forces are also influenced by joint articular geometry, and "reasonable" joint contact forces will be assumed to be oriented in the direction parallel to the surface normal, and obviously only compressive. A "reasonable" muscle or ligament force simply corresponds to a tensile force between zero and some
physiologically maximum value acting along the line of action of the particular muscle or ligament.

In summary, the fundamental difference between the knee joint model in this thesis and all previous in vivo models is that the joint articular geometry is included such that it can influence both the kinematics and dynamics of the joint. Only Wismans' model (66) of the in vitro knee shares these characteristics, so in a sense this model can be considered an extension of that model for in vivo analysis.

1.3.2 A Knee Joint Model Designed for Stability Analyses

As mentioned in section 1.2.2, a limitation of muscle force optimization methods is that they have difficulty estimating muscular co-contraction. The basis for muscle force optimization is an assumption that joint force and moment equilibrium is established using a minimum energy configuration (or some other objective function). Muscular co-contraction is energetically inefficient, but it may be observed during many apparently efficient activities such as walking. Muscle force estimation methods which use an assumption of minimum energy may therefore be inappropriate. Since the predicted muscle and joint force estimates are relatively insensitive to the choice of objective function [49], then perhaps the limitation of these methods is that they are missing some additional required constraint in additional to equilibrium.

Everyday experience demonstrates that muscular co-contraction is related to the stiffness or stability of joints. For example, if one simultaneously maximally contracts the biceps and triceps muscles, then the elbow becomes substantially stiffer. Perhaps there exists some level of joint stiffness which the human control system would always prefer to maintain. If this is indeed the case, then one might assume that a better estimate of muscle and joint forces would be based on a method which minimizes energy such that joint force and moment equilibrium is established while at the same time maintaining some minimum level of joint stiffness or stability (where stability is to be defined later in this thesis). Following this assumption, a second goal of this thesis is to develop a knee joint model which includes a minimum joint stiffness or stability constraint which influences the muscle
force optimization results.

1.3.3 Purpose of this Thesis

Following the preceding discussion, the purpose of this thesis can be summarized as follows.

The purpose of this thesis is to develop a mathematical model of the human knee joint which allows an examination of in vivo relationships between geometry, kinematics, and dynamics. Specifically, the model will estimate knee kinematics and muscle and joint forces given measurements of the motion of the lower extremity segments and foot-floor force and moment vector interactions. The model will be used not only to estimate joint kinematics and forces but also to examine sensitivity of the results to changes in selected modeling and processing parameters. The sensitivity analysis will be performed for both a force equilibrium analysis and a stability analysis of the knee joint during the stance phase of gait.

1.4 Summary of Chapters

This thesis is organized into eleven chapters. Introductory and explanatory material is presented first including the derivation of relevant equations. Then, more specifics of the knee model geometry, kinematics, and dynamics are provided. Although Chapters 6 through 8 are presented separately, their results are actually very much inter-related, as emphasized in Chapter 9. A more specific description of the contents of the following ten chapters is presented below.

Chapter 2 introduces the problem of estimating joint and muscle forces as an inverse control problem. The dynamic equations which these forces must satisfy are presented, and previous methods for estimating joint and muscle forces are reviewed. Assumptions and limitations of these methods are explained, along with a brief description of the methods to be employed in this thesis to overcome some of these limitations.
Chapter 3 defines joint stability as it is considered in this thesis. Specifically, the requirements on muscle forces are presented which must be satisfied in order for a joint to be in a state of stable equilibrium. A simple model for muscle stiffness is proposed for performing this analysis, and a model of the human elbow joint is used to demonstrate that the requirement of stability of equilibrium may necessitate muscular co-contraction under some loading conditions. An experiment is described which produces results qualitatively very similar to the elbow joint model stability predictions. Finally, the requirements for applying the stability analysis to the human knee joint are provided.

Chapter 4 presents a description of the system used to obtain the kinematic and dynamic data required for the knee model. The TRACK data acquisition system is described, along with extensions for making the measurements more useful for analyzing the lower extremity. Characteristics of the kinematic data are presented, along with descriptions of processing methods which are useful for defining anatomical coordinate systems and joint angles.

Chapter 5 summarizes the separate components of the knee model and explains how they are related. Requirements for geometry, kinematics, and dynamics of the knee model are provided such that a stability of equilibrium analysis may be performed. The assumptions and approximations used in the development of the knee model are enumerated and explained. The requirements for incorporating the knee model into a four-segment rigid body model of the lower extremity are described. With this background material covered, a more detailed summary of Chapters 6 through 9 is presented to emphasize the relationships between knee model geometry, kinematics, dynamics, and muscle and joint forces.

Chapter 6 explains the procedures and results used for measuring cadaver tibia and femur articular geometry and anatomical landmark locations to estimate the articular surface geometry of a human knee joint in vivo. A review of previous methods used to estimate cadaveric knee joint geometry is presented, along with a proposed method which is much simpler, more flexible and nearly as accurate. The method for modeling the joint surfaces is described, along with additional processing methods for identifying the valid regions of
data and insuring the convexity of the femoral condylar articular surfaces. Definitions for anatomical coordinate systems for the tibia and femur are presented which are based on the locations of the anatomical landmarks and a simple kinematic measurement.

Chapter 7 begins with a description of the problem of soft tissue motion in biomechanical analysis and describes the methods employed in this research to estimate and minimize these errors in kinematic data. The problem is reduced to using the measured lower extremity kinematic data along with estimated articular surface geometry and an assumption of contact on both condylar surfaces to produce a set of improved kinematic data plus estimates of the soft tissue motion errors. This method of kinematic data correction is divided into three procedures: (1) estimating the approximate locations of the tibia and femur from the instantaneous kinematic measurements and anatomical landmark coordinate information; (2) calculating the optimal average translation and rotation of the tibia and femur relative to their anatomical coordinate systems to minimize geometric incompatibilities over an entire set of data; and (3) calculating the optimal deviations from the average translations and rotations in order to guarantee geometric compatibility while minimizing changes in the kinematic data. Results are presented of raw and corrected kinematic data and the corresponding estimates of soft tissue motion errors.

Chapter 8 presents the methods and results for estimating the muscle, ligament, and joint forces which satisfy the dynamic equations while minimizing a specified optimization criterion. The equations for dynamic equilibrium and stability of equilibrium developed in Chapters 2 and 3 are reviewed with an emphasis on the requirements for applying these equations to the lower extremity. Results are provided for the muscle and joint force predictions during the stance phase of the gait cycle with equilibrium and stability constraints applied at various degrees of freedom of the lower extremity. The problem of estimating ligament forces given the displacements of the origins and insertions is addressed. Difficulties in satisfying equilibrium for all of the degrees of freedom of the knee joint are explained in terms of the method for estimating knee kinematics without regard to dynamics.
Chapter 9 describes a method of iteratively estimating knee joint kinematics and joint and muscle forces such that the equilibrium constraints may be satisfied for all degrees of freedom of the knee. The necessity of a three dimensional knee joint model to include estimates of joint geometry, kinematics, and dynamics in order to realistically estimate either of the latter two quantities is emphasized. The results of this method are compared to the results of Chapters 7 and 8, and a method is proposed for simultaneously estimating the knee joint kinematics and joint and muscle forces.

Chapter 10, the conclusion, includes a list of specific accomplishments of this thesis. The equilibrium and stability analyses are critiqued and interpreted in terms of benefits and possible limitations of the model and required extensions. Recommendations are made for future work in this and related areas of research.
Chapter 2

MUSCLE FORCE ESTIMATION AS A TRAJECTORY ANALYSIS PROBLEM

2.1 Introduction

The goal of this thesis is to estimate joint and muscle forces during the stance phase of the gait cycle in vivo. This problem is one of analysis rather than design in that the actual muscle and joint forces are already prescribed and will not be changed in any way by the force estimates. Although herein the muscle forces will be estimated using information about the kinematics, inertial properties, and actions of the actuators (i.e. muscles), this analysis is fundamentally different from a typical robotics design problem in trajectory control.

A typical application of a trajectory control design problem in robotics can be summarized with the following scenario. You are placed in a room with a robot and given estimates of its inertial properties, behavior of its environment, actuator contributions to loading the various degrees of freedom of the robot as a function of the control inputs, and a prescribed trajectory for the robot (or at least its end effector) to follow. Your assignment is to design a set of control commands and a feedback control law which will enable the robot to follow the trajectory, presumably with some degree of stability.

The type of problem encountered in muscle and joint force estimation is one of analysis rather than design. Referring to the above scenario, you are again placed in a room with a robot and given estimates of its inertial properties, behavior of its environment, and actuator contributions to loading the various degrees of freedom of the robot as a function of actuator inputs. You are then allowed to observe the robot perform some maneuver, and are provided with estimates of the resulting trajectory. Your assignment is to come up with the set of control commands which were used to enable the robot to follow the trajectory. After making your estimates, you will be allowed to compare them with the actual control
command signals. This problem will be referred to as a trajectory analysis problem.

For a robot with a single actuator controlling each degree of freedom, the trajectory analysis problem may not be difficult to solve. Once the kinematics are supplied, they can be substituted into the dynamic equations and the required set of control torques can be calculated directly. Small errors in the estimates of the kinematics, inertial properties, etc. will certainly cause some errors in the estimates, but they would be expected to agree very well with the actual command signals.

Next, consider the case in which the robot has many more actuators than degrees of freedom. In this case, the dynamic equations alone are insufficient for calculating the actuator commands. You are forced to make some assumptions to make your best estimate of the actuator commands from among all of the possible sets of commands which satisfy dynamic equilibrium. You might assume that the robot had been optimally controlled to minimize control actions or some other reasonable penalty function and use this information to select your best estimate. This situation is now similar to the typical procedure for estimating muscle forces given estimates of inertial properties, kinematics, and muscle moment arms, with the exception that the relationship between the neural control command input to a muscle (as measured by EMG) and the output force is not well established.

The remainder of this chapter develops the equations which may be used to estimate muscle and joint forces by assuming that the muscles are used optimally to satisfy the dynamic equations of motion. However, before continuing into the development of the equations for estimating muscle and joint forces, it is worthwhile to consider possible extensions to the trajectory analysis problem scenario. Assume now that you had been permitted to observe the robot perform the maneuver not once but rather many times, and in the presence of small disturbances by the environment. In this case it might be reasonable to assume that the robot had been controlled such that it was stable. Would this provide any additional information which you could use to improve your estimates of control commands? What if you were supplied with the feedback law? These questions will be addressed in this chapter and in Chapter 3. Another possible extension might be that you were provided with poor estimates of the kinematics for some of the degrees of freedom.
Chapter 9 will address one possible solution to this problem.

2.2 Four Segment Model of the Lower Extremity

For the purposes of dynamic analysis, a single leg of the lower extremity will be modeled as a set of four segments (foot, shank, thigh, and pelvis), each of which is approximated as a rigid body (figure 2.1). The segments are assumed to interact at their ends, due only to muscle, ligament, and joint contact forces. For convenience in writing the equations, the segments are numbered such that $S=1, 2, 3, 4$ refers to the foot, shank, thigh and pelvis, respectively. Each of the segments will be assumed to have a mass $m^S$ and a mass moment of inertia tensor $I^S$, where the latter quantity will in general depend on the configuration of the lower extremity. In the lower extremity model of this thesis, the pelvis effects will be handled as a set of forces and moments which act on the femur at the hip joint, and thus the inertial properties of segment $S=4$ (i.e. the pelvis) will have no effect on the dynamic equations.

In order to apply a Lagrangian analysis to the lower extremity model, a set of generalized coordinates must be selected. By definition, a set of generalized coordinates is a set of the minimum number of parameters required to completely specify the kinematic configuration of a body. For example, a rigid body in space requires six parameters to completely specify its position and orientation, and three parameters are required to specify the position and orientation of the hip joint (i.e. the pelvis relative to the thigh). The set of

![Figure 2.1: Lower Extremity Model](image-url)
generalized coordinates \( q \) used to analyze the kinematics of the lower extremity will be assigned as follows:

\[
q^T = \begin{bmatrix}
q_{1/0}^T & q_{2/1}^T & q_{3/2}^T & q_{4/3}^T
\end{bmatrix}
\]

(2.1)

where \( q_{1/0} \) specifies the six generalized coordinates which determine the position and orientation of the foot relative to the fixed inertial global coordinate system (GCS), \( q_{2/1} \) are the two generalized coordinates which specify the position and orientation of the tibia relative to the foot (i.e. the ankle joint), \( q_{3/2} \) specifies four generalized coordinates for the knee, and \( q_{4/3} \) specifies three generalized coordinates for the hip. The fifteen generalized coordinates \( q \) represent the nine degrees of freedom for the three joints plus an additional six coordinates for the position and orientation of the entire lower extremity. The actual coordinates which are selected for each joint will be described in Chapter 5.

Given the coordinates defined above, the position and orientation of all four of the lower extremity segments relative to the GCS are completely specified by \( q \) for all points in time. Furthermore, the translational and rotational velocities of the segments relative to the GCS depend only on \( q \) and \( \dot{q} \) such that

\[
\begin{bmatrix}
v_c^S \\
\omega_c^S
\end{bmatrix} = J^S q = \begin{bmatrix}
J_L^S \\
J_A^S
\end{bmatrix} \dot{q}
\]

(2.2)

where \( v_c^S \) is the translation velocity of the center of mass of segment \( S \), \( \omega_c^S \) is the angular velocity of segment \( S \), and \( J^S \) is by definition the Jacobian for segment \( S \). \( J_L^S \) and \( J_A^S \) are the matrices which determine the instantaneous translational and angular velocities of segment \( S \) given \( \dot{q} \), the rate of change of the generalized coordinate vector. Note that in these equations \( J^S \) is a function of \( q \).

By separately considering the columns of the Jacobian, the translational and angular
velocities of each segment can instantaneously be expressed as linear combinations of each of the time rates of change of the fifteen generalized coordinates.

\[ v^S_C = \sum_{k=1}^{N} J^S_{lk} q_k \]  
(2.3)

\[ \omega^S = \sum_{k=1}^{N} J^S_{Ak} \dot{q}_k \]  
(2.4)

2.3 Dynamic Equations Applied to the Lower Extremity

Now that some preliminary expressions have been provided for the inertial properties and translational and rotational velocities for each of the segments, it is possible to generate a set of dynamic equations for the lower extremity. The equations will be derived based on the notation of Asada and Slotine for dynamics of robotic systems [7]. Consider an instance during the stance phase of gait in which the foot is in contact with the floor and the interaction at the hip is represented by a net force and moment vector of segment 4 acting on segment 3. In the equations listed below, \( M \) represents the number of segments which are completely included in the model. Since segment 4 is included only in terms of the loads which it applies on segment 3, then for this lower extremity model of the foot, shank, and thigh, \( M = 3 \).

In order to write out Lagrange's equations of motion it is first necessary to evaluate the kinetic energy \( T \) and potential energy \( V \) for the lower extremity model. Using the expressions for translational and rotational velocities of the segments as defined in section 2.2, the expressions for kinetic and potential energy are as follows:

\[ T = \sum_{s=1}^{M} \left\{ \frac{1}{2} m^S v^S_C v^S_C + \frac{1}{2} \omega^S \omega^S + \frac{1}{2} \omega^S T \omega^S \right\} \]  
(2.5)

\[ V = \sum_{s=1}^{M} m^S g^T x^S_C \]  
(2.6)

In equation (2.6), \( g \) represents the acceleration of gravity and \( x^S_C \) represents the GCS vector
to the center of mass of segment $S$.

Using the previous definitions of the Jacobian for each segment, the kinetic energy can be written in a slightly different form.

$$
T = \frac{1}{2} \sum_{s=1}^{M} \left\{ m^s q^T J_s^L J_s^T q^s + q^T J_s^T I^s J_s^q q^s \right\} 
$$

(2.7)

Using a notation similar to Asada and Slotine [7], this expression can be simplified.

$$
T = \frac{1}{2} q^T \left\{ \sum_{s=1}^{M} H^S \right\} q = \frac{1}{2} q^T H q 
$$

(2.8)

$$
H^S = m^s J_s^L J_s^T + J_s^T I^s J_s^q
$$

(2.9)

In equation (2.8), $H$ is the "lower extremity inertia tensor", a symmetric matrix which represents the contributions of the foot, shank and thigh to inertia in the directions of each of the degrees of freedom. Although $q$ has been defined to include 15 generalized coordinates, only a 12 by 12 portion of $H$ will be non-zero (and positive definite), as the inertial properties of the pelvis are not included in the dynamic equations as written.

Now that the kinetic and potential energies have been defined in terms of the generalized coordinates and their derivatives with respect to time, Lagrange's equations can be used to generate the necessary dynamic equations for the lower extremity model.

$$
L = T - V 
$$

(2.10)

$$
\frac{d}{dt} \left( \frac{\partial L}{\partial q_i} \right) - \frac{\partial L}{\partial q_i} = Q_i 
$$

(2.11)

In equation (2.11), $L$ is the Lagrangian and $Q_i$ is the generalized force which represents the amount of work which would be performed by all of the external forces (other than those accounted for in the potential energy) due to a unit displacement in the direction of generalized coordinate $q_i$. The external forces which may contribute to the generalized
forces include the force plate and hip forces which act at the two ends of the kinematic chain and the muscle and ligament forces.

By substituting equations (2.6) and (2.7) into equation (2.11), the dynamic equations can be written in the following form:

\[ \sum_{j=1}^{N} H_{ij}^S \dot{q}_j + \sum_{k=1}^{N} \sum_{j=1}^{N} h_{ijk} + G_i^S \]

\[ \Delta Q_i^S = \sum_{j=1}^{N} H_{ij}^S \dot{q}_j + \sum_{k=1}^{N} \sum_{j=1}^{N} h_{ijk} + G_i^S \]  

\[ h_{ijk} = \frac{\partial H_{ij}^S}{\partial q_k} - \frac{1}{2} \frac{\partial H_{jk}^S}{\partial q_i} \]

\[ G_i^S = m^S g^T J_{Li}^S \]  

In these equations, \( N_{\text{max}}(M) \) represents the number of the generalized coordinates which affect the position and orientation of segment \( M \) (in this case \( N_{\text{max}}(3) = 12 \)), \( G_i^S \) represents required generalized forces due to conservative forces acting on segment \( S \), and \( \Delta Q_i^S \) represents the total contribution due to conservative and inertial forces. These equations can be evaluated once expressions for the \( H \) matrix (which depends on inertial properties plus the segment Jacobians) and the generalized forces have been obtained. Sections 2.3.1 and 2.3.2 will provide the necessary expressions for the segment Jacobians and generalized forces.

2.3.1 Evaluation of Segment Jacobians

Before evaluating the segment Jacobians, it is necessary to specify exactly how the segment positions and orientations are assumed to depend on the generalized coordinates \( q \). Consider a vector \( r_p^1 \) which represents the coordinates of a specific point \( P \) in space as measured in the coordinate system associated with segment \( 1 \) (i.e. the foot). The transformation from \( r_p^1 \) to \( x_p \), the GCS coordinates of this same point in space, is by convention written in the following form:
In this equation \( R_{1/0} \) represents the rotation matrix of the foot relative to the GCS and \( x_{1/0} \) represents the translation vector of the foot relative to the GCS (figure 2.2).

Similarly, the transformations between the coordinates of this same point as measured in the coordinate systems of each of the four segments will be written in the following form:

\[
x_p = R_{1/0} \mathbf{r}_p^1 + x_{1/0}
\]  
(2.16)

\[
\mathbf{r}_p^1 = R_{2/1} \mathbf{r}_p^2 + \mathbf{r}_{2/1}
\]  
(2.17)

\[
\mathbf{r}_p^2 = R_{3/2} \mathbf{r}_p^3 + \mathbf{r}_{3/2}
\]  
(2.18)

\[
\mathbf{r}_p^3 = R_{4/3} \mathbf{r}_p^4 + \mathbf{r}_{4/3}
\]  
(2.19)

In equations (2.16) through (2.19), \( R_{1/0}, R_{2/1}, R_{3/2}, \) and \( R_{4/3} \) are functions of \( q_{1/0}, q_{2/1}, q_{3/2}, \) and \( q_{4/3} \), respectively, as defined in equation (2.1). \( x_{1/0}, x_{2/1}, x_{3/2}, \) and \( x_{4/3} \) similarly are functions of \( q_{1/0}, q_{2/1}, q_{3/2}, \) and \( q_{4/3} \).

The segment Jacobians represent the dependence of the segment positions and orientations (in GCS coordinates) on the generalized coordinates. Thus it is useful to define the transformations between each segment's coordinate system and the GCS.

Figure 2.2: Segment Transformations
\[ x_p = R^S r_p^S + x^S \]  

(2.20)

In this equation, \( R^S \) represents the 3 by 3 rotation matrix of segment \( S \), \( x^S \) is the location of the origin of the coordinate system of segment \( S \) relative to the origin of the GCS, and \( r_p^S \) represents the coordinates of a point as observed in the coordinate frame of segment \( S \).

Equations (2.16) through (2.19) can be combined with equation (2.20) to establish the segment rotation matrix \( R^S \) for each of the four segments in terms of \( q \).

\[
\begin{align*}
R^1 & = R_{1/0} \\
R^2 & = R_{1/0} R_{2/1} \\
R^3 & = R_{1/0} R_{2/1} R_{3/2} \\
R^4 & = R_{1/0} R_{2/1} R_{3/2} R_{4/3}
\end{align*}
\]

(2.21) to (2.24)

Expressions for the translation vectors \( x^S \) can be obtained using these same equations.

\[
\begin{align*}
x^1 & = x_{1/0} \\
x^2 & = R^1 x_{2/1} + x^1 \\
x^3 & = R^2 x_{3/2} + x^2 \\
x^4 & = R^3 x_{4/3} + x^3
\end{align*}
\]

(2.25) to (2.28)

The expressions for \( J^S_{\ell} \), which are used to calculate velocities of the segment centers of mass from \( q \), can be obtained in a very straightforward manner. Let \( r_c^S \) be the fixed vector from the origin of segment \( S \) to its center of mass as measured in the coordinate system of segment \( S \). It follows by direct application of equation (2.20) that \( J^S_{\ell} \), the vector which represents the change of the GCS coordinates of this point with respect to a unit change in generalized coordinate \( q_\ell \), is given by the following expression.
\[
J^S_{yk} = \frac{\partial x^S_C}{\partial \dot{q}_k} = \frac{\partial R^S_C}{\partial \dot{q}_k} r^S_C + \frac{\partial x^S}{\partial \dot{q}_k} 
\]  
(2.29)

The expressions for \(J^S_{yk}\), which are used to calculate angular velocities of the segments from \(\dot{q}\), requires more analysis. Consider a point \(P\) which is fixed in segment \(S\), and has coordinates \(r^S_P\) as measured in the coordinate frame of segment \(S\). The position and velocity vectors in the GCS can be obtained using equation (2.20) and differentiation, as follows.

\[
x_P = R^S_0 r^S_p + x^S 
\]  
(2.30)

\[
\dot{x}_p = \dot{x}^S + \dot{R} r^S_p = \dot{x}^S + \left\{ \sum_{k=1}^{N} \frac{\partial R^S}{\partial \dot{q}_k} \dot{q}_k \right\} r^S_p 
\]  
(2.31)

An alternative method for calculating the GCS coordinates of the velocity of point \(P\) is to use the angular velocity of segment \(S\).

\[
\dot{x}_p = \dot{x}^S + \omega^S \times \left( R^S_0 r^S_p \right) 
\]  
(2.32)

Recall equation (2.4), and express the angular velocity vector in terms of its three components.

\[
\omega^S = \begin{bmatrix}
\omega^S_1 \\
\omega^S_2 \\
\omega^S_3
\end{bmatrix} = \sum_{k=1}^{N} J^S_{yk} \dot{q}_k 
\]  
(2.33)

Comparison of equations (2.31) and (2.33), along with the definition of vector cross product, leads to the following requirement for these two expressions to be equal for an arbitrary point \(P\) fixed with respect to segment \(S\).
Both sides equation of (2.34) can now be post-multiplied by the transpose of the rotation matrix of segment S. Since a rotation matrix must be orthonormal (i.e. its transpose is the same as its inverse), then the dependence of the left side of the equation on the rotation matrix disappears, and an expression for the dependence of the angular vector components on the time rates of change of the generalized coordinates results.

\[
\begin{bmatrix}
0 & -\omega_3^s & \omega_2^s \\
\omega_3^s & 0 & -\omega_1^s \\
-\omega_2^s & \omega_1^s & 0
\end{bmatrix}
= \sum_{k=1}^{N} \left\{ \frac{\partial R^S}{\partial q_k} \right\} \dot{q}_k 
\]  

(2.35)

Finally, the terms can be rearranged to give an explicit expression for the portion of the segment Jacobian which is required for calculating angular velocities.

\[
J_{Ak}^S = \begin{bmatrix}
R_{21}^S \frac{\partial R_{31}^S}{\partial q_k} + R_{22}^S \frac{\partial R_{32}^S}{\partial q_k} + R_{23}^S \frac{\partial R_{33}^S}{\partial q_k} \\
R_{31}^S \frac{\partial R_{11}^S}{\partial q_k} + R_{32}^S \frac{\partial R_{12}^S}{\partial q_k} + R_{33}^S \frac{\partial R_{13}^S}{\partial q_k} \\
R_{11}^S \frac{\partial R_{21}^S}{\partial q_k} + R_{12}^S \frac{\partial R_{22}^S}{\partial q_k} + R_{13}^S \frac{\partial R_{23}^S}{\partial q_k}
\end{bmatrix}
\]

(2.36)

Note that for a rotary joint, \( J_{Ak}^S \) simply represents the GCS coordinates of the unit vector in
the direction of increasing \( q_k \) (using the right-hand rule). Unfortunately, the kinematics of the human knee are rather complex and cannot be modeled as a series of rotary and translational degrees of freedom, but rather some degrees of freedom will represent combinations of translations and rotations.

### 2.3.2 Evaluation of Generalized Forces

Now that the segment Jacobians have been defined, the expressions on the left side of equation (2.12) can be completely evaluated except for one term, the inertia tensor of each segment. Let \( I^S_0 \) represent the mass moment of inertia of segment \( S \) about its center of mass as expressed in its own coordinate system. This tensor remains fixed for all times, and can be expressed in terms of the GCS using the following equation:

\[
I^S = R^S I^S_0 R^{S^T}
\] (2.37)

Now that all terms of the left-hand side of equation (2.12) have been evaluated, it remains to evaluate the right-hand side of the equation, the generalized forces. Recall that the generalized force \( Q_i \) represents the amount of work which would be performed by all of the external forces due to a unit displacement of generalized coordinate \( q_i \). In the lower extremity model containing \( M \) segments, there are four separate types of contributions to the generalized forces: (1) \( Q_{FPi} \), contributions due to foot-floor reactions at the force plate; (2) \( Q_{NETi}^{M+1} \), contributions due to the net force and moment vectors of segment \( M+1 \) acting on segment \( M \) (e.g. the pelvis acting on the femur); (3) \( Q_{Mi} \), contributions due to the muscles; and (4) \( Q_{Li} \), contributions due to the ligaments. If generalized coordinates are chosen carefully, then the joint forces cannot do any work and the frictional forces are negligible. Contributions to generalized forces due to joint capsules, skin, and other soft tissues are expected to be much smaller in magnitude than muscle and ligament contributions and will be ignored in this analysis. Hence, the total generalized force is a sum of these four terms.

\[
Q_i = Q_{FPi} + Q_{NETi}^{M+1} + Q_{Mi} + Q_{Li}
\] (2.38)
In evaluating the contributions of the force plate force and moment vectors to the generalized forces, the equivalent reaction force and moment vectors of the foot acting on the force plate will be calculated. If $F_{FP}$ and $M_{FP}$ represent the force and moment vectors of the force plate acting on segment 1, and $r_{FP}^1$ represents the vector from the origin of segment 1 to the center of pressure of the force plate, then the net force and moment vectors of the foot acting on the force plate, $F_{NET}^1$ and $M_{NET}^1$, can be calculated as follows.

\[
F_{NET}^1 = -F_{FP}
\]

\[
M_{NET}^1 = -M_{FP} - \left( R_{FP}^1 r_{FP}^1 \right) \times F_{FP}
\]

Given these expressions, then the contribution to the generalized force due to the force plate force and moment vectors can be calculated. First consider $P_{FP}$, the total amount of power that would be added to segment 1 due to the force plate force and moment vectors if it were subject to a translational and angular velocity.

\[
P_{FP} = -F_{NET}^1 v^1 - M_{NET}^1 \omega^1
\]

It is also possible to relate this power to the generalized force terms as follows.

\[
P_{FP} = \sum_{i=1}^N Q_{FP_i} \dot{q}_i
\]

By equating terms, equations (2.41) and (2.42) can be used along with equation (2.4) to calculate the contributions to the generalized forces due to the force plate loads.

\[
Q_{FP_i} = -F_{NET}^1 \frac{\partial x_i^1}{\partial q_i} - M_{NET}^1 J_{Ai}^1
\]

Similar reasoning can be used to evaluate the generalized force contributions due to the
force and moment vector of segment M+1 acting on segment M, \( F_{\text{NET}}^{M+1} \) and \( M_{\text{NET}}^{M+1} \), respectively. Note as previously that in the case of the lower extremity model M=3, so this represents the loads of the pelvis acting on the thigh. Assume that the moment vector is defined with respect to the origin of coordinate system M+1. Then the contributions to generalized forces due to the net force and moment vector can be written as follows.

\[
Q_{\text{NET}i}^{M+1} = F_{\text{NET}}^{M+1T} \frac{\partial x^{M+1}}{\partial q_i} + M_{\text{NET}}^{M+1T} J_A^M
\]  

Finally, the contributions to the generalized forces due to the muscles and ligaments must be evaluated. At any point in time, the power which the \( k \)th muscle is producing is related to its tensile force \( F_{M_k} \) and its instantaneous rate of change of length \( L_{M_k} \) by the following expression.

\[
P_{M_k} = -F_{M_k} \dot{L}_{M_k} = -F_{M_k} \sum_{i=1}^{N} \frac{\partial L_{M_k}}{\partial q_i} \dot{q}_i
\]  

From this expression, it follows that the total contribution of all of the muscles to the generalized force \( Q_i \) is given by

\[
Q_{Mi} = -\sum_{k=1}^{NM} F_{M_k} \frac{\partial L_{M_k}}{\partial q_i}
\]  

where NM refers to the total number of muscles.

Using the same reasoning, the contribution to the generalized forces due to the ligaments can be written with a nearly identical expression.

\[
Q_{Li} = -\sum_{k=1}^{N_L} F_{L_k} \frac{\partial L_{L_k}}{\partial q_i}
\]
where NL refers to the total number of ligaments. $F_{Lk}$ is the tensile force on the ligament, and $L_{Lk}$ is the length of the ligament.

2.3.3 Solution of the Dynamic Equations

Now that all of the terms of the right-hand side of equation (2.12) have been evaluated, the dynamic equations for the lower extremity can be written as follows

$$\sum_{k=1}^{NM} F_{Mk} \frac{\partial L_{Mk}}{\partial q_i} + \sum_{k=1}^{NL} F_{Lk} \frac{\partial L_{Lk}}{\partial q_i} = F_{NET}^{M+1} \frac{\partial x^{M+1}}{\partial q} + M_{NET}^{M+1} J_{Ai}^{M+1}$$

$$- F_{NET}^{T} \frac{\partial x^{1}}{\partial q} - M_{NET}^{T} J_{Ai}^{1} - \sum_{s=1}^{M} \Delta Q_{i}^{s}, \quad i \leq N_{max}(M)$$  (2.43)

where as before $\Delta Q_i$ are the generalized forces. Note that if they are applied to the $M=3$ segment model, then in this form the equations can only be used to address dynamic equilibrium for the first $N_{max}(M) = 12$ degrees of freedom (i.e. 6 dof for the foot relative to the GCS, 2 dof for the ankle, and 4 dof for the knee). These equations cannot be used to generate dynamic equations for the hip unless the pelvis is modeled with known inertial properties and forces and moments acting on its proximal end. The method for generating dynamic equations for the dof of the hip is presented later in this chapter.

The ultimate objective of this chapter is to derive the constraints on muscle forces in order that the dynamic equations may be satisfied. In the application of these equations to estimating muscle and joint forces during gait, the force and moment vector and center of pressure of the foot-floor reaction will be measured using a force plate. The net force and moment vector of the pelvis on the thigh, however, will be unknown. Fortunately, these vectors can easily be solved for by observing that the contributions of the muscles and ligaments to the generalized forces are exactly zero for the first six generalized coordinates (i.e. $q_{1/6}$). This is equivalent to an observation that muscle and ligament forces only change if the coordinates which affect the relative coordinates of the joints are changed. Using this information, six equations can be generated for the six unknown components of
the net force and moment vector of segment M+1 acting on segment M.

\[
\left[ \frac{\partial x^{M+1}}{\partial q_i} J^{M+1}_{Ai} \right] \left[ \begin{array}{c} F^{M+1}_{NET} \\ M^{M+1}_{NET} \end{array} \right] = F^{1}_{NET} \left( \frac{\partial x^1}{\partial q_i} + M^{1}_{NET} J^{1}_{Ai} + \sum_{s=1}^{M} \Delta Q^s_i \right) i \leq 6 \quad (2.49)
\]

Once these equations have been solved for the net force and moment vector of segment M+1 acting on segment M, then equation (2.48) can be used to generate the remaining equations in terms of the muscle and ligament forces and length derivatives.

Thus far emphasis has been placed on developing dynamic equations for an M segment model for the lower extremity. Note, however, that equation (2.49) is not valid exclusively for any specific value of M, but rather is valid for any value of M as long as all of the \( \Delta Q_i^S \) terms can be evaluated. Therefore, equation (2.49) can also be used to evaluate the net force and moment vector of segment 2 acting on segment 1, where the moment vector corresponds to the moment about the origin of the coordinate system of segment 2 (i.e. the loading across the ankle joint). Furthermore, it is easy to see that if \( F^{1}_{NET} \) and \( M^{1}_{NET} \) are defined as above, then the net force and moment vector of segment M+1 acting on segment M can be calculated from the following six equations (for \( i = 1,2,\ldots,6 \)):

\[
\left[ \frac{\partial x^{M+1}}{\partial q_i} J^{M+1}_{Ai} \right] \left[ \begin{array}{c} F^{M+1}_{NET} \\ M^{M+1}_{NET} \end{array} \right] = \left[ \frac{\partial x^{M}}{\partial q_i} J^{M}_{Ai} \right] \left[ \begin{array}{c} F^{M}_{NET} \\ M^{M}_{NET} \end{array} \right] + \Delta Q^M_i \quad (2.50)
\]

Therefore, although it is not a necessary part of the calculations, the net force and moment vector can be calculated across any of the lower extremity joints by iteratively applying equation (2.50).

Finally, it has been mentioned that the equations can thus far only be used to examine equilibrium for the first \( N_{max}(M) \) generalized coordinate directions (i.e. not including the
degrees of freedom of the hip). It is useful now to number the joints according to the following scheme: joint M connects segment M and segment M+1. Therefore, the motion of joint M is affected by generalized coordinates q_i such that \( N_{\text{max}}(M) < i \leq N_{\text{max}}(M+1) \). If all \( N_{\text{max}}(M+1) \) coordinates are held fixed except for a virtual displacement in coordinate q_i, such that q_i affects the motion of joint M, then the virtual work of segment M acting on segment M+1 can be written in terms of the net loading across the joint.

\[
\delta W = \left\{ -F_{\text{NET}}^{M+1} \frac{\partial x^{M+1}}{\partial q_i} - M_{\text{NET}}^{M+1} J_{Ai}^{M+1} \right\} \delta q_i, \quad \text{for joint M} \quad (2.51)
\]

This expression is only valid for the degrees of freedom of joint M. Note that this virtual work can only be produced by the muscles and ligaments, because the joint forces do no work. Therefore, the following expression must also hold.

\[
\delta W = \left\{ -\sum_{k=1}^M F_{Mk} \frac{\partial L_{Mk}}{\partial q_i} - \sum_{k=1}^N F_{Lk} \frac{\partial L_{Lk}}{\partial q_i} \right\} \delta q_i, \quad \text{for joint M} \quad (2.52)
\]

Finally, for the degrees of freedom of joint M, these two expressions can be combined to produce an expression in terms of the muscle and ligament forces and the net loading across the joint.

\[
\sum_{k=1}^M F_{Mk} \frac{\partial L_{Mk}}{\partial q_i} + \sum_{k=1}^N F_{Lk} \frac{\partial L_{Lk}}{\partial q_i} = F_{\text{NET}}^{M+1} \frac{\partial x^{M+1}}{\partial q_i} + M_{\text{NET}}^{M+1} J_{Ai}^{M+1} \quad (2.53)
\]

Note that this equation differs from equation (2.46) in that equation (2.46) is only applicable to the degrees of freedom below joint M whereas equation (2.53) is applicable exclusively for the degrees of freedom of joint M.

In summary, the dynamic equations can be derived by first using equation (2.49) to calculate the net loading across the hip joint, then using equation (2.46) to evaluate the constraints on muscle and ligament forces for the first 12 degrees of freedom, and finally
using equation (2.53) to evaluate the muscle constraints for the hip joint. An alternative approach is to use equations (2.49) and (2.53) to sequentially generate the equations for each of the joints. Another possible calculation method would be to simply use equation (2.46) with \( M = 4 \) and some arbitrary values for inertial properties of segment 4, because they will not affect the dynamic results at the ankle, knee, and hip joints. Although 15 equations will result, only 9 of these will involve muscle and ligament forces, as one would expect from the nine degree of freedom lower extremity model. The final result of all of this analysis is thus a set of 9 linear equations in the muscle and ligament forces.

2.3.4 Why a Lagrangian Analysis?

Prior muscle force optimization analyses have typically used a Newtonian formulation for evaluating the translational and angular accelerations of each segment, multiplying by the appropriate inertial properties, and then evaluating the required net forces and moments which must have produced the observed kinematics. The equations presented here appear to be more complex. What is the reason for selecting this analysis, and is it really more complex?

First, the equations here are simply written in a general form and will produce exactly the same results as a Newtonian analysis applied to the same model. Anticipating that the knee will be modeled as a four degree of freedom joint with complex kinematics, the use of a Lagrangian analysis eliminates the need to include the joint forces in the analysis (because they do no work). Furthermore, the joint stiffness and stability analysis described in the next chapter becomes much easier to implement using a set of generalized coordinates as the independent variables.

A second reason for using a Lagrangian analysis is that it is very straightforward to use kinematic and dynamic equations which are compatible and correspond to the same model. In all previous muscle optimization analyses, the dynamic equations were solved without regard to the kinematics of the joints, except for assuming locations of joint centers. By explicitly writing the equations in terms of the kinematic model of the joints, all forces may

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be accounted for either due to muscles, ligaments or other soft tissues.

2.4 Muscle Force Estimation

All of the solution schemes suggested in section 2.3.3 to generate the dynamic equations for the nine degrees of freedom of the joints result in equations of the following form

\[ \sum_{k=1}^{NM} F_{Mk} \frac{\partial L_{Mk}}{\partial q_i} + \sum_{k=1}^{NL} F_{Lk} \frac{\partial L_{Lk}}{\partial q_i} = \text{right-hand side expression} \quad (2.54) \]

where the "right-hand side expression" consists of inertial and external contributions to the generalized forces. The equations are linear in muscle and ligament forces, where the coefficients are derivatives of muscle and ligament lengths with respect to the generalized coordinates. Thus before using the equations to estimate the muscle and joint forces, it is necessary to evaluate the derivatives of the muscle and ligament lengths.

2.4.3 Calculation of Muscle Lengths and Derivatives

In this, as in all prior muscle force optimization analyses, an assumption will be made that the muscles act in straight lines from their insertions to their origins, although for some of the muscles additional points will be selected as effective origins and insertions where the muscles may wrap around some of the bones. The actual model of muscle geometry will be described in more detail in Chapter 5. Here consider that a muscle has an origin \( r_{O}^{SO} \) fixed in the coordinate system of segment SO and in insertion \( r_{I}^{SI} \) fixed in the coordinate system of segment SI. Using these definitions, then the GCS coordinates of the muscle origin and insertion, and their derivatives with respect to the generalized coordinates, are given by the following four equations.

\[ x_{O} = R_{O}^{SO} r_{O}^{SO} + x_{SO} \quad (2.55) \]
\[ x_{I} = R_{I}^{SI} r_{I}^{SI} + x_{SI} \quad (2.56) \]
Given these definitions, then the square of the length of the muscle can be written as

\[ L_M^2 = (\mathbf{x}_O - \mathbf{x}_I)^T (\mathbf{x}_O - \mathbf{x}_I) \]  

(2.59)

By differentiating both sides of this equation with respect to \( q_i \), the derivatives of the muscle length can be written in a relatively simple form.

\[ 2 L_M \frac{\partial L_M}{\partial q_i} = 2 (\mathbf{x}_O - \mathbf{x}_I)^T \left( \frac{\partial \mathbf{x}_O}{\partial q_i} - \frac{\partial \mathbf{x}_I}{\partial q_i} \right) \]  

(2.60)

\[ \frac{\partial L_M}{\partial q_i} = (\mathbf{x}_O - \mathbf{x}_I)^T \left( \frac{\partial \mathbf{x}_O}{\partial q_i} - \frac{\partial \mathbf{x}_I}{\partial q_i} \right) / L_M \]  

(2.61)

For a purely rotational degree of freedom, the calculated derivative value of the muscle length will be equal to the distance to the rotational axis, as would be required for this calculation method to produce the same results as using vector cross products and explicitly using a unit vector in the direction of the line of action of the muscle. The advantage of using this method is that it also works very well for degrees of freedom which are neither purely rotational or purely translational, as may occur in the kinematics of the knee joint. If ligaments are assumed to act along straight lines between their origins and insertions, then their lengths and derivatives can be calculated using the same scheme.

2.4.2 Muscle Force Optimization

Now comes the part of the analysis which is similar to the trajectory analysis problem
described in section 2.1. Given that there are many more than nine muscles which cross the lower extremity joints (36 in the model used in this thesis), what assumptions are made in order to select the optimal set of muscle forces from all of the feasible solutions? In all muscle force optimization analyses, including the method to be employed in this thesis, the set of equilibrium relations is used as constraints on the muscle forces and then some objective function is used to select the optimal set of muscle forces. As previously mentioned in section 1.2.2, typical choices for the objective function are sums of squares or cubes of muscle forces or stresses, or some function related to energy. The choice of the penalty function does not seem to affect the results, so the determining factor to the output muscle force estimates is the set of equilibrium constraints on the muscle forces. This section will describe some of the differences between the equilibrium constraints that will be generated in this thesis and in previous muscle force optimization analyses.

The fundamental difference between the equilibrium constraint equations derived in this work and in previous studies is that as much as possible the joint geometry and kinematics will be allowed to affect the equilibrium equations. In previous muscle force optimization analyses, inverse Newtonian analysis is used to calculate the net force and moment acting across the joints, and then a joint center is selected with the assumption that the muscles produce all of the net moments about that joint center. This assumption (i.e. that joint forces cannot produce a moment about the presumed center of the joint) is really only valid for the hip joint as it is reasonably modeled as a ball-and-socket joint. For the knee joint, the assumption that the joint forces produce no moment about the "joint center" may actually result in a very large error, since the force on the medial compartment of the knee alone has been estimated to be 2.0 BW or more. Furthermore, since the axis of rotation of the knee may change significantly throughout the gait cycle, any attempt to select a single joint center for all times will incur errors. Only Cheng [14] has attempted to include a somewhat realistic estimate of joint contact force locations in the knee in a muscle force optimization analysis for one time in the gait cycle.

A second major difference between this analysis and previous studies is that dynamic equilibrium will be required to be satisfied for all of the degrees of freedom of the kinematic model. In previous muscle force optimization analyses, nine equations result from the
requirement of moment equilibrium at each of the three joints, but then only selected degrees of freedom are required to satisfy equilibrium. For example, only the flexion moment equilibrium is required to be satisfied at the knee joint. If there is a valid reason to assume that some elements of a joint limit motion in a particular direction, then it may be reasonable to eliminate that equilibrium equation. If the muscles cannot move the joint in that direction, then that degree of freedom should also be eliminated from the kinematic model. Using a kinematic scheme which allows significant changes in the abduction or rotation angles of the knee joint but does not attempt to enforce equilibrium in these directions is inconsistent. One of the reasons only certain degrees of freedom are required to satisfy equilibrium in these models is that it may not be possible to satisfy equilibrium for all of the degrees of freedom. When all of the equilibrium equations are evaluated, feasible solutions to satisfy both equilibrium and constraints on maximum muscle forces may not occur. Although this problem may be partially due to the approximations in modeling muscle lines of action, at the very least all discrepancies must be documented when equilibrium violations occur.

It should be pointed out that most muscle force optimization analyses have attempted to include all three of the lower extremity joints. However while Cheng has attempted to include the knee joint forces in his optimization model, the optimization has been applied only to a single joint model (i.e. a model of the knee). Of the thirteen muscles which were included in Cheng’s model, nine also cross either the hip or the ankle joint. Therefore, these muscles must also contribute to the equilibrium equations at those joints. If this information is not considered, then the muscle forces may either be underestimated or possibly a set of muscle forces may be predicted that does not allow equilibrium to be satisfied at the other joints.

In summary, the muscle force optimization analysis in this thesis will include all three of the lower extremity joints, and will use a dynamic analysis method which is compatible with the kinematics and joint geometry such that equilibrium may be established for all of the degrees of freedom in the model.
2.5 Trajectory Analysis and Stability

Recall again the trajectory analysis problem. This time consider a specific example, that of an inverted pendulum. The dynamic equations for an inverted pendulum of length L and mass m, with an applied moment M can be written as:

\[ m L^2 \frac{d^2 \theta}{dt^2} - m g L \sin \theta = M \]  (2.61)

Assume that the pendulum is being controlled with the following algorithm.

\[ M = m L^2 \frac{d^2 \theta_d}{dt^2} - m g L \sin \theta_d - K_p(t) \left( \theta - \theta_d \right) - K_v(t) \left( \frac{d\theta}{dt} - \frac{d\theta_d}{dt} \right) \]  (2.62)

where \( \theta_d \) is the desired trajectory. This control is a combination of a feedforward term which inverts the plant and a PD feedback term. For small errors in the trajectory, the closed-loop dynamic equations for the error can be written as:

\[ \Delta \theta = \theta - \theta_d \]  (2.63)

\[ m L^2 \frac{d^2 \Delta \theta}{dt^2} + K_v(t) \frac{d\Delta \theta}{dt} + \left( K_p(t) - m g L \cos \theta_d \right) \Delta \theta = 0 \]  (2.64)

If \( K_v(t) \) is a small positive constant, then what constraints are there on \( K_p(t) \) for this system to remain stable?

This apparently trivial example points out the difficulties with trajectory analysis as opposed to trajectory control. If this were a trajectory control problem, and one were attempting to make this dynamically stable (assume that steady-state error is not a concern) then \( K_p(t) \) could be selected as shown below to guarantee stability of the system.

\[ K_p(t) = m g L \cos \theta_d + \beta^2, \ \beta = \text{constant} \]  (2.65)
since it would transform equation (2.64) into one with all positive constant coefficients. Hence, this is a sufficient condition for stability. For a trajectory analysis problem, a necessary condition for stability is required.

The difficulty with the trajectory analysis problem is that, except for the special case of small oscillations about an equilibrium configuration, nonlinear stability theory deals primarily with sufficient conditions rather than necessary conditions for stability. For example, the construction of a Lyapunov function [?] which decreases for all time is a sufficient condition for stability, but this analysis cannot be used to decide that a nonlinear system is necessarily unstable. Therefore, in the example problem if the value of $\theta_d$ changes with time, then strictly speaking you cannot make any decisions at all about necessary values for $K_p(t)$. If $\theta_d$ is a constant and $K_p(t)$ is selected to be a constant, then a necessary and sufficient condition for stability is $K_p > m g L \cos \theta_d$. Given this information, then for the case where $\theta_d$ changes relatively slowly with time it is a reasonable approximation to estimate that $K_0 \geq m g L \cos \theta_d$. At the very least, the value of $K_0(t)$ would not be expected to be significantly smaller than this value.

Now consider that there exist two actuators crossing the joint at the base of the pendulum (figure 2.3) and that they can exert a moment $M$ and stiffness $K_0$, such that

$$M = R (F_1 - F_2),$$  \hspace{1cm} (2.66)

$$K_0 = R (F_1 + F_2).$$  \hspace{1cm} (2.67)

If you were to estimate the moments due to the two actuators using only the equilibrium equation (2.61), then you would estimate that the actuator that produced the moment with the correct sign would be the only one used at that time. However, if you knew that inertial effects were not significant in the problem, then it might be reasonable to assume that for each point in time
\[ K_0 = R(F_1 + F_2) > mgL \cos \theta_d \] (2.68)

which might affect your estimates for \( F_1 \) and \( F_2 \).

Although this example is not directly related to muscle force optimization, it is conceptually equivalent to using the biceps and triceps muscles to support a load across the elbow joint. The idea of using an approximate stability analysis to effect selection of forces in a trajectory analysis problem is relevant to the stability analysis of human joints discussed in Chapter 3. The fundamental assumption arising from this analysis is the following: although not a necessary condition for stability, a reasonable assumption for a system in which inertial effects are not substantial is that it have at least enough stiffness to be stable, were it in exactly the same kinematic configuration and not moving.

Figure 2.3: Inverted Pendulum with Two Actuators
Chapter 3

STABILITY ANALYSIS
APPLIED TO HUMAN JOINTS

3.1 Introduction

Co-contraction of agonist-antagonist muscle groups is observed during many activities involving either the lower or upper extremity. While for some activities co-contraction appears to be used intentionally to increase the stiffness of a joint, antagonist muscle activity is also observed during motions for which it is not immediately obvious why additional joint stiffness would be desired at the expense of energy inefficiency. Furthermore, exactly how joint stiffness is increased by co-contraction over a wide range of force levels, or even more fundamentally how joint stiffness should be defined and calculated for a quasi-static activity have not been rigorously addressed.

The purpose of this chapter is to develop a quantitative method for predicting minimal antagonist co-contraction forces during quasi-static activities using a stability of equilibrium analysis. This analysis requires a precise definition of how individual muscles contribute to joint stiffness as a function of muscle length, moment arm, maximum force and activation level. The stability analysis is applied to a six-muscle model of the human elbow joint and validated using a simple experiment of supporting vertical loads over a range of elbow angles. Finally, the requirements for applying this analysis to a model of the human knee joint are discussed.

3.2 Stiffness of a Mechanical System in Static Equilibrium

By the Principle of Virtual Work, for a mechanical system which is in static equilibrium, the total work done by all of the forces must be exactly zero for any arbitrary infinitesimal virtual displacement which is compatible with the geometric constraints. If a set of generalized coordinates is used, then by definition all changes in these coordinates
are compatible with the geometrical constraints. Therefore, the Principle of Virtual Work defines a system to be in static equilibrium if and only if zero total work is done by the forces due to any combination of arbitrary infinitesimal virtual displacements in the generalized coordinate directions. Consider a mechanical system which has both conservative and non-conservative forces acting on it. Now define $V$ as the potential energy stored in the system due to actions of some of the conservative forces, and define $Q_i$ as the generalized force in the direction of generalized coordinate $q_i$ due to all remaining forces. With these two definitions, the requirement of the Principle of Virtual Work for static equilibrium can be written in the following form:

$$\delta W = \sum_{i=1}^{N} Q_i \delta q_i - \sum_{i=1}^{N} \frac{\partial V}{\partial q_i} \delta q_i = \sum_{i=1}^{N} \left[ Q_i - \frac{\partial V}{\partial q_i} \right] \delta q_i = 0 \quad (3.1)$$

Since this must be true for arbitrary $\delta q_i$, then the following must also be true.

$$Q_i - \frac{\partial V}{\partial q_i} = 0, \quad i = 1, 2, \ldots, N \quad (3.2)$$

Now consider a mechanical system which satisfies static equilibrium as expressed in equation (3.2) but then has additional infinitesimal loads placed on it. The loads are applied such that each generalized coordinate direction $q_j$ has its coordinate changed by $\Delta q_j$. What are the additional generalized forces required in each direction in order to accomplish this change in kinematic configuration?

In order to answer this question, it is useful to define two additional quantities. First define $Q_{Vi}$ as the force component in direction $q_i$ due to the conservative forces which are included in $V$.

$$Q_{Vi} = -\frac{\partial V}{\partial q_i} \quad (3.3)$$
Then define $Q_{\text{DIST}i}$ as the disturbance force in the direction of $q_i$. By requiring static equilibrium at the new kinematic configuration, the following must hold for all infinitesimal disturbance forces.

$$Q_{\text{DIST}i} + Q_i + Q_{Vi} = 0$$

(3.4)

Note that this equation reduces to equation (3.2) when no disturbance forces are present.

Next, apply a Taylor series expansion to get expressions for $Q_i$ and $Q_{Vi}$ for small deviations in the generalized coordinates.

$$Q_i = Q_i^0 + \sum_{j=1}^{N} \left[ \frac{\partial Q_i}{\partial q_j} \right]_0 \Delta q_j + \text{higher order terms}$$

(3.5)

$$Q_{Vi} = \left[ -\frac{\partial V}{\partial q_i} \right]_0 + \sum_{j=1}^{N} \left[ -\frac{\partial^2 V}{\partial q_i \partial q_j} \right]_0 \Delta q_j + \text{higher order terms}$$

(3.6)

Now substitute expressions (3.5) and (3.6) into equation (3.4).

$$Q_{\text{DIST}i} + \left[ Q_i - \frac{\partial V}{\partial q_i} \right]_0 + \sum_{j=1}^{N} \left[ \frac{\partial Q_i}{\partial q_j} - \frac{\partial^2 V}{\partial q_i \partial q_j} \right]_0 \Delta q_j + \text{h.o.t.} = 0$$

(3.7)

Finally, observe that the second term in the left-hand side expression is identically zero by equation (3.2), and solve for $Q_{\text{DIST}i}$ up to first order in $\Delta q_i$.

$$Q_{\text{DIST}i} = \sum_{j=1}^{N} \left[ \frac{\partial^2 V}{\partial q_i \partial q_j} - \frac{\partial Q_i}{\partial q_j} \right] \Delta q_j$$

(3.8)

Now define a generalized stiffness matrix $[K_{ij}]$, which will also be called the joint stiffness matrix, such that the following relationship is satisfied between application of
additional generalized forces $Q_{DIST_i}$ and the resulting generalized displacements $q_j$:

$$Q_{DIST_i} = \sum_{j=1}^{N} K_{ij} \Delta q_j$$  \hspace{1cm} (3.9)$$

By comparison of equations (3.8) and (3.9) the terms of the joint stiffness matrix can be evaluated using the following expression.

$$K_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j} - \frac{\partial Q_i}{\partial q_j}$$  \hspace{1cm} (3.10)$$

Nowhere in this analysis has it been assumed that the generalized forces are all conservative, so it is possible for the joint stiffness matrix generated using equation (3.10) to be asymmetric. The consequences of this possibility will be discussed later in this chapter.

### 3.3 Definition of Stability for Static Mechanical Systems

For the purpose of this thesis, two different but equivalent definitions for stability of a static mechanical system will be established. A system will be considered to be in a state of stable equilibrium if the following two equivalent conditions are satisfied:

1. for any small arbitrary disturbance load applied, the system returns to its unperturbed kinematic configuration when the load is removed.

2. positive work is required to displace the system a finite amount in any direction.

Condition (1) clearly requires that the system be in static equilibrium before any additional loads are applied. In terms of the stiffness matrix defined in the preceding section, either condition requires as a necessary condition that the joint stiffness matrix $K$ be positive semi-definite. Note that the semi-definiteness of the joint stiffness matrix is a necessary but
not sufficient condition for stability. If the joint stiffness matrix is singular, then higher order derivatives of the generalized forces and potential energy would have to be evaluated to determine the stability of the system. If the joint stiffness matrix is positive definite, then the system is stable.

To further explain the difference between the sufficient and necessary conditions for stability, consider a nonlinear spring which exerts a force $F$ in the direction opposite to a displacement $x$. If this system is in equilibrium for $x=0$, then it must be true that $F=0$ when $x=0$. The work which must be performed to displace the system from equilibrium can be written as

$$W(x) = \int_0^x F(\lambda) \, d\lambda$$

(3.11)

Using a Taylor series expansion, the work could also be written as

$$W(x) = W(0) + x \left( \frac{dW}{dx} \right)_0 + \frac{x^2}{2} \left( \frac{d^2W}{dx^2} \right)_0 + \frac{x^3}{3!} \left( \frac{d^3W}{dx^3} \right)_0 + \ldots$$

(3.12)

Substituting equation (3.11) into (3.12) allows the work to be expressed as a function of the spring force.

$$W(x) = W(0) + x F(0) + \frac{x^2}{2} \left( \frac{dF}{dx} \right)_0 + \text{higher order terms}$$

(3.13)

In order for this system to be stable by definition (2) above, then $W$ must be positive for all non-zero values of $x$. Noting that $W(0)=0$ and $F(0)=0$, a necessary condition can be written for the linearized spring stiffness $K$ such that $W$ is positive for arbitrarily small displacements in $x$. 

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\[ K = \left[ \frac{dF}{dx} \right]_0 \geq 0 \]  \hspace{1cm} (3.14)

If \( K \) is positive, then the system will be stable for small displacements independent of higher order derivatives of \( F \) with respect to \( x \). If \( K \) is exactly zero, then the higher order derivatives of \( F \) will determine the stability. Thus \( K > 0 \) is a sufficient but not necessary condition for stability, while \( K \geq 0 \) is a necessary but not sufficient condition for stability.

Consider the example of a stationary inverted pendulum (discussed in section 2.5). In this case the generalized force in the direction of increasing \( \theta \) is \( M \), and the potential energy \( V \) is equal to \( m g L \cos \theta \), so from equation (3.2) the requirement for static equilibrium is:

\[ M - m g L \cos \theta = 0 \]  \hspace{1cm} (3.15)

Equation (3.10) can be used to calculate the joint stiffness matrix of the pendulum system to additional applied moments. In this single degree of freedom case the joint stiffness matrix reduces to a scalar joint stiffness value, \( K \).

\[ K = -m g L \sin \theta - \frac{dM}{d\theta} \]  \hspace{1cm} (3.16)

The requirement for stability for this system is simply that the stiffness to external disturbances be non-negative. Therefore the requirement for stability in this example is:

\[ -m g L \cos \theta - \frac{dM}{d\theta} \geq 0 \]  \hspace{1cm} (3.17)

To compare this result with that of equation (2.68), use equation (2.62) to evaluate \( \frac{dM}{d\theta} \):

\[ \frac{dM}{d\theta} = -K_0(t) \]  \hspace{1cm} (3.18)
This expression can be substituted into equation (3.17) to obtain an identical requirement for stability as that obtained in equation (2.68) using a different analysis technique.

\[ K_0(t) \geq m g L \cos \theta_d \]  \hspace{1cm} (3.19)

3.4 Stability for Quasi-Static Systems

By rearranging terms, the requirements for static equilibrium (i.e. equation (3.2)) and dynamic equilibrium (i.e. equation (2.11)) can be written in very similar forms.

Requirement for Static Equilibrium

\[ \frac{\partial V}{\partial q_i} - Q_i = 0 \]  \hspace{1cm} (3.20)

Requirement for Dynamic Equilibrium

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial q_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} - Q_i = 0 \]  \hspace{1cm} (3.21)

The two expressions are identical when the kinetic energy \( T \) goes to zero.

Now recall the requirement for stability of static equilibrium.

\[ K_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j} - \frac{\partial Q_i}{\partial q_j}, \text{ where } K \geq 0 \]  \hspace{1cm} (3.22)

Can a similar requirement be applied to dynamic systems? Following the argument in section 2.5, a necessary condition for dynamic stability cannot be established easily. However, if the inertial contributions to the loads are small, then it may be reasonable to apply an approximate stability requirement to a quasi-static system as shown below.
Requirement for Stability of Quasi-Static Equilibrium

\[ K_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j} - \frac{\partial Q_i}{\partial q_j}, \text{ where } K \geq 0 \]  

(3.23)

Note that equation (3.23) is only an approximate relation. This is equivalent to saying that the stiffness values calculated with this formula are not large negative values.

3.5 Stability Analysis Applied to the Lower Extremity Model

In order to apply the stability analysis to the lower extremity, first write the equations for dynamic equilibrium in the same form as the equation (3.21) (i.e. with the generalized force contributions with a negative sign).

\[ \sum_{s=1}^{M} \Delta Q_i^s - Q_{FPI} - Q_{M+1}^{NETi} - Q_{Mi} - Q_{Li} = 0, \text{ for } q_i \text{ below joint } M \]  

(3.24)

Now equation (3.23) can be used to generate the approximate constraint for the stability analysis. Note that this equation follows directly from (3.24), which was based on equation (2.12) and (2.38) and is only valid for the degrees of freedom below joint M (i.e. below the hip joint).

Requirement for Stability of Quasi-Static Equilibrium for Lower Extremity

\[ K_{ij} = \frac{\partial}{\partial q_j} \left\{ \sum_{s=1}^{M} G_i^s - Q_{FPI} - Q_{M+1}^{NETi} - Q_{Mi} - Q_{Li} \right\}, \text{ where } K \geq 0 \]  

(3.25)

This equation can also be written in a different form using equation (2.53) instead of (2.12) and (2.38). Note that this equation is valid only for the \( q_i \) which affect the relative motion of joint M.
By carrying out the differentiation process some informative results can be obtained from equation (3.26):

\[
K_{ij} = \frac{\partial F_{\text{NET}}^{M+1}}{\partial q_j} \frac{\partial q^{M+1}}{\partial q_i} + F_{\text{NET}}^{M+1} \frac{\partial^2 \mathbf{J}_{\text{Ai}}^{M+1}}{\partial q_i \partial q_j} + \frac{\partial M_{\text{NET}}^{M+1} \mathbf{J}_{\text{Ai}}^{M+1}}{\partial q_j} + M_{\text{NET}}^{M+1} \frac{\partial \mathbf{J}_{\text{Ai}}^{M+1}}{\partial q_j} - \frac{\partial Q_{\text{Mi}}}{\partial q_i} - \frac{\partial Q_{\text{Li}}}{\partial q_i}, \quad \text{where } K \geq 0
\]  

(3.27)

Several observations can be made regarding equation (3.27). This equation represents the joint stiffness matrix terms for a net force and moment applied across a joint produced by muscle, ligament and joint forces. For a purely static set of data for the lower extremity the net force acting across the joint would not depend on the kinematic configuration, but rather only on the segment masses (i.e. net force across a joint would be the weight of the body above the joint). Hence the first term in the expression would be exactly zero and the second term would contribute symmetrically to the joint stiffness matrix. Although it is not straightforward to show in general, the two moment terms would also contribute symmetrically to the joint stiffness matrix as long as the moment was caused by a conservative force (such as gravity).

It is interesting to contrast the moment terms with the force terms in that a constant applied moment across the joint (i.e independent of changes in the generalized coordinates) would produce an asymmetric joint stiffness matrix, because the Jacobian associated with angular velocities is not an exact differential. An asymmetric stiffness matrix does not represent a conservative force system. Obviously the order of rotations of angles in three-dimensional space affects the amount of work done when a constant applied moment is applied. For a planar analysis, angular displacements can add as vectors and this phenomenon would not be observed.
3.6 Muscle Force and Stiffness Requirements for Stability

In order to explain requirements on muscle forces and stiffnesses due to the stability analysis, it is useful to transform equation (3.27) into a simplified form.

\[ K_{ij} = K_{NETij} - \frac{\partial Q_{Mi}}{\partial q_j} - \frac{\partial Q_{Li}}{\partial q_j} \text{, where } K \geq 0 \]  

(3.28)

In this equation, \( K_{NETij} \) represents the contributions to the joint stiffness matrix due to the net joint force and moment vectors. The derivatives of the contributions to the generalized forces due to the muscles and ligaments must now be evaluated.

Equation (2.46) can be differentiated with respect to \( q_j \) to get an expression for the contribution of the muscles to the joint stiffness matrix in terms of muscle forces and derivatives of muscle forces with respect to the generalized coordinates.

\[ -\frac{\partial Q_{Mi}}{\partial q_j} = - \frac{\partial}{\partial q_j} \left\{ \sum_{k=1}^{NM} F_{Mk} \frac{\partial L_{Mk}}{\partial q_i} \right\} \]  

(3.29)

\[ -\frac{\partial Q_{Mi}}{\partial q_j} = \sum_{k=1}^{NM} \left\{ \frac{\partial F_{Mk}}{\partial q_j} \frac{\partial L_{Mk}}{\partial q_i} + F_{Mk} \frac{\partial^2 L_{Mk}}{\partial q_i \partial q_j} \right\} \]  

(3.30)

But muscle force depends on the generalized coordinates only through the dependence on its length. Hence this information can be used along with a definition for muscle stiffness \( K_{Mk} \) to obtain an expression for the contribution of muscles to the total joint stiffness matrix as a function of the muscle force, stiffness, and partial derivatives of its length.

\[ \frac{\partial F_{Mk}}{\partial q_j} = \frac{\partial F_{Mk}}{\partial L_{Mk}} \frac{\partial L_{Mk}}{\partial q_j} = K_{Mk} \frac{\partial L_{Mk}}{\partial q_j} \]  

(3.31)
\[ K_{Mk} = \frac{\partial F_{Mk}}{\partial L_{Mk}} \]  

\[ \ldots = \sum_{k=1}^{NM} \left\{ K_{Mk} \frac{\partial L_{Mk}}{\partial q_i} \frac{\partial L_{Mk}}{\partial q_j} + F_{Mk} \frac{\partial^2 L_{Mk}}{\partial q_i \partial q_j} \right\} \]  

The contribution of the ligaments to the joint stiffness matrix can be evaluated using the same procedure as for the muscles. The final expression for the joint stiffness matrix can then be obtained by substitution of the expressions for muscle and ligaments contributions into equation (3.28).

\[ K_{ij} = K_{NETij} + \sum_{k=1}^{NM} \left\{ K_{Mk} \frac{\partial L_{Mk}}{\partial q_i} \frac{\partial L_{Mk}}{\partial q_j} + F_{Mk} \frac{\partial^2 L_{Mk}}{\partial q_i \partial q_j} \right\} 
+ \sum_{k=1}^{NL} \left\{ K_{Lk} \frac{\partial L_{Lk}}{\partial q_i} \frac{\partial L_{Lk}}{\partial q_j} + F_{Lk} \frac{\partial^2 L_{Lk}}{\partial q_i \partial q_j} \right\} \]  

Second partial derivatives of muscle lengths are required for evaluating the terms of the joint stiffness matrix. These derivative terms can be evaluated in a similar manner to the first derivative terms by differentiating both sides of equation (2.60). In the following equations, \( \Delta x_M \) represents the GCS coordinates of a vector from the muscle insertion to the muscle origin.

\[ \Delta x_M = x_O - x_I \]  

\[ \frac{\partial}{\partial q_j} \left\{ L_M \frac{\partial L_M}{\partial q_i} \right\} = \frac{\partial}{\partial q_j} \left\{ \Delta x_M^T \frac{\partial \Delta x_M}{\partial q_i} \right\} \]  

\[ \frac{\partial^2 L_M}{\partial q_i \partial q_j} = \frac{1}{L_M} \left\{ \frac{\partial \Delta x_M^T}{\partial q_j} \frac{\partial \Delta x_M}{\partial q_i} + \Delta x_M^T \frac{\partial^2 \Delta x_M}{\partial q_i \partial q_j} - \frac{\partial L_M}{\partial q_i} \frac{\partial L_M}{\partial q_j} \right\} \]
3.7 Muscle Stiffness Model for Stability Analysis

For this quasi-static stability of equilibrium analysis, how should the value of muscle stiffness, \( K \), be estimated for each muscle? Muscle stiffness obviously corresponds to a rate of change of force with a change in length. Unfortunately, for muscles this rate of change of force depends very strongly on several factors: (1) the rate at which the length is changed; (2) the absolute muscle length; and (3) the innervation level.

If a muscle is activated at a constant level and lengthened rapidly a short amount (< 1 percent) then it is very stiff. Furthermore the measured stiffness value, which is called short-range stiffness, is proportional to the muscle force [53]. The magnitude of this stiffness is approximated by

\[
\frac{K}{F_{\text{max}} / L_0} = 30 \text{ to } 50 \frac{F}{F_{\text{max}}} \tag{3.38}
\]

where \( F \) is the muscle force, \( F_{\text{max}} \) is the maximum force which the muscle can produce, and \( L_0 \) is the muscle's characteristic length (rest length).

If a muscle is maximally activated and allowed to reach a steady-state level of force at a range of different lengths, then the resulting plot of force versus length is qualitatively as shown in figure 3.1 [41]. Of the three curves shown, the passive force is measured with no activation, the active force is the additional amount due to maximal activation, and the total force is the sum of the passive and active components.

A stiffness value corresponding to how the maximum force changes with length could be calculated by differentiating this curve. For muscle forces less than the maximum, it would be necessary to calculate a stiffness value, \( K \), as a function of \( F, F_{\text{max}}, L \) and \( L_0 \). Unfortunately, the force-length relationship of muscles has not been well established for activation levels which fall between zero and the maximum. It would therefore be necessary to assume that the active component of muscle force as a function of length.
Figure 3.1: Muscle Force at Maximum Activation

scales with increased activation according to some simple model. Note that stiffness values calculated using the force-length curve are extremely sensitive to estimates of muscle rest length and may even predict negative stiffness values for some combinations of activation levels and length.

For low muscle activation levels, it has been shown that a bilinear muscle model is a reasonable approximation to the force-length-activation relationship \([46]\) (figure 3.2). Using this model, the force is estimated by

\[
F = \left( F_0 + \kappa L \right) U
\]

where \(L\) is the muscle length (with respect to some offset value), \(U\) is the neural activation level (between 0 and 1), and \(F_0\) and \(\kappa\) are constants. For this model, the stiffness at a specified muscle length is proportional to the force and can be calculated by

\[
K = \frac{\kappa F}{F_0 + \kappa L}
\]
Figure 3.2: Bilinear Model of Muscle Force

Now that several different methods for estimating muscle stiffnesses have been described, which one should be used in the quasi-static stability of equilibrium analysis of human joints? As has been mentioned, short-range stiffness is only appropriate for rapid changes in length. For a static stability analysis, disturbances are assumed to be infinitely slow. Given enough time, an isolated muscle at constant activation which initially has its length changed very rapidly will reach a steady-state force level which corresponds to the force-length curve.

Is it then reasonable to use the slope of the force-length curve, or possibly the bilinear approximation, to estimate the stiffness values? All of the curves presented thus far for approximating the force-length relationship of muscles have been based on results from either isolated muscles in vitro (without reflex mechanisms) or isometric in vivo experiments. The stiffness levels which would result from differentiating these force-length curves correspond to how difficult it would be to change the length of a muscle very slowly at constant activation levels.

The stiffness value which is required for a quasi-static stability analysis is related to how the force on a muscle would change with length if a subject were attempting to maintain a fixed kinematic configuration. Because of the reflex feedback mechanisms which are present in muscles in vivo, the activation level of a particular muscle would not
necessarily remain constant if the muscle length is changed while a person is attempting to maintain a static posture (i.e. attempting to maintain constant muscle length). Since stiffness at constant activation level is not the same as stiffness for a constant desired muscle length, then the stiffness values calculated using the force-length relationship (or bilinear approximation of this relationship) are not appropriate for this analysis.

How can the stiffness for a constant desired muscle length be estimated? An experiment performed by Hoffer and Andreassen [26] was designed to measure the total incremental stiffness of cat soleus muscle. These experiments were feasible because cats decerebrated at the periamillary level will spontaneously activate the muscles about the ankle with loads that cover a considerable range of the maximum possible muscle forces and vary slowly with time (time constant about 10 seconds). The soleus muscle was prepared with reflex mechanisms intact such that its length could be modulated while its force could be simultaneously measured. As the force level on the muscle was spontaneously varied at a fixed muscle length, a small (1 mm) rapid (50 msec) change in length was imposed. The muscle force reached steady-state at a new level within 200 msec. The muscle was returned to its unperturbed length, and then the procedure was repeated again about 1 second later when the muscle was now at a new force level. The total incremental stiffness of the soleus muscle was calculated simply by dividing the change in steady-state force by the prescribed length change. By perturbing the system during different points in the spontaneous loading cycle, stiffness values were measured over a considerable range of force levels. Similar procedures were performed at several different muscle lengths which corresponded to the entire physiological range of motion. The resulting calculated total stiffness values appear to be nearly invariant of muscle length (figure 3.3).

How do the results of an experiment designed to measure the total incremental stiffness of cat soleus muscle relate to a quasi-static stability analysis of human joints? Since the rate of change of spontaneous force on the muscle in the experiments was very slow (compared to how quickly the stiffness values were measured), then the stiffness values
Figure 3.3: Total Incremental Stiffness Measurements

obtained are approximately the same as would have been obtained had an intact cat been attempting to maintain a posture corresponding to the unperturbed muscle length and average force level. Thus the measured total incremental stiffness values are equivalent to measurements of stiffness at constant desired muscle length. In the remainder of this chapter, total incremental stiffness will be used to denote stiffness at constant desired muscle length.

In order to transform the results such that they can be applied to human joints, two propositions are required.

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1This figure was adapted from Höffer and Andreassen [25] by scanning figure 5 from the paper and then manually editing to remove extraneous marks in the scanned image which did not appear to represent data points.
Proposition 1:

the total incremental stiffness of a muscle is independent of muscle length, depending only on the force.

Proposition 2:

total incremental stiffness can be approximately scaled for different muscles by non-dimensionalizing it with muscle maximum force and characteristic length such that

$$\frac{K}{F_{\text{max}}/L_0} = f\left(\frac{F}{F_{\text{max}}}\right)$$

(3.41)

Proposition 1 follows directly from the results of the cat soleus muscle experiments of Hoffer and Andreassen [26]. If this proposition is correct, then, independent of the complicated relationships between muscle activation, length, and force, the total incremental stiffness for a muscle will depend only on the force.

Proposition 2 is a result of dimensional analysis, taking into account the observation that different mammalian muscles have similar numbers of Golgi tendon organs and muscle spindles per unit volume or per unit physiological cross-sectional area. A possible significant source of error in this approximation may be due to different tendon properties for muscles which have similar maximum forces. The tendon properties affect the passive stiffness and thus the total incremental stiffness of the muscle. Unfortunately, data are not available for measured total incremental stiffness values for different muscles, and thus some approximation must be assumed.

Assuming that these two propositions are correct, then what is a reasonable estimate for the functional relationship between total incremental stiffness and muscle force? An estimate for the maximum force for cat soleus muscle (4000 g), a quadratic function was fit to each of the six reported stiffness versus force curves (figure 3.3) and then the estimated
quadratic coefficients were averaged. The resulting function for the approximate non-dimensional total incremental stiffness is given by

\[ k \equiv -15.64 f^2 + 13.24 f + 2.41 \]  \hspace{1cm} (3.42)

\[ k = \frac{K}{F_{\text{max}} / L_0} \]  \hspace{1cm} (3.43)

\[ f = \frac{F}{F_{\text{max}}} \]  \hspace{1cm} (3.44)

### 3.8 Example: Stability Analysis Applied to the Human Elbow Joint

Note in equation (3.42) that for zero muscle force this model estimates a significant non-zero muscle stiffness (i.e. almost half of its maximum value). In contrast, both short-range stiffness and the bilinear stiffness model (which exclude passive joint properties) correspond to zero stiffness at zero force.

Until now this chapter has been purely theoretical as far as the stability analysis is concerned. In order to test the stability analysis on a human joint but avoid some of the complexities of the multiple degree of freedom kinematics of the knee, the method was first applied to the human elbow joint. The elbow is much more accurately modeled as a single degree of freedom joint than the knee. Application of the stability analysis to the elbow not only allowed development of intuition about some of the different factors in the analysis but will also allowed testing of the hypotheses about the total incremental stiffness model for the stability analysis.

A simple experiment was designed to quantitatively test the stability analysis method. Some motivation for this experiment comes from work by Murray [46], in which he observed more co-contraction near the ends of the range of elbow motion compared to the upright position when supporting a vertical load (a weight) with the upper arm horizontal. Murray concluded that his bilinear muscle model could not account for this phenomenon. A possible reason for this will be explained later in this chapter.
Consider the case in which the upper arm is held stationary in a horizontal plane and a vertical load P is supported at an angle \( \theta \) from full extension (figure 3.4).

![Diagram of arm position for critical load measurement]

**Figure 3.4: Arm Position for Critical Load Measurement**

In this case, then generalized force \( Q_\theta \) in the direction of \( \theta \) due to the vertical load P can be written as

\[
Q_\theta = -D \cdot P \cos \theta + \sum_{k=1}^{NM} F_{Mk} \frac{dL_{Mk}}{d\theta}
\]  \( (3.45) \)

where \( D \) is the distance between the elbow and the point of application of the vertical load. The equilibrium equation can then be obtained using equation (3.20), and the equation for stability of equilibrium is obtained from equation (3.22). For this analysis the mass of the arm is neglected, so the potential energy \( V \) is exactly zero.

**Requirement for Static Equilibrium**

\[
Q_\theta - \frac{dV}{d\theta} = -D \cdot P \cos \theta - \sum_{k=1}^{NM} F_{Mk} \frac{dL_{Mk}}{d\theta} = 0
\]  \( (3.46) \)

**Requirement for Stability of Static Equilibrium**

\[
K_{\theta\theta} = \frac{d^2V}{d\theta^2} - \frac{dQ_\theta}{d\theta} = -D \cdot P \sin \theta + \sum_{k=1}^{NM} \left\{ K_{Mk} \left( \frac{dL_{Mk}}{d\theta} \right)^2 + F_{Mk} \frac{d^2L_{Mk}}{d\theta^2} \right\} \]  \( (3.47) \)
The equations can be simplified by noting that the derivatives of the muscle lengths are related to the muscle moment arms $R_{Mk}$, where there is a sign difference for the flexor and extensor muscles.

**Flexor Muscles:**

$$\frac{dL_{Mk}}{d\theta} = -R_{Mk} \quad (3.48)$$

**Extensor Muscles:**

$$\frac{dL_{Mk}}{d\theta} = +R_{Mk} \quad (3.49)$$

With these definitions, the equations for equilibrium and stability of equilibrium can be simplified. In order to make the equations less complicated, all subscripts "Mk" which refer to muscle k will be eliminated and simply implied in all appropriate terms.

**Requirement for Static Equilibrium**

$$\sum_{k=1}^{\text{NFLEX}} FR - \sum_{k=1}^{\text{NEXT}} FR = DP \cos\theta \quad (3.50)$$

**Requirement for Stability of Static Equilibrium**

$$\sum_{k=1}^{\text{NFLEX}} \left\{-F \frac{dR}{d\theta} + KR^2\right\} + \sum_{k=1}^{\text{NEXT}} \left\{F \frac{dR}{d\theta} + KR^2\right\} \geq DP \sin\theta \quad (3.51)$$

In these equations, NFLEX and NEXT refer to the number of flexor and extensor muscles, respectively.

**3.8.1 Theoretical Predictions of Critical Load**

The purpose of this example is to try to show how co-contraction may be required to maintain a non-negative joint stiffness. In order to test this concept, a simple experiment is designed to examine the maximum load $P$ which can be supported as a function of the elbow angle $\theta$. In order to solve for the maximum vertical load $P$, which will also be called the critical load, the values of several parameters must be estimated.
The value of the constant D is assumed known, as are the values of R and $\frac{dR}{d\theta}$ for each muscle as a function of $\theta$. The model used here includes six muscles, three extensors and three flexors, as listed below.

**Extensor muscles:**
- long head of triceps
- medial head of triceps
- lateral head of triceps

**Flexor muscles:**
- biceps
- brachialis
- brachio-radialis

Thus the only remaining unknowns are the vertical load, P, and the individual muscle forces, F, and stiffnesses, K. Obviously this six-muscle model is redundant and thus does not yield a unique solution without some further assumptions. Rather than using a complex scheme for distributing the muscle forces via some type of optimization, an assumption will be made here that at each elbow angle, exactly one muscle has a non-zero force. Note that, depending on the model of the muscle stiffness used, this does not imply that only one muscle can contribute to satisfying the stability constraint.

In solving the equations over a range of flexion angles, three different conditions may be necessary to satisfy equilibrium: (1) a net flexion moment is required ($\theta < 90$ degrees), thus one flexor muscle will have a non-zero load; (2) no moment is required ($\theta = 90$ degrees), thus zero force on all muscles; and (3) a net extension moment is required ($\theta > 90$ degrees), thus one extensor muscle will have a non-zero load. The equations which are used to solve for the non-zero muscle force and the critical load are obtained by setting the appropriate muscle forces to zero. In solving for the critical load, the system is assumed to be on the border between stability and instability, and thus the stability inequality constraint.
is replaced by an equality constraint. The resulting equations for all three cases are shown below.

**case (1):** \( \theta < 90 \text{ degrees, flexion moment required} \)

\[
-F \left( R \tan \theta + \frac{dR}{d\theta} \right)_{\text{FLEXOR}} + \sum_{k=1}^{NM} K R^2 = 0 \quad (3.52)
\]

\[
P_{\text{crit}} = \frac{(F R)_{\text{FLEXOR}}}{D \cos \theta} \quad (3.53)
\]

**case (2):** \( \theta = 90 \text{ degrees, no moment required} \)

\[
P_{\text{crit}} = \frac{\sum_{k=1}^{NM} K R^2}{D} \quad (3.54)
\]

**case (3):** \( \theta > 90 \text{ degrees, extension moment required} \)

\[
F \left( R \tan \theta + \frac{dR}{d\theta} \right)_{\text{EXT}} + \sum_{k=1}^{NM} K R^2 = 0 \quad (3.55)
\]

\[
P_{\text{crit}} = \frac{-(F R)_{\text{EXT}}}{D \cos \theta} \quad (3.56)
\]

These equations appear rather easy to solve. The only remaining decisions are to estimate \( R, \frac{dR}{d\theta} \), and \( D \), decide which of the flexor or extensor muscles is assumed to have a non-zero force, and to use the total incremental stiffness model for \( K \) for all of the muscles.
3.8.1.1 Selection of Muscle Moment Arms

The values of R as a function of θ for the extensor muscles are taken from Amis et al [2]. The values of R as a function of θ for the flexor muscles are averages of moment arm values reported by Amis et al [2] and Braune and Fischer [10]. Natural cubic splines [20] are used to approximate the dependence on θ in order to enable estimation of dR / dθ values. Plots of the R values used for all six muscles are shown in figures 3.5 and 3.6.

3.8.2 Model Estimates of Critical Load

In order to estimate the critical load using the model, it will be useful to first write the expression for the total incremental stiffness in a more convenient form.

\[
K_0 = \frac{F_{\text{max}}}{L_0}
\]  

(3.57)

\[
K = K_0 \left( C_0 + C_1 f + C_2 f^2 \right)
\]  

(3.58)

Figure 3.5: Assumed Elbow Flexor Moment Arms
This equation for $K$ can be substituted into the previous equations to solve for the critical load for each of the three cases.

**case (1):** ($\theta < 90$ degrees, flexion moment required)

\[
F^2 \left( \frac{C_1 R^2}{F_{\text{max}} L_0} \right)_{\text{FLEX}} + F \left( \frac{C_1 R^2}{L_0} - R \tan \theta \cdot \frac{dR}{d\theta} \right)_{\text{FLEX}} + \sum_{k=1}^{NM} C_0 K_0 R^2 = 0 \quad (3.59)
\]

\[
P_{\text{crit}} = \frac{(FR)_{\text{FLEX}}}{D \cos \theta} \quad (3.60)
\]

**case (2):** ($\theta = 90$ degrees, no moment required)

\[
P_{\text{crit}} = \sum_{k=1}^{NM} C_0 K_0 R^2 \quad (3.61)
\]

**case (3):** ($\theta > 90$ degrees, extension moment required)
Values for $L_0$ for each of the six muscles are taken from Kapandji [32] and Amis et al [2].

D is obtained by measurement for the experimental subject (see next section).

The values for $F_{max}$ for the six muscles are estimated by the following procedure. First, an assumption is made that the maximum force of each muscle is proportional to the physiological cross-sectional area as reported by An et al [3]. Then the maximum flexion load which can be supported with the upper arm vertical and the elbow angle at 90 degrees is used to determine the scale factor by assuming that all three of the elbow flexor muscles are at their maximum forces. This scale factor is then used to estimate the maximum force not only for the flexor muscles but also for the extensor muscles.

With all of the parameters ($F_{max}$, $L_0$, $R$ and $\frac{dR}{d\theta}$) defined as described in this and the preceding section, the critical load is then solved for over a range of flexion angles from 0 to 130 degrees. In order to solve the equations, it is necessary to select one flexor and one extensor muscle which will be assumed to support the entire required moment. The results of the model predictions of critical load are similar for each of the three flexor muscles assumed active except in the region near 90 degrees. The results shown in figure 3.7 for the brachialis and long head of triceps as the only muscles with possible non-zero forces. Note that a discontinuity in slope occurs at 90 degrees where the required net moment changes from the flexion to the extension direction. For elbow angles less than 90 degrees the slope is affected by stiffness properties of the flexor muscles while for larger elbow angles the slope depends on properties of the extensors muscles.
3.8.3 Experimental Measurement of Critical Load

A simple experiment was performed to measure the critical load as a function of elbow angle in order to compare with the model predictions. With the subject seated, the upper arm was held approximately in a horizontal plane with the elbow resting on a cushioned support (figure 3.4). An orthoplast cast was made of the forearm and wrist such that a dumbbell could be supported without requiring any gripping force. A vertical wall was placed next to one side of the dumbbell to provide stability for out-of-plane motions at large vertical loads. EMG electrodes were placed over the biceps and triceps muscles.

Vertical loads between 2 and 37 lb were applied in 2.5 lb increments. Prior observations have shown that coactivation is present at the extremes of elbow angles even without loads. Therefore, for each load, the elbow was fully extended (significant extensor coactivation was always observed at full extension) and then slowly flexed until the level of EMG activity for the extensor muscles was no longer above a small threshold value, at which point the elbow angle was measured using a protractor. The same procedure was used with the arm flexed to about 135 degrees and then extended until EMG activity of

![Graph of Elbow Angle vs. Critical Load](image-url)

**Figure 3.7: Theoretical Prediction of Critical Load**
the flexor muscles was no longer present. The results of this experiment are as shown in figure 3.8 (superimposed on the model predictions).

3.8.4 Discussion of Elbow Model Results

Despite the many estimated parameters and assumptions used in the analysis, the results of the experiment agree quite favorably with the model predictions. Although some of this agreement may be fortuitous, the qualitative prediction of the model with a maximum critical load occurring at an elbow angle near 90 degrees is well-supported by the results of the simple experiment. There appears to be a 10 to 15 degree shift between the model predictions and the measurements. This shift may be due to errors in the muscle stiffness model or in estimates of the muscle moment arms.

Although the quantitative agreement between the predicted and measured critical values is not perfect, the qualitative agreement is good. Both short-range stiffness and the bilinear approximation to muscle stiffness are proportional to muscle force, and thus both models predict zero stiffness for zero muscle force. Thus at 90 degrees, where no net moment is required, these models predict that no load can be supported without co-contraction (see equation (3.54)). The critical load estimated using these muscle stiffness models would

![Figure 3.8: Experimental Measurement of Critical Load](image-url)
produce "U-shaped" curves for critical load versus elbow angle, with a minimum at 90 degrees where the experimentally measured critical values are at a maximum.

A major problem with using the slope of the force-length relationship is that it is very sensitive to selection of muscle characteristic length, L_0. For this type of model, a shorter value of L_0 allows passive stiffness to provide additional force and stability contributions, but it also predicts that increased co-contraction destabilizes the joint (since the slope of the active component of muscle force is negative for lengths greater than L_0). If the muscle length is assumed to always be less than L_0, then difficulty arises in generating enough force and stability to satisfy stable equilibrium in the range between 40 and 80 degrees.

Certainly more experiments could be performed to verify the results and to provide better estimates for some of the parameters, but the general agreement of the elbow model and the initial experiment is quite good. Some of the possible sources of error for the experiment include the following. The estimate of when the co-contraction was no longer present was certainly affected by using only relatively crude EMG detection to identify significant muscle activity, rather than collecting the EMG data and processing it. The shoulder was not held in a perfectly stationary position and thus may have moved for some of the heavier vertical loads in order for the subject to gain a slightly greater mechanical advantage, and thus there may be some errors in the elbow angles. The measurement of the elbow angles was also crude, using a simple protractor. Angle errors could have been about 5 degrees or more. The maximum moment which could be generated by the flexor muscles was only estimated.

The purpose of this simple elbow experiment was to test the quantitative method for predicting minimal antagonist co-contraction forces during quasi-static activities using a stability of equilibrium analysis. The derived analysis together with the method of estimating total incremental muscle stiffnesses shows much promise for estimating when co-contraction forces are necessary. Further experiments could be designed to test the ability of the analysis to estimate not only when co-contraction is necessary but how large the required co-contraction forces are.
3.9 Requirements for Stability Analysis of the Human Knee

One of the goals of the elbow study was to develop a model for estimating how individual muscles contribute to the stiffness of the joint. The total joint stiffness describes how difficult it would be to displace the entire elbow-weight system. A stability of equilibrium constraint is equivalent to a constraint that the total joint stiffness must be positive. For a multiple degree of freedom joint such as the knee, this scalar stiffness value is replaced by a stiffness matrix which must be positive semi-definite for stability. Although a quasi-static stability analysis of the knee is much more complex than a similar analysis for the elbow, the analysis may provide some very interesting results.

In order to apply this type of stability analysis to the human knee joint, several criteria must be satisfied:

1. Data must be available suitable for reasonable modeling as a quasi-static loading situation.
2. The first two derivatives of muscle lengths must be estimated. This in turn will require a model for the continuous curvature of the surfaces for the tibia and femur.
3. The loading across the knee joint must be established.
4. The changes in the net force and moment vectors acting across the knee joint as a function of the generalized coordinates must be estimated.
Chapter 4

KINEMATIC AND DYNAMIC DATA ACQUISITION

4.1 TRACK Data Acquisition System

All of the kinematic and dynamic data in this thesis was obtained using the TRACK (Telemetered Real-time Acquisition and Computation of Kinematics) data acquisition system developed at MIT [6], [15]. More specific details of the current software and hardware of this system are described by Mansfield [39], but a brief description is provided here to cover the necessary background information for discussing the measurement of lower extremity kinematics in more detail and to explain some of the systematic errors in the kinematic measurements.

The kinematic measurement system consists of a pair of opto-electronic cameras which have lateral-photorecffect diodes located in their image planes. The cameras are sensitive to infrared light, and are synchronized to sequentially sample the short (50 microsecond) duration output of a set of infrared light-emitting diodes (LEDs). Each camera produces two measurements for each sample, x and y, corresponding to the two angles of the LED marker with respect to the axis of the camera in two perpendicular directions. The value of each of these two signals is approximately equal to an intensity-weighted sum of all of the infrared light incident on the detector in the corresponding direction. Because of the scheme of using intensity-weighted averaging to obtain an estimate of an active LED marker, it is possible for reflected light to cause shifts in the measured position. These reflection errors may be substantially larger than the resolution or accuracy of the cameras if care is not taken to minimize them as much as possible.

The x and y values measured by each camera are equivalent to an estimate of a unit vector fixed in each of the camera’s coordinate systems from the focal point to the LED marker. The information from both cameras can then be combined with estimates for their
positions and orientations relative to a defined global coordinate system (GCS) to estimate the GCS coordinates of the marker.

Two types of calibration are used to reduce the errors in this geometric reconstruction process. Each of the cameras is internally calibrated to estimate the combined optical and electronic nonlinearities as a function of measured position. A two-dimensional lookup table of error values is generated, and interpolation is used to estimate and subtract off the error terms from each measurement. The patterns of these error values for the two cameras are very stable, so the internal calibration procedure is only required to be performed once for each camera (although the cameras are recalibrated periodically to verify this).

A secondary external calibration procedure must be performed each time the locations of the cameras are changed. This procedure combines the known coordinates and camera measurements of a number of LED markers to estimate the position and orientation of each camera relative to a defined GCS. The procedure has been fully automated and only requires a few minutes to perform prior to kinematic data collection.

In addition to the kinematic measurement system, a piezoelectric force plate is used to measure the force and moment vector associated with the foot-floor interaction during gait. The information measured by the force plate includes the three components of the force vector, the moment component about a vertical axis, and a location of the center of pressure of the load. The center of pressure is defined as that location on the surface of the force plate about which the loading is equivalent to an applied force plus a moment vector with its only nonzero component in the vertical direction. In order to be able to combine force plate and kinematic measurements, the external calibration procedure described above has been designed to locate the position and orientation of the force platform in the same coordinate system as the two cameras.

4.1.1 Processing Methods for the Kinematic Data

In contrast to nearly all other kinematic data acquisition systems used for gait analysis the TRACK system software and hardware has been designed to measure kinematics using
a rigid body approach. Other systems estimate lower extremity kinematics by placing markers at assumed joint center locations and then effectively "connecting the dots" between the measured coordinates of the joint centers. This type of analysis is equivalent to a stick figure model of the lower extremity, although the link lengths in fact change with time. Other measurements are sometimes added to estimate axial rotations of the anatomical segments. This type of kinematic analysis will produce errors when applied to joints which do not have constant joint centers (e.g. the knee). Even for joints such as the hip, which are accurately modeled as having a constant joint center, this type of analysis will produce errors if the marker is not placed exactly over the joint center (which is anatomically impossible for 3D data). Positioning errors of markers relative to the actual location of joint centers can easily be in excess of a centimeter.

The analysis method employed by the TRACK system groups LED markers into arrays of known relative geometry and then treats each array as a rigid body. The measured GCS coordinates of the individual LED markers are combined with the known geometry information to provide the best least-squares estimate for the position and orientation of the array. Arrays are attached as rigidly as possible to the individual segments of the lower extremity (i.e. foot, shank, thigh and pelvis) and used to measure the complete six degree of freedom position and orientation of each segment with no assumptions about joint kinematics. This method, as well as the method of placing markers at assumed joint locations, produces estimates of lower extremity kinematics which are affected by the motion of the markers relative to the underlying bones. Chapter 7 will address this problem of soft tissue motion in more detail.

The output of the TRACK kinematic processing is an estimate of the position and orientation of each of the arrays of LED markers for each time frame. Smoothing routines, which use cubic or higher order splines, are used to reduce the noise in the position estimates and also to estimate the first two derivatives of the kinematic data with respect to time. Note that the output of the system is the kinematics of the array and not of the anatomical segment to which the array is attached. The kinematics of the segments can be estimated using either of two methods. First, if static data is collected with arrays mounted
on each of the lower extremity segments and the subject stands in a neutral position, then
the "known" geometry which is used in later analyses can be based on this static data. The
measured relative motion between joints will then reflect changes from the neutral position,
and flexion, abduction, and rotation angles can be easily calculated. An alternative
approach is to use the coordinates of additional bony landmarks to estimate the positions
and orientations of the underlying bones relative to the attached arrays. For the purposes of
this thesis, the former approach will be used to estimate the motion of the pelvis and the
foot, while the latter technique will be applied to estimate kinematics of the tibia and femur.

The position and orientation information produced by the TRACK software system is
an estimate of the six degrees of freedom of the array relative to the GCS and can be
represented using many different but equivalent forms (e.g. a quaternion, a translation
vector plus three Euler angles, etc.). In the remainder of this chapter, the estimated
kinematics of each array will be assumed to be in the form of a translation vector $\mathbf{x}^A$ plus a
rotation matrix $\mathbf{R}^A$ such that the coordinates $\mathbf{r}_P^A$ of a point $P$ fixed relative to the arrays
coordinate system are transformed into GCS coordinates of the same point $P$ using the
following equation.

\[ \mathbf{x}_P = \mathbf{R}_P^A \mathbf{r}_P^A + \mathbf{x}^A \]  

(4.1)

The estimates of $\mathbf{R}^A$ and $\mathbf{x}^A$ will be related to $\mathbf{R}^S$ and $\mathbf{x}^S$, the kinematics of the underlying
anatomical segment, later in this chapter.

4.1.2 Characteristics of the Kinematic Data

As previously mentioned the output of each of the cameras consists of two
measurements, $x$ and $y$, which are related to the angle between the ray from the focal point
of the camera to the LED marker and the optical axis of the camera. The half-angle of the
field of view is approximately fifteen degrees and the camera measurements are recorded
with 12-bit resolution (i.e. 1 part in 4096) so the resolution of each camera is
approximately 0.007 degrees. The position of a particular marker relative to the cameras (especially
the distance). However, all of the gait data for this thesis was collected at or near the force plate, and thus the resolution can be described in terms of distances with the requirement that this value is valid only for a marker located near the force plate. For gait analyses, the cameras are located about 3 meters from the force plate, so the angular resolution corresponds to a positional resolution of about 0.4 mm for each camera. For an LED marker fixed in space, the measured RMS level of noise in the positional data was approximately 0.5 mm. The noise from measuring the positions of separate LEDs on the same array are not highly correlated, so the RMS noise value for the position of a four-LED array is about 0.3 mm.

An estimate of the accuracy of the kinematic data can be inferred from the results of the external calibration procedure. Using the cameras in the Newman Laboratory for Biomechanics and Human Rehabilitation at MIT, the accuracy was found to be significantly worse than for the cameras used in the Biomotion Lab at Massachusetts General Hospital (MGH) even though all cameras are the same model by the same manufacturer. This is presumably because the MIT cameras were prototype models. For this reason, all data in this thesis was obtained using the MGH cameras. For these cameras, the positional accuracy over a field of view of 2 meters is approximately 2 mm, as long as the LED marker is not allowed to get near the edge of the viewing volume. This value represents an estimate of the error between any two points in the field of view. For small motions within only a fraction of the viewing volume, the positional errors would be expected to be significantly less than 2 mm, and would scale roughly as 0.1 percent of the distance between any two locations.

4.2 Coordinate Measurement Procedures

For the purposes of measuring the geometry and anatomical landmark data for this thesis, it was useful to develop a method for using the TRACK system to locate points in the GCS as accurately as possible. Therefore, a special device was constructed which includes two locations for pointing and an attached LED array (figure 4.1). If \( p \) represents the array coordinates of the pointer location, then the GCS coordinates of this point can be obtained using equation (4.1).
Thus far the arrays have been described as consisting of groups of individual LEDs, but actually each marker corresponds to a cluster of three LEDs (figure 4.1) which were powered simultaneously and measured by the cameras as a single data point. For each array the LEDs were arranged in a planar configuration to minimize reflection errors. The distance between the LEDs in each cluster was about 5 mm, so the measurement of each camera represented a "bright spot" on the detector corresponding to an infrared source about 5 mm across. Exactly which point was measured by the cameras relative to the array geometry could not be directly measured. Any errors in the actual point observed by the cameras would affect the accuracy of the estimate of the pointer location relative to the array coordinate system. For this reason it was decided to try to use some kinematic data to estimate the array coordinates of the pointer location.
Consider a set of kinematic data such that the array is moved around with the pointer location physically constrained to be fixed in the GCS (i.e., the pointer endpoint is located in a small depression in a flat surface). If N frames of data are collected for this motion, and the estimated location of the endpoint in the array coordinate system is \( \mathbf{r}_p^A \), then N values of \( x_{pk} \), the GCS coordinates of this point, can be obtained as follows.

\[
x_{pk} = R_k^A \mathbf{r}_p^A + x_k^A
\]

(4.2)

where \( R_k^A \) and \( x_k^A \) are the measured rotation matrix and position vector, respectively, for array A for each frame k. For a perfect estimate and perfect kinematic data, all of the \( x_{pk} \) values would be identical. Therefore, deviations in \( x_{pk} \) are related to the accuracy of the estimate of \( \mathbf{r}_p^A \). It is therefore useful to define two additional quantities. \( \overline{x}_p \) is the mean calculated position of the estimated pointer location, and \( \sigma^2 \) is an estimate of the variance in the \( x_{pk} \) values.

\[
\overline{x}_p = \frac{1}{N} \sum_{k=1}^{N} \left( R_k^A \mathbf{r}_p^A + x_k^A \right)
\]

(4.3)

\[
\sigma^2 = \frac{1}{N} \sum_{k=1}^{N} \left( x_{pk} - \overline{x}_p \right)^T \left( x_{pk} - \overline{x}_p \right)
\]

(4.4)

Since the goal of this analysis is to find the best estimate for \( \mathbf{r}_p^A \), then it makes sense to minimize \( \sigma^2 \), as defined in equation (4.4), with respect to the three components of \( \mathbf{r}_p^A \).

This procedure was implemented and applied to motions of the pointer array for both of its pointing locations. For each of the endpoint locations, root-mean square (RMS) values (i.e., square root of the minimum \( \sigma^2 \) value) were approximately 0.2 mm for any given data set, and differences in predicted values of \( \mathbf{r}_p^A \) were also about 0.2 mm. The best results were obtained by making sure to include large angular deviations about two perpendicular axes in the motion of the pointer array. If the motion includes large deviations only about a single axis, then the resulting variance of the estimated point is small, but the test to test
variations in $p^A$ grow considerably larger. The variations in the best estimate of $p^A$ from test to test can be accounted for by two factors: (1) neither the depression in the flat surface nor the pointer endpoint is a perfectly sharp point, and (2) the use of the cameras to measure the kinematics only allowed motions such that all LEDs could be seen by both cameras for all frames.

Once the pointer endpoint coordinates $p^A$ have been estimated, then the assumed geometry of the pointer array can be translated such that the pointer endpoint is coincident with the origin of the coordinate system of the array. With this new assumed geometry used, the measured position of the array as expressed in equation (4.1) will return the estimated pointer location as the translation vector $x^A$ of the array in the GCS.

4.3 Measuring Kinematics of Lower Extremity Segments

The kinematic information produced by the TRACK system and as expressed by equation (4.1) represents the position and orientation of each array in the GCS. In order to apply this system to the lower extremity, it is necessary to estimate the kinematics of the underlying anatomical segments rather than just the attached arrays. For example, calculations of flexion and rotation angles in appropriate anatomical coordinate systems would be helpful in interpreting any kinematic or dynamic results. This section will describe the procedures used to transform the array kinematic information into an anatomical coordinate system.

Before describing the methods for transforming the array kinematics to anatomically based kinematics, the specific mounting procedures for the arrays on the segments will be described. For the purposes of analyzing the kinematics and dynamics of the lower extremity, LED arrays were attached to the foot, shank, thigh and pelvis via polyethylene molds which were designed at the MGH Biomotion Lab to minimize soft tissue motion while allowing relatively unconstrained motion at the joints. The molds were formed through a multi-stage process which involves casting the limb of a subject, forming a plaster positive of the lower limb from the cast, and then heating the polyethylene while
using a vacuum to make it conform to the plaster positive. For the foot, shank, and pelvis these molds were then fit relatively tightly to the superficial bony surfaces and are held in place by tightly bound elastic bandages. When mounted in this manner, the molds fit snug but unconfining, and little motion appeared to occur between the attached LED arrays and the underlying skeletal structures.

Soft tissue motion is particularly a problem for measuring the motion of the thigh, where only the epicondyles of the femur and the greater trochanter are close to the surface. If an array-supporting mold is used which conforms tightly to the epicondyles and encloses the posterior part of the knee region, then knee motion is constrained. Therefore, the thigh molds were designed to apply some pressure to the epicondyles but the straps are applied several centimeters above the knee region of the back of the thigh. With the thigh mold attached as described, the allowable soft tissue motion of the array relative to the femur still appeared to be significant in axial translation, but this method functioned qualitatively better than other array mounting procedures tested. Motion of the thigh array relative to the femur also seemed to occur during extreme flexion angles or contraction of the thigh muscles. An approach will be described in Chapter 7 to estimate and remove these soft tissue motion errors by using information about the articular geometry of the knee joint.

4.3.1 Coordinate System Definitions

The global coordinate system (GCS) has been mentioned several times throughout this thesis as the coordinate system which is fixed in the laboratory frame. By convention, the TRACK software system defines the global x-axis as parallel to the optical bench which supports the cameras, the y-axis as parallel to the positive vertical direction, and the z-axis as pointing perpendicularly away from the optical bench which supports the cameras (figure 4.2). LED arrays are mounted on the lateral aspects of the lower extremity segments and subjects walk parallel to the line between the cameras. For measuring the motion of the right side, the subject walks in the negative global x-axis direction; measuring kinematics of the left side, the subject walks in the positive x-axis direction. In either case, the z-axis always points in the medial direction and the y-axis is vertical. In order to
simplify the following analysis as much as possible, all coordinate system definitions and joint angles will be defined for a right leg. With very little effort, the necessary sign changes could be made to apply the angle and coordinate system definitions for application to the left side.

For each lower extremity segment, there are three coordinate systems which are relevant to describing the motion of that segment: (1) the global coordinate system (GCS); (2) the coordinate system defined for the array; and (3) the anatomical coordinate system. The TRACK software is designed to measure the transformations between the first two coordinate systems mentioned via equation (4.1). But a useful kinematic analysis of the lower extremity requires estimates of the anatomical motion. Therefore, it is necessary to estimate the transformation between the anatomical and array coordinate systems, which can then be combined with the measured motion of the arrays to calculate the motion of the underlying anatomical structures (neglecting soft tissue motion).

Assume that for a particular anatomical segment, the following position vector and rotation matrix have been estimated for transforming vectors from the coordinate system of the anatomical segment to the coordinate system of the array.
\[ r_p^A = R_{S/A}^S r_p^S + x_{S/A} \]  \hspace{1cm} (4.5)

By convention, \( r_p^S \) refers to the coordinates of the point \( P \) relative to the coordinate system of the anatomical segment, while \( r_p^A \) specifies the coordinates relative to the coordinate system of the array. Combining this relationship with equation (4.1), the transformation from the coordinate frame of the anatomical segment to the GCS can be written as follows.

\[ x_p = R^S r_p^S + x^S \]  \hspace{1cm} (4.6)

\[ R^S = R^A R_{S/A} \]  \hspace{1cm} (4.7)

\[ x^S = R^A x_{S/A} + x^A \]  \hspace{1cm} (4.8)

For the purpose of maintaining consistency with the TRACK system convention, it makes sense to define anatomical segment coordinate systems which are very closely aligned with the GCS when the subject is standing on the force plate in a neutral position and facing in the typical walking direction (i.e. negative GCS x-axis for the right side data). One method for satisfying this requirement has already been mentioned: collect static data with the subject in the anatomically neutral position, and then use the measured coordinates of the individual LED markers as the "known" array geometry input to the TRACK routines for calculating array positions and orientations. This procedure would result in having the \( R^A \) matrix very close to the identity matrix for the static data, and hence all array axes would be approximately aligned with the GCS. For this special case one should assign the relative rotation matrix \( R_{S/A} \) to be the identity matrix.

If the "known" geometry values are not adjusted in the TRACK processing scheme, then an alternative method for aligning the anatomical and GCS axes during the static data is to force the rotation matrix of the anatomical segment \( R^S \) as defined in equation (4.7) to be the identity matrix. This condition could be satisfied with the following requirement.

\[ R_{S/A} = \left( R_{STATIC} \right)^T \]  \hspace{1cm} (4.9)
Using either one of these methods would allow estimates of the orientation of the anatomical segment for all frames. However, the translations of the segments would still not have any anatomically significant interpretation. If the pointer array is used with the subject in the anatomically neutral position to locate the GCS coordinates of some point \( x_p \) which would be a useful segment origin, then equation (4.8) could be applied to solve for the required value of \( x_{S/A} \), the location of the anatomical segment origin in the coordinate system of the attached array.

\[
x_{S/A} = \left( R_{\text{STATIC}}^A \right)^T (x_p - x_{\text{STATIC}}^A)
\]

(4.10)

Once \( R_{S/A} \) and \( x_{S/A} \) have been evaluated, then equations (4.7) and (4.8) can be combined with the measured values of \( R^A \) and \( x^A \) to calculate the positions and orientations of the underlying anatomical segment for all subsequent data sets which are collected without moving the array attachments.

In order to use equation (4.10) to solve for \( x_{S/A} \), the origin of the anatomical segment must be selected. But which points should be selected as the origins of the anatomical segments? For a kinematic analysis of the lower extremity, the relative motion of adjacent segments (i.e. the joint motion) is of primary interest. Therefore, the segment origins should be selected such that translations which are of interest can easily be obtained by examining the relative transformations between two segments which surround a joint. The hip joint is a ball-and-socket joint and thus the measured translations between the femur and the pelvis do not represent any useful anatomical information. For the purposes of this thesis translations at the ankle joint are not of interest. But the translations at the knee joint are not only relevant to this thesis but are an integral part of it. For this reason it makes sense to select segment coordinate system origins for the tibia and femur which make the calculated translations of the femur relative to the tibia assume easily interpretable meaning.

If either of the coordinate system origins of the tibia or femur is located far from the axis of rotation of the knee joint, then the coordinates of the origin of the femur coordinate system as observed in the tibia coordinate system will be highly correlated with changes in
joint flexion (figure 4.3). The origins of the tibia and femur should be selected as close to the knee joint axis as possible in order to decouple the measured relative translations of the knee from the rotations. Unfortunately, the axis of the knee joint is not fixed in the coordinate system of either the femur or tibia but rather changes as the knee is articulated. For this reason, the coordinate system origins for the tibia and femur were defined by the following criteria. During static data acquisition in the anatomically neutral position, the coordinate systems should be coincident, with the z coordinate corresponding to the midpoint of the femoral epicondyles. The x and y coordinates of the origins should lie on the average axis of rotation, which is defined using the differences in segment positions and orientations from a set of static data with the knee fully extended and a similar set of static data with the knee flexed 45 degrees. The reason for not using a flexion angle greater than 45 degrees is related to the apparent problem with soft tissue motion on the thigh for large knee flexion angles. With the origins of the coordinate systems of the tibia and femur defined as described, then a perfect hinge joint knee would produce zero measured translations. Using these placements allows interpretation of the translations of the femur relative to the tibia as estimates of how much the knee deviated from a perfect hinge joint during the observed motion.

4.3.2 Definitions of Anatomical Angles

Assume now that the above procedures have been implemented and that for a particular

Figure 4.3: Dependence of Knee Translations on Flexion

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point in time $x^{\text{FEM}}$ and $R^{\text{FEM}}$, the position vector and rotation matrix for the femur relative to the GCS, and $x^{\text{TIB}}$ and $R^{\text{TIB}}$, the position vector and rotation matrix for the tibia relative to the GCS, are available. It is then possible to calculate the relative transformation between the two segments as follows.

$$R^{\text{TIB}}_p = R^{\text{FEM/TIB}}_p R^{\text{FEM}}_p + x^{\text{FEM/TIB}}$$  \hspace{1cm} (4.11)

$$R^{\text{FEM/TIB}} = R^{\text{TIB}}_p R^{\text{FEM}}_p$$  \hspace{1cm} (4.12)

$$x^{\text{FEM/TIB}} = R^{\text{TIB}}_p (x^{\text{FEM}} - x^{\text{TIB}})$$  \hspace{1cm} (4.13)

The $x^{\text{FEM/TIB}}$ vector can be interpreted as translations of the femur on the tibia. But in order for the rotation matrix $R^{\text{FEM/TIB}}$ to be useful for interpreting the kinematics, it should be transformed into a set of three angles. Clinicians are most familiar with flexion, abduction, and rotation angles, and therefore some angles comparable to these would probably be the best choice.

When the subject is standing in the neutral position, knee abduction corresponds to a negative rotation of the femur relative to the tibia about the anatomical x-axis, knee external rotation corresponds to a positive rotation of the femur relative to the tibia about the anatomical y-axis, and knee flexion corresponds to a negative rotation of the femur relative to the tibia about the anatomical z-axis (figure 4.4). When the knee is in some other orientation, knee flexion is defined as a rotation about the medial-lateral axis of the femur, and external rotation is defined as a rotation about the long axis of the tibia. In order to satisfy both of these conditions, a set of three Euler angles is used which corresponds to first a rotation $\theta_{\text{FLX}}$ about the negative z-axis, then a rotation $\theta_{\text{ABD}}$ about the negative x-axis, and finally a rotation $\theta_{\text{EXT}}$ about the positive y-axis. The appropriate set of joint angles to describe the rotational transformation from the femur to the tibia coordinate systems, which has also been suggested by Grood and Suntay [22], is thus defined by the following equation.
Figure 4.4: Anatomical Angle Definitions

\[ \mathbf{R}_{\text{FEM/TIB}} = \begin{bmatrix} \cos \theta_{\text{EXT}} & 0 & -\sin \theta_{\text{EXT}} \\ 0 & 1 & 0 \\ \sin \theta_{\text{EXT}} & 0 & \cos \theta_{\text{EXT}} \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_{\text{ABD}} & -\sin \theta_{\text{ABD}} \\ 0 & \sin \theta_{\text{ABD}} & \cos \theta_{\text{ABD}} \end{bmatrix} \times \begin{bmatrix} \cos \theta_{\text{FLX}} & -\sin \theta_{\text{FLX}} & 0 \\ \sin \theta_{\text{FLX}} & \cos \theta_{\text{FLX}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

(4.14)

When the right hand side of this expression is evaluated, then the flexion, abduction, and external rotation angles can be expressed as functions of the rotation matrix of the femur relative to the tibia as calculated using the following formulae:

\[ \theta_{\text{FLX}} = \tan^{-1} \left( \frac{-\mathbf{R}_{\text{FEM/TIB}}_{21}}{\mathbf{R}_{\text{FEM/TIB}}_{22}} \right) \]

(4.15)

\[ \theta_{\text{ABD}} = \sin^{-1} \left( \frac{\mathbf{R}_{\text{FEM/TIB}}_{23}}{\mathbf{R}_{\text{FEM/TIB}}_{22}} \right) \]

(4.16)

\[ \theta_{\text{EXT}} = \tan^{-1} \left( \frac{\mathbf{R}_{\text{FEM/TIB}}_{13}}{\mathbf{R}_{\text{FEM/TIB}}_{13}} \right) \]

(4.17)
The analysis shown can be used to calculate the joint angles of the hip and ankle as well as the knee joint, by calculating the rotation matrix of the segment proximal to the joint relative to the more distal segment, but the sign of the calculated flexion angle must be changed. If the kinematic data corresponds to the left side, then the signs of the calculated flexion and external rotation angles would have to be reversed for all three joints.

The anatomical coordinate systems as defined above would be useful for comparing the kinematics and dynamics of two data sets without having to express the results in the GCS. If a subject walks twice in the same way but in a different direction relative to the GCS, any kinematic or dynamic results expressed in the anatomical coordinate systems would not be affected. Thus the calculated anatomical angles plus the anatomical coordinates of the net force and moment vector of one segment acting on an adjacent segment would remain invariant to the direction of walking.
Chapter 5

OVERVIEW OF THE KNEE MODEL

5.1 Introduction

Recall from Chapter 1 that the purpose of this thesis is to develop a mathematical model of the human knee joint which allows an examination of relationships between geometry, kinematics, and dynamics of the joint in vivo and to examine the sensitivity of the results to selected modeling and processing parameters. Now that the background material and associated equations have been presented, more details of the knee model can be addressed. The following four chapters deal with specific aspects of the knee model. The purpose of this chapter is to present an overview of the knee model, including an explanation of how the geometry, kinematics, dynamics, and muscle and joint forces are all related and combined to create a three-dimensional mathematical model of the human knee joint. Requirements for geometry, kinematics, and dynamics of the knee model are provided such that a stability of equilibrium analysis may be performed. The assumptions and approximations used in the development of the knee model are enumerated and explained, and the requirements for incorporating the knee model into a four-segment rigid body model of the lower extremity are described.

5.2 Functional Description of the Knee Model

What has thus far been referred to as the "knee model" is actually a set of methods and analyses for combining measured lower extremity kinematics and floor platform force and moment information to estimate the lower extremity kinetics and from these the muscle, ligament and joint contact forces of the knee joint in vivo. These methods do not therefore define a dynamic model (i.e. one which predicts the kinematics for different forces) according to the typical engineering definition but rather are only applied to estimate the kinematics and forces for specific sets of in vivo data. Nevertheless, the term knee model will be used in this thesis as a much more compact description than "knee kinematic
analysis and force estimation procedures."

To explain the overall concept of the knee model developed in this thesis, the analysis procedures will be described first in terms of specific input data needed to produced certain output data. The actual procedures performed to transform the data will be explained after the functional description is presented.

A set of kinematic and force plate data corresponding to the stance phase of gait for a healthy, normal adult subject is collected using the TRACK system. Now consider the knee model which requires the following input data for each point in time:

1. measured six degree of freedom kinematics of four arrays attached to the foot, shank, thigh and pelvis, in the form of array position vectors and rotation matrices
2. estimated articular surface geometry of the tibia and femur at the knee joint for the same subject on which the kinematics are measured
3. estimated transformations between each array coordinate system and the corresponding anatomical segment on which the array is mounted
4. estimated locations of the ankle relative to the foot and tibia
5. estimated locations of the hip relative to the pelvis and femur
6. estimated limits on soft tissue motion for each degree of freedom of the thigh and shank (i.e. how much the thigh and shank arrays can be expected to move relative to the femur and tibia, respectively)
7. measured force plate force and moment vectors
8. estimated inertial properties for each anatomical segment
9. estimated muscle origin and insertion locations relative to the coordinate system of the appropriate anatomical segments, plus maximum force and rest length for each muscle
10. estimated knee ligament origin and insertion locations relative to the coordinate systems of the femur and tibia, plus ligament force-length characteristics and rest lengths
Note that much of what is called "input data" actually represents separate analyses for estimating joint geometry, transformations between arrays and anatomical segments, etc.

The output from the knee model, corresponding to the described set of input data, includes the following:

1. kinematics for the femur and tibia consistent with the constraint that the estimated articulating surfaces are in contact in both the medial and lateral compartments
2. soft tissue motion deviations from the measured kinematic data of the thigh and shank
3. joint contact forces acting on each condyle of the femur
4. muscle forces for all of the lower extremity muscles
5. joint contact forces for the the hip joint and a joint contact reaction for the ankle (which includes a force vector and one moment component)
6. forces and strains acting on the major knee ligaments
7. an estimated joint stiffness matrix for the lower extremity

Now that the input and output data have been described, the principles of incorporated into the model can be discussed. Briefly, the knee model adjusts the measured kinematics of the tibia and femur subject to the constraint that both condyles of the femur are in contact with the tibia. Since there exists an infinite set of kinematics compatible with this requirement, specific criteria must be applied to select an "optimal" set of kinematics. The improved set of kinematics is selected based on minimizing a penalty function which is equal to weighted sums of squares of changes in the tibia and femur degrees of freedom (where the weights are based on expected kinematic errors) plus sums of squares of errors in GCS coordinates of the hip and ankle joint centers as predicted by the two segments adjoining each of the two joints. The improved kinematics are used to calculate the associated equations for dynamic equilibrium which are then input into a muscle force optimization algorithm to predict the individual muscle, ligament, and joint contact forces. Iteration may be used to alternately adjust the knee kinematics and the muscle forces such that dynamic equilibrium is satisfied.
The knee model has been described as producing a certain output for a set of specified inputs. The purpose of this thesis is not only to apply the knee model to specific sets of gait data but also to test its sensitivity to certain modeling and processing parameters. The knee model may produce different estimated results for the same input data as the parameters are changed. The modeling and processing parameters which will be varied in the knee model have been selected for either of two reasons: (1) previous lower extremity or knee models have shown the parameters to be significant in muscle and joint force analyses or (2) parameters new to this model which could significantly affect the results. Since both the locations of assumed insertions of the knee extensor muscles and material properties and rest length estimates of the ligaments have both shown possible significant effects on the results in previous knee models, sensitivity of the knee model predictions to these parameters will be included in the analysis. A major feature of this research is the introduction of stability constraints in the muscle force optimization analyses, including testing the applicability of the quasi-static assumption in such analyses; thus the knee model will be tested for sensitivity to the inclusion of this constraint.

5.3 Muscles and Ligaments in the Knee Model

Among the required input data described in section 5.2, items 9 and 10 involve the muscle and ligament origins, insertions, and material properties. The methods of modeling the muscles and ligaments for the purposes of estimating joint and muscle forces are described in this section. Both muscles and ligaments are assumed to act along straight lines between their origins and insertions, an assumption common to all of the knee and lower extremity models mentioned previously. For those muscles which clearly bend around a bony protuberance, the effective attachment locations for either the origin or insertion will be defined to be at a fixed location which best approximates the path of the muscle across the joint. The quadriceps muscles are assumed to insert into the tibia via the patellar ligament at an angle with is linearly related to the knee joint flexion angle, with the angular dependence estimated from Yamaguchi and Zajac [68] and van Eijden et al [63]. Patriarco [48] estimated attachment points for thirty-six lower extremity muscles. The approximate centroid of attachment for each muscle was found through observation and measurement on dissected cadavers and referenced to bony landmarks. The locations for
muscle origins and insertions required for this thesis were obtained by transforming the values reported by Patriarco into coordinates compatible with the TRACK definitions of axis directions (i.e. for a right leg, x-axis is posterior, y-axis is superior, and z-axis is medial) and scaling by characteristic segment lengths. Figure 5.1 shows the lines of action of the set of 36 muscles in the model. Muscle maximum forces were estimated using values of physiological cross sectional area (PCSA) reported by Brand et al [8] and assuming a maximum effective stress level of 30 N/cm² for each muscle. Muscle rest lengths were estimated by assuming each muscle to be at its rest length with the knee in full extension and the subject standing in an anatomically neutral posture.

Four major ligaments of the knee are included in the model: the anterior cruciate ligament, the posterior cruciate ligament, the medial collateral ligament, and the lateral

![Figure 5.1: Muscle Lines of Action](image)

front view    side view
collateral ligament. Force-length characteristics of these ligaments were taken from Wismans et al. [66] in vitro knee model, which assumed a quadratic relationship between force and length for ligament lengths longer than the rest lengths. The quadratic coefficients for each ligament were approximated from results of a study by Trent et al. [61], and estimates of ligaments strains for determining ligament rest lengths were based on a study by Brantigan and Voshell [9]. Since Trent et al. actually reported large variations in ligament force-length curves for corresponding ligaments of different knees, the estimated force-length relationships for the ligaments may not be very accurate. However, Wismans et al. used these same estimates in a three-dimensional model of the knee and thereby approximated in vitro knee behavior previously reported in the literature. Because of a lack of confidence in the ligament values, an alternative formulation of the knee model includes the ligaments as pseudo-muscles in the muscle force optimization analysis in the sense that their forces will be estimated using optimization rather than calculated using an assumed force-length relationship. In this case, the maximum force assigned to each ligament is estimated from ligament ultimate strengths reported by Trent et al. [61] and the cost function is a weighted sum of the ligament forces squared. Ligament origins and insertions were measured on a cadaver knee in conjunction with the measurements of tibia and femur articular geometry. The cadaver geometry measurements are described in more detail in Chapter 6.

5.4 Knee Model as Part of Lower Extremity Model

Since many of the muscles which cross the knee joint also cross either the hip or the ankle (9 out of 13 of the muscles used for this thesis), additional constraint equations must be evaluated for equilibrium (and possibly stability) for the degrees of freedom associated with relative motion of the two related joints. For this reason, the knee joint was incorporated as part of a four segment rigid body model of the lower extremity. This section describes the requirements for applying a kinematic and dynamic analysis to this lower extremity model. Specifically, details will be presented for using the TRACK system measurements of position vectors and rotation matrices of the foot, shank, thigh, and pelvis arrays to estimate the 15 coordinates associated with the lower extremity as described in section 2.2 (i.e. $q_{1/0}$, $q_{2/1}$, $q_{3/2}$, and $q_{4/3}$).
Recent models of the human ankle joint include two rotary degrees of freedom, at the talar and subtalar joints [30], [37]. The motions associated with these two degrees of freedom are approximately equivalent to flexion and abduction, respectively, as defined in section 4.3.2. Motions of the ankle joint in the abduction and adduction directions are by convention called eversion and inversion. For the purposes of describing ankle kinematics, very little motion occurs about the anatomical y-axis of the foot (i.e. the vertical direction in an anatomically neutral position), while significant rotations may occur about the other two anatomical axes. Exactly which combination of bony contact and soft tissue forces are responsible for limiting rotations about the y-axis axis of the foot is not clear, but many independent kinematic studies of the ankle have observed this phenomenon [30]. Therefore, for the purposes of kinematic and dynamic analysis applied to the lower extremity, an assumption will be made that the ankle is a two degree of freedom joint. The hip, on the other hand, is accurately modeled as a three degree of freedom ball-and-socket joint with its center of rotation located at the center of the femoral head. The knee joint will be modeled as a four degree-of-freedom joint, and will be described in much more detail in Chapter 7.

5.4.1 Transformation of TRACK Data to Fit the Kinematic Model

The TRACK system software measures the complete six degrees of freedom associated with the position vector and rotation matrix of each array. Once the 24 degrees of freedom have been measured by the TRACK system (for the four segments), these measurements must be used to estimate the 15 degrees of freedom for the complete lower extremity model. Assume for now that the transformations from array coordinates to anatomical coordinates have been estimated for all four lower extremity segments. Furthermore, assume that an estimate for the position of the center of the ankle joint is known both relative to the foot coordinate system (\( r_{\text{ANKLE}}^1 \)) and the tibia coordinate system (\( r_{\text{ANKLE}}^2 \)). Similarly, assume that an estimate for the position of the center of the hip joint is known both relative to the pelvis coordinate system (\( r_{\text{HIP}}^4 \)) and the femur coordinate system (\( r_{\text{HIP}}^3 \)). By a procedure which will be described in more detail in Chapter 7, this joint center information can be used with the measured TRACK kinematic data to obtain improved
estimates for the tibia and femur kinematics. Assuming these improvements have been
made in the kinematic data, then the data available for estimating the 15 coordinates for the
lower extremity can be summarized as follows:

\[
\begin{array}{lll}
\text{foot:} & R^1 & x^1 & r_{\text{ANKLE}}^1 \\
\text{tibia:} & R^2 & x^2 & r_{\text{ANKLE}}^2 \\
\text{femur:} & R^3 & x^3 & r_{\text{HIP}}^3 \\
\text{pelvis:} & R^4 & x^4 & r_{\text{HIP}}^4 \\
\end{array}
\]

where \( R^S \) and \( x^S \) are the rotation matrix and position vector of segment \( S \), respectively, and
the joint center location terms have already been defined.

The kinematic model for the lower extremity assumes that the only motions which
occur at the ankle and hip joints are rotations about the respective joint centers. But if the
estimated ankle joint centers, \( r_{\text{ANKLE}}^1 \) and \( r_{\text{ANKLE}}^2 \), relative to the foot and tibia coordinate
systems are both transformed into the GCS (global coordinate system) using the
appropriate segment rotation matrices and position vectors, they will in general specify two
different points. Differences in the locations of these two points can be attributed to errors
in the measured segmental kinematics plus errors in the joint center estimates. This
problem is resolved by selecting the tibia's estimate of the GCS coordinates of the ankle
joint center as "correct". Therefore, all errors will be assigned to the estimated GCS
coordinates of the ankle joint using the segment rotation matrix and position vector of the
foot, as shown in the following three equations.

\[
\begin{align*}
x_{\text{ANKLE}} &= R^2 r_{\text{ANKLE}}^2 + x^2 & (5.1) \\
x_{\text{ANKLE}} &= R^1 r_{\text{ANKLE}}^1 + x^1 & (5.2) \\
\Delta x_{\text{ANKLE}} &= R^1 r_{\text{ANKLE}}^1 + x^1 - x_{\text{ANKLE}} & (5.3)
\end{align*}
\]

An approximation is then made that all of the errors in the ankle joint are due to positional
errors of the foot segment (as opposed to rotational errors). The only justification for
making this approximation is that no obvious method exists for assigning angular errors. Using this approximation, the position vector of the foot coordinate system is adjusted to account for the assigned position errors.

\[ x_{\text{NEW}}^1 = x^1 - \Delta x_{\text{ANKLE}} \] (5.4)

A similar procedure is then applied to adjust the position vector of the pelvis to guarantee that both the pelvis and femur joint center estimates coincide in the GCS.

\[ x_{\text{HIP}} = R^3 r_{\text{HIP}}^3 + x^3 \] (5.5)
\[ x_{\text{HIP}} = R^4 r_{\text{HIP}}^4 + x^4 \] (5.6)
\[ \Delta x_{\text{HIP}} = R_4 r_{\text{HIP}}^4 + x^4 - x_{\text{HIP}} \] (5.7)
\[ x_{\text{NEW}}^4 = x^4 - \Delta x_{\text{HIP}} \] (5.8)

Note that these procedures not only produce data which is compatible with the kinematic model but also estimate errors in the ankle and hip locations. These procedures insure that the hip joint is always at a fixed location relative to the femur, which is consistent with the kinematic and dynamic models. If, for example, rather than selecting the femur's estimate of the GCS coordinates of the hip as correct the average of the two generated GCS values was used, the location of the hip joint center relative to the femur's coordinates would change with time. This situation would be inconsistent with the assumption of only three degrees of freedom of motion for the pelvis relative to the femur.

**5.4.2 Assignment of Coordinates for Lower Extremity Model**

Now that the positions and orientations of the four segments have been corrected and are compatible with the kinematic assumptions, coordinates can be assigned to the degrees of freedom of the lower extremity model. It is useful to first define a function \( \mathcal{R} \) which specifies a rotation matrix as a function of three angles, \( \theta_x, \theta_y, \) and \( \theta_z \). 

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Additionally, we can define three functions $f_{\theta_x}$, $f_{\theta_y}$, and $f_{\theta_z}$, which operate on a rotation matrix and return an angle.

$$f_{\theta_x}(R) = \sin^{-1}(R_{23})$$

(5.10)

$$f_{\theta_y}(R) = \tan^{-1}\left(\frac{R_{13}}{R_{33}}\right)$$

(5.11)

$$f_{\theta_z}(R) = \tan^{-1}\left(\frac{R_{21}}{R_{22}}\right)$$

(5.12)

Note that with these definitions, as long as $\theta_x$ and $\theta_y$ have magnitudes less than 90 degrees, the following relations are also true.

$$\theta_x = f_{\theta_x}(\mathcal{R}(\theta_x, \theta_y, \theta_z))$$

(5.13)

$$\theta_y = f_{\theta_y}(\mathcal{R}(\theta_x, \theta_y, \theta_z))$$

(5.14)

$$\theta_z = f_{\theta_z}(\mathcal{R}(\theta_x, \theta_y, \theta_z))$$

(5.15)

In all of these equations, $\theta_x$, $\theta_y$, and $\theta_z$ correspond to the abduction angle, external rotation angle, and flexion angle, respectively, as defined in Chapter 4 in equations (4.14) through (4.17).

Using equations (2.21) through (2.28) it is possible to calculate all of the relative position vectors ($x_{1/0}$, $x_{2/1}$, $x_{3/2}$, and $x_{4/3}$) and rotation matrices ($R_{1/0}$, $R_{2/1}$, $R_{3/2}$, and $R_{4/3}$).
from the segment position vectors $x^S$ and rotation matrices $R^S$.

\[
R_{1:0} = R^1 
\]

\[
R_{2:1} = R^1 R^2 
\]

\[
R_{3:2} = R^2 R^3 
\]

\[
R_{4:3} = R^3 R^4 
\]

Finally, then, the fifteen coordinates which describe the kinematic of the lower extremity can be assigned.

\[
q_{1/0}^T = [x_{1/0} \quad y_{1/0} \quad z_{1/0} \quad f_{\theta x}(R_{1/0}) \quad f_{\theta y}(R_{1/0}) \quad f_{\theta z}(R_{1/0})] 
\]

\[
q_{2/1}^T = [f_{\theta x}(R_{2/1}) \quad f_{\theta z}(R_{2/1})] 
\]

\[
q_{3/2}^T = [x_{3/2} \quad z_{3/2} \quad f_{\theta y}(R_{3/2}) \quad f_{\theta z}(R_{3/2})] 
\]

\[
q_{4/3}^T = [f_{\theta x}(R_{4/3}) \quad f_{\theta y}(R_{4/3}) \quad f_{\theta z}(R_{4/3})] 
\]

In these equations, $x$, $y$, and $z$ are the components of the relative position vector $x$. 

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Recalling from equation (2.1) the definition for $q$,

$$q^T = \begin{bmatrix} q_{1/0}^T & q_{2/1}^T & q_{3/2}^T & q_{4/3}^T \end{bmatrix}$$

(5.28)

then the dependence of the lower extremity kinematics on $q$ can be written explicitly. In the following equations, the origin of the coordinate system for the foot is located at the center of the ankle joint and the origin of the coordinate system for the pelvis is located at the center of the hip joint. Note that the dependence of $R_{2/1}$ and $x_{2/1}$ on $q$ (i.e. the kinematics of the knee) will not be presented until Chapter 7.

$$R_{1/0}(q) = R(q_4, q_5, q_6)$$

(5.29)

$$x_{1/0}^T(q) = \begin{bmatrix} q_1 & q_2 & q_3 & . \end{bmatrix}$$

(5.30)

$$R_{2/1}(q) = R(q_7, f_{by}(R_{2/1}), q_8)$$

(5.31)

$$x_{3/2}^T(q) = -R_{2/1}(q) r_{ANKLE}^2$$

(5.32)

$$R_{4/3}(q) = R(q_{13}, q_{14}, q_{15})$$

(5.33)

$$x_{4/3}(q) = r_{HIP}^3$$

(5.34)

5.4.3 Estimated Inertial Properties for Lower Extremity Model

Application of the dynamic equations derived in Chapter 2 to the lower extremity model requires estimates of the inertial properties for the foot, shank, and thigh. Contini [16] has published results for estimating segment masses, rotation inertias, and segment centers of masses as functions of body weight and segment lengths. Procedures for automating this estimation process for application to the lower extremity have been implemented by Carlson [11]. These procedures were slightly modified in order to estimate the required estimated inertial parameters for the lower extremity model in this thesis. The segment length for the foot was estimated by direct measurement, while segment lengths for the shank and thigh were estimated using TRACK measured distances between the ankle, knee, and hip joints. These segment lengths were then used to estimate $r_C^5$, the locations of
the segment centers of mass in the anatomical coordinate systems. The mass of the subject for which the kinematic data was collected was measured using the force plate, and then segment masses $m_s$ and inertia tensors $I_0$ were defined using modified version of the routines of Carlson.

5.5 Summarized Assumptions and Approximations in the Knee Model

It is useful to summarize the major assumptions and approximations which are made in applying the model to estimate muscle and joint forces during the stance phase of gait. Some approximations related to how the stability analysis is implemented cannot be discussed without the accompanying explanations in Chapter 8, but the remaining assumptions are listed below. The following assumptions and approximations are used in the analysis of this thesis.

1. During weight-bearing activities, and particularly during the stance phase of gait, both condyles of the femur are in contact with the articulating surface of the tibia.
2. For the purposes of a kinematic and dynamic analysis of the knee, the articular surfaces of the tibia and femur can be approximated as rigid bodies.
3. For the purposes of a dynamic analysis, the foot, shank, and thigh are approximated as rigid bodies.
4. The lower extremity can be modeled as a nine degree of freedom kinematic system, with two degrees of freedom at the ankle, four degrees of freedom at the knee, and three degrees of freedom at the hip.
5. Muscles and ligaments act along straight lines.
6. Maximum muscle forces are proportional to the physiological cross sectional areas, with the constant of proportionality as $30 \text{ N/cm}^2$.
7. Muscles, ligaments, and joint contact forces are the dominant contributors to satisfying dynamic equilibrium, and thus the contributions of other soft tissues such as the menisci of the knee and the joint capsule can be neglected.
(8) The TRACK arrays do not move very much relative to the underlying motion of the corresponding bones.

(9) For the purposes of a quasi-static stability analysis, total incremental stiffness is independent of muscle length, and depends only on muscle force, maximum muscle force, and muscle rest length.

(10) Human joints are frictionless.

(11) The force corresponding to an integrated pressure distribution in the medial or lateral compartment of the knee joint is directed normal to the surface at the point of contact.
Chapter 6

KNEE MODEL GEOMETRY

6.1 Introduction

One of the major goals of this thesis was to investigate how knee geometry affects the kinematics and dynamics of the joint. This investigation requires the development of a method to estimate the in vivo articular surface geometry of the femur and tibia for an experimental subject whose kinematics will be measured. This chapter presents an explanation of the methods developed in this thesis for measuring cadaver knee geometry and how this information is transformed to estimate the articular geometry of a subject in a gait analysis experiment.

Huiskes et al [28] have reported a method of using a pair of cameras and a grid projected onto the articular surfaces of a cadaver femur and tibia to estimate the three-dimensional coordinates of approximately 400 points on each surface with an accuracy of 0.2 mm. The coordinates are estimated by manual digitizing the recorded images from the cameras and using stereophotogrammetric reconstruction methods. For the femur, three separate orientations were required along with the measurement of additional marker locations to map the geometry of the entire curved surfaces of the condyles. The measured points were then incorporated into a geometric model using bicubic spline patches as described by Coons [20].

Rushfeldt [54] developed an ultrasonic method for measuring the geometry of the femoral head and the corresponding acetabulum of the human hip joint in vitro which not only determined the geometry of the articular surfaces but also of the underlying cartilage-bone interface, and thus the cartilage layer thickness could be estimated. The accuracy of this method has been reported to be less than 2 μm [60]. The scanning method for measuring the geometry of a cadaver hip joint is relatively straightforward due to the almost perfect sphericity of the joint. Murphy [45] has recently completed the construction of a
device in the Newman Biomechanics Lab at MIT for measuring the geometry of a cadaver knee using ultrasonic methods similar to those employed by Rushfeldt. However, due to the complex geometry of the knee joint, the knee geometry measurement system has included a full six degrees of freedom for controlling the ultrasonic transducer position and orientation. When this device is fully functioning it will produce the most accurate measurement of knee geometry to date, but as of yet the scanning control algorithms have not been completed.

In light of the fact that the ultrasonic geometry measurement system for the knee developed in this laboratory would soon be completed, the possibility of developing a system similar to that of Huiskes et al for measuring knee geometry was eliminated. A decision was made to develop a system for measuring knee geometry which would be simple to implement, automatic, and able to produce results of high enough quality to be used in the knee model for the purposes of kinematic and dynamic analysis. Furthermore, the system would be developed such that the algorithms for calculating and storing the knee model geometry would be able to later incorporate the ultrasonic measurements of knee geometry when they become available.

The method used in this thesis for measuring the geometry of a cadaver knee is based on the coordinate measurement procedures as described in section 4.2. This method is used to measure not only the articular geometry of the tibia and femur but also the coordinates of a set of anatomical landmarks on each of the bones and the origins and insertions of the major knee ligaments. The procedures for acquiring the coordinate measurements, using them to model the geometry of the knee, and defining appropriate anatomical coordinate systems will be discussed in this chapter.

6.2 Method of Knee Geometry Measurement

The coordinate measurement procedures require the use of the TRACK kinematic measurement system and the pointer array as described in section 4.2. In order to obtain the best possible results for the coordinate measurements, the cameras are moved in from their standard positions used for gait analysis such that the center of the viewing volume is
about 2 meters from each camera. This change in camera geometry allows for better resolution and accuracy (both of which approximately scale with the distance from the cameras to the markers) while maintaining a large enough viewing volume to measure the geometry of an entire human femur.

A cadaver tibia or femur is then rigidly mounted approximately in the center of the viewing volume and the coordinate measuring array is moved around the articulating surface by hand, keeping the pointer endpoint in contact with the articular surface with minimal applied force (figure 6.1). Kinematic data is collected for 100 seconds at 20 Hz while the pointer is moved systematically around the surface several times. Thus 2000 approximately uniformly distributed points are generated for each articulating surface. The medial and lateral plateaus of the tibia, as well as the medial and lateral condyles of the femur, are considered as separate articulating surfaces for the purposes of measuring the coordinates. Prior to fixing the cadaver tibia or femur in place, a second-array is attached to the bone to allow later referencing of the geometric coordinates to the array coordinate system. Each cadaver bone is fixed with the anatomical z-axis approximately parallel to the z-axis of the GCS, and thus the the long axis of the bone (i.e. the y-axis) is located in the x-y plane of the GCS.

Without changing the position or orientation of the bone, a set of static data is collected

Figure 6.1: Setup for Geometry Measurement
for the secondary array. The coordinate measuring array is then used to collect several sets of static data with the pointer endpoint located on a set of anatomical landmarks and estimated ligament origins and insertions. The coordinates of the following anatomical landmarks are measured for the tibia (which is mounted with the fibula still attached): medial malleolus, lateral malleolus, tibial tuberosity, and head of the fibula. For the femur, the coordinates of the medial and lateral epicondyles and the greater trochanter are measured. In order to be consistent with the ligament model of Wismans et al [66], a single point is selected for the origins and insertions of the anterior and posterior cruciate and lateral collateral ligaments, while both anterior and posterior attachments are estimated for the medial collateral ligament.

6.2.1 Measurement of Average Axis of Rotation

After the articular surface geometry and anatomical landmark measurements are made for the femur and tibia individually, the bones are removed from their rigid fixtures and two additional data sets are collected using the femur and tibia secondary arrays. One set of data is collected with the condyles of the femur resting on the tibial articulating surfaces and the knee in an approximately anatomically neutral position (i.e. flexion angle approximately zero). The second data set corresponds to approximately 45 degrees of knee flexion and the condyles of the femur again in contact with the tibia in both the medial and lateral compartments. Since the knee (and particularly a cadaver knee without any ligaments) is not a single degree of freedom joint, the joint translations and rotations are not functions of the flexion angle alone. The two sets of static data are therefore collected with the independent joint translations and rotations selected such that the apparent axial rotations are minimal and the femoral condyles are approximately centered on the corresponding tibia articular surfaces. The purpose of collecting these two additional data sets is to approximate the position and orientation of an average axis of rotation of the knee in the same coordinate system as the articular surface geometry and anatomical landmarks.

The procedure for calculating the average axis of rotation from the TRACK measurements of these two data sets will now be described. Define $x_{\text{EXT}}^2$ and $x_{\text{EXT}}^3$ as the
position vectors and $R_{\text{EXT}}^2$ and $R_{\text{EXT}}^3$ as the rotation matrices for the tibia and femur arrays relative to the GCS for the fully extended data, and define $x_{\text{FLX}}^2$, $x_{\text{FLX}}^3$, $R_{\text{FLX}}^2$, and $R_{\text{FLX}}^3$ similarly for the flexed data. Next use equations (5.18) and (5.22) to calculate $x_{\text{EXT}}^3$, $R_{\text{EXT}}^3$, $x_{\text{FLX}}^3$, and $R_{\text{FLX}}^3$, the position vectors and rotations matrices of the femur array relative to the tibia array, for the two data sets. For a point $r^3_p$ which is fixed relative to the femur, it will be measured to have two different positions $r^2_{\text{EXT}}$ and $r^2_{\text{FLX}}$ for the extension and flexion data relative to the tibia coordinate system as shown in the following two equations.

$$r^2_{\text{EXT}} = R_{\text{EXT}}^3 r^3_p + x_{\text{EXT}}^3$$  \hspace{1cm} (6.1)

$$r^2_{\text{FLX}} = R_{\text{FLX}}^3 r^3_p + x_{\text{FLX}}^3$$  \hspace{1cm} (6.2)

The transformation of the tibial coordinates of point P from the extended position to the flexed position of the knee can then be written as follows.

$$r^2_{\text{FLX}} = R^2_{\text{AVG}} r^2_{\text{EXT}} + x^2_{\text{AVG}}$$  \hspace{1cm} (6.3)

$$R^2_{\text{AVG}} = R_{\text{FLX}} R_{\text{EXT}}^T$$  \hspace{1cm} (6.4)

$$x^2_{\text{AVG}} = x_{\text{FLX}}^3 - R^2_{\text{AVG}} x_{\text{EXT}}^3$$  \hspace{1cm} (6.5)

In order to transform this position vector and rotation matrix into an axis of rotation, consider the rotation matrix $R$ which would result from the rotation by an angle $\theta$ about an axis defined by the unit vector $u$ with components $u_1$, $u_2$, and $u_3$ [20].

$$R = \begin{bmatrix}
    u_1^2 \cos(1-u_1^2) & u_1 u_2 (1-\cos\theta) - u_3 \sin\theta & u_3 u_1 (1-\cos\theta) + u_2 \sin\theta \\
    u_1 u_2 (1-\cos\theta) + u_3 \sin\theta & u_2^2 + \cos(1-u_2^2) & u_2 u_3 (1-\cos\theta) - u_1 \sin\theta \\
    u_3 u_1 (1-\cos\theta) - u_2 \sin\theta & u_2 u_3 (1-\cos\theta) + u_1 \sin\theta & u_3^2 + \cos(1-u_3^2)
\end{bmatrix}$$  \hspace{1cm} (6.6)
Given the rotation matrix $R$, the equivalent rotation angle $\theta$ and unit vector $u$ can be calculated using the following two equations.

$$\cos \theta = \frac{1}{2} \left( R_{11} + R_{22} + R_{33} - 1 \right) \quad (6.7)$$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix} \sin \theta \quad (6.8)$$

Equations (6.7) and (6.8) should therefore be applied to the rotation matrix $R_{2}^{AVG}$ to calculate the values for $\theta$ and $u$. Then for any point $P$ fixed in the femur and with coordinates $r_P^2$ defined with respect to the tibia coordinate system for the extension data, the tibia coordinates in the flexed position will be given by both sides of the following equation, where $r_{AVG}^2$ represents a point on the average axis of rotation with respect to the coordinate system of the tibia, and $d$ represents the translation along the axis.

$$R_2^{AVG} r_P^2 + x_{AVG}^2 = r_{AVG}^2 + (r_P^2 \cdot r_{AVG}^2) \cos \theta + u \times (r_P^2 \cdot r_{AVG}^2) \sin \theta + du \quad (6.9)$$

This equation can be rearranged into a simpler form by considering the point $P$ which is fixed in the femur coordinate system but is coincident with the tibia coordinate system origin during the extension static data.

$$r_{AVG}^2 (1 - \cos \theta) - (u \times r_{AVG}^2) \sin \theta = x_{AVG}^2 - du \quad (6.10)$$

Finally the coordinates of a point on the axis can be solved for using the component of the motion which is perpendicular to the direction of the unit vector $u$. 

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The point on the average axis of rotation $r^2_{AVG}$ and the unit vector $\mathbf{u}$ in the direction of the axis are both defined with respect to the coordinate system of the tibia array. The information can easily be transformed into the coordinate system of the femur using the position vector and rotation matrix of the tibia relative to the femur in either the flexion or extension static data.

6.3 Modeling the Articular Surfaces

How can the information from the measured GCS coordinates of the points on the articulating surfaces of the tibia and femur be used to generate mathematical models of these surfaces? The most common methods for modeling geometric surfaces all require the selection of two parametric coordinates, often called $u$ and $v$, which have monotonically increasing values in two directions such that they form a grid across the surface. The modeling of the surface then involves deciding which functions of $u$ and $v$ will be used to approximate the surface. The development of a geometric model of a surface thus requires two separate steps: (1) the selection of a set of parametric coordinates and (2) the selection of how the coordinates of the points on the surface are allowed to depend on these parametric coordinates. No unique solution exists to the selection of an appropriate set of parametric coordinates to describe a surface, but there must exist a one-to-one correspondence between points in parametric $(u,v)$ space and points on the surface. Using conventional notation, $\mathbf{r}$ represents the $(x,y,z)$ coordinates of a point on a surface and $(u,v)$ represents the corresponding parametric coordinates. Thus, a model of the surface would require the specification of the functional dependence of $\mathbf{r}$ on $u$ and $v$ such that:

$$
\begin{bmatrix}
1 - \cos\theta & u_3\sin\theta & -u_2\sin\theta \\
-u_3\sin\theta & 1 - \cos\theta & u_1\sin\theta \\
u_2\sin\theta & -u_1\sin\theta & 1 - \cos\theta
\end{bmatrix}
\begin{bmatrix}
r^2_{AVG} \\
x^2 \\
y^2 \\
z^2
\end{bmatrix} = \mathbf{x}^2_{AVG} = x^2_{AVG} - (x^2_{AVG}^T \mathbf{u}) \mathbf{u}
$$
The selection of a set of parametric coordinates appropriate for modeling the articulating surfaces of the tibia is straightforward. One reasonable choice is a set of coordinates which defines the plane perpendicular to the long axis of the tibia (e.g. anatomical x-axis and z-axis directions), because with this set of parametric coordinates the articular surface geometry is reasonable approximated as a function of these two coordinates. The parametric coordinates used to model the geometry of each of the articulating surfaces on the tibia can thus be represented as follows.

\[
r = r(u, v)
\]

(6.13)

\[
u = \frac{x - x_{\text{MIN}}}{x_{\text{MAX}} - x_{\text{MIN}}} \quad \text{and} \quad v = \frac{z - z_{\text{MIN}}}{z_{\text{MAX}} - z_{\text{MIN}}}
\]

(6.14)

In these equations, \(x_{\text{MAX}}\) and \(x_{\text{MIN}}\) represent the maximum and minimum values measured for points on the articulating surface. Using these definitions will then allow the specification of any point on one of the articulating surfaces of the tibia by selecting a pair of values for \(u\) and \(v\) between 0 and 1 and substituting them into the following three expressions.

\[
x = x_{\text{MIN}} + u(x_{\text{MAX}} - x_{\text{MIN}})
\]

(6.15)

\[
y = y(u, v)
\]

(6.16)

\[
z = z_{\text{MIN}} + v(z_{\text{MAX}} - z_{\text{MIN}})
\]

(6.17)

For the femur the anatomical x and z coordinate directions are inappropriate for a set of parametric coordinates since the points on the curved surface of the condyles cannot be assumed to be functions of any set of planar variables (Huiskes et al [28] required three planes of data to model the geometry of the condyles). A much better choice for a set of parametric coordinates to model the condyles of the femur is the distance \(z\) along the medial-lateral direction and a rotation angle \(\Theta\) about a medial-lateral axis which passes approximately through both epicondyles of the femur (figure 6.2). The parametric
Figure 6.2: Coordinates for Femur Geometry

coordinates used to model the geometry of the articulating surfaces on the condyles can thus be represented as follows.

\[
\begin{align*}
    u &= \frac{z - z_{\text{MIN}}}{z_{\text{MAX}} - z_{\text{MIN}}} \quad \text{and} \quad v = \frac{\theta - \theta_{\text{MIN}}}{\theta_{\text{MAX}} - \theta_{\text{MIN}}} \\
    x &= r(u, v) \cos(\theta_{\text{MIN}} + v(\theta_{\text{MAX}} - \theta_{\text{MIN}})) \\
    y &= r(u, v) \sin(\theta_{\text{MIN}} + v(\theta_{\text{MAX}} - \theta_{\text{MIN}})) \\
    z &= z_{\text{MIN}} + u(z_{\text{MAX}} - z_{\text{MIN}})
\end{align*}
\]  

(6.18)

Using these parametric coordinates, an appropriate geometric model is that the distance \( r \) from the axis through the condyles to the points on the surface is a function of \( u \) and \( v \). Hence, the following equations may be used for generating the coordinates of points on the surface of the condyles.

\[
\begin{align*}
    x &= r(u, v) \cos(\theta_{\text{MIN}} + v(\theta_{\text{MAX}} - \theta_{\text{MIN}})) \\
    y &= r(u, v) \sin(\theta_{\text{MIN}} + v(\theta_{\text{MAX}} - \theta_{\text{MIN}})) \\
    z &= z_{\text{MIN}} + u(z_{\text{MAX}} - z_{\text{MIN}})
\end{align*}
\]  

(6.19) (6.20) (6.21)

Now that the parametric coordinates have been selected for the surfaces of the tibia and femur, the next step in defining the geometry of the surfaces is generating sets of points which can be used to estimate the functional dependence \( y(u, v) \) and \( r(u, v) \) as previously defined. Therefore, the geometric data for the tibia and femur are processed using the following three steps:
1. The TRACK GCS coordinates of the anatomical landmarks are used to define an anatomical coordinate system (as will be described in section 6.4) and the coordinates of the sampled points of the articular surface are transformed into the defined anatomical coordinate system.

2. For each articulating surface of the tibia, equation (6.14) is used to calculate 2000 pairs of (u,v) values which correspond to the dependent anatomical y coordinates of the points on the surface. For each condyle of the femur, equation (6.18) is used to calculate 2000 pairs of (u,v) values and the dependent values of r for each point are calculated using

\[ r = \sqrt{(x - x_{AXIS})^2 + (y - y_{AXIS})^2} \]  

(6.22)

where \( x_{AXIS} \) and \( y_{AXIS} \) are the anatomical x and y coordinates of the medial-lateral axis (i.e. anatomical z-axis) which passes approximately through both epicondyles.

3. The 2000 sets of dependent variables (i.e. y for the tibia and r for the femur) and corresponding independent variables (u,v) are used to calculate the functional relationship between the dependent and independent variables; and thus to define the geometry of the articulating surfaces as expressed in equations (6.15) through (6.17) for the tibia and (6.19) through (6.21) for the femur.

The third step as described above requires an assumption of the form of the allowed functional relationship between the dependent and independent variables. To simplify the discussion, define \( f \) as the dependent variable (i.e. y for the tibia and r for the femur) which is to be estimated for all 2000 points on the surface. Then what methods will be used to define the relationship between the dependent variable \( f \) and the independent variables \( u \) and \( v \)? As previously mentioned, Huiskes et al [28] used bicubic spline patches to model the
geometry of the data. Unfortunately, in order to use bicubic spline patches, the coordinates of a surface must be available in a form which corresponds to (at least approximately) regularly sampled intervals in two parametric coordinate directions. The data generated using the TRACK pointer array corresponds instead to points randomly distributed on the surfaces.

In order to transform the randomly distributed points on each articulating surface into a geometric model, a best-fit bicubic approximation to the surface is estimated. The 2000 data points were used for estimating the surface, a best-fit bicubic functional relation between the dependent variable f and the independent variables u and v. The most general form of a bicubic relationship can be written by the following expression.

\[
f(u, v) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 1 \\ u \\ v \\ u^2 \end{bmatrix} \]

Note that the right side of this expression is linear in each of the coefficients \( a_{ij} \). Since the values of \( u, v, \) and \( f \) have been measured at each of the 2000 points on the surface, then it is very straightforward to use the definition of the surface in equation (6.23) to generate 2000 equations which can be used to solve for the optimal values of the sixteen constant coefficients \( a_{ij} \) which define the surface. The equations are then solved using linear least squares via singular value decomposition methods [52] to minimize the sum of \( \Delta f^2 \) for the 2000 points on the surface.

\[
\sum_{i=1}^{4} \sum_{j=1}^{4} a_{ij} u^{i-1} v^{j-1} = f
\]

\[
\begin{bmatrix} 1 & v & v^2 & v^3 & u & uv & \ldots & u^3 v^3 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ \ldots \\ a_{44} \end{bmatrix} = f
\]
The sixteen constant coefficients which result from the least squares solution of equation (6.25) for all of the points on the surface can then be used to not only define the surface but also to estimate the errors in the approximation (i.e. $\Delta f^2$).

In general no constraints are placed on the set of equations (6.25) when they are solved for the least-squares best fit solution. However, it is observed that the surface generated by the optimal coefficients does not appear to represent anything anatomical in the region for which data has not been collected. In other words, the bicubic surfaces should not be used for extrapolation purposes. This was initially noticed when displaying the resulting geometric models as wire frame surfaces. However, another observation was also discovered related to this phenomenon. The second derivatives of the surface (first derivatives are related to surface normals and second derivatives are related to curvature) may not be accurately estimated near the edges of the $(u,v)$ region over which data was collected. Specifically, some data points near the edge of a femoral condyle data were estimated to be slightly concave. Two additional analyses were implemented to resolve these problems.

Once the estimated coefficients for the geometry of the femoral condyles were available, they were used as the initial estimate for a routine which slightly modified the coefficients so that the Gaussian curvature [20] was of the same sign for all 2000 points on the surface. This was accomplished using a nonlinear gradient search algorithm (a published Fortran routine FMFP [35] which was translated into C) which used the same penalty function as the linear least squares analysis but which included a "convexity" constraint. A convex object will have Gaussian curvature with the same sign on the entire surface. As will be shown in section 6.5, this additional constraint increased the errors in the fit of the bicubic approximation to the data only very slightly.

In general, the inaccuracies of the data for extrapolation may seem to only be a problem for display purposes. However, the bottom line is that the "estimates" of the geometry in the extrapolation region should not be used for any purpose, particularly for correcting kinematics or affecting muscle or joint force estimates. Therefore, a method was
implemented to identify the valid region of \((u,v)\) values by finding the smallest convex boundary which contained all of the sampled \((u,v)\) points. With the boundary for valid data defined in this manner, deciding whether or not a \((u,v)\) point is within the valid data region consists only of evaluating a series of scalar products for all of the vertex edges which define the boundary.

Although the articulating surfaces were reasonably modeled using a single bicubic approximation, one might consider possible improvements in the surface models if a set of bicubic patches were used. A procedure was implemented to rearrange the sampled \((u,v)\) points into a set of triangular patches and then to linearly interpolate for regularly spaced intervals in the \(u\) and \(v\) directions. The data points were then used to define a Ferguson surface \([20]\) (i.e. one special case of bicubic spline patches which guarantees continuous second derivatives across the boundaries between patches). Note that by definition, the bicubic spline surface passes exactly through each of the data points in the grid. However, recalling from section 4.3 that the RMS noise of the pointer array is about 0.2 mm, the measured TRACK coordinates are clearly not perfect. 2D FFT routines \([52]\) were therefore implemented to spatially filter the \(x\), \(y\), and \(z\) coordinates of each surface. Then after all of the work described in this paragraph had been completed, the ultimate purpose for estimating the articular geometry was reviewed. The estimated articular geometry is intended to be used in conjunction with the kinematic and dynamic data for muscle force optimization including the possible addition of stability constraints. The major assumption related to the kinematic data is intended to be that the articular surfaces of the tibia and femur are in contact at exactly one point for each condyle of the femur. For this purpose, therefore, a "wavy" articular surface for either the tibia or femur is inappropriate, because it would almost assuredly guarantee that multiple point contact could would occur on a single condyle. Another possible problem could occur if the surface of the femur near the contact point were locally concave, because this could lead to an entire closed contour of contact. For these reasons, all of the methods described in this paragraph were abandoned. Hence, for the purposes of the knee model, the single bicubic approximation to the articular geometry works better than bicubic patches.
6.4 Definition of Anatomical Coordinate Systems

The knee geometry measurements as previously described include the identification of the GCS coordinates of several anatomical landmarks for the tibia and femur. The purpose of these landmarks is twofold: (1) to allow the definition of an anatomical coordinate system for transforming the measured points on the articular surfaces into a more convenient form and (2) to allow these same anatomical landmarks to be referenced with respect to the arrays in the in vivo kinematic experiments for scaling and transforming the cadaver knee geometry data. Each of the landmarks used in the cadaver geometry experiments has therefore been selected because its location could be accurately estimated (at least in some directions) on a living subject.

The four anatomical landmarks for the tibia include the medial and lateral malleoli, the tibial tuberosity, and the head of the fibula. The coordinate system associated with these four landmarks is defined as follows. The y-axis direction is defined from the midpoint of the two malleoli to the combined average of all the data points on the tibia articulating surfaces. The z-axis direction is defined by the assumption that the tibia has been mounted to have its medial axis aligned with the GCS z-axis during the TRACK measurement of the geometry data. The location for the anatomical z-coordinate to be equal to zero is defined by the average of the articular geometry points as previously described. The location for the anatomical x- and y-coordinates is defined using the average axis of rotation as defined in section 6.2.1. The definition of the coordinate system origin for the tibia as herein described actually corresponds to a point which is not in the tibia at all but is centered above the condyles at the average axis of rotation of the knee. The reasons for selecting the coordinate system origin of the tibia to be located at this point have to do with an attempt to decouple the translational motion from the rotational motion as described in more detail in section 4.3.1.

The anatomical landmarks for the femur are the medial and lateral epicondyles and the greater trochanter. The anatomical y-axis for the femur is defined to be parallel to the vector from the lateral femoral epicondyle to the greater trochanter. The anatomical z-axis
direction is defined by the assumption that the femur has been mounted to have its medial axis aligned with the GCS z-axis during the TRACK measurement of the geometry data. The location for the anatomical z-coordinate to be equal to zero is defined by the midpoint of the medial and lateral epicondyles. The location for the anatomical x- and y-coordinates is defined using the average axis of rotation as defined in section 6.2.1.

How are the coordinate system definitions used to calculate position vectors and rotation matrices which can then transform the tibia and femur data in the GCS to anatomical coordinates? Define $u_x$, $u_y$, and $u_z$ as the GCS coordinates of the unit vectors in the direction of the defined anatomical x, y, and z-axes, respectively. Furthermore, define $x_0$ as the GCS coordinates of the desired anatomical coordinate system origin. With these definitions, the position vector $x_{\text{GCS/ANAT}}$ and rotation matrix $R_{\text{GCS/ANAT}}$ for transforming the GCS coordinates into anatomical coordinates can be defined as follows.

$$x_{\text{ANAT}} = R_{\text{GCS/ANAT}} x_{\text{GCS}} + x_{\text{GCS/ANAT}}$$  \hspace{1cm} (6.26)

$$R_{\text{GCS/ANAT}} = \begin{bmatrix} u_x^T \\ u_y^T \\ u_z^T \end{bmatrix}$$  \hspace{1cm} (6.27)

$$x_{\text{GCS/ANAT}} = \cdot R_{\text{GCS/ANAT}} x_0$$  \hspace{1cm} (6.28)

Once the anatomical coordinate system has been defined, the measured coordinates of the GCS data is transformed into anatomical coordinates, including the articular surface geometry, the anatomical landmark positions, and the ligament origins and insertions. Finally, then, the data can be stored for use in the kinematic and dynamic analyses.

In order to store the geometry information for the tibia, the following are stored for each articulating surface: $x_{\text{MIN}}$, $x_{\text{MAX}}$, $z_{\text{MIN}}$, $z_{\text{MAX}}$, $(u,v)$ boundary vertices, and the 16 constants coefficients $[a_{ij}]$. Additionally, the transformed coordinates of the anatomical landmarks and axis of rotation information are also stored.
For the femur, the following are stored for each condyle: \( z_{\text{MIN}}, z_{\text{MAX}}, \theta_{\text{MIN}}, \theta_{\text{MAX}} \), \((u,v)\) boundary vertices, and the 16 constants coefficients \([a_{ij}]\). Additionally, the transformed coordinates of the anatomical landmarks and axis of rotation information are stored.

In order to allow the use of the cadaver geometry data for estimating the in vivo articular geometry of either a right or a left leg, the data is stored for both a left leg and a right leg using the data from a single cadaver knee and an assumption of symmetry. With the anatomical coordinate systems defined as in this section, the best way to create a leg for the opposite side is simply to reverse the signs of all of the \( x \) values. Instead of simply changing the signs of the \( x_{\text{MIN}} \) and \( x_{\text{MAX}} \) values, an additional transformation matrix is stored with the geometry data. The rotation matrix for the opposite side leg corresponds to a left-handed coordinate system in that the \( x \)-values are reversed. The reason for using the additional rotation matrix in place of directly changing the signs in the data is that unit normal directions are conserved (i.e. unit normals can always point out of the surface).

### 6.5 Results of Knee Geometry Measurements

Results will be presented for the application of the described methods to a single cadaver tibia and femur corresponding to a left leg. Using 2000 data points for each articulating surface, the linear least squares analysis to estimate the coefficients to define each surface requires only about 20 seconds on a SUN 3. However, the nonlinear least squares with the convexity constraints requires about 30 minutes to converge to the optimal solution. For the purposes of display or even kinematic and dynamic analysis the linear least squares solution may be sufficient because as will be shown in Chapter 7 the kinematic data does not generally correspond to contact near the edges of the valid region of data. Nevertheless, since the process only has to be performed once, enforcing the convexity of the femur is probably worth the 30 minute wait. Thus, the convexity constraint was included for the geometric model of the femur in this thesis.

Figure 6.3 shows a top view of the tibia model geometry, while figure 6.4 shows the data as viewed from the front. Only that portion of the bicubic surface lying inside of the
Figure 6.3: Tibia Geometry, Top View

Figure 6.4: Tibia Geometry, Front View

measured points is displayed. Figure 6.5 shows the tibia geometry including the defined anatomical landmarks.

Figure 6.6 shows the femur model geometry data. In this figure the data has been rotated about the medial-lateral axis (z-axis) of the femur to expose as much of the surface as possible. The anatomical landmarks and the coordinate system origin of the femur are displayed in figure 6.7.

For the results displayed here, the RMS errors (calculated using distances between the 2000 raw data points and the bicubic surface model) were approximately 0.65 mm for the
Figure 6.5: Tibia Anatomical Landmarks

Figure 6.6: Femur Geometry
medial articulating surface of the tibia and 0.71 mm for the lateral articulating surface. RMS errors for the femur were 0.81 mm for the medial condyle and 0.84 for the lateral condyle using the linear least squares estimate. Using the nonlinear least squares solution including the requirement of convexity for the condyles of the femur, the RMS errors increase slightly to 0.86 mm and 0.88 mm.

Overall, this system of geometry measurement works extremely well, especially since it was only designed to be a temporary replacement for a more accurate system. The system is fully automatic, generates articular geometry estimates which can be compactly stored, and produces data with errors not that much larger than the method employed by Huiskes et al [28]. Note that the error values reported here were actually the accumulation of measurement errors and modeling errors, whereas Huiskes et al only reported measurement errors. They did not really use a geometric model, but rather passed a bicubic surface exactly through their data points. For the purposes of later analyses to which the geometry estimates will be employed, a model of the surface which employs bicubic spline
patches which pass exactly through the points on the surface is inadequate. Small errors in the single bicubic approximation plus possible errors due to the pointer coming off of the surface slightly account for the RMS errors of 0.8 mm compared to the RMS noise in the endpoint position of 0.2 mm and the variation of the estimated endpoints coordinates of 0.2 mm. At the very least, the use of a method similar to this one would eliminate the need for the tedious manual digitization which the method of Huiskes et al requires.
Chapter 7

KNEE MODEL KINEMATICS

7.1 Introduction: The Problem of Soft Tissue Motion

As has been briefly mentioned in Chapter 4, all motion analysis systems which use externally mounted markers or goniometers to measure the kinematics of the lower extremity are sensitive to errors caused by relative motion between the external fixtures and the underlying bones. These errors are called soft tissue motion errors. Most researchers who use kinematic measurement systems for gait analysis are aware of the problem of soft tissue motion, and some have attempted to minimize the errors through physical mounting procedures, but few have attempted to assess the magnitude of these errors during in vivo gait experiments. van Weeren and Barneveld [64] have measured skin motion of markers relative to the underlying skeletal structure for a walking horse and have observed errors of about 4 cm. Murphy [45] has collected kinematic data from a single subject during gait using LED arrays mounted both on intracortical bone pins and on the surface of the segments. In no case has any procedure been reported in the literature for estimating and removing the kinematic errors caused by soft tissue motion. This chapter presents the method employed in this thesis for correcting kinematic measurements of the lower extremity by using estimates of articular geometry to assess and subtract off soft tissue motion errors. The result of application of these methods is a set of kinematics for the lower extremity which corresponds to a four degree of model of the knee.

7.2 Overview of Kinematic Correction Procedures

In order to be able to apply a dynamic analysis to the human knee joint for estimating muscle and joint contact forces which are consistent with the knee geometry, it is necessary to estimate where the articulating surfaces of the tibia and femur are located in space. However, the estimated positions and orientations of the joint geometry will only be useful for a dynamic analysis if the surfaces of the two bones are in contact. More specific
reasons for assuming point contact of each of the femoral condyles on the tibia will be discussed in section 7.4, but for now this will simply be stated as an assumption. This assumption, along with an estimate of the in vivo geometry for a subject, limits the motions which are allowed for the relative motion between the tibia and the femur. Specifically, as will be discussed later in this chapter, the requirement of joint contact at two locations reduces the kinematic model of the knee from six degrees of freedom to four degrees of freedom. Procedures will be discussed in this chapter for using the measured knee kinematics to select the best set of four degrees of freedom for the model.

Why should the kinematic model for the knee joint have to be assumed? Should it not be possible to simply measure the kinematics of the knee joint in vivo and thereby measure the number of degrees of freedom via some type of functional analysis? There are two difficulties with this approach. The first is that the soft tissue motion errors are systematic, and very repeatable. Whatever causes the array to move relative to the underlying bones, whether it is muscle contraction or simply the elasticity of the skin, does not change much from cycle to cycle (i.e. when the TRACK system is used to measure the relative motion of arrays strapped to the tibia and femur, the results are very repeatable). Hence an estimate of the degrees of freedom using this type of approach would correspond to motions of the bone-soft tissue complex, which will not accurately represent the motion of the skeletal structure. The second difficulty with a functional analysis approach is that this method can really only be used to solve for values of parameters of assumed functional relationships between the independent and dependent variables. There does not appear to be any straightforward method for estimating reasonable function forms a priori for knee kinematics.

Another possible approach for specifically examining knee kinematics in vivo might be to use the measured kinematics to generate a surface via the set of instantaneous axes of rotation for the joint. Two sets of surfaces could be generated in this manner: one each for the tibia and femur. Presumably then the resulting surfaces could in some manner be compared to estimates of the actual in vivo geometry to estimate soft tissue motion errors. Unfortunately, the axes of rotation would only be directly comparable to the surface geometry if no sliding occurred in the joint. Since significant amounts of sliding occur in
the knee joint, this approach cannot be used.

As approached in this thesis, the problem of soft tissue motion is reduced to using the measured lower extremity kinematic data along with estimated articular surface geometry and an assumption of contact on both condylar surfaces to produce a set of improved kinematic data plus estimates of the soft tissue motion errors. This method of kinematic data correction will be broken down into three procedures: (1) estimating the approximate locations of the tibia and femur from the instantaneous kinematic measurements and anatomical landmark coordinate information; (2) calculating the optimal average translation and rotation of the tibia and femur relative to their anatomical coordinate systems to minimize geometric incompatibilities over an entire set of data; and (3) calculating the optimal deviations from the average translations and rotations in order to guarantee geometric compatibility while minimizing changes in the kinematic data. Each of these procedures will be described in more detail throughout this chapter.

7.3 Estimating Tibia and Femur Kinematics Using Anatomical Landmarks

The procedure for using the anatomical landmark information to estimate the position and orientation of the tibia and the femur requires two steps: measuring the coordinates of the anatomical landmarks relative to the coordinate system of an array and using this information to scale and transform the articular geometry from the cadaver so that it approximates the articular surfaces of the subject of an in vivo gait analysis. Before explaining the specific procedures used to accomplish these steps, the required kinematic data for all of the analyses performed in this chapter will be described.

Five different LED arrays are used for the kinematic data acquisition process. For the purpose of consistency with preceding chapters, the arrays mounted to the foot, shank, thigh, and pelvis will be referred to as array numbers A1, A2, A3, and A4, respectively. Additionally the pointer array, as described in section 4.2, will be referred to as array number A5. Prior to collecting any kinematic data, the four arrays are attached to the lower extremity segments using mounting procedures discussed in section 4.3. A set of static
TRACK data is collected with the subject standing in an anatomically neutral position and another set of static data is collected with the knee flexed approximately 45 degrees. For the fully extended static data, the subject is oriented to make the anatomical coordinates align with the GCS coordinate system (i.e., the subject faces in either the positive or negative GCS x-axis direction, depending on whether the right or left side is instrumented). In separate static data sets the pointer array is used to sequentially locate all of the anatomical landmarks for the tibia and femur. No assumption is made that the subject is standing in the same posture for different landmark measurements, and furthermore no constraints are placed on the positions or orientations of the lower extremity segments during these static data sets (except of course that the posture must indeed be static). Subsequent sets of gait data or other kinematic data, including accompanying force plate measurements, are then collected without changing the mounts of the LED arrays to the anatomical segments.

Two additional data sets are collected with the position and orientation of the tibia and femur stabilized as much as possible and the array attached to the shank and thigh manually moved around to estimate maximum expected soft tissue motion in each direction. Finally a data set is collected in which the subject moves the ankle as much as possible about the two anatomical axes, and a similar data set is collected for motion at the hip. The kinematic data sets described in this paragraph will be used for the kinematic optimization procedures explained in section 7.4.4 and will have no effect whatsoever on the initial calculated transformations for the tibia and femur based on anatomical landmark positions.

The purpose of the procedures described in sections 7.3.1 through 7.3.3 is to provide estimates for the positions and orientations of the tibia and femur articular geometry during the sets of in vivo kinematic and dynamic data. These estimates will include an implicit assumption that the TRACK measurements of the array kinematics directly measure the underlying skeletal motion (i.e., no soft tissue motion occurs). The soft tissue motion errors will be estimated and removed in subsequent procedures when the geometric compatibility constraint is enforced.
7.3.1 Scaling the Geometric Data

In order to use the measured cadaver geometry data for a subject of a gait analysis experiment it must first be scaled appropriately using scaling factors determined by distances between the anatomical landmarks. Lew and Lewis [38] studied several different scaling methods for transforming the anatomical landmarks from one human cadaver knee to another and concluded that using different scaling factors in the three anatomical axis directions produced smaller errors in the transformed landmark coordinates than using a single scaling factor for all directions. Unfortunately no externally identifiable bony landmarks exist which can be used to accurately estimate a geometric scaling factor for the anatomical x-axis direction near the knee.

Two scaling factors were therefore used to scale the anatomical landmark data, one in the anatomical y-axis direction and another for the anatomical x- and z-axis directions. The scaling factor for the y-direction for the tibia anatomical landmark data was selected using the distance between the midpoint of the medial and lateral malleoli and the midpoint of the tibial tuberosity and head of the fibula. The scaling factor for the y-direction for the femur anatomical landmark data was selected using the distance between the midpoint of the lateral epicondyle and the greater trochanter. The scaling factor for the x- and z-directions for the anatomical landmarks for both the tibia and the femur was selected using the distance between the medial and lateral epicondyles of the femur.

Note that using different scaling factors in the two directions should not be applied to articular geometry data. If the differential scaling method described by Lewis and Lew (and originally used by Morrison [43]) were used to scale a complete human cadaver femur, including all articular geometry, then the femoral head would no longer be predicted to be spherical for any scaled femur. It is therefore most appropriate to use a single scaling factor for articular geometry data. For the subject of the experiments of this thesis, the two scaling factors were within 3 percent of each other and thus scaling with one or two factors would not change the geometry dramatically. Future studies could be conducted to examine optimal scaling methods for articular geometry. For the purposes of this thesis,
the distance between the femoral epicondyles is used to scale all three coordinates of the articulating surfaces of the tibia and femur.

The procedure for scaling the cadaver geometry is thus summarized as follows: (1) use the measured distances between the *in vivo* anatomical landmarks and the corresponding cadaver anatomical landmarks to calculate the scaling factors; (2) scale the anatomical landmarks of the tibia and femur cadaver data using two different scaling factors for each bone; and (3) scale the articular geometry data using a single scaling factor in all three directions. Note that the origins of the coordinate systems for the tibia and femur have been assigned to be located on the average axis of rotation of the knee. Since the articular geometry coordinates are uniformly scaled, using the same scaling factor for both the tibia and the femur, then the anatomical coordinate system origins of the scaled tibia and femur should still be expected to be located on the average axis of rotation. In addition to scaling the anatomical landmark and articular geometry data, muscle and ligament origins and insertions in the femur and tibia are scaled using similar procedures.

### 7.3.2 Initial Estimates of Transformations Using Anatomical Landmarks

The scaled articular geometry and anatomical landmark data measured on the cadaver is stored in an anatomical coordinate system. The purpose of the next step of the kinematic data processing is to translate and rotate the anatomical landmark data so that it is aligned as well as possible with the array coordinates of the anatomical landmarks measured *in vivo*. This requires using the sets of static anatomical landmark TRACK data to transform the GCS coordinates of the individual anatomical landmarks into the coordinate system of the corresponding array. For example, let \( x^A \) and \( R^A \) represent the position vector and rotation matrix of the shank array relative to the GCS and \( x^A \) represent the position vector of the pointer endpoint relative to the GCS. For this anatomical landmark, the coordinates \( r^A \) of the anatomical landmark (i.e. the pointer endpoint) relative to the coordinate system of the shank array would be given by the following equation.

\[
r^A = R^A^T (x^A - x^A^2)
\]  

(7.1)
Hence all of the measured anatomical landmarks are transformed into array coordinates of the thigh and shank using this procedure. Next, the position vector and rotation matrix which best transform the scaled tibia cadaver anatomical landmark data to be aligned with the shank array coordinates of the measured anatomical landmarks are calculated using a routine developed by Schut [56]. This routine solves for the position vector and rotation matrix by minimizing squares of angular errors between a set of measured and known coordinates of a set of points, and is the same algorithm used by the TRACK system for calculating array positions and orientations. If \( \mathbf{r}^{A2} \) represents the measured shank array coordinates of an anatomical landmark and \( \mathbf{r}^{2} \) represents the scaled anatomical coordinates of the corresponding point on the cadaver tibia, then this procedure estimates values for \( R_{2/A2} \) and \( \mathbf{x}_{2/A2} \) such the following equation is approximately satisfied for each of the anatomical coordinates of the tibia.

\[
r^{A2} = R_{2/A2} \mathbf{r}^{2} + \mathbf{x}_{2/A2} \tag{7.2}
\]

Since the scaled cadaver anatomical landmark data was initially stored in the anatomical coordinate system, then the resulting position vector \( \mathbf{x}_{2/A2} \) and rotation matrix \( R_{2/A2} \) define the transformation between the anatomical coordinate system of the scaled tibia and the coordinate system of the shank array. The position vector \( \mathbf{x}_{3/A3} \) and rotation matrix \( R_{3/A3} \) of the scaled femur relative to the coordinate system of the thigh array are calculated using similar procedures.

### 7.3.3 Improved Estimates Using Average Axis of Rotation

Approximate for now that no soft tissue motion occurs for the thigh and shank arrays relative to the femur and tibia, respectively. Then the transformations between anatomical coordinates and array coordinates as defined by \( \mathbf{x}_{2/A2} \), \( R_{2/A2} \), \( \mathbf{x}_{3/A3} \), and \( R_{3/A3} \) should be constant for all kinematic configurations of the lower extremity. Specifically, these transformations will be valid for the static data sets which correspond to the anatomically neutral position and 45 degrees of knee flexion. It may therefore be possible to improve the estimated transformations between the anatomical and array coordinate systems by
assuming that the \textit{in vivo} average axis of rotation is approximately in the same location relative to the articular geometry as the \textit{in vitro} average axis of rotation.

Using the average axis of rotation information to alter the estimated position vectors of the tibia relative to the shank array and the femur relative to the thigh array may not appear to be an "improvement" in the estimated transformations. Why would one expect the average axis of rotation, which was estimated for the cadaver knee without any muscles or ligaments, to be more accurate than the anatomical landmarks? By examining the set of anatomical landmarks, it is obvious that for each one the measured anatomical coordinates should be expected to be have substantially smaller errors in some directions than in others. For example, the anatomical z-coordinates of the medial and lateral epicondyles (which define the width of the femur at the knee) can be measured with very small errors, while the x- and y-coordinates could easily be expected to have errors of several millimeters, or perhaps even almost a centimeter. In general for any of the anatomical landmarks used, the coordinates in the direction of the normal to the surface will have smaller errors than the coordinates parallel to the surface. For this reason, using the anatomical coordinates to estimate the position of the tibia or femur in the anatomical x- or y-directions could result in relatively large errors. Fortunately this does not invalidate the entire procedure of using the anatomical landmarks for estimating segment positions and orientations. Since the sets of anatomical coordinates selected for the tibia and femur both contain points which have significantly different anatomical y-coordinates, then the estimated direction for the anatomical y-axis will be reasonably accurate. The rotation of the tibia or femur about the anatomical y-axis cannot be estimated accurately using anatomical landmarks.

It appears then that only two of the six degrees of freedom of the transformations between anatomical and array coordinates can be accurately estimated using the anatomical landmarks (i.e. the x-axis rotation or abduction angle and the z-axis rotation or flexion angle). Additional methods must therefore be used to produce accurate estimates for the other four directions. Recall from Chapter 6 that the origin of the femur coordinate system was selected to have an anatomical z-coordinate equal to the average z-value of the two epicondyles. This constraint can be enforced for the \textit{in vivo} data by changing the position vector $\mathbf{x}_{3/A3}$ such that the origin of the femur coordinate system relative to the thigh array is
located at the midpoint of the epicenters.

\[ \mathbf{x}_{3:A3} = \frac{1}{2} \mathbf{x}_{\text{MED EPI} \cdot A3} + \mathbf{x}_{\text{LAT EPI} \cdot A3} \]  

(7.3)

Also recall from Chapter 6 that the anatomical coordinate system of the femur was defined such that the z-axis direction corresponds to the expected GCS z-axis during an anatomically neutral stance. The anatomically neutral static data can be used to estimate the required rotation angle of the femur about its anatomical y-axis direction to align the anatomical z-axis direction such that it is as close as possible to the GCS z-axis. This can be performed by first calculating the femur coordinates of a unit vector in the GCS x-axis direction \( \mathbf{u}_x \), and then calculating the required rotation angle \( \Delta \theta_y \) about the anatomical y-axis to make the femur's anatomical z-axis perpendicular to this vector as follows.

\[ \mathbf{u}_x = R_{3/A3}^T R_{A3}^{3T} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]  

(7.4)

\[ \Delta \theta_y = \tan^{-1} \left( \frac{-x_{13}}{u_{x1}} \right) \]  

(7.5)

\[ R_{3/A3} = \begin{pmatrix} R_{3/A3}^{3} \\ 0 \end{pmatrix} \begin{bmatrix} \cos \Delta \theta_y & 0 & \sin \Delta \theta_y \\ 0 & 1 & 0 \\ -\sin \Delta \theta_y & 0 & \cos \theta_y \end{bmatrix} \]  

(7.6)

Finally, the flexed and extended sets of static data for the knee can be used to define an average axis of rotation for the knee in the coordinate system of the thigh array (as outlined in section 6.2.1). Let this axis information then be transformed into the anatomical coordinate system of the femur, and define \( x_{\text{AXIS}} \) and \( y_{\text{AXIS}} \) as the calculated x- and y-coordinates of the axis of rotation such that the anatomical z-coordinate is zero. The coordinates of \( x_{3/A3} \) can then be altered to force the coordinate system origin of the femur
anatomical coordinate system to lie along the average axis of rotation using the following equation.

\[
x_{3/A3} = \begin{bmatrix} x_{\text{axis}} \\ y_{\text{axis}} \\ 0 \end{bmatrix} - R_{3/A3} \begin{bmatrix} x_{3/A3} \\ y_{3/A3} \\ 0 \end{bmatrix}
\] (7.7)

The equations have therefore been completed for using the axis of rotation information to update the position vector \(x_{3/A3}\) and the rotation matrix \(R_{3/A3}\) of the femur relative to the thigh array. Anatomical landmarks for the tibia can be used to update the estimates for the position vector \(x_{2/A2}\) and the rotation matrix \(R_{2/A2}\) of the tibia relative to the shank array in a similar manner. However, a difference exists in that no tibial anatomical landmarks near the knee can be used to accurately estimate the location of the anatomical z-coordinate for the origin of the tibia. This coordinate is therefore estimated using the anatomically neutral static data. The anatomical z-coordinate of the femur relative to the tibia during the in vivo anatomically neutral data \((z_{\text{in vivo}}^{3/2})\) is assigned to be equal to the scaled anatomical z-coordinate of the femur relative to the tibia during the anatomically neutral data for the cadaver knee \((z_{\text{cadaver}}^{3/2})\). Therefore, equation (7.3) for the femur is replaced by the following equation for the tibia.

\[
x_{2/A2} = \begin{bmatrix} x_{2/A2} \\ y_{2/A2} \\ 0 \end{bmatrix} - R_{2/A2} \begin{bmatrix} 0 \\ 0 \\ z_{\text{in vivo}}^{3/2} - z_{\text{cadaver}}^{3/2} \end{bmatrix}
\] (7.8)

The equivalent equations to (7.4) through (7.7) are then applied to correct the other three degrees of freedom of the estimated transformation between the tibia and the shank array.

The analysis of this section includes the approximation that soft tissue motion does not occur between the thigh and shank arrays and the femur and tibia. Although soft tissue motion actually does occur, the analysis is useful for improving the estimated locations of the origins of the tibia and femur such that they are approximately coincident throughout
much of the range of motion of the knee. Errors in estimated coordinates of the anatomical landmarks of the tibia and femur in the axial direction could be as large as 1 cm. Using the axis of rotation information, estimated translations of the tibia and femur in the axial direction can be improved. In the next section, the geometry of the tibia and femur articulating surfaces are used to improve the estimated transformations, and thus the calculations using the knee axis are only used as initial estimates which are assumed to be better than using the anatomical landmark measurements alone.

7.4 Estimating Kinematics Using Geometric Compatibility

The position vectors and rotation matrices of the tibia and femur anatomical coordinate systems relative to the coordinate systems of the shank and thigh arrays have been estimated using the anatomical landmark and average axis of rotation information. Therefore, for any measurements of the kinematics of the shank and thigh arrays, the positions and orientations of the thigh and femur relative to the GCS can be estimated using the following equations.

\[ R_2^2 = R_{2/A2}^A \]
\[ x_2^2 = x_{2/A2}^A + R_{2/A2}^A x_{2/A2} \]
\[ R_3^3 = R_{3/A3}^A \]
\[ x_3^3 = x_{3/A3}^A + R_{3/A3}^A x_{3/A3} \]

Note that these transformations apply not only to the scaled anatomical landmarks of the cadaver data but also to the coordinates of the articulating surfaces. For any set of kinematic data, equations (7.9) through (7.12) can thus be used to estimate the position and orientation of the articulating surfaces of the tibia and femur relative to the GCS. One of the major kinematic assumptions in this thesis is that the condyles of the femur are in contact with the articulating surfaces of the tibia during weight-bearing activities. A set of knee kinematics which corresponds to contact of the medial and lateral articulating surfaces is compatible with the knee model geometry. Sections 7.4.1 through 7.4.5 will describe
methods for improving the estimated kinematics for the tibia and femur by enforcing a geometry compatibility requirement for the kinematic data.

Before explaining the details of the procedures to guarantee compatibility of the knee model kinematics and geometry, the assumption of contact between the condyles of the femur and the articulating surfaces of the tibia will be discussed. First of all note that this assumption is only intended to be applied during substantial load-bearing of the joint. Therefore it may not be appropriate to make this assumption during the swing phase of gait or during other activities in which the knee joint does not support large compressive loads. Even if the loads on both femoral condyles are assumed to be large, one may still worry about the requirement of the joint surfaces being in contact since much of the load on the articulating surfaces may be distributed through the menisci. However, of experiments reported in the literature conducted to study loading of the meniscus of the knee during load bearing activities, all of them have estimated a substantial proportion of the load to be supported by direct contact of the femur on the tibia. The meniscus may therefore serve to distribute a significant portion of the load, but at least part of it is due to direct contact between the opposing articulating surfaces of the femur and tibia. Hence, the assumption of the tibia and femur being in contact in both the medial and lateral compartments of the knee for load-bearing activities appears to be very reasonable.

7.4.1 Requirements for Geometric Compatibility

Exactly what are the requirements for a set of kinematics to be compatible with the knee model geometry? Consider a single frame of measured knee kinematic data represented by the values of $x^2$, $R^2$, $x^3$, and $R^3$, as defined in equations (7.9) through (7.12). For this frame, the position and orientation of the femur relative to the tibia can be expressed using the following equations,

$$r^2 = R_{3/2} r^3 + x_{3/2}$$

$$R_{3/2} = R^2 T R^3$$

(7.13)

(7.14)
\[ x_{3/2} = R_{3/2}^T x^3 - x^2 \]  

(7.15)

where \( x^2 \) and \( x^3 \) represent the tibia and femur coordinates, respectively, of the same point in space. By definition, the relative kinematics of the knee specified by \( x_{3/2} \) and \( R_{3/2} \) depend on the generalized coordinates \( q_{3/2} \) as shown below.

\[ x_{3/2} = x_{3/2} ( q_{3/2} ) \]  

(7.16)

\[ R_{3/2} = R_{3/2} ( q_{3/2} ) \]  

(7.17)

In general, up to six parameters are required to completely specify the position vector and rotation matrix of one rigid body relative to the coordinate system associated with another rigid body. The most convenient form these six parameters is a set of three translations and three rotation angles. By definition, the number of parameters required to specify the kinematic configuration between two segments determines the number of degrees of freedom for the relative motion. If kinematic constraints exist which limit the allowed relative motion, then the number of degrees of freedom will be less than 6. For the knee joint the two kinematic constraints for condylar contact on the tibia will reduce the number of degrees of freedom of the knee joint from six to four, and hence only four independent parameters may be selected to specify the joint kinematics. As previously mentioned in section 5.4.2, the four independent parameters, or generalized coordinates, used for specifying the relative motion of the knee joint in this thesis will be as follows:

\[ q_{3/2}^T = [ x_{3/2} \quad z_{3/2} \quad \theta_y_{3/2} \quad \theta_z_{3/2} ] \]  

(7.18)

where \( x_{3/2} \) and \( z_{3/2} \) are components of \( x_{3/2} \), and \( \theta_y_{3/2} \) and \( \theta_z_{3/2} \) are obtained from \( R_{3/2} \) using equations (5.14) and (5.15). This choice of independent coordinates requires the designation of \( y_{3/2} \) and \( \theta_x_{3/2} \) as dependent coordinates. In terms of the anatomical coordinates and angles defined in Chapter 4, the independent coordinates represent the medial-lateral and anterior-posterior translations of the femur relative to the tibia plus the
external rotation and flexion angles of the knee. The dependent coordinates correspond to the translation of the femur parallel to the long axis of the tibia plus the abduction angle of the knee.

In order to guarantee compatibility of the knee model kinematics and geometry, the dependent values of $y_{3/2}$ and $\theta_{x_{3/2}}$ must be selected to insure that the condyles are in contact with the tibia. However, selecting a point on the surface for either the femur or tibia articulating surfaces requires the specification of the two parametric coordinates. Altogether, eight parametric coordinates must be specified to choose points on the medial and lateral articulating surfaces of the tibia and femur. To examine geometric compatibility it is useful to define a vector $s$ which includes all eight of the required parametric coordinates plus the two dependent kinematic coordinates of the knee.

$$s^T = \begin{bmatrix} u_{t1} & v_{t1} & u_{t2} & v_{t2} & u_{f1} & v_{f1} & u_{f2} & v_{f2} & y_{3/2} & \theta_{x_{3/2}} \end{bmatrix} (7.19)$$

In equation (7.19) $u_{t1}$ represents the parametric $u$ coordinate for the femur for condyle 1 (i.e. the medial condyle), and the other seven parametric coordinates are similarly defined. Given this definition of $s$, the specification of values for both $q_{3/2}$ and $s$ will determine the positions of four points relative to the coordinate system of the tibia: one point on each of the articulating surfaces of the tibia and one point on each of the femoral condyles.

Geometric compatibility requires that the two points on the femoral condyles have the same coordinates as the corresponding two points on the articulating surfaces of the tibia, but this requirement alone is insufficient to guarantee geometric compatibility. A sufficient condition for geometric compatibility is that the two surfaces are tangent at the contact points. Define $r_1^t$ and $r_2^t$ as the tibia coordinates of the points on the articulating surfaces of the tibia which are specified by parametric coordinates $(u_{t1}, v_{t1})$ and $(u_{t2}, v_{t2})$, and define $n_1^t$ and $n_2^t$ as the corresponding unit vectors in the directions normal to the surfaces. Then define $r_1^f$, $r_2^f$, $n_1^f$, and $n_2^f$ similarly for the points and unit normals relative to the femur coordinate system which depend on $(u_{f1}, v_{f1})$ and $(u_{f2}, v_{f2})$. With these definitions the
requirement for geometric compatibility can be expressed mathematically using the following four equations.

\[
\begin{align*}
    r_1' &= R_{3/2} r_1' + x_{3/2} & (7.20) \\
    r_2' &= R_{3/2} r_2' + x_{3/2} & (7.21) \\
    n_1' &= \pm R_{3/2} n_1' & (7.22) \\
    n_2' &= \pm R_{3/2} n_2' & (7.23)
\end{align*}
\]

For a parametric surface, the unit normal is defined using the cross product of the two tangent vectors to the surface in the \(u\) and \(v\) directions. For example, the unit vector \(n_1'\) can be calculated from derivatives of the position vector \(r_1'\) as shown below.

\[
    n_1' = \frac{\frac{\partial r_1'}{\partial u_{11}} \times \frac{\partial r_1'}{\partial v_{11}}}{\left| \frac{\partial r_1'}{\partial u_{11}} \times \frac{\partial r_1'}{\partial v_{11}} \right|}
\]  

\(n_1'\) is defined as the cross product of two tangent vectors, then it will be parallel to another unit vector if and only if the two tangent vectors are perpendicular to that vector. Therefore, equation (7.22) can be replaced by a pair of equations involving the tangent vectors for the tibia in place of \(n_1'\).

\[
\begin{align*}
    \left( \frac{\partial r_1'}{\partial u_{11}} \right)^T R_{3/2} n_1' &= 0 & (7.25) \\
    \left( \frac{\partial r_1'}{\partial v_{11}} \right)^T R_{3/2} n_1' &= 0 & (7.26)
\end{align*}
\]

Equations (7.20) through (7.23) can therefore be written in a more convenient form using the equations (7.27) through (7.32) as shown below.
\[
\begin{align*}
&f_1 f_2 f_3^T = -r_1^f + R_{3/2} r_1^f + x_{3/2} = 0 \quad (7.27) \\
f_4 f_5 f_6^T = -r_2^f + R_{3/2} r_2^f + x_{3/2} = 0 \quad (7.28) \\
f_7 = (\frac{\partial r_1^f}{\partial u_{11}})^T R_{3/2} n_1^f = 0 \quad (7.29) \\
f_8 = (\frac{\partial r_1^f}{\partial v_{11}})^T R_{3/2} n_1^f = 0 \quad (7.30) \\
f_9 = (\frac{\partial r_2^f}{\partial u_{12}})^T R_{3/2} n_2^f = 0 \quad (7.31) \\
f_{10} = (\frac{\partial r_2^f}{\partial v_{12}})^T R_{3/2} n_2^f = 0 \quad (7.32)
\end{align*}
\]

Notice that the right-hand side expressions are all functions of \(q_{3/2}\) and \(s\). Therefore, the requirements for geometric compatibility in equations (7.27) through (7.32) can be combined into a single vector equation, which represents ten scalar equations in \(q_{3/2}\) and \(s\).

\[f(\ q_{3/2}, s\ ) = 0 \quad (7.33)\]

7.4.2 Adjusting the Data to Guarantee Geometric Compatibility

If \(q_{3/2}\) truly represents a set of generalized coordinates for the knee joint, then not only \(y_{3/2}\) and \(\theta_{x3/2}\) but also all of the parametric coordinates should be dependent on \(q_{3/2}\). Thus, the entire \(s\) vector must a function of \(q_{3/2}\):

\[s = s(\ q_{3/2}\ ) \quad (7.34)\]
If this is true, then it should be possible to solve for the values of \( s \) which will satisfy geometric compatibility for arbitrary values of \( q_{3/2} \). In order to estimate the value of \( s \), a Taylor series expansion of \( f \) is written as follows (assuming that \( q_{3/2} \) is held constant):

\[
f = f_0 + \left( \frac{\partial f}{\partial s} \right)_0 \Delta s + \text{higher order terms} = 0 \tag{7.35}
\]

where the coefficient which premultiplies \( \Delta s \) is actually a 10 by 10 matrix. If some initial estimate \( s_0 \) can be obtained for \( s \) such that equation (7.33) is approximately satisfied, then Newton-Raphson iteration [52] can be used to solve for the final value of \( s \) by repeatedly using the following pair of equations until convergence is obtained.

\[
\Delta s = \left( \left( \frac{\partial f}{\partial s} \right)_0 \right)^{-1} f_0 \tag{7.36}
\]

\[
s = s_0 + \Delta s \tag{7.37}
\]

An initial estimate for \( s_0 \) is obtained by the following procedure:

1. Select \( u_{11}, v_{11}, u_{12} \) and \( v_{12} \) equal to 0.50 (i.e. estimate that the contact point is approximately in the middle of each of the articulating surfaces of the tibia)
2. Select \( u_{x1}, v_{x1}, u_{x2} \) and \( v_{x2} \) to satisfy equations (7.29) through (7.32)
3. Use the initial values of \( y_{3/2} \) and \( \theta_{x3/2} \) as defined by \( x_{3/2} \) and \( R_{3/2} \)

Using the iterative procedure outlined above to solve for \( s \) as a function of \( q_{3/2} \) may cause some problems when the initial estimates for \( x_{3/2} \) and \( R_{3/2} \) are not accurate. For example if large medial-lateral or anterior-posterior translation errors exist in the initial data, then during the Newton-Raphson iteration process values of \( s \) may be estimated which correspond to points outside of the valid region of data for one of the parametric surfaces. When the parametric coordinates are located outside of the valid region the iteration procedure is halted, and the kinematic data for that frame is labeled as invalid. For high quality kinematic data this is not a problem, but for activities in which significant soft tissue
motion occurs in cases where the transformations between the femur and tibia and their corresponding arrays have large errors. Convergence may not be obtained. For the purposes of making an initial estimate of the changes required in the kinematics to insure geometric compatibility, both s and q_{3/2} should be modified.

For this reason, the three-step procedure presented above for estimating an initial value of s_0 should be appended with a fourth step. After estimating s_0 by the previously described methods, expressions (7.27) and (7.28) should be evaluated. Then the position vector x_{3/2} (which affects both s and q_{3/2}) should be updated to make the average error of the points on the femur articulating surface relative to the corresponding points on the tibia articulating surface exactly zero.

$$\begin{align*}
x_{3/2} &= x_{3/2, 0} - \frac{1}{2}
\begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5 \\
f_6
\end{bmatrix}
+ \begin{bmatrix}
f_7 \\
f_8 \\
f_9 \\
f_{10} \\
f_{11} \\
f_{12}
\end{bmatrix}
\end{align*}$$

(7.38)

After the position vector of the femur relative to the tibia is changed as in equation (7.38) the Newton-Raphson iterative procedure should be used to converge to a geometrically compatible set of kinematics.

7.4.3 Estimating Derivatives for Dependent Variables

For the purposes of applying the kinematic data to evaluate dynamic equilibrium and stability of equilibrium, it is necessary to calculate the first two derivatives of the dependent variables of the knee \(y_{3/2} \text{ and } \theta_{3/2}\) with respect to the independent variables \(q_{3/2}\). A Taylor series expansion can be used to expand \(f\) about \((q_{3/2})_0\) and \(s_0\) as follows.

$$f = f_0 + \left(\frac{\partial f}{\partial q_{3/2}}\right)_0 \Delta q_{3/2} + \left(\frac{\partial f}{\partial s}\right)_0 \Delta s + \text{higher order terms} = 0$$

(7.39)
Note that this expression is different from equation (7.35) because $q_{3/2}$ is also allowed to vary in this case. Now use the fact that $s$ depends on $q_{3/2}$ to write this expression in a slightly different form.

$$f = f_0 + \left( \frac{\partial f}{\partial q_{3/2}} \right)_0 + \left( \frac{\partial f}{\partial s} \right)_0 \left( \frac{\partial s}{\partial q_{3/2}} \right)_0 \Delta q_{3/2} + \text{h.o.t.} = 0 \quad (7.40)$$

If the initial conditions $(q_{3/2})_0$ and $s_0$ correspond to geometric compatibility, then all components of $f_0$ will be exactly zero. In that case, the only way that equation (7.40) can be true for all arbitrary infinitesimal changes in $\Delta q_{3/2}$ is for the following condition to hold.

$$\left( \frac{\partial s}{\partial q_{3/2}} \right)_0 = - \left[ \left( \frac{\partial f}{\partial s} \right)_0 \right]^{-1} \left( \frac{\partial f}{\partial q_{3/2}} \right)_0 \quad (7.41)$$

Since $s$ includes the dependent degrees of freedom, then the required first derivative terms have been calculated. Second derivatives of $s$ with respect to $q_{3/2}$ can be calculated in a similar manner, but the equations are much more complex and will not be presented here. However, one important point to be made regarding derivatives of $s$ with respect to $q_{3/2}$. They all require the the same 10 by 10 matrix to be inverted for solution. The matrix $A$ as defined below must therefore be non-singular for derivatives to be calculated.

$$A \left( q_{3/2}, s \right) = \left( \frac{\partial f}{\partial s} \right)_0 \quad (7.42)$$

The actual matrix inversion procedure does not have out be carried out, as Gaussian elimination or singular value decomposition methods can be used to solve equations (7.36) and (7.41). But the the $A$ matrix is singular, then in general no solutions will exist which satisfy all of the equations. Since this matrix is required to be non-singular for convergence to occur toward geometric compatibility and for partial derivatives to be calculated, then the minimum singular value of the $A$ matrix is a measure of the validity of the selected four degrees of freedom knee model. At least instantaneously, $q_{3/2}$ is an
appropriate choice of four generalized coordinates for the knee if and only if the A matrix is non-singular. It should be pointed out here that one of the reasons for selecting the four independent variables in equation (7.18) was that the resulting A matrix is not singular for the kinematic and geometry data of this thesis. Therefore, the knee kinematics are appropriately modeled using this set of four degrees of freedom.

7.4.4 Improved Estimates for Average Tibia and Femur Transformations

If a set of kinematic data is modified to insure geometric compatibility as described in section 7.4.2, then in general a different adjustment to the knee kinematics will be estimated for each frame of data. It would be useful to be able to modify the estimated transformations of the tibia and femur relative to the shank and thigh arrays in order to minimize the average errors over all frames of a set of data (or several sets of data). A procedure has thus been implemented to allow the tibia and femur translations to be modified in the anatomical x- and y-directions such that distances between the points on the tibia and femur (as measured by equations (7.28) and (7.29)) are minimized. The procedure first requires convergence to a geometrically compatible kinematic configuration for each frame. This process is performed only to calculate the eight parametric coordinates on the articulating surfaces, and thus the tibia and femur coordinates of the estimated contact points $r^t_1, r^t_2, r^f_1, \ldots, r^f_2$.

The allowance for an adjustment in the anatomical x- and y-coordinates of the tibia and femur coordinate system origins corresponds to the following equations.

\[
x = R^2 \left\{ r^2 + \begin{bmatrix} \Delta x_2 \\ \Delta y_2 \\ 0 \end{bmatrix} \right\} + x^2 \quad (7.43)
\]
In these equations, \( \Delta x_2 \) and \( \Delta y_2 \) are the allowed changes in the tibia transformation, and \( \Delta x_3 \) and \( \Delta y_3 \) are the allowed changes in the femur transformation. Since changes are only allowed in the translations, then the relative rotation matrix \( R_{3/2} \) does not change. The expression for \( x_{3/2} \) in equation (7.45) is identical to equation (7.15) except for the additional terms due to the changes in the tibia and femur transformations.

The expression (7.45) is then substituted into equations (7.27) and (7.28) along with the estimated values of \( r_1^t, r_2^t, r_1^f, \) and \( r_2^f \). The sums of the squares of the errors between the points on the tibia and femur articulating surfaces are then minimized with respect to \( \Delta x_2, \Delta y_2, \Delta x_3 \), and \( \Delta y_3 \) using a nonlinear gradient search technique. After the minimization is completed, the optimized values of \( \Delta x_2 \) and \( \Delta y_2 \) are added to \( x^2 \) and the optimized values of \( \Delta x_3 \) and \( \Delta y_3 \) are added to \( x^3 \). Thus the average transformation between the tibia and femur and their corresponding arrays are adjusted to minimize average contact point position errors in the kinematic data.

It only makes sense to apply this method to a set of kinematic data with relatively large variations in the knee flexion angle. If, for example, the procedure is applied to only a single frame of data then changes in \( \Delta x_2 \) and \( \Delta y_2 \) will produce nearly the same results as changes in \( \Delta x_3 \) and \( \Delta y_3 \). Note that this procedure will not only estimate optimal changes in the translations between the anatomical and array coordinate systems of the tibia and femur but also measure the errors in the average axis of rotation estimates for the anatomical x- and y-coordinates of the segment origins.
7.4.5 Estimating Hip and Ankle Joint Center Locations

In the section 7.4.6, a method for adjusting the kinematic data as little as possible to insure geometric compatibility will be presented. One of the requirements for this optimization procedure is knowledge of the estimated hip and ankle locations relative to the adjacent segments. This section will very briefly explain the procedures for estimating the locations of the joint center of the ankle relative to the foot and tibia coordinate systems \( r_{\text{ANKLE}}^1 \) and \( r_{\text{ANKLE}}^2 \) and the the joint center of the hip relative to the femur and pelvis coordinate systems \( r_{\text{HIP}}^3 \) and \( r_{\text{HIP}}^4 \). Note that this information is also required for defining the generalized coordinates for the lower extremity as explained in section 5.4.

After the average transformations of the tibia and femur relative to the shank and thigh arrays have been estimated as accurately as possible using the methods described above, this information can be used to transform any set of kinematic data into estimates for the position vectors \( (x_1, x_2, x_3, \text{and } x_4) \) and the rotation matrices \( (R_1, R_2, R_3, \text{and } R_4) \) for the four lower extremity segments relative to the GCS. By considering the motion of the tibia relative to the foot during a set of kinematic data in which a subject manually rotates the ankle as much as possible about two different axes, the location of the ankle joint center relative to the tibia coordinate system \( r_{\text{ANKLE}}^2 \) may be estimated. The estimate for the joint center is obtained using similar procedures to those which were used to estimate the location of the pointer endpoint relative to its array (as described in section 4.2), except the motion of segment 2 relative to segment 1 is used as shown in the equations below. In these equations, the subscript \( k \) is used to specify the \( k \)th frame of the kinematic data set.

The first step of the estimation of the ankle joint center relative to the tibia requires calculating the position vector \( x_{2/1k} \) and rotation matrix \( R_{2/1k} \) for the tibia relative to the foot for each frame.

\[
R_{2/1k} = R_k^T R_k^2
\]  \hspace{1cm} (7.46)

\[
x_{2/1k} = R_k^T (x_k^2 - x_k^1)
\]  \hspace{1cm} (7.47)
Next a value for $r^2_{\text{ANKLE}}$ is assumed, and the coordinates of the ankle joint center are calculated relative to the foot for each frame, along with the mean and variance.

\[
\begin{align*}
\overline{r^1_{\text{ANKLE},k}} &= R_{2:1k} r^2_{\text{ANKLE}} + x_{2:1k} \\
\overline{r^1_{\text{ANKLE}}} &= \frac{1}{N} \sum_{k=1}^{N} r^1_{\text{ANKLE},k} \\
\sigma^2 &= \frac{1}{N} \sum_{k=1}^{N} \left( r^1_{\text{ANKLE},k} - \overline{r^1_{\text{ANKLE}}} \right) \left( r^1_{\text{ANKLE}} - \overline{r^1_{\text{ANKLE}}} \right)
\end{align*}
\]

(7.48)  
(7.49)  
(7.50)

Finally, the variance $\sigma^2$ as defined in equation (7.50) is minimized with respect to $r^2_{\text{ANKLE}}$ to estimate the optimal location of the ankle relative to the coordinate system of the tibia. Similar procedures are then applied to estimate $r^1_{\text{ANKLE}}$, $r^3_{\text{HIP}}$, and $r^4_{\text{HIP}}$.

### 7.4.6 Improving Instantaneous Kinematic Estimates via Optimization

The transformations for the tibia and femur relative to the shank and thigh arrays have been estimated to minimize geometric incompatibility errors over all of the data sets. But in order to apply the kinematic data to a dynamic analysis of the knee the geometric compatibility must be satisfied exactly. Therefore, a procedure is implemented to adjust the estimated position and orientation of the tibia and femur for each frame so that the kinematics are compatible with the articular geometry.

The procedure for adjusting the kinematics involves modifying the position and orientation of the tibia and femur just enough to satisfy geometric compatibility. In other words, that set of compatible knee kinematics which is "closest" to the TRACK kinematic measurements is selected as the improved estimate for the actual motion of the tibia and femur. The "closest" set of kinematics is selected based on minimizing a penalty function which is equal to weighted sums of squares of changes in the tibia and femur degrees of freedom (where the weights are based on expected kinematic errors) plus sums of squares of errors in GCS coordinates of the hip and ankle joint centers as predicted by the two
segments adjacent to each of the two joints.

In order to improve measured knee kinematics using the methods described above, estimates must be obtained for expected errors in the measurements of the degrees of freedom of the tibia and femur. Two sources of error will contribute to deviations in measured knee kinematics. Noise in the TRACK kinematic data will cause errors in the transformation between the array and the GCS. Soft-tissue motion will cause errors in the transformation between the tibia and femur and the corresponding arrays.

For the purposes of explaining how these two sources of error are combined, consider a generic anatomical segment S and its associated array A. The transformation from the coordinate system of the segment to the GCS \( x^S \) and \( R^S \) will include the transformation from the segment to the array \( x_{S/A} \) and \( R_{S/A} \) and the transformation from the array to the GCS \( x^A \) and \( R^A \) as shown below. These three transformations depend on \( q^S \), \( q^A \), and \( q_{S/A} \), respectively.

\[
R^S = R^A R_{S/A} \\
x^S = x^A + R^A x_{S/A}
\]

The coordinates of \( q^S \), \( q^A \), and \( q_{S/A} \) are defined using three translations and three angles, where the angles are as defined using equations (5.10) through (5.12).

\[
[q^S]_T = [x^S]^T \begin{bmatrix} \theta_x^S & \theta_y^S & \theta_z^S \end{bmatrix}
\]

(7.53)

Derivatives of \( q^S \) with respect to the 12 degrees of freedom \( q_i \) can be calculated as shown below, where \( q_i \) represents any element of \( q^A \) or \( q_{S/A} \).

\[
\left[ \frac{\partial q^S}{\partial q_i} \right]^T = \left[ \frac{\partial x^S}{\partial q_i} \frac{\partial \theta_x^S}{\partial q_i} \frac{\partial \theta_y^S}{\partial q_i} \frac{\partial \theta_z^S}{\partial q_i} \right]
\]

(7.54)
With these definitions, small variations in \( \frac{\partial \Theta}{\partial q_i} \) can be related to deviations in \( dq^A \) and \( dq_{S/A} \). For this particular application, noise in measured TRACK kinematics is completely unrelated to soft tissue motion. Hence the deviations in \( q^A \) and \( q_{S/A} \) are independent, and a covariance matrix \( C^S \) for errors in \( q^S \) can be written in terms of covariance matrices \( C^A \) and \( C_{S/A} \).

\[
\frac{\partial \Theta}{\partial q_i} = \frac{\partial R_{23}^S}{\partial q_i} \sqrt{1 - (R_{23}^S)^2}
\]

\[
\frac{\partial \Theta_i}{\partial q_i} = \frac{R_{33}^S \frac{\partial R_{13}^S}{\partial q_i} + R_{13}^S \frac{\partial R_{23}^S}{\partial q_i}}{(R_{13}^S)^2 + (R_{33}^S)^2}
\]

With these definitions, small variations in \( dq^S \) can be related to deviations in \( dq^A \) and \( dq_{S/A} \). For this particular application, noise in measured TRACK kinematics is completely unrelated to soft tissue motion. Hence the deviations in \( q^A \) and \( q_{S/A} \) are independent, and a covariance matrix \( C^S \) for errors in \( q^S \) can be written in terms of covariance matrices \( C^A \) and \( C_{S/A} \).

\[
dq^S = \frac{\partial dq^S}{\partial q^A} dq^A + \frac{\partial dq^S}{\partial q_{S/A}} dq_{S/A} = J^A dq^A + J_{S/A} dq_{S/A}
\]

\[
C^S = J^A C^A J^{A^T} + J_{S/A} C_{S/A} J_{S/A}^T
\]
One more definition is required before the penalty function for minimization can be completely defined. As mentioned above, the penalty function will include terms related to ankle and hip position errors. These errors measure how well the tibia and foot transformations and estimated ankle locations specify the same point in the GCS, and a similar set of errors is specified for the hip. The definitions for the ankle and hip errors are shown in the following equations:

\[
\begin{align*}
\Delta x_{\text{ANKLE}} &= \left( R^2 r^2_{\text{ANKLE}} + x^2 \right) - \left( R^1 r^1_{\text{ANKLE}} + x^1 \right) \\
\Delta x_{\text{HIP}} &= \left( R^3 r^3_{\text{HIP}} + x^3 \right) - \left( R^4 r^4_{\text{HIP}} + x^4 \right)
\end{align*}
\]

(7.60) (7.61)

An explicit expression for the penalty function to be minimized by the kinematic correction procedure is shown below. In this equation the scalar constant \( W_{\text{JOINTS}} \) is a weighting factor which can be used to change the amount of influence the ankle and hip position errors may have on the results.

\[
f = \Delta q^T W^2 \Delta q^2 + \Delta q^T W^3 \Delta q^3 + W_{\text{JOINTS}} \left( \Delta x^T_{\text{ANKLE}} \Delta x_{\text{ANKLE}} + \Delta x^T_{\text{HIP}} \Delta x_{\text{HIP}} \right)
\]

(7.62)

\[
W^2 = \left[ \text{diag } C^2 \right]^{-1}
\]

(7.63)

\[
W^3 = \left[ \text{diag } C^3 \right]^{-1}
\]

(7.64)

The penalty function is minimized using a nonlinear gradient search minimization routine. Two separate implementations of the kinematic correction method were tested. In the first method, only 10 degrees of freedom are used in the optimization (six for the position and orientation of the tibia plus four more for the degrees of freedom of the knee). In this method, geometric compatibility is guaranteed at each step of the search. This method is slow and has difficulty when the dependent values \( s \) (as defined in equation (7.19)) are estimated to be in the invalid regions of the articular surfaces.

A second method for obtaining the optimal solution was implemented which allowed 20
degrees of freedom. For this approach, the $s$ vector is not assumed to be dependent on the estimated independent kinematic variables. Instead, an additional term is included in the function to be minimized so that it severely penalizes geometric incompatibility (as expressed in equations (7.27) to (7.32)). Terms are also included to keep the parametric coordinates within the valid regions of articular surface data. This approach works much faster than the initial method implemented.

Once the kinematic optimization has been completed, improved estimates are available for the positions and orientations of the tibia and femur relative to the GCS. This information is then combined with the TRACK kinematic measurements of the foot and pelvis using the methods described in section 5.4.2 in order to calculate all fifteen generalized coordinates of the $q$ vector (i.e. nine degrees of freedom for the ankle, knee, and hip joints plus six degrees of freedom for the position and orientation of the entire lower extremity structure). For each frame the kinematic data is stored as the final estimated $q$ vector and the associated vector of dependent values $s$. Additionally, the modifications which have been applied to each coordinate of the kinematics of the tibia and femur (i.e. estimates of the soft tissue motion) are also stored.

### 7.5 Results of Kinematic Correction Procedures

This section will present results from raw and corrected kinematic gait data from a single subject during the stance phase of gait. Effects of different steps in the procedure on the estimated kinematics will be emphasized. Although strains on the knee ligaments can be calculated using the kinematic data, the estimated ligament strains and forces will not be discussed until Chapters 8 and 9.

Using the procedures outlined in this chapter, the TRACK system is used to measure the position and orientation of the four lower extremity segments. The anatomical landmark and average axis of rotation information is applied to the static neutral and flexed position data to obtain initial estimates for the transformations between the tibia and femur and the shank and thigh arrays. These transformations can then be used to estimate the
positions of the tibia and femur during gait data. Kinematic data estimated by this procedure typically corresponds to either penetration of the femur by the tibia (figure 7.1) or a gap between the articulating surfaces. In this figure and all similar figures to follow, the medial condyle is on the right and the lateral condyle is on the left (i.e. front view of a right knee).

When the geometric compatibility information is used to minimize RMS position errors by allowing changes in the anatomical x- and y-coordinates of the segment origins, the RMS errors for the entire stance phase of gait are reduced from about 8 mm to 6 mm. Assuming that the joint is actually in contact during the collected gait data, then the remaining 6 millimeters of error must be due to either soft tissue motions or errors in the estimates for the in vivo geometry using the scaled cadaver data.

**Figure 7.1: Uncorrected Kinematic Data**

During the maneuvers designed to include large ranges of motion about two axes of the ankle and hip, the optimal joint centers include estimated RMS errors of 5 mm and 7 mm, respectively. Recall that the values of ankle joint errors represent differences in estimated locations using TRACK measurements of the kinematics of the arrays attached to the foot.
and shank. Similarly, errors for the hip are calculated using TRACK measurements of the thigh and pelvis arrays. These joint center locations are estimated without accounting for soft tissue motion. Since the errors are approximately the same size as the estimated soft tissue motion errors for the knee, most of the error may be due to soft tissue motion. Furthermore, since two arrays are involved in the estimates of each joint center, the magnitude of the errors is remarkably small. For the ankle joint some of the error may be due to small translations which are reported to occur in conjunction with the two rotations [30]. Since the kinematic model in this thesis assumes pure rotation about axes which intersect at a point.

Figure 7.2 demonstrates that the kinematic correction procedure successfully produces knee positions and orientations compatible with the knee model geometry, as the corrected data shown corresponds to the raw data of figure 7.1. By definition, geometric compatibility has been satisfied for corrected data, and therefore the location of the contact points and direction of the unit vectors can be calculated and displayed with the articular geometry data. Since the normals of the tibia and femur are parallel when the surfaces are in contact at a single point, only the outward surface normal of the tibia is displayed (figure 7.2). Notice that the articular geometry of the medial condyle of the femur is not significantly different from the geometry of the opposing tibial surface. Therefore the estimated contact point could change significantly without affecting the kinematics very much. This consequences of this point as it relates to joint contact forces and equilibrium will be discussed further in Chapters 8 and 9.

In all of the following figures of this chapter, translations of the knee represent the motion of the femur relative to the tibia. Therefore, anterior translation represents the femur moving forward on the tibia, medial translation represents the femur moving toward the center of the body, and axial translation represents motion of the femur in the upward direction along the long axis of the tibia. Figure 7.3 shows the estimated anterior translation of the knee for three separate sets of TRACK kinematic data. The corrected kinematic data is very repeatable. Variations in estimated translations between data sets are less than about 2 mm (figure 7.3), and variations in rotation angles are less than 1 degree.
Figure 7.2: Corrected Kinematic Data

Figure 7.3 Knee Anterior Translations
Figures 7.4 through 7.9 show the estimated corrected knee kinematics and the corresponding required adjustments in the tibia and femur kinematic data for a single set of kinematic data during the stance phase of gait. Each figure contains three curves. The curves labelled "tibia" and "femur" display the amount of correction which was added to the segment kinematics to provide geometric compatibility. The remaining curve represents the final estimated knee kinematics. The mean value of each of the correction curves is a measure of the errors in the estimated average transformation between the anatomical segment and the corresponding array. The assumed transformation between the array and the underlying anatomical segment could be adjusted to account for these errors, and thus only variations in the corrections are relevant. The values of corrected knee kinematics for each degree of freedom have been adjusted to correspond to zero during a set of static data collected with the subject in an anatomically neutral position.

For all three estimated rotations, the corrections applied to the tibia and femur data are substantially smaller than the estimate changes in the angles during the stance phase. The flexion angle of the knee changes by more than 30 degrees during stance (figure 7.4), and
Figure 7.5: Knee Abduction and Errors

Figure 7.6: Knee External Rotation and Errors
Figure 7.7: Knee Anterior Translation and Errors

Figure 7.8: Knee Axial Translation and Errors
thus the estimated corrections are almost negligible. Estimated variations in knee abduction are small (figure 7.5), whereas the knee is estimated to internally rotate by about 12 degrees during the stance phase of gait (figure 7.6).

Figure 7.7 demonstrates that the knee motion estimated by the kinematic correction procedure does not correspond to a simple hinge joint, as the origin of the femur moves forward on the tibia about 13 mm between heel strike and toe-off. Estimated axial translations of the knee are smaller than the applied corrections (figure 7.8). The corrections applied to the tibia and femur kinematic data in this direction are approximately equal and opposite and are used to correct for TRACK measurements which correspond to penetration of the tibia by the femur near heel strike (figure 7.1) and a gap between the articulating surfaces at 80 percent of stance phase. The required soft tissue motion errors to insure that the condyles are in contact with the tibia are up to 6 mm in the axial direction. Possible reasons for estimated corrections of this size in the axial direction include differences between the cadaver and in vivo geometry and actual soft tissue motion in this
direction which occurs near heel strike. With the implemented array mounting methods, the arrays can be manually displaced several millimeters in the axial direction. Figure 7.8 demonstrates that the estimated corrections in medial translations are small compared to the almost 8 mm of motion estimated for this direction.

Given that the repeatability of the data is approximately 2 mm and the corrections applied to the data are about 6 mm, then one may estimate that the overall accuracy of this data is somewhere between 2 mm and 6 mm. The signal-to-correction ratio for knee translations is only about 2 to 1, so estimated error bounds for the knee kinematics are the same order of magnitude as the estimated translations. By default, these estimated translations are better than knee translations estimated using a kinematic measurement system which places markers at the assumed knee joint center, since by definition the system assumes all translations to be exactly zero.

This the first time that soft tissue motion errors have been estimated using assumptions about articular geometry. The only other study conducted to estimate soft tissue motion errors during gait data included comparisons of measured kinematics using intracortical bone pins and externally mounted arrays [45]. The kinematic correction procedures developed in this thesis could be applied during any standard gait analysis with no requirement of bone pins. The measurements should not be expected to be as accurate as bone pins, but at the very least a quantitative estimate of the soft tissue motions provides an measure of the quality of the kinematic data. A requirement for applying these methods is the use of a rigid body approach to measuring kinematics, rather than the method of placing markers at the presumed joint centers. Unfortunately, most gait analysis systems throughout the world use the method of placing markers at joint centers. The TRACK system is able to measure the kinematics of rigid bodies accurately. However, when applying the system to measure kinematics of the lower extremity an assumption is necessary that the soft tissue motion errors are small. The analysis developed here enables some quantitative estimation and removal of these errors for the first time.

The purpose of the development of the kinematic correction procedure for this thesis was to provide a set of compatible kinematic and articular geometry data which could be
used for a dynamic analysis of the knee joint. However, the information which comes out of this kinematic analysis could also be appropriate for several additional applications:

1. analysis of accuracy of the TRACK system
2. determination of which array mounting procedures give the least soft-tissue motion errors
3. analysis of whether or not there is a screw-home mechanism in the knee
4. determination (possibly, if data is high quality) of whether or not the knee can be well modeled as a single degree of freedom joint.
Chapter 8

KNEE MODEL MUSCLE, LIGAMENT, AND JOINT FORCES

8.1 Introduction

Chapters 6 and 7 have presented methods and results for calculating a set of kinematics compatible with a nine degree of freedom model of the lower extremity, including four degree of freedom knee kinematics which are compatible with the estimated articular surface geometry. These methods have been applied to TRACK kinematic measurements during the stance phase of gait. The purpose of the procedures described in this chapter is to use the resulting kinematics and geometry in a dynamic analysis in order to calculate net joint forces and moments and then to estimate how these net joint loads are produced by the muscle, ligament, and joint contact forces.

Specific requirements for incorporating the dynamic equilibrium and stability constraints into a muscle force optimization analysis are explained in sections 8.2 through 8.4. The muscle force optimization procedures are described in section 8.5, including discussion of some potential problems which may be encountered. Results are presented for the calculated net joint loads as well as for estimated muscle, ligament, and joint contact forces. A limitation of these procedures is discussed and used as motivation for further analyses which are described in Chapter 9.

8.2 Derivatives of Segment Transformations

The output from the kinematic correction procedures described in Chapter 7 includes estimates for $q$, the fifteen generalized coordinates for the lower extremity model, and $s$, the eight parametric coordinates and two dependent kinematic coordinates of the knee, for each frame. After a complete set of kinematic data has been processed, these vectors are smoothed and the first two time derivatives of $q$ are calculated. Before all of the terms
required for a dynamic and stability analysis can be evaluated, the first two derivatives of the segment position vectors and rotation matrices with respect to $q$ must be calculated.

Recall from section (5.4.2) that the 15 generalized coordinates for the lower extremity are assigned as shown below. In equation (8.1), the vector $q_{1/0}$ specifies motions of the foot relative to the GCS, while the vectors $q_{2/1}$, $q_{3/2}$, and $q_{4/3}$ specify the relative motions of the ankle, knee, and hip joints, respectively.

$$
q^T = q_{1/0}^T q_{2/1}^T q_{3/2}^T q_{4/3}^T
$$

(8.1)

$$
q_1 \quad q_2 \quad q_3 \quad q_4 \quad q_5 \quad q_6 = \begin{bmatrix} x_{1/0}^T & \theta_{x1/0} & \theta_{y1/0} & \theta_{z1/0} \end{bmatrix}
$$

(8.2)

$$
q_7 \quad q_8 = \begin{bmatrix} \theta_{x2/1} & \theta_{z2/1} \end{bmatrix}
$$

(8.3)

$$
q_9 \quad q_{10} \quad q_{11} \quad q_{12} = \begin{bmatrix} x_{3/2} & z_{3/2} & \theta_{y3/2} & \theta_{z3/2} \end{bmatrix}
$$

(8.4)

$$
q_{13} \quad q_{14} \quad q_{15} = \begin{bmatrix} \theta_{x4/3} & \theta_{y4/3} & \theta_{z4/3} \end{bmatrix}
$$

(8.5)

The relative position vectors ($x_{1/0}$, $x_{2/1}$, and $x_{4/3}$) and rotation matrices ($R_{1/0}$, $R_{2/1}$, and $R_{4/3}$) which describe the motions of the foot, ankle, and hip are obtained by evaluating equations (5.29) through (5.34). The relative position vector $x_{3/2}$ and rotation matrix $R_{3/2}$ of the knee is evaluated using the following two equations,

$$
x_{3/2}^T = q_9 \quad s_9 \quad q_{10}
$$

(8.6)

$$
R_{3/2} = \mathcal{R} \left( s_{10}, q_{11}, q_{12} \right)
$$

(8.7)

where the function $\mathcal{R}$ has been defined in equation (5.9).

The position vectors ($x^1$, $x^2$, $x^3$, and $x^4$) and rotation matrices ($R^1$, $R^2$, $R^3$, and $R^4$) for the four lower extremity segments are obtained by substituting the calculated relative transformations into equations (2.21) through (2.28). Derivatives of the segment
transformations with respect to \( q \) can then be obtained using derivatives of the relative transformations and applying the chain rule. Therefore, in order to evaluate the required terms, the derivatives with respect to \( q \) of \( x_{1,0}, x_{2,1}, x_{3,2}, \text{ and } x_{4,3} \) plus \( R_{1,0}, R_{2,1}, R_{3,2}, \) and \( R_{4,3} \) must be available. Most of the required derivatives may be obtained in a straightforward manner using the definitions in equations (5.29) through (5.34). However, the two dependent coordinates \( s_9 \) and \( s_{10} \) in equations (8.6) and (8.7) increase the complexity of derivative calculations for the position vector \( x_{3/2} \) and rotation matrix \( R_{3/2} \) of the knee.

Notice in equations (8.6) and (8.7) that the definitions for \( x_{3/2} \) and \( R_{3/2} \) each include a single dependent variable. To explain how the derivatives are calculated for these terms, define \( f \) as one of the elements of \( x_{3/2} \) or \( R_{3/2} \) and \( s \) as the dependent variable (i.e. \( s_9 \) or \( s_{10} \), respectively). Using this notation, the dependence of \( f \) on \( q \) can be written in the following form.

\[
f = f(\ q, \ s(q) )
\]

Next define an operator \( \partial / \partial q_i \) as a differentiator which only recognizes explicit dependence on \( q_i \). With this definition, the first two derivatives of function \( f \) with respect to \( q \) can be written using the following two equations.

\[
\frac{\partial f}{\partial q_i} = \frac{\partial f}{\partial q_i} + \frac{\partial f}{\partial s} \frac{\partial s}{\partial q_i} \tag{8.9}
\]
\[
\frac{\partial^2 f}{\partial q_i \partial q_j} = \frac{\partial^2 f}{\partial q_i \partial q_j} + \frac{\partial^2 f}{\partial q_i \partial s} \frac{\partial s}{\partial q_j} + \frac{\partial^2 f}{\partial s \partial q_i} \frac{\partial s}{\partial q_j} + \frac{\partial f}{\partial s} \frac{\partial^2 s}{\partial q_i \partial q_j} \tag{8.10}
\]

This procedure can be applied to calculate the derivatives of \( x_{3/2} \) and \( R_{3/2} \) with respect to \( q \) by first calculating derivatives for the explicit dependence on \( q_{3/2} \) and then adjusting these values using equations (8.9) and (8.10) plus derivatives of \( s_9 \) and \( s_{10} \) with respect to \( q_{3/2} \). Methods for calculating derivatives of the dependent kinematic coordinates of the knee (i.e. \( s_9 \) and \( s_{10} \)) with respect to \( q_{3/2} \) were presented in section 7.4.3.
8.3 Knee Model Dynamics

Recall equation (2.53), which represents the requirement for dynamic equilibrium to be satisfied in the direction of generalized coordinate q_i.

\[
\sum_{k=1}^{NM} F_{Mk} \frac{\partial L_{Mk}}{\partial q_i} + \sum_{k=1}^{NL} F_{Lk} \frac{\partial L_{Lk}}{\partial q_i} = F_{\text{NET}}^T \frac{\partial x_{M+1}^T}{\partial q_i} + M_{\text{NET}}^T J_{Ai}^{M+1} \tag{8.11}
\]

The methods for calculating derivatives of segment positions and orientations have been described in section 8.2. The segment transformations and derivatives can be used to calculate the segment Jacobians using equations (2.29) and (2.36). This information can then be used with the estimated time derivatives of \( q \) and the measured force plate force and moment vector to calculated the net force and moment vectors acting across joint M via equations (2.12) through (2.15) and equation (2.50). The derivatives of muscle and ligament lengths may be calculated using equation (2.61). Therefore, all of the terms in equation (8.11) except for the muscle and ligament forces, \( F_{Mk} \) and \( F_{Lk} \) respectively, may be evaluated. Equation (8.11) is thus a linear constraint on muscle and ligament forces so that equilibrium will be satisfied in direction q_i.

In standard dynamic analyses of the lower extremity which use Cartesian coordinates and a Newtonian analysis to establish constraints on muscle forces, the coefficient of a particular muscle force in the linear equation is the distance from the rotational axis to the point of application of the load (i.e. the muscle moment arm). The approach here, although computationally different, will produce the same equation for dynamic equilibrium for a direction q_i which represents pure rotation about an axis. To verify that the methods in this thesis were implemented correctly, the coefficients of the muscle forces in the dynamic equations were compared using derivatives of muscle lengths and conventional vector cross products, and produced essentially the same results (within expected computational limits). However, for the degrees of freedom of the knee joint, which do not represent either pure translations or pure rotations, the muscle coefficients for the rotational degrees of freedom were slightly different due to the effects of the dependent variables on the derivatives.
One of the major reasons for using a set of generalized coordinates and a Lagrangian formulation of the dynamics was to avoid including the joint reaction forces in the dynamic equations. If a Newtonian analysis had been performed, then additional procedures would have been necessary to account for the joint reaction forces or at least to only consider dynamic equilibrium in directions to which the joint forces do not contribute. Although the method of using derivatives of muscle and ligament lengths in the dynamic equations may seem somewhat complex, it accounts for the joint contact forces from the beginning and avoids the additional complexities which appear later in the Newtonian analysis.

Equation (8.11) can be used to generate the constraints on muscles and ligament forces in order to satisfy dynamic equilibrium. These constraint equations will be applied to a muscle force optimization analysis in section 8.5. A discussion of exactly how ligament forces are included in the dynamic equations will be presented in section 8.3.1. Although it is true that joint contact forces are not present in the dynamic equations, estimating these forces during the stance phase of gait is one of the primary goals of this thesis. Section 8.3.2 will therefore present equations for calculating joint contact forces from the results of the muscle force optimization analysis.

8.3.1 Ligament Forces

The methods used to model the knee ligaments for the purposes of a dynamic analysis have been presented in section 5.3, and they will be briefly reviewed here. In a model designed to predict knee behavior in vitro, Wismans et al. [66] assumed the force on a ligament to have a quadratic dependence on the ligament length.

\[ F_L = C_L \cdot \left( L_L - L_{L0} \right)^2, \quad L_L > L_{L0} \] (8.12)

In the case for which the ligament length is less than the specified rest length, the force is assumed to be zero. The values of \( C_L \) were estimated from results of Trent et al. [61] and the values for \( L_{L0} \) were estimated from Brantigan and Voshell [9]. Wismans et al. actually presented estimated ligament strains at zero degrees of knee flexion. For the purposes of
this thesis, the frame of the gait cycle which corresponds to a knee flexion angle closest to zero is used to convert the estimated strain value to an estimated ligament rest length. Ligament lengths are calculated using the scaled origins and insertions measured on the cadaver knee and a straight-line approximation.

Using the methods described above, the ligament forces may be calculated directly from the results of the kinematics, without regard to dynamics. Unfortunately, applying these methods to three sets of corrected kinematic data from the stance phase of gait produced estimated ligament strains for the posterior cruciate ligament of up to 0.25 (figure 8.1), a strain value which exceeds failure limits reported by Trent et al. The strains on the other three major knee ligaments are within physiological limits. Although some of the excessive strain may be attributable to errors in the strain offset value used to estimate $L_{0}$, the entire procedure of estimating ligament forces based solely on kinematics must be questioned. For this reason, an alternative scheme was implemented for estimating ligament forces by including each ligament as a "pseudo-muscle" in the muscle force optimization analysis. The muscle force optimization routine is supplied with data which corresponds to five

Figure 8.1: Ligament Strains
additional muscles (i.e. the five knee ligaments). The required "muscle" length derivatives have been calculated for the ligaments, and maximum forces for each ligament are estimated from measurements by Trent et al.

The calculation of large strain values in the posterior cruciate ligament may be attributed to a combination of errors in the ligament attachment points and errors in the kinematics of the knee model. Since the ligament origins and insertions have been accurately measured on a cadaver knee and the calculated strain values are relatively insensitive to the attachment locations, then most of the errors must be due to kinematic errors. Specifically, errors in the anterior-posterior displacements of the knee will affect the calculated cruciate ligament strains. As discussed further in chapter 9, these errors can only be substantially reduced if the knee kinematics are calculated by including information about joint forces, including those contributed by the ligaments.

**8.3.2 Joint Forces**

In order to discuss how the joint contact forces are calculated, it is useful to write the equilibrium equations in terms of the net force and moment vectors of the femur acting on the tibia ($F_{NET}^3$ and $M_{NET}^3$).

\[
F_{NET}^3 = F_{C1} u_{C1} + F_{C2} u_{C2} + \sum_{k=1}^{NM} F_{Mk} u_{Mk} + \sum_{k=1}^{NL} F_{Lk} u_{Lk} \tag{8.13}
\]

\[
M_{NET}^3 = F_{C1} (r_{C1} \times u_{C1}) + F_{C2} (r_{C2} \times u_{C2}) + \sum_{k=1}^{NM} F_{Mk} (r_{Mk} \times u_{Mk}) + \sum_{k=1}^{NL} F_{Lk} (r_{Lk} \times u_{Lk}) \tag{8.14}
\]

In these equations, $F_{C1}$ and $F_{C2}$ are the magnitudes of the joint contact forces, $u_{C1}$ and $u_{C2}$ are the associated unit vectors pointing down into the tibia, and $r_{C1}$ and $r_{C2}$ are the vectors from the origin of the femur coordinate system to the contact points. $F_{Mk}$ represents the force on muscle $k$, $u_{Mk}$ represents the unit vector from the insertion to the origin, and $r_{Mk}$ is the vector from the origin of the femur coordinate system to the muscle origin. $F_{Lk}$, $u_{Lk}$, and $r_{Lk}$ are similarly defined for the ligaments.
If the muscle and ligament forces are estimated using muscle force optimization methods, then equations (8.13) and (8.14) can be used to calculate the joint contact forces. When used to solve for the joint contact forces, the equations can be written in the following form:

\[
\begin{bmatrix}
  u_{C1} & u_{C2} \\
  (r_{C1} \times u_{C1}) & (r_{C2} \times u_{C2})
\end{bmatrix}
\begin{bmatrix}
  F_{C1} \\
  F_{C2}
\end{bmatrix} =
\begin{bmatrix}
  \Delta F \\
  \Delta M
\end{bmatrix}
\]  

(8.15)

Note that equation (8.15) actually represents six scalar equations for the two unknown joint contact forces. As long as equilibrium is satisfied for all four degrees of freedom of the knee joint, then the six equations can be satisfied exactly (neglecting computer roundoff errors). However, if equilibrium is not satisfied for all four directions of the knee model, then equation (8.15) will have no solutions.

Later in this chapter reasons will be given for not being able to satisfy equilibrium for all four degrees of freedom of the knee model for some sets of kinematics and net joint loads. The degrees of freedom which cause the most problems in this regard are the anatomical x- and z-translation directions, which correspond to anterior-posterior forces and medial-lateral forces, respectively. Several choices exist for how the joint contact forces should be estimated when equilibrium is not satisfied for all four degrees of freedom of the knee. The first and most obvious choice is to only calculate contact forces when equilibrium is completely satisfied. Another choice might be to solve equation (8.15) by a least-squares approach to get the "best possible" solution to all six equations. However, because large errors may occur in the anatomical x- and z-directions and the unit vector components of the joint contact forces are small in these directions, then very large contact forces may be estimated by this method. The approach used in this thesis is to choose the two equilibrium directions in which the contact forces contribute significantly toward satisfying equilibrium. In the anatomical coordinate system of the tibia, the joint contact forces would be expected to contribute significantly to forces in the y-axis direction (i.e. long axis of the tibia) and moments about the x-axis (i.e. abduction moment). Therefore, the knee joint contact forces are selected to satisfy equilibrium in these two directions.
Regardless of the method selected to estimate the joint contact forces, equation (8.15) should be evaluated to determine the equilibrium errors in all six directions. Note that the specification of equations for the joint contact forces must also be supplied to the muscle force optimization analysis so that it does not allow negative joint contact forces to be calculated.

8.4 Stability Analysis

The requirement of dynamic equilibrium has been shown to be a linear equality constraint on muscle and ligament forces. This constraint is supplied to the muscle force optimization routine for limiting the set of muscle forces which it can select as the optimum solution. Additional constraints on the muscle forces are also required for requirements of joint stability. The constraint of a positive definite joint stiffness matrix \( K \) is, however, nonlinear in the muscle forces and expressed as an inequality rather than equality equation. Combining equations (3.26) and (3.34) produces the following equation for element \( K_{ij} \) of the joint stiffness matrix.

\[
K_{ij} = \frac{\partial F_{NET}^{M-1} }{\partial q_j} \frac{\partial x_i^{M+1} }{\partial q_j} + \frac{\partial M_{NET}^{M-1} }{\partial q_j} \frac{\partial M_x^{M+1} }{\partial q_j} + \frac{\partial M_{NET}^{M-1} }{\partial q_j} \frac{\partial M_{NET}^{M-1} }{\partial q_j} J_{Ai}^{M-1} + \frac{\partial M_{NET}^{M-1} }{\partial q_j} \frac{\partial M_{NET}^{M-1} }{\partial q_j} J_{Ai}^{M-1} \\
+ \sum_{k=1}^{N_M} \left\{ K_{Mk} \frac{\partial L_{Mk}}{\partial q_k} \frac{\partial L_{Mk}}{\partial q_j} \right\} + \sum_{k=1}^{N_L} \left\{ K_{Lk} \frac{\partial L_{Lk}}{\partial q_k} \frac{\partial L_{Lk}}{\partial q_j} \right\} + \sum_{k=1}^{N_L} \left\{ K_{Lk} \frac{\partial L_{Lk}}{\partial q_k} \frac{\partial L_{Lk}}{\partial q_j} \right\}
\]

(8.16)

All of the terms in equation (8.16) have been evaluated except for the derivatives of the net force and moment with respect to changes in the generalized coordinates \( q \) and the unknown muscle forces and stiffnesses. Using the quasi-static assumption for the stability analysis, the stiffness matrix \( K \) is required to be positive definite. A matrix \( K \) is positive definite if and only if the following condition holds for any arbitrary non-zero vector \( \Delta q \).
\[ \Delta q^T K \Delta q = \Delta q^T \left( \frac{1}{2} K + K^T \right) \Delta q \geq 0, \Delta q \neq 0 \] (8.17)

For a symmetric real matrix, the requirement of positive definiteness can be expressed in terms of determinants of the submatrices [59].

\[
\begin{align*}
K_{11} & > 0 \\
\begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} & > 0 \\
K > 0 \text{ iff } & \\
\begin{vmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{vmatrix} & > 0 \\
\text{etc.} &
\end{align*}
\] (8.18)

Constraints for a positive definite joint stiffness matrix are therefore established by first evaluating equation (8.16) for each \( K_{ij} \) value, and then using equations (8.17) and (8.18) to generate the nonlinear inequality constraints on the muscle forces and stiffnesses. Note that the muscle stiffnesses can be expressed in terms of muscle forces using equation (3.42) for the assumed relationship between total incremental stiffness and muscle force which was approximately scaled from results by Hoffer and Andreassen [26].

### 8.4.1 Changes of Net Joint Loads with Kinematics

As mentioned above, the expression for the elements of the joint stiffness matrix \( K_{ij} \) cannot be evaluated without estimating derivatives of the net joint loads with respect to the generalized coordinates \( q \). During the double-support part of the stance phase of gait, the "opposite" leg (i.e. the one which does not have its kinematics measured and is not on the
force plate) is also in contact with the floor. During this period of time, exactly how the loading across the joints would vary with the degrees of freedom cannot be analyzed unless a lower extremity model includes both legs. Furthermore, if two dynamic systems are coupled together to form a single system, the stability of the total system is not dependent on the stability of the individual component systems. In the case of double-support for the lower extremity, even if stability analyses determine that each half is unstable the entire two-leg model may actually correspond to a stable system. One may choose, therefore, to either apply the stability analysis only to the single-support part of the stance phase or to apply it to the entire support phase and interpret the results with these concepts in mind. The latter approach is used in this thesis. During some parts of double support, one might conjecture that the control system for the lower extremity is designed to maintain stability for each side individually. In any event, the muscle force optimization routines will be applied both with and without stability constraints, so one may as well examine effects on the complete stance phase and then carefully interpret the results. Since for each side the stance phase is approximate 65 percent of the gait cycle, then only the middle half of stance phase corresponds to single-support. During the heel-strike and toe-off portions of the stance phase, the opposite leg is also in contact.

During the single-support portion of the stance phase of gait, how would one expect the net joint loads to vary with changes in the generalized coordinates? For a quasi-static stability analysis, the only contributions to net joint loads are segment masses. In a truly static case, the net force vector across a joint would act vertically downward with a magnitude equal to the entire body weight excluding the portion below the joint. This force vector would be completely independent of the kinematic configuration. The moment vector would depend on the mass above the joint \( m_{\text{ABOVE}} \) and the vector to the corresponding center of mass \( r_{\text{CG,ABOVE}} \). Therefore, during the double support portion of the stance phase of the gait cycle, the following approximations will be assumed for the dependence of the net joint loads on the generalized coordinates \( q \).

\[
\frac{\partial F_{\text{NET}}^M}{\partial q_i} = 0
\]  

(8.19)
When applying equation (8.20) to the double-support part of the stance phase, if both legs are assumed to be stable then it does not seem reasonable that each leg would be controlled to stabilize the entire body mass. A more reasonable assumption might be that each leg stabilizes that portion of the mass which it instantaneously supports. For example immediately after heel strike, when the foot supports only small vertical loads, the joints may not be expected to stabilize deviations of the total center of mass. For the purposes of this analysis, equation (8.20) will be applied to the double-support portion of stance using the net force vector across the joint rather than the weight above the joint. Exactly how the body weight is transferred from one leg to the other via stabilization mechanisms during the double-support part of the stance phase can only be addressed with a two-leg lower extremity model which is beyond the scope of this thesis. It is interesting to note, however, that pressure measurements from an instrumented hip prosthesis have shown considerable joint loads occurring before heel strike [12], and thus increased joint stabilization is probably not related to joint loads directly.

In order to estimate the total body weight and the location of the total body center of mass for use in equation (8.20), an additional set of static kinematic data is collected with the subject standing on the force plate. The total mass can be estimated using the measured force vector. The location of the total body center of mass relative to the pelvis can be estimated using the measured force and moment vectors plus an estimate for the vertical component of the center of mass location. The individual segment masses and center of mass locations are calculated as described in section 5.4.3.

8.4.2 Assumptions and Approximations in the Stability Analysis

The stability analysis described here and in Chapter 3 includes several approximations and limitations. These will be summarized here for reference later in this chapter and in Chapter 9 when the results are discussed. Approximations related to quasi-static
assumptions include both the requirement for stability of quasi-static equilibrium (equation (3.23)) and the method of estimating the dependence of joint loads on the kinematic configuration (equations (8.19) and (8.20)). Modeling total incremental stiffness as a quadratic function of muscle force is an additional approximation (equation (3.42)). For the purposes of analyzing the stance phase of gait, the limitation of these methods is that they are strictly applicable only to the single-support portion of the stance phase. These restrictions will obviously have an impact on how the results of adding stability constraints to a muscle force optimization analysis are evaluated.

8.5 Muscle Force Optimization

The purpose of the muscle force optimization procedures is to select the "best" set of muscle forces which satisfy all of the required constraints equations. In order to avoid unnecessary wordiness, "optimization" will always refer to muscle force optimization throughout this section. The general form of the optimization problem and the constraint equations are listed below. Several aspects of the optimization procedures and output results are then explained in sections 8.5.1 and 8.5.2.

The purpose of the optimization procedures is to minimize

\[
\text{penalty function} = \sum_{k=1}^{NM} \left( \frac{F_{Mk}}{F_{M,MAXk}} \right)^p
\]  

(8.21)

where \(F_{Mk}\) is the force on muscle \(k\), \(F_{M,MAXk}\) is the maximum force which muscle \(k\) can produce, and \(p\) is an integral power, either two or three. Since the maximum muscle force is assumed to be proportional to physiological cross section area, the penalty function to be minimized is the sum of the muscle stresses squared or cubed. This form of cost function is selected because it has been used by most previous muscle force optimization models based on muscle endurance and other criteria.

The constraints which the muscle forces must satisfy are summarized using the
following equations.

\[ a^T_{\text{EQ}_k} F_M = b_{\text{EQ}_k}, \quad k = 1, 2, \ldots, \text{nEQ} \]  
(8.22)

\[ 0 \leq F_{Mk} \leq F_{M,k,\text{MAX}}, \quad k = 1, 2, \ldots, \text{NM} \]  
(8.23)

\[ a^T_{\text{CK}_k} F_M \geq b_{\text{CK}_k}, \quad k = 1, 2 \]  
(8.24)

\[ f_{\text{STAB}_k}( F_M ) \geq 0, \quad k = 1, 2, \ldots, \text{nSTAB} \]  
(8.25)

In equation (8.22) \( F_M \) is the vector of muscle forces, and \( a_{\text{EQ}_k} \) and \( b_{\text{EQ}_k} \) define the equilibrium constraints for the muscle forces. If equilibrium is required to be satisfied for all of the degrees of freedom of the lower extremity, then the number of equality constraints \( \text{nEQ} \) is nine. In equation (8.24) \( a_{\text{CK}_k} \) and \( b_{\text{CK}_k} \) define the non-negativity constraints for the knee contact forces. Equation (8.25) represents the \( \text{nSTAB} \) nonlinear stability constraints for a positive definite stiffness matrix. For an optimization analysis in which the ligament material properties are estimated and equilibrium and stability constraints are required for each degree of freedom, then \( \text{nEQ} \) and \( \text{nSTAB} \) would both equal nine while \( \text{NM} \) would equal 36, the number of muscles in the lower extremity model. The actual values of \( \text{nEQ} \), \( \text{NM} \), and \( \text{nSTAB} \) which are used in the optimization analysis may be altered to model ligaments as pseudo-muscles or to require equilibrium or stability satisfied for only a limited number of degrees of freedom.

8.5.1 Optimization Procedures

Before a nonlinear search procedure is applied to the penalty function, the constraint equations are transformed into a more convenient form. Using the equilibrium equation (8.22), \( \text{nEQ} \) of the muscle forces are expressed as linear combinations of the remaining muscle forces using Gaussian elimination. The best muscle forces to eliminate are selected on the basis of the pivot elements during the Gaussian elimination procedure. The remaining muscle forces are non-dimensionalized using the maximum muscle forces as scaling factors.
Equation (8.23) is then used to derive two sets of inequality constraints, one for the eliminated muscles and one for the remaining muscles. Note that non-dimensionalizing simplifies the constraints on the remaining muscles.

\[ 0 \leq c_{EQk}^T f_M + d_{EQk} \leq 1, \quad k = 1, 2, \ldots, nEQ \]  \hspace{1cm} (8.27)

\[ 0 \leq f_{Mk} \leq 1, \quad k = 1, 2, \ldots, (NM - nEQ) \]  \hspace{1cm} (8.28)

The terms in equation (8.27) are the eliminated dimensionless muscle forces expressed in terms of the remaining dimensionless muscle forces. Knee joint contact force and stability requirements can then also be written in terms of the dimensionless muscle forces \( f_M \).

\[ c_{Ck}^T f_M \geq d_{Ck}, \quad k = 1, 2 \]  \hspace{1cm} (8.29)

\[ g_{STABk}(f_M) \geq 0, \quad k = 1, 2, \ldots, nSTAB \]  \hspace{1cm} (8.30)

Finally, the penalty function is expressed in terms of the dimensionless muscle forces.

\[ \text{penalty function} = \sum_{k=1}^{NM - nEQ} f_{Mk}^p + \sum_{k=1}^{nEQ} (c_{EQk}^T f_M + d_{EQk})^p \]  \hspace{1cm} (8.31)

The reduced minimization problem described by equations (8.27) through (8.31) includes only \((NM-nEQ)\) degrees of freedom compared to \(NM\) degrees of freedom for the original problem. The best solution to the reduced optimization problem is obtained using a nonlinear gradient search minimization routine [35]. Reasonable initial estimates for the final muscle dimensionless force values substantially improve convergence rate. For this reason, a linear programming solution to the problem is estimated using the Simplex algorithm [52]. A linear programming solution may be obtained very quickly, but the
algorithm is not designed to handle nonlinear constraints or penalty functions. For this reason, the nonlinear stability constraints are excluded from the linear programming problem, and the penalty function is approximated using the equation below:

\[
\text{linearized penalty function} = \sum_{k=1}^{nM} f_{Mk} + \sum_{k=1}^{nEQ} c_{EQk}^T f_{M} + d_{EQk} \quad (8.32)
\]

If a solution to the linear programming problem exists, then it is used as the initial estimate for the dimensionless forces for the nonlinear optimization. If no feasible solution exists for the linear problem then no solution can exist for the nonlinear problem, because both problems include equations (8.27) through (8.29) as linear inequality constraints. The output results for the case of an infeasible solution are labelled invalid.

In the nonlinear minimization procedure, additional terms are included in the penalty function when the inequality constraints of equations (8.27) through (8.30) are violated. The impossibility of satisfying equilibrium for all degrees of freedom of the lower extremity for some data sets has been briefly mentioned and will be discussed further in sections 8.6 and 8.7. For now it should be mentioned that terms are also added to the penalty function in case of errors in the equilibrium equations which have not been constrained to be satisfied exactly. The user is required to specify a weighting factor which determines how much these errors will affect the penalty function.

8.5.2 Output of the Optimization Methods

When a feasible solution exists to the optimization problem and convergence is obtained, the final values of the dimensionless muscle forces are used to calculate the eliminated dimensionless muscle forces using the expression in equation (8.27). All of the muscle forces are then scaled using the maximum force values. The knee joint contact loads are then calculated using the expressions derived from equation (8.15). Errors in the equilibrium equations for the four degrees of freedom of the knee are also calculated and stored, as well as the final estimated joint stiffness matrix. The processing information, including all user-selected weighting factors, file names, frame numbers, etc. are recorded.
in the header portion of the output file.

An additional data compression program was developed which combines the output from the raw and corrected kinematics, dynamics, and muscle force optimization analyses into a single file. This file is stored in a format which allows plotting any number of variables with respect to time or with respect to any of the other variables. The format is also compatible with a graphics program for displaying the articular surface geometry, anatomical landmarks, and ligament lines of action corresponding to the in vivo kinematic data. Two or more compressed files can then be merged for comparing the results of kinematic and dynamic analyses of different data sets (with time scaled by percent of stance phase) or for the same data set processed with different parameters. The kinematic and dynamic data which can be displayed are summarized below.

Kinematic Data:
- corrected knee kinematics
- knee contact point locations
- knee contact point unit normal directions
- corrections in tibia kinematics (soft tissue motion estimates)
- corrections in femur kinematics (soft tissue motion estimates)
- ligament strains

Dynamic Data:
- muscle forces
- force plate forces and moments
- knee joint net force and moment vectors
- knee joint contact forces
- errors in knee dynamic equilibrium
- knee joint stiffness matrix
- ligament forces
8.6 Results

The results of the dynamic analysis of the knee joint include estimates of all 36 muscle forces, five ligament forces, and two joint contact forces for the knee joint. Additionally, the results include estimates of hip and ankle joint forces. As the major emphasis of this thesis is on the knee joint, only those results which are directly to the knee will be included. Thus, the ankle and hip joint loads will not be presented, and only the thirteen muscles which cross the knee will be included in the discussion. The results shown in this chapter correspond to the sets of kinematic data presented in Chapter 7, and thus correspond to the stance phase of gait for one subject for three different trials.

8.6.1 Net Joint Forces and Moments

A typical set of the measured net forces and moments which act across the knee throughout the stance phase of gait are shown in figures 8.2 and 8.3. By convention, the net force and moment represent the total loads due to the femoral condyles, muscles, and ligaments acting on the proximal portion of the tibia. The results are calculated in the coordinate system of the tibia to make them independent of the direction of walking. Therefore, the large anterior force late in the stance phase (figure 8.2) is due to the vertical load across the joint while the tibia is leaning forward prior to toe-off. The forces are dominated by the vertical component of the force plate ground reaction force.

An important note concerning the knee moment components should be made here. Andriacchi and Strickland [5] measured the net moments across the lower extremity joints for over 300 subjects and classified them according to percentages of subjects who exhibited different moment patterns. For the knee joint, many subjects were measured to produce a large positive flexion moment and then a smaller negative component in this direction during the stance phase of gait. Other subjects were observed to have almost no positive flexion moments throughout the stance phase but a large negative component (i.e. an extension moment) near toe-off. Figure 8.3 demonstrates that the subject of these experiments exhibited the latter type of moment patterns. This point will be repeated later when the results of the analyses in this thesis are compared to results of other researchers.
Figure 8.2: Knee Joint Net Forces

Figure 8.3: Knee Joint Net Moments
8.6.2 Muscle and Joint Forces

Figure 8.4 shows the estimated forces for the thirteen knee muscles during the stance phase of gait. Included in the muscle plots are horizontal lines located at a force of one body-weight which represent the temporal patterns of myoelectric (EMG) activity for the leg muscles during gait as reported by the University of California at Berkeley [62]. The agreement between the estimated muscle forces and the reported temporal EMG patterns is reasonable for many of the muscles, but the Vastus Lateralis, an extensor muscle, has large forces later in stance which are missing in the corresponding EMG pattern. This may be due to the knee flexion moment pattern described in the previous section. Obviously, the flexion moment will dominate the activity of the flexor and extensor muscles of a joint. Since there is such a large variation in moment patterns, then one should not expect all muscle force patterns to be similar.

Figures 8.5 and 8.6 show the estimated forces on the medial and lateral condyles for

Figure 8.4: Knee Muscle Forces
Figure 8.5: Knee Medial Contact Force

Figure 8.6: Knee Lateral Contact Force
three separate sets of kinematic and dynamic TRACK data. Figure 8.7 includes the forces on the medial and lateral condyles as well as the vertical force on the force plate for one set of data. The body weight of the subject of these experiments is approximately 715 N, so the peak load on the medial condyle corresponds to about 2.5 time body weight (BW) and the lateral condyle has an estimated peak load of almost 2.0 BW. Near toe-off, this subject is estimated to have a total joint contact load of almost 4.5 BW. Recall from Chapter 1 that previous knee models have estimated knee joint loads between 3.0 BW and 9.0 BW for the stance phase of gait [14], [24], [42], [43]. Cheng [14] used a knee model somewhat similar to the one developed in this thesis with 13 muscles to estimate the total joint load at one point in the stance phase of gait to be 4.3 BW, with 3.0 BW on the medial condyle.

![Figure 8.7: Knee and Foot-Floor Forces](image-url)
8.6.3 Ligament Forces

The collateral ligaments are estimated to have very small loads during the stance phase. Estimated forces on the anterior and posterior cruciate ligaments are shown in Figure 8.8. These forces are calculated by including the ligaments in the muscle force optimization such that their forces are estimated along with those of the muscles. Relative to the strengths of the knee muscles, the maximum forces on the ligaments are small. As the cruciate ligaments are close to the center of the knee joint, their forces do not contribute significantly to the joint moments. In the muscle force optimization analysis the cruciate ligaments serve the purpose of helping to satisfy equilibrium, particularly in the anterior-posterior direction. If one compares the results of the ligament forces in Figure 8.8 with the ligament strains shown in Figure 8.1, it is apparent that the cruciate ligaments are estimated to have the largest forces when their lengths are the smallest. One could plot the ligament forces versus strains to estimate the stiffness. Unfortunately in this case the cruciate ligaments would be predicted to have negative stiffness values. The knee model is therefore not

![Figure 8.8: Cruciate Ligament Forces](image-url)
accurate in estimating forces in the anterior-posterior direction. However, considering that the total joint load is estimated to exceed 3000 N, then the 300 N maximum force on the anterior cruciate ligament could be accounted for by a change of about 5 degrees in the direction of this force. More discussion of equilibrium errors will follow later in this chapter and in Chapter 9.

8.6.4 Effects of Hip and Ankle Constraints on Knee Forces

The muscle force optimization procedures require equilibrium to be satisfied for all nine degrees of freedom of the lower extremity. Is it absolutely essential to include the hip and ankle joints in this analysis? Figure 8.9 compares two results of estimated knee joint forces for the same set of kinematic and force plate data. One set of data was processed requiring equilibrium to be satisfied for all nine lower extremity degrees of freedom, while the other set was processed only requiring equilibrium for the four degrees of freedom of the knee. The lower two curves in the figure are the forces on the lateral condyle and show

![Figure 8.9: Effects of Hip and Ankle Constraints](image)
a substantial increase in joint load at 20 percent of the stance phase when ankle and hip constraints are included. The upper two curves show similar results for the medial condyle. Eliminating the requirements of equilibrium at the ankle and hip reduces the estimated joint contact loads by about 30 percent. Therefore, an analysis which does not include the hip and ankle constraints may have large errors. The knee model used by Cheng [14] included only equilibrium at the knee yet still produced joint loads of very similar magnitude to those estimated by the knee joint model of this thesis. Apparently, significant differences must exist between the muscle moment arms or net joint loads assumed by Cheng and measured in this thesis.

8.6.5 Effects of Stability Constraint

During the central part of the stance phase of gait (which corresponds to single limb support) the joint stiffness in the flexion direction of the knee is estimated to be negative, especially during the peak loads. Figure 8.10 displays the results of including a constraint

![Graph showing knee joint flexion stiffness](image)

**Figure 8.10: Knee Joint Flexion Stiffness**
to require the flexion stiffness to be non-negative. Two different curves are shown in this figure. The bottom curve represents the estimated joint stiffness with no stability constraints in the muscle force optimization. The other curve represents the flexion stiffness when a stability constraint is included. Unfortunately, no set of muscle forces was found which could satisfy equilibrium and also produce a non-negative flexion stiffness value. Recall from Chapter 3 that the assumed relationship between total incremental stiffness and force of muscles is not monotonic but rather falls off after reaching a peak. For this reason, increasing the co-contraction level above the already increased level would not be predicted to increase the joint stiffness.

Figure 8.11 shows the corresponding predicted joint loads for these same three data sets. These curves are presented to demonstrate that the addition of the stability constraint does require additional muscular co-contraction and thus increases the joint loads. Given the results of figure 8.10, the stability analysis clearly does not work completely. The assumed relationship between muscle stiffness and force is unsatisfactory. This is not completely surprising since the function was extrapolated from the results of a single
muscle from the ankle of a cat and scaled for all different muscles using maximum force and rest length. Numerically, this procedure does not perform well. However, the author is convinced that this approach is conceptually sound and that it could provide interesting results if if appropriate values for muscle stiffnesses are obtained.

8.6.6 Equilibrium Errors

The muscle force optimization procedures are designed to satisfy equilibrium at all degrees of freedom of the lower extremity joints. Unfortunately, a feasible solution does not always exist. In this case, the closest solution is selected by penalizing equilibrium errors. The two degrees of freedom of the lower extremity which typically present difficulties in satisfying equilibrium are the medial-lateral and anterior-posterior translations of the knee joint. Figure 8.12 shows the equilibrium errors associated with a typical data set. The large force equilibrium error in the medial direction can be explained by considering the joint contact force directions. Figure 8.13 shows the estimated knee kinematics for the largest medial force error. Note that the unit normal associated with the medial force (i.e. the one on the right in the figure) has a significant component in the medial direction. Since the joint contact force on this condyle is instantaneously about 2000 N, then for each 1 degree of change in the unit normal direction the medial force will change by about 35 N. Note also that the medial condyle and medial tibial plateau are relatively congruent in that only a small displacement of the femur would correspond to a fairly large change in the contact normal direction. Unfortunately, thus far the knee kinematics have been estimated completely ignoring the dynamics and joint loads. Therefore, the estimated location of the joint contact point has been obtained without regard to dynamics. Chapter 9 describes methods for improving the estimated kinematics by considering the effects on equilibrium errors.
Figure 8.12: Force Equilibrium Errors

Figure 8.13: Kinematics with Medial Force Errors
8.6.7 Sensitivity to Quadriceps Insertions

The quadriceps muscles, including the rectus femoris and the three vasti muscles, insert into the tibia through the patella and patellar ligament. For the subject of the experiments in this thesis, these muscles are estimated to have the largest loads. In order to determine the sensitivity of the results to the estimated insertion location (i.e. the tibial tuberosity), the muscle force optimization procedures were repeated with the assumed quadriceps muscle insertions translated by 4 mm in either the anterior or lateral direction. Figure 8.14 shows results of the joint contact forces estimates for changing the anterior component of the insertion by 4 mm. Figure 8.15 shows similar results except for a 4 mm lateral shift of the insertion. For the anterior shift, the flexion moment arms of the extensor muscles is increased and the required maximum joint loads decreases slightly. For the lateral shift of the insertion location, the total joint load remains about the same, but the percentage of the load carried by the medial condyle is significantly reduced and is transferred to the lateral condyle. Since the medial-lateral location of the insertion of the patellar ligament is relatively difficult to estimate, the percentages of the knee joint load supported by each of

![Graph showing knee contact forces (N) vs. percent of stance phase.]

Figure 8.14: Sensitivity to Anterior Displacement of Quadriceps
Figure 8.15: Sensitivity to Lateral Displacement of Quadriceps

The two contact forces may be significantly affected by estimated insertions. However, the total joint load is insensitive to these changes and thus can be estimated accurately.
Chapter 9

SIMULTANEOUS ESTIMATION OF KNEE MODEL KINEMATICS AND MUSCLE AND JOINT FORCES

9.1 Introduction

In Chapter 8, the difficulties encountered in satisfying equilibrium for all degrees of freedom of the lower extremity model were attributed to small errors in the kinematic estimates which may produce large errors in the directions of the knee joint contact forces. A method is presented in this chapter for estimating knee kinematics which include corresponding joint contact force directions that minimize errors in dynamic equilibrium. An iterative procedure is then presented for alternately estimating kinematics and muscle and joint forces until convergence is obtained. The conclusion associated with this analysis is that in vivo knee kinematics should be estimated including not only geometrical constraints but constraints of dynamic equilibrium as well.

9.2 Equilibrium Equations for Net Force and Moment Vectors

The force and moment equilibrium equations for the foot are given below.

\[ \mathbf{F}_{FP} + m^1 \mathbf{g} + \mathbf{F}_{NET}^2 = m^1 \mathbf{a}^1 \]  
\[ \mathbf{M}_{FP} + \left( \mathbf{x}_{FP} - \mathbf{x}_{CG}^1 \right) \times \mathbf{F}_{FP} + \mathbf{M}_{NET}^2 + \left( \mathbf{x}^2 - \mathbf{x}_{CG}^1 \right) \times \mathbf{F}_{NET}^2 = I^1 \mathbf{a}^1 + \omega^1 \times I^1 \omega^1, \]  

In these equations, \( \mathbf{F}_{FP} \) and \( \mathbf{M}_{FP} \) are the force and moment vector which act on the foot at location \( \mathbf{x}_{FP} \), \( \mathbf{F}_{NET}^2 \) and \( \mathbf{M}_{NET}^2 \) are the net force and moment vectors acting on the foot at the origin of the coordinate system of segment 2, and \( \mathbf{a}^1 \), and \( \omega^1 \) are the translational and angular accelerations of the foot (segment 1).

These equations may be written in a slightly different form to express the dependence
of \( F_{NET}^2 \) and \( M_{NET}^2 \) on the other terms.

\[
F_{NET}^2 = m^1 a^1 - m^1 g - F_{FP} \quad (9.3)
\]

\[
M_{NET}^2 = I^1 \alpha^1 + \omega^1 \times (I^1 \omega^1) - (x^2 - x_{CG}^1) \times F_{NET}^2 - M_{FP} - (x_{FP} - x_{CG}^1) \times F_{FP} \quad (9.4)
\]

Equations similar to (9.3) and (9.4) can be written for \( F_{NET}^3 \) and \( M_{NET}^3 \), the net force and moment vector of segment 3 acting on segment 2. By convention, these net loads across the knee joint are assumed to act at the origin of segment 3.

\[
F_{NET}^3 = m^2 a^2 - m^2 g + F_{NET}^2 \quad (9.5)
\]

\[
M_{NET}^3 = I^2 \alpha^2 + \omega^2 \times (I^2 \omega^2) - (x^3 - x_{CG}^2) \times F_{NET}^3 + M_{NET}^2 + (x^2 - x_{CG}^2) \times F_{NET}^2 \quad (9.6)
\]

By combining equations (9.3) and (9.5), an expression for the net force across the knee joint as a function of segment accelerations and the force plate force can be obtained.

\[
F_{NET}^3 = m^2 (a^2 - g) + m^1 (a^1 - g) - F_{FP} \quad (9.7)
\]

Before writing a similar equation for the net moment vector across the knee joint, two preliminary expressions should be reviewed from previous chapters. The GCS coordinates of the centers of mass for segments 1 and 2 can be obtained from their anatomical coordinates \( r_{CG}^1 \) and \( r_{CG}^2 \) plus the segment translations \( x^1 \) and \( x^2 \) and rotations \( R^1 \) and \( R^2 \). Note that by convention in this thesis, the origin of the coordinate system of segment 1 is located at the ankle joint.

\[
x_{CG}^1 = R^1 r_{CG}^1 + x^1 = R^2 R^1 r_{CG}^1 + R^2 r_{ANKLE}^1 + x^2 \quad (9.8)
\]

\[
x_{CG}^2 = R^2 r_{CG}^2 + x^2 \quad (9.9)
\]
Equations (9.4) and (9.6) can be combined and substituted into equations (9.8) and (9.9) to produce the following equations for the net moment acting across the knee joint.

\[ M_{\text{NET}}^3 = M_0 + R^2 \dot{M} + x^2 \times \hat{F} \cdot x^3 \times F_{\text{NET}}^3 \]  \hfill (9.10)

\[ M_0 = I^2 \alpha^2 + \omega^x \times (I^2 \omega^s) + I^1 \alpha^1 + \omega^1 \times (I^1 \omega^1) - M_{FP} \cdot \mathbf{x}_{FP} \times F_{FP} \]  \hfill (9.11)

\[ \dot{M} = r_{\text{CG}}^2 \times \left( F_{\text{NET}}^3 - F_{\text{NET}}^2 \right) + \left( r_{\text{ANKLE}}^2 + R^2 R^1 r_{\text{CG}}^1 \right) \times \left( F_{\text{NET}}^2 + F_{FP} \right) \]  \hfill (9.12)

\[ \hat{F} = F_{\text{NET}}^3 + F_{FP} \]  \hfill (9.13)

9.3 Equilibrium Equations for Muscle, Ligament, and Joint Forces

In the knee model of this thesis, the net force and moment vectors acting across the knee joint are produced by three contributions: muscle, ligament and joint contact forces. By defining \( u_{Mk} \) as the unit vector from the insertion to the origin of muscle \( k \), \( u_{Lk} \) as the unit vector from the insertion to the origin of ligament \( k \), and \( u_{Ck} \) as the unit vector of knee contact force \( k \) pointing down into the tibia, the following expression can be written for the net knee joint force.

\[ F_{\text{NET}}^3 = \sum_{k=1}^{NM} F_{Mk} u_{Mk} + \sum_{k=1}^{NL} F_{Lk} u_{Lk} + \sum_{k=1}^{NC} F_{Ck} u_{Ck} \]  \hfill (9.14)

Defining \( x_{Mk}^O \) and \( x_{Lk}^O \) as the GCS coordinates of the origin of muscle \( k \) and ligament \( k \), respectively, plus \( x_{Ck}^O \) as the GCS coordinates of knee joint contact force \( k \) allows the expression of the net moment across the knee joint in a similar form.

\[ M_{\text{NET}}^3 = \sum_{k=1}^{NM} F_{Mk} \left( x_{Mk}^O - x^3 \right) \times u_{Mk} + \sum_{k=1}^{NL} F_{Lk} \left( x_{Lk}^O - x^3 \right) \times u_{Lk} + \sum_{k=1}^{NC} F_{Ck} \left( x_{Ck}^O - x^3 \right) \times u_{Ck} \]  \hfill (9.15)
Several terms in equations (9.14) and (9.15) depend on the position vectors \( \mathbf{x}^2 \) and \( \mathbf{x}^3 \) and rotation matrices \( R^2 \) and \( R^3 \) of segments 2 and 3 (i.e. the tibia and femur). The unit vectors of the muscles obviously depend on the muscle origins and insertions.

\[
\mathbf{u}_{\mathbf{Mk}} = \frac{\mathbf{x}_{\mathbf{Mk}}^O \cdot \mathbf{x}_{\mathbf{Mk}}^I}{\| \mathbf{x}_{\mathbf{Mk}}^O \cdot \mathbf{x}_{\mathbf{Mk}}^I \|}
\]

(9.16)

By defining \( \mathbf{p}_{\mathbf{Mk}}^O \) as the instantaneous femur coordinates of the origin of muscle \( k \) and \( \mathbf{p}_{\mathbf{Mk}}^I \) as the tibia coordinates of the corresponding insertion, the GCS coordinates of the muscle origins and insertions can be expressed as shown below.

\[
\mathbf{p}_{\mathbf{Mk}}^O = R^3 \mathbf{T} \left( \mathbf{x}_{\mathbf{Mk}}^O \cdot \mathbf{x}^3 \right)
\]

(9.17)

\[
\mathbf{p}_{\mathbf{Mk}}^I = R^2 \mathbf{T} \left( \mathbf{x}_{\mathbf{Mk}}^I \cdot \mathbf{x}^2 \right)
\]

(9.18)

\[
\mathbf{x}_{\mathbf{Mk}}^O = R^3 \mathbf{p}_{\mathbf{Mk}}^O + \mathbf{x}^3
\]

(9.19)

\[
\mathbf{x}_{\mathbf{Mk}}^I = R^2 \mathbf{p}_{\mathbf{Mk}}^I + \mathbf{x}^2
\]

(9.20)

Equations comparable to (9.16) through (9.20) could similarly be obtained for the ligaments. The GCS coordinates of the unit vector \( \mathbf{u}_{\mathbf{Ck}} \) and location \( \mathbf{x}_{\mathbf{Ck}} \) of the joint contact force can be written in terms of the femur coordinates \( \mathbf{n}_{\mathbf{k}} \) and \( \mathbf{r}_{\mathbf{f}} \) of these two same quantities.

\[
\mathbf{u}_{\mathbf{Ck}} = R^3 \mathbf{n}_{\mathbf{k}}
\]

(9.21)

\[
\mathbf{x}_{\mathbf{Ck}} = R^3 \mathbf{r}_{\mathbf{f}} + \mathbf{x}^3
\]

(9.22)

Substitution of equations (9.17) through (9.22) into equation (9.15) produces an equation which shows more of the dependence of the knee joint net moment vector on the rotation...
matrix of segment 3. Note, however, that for simplicity the following equation does not explicitly show the dependence of the muscle and ligament unit vectors on the rotation matrices and position vectors.

\[
M_{NET}^3 = \sum_{k=1}^{NM} F_{MK} \left( R^3 \hat{p}_{MK}^O \times u_{MK} \right) + \sum_{k=1}^{NL} F_{LK} \left( R^3 \hat{p}_{LK}^O \times u_{LK} \right) + \sum_{k=1}^{2} F_{Ck} R^3 r_{nk} \times n_{nk} \quad (9.23)
\]

### 9.4 Equilibrium Errors for the Knee Joint

Equations (9.7) and (9.14) are both expressions of the same quantity, the GCS coordinates of the net force vector acting across the knee joint. Similarly, equations (9.10) and (9.15) are both expressions for the GCS coordinates of the net knee joint moment calculated about the origin of the coordinate system of the femur. These expressions must therefore be exactly equal if equilibrium is satisfied for each degree of freedom of the knee. Although in reality equilibrium is always satisfied, for specified values of muscle, ligament and joint contact forces and assumed joint kinematics the calculated equilibrium equations may not be exactly satisfied for all of the degrees of freedom. The equations below show the equilibrium errors in the net force and moment vector acting across the knee joint as a function of previously defined terms.

\[
\Delta F_{NET}^3 = \left( F_{NET}^3 \right)_0 - \sum_{k=1}^{NM} F_{MK} u_{MK} - \sum_{k=1}^{NL} F_{LK} u_{LK} - \sum_{k=1}^{2} F_{Ck} R^3 n_{nk} \quad (9.24)
\]

\[
\Delta M_{NET}^3 = M_0 + R^2 \hat{M} + x^2 \times \hat{F} - x^3 \times \left( F_{NET}^3 \right)_0 - \sum_{k=1}^{2} F_{Ck} R^3 r_{nk} \times n_{nk},
\]

\[
- \sum_{k=1}^{NM} F_{MK} \left( R^3 \hat{p}_{MK}^O \times u_{MK} \right) - \sum_{k=1}^{NL} F_{LK} \left( R^3 \hat{p}_{LK}^O \times u_{LK} \right) \quad (9.25)
\]

Consider the case in which the ankle and hip joints are fixed and the kinematics of the tibia and femur are allowed to change by small amounts. The force plate force and moment vectors and center of pressure are assumed to remain fixed at their measured values. Assume that changes in the segment translational and rotational accelerations are negligible.

In this case, the following terms would remain constant: \(M_0, \hat{M}, \hat{F}, F_{NET}^3, \hat{p}_{MK}^O\) and \(p_{LK}^O\)
If additional values were specified for muscle and ligament forces, then the error terms of equations (9.24) and (9.25) would depend on the tibia and femur kinematics, the parametric coordinates of the articulating surfaces, and the condyle contact force magnitudes $F_C$. Recalling the definitions of $q$ (generalized coordinates for the lower extremity) and $s$ (eight parametric coordinates plus two dependent kinematic coordinates of the knee) from Chapter 7, equations (9.24) and (9.25) can be expressed in the following equations as functions of $q$, $s$, and $F_C$.

\[
\Delta F_{NET}^3(q, s, F_C) = F_{NET}^3 - \sum_{k=1}^{NM} F_{Mk} u_{Mk} - \sum_{k=1}^{NL} F_{Lk} u_{Lk} - \sum_{k=1}^{2} F_{Ck} R^3_{nk}(9.26)
\]

\[
\Delta M_{NET}^3(q, s, F_C) = M_0 + R^2_M + x^2 \times \hat{F} \cdot x^3 \times F_{NET}^3 - \sum_{k=1}^{2} F_{Ck} R^3_{nk} \times n_{nk}(9.27)
\]

### 9.5 Iterative Estimation of Kinematics and Muscle and Joint Forces

Assume that the procedures for estimating kinematics for the lower extremity model have been performed as described in Chapter 7. Additionally, assume that muscle and joint contact forces have been estimated using the methods of Chapter 8 without requiring equilibrium to be exactly satisfied for all four degrees of freedom of the knee joint. This would typically be the case if no feasible solutions exist which satisfy all of the equality and inequality constraints in the optimization procedures. Note that the dynamic calculation routines determine the net force and moment vectors across the ankle and knee joints as required intermediate values in the calculations. Using this information, the values of $\hat{M}$, $\hat{F}$, $F_{NET}^3$, $\hat{p}_{Mk}$, and $\hat{p}_{Lk}$ could be evaluated for the estimated lower extremity kinematics. $M_0$ could then be evaluated by substituting the values of these terms into equation (9.10). All of the information would therefore be available to evaluate the equilibrium errors as defined in equations (9.26) and (9.27).
If the estimated muscle and ligament forces are held constant, then the errors associated with equations (9.26) and (9.27) would represent incompatibility of the knee kinematics and dynamic equilibrium equations. For different sets of knee kinematics specified by $q$ and $s$, the equilibrium errors would in general be different. It is therefore possible to calculate a value for an equilibrium error penalty function associated with each set of estimated kinematics. Using this approach, additional terms may be included in the penalty function of Chapter 7 for selecting the best set of geometrically compatible kinematics, as symbolically shown below.

$$f_{\text{TOTAL}}(q,s,F_C) = f_{\text{KINEMATICS}}(q,s) + f_{\text{EQUILIBRIUM}}(q,s,F_C) \quad (9.28)$$

Notice in equation (9.28) that the augmented penalty function depends on two additional variables, the magnitudes of the knee joint contact forces. Although it would definitely be possible to hold these two joint forces constant and just optimize for the kinematics, inclusion of the joint forces only minimally increases the complexity of the optimization procedures. Furthermore, if the joint contact force magnitudes are maintained constant, then changes in the contact force directions may cause large equilibrium errors. As one of the major reasons for adding these terms is to force the kinematic correction method to improve its estimates of joint contact force directions, the joint contact forces must be allowed to vary. Therefore, a kinematic correction procedure was implemented which includes additional penalty function terms associated with equilibrium errors as defined by equations (9.26) and (9.27). The number of degrees of freedom of the nonlinear optimization problem is increased from 20 ($q$ and $s$) to 22 ($q$, $s$, and $F_C$). The output of this procedure is an estimate of lower extremity kinematics which is both compatible with the articular geometry and as close as possible to satisfying the force and moment equilibrium equations for the knee.

Once the updated kinematics are available, then they are used in the dynamics and muscle force optimization analyses to estimate the muscle and joint contact forces for the slightly changed kinematics. The updated muscle force estimates can then be used in another iteration of the kinematic correction procedure which penalizes equilibrium errors.

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These two methods can be used iteratively until the joint contact forces estimated by both methods are similar. As these procedures are iteratively applied, it may eventually be possible to include constraints in the muscle force optimization routine to require equilibrium to be satisfied for additional degrees of freedom of the knee. Specific results of applying the described iterative methods to kinematic and dynamic data corresponding to the stance phase of gait are presented in section 9.6.

9.6 Results of Iterative Estimation of Kinematics and Forces

Figures 9.1 and 9.2 show the improvements made by including the iteration procedure for estimating kinematics to minimize equilibrium errors. Although this procedure has been described as iterative, the results actually converge almost completely in one iteration. Judging from these two figures, equilibrium errors are much easier to correct in the medial-lateral direction by this method than are errors in the anterior-posterior direction. Figure 9.3 shows the improved knee kinematics for the frame in figure 9.2 which corresponds to

![Graph showing reduction of anterior force equilibrium errors](image)

**Figure 9.1: Reduction of Anterior Force Equilibrium Errors**
Figure 9.2: Reduction of Medial Force Equilibrium Errors

Figure 9.3: Kinematics for Reduced Equilibrium Errors
the largest medial force error. This is also the same frame of data as in figure 8.13. Note that the medial component of the unit normal has been substantially reduced.

If this procedure works so well for medial equilibrium errors, then why can it not work completely for anterior force errors? Two major factors restrict the ability of this method to eliminate errors in anterior force equilibrium. First, the muscle forces which are estimated in the muscle force optimization have been calculated to guarantee flexion moment equilibrium for the initial joint contact locations. Therefore, if the muscle forces are held constant and the estimated contact location is moved in the anterior direction, errors will occur in the flexion moment equilibrium. Since errors in the moments are penalized as well as errors in the forces, very little anterior translation will be allowed to occur. Another reason for the inability of the methods to converge in the anterior direction is that the curvature of the tibial surface is much smaller in the anterior-posterior direction than it is in the medial-lateral direction. Thus a relatively large translation is required to significantly affect the anterior component of the joint contact load.

9.7 Simultaneous Estimation of Kinematics and Forces

Problems with estimating knee joint kinematics without regard to dynamics have been described in Chapter 8, and a method to remedy this problem has been presented in this chapter. Why not just simultaneously estimate the joint kinematics and joint and muscle forces which somehow minimize required changes in the kinematics and also satisfy equilibrium for all of the degrees of freedom of the lower extremity? In this way, the problem of equilibrium errors could be completely avoided. In order to at least analyze the feasibility of this type of approach, procedures were implemented to simultaneously estimate the kinematics and joint forces of the knee without regard to equilibrium at the ankle or hip joints. As mentioned in Chapter 8, excluding the equilibrium constraints at the ankle and hip may produce joint contact forces underestimated by about 30 percent. Therefore, this procedure was implemented only as a proof of concept with the results compared to the standard kinematics and muscle force optimization analyses without the ankle or hip constraints.
Figures 9.4 and 9.5 demonstrate that this method is able to substantially reduce the equilibrium errors not only in the medial direction but also in the anterior direction. Due to time constraints only a preliminary version of this algorithm was implemented, and it was not implemented to be robust. Hence, in figures 9.4 and 9.5 one may notice that the estimated optimal equilibrium errors actually increase at the very end of the data set. This is due both to not carefully selecting convergence parameters and to allowing the routine to search in the invalid region of the geometry data. If care is taken to avoid these problems, then an implementation could be developed to simultaneously estimate the knee joint kinematics and dynamics and account for all equilibrium errors.

![Graph showing further reduction of anterior force errors over the percent of stance phase.](image)

**Figure 9.4: Further Reduction of Anterior Force Errors**
Figure 9.5: Further Reduction of Medial Force Errors
Chapter 10

CONCLUSION

10.1 Conclusions

The objective of this thesis was to develop a mathematical model of the human knee joint which allows an examination of relationships between geometry, kinematics, and dynamics of the joint in vivo and also to examine sensitivity of the results to changes in selected modeling and processing parameters. This goal has been accomplished by the development of a set of computer programs for combining measurements of lower extremity kinematics and foot-floor interactions to estimate kinematics and muscle and joint forces for a nine degree of freedom lower extremity model, including a four degree of freedom knee joint model.

The major accomplishments of this thesis include the following:

- the development of a simple, accurate, and automatic method for using a kinematic measurement system to generate a geometric model of the articular surfaces of a cadaver tibia or femur, including scaling from in vitro measurements to a model of the in vivo geometry

- the development of a novel method to concurrently estimate and remove soft tissue motion errors of in vivo knee kinematics by including geometric constraints

- the development of a three-dimensional, four degree of freedom model of the human knee joint which includes muscles, ligaments, and articular surface geometry and which can be applied directly to in vivo kinematic and dynamic measurements
• the demonstration of both the necessity and the feasibility of simultaneously estimating kinematics and dynamics for a truly three-dimensional model of the human knee joint

• the derivation of a joint stiffness matrix and a stability analysis for human joints, including specifically how muscles must be included

• the application of this stability analysis to the human elbow joint to demonstrate relationships between required co-activation and joint stiffness and stability

The major emphasis of this thesis was obviously the development of the knee joint model and its application to sets of in vivo gait data. In contrast to all previous models of the knee joint which have been applied to in vivo data, the joint kinematics in this thesis are not estimated from cadaver motions but rather from direct measurements of a walking subject and a single reasonable assumption (i.e. that both condyles of the femur are in contact with the tibia during the stance phase of gait). This novel approach not only allows application of the knee joint model to a wide range of activities but also provides estimates of the quality of the kinematic data. Furthermore, the inclusion of the articular surface geometry in the knee model is not simply an equivalent alternative to estimating joint contact force locations as in previous knee models. The dependence of the joint contact locations and directions on the knee kinematics cannot be accurately estimated from any model which excludes joint geometry. Therefore, a joint stiffness or stability analysis should not be applied to any knee joint model without geometrical constraints.

One of the requirements for the knee joint model developed in this thesis is a model of the articular surface geometry of the tibia and femur. The methods developed in this thesis to obtain estimates of the geometry for the knee joint, including both the coordinate measurement procedure and the bicubic surface approximation, are nearly as accurate as much more complex methods which have been described in the literature. These methods could easily be implemented by anyone with a kinematic measurement system. The bicubic models of the surfaces are reasonably accurate and very compact representations of the surfaces compared to storing the coordinates of a complete mesh on the surface. As has
been explained in Chapter 6, for the purposes of a stability analysis the bicubic approximation is preferable for joint contact and stability calculations.

When applied to kinematic and foot-floor reaction measurements during the stance phase of gait, the knee model estimates peak loads of approximately 3.0 times body weight (BW) on the medial condyle and 1.5 BW on the lateral condyle. These estimates of the two joint contact forces are more sensitive to changes in medial-lateral locations of the assumed extensor muscle insertions than they are to equal changes in the anterior direction (i.e. the extension moment arms). However, the estimated total joint load is relatively insensitive to muscle attachments. Extremely large errors (30 percent) in the contact forces were estimated when the constraints of the hip and ankle joints were ignored.

The joint loads produced by this model are similar to estimates of previous knee joint models except the maximum joint force is estimated here to occur much later in the stance phase (i.e. near the second peak of the vertical foot-floor force rather than the first peak). A major factor in lower extremity muscle force optimization analyses is the assumed or measured knee flexion moment. Healthy normal human subjects walk with a large variety of different knee flexion moment patterns which are not even qualitatively similar (i.e. same number of peaks and valleys in the curves, etc) [5]. Hence it is not surprising that a knee model which is applied to the kinematic and dynamic gait data of an individual subject would produce results different from those which assume a stereotypical knee flexion moment pattern. For this reason, the dynamics and kinematics presented in this thesis are not intended to represent knee joint force or displacements for all human subjects but rather only for one subject. If the method were employed for another subject, slight variations may be expected both in the kinematics and dynamics.

The accuracy of the knee kinematic estimation procedure is between 2.0 and 6.0 mm, where the lower limit refers to the repeatability of the kinematic measurements and the upper limit corresponds to the maximum required changes in the kinematic measurements to obtain geometric compatibility. Since absolute measurements of the skeletal kinematics are not available for comparison, the upper limit is only an estimate which would be
affected by the assumed joint geometry. Judging from the large posterior cruciate ligament strain values, errors in the anterior-posterior displacements may be significant, but they cannot be estimated using these methods. Researchers studying lower extremity kinematics have reported the ability of motion measurement systems to track the coordinates of markers in space, but they have not reported accuracy of measurement of skeletal motion because soft tissue motion is so difficult to estimate. The method employed in this thesis for estimating soft tissue motion is not a direct measurement and requires an assumption of joint contact, but it is also non-invasive and requires only minimal additional measurements compared to standard gait analysis methods. Estimation of soft tissue motion by including joint geometry has never been accomplished previously. The knee kinematic data obtained in this thesis is likely the most accurate in vivo measurement of in vivo knee kinematics to date. Considering that nearly all other gait analysis systems measure knee motion by placing markers at presumed joint center locations and that in vivo knee motion includes translations in excess of 1 cm (e.g. see figure 7.7), even neglecting soft tissue motion these systems cannot match the accuracy of the knee kinematics estimated herein.

One of the most important conclusions of this thesis is that a three-dimensional model of the human knee joint should allow dynamics to influence the kinematic estimates if accurate equilibrium relations are desired for all of the degrees of freedom of the joint. Only two previous knee joint models have been applied to in vivo data which attempted to satisfy equilibrium in all directions. Cheng [14] estimated joint contact force locations and assumed stereotypical joint loads for one instant during the stance phase of gait. His model was unable to satisfy equilibrium for all of the degrees of freedom, so he selected the two most important directions to satisfy using the joint contact loads (i.e. the vertical force and the abduction moment). Garg [21] was able to satisfy equilibrium for all directions but did not present the calculated ligament forces during gait. In her analysis, the locations and directions of the contact forces were fixed in the equations to solve for the muscle and joint forces. Throughout much of the gait cycle only a single equivalent muscle was assumed to be active. Judging from the results of this thesis, the calculated ligament forces would very likely have been above physiologically maximum levels in order to satisfy equilibrium for all six directions. Methods were developed in this thesis to minimize equilibrium errors in the medial and anterior force directions and then to adjust the kinematics to further reduce
the errors. Additionally, a method for a simplified model of the knee (which neglected hip and ankle constraints) was implemented to demonstrate the feasibility of estimating knee joint kinematics and dynamics simultaneously. Only when these procedures for optimally satisfying equilibrium are included in the knee joint model can all net joint loads be accounted for by muscle, ligament, and joint forces.

If one considers a possible control strategy for the knee joint, the prospect of allowing the joint loads to influence small changes in the knee kinematics makes sense. The methods employed in this thesis to estimate knee kinematics and muscle and joint forces are similar to solving an optimal control problem. Given a nine degree of freedom model for the lower extremity and specified locations of the ankle and hip joints throughout the stance phase of the gait cycle, then what kinematics and muscle forces should be prescribed for the knee to minimize energy (or muscle forces)? This problem is fundamentally different from the standard muscle force optimization procedures which assume that the kinematic trajectory is fixed and that only the muscle forces can be selected. By altering the kinematics very slightly, not only the required muscle forces but also the resulting joint contact loads can be reduced. It does not seem reasonable that a control system would select a set of knee kinematics which requires larger muscle forces and joint loads than another set which accomplishes the same task (i.e. motion of the pelvis relative to the foot).

The allowance of slight variations in knee kinematics is not the only difference between the muscle force optimization in this thesis and previous methods. A solution to an optimal control problem would presumably correspond to a set of control commands which would make the lower extremity stable. A stability constraint has not previously been included in any lower extremity model. Murray [46] applied a stability analysis to examine co-contraction in the human elbow joint, but his model was unable to account for the experimental measurements. Although the stability analysis developed in this thesis was incapable of satisfying positive knee flexion stiffness throughout the stance phase of gait, it was able to demonstrate that a requirement of non-negative stiffness could require muscular co-contraction (i.e. knee joint forces increased with stability constraint in figure 8.11). Furthermore, when applied to the human elbow joint, the stability analysis results appear to
be promising. Even more fundamentally, the equations and analysis developed in Chapter 3 of exactly how joint stiffnesses should be defined and how muscles contribute to this stiffness have not been completely described previously.

In summary, the knee joint model and the complete lower extremity model developed in this thesis are the most comprehensive to date. Only Wismans et al [66] have previously attempted to include joint geometry in a knee model, but this was an in vitro study. All knee joint models which have been applied to in vivo data have only required equilibrium to be satisfied for a limited number of degrees of freedom and/or have excluded equilibrium constraints of the hip and the ankle joints. The results of the model are limited slightly by equilibrium errors, but methods have been presented for reducing or possibly eliminating these errors. The many developments of this thesis represent a substantial advance toward producing a model which is capable of accurate estimation of knee joint kinematics and ligament, muscle and joint contact forces during a variety of weight-bearing activities.

10.2 Recommendations for Future Research

The major limitations of the knee model appear to be inaccurate estimates of ligament material properties and rest lengths and the inability of the model to satisfy equilibrium in the anterior direction (these two limitations are not independent). If methods become available for more accurately modeling ligament forces, the model should be updated to include the improved ligament models. Besides the cruciate ligaments, another major contributor to force equilibrium in the anterior direction of the knee is the quadriceps muscles. In the current model, the patellar ligament (through which all four quadriceps muscles are attached to the tibia) is assumed to insert into the tibia at an angle which is linearly related to the knee joint flexion angle. Since the basis of this angle information is cadaver measurements, and since the knee model does not assume that in vivo and cadaver knees have the same kinematics, then the model may be improved if the attachment angle is allowed to be defined directly by the kinematics. This improvement would require adding the patella to the model, possibly via additional articular geometry measurements of the patella and the anterior portion of the distal femur. Note that in the current model the patellar ligament is assume to insert into the tibia at a 20 degree angle at zero flexion angle.
Small changes in this angle could have significant changes in the anterior knee forces.

The suggested methods for simultaneously estimating knee joint kinematics and muscle and joint forces should be implemented. The resulting model would then be able to completely satisfy force and moment equilibrium for the knee joint using only muscle, joint, and ligament forces. No current knee model (except the one in this thesis) attempts to account for equilibrium for all degrees of freedom of the knee.

Certainly, the knee model should be applied to more sets of kinematic and dynamic data for a variety of weight bearing activities. A very accurate kinematic measurement system should be used to get the best accuracy possible. Depending on the accuracy of the data, the results of experiments could answer some important questions about the knee: (1) is the knee really a complex single degree of freedom joint (as assumed in much of the literature) ? (2) how repeatable are the knee kinematics and dynamics from subject to subject ? (3) which array mounting procedures produce the smallest estimated soft tissue motion errors ?

Improved methods for using measured cadaver data to estimate geometric data for a gait analysis subject should be developed. Scaling methods may possibly be improved by measuring the geometry and anatomical landmarks of a number of cadaver knees. There are at least two special cases in which the actual subject's geometry data can be ascertained with some degree of accuracy. When a total knee prosthesis is implanted, the joint surfaces should be known exactly, or at least could be measured with precision before implantation. Also, CT or MRI scanning methods could possibly be used to estimate the geometrical surfaces. Although these would have a difficult time establishing the surface geometry of the tibia with any degree of accuracy, they would at least be helpful in establishing appropriate scaling factors for the joint. Finally, when the data from the ultrasonic geometry measurement system is available, then it can be used in place of the current data. This will require estimation of the transformations between the position of the knee joint being measured and some externally identifiable landmarks, but methods could be developed to obtain the estimate.
The stability analysis developed in this thesis as applied to human joints should be examined in more detail. Because of the many assumptions required for applying the analysis to the knee (i.e. quasi-static, both sides are maintained stable at all times, etc.), the lack of a complete solution is not surprising. However the results of applying the analysis to the elbow joint are promising. Even for this joint, however, the analysis as it is currently implemented requires the estimation of the total incremental stiffness values for each muscle. Rather than scaling results from an ankle muscle of a cat, it would be recommended to attempt to make some physical measurements so that the stiffnesses of elbow muscles could be more directly estimated. If a device is constructed to independently control the applied load and stiffness acting on the elbow, and EMG is measured for both the biceps and triceps muscles, then relationships between muscle stiffness and force and length may be examined. For example, the amount of co-contraction of biceps and triceps muscles may possibly be increased at a constant applied net joint moment if varying levels of negative joint stiffness are added to the joint. If these experiments are carried out, then some of the following issues may be addressed: (1) what is the relationship between joint moment, stiffness, and muscle activation levels? (2) if negative stiffness is added, will the muscles automatically co-contract? (3) can these experiments be used to infer the dependence of muscle stiffnesses on length and force? (4) if so, do the stiffnesses actually decrease after a certain load is exceeded? (5) how do the stiffness versus force relationships vary from subject to subject?

These questions are fundamental to the analysis of muscle forces applied to all human joints. The development of the stability analysis was not the fundamental objective of this thesis, and it certainly did not produce the most successful results when applied to the knee joint. The success of applying the stability analysis to the elbow joint, however, will provide motivation for researchers to work in this area. At the very least, the author personally plans to pursue research in this area in order to answer some of the questions raised above.
REFERENCES


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