POLYHEDRAL STRUCTURE OF THE 
PRODUCT CYCLING PROBLEM 
WITH CHANGEOVER COSTS 

by 

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Submitted to the Sloan School of Management 
in partial fulfillment of the requirements 
for the degree of Doctor of Philosophy 
at the 

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Abstract

The product cycling problem, a core model in production planning, has been the subject of extensive research. Yet the problem continues to pose significant modeling and computational issues. The model describes the production operations for multiple products that incur a fixed cost (and/or time) when setting up a facility for the manufacture of any product. This thesis, building upon prior results of Magnanti and Vachani (1987), investigates the polyhedral structure of the model and related modeling and computational issues. In particular, we identify classes of facets and solve the associated separation problem. Using ideas developed by Martin (1987), we also reformulate the problem in a higher dimensional space with a polynomial number of variables and constraints.

We also extend our results to the multi item problem and to the multi machine model. The inequalities can be viewed as a progressive series of generalizations from the partitioning inequalities identified by Magnanti and Vachani (1987) for the single machine, single item problem, to the generalized partitioning inequalities and the skip inequalities for this problem, to the multi item inequalities for the
single machine, multi item problem, and finally to the multi machine, multi item problem. We show that a small subset of the inequalities for the single item problem are sufficient to ensure that we obtain optimal integer solutions for some cost structures that are encountered in practice.

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Chapter 1 Introduction.

Motivation

A production management system is concerned with the total flow of goods from the acquisition of raw materials to the delivery of finished products to the final customer. The planning process needs to address many wide ranging issues: for example, (i) the location of plants and warehouses, (ii) plant design, (iii) capacity planning, (iv) product specifications, and (v) production scheduling. Researchers and practitioners often view production planning as a hierarchy of decisions, with each set of decisions at an aggregate level imposing constraints on more detailed lower level decisions. Researchers have proposed various frameworks for the entire planning process (see for example, Hax and Candea (1984), Nahmias (1989) and Schroeder (1989)).

Within this range of functions, we are concerned with a production scheduling problem. Scheduling is an operational function within the confines of given capacities and product specifications. We can broadly classify scheduling problems as either deterministic or stochastic. Stochastic models include various uncertainties associated with allowable schedules, for example, uncertainties in availability of resources and in job durations. Deterministic models are more amenable to combinatorial analysis, and researchers have studied the problem extensively. We study a deterministic model that is appropriate for many practical situations. Moreover, as we indicate later, the problem poses some interesting theoretical issues.
Lawler, Lenstra and Rinnooy Kan (1982) classify deterministic models along three dimensions: machine environment, job and scheduling characteristics, and optimality criteria. For example, the production environment might have a single machine, a number of identical machines, or a number of different types of machines. Job and scheduling characteristics might include the number of operations for a job, the release date by which the job becomes available, the due date for the job, whether preemption of the job by another job is allowed, and whether there are precedence constraints between jobs. Our objective might be to minimize the sum of completion times, or to minimize the maximum completion time, or some other cost function.

We study a single machine, dynamic, deterministic problem, focusing primarily on an environment with a single machine. All jobs are available at time zero (in other words, the release date for all jobs is zero), and a due date is specified for each job. Preemption is not allowed and there are no precedence constraints. The problem is important for several reasons. First, the model is tractable and is a building block in our understanding of more complicated systems. Secondly, the single machine problem quite often appears as a subproblem in a larger scheduling problem. At other times, it might be appropriate to aggregate the facilities and treat them as a single resource.

One of the key issues in scheduling is the effective allocation of shared resources to multiple products. Several models have been proposed in the literature to address this issue. We study the product cycling problem, which is one of the prototypical models in the production planning literature. The model can be described as follows: several products need to be scheduled on a common production facility
to meet given demand for the products over a finite time horizon. The model assumes that we incur a fixed cost when we switch production from one product to another and that we can produce only one product at a time. Another widely studied model is the lot sizing problem, which assumes that we incur a fixed cost every time we produce and that several products can be produced in each time period. If the length of the time periods is large, as for example in a long term planning system, we can assume that we produce several products in each time period. The lot sizing model might be appropriate in this situation. On the other hand, the product cycling model is appropriate in many situations with small time periods and a short term planning horizon. The model has several applications, for example:

1) Scheduling production of different chemicals through an expensive chemical processor. For example, a facility might manufacture different types of soap. If we switch production from one soap to another, we need to clean out the facility and thus incur a fixed cost. Another example is a facility that produces different colors of paint.

2) A facility for producing printed circuit boards might include a machine that places a set of components on to a board. Typically, the facility will be producing different types of boards, each with a different set of components. If we switch from one product to another, we need to set up the machine for the new set of components and thus incur a fixed cost.

3) The model is also appropriate for assembly lines producing a few models of the same product and a fixed cost for setting up the line for a particular model.
Leachman and Gascon (1988) describe the use of such a scheduling model in a decision support system developed for a large consumer products manufacturer. Thus the product cycling problem is of interest not only because it is a prototypical model in the literature but also because it frequently arises in practice.

Another motivation for our study is that most of the earlier researchers have used solution methods that have not performed well in practice. They have focused on dynamic programming or Lagrangean relaxation methods. Since the problem is NP-hard, the running time of all dynamic programming methods increases exponentially with the number of time periods and products. Hence the use of these methods is limited to small problems. Lagrangean relaxation techniques have not been successful either; computational results with this solution approach have not been encouraging (see for example, Karmarkar, Kekre and Kekre (1987)).

On the other hand, recent research has suggested that solution methods based on a cutting plane approach might provide better results (see for example, Crowder and Padberg (1980), Padberg and Hong (1980), and Crowder, Johnson and Padberg (1983) who study several other large scale combinatorial optimization problems). Wolsey (1988) used a cutting plane method that performed well for the uncapacitated version of our model. Magnanti and Vachani (1987) developed a solution technique based on cutting planes for the constant capacity case. This approach performed well on problems having up to 20 time periods and 5 products. We extend the work of Magnanti and Vachani, and propose to study related models with set up and changeover times, machine idle time and varying capacity in each time period. Another advantage of
studying the polyhedral structure of this model is that it might form a substructure of a larger problem. Facets of the subproblem could provide strong valid inequalities for the larger problem. For example, Crowder, Padberg and Johnson (1983) showed that minimal cover cuts from single constraints of a zero-one program are effective in solving large scale 0-1 programs. Barany, Van Roy and Wolsey (1984a, 1984b) have also reported good computational results for a multi-item capacitated lot-sizing problem using facets of the single item problem.

**Thesis Objectives and Overview.**

In this thesis we propose to study the polyhedral structure of the product cycling problem. Our aim is to develop a better understanding of the problem and see how it relates to other similar models. We also wish to develop efficient cutting plane methods for solving the problem and see if it is useful in solving more complex models. The objectives can be described in detail as follows:

- to identify facets for the problem, providing a proof of their validity and showing that they are facets. By identifying new facets, we will be extending the prior research of Magnanti and Vachani (1987). We also identified inequalities for the multi item problem that are not trivial generalizations of the inequalities for the single item problem.

- to develop a cutting plane based method that uses the facet inequalities.

- to investigate if we can completely characterize the single product case. This issue is of theoretical interest in itself. But, equally important, it should provide modeling and formulation
insight that we can use to study more general and related problems. For example, by solving the separation problem for the facets we obtain a reformulation with a polynomial (in the number of time periods) number of variables and constraints. Since the original problem has an exponential number of facets, the reformulation provides a compact representation that could be solved as a linear program. This reformulation could be useful as a subproblem in the multi-product case. We have shown that for certain cost structures encountered in practice, a small subset of the inequalities is sufficient to guarantee that we obtain optimal integer solutions.

- to extend our results to a related multi machine model. Here again, we identify classes of inequalities that are not trivial generalizations of the single machine problem.

The thesis is organized as follows. The next section provides a literature survey. In Chapter 2 we formulate the problem, describe some inequalities and show that they are valid. In Chapter 3 we prove that the inequalities are facets if they satisfy some additional conditions. In Chapter 4 we solve the separation problem and specify a polynomial reformulation of the problem. We also show that the dual of the reformulated problem is a network flow problem with side constraints. In Chapter 5 we show that for certain cost structures encountered in practice, a small subset of inequalities is sufficient to guarantee that we obtain optimal integer solutions. In Chapter 6 we generalize the single item inequalities to the multi item case and derive new classes of inequalities using the fact that only one item can be produced in one period. In Chapter 7, we extend our model to the multi machine
problem, generalize the single machine inequalities to the multi machine problem, and derive new classes of inequalities for the problem. Finally, in Appendix 1, we show that the single item, single machine problem can be solved as a shortest path.

Literature Survey

Recent research has provided increasing evidence that many integer programming problems can be solved optimally using results about the problems' underlying polyhedral structure. Crowder and Padberg (1980) and Padberg and Hong (1980) were among the first researchers to successfully apply cutting plane approaches to NP-complete problems. They solved large traveling salesman problems using a combination of cutting planes and branch and bound techniques. Crowder, Johnson and Padberg (1983) solved large 0-1 real world problems and Van Roy and Wolsey (1984) solved mixed integer 0-1 problems using strong valid inequalities as cutting planes and reported good computational results. Johnson, Kostreva and Suhl (1985) solved a strategic planning problem using a combination of cutting planes and branch and bound techniques. Van Roy and Wolsey (1987) developed another method for mixed 0-1 integer programming problems where they add strong cutting planes when necessary using an algorithm that automatically identifies a violated inequality and reformulates the problem. Groetschel and Holland (1984) implemented a strong cutting plane algorithm for the matching problem and reported that their algorithm performed better than the polynomial time algorithm of Edmonds (1965) for large scale problems. This brief summary provides only a glimpse of the many successful applications of cutting plane approaches. Later we cite some applications in manufacturing as well.
For more information, the reader might consult the survey of computational uses of polyhedral methods by Groetschel (1985) and Hoffman and Padberg (1987), the recent book by Nemhauser and Wolsey (1988), and the collection of articles in two specialized issues of the journal Mathematical Programming edited by Padberg and Rinaldi (1989).

Researchers have studied the product cycling problem and its variants fairly extensively. Magnanti and Vachani (1987) provide a comprehensive survey of this problem. As they pointed out, research on this problem has mainly been focused on the constant demand, infinite horizon case, and on dynamic programming approaches to the general, dynamic, deterministic model. The constant demand, infinite horizon model assumes a cyclic pattern of production that repeats over time. It is therefore appropriate when the demand pattern over a long range time horizon is known and is fairly stable. Elmaghraby (1978) calls this model the Economic Lot Scheduling Problem (ELSP) and surveys methods proposed for solving it. Most of the methods use some variant of the EOQ model to obtain a first approximation to the solution, and then adjust the solution to conform to the capacity restrictions. Although these methods perform well for the ELSP, they do not perform well for the general case. See, for example, Bomberger (1966), Madigan (1968), Stankard and Gupta (1969), Hodgson (1970), Doll and Whybark (1973), Goyal (1973) and Haessler (1979).

The general, dynamic, deterministic, case is NP-hard. The running time of all the dynamic programming methods therefore increases exponentially with the number of time periods and products. Glassey (1968) and Tenzer (1969) considered the special case of unit changeover costs. Mitumori (1972) and Gascon and Leachman (1988) solve the
general case with non-sequence dependent costs. Driscoll and Emmons (1977) presented an algorithm that allows sequence dependent changeover costs.

Researchers have also used other approaches to solve the problem. Geoffrion and Graves (1976) proposed a quadratic assignment approach for the multi-machine problem with sequence dependent changeover costs. Schrage (1982) suggested a linear programming based method. Karmarkar, Kekre and Kekre (1987) studied the single item version of the problem. Karmarkar and Schrage (1985) proposed a formulation that is similar to the one we study and solved it using Lagrangean relaxation. However, they report that the computational results are not very encouraging. Eppen and Martin (1987) reformulated the single-item version of the problem as a shortest path problem and report good computational results. Fleischmann (1990) studied the lot-sizing problem using a Lagrangean relaxation based method, and reported good computational results.

Very few researchers have used a polyhedral cutting plane approach for the product cycling problem. Wolsey (1988) studied the uncapacitated version of the problem and obtained valid inequalities that were useful in developing an efficient solution method. Magnanti and Vachani (1987) proposed a formulation similar to the one we study and developed a solution method based on a cutting plane approach using certain facets that they identified. They reported good computational results for problems with up to 20 time periods and 5 products.

The lot sizing problem is a related scheduling problem that researchers have studied using a polyhedral approach. Barany, Van Roy
and Wolsey (1984b) characterized the uncapacitated lot sizing problem and reported good computational results for the multi-item capacitated problem using facets of the single item problem. Pochet and Wolsey (1988) investigated the lot sizing model with backlogging. They also provided some reformulations of the problem. Leung, Magnanti and Vachani (1988) studied the single-item capacitated version of the dynamic economic lot sizing problem and showed that these inequalities define facets. Pochet (1988) developed valid inequalities for the same problem, and showed independently that these inequalities are facets when the capacities are equal.

Vergin (1978), Graves (1980) and Leachman and Gascon (1988) have studied the stochastic version of the problem.
Chapter 2. Polyhedral Structure

Problem Formulation

We now describe a single machine, multi-product, production planning model as formulated by Magnanti and Vachani (1987). The model has several underlying assumptions:

(i) The changeover time is zero, i.e., switching from the production of one product to another, we require no time to set up the machine. This assumption is realistic if the machine is set up during idle periods, for example, between one day's production and the next, or if the set up time is a small fraction of the production time.

(ii) We can maintain a machine setup even if the machine is idle in a particular period. This assumption is reasonable in many situations such as the production of chemicals; in this setting, once the equipment has been cleaned for a particular product, it can be used for the production of the same chemical even after an idle period.

(iii) The changeover costs are not sequence dependent. In other words, the changeover cost in any period does not depend on which product was produced in the previous period.

(iv) We use a discrete production policy i.e., in any production period, the production level is either zero or equal to capacity. This assumption is reasonable if it is expensive to set up the machine, or if it is very expensive to run the facility at less than full capacity.
Moreover, this policy has the advantage that it may be easier to implement and control.

The problem can then be described as follows. Time is discretized into time periods 1 through T. We produce P products with unit demand at times $t_{p1}, t_{p2}, \ldots, t_{pn_p}$ for product p. Thus the total demand for product p is $n_p$. We also assume that the production capacity in each period is the same. The 0-1 variable $w_{pi}$ is equal to 1 if we produce product p in period i and is zero otherwise. The cost of producing a unit of product p at time i is $h_{pi}$. In order to produce at time i, we must set up the machine: the variable $y_{pi}$ is equal to 1 if the machine is set up to produce product p and is zero otherwise. The setup cost is $s_{pi}$. We also incur a changeover cost of $k_{pi}$ if the machine is not set up at time i-1 and we turn it on at time i. The variable $z_{pi}$ is equal to one if we incur the changeover cost and is zero otherwise. The demand in period i is denoted by $d_{pi}$; as shown in Magnanti and Vachani (1987), without loss of generality we can assume that $d_{pi}$ is either 0 or 1. The scheduling problem, which we call CSP (for the Changeover Cost Scheduling Problem) can be formulated as follows:

(CSP). minimize $u = \sum_{p=1}^{P} \sum_{i=1}^{T} \{ h_{pi}w_{pi} + s_{pi}y_{pi} + k_{pi}z_{pi} \}$

subject to

$$\sum_{j=1}^{i} w_{pj} \geq \sum_{j=1}^{i} d_{pj} \quad \text{for all p, i} \quad (i)$$

$$\sum_{j=1}^{T} w_{pj} = n_p \quad \text{for all p} \quad (ii)$$

$$w_{pi} - y_{pi} \leq 0 \quad \text{for all p, i} \quad (ii)$$

$$z_{pi} + y_{p,i-1} - y_{pi} \geq 0 \quad \text{for all p, i} \quad (iii)$$
\[ \sum_{p=1}^{P} y_{pi} \leq 1 \quad \text{for all } i \quad (iv) \]

\[ w_{pi} \leq 1, \, y_{pi} \leq 1, \, z_{pi} \leq 1 \quad \text{for all } p, i \quad (v) \]

\[ w_{pi}, y_{pi}, z_{pi} \geq 0 \quad \text{and integer.} \quad (vi) \]

We let CSP(L) denote the linear programming relaxation of CSP and let F(CSP) be the set of feasible (integer) solutions for CSP. Constraints (i) are the demand constraints. Constraints (ii) ensure that we can produce only if the machine is set up. Constraints (iii) state that if the machine is set up at time \( i \) (i.e., \( y_{pi} = 1 \)), then either the machine was set up at time \( i-1 \) or we turned it on (incurring a changeover) at time \( i \). Constraints (iv) ensure that we produce only one product in any period.

To facilitate our discussion we focus on the single product version of the problem. Although a dynamic programming algorithm will solve this problem in polynomial time, we have studied valid inequalities for the problem. There were two motivations for doing so. First, these inequalities can be generalized to the the multi-product problem (which is NP-complete) or for problem settings with arbitrary demands and varying production capacity over time. Second, we would like to find a complete characterization of the convex hull of feasible solutions for the single product version, and therefore, hopefully, obtain insight concerning the combinatorial structure of this problem.

Let SCSP denote the single product version of the problem and SCSP(L) denote the linear programming relaxation of SCSP. Let F(SCSP) be the set of feasible (integer) solutions for SCSP. Finally, let \( C \) denote the convex hull of F(SCSP). Since the model has only one product, we
drop the subscript p on all variables. The demands occur in the n periods $t_1, t_2, \ldots, t_n$. Since there is no need to produce in periods after $t_n$, we assume that $t_n = T$. The constraint $\sum_{i=1}^{t_k} w_i \geq \sum_{i=1}^{t_k} d_i$ implies the constraints $\sum_{i=1}^{t} w_i \geq \sum_{i=1}^{t} d_i$ for $t = t_k + 1$ through $t_{k+1} - 1$ because we incur no demand between periods $t_k + 1$ and $t_{k+1}$. Hence we can drop the demand constraints for all periods except the periods $t_1, t_2, \ldots, t_n$. Note that if demand equals 1 unit in periods 1 through $j$ and zero units in period $j+1$, then $y_i = w_i = 1$ for all periods 1 through $j$. Hence the problem essentially reduces to one from periods $j+1$ through $T$. Therefore, to exclude uninteresting cases, we assume that $t_1 \geq 2$.

Motivation

We can summarize the motivation for our study as follows:

- The model is applicable in many practical situations.
- It is a prototypical model in the production planning literature, and many researchers have studied it.
- Most of the earlier solution approaches have not provided encouraging results.
- Polyhedral approaches have performed well on a variety of other problems.
- The polyhedral structure of the model might prove useful as a substructure of a larger model.
- There are some interesting theoretical issues regarding the polyhedral structure of the problem.
Introduction to valid inequalities

To motivate the discussion, let us first consider the following example:

Figure 2.1 Gap between the IP and the LP solutions.

Figure 2.1 depicts the integer and linear programming solutions to a single item problem with demands in periods $t_1$ and $t_2 = T$. Assume that the turn on cost $k$ per period is very large compared to the setup or production costs. The (optimal) integer solution might be to turn the machine on at the beginning of period $t_1$ (as indicated by the vertical arrow), keep it on through period $t_1+1$ (indicated by the horizontal arrow), and produce in periods $t_1$ and $t_1+1$ (indicated by the shaded area). However, if we solve the problem as a linear program, we obtain a lower cost solution by setting $z_1 = 2/T$. Thus we 'partially' turn the machine on in period 1 by $2/T$ units, keep the machine on until period $T$, and produce $2/T$ units in each period.
In order to eliminate the fractional solution from the linear programming relaxation, we need to introduce some additional valid inequalities. We first introduce those discovered by Magnanti and Vachani (1987). Although these inequalities cut off some fractional solutions, we might still obtain significant gaps between the value of the linear programming solution and the value of the optimal integer solution. After introducing the inequalities, we then indicate why we might need to introduce some different types of inequalities to obtain better solutions.

The following inequality cuts off this fractional solution:

$$\sum_{i=1}^{t_1} z_i \geq 1.$$ 

The inequality states that the machine must be turned on at least once in the interval \( \{1, \ldots, t_1\} \), which is certainly true because we need to produce at least once in this interval to meet the demand in period \( t_1 \). Note that we can tighten the inequality by replacing \( z_1 \) by \( y_1 \) (recall that \( z_1 \geq y_1 \) is one of the original constraints), to obtain \( y_1 + \sum_{i=2}^{t_1} z_i \geq 1 \). This inequality states that either the machine is set up in period 1 or we turn the machine on in one of the periods 2 through \( t_1 \). Again, the inequality is valid because we must meet the demand in period \( t_1 \).

We notice one fact about the inequality. Suppose we turn on the machine in period 1, and keep it on and produce in period 2. The lefthand side does not account for the one unit of production in period 2 even though we produce in that period. However, to compensate for this, the inequality does have a contribution of one unit in period 1. This observation shows that the inequality \( w_1 + \sum_{i=2}^{t_1} z_i \geq 1 \) is not valid.

Let us now consider the following inequality:
assuming that \( t_1 \geq 3 \). This inequality states that either we produce in period 1, or set up the machine in period 2, or turn on the machine in one of the period 3 through \( t_1 \). This inequality is also valid. All the inequalities considered so far have the following properties: (i) the inequality contains exactly one of the variables \( w_i, y_i \) or \( z_i \) for each time period \( i \), and (ii) if variable \( w_i \) is in the inequality, then variable \( z_{i+1} \) is not in the inequality. Otherwise, as shown earlier, we can turn the machine on in period \( i \), and keep it on and produce in period \( i+1 \). The inequality will not account for the production either directly through a production variable or indirectly through a set up variable \( y_i \) or changeover variable \( z_i \). We depict the inequalities as follows:

![Diagram](image)

**Figure 2.2** Pictorial representation of valid inequalities.

We can generalize this development as follows. We partition the interval \( \{1, \ldots, t_1\} \) into the subsets \( W, Y \) and \( Z \), letting \( W \) consist of indices for the variables \( w_i \), \( Y \) consist of indices for the variables \( y_i \), and
\( Z \) consist of indices for the variables \( z_i \). We can then write the following inequalities:

\[
\sum_{i \in W} w_i + \sum_{i \in Y} y_i + \sum_{i \in Z} z_i \geq 1,
\]

and impose the condition that if \( i \in W \), then \( i+1 \notin Z \). A slight generalization of the earlier argument shows that these inequalities are valid.

Suppose we now consider demand up to period \( t_q \) for \( q \leq n \). Let \( j \) be any period in the interval \( \{t_{q-1}+1, \ldots, t_q\} \). We partition the interval \( \{j, \ldots, t_q\} \) into the subsets \( W, Y \) and \( Z \). We can write the following inequalities:

\[
\sum_{i=1}^{j-1} w_i + y_j + \sum_{i \in W} w_i + \sum_{i \in Y} y_i + \sum_{i \in Z} z_i \geq q,
\]

and again impose the condition that if \( i \in W \) then \( i+1 \notin Z \).

We indicate why this inequality is valid. One of the original constraints is \( \sum_{i=1}^{t_q-1} w_i \geq q-1 \). Since \( j \geq t_{q-1}+1 \), any feasible solution satisfies the inequality \( \sum_{i=1}^{j-1} w_i \geq q-1 \). If \( \sum_{i=1}^{j-1} w_i = q-1 \), we need to show that \( y_j + \sum_{i \in W} w_i + \sum_{i \in Y} y_i + \sum_{i \in Z} z_i \geq 1 \). We have previously shown that this inequality is valid because we must produce at least once in the interval \( \{j, \ldots, t_q\} \) to meet the demand in period \( t_q \).

The next figure shows some valid inequalities for \( q=2 \).
Magnanti and Vachani (1987) have also identified these inequalities. The following example with demands in periods $t_1$, $t_2$ and $t_3 = T$ shows that we might obtain fractional solutions even after introducing all the earlier inequalities.

Figure 2.4 Linear programming fractional solution after introducing some valid inequalities.
The figure shows a fractional solution that satisfies all the inequalities discussed so far: 
\[ z_{t_1} = y_{t_1} = y_{t_1+1} = w_{t_1} = 1, \quad y_{t_1+2} = 2/3, \quad y_{t_1+3} = y_{t_1+4} = y_{t_1+5} = 1/3, \quad w_{t_1+1} = 1/3, \quad w_{t_1+2} = 2/3, \quad w_{t_1+3} = w_{t_1+4} = w_{t_1+5} = 1/3. \]
Let us assume that the turn on and setup costs are constant over all periods, and that the production cost is \((T-i)H\) in period \(i\). The cost of this solution is \(k + 3.67s + 3(T-t_1)H - 5.67H\). The optimal integer solution might be 
\[ z_{t_1} = y_{t_1} = y_{t_1+1} = w_{t_1} = 1, \quad w_{t_1+1} = w_{t_1+2} = y_{t_1+1} = y_{t_1+2} = 1. \]
The cost of this solution is \(k + 3s + 3(T-t_1)H - 3H\). The linear programming solution is less than the integer solution if \(0.67s < 2.67H\).

The following inequality cuts off this fractional solution:

\[
\sum_{i=1}^{t_1+2} w_i + y_{t_1+3} + (y_{t_1+4} + z_{t_1+4}) + z_{t_1+5} + \sum_{i=t_1+6}^{T} w_i \geq 3. \quad (2.1)
\]

We show later that this inequality is valid. The lefthand side of the inequality for the fractional solution in Figure 2.4 equals 22/3.

Before we proceed further, let us first define the following terms:

\textit{jth demand interval}. Recall that \(t_j\) denotes the period at which the jth demand occurs. The \textit{jth demand interval} is the interval \([t_{j-1}+1, t_{j-1}+2, \ldots, t_j]\).

\textit{Contribution}. We say that the sum of the terms on the lefthand side of the inequality associated with some sequence of machine operations (or some set of time periods) is the \textit{contribution} of that set of operations (or time periods). For example, suppose we turn the machine on at period 2 and keep it on until period 5, producing in periods 3 and 4. Suppose the inequality in the interval from period 2 through 5 has the form:

\[
\ldots + w_2 + y_3 + z_4 + w_5 + \ldots.
\]
The contribution of this set of operations is 1, since \( w_2 = 0, y_3 = 1, z_4 = 0 \) and \( w_5 = 0 \).

We say that we turn on the machine in period \( i \), if \( z_i = 1 \) and thus we incur the changeover cost in this period.

Notice that this inequality includes production variables \( w \) for each period in the first and third demand intervals \((1, ... , t_1)\) and \((t_2+1, ... , T)\) and partitions the intermediate interval \((t_1+1, ... , t_2)\) so that each period in the interval contains a term \( w, y, z \) or \( y+z \). As we will see, the partitions of the earlier intervals is generally more complex than the partitions of the final demand interval.

The earlier inequalities partition only the interval \([j, ... , t_q]\) for \( t_{q-1}+1 \leq j \leq t_q \). We now consider the following questions:

1) Can we partition the entire interval \([1, ... , t_q]\) into appropriate subsets and obtain valid inequalities that are not implied by the inequalities obtained so far?

2) Is it necessary to partition the interval \([1, ... , t_q]\) or can we partition some subset of the interval to obtain further generalizations of the inequalities?

To generalize the results discussed to this point, we would like to develop a more extensive set of valid inequalities that account for production in the first \( q \) production intervals for some \( q \leq n \). We will write inequalities in the form

\[
\alpha w + \beta y + \gamma z \geq q. \tag{PI}
\]

We will use these inequalities to model the fact that in the first \( q \) demand intervals we must produce at least \( q \) times. The coefficients \( \alpha_i, \beta_i, \) and \( \gamma_i \) will be zero for \( t > t_q \). As we have seen in our previous examples, we partition the last demand interval, \( \alpha_i = 1, \beta_i = 0, \) and \( \gamma_i = 0 \)
for all \( i \leq t_{q-1} \), and one of the coefficients \( \alpha_i, \beta_i \) and \( \gamma_i \) equals 1 (the others are zero) for each period in the final demand interval \((t_{q-1}+1, \ldots, t_q)\) covered by the inequality. However, if we partition intermediate demand intervals, we need not restrict the coefficients are not restricted to be 0 or 1, and we might also include terms like \( y+z \) indicated in the inequality (2.1).

Some motivation.

The initial demand equations
\[
\sum_{i=1}^{t_{q-1}} w_i \geq q \quad \text{for } 1 \leq q \leq n-1
\]
and
\[
\sum_{i=1}^{T} w_i = n,
\]
contain only the production variables \( w \). As we have seen at the outset of the section, the linear program obtained by relaxing the integrality conditions in the original problem formulation SCSP does not properly account for the interaction between the turn on variables \( z \) and the production and setup variables \( w \) and \( y \). Therefore we need to define additional valid inequalities that couple the problem variables more tightly. One way to do so would be to replace some of the \( w \) variables by \( y \)'s, and \( z \)'s. Suppose we replace \( w_i \) in the last demand interval by \( z_i \), i.e., for \( i \geq t_{q-1}+2 \). We can turn the machine on in period \( i-1 \), not produce in this period, keep the machine on until period \( i \), and produce in period \( i \). Since the variable \( w_i \) is no longer included in the inequality, period \( i \) contributes zero units to the lefthand side of the inequality. If period \( i-1 \) does not contain either \( y_{i-1} \) or \( z_{i-1} \), then period \( i-1 \) also does not contribute anything to the lefthand side of the inequality. Hence, we 'do not account' for this unit of production on the lefthand side of the inequality. If we produce exactly \( q-1 \) times up to period \( t_{q-1} \), and once in period \( i \) using the method described, then the
lethand side of the inequality is equal to \( q-1 \), and we cannot write a valid inequality with \( q \) on the right-hand side.

We therefore impose the condition that if period \( i \in Z \), and \( i \in \{ t_{q-1}+2, \ldots, t_q \} \), then \( i-1 \in Y \cup Z \). Notice that if we use the production plan described for producing in period \( i \), then \( y_{i-1} = z_{i-1} = 1 \), and since \( i-1 \in Y \cup Z \), the lefthand side increases by one unit in period \( i-1 \). Thus, even though we produce in period \( i \), the lefthand side increases by one unit in period \( i-1 \). We say that period \( i-1 \) \textit{compensates} for period \( i \in Z \). Notice that we can replace any number of \( w_i \)'s in the interval \( \{ t_{q-1}+2, \ldots, t_q \} \) by \( z_i \), provided we impose the condition that if \( i \in Z \), then \( i-1 \in Y \cup Z \).

In the following development we partition the intermediate intervals \( \{ t_{j-1}+1, \ldots, t_j \} \) for \( j < q \). The basic idea is the same. We replace some term \( w_i \) by \( z_i \). To compensate for this exchange, we need to introduce terms of the type \( y+z \) in periods prior to period \( i \). For example, in inequality (2.1), we compensate for \( z_{t_1+5} \) by introducing the terms \( y_{t_1+3}+z_{t_1+4} \). Notice that in the second demand interval, we can produce twice to meet the final demand up to \( t_3 = T \). Hence we can turn on the machine in period \( i-2 \), keep it on until period \( i \), and produce twice, once in period \( i-1 \) and once in period \( i \). Alternately, we can turn the machine on in period \( i-1 \), and produce in periods \( i-1 \) and \( i \). We therefore need the terms \( y_{i-2}+z_{i-1} \). We specify the precise length of a sequence of terms \( y+z \) depending on the demand interval in which it occurs.

\textit{Inequalities with the terms} \( y_i+z_i \)
Let us consider the first question. Suppose $q=2$, and we write the following inequality:

$$w_1 + y_2 + z_3 + \sum_{i=4}^{t_2} w_i \geq 2.$$  

This inequality is not valid as shown by the following production schedule that is feasible for demand up to period $t_2$: turn the machine on in period 2, keep it on until period 3, and produce in both periods. The lefthand side of the inequality is equal to 1, whereas the righthand side is 2. Even though we produce in period $3 \in \mathbb{Z}$, this period does not contribute a unit to the lefthand side of the inequality. Thus, we must compensate for this in some other period. Suppose we modify the inequality as follows:

$$w_1 + (y_2 + z_2) + z_3 + \sum_{i=4}^{t_2} w_i \geq 2.$$  

The schedule now satisfies the inequality because period 2 contributes two units to the lefthand side. However, the following schedule violates the inequality: turn the machine on in period 1, keep it on until period 3, and produce in periods 1 and 3. If we replace $w_1$ by $y_1$, then the schedule satisfies the inequality. In fact, we show later that this modified inequality is valid. This discussion suggests that $w_{i-1}$ must not precede the term $(y_1 + z_i)$. Suppose we replace $(y_2 + z_2)$ by $(w_2 + z_2)$ or by $2z_2$. The modified schedule then violates the inequality. This observation indicates that we cannot replace the term $(y_2 + z_2)$ by some other term and obtain a tighter valid inequality.

The following figure shows an inequality with $t_1 = 4$ and $t_2 = 7$:
Suppose we extend the inequality to \( q = 3 \). Consider the following inequality:

\[
y_1 + (y_2 + z_2) + z_3 + \sum_{i=4}^{t_3} w_i \geq 3.
\]

If we produce in periods 1, 2 and 3, the lefthand side has value 2 whereas the righthand side is 3. Even though \((y_2 + z_2)\) precedes \(z_3\), we can start production in period 1, and no period contributes two units to the lefthand side to compensate for period 3. Therefore, we need the term \((y_1 + z_1)\) in period 1. In other words, the inequality becomes

\[
(y_1 + z_1) + (y_2 + z_2) + z_3 + \sum_{i=4}^{t_3} w_i \geq 3.
\]

We show later that this inequality is valid.

We can generalize these ideas as follows. Let \( YZ \) denote the subset of indices for the terms \((y_m + z_m)\). Suppose period \( i \in Z \) for \( i \geq 2 \), and we can construct a feasible solution that produces in period \( i \) with \( z_i = 0 \).

We then need to compensate for this period by including the terms \((y_m + z_m)\) for some interval preceding period \( i \). For example, if \( i \leq t_1 \) and \( q = 2 \), then \((y_{i-1} + z_{i-1})\) is in the inequality. On the other hand, if \( i \leq t_1 \) and \( q = 3 \), then \((y_{i-2} + z_{i-2}) + (y_{i-1} + z_{i-1})\) is in the inequality. In general, whenever \( t_j + 1 \leq i \leq t_{j+1} \), and \( i \in Z \), the length of the interval preceding \( i \) and containing the terms \((y_m + z_m)\) must be at least \( q - j - 1 \). Moreover, the term
$w_{m-1}$ must not precede the term $(y_m + z_m)$. We state these ideas more precisely later.

To summarize the discussion so far, suppose period $i$ belongs to the $(j+1)$st demand interval. In any feasible solution, we must produce at least $j$ times up to period $t_j$ and might need to produce $q-j$ times after $t_j$ to meet the demand up to period $t_q$. Then we can write the following terms from periods $i$ through $i+q-j$:

$$y_i + (y_{i+1} + z_{i+1}) + \ldots + (y_{i+q-j-1} + z_{i+q-j-1}) + z_{i+q-j}$$

**Required conditions for validity of the inequalities.**

We now derive the feasibility conditions for a valid inequality of the form $\sum_{i \in W} w_i + \sum_{i \in Y} y_i + \sum_{i \in Z} z_i + \sum_{i \in YZ} (y_i + z_i) \geq q$. We will provide motivation by first describing some special cases. Suppose $k^* \in YZ$ for some $k^* \leq t_1$. If any period $i \in Z$ is in the interval $(t_{q-1} + 2, \ldots, t_q)$, then period $i-1 \in W$. Otherwise, we can produce in the first $q-1$ periods starting at the period $k^* \leq t_1$, with $k^* \in YZ$. This partial solution gives a contribution of $q-1$. We then turn the machine on in period $i-1$ and keep it on and produce in period $i$. The contribution of these operations is zero. This feasible solution violates the inequality.

Next we consider a period $i \in Z$ in the $(q-1)$st demand interval. Then $i-1 \in YZ$ and $i-2 \in W$. To see this result, suppose that $i-2 \in W$. We construct a valid solution that is violated by the inequality: namely, we produce in the first $q-2$ periods starting at a period $k^*$. This production sequence gives a contribution of $q-2$. We then turn the machine on in period $i-2$, keep it on until period $i$ and produce in periods $i-1$ and $i$. The contribution of these operations is 1. The inequality is therefore violated.
Now suppose that \( i-1 \notin YZ \). We use the same solution except that we turn the machine on for the second time in period \( i-1 \), keep it on until period \( i \), and produce in both periods. The contribution of these operations is 1 and the inequality is violated.

Moreover, if \( 2 \leq i \leq q \), and \( i \in Z \), then \( j \notin YZ \) for all \( j \leq \min(t_1, i-1) \). Otherwise the following feasible solution violates the inequality: we turn the machine on in a period \( j \) satisfying the conditions \( j \notin YZ \) and \( j \leq \min(t_1, i-1) \) and keep it on and produce in each of the periods \( j \) through \( q+j-1 \). Since \( i \in Z \) is a production period and contributes nothing, the total contribution of this feasible solution is at most \( q-1 \).

If all periods \( k \leq t_1 \) belong to \( YZ \), then we show later that (PI) can be written as a linear combination of other inequalities each containing some period \( k \leq t_1 \) with \( k \notin YZ \).

We can generalize these results to situations in which \( i \in Z \) lies in any demand interval \( j \) as follows. If period \( i \) is in demand interval \( j+1 \), for \( 0 \leq j \leq q-1 \), then we can write the terms \( y_i+(y_{i+1}+z_{i+1})+\ldots+(y_{i+q-j-1}+z_{i+q-j-1})+z_{i+q-j} \) in the periods \( i \) through \( i+j \).

As an illustration, consider the following example: demands occur in periods \( t = 10, 12, 15, 18 \) and 20. Some valid inequalities would be:

\[
\begin{align*}
\sum_{i=2}^{10} z_i & \geq 1 & q = 1 \\
\sum_{i=1}^{10} w_i + y_{11} + z_{12} & \geq 2 & q = 2 \\
w_1 + w_2 + y_3 + y_4 + z_4 + y_5 + z_5 + z_6 + \sum_{i=7}^{13} w_i + y_{14} + z_{15} & \geq 3 & q = 3.
\end{align*}
\]

*Inequalities with the terms \( c_i z_i \), for \( c_i > 1 \).*
We generalize these inequalities further. So far we have chosen all the coefficients \( \alpha, \beta \) and \( \gamma \) in (PI) to have values of either 0 or 1. We now introduce coefficients \( c_i \geq 1 \) for periods \( i \in \mathbb{Z} \). For the example above, we can write the following inequality:

\[
y_1 + (y_2 + z_2) + (y_3 + 2z_3) + 3z_4 + 2z_5 + z_6 + \sum_{i=7}^{15} w_i \geq 3.
\]

This inequality has a sequence of periods 4 through 6 with the terms \( 3z_4, 2z_5 \) and \( z_6 \). If we do not produce in periods 4, 5 or 6, the inequality is satisfied, because each of the other periods includes either \( w_i \) or \( y_i \). If we produce in period \( 4 \leq i^* \leq 6 \), we must turn on the machine in some period \( i \leq i^* \), and keep it on until period \( i^* \). If we turn it on in one of the periods 1 through 4, the inequality is satisfied. If we turn it on in period 5, the lefthand side of the inequality increases by two units, and we can produce at most twice in the interval \( \{4, 5, 6\} \). If we turn it on in period 6, we can produce at most once in the interval \( \{4, 5, 6\} \). Thus, the lefthand side of the inequality increases by at least one unit for each unit of production. Since we must produce at least three times up to period 15, the inequality is valid.

We note that period 4, 5 and 6 are in \( \mathbb{Z} \). We can turn on the machine in some period prior to 4, keep it on until period 6, and produce in periods 4, 5 and 6. These periods contribute nothing to the lefthand side of the inequality because they are not in \( W, Y \) or \( YZ \). To compensate for this, we must introduce the terms \( y_1 + (y_2 + z_2) + (y_3 + 2z_3) \). Whether we turn the machine on in period 1, 2 or 3, keep it on until period 6, and produce in periods 4, 5 and 6, the lefthand side of the inequality contributes three units in periods 1 through 3.

Some other examples of these types of inequalities are:
\[ y_1 + (y_2 + z_2) + (y_3 + 2z_3) + (y_4 + 3z_4) + (y_5 + 4z_5) \ldots \]
\[ + 5z_6 + 4z_7 + 3z_8 + 2z_9 + z_{10} + \sum_{i=11}^{20} w_i \geq 5 \]
\[ \sum_{i=1}^{10} w_i + y_{11} + (y_{12} + z_{12}) + (y_{13} + 2z_{13}) + (y_{14} + 3z_{14}) \ldots \]
\[ + 4z_{15} + 3z_{16} + 2z_{17} + z_{18} + \sum_{i=19}^{20} w_i \geq 5 \]

These inequalities have the following structure. If period \( i \) is in demand interval \( (j+1) \), we can write the sequence

\[(q-j)z_i + (q-j-1)z_{i+1} + \ldots + z_{i+q-j-1}\]  

(i)  

Period \( i \) is in turn preceded by

\[ y_{i-(q-j)} + (y_{i-(q-j)+1} + z_{i-(q-j)+1}) + (y_{i-(q-j)+2} + 2z_{i-(q-j)+2}) + \ldots + (y_{i} + (q-j-1)z_{i-1}). \]

We can generalize these inequalities if the sequence (i) has an arbitrary length less than \( (q-j) \). For example we can write

\[ y_1 + (y_2 + z_2) + (y_3 + 2z_3) + (y_4 + 3z_4) + (y_5 + 4z_5) \ldots \]
\[ + 4z_6 + 3z_7 + 2z_8 + z_9 + \sum_{i=10}^{20} w_i \geq 5 \]
\[ y_1 + (y_2 + z_2) + (y_3 + 2z_3) + (y_4 + 2z_4) + (y_5 + 2z_5) + 2z_6 + z_7 + \sum_{i=8}^{20} w_i \geq 5 \]

We show why the second inequality is valid. Since either \( y_i \) or \( w_i \) is in the inequality in all periods except 6 and 7, the inequality is satisfied if we do not produce in periods 6 and 7. If we produce in periods 6 and 7, and turn on the machine in period 6, the inequality is satisfied. If we do not turn on the machine in period 6, we must turn it on in an earlier period. If we turn it on in periods 1 or 2, the inequality is satisfied. If we turn it on in period 3, then period 3 contributes three units and the inequality is satisfied. If we turn it on in period 4, we can
produce at most 4 times from periods 4 through 7, and hence the inequality is satisfied. Similarly, if we turn it on in period 5, we can produce at most three times from periods 5 to 7, and the inequality is satisfied. The inequality is therefore valid.

The following figure shows a valid inequality:

\[
y + (y + z) + (y + 2z) + (y + 3z) + 4z + 3z + 2z + z + w + w + w + w \geq 4
\]

**Figure 2.6.** Partitions with coefficients greater than 1.

We can summarize this as follows. For any period \(i\) belonging to the \((j+1)\)st demand interval, we can write the following terms from periods \(i\) through \(i+(q-j)+t\), for \(t \leq q-j-1\):

\[
y_i + (y_{i+1} + z_{i+1}) + \ldots + (y_{i+q-j-1} + z_{i+q-j-1}) + (t+1)z_{i+q-j} + tz_{i+q-j+1} + \ldots + z_{i+q-j+t}
\]

The coefficients for periods \(i \in (i+1 \ldots i+q-j-1)\) are determined by the following equations: \(c_i = i' - i\) if \(t+1-i' \geq q-j\), and \(c_i = t+1\) if \(t+1-i' < q-j\).

We can generalize these inequalities further. For our example we can write down the following inequality:

\[
y_1 + (y_2 + z_2) + (y_3 + z_3) + (y_4 + 2z_4) + (y_5 + 3z_5) + 3z_6 + (y_7 + 2z_7) + 2z_8 + z_9 + \sum_{i=9}^{20} w_i \geq 5.
\]
This inequality splits the sequence \(3z_6 + 2z_7 + z_8\) into the sequence \(3z_6 + (y_7 + 2z_7) + 2z_8 + z_9\). The basic idea is as follows: period 6 does not have either a \(w_i\) or a \(y_i\) in it. For periods \(i \leq 5\), the sum of the coefficient of \(z_i\) and the coefficients of \(y\) from \(i\) through 5 is at least \(6-i\). This ensures that if we turn on the machine in period \(i\), keep it on until period 6 and then produce, the lefthand side increases by at least \(6-i\). Similarly, periods 8 and 9 do not have either \(w_i\) or \(y_i\) in them. For periods \(i \leq 7\), the sum of the coefficient of \(z_i\) and the coefficients of \(y\) from \(i\) through 7 is at least \(\min(5, 10-i)\). The inequality is therefore valid.

We can summarize these results as follows. If there is a sequence of terms beginning in period \(t+1\) in the first demand interval, without \(w_i\) or \(y_i\) in them, i.e., with terms \(c_{t+1}z_{t+1} + \ldots + c_{t^*}z_{t^*}\), then for any period \(i \leq t^*\), the sum of the coefficient of \(z_i\) and the coefficients of \(y\) from \(i\) through \(t\) is at least \(\min(q, t^*+1-i)\), where \(q\) is the righthand side of the inequality. Note that this implies that the coefficient of \(z_i\) for \(t+1 \leq i \leq t^*\) is at least \(\min(q, t^*+1-i)\).

Some more examples of this type of inequality are:

\[
y_1 + (y_2 + z_2) + (y_3 + z_3) + (y_4 + 2z_4) + (y_5 + 3z_5) + \ldots + 4z_6 + (y_7 + 3z_7) + 3z_8 + 2z_9 + z_{10} + \sum_{i=11}^{20} w_i \geq 5
\]

\[
y_1 + (y_2 + z_2) + (y_3 + z_3) + (y_4 + z_4) + (y_5 + 2z_5) + \ldots + 3z_6 + (y_7 + 2z_7) + (y_8 + 2z_8) + 2z_9 + z_{10} + \sum_{i=11}^{20} w_i \geq 5
\]

\[
y_1 + (y_2 + z_2) + (y_3 + z_3) + (y_4 + z_4) + (y_5 + z_5) + \ldots + 2z_6 + (y_7 + z_7) + (y_8 + z_8) + (y_9 + z_9) + z_{10} + \sum_{i=11}^{20} w_i \geq 5
\]

We can generalize these inequalities in yet another way. For example, we can string together a sequence of 5z's of arbitrary length if the righthand side is 5:

\[
y_1 + (y_2 + z_2) + (y_3 + 2z_3) + (y_4 + 3z_4) + (y_5 + 4z_5) + \ldots
\]
Inequalities with the terms \((w_i+z_i)\).

We consider another type of inequality for our example:

\[
y_1 + (y_2 + z_2) + (y_3 + z_3) + (y_4 + z_4) + (y_5 + z_5) + (w_6 + z_6) + \sum_{i=8}^{20} w_i \geq 5
\]

In this inequality, we insert the term \((w_6 + z_6)\) between \((y_5 + z_5)\) and \(z_7\). It is easy to see that the inequality is valid: if we do not produce in period 7, the inequality is satisfied. If we do, and turn on the machine in period 7, the inequality is satisfied. If we do not turn on the machine in period 7, then for any period \(i \leq 5\), the sum of the coefficients of \(y\) and \(z\) from \(i\) through 5 is at least \(\min(5, 7-i)\). Therefore, if we turn on the machine in period \(i\) and produce in periods \(i\) through 7, the lefthand side contributes at least \(\min(5, 7-i)+1\), where the extra unit comes from production in period 6. The inequality is therefore valid.

We can introduce a sequence of terms of the type \((w_i+z_i)\). For example,

\[
y_1 + (y_2 + z_2) + (y_3 + z_3) + (y_4 + z_4) + (y_5 + z_5) + (w_6 + z_6) + (w_7 + z_7) + (w_8 + z_8) + \sum_{i=10}^{20} w_i \geq 5.
\]

We can also introduce the terms anywhere between \(y_1\) and the period with a solitary \(z_i\):

\[
y_1 + (y_2 + z_2) + (w_3 + z_3) + (y_4 + z_4) + (w_5 + z_5) + (y_6 + z_6) + (y_7 + z_7) + (w_8 + z_8) + \sum_{i=10}^{20} w_i \geq 5.
\]

The structure of the inequality is the same as the earlier ones, except that we now introduce a sequence of terms \((w_i+z_i)\) of arbitrary length in between a solitary \(y\) and a solitary \(z\).

Inequalities with the terms \((w_i+c_i z_i)\) for \(c_i > 1\).
We can combine the terms \((w_i + c_i z_i)\) with the inequalities described earlier as shown in the following example.

\[
y_1 + (w_2 + z_2) + (y_3 + z_3) + (w_4 + 2z_4) + (w_5 + 2z_5) + (y_6 + 2z_6) + (y_7 + 3z_7) + (w_8 + 4z_8) + (y_9 + 4z_9) + (w_{10} + 4z_{10}) + 4z_{11} + 3z_{12} + 2z_{13} + z_{14} + \sum_{i=15}^{20} w_i \geq 5
\]

We can string together an arbitrary length of a sequence of \((w_i + c_i z_i)\)'s. In this example, we have the sequence \((w_4 + 2z_4) + (w_5 + 2z_5)\) preceding \((y_6 + 2z_6)\), the term \((w_8 + 4z_8)\) preceding \((y_9 + 4z_9)\), and the term \((w_{10} + 4z_{10})\) preceding \(4z_{11}\).

We briefly indicate why we do not introduce inequalities with terms of the type \((c_1 y_i + c_2 z_i)\) with \(c_1 > 1\). Consider the following inequality:

\[
y_1 + (y_2 + z_2) + (y_3 + z_3) + (y_4 + z_4) + (y_5 + z_5) + z_6 + (4y_7 + z_7) + 5z_8 + 4z_9 + 3z_{10} + 2z_{11} + z_{12} + \sum_{i=13}^{20} w_i \geq 5
\]

This inequality contains the term \((4y_7 + z_7)\). However, we can obtain a tighter inequality. If we sum the inequalities\( z_i + y_{t-1} - y_t \geq 0 \) for \( i \leq t \leq i^* \), we obtain \( y_i + \sum_{t=i+1}^{i^*} z_t \geq y_{i^*} \). We use the following inequalities:

\[
y_7 + z_8 \geq y_8.
y_7 + z_8 + z_9 \geq y_9.
y_7 + z_8 + z_9 + z_{10} \geq y_{10}.
\]

If we replace the quantities on the lefthand side of these inequalities by the righthand side in our original inequality, we obtain

\[
y_1 + (y_2 + z_2) + (y_3 + z_3) + (y_4 + z_4) + (y_5 + z_5) + z_6 + (y_7 + z_7) + (y_8 + 2z_8) + (y_9 + 2z_9) + (y_{10} + 2z_{10}) + 2z_{11} + z_{12} + \sum_{i=13}^{20} w_i \geq 5
\]

This is a tighter valid inequality.

Summary
We now describe one class of valid inequalities, which we call the partitioning inequalities (PI). Later, we generalize them to obtain another class of inequalities. Subsequently, we show that if we impose certain additional conditions on the inequalities, they are all facets.

Consider the demand up to time $t = t_q$. Let $L = \{1, 2, \ldots, t_q\}$, and let $W, Y, Z, YZ$ and $WZ$ be disjoint subsets of $L$ that partition $L$: that is, $W \cup Y \cup Z \cup YZ \cup WZ = L$.

The inequality extends up to time $t_q$ (the variables in the inequality have indices up to $t_q$). The inequality can be written as follows:

$$
\Sigma_{i \in W} w_i + \Sigma_{i \in Y} y_i + \Sigma_{i \in Z} c_i z_i + \Sigma_{i \in YZ} (y_i + c_i z_i) + (\Sigma_{i \in WZ} w_i + c_i z_i) \geq q
$$

$$
q = 1, \ldots, n. \quad \text{(PI)}
$$

The set $W$ consists of indices for the variables $w_i$, the set $Y$ consists of indices for variables $y_i$, $Z$ consists of indices for the variables $c_i z_i$, for $c_i \geq 1$, $YZ$ consists of indices for the terms $y_i + c_i z_i$, and $WZ$ consists of indices for the terms $w_i + c_i z_i$. To ensure validity, we impose the following condition:

**Compensation Condition.** For any period $i^*$ containing a term of the type $c_i z_i$, $c_i \geq 1$, and any other period $i \leq i^*$ in the $(j+1)$st demand interval, the sum of the coefficient of $z_i$ and the coefficients of $y_i$ for $i \leq t \leq i^*$ is at least $\min(q-j, i^*+1-i)$.

Suppose the condition is not satisfied. Let $(j^*+1)$ be the first demand interval for which the condition is not satisfied.

For any period $i^*$ in $Z$ and for any $i' \leq i^*$ in demand interval $j < j^*$, suppose that the sum of the coefficients of $z_i$, and the coefficients of
$y_t$ for $i' \leq t \leq i''$ is exactly equal to $\min \{ q-j, i''+1-i' \}$. Further, suppose that if there is no period in $Z$ before period $t_j$, then there are no periods in $YZ$ or in $WZ$ before $t_j$. If $q-j* \leq i*+1-i$, we produce in periods 1 through $j*$. If $q-j* > i*+1-i$, we produce in periods 1 through $q-(i*+1-i)$. We then turn on the machine in period $i$ and produce $\min (q-j*, i*+1-i)$ times. The solution violates the inequality, showing that the condition is necessary.

For example, consider the following inequality:

$$\sum_{i=1}^{10} w_i + y_{11} + (y_{12} + z_2) + (y_{13} + z_{13}) + (y_{14} + z_{14}) + 2z_{15} + z_{16} + \sum_{i=17}^{20} w_i \geq 5.$$ 

Assume $t_3 \geq 16$. For $i*=16$ and $i=14$, the sum of the coefficients of $z_{14}$, and the coefficients of $y$ from 14 through 16 is 2. Since $i*$ lies in the 3rd demand interval, the quantity $\min (q-j, i*+1-i) = \min (5-2, 16+1-14) = 3$. The following feasible solution violates the inequality: $z_1 = y_1 = y_2 = w_1 = w_2 = 1$, and $z_{14} = y_{14} = y_{15} = y_{16} = w_{14} = w_{15} = w_{16} = 1$.

We now show that the partitioning inequalities of Magnanti and Vachani (which we call inequalities (pi)) can be viewed as a special case of our inequalities (PI). Inequalities (pi) are of the form:

$$\sum_{i=1}^{j-1} w_i + \sum_{i \in W} w_i + \sum_{i \in Y} y_i + \sum_{i \in Z} z_i \geq q$$

where $t_{q-1}+1 \leq j \leq t_q$, and the set $W$ does not include periods up to $j-1$.

The inequalities (pi) confined periods $i \in Z$ to the interval $\{t_{q-1}+2, \ldots, t_q\}$, whereas inequalities (PI) allow $i \in Z$ anywhere in the inequality. The conditions imposed on (pi) were the following:
i) period \( j \in \mathbb{Z} \), and

ii) if \( i \in W \), then \( i+1 \in \mathbb{Z} \).

Since we include the periods up to \( j-1 \) in \( W \), condition (ii) implies (i), which is the same as condition 1 of (PI): that is, if \( i \) lies in the interval \( \{t_{q-1}+2, \ldots, t_q\} \) and \( i \in \mathbb{Z} \), then \( i-1 \in W \).
Generalized Valid Inequalities

We can generalize inequalities (PI) to obtain another class of inequalities, which we call the 'skip' inequalities (SI). These inequalities differ from all of those developed so far in one important respect: they can 'skip' time periods. That is, the inequalities need not contain any variable $y_j$, $w_j$ or $z_j$ for some time periods $j$. We consider the demand up to time $t = t_q$. We define the disjoint subsets $W$, $Y$, $Z$, $YZ$ and $WZ$ as before. But now, we no longer require that these subsets form a partition of the set $L = \{1, \ldots, t_q\}$; instead we simply require that $W \cup Y \cup Z \cup YZ \cup WZ \subset L$. If $j \in L \setminus (W \cup Y \cup Z \cup YZ \cup WZ)$, we will say that the inequality has skipped period $j$. Otherwise, we will say that period $j$ is in the inequality. We can describe the inequalities as follows:

Consider the demand up to time $t = t_q$. Let the cardinality of $W \cup Y \cup Z \cup YZ \cup WZ$ be $t_q - b$, which means that the inequality has skipped $b$ time periods. Let $L = \{1, 2, \ldots, t_q\}$. The inequality is of the form:

$$\sum_{i \in W} w_i + \sum_{i \in Y} y_i + \sum_{i \in Z} c_i z_i + \sum_{i \in YZ} (y_i + c_i z_i) + \sum_{i \in WZ} (w_i + c_i z_i) \geq q - b$$

$$b \leq q, \quad q = 1, \ldots, n. \quad \text{(SI)}$$

We need to impose some conditions to ensure that these inequalities are valid. These conditions are similar to the ones we imposed on inequalities (PI). Consider the following example with demands in periods 10, 20, 30, 40 and 50. A valid inequality is:

$$y_2 + \sum_{i=2}^{20} z_i \geq 1.$$
This inequality skips period 1. The following argument shows that it is a valid inequality. We must produce at least twice up to period 20. We can produce at most one unit in period 1. Hence we must produce at least once in periods 2 through 20. We must therefore set up the machine in period 2 or turn it on in one of the periods 3 through 20. Hence \( y_2 + \sum_{i=2}^{12} z_i \geq 1 \). Notice that if we replace \( y_2 \) by \( z_2 \) the inequality is no longer valid. We can turn on the machine in period 1 and produce in periods 2 and 3. The lefthand side of the inequality is 0, and the solution violates it.

Another example of a valid inequality is:

\[
y_2 + (y_3 + z_3) + (y_4 + 2z_4) + (y_5 + 3z_5) + 4z_6 + 3z_7 + 2z_8 + z_9 + \sum_{i=10}^{50} w_i \geq 4.
\]

This inequality extends up to the 5th demand in period 20. But we skip period 1. Hence we reduce the righthand side of the inequality from 5 to 4. We also reduce the length of the sequence \( y_2 + (y_3 + z_3) + (y_4 + 2z_4) + (y_5 + 3z_5) \) to 4 periods, since the righthand side is 4.

Consider the following two inequalities:

\[
\sum_{i=1}^{10} w_i + y_{12} + (y_{13} + z_{13}) + (y_{14} + 2z_{14}) + 3z_{15} + 2z_{16} + z_{17} + \sum_{i=18}^{50} w_i \geq 4 \quad \text{(i)}
\]

\[
\sum_{i=2}^{10} w_i + y_{11} + (y_{12} + z_{12}) + (y_{13} + z_{13}) + (y_{14} + 3z_{14}) + \ldots
\]

\[
+ 3z_{15} + 2z_{16} + z_{17} + \sum_{i=18}^{50} w_i \geq 4 \quad \text{(ii)}
\]

In the first inequality we skip period 11, which occurs in the second demand interval. Since we must produce once up to period 10, \( \sum_{i=1}^{10} w_i \geq 1 \). The remaining terms from periods 12 through 50 must sum to at
least 3 in every feasible solution if the inequality is valid. Hence the length of the sequence \( y_{12} + (y_{13} + z_{13}) + (y_{14} + 2z_{14}) \) is at least three. Otherwise we can turn on the machine in period 11 and produce 4 times from 11 through 14. This solution would violate the inequality. In the second inequality, we skip period 1 in the first demand interval. If we produce in this period, the lefthand side does not increase. Hence the length of the sequence \( y_{11} + (y_{12} + z_{12}) + (y_{13} + z_{13}) + (y_{14} + 3z_{14}) \) is at least 4.

In general, if we consider an interval of periods \( i, i+1, \ldots, i^* \), and period \( i \) is in the \((j+1)\)st demand interval, we first determine the minimum contribution of the terms 1 through \( t_j \) in any feasible solution. This computation permits us to determine the minimum quantity that the terms \( t_{j+1} \) through \( t_q \) must contribute. Let us denote this quantity by \( W_j \). If we let \( b_j \) denote the number of periods skipped up to demand period \( t_j \), we later show that in a tight inequality, \( W_j = \max(i-b_i; i \leq j) \). For example, in inequality (i), the minimum up to period 10 is 1, whereas in inequality (ii) it is zero.

In general, suppose that we skip \( b_1 \) periods up to \( t_1 \). If \( b_1 = 0 \), then in any feasible solution, we require that the contribution of the terms up to \( t_1 \) to be at least 1. However, we may skip two (or more) periods in demand interval 2. Then the sum of the terms up to \( t \) is not 2-2=0, but \( \max(1-0, 2-2) = 1 \). It is easy to see that in general, the contribution of the terms up to \( t_j \) is \( \max(i-b_i; i \leq j) \). The remaining terms from period \( t_{j+1} \) through \( t_q \) must contribute \( q-b-W_j \). Notice that the condition we are really trying to impose is the following: suppose we turn on the machine in period \( i \) in demand interval \( j+1 \), keep it on until period \( i^* \), and produce in some of the periods \( i \) through \( i^* \). We want to ensure
that if we produce $k \leq i^*+1-i$ times in the interval $\{i, \ldots, i^*\}$, in periods that are not skipped, then the interval contributes at least $\min \{k, q-b-\max \{i-b_i: i \leq j\}\}$ to the lefthand side. In other words, we want to account for each production that occurs in a period that is not skipped.

We can state the following feasibility condition for a skip inequality extending up to period $t_q$ and skipping $b$ periods:

**Skip Condition.** For any period $i^*$, containing a term of the type $c_{i^*}z_{i^*}$, $c_{i^*} \geq 1$, and any other period $i \leq i^*$ in the $(j+1)$st demand interval, the sum of the coefficient of $z_i$ and the coefficients of $y_t$ from $i$ through $i^*$ is at least $\min(q-b-\max \{(i-b_i: i \leq j), 0\}, i^*+1-i)$

Let $j^*+1$ be the first demand interval in which the condition is not satisfied. For $i'' \in Z$ and $i'' \leq i''$ in demand interval $j \leq j^*$, let the coefficient of $z_{i''}$ and the coefficients of $y_t$ for $i' \leq t \leq i''$ be exactly equal to $\min$ $(q-b-\max \{(i-b_i: i \leq j), 0\}, i''+1-i')$. Further, if there is no period in $Z$ up to period $t_q$ then let there be no period in $YZ$ or $WZ$ up to period $t_j$. If $q-b-\max \{(i-b_i: i \leq j), 0\} \leq i^*+1-i$, we turn on the machine in period 1, and produce in the first $q-b-\max \{(i-b_i: i \leq j), 0\}$ periods that are not skipped. If $q-b-\max \{(i-b_i: i \leq j), 0\} > i^*+1-i$, we produce in the first $q-b-(i^*+1-i)$ periods that are not skipped. We then turn on the machine in period in period $i$ and produce for $\min (q-b-\max \{(i-b_i: i \leq j), 0\}, i^*+1-i)$ periods. We can also produce once in each of the $b$ skipped periods. This feasible solution violates the inequality. Hence the condition is necessary.

For example, consider the following inequality:
\[ \sum_{i=1}^{10} w_i + y_{12} + (y_{13} + z_{13}) + (y_{14} + z_{14}) + 2z_{15} + z_{16} + \sum_{i=17}^{50} w_i \geq 4. \]

We skip period 11. Assume \( t_2 \geq 16 \). For \( i^* = 16 \) and \( i=14 \), \( i \) is in the 2nd demand interval, and let \( j^* = 1 \). The quantity \( \max(i-b_i; i \leq j^*) = 1 \). The quantity \( \min(q-b-\max((i-b_i; i \leq j), 0), i^*+1-i) = \min(5-1-1, 16+1-14) = 3 \). However, the sum of the coefficients of \( z_{14}, y_{14}, y_{15} \) and \( y_{16} \) is 2. The following feasible solution violates the inequality: \( z_1 = y_1 = w_1 = 1, z_{11} = y_{11} = w_{11} = 1, z_{14} = y_{14} = y_{15} = y_{16} = w_{14} = w_{15} = w_{16} = 1 \).

Some valid skip inequalities are:

\[ \sum_{i=1}^{10} w_i + y_{13} + (y_{14} + z_{14}) + 2z_{15} + z_{16} + \sum_{i=17}^{50} w_i \geq 3 \]

skips periods 11 and 12

\[ \sum_{i=1}^{20} w_i + y_{23} + z_{24} + z_{25} + z_{26} + \sum_{i=27}^{50} w_i \geq 3 \]

skips periods 21 and 22
Examples

The following examples illustrate the necessity of different types of inequalities. We include all the inequalities described so far in a linear programming relaxation of the original integer program. Each example has an integer optimal solution when solved as a linear program. However, when we remove any particular inequality, the linear program has a fractional solution whose objective function value is less than the integer optimal solution.

We consider a single product problem with two demands in periods $t_1=5$, and $t_2=T=7$. All the examples have 7 time periods, but we vary the cost structure in each example.

*Inequality extending up to the first demand period $t_1=5$.*

**Example 1.**

Necessity of the inequality $w_1y_2+z_3y_4+z_5 \geq 1$.

Consider the following cost structure:

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn on $(k_i)$</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Setup $(s_i)$</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Production $(h_i)$</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The optimal integer objective function value is 10, and an optimal solution is $z_2=y_2=y_3=w_2=w_3=1$. The linear program with all the
inequalities also has the same solution. However, if we remove the indicated inequality from the linear program, the optimal objective function becomes 6.67, and the optimal solution is fractional. Moreover, the solution satisfies all other inequalities.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z_i )</td>
<td>0</td>
<td>2/3</td>
<td>0</td>
<td>2/3</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( y_i )</td>
<td>0</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
<td>2/3</td>
<td>0</td>
</tr>
<tr>
<td>( w_i )</td>
<td>0</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
<td>2/3</td>
<td>0</td>
</tr>
</tbody>
</table>

*Skip Inequalities.*

**Example 2.**

Necessity of the inequality \( y_2 + z_3 + z_4 + z_5 + z_6 + z_7 \geq 2. \)

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn on (( k_i ))</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Setup (( s_i ))</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Production (( h_i ))</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The linear program has the following (optimal) integer solution: \( z_1 = y_1 = y_2 = w_1 = w_2 = 1. \) The objective function value is 10. If we remove the indicated inequality from the linear program, the objective function
value becomes 5, and we obtain the following fractional solution, which satisfies all the other inequalities.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_i$</td>
<td>1</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$y_i$</td>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>$w_i$</td>
<td>1</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Example 3.

Necessity of the inequality $w_1 + w_2 + w_3 + y_5 + z_6 + z_7 \geq 2$.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn on ($k_i$)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Setup ($s_i$)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Production ($h_i$)</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The linear program has the following (optimal) integer solution: $z_4 = y_4 = y_5 = w_4 = w_5 = 1$. The objective function value is 10. If we remove the indicated inequality from the linear program, the objective function value becomes 5, and we obtain the following fractional solution, which satisfies all the other inequalities.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_i$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$y_i$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>
\[ w_i \quad 0 \quad 0 \quad 0 \quad 1 \quad 0.5 \quad 0 \quad 0.5. \]

The turn on, setup and capacity constraints

Example 4.

Next, we illustrate the necessity of the turn on constraints \( z_i + y_{i-1} - y_i \geq 0 \), the setup constraints \( y_i \geq w_i \), and the capacity constraints \( y_i \leq 1, z_i \leq 1 \). The following example shows that if we remove the constraint \( z_3 + y_2 - y_3 \geq 0 \) from the linear program, we obtain a fractional optimal solution.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn on (k_i)</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Setup (s_i)</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Production (h_i)</td>
<td>10</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

The linear program has the following (optimal) integer solution: \( z_1 = y_1 = y_2 = w_1 = w_2 = 1 \). The objective function value is 20. If we remove the indicated inequality from the linear program, the objective function value becomes 12.5, and we obtain the following fractional solution, which satisfies all the other inequalities.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z_i )</td>
<td>1/4</td>
<td>1</td>
<td>0</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
<td>0</td>
</tr>
</tbody>
</table>
\[ \begin{align*}
\ y_i & \quad 1/4 \quad 1/4 \quad 1 \quad 1/4 \quad 1/4 \quad 1/2 \quad 0 \\
\ w_i & \quad 1/4 \quad 0 \quad 1 \quad 1/4 \quad 1/4 \quad 1/4 \quad 0 \\
\end{align*} \]

Example 5.

The necessity of the constraint \( y_3 \geq w_3 \).

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn on ( (k_i) )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Setup ( (s_i) )</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Production ( (h_i) )</td>
<td>10</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

The linear program has the following (optimal) integer solution: \( z_1 = y_1 = y_2 = w_1 = w_2 = 1 \). The objective function value is 20. If we remove the indicated inequality from the linear program, the objective function value becomes 10, and we obtain the following fractional solution, which satisfies all the other inequalities.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z_i )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( y_i )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( w_i )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Example 6.

The necessity of the constraint \( y_2 \leq 1 \).
<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn on ($k_i$)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Setup ($s_i$)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Production ($h_i$)</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

The linear program has the following (optimal) integer solution: $z_1 = y_1 = y_2 = w_1 = w_2 = 1$. The objective function value is 10. If we remove the indicated inequality from the linear program, the objective function value becomes 0, and we obtain the following fractional solution, which satisfies all the other inequalities.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_i$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$y_i$</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$w_i$</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The two interval inequalities extending up to $T=7$.

Example 7.

Necessity of the inequality $w_1 + w_2 + w_3 + w_4 + w_5 + y_6 + z_7 \geq 2$.

Consider the following cost structure:

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
</table>
Turn on \( (k_i) \) \[ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 10 \]

Setup \( (s_i) \) \[ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 10 \quad 0 \]

Production \( (h_i) \) \[ 10 \quad 10 \quad 10 \quad 10 \quad 10 \quad 0 \quad 0 \]

The optimal integer objective function value is 20, and an optimal solution is \( z_5 = y_5 = y_6 = w_5 = w_6 = 1 \). The linear program with all the inequalities also has the same solution. However, if we remove the indicated inequality from the linear program, the optimal objective function becomes 15, and the optimal solution is fractional. Moreover, the solution satisfies all other inequalities.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z_i )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( y_i )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>( w_i )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

**Example 8.**

Necessity of the inequality \( y_1 + (y_2 + z_2) + z_3 + w_4 + w_5 + w_6 + w_7 \geq 2 \).

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn on ( (k_i) )</td>
<td>0</td>
<td>10</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Setup ( (s_i) )</td>
<td>10</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Production ( (h_i) )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>
The linear program has the following (optimal) integer solution:
\( z_1 = y_1 = y_2 = w_1 = w_2 = 1 \). The objective function value is 20. If we remove the indicated inequality from the linear program, the objective function value becomes 15, and we obtain the following fractional solution, which satisfies all the other inequalities.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z_i )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( y_i )</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( w_i )</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Example 9.

Necessity of the inequality \( y_1 + (y_2 + z_2) + (w_3 + z_3) + z_4 + w_5 + w_6 + w_7 \geq 2 \).

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn on ( (k_i) )</td>
<td>0</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Setup ( (s_i) )</td>
<td>10</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Production ( (h_i) )</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

The linear program has the following (optimal) integer solution:
\( z_1 = y_1 = y_2 = w_1 = w_2 = 1 \). The objective function value is 20. If we remove the indicated inequality from the linear program, the objective function value becomes 15, and we obtain the following fractional solution, which satisfies all the other inequalities.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
</table>


\[ z_i \quad 1 \quad 0 \quad 0.5 \quad 0 \quad 0 \quad 0.5 \quad 0 \]
\[ y_i \quad 0.5 \quad 0.5 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \]
\[ w_i \quad 0.5 \quad 0.5 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \]

**Example 10.**

Necessity of the inequality \( y_1 + (w_2 + z_2) + (y_3 + z_3) + z_4 + w_5 + w_6 + w_7 \geq 2 \).

\[
\begin{array}{cccccccc}
\text{Period} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\text{Turn on} (k_j) & 0 & 10 & 10 & 10 & 0 & 0 & 0 \\
\text{Setup} (s_i) & 10 & 0 & 10 & 0 & 0 & 0 & 0 \\
\text{Production} (h_i) & 0 & 10 & 10 & 0 & 10 & 10 & 10 \\
\end{array}
\]

The linear program has the following (optimal) integer solution: \( z_1 = y_1 = y_2 = w_1 = w_2 = 1 \). The objective function value is 20. If we remove the indicated inequality from the linear program, the objective function value becomes 15, and we obtain the following fractional solution, which satisfies all the other inequalities.

\[
\begin{array}{cccccccc}
\text{Period} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\text{Period} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\text{z_i} & 1 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\
\text{y_i} & 0.5 & 0.5 & 0.5 & 1 & 0 & 0 & 0 \\
\text{w_i} & 0.5 & 0 & 0.5 & 1 & 0 & 0 & 0 \\
\end{array}
\]

**Example 11.**
Necessity of the inequality \( y_1 + (w_2 + z_2) + (y_3 + z_3) + (w_4 + z_4) + z_5 + w_6 + w_7 \geq 2 \).

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn on ((k_i))</td>
<td>0</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Setup ((s_i))</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Production ((h_i))</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

The linear program has the following (optimal) integer solution: \( z_1 = y_1 = y_2 = w_1 = w_2 = 1 \). The objective function value is 20. If we remove the indicated inequality from the linear program, the objective function value becomes 16.67, and we obtain the following fractional solution, which satisfies all the other inequalities.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z_i )</td>
<td>2/3</td>
<td>0</td>
<td>0</td>
<td>2/3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( y_i )</td>
<td>2/3</td>
<td>2/3</td>
<td>1/3</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( w_i )</td>
<td>2/3</td>
<td>0</td>
<td>1/3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Example 12.

Necessity of the inequality \( y_1 + (y_2 + z_2) + 2z_3 + 2z_4 + z_5 + w_6 + w_7 \geq 2 \).

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn on ((k_i))</td>
<td>0</td>
<td>10</td>
<td>20</td>
<td>20</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Setup ((s_i))</td>
<td>10</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Production \((h_i)\) 0 0 0 0 0 10 10.

The linear program has the following (optimal) integer solution: \(z_1 = y_1 = y_2 = w_1 = v_2 = 1\). The objective function value is 20. If we remove the indicated inequality from the linear program, the objective function value becomes 16, and we obtain the following fractional solution, which satisfies all the other inequalities.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>(z_i)</td>
<td>2/5</td>
<td>1/5</td>
<td>0</td>
<td>0</td>
<td>2/5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(y_i)</td>
<td>2/5</td>
<td>3/5</td>
<td>3/5</td>
<td>3/5</td>
<td>4/5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(w_i)</td>
<td>2/5</td>
<td>2/5</td>
<td>0</td>
<td>2/5</td>
<td>4/5</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Example 13.

Necessity of the inequality \(y_1 + (y_2 + z_2) + 2z_3 + (w_4 + z_4) + z_5 + w_6 + w_7 \geq 2\).

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn on ((k_i))</td>
<td>0</td>
<td>10</td>
<td>20</td>
<td>10</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Setup ((s_i))</td>
<td>10</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Production ((h_i))</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

The linear program has the following (optimal) integer solution: \(z_1 = y_1 = y_2 = w_1 = w_2 = 1\). The objective function value is 20. If we remove the indicated inequality from the linear program, the objective function value becomes 16, and we obtain the following fractional solution, which satisfies all the other inequalities.
Example 14.

Necessity of the inequality \( y_1 + (w_2 + z_2) + (y_3 + z_3) + 2z_4 + z_5 + w_6 + w_7 \geq 2. \)

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z_i )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2/5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( y_i )</td>
<td>3/5</td>
<td>3/5</td>
<td>3/5</td>
<td>4/5</td>
<td>4/5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( w_i )</td>
<td>2/5</td>
<td>2/5</td>
<td>2/5</td>
<td>0</td>
<td>4/5</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Turn on (\( k_j \))

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_j )</td>
<td>0</td>
<td>10</td>
<td>10</td>
<td>20</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Setup (\( s_i \))

<table>
<thead>
<tr>
<th>Period</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_i )</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Production (\( h_i \))

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h_i )</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
</tr>
</tbody>
</table>

The linear program has the following (optimal) integer solution: \( z_1 = y_1 = y_2 = w_1 = w_2 = 1. \) The objective function value is 20. If we remove the indicated inequality from the linear program, the objective function value becomes 17.14286, and we obtain the following fractional solution, which satisfies all the other inequalities.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z_i )</td>
<td>5/7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2/7</td>
<td>3/7</td>
<td>0</td>
</tr>
<tr>
<td>( y_i )</td>
<td>5/7</td>
<td>5/7</td>
<td>3/7</td>
<td>3/7</td>
<td>4/7</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( w_i )</td>
<td>4/7</td>
<td>0</td>
<td>2/7</td>
<td>2/7</td>
<td>4/7</td>
<td>2/7</td>
<td>0</td>
</tr>
</tbody>
</table>
Example 15.

Necessity of the inequality \( y_1 + (y_2 + z_2) + z_3 + (y_4 + z_4) + z_5 + w_6 + w_7 \geq 2 \).

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn on ((k_i))</td>
<td>0</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Setup ((s_i))</td>
<td>10</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Production ((h_i))</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

The linear program has the following (optimal) integer solution: \( z_1 = y_1 = y_2 = w_1 = w_2 = 1 \). The objective function value is 20. If we remove the indicated inequality from the linear program, the objective function value becomes 16.67, and we obtain the following fractional solution, which satisfies all the other inequalities.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z_i )</td>
<td>2/3</td>
<td>0</td>
<td>2/3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( y_i )</td>
<td>1/3</td>
<td>1/3</td>
<td>1</td>
<td>1/3</td>
<td>1/3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( w_i )</td>
<td>1/3</td>
<td>1/3</td>
<td>2/3</td>
<td>1/3</td>
<td>1/3</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Example 16.

Necessity of the inequality \( y_1 + (y_2 + z_2) + z_3 + (y_4 + z_4) + 2z_5 + z_6 + w_7 \geq 2 \).

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn on ((k_i))</td>
<td>0</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>20</td>
<td>10</td>
<td>0</td>
</tr>
</tbody>
</table>
Setup \((s_i)\) \hspace{1cm} 10 \hspace{0.2cm} 10 \hspace{0.2cm} 0 \hspace{0.2cm} 10 \hspace{0.2cm} 0 \hspace{0.2cm} 0 \hspace{0.2cm} 0 \hspace{0.2cm} 0  \\
Production \((h_i)\) \hspace{1cm} 0 \hspace{0.2cm} 0 \hspace{0.2cm} 0 \hspace{0.2cm} 0 \hspace{0.2cm} 0 \hspace{0.2cm} 0 \hspace{0.2cm} 0 \hspace{0.2cm} 10.

The linear program has the following (optimal) integer solution: \(z_1=y_1=y_2=w_1=w_2=1\). The objective function value is 20. If we remove the indicated inequality from the linear program, the objective function value becomes 16.67, and we obtain the following fractional solution, which satisfies all the other inequalities.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>(z_i)</td>
<td>1/3</td>
<td>0</td>
<td>2/3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(y_i)</td>
<td>1/3</td>
<td>1/3</td>
<td>1</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
<td>0</td>
</tr>
<tr>
<td>(w_i)</td>
<td>1/3</td>
<td>1/3</td>
<td>2/3</td>
<td>1/3</td>
<td>0</td>
<td>1/3</td>
<td>0</td>
</tr>
</tbody>
</table>

Example 17.

Necessity of the inequality \(y_1+(y_2+z_2)+z_3+(y_4+z_4)+(w_5+z_5)+z_6+w_7 \geq 2\).

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn on ((k_i))</td>
<td>0</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>Setup ((s_i))</td>
<td>10</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Production ((h_i))</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>0</td>
<td>10</td>
</tr>
</tbody>
</table>

The linear program has the following (optimal) integer solution: \(z_1=y_1=y_2=w_1=w_2=1\). The objective function value is 20. If we remove the indicated inequality from the linear program, the objective function
value becomes 17.5, and we obtain the following fractional solution, which satisfies all the other inequalities.

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_i$</td>
<td>1/4</td>
<td>0</td>
<td>3/4</td>
<td>0</td>
<td>1/4</td>
<td>0</td>
<td>1/4</td>
</tr>
<tr>
<td>$y_i$</td>
<td>1/4</td>
<td>1/4</td>
<td>1</td>
<td>1/4</td>
<td>1/2</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>$w_i$</td>
<td>1/4</td>
<td>1/4</td>
<td>3/4</td>
<td>1/4</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
</tr>
</tbody>
</table>

We can summarize the data as follows. For each inequality, we use the following cost structure: if variable $w_i$ (or $y_i$ or $z_i$) belongs to the inequality, then we set the changeover cost $k_i$ (or the setup cost $s_i$, or the production cost $h_i$) equal to 10. Otherwise, we set the cost equal to 0. In all cases, the linear programming solution after including all the valid inequalities was equal to the optimal integer solution.

<table>
<thead>
<tr>
<th>Example</th>
<th>IP solution</th>
<th>LP solution after relaxing constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>6.67</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>12.5</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>----</td>
<td>---</td>
</tr>
<tr>
<td>6</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>8</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>9</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>10</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>11</td>
<td>20</td>
<td>16.67</td>
</tr>
<tr>
<td>12</td>
<td>20</td>
<td>16</td>
</tr>
<tr>
<td>13</td>
<td>20</td>
<td>16</td>
</tr>
<tr>
<td>14</td>
<td>20</td>
<td>17.14</td>
</tr>
<tr>
<td>15</td>
<td>20</td>
<td>16.67</td>
</tr>
<tr>
<td>17</td>
<td>20</td>
<td>17.5</td>
</tr>
</tbody>
</table>
Proof of Validity

Proposition 1. Inequality (PI) is valid if and only if it satisfies the compensation condition.

Proof.

We have already established the necessity of the condition.

Consider any feasible solution. We show that it must satisfy the inequality. The essential idea of the proof is to account for each unit of production up to the qth production, i.e., each unit of production up to the qth unit contributes at least one unit to the lefthand side, either in the period in which we produce, or by compensating in some earlier period.

If we produce in any period belonging to W, Y or YZ, then the period contributes one unit to the lefthand side. Suppose we produce in some period \(i^*\) belonging to Z. If we turn the machine on in period \(i^*\), then the period contributes at least one unit to the lefthand side. If we do not turn the machine on in period \(i^*\), then we must turn it on in some period \(i < i^*\), and keep it on until period \(i^*\). Suppose period \(i\) belongs to the \(j+1\)st demand interval. Then the compensation condition ensures that the contribution of the terms in periods \(i\) through \(i^*\) is at least \(\min(i^*+1-i, q-j)\). Since we must produce at least \(j\) times up to \(t_j\) in any feasible solution, we need to produce at most \(q-j\) times from period \(i\) onwards to meet the demand up to \(t_q\). Notice that we can produce at most \(i^*+1-i\) times in periods \(i\) through \(i^*\). Hence the lefthand side increases by at least one unit for each unit of production up to \(q\) units.
For example, suppose we have demands in periods 5, 10, 15, 20 and 25, and we consider the following feasible schedule: turn the machine on in period 4, and produce in periods 4 through 8. Consider the following valid inequality:

\[ y_1 + (y_2 + z_2) + (y_3 + 2z_3) + (y_4 + 3z_4) + 4z_5 + 3z_6 + 2z_7 + z_8 + \sum_{i=9}^{20} w_i \geq 4. \]

Periods 5, 6 and 7 contribute nothing to the lefthand side of the inequality, but period 4 compensates by contributing 4 units, since \( y_4 = z_4 = 1 \).

In Appendix 2, we show how to use integer programming aggregation and rounding arguments to establish the validity of certain versions of the inequalities.

**Proposition 2.** Any inequality (SI) is valid if and only if it satisfies the skip condition.

**Proof**

We have already established the necessity of the conditions.

We use an argument similar to the one we used for the partitioning inequalities. We account for each production in an unskipped period up to the qth production.

If we produce in a skipped period, the period contributes nothing to the lefthand side of the inequality. However, we 'account' for it by reducing the righthand side of the inequality by b, the total number of skipped periods up to \( t_q \). If we produce in a period in W, Y or YZ,
then the period contributes at least one unit to the lefthand side of the inequality. If we produce in a period in $Z$, and we turn the machine on in that period, then the period contributes at least one unit to the lefthand side. If we produce in period $i^* \in Z$, and we do not turn the machine on in that period, then we must turn it on in some period $i < i^*$, and keep it on until period $i^*$. Suppose period $i$ belongs to the $j+1$st demand interval. Then the skip condition ensures that periods $i$ through $i^*$ contribute at least \( \min(q-b-\max((j'-b_{j'}: j' \leq j), 0), i^*+1-i) \) (recall that $b$ is the number of periods skipped up to $t_q$, and $b_j$ is the number of periods skipped up to $t_j$).

We show by induction that any feasible solution contributes at least \( \max((j'-b_{j'}: j' \leq j), 0) \) units to the lefthand side of the inequality for periods up to $t_{j'}$, for $1 \leq j \leq q$.

1) Demand interval hypothesis. Any feasible solution contributes at least \( \max((j'-b_{j'}: j' \leq j), 0) \) units to the lefthand side of the inequality for periods up to $t_{j'}$ for $1 \leq j \leq q$.

2) Contribution hypothesis. Each unit of production up to $q-b$ units in unskipped periods contributes at least 1 unit to the lefthand side of the inequality for $1 \leq j \leq q$.

As shown earlier, we need to consider only the case when we produce in period $i^* \in Z$ by turning the machine on in period $i < i^*$. For $j = 1$, we skip $b_1$ periods up to $t_1$. If $b_1 \geq 1$, then it is true that every feasible solution contributes \( \max((1-b_1), 0) = 0 \) units for periods up to $t_1$. If $b_1 = 0$, then we do not skip any periods up to $t_1$. From the skip condition, since period $i$ lies in the $1$st demand interval, periods $i$
through \( i^* \) contribute at least \( \min (q-b, \ i^*+1-i) \) units, which is at least equal to 1. Further, each unit of production up to \( q-b \) units in unskipped periods contributes at least 1 unit to the lefthand side of the inequality. Hence the inductive hypotheses are true for \( j = 1 \).

Suppose the inductive hypotheses are true for arbitrary \( j < q \). We show that they are true for \( j+1 \). Notice that if we produce in any skipped periods in the \( j+1 \)st demand interval, then \( \max ((j'-b_{j'}: j' \leq j+1), 0) = \max ((j'-b_{j'}: j' \leq j), 0) \), and hence the demand interval hypothesis is satisfied.

If we produce \( j+1 \) or more times up to \( t_{j'} \) then the hypotheses are certainly true. If we produce exactly \( j \) times up to \( t_{j'} \) then we need to produce at most \( q-j \) times starting in the demand interval \( j+1 \) to meet the demand up to \( t_q \). Consider the case when we produce in period \( i^* \in Z \) by turning the machine on in period \( i < i^* \). From the skip condition, periods \( i \) through \( i^* \) contribute at least \( \min (q-b-\max ((j'-b_{j'}: j' \leq j), 0), i^*+1-i) \) units. We show that \( q-j \geq q-b-\max ((j'-b_{j'}: j' \leq j), 0) \).

\[
\begin{align*}
b + \max ((j'-b_{j'}: j' \leq j), 0) & \geq b + j - b_j \\
& \geq j \\
\text{Hence } q-j & \geq q-b-\max ((j'-b_{j'}: j' \leq j), 0).
\end{align*}
\]

If \( i^*+1-i < (q-b-\max ((j'-b_{j'}: j' \leq j), 0), 0) \), then the contribution is \( i^*+1-i \), and each unit of production contributes at least one unit. If \( (q-b-\max ((j'-b_{j'}: j' \leq j), 0)) \leq i^*+1-i \), then periods 1 through \( t_j \) contribute \( \max ((j'-b_{j'}: j' \leq j), 0) \) units, and periods \( i \) through \( i^* \) contribute \( q-b-\max ((j'-b_{j'}: j' \leq j), 0) \) units, and hence the inequality is satisfied. Thus both the hypotheses are satisfied for \( j+1 \).
Since the demand interval hypothesis is true for $j = q$, any feasible solution contributes at least $\max \left( \left\{ j'-b_j : j' \leq q \right\}, 0 \right)$ units. Since $\max \left( \left\{ j'-b_j : j' \leq q \right\}, 0 \right) \geq q-b$, the inequality is valid.
Chapter 3. Facets

It is possible to tighten the inequalities described in Chapter 2 if we impose certain additional conditions. In this chapter, we describe these conditions and show that the inequalities are facets.

Let us consider the skip condition. We wish to obtain a stronger condition that will enable us to show that some of the skip inequalities (which are generalizations of the partitioning inequalities) are facets. Suppose for some period $i$ in the $(j+1)$st demand interval, the sum of the coefficient of $z_i$ and the coefficients of $y$ from $i$ through $i^*$ is greater than $(q-b_{\max(i-b_i; i \leq j)}$ for $i^* \in Z$, $i^* = i+(q-b_{\max(i-b_i; i \leq j)}-1$. For example, suppose we have demands in periods 10, 20, 30, 40 and 50, and we consider the following inequality

$$y_1+(y_2+z_2)+(y_3+2z_3)+(y_4+4z_4)+4z_5...$$

$$+3z_6+2z_7+z_8+w_9+w_{10}+\sum_{i=12}^{50} w_i \geq 4,$$

which skips period 11. Then for $i=4$, the sum of the coefficient of $z_4$ and the coefficients of $y$ from 4 through 7 is equal to 5. Hence we can reduce the coefficient of $z$ in period 4 to 3. For $i = 6$, the sum of the coefficient of $z_6$ and the coefficients of $y$ from 6 through 10 is only equal to 3. But then the periods 9 and 10 are not in $Z$.

Similarly in the following example

$$y_1+(y_2+z_2)+(y_3+z_3)+(y_4+2z_4)+(y_5+3z_5)...$$
\[ +3z_6 + (y_7 + 2z_7) + 2z_8 + z_9 + \sum_{i=10}^{50} w_i \geq 5, \]

the sum of the coefficients of \( z_i \) and the coefficients of \( y \) from \( i \) through \( i+4 \) is 5 which is equal to the right-hand side of the inequality.

So we can state the following condition:

**Condition 1.** For any period \( i \) in the \((j+1)\)st demand interval, if \( i^* \) is in \( \mathbb{Z} \), and \( i^* = (q-b{-}\text{-max}(i-b_i, i \leq j)) \), the sum of the coefficient of \( z_i \) and the coefficients of \( y \) from \( i \) through \( i^* \) is exactly equal to \((q-b{-}\text{-max}(i-b_i, i \leq j))\). On the other hand, if \( i^* \) is in \( \mathbb{Z} \) and \( i^* < (q-b{-}\text{-max}(i-b_i, i \leq j)) \), then the sum of the coefficient of \( z_i \) and the coefficients of \( y \) from \( i \) through \( i^* \) is exactly equal to \((i^*+1-i)\).

We next show that the coefficients of \( y_i \) and \( w_i \) must be either 1 or zero (if the period is skipped). Consider the following inequality:

\[ w_1 + 3y_2 + (y_3 + 3z_3) + (y_4 + 4z_4) + 5z_5 + 4z_6 + 3z_7 + 2z_8 + z_9 + \sum_{i=10}^{50} w_i \geq 5. \]

In period 2, the coefficient of \( y \) is 3. We can use the inequalities

\[ y_2 + z_3 + z_4 + z_5 + z_6 \geq y_6 \quad \text{and} \]

\[ y_2 + z_3 + z_4 + z_5 \geq y_5 \]

and obtain the tighter valid inequality

\[ w_1 + y_2 + (y_3 + z_3) + (y_4 + 2z_4) + (y_5 + 3z_5) + (y_6 + 3z_6) + 3z_7 + 2z_8 + z_9 + \sum_{i=10}^{50} w_i \geq 5. \]

In general, suppose that for some period \( i \in Y \cup YZ \) in the \((j+1)\)st demand interval the coefficient of \( y_i \) is \( c>1 \). Then by condition 1, the sum of the coefficient of \( z_i \) and the coefficients of \( y \) from \( i \) through \( i^* \) is exactly equal to \( \min (q-b{-}\text{-max}(i-b_i, i \leq j), i^*+1-i) \) for some \( i^* \geq i, i^* \in \mathbb{Z} \). If more
than one value of \( i^* \) satisfies the condition, then we choose the minimum value. We can then use the inequalities

\[
y_1 + z_{i+1} + \ldots + z_{i+c-2} + z_{i+c-2} \geq y_{i+c-2}
\]
\[
y_1 + z_{i+1} + \ldots + z_{i+c-3} + z_{i+c-3} \geq y_{i+c-3}
\]
\[
\vdots
\]
\[
y_1 + z_{i+1} + \ldots + z_{i} \geq y_{i}
\]

and obtain the tighter valid inequality. Similarly, we show that the coefficient of \( w_i \) must be 0 or 1. If it is greater, we can simply reduce the coefficient to 1 because from condition 1, the sum of the coefficient of \( z_i \) and the coefficients of \( y \) from \( i+1 \) to any period \( i^* \in Z \) is sufficient to ensure validity. Thus we can state the following condition:

**Condition 2.** The coefficient of \( y_i \) for a period \( i \in Y \cup YZ \) and the coefficient of \( w_i \) for a period \( i \in W \cup WZ \) is equal to 1.

We now show that if \( t=1 \in YZ \cup WZ \) then the inequality can be tightened. Consider the following example:

\[
(y_1 + z_1) + (y_2 + 2z_2) + (y_3 + 3z_3) + (y_4 + 3z_4) + 3z_5 + 2z_6 + z_7 + \sum_{i=8}^{50} w_i \geq 5
\]

We use the relation \( z_1 + z_2 + \ldots + z_5 \geq y_5 \) (which in turn is derived by summing the inequalities \( z_1 \geq y_1, z_2 + y_1 - y_2 \geq 0, \ldots, z_5 + y_4 - y_5 \geq 0 \)) to tighten the inequality and obtain:

\[
y_1 + (y_2 + z_2) + (y_3 + 2z_3) + (y_4 + 2z_4) + (y_5 + 2z_5) + 2z_6 + z_7 + \sum_{i=8}^{50} w_i \geq 5
\]

In general, suppose \( i \in Z \) for \( i \leq q - b \). Then if \( = q - b \), we have a subsequence of the form:

\[
(y_1 + z_1) + (y_2 + 2z_2) + \ldots + (y_{q - b - 1} + cz_{q - b - 1}) + cz_{q - b} + (c - 1) z_{q - b + 1} + \ldots
\]
Using the inequality \( z_i + y_{i-1} \geq y_i \), we replace all \( z_i + y_{i-1} \) by \( y_i \), for \( i = 1, \ldots, q - b \) and obtain the following subsequence:

\[
y_1 + (y_2 + z_2) + \ldots + (y_{q-b-1} + (c-1)z_{q-b-1}) + (y_{q-b} + (c-1)z_{q-b}) + (c-1)z_{q-b+1} + \ldots
\]

This gives us a tighter valid inequality. To see that it is still valid, notice that if we turn the machine on in period \( i \leq q - b \), and keep it on until period \( i^* \geq q - b + 1 \), we can produce at most \( i^* + 1 - i \) times. But the sum of the coefficients of \( z_i \) and the coefficients of \( y \) from \( i \) through \( i^* \) is at least \( \min(q, i^* + 1 - i) \). Similarly, if we turn the machine on in period \( q - b < i \leq q - b + c - 1 \), we can produce at most \( q - b + c - i \) times up to period \( q - b + c - 1 \). But this is the coefficient of \( z_i \). Hence the inequality is valid.

If \( i \in Z \) and \( i < q - b \), then consider the following example:

\[
(y_1 + 2z_1) + (y_2 + 3z_2) + (y_3 + 4z_3) + 5z_4 + 4z_5 + 3z_6 + 2z_7 + z_8 + \sum_{i=9}^{50} w_i \geq 5.
\]

We use the inequalities \( z_1 + z_2 + z_3 + z_4 + z_5 \geq y_5 \), and \( z_1 + z_2 + z_3 + z_4 \geq y_4 \) and obtain

\[
y_1 + (y_2 + z_2) + (y_3 + 2z_3) + (y_4 + 3z_4) + (y_5 + 3z_5) + 3z_6 + 2z_7 + z_8 + \sum_{i=9}^{50} w_i \geq 5.
\]

We also require that \( 1 \in Z \). Otherwise we can replace \( z_1 \) by \( y_1 \) and since \( z_1 \geq y_1 \), we obtain a tighter inequality.

We therefore state the following condition:

**Condition 3.** Period \( i \in Z \) for \( i \leq q - b \).
We now consider period \( i \in Y \). If period \( i+1 \) is skipped or belongs to \( W \cup Y \), we can replace \( y_i \) by \( w_i \) and still have a valid inequality. For example, the following inequality

\[
y_1 + (y_2 + z_2) + (y_3 + 2z_3) + (y_4 + 3z_4) + (y_5 + 3z_5) + 3z_6 + 2z_7 + z_8 + y_9 + \sum_{i=10}^{50} i \geq 5.
\]

satisfies all the previous conditions. But we can tighten it by replacing \( y_9 \) by \( w_9 \). Since \( y_i \geq w_i \), we see that the inequality is dominated. Hence we need the following condition:

**Condition 4.** If \( i \in Y \) then \( i+1 \in Y \cup W \cup Z \). In particular, \( t_q \in Y \).

We next consider the location of the skipped periods. For example, suppose all the skipped periods are in the last demand interval \( (q) \), and the number of skipped periods is \( b \geq 1 \). The sum of the terms in the inequality up to \( t_{q-1} \) for any feasible solution is \( q-1 \geq q - b \). We can therefore drop all variables with indices greater than \( t_{q-1} \) from the inequality. It is therefore clear that we must impose some condition on the location of the skipped periods. If the number of skipped periods after period \( t_j \) is \( q-j \) or more, it means that the right hand side is at most \( q-(q-j) \). Moreover the number of skipped periods up to \( t_j \) is at most \( b-(q-j) \), and the sum of the terms in the inequality up to \( t_j \) is at least \( j-(b-(q-j)) = q-b \). Therefore, we can simply drop all variables with indices more than or equal to \( t_j+1 \) and still have a valid inequality.

For example, in the following inequality:

\[
y_1 + (y_2 + z_2) + z_3 + \sum_{i=4}^{10} w_i + \sum_{i=12}^{30} w_i + \sum_{i=33}^{50} w_i \geq 2,
\]

we skip periods 11, 31, and 32. However, for \( j=3 \), we skip \( (q-j) = (5-3) = 2 \) periods after \( t_j \). Hence we can drop all the terms in this inequality with indices greater than \( t_3 = 30 \).
Therefore we impose the following condition:

**Condition 5.** The number of periods skipped from $t_j + 1$ through $t_q$ must be strictly less than $q - j$, for $j = 0, \ldots, q - 1$.

Some inequalities can be derived from others. For example, the inequality $\sum_{i=2}^{t_2} w_i \geq 1$ can be derived from the inequalities $\sum_{i=1}^{t_2} w_i \geq 2$ and $1 \geq w_1$. On the other hand, the inequality $y_2 + \sum_{i=3}^{t_2} z_i \geq 1$ cannot be derived from other inequalities (we show later that it is a facet). This indicates that if we skip some periods up to some demand period $t_j$, then there must be some period in $Z$ before $t_j$.

We state this requirement more precisely in the following discussion. Suppose we skip $b_j$ periods up to $t_j$ that $b_j \leq j$, and that $i \in W$ for $i \leq t_j$ or $i$ is skipped. For example consider the following inequality:

$$\sum_{i=3}^{30} w_i + y_{31} + (y_{32} + z_{32}) + z_{33} + \sum_{i=34}^{50} w_i \geq 3 \text{ (skipping 1 and 2)}$$

We can replace it by the following inequalities:

$$\sum_{i=1}^{30} w_i + y_{31} + (y_{32} + z_{32}) + z_{33} + \sum_{i=34}^{50} w_i \geq 5$$

$$1 \geq w_1$$

$$1 \geq w_2$$

Let us denote the terms in the inequality (SI) after $t_j$ as (SI'). Then the inequality (SI) can be written as a linear combination of the following inequalities:

$$\sum_{i=1}^{t_j} w_i + (SI') \geq q - b + b_j \text{ for } b_j \leq j \quad \text{(i)}$$
1 \geq w_s \quad \text{for each skipped period up to } t_j.

We show that the first inequality is valid. Suppose we produce W(j) times up to period \( t_{j'} \) (notice that \( W(j) \geq j \)) then the contribution of the terms up to \( t_j \) is \( W(j)-b_j \). Therefore the contribution of the terms after \( t_j \) in the original skip inequality is at least \( q-b-(W(j)-b_j) \). But in inequality (i), the sum of the terms up to \( t_j \) is \( W(j) \). The sum of the terms after \( t_j \) remains \( q-b-(W(j)-b_j) \) since we do not change these terms. Hence it must be at least \( q-b+b_j \).

We indicate why we need the condition \( b_j \leq j \). Consider the following inequality in which \( b_1 = 2 \), i.e., we skip two periods up to \( t_1 \):

\[
\sum_{i=3}^{10} w_i + y_{11} + (y_{12} + z_{12}) + (y_{13} + z_{13}) + z_{14} + \sum_{i=15}^{50} w_i \geq 3 \quad \text{(skipping 1 and 2)}.
\]

This inequality cannot be tightened by adding the terms \( w_1 + w_2 \) on the lefthand side and increasing the righthand side to 5 because that does not yield a valid inequality. A feasible solution that violates the inequality is: \( z_{11} = y_{11} = w_1 = 1 \), and \( z_{12} = y_{12} = y_{13} = y_{14} = w_{11} = w_{12} = w_{13} = w_{14} = 1 \).

Hence we impose the following condition:

**Condition 6.** If \( 0 < b_j \leq j \) periods are skipped up to period \( t_j \), then \( i \in Y \cup Z \cup YZ \cup WZ \) for some \( i \leq t_j \).

Consider the situation when \( q=n \). If \( YZ \cup Y \cup Z \cup WZ = \emptyset \), then by the previous condition no periods are skipped. Hence the inequality reduces to

\[
\sum_{i=1}^{T} w_i \geq n.
\]
But this is implied by $\sum_{i=1}^{T} w_i = n$. Hence we impose the condition that if $q=n$, then $YZ \cup Y \cup Z \cup WZ \neq \emptyset$.

For $q=n$, we first argue that $|Y| \geq 1$. If $Z = \emptyset$, then from condition 1 $YZ \cup Y \cup WZ = \emptyset$. Hence, $YZ \cup Y \cup Z \cup WZ = \emptyset$, which contradicts the previous condition. Therefore $Z \neq \emptyset$. Further, from condition 3, period $1 \in YZ \cup WZ$. Therefore, the first $i \in Z$ (i.e., $i \leq i'$ for all $i' \in Z$) is preceded by a sequence in $YZ \cup WZ$, which in turn is preceded by $i \in Y$. Therefore $|Y| \geq 1$. Suppose that $|Y| > 1$. We show that (SI) is a combination of other inequalities and hence cannot be a facet. Consider the following example: suppose demands occur in periods 4 and 8. Let us look at an inequality with $|Y| = 2$ and no skipped periods:

$$w_1 + y_2 + (y_3 + z_3) + z_4 + w_5 + w_6 + y_7 + z_8 \geq 2$$

We can obtain this inequality as a linear combination of the following inequalities and hence it is not a facet:

$$2 = w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + w_7 + w_8$$

$$w_1 + y_2 + (y_3 + z_3) + z_4 + w_5 + w_6 + w_7 + w_8 \geq 2$$

$$w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + y_7 + z_8 \geq 2$$

We can generalize this observation as follows. Suppose that $|Y| > 1$. Each $t \in Y$ is followed by a sequence of periods in $YZ \cup WZ$, which in turn is followed by periods in $Z$. Let us denote the last period in $Z$ in this sequence by $t^*$, and the terms in periods $t$ through $t^*$ by $(SI(t, t^*))$, and the terms in the other periods by $(SI) \setminus (SI(t, t^*))$.

We can write this inequality as a linear combination of the following inequalities (and hence cannot be a facet):
\[ n = \sum_{i=1}^{T} w_i \] written \(|Y| - 1\) times.

\[ \sum_{i=1, \text{ } i \notin \{t, \ldots, t^*\}}^{T} w_i + (\text{SI}(t, t^*)) \geq n \] for each \( t \in Y \).

\[ 1 \geq w_i \] for each skipped period \( i \).

For example, suppose demands occur in periods 10, 20, 30, 40 and 50, and \( n = 5 \). Consider the following inequality:

\[
y_1 + (y_2 + z_2) + (y_3 + z_3) + (y_4 + z_4) + (y_5 + z_5) + z_6 + \sum_{i=7}^{10} w_i + \ldots \]

\[
y_{11} + (y_{12} + z_{12}) + (y_{13} + z_{13}) + (y_{14} + z_{14}) + z_{15} + \sum_{i=16}^{50} w_i \geq 5.
\]

We can write this inequality as a linear combination of the following inequalities:

\[ 5 = \sum_{i=1}^{50} w_i \]

\[ y_1 + (y_2 + z_2) + (y_3 + z_3) + (y_4 + z_4) + (y_5 + z_5) + z_6 + \sum_{i=7}^{50} w_i \geq 5 \]

\[ \sum_{i=1}^{10} w_i + y_{11} + (y_{12} + z_{12}) + (y_{13} + z_{13}) + (y_{14} + z_{14}) + z_{15} + \sum_{i=16}^{50} w_i \geq 5. \]

Hence we impose the following condition:

\textbf{Condition 7.} If \( q = n \), then \( Y \cup Z \cup YZ \cup WZ \neq \emptyset \), and \(|Y| = 1\).

Finally, we consider special cases. Suppose \( t_1 = 2 \), and the inequality (SI) has \( y_1 + (y_2 + z_2) \) in the first two periods. Let us denote the terms in the inequality after period 2 by (SI'). The inequality can be written as the linear combination of the following two inequalities:

\[ y_1 + z_2 \geq 1 \]

\[ y_2 + (\text{SI'}) \geq q - b - 1 \]
The second inequality is valid because there is one more skipped period (it skips period 1). Therefore the right hand side is reduced to $q-b-1$. Since (SI) can be written as a linear combination of other inequalities, it cannot be a facet. Consider another example. Suppose $t_1=3$, and the inequality contains the terms $y_1 + (y_2 + z_2) + (y_3 + 2z_3)$ in the first three periods. Let us denote the terms in the inequality after period 3 by (SI'). Then we can replace this inequality by the following two inequalities:

$$y_1 + z_2 + z_3 \geq 1$$

$$y_2 + (y_3 + z_3) + (SI') \geq q-b-1$$

For example, we can replace the following inequality (demands in periods 3, 10, 15 and 20)

$$y_1 + (y_2 + z_2) + (y_3 + 2z_3) + (y_4 + 2z_4) + 2z_5 + z_6 + \sum_{i=7}^{20} w_i \geq 4$$

by

$$y_1 + z_2 + z_3 \geq 1$$

and

$$y_2 + (y_3 + z_3) + (y_4 + 2z_4) + 2z_5 + z_6 + \sum_{i=7}^{20} w_i \geq 3 \text{ (skipping period 1)}$$

We therefore impose the following condition:

**Condition 8.** The inequality does not contain the terms $y_1 + (y_2 + z_2) + (y_3 + 2z_3) + \ldots + (y_{t_1} + (t_1-1)z_{t_1})$ in the first $t_1$ periods.

Otherwise we can replace this inequality by the inequalities

$$y_1 + z_2 + \ldots + z_{t_1} \geq 1$$

and

$$y_2 + (y_3 + z_3) + \ldots + (y_{t_1} + (t_1-2)z_{t_1}) + \text{(terms after period } t_1) \geq q-b-1.$$
We impose one more condition. Suppose we have demands in two consecutive periods, \( t_q \) and \( t_{q+1} = t_q + 1 \). If \( Y \cup Z \cup YZ \cup WZ = \emptyset \) for an inequality extending up to \( t_{q'} \), then the inequality reduces to \( \sum_{i=1}^{t_{q+1}} w_i \geq q \). But this is implied by the constraints \( \sum_{i=1}^{t_{q+1}} w_i \geq q + 1 \) and \( 1 \geq w_{i_{q+1}} \). Hence we impose the following condition:

**Condition 9.** If demands occur in two consecutive periods, \( t_q \) and \( t_{q+1} = t_q + 1 \), then for an inequality extending up to \( t_q \), \( Y \cup Z \cup YZ \cup WZ \neq \emptyset \).

The following theorem shows that the conditions are not only necessary, but also sufficient to define facets.

**Theorem 1.** The inequalities (SI) and (PI) are facets of \( C \) if and only if \( W, Y, Z, WZ \) and \( YZ \) satisfy the conditions 1 through 9.

**Proof.**

We have already established the necessity of the conditions.

**Sufficiency of the conditions.**

Let \( C^* = \{ (w, y, z) \in C : (w, y, z) \text{ satisfies (SI) at equality} \} \). To show that (SI) is a facet, we let \( \alpha w + \beta y + \gamma z = \delta \) represent an arbitrary equation that is satisfied by all \((w, y, z) \in C^*\). We show that \( \alpha w + \beta y + \gamma z = \delta \) must be a linear combination of

\[
\sum_{i \in W} w_i + \sum_{i \in Y} y_i + \sum_{i \in Z} c_i z_i + \sum_{i \in YZ} (y_i + c_i z_i) + \sum_{i \in WZ} (w_i + c_i z_i) = q - b
\]

and the only equality in SCSP which is

\[
\sum_{i=1}^{T} w_i = n.
\]
In the proof, we need to relate the values $\alpha$, $\beta$, $\gamma$ and $\delta$ to each other. We do this by generating different solutions in $C^*$, and comparing the resultant equations. For example, consider the inequality $\sum_{i=1}^{T_l} w_i \geq 1$. Two alternate solutions in $C^*$ are $z_1 = y_1 = w_1 = 1$, and $z_2 = y_2 = w_2 = 1$. These solutions yield the equations $\alpha_1 = \delta$ and $\alpha_1 = \delta$. From this we can say that $\alpha_1 = \alpha_2$.

We first define two solutions in $C^*$ which we call the $k^0$ solution and the $k(i^*)$ solutions. We shall use these two solutions as the basis for generating alternate solutions in $C^*$. The $k^0$ solution is the following: Define period $k^0$ as follows:

$$k^0 \cdot c = q \cdot b$$

where $c = \text{number of skipped periods upto } k^0$.

Note that $k^0$ is not uniquely defined. For example let $k'$ satisfy equation (1). Then if the period $k'+1$ is skipped, then $k'+1$ also satisfies the equation. We define $k^0$ to be the minimum of all $k'$ that satisfy equation (1).

We turn the machine on at $t=1$ and maintain the set up until time $k^0$ and produce in each period up to $k^0$. Since by condition 3 period $t=1 \in YZWZW\infinite Z$, this production sequence contributes $q \cdot b$ to the lefthand side of inequality (SI). If $c<b$, we produce in each of the remaining $b-c$ skipped periods. These productions do not contribute to the lefthand side. For demand beyond $t_{q'}$, we produce after $t_q$. Let us see if this solution is feasible. For any $j = 1, \ldots, q$, suppose $k^0 \geq t_j$, where $t_j$ is the time at which the $j$th demand occurs. Since we produce in all the periods up to $k^0$, the demand up to time $t_j$ is satisfied. If $k^0 < t_j$, by
condition 5, the number of skipped periods after demand \( j \) is strictly less than \( q-j \). Consequently, we produce at most \( q-j-1 \) times after time \( t_j \) and therefore, we produce at least \( j+1 \) times up to time \( t_j \). Hence we have a feasible solution in \( C^* \).

The following example illustrates the \( k^0 \) solution. Suppose there are 5 demands in periods 10, 20, 30, 40 and 50, and we consider the following inequality which skips period 3 and extends up to period 40:

\[
w_1 + w_2 + y_4 + (y_5 + z_5) + (w_6 + 2z_6) + (y_7 + 2z_7) + 3z_8 + 2z_9 + z_{10} + \sum_{i=11}^{20} w_i + \ldots + y_{21} + (y_{22} + z_{22}) + 2z_{23} + z_{24} + \sum_{i=25}^{40} w_i \geq 3.
\]

The \( k^0 \) solution is \( z_1 = y_1 = y_2 = y_3 = y_4 = w_1 = w_2 = w_3 = w_4 = 1 \), and \( z_{41} = y_{41} = w_{41} = 1 \).

We define the \( k(i^*) \) solution as follows. Suppose period \( i^* \in Z \) is in the \( (j^*+1) \)st demand interval and that \( i^*-1 \notin Z \). Then we define the period \( k(i^*) \) as follows:

\[ k(i^*) - c = \max(j-b_j; j \leq j^*) \]

where \( c \) is the number of periods skipped up to period \( k(i^*) \), and \( b_j \) is the number of periods skipped up to \( t_j \). If more than one period \( k(i^*) \) satisfies the equation, we choose the minimum of all such values. In our example, for period \( i^* = 23 \) in the 3rd demand interval, \( k(23) = 2-1 = 1 \).

We produce in periods 1 through \( k(i^*) \) and in all the skipped periods. Let \( i' \) be the first period before \( i^* \) so that the sum of the coefficient of \( z \) and the coefficients of \( y \) from \( i' \) through \( i^* \) is exactly \( q-b-\max(j-b_j; j \leq j^*) \). We then produce from period \( i' \) to \( i^* \) in those periods that are in \( Y \cup YZ \). In our example, we produce in period 1, in the
skipped period 3, and in periods 21 and 22. We also produce in periods beyond \( t_q \) to meet the rest of the demand. For example, we can produce in period 41.

We show that the \( k(i^*) \) solution is feasible. If demand period \( t_j \leq k(i^*) \), the demand up to period \( t_j \) is clearly satisfied. If \( k(i^*) < t_j < i^* - \max (j-b_j; j \leq j^*) \), the total production up to \( t_j = k(i^*) + (\# \text{ of skipped periods from } k(i^*)+1 \text{ through } t_j) = \max (j-b_j; j \leq j^*) + b_j \geq j \). Hence the solution satisfies the demand up to period \( t_j \). For demand beyond \( i^* \), i.e., for \( t_j > i^* \), the total production up to \( t_j \) is \((q-b) + (\# \text{ of skipped periods up to } t_j) = q-b+b_j \geq j \) from condition 9. Hence the solution is feasible.

1) \( \gamma_i = 0 \) for all \( i \in (Y \cup W \cup Z) \).

a) First consider \( i > t_q \). We construct a solution in \( C^* \) and show that whether \( z_i = 1 \) or \( z_i = 0 \), the solution still remains in \( C^* \). This result would imply that \( \gamma_i = 0 \) for \( i > t_q \).

We modify the \( k^0 \) solution by producing up to period \( k^0-1 \) and in period \( t_q \). The rest of the productions occur after \( t = t_q \), and we maintain the setup from \( t_q \) onwards as long as necessary for meeting the rest of the demand. By using arguments similar to those used earlier, we can show that we produce at least \( j \) times up to \( t_j \) for any \( t > k^0 \). Therefore this is a feasible solution. For any time \( i > t_q \), we can set \( z_i = 0 \) or 1 and still be in \( C^* \). Hence \( \gamma_i = 0 \) for all \( i > t_q \).

In our inequality, we produce in periods 1 through 4 in the \( k^0 \) solution. We modify it by producing in periods 1 through 3, and setting \( z_{40} = y_{40} = w_{40} = y_{41} = w_{41} = 1 \). We can then set \( z_i = 0 \) or 1 for \( i > 40 \). Hence the coefficient \( \gamma_i \) of \( z_i \) must be zero.
b) Consider $i \leq t_q$. Suppose $i \in W \cup Y$. We use the $k^0$ solution. For any period $i \neq 1$ and $i \in W \cup Y$, we can set $z_i = 0$ or 1. So $\gamma_i = 0$ for $i \neq 1$, $i \in W \cup Y$. For $i=1$ we construct any solution in $C^*$ with $z_1 = y_1 = w_1 = 0$. For example, we can shift production in period 1 in the $k^0$ solution to the first period after $k^0$ that is not skipped. Since the first demand occurs at $t \geq 2$, we still have a feasible solution. By condition 2, period $1 \in Z \cup YZ \cup WZ$. Moreover, if $2 \in YZ \cup WZ$, then by condition 1, the sum of the coefficient of $z_2$ and the coefficients of $y$ from 2 through $i^*$ is exactly equal to $\min(q-b, i^*-1)$ for some $i^* \geq i$, $i^* \in Z$. However, by condition 2, $i^* \in Z$ for $i^* \leq q-b$. Hence $q-b+1 \in Z$. The left hand side does not increase by one unit at $q-b+1$ even though we produce at that time. Therefore, the solution is in $C^*$. Whether we set $z_1 = 0$ or 1, the solution is in $C^*$. Hence $\gamma_1 = 0$.

In our example, we can set $z_i = 0$ or 1 for $i \geq 2$, $i \in W \cup Y$ in the $k^0$ solution. We can modify the solution by turning the machine on in period 2, and producing in periods 2 through 5. We can then set $z_1 = 0$ or 1.

Suppose period $i$ is skipped and lies in the $j+1$st demand interval. Then from condition 6, $Z \neq \emptyset$. Let $i^* \in Y \cup Z \cup YZ \cup WZ$ be the minimum of all periods in $Y \cup Z \cup YZ \cup WZ$ belong to the $(j^*+1)$st demand interval. Thus there are no periods in $Y \cup Z \cup YZ \cup WZ$ up to period $t_{j^*}$. From condition 5, there are no skipped periods up to $t_{j^*}$, and hence the skipped period $i \geq t_{j^*}$. Moreover, the value of $k(i^*)$ in the $k(i^*)$ solution is $\max(j'-b_j: j' \leq j^*) = j^*$. We can extend the second sequence of productions starting in period $i^*- (q-b) - j^*$ to period $i^* \in Z$ without increasing the lefthand side. Hence, if $w_i = 1$, we can shift production in
period $i$ to period $i^*$. Whether we set $z_i=0$ or 1, the solution is in $C^*$, and hence $\gamma_i=0$.

In our example, $i^*=8$, and in the $k(i^*)$ solution, we produce in periods 3, 4, 5 and 7. Alternately, we can produce in periods 4, 5, 7 and 8. We can now set $z_3=0$ or 1 and still have a solution in $C^*$.

Note that for skipped periods, we can set $z_i=y_i=0$ or 1 in the $k(i^*)$ solution and hence $\beta_i=0$ as well.

2) $\beta_i = 0$ for all $i \notin (YZ \cup Y)$.

a) Suppose $i > t_q$. Suppose that demand at $t_{q+1} = 0$, i.e., $t_{q+1} > t_q + 1$. We can always choose a solution in $C^*$ with $z_i = y_i = w_i = 0$. Whether $z_i = y_i = 0$ or 1, the solution is in $C^*$. Hence $\beta_i + \gamma_i = 0$. Since $\gamma_i = 0$ for $i > t_q'$, this shows that $\beta_i = 0$. Suppose $t_{q+1} = t_q + 1$. Then by condition 9, $YZ \cup WZ \cup Y \cup Z \neq \emptyset$. Let $i^* \in Z$ be the minimum of all periods in $Z$. Consider the $k(i^*)$ solution. We modify it to allow one extra unit of production in period $i^*$. Since this produces $q+1$ times up to $t_q'$, we can set $z_i = y_i = w_i = 0$ for $i > t_q$. Another solution is to set $z_i = y_i = 1$. We have shown that $\gamma_i = 0$ for $i > t_q'$. Therefore $\beta_i = 0$ for $i > t_q'$.

In our example, we produce in periods 3, 4, 5, 7 and 8. We can then set $z_i = y_i = 0$ or 1 for $i > 40$.

b) Suppose $i \leq t_q$. By assumption $i \notin YZ \cup Y$. If $i$ is a skipped period, we showed earlier that $\beta_i = 0$. If $i \in Z$ then $Y \cup Z \cup YZ \cup WZ \neq \emptyset$, and there is a $k(i^*)$ solution for some period $i^* \leq i$, and $i^* \in Z$, $i^* - 1 \notin Z$. We can set $y_{i^*} = \ldots, y_i = 1$, or alternately, set $y_i = 0$. Hence $\beta_i = 0$. If $i \in W$ and $w_i = 0$, then whether $z_i = y_i = 0$ or 1, period $i$ does not contribute to the left hand side. Hence $\beta_i = 0$. If $i \in W$ and $w_i = 1$, then $i$ is in the first sequence of
productions from 1 through k(i), and therefore i\leq k(i). Hence we can shift production from i to the first unskipped period after k(i), and still have a feasible solution in C^∗. Thus β_i = 0. If i ∈ WZ, then for some i^* > i, i^* ∈ Z and i^* - 1 ∈ Z. We consider the k(i^*) solution, which produces in periods 1 through k(i^*), and a sequence of periods preceding i^* except in periods in WZ. We can modify it by shifting production from i+1 to i^*-1, to unskipped periods starting from k(i^*)+1. The second sequence of productions therefore ends in period i-1. we can now set y_i=0 or 1. Thus β_i = 0.

In our example, if i = 8, 9 or 10, and i ∈ Z, we can produce in periods 3, 4, 5, 7 and period 41. We can then set y_8 = y_9 = y_{10} = 1. We can also set y_8 = y_9 = 1 and y_{10} = 0. This shows that β_{10} = 0. We can also set y_8 = 1 and y_9 = y_{10} = 0. This shows that β_9 = 0. Finally, we can set y_3 = 0. This shows that β_3 = 0.

For i = 23 or 24 ∈ Z, we produce in periods 1, 3, 21 and 22. We can then set y_{23} = y_{24} = 1, or y_{23} = 1 and y_{24} = 0. This shows that β_{24} = 0. We can also set y_{23} = 0. This shows that β_{23} = 0.

For i = 1 or 2 ∈ W, we produce in periods 1 through 4. We can then shift production from 1 to 5, and turn on the machine in period 2. We can then set y_1 = 0 or 1. This shows that β_1 = 0. Similarly, we can shift production from period 2 to 5. This solution corresponds to z_1 = y_1 = w_1 = 1, and z_3 = y_3 = y_4 = y_5 = w_3 = w_4 = w_5 = 1. We can then set y_2 = 0 or 1. This shows that β_2 = 0.

For i = 6 ∈ WZ, we produce in periods 2 through 5. We can then set y_6 = 0 or 1. This shows that β_6 = 0.

3) \(α_i = α^∗ \) for \(i ∈ W ∪ WZ\) and \(α_i = α\) for \(i ∈ W \cup WZ\).
To show that $\alpha_i = \alpha_{i'}$ we pick two solutions in $C^*$, one with $w_i = 1$, $w_j = 0$ and the other with $w_i = 0$, $w_j = 1$. All other variables have the same value in both solutions. This shows that $\alpha_i = \alpha_{i'}$.

First consider $i \in W \cup WZ$. Suppose $i > t_q$. If $t_{q+1} > t_{q+1}$, then we can always pick a solution in $C^*$ with $w_i = 0$, $w_i = 1$ for $i, i' > t_q$. Alternately, we could produce in period $i$, and set $w_i = 0$. Hence $\alpha_i = \alpha_{i'}$.

If $i \in Y \cup YZ$, and $i$ is in the $(j+1)$st demand interval, then from condition 1, there exists some $i^* > i$, $i^* \in Z$. We can modify the $k(i^*)$ solution by shifting production from period $i$ to period $i^*$, and still have a feasible solution in $C^*$. For example, in our inequality, we can shift production from any of the periods 4 through 6 to period 7 in the $k^0$ solution or from any of the periods 22 or 23 to period 24 in the $k(i^*)$ solution. If there is more than one period $i^* \in Z$, and $i^* - 1 \in Z$, (as in our example, periods 7 and 24), we pick the minimum of all such $i^*$, and consider the associated $k(i^*)$ solution. We can shift production from period $i^* - 1$ to any other period $i \in Y$, $i > i^*$ preceding some other period $i' \in Z$, with $i' - 1 \in Z$. This solution is also in $C^*$. For example, in our inequality, we can shift production from period 6 of the $k^0$ solution to period 22. This shows that $\alpha_i$ is the same for all periods $i \in Y \cup YZ$.

If $i \in Z$, then for some $i^* \leq i$, $i^* \in Z$ and $i^* - 1 \in Z$. We can modify the associated $k(i^*)$ solution by setting $w_{i^* - 1} = 0$, $y_{i^*} = \ldots y_i = 1$, and $w_i = 1$. This solution is also in $C^*$. This shows that $\alpha_i$ is the same for all periods $i \in Y \cup Z \cup YZ \cup WZ$. In our inequality, we can set $w_6 = 0$, $y_7 = y_8 = y_9 = 1$, and $w_9 = 1$.

If period $i$ is skipped, then from condition 5, $Z \neq \emptyset$. Let $i^*$ be the minimum of all periods $i^* \in Z$, $i^* - 1 \in Z$, and let $i^*$ be in the $(j^* + 1)$st
demand interval. Hence there are no periods in $Z$ before $t_{j^*}$. From condition 6, there are no skipped periods before $t_{j^*}$. Hence period $i > i^*$. We can modify the $k(i^*)$ solution by shifting the production from period $i$ to period $i^*$. Since the $k(i^*)$ solution is feasible, and since we are modifying it by shifting one unit of production to an earlier period, the new solution is also feasible. Moreover, it is still in $C^*$. Hence $\alpha_i$ for all skipped periods is the same as $\alpha_{i^*}$. Hence $\alpha_i$ is the same for all periods $i \leq t_q$, and $i \in W \cup WZ$.

Notice that we can modify the $k(i^*)$ solution by shifting production from any period after $t_q$ to period $i^*$. Hence $\alpha_i = \alpha$ for all periods $i \in W \cup WZ$.

Next we consider a period $i \in W \cup WZ$. If $Z = \Phi$, then from condition 1, $Y \cup YZ \cup WZ = \Phi$, and from condition 6, there are no skipped periods. Hence all periods are in $W$. For any period $i \geq q$, we can produce from periods 1 through $q-1$, and in period $i$. This shows that $\alpha_i$ is the same for all periods $i \geq q$. For $i \leq q-1$, we can produce in all periods 1 through $q$ except $i$, and in period $t_q$. Hence $\alpha_i = \alpha^*$ for all periods $i \in W$ if $WZ = \Phi$.

If $Z \neq \Phi$, consider the $k(i^*)$ solution, where $i^*$ is the minimum of all periods in $Z$, and $i^*$ is in the $(j^*+1)$st demand interval. As shown earlier, there are no skipped periods before $i^*$, and all periods before $t_j$ are in $W$. Hence $k(i^*) = j$. We can modify the solution by shifting production from period $i^*-1$ to any period $i \in W$ with $w_i = 0$. If $i \in W$, and $w_i = 1$, then $i < i^*$. We can modify the $k(i^*)$ solution by shifting production from period $i^*-1$ to $i$. Hence $\alpha_i = \beta_{i^*-1}$ for all such periods $i \in W$. Finally, we can shift production from any period $i \in W$ with $w_i = 1$.
to period $k(i^*)+1$ which must be in $W$ since all periods before interval $j^*$ are in $W$. This shows that $\alpha_i = \alpha^*$ for all periods $i \in W$.

For $i \in WZ$, we can modify the $k(i^*)$ solution by shifting production from period $i^*-1$ to period $i$.

In our example, the $k(i^*)$ solution for the minimum $i^*$ is to produce in periods 3, 4, 5 and 7. We can modify this solution by shifting one unit of production from period 7 to either 1, 2 or 6. Hence $\alpha_1 = \alpha_2 = \alpha_6 = \alpha^*$.

Therefore the inequality has the form:

$$\alpha^* \sum_{i \in W \cup WZ} \alpha \sum_{i \in W \cup WZ} \beta_i \gamma_i + \sum_{i \in Y \cup YZ} \beta_i \gamma_i \geq \delta$$

4) $\beta_i = \beta$ for all $i \in Y \cup YZ$, and $\gamma_i = c \beta$, where $c$ is the coefficient of $z_i$ for $i \in Z \cup YZ \cup WZ$ in the inequality (SI).

Before we establish this, we refer to our example inequality. For the periods 4 through 10, we wish to relate the following 9 quantities:

$\beta_4$, $\beta_5$, $\gamma_5$, $\gamma_6$, $\beta_7$, $\gamma_7$, $\gamma_8$, $\gamma_9$, $\gamma_{10}$.

We formulate 10 equations based on feasible solutions in $C^*$. These solutions produce in period 3, in three of the periods 4 through 40, and in any period after 40. These solutions are the following:

1) Produce in periods 4, 5, and 7. The lefthand side of the inequality is therefore $\beta_4 + \beta_5 + \beta_7 + 5\alpha$. Let us define l.h.s. = $\beta_4 + \beta_5 + \beta_7 + 5\alpha$.

2) Produce in periods 4, 5 and 10. The lefthand side of the inequality is therefore $\beta_4 + \beta_5 + \gamma_{10} + 5\alpha$. This establishes that $\beta_7 = \gamma_{10}$.
3) Produce in periods 4, 9 and 10. The lefthand side of the inequality is therefore $\beta_4+\gamma_9+5\alpha$. This establishes that $\beta_5+\beta_7 = \gamma_9$.

4) Produce in periods 8, 9 and 10. The lefthand side of the inequality is therefore $\gamma_9+5\alpha$. This establishes that $\beta_4+\beta_5+\beta_7 = \gamma_8$.

We modify each of the solutions 2, 3 and 4, to produce in period 11 as follows:

5) Produce in periods 4, 5 and 11. The lefthand side of the inequality is therefore $\alpha^*+\beta_4+\beta_5+4\alpha$. The lefthand side of the inequality from solution 1 is $\beta_4+\beta_5+\beta_7+5\alpha$. This establishes that $\beta_7 = \alpha^*-\alpha$. In solution 2 we established that $\beta_7 = \gamma_{10}$.

6) Produce in periods 4, 10 and 11. The lefthand side of the inequality is therefore $\alpha^*+\beta_4+\gamma_{10}+4\alpha$. The lefthand side of the inequality from solution 3 is $\beta_4+\gamma_9+5\alpha$. This establishes that $\gamma_9 = 2(\alpha^*-\alpha)$.

7) Produce in periods 9, 10 and 11. The lefthand side of the inequality is therefore $\alpha^*+\gamma_9+4\alpha$. The lefthand side of the equation from solution 4 is $\gamma_8+5\alpha$. This establishes that $\gamma_8 = 3(\alpha^*-\alpha)$.

Thus we have established the following:

\[
\beta_7 = \gamma_{10} = \alpha^*-\alpha
\]

\[
\beta_5+\beta_7 = \gamma_9 = 2(\alpha^*-\alpha)
\]

\[
\beta_4+\beta_5+\beta_7 = \gamma_8 = 3(\alpha^*-\alpha).
\]

Hence $\beta_4 = \beta_5 = \beta_7 = (\alpha^*-\alpha)$, and lefthand side $= 3\alpha^*+2\alpha$.

8) We can turn the machine on in period 5, and produce in 5, 7 and 8. Hence $\beta_5+\gamma_5+\beta_7+5\alpha = \text{l.h.s.} = 3\alpha^*+2\alpha$. Hence $\gamma_5 = \alpha^*-\alpha$. 


9) We can turn the machine on in period 6, and produce in 7, 8 and 9. Hence \( \gamma_6 + \beta_7 + 5\alpha = \text{l.h.s.} = 3\alpha^* + 2\alpha \). Hence \( \gamma_6 = 2(\alpha^* - \alpha) \).

10) Alternatively, we can turn the machine on in period 7, and produce in 7, 8 and 9. Hence \( \gamma_7 + \beta_7 + 5\alpha = \text{l.h.s.} = 3\alpha^* + 2\alpha \). Hence \( \gamma_7 = 2(\alpha^* - \alpha) \).

This result establishes that the coefficient of \( y_i \) is \( \alpha^* - \alpha \) and that the coefficient of \( z_i \) is \( c(\alpha^* - \alpha) \) if the coefficient of \( z_i \) in the skip inequality (SI) is \( c \). We can similarly establish this for the periods 21 through 24. Finally, we note that the righthand side \( \delta = 3\alpha^* + 2\alpha \).

Suppose we represent the inequality by \( \alpha w + \beta y + \gamma z \geq \delta \). Our prior results show that we can write the inequality as follows:

\[
\alpha^* \sum_{i \in W \cup WZ} w_i + (\alpha^* - \alpha) \sum_{i \in Y \cup YZ} y_i + \ldots = \gamma^* \sum_{i \in Z \cup YZ \cup WZ} c_i z_i = 3\alpha^* + 2\alpha.
\]

Since this equality is a linear combination with weight \( (\alpha^* - \alpha) \) of our example inequality and weight \( \alpha \) times the equality \( \sum_{i=1} w_i = 5 \), it is a facet.

For the general case also we use the same approach: we set up equations based on feasible solutions in \( C^* \), and solve them to relate the values of \( \alpha_i, \beta_i, \) and \( \gamma_i \). Suppose that period \( i_0 \) is in the \((j+1)\)st demand interval, and that \( i_0 \in Y \). Then period \( i_0 \) is followed by a sequence of periods in \( YZ \cup WZ \cup Z \). Let \( i^0 \) be the first period after \( i_0 \) satisfying the conditions that \( i^0 \in Z \), and \( i^0 + 1 \in W \cup Y \). Thus, in our example, \( i_0 = 4, i^0 = 10 \) is one set of values, and \( i_0 = 21, i^0 = 24 \) is another set of values.
Let us denote \( K = (q-b) \text{max} (i-b_i : i \leq j) \). We produce in all the skipped periods up to \( t_j \) and if \( b_j < j \), we produce in \( j-b_j \) unskipped periods so that the contribution of the terms in the inequality up to \( t_j \) is \( \text{max} (i-b_i : i \leq j) \). While describing the \( k(i^*) \) solution we showed that it is always possible to find such a solution. We produce \( K \) times in the periods \( i_0 \) through \( i^0 \). From condition 1, the sum of the coefficient of \( z_i \) and the coefficients of \( y \) from \( i \) through \( i+K-1 \) is equal to \( K \) if \( i \leq i^0-K+1 \), and is equal to \( i^0+1-i \) if \( i > i^0-K+1 \).

Finally, we produce \( n-q \) times in periods after \( t_q \).

To complete the proof, we show the following:

1) \( \beta_i = \beta \) for all \( i_0 \leq i \leq i^0, \quad i \in Y \cup YZ \).

2) \( \gamma_i = c_i \beta \) for all \( i_0 \leq i \leq i^0, \quad i \in Z \cup YZ \) where \( c_i \) is the coefficient of \( z_i \) in (S).

3) \( \beta = \alpha^*-\alpha \).

4) If there is any other interval similar to the interval \( (i_0 \ldots i^0) \), then the first three conditions apply to this interval as well.

5) The righthand side \( \delta \) of the inequality is equal to \( (q-b)(\alpha^*-\alpha) + n\alpha \).

Therefore, the equation

\[ \alpha w + \beta y + \gamma z = \delta \]

can be written as a linear combination of

\[ \sum_{i \in W \cup WZ} w_i + \sum_{i \in Y \cup YZ} y_i + \sum_{i \in Z \cup YZ \cup WZ} c_i z_i = (q-b) \text{ with weight } (\alpha^*-\alpha) \]
and
\[ \sum_{i=1}^{T} w_i = n \quad \text{with weight } \alpha \]

and hence (SI) is a facet.

We first assume that \( WZ = \emptyset \). Later we generalize the proof to the case when \( WZ \neq \emptyset \).

Consider two periods \( i_1 - 1 \) and \( i_2 + 1 \) in \( Z \) with the intervening periods from \( i_1 \) through \( i_2 \) in \( \mathcal{Y} \cup YZ \cup WZ \) for \( i_0 \leq i_1 \leq i_2 \leq i^0 \). In other words, none of these periods are in \( Z \) or \( W \). For any period \( i_1 \leq i \leq i_2 \), we consider the following three feasible solutions in \( C^* \):

1) Produce in periods \( i_2 - K + 1 \) through \( i_2 \).

2) Produce in periods \( i_2 - K + 1 \) through period \( i - 1 \). We also turn the machine on in period \( i^0 + i - i_2 \) and produce in periods \( i^0 + i - i_2 \) through \( i^0 \).

3) Produce in periods \( i_2 - K + 1 \) through period \( i - 1 \). We also turn the machine on in period \( i^0 + i - i_2 + 1 \) and produce in periods \( i^0 + i - i_2 + 1 \) through \( i^0 \). This solution gives a total production of \( K - 1 \) units in the interval \( \{i_0, \ldots, i^0\} \). We also produce in period \( t < i_0 \), or \( t > i^0 \). If \( i^0 < t_q \), then we can choose \( t = i^0 + 1 \). Since by assumption \( i^0 + 1 \in \mathcal{W} \cup \mathcal{Y} \), this solution is in \( C^* \). Suppose \( i^0 = t_q \) and there is at least one period before \( i_0 \) in which we do not produce. If \( i'' \) is the last period before \( t \) in which we produce, we can choose \( t \) as the first unskipped period after \( i'' \). This solution is also in \( C^* \). Let us denote the coefficient of period \( t \) by \( c(t) \).

We treat the case \( i^0 = t_q \) and no period before \( i_0 \) in which we do not produce separately.
For notational simplicity, we let \( i' \) denote the period \( i^0+i-2 \). Notice that \( i \geq i_2-K+1 \), and hence \( i' \geq i^0-K+1 \). We obtain the following equation if we compare solution 1 and 2:

\[
\beta_i + \ldots + \beta_{i_2} = \gamma_i + \beta_i + \ldots + \beta_{i_0}. \tag{i}
\]

We obtain the following equation if we compare solution 2 and 3:

\[
\alpha + \beta_i + \ldots + \beta_{i_2} = \gamma_{i+1} + \beta_{i+1} + \ldots + \beta_{i_0} + \alpha(t) \tag{ii}
\]

Hence

\[
\alpha + \beta_i + \ldots + \beta_{i_2} = \gamma_{i+1} + \beta_{i+1} + \ldots + \beta_{i_0} + \alpha(t)
\]

\[
= \beta_{i+1} + \ldots + \beta_{i_2} + \alpha(t)
\]

and thus

\[
\beta_i = \alpha(t) - \alpha = \beta. \tag{iii}
\]

Equation (iii) shows that the value of \( \beta_i \) is the same in all periods \( \in Y \cup YZ \) in the interval \( \{i_0, \ldots, i^0\} \).

We also obtain the following equations from (i) and (ii):

\[
\gamma_{i+1} + \beta_{i+1} + \ldots + \beta_{i_0} + \alpha(t) = \alpha + \beta_i + \ldots + \beta_{i_2}
\]

\[
= \alpha + \gamma_i + \beta_{i+1} + \ldots + \beta_{i_0}.
\]

Hence

\[
(\gamma_i + \beta_{i+1}) - \gamma_{i+1} = \alpha(t) - \alpha = \beta. \tag{iv}
\]

For \( i = i_2 \), \( \beta_{i_0+i-i_2} = \beta_{i_0} = 0 \) since by assumption \( i^0 \) is the first period after \( i_0 \) for which \( i^0+1 \in W \cup Y \), and from condition 1, the coefficient of \( z_{i_0} \) in period \( i^0 \) is 1, and that of \( y_{i_0} \) is 0. Hence from (iv) we obtain \( \gamma_{i_0} = \beta \).

In general, \( \gamma_i = (i^0+i-1') \beta - \sum_{t=i'}^{i_0} \beta_t \). \tag{v}
Since the length of the interval \( \{i_1, \ldots, i_2\} \) is at most \( K \), by condition 1, equation (iv) applies only to those periods \( i' \) for which \( i^{0+1-i'} \leq K \). Moreover, from condition 1, the sum of the coefficient of \( z_{i'} \) and the coefficients of \( y \) from \( i' \) through \( i^0 \) is \( i^{0+1-i'} \) if \( i^{0+1-i'} \leq K \). For these periods we therefore have \( c_{i'} + s_{i'} = i^{0+1-i'} \) or

\[
    c_{i'} = i^{0+1-i'} - s_{i'} \tag{vi}
\]

where \( s_{i'} \) is the sum of the coefficients of \( y \) from \( i' \) through \( i^0 \).

If we substitute the value of \( s_{i'} \) in (v), we obtain \( \gamma_{i'} = (i^{0+1-i'})\beta - s_{i'}\beta = (i^{0+1-i'} - s_{i'})\beta = c_{i'}\beta \). Hence we have established that

\[
    \gamma_{i'} = c_{i'}\beta \text{ for } i' \geq i^{0-K+1}, \ i' \in \mathbb{Z}.
\]

For \( i_0 \leq i' \leq i^{0-K} \), we use the following solutions:

4) Turn the machine on in period \( i' \) and produce in periods \( i' \) through \( i'+K-1 \).

5) Turn the machine on in period \( i_0 \) and produce in periods \( i_0 \) through \( i_0+K-1 \). Since \( i_0 \in Y \) by assumption, from condition 1, the periods \( \{i_0, \ldots, i_0+K-1\} \) are in \( YZ \).

We therefore obtain

\[
    \gamma_{i'} + \sum_{t=i'}^{i'+K-1} \beta_t = K\beta \tag{vii}
\]

Since \( i' \leq i^{0-K} \), from condition 1, the sum of the coefficients of \( z_{i'} \) and the coefficients of \( y \) from \( i' \) through \( i'+K-1 \) is \( K \). Hence \( c_{i'} + s_{i'} = K \), where \( c_{i'} \) is the coefficient of \( z_{i'} \) and \( s_{i'} \) is the sum of the coefficients of \( y \) from \( i' \) through \( i'+K-1 \). We can therefore write (vii) as

\[
    \gamma_{i'} = K\beta - \sum_{t=i'}^{i'+K-1} \beta_t
\]
\[ = (K-s_i)\beta \]
\[ = c_i\beta \]  \hspace{1cm} (viii)

Suppose there are two intervals \( \{i_0, \ldots, i^0\} \) and \( \{i'_0, \ldots, i^{0'}\} \). Then we can produce \( K \) times in the interval \( \{i_0, \ldots, i^0\} \) or shift production of one unit to the period \( i'_0 \). This possibility shows that \( \beta_{i_0} = \beta \).

We generalize this result to the case when \( WZ \neq \emptyset \). If we modify the solutions to produce only in periods in \( Y \cup Z \), then the relations \( \beta_i = \beta \), \( \beta = (\alpha^*-\gamma) \) and \( \gamma_i = c_i\beta \) still hold. For the coefficient of \( z_i \) in periods \( i \in WZ \), we note from condition 1 that if we turn the machine on in period \( i \), and produce \( \min(K, i^0+1-i) \) times in periods in \( YZ \), then the solution is in \( C^* \). Alternately, we can turn the machine on in period \( i' \) where \( i' \) is the first period after \( i \) in \( YZ \cup Z \), and produce in the same periods. Thus \( \gamma_i = \gamma_{i'} = c_i\beta \). However, from condition 1, \( c_i = c_{i'} \). Hence \( \gamma_i = c_i\beta \).

Finally, consider any solution in \( C^* \). Suppose we produce \( k \) times in periods in \( W \cup WZ \) and \( n-k \) times in periods not in \( W \cup WZ \). Then we obtain the following equation:

\[ k\alpha^*+(n-k)\gamma+(q-b-k)\beta = \delta. \]

If we use the substitution \( \beta = \alpha^*-\gamma \) we obtain

\[ \delta = (q-b)(\alpha^*-\gamma)+n\gamma. \]

We have thus established that (SI) is a facet.
Chapter 4. Separation Problem

In this chapter we solve the separation problem for the inequalities (P1). We pattern the development after Wolsey (1988), who used a similar approach to solve the separation problem for the uncapacitated version of this problem. A similar approach for solving the separation problem has also been described by Martin (1987). The basic idea is to formulate the separation problem as an integer program, and show that the constraints describe a network flow problem (for our problem instance, the separation problem can be solved as a shortest path problem). Therefore the linear programming relaxation of the separation problem gives us the same optimal solution as the integer program.

There are two advantages to using a linear programming based approach as opposed to a dynamic programming approach. First, as we show later, it enables us to reformulate the original problem more compactly. Second, this reformulation enables us to prove that our inequalities guarantee integer solutions for certain types of objective functions.

The separation problem would be useful in any cutting plane method that we might develop for the problem. To motivate the discussion, consider the problem with demands at periods t=3 and t=5.

The two demand case.

The facets extending to t=3 are the following:
\[ w_1 + w_2 + w_3 \geq 1 \]
\[ w_1 + y_2 + z_3 \geq 1 \]
\[ y_1 + z_2 + z_3 \geq 1 \]
\[ y_1 + z_2 + w_3 \geq 1. \]

We write a separation problem for these facets. Suppose we are given a solution \((w^*, y^*, z^*)\) to the linear programming relaxation of SCSP. We want to write a separation problem that finds out if this solution violates any of these facets. In these facets, each \(z_i\) must be preceded by a \(y_{i-1}\) or a \(z_{i-1}\). Consider the following problem:

\[
\text{(Sep)} \quad \min \sum_{i=1}^{3} (w_i^*a_i + y_i^*b_i + z_i^*c_i) - \delta
\]

subject to

\[ a_i + b_i + c_i = \delta \quad (1) \]

\[ b_i + c_i - c_{i+1} \geq 0 \quad (2) \]

\[ a, b, c, \delta \in (0, 1) \text{ and integer.} \]

We show later that this problem can be cast as a network flow problem, and hence has integer optimal solutions even if it is solved as a linear program. \(a_i, b_i\) and \(c_i\) are weights on the solution values \(w_i^*, y_i^*\) and \(z_i^*\) respectively. Constraint (1) implies that if \(\delta = 1\), then for each period \(i\), one of the variables \(a_i, b_i\) or \(c_i\) is equal to 1. Constraint (2) ensures that if \(c_{i+1} = 1\) (i.e., if \(z_{i+1}\) has a weight of 1), then either \(b_i\) or \(c_i\) is equal to 1. In other words, \(z_{i+1}\) is preceded by \(z_i\) or \(y_i\). This
condition ensures that we do not obtain an invalid inequality like $w_1+w_2+z_3 \geq 1$ by setting $a_1=b_2=c_3=1$.

We use the following substitution to show that the problem can be cast as a network flow problem:

$$r_i = b_i + c_{i-1} - c_{i+1} \geq 0 \text{ for } i=1,2.$$  

Then the constraints can be written as follows:

$$a_1 + b_1 + c_1 - \delta = 0 \quad (3)$$

$$r_i = -b_i - c_1 + c_{i+1} = 0 \quad (4)$$

$$r_i - a_i + a_{i+1} + b_{i+1} = 0 \quad (5)$$

$$-a_3 - b_3 - c_3 + \delta = 0 \quad (6)$$

Constraints (5) state that $b_i + c_{i-1} - c_{i+1} = -a_i + a_{i+1} + b_{i+1}$ or equivalently that $a_i + b_i + c_i - a_{i+1} - b_{i+1} - c_{i+1} = 0$. These constraints clearly have a network form as shown in Figure 4.

Next we consider facets that skip one period. Those that skip period one are:

$$y_2 + z_3 + z_4 + z_5 \geq 1$$

$$y_2 + z_3 + z_4 + w_5 \geq 1$$

$$y_2 + z_3 + w_4 + w_5 \geq 1$$

$$w_2 + y_3 + z_4 + z_5 \geq 1$$

$$w_2 + y_3 + z_4 + w_5 \geq 1.$$

Facets that skip period 2 are:
\[ w_1 + y_3 + z_4 + z_5 \geq 1 \]
\[ w_1 + y_3 + z_4 + w_5 \geq 1. \]

The facet that skips period 3 is:
\[ y_1 + z_2 + w_4 + w_5 \geq 1. \]

Skipping any period after \( t=3 \) gives redundant inequalities. For example, \( y_1 + z_2 + z_3 + w_5 \geq 1 \) is implied by \( y_1 + z_2 + z_3 \geq 1. \)

These facets also have the same form as the earlier ones: any \( i \in \mathbb{Z} \) must be preceded by \( i-1 \in Y \cup Z \). The separation problem for the facets that skip period 1 is:

\[
\min \sum_{i=2}^{5} (w_i \cdot a_i + y_i \cdot b_i + z_i \cdot c_i) - \delta
\]

subject to

\[ a_i + b_i + c_i = \delta \quad i=2, ..., 5 \]
\[ b_i + c_i - c_{i+1} \geq 0 \quad i=2, ..., 4 \]
\[ c_2 = 0 \]

\[ a, b, c, \delta \in (0, 1) \text{ and integer.} \]

If \( z_2 \) is in the inequality, producing in periods 1 and 2 gives us a feasible solution that contributes zero, and thus violates the inequality. Hence we have to specify that \( \gamma_2 = 0 \). Substituting \( r_i = b_i + c_i - c_{i+1} \), we can show that this separation problem also has a network structure. Figure 5 shows the corresponding network.

Similarly, the separation problems associated with facets that skip periods 2 and 3 also have a network structure.
Now we consider the last type of facets: those that do not skip any period. We consider a general two demand problem. Since there are many different types of terms in this inequality, we specify the associated variables for each term in the separation problem. Consider the following inequality:

\[ y_1 + (y_2 + z_2) + (w_3 + z_3) + (w_4 + z_4) + z_5 + \sum_{i=6}^{12} w_i \geq 2 \]

We associate variable \( a_i \) with the term \( w_i \), \( b_i \) with \( y_i \), \( c_i \) with \( z_i \), \( e_i \) with \((y_i + z_i)\) and \( g_i \) with \((w_i + z_i)\). The term \((w+z)\) in period \( i+1 \) requires either \((w+z)\) or \((y+z)\) in period \( i \). In the next inequality, the terms \((w+z)\) occur before the term \((y+z)\):

\[ y_1 + (w_2 + z_2) + (w_3 + z_3) + (y_4 + z_4) + z_5 + \sum_{i=6}^{12} w_i \geq 2. \]

In this inequality, the term \((w+z)\) in period \( i+1 \) requires either \((w+z)\) or \( y \) (or \( z \)) in period \( i \). To distinguish \((w+z)\) occurring before and after \((y+z)\), we associate the variables \( g_i \) with the terms \((w+z)\) in this inequality, and the superscript refers to the fact that \((w+z)\) occurs before the first \((y+z)\).

Consider the following inequality:

\[ y_1 + (y_2 + z_2) + 2z_3 + 2z_4 + 2z_5 + z_6 + \sum_{i=7}^{12} w_i \geq 2. \]

We associate the variable \( m_5 \) with the term \( 2z_5 \) in period 5. We distinguish the last term in the sequence \( 2z_3, 2z_4, 2z_5 \) from the variables in periods 3 and 4, which we denote by \( m_3^0 \) and \( m_4^0 \) respectively. Finally, we associate the variable \( e_2^0 \) with the term \((y_2 + z_2)\) in period 2: this notation permits us to distinguish the term \((y+z)\) occurring before \( 2z \) from the term \((y+z)\) occurring before \( z \). We can summarize the terms and their associated variables as follows:
Term Variable

\( w_i \) \( a_i \)

\( y_i \) \( b_i \)

\( z_i \) \( c_i \)

\( (y_i+z_i) \) \( e_i \) if it precedes \( z_{i+1} \)

\( e^0_i \) if it precedes \( 2z_{i+1} \)

\( (w_i+z_i) \) \( g_i \) if it occurs after \( (y_t+z_t) \) for some \( t > i \)

\( g^1_i \) if it precedes \( (y_t+z_t) \) for some \( t > i \)

\( 2z_i \) \( m_i \) if it precedes \( z \)

\( m^0_i \) if it precedes \( 2z \)

The following example illustrates a sequence containing all types of terms:

\[ y_1 + (w_2+z_2) + (y_3+z_3) + 2z_4 + 2z_5 + z_6 + (y_7+z_7) + (w_8+z_8) + z_9. \]

We associate the following variables with each period:

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variable</td>
<td>( b_1 )</td>
<td>( g^1_2 )</td>
<td>( e^0_3 )</td>
<td>( m^0_4 )</td>
<td>( m_5 )</td>
<td>( c_6 )</td>
<td>( e_7 )</td>
<td>( g_8 )</td>
<td>( c_9 )</td>
</tr>
</tbody>
</table>

Given a solution \((w^*, y^*, z^*)\) to the linear programming relaxation of SCSP, we wish to find out if they violate any of these inequalities. Consider the following separation problem:

\[
\min \sum_{i=1}^{T}(w_i a_i + y_i b_i + z_i c_i + (y_i^* + z_i^*)(e_i + e^0_i)).
\]
\[ (w_i^*+z_i^*)(g_i+g_i^1)+2z_i^*(m_i+m_i^0)-2\delta \quad (1) \]

subject to

\[ a_i+b_i+c_i+e_i^0+g_i+g_i^1+m_i+m_i^0-\delta = 0 \quad i \leq t_1 \quad (2) \]

\[ a_i+b_i+c_i+ -\delta = 0 \quad i > t_1 \quad (3) \]

\[ m_i +g_i + e_i - c_{i+1} - g_{i+1} \geq 0 \quad i \leq t_1 \quad (4) \]

\[ m_i^0 + e_i^0 - m_{i+1}^0 - m_{i+1} \geq 0 \quad i \leq t_1-1 \quad (5) \]

\[ b_i + c_i + g_i^1 - e_{i+1} - e_{i+1}^0 - g_{i+1}^1 \geq 0 \quad i \leq t_1-1 \quad (6) \]

\[ b_i + c_i - c_{i+1} \geq 0 \quad i \geq t_1+1 \quad (7) \]

\[ \delta \leq 1 \quad (8) \]

\[ a, b, c, e, e^0, g, g^1, m, m^0, \delta \geq 0 \text{ and integer} \quad (9) \]

Constraints (2) state that each time period \( i \leq t_1 \), contains one of the terms \( w_i, y_i, z_i, (y_i+z_i), (w_i+z_i) \) or \( 2z_i \). Constraints (3) state that we need one of the terms \( w_i, y_i \) or \( z_i \) in each of the time periods after \( t_1 \). We require this condition because the terms \((y_i+z_i), (w_i+z_i)\) and \( 2z_i \) do not occur in periods \( i \geq t_1+1 \). Constraints 4, 5 and 6 refer to periods \( i \leq t_1 \). Constraints (4) state that if the inequality contains the terms \( z_{i+1} \) or \((w_{i+1}+z_{i+1})\) in period \( i+1 \), it must also contain the terms \((y_i+z_i), (w_i+z_i)\) or \( 2z_i \) in period \( i \). Constraint (5) states that if the inequality contains the terms \( 2z_{i+1} \) in period \( i+1 \), it must contain the terms \((y_i+z_i)\) or \( 2z_i \) in period \( i \). Constraint (6) states that if \((y_{i+1}+z_{i+1})\) or \((w_{i+1}+z_{i+1})\) belong to the inequality in period \( i+1 \), then the inequality contains the terms \((w_i+z_i)\) or \( y_i \) or \( z_i \) in period \( i \). Constraint (7) refers to periods...
after \( t_1+1 \). It states that if the inequality contains the term \( z_{i+1} \) in period \( i+1 \), it must contain \( y_i \) or \( z_i \) in period \( i \).

We show that the separation problem has a network structure. Notice that each variable appears at most twice in each of the constraints 4 through 7, once with a positive sign and once with a negative sign. We can cast the problem in the form of a network flow constraint matrix by using the following substitutions.

\[
\begin{align*}
    r_i &= m_i + e_i + g_i - c_{i+1} - g_{i+1} \geq 0 & i \leq t_1 \\
    s_i &= m^0_i + e^0_i - m^0_{i+1} - m_{i+1} \geq 0 & i \leq t_1-1 \\
    u_i &= b_i + c_i + g^1_i - e_{i+1} - e^0_{i+1} - g^1_{i+1} \geq 0 & i \leq t_1-1 \\
    u_i &= b_i + c_i - c_{i+1} \geq 0 & i \geq t_1+1
\end{align*}
\]

The problem can be rewritten as follows:

For \( i \leq t_1 \),

\[
\begin{align*}
    a_1 + b_1 + c_1 + e_1 + e^0_1 + g_1 + g^1_1 + m_1 + m^0_1 - \delta &= 0 & (i) \\
    r_i + c_{i+1} - e_i - g_i + g_{i+1} - m_i &= 0 & (ii) \\
    s_i - e^0_i + m_{i+1} - m^0_i - m^0_{i+1} &= 0 & (iii) \\
    u_i - b_i - c_i + e_{i+1} + e^0_{i+1} - g^1_i + g^1_{i+1} &= 0 & (iv) \\
    -r_i - s_i - a_i - a_{i+1} + a_{i+1} + b_{i+1} &= 0 & (v)
\end{align*}
\]

For \( i \geq t_1+1 \):

\[
\begin{align*}
    -u_i - a_i + a_{i+1} + b_{i+1} &= 0 & (vi) \\
    u_i - b_i - c_i + c_{i+1} &= 0 & (vii)
\end{align*}
\]
$$-a_T -b_T -c_T + \delta = 0 \text{ (viii)}$$

Constraints (i) and (viii) follow directly from constraints (2) and (3) of the separation problem. Constraints (ii), (iii) and (iv) and (vii) follow from the substitutions. Constraint (v) states that

$$a_i + b_i + c_i + e_i + e_0 + g_i + g^1 + m_i + m_0 \ldots$$

$$- (a_{i+1} + b_{i+1} + c_{i+1} + e_{i+1} + e_{0_{i+1}} + g_{i+1} + g^1_{i+1} + m_{i+1} + m_0_{i+1}) = 0$$

for \( i \leq t_1 \). This result follows from constraint (2) of the separation problem. Constraint (vi) states that \( a_i + b_i + c_i - (a_{i+1} + b_{i+1} + c_{i+1}) = 0 \) for \( i \geq t_1 + 1 \). These constraints clearly have a network optimal structure. Therefore the separation problem always has an integer optimal solution to even if we is solved as a linear program.

We also wish to include the setup constraints \( y_i \geq w_i \) in the separation problem. If we add the constraint \( y_t \geq w_t \) and the equality \( \sum_{i=1}^{T} w_i = 2 \), we obtain \( \sum_{i=1, i \neq t}^{T} w_i + y_t \geq 2 \), which is therefore an equivalent alternate form of the inequality \( y_t \geq w_t \). However, the constraints of the separation problem allow us to set \( a_i = 1 \) for \( i \neq t \) and \( b_t = 1 \). Hence, if \( y_t < w_t \), then \( \sum_{i=1, i \neq t}^{T} w_i + y_t < 2 \) and the separation problem has an optimum value less than zero. In other words, the separation problem detects any violated inequalities of the form \( y_t \geq w_t \).

To combine all the separation problems into one problem, we use double subscripts on all variables. We define the subscripts as follows:
\(a_{i1}, b_{i1}, c_{i1}\) and \(\delta_1\): variables for the separation problem for facets extending up to period \(t_1\).

\(a_{i2}, b_{i2}, c_{i2}, e_{i2}, e_{i2}^0, g_{i2}, g_{i2}^1, m_{i2}, m_{i2}^0,\) and \(\delta_2\): variables for the separation problem for facets extending up to period \(T\) with no periods skipped.

\(a_{is}, b_{is}, c_{is}\) and \(\delta_s\): variables for the separation problem for facets extending up to period \(T\) with period 's' skipped for \(s=1, \ldots, t_1\).

The following formulation corresponds to the complete separation problem:

\[
\text{(SEP)}
\]

\[
\min \xi = \sum_{i=1}^{t_1} (w_i^*a_{i1} + y_i^*b_{i1} + z_i^*c_{i1}) + \sum_{s=1}^{t_1} (\sum_{i=1, i \neq s}^{T} w_i^*a_{is} + y_i^*b_{is} + z_i^*c_{is}) + \\
+ \sum_{i=1}^{T} (w_i^*a_{i2} + y_i^*b_{i2} + z_i^*c_{i2} + (y_i^* + z_i^*)(e_{i2} + e_{i2}^0) + (w_i^* + z_i^*)(g_{i2} + g_{i2}^1)) \\
+ 2z_i^*m_{i1} + m_{i2}^0 - \delta_1 - \sum_{s=1}^{t_1} \delta_s - 2\delta_2
\]

subject to

\[
a_{i1} + b_{i1} + c_{i1} = \delta_1
\]

\[
b_{i1} + c_{i1} - c_{i+1,1} \geq 0
\]

\[
a_{i2} + b_{i2} + c_{i2} + e_{i2} + e_{i2}^0 + g_{i2} + g_{i2}^1 + m_{i2} + m_{i2}^0 - \delta_2 = 0 \quad i \leq t_1
\]

\[
a_{i2} + b_{i2} + c_{i2} - \delta = 0 \quad i > t_1
\]

\[
m_{i2} + g_{i2} + e_{i2} - c_{i+1,2} - g_{i+1,2} \geq 0 \quad i \leq t_1
\]

\[
m_{i2}^0 + e_{i2}^0 - m_{i+1,2}^0 - m_{i+1,2} \geq 0 \quad i \leq t_1 - 1
\]

\[
b_{i2} + c_{i2} + g_{i2}^1 - e_{i+1,2} - e_{i+1,2}^0 - g_{i+1,2}^1 \geq 0 \quad i \leq t_1 - 1
\]
\[ \begin{align*}
  b_{i2} + c_{i2} - c_{i+1,2} & \geq 0 & \text{for } i \geq t_1 + 1 & \quad (18) \\
  a_{is} + b_{is} + c_{is} & = \delta_s & \text{for } i \leq T; \ s \leq t_1, i \neq s & \quad (19) \\
  b_{is} + c_{is} - c_{i+1,s} & \geq 0 & \text{for } i \leq T; \ s \leq t_1, i \neq s & \quad (20) \\
  c_{s+1,s} & = 0 & & \quad (21)
\end{align*} 

\]

Note that we have dropped the integrality constraint.

Observe that if we set all the variables to zero, we obtain a feasible solution with \( \zeta = 0 \). Hence the objective function is not greater than zero.

**Lemma 1.** The separation problem (SEP) has an optimum value of zero if and only if any solution \((w^*, y^*, z^*)\) to the linear programming relaxation of SCSP satisfies all the inequalities.

**Proof.**

Suppose the solution violates some facet. Then by giving a weight of 1 to each variable in the facet, we obtain a solution to the separation problem that has a value less than 0.

Suppose the solution satisfies all the facets. Since the constraint matrix of problem (SEP) has a network structure, some optimal solution has integer values for the variables \(a, b, c, e, f, g, k\) and \(\delta\). Thus the terms in the objective function with positive coefficients correspond to the lefthand side of some valid inequality. Since the solution satisfies all inequalities, the objective function has a nonnegative value. Setting all
variables in the separation problem to zero gives a solution of value zero.

Note that if \( \sum_{i=1}^{T} w^*_i > 2 \) and the solution satisfies all the other inequalities, the optimal solution value is still zero. The separation problem therefore cannot detect whether \( \sum_{i=1}^{5} w^*_i > 2 \), but can detect it if \( \sum_{i=1}^{5} w^*_i < 2 \).

Since the optimal value is zero if the solution satisfies all the inequalities, we can drop the constraint that \( \delta \leq 1 \). (If any facet is violated, the optimal solution is then unbounded).

Reformulation.

The dual of (SEP) has a feasible solution if and only if every feasible solution to the linear programming relaxation satisfies all the facets. If we add the remaining facets \( \sum_{i=1}^{T} w_i = 2, y_i - w_i \leq 0, z_i + y_i - y_{i-1} \geq 0, y_i \leq 1 \) and \( z_i \leq 1 \) from SCSP to the dual, we obtain a reformulation (R) of SCSP as follows:

\[
\begin{align*}
\text{(R) dual var.} \\
a_{i1} : & \quad p_{i1} \leq w_i \quad i \leq t_1 \\
a_{is} : & \quad p_{is} \leq w_i \quad i=1, \ldots, T; s \leq t_1, i \neq s \\
a_{i2} : & \quad p_{i2} \leq w_i \quad i \leq T \\
b_{i1} : & \quad p_{i1} + u_{i1} \leq y_i \quad i \leq t_1 \\
b_{is} : & \quad p_{is} + u_{is} \leq y_i \quad i=1, \ldots, T; s \leq t_1, i \neq s
\end{align*}
\]
\[ b_{i2}^i: \quad p_{i2} + u_{i2} \leq y_i \quad i \leq T \]

\[ c_{i1}^i: \quad p_{i1} + u_{i1} - u_{i-1,1} \leq z_i \quad i \leq t_1 \]

\[ c_{is}^i: \quad p_{is} + u_{is} - u_{i-1,s} \leq z_i \quad i=1, \ldots, T; s \leq t_1, i \neq s \]

\[ c_{i2}^i: \quad p_{i2} + u_{i2} - u_{i-1,2} \leq z_i \quad i \leq t_1 \]

\[ p_{i2} + u_{i2} - u_{i-1,2} \leq z_i \quad i \geq t_1 + 1 \]

\[ e_{i2}^i: \quad p_{i2} + r_{i2} - u_{i-1,2} \leq y_i + z_i \quad i \leq t_1 \]

\[ e_{0i2}^i: \quad p_{i2} + s_{i2} - u_{i-1,2} \leq y_i + z_i \quad i \leq t_1 \]

\[ g_{i2}^i: \quad p_{i2} + r_{i2} - r_{i-1,2} \leq w_i + z_i \quad i \leq t_1 \]

\[ g_{1i2}^i: \quad p_{i2} + u_{i2} - u_{i-1,2} \leq w_i + z_i \quad i \leq t_1 \]

\[ m_{i2}^i: \quad p_{i2} + r_{i2} - s_{i-1,2} \leq 2z_i \quad i \leq t_1 \]

\[ m_{0i2}^i: \quad p_{i2} + s_{i2} - s_{i-1,2} \leq w_i + z_i \quad i \leq t_1 \]

\[ \delta_{i:} \quad \sum_{i=1}^{t_1} p_{i1} = 1 \]

\[ \delta_{s2:} \quad \sum_{i=1, i \neq s}^{T} p_{is} = 1 \]

\[ \delta_{2:} \quad \sum_{i=1}^{T} p_{i2} = 2 \]

\[ \Delta: \quad \sum_{i=1}^{T} w_i = 2 \]

\[ \sigma_{i:} \quad y_i - w_i \geq 0 \quad i \leq t_1 \]

\[ \mu_{i:} \quad z_i + y_i - y_{i-1} \geq 0 \quad i \leq T \]

\[ \theta_{i:} \quad y_i \leq 1 \quad i \leq T \]

\[ \phi_{i:} \quad z_i \leq 1 \quad i \leq T \]
Note that the dual of this problem has the same network constraints as the separation problem and the following additional constraints:

primal
variables
\begin{align*}
z_i: & \quad c_{i1} + c_{i2} + e_{i2} + e_0^{i2} + g_{i2} + g_1^{i2} + 2m_{i2} + 2m_0^{i2} + \sum_{s=1}^{t_1} c_{is} + \mu_i - \phi_i \leq k_i \\
y_i: & \quad b_{i1} + b_{i2} + e_{i2} + e_0^{i2} + \sum_{a=1}^{t_1} b_{is} + \sigma_i + \mu_{i+1} - \mu_i - \theta_i = s_i \\
w_i: & \quad a_{i1} + a_{i2} + g_{i2} + g_1^{i2} + \sum_{s=1}^{t_1} a_{is} - \sigma_i \leq h_i
\end{align*}

The dual of the reformulation (R) is therefore a network flow problem with these complicating side constraints.

Note that in general the reformulation has a polynomial (in the number of time periods) number of variables and constraints, whereas the original problem with all the facets has an exponential number of constraints. Therefore this reformulation is a more compact representation of the linear programming relaxation of the original problem (SCSP) with all the facets. Moreover, it might be useful to investigate whether R has integer extreme points. If so, we would be able to show that the facets completely describe the feasible hull of solutions, because the reformulation (R) is equivalent to the linear programming relaxation of SCSP with all the facets.

Separation problem for the general case
We can extend this discussion to the general case with \( n \) units of demands. To motivate the discussion, consider the following inequality from our example:

\[
y_1 + (y_2+z_2) + (y_3+z_3) + (y_4+z_4) + (y_5+z_5) + z_6 + \sum_{i=7}^{20} w_i \geq 5.
\]

We associate the following variables in the separation problem with each term:

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term</td>
<td>( y )</td>
<td>( y+z )</td>
<td>( y+z )</td>
<td>( y+z )</td>
<td>( y+z )</td>
<td>( z )</td>
</tr>
<tr>
<td>Variable</td>
<td>( b )</td>
<td>( e^4(1) )</td>
<td>( e^3(1) )</td>
<td>( e^2(1) )</td>
<td>( e^1(1) )</td>
<td>( m(1) )</td>
</tr>
</tbody>
</table>

The variable \( e^1(1) \) in period 5 refers to the term \( (y+z) \). The quantity in brackets in the variable \( e^1(1) \) refers to the coefficient of \( z \) which is 1. In the other inequalities, the coefficient might be \( c > 1 \), and we would denote the associated variable by \( e^k(c) \). The superscript in the variable \( e^1(1) \) refers to the first \( (y+z) \) preceding the term \( z \) in period 6, the variable \( e^2(1) \) refers to the term in period 4 preceding the first \( (y+z) \) in period 5. Similarly, \( e^3(1) \) precedes \( e^2(1) \), and \( e^4(1) \) precedes \( e^3(1) \).

In general, for an inequality extending up to period \( t_q \), if the sequence \( y_i, (y_{i+1}+z_{i+1}), (y_{i+2}+z_{i+2}) \) begins in demand interval \( (j+1) \), the length of the sequence is \( q-j \), and hence the superscript \( k \) varies from 1 through \( q-j-1 \).

Consider another inequality from our example:

\[
y_1 + (y_2+z_2) + (y_3+2z_3) + (y_4+3z_4) + (y_5+3z_5) + 3z_6 + 2z_7 + z_8 + \sum_{i=9}^{20} w_i \geq 5.
\]
This inequality has a sequence $3z$, $2z$, $z$. The term $3z$ in period 5 is preceded by the term $(y+3z)$ in two periods 5 and 6.

We associate the following variables with each term:

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
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<th>5</th>
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<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term</td>
<td>$y$</td>
<td>$(y+z)$</td>
<td>$(y+2z)$</td>
<td>$(y+3z)$</td>
<td>$(y+3z)$</td>
<td>$3z$</td>
<td>$2z$</td>
<td>$z$</td>
</tr>
<tr>
<td>Variable</td>
<td>$b$</td>
<td>$e^1(1)$</td>
<td>$e^1(2)$</td>
<td>$e^2(3)$</td>
<td>$e^1(3)$</td>
<td>$m(3)$</td>
<td>$m(2)$</td>
<td>$m(1)$</td>
</tr>
</tbody>
</table>

For example, $e^1(3)$ in period 5 refers to a $(y+3z)$ occurring just before $3z$ in period 7, and $e^2(3)$ refers to $(y+3z)$ in period 4 occurring just before $e^1(3)$. Further, we refer to the terms in period 6, 7 and 8 by $m(3)$, $m(2)$ and $m(1)$. We now calculate the length of a sequence of $(y+cz)$'s. If the coefficient is equal to 1, the length is 5. In general, for an inequality extending up to $t_q$, and a sequence starting in the $(j+1)$st demand interval, the length equals $q-j$. In this example, the sequence $(y+3z)$ has length 2. In general, for terms $(y+cz)$ it is equal to $q-j-c$. The coefficient $c$ varies from 1 through $q-j-1$ for the term $(y+cz)$, and from 1 through $q-j$ for the term $cz$.

Consider yet another inequality:

$$y_1+(y_2+z_2)+(y_3+2z_3)+(y_4+3z_4)+(y_5+3z_5)+(w_6+3z_6)+(w_7+3z_7)+(w_8+3z_8)\ldots$$
$$+3z_9+2z_{10}+z_{11}+\sum_{i=12}^{20}w_i \geq 5.$$

We refer to the terms in periods 6, 7 and 8 by $g_i(3)$, where the quantity 3 in brackets refers to the coefficient of $z$ in these periods. In the following inequality, the terms $(w+3z)$ occur just before $(y_8+3z_8)$:

$$y_1+(y_2+z_2)+(y_3+2z_3)+(y_4+3z_4)+(w_5+3z_5)+(w_6+3z_6)+(w_7+3z_7)+(y_8+3z_8)\ldots$$
$$+3z_9+2z_{10}+z_{11}+\sum_{i=12}^{20}w_i \geq 5.$$
We associate the variables $g_{i}(3)$ with the terms in the periods 5, 6 and 7. The superscript is the same as the superscript for the variable $e_{i}(3)$ associated with $(y_{8} + 3z_{8})$. In the following example, we associate the variables $g_{i}(3)$ and $e_{i}(3)$ with periods 4 and 5, $g_{i}(3)$ and $e_{i}(3)$ with periods 6 and 7, and $g_{i}(3)$ with period 8:

$$y_{1} + (y_{2} + z_{2}) + (y_{3} + 2z_{3}) + (w_{4} + 3z_{4}) + (y_{5} + 3z_{5}) + (w_{6} + 3z_{6}) + (y_{7} + 3z_{7}) + (w_{8} + 3z_{8}) \ldots + 3z_{9} + 2z_{10} + z_{11} + \sum_{i=1}^{20} w_{i} \geq 5.$$ 

Consider the following inequalities:

$$y_{1} + (y_{2} + z_{2}) + (y_{3} + 2z_{3}) + (y_{4} + 3z_{4}) + (y_{5} + 4z_{5}) + 5z_{6} + 5z_{7} + 5z_{8} \ldots + 4z_{9} + 3z_{10} + 2z_{11} + z_{12} + \sum_{i=13}^{20} w_{i} \geq 5$$

$$y_{1} + (y_{2} + z_{2}) + (y_{3} + 2z_{3}) + (y_{4} + 3z_{4}) + (y_{5} + 4z_{5}) + 4z_{6} + 3z_{7} + 2z_{8} \ldots + z_{9} + \sum_{i=10}^{20} w_{i} \geq 5$$

We distinguish the term $(y_{5} + 4z_{5})$ in the two inequalities by associating the variables $e_{i}(4)$ and $e_{i}(4)$ with the first and the second inequality respectively. We also associate the variables $m_{i}(5)$ with periods 6 and 7, and the variable $m_{i}(5)$ with period 8 in the first inequality. This distinguishes the last term in the sequence $5z_{6}$, $5z_{7}$, $5z_{8}$. In general, for an inequality extending up to $t_{q}$ and for a sequence of $(y + cz)$'s starting in demand interval $(j+1)$, we associate $e_{i}(q-j-1)$ if the period $i+1$ contains the term $qz_{i+1}$, and the variable $e_{i}(q-j-1)$ if the period $i+1$ contains the term $(q-1)z_{i+1}$.

The following table gives the terms and their associated variables.

<table>
<thead>
<tr>
<th>Term</th>
<th>Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w$</td>
<td>$a$</td>
</tr>
</tbody>
</table>
\[ y \quad \text{b} \]

\[(y+z) \quad e^k(1) \text{ for } 1 \leq k \leq q-j-1; \text{ and } 0 \leq j \leq q-1 \]

\[(y+2z) \quad e^k(2) \text{ for } 1 \leq k \leq q-j-1; \text{ and } 0 \leq j \leq q-1 \]

\[(y+cz) \quad e^k(c) \text{ for } 1 \leq k \leq q-j-c; \quad 1 \leq c \leq q-j-1; \text{ and } 0 \leq j \leq q-1 \]

\[(y+(q-1)z) \quad e^1(q-j-1) \text{ if period } i+1 \text{ contains the term } (q-j-1)z \]

\[e^0(q-j-1) \text{ if period } i+1 \text{ contains the term } (q-j)z \]

\[cz \quad m(c) \text{ for } c \leq q-j-1; \text{ and } j \leq q \]

\[qz \quad m(q-j) \text{ if period } i+1 \text{ contains the term } (q-j-1)z \]

\[m^0(q-j) \text{ if period } i+1 \text{ contains the term } (q-j)z \]

\[(w+cz) \quad g^k(c) \text{ if i is between } e^{k+1}(c) \text{ and } e^k(c), 0 \leq k \leq q-j-c-1 \]

\[g^{q-j+c}(c) \text{ if i is between } e^{1}(c-1) \text{ and } e^{q-j+c}(c) \text{ for } k=q-j-c \]

For the Partition Inequalities (PI), can write the following separation problem for inequalities extending up to \(t_q\). The index \(j\) varies from 0 through \(q-1\), and each demand interval \(\{t_{j-1}+1, ..., t_j\}\) has a different set of variables associated with it.

\[
\text{Min } \zeta = \sum_{i=1}^{t_q} (w_i^*a_i + y_i^*b_i^* + z_i^*c_i)....
\]

\[+ \sum_{j=0}^{q-1} \sum_{i=t_j+1}^{t_{j+1}} (\sum_{c=1}^{q-j-1} (y_i^*+cz_i^*) (\sum_{k=1}^{q-j-c} e^k_i(c)))....
\]

\[+(y_i+(q-1)z_i)e^0_i(q-j-1)+\sum_{c=1}^{q-j-1} (w_i^*+cz_i^*) (\sum_{k=1}^{q-j-c} g^{q-j+c})....
\]
\[ + \sum_{c=1}^{q-j-1} (w_i + cz_i) g_i(c) + \sum_{c=1}^{q-j} \sum_{c=1}^{s} m_i(c) + (q-j) z_i m_i(q-j)) - q \delta \]

subject to:

\[ m_{i}(c+1) + g_i(c) + e_{i}^{1}(c) - m_{i+1}(c) - g_{i+1}(c) \geq 0 \]

\[ 0 \leq j \leq q-1, \quad t_{j}+1 \leq i \leq t_{j+1}, \quad 1 \leq c \leq q-j-1 \quad (1.ii) \]

\[ m^{0}_{i}(q-j) + e_{i}^{0}(q-j-1) - m_{i+1}(q-j) - m^{0}_{i+1}(q-j) \geq 0 \]

\[ 0 \leq j \leq q-1, \quad t_{j+1} \leq i \leq t_{j+1}, \quad c = q-j \quad (1.iii) \]

\[ g^{k}_{i}(c) + e^{k+1}_{i}(c) - e^{k}_{i+1}(c) - g^{k}_{i+1}(c) \geq 0 \]

\[ 0 \leq j \leq q-1, \quad t_{j+1} \leq i \leq t_{j+1}, \quad 1 \leq c \leq q-j-2, \quad 1 \leq k \leq q-j-c-1 \quad (1.iv) \]

\[ g_{i}^{q-j}(c) + e_{i}^{q-j}(c+1) - e^{q-j}_{i+1}(c) - g^{q-j}_{i+1}(c) \geq 0 \]

\[ 0 \leq j \leq q-1, \quad t_{j+1} \leq i \leq t_{j+1}, \quad 2 \leq c \leq q-j-2, \quad k = q-j-c \quad (1.v) \]

\[ g_{i}^{1}(q-j-1) + e_{i}^{2}(q-j-2) - e^{1}_{i+1}(q-j-1) - e^{0}_{i+1}(q-j-1) - g^{1}_{i+1}(q-j-1) \geq 0 \]

\[ 0 \leq j \leq q-1, \quad t_{j+1} \leq i \leq t_{j+1}, \quad c = q-j-1, \quad k = q-j-c \quad (1.vi) \]

\[ b_{i} + m_{i}(1) + g_{i}^{q-j+1}(1) - e^{q-j}_{i+1}(1) - g^{q-j+1}_{i+1}(1) \geq 0 \]

\[ 0 \leq j \leq q-1, \quad t_{j+1} \leq i \leq t_{j+1}, \quad c = 1, \quad k = q-j-c \quad (1.vii) \]

\[ a_{i} + b_{i} + c_{i} + \sum_{c=1}^{q-j-1} \sum_{k=1}^{c} (e^{k}_{i}(c) + g^{k}_{i}(c)) \ldots . \]

\[ + e^{0}_{i}(c) + g_{i}(c) + \sum_{c=1}^{q-j} m_{i}(c) + m^{0}(q-j) - \delta = 0 \]

\[ 0 \leq j \leq q-1, \quad t_{j}+1 \leq i \leq t_{j+1} \quad (1.viii) \]

\[ \delta \leq 1, \quad a, b, c, e, g, m, \delta \geq 0 \quad (1.ix) \]

We use the following inequalities to illustrate the necessity of these constraints:

\[ y_{1} + (y_{2} + z_{2}) + (y_{3} + 2z_{3}) + (y_{4} + 3z_{4}) + (y_{5} + 3z_{5}) + 3z_{6} + 2z_{7} + z_{8} + \sum_{i=9}^{20} w_{i} \geq 5 \quad (a) \]

\[ y_{1} + (y_{2} + z_{2}) + (y_{3} + 2z_{3}) + (y_{4} + 3z_{4}) + (y_{5} + 4z_{5}) + 5z_{6} + 5z_{7} \ldots \]
\[ +4z_8+3z_9+2z_{10}+z_{11}+\sum_{i=12}^{20} w_i \geq 5 \] (b)

\[ y_1+(y_2+z_2)+(y_3+2z_3)+(y_4+3z_4)+(y_5+4z_5)+4z_6+3z_7 \ldots \]

\[ +2z_8+z_9+\sum_{i=10}^{20} w_i \geq 5 \] (c)

\[ y_1+(y_2+z_2)+(y_3+2z_3)+(w_4+3z_4)+(y_5+3z_5)+(w_6+3z_6)+(y_7+3z_7) \ldots \]

\[ +(w_8+3z_8)+3z_9+2z_{10}+z_{11}+\sum_{i=12}^{20} w_i \geq 5 \] (d)

Constraint (1.ii) ensures that \((c+1)z_i, (w_i+cz_i), (y_i+cz_i)\) in period \(i\) precedes \(cz_{i+1}\) or \((w_{i+1}+cz_{i+1})\) in period \(i+1\). For example, if \(c=3\), and period \(i+1\) contains \(3z_{i+1}\) or \((w_{i+1}+3z_{i+1})\), then period \(i\) contains \(4z_1\) or \((w_i+3z_{i+1})\), or \((y_i+3z_i)\). Inequalities (a), (b) and (d) ensure this condition. This constraint holds for \(c \leq q-j-1\), and if \(c = q-j-1\), then constraint (1.iii) holds.

Constraint (1.iii) ensures that \((q-j)z_i\) or \((y_i+(q-j-1)z_i)\) in period \(i\) precedes \((q-j)z_{i+1}\) in period \(i+1\). For example, if \(q=5\) and \(j=0\), and if period \(i+1\) contains \(5z_{i+1}\), then period \(i\) contains \(5z_1\) or \(y_i+4z_i\). Inequality (b), where \((y_5+4z_5)\) precedes \(5z_6\), or \(e_5^0(4)\) precedes \(m_6^0(5)\), and \(5z_6\) precedes \(5z_7\), or \(m_6^0(5)\) precedes \(m_7^0(5)\) ensures this condition.

Constraint (1.iv) ensures that \((w_i+cz_i)\) or \((y_i+cz_i)\) in period \(i\) precedes \((w_{i+1}+cz_{i+1})\) or \((y_{i+1}+cz_{i+1})\) in period \(i+1\). For example, if \(c=3\), \(q=5\) and \(j=0\), and if period \(i+1\) contains \((w_{i+1}+3z_{i+1})\) or \((y_{i+1}+3z_{i+1})\) period \(i+1\), then period \(i\) contains \((w_i+3z_i)\) or \((y_i+3z_i)\). In inequality (d), \((w_6+3z_6)\) precedes \((y_7+3z_7)\), and \((w_4+3z_4)\) precedes \((y_7+3z_7)\). This is ensured in the constraints by \(g_1^6(3)\) preceding \(e_1^7(3)\), and \(g_2^4(3)\) preceding \(e_2^5(3)\). Further, we restrict the superscript in \(e_i^k(3)\) to \(q-j-c-1 (=1)\). Hence we have at most 2 \((y_i+3z_i)\)'s. This is shown in inequalities (a) and (d). If \(k=q-j-c\), then constraint (1.v) holds.
Constraint (1.v) ensures that \((w_i + cz_i)\) or \((y_i + (c-1)z_i)\) in period \(i\) precedes \((w_i + cz_i)\) or \((y_i + cz_i)\) if \(k=q-j-c\). For example, in inequality (a), \((y_3 + 2z_3)\) precedes \((y_4 + 3z_4)\). In inequality (d) \((y_3 + 2z_3)\) precedes \((w_4 + 3z_4)\), which in turn precedes \((y_5 + 3z_5)\).

Constraint (1.vi) ensures that \((w_i + (q-j-1)z_i)\) or \((y_i + (q-j-2)z_i)\) in period \(i\) precedes \((w_i + (q-j-1)z_i)\) or \((y_i + (q-j-1)z_i)\). In inequality (b) \((y_4 + 3z_4)\) precedes \((y_5 + 4z_5)\), or \(e_4^2(3)\) precedes \(e_5^0(4)\). In inequality (c) \((y_4 + 3z_4)\) precedes \((y_5 + 4z_5)\), or \(e_4^2(3)\) precedes \(e_5^1(4)\).

Constraint (1.vii) ensures that \(y_i\) or \(z_i\) or \((w_i + z_i)\) in period \(i\) precedes \((y_i + z_i)\) or \((w_i + z_i)\). In inequality (a) \(y_1\) precedes \((y_2 + z_2)\).

Constraint (1.viii) ensures that in each period, we choose one of the variables corresponding to \(w_i\), \(y_i\), \(cz_i\), \((w_i + cz_i)\) or \((y_i + cz_i)\).

Notice that each variable appears at most twice in constraints (1.ii) to (1.vii). By using substitutions similar to those used for the separation problem for the two demand case, we can show that the constraints define a network. Therefore, the separation problem always has an integer optimal solution. Hence we can solve the problem as a linear program.

**Separation Lemma.** The optimum objective function value of the separation problem is zero if and only if the solution \((w^*, y^*, z^*)\) of the linear programming relaxation of SCSP satisfies all the partitioning inequalities (PI).

**Proof.** If some inequality is violated, then we can set \(\delta\) and the variables associated with the inequality in the separation problem to 1.
This is a feasible solution to the separation problem with an objective function value strictly less than zero.

Suppose all the inequalities are satisfied. Since there always exists an optimum integer solution and $\delta \leq 1$, all the variables are either 0 or 1 in the integer solution. Hence the set of positive variables in an integer solution to the separation problem correspond to a partitioning inequality. Since all the inequalities are satisfied, the objective function is at least equal to zero. Notice that we can set all the variables in the separation problem to zero. Hence the optimum objective function value is zero.

**Corollary.** If we drop the constraint $\delta \leq 1$ from the separation problem, then the dual has a feasible solution if and only if all the partitioning inequalities are satisfied.

If any inequality is violated, then we can arbitrarily increase $\delta$ and the objective function is unbounded from below. Hence the dual is infeasible. If all the inequalities are satisfied, then the optimal objective function is zero, and hence there exists a feasible dual solution.
Chapter 5. Characterization

The single machine, multi item problem is NP-complete. However, the single machine, single item problem can be solved in polynomial time. We wish to investigate whether the partitioning and skip inequalities completely characterize the polyhedron for the single item, single machine problem. This result would not only be interesting in itself, but would also allow us to use a linear programming based approach to solve the multi item case if we use the single item problem as a substructure.

However, it seems to be difficult to prove the characterization result for the general case. In this chapter we show that for certain cost structures that might reasonably occur in practice, a subset of the partition inequalities is sufficient to guarantee optimal integer solutions. These results would help us to obtain a better understanding of the underlying polyhedral structure that might eventually be useful in proving the general characterization result. They also help to explain why Magnanti and Vachani (1987) obtained near optimal solutions in their computational work.

The one demand case

We first consider the case when there is only one demand in period T. We relax the integrality constraints on the original formulation, and add the partitioning inequalities (PI) to this relaxation. Let us call this problem P. We wish to show that P has integer extreme points and hence that the partitioning inequalities together with the
original inequalities \( w_i \leq y_{i-1} + y_i \leq 1, y_i \leq 1 \) and \( z_i \leq 1 \) completely characterize the convex hull of feasible solutions of the original integer program.

We initially restrict attention to objective functions with nonnegative cost coefficients.

We write a separation problem for the partitioning inequalities (PI). Suppose we are given a solution \((w^*, y^*, z^*)\) to the linear programming relaxation of SCSP (recall that SCSP refers to our original integer program and stands for Single product Changeover Scheduling Problem). We want to write a separation problem that finds out if this solution violates any of these facets. In these facets, each \( z_i \) must be preceded by a \( y_{i-1} \) or a \( z_{i-1} \). Consider the following problem:

\[
\begin{align*}
\text{(Sep)} & \quad \min \sum_{i=1}^{T} (w_i^* a_i + y_i^* b_i + z_i^* c_i) - \delta \\
\text{subject to} & \quad \text{(dual var)} \\
\quad & \quad \begin{aligned}
\quad & \quad p_i: \quad a_i + b_i + c_i = \delta \\
\quad & \quad r_i: \quad b_i + c_i - c_{i+1} \geq 0 \\
\end{aligned} \\
& \quad \begin{aligned}
\quad & \quad \delta \leq 1 \\
& \quad a, b, c, \delta \in (0, 1) \text{ and integer.}
\end{aligned}
\end{align*}
\]

We showed earlier that this problem can be cast as a network flow problem, and hence has integer optimal solutions even if it is solved as a linear program. \( a_i \), \( b_i \) and \( c_i \) are weights on the solution values \( w_i^* \), \( y_i^* \) and \( z_i^* \) respectively. Constraint (1) implies that if \( \delta = 1 \), then for each period \( i \), one of the variables \( a_i \), \( b_i \) or \( c_i \) is equal to 1. Constraint (2) ensures that if \( c_{i+1} = 1 \) (i.e., if \( z_{i+1} \) has a weight of 1), then either \( b_i \) or \( c_i \) is equal to 1. In other words, \( z_{i+1} \) is preceded by \( z_i \) or
$y_i$. This restriction ensures that we do not obtain an invalid inequality like $w_1+w_2+\sum_{i=3}^{T} z_i \geq 1$ by setting $a_1=a_2=c_3=...=c_T=1$.

We also showed earlier that the separation problem has an optimal objective function value of zero if and only if the solution $(w^*, y^*, z^*)$ satisfies all the inequalities. If we drop the constraint $\delta \leq 1$, the optimal value of the objective function is unbounded from below if the solution violates any of the inequalities. Therefore, the dual of the separation problem has a feasible solution if and only if the solution $(w^*, y^*, z^*)$ satisfies all the partitioning inequalities.

The dual of the separation problem is:

dual var

\begin{align*}
a_i: & \quad p_i \leq w_i^* \\
b_i: & \quad p_i + r_i \leq y_i^* \\
c_i: & \quad p_i + r_i - r_{i-1} \leq z_i^* \\
\delta: & \quad \sum_{i=1}^{T} p_i = 1 \\
r \geq 0
\end{align*}

These constraints define a polyhedron that has a feasible solution if and only if the solution satisfies all the partitioning inequalities. We can therefore reformulate $P$ as follows:

\[(R) \quad \min \sum_{i=1}^{T} (h_i w_i + s_i y_i + k_i z_i) \]

subject to

\begin{align*}
dual \text{ var} \\
a_i: & \quad p_i \leq w_i \\
b_i: & \quad p_i + r_i \leq y_i \\
c_i: & \quad p_i + r_i - r_{i-1} \leq z_i \\
\delta: & \quad \sum_{i=1}^{T} p_i = 1 \\
\Delta: & \quad S T_{i=1} w_i = 1
\end{align*}
\[ \sigma_i: \quad w_i \leq y_i \]
\[ \mu_i: \quad z_i + y_{i-1} - y_i \geq 0 \]
\[ \theta_i: \quad y_i \leq 1 \]
\[ \phi_i: \quad z_i \leq 1 \]
\[ r, w, y, z \geq 0 \]

The dual of this reformulation (R) is:

(DR) \[ \max \delta + \Delta \sum_{i=1}^{T} \theta_i - \sum_{i=1}^{T} \phi_i \]

subject to (dual var)

\[ p_i: \quad a_i + b_i + c_i = \delta \] \hspace{1cm} (i)
\[ r_i: \quad b_i + c_i - c_{i+1} \geq 0 \] \hspace{1cm} (ii)
\[ w_i: \quad \Delta + a_i - \sigma_i \leq h_i \] \hspace{1cm} (iii)
\[ y_i: \quad b_i + \mu_{i+1} - \mu_i + \sigma_i - \theta_i \leq s_i \] \hspace{1cm} (iv)
\[ z_i: \quad c_i + \mu_i - \phi_i \leq k_i \] \hspace{1cm} (v)

\( a, b, c, \delta, \sigma, \mu, \theta, \phi \geq 0 \) and integer.

We now use a dual algorithm that gives integer solutions, and use it to generate a primal feasible solution for (R) that satisfies complementary slackness. The algorithm shows that we obtain an optimal integer solution even if we set the dual variables \( \Delta, \sigma, \mu, \theta \) and \( \phi \) to zero provided we assume that the costs are nonnegative, and that there is only one demand. Therefore we rewrite the formulation as follows:

(DR) \[ \max \delta \]

subject to (dual var)

\[ p_i: \quad a_i + b_i + c_i = \delta \] \hspace{1cm} (i)
\[ r_i: \quad b_i + c_i - c_{i+1} \geq 0 \quad (ii) \]

\[ w_i: \quad a_i \leq h_i \quad (iii) \]

\[ y_i: \quad b_i \leq s_i \quad (iv) \]

\[ z_i: \quad c_i \leq k_i \quad (v) \]

\[ a, b, c, \delta \geq 0 \text{ and integer.} \]

The algorithm uses the following facts:

1) Since \( a_i + b_i + c_i = \delta \), the value of the quantity \( a_i + b_i + c_i \) is the same for all time periods.

2) If we sum the equations (iii), (iv) and (v), we obtain the following inequality:

\[ a_i + b_i + c_i \leq k_i + s_i + h_i \]

Therefore, \( \delta \leq k_i + s_i + h_i \), and hence \( \delta \leq \min_i \{k_i + s_i + h_i\} \).

Let \( \delta(i) \) denote the value of \( \delta \) in period \( i \). Initially we set \( \delta(1) = k_1 + s_1 + h_1 \). Since \( \delta \leq \min_i \{k_i + s_i + h_i\} \), clearly \( \delta(i+1) \leq \delta(i) \). The algorithm moves from period \( i \) to period \( i+1 \) and updates the value of \( \delta(i) \), and proceeds to the last period \( T \).

Initially, \( c_1 = k_1, b_1 = s_1, a_1 = h_1 \).

The constraint \( b_1 + c_1 - c_2 \geq 0 \) restricts the value of \( c_2 \) to \( k_1 + s_1 \). The constraint \( c_2 \leq k_2 \) restricts the value of \( c_2 \) to \( k_2 \). The maximum value of \( c_2 = \min(k_1 + s_1, k_2) \). The maximum value that \( \delta(2) \) can potentially have is therefore \( c_2 + s_2 + h_2 \). If \( \delta(1) \leq c_2 + s_2 + h_2 \), we set \( \delta(2) = \delta(1) \), and

\[ b_2 = \min \{s_2, \delta(2) - b_2\} \]
\[ a_2 = \delta(2)-c_2 - c_2. \]

If \( \delta(1) > c_2 + s_2 + h_2 \), we set the value of \( \delta(2) = c_2 + s_2 + h_2 \), and set

\[ b_2 = s_2 \text{ and } a_2 = h_2. \]

In general, we make the following computations:

\[ c_{i+1} = \min\{k_{i+1}, b_i + c_i\} \quad (1.1) \]
\[ \delta(i+1) = \min\{\delta(i), c_{i+1} + s_{i+1} + h_{i+1}\} \quad (1.2) \]
\[ b_{i+1} = \min\{s_{i+1}, \delta(i+1) - c_{i+1}\} \quad (1.3) \]
\[ a_{i+1} = \delta(i+1) - b_{i+1} - c_{i+1}. \quad (1.4) \]

We continue until we obtain \( \delta(T), c_T, b_T, a_T \). Note that the quantity \( (a_i + b_i + c_i) \) equals the current value of \( \delta(i) \). Since \( \delta(T) \leq \delta(i) \), for \( i < T \), the values of \( a_i, b_i \), and \( c_i \) we obtained might no longer be valid, because the quantity \( (a_i + b_i + c_i) \) should be the same in all time periods. We then backtrack from period \( T-1 \) to \( 1 \) and obtain the new values of \( a_i, b_i \), and \( c_i \).

We first note that the quantities \( a_i + b_i + c_i \) we obtained in the first pass as we went from period \( 1 \) to period \( T \) are all greater than or equal to \( \delta(T) \). We set the quantities \( a_i, b_i \), and \( c_i \) in the backtracking pass as follows:

If \( \delta(T) = \delta(T-1) \), then we retain the values of \( c_{T-1}, b_{T-1}, a_{T-1} \). If \( \delta(T) < \delta(T-1) \), and \( c_T = b_{T-1} + c_{T-1} \), then we retain the values of \( b_{T-1} \) and \( c_{T-1} \) and set \( a_{T-1} = \delta(T) - c_{T-1} - b_{T-1} \). We need to show that the new value of \( a_{T-1} \) is feasible. The value of \( a_{T-1} \) in the forward pass is \( a_{T-1} = \delta(T-1) - b_{T-1} - c_{T-1} \), and since \( \delta(T) < \delta(T-1) \), the new value of \( a_{T-1} \) is lower than the old value, and hence \( a_{T-1} < h_{T-1} \). Therefore it is feasible.

If \( \delta(T) < \delta(T-1) \), and \( c_T = k_T \), then we set
\[ a_{T-1} = \min \{ \delta(T) - c_T, h_{T-1} \} \]
\[ b_{T-1} = \min \{ \delta(T) - a_{T-1} , b_{T-1} \} \]
\[ c_{T-1} = \delta(T) - a_{T-1} - b_{T-1} . \]

In general, we do the following:

If the value of \( \delta(i) = \delta(T) \), then we retain the values of \( a_i, b_i \) and \( c_i \). If \( \delta(i) > \delta(T) \), and \( c_{i+1} \) (from the return pass) is equal to \( b_i + c_i \) (from the forward pass), then we retain the values of \( b_i \) and \( c_i \) and set \( a_i = \delta(T) - b_i - c_i \). If \( \delta(i) > \delta(T) \), and \( c_{i+1} = k_{i+1} \), then we set

\[ a_i = \min \{ \delta(T) - c_{i+1}, h_i \} \quad (1.5) \]
\[ b_i = \min \{ \delta(T) - a_i, b_i \} \quad (1.6) \]
\[ c_i = \delta(T) - a_i - b_i \quad (1.7) \]

We also set \( \mu_i = 0 \) for all \( i \). We can set \( \mu_i = 0 \) only for the one demand case. In the two demand case, we need to find the values of \( \mu_i \).

We next compute \( k_i = k_1 - c_i, s_i = s_i - b_i, h_i = h_i - a_i \) for all \( i \). Before we obtain a primal feasible solution, we note the following about the values of \( c_i, b_i, a_i \):

We identify the period of production \( i^* \), and the period \( i_0 \) in which we turn on the machine, and show that for the intervening periods from \( i_0 \) through \( i^* \), the slack on the constraint \( b_i \leq s_i \) is zero, so that we can set \( y_i = 1 \).

1) \( s_i^* + h_i^* = 0 \) for at least one period \( i^* \).

Let \( i^* \) be defined as follows: \( \delta(i^*) = \delta(T) \), \( i^* \leq T \), and \( \delta(i^* - 1) > \delta(i^*) \). If no such \( i^* \) exists, we define \( i^* = 1 \).
In the forward pass, from equation (1.2), we get 
\[ \delta(i^*) = a_{i^*} + b_{i^*} + c_{i^*}. \]

From equations (1.3) and (1.4), we get \( b_{i^*} = s_{i^*}, \ a_{i^*} = h_{i^*} \). In the return pass, since \( \delta(i^*) = \delta(T) \), the values of \( a_i, b_i, c_i \) do not change for \( i \geq i^* \). Hence \( s_{i^*} + h_{i^*} = 0 \).

2) If for some \( i \leq i^*, \ k_t > 0 \) for all \( i \leq t \leq i^* \), then \( s_t = 0 \).

Suppose that \( k_{i^*} > 0 \). Since \( c_i \) is the same in the forward and the return pass, the final value of \( c_i \) is \( \min\{k_{i^*}, b_{i^*} + c_{i^*} - 1\} \). Since \( k_{i^*} = k_{i^*} - c_{i^*} > 0 \), hence \( c_{i^*} = b_{i^*} + c_{i^*} - 1 \). This result implies that the values of \( b_{i^*} \) and \( c_{i^*} \) do not change after the return pass: this result follows from the paragraph preceding equation 1.5. Hence \( b_{i^*} = \min\{s_{i^*} - 1, \delta(i^*-1) - c_{i^*} \} \).

If \( b_{i^*} < s_{i^*} \), then \( b_{i^*} = \delta(i^*-1) - c_{i^*} \). But then \( c_{i^*} = b_{i^*} + c_{i^*} = \delta(i^* - 1) \). This result contradicts our assumption that \( \delta(i^*-1) > \delta(i^*) \). Hence \( b_{i^*} \) and \( c_{i^*} < \delta(i^*-1) \) and so \( b_{i^*} = s_{i^*} \). Moreover, since \( b_{i^*} + c_{i^*} < \delta(i^*-1) \), \( a_{i^*} > 0 \).

Similarly, if for some \( i \leq i^*, \ k_t > 0 \) for all \( i \leq t \leq i^* \), then \( c_t = b_{t-1} + c_{t-1} \) for all \( i \leq t \leq i^* \). If \( b_{t-1} + c_{t-1} = \delta(t-1) \), then \( c_t = \delta(t-1) \). But \( c_t \leq c_{t+1} \), which follows from the equation \( b_{t-1} + c_{t-1} = c_t \). Hence \( \delta(t-1) \leq g_i \leq \delta(i^*) \). But this result contradicts the fact that \( \delta(t-1) \geq \delta(i^*-1) > \delta(i^*) \). Therefore \( b_{t-1} + c_{t-1} < \delta(t-1) \), and hence \( b_{t-1} = \min\{s_{t-1}, \delta(t-1) - c_{t-1} \} = s_{t-1}, \) and \( s_{t-1} = 0 \).

Finally, we note that if \( k_i > 0 \) for all \( 2 \leq t \leq i^* \), then \( c_2 = b_1 + c_1 \) and since the value of \( b_1 \) and \( c_1 \) does not change in the return pass, therefore \( c_1 = k_1 \) and \( k_1 = 0 \). Therefore we can claim that:

3) \( k_{i_0} = 0 \) for some \( i_0 \leq i^* \).

We define \( i^0 \) as the first period preceding \( i^* \) with \( k_{i_0} = 0 \). If \( k_{i^*} = 0 \), then \( i^0 = i^* \).

The primal solution is as follows:
\( p_{i^*} = w_{i^*} = 1, \ y_i = 1 \) for \( i^0 \leq i \leq i^* \), \( r_i = 1 \) for \( i^0 \leq i \leq i^*-1 \) and \( z_{i_0} = 1 \). All other primal variables are set to zero. We now show that this solution is primal feasible and that it satisfies the complementary slackness conditions.

First, we check the constraints \( p_i \leq w_i \). All these constraints are satisfied at strict equality and hence they satisfy complementary slackness conditions as well.

Next, we check the constraints \( p_i + r_i \leq y_i \). For \( i \leq i^0 - 1 \), and \( i \geq i^* + 1 \), \( p_i = r_i = y_i = 0 \) and hence these constraints satisfy complementary slackness conditions. For \( i^0 \leq i \leq i^*-1 \), \( p_i = 0 \), \( r_i = 1 \), and \( y_i = 1 \). Hence these equations are also satisfied at equality. For \( i = i^* \), \( p_i = 1 \), \( r_i = 0 \), and \( y_i = 1 \). This equation is also satisfied at equality.

We now check the constraints \( p_i + r_i - r_{i-1} \leq z_i \). For \( i \leq i^0 - 1 \), and \( i \geq i^* + 1 \), \( p_i = r_i = z_i = 0 \). Moreover, \( r_{i^*} = 0 \) and hence these constraints are satisfied at equality and satisfy the complementary slackness conditions. For \( i^0 + 1 \leq i \leq i^*-1 \), \( p_i = 0 \), \( r_i = 1 \), and \( z_i = 0 \). Moreover, \( r_i = 1 \) and hence these equations are also satisfied at equality. For \( i = i^0 \), \( r_i = 1 \), \( r_{i0} = 0 \), \( p_i = 0 \) and \( z_{i0} = 1 \). Hence this equation is also satisfied as an equality. For \( i = i^* \), \( p_i = 1 \), \( r_i = 0 \), \( z_i = 0 \), and \( r_{i-1} = 1 \). Hence this equation is also satisfied as an equality.

We then check the equation \( \sum_{i=1}^{T} p_i = 1 \). The solution satisfies this equation because \( p_{i^*} = 1 \) and \( p_i = 0 \) for all other \( i \). Since \( \mu_i = 0 \), the inequalities \( z_i + y_{i-1} - y_i \geq 0 \) satisfy the complementary slackness conditions as well.

Similarly, the solution satisfies the equations \( \sum_{i=1}^{T} w_i = 1, \ w_i \leq y_i, \ z_i + y_{i-1} - y_i \geq 0, \ y_i \leq 1, \ z_i \leq 1 \), and since we set the corresponding dual
variables $\Delta, \sigma, \mu, \theta$, and $\phi$ to zero, they also satisfy complementary slackness.

Dual complementary slackness.

We set $r_i = 1$ only for the periods $i^0 \leq i \leq i^*-1$. But for these periods, $c_t = b_{t-1} + c_{t-1}$. This result ensures that the solution satisfies the complementary slackness condition on the constraint $b_{t-1} + c_{t-1} \geq c_t$.

We set $w_i = 1$ for $i = i^*$, but then $h_{i^*} = 0$ and so the solution satisfies the constraint $a_i \leq h_i$ as an equality.

We set $y_i = 1$ for $i^0 \leq i \leq i^*$. But then $s_i = 0$, ensuring that the solution satisfies the constraint $b_i \leq s_i$ as an equality.

Finally, we set $z_i = 1$ for period $i^0 = i$, and by definition, $k_{i0} = 0$. This result ensures that the solution satisfies the constraint $c_i \leq k_i$ as an equality.

Hence we obtain an optimal integer solution.

The two demand case

We now consider the problem with two demands in periods $t_1$ and $t_2 = T$. We have earlier described the inequalities extending up to the second demand period.

We assume the following cost structure: the changeover cost $k_i$ is constant for all periods. Similarly, the setup cost is the same in all time periods. Finally, the holding cost in period $i$ is defined by $(T-i)H$. The production cost is also assumed to be constant for all time periods. Since we must produce exactly $n$ times, the production cost is not a
variable, and hence we need not consider it explicitly. These restrictions on the objective function cost coefficients are reasonable since the changeover or setup costs do not change over a relatively short time horizon. Further, if the holding cost per period $H$ is constant, then the holding cost becomes as $(T-i)H$.

We show that a small subset of the inequalities are sufficient to obtain integer solutions for this problem, namely

$$\sum_{i=1}^{t_1} w_i + \sum_{i\in W} w_i + \sum_{i\in Y} y_i + \sum_{i\in Z} z_i^+ \geq 2$$ (i)

which partitions only the second demand interval $[t_1+1, \ldots, T]$ into the subsets $W, Y$ and $Z$ and impose the conditions that if $i\in W$ then $i+1\in Z$, and that $t_1+1\in Z$. These are the inequalities discovered by Magnanti and Vachani(1987). We also need the inequalities extending up to $t_1$:

$$\sum_{i\in W} w_i + \sum_{i\in Y} y_i + \sum_{i\in Z} z_i^+ \geq 1$$ (ii)

In other words we do not require the skip inequalities, or inequalities with the terms $(y+z), (w+z)$ and $2z$. In fact, we do not need the constraints $z_i+y_{i-1}y_i \geq 0$, the constraints $y_i \geq w_i$, or the capacity constraints $y_i \leq 1$, and $z_i \leq 1$, and the equality $\sum_{i=1}^T w_i = 2$.

The separation problem for these inequalities can be written as follows:

\begin{align*}
\text{(Sep2)} \quad \min & \sum_{i=1}^{t_1} (w_i a_{i1} + y_i b_{i1} + z_i c_{i1}) \ldots \\
& + \sum_{i=1}^T (w_i a_{i2} + y_i b_{i2} + z_i c_{i2}) - \delta_1 - 2\delta_2.
\end{align*}

subject to

\begin{align*}
\text{(dual var)} \\
p_{i1} & : \quad a_{i1} + b_{i1} + c_{i1} - \delta_1 = 0 \quad i \leq t_1 \quad (1) \\
r_{i1} & : \quad b_{i1} + c_{i1} - c_{i+1,1} \geq 0 \quad i \leq t_1 - 1 \quad (2)
\end{align*}
\[ p_{i1}: \quad a_{i1} - \delta_2 = 0 \quad i \leq t_1 \quad (3) \]
\[ a_{i1} + b_{i2} + c_{i1} - \delta_2 = 0 \quad i > t_1 \quad (4) \]
\[ r_{i1} : \quad b_{i2} + c_{i1} - c_{i+1,1} \geq 0 \quad i \geq t_1 + 1 \quad (5) \]
\[ c_{t_1+1,2} = 0 \quad (6) \]
\[ a_{i1}, b_{i1}, c_{i1}, a_{i2}, b_{i2}, c_{i2}, \delta_1, \delta_2 \geq 0 \]

We showed earlier that this separation problem can be cast as a network flow problem, and hence has integer extreme points. The dual of the reformulation of the problem is:

\[
\begin{align*}
\min & \quad 2\delta_2 + \delta_1 \\
\text{subject to} & \\
\text{(dual var)} & \\
p_{i1} & : \quad a_{i1} + b_{i1} + c_{i1} - \delta_1 = 0 \quad i \leq t_1 \quad (1.1) \\
r_{i1} & : \quad b_{i1} + c_{i1} - c_{i+1,1} \geq 0 \quad i \leq t_1 - 1 \quad (1.2) \\
p_{i2} & : \quad a_{i2} - \delta_2 = 0 \quad i \leq t_1 \quad (1.3) \\
a_{i1} + b_{i2} + c_{i2} - \delta_2 = 0 \quad i > t_1 \quad (1.4) \\
r_{i2} & : \quad b_{i2} + c_{i2} - c_{i+1,2} \geq 0 \quad i \geq t_1 + 1 \quad (1.5) \\
c_{t_1+1,2} & = 0 \quad (1.6) \\
\end{align*}
\]

Capacity constraints:
\[
\begin{align*}
z_i & : \quad c_{i1} + c_{i2} \leq k \quad i \leq T \quad (1.7) \\
y_i & : \quad b_{i1} + b_{i2} \leq s \quad i \leq T \quad (1.8) \\
w_i & : \quad a_{i1} + a_{i2} \leq (t-i)H \quad i \leq T \quad (1.9) \\
w, y, z, a_{i1}, b_{i1}, c_{i1}, a_{i2}, b_{i2}, c_{i2}, \delta_1, \delta_2 \geq 0 \\
\end{align*}
\]

The optimal integer solution has the following properties:

(i) We produce for the first time in period \( t_1 \). Since the changeover costs and setup costs are the same in each period, the only difference in cost between periods is due to the production cost \((T-i)H\). Therefore we
would like to postpone production as much as possible. Since we cannot postpone it beyond the first demand period $t_1$, we produce for the first time by turning on the machine in this period.

(ii) We produce for the second and last time in period $t_1+1$ or $T$, (and not in any of the intervening periods $t_1+2$ through $T-1$). Suppose we produce in some period $t_1+1 < i < T$. Consider the two cases:

(a) $s \leq H$. Then we can obtain a better solution by keeping the machine on from periods $i+1$ through $T$ and producing in period $T$. The setup and production cost in period $i$ is $s+(T-i)H$, and in period $T$ it is $(T+1-i)s+(T-T)H=(T+1-i)s$. Since $s \leq H$, therefore $(T+1-i)s \leq s+(T-i)H$. Hence we can find an alternate optimal solution by producing in period $T$.

(b) $s > H$. Here again we consider two cases: (1) We turn the machine for a second time in period $i$. In this case we obtain a better solution by turning it on in period $T$, since the production cost in period $T$ is lower than that in period $i$. (2) We keep the machine on from period $t_1$ through $i$. In this case we obtain a better solution by producing in period $t_1+1$, and shutting the machine off from period $t_1+2$ onwards, since the cost $s+(T-t_1-1)H$ in period $t_1+1$ is lower than the cost $(i-t_1)s+(T-i)H$ in period $i$.

Graphically we can show the different scenarios as follows:
Figure 5.1. Feasible solutions for the two demand problem.

In figures 5.1(a) and 5.1(c), the periods $t_1$ and $T$ are far apart, and it is more economical to turn the machine on twice rather than keep it on from $t_1$ through $T$. In figure 5.1(b), we keep the machine on from $t_1$ through $T$, and in figure 5.1(d), we produce in the periods $t_1$ and $t_1+1$.

We can now write down a dual feasible solution with the same objective function value as the optimal integer solution. We consider the following two cases:

1. $s \leq H$. Then we produce for the second time in period $T$.

   (a) *If we turn the machine on for a second time*, then the following condition must hold: the cost of keeping the machine on
from period \( t+1 \) through period \( T \) is greater than the cost of turning the machine on for the second time. *Hence* \((T-t_1-1)s \geq k \). We set the dual variables as follows. We first set the values for the variables associated with the inequalities extending up to period \( T \). Then we set the values for the variables associated with the variables extending up to \( t_1 \). We set

\[
\delta_2 = k+s \quad \text{and} \quad \delta_1 = (k+s+(T-t_1)H) - (k+s) = (T-t_1)H.
\]

This solution gives us a dual objective function value of \( 2\delta_2 + \delta_1 = 2k+2s+(T-t_1)H \) which is the same as the optimal integer objective function value of the primal. We set the other dual variables for the two interval inequalities as follows:

\[
a_{12} = 0, \quad b_{12} = s, \quad c_{12} = k
\]

for periods \( T-1 \geq i \geq t_1+1 \)

\[
c_{12} = [k - (T-i)s]^+, \quad b_{12} = c_{i+1,2} - c_{i2}, \quad a_{12} = \delta_2 - b_{12} - c_{12}.
\]

for periods \( i \leq t_1 \)

\[
a_{12} = \delta_2.
\]

We show that variables are dual feasible. First we note that the solution satisfies the constraints (1.3) and (1.4) for the two interval inequalities since \( a_{12} + b_{12} + c_{12} = \delta_2 \) for \( i \geq t_1 +1 \) and \( a_{12} = \delta_2 \) for \( i \leq t_1 \). Similarly it satisfies the constraint (1.5) because \( b_{12} = c_{i+1,2} - c_{i2} \) and hence \( b_{12} + c_{i2} - c_{i+1,2} \geq 0 \). We show that the solution also satisfies the constraint (1.6) \( c_{t_1+1,2} = 0 \). Since \( c_{12} = [k - (T-i)s]^+ \), and since as we showed earlier \( (T-t_1-1)s \geq k \), therefore \( c_{t_1+1,2} = 0 \).

We check the capacity constraints. For periods \( i \geq t_1+1 \):

The solution clearly satisfies the capacity constraint \( c_{i2} \leq k \). If \( k - (T-i)s \geq 0 \), then \( c_{i2} = k - (T-i)s \), and \( c_{i+1,2} = k - (T-i-1)s \). Hence \( b_{12} = s \), and the solution
satisfies the capacity constraint \( b_{i2} \leq s \). If \( k - (T-i)s < 0 \), then \( k - (T-i-1)s \leq s \). But \( c_{i+1,2} = k - (T-i-1)s \) and \( c_{i2} = 0 \). Hence \( b_{i2} = c_{i+1,2} - c_{i2} \leq s \). Finally, we check the capacity constraint \( a_{i2} \leq (T-i)H \).

The lefthand side \( a_{i2} = \delta_2 - b_{i2} - c_{i2} = \delta_2 - c_{i+1,2} = k + s - [k - (T-i-1)s]^+ \). If \( k - (T-i-1)s \geq 0 \), then the lefthand side \((T-i)s\). By assumption \( s \leq H \), hence the solution satisfies the constraint. If \( k - (T-i-1)s < 0 \), then the righthand side \((T-i)H \geq (T-i-1)s + s \geq k + s = a_{i2} \).

Hence the solution satisfies the constraint.

Thus far we have obtained a dual feasible set of variables for the two interval inequalities. We set the variables for the one interval inequalities as follows:

\[
\delta_1 = (T-t_1)H, \\
c_{i1} = k, \quad b_{i1} = s, \quad a_{i1} = (T-t_1)H - a_{t_2} \quad \text{for all } i \leq t_1.
\]

We show that \( a_{t_1} + b_{t_1} + c_{t_1} = \delta_1 \). We first obtain the value of \( a_{t_2} \). Since \( c_{t_1+1,2} = 0 \), hence \( c_{t_2} = 0 \), and so \( b_{t_2} = 0 \). Hence \( a_{t_2} = \delta_2 = k + s \).
Therefore \( a_{t_1} + b_{t_1} + c_{t_1} = (T-t_1)H - a_{t_2} + b_{t_1} + c_{t_1} \)
\[
= (T-t_1)H - (k + s) + b_{t_1} + c_{t_1} \]
\[
= (T-t_1)H \]
\[
= \delta_1.
\]

This result also shows that the solution satisfies the constraints (1.1) \( a_{i1} + b_{i1} + c_{i1} - \delta_1 = 0 \).
It also satisfies the constraints (1.2) \( b_{i1} + c_{i1} - c_{i+1,1} \geq 0 \) since \( c_{i1} = k \). The capacity constraints \( c_{i1} + c_{i2} \leq k \) and \( b_{i1} + b_{i2} \leq s \) follow trivially since \( c_{i2} = 0 \) and \( b_{i2} = 0 \). Finally, we need to show that \( a_{i1} + a_{i2} \leq (T-i)H \).

The lefthand side \( a_{i1} + a_{i2} \)
\[
= (T-t_1)H \]
\[
\leq (T-i)H \quad \text{for } i \leq t_1.
\]
Hence we have obtained a dual feasible solution with the same objective function value as the optimal primal integer solution.

The following example illustrates this result. Suppose we have two demands in periods 2 and 7. Let $k=20$, $s=5$ and $H=10$. The costs of production are therefore 60, 50, 40, 30, 20, 10 and 0. The optimal solution is to turn on the machine in periods 2 and 7 and produce in each of these periods. The objective function value is $(20+5+50)+(20+5)=100$. The variable $\delta_2 = (20+5) = 25$, the cost of producing the second unit, and $\delta_1 = (20+5+50)-(20+5) = 50$, the cost of producing the first unit minus the cost of producing the second unit. The other variables have the following values:

Two interval inequalities:

$$
\begin{align*}
\text{Period} & \quad 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
c_{i2} & 0 & 0 & 0 & 5 & 10 & 15 & 20 \\
b_{i2} & 0 & 0 & 5 & 5 & 5 & 5 & 5 \\
a_{i2} & 25 & 25 & 20 & 15 & 10 & 5 & 0.
\end{align*}
$$

One interval inequalities:

$$
\begin{align*}
\text{Period} & \quad 1 & 2 \\
c_{i1} & 20 & 20 \\
b_{i1} & 5 & 5 \\
a_{i1} & 25 & 25.
\end{align*}
$$

(b) If we do not turn on the machine for a second time, then the cost of turning the machine on is greater than keeping the machine on from periods $t_1+1$ through $T-1$. Hence $k \geq (T-t_1-1)s$. Then we set the values for the variables associated with the variables extending up to $T$. We set

$$
\delta_2 = (T-t_1)s \quad \text{and} \quad \delta_1 = (k+s+(T-t_1)H) - (T-t_1)s.
$$
This solution gives us a dual objective function value of \(2\delta_2 + \delta_1 = k + s + (T-t_1)s + (T-t_1)H\) which is the same as the optimal integer objective function value of the primal. We set the other dual variables for the two interval inequalities as follows:

\[ a_{T_2} = 0, \quad b_{T_2} = s, \quad c_{T_2} = k \]

for periods \(T-1 \geq i \geq t_1 + 1\)

\[ c_{i_2} = [(i-t_1-1)s]^+, \quad b_{i_2} = c_{i+1,2} - c_{i_2}, \quad a_{i_2} = \delta_2 - b_{i_2} - c_{i_2}. \]

for periods \(i \leq t_1\)

\[ a_{i_2} = \delta_2. \]

It is easy to show that these variables are dual feasible. First we check the constraint \(a_{i_2} + b_{i_2} + c_{i_2} = \delta_2\), which is clearly satisfied. Similarly the solution also satisfies the constraint \(b_{i_2} + c_{i_2} \geq c_{i+1,2}\). Note that for \(i = t_1 + 1\), \(c_{i_2} = 0\). Next we check the capacity constraints \(c_{i_2} \leq k\). For \(i = T\), \(c_{T_2} = (T-t_1-1)s\), and since we keep the machine on from period \(t_1\) through period \(T\), \((T-t_1-1)s \leq k\). For \(i \leq T\), \(c_{i_2} \leq c_{T_2} \leq k\). If \(c_{i_2} = (i-t_1-1)s\) (when \(c_{i_2} = (i-t_1-1)s \geq 0\)), then \(b_{i_2} = c_{i+1,2} - c_{i_2} = s\), and hence the solution satisfies the constraint \(b_{i_2} \leq s\). If \(c_{i_2} = 0\), then \((i-t_1-1)s \leq 0\), and hence \(c_{i+1,2} = (i-t_1)s \leq s\). Hence the solution satisfies the constraint \(b_{i_2} \leq s\). Finally, we check the constraint \(a_{i_2} \leq (T-i)H\). For \(i \geq t_1 + 1\),

\[ a_{i_2} = \delta_2 - b_{i_2} - c_{i_2} \]

\[ = \delta_2 - c_{i+1,2}. \]

\[ = (T-t_1)s - [(i-t_1)s]^+ \]

\[ \leq (T-t_1)s - (i-t_1)s \]

\[ = (T-i)s \]

\[ \leq (T-i)H. \]

For \(i \leq t_1\), \(a_{i_2} = \delta_2 = (T-t_1)s \leq (T-t_1)H \leq (T-i)H\). Hence the capacity constraint \(a_{i_2} \leq (T-i)H\) is satisfied.
Then we set the variables for the inequalities extending up to $t_1$.

$$
\delta_1 = (k+s+(T-t_1)H) - (T-t_1)s
$$

$$
c_{i1} = k, \quad b_{i1} = s, \quad a_{i1} = (T-t_1)H - a_{t12}\text{ for all } i \leq t_1.
$$

Clearly the solution satisfies all the dual constraints.

The following example illustrates this solution. Suppose we have two demands in periods 2 and 7. Let $k = 30$, $s = 5$ and $H = 10$. This data is the same as that for the example we used earlier, except that we change the cost $k$ from 20 to 30. The optimal solution is to turn on the machine in periods 2, keep it on until period 7 and produce in periods 2 and 7. The objective function value is $(30+5+50)+(5*5)=110$.

The variable $\delta_2 = 25$, the cost of producing the second unit, and $\delta_1 = (30+5+50)-(25) = 60$, the cost of producing the first unit minus the cost of producing the second unit. The other variables have the following values:

### Two interval inequalities:

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{i2}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>10</td>
<td>15</td>
<td>20</td>
</tr>
<tr>
<td>$b_{i2}$</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$a_{i2}$</td>
<td>25</td>
<td>25</td>
<td>20</td>
<td>15</td>
<td>10</td>
<td>5</td>
<td>0</td>
</tr>
</tbody>
</table>

### One interval inequalities:

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{i1}$</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$b_{i1}$</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>$a_{i1}$</td>
<td>25</td>
<td>25</td>
</tr>
</tbody>
</table>
(2) \( s > H \). If we produce for the second time in period \( T \), then the cost of producing in period \( t_1+1 \) is more than the cost of turning the machine on in period \( T \). In other words, \( s+(T-t_1-1)H \geq k+s \), or \( (T-t_1-1)H \geq k \). We set the dual variables as follows:

\[ \delta_2 = k+s, \quad \delta_1 = (T-t_1)H. \]

The dual objective function value is \( 2\delta_2 + \delta_1 = 2k+2s+(T-t_1)H \), which is the same as the primal integer optimal solution. The other dual variables for the two interval inequalities are set as follows:

\[ a_{t_2} = 0, \quad b_{t_2} = s, \quad c_{t_2} = k \]

for periods \( T-1 \geq i \geq t_1+1 \)

\[ c_{i_2} = [k - (T-i)H]^+, \quad a_{i_2} = \min(\delta_2, (T-i)H) \quad b_{i_2} = \delta_2 - a_{i_2} - c_{i_2}. \]

for periods \( i \leq t_1 \)

\[ a_{i_2} = \min(\delta_2, (T-i)H), \quad b_{i_2} = \delta_2 - a_{i_2}. \]

The solution clearly satisfies the constraint \( a_{i_2} + b_{i_2} + c_{i_2} = \delta_2 \). We check the constraint \( b_{i_2} + c_{i_2} - c_{i+1,2} \geq 0 \). If \( c_{i+1,2} = 0 \), then the solution satisfies the constraint. If \( c_{i+1,2} = k - (T-i-1)H \), then

\[ b_{i_2} + c_{i_2} = \delta_2 - a_{i_2} \geq \delta_2 - (T-i)H \]

\[ = k+s - (T-i)H \]

\[ \geq k - (T-i-1)H. \]

and hence the solution satisfies the constraint. Moreover \( c_{t_1+1,2} = 0 \) since \( (T-t_1-1)H \geq k \).

We next check the capacity constraints for periods \( i \geq t_1+1 \). The capacity constraints \( c_{i_2} \leq k \) and \( a_{i_2} \leq (T-i)H \) are clearly satisfied. For the capacity constraint \( b_{i_2} \leq s \), \( b_{i_2} = \delta_2 - a_{i_2} - c_{i_2} \). If \( a_{i_2} = \delta_2 \), then \( b_{i_2} = 0 \), and the solution satisfies the capacity constraint. If \( a_{i_2} = (T-i)H \), then \( (T-i)H \leq \delta_2 = k+s \). If \( c_{i_2} = (k-(T-i)H) \), then \( b_{i_2} = s \). If \( c_{i_2} = 0 \), then \( k-(T-i)H \leq 0 \). Hence
(k+s) - (T-i)H ≤ s. Hence \( \delta_2 - a_{i2} - c_{i2} = (k+s) - (T-i)H - 0 \leq s \), and the solution satisfies the capacity constraint.

For periods \( i \leq t_1 \), \( c_{i2} = 0 \), and so the solution satisfies the capacity constraint \( a_{i2} \leq (T-i)H \). For the capacity constraint \( b_{i2} \leq s \), notice that it is satisfied for \( i = t_1 + 1 \). For \( i \leq t_1 \), \( a_{i2} \geq a_{i+1,2} \) since \( (T-i)H \geq (T-i-1)H \). Hence \( b_{i2} \leq b_{i+1,2} \), and so the solution satisfies the capacity constraints.

The following example illustrates this solution. Suppose we have two demands in periods 2 and 7. Let \( k = 30 \), \( s = 20 \) and \( H = 10 \). This data is the same as that for the example earlier, except that we change the cost \( s \) from 5 to 20. The optimal solution is to turn on the machine in periods 2 and 7 and produce in each of these periods. The objective function value is \((30+20+50)+(30+20)=150\).

The variable \( \delta_2 = 50 \), the cost of producing the second unit, and \( \delta_1 = (30+20+50)-(50) = 50 \), the cost of producing the first unit minus the cost of producing the second unit. The other variables have the following values:

**Two interval inequalities:**

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_{i2} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>( b_{i2} )</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>( a_{i2} )</td>
<td>50</td>
<td>50</td>
<td>40</td>
<td>30</td>
<td>20</td>
<td>10</td>
<td>0</td>
</tr>
</tbody>
</table>

**One interval inequalities:**

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_{i1} )</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>( b_{i1} )</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>( a_{i1} )</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
If we produce for the second time in period \(i+1\), then the cost of production in period \(t_1\) is less than that in period \(T\). In other words, \(s+(T-t_1-1)H \leq k+s\), or \((T-t_1-1)H \leq k\). We set the dual variables as follows:

\[
\delta_2 = s+(T-t_1-1)H, \quad \delta_1 = (k+s+(T-t_1)H) - (s+(T-t_1-1)H) = k+H.
\]

The dual objective function value is \(k+2s+(T-t_1)H+(T-t_1-1)H\), which is the same as the primal optimal integer solution. We set the variables for the inequalities extending up to \(T\) as follows:

\[
a_{T2} = 0, \quad b_{T2} = s, \quad c_{T2} = (T-t_1-1)H.
\]

for periods \(T-1 \geq i \geq t_1+1\)

\[
c_{i2} = [c_{T2} - (T-i)H]^+ = [(i-t_1-1)H]^+
\]

\[
a_{i2} = \min(\delta_2, (T-i)H)
\]

\[
b_{i2} = \delta_2 - a_{i2} - c_{i2}.
\]

for periods \(i \leq t_1\)

\[
a_{i2} = \min(\delta_2, (T-i)H), \quad b_{i2} = \delta_2 - a_{i2}.
\]

The solution clearly satisfies the constraint \(b_{i2} + a_{i2} + c_{i2} = \delta_2\). We check the constraint \(b_{i2} + c_{i2} \geq c_{i+1,2}\) for \(i \leq t_1\). If \(c_{i+1,2} = 0\), the constraint is satisfied. If not, then \(c_{i+1,2} = (i-t_1)H\), and

\[
b_{i2} + c_{i2} = \delta_2 - a_{i2}
\]

\[
= s+(T-t_1-1)H - a_{i2}
\]

\[
\geq s+(T-t_1-1)H - (T-i)H
\]

\[
\geq s+i - (t_1-1)H
\]

\[
\geq (i - t_1)H
\]

\[
= c_{i+1,2}.
\]

Hence the solution satisfies the constraint. For the capacity constraint \(b_{i2} \leq s\), if \(a_{i2} = \delta_2\), then \(b_{i2} = 0\), and the solution satisfies the capacity
constraint. If \( a_{i2} = (T-i)H \), then \( (T-i)H \leq \delta_2 = s + (T-t_1-1)H \). If \( c_{i2} = (i-t_1-1)H \), then \( b_{i2} = s \). If \( c_{i2} = 0 \), then \( (i-t_1-1)H \leq 0 \). Hence \( ((T-t_1-1)H+s) - (T-i)H \leq s \). Hence \( \delta_2 - a_{i2} - c_{i2} = ((T-t_1-1)H+s) - (T-i)H \leq s \), and the solution satisfies the capacity constraint.

For periods \( i \leq t_1 \), \( c_{i2} = 0 \), and the solution clearly satisfies the capacity constraint \( a_{i2} \leq (T-i)H \). For the capacity constraint \( b_{i2} \leq s \), notice that it is satisfied for \( t_1+1 \). For \( i \leq t_1 \), \( a_{i2} \geq a_{i+1,2} \) since \( (T-i)H \geq (T-i-1)H \). Hence \( b_{i2} \leq b_{i+1,2} \), and so the solution satisfies the capacity constraints.

The following example illustrates this solution. Suppose we have two demands in periods 2 and 7. Let \( k = 50 \), \( s = 20 \) and \( H = 10 \). The data is the same as that for the earlier example, except that we change the cost \( k \) from 30 to 50. The optimal solution is to turn on the machine in periods 2 and produce in periods 2 and 3. The objective function value is \( (50+20+50)+(40+20)=180 \).

The variable \( \delta_2 = 60 \), the cost of producing the second unit, and \( \delta_1 = (50+20+50)-(60) = 60 \), the cost of producing the first unit minus the cost of producing the second unit. The other variables have the following values:

Two interval inequalities:

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<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_{i2} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
<td>30</td>
<td>30</td>
<td>40</td>
</tr>
<tr>
<td>( b_{i2} )</td>
<td>0</td>
<td>10</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>( a_{i2} )</td>
<td>60</td>
<td>50</td>
<td>40</td>
<td>30</td>
<td>20</td>
<td>10</td>
<td>0</td>
</tr>
</tbody>
</table>

One interval inequalities:

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_{i1} )</td>
<td>10</td>
<td>50</td>
</tr>
</tbody>
</table>
\[ \begin{align*}
  b_{i1} & \quad 10 \quad 10 \\
  a_{i1} & \quad 0 \quad 0.
\end{align*} \]

Therefore, we can claim the following lemma:

**Lemma (Two demands).** If we have two demands, and if the (i) changeover and setup costs do not change from period to period, and (ii) the inventory holding cost per period is constant, then the inequalities (PI), which partition only the last demand interval are sufficient to obtain optimal integer solutions.

**General case.**

For the general problem with \( n \) demands in periods 1, ..., \( T \), we assume the same cost structure: constant changeover cost \( k \), setup cost \( s \), and per period inventory holding cost \( H \). We show that a subset of the inequalities are sufficient to find optimal integer solutions. We consider the case when \( s \leq H \). Then the optimal solution has the following form:

1) **We produce only in periods** \( t_1, t_2, \ldots, T \). As shown earlier, we produce for the first time in period \( t_1 \) because the changeover and setup costs \( k \) and \( s \) are constant for every period, and the production cost decreases by \( H \) every period. Since \( s \leq H \), therefore \((t_2-t_1-1)s+(T-t_2)H \leq (T-t_1-1)H\). Hence the cost of maintaining the setup from period \( t_1+1 \) through \( t_2 \) and producing for the second time in period \( t_2 \) is less than the cost of producing in period \( t_1+1 \). Similarly, for each succeeding production \( j \), the cost of producing in period \( t_{j-1}+1 \) is more than that of producing in period \( t_j \). Hence we produce in period \( t_j \).

2) If \((t_j-t_{j-1}-1)s > k\), then instead of maintaining the setup from period \( t_{j-1}+1 \) to \( t_j \), we can turn the machine off in period \( t_{j-1}+1 \) and turn
it on again in period $t_j$. Thus we turn the machine on in period $t_j$ if and only if $(t_j - t_{j-1})s > k$. Otherwise we keep the machine on from period $t_{j-1} + 1$ through $t_j$.

We show that the subset of partitioning inequalities (PI) that partition the last demand interval $\{t_{q-1} + 1, \ldots, t_q\}$ are sufficient to ensure that we get optimal integer solutions for the SCSP. The dual of the reformulation is

$$\min \sum_{q=1}^{n} \delta_q$$

subject to

(dual var)

$$p_{iq} - a_{iq} = 0 \quad 1 \leq i \leq t_{q-1}, \quad 1 \leq q \leq n$$

$$c_{t_{q-1} + 1} \geq 0 \quad 1 \leq q \leq n \quad (2.1)$$

$$a_{iq} = 0 \quad 1 \leq q \leq n$$

$$b_{iq} + c_{iq} - c_{i+1,q} \geq 0 \quad t_{q-1} + 1 \leq i \leq t_q, \quad 1 \leq q \leq n \quad (2.2)$$

$$c_{t_{q-1} + 1,q} = 0 \quad 1 \leq q \leq n \quad (2.3)$$

Capacity constraints:

$$z_i: \sum_{q=1}^{n} c_{iq} \leq k \quad i \leq T, \quad 1 \leq q \leq n \quad (2.4)$$

$$y_i: \sum_{q=1}^{n} b_{iq} \leq s \quad i \leq T, \quad 1 \leq q \leq n \quad (2.5)$$

$$w_i: \sum_{q=1}^{n} a_{iq} \leq (t-i)H \quad i \leq T, \quad 1 \leq q \leq n \quad (2.6)$$

$$w, y, z, a_{iq}, b_{iq}, c_{iq}, a_{iq}, b_{iq}, c_{iq}, \delta_q \geq 0$$

We now specify a dual feasible solution that has the same objective function value as the primal optimal integer solution. The variables $\delta_q$ have the following values:

$$\delta_n = \min (k+s, (T-t_{n-1})s)$$

For $1 \leq q \leq n-1$,

$$\delta_q = \min (k+s, (t_q-t_{q-1})s) + (T-t_q)H - \min (k+s, (t_{q+1}-t_q)s) + (T-t_{q+1})H$$
Notice that \( \sum_{j=q}^{n} \delta_j = \min \{ k+s, (t_q-t_{q-1})s \} + (T-t_q)H \). Hence we can also write
\[
\delta_q = \min \{ k+s, (t_q-t_{q-1})s \} + (T-t_q)H - \sum_{j=q+1}^{n} \delta_j
\]

The value of the objective function is \( \sum_{q=1}^{n} q \delta_q \) which is equal to the cost of the optimal primal objective function value.

We specify the other dual variables. For inequalities extending up to \( T \), the values are
\[
\begin{align*}
  c_{Tn} &= \delta_n - s \\
  b_{Tn} &= s \\
  a_{Tn} &= 0
\end{align*}
\]

for \( t_{n-1}+1 \leq i \leq T \),
\[
\begin{align*}
  c_{in} &= [c_{Tn}-(T-i)s]^+ \\
  b_{in} &= s \\
  a_{in} &= \delta_n - c_{in} - b_{in}
\end{align*}
\]

for \( 1 \leq i \leq t_{n-1} \),
\[
\begin{align*}
  c_{in} &= 0 \\
  b_{in} &= 0 \\
  a_{in} &= \delta_n
\end{align*}
\]

We show that this solution is feasible. Consider the capacity constraints. We know that \( \delta_n = \min \{ k+s, (T-t_{n-1})s \} \). For period \( T \),
\[
\begin{align*}
  c_{Tn} &= \delta_n - s \\
  &\leq \min \{ k+s, (T-t_{n-1})s \} - s \\
  &\leq k.
\end{align*}
\]

Hence the solution satisfies the capacity constraint (2.4) for period \( T \). Further, \( c_{in} \leq c_{Tn} \). Hence the solution satisfies the constraint for all
periods. The solution satisfies the capacity constraint (2.5). Consider constraint (2.6).

For \( t_{n-1} + 1 \leq i \leq T \),

\[
    a_{in} = \delta_n - c_{in} - b_{in}
    = \delta_n - (c_{Tn} - (T-i)s) - s
    = (\delta_n - c_{Tn}) + (T-i)s - s
    = (T-i)s
    \leq (T-i)H.
\]

Hence the solution satisfies the capacity constraints.

For \( i = t_{n-1} \),

\[
    a_{in} = \delta_n
    = \min(k+s, (T-t_{n-1})s)
    \leq (T-t_{n-1})s
    \leq (T-t_{n-1})H.
\]

Hence the solution satisfies the capacity constraint (2.6) for \( i = t_{n-1} \). Since \( h_{i+1,n} = (T-i-1)H \leq h_{in} = (T-i)H \), the solution satisfies the capacity constraints for all periods.

Constraint (2.1) is clearly satisfied. Consider the constraint \( c_{t_{n-1}+1,n} = 0 \). We set

\[
    c_{t_{n-1}+1,n} = [c_{Tn} - (T-t_{n-1}-1)s]^+
    = [\delta_n - s - (T-t_{n-1}-1)s]^+
    = [\min(k+s, (T-t_{n-1})s) - s - (T-t_{n-1}-1)s]^+
    \leq [(T-t_{n-1})s - s - (T-t_{n-1}-1)s]^+
    = 0.
\]

Hence \( c_{t_{n-1}+1,n} = 0 \).
Consider the constraint \( b_{in} + c_{in} - c_{i+1,n} \geq 0 \). For \( i \geq t_{n-1} + 1 \), \( c_{in} = [c_{in} - (T_i) s]^+ \) and hence \( c_{i+1,n} - c_{in} \leq s \). Since \( b_{in} = s \), the solution satisfies the constraint.

For the other inequalities, we set the values as follows:

For \( q = 1, \ldots, n-1 \),
\[
\begin{align*}
a_{tq}{q} &= (T-t_q) H - \sum_{j=q+1}^{n} a_{tj} \\
b_{tq}{q} &= s \\
c_{tq}{q} &= \delta_q - a_{tq}{q} - b_{tq}{q}
\end{align*}
\]

for \( t_{q-1} + 1 \leq i \leq t_q' \),
\[
c_{iq} = [c_{tq}{q} - (t_q - i)s]^+.
\]
\[
b_{iq} = s \\
a_{iq} = \delta_q - c_{iq} - b_{iq}
\]

for \( 1 \leq i \leq t_{q-1} \),
\[
c_{iq} = 0 \\
b_{iq} = 0 \\
a_{iq} = \delta_q
\]

We show that this solution is feasible. It clearly satisfies constraints (2.1) for all \( q \). Similarly, it satisfies constraints (2.2) because \( c_{i+1,q} - c_{iq} \leq s \), and \( b_{iq} = s \) for \( t_{q-1} + 1 \leq i \leq t_q' \) and \( c_{iq} = b_{iq} = 0 \) for \( i \leq t_{q-1} \). We show that \( c_{t_{q-1} + 1,q} = 0 \).

\[
c_{iq} = [c_{tq}{q} - (t_q - i)s]^+ \\
= [\delta_q - a_{tq}{q} - b_{tq}{q} - (t_q - i)s]^+ \\
= [\delta_q - (T-t_q) H + \sum_{j=q+1}^{n} a_{tj} - s - (t_q - i)s]^+ \\
= [\sum_{j=q}^{n} \delta_q - (T-t_q) H -(t_q - i + 1)s]^+ \\
= [\min \{k + s, (t_q - t_{q-1})s\} + (T-t_q) H - (T-t_q) H -(t_q - i + 1)s]^+
\]
\[ \leq [(t_q-t_{q-1})s-(t_q-i+1)s]^+ \]

Thus for \( i = t_{q-1}+1 \), \( c_{t_{q-1}+1, q} \leq [(t_q-t_{q-1})s-(t_q-t_{q-1})s]^+ = 0. \]

Consider the capacity constraint (2.4). In any interval \( \{t_{q-1}+1, \ldots, t_q\} \), only the variables \( c_{i, q} \) are nonzero, because for \( q' > q \), \( c_{i, q} = 0 \), and for \( q' < q \), the inequality does not extend beyond \( t_{q-1} \). Thus,

\[
\sum_{q=1}^{n} c_{i, q} = c_{i, q}^n = [c_{t_q, q} - (t_q-i)s]^+
= [\delta_q - a_{t_q, q} - b_{t_q, q} - (t_q-i)s]^+
= [\delta_q - (T-t_q)H + \sum_{j=q+1}^{n} a_{t_j, q} - (t_q-i)s]^+
= [\sum_{j=q}^{n} \delta_q -(T-t_q)H - (t_q-i+1)s]^+
= [\min \{k+s, (t_q-t_{q-1})s\} - (T-t_q)H - (t_q-i+1)s]^+
= [\min \{k+s, (t_q-t_{q-1})s\} - (t_q-i+1)s]^+.
\]

For \( i = t_q \), \( c_{t_q, q} = [\min \{k+s, (t_q-t_{q-1})s\} - s]^+ \leq (k+s) - s = k \). Hence the solution satisfies the capacity constraint for \( i = t_q \). Since \( c_{i, q} \leq c_{t_q, q}^* \) the solution satisfies the constraint for all periods \( i \).

The solution clearly satisfies the capacity constraints (2.5). Consider the constraints (2.6). In any interval \( \{t_{q-1}+1, \ldots, t_q^*\} \),

\[
\sum_{q=1}^{n} a_{i, q} = \sum_{q > q^*} a_{i, q}
= \sum_{q > q^*} \delta_q + a_{i, q}^*
= \sum_{q > q^*} \delta_q - c_{i, q}^* - b_{i, q}^*
= \min \{k+s, (t_q-t_{q-1})s\} + (T-t_q)H - \min \{k+s, (t_q-t_{q-1})s\} - (t_q-i+1)s]^+ - s
\leq \min \{k+s, (t_q-t_{q-1})s\} + (T-t_q)H - \min \{k+s, (t_q-t_{q-1})s\} - (t_q-i+1)s - s
= (T-t_q)H + (t_q-i)s
\leq (T-t_q)H + (t_q-i)H
= (T-i)H.
\]
Hence the capacity constraints are also satisfied, and we have a dual feasible solution. We have thus proved the following theorem:

**Theorem 1** (Characterization). *If the changeover and setup costs do not vary from period to period, and if the setup cost $s \leq H$, the holding cost per period, then the subset of the partitioning inequalities (PI) that partition the interval $(t_{q-1} + 1, \ldots, t_q)$ are sufficient to ensure that we obtain optimal integer solutions.***

**Example.** Suppose we have 5 demands in periods 3, 6, 10, 17 and 20. Suppose the changeover cost $k=30$, the setup cost $s=5$, and the per unit holding cost is $H=10$, so that the production cost in period $i$ is $(20-i)10$.

The optimal solution is:

Turn on the machine in period 3, keep it on until period 10, and produce in periods 3, 6 and 10. Turn on the machine again in period 17 and keep it on until period 20, and produce in periods 17 and 20. The cost can be computed as follows:

Cost of first production = $30+5+170 = 205$
Additional cost of second prod. = $155$
Additional cost of third prod. = $120$
Cost of fourth prod. = $30+5+30 = 65$
Additional cost of fifth prod. = $15$
Total optimal cost = $560$.

The dual variables have the following values:

$\delta_5 =$ cost of 5th production = 15
$\delta_4 =$ cost of 4th prod. - cost of 5th prod. = 50
\( \delta_3 = \text{cost of 3rd prod. - cost of 4th prod.} = 55 \)
\( \delta_2 = \text{cost of 2nd prod. - cost of 3rd prod.} = 35 \)
\( \delta_1 = \text{cost of 1st prod. - cost of 2nd prod.} = 50 \)

Dual objective function value = \( 5\delta_5 + 4\delta_4 + 3\delta_3 + 2\delta_2 + \delta_1 = 560 \).

The following table specifies the values of the dual variables in each period (blank spaces indicate that the inequality does not extend up to that period):

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Future Directions.

We might extend these results to the following cases:

1) \( s > H \). The following example shows that the subset of inequalities considered so far are not sufficient to obtain optimal integer solutions.

\[
\text{Changeover cost } K = 20 \\
\text{Setup cost } s = 10 \\
\text{Per unit holding cost } H = 5.
\]

Demands in periods 2, 4 and 8. The optimal solution is to turn on the machine in period 2, and produce in periods 2, 3 and 4. The cost of this solution is 125. If we use the cuts used so far we obtain an objective function value of 115. However, if we use all the inequalities, we obtain an integer solution.

Thus we might investigate whether we obtain integer solutions if we include some more inequalities.

2) Identify other cost structures for which we obtain integer solutions

3) Investigate if the inequalities characterize the polyhedron. It might be useful to see if we can transform the problem further before we can establish a characterization of the integer polyhedron of feasible (integer) solutions.
Chapter 6. Multi Item Inequalities

Inequalities for the multi item problem.

The inequalities (PI) and (SI) for the single item problem are also valid for the multi item problem. We now extend our results to obtain new valid inequalities for the multi item problem. To motivate the discussion we first consider the two item problem.

The two item problem

Consider the following example: we have two items with demand for item 1 occurring in period 2 and for item 2 in period 3. We may have additional demand beyond period 3, but we do not consider it at this point. Let \( w_{1i}, y_{1i} \) and \( z_{1i} \) for \( i=1,2 \) denote the variables for item 1 and \( w_{2i}, y_{2i} \) and \( z_{2i} \) for \( i=1,2,3 \) denote the variables for item 2.

The inequality \( y_{2,1} + z_{2,2} + z_{2,3} \geq 1 \) is valid for item 2. However we can tighten it to \( w_{2,1} + z_{2,2} + z_{2,3} \geq 1 \). This is not a valid inequality for the single item problem as shown by the following feasible solution: we turn the machine on in period 1, and keep it on and produce in period 2. This contributes zero and the inequality is violated. However, this solution is not feasible for the two item example. The solution maintains the set up for item 2 in periods 1 and 2. This does not allow us to produce item 1 before period 3, whereas demand for item 1 occurs in period 2.

In fact, it is easy to see that the inequality is valid. If we produce item 1 in period 1, we must turn on the machine for item 2 in either period 2 or 3. Hence \( z_{2,2} + z_{2,3} \geq 1 \). On the other hand, if we produce
item 1 in period 2, we must produce item 2 in period 1 or 3. Hence \( w_{2,1} + z_{2,3} \geq 1 \). The inequality is therefore valid.

Suppose we shift demand for items 1 and 2 by one period, so that demands occur in periods 3 and 4. The inequality \( w_{2,1} + z_{2,2} + z_{2,3} + z_{2,4} \geq 1 \) for item 2 is not valid: we can turn on the machine in period 1 for item 2, and keep it on and produce item 2 in period 2. We produce item 1 in period 3. This solution violates the inequality. To get over this difficulty, we can introduce the variable \( w_{1,3} \) in the inequality to obtain \( w_{2,1} + z_{2,2} + (w_{1,3} + z_{2,3}) + z_{2,4} \geq 1 \). Alternatively, we can introduce the variable \( z_{1,3} \) to obtain \( w_{2,1} + z_{2,2} + (z_{1,3} + z_{2,3}) + z_{2,4} \geq 1 \). Product 1 must be produced in period 1, 2 or 3. By considering these three cases, we can easily show that both the inequalities are valid.

Let us generalize this a little further. Suppose demands for items 1 and 2 occur in periods \( t_{1,1} \) and \( t_{2,1} \) respectively. Consider the following inequality:

\[
 w_{2,1} + \sum_{i=2}^{t_{2,1}} z_{2i} + \sum_{i=3}^{t_{1,1}} w_{1i} \geq 1.
\]

If we produce item 1 in any of the periods 3 through \( t_{1,1} \), the inequality is satisfied. If we produce item 1 in period 1, then \( \sum_{i=2}^{t_{2,1}} z_{2i} \geq 1 \) because we need to turn on the machine for item 2 at least once in the interval \( \{2, \ldots, t_{2,1}\} \). If we produce item 1 in period 2, then \( w_{2,1} + \sum_{i=3}^{t_{2,1}} z_{2i} \geq 1 \) because we either produce item 2 in period 1 or turn on the machine at least once in the interval \( \{3, \ldots, t_{2,1}\} \). Hence the inequality is valid.

We point out a key feature of the inequality. It is possible to produce item 2 without obtaining any contribution by turning the
machine on in period 1, keeping it on until some period \(2 \leq i \leq t_{1,1}-1\), and producing in period \(i\). However, we need to produce item 1 in one of the periods \(i+1\) through \(t_{1,1}\) and the presence of the terms \(\sum_{i=3}^{t_{1,1}} w_{1i}\) ensures that the inequality is satisfied.

Suppose we consider the inequality \(\sum_{i=1}^{t_{1,1}} w_{1i} \geq 1\). This inequality is one of the original demand constraints. If we drop the terms \(w_{1,1}+w_{1,2}\) and replace them by \(w_{2,1}+\sum_{i=2}^{t_{2,1}} z_{2i}\) we obtain the inequality we are considering. If \(w_{2,1}+\sum_{i=2}^{t_{2,1}} z_{2i} \geq w_{1,1}+w_{1,2}\), then the inequality is certainly not tight. However, this is not true because both \(w_{1,1}\) and \(w_{1,2}\) might be equal to 1 if we have demand beyond \(t_{2,1}\), and \(w_{2,1}+\sum_{i=2}^{t_{2,1}} z_{2i}\) might be equal to 1.

In fact, we can partition the interval \([3, \ldots, t_{1,1}]\) into subsets \(W_1, Y_1\) and \(Z_1\) as in the single item case and impose the condition that if \(i \in W_1\), then \(i+1 \notin Z_1\). Similarly, we can partition the interval \([1, \ldots, t_{2,1}]\) into subsets \(W_2, Y_2\) and \(Z_2\). The following example illustrates this: demand for item 1 occurs in period 12, and demand for item 2 occurs in any period after that, say period 15. We can write down the following valid inequality:

\[
\sum_{i=1}^{3} w_{2i} + \sum_{i=4}^{7} z_{2i} + \sum_{i=8}^{9} w_{2i} + y_{2,10} \sum_{i=11}^{15} z_{2i} + \sum_{i=1}^{2} w_{1i} + w_{1,5} + y_{1,6} + \sum_{i=7}^{12} z_{1i} \geq 1.
\]

Here we partition the interval \([1, \ldots, 15]\) into subsets \(W_2, Y_2\) and \(Z_2\) with \(W_2 = \{1, 2, 3, 8, 9\}\), \(Y_2 = \{10\}\) and \(Z_2 = \{4, \ldots, 7\} \cup \{11, \ldots, 15\}\). For item 1, we skip the periods 3 and 4 (note that \(3 \in W_2\) and \(4 \in Z_2\)) from the interval \([1, \ldots, 12]\) and partition these periods into subsets \(W_1, Y_1\) and \(Z_1\) so that if \(i \in W_1\), then \(i+1 \notin Z_1\).
The following argument shows that the inequality is valid. If we produce item 1 in any of the periods 1, 2, or 5 through 12, the inequality is satisfied. If we produce it in period 3, then \[ \sum_{i=1}^{2} w_{2i} + \sum_{i=4}^{7} z_{2i} + \sum_{i=8}^{9} w_{2i} + y_{10} + \sum_{i=11}^{15} z_{2i} \geq 1, \] because we must produce item 2 at least once in one of the periods 1, 2, or 4 through 15. Similarly, if we produce item 1 in period 4, we must produce item 2 at least once in one of the periods 1, 2, 3, or 5 through 15.

We again point out a feature of the inequality. Suppose period \( j_1 \in Z_2 \), \( j_1 + 1 \) through \( j_2 \) belongs to \( W_2 \), and \( j_2 + 1 \) through \( j'_2 \) belongs to \( Z_2 \). Furthermore, let \( j'_2 \geq t_{1,1} \), i.e., we have a sequence of \( w_{2i} \)'s followed by a sequence of \( z_{2i} \)'s before the first demand for item 1. We can turn on the machine in one of the periods \( j_1 + 1 \) through \( j_2 \), keep it on until one of the periods \( j_2 + 1 \) through \( j'_2 \), and produce in that period. This set of operations contributes nothing to the inequality. We can compensate for this by introducing the variables \( w_{1i} \) for item 1 in all periods 1 through \( t_{1,1} \) except \( j_2 \) and \( j_2 + 1 \). This ensures that if producing item 2 does not contribute to the inequality, then item 1 contributes one unit. We show later that this defines a valid inequality.

Another fact is to be noted. Suppose there are two sets of periods \( \{j, j+1\} \) and \( \{k, k+1\} \) with \( j, k \in W_2 \) and \( j+1, k+1 \in Z_2 \). Further, suppose that all periods in the set \( \{1, \ldots, t_{1,1}\} \setminus \{j, j+1\} \) belong to \( W_1 \cup Y_1 \cup Z_1 \). We can produce item 2 by turning the machine on in period \( k \), and keeping it on and producing in period \( k+1 \). We also produce item 1 in one of the periods \( j \) or \( j+1 \). This solution contributes nothing to the inequality, and hence the inequality is not valid. Therefore, there can be at most one set of periods \( \{j, j+1\} \) with \( j \in W_2 \) and \( j+1 \in Z_2 \). Further, if \( i+1 \in Z_2 \) for every \( i \in W_2 \), then we do not need the terms for product 1. Therefore,
j \in W_2 \text{ and } j+1 \in Z_2 \text{ for exactly one period } j \text{ if we want a tight inequality containing terms for both the items.}

We can describe these inequalities as follows:

$$\sum_{i \in W_2} w_{2i} + \sum_{i \in Y_2} y_{2i} + \sum_{i \in Z_2} z_{2i} + \sum_{i \in W_1} w_{1i} + \sum_{i \in Y_1} y_{1i} + \sum_{i \in Z_1} z_{1i} \geq 1,$$

and impose the following conditions:

(i) $W_2, Y_2 \text{ and } Z_2 \text{ partition the interval } \{1, \ldots, t_{2,1}\}$

(ii) Period $j \in W_2 \text{ and } j+1 \in Z_2 \text{ for exactly one period } j \text{ in the interval } \{1, \ldots, t_{2,1}\}$.

(iii) If period $i \in W_2$, then $i+1 \notin Z_2 \text{ for all } i \neq j$.

(iv) $W_1, Y_1 \text{ and } Z_1 \text{ partition the set } \{1, \ldots, t_{1,1}\} \setminus \{j, j+1\}$.

(v) If period $i \in W_1$, then $i+1 \notin Z_1$, i.e., if $w_{1i}$ is in the inequality, then $z_{1i}$ is not in the inequality.

Note that we obtain similar inequalities if we partition the interval $\{1, \ldots, t_{1,1}\}$ into $W_1, Y_1 \text{ and } Z_1 \text{ and the set } \{1, \ldots, t_{2,1}\} \setminus \{j, j+1\}$ into $W_2, Y_2 \text{ and } Z_2 \text{ for some } j \in W_1 \text{ and } j+1 \in Z_1$.

Generalizing the inequalities for the two item problem.

We now consider the demand for item 1 up to period $t_{1,m}$ and for item 2 up to $t_{2,p}$. Let us assume that $t_{2,p-1} < t_{1,m} < t_{2,p}$. We partition the interval $\{t_{2,p-1}+1, \ldots, t_{2,p}\}$ into subsets $W_2, Y_2 \text{ and } Z_2$, and impose the following conditions:

(i) Period $j \in W_2 \text{ and } j+1 \in Z_2 \text{ for exactly one period } j \text{ in the interval } \{t_{2,p-1}+1, \ldots, t_{2,p}\}$. 

(ii) If period $i \in W_2$, then $i+1 \notin Z_2$ for all $i \neq j$.

We next partition the set $\{t_{1,m-1}+1, \ldots, t_{1,m}\} \setminus \{j+1\}$ into subsets $W_1$, $Y_1$ and $Z_1$, and impose the condition that if $i \in W_1$, then $i+1 \notin Z_1$. Consider the following inequality, which we call inequality (pi2) for partitioning inequality for the two item problem:

$$
\sum_{i=1}^{t_2,w-1} w_{2i} + \sum_{i \in W_2} w_{2i} + \sum_{i \in Y_2} y_{2i} + \sum_{i \in Z_2} z_{2i} + \sum_{i=1}^{t_1,m-1} w_{1i} + \sum_{i \in W_1} w_{1i} + \sum_{i \in Y_1} y_{1i} + \sum_{i \in Z_1} z_{1i} \geq p + m - 1.
$$

We briefly argue that this inequality is valid. Later we show this more rigorously. In any feasible solution, $\sum_{i=1}^{t_2,w-1} w_{2i} \geq p - 1$ and $\sum_{i=1}^{t_1,m-1} w_{1i} \geq m - 1$. If both these inequalities are satisfied at equality, then we need to show that $\sum_{i \in W_2} w_{2i} + \sum_{i \in Y_2} y_{2i} + \sum_{i \in Z_2} z_{2i} + \sum_{i \in W_1} w_{1i} + \sum_{i \in Y_1} y_{1i} + \sum_{i \in Z_1} z_{1i} \geq 1$, for inequality (pi2) to be valid. We need to produce item 1 at least once in the interval $\{t_{1,m-1}+1, \ldots, t_{1,m}\}$ to satisfy the demand in period $t_{1,m}$. If we produce in one of the periods $W_1 \cup Y_1 \cup Z_1$, the inequality is satisfied: we showed this for the single item inequalities (PI). If we produce item 1 in period $j$, then we show that item 2 contributes one unit to the inequality. We must produce item 2 at least once in the set $\{t_{2,p-1}+1, \ldots, t_{2,p}\} \setminus \{j\}$ to meet the demand in period $t_{2,p}$. Note that this set is split into two intervals $\{t_{2,p-1}+1, \ldots, j+1\}$ and $\{j+1+1, \ldots, t_{2,p}\}$ because we produce item 1 in period $j$. Since the machine is on for item 1 in period $j$, and $\sum_{b=1}^{P} y_{bi} \leq 1$ for all periods $i$ (recall that $\sum_{b=1}^{P} y_{bi} \leq 1$ is one of the original constraints of the multi item problem), we cannot keep the machine on for the item 2 in period $j$. Therefore, it is not possible to turn on the machine for item 2 in some period before
period \( j \), keep it on until some period \( j' \geq j+1 \), and produce item 2 in period \( j' \). Hence item 2 must contribute one unit to the inequality.

**The multi item problem.**

We consider the following three item example: \( t_{1,1}=8, t_{2,1}=10 \) and \( t_{3,1}=15 \). We write down the following inequality and show that it is valid:

\[
\sum_{i=1}^{6} w_{3i} + \sum_{i=7}^{15} z_{3i} + \sum_{i=1}^{3} w_{2i} + \sum_{i=4}^{5} z_{2i} + \sum_{i=8}^{10} z_{2i} + \ldots + \sum_{i=1}^{2} w_{1i} + w_{1,5} + z_{1,8} \geq 1.
\]

Item 3 partitions the interval \( \{1, \ldots, 15\} \), and has exactly one set of two consecutive periods, 6 and 7 with \( 6 \in W_3 \) and \( 7 \in Z_3 \). Item 2 partitions the set \( \{1, \ldots, 10\} \setminus \{6, 7\} \) and has exactly one set of two consecutive periods, 3 and 4 with \( 3 \in W_2 \) and \( 4 \in Z_2 \). Item 1 partitions the set \( \{1, \ldots, 8\} \setminus \{3, 4, 6, 7\} \) ensuring that if period \( i \in W_1 \), then \( i+1 \in Z_1 \).

We must produce item 3 at least once in one of the periods 1 through 15. Producing item 3 contributes 1 unit unless we turn on the machine for item 3 in some period \( k_3 \leq 6 \), keep it on until some period \( k_3' \geq 7 \), and produce in period \( k_3' \). If so, we show that items 1 or 2 contribute one unit to the inequality. We must now produce item 2 in one of the periods 1 through \( k_3-1 \) if \( k_3 > 1 \), or in one of the periods \( k_3'+1 \) through 10 if \( k_3' < 10 \). This contributes one unit unless we turn on the machine for item 2 in some period \( k_2 \leq 3 \), keep it on until some period \( k_2' \geq 4 \), and produce in that period. If so, we show that item 1 contributes one unit to the inequality. We must produce item 1 in one of the periods 1 through \( k_2-1 \) if \( k_2 > 1 \), or in one of the periods \( k_2'+1 \)
through 8 if \( k_2 < 8 \). Hence
\[
\sum_{i=1}^{2} w_{i1} + \sum_{i=5}^{8} z_{1i} \geq \sum_{i=1}^{k_2-1} w_{i1} + \sum_{i=k_2+1}^{8} z_{1i} \geq 1,
\]
and the inequality is valid.

Let us consider the three item problem with demands in periods \( t_{1,1}, t_{2,1} \) and \( t_{3,1} \) for items 1, 2 and 3 respectively. We partition the interval \( \{1, ..., t_{3,1}\} \) into the subsets \( W_3, Y_3 \) and \( Z_3 \) and impose the following conditions:

(i) period \( j_3 \in W_3 \) and \( j_3 + 1 \in Z_3 \) for exactly one period \( j_3 \) in the interval \( \{1, ..., t_{3,1} - 1\} \).

(ii) if period \( i \in W_3 \), then \( i + 1 \notin Z_3 \) for all \( i \neq j_3 \).

Next, we partition the set \( \{1, ..., t_{2,1}\} \backslash \{j_3, j_3 + 1\} \) into subsets \( W_2, Y_2 \) and \( Z_2 \) and impose the following conditions:

(iii) period \( j_2 \in W_2 \) and \( j_2 + 1 \in Z_2 \) for exactly one period \( j_2 \) in the set \( \{1, ..., t_{2,1} - 1\} \backslash \{j_3, j_3 + 1\} \).

(iv) if period \( i \in W_2 \), then \( i + 1 \notin Z_2 \) for all \( i \neq j_2 \).

Finally, we partition the set \( \{1, ..., t_{1,1} - 1\} \backslash \{j_2, j_2 + 1, j_3, j_3 + 1\} \) into subsets \( W_1, Y_1 \) and \( Z_1 \) and impose the following condition:

(v) if period \( i \in W_1 \), then \( i + 1 \notin Z_1 \).

Then we can write down the following inequalities:
\[
\sum_{i \in W_3} w_{3i} + \sum_{i \in Y_3} y_{3i} + \sum_{i \in Z_3} z_{3i} + \sum_{i \in W_2} w_{2i} + \sum_{i \in Y_2} y_{2i} + \sum_{i \in Z_2} z_{2i} + \sum_{i \in W_1} w_{1i} + \sum_{i \in Y_1} y_{1i} + \sum_{i \in Z_1} z_{1i} \geq 1.
\]
We can generalize these inequalities to the $p$ item problem with demands in periods $t_{1,1}, t_{2,1}, \ldots, t_p, 1, t_{1,2}, t_{2,2}, \ldots, t_p, 2, \ldots, t_{1,q_1}, t_{2,q_2}, \ldots, t_{p,q_p}$. We partition the interval $\{t_{p,q_p}^{-1}+1, \ldots, t_p q_p\}$ into the subsets $W_p$, $Y_p$ and $Z_p$ and impose the following conditions:

(i) period $j_p \in W_p$ and $j_p+1 \in Z_p$ for exactly one period $j_p$ in the interval $\{t_{p,q_p}^{-1}+1, \ldots, t_p q_p\}$.

(ii) if period $i \in W_p$, then $i+1 \not\in Z_p$ for all $i \neq j_p$.

We partition the set $\{1, \ldots, t_{r,1}\} \setminus \bigcup_{k=r+1}^{P} \{j_k, j_k+1\}$ for $2 \leq r \leq p-1$ into subsets $W_r$, $Y_r$ and $Z_r$ and impose the following conditions:

(iii) period $j_r \in W_r$ and $j_r+1 \in Z_r$ for exactly one period $j_r$ in the interval $\{t_{r,q_r}^{-1}+1, \ldots, t_r q_r\} \setminus \bigcup_{k=r+1}^{P} \{j_k, j_k+1\}$.

(iv) if period $i \in W_r$, then $i+1 \not\in Z_r$ for all $i \neq j_r$.

We partition the set $\{t_{1,q_1}^{-1}+1, \ldots, t_{1,q_1}\} \setminus \bigcup_{k=2}^{P} \{j_k, j_k+1\}$ into the subsets $W_1$, $Y_1$ and $Z_1$ and impose the following condition:

(v) if period $i \in W_1$, then $i+1 \not\in Z_1$ for all $i \neq j_1$.

We show later that the following inequalities are valid:

$$\sum_{r=1}^{P} \sum_{i=1}^{t_r, q_r^{-1}} w_{ri} + \sum_{r=1}^{P} \left[ \sum_{i \in W_r} w_{ri} + \sum_{i \in Y_r} y_{ri} + \sum_{i \in Z_r} z_{ri} \right] \geq \sum_{r=1}^{P} (q_r-1)+1$$

(MPI)

**Theorem 1.** (Multi-item). The multi-product inequalities (MPI) are valid.

**Proof.**
For any product \( r \), the terms \( \sum_{i=1}^{t_{rqr}} w_{ri} \) add up to at least \((q_r-1)\) in any feasible solution. Hence the terms \( \sum_{r=1}^{p} \sum_{i=1}^{t_{rqr}} w_{ri} \) add up to at least \( \sum_{r=1}^{p} (q_r-1) \) in any feasible solution.

Therefore, we need to show that the terms in the last demand intervals add up to at least 1. If \( \sum_{r=1}^{p} \sum_{i=1}^{t_{rqr}} w_{ri} = \sum_{r=1}^{p} (q_r-1)+1 \), the inequality is valid. Otherwise \( \sum_{i=1}^{t_{rqr}} w_{ri} = (q_r-1) \) for each product \( r \).

The terms \( \sum_{i\in W_p} w_{pi} + \sum_{i\in Y_p} y_{pi} + \sum_{i\in Z_p} z_{pi} \), where \( W_p, Y_p \) and \( Z_p \) partition the interval \( [t_{p,q_p+1}, \ldots, t_{p,q_p}] \) add up to 1 unit unless we turn on the machine in period \( j_p \) and produce in period \( j_p+1 \).

If we do, then we cannot produce any other product in periods \( j_p, j_p+1 \). Similarly, the terms \( \sum_{i\in W_r} w_{ri} + \sum_{i\in Y_r} y_{ri} + \sum_{i\in Z_r} z_{ri} \), where \( W_r, Y_r \) and \( Z_r \) partition the interval \( [t_{r,q_r+1}, \ldots, t_{r,q_r}] \) add up to 1 unit unless we turn on the machine in period \( j_r \) and produce in period \( j_r+1 \), for \( 2 \leq r \leq p-1 \). If each of these add up to zero, then we turn on the machine for product \( r \) in period \( j_r \) and produce in period \( j_r+1 \). Thus we cannot produce product 1 in any of the periods \( \bigcup_{k=2}^{p} \{ j_k, j_k+1 \} \). So we must produce product 1 in one of the periods

\[ S_1 = (t_{1,q_1+1}, \ldots, t_{1,q_1}) \cup \bigcup_{k=2}^{p} \{ j_k, j_k+1 \}. \]

Since the partition of \( S_1 \) into \( W_1, Y_1 \) and \( Z_1 \) satisfies condition 5, then, as shown earlier for the single product inequalities,

\[ \sum_{i\in W_1} w_{1i} + \sum_{i\in Y_1} y_{1i} + \sum_{i\in Z_1} z_{1i} \geq 1. \]

Hence the inequality is valid.

Example. Suppose we have three products, with the following demands:
Periods

Product 1  5  10  20

Product 2  6  11  21

Product 3  7  12  22

The following inequality is valid:

\[ \sum_{i=1}^{10} w_{1i} + \sum_{i=11}^{20} z_{1i} + \ldots \]
\[ + \sum_{i=15}^{19} w_{2i} + \sum_{i=16}^{21} z_{2i} + \ldots \]
\[ + \sum_{i=14}^{18} w_{1i} + w_{1,17} + y_{1,18} + \sum_{i=19}^{22} z_{1i} \geq 7. \]

Further Generalizations of the Inequalities.

Let us consider the two item problem again. So far we have restricted the partitions \( W_2, Y_2 \) and \( Z_2 \) to the demand interval \( \{ t_{2,p-1} + 1, \ldots, t_{2,p} \} \). We now extend this the entire interval \( \{1, \ldots, t_{2,p}\} \). Let us consider the following example: demands for item 1 occur in periods 5 and 15, and for item 2 in periods 10 and 20. Consider the following inequality:

\[ \sum_{i=1}^{3} w_{2i} + \sum_{i=4}^{10} z_{2i} + \sum_{i=4}^{20} z_{2i} + \sum_{i=1}^{2} w_{1i} + \sum_{i=5}^{15} w_{1i} \geq 2. \]

Item 1 skips periods 3 and 4, and item 2 has one set of two consecutive periods 3 and 4 with \( 3 \in W_2 \) and \( 4 \in Z_2 \). The following argument shows that the inequality is valid. Item 2 contributes zero if we turn on the machine for item 2 in some period \( k \leq 3 \), keep it on until some period \( k' \geq 5 \), and produce twice in the interval \( \{4, \ldots, k'\} \). However, as shown earlier, item 1 must contribute 2 units in this case.
Note that the expression $\sum_{i=1}^3 w_{2i} + \sum_{i=4}^{20} z_{2i}$ could be equal to zero for some feasible solutions: we turn on the machine for item 2 in period 3, keep it on until period 5, and produce in periods 4 and 5. At the same time, the expression $\sum_{i=1}^2 w_{1i} + \sum_{i=5}^{15} w_{1i}$ might be equal to zero if we produce in periods 3 and 4. However, if we sum the two expressions, the sum must be equal to 2. This suggests that the inequality might be tight.

Recall the partitioning inequalities (PI) for the single item problem. For any period $i^*$, containing a term of the type $c_{i^*} z_{i^*}$, $c_{i^*} \geq 1$, and any other period $i \leq i^*$, in the $(j+1)$st demand interval, the sum of the coefficient of $z_i$ and the coefficients of $y_t$ for $i \leq t \leq i^*$ is at least $\min\{q-j, i^*+1-i\}$.

For example, if item 2 has demands occurring in periods 5, 10 and 15, a partitioning inequality (PI) is:

$$w_{2,1} + y_{2,2} + (y_{2,3} + z_{2,3}) + (y_{2,4} + z_{2,4}) + z_{2,5} + \sum_{i=6}^{15} w_{2i} \geq 3.$$  

We modify the left hand side as follows:

$$\sum_{i=1}^4 w_{2i} + z_{2,5} + \sum_{i=6}^{15} w_{2i}.$$  

The sum of the modified terms might be only two instead of three, if we turn on the machine in period 3 and produce in periods 3, 4 and 5. Moreover, the modified terms must equal at least 2 because we can miss at most one unit of contribution in period 5. To compensate for this, we introduce terms for item 1. Suppose item 1 has demands occurring in periods 7, 12, 17, and 22. We can introduce the terms $\sum_{i=1}^3 w_{1i} + \sum_{i=6}^{17} w_{1i}$ and obtain the following valid inequality:

$$\sum_{i=1}^4 w_{2i} + z_{2,5} + \sum_{i=6}^{15} w_{2i} + \sum_{i=1}^3 w_{1i} + \sum_{i=6}^7 w_{1i} \geq 3.$$
Alternately, we can introduce the terms $\sum_{i=1}^{3} w_{1i} + \sum_{i=6}^{12} w_{1i}$ and obtain the following valid inequality:

$$\sum_{i=1}^{4} w_{2i} + z_{2,5} + \sum_{i=6}^{15} w_{6i} + \sum_{i=1}^{3} w_{1i} + \sum_{i=6}^{12} w_{1i} \geq 4.$$  

In fact, we can skip periods for item 2 as well as shown by the following inequality:

$$\sum_{i=1}^{4} w_{2i} + z_{2,5} + y_{2,7} + \sum_{i=8}^{15} z_{2i} + \sum_{i=1}^{3} w_{1i} + w_{1,7} \geq 2.$$  

Item 2 skips period 6 and to compensate for this we reduce the right hand side to 2. We also skip periods 4, 5 and 6 for item 1. The terms for item 2 contribute 2 units unless we produce in periods 5 and 6 without turning on the machine in period 5 (in which case item 2 contributes only one unit). This is possible only if we turn on the machine for item 2 in some period $k \leq 4$ and keep it on until period 6. Therefore, we must produce item 1 in one of the periods 1, 2, 3 or 7 to meet the demand in period 7, and hence $\sum_{i=1}^{3} w_{1i} + w_{1,7} \geq 1$. The inequality is therefore valid.

Let us summarize the discussion to provide some insight into the inequalities. If period $i \in W_2$ and $i+1 \in Z_2$, then period $i+1$ does not contribute to the inequality for some feasible solution for item 2. To compensate for this, we introduce terms for item 1 skipping the two periods $i$ and $i+1$.

We generalize these inequalities, (which we call inequalities (GI) for General Inequalities), in the following directions:
1) Instead of introducing the terms $w_{2i}$, we can replace them by terms with the general structure of the skip inequalities for item 2.

2) We can extend this to the multi item case with $P$, the number of products greater than 2.

A general principle that the partitioning inequalities (PI) for the single item problem satisfied was that if we produce $j$ times in any sequence of periods, then the lefthand side increases by $j$ units. for the skip inequalities, we modified it so that if we produce $j$ times in sequence, and $j'$ of these are not skipped, then the lefthand side must increase by at least $j'$. However, for the multi-item problem, we do not require that the lefthand side increases by $j'$. We compensate by introducing terms for other products.

For the two item problem, suppose there is some period $i^* \in \mathbb{Z}$, and for some period $i \leq i^*$, in the $(j+1)$st demand interval of item 1, the sum of the coefficient of $z_{1i}$ and the coefficient of $y$ from $i$ through $i^*$ is $K < q-j$. We assume that $WZ = \Phi$. Then we can turn on the machine for item 1 in period $i$ and produce $q-j$ times. This increases the lefthand side by only $K$ units. In the skip inequalities, we compensated for this by reducing the righthand side. For the two item problem, we can introduce terms for item 2 which must compensate for $q-j-K$ units. We can skip some of the periods $i$ through $i+q-j$ for item 2, and ensure that the rest of the terms add up to at least $q-j-K$ units.

Let us first consider an inequality which extends to the $q$th demand for item 1, or the $p$th demand for item 2, whichever occurs later, i.e., $\max(t_{1q}, t_{2p})$. (1) $w_{1,1} + z_{1,2} + z_{1,3} + \sum_{i=4}^{35} w_{1i} + \sum_{i=3}^{30} w_{2i} \geq 5$
If we produce item 1 in periods 1 through 4, it contributes 2 units. However, item 2 contributes 3 units in periods 5 through 30. If we produce item 1 in periods 2 through 5, it contributes 3 units. But then item 2 must contribute 2 units in periods 6 through 30. Similarly, if we produce item 1 in periods 3 through 6, it contributes 4 units. Item 2 contributes one unit in periods 7 through 30. Finally, if we produce item 1 in periods 4 through 7, it contributes 4 units. However, item 2 contributes one unit either in period 3 or in periods 8 through 30. The inequality is therefore valid.

Notice that we can replace $\sum_{i=3}^{30} w_{2i}$ by $\sum_{i=4}^{40} w_{2i}$. However we cannot replace it by $\sum_{i=2}^{20} w_{2i}$ or $\sum_{i=1}^{10} w_{2i}$, because we can produce item 1 in periods 1 through 4, and item 2 in periods 5 and 6. This violates the inequality.

Another example is

\begin{equation}
(2) \quad w_{1,1} + z_{1,2} + z_{1,3} + \sum_{i=4}^{35} w_{1i} + \sum_{i=3}^{30} w_{2i} \ldots
\end{equation}

\begin{equation*}
+ y_{2,11} + (y_{2,12} + z_{2,12}) + z_{2,13} + \sum_{i=14}^{30} w_{2i} \geq 5.
\end{equation*}

We can therefore derive the following condition for an inequality extending up to maximum $(t_{1q}, t_{2p})$, with righthand side $Q \leq p+q$. We define the sets $W_1, Y_1, Z_1, WZ_1$ and $YZ_1$ for item 1, and the sets $W_2, Y_2, Z_2, WZ_2$ and $YZ_2$ for item 2 in the same manner as we did for the single item inequalities. We assume that no periods are skipped for item 1.
Suppose period $i$ is in demand interval $j+1$ for item 1 and in interval $j'+1$ for item 2. Suppose that period $i \in W$ and that periods $i+1, \ldots, i'$ are in $Z_1$. Then we can turn the machine on in period $i$, and produce item 1 from $i+1$ through $i'$, and the lefthand side does not increase. We 'lose' contribution from the periods $i+1$ through $i'$. Let $K= (i'-i)$. Then the terms for item 1 contribute $q\cdot\min(K, q-j)$ units if we produce in periods $i+1$ through $i'$. We denote the interval $i$ through $i'$ as a 'skip interval', skipping $s_i = \min(K, q-j)$ units. We allow only one skipped interval in each demand interval. The total number of skipped units is $s = \min(q, \sum s_i)$ where $\sum s_i$ is the sum of $s_i$ over each skipped interval $i$.

For item 2, we write any valid inequality. However, we can skip some periods. For each skipped interval of item 1, we lose $s_i$ units only if we produce in periods $i+1$ through $i'$. Hence we cannot produce item 2 in these periods, and can drop $s_i$ periods from $i$ through $i+s_i-1$. We show later that this yields a valid inequality.

Notice that for items 1 and 2, we can generalize these inequalities if we allow the terms to have the general structure of the skip inequalities for the single produce case except in the skip intervals.

**Lemma (two-item).** The following inequalities are valid:

$$
\sum_{i \in W_1} w_{1i} + \sum_{i \in Y_1} y_{1i} + \sum_{i \in Y_1} y_{1i} + \sum_{i \in Z_1} (y_{1i} + c_{1i} z_{1i}) + \sum_{i \in Z_1} c_{1i} z_{1i} \ldots
$$

$$
\sum_{i \in W_2} w_{2i} + \sum_{i \in Y_2} y_{2i} + \sum_{i \in Y_2} y_{2i} + \sum_{i \in Z_2} (y_{2i} + c_{2i} z_{2i}) + \sum_{i \in Z_2} c_{2i} z_{2i} \geq (p+q) - s
$$

(TPI)

where
1) The sets \( W_1, Y_1, YZ_1 \) and \( Z_1 \) partition the interval \( \{1, \ldots, t_{1,q}\} \) and satisfy condition 1 for the single item inequalities extending up to \( t_{1,q} \) for all periods except the skipped intervals.

2) For each skipped interval starting in period \( i \) in demand interval \( j'+1 \) for item 2, we skip \( s_i \) periods starting in period \( i \). The sets \( W_2, Y_2, YZ_2 \) and \( Z_2 \) partition the remaining periods from \( \{1, \ldots, t_{2,p}\} \), and satisfy the condition 1 for the single item inequalities extending up to \( t_{2,p} \) for all periods except the skipped intervals.

3) The quantity \( s = \min(q, \sum s_i) \), where \( s_i \) is the number of skipped periods in the skipped interval starting in period \( i \), and there is at most one skipped interval in each demand interval.

**Proof.**

The terms for item 1 contribute \( q \) units to the lefthand side of the inequality unless we produce in a skipped interval. Suppose we produce item 1 in some subset \( S \) of skipped intervals. Then item 1 terms contribute \( \sum_{i \in S} s_i \) units less. But then we cannot produce item 2 in these skipped interval, and the rest of the terms for item 2 must contribute at least \( q-s-\sum_{i \in S} s_i \) units. Hence the inequality is valid.

**Example.**

Suppose we need to produce item 1 in periods 10, 20, 30 and 40, and item 2 in periods 15, 25, 35, 45 and 55. Then a valid inequality is

\[
\sum_{i=1}^{10} w_{i1} + z_{1,11} + z_{1,12} + \sum_{i=13}^{20} w_{i1} + z_{1,21} + \sum_{i=22}^{40} w_{i1} + \ldots
\]

\[
\sum_{i=1}^{9} w_{i2} + \sum_{i=12}^{19} w_{i2} + \sum_{i=21}^{55} w_{i2} \geq 6.
\]
Item 1 has two skipped intervals starting in period 10 and 20. The first one has a length $s_1=2$, and the second one has a length $s_2=1$. For item 2, we skip periods 10 and 11, and period 20.

Multi-item inequalities.

We can generalize these inequalities to an arbitrary number of products 1, ..., $P$. The inequality extends up to $\max(t_{qp} : 1 \leq p \leq P)$, where $q_p$ is the $q$th demand for item $p$. We allow skip intervals for each of the products 1 through $P-1$. If product $p$ has a skip interval of length $s_{ip}$ in demand interval $j_p+1$, then for item $p+1$, we skip $s_{ip}$ periods starting in period $i$. We allow at most one skip interval for an item in each demand interval. Let $s = \sum_{p=1}^{P} \sum_{i} s_{ip}$ denote the total number of skipped periods. We state without proof the following valid inequalities, which we call the multi item skip inequalities (MSI):

$$\sum_{p=1}^{P} [\sum_{i \in W_p} w_{pi} + \sum_{i \in Y_p} y_{pi} + \sum_{i \in YZ_p} (y_{pi} + c_{pi} z_{pi}) + \sum_{i \in Z_p} c_{pi} z_{pi}]... \geq \sum_{p=1}^{P} q_p - s$$

(1)

where

1) The sets $W_1$, $Y_1$, $YZ_1$ and $Z_1$ partition the interval $\{1, ..., t_{1,q}\}$ and satisfy condition 1 for the single item inequalities extending up to $t_{1,q}$ for all periods except the skipped intervals.

2) For each skipped interval starting in period $i$ in demand interval $j_p+1$ for item 2, we skip $s_{ip}$ periods starting in period $i$. The sets $W_p$, $Y_p$, $YZ_p$ and $Z_p$ partition the remaining periods from $\{1, ..., t_{p,q_p}\}$, and
satisfy the condition 1 for the single item inequalities extending up to \( t_{p,q_p} \) for all periods except the skipped intervals.

3) The quantity \( s = \min(q, \sum_{p=1}^{P} \sum_{i} s_{ip}) \), where \( s_i \) is the number of skipped periods in the skipped interval starting in period \( i \), and there is at most one skipped interval in each demand interval.

Examples.

Consider the following 4 item problem. Against each item, the periods in which demands occur are shown.

<table>
<thead>
<tr>
<th>Item</th>
<th>1</th>
<th>20</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>25</td>
<td>45</td>
<td>65</td>
<td>85</td>
<td>105</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>50</td>
<td>70</td>
<td>90</td>
<td>110</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>35</td>
<td>55</td>
<td>75</td>
<td>95</td>
<td>115</td>
<td></td>
</tr>
</tbody>
</table>

A valid inequality is

\[
\begin{align*}
&w_{1,1} + z_{1,2} + z_{1,3} + z_{1,4} + z_{1,5} + \sum_{i=6}^{100} w_{1i} \cdots \\
&+ w_{2,6} + z_{2,7} + z_{2,8} + z_{2,9} + z_{2,10} + \sum_{i=11}^{105} w_{2i} \cdots \\
&+ w_{3,11} + z_{3,12} + z_{3,13} + z_{3,14} + z_{3,15} + \sum_{i=16}^{110} w_{3i} \cdots \\
&+ \sum_{i=16}^{115} w_{4i} \geq 8.
\end{align*}
\]

Items 1, 2 and 3 skip 3 periods starting in periods 1, 2 and 3 respectively. If we produce item 1 in periods 1 through 5, item 2 in periods 6 through
10 and item 3 in periods 11 through 15, the lefthand side increases to 3 units. We must produce item 4 in periods 16 through 115 and hence item 4 contributes 5 units.

Another valid inequality is

\[
\begin{align*}
\sum_{i=2}^{5} z_{1i} + y_{1,6} + (y_{1,7} + z_{1,7}) + (y_{1,8} + 2z_{1,8}) + (y_{1,9} + 3z_{1,9}) + (y_{1,10} + 4z_{1,11}) & \\
+ 5z_{1,12} + 4z_{1,13} + 3z_{1,14} + 2z_{1,15} + z_{1,16} + \sum_{i=17}^{100} w_{1i} & \\
+ w_{2,6} + \sum_{i=7}^{10} z_{2i} + \sum_{i=11}^{105} w_{2i} & \\
w_{3,11} + \sum_{i=12}^{15} z_{3i} + \sum_{i=16}^{110} w_{2i} & \\
+ \sum_{i=16}^{115} w_{2i} & \geq 8.
\end{align*}
\]

Future Directions.

The Multi-item Partitioning Inequalities (MPI) can be generalized in several ways. We point out some of the directions in which we can proceed to discover new valid inequalities:

1) Consider two inequalities, one for item 1 and another for item 2, both having the general form of the skip inequalities (SI) described for the single item problem. Let the right hand side of the inequality for item 1 be \( q \) and of the other one be \( p \). Suppose we write down a skip inequality for item 1. We can modify it by violating condition 1 for some period \( i^* \in Z \). In other words, for some period \( i^* \in Z \), and \( i \leq i^* \), we allow the sum of the coefficient of \( z \) and the coefficients of \( y \) to be less than \( K = (q-b_{i^*} : i \leq j) \). Suppose it is equal to \( K - 1 \). Then we can turn the machine on in period \( i \), and produce item 1 \( K \) times from period \( i \) through \( i + K - 1 \). To compensate for this, we introduce terms for item 2, which satisfy condition 1, but skip some of the terms \( i \) through \( i^* \).
An example of this is item 1, with demands in periods 5, 15 and 25 and 35; and item 2 with demands in periods 10, 20, 30 and 40. Consider the item 1 inequality:

\[ y_{1,1} + (y_{1,2} + z_{1,2}) + (y_{1,3} + 2z_{1,3}) + (y_{1,4} + 3z_{1,4}) + 4z_{1,5} + 3z_{1,6} + 2z_{1,7} + z_{1,8} + \sum_{i=9}^{40} w_{1i} \geq 4. \]

We can modify it by decreasing the length of the sequence in \( Y \cup YZ \) to three periods, and introduce terms for item 2 from period 8 through 40.

\[ y_{1,1} + (y_{1,2} + z_{1,2}) + (y_{1,3} + 2z_{1,3}) + 4z_{1,4} + 4z_{1,5} + 3z_{1,6} + 2z_{1,7} + z_{1,8} \ldots \]

\[ + \sum_{i=9}^{35} w_{1i} + \sum_{i=8}^{40} w_{2i} \geq 4. \]

Item 1 contributes 4 units unless we turn the machine on in periods 1, 2 or 3, and produce 4 times. But then we must produce at least 4 times up to period 40 for item 2, and hence at least once in periods 8 through 40. The inequality is therefore valid.

Notice that we can tighten the inequality by reducing the coefficients of \( z_{1,4} \) as follows:

\[ y_{1,1} + (y_{1,2} + z_{1,2}) + (y_{1,3} + 2z_{1,3}) + 3z_{1,4} + 4z_{1,5} + 3z_{1,6} + 2z_{1,7} + z_{1,8} \ldots \]

\[ + \sum_{i=9}^{35} w_{1i} + \sum_{i=8}^{40} w_{2i} \geq 4. \]

Instead of skipping periods 1 through 7 for item 2, we can skip any 7 of the periods 1 through 8. However, we need to adjust the coefficients of \( z_{1i} \) appropriately. For example, if we skip periods 2 through 8 for item 2, we can reduce the coefficient of \( z_{1,5} \) to 3 as well.
Notice that we can also introduce the terms $\sum_{i=7}^{30} w_{2i}$ for item 2 if we set the coefficient of $z_{1,4}$ to 4. Item 1 contributes 4 units unless we turn the machine on for item 1 in one of the periods 1, 2 or 3, and produce. In that case, we must produce item 2 at least 3 times up to period 30 to meet the demand, and hence at least once in the periods 7 through 30.

Similarly, we can also generalize the terms for item 2 as well to the form of the skip inequalities (SI).

2) We can extend the inequalities to the multi-item problem.
Chapter 7. The Multi Machine Problem

Multi machine model.

We can extend our model to include more than one machine. Suppose we need to produce $P$ products on $M$ machines, over a time horizon of $T$ units. Let $d_{pi}$ denote the demand for product $p$ in period $i$, and $d_{pi}$ is either 0 or 1. Let $n_p$ denote the total demand for product $p$. We can formulate the problem as follows:

Minimize $\sum_{m=1}^{M} \sum_{p=1}^{P} \sum_{i=1}^{T} (k_{mpi}z_{mpi} + s_{mpi}y_{mpi} + h_{mpi}w_{mpi})$

subject to

$\sum_{m=1}^{M} \sum_{i=1}^{t} w_{mpi} \geq \sum_{i=1}^{t} d_{pi}$ for all $p, t$ (i)

$\sum_{m=1}^{M} \sum_{i=1}^{T} w_{mpi} = n_p$ for all $p$ (ii)

$y_{mpi} \geq w_{mpi}$ for all $m, p, i$ (iii)

$z_{mpi} + y_{mp,i-1} - y_{mpi} \geq 0$ for all $m, p, i$ (iv)

$\sum_{p=1}^{P} y_{mpi} \leq 1, \sum_{p=1}^{P} z_{mpi} \leq 1$ for all $m, p, i$ (v)

$w_{mpi} \geq 0, y_{mpi} \geq 0, z_{mpi} \geq 0$ for all $m, p, i$ (vi)

$w_{mpi}, y_{mpi}, z_{mpi}$ integer for all $m, p, i$ (vii)
Constraints (i) and (ii) are the demand constraints. Constraint (iii) states that if we produce item \( p \) in period \( i \) on machine \( m \), then we must setup machine \( m \) in period \( i \) for item \( p \). Constraint (iv) is analogous to the single machine problem, and requires that if machine \( m \) is setup for item \( p \) in period \( i \), then it is either turned on in period \( i \), or is setup in period \( i-1 \). Constraint (v) states that for machine \( m \), we can setup the machine for at most one item \( p \), or turn it on for at most one item \( p \).

**Partitioning Inequalities.**

We consider the single machine inequalities with only one item in each inequality. For example,

\[
y_{p1} + \sum_{i=2}^{t_1} z_{pi} \geq 1.
\]

For notational simplicity, we drop the subscript \( p \), and write

\[
y_{1} + \sum_{i=2}^{t_1} z_{i} \geq 1.
\]

One set of valid inequalities is obtained by generalizing the single machine inequalities. For example, the constraint for item \( p \) of the single machine problem can be generalized to the multi-machine case as follows:

\[
\sum_{m=1}^{M} y_{m1} + \sum_{m=1}^{M} \sum_{i=2}^{t_1} z_{mi} \geq 1.
\]

The inequality states that we must setup one of the machines in period 1, or, turn on one of the machines in one of the periods 2 through \( t_1 \), the period in which the first demand for item \( p \) occurs.

Consider any partitioning inequality (PI) for the single machine. Recall that we do not skip any periods in these inequalities,
and that it has variables for only one product in it. We can generalize it to the multi machine case if we replace \( w_i \) by \( \sum_{m=1}^{M} w_{mi} \), \( y_i \) by \( \sum_{m=1}^{M} y_{mi} \) and \( c_{i}z_{i} \) by \( c_{i}(\sum_{m=1}^{M} z_{mi}) \). We use the following argument to show that the new inequality is valid.

Since (PI) is valid for the single machine case, from condition 1 we obtain that if we turn on the machine in period \( i \) in demand interval \( j+1 \), and produce from \( i \) through \( i^* \), the terms from \( i \) through \( i^* \) sum to at least \( \min(q-j, i^*+1-i) \). To see this, notice that if we produce in a period \( i^* \in W_m \cup Y_m \cup Y_Z_m \cup W_Z_m \), the lefthand side increases by at least one unit. If we produce in period \( i^* \in Z_m \), and turn the machine on in period \( i^* \), the lefthand side increases by at least one unit. If \( i^* \in Z_m \) and we do not turn the machine on in period \( i^* \), we must turn it on in some period \( i \leq i^* \). Then condition 1 of the skip inequalities for the single machine problem ensures that the sum of the terms from \( i \) through \( i^* \) is \( \min(q-j, i^*+1-i) \).

For \( j=0 \), this is \( \min(q, i^*+1-i) \). Hence, for each period of production in the first demand interval, the lefthand side increases by at least one unit, up to a maximum of \( q \), the righthand side. Suppose it is true for arbitrary \( j \leq q \) that the sum of the terms for each period of production up to \( t_j \) increases by at least one unit up to a maximum of \( q \). We need to show that it is true for \( j+1 \). Suppose we produce \( k \leq q \) times up to \( t_j \). By the inductive hypothesis, the sum of the terms up to \( t_j \) is at least \( k \). Also in any feasible solution \( k \geq j \). Hence \( q-k \leq q-j \). Hence if we produce \( q-k \) times in interval \( j+1 \), the lefthand side goes up by at least \( q-k \) units by condition 1. Hence the inequality is satisfied. Notice that the inductive hypothesis is true for each machine, and since we
must produce at least $q$ times up to $t_j$ in any feasible solution, the inequality is valid.

We can therefore state the following theorem for the multi-machine partitioning inequalities (MMPI):

**Theorem (MMPI).** For any partitioning inequality (PI) for the single machine problem, if we replace each variable $w_{pi}$ by $\sum_{m=1}^{M} w_{mpi}$, $y_{pi}$ by $\sum_{m=1}^{M} y_{mpi}$ and $c_{pi}z_{pi}$ by $c_{pi}\sum_{m=1}^{M} z_{mpi}$, we obtain a valid inequality for the multi-machine problem.

We can generalize the inequalities. Consider the following example, with 2 machines $m$ and $m'$:

$$\sum_{i=1}^{t_1} w_{mpi} + [w_{m'p1} + y_{m'p2} + \sum_{i=1}^{t_1} z_{m'pi}] \geq 1.$$ 

The terms for machine $m$ are $\sum_{i=1}^{t_1} w_{mpi}$ which is valid if we had only machine $m$. The terms for machine $m'$ are $[w_{m'p1} + y_{m'p2} + \sum_{i=1}^{t_1} z_{m'pi}]$ which is valid if we had only machine $m'$. We can combine these two and obtain a valid inequality for the two machine problem.

Thus we can generalize the inequalities (MMPI) as follows. Suppose the inequality extends up to period $t_q$. For each machine, we write terms corresponding to a single machine, single item partitioning inequality (PI). Notice that we do not require that the partitioning inequality on each machine is the same. We sum the terms for all the machines and obtain a valid inequality. We denote this by the generalized (MMPI) inequalities.

**Corollary (MMPI).** The generalized MMPI inequalities are valid.
Skip Inequalities.

Consider the following skip inequality for the single machine problem.

\[ y_2 + \sum_{i=3}^{t_2} z_i \geq 1. \]  

(skips period 1)

Let us try to generalize it to the multi machine case as follows:

\[ \sum_{m=1}^{M} y_{m2} + \sum_{m=1}^{M} \sum_{i=3}^{t_2} z_{mi} \geq 1. \]

This inequality is not valid because we can produce one unit each on two machines in period 1. This feasible schedule violates the inequality.

However, suppose we have 2 machines, and we want to produce 3 units of item p. Then we can write

\[ \sum_{m=1}^{2} y_{m2} + \sum_{m=1}^{2} \sum_{i=3}^{t_3} z_{mi} \geq 1. \]

This inequality is valid because we can produce at most two units in period 1, one on each machine. But then we must produce at least one unit in periods 3 through t_3, and hence the inequality is valid. Notice that we need not skip the same period on both machines. Another valid inequality that skips period 1 on machine 1, and period 2 on machine 2 is

\[ (y_{1,2} + z_{1,3}) + (w_{2,1} + y_{2,3}) + \sum_{m=1}^{2} \sum_{i=4}^{t_3} z_{mi} \geq 1. \]

The skip inequalities for the single machine problem were based on the following argument. If we skip b periods, and we need to
produce \( q > b \) times, then we must produce at least \( q-b \) times in the remaining periods. If we generalize it to the multi-machine case, then we use the following argument. If we skip \( b_m \) periods on machine \( m \), then we can produce \( \sum_{m=1}^{M} b_m \) times in these periods. If we need to produce \( q > \sum_{m=1}^{M} b_m \) times, then we must produce \( q \sum_{m=1}^{M} b_m \) times in the remaining periods.

We can therefore describe the multi-machine skip inequalities (MMSI) as follows. We drop the subscript \( p \) for item \( p \) because in each inequality we have variables only for item \( p \). Suppose we consider demand up to \( t_q \) for item \( p \), and we skip \( b_m \) periods on machine \( m \). Then we reduce the righthand side to \( q \sum_{m=1}^{M} b_m \). Notice that if \( q \leq \sum_{m=1}^{M} b_m \) then the inequality is redundant, because the righthand side is at most zero.

We describe the multi-machine skip inequalities (MMSI). For each machine, we write terms up to \( t_q \) that satisfy the conditions for the single machine skip inequalities (SII), for a righthand side of \( \text{rhs} = q \sum_{m=1}^{M} b_m \), and as though we skip \( \sum_{m=1}^{M} b_m \) periods. In other words, though we skip only \( b_m \) periods on machine \( m \), we modify the condition for the skip inequalities as follows.

**Condition (MMSI).** For any period \( i^* \), containing a term of the type \( c_{i^*,m} z_{i^*,m} \), \( c_{i^*,m} \geq 1 \), and any other period \( i \leq i^* \), in the \((j+1)st\) demand interval, the sum of the coefficient of \( z_{i,m} \) and the coefficients of \( y_{i,m} \) from \( i \) through \( i^* \) is at least \( \min(\text{rhs} - \max (j^* - \sum_{m=1}^{M} b_{m^*} : j^* \leq j), i^*+1-i) \).
where we define $\sum_{m=1}^{M} b_{mj}$ as the number of periods skipped on all machines up to $t_j$.

We sum the terms for each machine, and this defines the lefthand side of the inequality.

**Theorem (MMSI).** The multi machine, single item skip inequalities (MMSI) are valid.

**Proof.**

We show that in any feasible schedule, for each unskipped period that we produce in, the lefthand side increases by at least one unit, up to a maximum of rhs.

Notice that condition (MMSI) also satisfies condition (SI) for the single machine skip inequalities (SI). Hence if we produce all the demand on one machine, the inequality is satisfied. Hence, for each unskipped period we produce in, the lefthand side increases by at least one unit up to a maximum of rhs units. This holds even if we produce on more than one machine. Hence the inequality is valid.

**Example.**

Suppose we have demands in periods 5, 10, 15, 20 and 25, and we have 2 machines. On machine 1 we skip period 1, and on machine 2 we skip period 16. A valid inequality is

$$y_{1,2} + (y_{1,3} + z_{1,3}) + (y_{1,4} + 2z_{1,4}) + 3z_{1,5} + 2z_{1,6} + z_{1,7} + \sum_{i=8}^{25} w_{1i} \quad \cdots$$

$$+ \sum_{i=1}^{15} w_{2i} + y_{2,17} + (y_{2,18} + z_{2,18}) + 2z_{2,19} + z_{2,20} + \sum_{i=21}^{25} w_{2i} \geq 3.$$
Multi-item, multi-machine Inequalities.

Let us call these the General Inequalities (GI). Consider the following 2 item, single machine inequality. We denote the two items by \( p \) and \( p' \).

\[
w_{p1} + \sum_{i=2}^{3} p_{i} z_{pi} + \sum_{i=3}^{3} w_{p'i} \geq 1.
\]

Item \( p \) contributes 1 unit unless we turn the machine on in period 1, and produce in period \( i \geq 2 \). But then item \( p' \) contributes 1 unit. Let us extend this to the multi machine case. Consider the following inequality:

\[
\sum_{m=1}^{M} [w_{mp1} + \sum_{i=2}^{3} z_{mpi} + \sum_{i=3}^{3} w_{mp'i}] \geq 1.
\]

It is not valid because we can choose some machine \( m^* \), turn it on in period 1, and produce item \( p \) in period 2. We can then produce item \( p' \) on some other machine in period 1 or 2. The lefthand side is then equal to 0. Therefore we modify it as follows:

\[
w_{m^*p1} + \sum_{i=2}^{3} z_{m^*pi} + \sum_{m=1,m\neq m^*}^{M} \sum_{i=1}^{3} w_{mpi} + \sum_{i=3}^{3} w_{m^*p'i} \geq 1.
\]

Notice that we single out machine \( m^* \), the terms for which are the same as those for the single machine inequality. However, for other machines we also introduce the terms \( \sum_{m=1,m\neq m^*}^{M} \sum_{i=1}^{3} w_{mpi} \) for item \( p \).
We can therefore describe the following class of inequalities. Consider any valid single machine, multi item inequality (MPI), with a righthand side of rhs. We retain the same terms for some machine \( m^* \), replacing each variable \( w_{pi}, y_{pi} \) and \( c_{pi}z_{pi} \) by \( w_{m^*pi}, y_{m^*pi} \) and \( c_{m^*pi}z_{m^*pi} \) respectively. For the other machines, we introduce any valid single machine, single item inequality (PI) or (SI) with a righthand side of rhs. This yields a valid multi machine, multi item inequality. We denote this by GMSI for generalized multi item skip inequalities.

**Lemma (GMSI).** The generalized multi item skip inequalities are valid.

However, we can generalize this still further. For example, if we had two machines, \( m \) and \( m' \) and three products \( p, p' \) and \( p'' \), we can write the following inequality:

\[
\sum_{i=1}^{t} w_{mpi} + w_{mp'i} + \sum_{i=2}^{t} z_{mp'i} + \sum_{i=3}^{t} w_{mp''i} + \sum_{i=1}^{t} w_{m'p1} + \sum_{i=2}^{t} z_{m'pi} + \sum_{i=3}^{t} w_{m'p'i} + \sum_{i=3}^{t} w_{m'p''i} \geq 1.
\]

If we produce item \( p \) on machine \( m \), the inequality is satisfied. Item \( p \) on machine \( m' \) contributes 1 unit unless we turn on machine \( m' \) in period 1 and produce item \( p \) in period \( i \geq 2 \). If we produce items \( p' \) or \( p'' \) on machine \( m' \), then the inequality is satisfied. On the other hand, if we produce them on machine \( m \), then item \( p' \) contributes 1 unit unless we turn on machine \( m' \) in period 1 and produce item \( p' \) in period \( i \geq 2 \). But then item \( p'' \) must contribute 1 unit.

In general, consider any single machine, multi item inequality (MPI). Suppose produce item \( p \) in periods \( i \) through \( i^* \) on machine \( m^* \). Suppose this does not satisfy the inequality. Then, since we need to
produce other products, in periods other than i through \( i^* \), the inequality is satisfied. However, in a multi machine problem, we can use the periods \( i \) through \( i^* \) on other machines. Hence we need to ensure that producing the other products on any of the machines satisfies the inequalities.

Consider two examples which illustrate this idea. Suppose we have 2 machines \( m \) and \( m' \), and 4 items.

\[
(w_{m1,1} + \sum_{i=2}^{1} z_{m1i}) + (\sum_{i=3}^{1} w_{m2i}) \ldots
\]
\[
+ (w_{m3,1} + \sum_{i=2}^{1} z_{m3i}) + (\sum_{i=3}^{1} w_{m4i}) \ldots
\]
\[
+ (w_{m',1,1} + \sum_{i=2}^{1} z_{m'i}) + (\sum_{i=3}^{1} w_{m'2i}) \ldots
\]
\[
+ (w_{m',3,1} + \sum_{i=2}^{1} z_{m'3i}) + (\sum_{i=3}^{1} w_{m'4i}) \geq 1 \quad (1.1)
\]

\[
(w_{m1,1} + \sum_{i=2}^{1} z_{m1i}) + (\sum_{i=3}^{1} w_{m2i}) \ldots
\]
\[
+ (w_{m3,1} + \sum_{i=2}^{1} z_{m3i}) + (\sum_{i=3}^{1} w_{m4i}) \ldots
\]
\[
+ (\sum_{i=3}^{1} w_{m'2i}) + (w_{m',1,1} + \sum_{i=2}^{1} z_{m'i}) + \sum_{i=3}^{1} (w_{m'4i} + \sum_{i=2}^{1} z_{m'4i}) \geq 1 \quad (1.2)
\]

Inequality (1.1) is valid, whereas (1.2) is not. In (1.2) we can produce items 2 and 4 on machine \( m \) in periods 1 and 2, and items 1 and 3 on machine \( m' \) in periods 1 and 2. This yields a lefthand side of 0, and violates the inequality. However, this yields a lefthand side of 2 in (1.1). We can confirm easily that (1.1) is valid. Graphically, we depict the situation as follows. A * represents a period in \( W \), a dash represents a period in \( Z \), and a blank space indicates that the period is skipped for that particular item.
Inequality (1.1)

Machine m

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3 onwards</th>
</tr>
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<tbody>
<tr>
<td>Item 1</td>
<td>*</td>
<td>-</td>
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<tr>
<td>Item 2</td>
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<td>Item 3</td>
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<tr>
<td>Item 4</td>
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Machine m'

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<th>3 onwards</th>
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</thead>
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<td>Item 4</td>
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Inequality (1.2)

Machine m

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Machine m'

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In inequality (1.2), we can produce items 2 and 4 on machine m in periods 1 and 2. We can also produce items 1 and 3 on machine m' in periods 1 and 2. Hence the inequality is not valid.

In general, if we produce some set of items $P_m$ on machine m, the lefthand side increases by 1 unit unless each of the items $p$ in $P_m$ is produced in a skipped period or by turning the machine on in period $i \in W_{mp}$ and keeping it on until period $t \geq i+1$, and producing in period $t \in Z_{mp}$. In that case, we need to ensure that producing the other items either on machine m, or some other machines, increases the lefthand side by at least 1 unit. Let us call period $i$ a transition period for item $p$ on machine m if $i \in W_{mp}$ and $i+1 \in Z_{mp}$.
We describe another class of inequalities. Suppose we have products $1, \ldots, P$, and a set of machines $1, \ldots, M$. For each item we have the terms $\sum_{m=1}^{M} \sum_{i=1}^{t_{p-1}} w_{mip} \cdot$

We divide the last demand interval into subsets $W_{mp}, Y_{mp}$ and $Z_{mp}$ for each machine and product. If we skip period $i$ on machine $m$ for item $p$, we call it a skip period.

We divide the items into groups of 2. If there is any item left over, it forms a group by itself. For each pair of items, we write a single machine 2 item inequality for each machine. If we sum these terms for all machines, we obtain a valid inequality, which we call the generalized inequalities (GI).

**Theorem (GI).** The following inequalities are valid

$$\sum_{m=1}^{M} \sum_{p=1}^{P} \sum_{i=1}^{t_{p-1}} w_{mip} + \sum_{m=1}^{M} \sum_{p=1}^{P} \sum_{i\in W_{mp}} w_{mip} \cdot ...$$

$$+ \sum_{m=1}^{M} \sum_{p=1}^{P} \sum_{i\in Y_{mp}} y_{mip} + \sum_{m=1}^{M} \sum_{p=1}^{P} \sum_{i\in Z_{mp}} z_{mip} \geq \sum_{p=1}^{P} (q_{p-1}) + 1$$

(GI)

where the items $1, \ldots, P$, the number of machines and the sets $W_{mp}$, $Y_{mp}$ and $Z_{mp}$ satisfy the following conditions:

1) We divide the items into pairs, which leaves at most one item without a pair. If an item is without a pair, we do not include that item in the inequality.

2) The number of machines is at most equal to $P/4 + P/2 - 1$. 

3) For each pair $p$ and $p'$, the terms for periods $(\min(t_{qp}^{-1}+1, t_{qp}^{-1}+1), \ldots, \max(t_{qp}, t_{qp}'))$ on a particular machine satisfy the conditions for a valid 2 item, single machine inequality.

4) The same set of periods is skipped for each period.

5) The terms on all machines are the same.

Proof.

We first specify what a single machine 2 item inequality is. We consider the last demand interval for a pair of items $p$ and $p'$: $(\min(t_{qp}^{-1}+1, t_{qp}^{-1}+1), \ldots, \max(t_{qp}, t_{qp}))$. For item $p$ we have one transition period, i.e., one period $i^* \in W_{mp}$ and $i^*+1 \in Z_{mp}$. For all other periods, if $i+1 \in Z_{mp}$, then $i \notin W_{mp}$. Moreover, none of the periods $(t_{qp}^{-1}+1, \ldots, t_{qp})$ are skipped for item $p$. For item $p'$, we skip periods $i^*$ and $i^*+1$, and for the other periods $(t_{qp}^{-1}+1, \ldots, t_{qp}) \setminus \{i, i+1\}$, we require that $i^*+2 \in Z_{mp}$, and that if $i+1 \in Z_{mp}$, then $i \notin W_{mp}$. Graphically, we can depict it as follows:

```
Item p  .... * - .......
```

```
Item p' .... .........
```

The * indicates that in that period, item $p$ has $i \in W_{mp}$, and the '−' indicates that $i+1 \in Z_{mp}$. The corresponding periods for item $p'$ are skipped.

The terms $\sum_{m=1}^{M} \sum_{p=1}^{P} \sum_{i=1}^{t_{qp}^{-1}} w_{mpi}$ sum up to at least $\sum_{p=1}^{P} (q_{p}^{-1})$ in any feasible solution. If they sum up to $\sum_{p=1}^{P} (q_{p}^{-1})+1$, the inequality is satisfied. Suppose $\sum_{m=1}^{M} \sum_{i=1}^{t_{qp}^{-1}} w_{mpi} = (q_{p}^{-1})$ for each item...
p. Then we must produce item p in the interval \( \{t_{p-1}+1, \ldots, t_p\} \) for each item p.

Consider any machine m. If both items from a pair are produced on a machine, they contribute at least 1 unit since they satisfy the conditions for a single machine 2 item inequality. Suppose one item from a pair is produced on one machine, and the other item on a different machine. This contributes at least 1 unit unless we produce one item by turning the machine on in a transition period \( i \in W_{mp} \) and producing in period \( i+1 \in Z_{mp} \), and the other item in a skipped period on another machine. We show that we cannot do this for all the pairs.

Since each pair satisfies the condition for a single machine 2 item inequality, one of the items p in the pair does not skip any period, and the other item p' skips at most two consecutive periods i and i+1, if period i is a transition period for the item p. From condition 4, the same set of periods is skipped for each period. Hence, the same set of two periods is skipped on all machines. From condition 2, the number of machines is at most \( P/4 + P/2 - 1 \). If we produce one item from each of the pairs in a skipped period, we use \( P/4 \) machines. If any of the remaining items is produced on one of these machines, it must be turned on in some unskipped period \( i \in Z_{mp} \), and hence the inequality is satisfied. Suppose the remaining \( P/2 \) items are produced in the remaining \( P/2 - 1 \) machines. If we produce one of these items, it contributes 1 unit unless we produce it by turning it on in a transition period \( i \in W_{mp} \) and produce in period \( t \geq i+1, t \in Z_{mp} \). But we can do this for at most \( P/2 - 1 \) items. Therefore at least one of the items must be
produced either in period $i \in W_{mp}$, or by turning the machine on in period $i \in W_{mp}$. In either case the inequality is satisfied.

Example.

Suppose we have 11 items. We form 5 pairs and discard one item, say the 11th item. Suppose we have $10/4 + 10/2 - 1 = 7$ machines. We can then write the following terms in the last demand interval for each pair $p$ and $p'$ and for each machine. To simplify notation, we denote the period $t_{qp} + 1$ by $t_p + 1$, and the period $\max_p (t_{qp} + 1)$ by $t + 1$. We also drop the subscript $m$ for machine.

$$(w_{p,t+1} + \sum_{i=t+2}^{t_{qp}+1} z_{pi}) + (\sum_{i=t+1}^{t+2} w_{pi}).$$

For each pair, the period $t+1$ is a transition period for one item, and the periods $t+1$ and $t+2$ are skipped for the other item.

We need to show that these terms contribute at least 1 unit if we produce each of the items once in the last demand interval. Consider the 5 items that skip periods $t+1$ and $t+2$ on each machine. We can produce them in these periods using 3 machines. If we produce the other 5 products on one of these machines, we must turn the machine on in some period $i \in Z_{mp}$, and hence the inequality is satisfied. If we do not, we must use one of the other 4 machines. An item $p$ contributes 1 unit unless we turn on a machine in period $t+1$, and produce in period $t+2$ or later. But we can do this on 4 machines for only 4 products. Hence at least one item must be produced by turning the machine on in some period $i \in Z_{mp}$. The inequality is therefore valid.
Comments.

It is clear that we can generalize these inequalities in various ways. For example:

1) Instead of pairs of products, we can consider a single machine inequality for an arbitrary number of items, and then generalize it.

2) We can relax the condition that the same periods must be skipped on each machine. However, we need to be careful. For example, if we consider inequality (1.1), we have two pairs \( \{1, 2\} \) and \( \{3, 4\} \). We skip periods 1 and 2 for items 2 and 4, and period 1 is a transition period for items 1 and 3 on both machines. If we shift the transition period for the pair \( \{3, 4\} \) to say, period 5, then the inequality is no longer valid: we can produce items 1 and 3 on machine \( m \) in periods following the transition period. We can produce item 2 on machine \( m' \) in period 1, and item 4 on machine \( m' \) in period 5. This feasible schedule violates the inequality.

3) Instead of the last demand interval, we can generalize the inequalities to other demand intervals.

4) Instead of restricting the transitions to \( i \in W_{mp} \) and \( i+1 \in Z_{mp} \), we can try transitions of a more general nature. For example, the following is a valid single machine 2 item inequality, and has a 'transition' in periods 1 and 2:

\[
(y_{p,1} + z_{p,2} + \sum_{i=3}^{t} w_{pi}) + (\sum_{i=3}^{t'} w_{p'i}) \geq 2.
\]
Item $p$ contributes 2 units unless we produce it in periods 1 and 2. But then, item $p'$ must contribute at least 1 unit.

We could generalize these inequalities to the multi machine problem.

However, the structure of the inequalities becomes more and more complex. Moreover, it is not clear if it is necessary to pursue these generalizations for most practical applications.
Chapter 8. Computational Results

In this chapter we use the valid inequalities to solve the product cycling problem. The algorithm we use is very simple. We use the partitioning inequalities to reformulate the problem as shown in Chapter 6, and use a standard linear programming package to solve it. We compare this solution with the linear programming relaxation of the original problem, and with the optimal integer solution.

The literature contains very little test data for the product cycling problem. Karmarkar and Schrage (1985) report computational experience for the continuous production policy version of the product cycling problem. In our model, we follow a discrete production policy in which we produce either zero or one unit in each period. In the continuous policy, we can produce any amount between zero and the production capacity. Karmarkar and Schrage use Lagrangean relaxation to solve problem instances of up to 4 products and 8 time periods. Magnanti and Vachani (1987) report computational results for the discrete version of the problem. They solve problem instances of up to 5 products and 15 time periods.

We use the same approach as Magnanti and Vachani to generate problem instances. For all problem instances, we assume that the initial inventory is zero, and that the machine is in the off state at the start of the time horizon.
The cost parameters $k_i$ and $s_i$ are the same for all the products and for all machines, and are constant over the time horizon. The inventory holding cost $h_i = 20(T-i)$. We showed in Chapter 7 that for this cost structure, the subset of the partitioning inequalities (PI) which partition only the last demand interval are sufficient to guarantee optimal integer solutions if $s_i \leq h_i$. We therefore do not test problem instances with $s_i > h_i$. We conjecture that for the case $s_i > h_i$, the partitioning inequalities (PI) are sufficient to guarantee optimal solutions. Our computational results confirm this result. We tested two categories of problems:

1) The single item, single machine problem. We tested problems of up to 100 time periods and 30 demands. The largest problem instance had 300 variables (100 each of the $w_i$ production variables, $y_i$ setup variables and $z_i$ changeover variables).

2) The two item, two machine problem. We tested problems of up to 100 time periods and 15 demands for each item. The largest problem instance had 1200 variables (300 variables for each item and machine).

For both the problem categories we used only a subset of the partitioning inequalities (PI). For any inequality extending up to $t_{q'}$ we partitioned only the last 5 demand intervals $t_{q-5}$ through $t_q$. Further, we used only those inequalities whose coefficients for the variables $z_i$ are 0, 1 and 2. We did not use any of the skip inequalities (SI), or the multi item or multi machine inequalities. For all the problem instances tested, we obtained optimal integer solutions.
We use a machine utilization of up to 30%. For example, a 10 period problem has three demands, and a 100 period problem has 30 demands. The computations were performed on a IBM 4341 computer using the GAMS package. Tables 8.1, 8.2 and 8.3 and Figures 8.1 and 8.2 summarize the computational results. Let $v(IP)$ and $v(LP)$ denote the optimal objective function values of the original integer program SCSP and the linear programming relaxation SCSP (LP) respectively. Let $v(p)$ denote the optimal objective function value of SCSP (LP) after including the inequalities discovered by Magnanti and Vachani (1987) and let $v(n)$ denote the optimal objective function value of SCSP (LP) after including the partitioning inequalities (PI). In Figure 8.1 we plot the optimal objective function values $v(IP)$, $v(LP)$, $v(p)$ and $v(n)$ divided by the total number of demands against the total number of demands. In Figure 8.2 we plot the percentage gaps between $v(IP)$ and the other problems namely, $v(LP)$, $v(p)$ and $v(n)$.

For the single item, single machine problem we computed optimal solutions for three different problems: (1) the linear programming relaxation, which we call the LP solution, (2) the linear programming relaxation along with the cuts discovered by Magnanti and Vachani (1987), which we call the previous cuts solution, and (3) the linear programming solution with the partitioning inequalities (PI), which we call the new cuts solution.

For the two machine, two product problems, we computed the optimal solution for the linear programming relaxation, and the solution after including the partitioning inequalities (PI).
Table 8.1

Computational Experience for
single machine, single item problems.

<table>
<thead>
<tr>
<th># of demands</th>
<th>Objective value of LP soln. $v(LP)$</th>
<th>Objective value with Prev. cuts $v(p)$</th>
<th>Objective value with New cuts $v(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>26.7</td>
<td>146.7</td>
<td>160</td>
</tr>
<tr>
<td>5</td>
<td>56.7</td>
<td>267</td>
<td>300</td>
</tr>
<tr>
<td>10</td>
<td>117.8</td>
<td>497</td>
<td>540</td>
</tr>
<tr>
<td>15</td>
<td>195.6</td>
<td>746.7</td>
<td>840</td>
</tr>
<tr>
<td>20</td>
<td>282.2</td>
<td>1047</td>
<td>1160</td>
</tr>
<tr>
<td>25</td>
<td>340</td>
<td>1277</td>
<td>1440</td>
</tr>
<tr>
<td>30</td>
<td>371</td>
<td>1496</td>
<td>1680</td>
</tr>
</tbody>
</table>

Notes: Constant turn on and setup cost

Production cost $= H^*(T-I)$
Table 8.2

Percentage gaps for the Single machine, Single Item Problem

<table>
<thead>
<tr>
<th># of Gap demands</th>
<th>LP soln. Gap ( \frac{(v(\text{IP}) - v(\text{LP})) \times 100}{v(\text{IP})} )</th>
<th>Prev. cuts Gap ( \frac{(v(\text{IP}) - v(p)) \times 100}{v(\text{IP})} )</th>
<th>New cuts</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>83.3</td>
<td>8.3</td>
<td>0.0</td>
</tr>
<tr>
<td>5</td>
<td>81.1</td>
<td>11.1</td>
<td>0.0</td>
</tr>
<tr>
<td>10</td>
<td>78.2</td>
<td>8.0</td>
<td>0.0</td>
</tr>
<tr>
<td>15</td>
<td>76.7</td>
<td>11.1</td>
<td>0.0</td>
</tr>
<tr>
<td>20</td>
<td>75.7</td>
<td>9.8</td>
<td>0.0</td>
</tr>
<tr>
<td>25</td>
<td>76.4</td>
<td>11.3</td>
<td>0.0</td>
</tr>
<tr>
<td>30</td>
<td>77.9</td>
<td>10.9</td>
<td>0.0</td>
</tr>
</tbody>
</table>
Figure 8.1. Objective function values
Figure 8.2. Percentage gap between the value of the optimal integer solution and the value of other solutions.
TABLE 8.3

Computational Experience with
Two Machine, Two Item Problems

Turn on cost $k = 100$

<table>
<thead>
<tr>
<th># of demands</th>
<th>LP soln.</th>
<th>% age gap</th>
<th>Cuts Soln.</th>
<th>Optimal soln.</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1453.3</td>
<td>15.51</td>
<td>1720</td>
<td>1720</td>
</tr>
<tr>
<td>5</td>
<td>4700</td>
<td>16.37</td>
<td>5620</td>
<td>5620</td>
</tr>
<tr>
<td>10</td>
<td>13569</td>
<td>5.77</td>
<td>14400</td>
<td>14400</td>
</tr>
<tr>
<td>15</td>
<td>32278</td>
<td>4.56</td>
<td>33820</td>
<td>33820</td>
</tr>
</tbody>
</table>

Higher turn on cost $k = 200$. 

<table>
<thead>
<tr>
<th># of demands</th>
<th>LP soln.</th>
<th>% age gap</th>
<th>Cuts Soln.</th>
<th>Optimal soln.</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1520</td>
<td>20.83</td>
<td>1920</td>
<td>1920</td>
</tr>
<tr>
<td>5</td>
<td>4820</td>
<td>20.72</td>
<td>6080</td>
<td>6080</td>
</tr>
<tr>
<td>10</td>
<td>13787</td>
<td>9.06</td>
<td>15160</td>
<td>15160</td>
</tr>
<tr>
<td>15</td>
<td>32633</td>
<td>6.07</td>
<td>34740</td>
<td>34740</td>
</tr>
</tbody>
</table>

Notes: Identical machines

Constant turn on and setup costs

Production cost = $H^*(T-I)$
Conclusions

1) We obtain optimal integer solutions for all the problem instances tested. This result is not surprising in view of our earlier conjecture that the partitioning inequalities (PI) guarantee integer solutions for the cost structures that we are testing. In fact, we included only a subset of the partitioning inequalities.

2) The gaps between the optimal objective function value of the linear programming relaxation and the optimal integer program objective function value are large for the single item, single machine problem, and vary between 75% and 83%. A small subset of the partitioning inequalities that partition only the last demand interval reduces the gap considerably to between 8% and 11%. However, we still obtain fractional solutions, and need to introduce more general partitioning inequalities to reduce the gaps to zero.

3) The gaps between the optimal objective function value of the linear programming relaxation and the optimal integer program objective function value are smaller for the two item, two machine problem, and vary between 4.5% and 21%. The percentage gap increases somewhat if we increase the turn on cost from 100 to 200. This result is not surprising because the linear programming solution tends to reduce the values of the turn on variables $z_i$ to fractions less than 1 to save on the turn on cost. The integer solution, on the other hand, tends to reduce the number of turn ons if we increase the turn on cost.
4) We were able to optimally solve much larger problem instances (with up to 1200 variables) as compared to Magnanti and Vachani (1987), who solved problems with up to 300 variables. There are probably two reasons for this: (i) we used the GAMS programming package, which can handle larger problem instances than the LINDO package used by Magnanti and Vachani, and (ii) we used a larger set of valid inequalities.
Chapter 9. Conclusions and Future Directions

We can summarize the contributions of this thesis as follows.

- We have studied a series of generalizations of the changeover cost problem, ranging from the single machine, single product problem to the single machine, multi product problem, and finally to the multi machine, multi product problem. We identified valid inequalities for each version of the problem. Some of these inequalities are direct generalizations from a simpler version of the problem. Other classes of inequalities are derived from considerations particular to the specific model being studied. The structure of these inequalities is complex, and we need to be able to identify precisely the conditions under which they are valid. Figure 9.1 gives an overview of the different models and valid inequalities.

- We have identified conditions under which some of the inequalities are tight, and proved that they are facets. These conditions relate to

(1) the length of the sequences containing terms other than the production variables \( w_i \), i.e., terms of the type \( c_i z_i \), \( y_i + c_i z_i \) and \( w_i + c_i z_i \). The length of these intervals varies depending on the demand interval in which they occur.
(2) The values of the coefficients for each term. We have to precisely identify the coefficients, especially when we cross over form one demand interval to the next.

(3) The number of periods we can skip, and the precise locations for when we can skip periods. For example, the inequality is not tight if we skip periods only in the last demand interval.

(4) Some special conditions for the first few time periods, restricting the type of terms that can occur in them if the inequality is tight.

(5) Boundary conditions when the inequality extends up to the last period T.

We have identified a total of nine conditions and shown that the inequalities satisfying these conditions are facets.

- We have solved the separation problem for different classes of inequalities, and derived an alternate formulation based on the separation problem. We showed that the separation problem can also be solved as a shortest path problem. The reformulation is more compact in the following sense. If we consider the linear programming relaxation of the original single item, single machine integer program with all of the valid inequalities, for problems with time horizon T, the formulation contains $3T$ variables and $O(2^T)$ constraints. The reformulation has $O(T^2)$ variables and $O(T^2)$ constraints.
- We have proved that for some special cases, a subset of the inequalities are sufficient to guarantee optimal integer solutions, even if we solve the problem as a linear program.

- We have shown computationally that a subset of the (PI) inequalities are effective in reducing the linear programming relaxation gap for certain instances of the changeover problem. Indeed, our computational results have shown that for a single item, single machine problem with a special (but practical) cost structure, a subset of the (PI) inequalities are effective in completely eliminating the linear programming relaxation gap.

Future directions.

Some questions arise naturally from the work in this thesis. First, we would like to extend the characterization results to more general cost structures. Specifically, we might first extend the result to situations with a turn on cost $k$, setup cost $s$, and per unit holding cost $H$ that are constant over the time horizon, and when $s > H$. We would also like to investigate whether the partitioning and skip inequalities completely characterize the convex hull of feasible solutions for the single item, single machine problem.

We could explore alternate formulations of the basic single machine, single item problem. For example, we could redefine the $z_i$ variable to mean that if $z_i = 1$, then we incur a turn on and a setup cost, whereas if $y_i = 1$, then we incur only a setup cost. Alternately, we
could redefine production variables as $w_{ij}$, so that $w_{ij} = 1$ if we produce in period $i$ for demand in period $j$.

We could extend the work to more complex models that include sequence dependent costs, changeover times (i.e., we might consider situations that require, say, $c$ units of time to changeover from one product to another) and setup times. Another natural extension is to the multi machine model with each machine having different production capacities. This model would require us to reformulate the problem somewhat because we cannot assume 0-1 demands in this case. We could also extend the model so that it captures more complex multi machine situations that requires us to process each job on more than one machine.

We could perhaps use some of the insight gained in proving the characterization results and try to apply them to two similar models: the uncapacitated version of this problem, and the lot sizing problem, (which allows us to produce more than one item in each period, and has a less complex cost structure involving only turn on and production costs and no setup costs).

Finally, we could explore the possibility of developing some common unifying framework to generate valid inequalities for related scheduling problems.
Figure 9.1. Overview of the different models and valid inequalities
Appendix 1.

Shortest Path Formulation.

We show how to solve SCSP as a shortest path problem. To motivate the discussion, we look at the problem with two demands at \( t=3 \) and 5 and nonnegative changeover, set up and production costs. The problem can be cast as the following shortest path problem between the BEGIN and END nodes:

\[ \text{BEGIN} \quad \text{END} \]

\[ \begin{array}{c}
\text{k(1)} \quad \text{k(2)} \quad \text{k(3)} \\
\text{s(1)} \quad 0 \quad \text{s(2)} \quad 0 \quad \text{s(3)} \\
1 \quad 10 \quad 2 \quad 20 \quad 3 \quad 30 \\
\text{h(1)} \quad \text{h(2)} \quad \text{h(3)} \\
\text{s(2)} \quad \text{k(4)} \quad \text{k(5)} \\
2 \quad 0 \quad 3 \quad 0 \quad 3 \quad 0 \\
\text{h(2)} \quad \text{h(3)} \quad \text{h(4)} \\
4 \quad 4 \quad 5 \quad 5' \\
\text{END} \\
\end{array} \]

\( k(i), s(i) \) and \( h(i) \) and 0 are arc costs. Nodes are indexed by numbers next to the node.

Figure 3. SCSP as a shortest path problem.

We can turn the machine on for the first time in any of the periods 1, 2 or 3. Thus the BEGIN node has three arcs \( z_{11}, z_{12} \) and \( z_{13} \) leaving it with changeover costs of \( k_1, k_2 \) and \( k_3 \) respectively. After we turn the machine on in period \( i \), we incur a set up cost of \( s_i \). This is
shown by the arc leaving node i and entering node (i0). After we set up the machine in period i, we can either produce in period i and incur a cost of \( h_i \) or go to the next period along an arc of zero cost, and then incur a cost of \( s_{i+1} \). Thus we have two arcs leaving node (i0) in the network.

The first time we produce is in period 1, 2 or 3. For example, if we produce in period 2, we are at node II in the network. We can either maintain the set up for the next period or turn the machine off and turn it on again in one of the periods 4 or 5. Thus node II has three arcs leaving it with costs of \( s_3, k_4 \) and \( k_5 \).

After we turn the machine on for the second time, we are at one of the nodes 3' through 5'. We have to set up the machine and so we have an arc leaving node i' and entering (i'0) for i'=3, ..., 5. On the other hand, if we produce in period i'-1 and keep the machine on in period i', we are at one of the nodes (2'0) through (5'0). At this stage we again have two choices: produce in period i' or keep the machine on for the next period. This is shown by two arcs leaving node (i'0). If we produce for the second time we reach the END node.

We define the following variables:

\[ z_{1i} = 1 \] if we turn the machine on for the first time in period i, for \( i=1, 2 \) and 3;

\[ = 0 \] otherwise.

The arc associated with this variable connects the BEGIN node to node i. The cost of this arc is \( k_i \).
$y_{1i} = 1$ if the machine is set up in period $i$ and we have not produced up to period $i-1$ for $i=1, ..., 3$;

$= 0$ otherwise.

The arc associated with this variable connects node $i$ to node $(i0)$. The cost of this arc is $s_i$.

$w_{1i} = 1$ if we produce for the first time in period $i$ for $i=1, 2$ and $3$;

$= 0$ otherwise.

The arc associated with this variable connects node $(i0)$ to one of the nodes $I, II$ or $III$. The cost of this arc is $h_i$.

$z_{2ij} = 1$ if the machine is turned on for the second time in period $i$ after we produce in period $j$, for $j=1, 2, 3$ and $i=j+2, ..., 5$;

$= 0$ otherwise.

The arc associated with this variable connects node $j$ to node $i$ for $j=I, II$ or $III$, and $i=3', 4'$ or $5'$. The cost of this arc is $k_i$.

$y'_{2i} = 1$ if the machine is set up in period $i$ after we produce in period $i-1$ for $i=2, ..., 4$;

$= 0$ otherwise.

The arc associated with this variable connects node $j$ to node $i$ for $j=I, II$ or $III$ and $i=(2'0), (3'0)$ or $(4'0)$. The cost of this arc is $s_i$.

$y_{2i} = 1$ if $y'_{2i} = 0$, we set up the machine in period $i$, and we produce in some period up to $i-1$;

$= 0$ otherwise.
The arc associated with this variable connects node $i$ to node $(i0)$ for $i=3', 4'$ or $5'$. The cost of this arc is $s_i$.

$$w_{2i}=1 \text{ if we produce for the second time in period } i \text{ for } i=2, \ldots, 5;$$

$$=0 \text{ otherwise.}$$

The arc associated with this variable connects node $i$ to the END node, for $i=(2'0), \ldots, (5'0)$. The cost of this arc is $h_i$.

Arcs $((i0), i+1)$ and $((i'0), (i+1)')$ have zero cost and represent slack variables. We do not define any variables for these arcs.

The problem can be formulated as follows:

$$\min \sum_{i=1}^{5} \left( k_i(z_{1i} + \sum_{j=1}^{i-2} z_{2ij}) + s_i(y_{1i} + y'_{2i} + y_{2i}) + h_i(w_{1i} + w_{2i}) \right)$$

subject to

$$\sum_{i=1}^{3} z_{1i} = 1$$

$$(y_{1i} - z_{1i}) \geq 0 \quad i=1, 2, 3.$$  

$$(y_{1i} - w_{1i}) \geq 0 \quad i=1, 2, 3.$$  

$$-w_{1j} + y'_{2,j+1} + \sum_{i=j+2}^{5} z_{2ij} = 0 \quad j=I, II, III.$$  

$$(y_{2i} - \sum_{j=1}^{i-2} z_{2ij}) \geq 0 \quad i=3', 4', 5'.$$  

$$(y_{2i} + y'_{2i} - w_{2i}) \geq 0 \quad i=2', \ldots, 5'.$$  

$$\sum_{i=2}^{5} w_{2i} = 1$$

All variables $\geq 0$. 


We can generalize this to the problem with n demands in periods $t_1, t_2, \ldots, t_n=T$. We define the following variables:

$z_{1i}=1$ if we turn the machine on for the first time in period $i$ for $i=1, \ldots, t_1$;

$=0$ otherwise.

$z_{kij}=1$ if we turn the machine on in period $i$ after we produce for the $(k-1)$th time in period $j$, for $k=2, \ldots, n$; $j=k-1, \ldots, t_{k-1}$; and $i=j+2, \ldots, t_k$;

$=0$ otherwise.

$y'_{ki}=1$ if the machine is set up in period $i$ after we produce for the $(k-1)$th time in period $i-1$, for $k=2, \ldots, n$ and $i=k, \ldots, t_k$;

$=0$ otherwise.

$y_{ki}=1$ if $y'_{ki}=0$ and the machine is set up in period $i$ after we produce $(k-1)$ times up to period $i-1$;

$=0$ otherwise.

$w_{ki}=1$ if we produce for the $k$th time in period $i$ for $k=1, \ldots, n$ and $i=k, \ldots, t_k$;

$=0$ otherwise.

The problem can be formulated as the following shortest path problem:

$$\min \sum_{i=1}^{T} (k_i(z_{1i} + \sum_{k=2}^{n} \sum_{j=k-1}^{t_{k-1}} z_{kij}) + s_i(\sum_{k=1}^{n} (y_{ki} + y'_{ki})) + h_i(\sum_{k=1}^{n} w_{ki}))$$
subject to

\[ \sum_{i=1}^{t_1} z_{1i} = 1 \]

\[ y_{1i} - z_{1i} \geq 0 \quad i=1, \ldots, t_1. \]

\[ y_{1i} - w_{1i} \geq 0 \quad i=1, \ldots, t_1. \]

\[ -w_{k-1,j} + y'_{k,j+1} + \sum_{i=j+2}^{t_k} z_{kij} = 0 \]

\[ y_{ki} - \sum_{j=k-1}^{i-2} z_{kij} \geq 0 \quad k=2, \ldots, n; \quad i=k, \ldots, t_k. \]

\[ y_{ki} + y'_{ki} - w_{ki} \geq 0 \quad k=2, \ldots, n; \quad i=k, \ldots, t_k. \]

\[ \sum_{i=n}^{T} w_{ni} = 1 \]

All variables \( \geq 0. \)

Note that this reformulation is possible only for nonnegative costs.
Appendix 2. A Polyhedral Approach to some Valid Inequalities

A Polyhedral Development

In this section we show how to use integer programming aggregation and rounding arguments to establish the validity of a special case of the (PI) inequalities, namely those in which $c_i = 1$, and $WZ = \Phi$. To establish the validity of the inequality

$$W \leq y_1 + (y_2 + z_2) + \ldots + (y_p + z_p) + z_{p+1}$$

by integer programming methods, we first write the inequalities

$$w_1 \leq y_1$$

$$w_2 \leq z_2 + y_1$$

$$w_3 \leq z_2 + z_3 + y_1$$

$$\ldots$$

$$w_p \leq z_2 + z_3 + \ldots + z_p + y_1$$

$$w_{p+1} \leq z_2 + z_3 + \ldots + z_{p+1} + y_1$$
which are valid because \( y_k \leq z_k + y_{k-1} \leq z_k + z_{k-1} + y_{k-2} \leq \ldots \leq z_k + z_{k-1} + \ldots + z_2 + y_1 \).

Adding the inequalities in (4) gives

\[
W \leq (p+1)y_1 + pz_2 + (p-1)z_3 + \ldots + 2z_p + z_{p+1}.
\]

But since the value of the left-hand side never exceeds \( p \), this inequality remains valid if we reduce the coefficient of \( y_1 \) to \( p \). Therefore, the inequality

\[
W \leq py_1 + pz_2 + (p-1)z_3 + \ldots + 2z_p + z_{p+1} \quad (4.1)
\]

is valid.

We can also establish \((p-1)\) other valid inequalities \((4.2)-(4.p)\). For each \( k, 2 \leq k \leq p \), if we add the valid inequalities

\[
\begin{align*}
  w_1 & \leq y_1 \\
  w_2 & \leq y_2 \\
  w_3 & \leq y_3 \\
  & \quad \ldots \\
  w_p & \leq y_p \\
  w_{p+1} & \leq z_{k+1} + z_{k+2} + \ldots + z_p + z_{p+1} + y_k
\end{align*}
\]

we obtain
\[ W \leq y_1 + y_2 + \ldots + y_p + z_{k+1} + \ldots + z_p + z_{p+1} + y_k. \] (4.k)

Notice that the variable \( z_t \) for any \( 2 \leq t \leq p \) appears in each inequality (4.k) for \( k \leq t-1 \) and it appears in the kth inequality of (3) for each \( k \geq t \). Therefore, adding the inequalities (4.k) for all \( k \geq 2 \) and (4.1) gives

\[
pW \leq (p-1)y_1 + p(y_1 + y_2 + \ldots + y_p) + p(z_2 + \ldots + z_p + z_{p+1})\]

or,

\[
W - [(p-1)/p]y_1 \leq y_1 + (y_2 + z_2) + (y_3 + z_3) + \ldots + (y_p + z_p) + z_{p+1}.
\]

Since the second term to the left of this inequality is strictly less than one and the righthand side is an integer, we can round up the lefthand side to the nearest integer and obtain the desired valid inequality (1).

As before, this inequality implies the (PI) inequalities for \( q=n \). By using disaggregation arguments, we can again extend the argument to cover situations when \( q<n \). We can use a similar approach to derive the general inequality (2).
Bibliography


