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1-factorizations of pseudorandom graphs

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Abstract

A 1-factorization of a graph G is a collection of edge-disjoint perfect matchings whose union is $E(G)$. In this paper, we prove that for any $\epsilon > 0$, an (n, d, λ) -graph G admits a 1-factorization provided that n is even, $C_0 \leq d \leq n - 1$ (where $C_0 = C_0(\epsilon)$ is a constant depending only on ϵ), and $\lambda \leq d^{1-\epsilon}$. In particular, since (as is well known) a typical random d -regular graph $G_{n,d}$ is such a graph, we obtain the existence of a 1-factorization in a typical $G_{n,d}$ for all $C_0 \leq d \leq n - 1$, thereby extending to all possible values of d results obtained by Janson, and independently by Molloy, Robalewska, Robinson, and Wormald for fixed d . Moreover, we also obtain a lower bound for the number of distinct 1-factorizations of such graphs G , which is better by a factor of $2^{nd/2}$ than the previously best known lower bounds, even in the simplest case where G is the complete graph.

1 Introduction

The *chromatic index* of a graph G , denoted by $\chi'(G)$, is the minimum number of colors with which it is possible to color the edges of G in a way such that every color class consists of a matching (that is, no two edges of the same color share a vertex). This parameter is one of the most fundamental and widely studied parameters in graph theory and combinatorial optimization, and in particular, is related to optimal scheduling and resource allocation problems and round-robin tournaments (see, e.g., [15], [28], [29]).

A trivial lower bound on $\chi'(G)$ is $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ denotes the maximum degree of G . Indeed, consider any vertex with maximum degree, and observe that all edges incident to this vertex must have distinct colors. Perhaps surprisingly, a classical theorem of Vizing [38] from the 1960s shows that $\Delta + 1$ colors are always sufficient, and therefore, $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$ holds for all graphs. In particular, this shows that one can partition all graphs into two classes: *Class 1* consists of all graphs G for which $\chi'(G) = \Delta(G)$, and *Class 2* consists of all graphs G for which $\chi'(G) = \Delta(G) + 1$. Moreover, the strategy in Vizing's original proof can be used to obtain a polynomial time algorithm to edge color any graph G with $\Delta(G) + 1$ colors ([31]). However, Holyer [18] showed that it is actually NP-hard to decide whether a given graph G is in Class 1 or 2. In fact, Leven and Galil [26] showed that this is true even if we restrict ourselves to graphs with all the degrees being the same (that is, to *regular graphs*).

Note that for d -regular graphs G (that is, graphs with all their degrees equal to d) on an even number of vertices, the statement ' G is of Class 1' is equivalent to the statement that G contains d edge-disjoint *perfect matchings* (also known as *1-factors*). A graph whose edge set decomposes as a disjoint union of perfect matchings is said to admit a *1-factorization*. Note that if G is a d -regular *bipartite* graph, then a straightforward application of Hall's marriage theorem immediately shows

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that G is of Class 1. Unfortunately, the problem is much harder for non-bipartite graphs, and it is already very interesting to find (efficiently verifiable) sufficient conditions which ensure that $\chi'(G) = \Delta(G)$. This problem is the main focus of our paper.

1.1 Regular expanders are of Class 1

Our main result shows that d -regular graphs on an even number of vertices which are ‘sufficiently good’ spectral expanders, are of Class 1. Before stating our result precisely, we need to introduce some notation and definitions. Given a d -regular graph G on n vertices, let $A(G)$ be its adjacency matrix (that is, $A(G)$ is an $n \times n$, 0/1-valued matrix, with $A(G)_{ij} = 1$ if and only if $ij \in E(G)$). Clearly, $A(G) \cdot \mathbf{1} = d\mathbf{1}$, where $\mathbf{1} \in \mathbb{R}^n$ is the vector with all entries equal to 1, and therefore, d is an eigenvalue of $A(G)$. In fact, as can be easily proven, d is the eigenvalue of $A(G)$ with largest absolute value. Moreover, since $A(G)$ is a symmetric, real-valued matrix, it has n real eigenvalues (counted with multiplicities). Let

$$d = \lambda_1 \geq \lambda_2 \geq \dots \lambda_n \geq -d$$

denote the eigenvalues of $A(G)$, and let $\lambda(G) := \max\{|\lambda_2|, |\lambda_n|\}$. With this notation, we say that G is an (n, d, λ) -graph if G is a d -regular graph on n vertices with $\lambda(G) \leq \lambda$. In recent decades, the study of (n, d, λ) graphs, also known as ‘spectral expanders’, has attracted considerable attention in mathematics and theoretical computer science. An example which is relevant to our problem is that of finding a perfect matching in (n, d, λ) -graphs for which, extending a result of Krivelevich and Sudakov [24], Cioabă, Gregory and Haemers [9] provided accurate spectral conditions for an (n, d, λ) -graph to contain a perfect matching. For much more on these graphs and their many applications, we refer the reader to the survey of Hoory, Linial and Wigderson [19], the survey of Krivelevich and Sudakov [24], and to the book of Brouwer and Haemers [7]. We are now ready to state our main result.

Theorem 1.1. *For every $\varepsilon > 0$ there exist $d_0, n_0 \in \mathbb{N}$ such that for all even integers $n \geq n_0$ and for all $d \geq d_0$ the following holds. Suppose that G is an (n, d, λ) -graph with $\lambda \leq d^{1-\varepsilon}$. Then, $\chi'(G) = d$.*

Remark 1.2. It seems plausible that with a more careful analysis of our proof, one can improve our bound to $\lambda \leq d/\text{poly}(\log d)$. Since we believe that the true bound should be much stronger, we did not see any reason to optimize our bound at the expense of making the paper more technical.

In particular, since the eigenvalues of a matrix can be computed in polynomial time, [Theorem 1.1](#) provides a polynomial time checkable sufficient condition for a graph to be of Class 1. Moreover, our proof gives a probabilistic polynomial time algorithm to actually find an edge coloring of such a G with d colors. Our result can be viewed as implying that ‘sufficiently good’ spectral expanders are easy instances for the NP-complete problem of determining the chromatic index of regular graphs. It is interesting (although, perhaps a bit unrelated) to note that in [3], Arora et al. showed that constraint graphs which are reasonably good spectral expanders are easy for the conjecturally NP-complete Unique Games problem as well.

1.2 Almost all d -regular graphs are of Class 1

The phrase ‘almost all d -regular graphs’ usually splits into two cases: ‘dense’ graphs and random graphs. Let us start with the former.

Dense graphs: It is well known (and quite simple to prove) that every d -regular graph G on n vertices, with $d \geq 2\lceil n/4 \rceil - 1$ has a perfect matching (assuming, of course, that n is even). Moreover, for every $d \leq 2\lceil n/4 \rceil - 2$, it is easily seen that there exist d -regular graphs on an even number of vertices that do not contain even one perfect matching. In a (relatively) recent breakthrough, Csaba, Kühn, Lo, Osthus, and Treglown [11] proved a longstanding conjecture of Dirac from the 1950s, and showed that the above minimum degree condition is tight, not just for containing a single perfect matching, but also for admitting a 1-factorization.

Theorem 1.3 (Theorem 1.1.1 in [11]). *Let n be a sufficiently large even integer, and let $d \geq 2\lceil n/4 \rceil - 1$. Then, every d -regular graph G on n vertices admits a 1-factorization.*

Hence, every ‘sufficiently dense’ regular graph is of Class 1. It is worth mentioning that they actually proved a much more general statement about finding edge-disjoint *Hamilton cycles*, from which the above theorem follows as a corollary.

Random graphs: As noted above, one cannot obtain a statement like [Theorem 1.3](#) for smaller values of d since the graph might not even have a single perfect matching. Therefore, a natural candidate to consider for such values of d is the random d -regular graph, denoted by $G_{n,d}$, which is simply a random variable that outputs a d -regular graph on n vertices, chosen uniformly at random from among all such graphs. The study of this random graph model has received much interest in recent years. Unlike the traditional binomial random graph $G_{n,p}$ (where each edge of the complete graph is included independently, with probability p), the uniform regular model has many dependencies, and is therefore much harder to work with. For a detailed discussion of this model, along with many results and open problems, we refer the reader to the survey of Wormald [40].

Working with this model, Janson [21], and independently, Molloy, Robalewska, Robinson, and Wormald [32], proved that a typical $G_{n,d}$ admits a 1-factorization for all fixed $d \geq 3$, where n is a sufficiently large (depending on d) even integer. Later, Kim and Wormald [23] gave a randomized algorithm to decompose a typical $G_{n,d}$ into $\lfloor \frac{d}{2} \rfloor$ edge-disjoint *Hamilton cycles* (and an additional perfect matching if d is odd) under the same assumption that $d \geq 3$ is fixed, and n is a sufficiently large (depending on d) even integer. For values of d of the form $\Theta(n)$, Kühn and Osthus proved in [25] that every ‘quasi-random’ regular graph has a Hamilton cycle decomposition, and hence, a 1-factorization. Moreover, Glock, Kühn and Osthus [16] also studied optimal edge-colorings in the dense quasi-random case when the underlying graph is not necessarily regular. Usually, the main problem with handling values of d which grow with n is that the so-called ‘configuration model’ (see [4] for more details) is not very useful in this regime.

Here, as an almost immediate corollary of [Theorem 1.1](#), we deduce the following, which together with the results of [21] and [32] shows that a typical $G_{n,d}$ on a sufficiently large even number of vertices admits a 1-factorization for all $3 \leq d \leq n - 1$.

Corollary 1.4. *There exists a universal constant $d_0 \in \mathbb{N}$ such that for all $d_0 \leq d \leq n - 1$, a random d -regular graph $G_{n,d}$ admits a 1-factorization asymptotically almost surely (a.a.s.).*

Remark 1.5. By asymptotically almost surely, we mean with probability going to 1 as n goes to infinity (through even integers). Since a 1-factorization can never exist when n is odd, we will henceforth always assume that n is even, even if we do not explicitly state it.

To deduce [Corollary 1.4](#) from [Theorem 1.1](#), it suffices to show that we have (say) $\lambda(G_{n,d}) = O(d^{0.9})$ a.a.s. In fact, the considerably stronger (and optimal, up to the choice of constant in the big-oh) bound that $\lambda(G_{n,d}) = O(\sqrt{d})$ a.a.s. is known. For $d = o(\sqrt{n})$, this is due to Broder, Frieze, Suen and Upfal [6]. This result was extended to the range $d = O(n^{2/3})$ by Cook, Goldstein, and

Johnson [10] and to all values of d by Tikhomirov and Youssef [36]. We emphasize that the condition on λ we require is significantly weaker and can possibly be deduced from much simpler arguments than the ones in the references above.

Finally, it is also worth mentioning that very recently, Haxell, Krivelevich and Kronenberg [17] studied a related problem in a random multigraph setting; it is interesting to check whether our techniques can be applied there as well.

1.3 Counting 1-factorizations

Once the existence of 1-factorizations in a family of graphs has been established, it is natural to ask for the number of *distinct* 1-factorizations that any member of such a family admits. Having a ‘good’ approximation to the number of 1-factorizations can shed some light on, for example, properties of a ‘randomly selected’ 1-factorization. We remark that the case of counting the number of 1-factors (perfect matchings), even for bipartite graphs, has been the subject of fundamental works over the years, both in combinatorics (e.g., [5], [12], [13], [34]), as well as in theoretical computer science (e.g., [37], [22]), and has led to many interesting results such as both closed-form as well as computational approximation results for the permanent of 0/1 matrices.

As far as the question of counting the number of 1-factorizations is concerned, much less is known. Note that for d -regular bipartite graphs, one can use estimates on the permanent of the adjacency matrix of G to obtain quite tight results. But quite embarrassingly, for non-bipartite graphs (even for the complete graph!) the number of 1-factorizations is unknown. The best known upper bound for the number of 1-factorizations in the complete graph is due to Linial and Luria [27], who showed that it is upper bounded by

$$\left((1 + o(1)) \frac{n}{e^2} \right)^{n^2/2}.$$

Moreover, by following their argument verbatim, one can easily show that the number of 1-factorizations of any d -regular graph is at most

$$\left((1 + o(1)) \frac{d}{e^2} \right)^{dn/2}.$$

On the other hand, the previously best known lower bound for the number of 1-factorizations of the complete graph ([8], [41]) is only

$$\left((1 + o(1)) \frac{n}{4e^2} \right)^{n^2/2},$$

which is off by a factor of $4^{n^2/2}$ from the upper bound.

An advantage of our proof is that it immediately gives a lower bound on the number of 1-factorizations which is better than the one above by a factor of 2 in the base of the exponent, not just for the complete graph, but for all sufficiently good regular spectral expanders with degree greater than some large constant. More precisely, we will show the following (see also the third bullet in Section 7)

Theorem 1.6. *For any $\epsilon > 0$, there exist $D = D(\epsilon), N = N(\epsilon) \in \mathbb{N}$ such that for all even integers $n \geq N(\epsilon)$ and for all $d \geq D(\epsilon)$, the number of 1-factorizations in any (n, d, λ) -graph with $\lambda \leq d^{0.9}$ is at least*

$$\left((1 - \epsilon) \frac{d}{2e^2} \right)^{dn/2}.$$

Remark 1.7. As discussed before, this immediately implies that for all $d \geq D(\epsilon)$, the number of 1-factorizations of $G_{n,d}$ is a.a.s. at least

$$\left((1 - \epsilon) \frac{d}{2e^2} \right)^{dn/2}.$$

1.4 Outline of the proof

It is well known, and easily deduced from Hall's theorem, that any regular bipartite graph admits a 1-factorization (Corollary 2.8). Therefore, if we had a decomposition $E(G) = E(H'_1) \cup \dots \cup E(H'_t) \cup E(\mathcal{F})$, where H'_1, \dots, H'_t are regular balanced bipartite spanning graphs, and \mathcal{F} is a 1-factorization of the regular graph $G \setminus \bigcup_{i=1}^t H'_i$, we would be done. Our proof of Theorem 1.1 will obtain such a decomposition constructively.

As shown in Proposition 5.1, one can find a collection of edge disjoint, regular bipartite spanning graphs H_1, \dots, H_t , where $t \ll d$ and each H_i is r_i regular with $r_i \approx d/t$, which covers 'almost all' of G . In particular, one can find an 'almost' 1-factorization of G . However, it is not clear how to complete an arbitrary such 'almost' 1-factorization to an actual 1-factorization of G . To circumvent this difficulty, we will adopt the following strategy. Note that $G' := G \setminus \bigcup_{i=1}^t H_i$ is a k -regular graph with $k \ll d$, and we can further force k to be even (for instance, by removing a perfect matching from H_1). Therefore, by Petersen's 2-factor theorem (Theorem 2.12), we easily obtain a decomposition $E(G') = E(G'_1) \cup \dots \cup E(G'_t)$, where each G'_i is approximately k/t regular. The key ingredient of our proof (Proposition 4.2) then shows that the H_i 's can initially be chosen in such a way that each $R_i := H_i \cup G'_i$ can be edge decomposed into a regular bipartite spanning graph, and a relatively small number of 1-factors.

The basic idea in this step is quite simple. Observe that while the regular graph R_i is not bipartite, it is 'close' to being one, in the sense that most of its edges come from the regular balanced bipartite spanning graph $H_i = (A_i \cup B_i, E_i)$. Let $R_i[A_i]$ denote the graph induced by R_i on the vertex set A_i , and similarly for B_i , and note that the number of edges $e(R_i[A_i]) = e(R_i[B_i])$. We will show that H_i can be taken to have a certain 'goodness' property (Definition 4.1) which, along with the sparsity of G'_i , enables one to perform the following process to 'absorb' the edges in $R_i[A_i]$ and $R_i[B_i]$: decompose $R_i[A_i]$ and $R_i[B_i]$ into the same number of matchings, with corresponding matchings of equal size, and complete each such pair of matchings to a perfect matching of R_i . After removing all the perfect matchings of R_i obtained in this manner, we are clearly left with a regular balanced bipartite spanning graph, as desired.

Finally, for the lower bound on the number of 1-factorizations, we show that there are many ways of performing such an edge decomposition $E(G) = E(H'_1) \cup \dots \cup E(H'_t) \cup E(\mathcal{F})$ (Remark 5.3), and there are many 1-factorizations corresponding to each choice of edge decomposition (Remark 4.3).

1.5 Notation

We use standard graph theory notation. For a graph G , we use $V(G)$ to denote the set of its vertices and $E(G)$ to denote the set of its edges. Often, we will write $G = (V(G), E(G))$. We will use $\delta(G)$ to denote the minimum degree of any vertex in G , and $\Delta(G)$ to denote the maximum degree of any vertex in G . For a vertex $v \in V(G)$, we denote the degree of v in G by $\deg_G(v)$. For a regular graph G , we will use $\deg(G)$ to denote the common degree of all of its vertices. For disjoint subsets X and Y of $V(G)$, we use $E_G(X, Y)$ to denote the collection of edges with one endpoint in X and the other endpoint in Y , and $e_G(X, Y)$ to denote $|E_G(X, Y)|$. Given a subset X of $V(G)$, we let $G[X]$ denote the graph induced by G on the vertex set X , and given disjoint subsets A and B of $V(G)$, we let $G[A, B]$ denote the bipartite graph with parts A and B and edge set $E_G(A, B)$. Given a

subset E' of $E(G)$, we let $G[E']$ denote the graph induced by the edges in E' . We will often denote a bipartite graph G with parts A and B and edge set E by $G = (A \cup B, E)$. We will use $\binom{V}{k}$ to denote the collection of subsets of a set V of size k . In particular, $\binom{V}{2}$ will denote subsets of V of size 2, and we will often denote such subsets by xy instead of $\{x, y\}$. Finally, we refer to the set $\{1, 2, \dots, n\}$ by $[n]$, and say that $c \in a \pm b$ or $c = a \pm b$ if $c \in [a - b, a + b]$.

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2 Tools and auxiliary results

In this section we have collected a number of tools and auxiliary results to be used in proving our main theorem.

2.1 Probabilistic tools

Throughout the paper, we will make extensive use of the following well-known concentration bounds due to Chernoff (see, e.g., Theorem 2.8 in [20]).

Lemma 2.1 (Chernoff's bounds). *Let $X := \sum_{i=1}^n X_i$, where $X_i \sim \text{Ber}(p_i)$ are independent, and let $\mathbb{E}(X) = \mu$. Then*

- $\Pr[X < (1 - a)\mu] < e^{-a^2\mu/2}$ for every $a > 0$;
- $\Pr[X > (1 + a)\mu] < e^{-a^2\mu/3}$ for every $0 < a < 3/2$.

Remark 2.2. These bounds also hold when X is hypergeometrically distributed with mean μ (see, e.g., Theorem 2.10 in [20]).

Before introducing the next tool to be used, we need the following definition.

Definition 2.3. Let $(A_i)_{i=1}^n$ be a collection of events in some probability space. A graph D on the vertex set $[n]$ is called a *dependency graph* for $(A_i)_i$ if A_i is mutually independent of all the events $\{A_j : ij \notin E(D)\}$.

The following is the so called Lovász local lemma, in its symmetric version (see, e.g., [1]).

Lemma 2.4 (Local lemma). *Let $(A_i)_{i=1}^n$ be a sequence of events in some probability space, and let D be a dependency graph for $(A_i)_i$. Let $\Delta := \Delta(D)$ and suppose that for every i we have $\Pr[A_i] \leq q$, such that $eq(\Delta + 1) < 1$. Then, $\Pr[\bigcap_{i=1}^n \bar{A}_i] \geq \left(1 - \frac{1}{\Delta+1}\right)^n$.*

We will also make use of the following asymmetric version of the Lovász local lemma (see, e.g., [1]).

Lemma 2.5 (Asymmetric local lemma). *Let $(A_i)_{i=1}^n$ be a sequence of events in some probability space. Suppose that D is a dependency graph for $(A_i)_i$, and suppose that there are real numbers $(x_i)_{i=1}^n$, such that $0 \leq x_i < 1$ and*

$$\Pr[A_i] \leq x_i \prod_{ij \in E(D)} (1 - x_j)$$

for all $1 \leq i \leq n$. Then, $\Pr[\bigcap_{i=1}^n \bar{A}_i] \geq \prod_{i=1}^n (1 - x_i)$.

2.2 Perfect matchings in bipartite graphs

Here, we present a number of results related to perfect matchings in bipartite graphs. The first result is a slight reformulation of the classic marriage theorem due to Hall (see, e.g., [35]).

Theorem 2.6. *Let $G = (A \cup B, E)$ be a balanced bipartite graph with $|A| = |B| = k$. Suppose $|N(X)| \geq |X|$ for all subsets X of size at most $k/2$ which are completely contained either in A or in B . Then, G contains a perfect matching.*

Moreover, we can always find a maximum matching in a bipartite graph in polynomial time using standard network flow algorithms (see, e.g., [39]).

The following simple corollaries of Hall's theorem will be useful for us.

Corollary 2.7. *Every r -regular balanced bipartite graph has a perfect matching, provided that $r \geq 1$.*

Proof. Let $G = (A \cup B, E)$ be an r -regular graph. Let $X \subseteq A$ be a set of size at most $|A|/2$. Note that as G is r -regular, we have

$$e_G(X, N(X)) = r|X|.$$

Since each vertex in $N(X)$ has degree at most r into X , we get

$$|N(X)| \geq e_G(X, N(X))/r \geq |X|.$$

Similarly, for every $X \subseteq B$ of size at most $|B|/2$ we obtain

$$|N(X)| \geq |X|.$$

Therefore, by [Theorem 2.6](#), we conclude that G contains a perfect matching. \square

Since removing an arbitrary perfect matching from a regular balanced bipartite graph leads to another regular balanced bipartite graph, a simple repeated application of [Corollary 2.7](#) shows the following:

Corollary 2.8. *Every regular balanced bipartite graph has a 1-factorization.*

In fact, as the following theorem due to Schrijver [34] shows, a regular balanced bipartite graph has many 1-factorizations.

Theorem 2.9. *The number of 1-factorizations of a d -regular bipartite graph with $2k$ vertices is at least*

$$\left(\frac{d!^2}{d^d}\right)^k.$$

The next result is a criterion for the existence of r -factors (that is, r -regular, spanning subgraphs) in bipartite graphs, which follows from a generalization of the Gale-Ryser theorem due to Mirsky [30].

Theorem 2.10. *Let $G = (A \cup B, E)$ be a balanced bipartite graph with $|A| = |B| = m$, and let r be an integer. Then, G contains an r -factor if and only if for all $X \subseteq A$ and $Y \subseteq B$*

$$e_G(X, Y) \geq r(|X| + |Y| - m).$$

Moreover, such factors can be found efficiently using standard network flow algorithms (see, e.g., [2]).

As we are going to work with pseudorandom graphs, it will be convenient for us to isolate some ‘nice’ properties that, together with [Theorem 2.10](#), ensure the existence of large factors in balanced bipartite graphs.

Lemma 2.11. *Let $G = (A \cup B, E)$ be a balanced bipartite graph with $|A| = |B| = n/2$. Suppose there exist $r, \varphi \in \mathbb{R}^+$ and $\beta_1, \beta_2, \beta_3, \gamma \in (0, 1)$ satisfying the following additional properties:*

(P1) $\deg_G(v) \geq r(1 - \beta_1)$ for all $v \in A \cup B$.

(P2) $e_G(X, Y) < r\beta_2|X|$ for all $X \subseteq A$ and $Y \subseteq B$ with $|X| = |Y| \leq r/\varphi$.

(P3) $e_G(X, Y) \geq 2r(1 - \beta_3)|X||Y|/n$ for all $X \subseteq A$ and $Y \subseteq B$ with $|X| + |Y| > n/2$ and $\min\{|X|, |Y|\} > r/\varphi$.

(P4) $\gamma \geq \max\{\beta_3, \beta_1 + \beta_2\}$

Then, G contains an $\lfloor r(1 - \gamma) \rfloor$ -factor.

Proof. By [Theorem 2.10](#), it suffices to verify that for all $X \subseteq A$ and $Y \subseteq B$ we have

$$e_G(X, Y) \geq r(1 - \gamma) \left(|X| + |Y| - \frac{n}{2} \right).$$

We divide the analysis into four cases:

Case 1 $|X| + |Y| \leq n/2$. In this case, we trivially have

$$e_G(X, Y) \geq 0 \geq r(1 - \gamma) \left(|X| + |Y| - \frac{n}{2} \right),$$

so there is nothing to prove.

Case 2 $|X| + |Y| > n/2$ and $|X| \leq r/\varphi$. Since $|Y| \leq |B| = n/2$, we always have $|X| + |Y| - \frac{n}{2} \leq |X|$. Thus, it suffices to verify that

$$e_G(X, Y) \geq r(1 - \gamma)|X|.$$

Assume, for the sake of contradiction, that this is not the case. Then, since there are at least $r(1 - \beta_1)|X|$ edges incident to X , we must have

$$e_G(X, B \setminus Y) \geq r(1 - \beta_1)|X| - e_G(X, Y) \geq r(\gamma - \beta_1)|X| \geq r\beta_2|X|.$$

However, since $|B \setminus Y| \leq |X|$, this contradicts (P2).

Case 3 $|X| + |Y| > n/2$ and $|Y| \leq r/\varphi$. This is exactly the same as the previous case with the roles of X and Y interchanged.

Case 4 $|X| + |Y| > n/2$ and $|X|, |Y| > r/\varphi$. By (P3), it suffices to verify that

$$2r(1 - \beta_3)|X||Y|/n \geq r(1 - \gamma) (|X| + |Y| - n/2).$$

Dividing both sides by $rn/2$, the above inequality is implied by the inequality

$$xy - \frac{(1 - \gamma)}{(1 - \beta_3)}(x + y - 1) \geq 0,$$

where $x = 2|X|/n$, $y = 2|Y|/n$, $x + y \geq 1$, $0 \leq x \leq 1$, and $0 \leq y \leq 1$.

Since $\frac{1-\gamma}{1-\beta_3} \leq 1$ by (P4), this is readily verified on the (triangular) boundary of the region, on which the inequality reduces to one of the following: $xy \geq 0$; $x \geq \frac{1-\gamma}{1-\beta_3}x$; $y \geq \frac{1-\gamma}{1-\beta_3}y$. On the other hand, the only critical point in the interior of the region is possibly at $x_0 = y_0 = \frac{1-\gamma}{1-\beta_3}$, for which we have $x_0y_0 - \frac{1-\gamma}{1-\beta_3}(x_0 + y_0 - 1) = \frac{1-\gamma}{1-\beta_3} \left(1 - \frac{1-\gamma}{1-\beta_3}\right) \geq 0$, again by (P4). \square

2.3 Matchings in graphs with controlled degrees

In this section, we collect a couple of results on matchings in (not necessarily bipartite) graphs satisfying some degree conditions. A *2-factorization* of a graph is a decomposition of its edges into 2-factors (that is, a collection of vertex disjoint cycles that covers all the vertices). The following theorem, due to Petersen [33], is one of the earliest results in graph theory.

Theorem 2.12 (2-factor Theorem). *Every $2k$ -regular graph with $k \geq 1$ admits a 2-factorization.*

The next theorem, due to Vizing [38], shows that every graph G admits a proper edge coloring using at most $\Delta(G) + 1$ colors.

Theorem 2.13 (Vizing’s Theorem). *Every graph with maximum degree Δ can be properly edge-colored with $k \in \{\Delta, \Delta + 1\}$ colors.*

2.4 Expander mixing lemma

When dealing with (n, d, λ) graphs, we will repeatedly use the following lemma (see, e.g., [19]), which bounds the difference between the actual number of edges between two sets of vertices, and the number of edges we expect based on the sizes of the sets.

Lemma 2.14 (Expander mixing lemma). *Let $G = (V, E)$ be an (n, d, λ) graph, and let $S, T \subseteq V$. Let $e(S, T) = |\{(x, y) \in S \times T : xy \in E\}|$. Then,*

$$\left| e(S, T) - \frac{d|S||T|}{n} \right| \leq \lambda \sqrt{|S||T|}.$$

3 Random partitioning

While we have quite a few easy-to-use tools for working with balanced bipartite graphs, the graph we start with is not necessarily bipartite (when the starting graph *is* bipartite, the existence problem is easy (see Corollary 2.8), and the counting problem is solved by [34] (see Theorem 2.9)). Therefore, perhaps the most natural thing to do is to partition the edges into ‘many’ balanced bipartite graphs, where each piece has suitable expansion and regularity properties. The following lemma is our first step towards obtaining such a partition.

Lemma 3.1. *Fix $a \in (0, 1)$, and let $G = (V, E)$ be a d -regular graph on n vertices, where d is a sufficiently large integer, and n is a sufficiently large even integer. Then, for every integer $t \in [d^{a/100}, d^{1/10}]$, there exists a collection $(A_i, B_i)_{i=1}^t$ of balanced bipartitions for which the following properties hold:*

(R1) *Let G_i be the subgraph of G induced by $E_G(A_i, B_i)$. For all $1 \leq i \leq t$ we have*

$$\frac{d}{2} - d^{2/3} \leq \delta(G_i) \leq \Delta(G_i) \leq \frac{d}{2} + d^{2/3}.$$

(R2) For all $e \in E(G)$, the number of indices i for which $e \in E(G_i)$ is $\frac{t}{2} \pm t^{2/3}$.

We will divide the proof into two cases – the *dense case*, where $\log^{1000/a} n \leq d \leq n - 1$, and the *sparse case*, where $d \leq \log^{1000/a} n$. The underlying idea is similar in both cases, but the proof in the sparse case is technically more involved as a standard use of Chernoff’s bounds and the union bound does not work (and therefore, we will instead use the local lemma).

Proof in the dense case. Let A_1, \dots, A_t be random subsets chosen independently from the uniform distribution on all subsets of V of size exactly $n/2$, and let $B_i = V \setminus A_i$ for all $1 \leq i \leq t$. We will show that with high probability, for every $1 \leq i \leq t$, (A_i, B_i) is a balanced bipartition satisfying (R1) and (R2).

First, note that for any $e \in E(G)$ and any $i \in [t]$,

$$\Pr[e \in E(G_i)] = \frac{1}{2} \left(1 + \frac{1}{n-1} \right).$$

Therefore, if for all $e \in E(G)$ we let $C(e)$ denote the set of indices i for which $e \in E(G_i)$, then

$$\mathbb{E}[|C(e)|] = \frac{t}{2} \left(1 + \frac{1}{n-1} \right).$$

Next, note that, for a fixed $e \in E(G)$, the events $A_i := ‘i \in C(e)’$ are mutually independent, and that $|C(e)| = \sum_i X_i$, where X_i is the indicator random variable for the event A_i . Therefore, by Chernoff’s bounds (Lemma 2.1), it follows that

$$\Pr \left[|C(e)| \notin \frac{t}{2} \pm t^{2/3} \right] \leq \exp \left(-\frac{t^{1/3}}{10} \right) \leq \frac{1}{n^3}.$$

Now, by applying the union bound over all $e \in E(G)$, it follows that the collection $(A_i, B_i)_{i=1}^t$ satisfies (R2) with probability at least $1 - 1/n$. Similarly, it is immediate from Chernoff’s bounds (Lemma 2.1) for the hypergeometric distribution that for any $v \in V$ and $i \in [t]$,

$$\Pr \left[d_{G_i}(v) \notin \frac{d}{2} \pm d^{2/3} \right] \leq \exp \left(-\frac{d^{1/3}}{10} \right) \leq \frac{1}{n^3},$$

and by taking the union bound over all such i and v , it follows that (R1) holds with probability at least $1 - 1/n$. All in all, with probability at least $1 - 2/n$, both (R1) and (R2) hold. This completes the proof. \square

Proof in the sparse case. Instead of using the union bound as in the dense case, we will use the symmetric version of the local lemma (Lemma 2.4). Note that there is a small obstacle with choosing *balanced* bipartitions, as the local lemma is most convenient to work with when the underlying experiment is based on independent trials. In order to overcome this issue, we start by defining an auxiliary graph $G' = (V, E')$ as follows: for all $xy \in \binom{V}{2}$, $xy \in E'$ if and only if $xy \notin E$ and there is no vertex $v \in V(G)$ with $\{x, y\} \subseteq N_G(v)$. In other words, there is an edge between x and y in G' if and only if x and y are not connected to each other, and do not have any common neighbors in G . Since for any $x \in V$, there are at most d^2 many $y \in V$ such that $xy \in E$ or x and y have a common neighbor in G , it follows that $\delta(G') \geq n - d^2 \geq n/2$ for n sufficiently large. An immediate application of Dirac’s theorem shows that any graph on $2k$ vertices with minimum degree at least k contains a perfect matching. Therefore, G' contains a perfect matching.

Let $s = n/2$ and let $M := \{x_1y_1, \dots, x_sy_s\}$ be an arbitrary perfect matching of G' . For each $i \in [t]$ let f_i be a random function chosen independently and uniformly from the set of all functions from $\{x_1, \dots, x_s\}$ to $\{\pm 1\}$. These functions will define the partitions according to the vertex labels as follows:

$$A_i := \{x_j \mid f_i(x_j) = -1\} \cup \{y_j \mid f_i(x_j) = +1\},$$

and

$$B_i := V \setminus A_i.$$

Clearly, $(A_i, B_i)_{i=1}^t$ is a random collection of balanced bipartitions of V . If, for all $i \in [t]$, we let $g_i: V(G) \rightarrow \{A, B\}$ denote the random function recording which of A_i or B_i a given vertex ends up in, it is clear – and this is the point of using G' – that for any $i \in [t]$ and any $v \in V(G)$, the choices $\{g_i(w)\}_{w \in N_G(v)}$ are mutually independent. This will help us in showing that, with positive probability, this collection of bipartitions satisfies properties (R1) and (R2).

Indeed, for all $v \in V(G)$, $i \in [t]$, and $e \in E(G)$, let $D_{i,v}$ denote the event that $d_{G_i}(v) \notin \frac{d}{2} \pm d^{2/3}$, and let A_e denote the event $|C(e)| \notin \frac{t}{2} \pm t^{2/3}$. Then, using the aforementioned mutual independence of $\{g_i(w)\}_{w \in N_G(v)}$ and Chernoff's bounds (Lemma 2.1), we have that

$$\Pr[D_{i,v}] \leq \exp\left(-d^{1/3}/4\right).$$

Moreover, using the independence of $g_i(v)$ and $g_i(w)$ for any $w \in N_G(v)$, the mutual independence of $\{f_i\}_{i \in [t]}$, and Chernoff's bounds (Lemma 2.1) shows that

$$\Pr[A_e] \leq \exp\left(-t^{1/3}/4\right).$$

In order to complete the proof, we need to show that one can apply the symmetric local lemma (Lemma 2.4) to the collection of events consisting of all the $D_{i,v}$'s and all the A_e 's. To this end, we first need to upper bound the number of events which depend on any given event.

Note that $D_{i,v}$ depends on $D_{j,u}$ only if $i = j$ and either $\text{dist}_G(u, v) \leq 2$ or $uv \in M$. Note also that $D_{i,v}$ depends on A_e only if an endpoint of e is within distance 1 of v either in G or in M . Therefore, it follows that any $D_{i,v}$ can depend on at most $3d^2$ events in the collection. Since A_e can depend on $A_{e'}$ only if e and e' share an endpoint in G or if any of the endpoints of e are matched to any of the endpoints of e' in M , it follows that we can take the maximum degree of the dependency graph in Lemma 2.4 to be $3d^2$. Since $3d^2 \exp(-t^{1/3}/4) = o_d(1)$, we are done. \square

4 Completion

In this section, we describe the key ingredient of our proof, namely the completion step. Before stating the relevant lemma, we need the following definition.

Definition 4.1. A graph $H = (A \cup B, E)$ is called (α, r, m) -good if it satisfies the following properties:

- (G1) H is an r -regular, balanced bipartite graph with $|A| = |B| = m$.
- (G2) Every balanced bipartite subgraph $H' = (A' \cup B', E')$ of H with $|A'| = |B'| \geq (1 - \alpha)m$ and with $\delta(H') \geq (1 - 2\alpha)r$ contains a perfect matching.

The motivation for this definition comes from the next proposition, which shows that a regular graph on an even number of vertices, which can be decomposed as a union of a good graph and a sufficiently sparse graph, has a 1-factorization.

Proposition 4.2. *For every $\alpha \leq 1/10$, there exists an integer r_0 such that for all $r_1 \geq r_0$ and m a sufficiently large integer, the following holds. Suppose that $H = (A \cup B, E(H))$ is an (α, r_1, m) -good graph. Then, for every $1 \leq r_2 \leq \alpha^5 r_1 / \log r_1$, every $r := r_1 + r_2$ -regular (not necessarily bipartite) graph R on the vertex set $A \cup B$, for which $H \subseteq R$, admits a 1-factorization.*

Proof. Let C be any positive integer and let $k = \lfloor 1/\alpha^4 \rfloor$. We begin by showing that any matching M in $R[X]$ for $X \in \{A, B\}$ with $|M| = C$ can be partitioned into k matchings M_1, \dots, M_k (some of which may be empty) such that no vertex $v \in V(H)$ is incident (in H) to more than αr_1 vertices in $\cup M_i$ for any $i \in [k]$. If $C < \alpha r_1 / 2$, then there is nothing to show. If $C \geq \alpha r_1 / 2$, consider an arbitrary partition of M into $\lceil C/k \rceil$ sets $S_1, \dots, S_{\lceil C/k \rceil}$ with each set (except possibly the last one) of size k . For each S_j , $j \in \lceil C/k \rceil$, choose a permutation of $[|S_j|]$ independently and uniformly at random, and let M_i denote the random subset of M consisting of all elements of $S_1, \dots, S_{\lceil C/k \rceil}$ which are assigned the label i . We will show, using the symmetric version of the local lemma (Lemma 2.4), that the decomposition M_1, \dots, M_k satisfies the desired property with a positive probability.

To this end, note that for any vertex v to have at least αr_1 neighbors in some M_i , it must be the case that the r_1 neighbors of v in H are spread throughout at least αr_1 distinct S_j 's. Let D_v denote the event that v has at least αr_1 neighbors in some matching M_i . Since v has at least \sqrt{k} neighbors in at most $r_1/\sqrt{k} \ll \alpha r_1$ distinct S_j 's, it follows that $\Pr[D_v] \leq k \binom{r_1}{\alpha r_1/5} (1/\sqrt{k})^{\alpha r_1/5} \ll k(1/\sqrt{k})^{\alpha r_1/10}$. Finally, since each D_v depends on at most $r^2 k$ many other D_w 's, and since $k^2 r^2 (1/\sqrt{k})^{\alpha r_1/10} < 1/e$, we are done.

Now, observe that $e(R[A]) = e(R[B])$. Indeed, as R is r -regular, we have for $X \in \{A, B\}$ that

$$rm = \sum_{v \in X} d_R(v) = 2e(R[X]) + e(R[A, B]),$$

from which the above equality follows. Moreover, $\Delta(R[X]) \leq r_2$ for all $X \in \{A, B\}$. Therefore, by Vizing's theorem, we can decompose $R[A]$ and $R[B]$ into exactly $r_2 + 1$ matchings $N_1^A, \dots, N_{r_2+1}^A$ and $N_1^B, \dots, N_{r_2+1}^B$, and it is readily seen that these matchings can be used to decompose $R[A]$ and $R[B]$ into $\ell \leq 2(r_2 + 1)$ matchings M_1^A, \dots, M_ℓ^A and M_1^B, \dots, M_ℓ^B such that $|M_i^A| = |M_i^B|$ for all $i \in [\ell]$. Indeed, consider the collection of $2r_2 + 2$ matchings $N_1^A, \dots, N_{r_2+1}^A, N_1^B, \dots, N_{r_2+1}^B$, and suppose (without loss of generality) that $|N_1^A| \geq \dots \geq |N_{r_2+1}^A|$, $|N_1^B| \geq \dots \geq |N_{r_2+1}^B|$, and $|N_1^A| \geq |N_1^B|$. Crucially, since $e(R[A]) = e(R[B])$, we have $|N_1^A| + \dots + |N_{r_2+1}^A| = |N_1^B| + \dots + |N_{r_2+1}^B|$. Let $M_1^B := N_1^B$, M_1^A be an arbitrary collection of $|N_1^B|$ edges of N_1^A , and $\bar{N}_1^A := N_1^A \setminus M_1^A$. Then, the collection of matchings $\bar{N}_1^A, N_2^A, \dots, N_{r_2+1}^A, N_2^B, \dots, N_{r_2+1}^B$ still satisfies the crucial condition $|\bar{N}_1^A| + |N_2^A| + \dots + |N_{r_2+1}^A| = |N_2^B| + \dots + |N_{r_2+1}^B|$, and the total number of matchings in this collection is at most $2r_2 + 1$, so that we may proceed by induction.

Next, by using the argument in the first paragraph of this proof, we can further decompose each $|M_i^X|$, $X \in \{A, B\}$ into at most k matchings each in order to obtain a collection of edge-disjoint matchings $M_1^A, \dots, M_s^A, M_1^B, \dots, M_s^B$ such that $\cup_{i=1}^s M_i^X = R[X]$ for $X \in \{A, B\}$, $s \leq 2k(r_2 + 1)$, $|M_i^A| = |M_i^B| \leq \alpha m / 10$ for all $i \in [s]$, and no vertex $v \in V(H)$ is incident (in H) to more than αr_1 vertices involved in any of the M_i^X .

Consider the following iterative procedure. Let $R_1 := R$, $H_1 := H$, delete the vertices in $\cup M_1^A \cup \cup M_1^B$, as well as any edges incident to them, from H_1 , and denote the resulting graph by $H'_1 = (A'_1 \cup B'_1, E'_1)$. Since $|A'_1| = |B'_1| \geq (1 - \alpha)|A|$ and $\delta(H'_1) \geq (1 - \alpha)r_1$ by the choice of M_1^A, M_1^B , it follows from (G2) that H'_1 contains a perfect matching M'_1 . Note that $M_1 := M_1^A \cup M_1^B$ is a perfect matching in R_1 . Next, set $R_2 := R_1 - M_1$ (deleting only the edges in M_1 , and not the vertices), $H_2 := H_1 - M'_1$ (deleting only the edges in M'_1 , and not the vertices), and repeat the above process.

In order to do this for s steps, we need to verify that (G2) can be applied to the resulting graphs H'_j for all $j \in [s]$. The condition $|A'_j| = |B'_j| \geq (1 - \alpha)|A|$ follows as before from $|M'_j{}^A| = |M'_j{}^B| \leq \alpha m/10$. Moreover, for all $j \in [s]$,

$$\delta(H'_j) \geq \delta(H) - \alpha r_1 - s \geq (1 - \alpha)r_1 - 2k(r_2 + 1) \geq (1 - \alpha)r_1 - 4\alpha^{-4}r_2 \geq (1 - 2\alpha)r_1,$$

where the final inequality follows from the assumption that $4r_2/\alpha^4 \ll \alpha r_1$. Hence, we can indeed apply (G2) for s steps.

Finally, after removing all the perfect matchings obtained via this procedure, we are left with a regular, balanced, *bipartite* graph, and therefore it admits a 1-factorization (Corollary 2.8). Taking any such 1-factorization along with all the perfect matchings that we removed gives a 1-factorization of R . \square

Remark 4.3. In the last step of the proof, we are allowed to choose an arbitrary 1-factorization of an r' -regular, balanced bipartite graph, where $r' \geq r_1 - r_2$. Therefore, using Theorem 2.9 along with the standard inequality $d! \geq (d/e)^d$, it follows that R admits at least $((r_1 - r_2)/e^2)^{(r_1 - r_2)m}$ 1-factorizations.

5 Finding good subgraphs which almost cover G

In this section we present a structural result which shows that a ‘good’ regular expander on an even number of vertices can be ‘almost’ covered by a union of edge disjoint good subgraphs.

Proposition 5.1. *For every $c \in (0, 1)$, there exists d_0 such that for all $d \geq d_0$ the following holds. Let $G = (V, E)$ be an (n, d, λ) -graph with $\lambda < d/4t^4$ where t is an integer in $[d^{c/100}, d^{c/10}]$. Then, G contains t distinct, edge disjoint $(\alpha, \bar{r}, \frac{n}{2})$ -good subgraphs W_1, \dots, W_t with $\alpha = \frac{1}{10}$ and $\bar{r} = \left\lfloor \frac{d}{t} \left(1 - \frac{16}{t^{1/3}}\right) \right\rfloor$.*

The proof of this proposition is based on the following technical lemma, which lets us apply Lemma 2.11 to each part of the partitioning coming from Lemma 3.1 in order to find large good factors.

Lemma 5.2. *For every $c \in (0, 1)$, there exists d_0 such that for all $d \geq d_0$ the following holds. Let $G = (V, E)$ be an (n, d, λ) -graph with $\lambda < d/4t^4$ where t is an integer in $[d^{c/100}, d^{c/10}]$, and let $\alpha = 1/10$. Then, there exists an edge partitioning $E(G) = E_1 \cup \dots \cup E_t$ for which the following properties hold:*

(S1) $H_i := G[E_i]$ is a balanced bipartite graph with parts (A_i, B_i) for all $i \in [t]$.

(S2) For all $i \in [t]$ and for all $X \subseteq A_i, Y \subseteq B_i$ with $|X| = |Y| \leq n/2t^2$ we have

$$e_{H_i}(X, Y) < d|X|/t^2.$$

(S3) For all $i \in [t]$ and all $X \subseteq A_i, Y \subseteq B_i$ with $|X| + |Y| > n/2$ and $\min\{|X|, |Y|\} > \frac{n}{2t^2}$,
$$e_{H_i}(X, Y) \geq 2\frac{d}{t} \left(1 - \frac{8}{t^{1/3}}\right) \frac{|X||Y|}{n}.$$

(S4) $d_{H_i}(v) \in \frac{d}{t} \pm \frac{8d}{t^{4/3}}$ for all $i \in [t]$ and all $v \in V(H_i) = V(G)$.

(S5) $e_{H_i}(X, Y) \leq (1 - 4\alpha)\frac{d}{t}|X|$ for all $X, Y \subseteq V(H_i)$ with $\frac{n}{2t^2} \leq |X| = |Y| \leq \frac{n}{4}$, and for all $i \in [t]$.

Before proving this lemma, let us show how it can be used to prove [Proposition 5.1](#).

Proof of Proposition 5.1. Note that each balanced bipartite graph H_1, \dots, H_t coming from [Lemma 5.2](#) satisfies the hypotheses of [Lemma 2.11](#) with

$$r = \frac{d}{t}, \varphi = \frac{2dt}{n}, \beta_1 = \beta_2 = \beta_3 = \frac{8}{t^{1/3}}, \gamma = \frac{16}{t^{1/3}}.$$

Indeed, (P1) follows from (S4), (P2) follows from (S2), (P3) follows from (S3) and (P4) is satisfied by the choice of parameters. Therefore, [Lemma 2.11](#) guarantees that each H_i contains an \bar{r} -factor, and by construction, these are edge disjoint.

Now, let W_1, \dots, W_t be any \bar{r} -factors of H_1, \dots, H_t . It remains to check that W_1, \dots, W_t satisfy property (G2). We will actually show the stronger statement that H_1, \dots, H_t satisfy (G2). Indeed, let $H'_i = (A'_i \cup B'_i, E'_i)$ be a subgraph of H_i with $A'_i \subseteq A_i, B'_i \subseteq B_i$ such that

$$|A'_i| = |B'_i| \geq (1 - \alpha)n/2$$

and

$$\delta(H'_i) \geq (1 - 2\alpha)\bar{r}.$$

Suppose H'_i does not contain a perfect matching. Then, by [Theorem 2.6](#), without loss of generality, there must exist $X \subseteq A'_i$ and $Y \subseteq B'_i$ such that

$$|X| = |Y| \leq |A'_i|/2 \leq n/4$$

and

$$N_{H'_i}(X) \subseteq Y.$$

In particular, by the minimum degree assumption, it follows that

$$e_{H'_i}(X, Y) \geq (1 - 2\alpha)\bar{r}|X|.$$

On the other hand, by [Lemma 2.14](#), we know that for such a pair,

$$e_{H'_i}(X, Y) \leq e_G(X, Y) \leq d|X|^2/n + \lambda|X|.$$

Thus, since $\lambda < d/4t^4$ by assumption, we must necessarily have that $|X| \geq n/2t^2$, which contradicts (S5). This completes the proof. \square

Proof of Lemma 5.2. Our construction will be probabilistic. We begin by applying [Lemma 3.1](#) to G to obtain a collection of balanced bipartitions $(A_i, B_i)_{i=1}^t$ satisfying Properties (R1) and (R2) of [Lemma 3.1](#), with c playing the role of a . Let $G_i := G[A_i, B_i]$, and for each $e \in E(G)$, let $C(e)$ denote the set of indices $i \in [t]$ for which $e \in E(G_i)$. Let $\{c(e)\}_{e \in E(G)}$ denote a random collection of elements of $[t]$, where each $c(e)$ is chosen independently and uniformly at random from $C(e)$. Let H_i be the (random) subgraph of G_i obtained by keeping all the edges $e \in E(G_i)$ with $c(e) = i$. Then, the H_i 's always form an edge partitioning of $E(G)$ into t balanced bipartite graphs with parts $(A_i, B_i)_{i=1}^t$.

It is easy to see that these H_i 's will always satisfy (S2). Indeed, if for any $X, Y \subseteq V(G)$ with $|X| = |Y|$, we have $e_{H_i}(X, Y) \geq d|X|/t^2$, then since

$$e_{H_i}(X, Y) \leq e_G(X, Y) \leq \frac{d|X|^2}{n} + \lambda|X|$$

by the expander mixing lemma (Lemma 2.14), it follows that

$$\frac{d}{t^2} \leq \frac{d|X|}{n} + \lambda,$$

and therefore, since $\lambda < d/4t^4$, we must have

$$|X| > \frac{n}{2t^2}.$$

We now provide a lower bound on the probability with which this partitioning also satisfies (S3) and (S4). To this end, we first define the following events:

- For all $v \in V(G)$ and $i \in [t]$, let $D_{i,v}$ denote the event that $d_{H_i}(v) \notin \frac{d}{t} \pm \frac{8d}{t^{4/3}}$.
- For all $i \in [t]$ and all $X \subseteq A_i, Y \subseteq B_i$ with $|X| + |Y| > n/2$ and $\min\{|X|, |Y|\} > \frac{n}{2t^2}$, let $A(i, X, Y)$ denote the event that $e_{H_i}(X, Y) \leq 2\frac{d}{t} \left(1 - \frac{8}{t^{1/3}}\right) \frac{|X||Y|}{n}$.

Next, we wish to upper bound the probability of occurrence for each of these events. Note that for all $i \in [t]$ and $v \in V(G)$, it follows from (R1) and (R2) that

$$\mathbb{E}[d_{H_i}(v)] \in \frac{d/2 \pm d^{2/3}}{t/2 \mp t^{2/3}} \in \frac{d}{t} \pm \frac{4d}{t^{4/3}}.$$

Therefore, by Chernoff's bounds (Lemma 2.1), we get that for all $i \in [t]$ and $v \in V(G)$,

$$\Pr[D_{i,v}] \leq \exp\left(-\frac{d}{t^{5/3}}\right). \quad (1)$$

Moreover, for all $i \in [t]$ and for all $X \subseteq A_i, Y \subseteq B_i$ with $|X| + |Y| > n/2$ and $\min\{|X|, |Y|\} > \frac{n}{2t^2}$, we have from the expander mixing lemma and (R2) that

$$\mathbb{E}[e_{H_i}(X, Y)] \geq 2\frac{d}{t} \left(1 - \frac{4}{t^{1/3}}\right) \frac{|X||Y|}{n}.$$

Therefore, by Chernoff's bounds (Lemma 2.1), we get that for $i \in [t]$ and all such X, Y ,

$$\Pr[A(i, X, Y)] \leq \exp\left(-\frac{d|X||Y|}{nt^{5/3}}\right). \quad (2)$$

Now, we apply the asymmetric version of the local lemma (Lemma 2.5) as follows: our events consist of all the previously defined $D_{i,v}$'s and $A(j, X, Y)$'s. Note that each $D_{i,v}$ depends only on those $D_{j,w}$ for which $\text{dist}_G(v, w) \leq 2$. In particular, each $D_{i,v}$ depends on at most td^2 many $D_{j,w}$. Moreover, we assume that $D_{i,v}$ depends on all the events $A(j, X, Y)$ and that each $A(j, X, Y)$ depends on all the other events. For convenience, let us enumerate all the events as $\mathcal{E}_k, k = 1, \dots, \ell$. For each $k \in [\ell]$, let x_k be $\exp(-\sqrt{d})$ if \mathcal{E}_k is of the form $D_{i,v}$, and x_k be $\exp(-\sqrt{d}|X||Y|/n)$ if \mathcal{E}_k is of the form $A(j, X, Y)$. To conclude the proof, we verify that

$$\Pr[\mathcal{E}_k] \leq x_k \prod_{j \sim k} (1 - x_j)$$

for all k . Indeed, if \mathcal{E}_k is of the form $D_{i,v}$, then we have (using the numerical inequality $1 - x \geq e^{-2x}$, which is valid for $x \in [0, 3/4]$) that for all d sufficiently large,

$$\begin{aligned}
e^{-\sqrt{d}} \left(1 - e^{-\sqrt{d}}\right)^{td^2} \prod_{j \in [t]} \prod_{n \geq x, y \geq \frac{n}{2t^2}} \left(1 - e^{-\sqrt{d}xy/n}\right)^{\binom{n}{x} \binom{n}{y}} &\geq e^{-\sqrt{d}} e^{-2td^2 e^{-\sqrt{d}}} \prod_{n \geq x, y \geq \frac{n}{2t^2}} \left(e^{-2e^{-\sqrt{d}xy/n}}\right)^{t \binom{n}{x} \binom{n}{y}} \\
&\geq e^{-\sqrt{d}} e^{-2td^2 e^{-\sqrt{d}}} \prod_{n \geq x, y} \exp\left(-2te^{-\sqrt{d}n/4t^4} \binom{n}{x} \binom{n}{y}\right) \\
&\geq e^{-\sqrt{d}} e^{-2td^2 e^{-\sqrt{d}}} \exp\left(-2te^{-\sqrt{d}n/4t^4} 2^{3n}\right) \\
&\gg e^{-3\sqrt{d}} \\
&\gg e^{-d/t^{5/3}} \\
&\geq \Pr[\mathcal{E}_k].
\end{aligned}$$

On the other hand, if \mathcal{E}_k is of the form $A(j, X, Y)$, then we have (using the same numerical inequality as above) that for d sufficiently large,

$$\begin{aligned}
e^{-\sqrt{d}|X||Y|/n} (1 - e^{-\sqrt{d}})^{nt} \prod_{i \in [t]} \prod_{n \geq x, y \geq \frac{n}{2t^2}} \left(1 - e^{-\sqrt{d}xy/n}\right)^{\binom{n}{x} \binom{n}{y}} &\geq e^{-\sqrt{d}|X||Y|/n} e^{-2e^{-\sqrt{d}}nt} \prod_{n \geq x, y \geq \frac{n}{2t^2}} \left(e^{-2e^{-\sqrt{d}xy/n}}\right)^{t2^{2n}} \\
&\geq e^{-\sqrt{d}|X||Y|/n} e^{-2e^{-\sqrt{d}}nt} \exp\left(-2te^{-\sqrt{d}n/4t^4} 2^{3n}\right) \\
&\geq e^{-\sqrt{d}|X||Y|/n} e^{-8t^5 e^{-\sqrt{d}}|X||Y|/n} \exp\left(-2te^{-\sqrt{d}n/4t^4} 2^{3n}\right) \\
&\gg e^{-3\sqrt{d}|X||Y|/n} \\
&\gg e^{-d|X||Y|/nt^{5/3}} \\
&\geq \Pr[\mathcal{E}_k],
\end{aligned}$$

where in the third line, we have used the assumption that $|X||Y| \geq (n/2t^2) \cdot (n/2t^2)$ and hence, $nt \leq 4t^5|X||Y|/n$. Therefore, by the asymmetric version of the local lemma, Properties (S3) and (S4) are satisfied with probability at least

$$\left(1 - e^{-\sqrt{d}}\right)^{nt} \prod_{j \in [t]} \prod_{n \geq x, y \geq \frac{n}{2t^2}} \left(1 - e^{-\sqrt{d}xy/n}\right)^{\binom{n}{x} \binom{n}{y}} \geq e^{-nt}.$$

To complete the proof, it suffices to show that the probability that (S5) is not satisfied is less than $\exp(-nt)$. To see this, fix $i \in [t]$ and $X, Y \subseteq V(H_i)$ with $n/2t^2 \leq |X| = |Y| \leq n/4$. By the expander mixing lemma, we know that

$$e_G(X, Y) \leq d|X|^2/n + \lambda|X| \leq d|X|/4 + \lambda|X|,$$

so by (R2) we get

$$\mathbb{E}[e_{H_i}(X, Y)] \leq \frac{d}{2t} \left(1 + \frac{4}{t^{1/3}}\right) |X|.$$

Therefore, by Chernoff's bounds (Lemma 2.1), it follows that

$$\Pr[e_{H_i}(X, Y) \geq (1 - 4\alpha)d|X|/t] \leq \exp(-d|X|/100t) \leq \exp(-dn/200t^3).$$

Applying the union bound over all $i \in [t]$, and all such $X, Y \subseteq V(G)$, implies that the probability for (S5) to fail is at most $\exp(-dn/400t^3) < \exp(-nt)/2$. This completes the proof. \square

Remark 5.3. The above proof shows that there are at least $\frac{1}{2} \exp(-nt) \left(\frac{t}{2} - t^{2/3}\right)^{nd/2}$ (labeled) edge partitionings satisfying the conclusions of [Lemma 5.2](#).

6 Proofs of [Theorem 1.1](#) and [Theorem 1.6](#)

In this section, by putting everything together, we obtain the proofs of our main results.

Proof of [Theorem 1.1](#). Let $c = \varepsilon/10$, and apply [Proposition 5.1](#) with $\alpha = 1/10$, c , and t being an odd integer in $[d^{c/100}, d^{c/10}]$ to obtain t distinct, edge disjoint $(\alpha, r, \frac{n}{2})$ -good graphs W_1, \dots, W_t , where $r = \lfloor \frac{d}{t} \left(1 - \frac{16}{t^{1/3}}\right) \rfloor$. Let $G' := G \setminus \bigcup_{i=1}^t W_i$, and note that G' is $r' := d - rt$ regular. After possibly replacing r by $r - 1$, we may further assume that r' is even. In particular, by [Theorem 2.12](#), G' admits a 2-factorization. By grouping these 2-factors, we readily obtain a decomposition of G' as $G' = G'_1 \cup \dots \cup G'_t$ where each G'_i is r'_i -regular, with $r'_i \in \frac{r'}{t} \pm t \leq 40 \frac{d}{t^{4/3}}$. Finally, applying [Proposition 4.2](#) to each of the regular graphs R_1, \dots, R_t , where $R_i := W_i \cup G'_i$, finishes the proof. \square

We will obtain ‘enough’ 1-factorizations by keeping track of the number of choices available to us at every step in the above proof.

Proof of [Theorem 1.6](#). Suppose that $\lambda \leq d^{1-\varepsilon}$ and let $c = \varepsilon/10$. Now, fix $\epsilon > 0$. Throughout this proof, $\epsilon_1, \dots, \epsilon_4$ will denote positive quantities which go to 0 as d goes to infinity. By [Remark 5.3](#), there are at least $\left(\frac{1 - \epsilon_1}{2}\right)^{nd/2}$ edge partitionings of $E(G)$ satisfying the conclusions of [Lemma 5.2](#) with $\alpha = 1/10$, c , and t an odd integer in $[d^{c/100}, d^{c/10}]$. For any such partitioning $E(G) = E_1 \cup \dots \cup E_t$, the argument in the proof of [Theorem 1.1](#) provides a decomposition $E(G) = E(R_1) \cup \dots \cup E(R_t)$. Recall that for all $i \in [t]$, $R_i := W_i \cup G'_i$, where W_i is an $(\alpha, r, n/2)$ -good graph with $r \geq \lfloor \frac{d}{t} \left(1 - \frac{16}{t^{1/3}}\right) \rfloor - 1$ and $E(W_i) \subseteq E_i$, and G'_i is an r'_i -regular graph with $r'_i \leq 40d/t^{4/3}$. In particular, by [Remark 4.3](#), each R_i has at least $\left(\frac{1 - \epsilon_2}{2\epsilon^2}\right)^{nd/2t}$ 1-factorizations. It follows that the multiset of 1-factorizations of G obtained in this manner has size at least $\left(\frac{1 - \epsilon_3}{2\epsilon^2}\right)^{nd/2}$.

To conclude the proof, it suffices to show that no 1-factorization $\mathcal{F} = \{F_1, \dots, F_d\}$ has been counted more than $(1 + \epsilon_4)^{nd/2}$ times above. Let us call an edge partitioning $E(G) = E_1 \cup \dots \cup E_t$ *consistent* with \mathcal{F} if $E(G) = E_1 \cup \dots \cup E_t$ satisfies the conclusions of [Lemma 5.2](#), and \mathcal{F} can be obtained from this partition by the above procedure. It is clear that the multiplicity of \mathcal{F} in the multiset is at most the number of edge partitionings consistent with \mathcal{F} , so that it suffices to upper bound the latter.

For this, the crucial thing to note is that if $E(G) = E_1 \cup \dots \cup E_t$ is consistent with \mathcal{F} , then at least $rt - r'_1 - \dots - r'_t \geq d - 57d/t^{1/3}$ of the perfect matchings in \mathcal{F} must have all of their edges in the same part E_i (for some $i \in [t]$) – this is because in the procedure described above, at least $rt - r'_1 - \dots - r'_t$ of the perfect matchings arise from 1-factorizations of bipartite subgraphs of W_1, \dots, W_t . Reversing this, we see that given \mathcal{F} , the number of edge partitions of $E(G) = E_1 \cup \dots \cup E_t$ consistent with \mathcal{F} is at most $\binom{d}{57d/t^{1/3}} t^d \cdot t^{57nd/2t^{1/3}}$, and observe that this last quantity can be expressed as $(1 + \epsilon_4)^{nd/2}$. \square

7 Concluding remarks and open problems

- In [Theorem 1.1](#), we proved that every (n, d, λ) -graph contains a 1-factorization, assuming that $\lambda \leq d^{1-\varepsilon}$ and $d_0 \leq d \leq n - 1$ for d_0 sufficiently large. As we mentioned after the statement, it seems reasonable that one could, by following our proof scheme with a bit more care, obtain a bound of the form $\lambda \leq d/\log^c n$. In [\[24\]](#), Krivelevich and Sudakov showed that if $d - \lambda \geq 2$

(and n is even) then every (n, d, λ) -graph contains a perfect matching (and this, in turn, was further improved in [9]). This leads us to suspect that our upper bound on λ is anyway quite far from the truth. It will be very interesting to obtain a bound of the form $\lambda \leq d - C$, where C is a constant, or even one of the form $\lambda \leq \varepsilon d$, for some small constant ε . Our proof definitely does not give it and new ideas are required.

- In [23], Kim and Wormald showed that for every fixed $d \geq 4$, a typical $G_{n,d}$ can be decomposed into perfect matchings, such that for ‘many’ prescribed pairs of these matchings, their union forms a Hamilton cycle (in particular, one can find a Hamilton cycle decomposition in the case that d is even). Unfortunately, our technique does not provide us with any non trivial information about this kind of problem, but we believe that a similar statement should be true in $G_{n,d}$ for all d .
- In [Theorem 1.6](#), we considered the problem of counting the number of 1-factorizations of a graph. We showed that the number of 1-factorizations in (n, d, λ) -graphs is at least

$$\left((1 - o_d(1)) \frac{d}{2e^2} \right)^{nd/2},$$

which is off by a factor of $2^{nd/2}$ from the conjectured upper bound (see [27]), but is still better than the previously best known lower bounds (even in the case of the complete graph) by a factor of $2^{nd/2}$. In joint work together with Sudakov [14], we have managed to obtain an optimal asymptotic formula for the number of 1-factorizations in d -regular graphs for all $d \geq \frac{n}{2} + \varepsilon n$. It seems possible that by combining the techniques in this paper and the one to come, one can obtain the same bound for (n, d, λ) -graphs, assuming that d is quite large. It would be nice, in our opinion, to obtain such a formula for all values of d .

- A natural direction would be to extend our results to the hypergraph setting. That is, let $H_{n,d}^k$ denote a k -uniform, d -regular hypergraph, chosen uniformly at random among all such hypergraphs. For which values of d does a typical $H_{n,d}^k$ admit a 1-factorization? How many such factorizations does it have? Quite embarrassingly, even in the case where H is the complete k -uniform hypergraph, no non-trivial lower bounds on the number of 1-factorizations are known. Unfortunately, it does not seem like our methods can directly help in the hypergraph setting.

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