AN INTEGRAL METHOD FOR QUASI THREE-DIMENSIONAL BOUNDARY LAYER ANALYSIS WITH ROTATIONAL EFFECTS

by

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(1987)

Submitted to the
Department of Aeronautics and Astronautics
in partial fulfillment of the requirements
of the Degree of

MASTER OF SCIENCE IN
AERONAUTICS AND ASTRONAUTICS

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

February 1990

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Dept. of Aeronautics and Astronautics, 9 February 1990

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Abstract

An integral boundary layer method is presented for the solution of quasi three-dimensional boundary layer problems. It is a method which can be of particular usefulness in certain rotating turbo-machinery problems since it presents a more economical alternative to the more expensive quasi three-dimensional finite difference approach. To develop the method, the compressible integral boundary layer equations with rotational and axial effects are derived from the Prandtl boundary layer equations becoming the momentum and shape parameter equations. Necessary functional dependencies are provided for laminar and turbulent closure of the system. The compressible integral boundary layer equations are applied to various test cases and are numerically solved. The results are reported, compared to theoretically expected and experimental calculations, and the accuracy of the method is discussed.

Thesis Supervisor: Mark Drela
Title: Assistant Professor of Aeronautics and Astronautics
Acknowledgements

I would like to thank MIT for giving me the opportunity to push myself and become a better person through my educational experience here. I am indebted to my professors for all their help and the knowledge they imparted on me. I am particularly grateful to Professor Mark Drela for his help and patience with me. Without him, this thesis would not exist right now.

I thank the good Lord for giving me the strength all these years to overcome all the difficulties I encountered, for giving me the wisdom I needed so much, and the emotional strength to go on when everything seemed so hard.

Words cannot express my thanks to my parents without whose love, support, financial and emotional, and without whose sacrifices, I would not be who I am today. I also am grateful for Rita whose decision to share her life with me gave me something to look forward to during the preparation of this thesis.
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\( \nu \)  
kinematic viscosity

\( \rho \)  
density

\( \rho_t \)  
stagnation density

\( \tau_w \)  
wall shear stress

\( \omega, \Omega \)  
rotational speed

1, 2, n  
subscripts and superscripts used to indicate various stations
1. INTRODUCTION

The numerical solution of a boundary layer problem rapidly and with good accuracy is such a desirable tool in aerodynamic design that it has been the goal of many researchers. The result of these efforts has been the development of various computational methods for boundary layers which can be divided into two classes, differential and integral.

Differential techniques require the solution of a system of the partial differential equations, or PDE's, which govern the boundary layer problem. The development of these techniques has accelerated in recent years following advances in computational capabilities as both computers and algorithms have become more powerful and more efficient.

Integral methods, requiring less in the way of computational effort, have been under investigation for many years. In certain cases, in fact, work done with integral methods has produced good results which are of accuracy comparable to that of differential methods. Papers by Whitfield [29], Green [15], Drela and Giles [12], and East et al [10] among others, demonstrate the usage of the integral methods in two-dimensional and axisymmetric boundary layer flows.

The integral method described in this thesis applies to quasi three-dimensional boundary layers, taking into account axial, radial, and rotational effects. It builds upon work done by Drela and presented in his Ph. D. thesis [9]. Drela describes a procedure
by which two-dimensional transonic aerodynamic design and analysis using the Euler equations can be done. In that method, viscous layers are accounted for in the inviscid calculation by the use of the displacement thickness model, a procedure commonly referred to as boundary layer coupling. Lighthill [19] has shown that the leading order effect of the viscous layers is to displace the inviscid flow away from the body or wake by a distance equal to the displacement thickness.

The coupling approach between the viscous and the inviscid regions presented in Drela is not the same as the classical semi-inverse or quasi-simultaneous methods developed previously. Instead of the usual space-marching boundary layer solution at every viscous-inviscid iteration, the boundary layer equations for all streamwise stations are included in a global Newton system and are solved together with the inviscid equations as a fully-coupled system. This approach has the advantage of not requiring an interaction law. Also, the computational cost is reduced since the very rapid convergence rate of the Newton algorithm minimizes the cost penalty for the viscous coupling to a negligible quantity.

This thesis extends the boundary layer integral method calculations from the two-dimensional to the quasi three-dimensional case. Now, in addition to the problems that could be investigated before, bodies that are rotating can also be considered. The radius $R_b$ of the body, the rotational speed $\omega$, and the variation in the streamtube thickness $h$ enter the calculations. As before, the displacement thickness approach can be used to
relate the inviscid and viscous layers (Appendix A shows this to be true), and the whole formulation can be incorporated into inviscid region solvers such as the ISES [9,11,13] transonic airfoil/cascade analysis/design code or the Denton time-marching algorithm [8], thereby extending them to other important applications.

One very important area of application for the quasi three-dimensional integral boundary layer method is rotating turbomachinery. In a turbine blade row, complex three-dimensional flow phenomena can occur such as horseshoe vortex formation, three-dimensional flow separation, tip-clearance flow, and shock-boundary layer interaction. These complex flow phenomena can be predicted with three-dimensional viscous flow calculations. Indeed, such work has been done (for examples see papers by Hah and Leybek [17], Weinberg, etal. [26], Dawes [7], etc.), and the results of these efforts undoubtedly provide an advance in current design technology.

This numerical approach to the calculation of complex three-dimensional flows using fully three-dimensional or, when possible, quasi three-dimensional finite difference algorithms, however, requires a significant amount of computational effort. Therefore, it would be very beneficial to develop a method that applies to certain rotating turbomachinery flows, but which is computationally fast, cheap, and quite accurate. The technique described in this thesis, based on the compressible integral boundary layer equations, seeks to be just that. Although, it does not account for all three-dimensional phenomena, it can be applied to cases of
boundary layers with an outer potential flow and is more economical than a quasi three-dimensional finite difference calculation.

The approach taken in this thesis is to first develop the integral momentum and shape parameter equations. That is done in chapter two. The conditions for laminar and turbulent closure are also developed there, and brief discussions of transition and the wake region are presented. In chapter three, the discretization of the equations and some computational aspects of the method are discussed. Chapter four compares the solutions to test problems obtained using this method to theoretically and experimentally calculated results. The conclusions follow in chapter five.
2. DESCRIPTION OF THE QUASI THREE-DIMENSIONAL BOUNDARY
LAYER

In this chapter, the equations that describe the quasi three-dimensional boundary layer are presented. They include the momentum and shape parameter compressible integral boundary layer equations with terms accounting for rotational and axial effects. These equations are valid for both laminar and turbulent boundary layers, as well as for free wakes. However, they contain more than two independent variables and certain assumptions have to be made and correlations to be provided in order to solve them.

The two dependent variables in the compressible boundary layer equations are selected to be the momentum thickness $\theta$ and the displacement thickness $\delta^*$. These are the same two dependent variables as in the two-dimensional case presented in Drela [9]. The known variables include the edge velocity $u_e$ and the edge Mach number $M_e$ which can be related to the inviscid flow. In addition to these however, in the quasi three-dimensional case, there are three more known quantities not found in the two-dimensional case, namely the rotational speed $\omega$, the surface radius of the body $R_b$, and the streamtube thickness $h$. The latter two variables are readily obtained from the geometry of the problem.

A third dependent variable and a third equation, also found in the two-dimensional case, come into the picture. For turbulent flows, these are the stress coefficient $C_t$ and the shear stress coefficient lag equation. For laminar flows, they are the
amplification factor $\hat{n}$ and the amplification equation. Thus, an additional unknown, but also an additional equation, are introduced to the problem.

Still, there remain five undefined variables: the skin friction coefficient $C_f$, the dissipation coefficient $C_d$, and the shape parameters $H^*$, $H^{**}$, and $H_p$. All of these variables except $H_p$ are also present in the two-dimensional case. This means that a functional dependency for $H_p$, in addition to the existing ones for $C_f$, $C_d$, $H^*$, and $H^{**}$, is necessary for closure.

2.1 The integral boundary layer equations

In order to derive the momentum and shape parameter compressible integral boundary layer equations, the well-known Prandtl compressible boundary layer equations are used as the starting point. For two dimensions these are:

\[
\text{mass:} \quad \frac{\partial (\rho u)}{\partial s} + \frac{\partial (\rho v)}{\partial n} = 0 \quad (2.1)
\]

\[
\text{momentum:} \quad \rho u \frac{\partial u}{\partial s} + \rho v \frac{\partial u}{\partial n} = \rho e u_e \frac{\partial u}{\partial s} + \frac{\partial \tau}{\partial n} \quad (2.2)
\]

\[
\text{enthalpy:} \quad \rho u \frac{\partial h_t}{\partial s} + \rho v \frac{\partial h_t}{\partial n} = \frac{\partial Q}{\partial n} \quad (2.3a)
\]

\[
Q = \frac{\mu_t}{Pr} \frac{\partial h_t}{\partial n} + \mu \left(1 - \frac{1}{Pr}\right) u \frac{\partial u}{\partial n} - \rho h' v' \quad (2.3b)
\]
The enthalpy profile can be fairly accurately related to just the velocity profile and the temperature recovery factor and so the enthalpy equation can be dropped from the calculations. This can be done because only adiabatic flows with near-unity Prandtl numbers will be considered.

To derive the momentum and shape parameter integral boundary layer equations for the quasi three-dimensional case, the scaled coordinates $\xi$ and $\eta$ are introduced. They are defined by the relations $ds=Rd\xi$ and $dn=Rd\eta$. If the rotational and axial effects are taken into account and the mass and momentum boundary layer equations are applied to a cube of dimensions $Rd\xi \times Rd\eta \times h$, then these equations can be written as:

\[
\text{mass:} \quad \frac{\partial}{\partial \xi} (p u R h) + \frac{\partial}{\partial \eta} (p v R h) = 0 \quad (2.4)
\]

\[
\text{momentum:} \quad \rho u \frac{\partial u}{\partial \xi} + \rho v \frac{\partial v}{\partial \eta} - \rho \omega^2 R \frac{\partial R}{\partial \xi} = \rho e u_e \frac{\partial u_e}{\partial \xi} - \rho_e \omega^2 R \frac{\partial R}{\partial \xi} + \frac{\partial \tau}{\partial \eta} \quad (2.5)
\]

When equations (2.4) and (2.5) are integrated across the boundary layer (see Appendix B for the detailed derivation), the momentum and shape parameter compressible integral boundary layer equations result:

\[
\frac{1}{\theta} \frac{d\theta}{ds} + \frac{1}{R h} \frac{d(R h)}{ds} = \frac{1}{\theta} \frac{C_f}{2} - \left( H + 2 - \frac{M_e^2}{R} \right) \frac{1}{u_e} \frac{du_e}{ds} \quad (2.6)
\]
\[
\frac{1}{H^*} \frac{dH^*}{ds} = \frac{1}{\theta} C_d - \frac{1}{\theta^*} \frac{2}{2} C_f - \left( \frac{2}{H^*} \frac{H^{**}}{2} + 1 - H \right) \frac{1}{u_e} \frac{du_e}{ds} \\
- \left( H_\rho - 2 \frac{H^{**}}{H^*} \right) \left( \frac{\omega R}{u_e} \right)^2 \frac{1}{R} \frac{dR}{ds}
\] (2.7)

In the above equations the following definitions are used:

\[
\delta^* = \int_0^\infty \left( 1 - \frac{\rho u}{\rho_e u_e} \right) R d\eta, \quad \delta_\rho = \int_0^\infty \left( 1 - \frac{\rho}{\rho_e} \right) R d\eta, \quad (2.8-9)
\]

\[
\theta = \int_0^\infty \left( 1 - \frac{u}{u_e} \right) \frac{\rho u}{\rho_e u_e} R d\eta, \quad C_f = \frac{2 \tau_w}{\rho_e u_e^2} \quad (2.10-11)
\]

\[
H = \frac{\delta^*}{\theta}, \quad H_\rho = \frac{\delta_\rho}{\theta} \quad (2.12-13)
\]

\[
H^* = \frac{\theta^*}{\theta}, \quad H^{**} = \frac{\delta^{**}}{\theta} \quad (2.14-15)
\]

\[
\delta^{**} = \int_0^\infty \left( 1 - \frac{\rho}{\rho_e} \right) \frac{u}{u_e} R d\eta, \quad \theta^* = \int_0^\infty \left[ 1 - \left( \frac{u}{u_e} \right)^2 \right] \frac{\rho u}{\rho_e u_e} R d\eta \quad (2.16-17)
\]

\[
C_d = \frac{1}{\rho_e u_e^3} \int_0^\infty \frac{\tau}{\eta} d\eta \quad (2.18)
\]

The quantity \( \delta^* \) is the displacement thickness, \( \delta^{**} \) is the density thickness, \( \theta \) is the momentum thickness, \( \theta^* \) is the kinetic energy thickness, and \( H, H^*, H^{**}, \) and \( H_\rho \) are shape parameters.
In comparing equation (2.6) to the two-dimensional integral momentum equation, two new terms can be seen. They are the ones that involve the variables \( Rh \) and \( \omega R \). Equation (2.7) also contains an additional term when compared to the two-dimensional shape parameter equation, the term with the same \( \omega R \) variables.

### 2.2 Functional dependencies for laminar flow

As was noted before, the compressible integral momentum equations apply to both laminar and turbulent flows. Certain relationships however, have to be developed for \( C_f \), \( C_d \), \( H^* \), \( H^{**} \), and \( H_\rho \). Since all of these except \( H_\rho \) are found in the two-dimensional case, a good reference for the the expressions used for most of these variables is Drela's thesis [9]. Because \( H_\rho \) is a new quantity to the quasi three-dimensional case, a correlation for it must be obtained elsewhere.

In order to develop the functional dependencies for the above variables for compressible, adiabatic flow, the following assumptions are made:

\[
H^* = H^*(H_k, M_e, Re_\theta)
\]

\[
H^{**} = H^{**}(H_k, M_e)
\]

\[
H_\rho = H_\rho(H_k, M_e)
\]

\[
C_f = C_f(H_k, M_e, Re_\theta)
\]

\[
C_d = C_d(H_k, M_e, Re_\theta)
\]
$H_k$ is the kinematic shape parameter which depends only on the velocity profile, and is defined with the density across the boundary layer assumed to be constant. Since compressible and incompressible velocity profiles have nearly the same shape, the assumption that the above variables depend on $H_k$ is a logical one.

$H_k$ for the quasi three-dimensional boundary layer flow is defined as:

$$
H_k = \frac{\int_0^\infty \left( 1 - \frac{u}{u_e} \right) R_d \eta \, d\eta}{\int_0^\infty \left( 1 - \frac{u}{u_e} \right) \frac{u}{u_e} R_d \eta}
$$

An empirical formula for $H_k$, expressed in terms of $M_e$, the edge Mach number, and the shape parameter $H$, has been developed by Whitfield [28]. Assuming adiabatic flow and a Prandtl number that is not equal to unity:

$$
H_k = \frac{H - 0.290 \frac{M_e^2}{1 + 0.113 M_e^2}}{1 + 0.113 M_e^2}
$$

(2.19)

Whitfield has demonstrated the accuracy of this expression up to $M_e=3$. In this thesis, as in the two-dimensional case, it is used for both laminar and turbulent flows.

Correlations for the other variables are calculated from the Falkner-Skan one parameter family of velocity profiles by Drela. The correlations for the variables presented in this thesis have
been very slightly modified since they were originally presented in Drela [9]. The result of these modifications is better numerical algorithm robustness.

\[
H_k^* = \begin{cases} 
1.515 + 0.076 \frac{(H_k - 4)^2}{H_k}, & H_k < 4 \\
1.515 + 0.020 \frac{(H_k - 4)^2}{H_k}, & H_k > 4 
\end{cases} \tag{2.20}
\]

\[
\frac{C_f}{2} \Re_\theta = \begin{cases} 
-0.067 + 0.01977 \frac{(7.4 - H_k)^2}{H_k - 1}, & H_k < 7.4 \\
-0.067 + 0.022 \left(1 - \frac{1.4}{H_k - 6}\right)^2, & H_k > 7.4 
\end{cases} \tag{2.21}
\]

\[
\frac{2C_d}{H^*} \Re_\theta = \begin{cases} 
0.207 + 0.00205 (4 - H_k)^{5.5}, & H_k < 4 \\
0.207 + 0.003 \frac{(H_k - 4)^2}{1 + 0.02 (H_k - 4)^2}, & H_k > 4 
\end{cases} \tag{2.22}
\]

\[
H^* = \frac{(H_k^* + 0.028 \text{ M}^2)}{(1 + 0.014 \text{ M}^2)} \tag{2.23}
\]

The last relation, developed by Whitfield [29], gives an additional small compressibility correction for the energy shape (2.20) equation.

The same reference [29] also gives the expression for the thickness shape parameter $H^{**}$. This parameter, which will be used for both laminar and turbulent flows, is negligible for low subsonic flows and has a small effect in transonic flows. It is given by:
\[ H^{**} = \left( \frac{0.064}{H_k - 0.8} + 0.251 \right) M_a^2 \]  \hspace{1cm} (2.24)

An expression for the new necessary correlation for $H_p$, introduced in the quasi three-dimensional case, is given by Swafford and Whitfield [22]:

\[ H_p = [0.185 \times H_k + 0.150] \, M_a^2 \]  \hspace{1cm} (2.25)

This relationship is used in both laminar and turbulent flows.

The functions for $H_k^*$, $C_f$, $C_d$, $H^{**}$, and $H_p$ are plotted in the figures that follow.

**Fig. 2.1** Laminar $H_k^*$ vs. $H_k$ correlation
Fig. 2.2 Laminar $C_f$ vs. $H_k$ correlation

Fig. 2.3 Laminar $C_d$ vs. $H_k$ correlation
Fig. 2.4 $H^{**}$ vs. $H_k$ correlation

Fig. 2.5 $H_p$ vs. $H_k$ correlation
The use of all the relationships in the two integral boundary layer equations results in two equations and three unknowns: $\theta$, $\delta^*$, and $u_e$. Either $u_e$ or $\delta^*$ can be specified. If $u_e$ is specified, then a direct boundary layer problem results. If $\delta^*$ is specified, then an inverse problem results. The accuracy of the results when the two equations have been used to solve incompressible flow problems, has been demonstrated by Drela. Figure 2.6, reproduced from his thesis, shows how favorably this method compares to the finite difference and Thwaites' methods.

![Graph showing comparisons](image)

**Fig. 2.6** Comparison of present formulation, finite difference, and Thwaites' methods for incompressible laminar flow.
2.3 Functional dependencies for turbulent flow

Relationships for $C_f$, $C_d$, $H^*$, and $H^{**}$ also have to be developed for turbulent flow. However, the case is not as simple as for laminar flow. Again, Drela's thesis is the primary reference for all the expressions used for these variables.

In turbulent flows, the presence of the Reynolds stresses must be related to the mean flow. Introduction of these stresses into the calculations makes the development of the needed expressions more complicated. Furthermore, since turbulent boundary layers have a two-layer structure with the thickness of each layer scaling differently depending on the local Reynolds number, a one-parameter velocity profile family is no longer adequate, and a two-parameter velocity profile becomes necessary.

In order to present a two-parameter turbulent velocity profile, an empirical relation for the skin friction coefficient is needed since the skin friction is the primary scaling parameter for the wall layer. The formula presented here has been developed by Swafford [23].

\[
F_C C_f = \frac{0.3 \, e^{-1.33 H_k}}{\left[ \log_{10} \left( \frac{Re_d}{F_C} \right) \right]^{1.74 + 0.31 \, H_k}} + 0.00011 \left[ \tanh \left( 4 - \frac{H_k}{0.875} \right) - 1 \right] \tag{2.26}
\]

where
\[ F_c = \left( 1 + 0.2 \, M_e^2 \right)^{0.5} \]

Figure 2.7 shows the above function for \( M_e = 0 \). It is reassuring to note that when figures 2.2 and 2.7 are compared, the turbulent and laminar curves are very similar for \( Re_\theta = 400 \), close to the minimum Reynolds number at which a turbulent boundary layer can exist.

![Diagram showing turbulent \( C_f \) vs. \( H_k \) correlation](image)

**Fig. 2.7** Turbulent \( C_f \) vs. \( H_k \) correlation

The \( H^+ \) expression for turbulent flows is based on an analytic relation of the velocity profile derived by Swafford [21]. Two asymptotic solutions, an inner one for the laminar sublayer and the
buffer layer, and an outer solution for the outer layer or wake are matched to produce the following velocity profile:

\[
\frac{u}{u_e} = \frac{u_t}{u_e} \frac{s}{0.09} \arctan(0.09 \; y^+) \]

\[
+ \left(1 - \frac{u_t}{u_e} \frac{s \pi}{0.18} \right) \tanh^{0.5} \left[ a \left( \frac{\eta}{\theta} \right)^b \right] \quad (2.27)
\]

where

\[
\frac{u_t}{u_e} = \left| \frac{C_f}{2} \right|^{0.5}, \quad s = \frac{C_f}{\left| C_f \right|}, \quad y^+ = \frac{Du_t}{\mu} \eta
\]

and \(a\) and \(b\) are constants which can be calculated for any \(\theta\) and \(\delta^*\) from the skin friction formula (2.26) and the subsequent substitution of (2.26) into the \(\theta\) and \(\delta^*\) definitions. The resulting coupled, non-linear 2x2 implicit system can be solved using the Newton method.

The expression for \(H_k^*\) can then be derived from (2.27):

\[
H_k^* = \begin{cases} 
1.505 + \frac{4}{Re_{\theta}} + \left(0.165 - \frac{1.6}{\sqrt{Re_{\theta}}} \right) \frac{(H_o - H_k)1.6}{H_k}, & H_k < H_o \\
1.505 + \frac{4}{Re_{\theta}} + (H_k - H_o)^2 \left( \frac{0.02}{H_k} + 0.007 \right) \ln(Re_{\theta}) \left[ H_k - H_o + \frac{4}{\ln(Re_{\theta})} \right]^{2}, & H_k > H_o 
\end{cases}
\]

\[
(2.28)
\]

where
\[ H_o = 3.0 + \frac{400}{Re_\theta} \]

The expression for \( H^*_k \) can then be substituted in (2.23) and a value for \( H_k \) can be obtained. The plot of \( H^*_k \) versus \( H \) is given in figure 2.8.

![Graph showing \( H^*_k \) vs. \( H_k \)](image)

**Fig. 2.8** Turbulent \( H^*_k \) vs. \( H_k \) correlation

The expression for \( C_d \) depends on the Reynolds stress distribution across the boundary layer and is harder to develop. The method used to derive the dissipation coefficient is described
in detail in Drela [9]. It is based on Clauser's equilibrium boundary layer concept (see reference [5]).

If \( \beta \) is a constant pressure gradient parameter given by

\[
\beta = \frac{\delta^*}{\tau_w} \frac{d\phi}{d\xi} = - \frac{2}{C_f} \frac{\delta^*}{u_e} \frac{du_e}{d\xi}
\]  

(2.29)

then \( G \), the modified shape parameter given by

\[
G = \frac{H_k - 1}{H_k} \frac{1}{\sqrt{C_f/2}}
\]

(2.30)
is also constant, and, for the self-prescribed flows described by Clauser, \( G \) is only a function of \( \beta \). An empirical relation for \( G \) is:

\[
G = 6.7 \sqrt{1 + 0.75 \beta}
\]

(2.31)

The equilibrium pressure gradient, given in terms of the shape parameter and the skin friction coefficient, is then,

\[
\frac{\delta^*}{u_e} \frac{du_e}{d\xi} = \frac{1}{0.75} \left[ \frac{C_f}{2} - \frac{(H_k - 1)^2}{6.7 H_k} \right] \quad \text{(equilibrium)}
\]

(2.32)

To derive the expression for \( C_d \), the equilibrium relations above are introduced into the integral shape parameter equation. Since \( C_f \) is primarily a function of \( H \), then, in equilibrium layers, \( H \), and therefore \( H^* \) (which was assumed to be a function of \( H \)) must be constant. If \( (H - 2H^*/H) = H_k \) and the streamwise derivative of
H* is set to zero in the shape parameter equation, then for equilibrium:

\[
\frac{\xi}{\theta} 2Cd \frac{2}{H^*} - \frac{\xi}{\theta} \frac{Cd}{2} \frac{d\xi}{d\xi} \frac{du_e}{ue} - (1 - H_k) \frac{\xi}{ue} \frac{du_e}{d\xi} = 0 \tag{2.33}
\]

The use of equation (2.32) eliminates the velocity gradient from the equation above, and the final expression for Cd becomes (Drela [9]):

\[
\frac{2Cd}{H^*} = \frac{Cd}{2} \left( \frac{4}{H_k} - 1 \right) \frac{1}{3} + 0.03 \left( \frac{H_k}{H_k} - 1 \right)^3 \tag{2.34}
\]

Note here, that (2.34) is assumed to apply to all turbulent flows although it was derived based on equilibrium flows. A plot of the above function is given in figure 2.9.
2.4 History effects

The dissipation coefficient involves a velocity gradient-weighted integral of the Reynolds stresses. After examining equation (2.34), it can be seen that these stresses only depend on the local boundary layer parameters. While this assumption holds when turbulence production in boundary layers and their dissipation mechanisms are close to equilibrium, experimental data (see Goldberg [14], Kline et al. [18]) suggests that the Reynolds
stresses are affected by the upstream history of the flow. Therefore, these history effects have to be considered.

Bradshaw and Feriss [1,2] have presented a stress-transport equation of the form

\[
\frac{u}{2a_1} \frac{\partial (-u'v')}{\partial \xi} + \frac{v}{2a_1} \frac{\partial (-u'v')}{\partial \eta} = (-u'v') \frac{\partial u}{\partial \eta} - \frac{(-u'v')^{1.5}}{L} - \frac{\partial (\text{diffusion})}{\partial \eta}
\]

Green et al [16] have shown that by assuming that the point of maximum Reynolds stress is representative of the Reynolds stress level for the entire boundary layer, at the maximum shear stress point, \( L \) is equal to the conventional mixing length, and neglecting normal convection, the above equation can be simplified to:

\[
\frac{\delta}{C_t} \frac{dC_t}{d\xi} = 2a_1 \frac{u_e}{u} \frac{\delta}{L} \left( C_{\text{eq}}^{0.5} - C_t^{0.5} \right) - \frac{\partial (\text{diffusion})}{\partial \eta} - \frac{2\delta}{u_e} \frac{\partial u}{\partial \xi}
\]

where

\[
C_t = \frac{1}{u_e^2} (-u'v')_{\text{max}}^{0.5}
\]

(2.35)

\[
\frac{\partial u}{\partial \eta} = \frac{1}{L} (-u'v')_{\text{max}}^{0.5}
\]

(2.36)

Thomas [24] and Drela [9] simplify the equation even further by discarding the last two terms based on the argument that the
remaining terms form the dominant balance. The final form of the equation then becomes:

\[
\frac{\delta}{C_t} \frac{dC_t}{d\xi} = K_C \left( C_{eq}^{0.5} - C_t^{0.5} \right) \tag{2.37}
\]

Green obtains a value of 5.6 for \( K_C \) using the values \( a_1 = 0.15 \), \( u_e/u = 1.5 \), and \( L/\delta = 0.08 \). Drela uses a value of \( K_C = 4.2 \) because it provides better results.

A connection between \( \delta \) and \( (C_t)_{eq} \) still has to be made. Drela takes an expression from Green, simplifies it, and obtains:

\[
\delta = \Theta \left( 3.15 - \frac{1.72}{H_k - 1} \right) + \delta^* \tag{2.38}
\]

For equilibrium flows then,

\[
C_d = \frac{C_f}{2} U_s + C_{eq} \left( 1 - U_s \right)^3 \tag{2.39}
\]

where

\[
U_s = \frac{U_s}{u_e} = \frac{H^*}{6} \left( \frac{4}{H_k} - 1 \right) \tag{2.40}
\]

\[
C_{eq} = \frac{H^*}{2} \frac{0.03}{1 - U_s} \left( \frac{H_k - 1}{H_k} \right)^3 \tag{2.41}
\]

The above expressions are the result of the requirement that (2.34) and (2.39) must become identical in equilibrium flows.
Finally, Drela [9] gives the resulting non-equilibrium formula for the dissipation coefficient as:

\[
\frac{2C_D}{H^*} = \frac{C_f}{2} \left( \frac{4}{H_k} - 1 \right) \frac{1}{3} + \frac{2}{H^*} C_T (1 - U_s) \quad (2.42)
\]

A physical explanation to equation (2.42) can be given by observing that there is a wall and a wake contribution to the dissipation coefficient. Both of these contributions are composed of a shear stress scale and a velocity scale. The wall contribution is determined only from the local boundary layer parameters. The wake contribution however, does not depend strictly on the local conditions. This is a modeling that is preferable to the local-equilibrium assumption which gives poor results for rapidly changing flows.

2.5 Comments about the wake

In flows where the wake region needs to be considered, only turbulent closure expressions have to be developed since laminar wakes are not found in any aerodynamic flows of interest. This is true because the disturbances that occur in a free wake grow exponentially and rapidly lead to turbulence.

The compressible integral momentum equations hold for this region if \( C_f \) in these equations is set to zero. Then the terms in the momentum and shape parameter equations that contain the skin friction coefficient can be discarded.
The closure relations needed for the turbulent wake then, are the same ones used for the turbulent boundary layer, but with the wall shear coefficient again set to zero. Drela [9] demonstrates that this approach works quite well for two-dimensional flows. If $C_f$ is set equal to zero, the dissipation coefficient is accurately described, and since a free wake is basically a turbulent boundary layer without the wall layer, the relations for $H_k^*$ and $H^{**}$ are the same as the ones used for the turbulent boundary layer closure. The lag equation (2.37) also remains unchanged.

2.6 Transition

The formulation of the boundary layer equations that govern the transition region is one of the two problems faced when considering the changeover from laminar to turbulent flows. The second problem faced is to determine the point where the onset of transition appears.

The approach taken in this thesis does not differ from the method used in the two-dimensional boundary layer problem and discussed in detail in Drela. Therefore, the treatment of transition from laminar to turbulent flow will not be presented in detail here, but just a brief explanation of how the two problems associated with transition are handled will be given. For further details, one is referred to Drela's Ph.D. thesis [9].

The problem of determining the governing boundary layer equations for the transition region is solved by deriving them from
simple intermittency-weighted averages of the laminar and turbulent relations developed earlier. To solve the second problem associated with transition, which is to determine the point of the onset of transition, an empirical assumption is made. This assumption is that transition begins to take place when the amplitude of infinitesimal wave-like disturbances has grown by more than a factor of \( e^9 \).

For a typical amplification ratio envelope,

\[
\ln \left( \frac{A}{A_0} \right) = \tilde{n} = \frac{\Delta n}{d \text{Re}_\theta} \{ \text{Re}_\theta - \text{Re}_{\theta_0}(H) \} \quad (2.43)
\]

where the slope \( \Delta n/d \text{Re}_\theta \) and the critical Reynolds number \( \text{Re}_{\theta_0} \) are given by

\[
\frac{\Delta n}{d \text{Re}_\theta} = 0.01 \left( \left\{ 2.4 H - 3.7 + \right. \right.
\]
\[
\left. + 2.5 \tanh \left[ 1.5 (H - 3.1) \right] \right\}^2 + 0.25 \right)^{0.5} \quad (2.44)
\]

\[
\log_{10} \text{Re}_{\theta_0} = \left( \frac{1.415}{H - 1} - 0.489 \right) \tanh \left( \frac{20}{H - 1} - 12.9 \right) +
\]
\[
+ \frac{3.295}{H - 1} + 0.440 \quad (2.45)
\]

If \( \xi \) is used as the spatial amplification coordinate for non-similar flows and the following two empirical relations are employed,
\[
\frac{\rho_e u_e \theta^2}{\mu_e \xi} = \alpha(H) = (6.54 \, H - 14.07) / H^2 
\]

(2.46)

\[
\xi \frac{d u_e}{d \xi} = \beta(H) = \left( 0.058 \, \frac{(H - 4)^2}{H - 1} - 0.068 \right) \frac{1}{\alpha(H)} 
\]

(2.47)

then,

\[
\frac{d n}{d \xi} = \frac{d n}{d \text{Re}_\theta} (H) \frac{\beta(H) + 1}{2} \frac{1}{\alpha(H)} \frac{1}{\theta} 
\]

(2.48)

Downstream of the instability point \( \xi_{cr} \), the amplification rate can be integrated:

\[
\tilde{n}(\xi) = \int_{\xi_{cr}}^{\xi} \frac{d n}{d \xi} \, d \xi 
\]

(2.49)

The condition for transition to take place is then that \( \tilde{n} = 9 \). Equation (2.49) is not the equation that is solved. Instead (2.48) is the equation that is discretized and which can be incorporated in the Newton system. This is a more robust procedure than using (2.49) to calculate the transition point at every iteration.
3. DISCRETIZATION AND NUMERICAL SOLUTION OF THE EQUATIONS

The compressible integral boundary layer equations still have to be discretized in order to be solved numerically. This chapter presents the discretized form of these equations and all of the other equations described in chapter two which are necessary to obtain a solution to the quasi three-dimensional flows being considered.

The computational aspect of the problem is discussed and some comments on the numerical algorithm are made. A better understanding of the program code used can be reached by looking at the complete program listing in appendix C.

3.1 Discretization of the equations

The momentum and shape parameter integral compressible boundary layer equations are the first ones to be discretized. The variables involved are considered at two adjacent stations which are denoted by the subscripts 1 and 2. Figure 3.1 which follows, depicts where the variables are located along the surface under consideration.
As in the two-dimensional case, the momentum equation is discretized using central two-point differences for accuracy. However, because large \( \Delta s/s \) values occur near the leading edge of a body, logarithmic differences, which are exact for similar flows, are used to minimize the discretization errors. Values for variables averaged between the adjacent points 1 and 2 are indicated by the subscript \( \text{a} \). Also, since \( C_d, C_f, H_p, \) and \( H^{*\*} \) are functionally dependent on other variables (for example \( C_d = C_d(H_{ka}, M_{ea}, Re_{\theta a}) \)), no subscript is attached to them.

Taking the momentum equation (2.6), applying the conditions stated above, and multiplying through by \( s_a \), the discretized form of the momentum equation arises:
\[
\frac{\ln(\theta_2/\theta_1)}{\ln(s_2/s_1)} + \frac{\ln(R_{2h2}/R_{1h1})}{\ln(s_2/s_1)} - \frac{s_a}{\theta_a} \frac{C_f}{2} + \left(\frac{H_a + 2 - M_e^2}{\ln(s_2/s_1)}\right) \frac{\ln(u_{e2}/u_{e1})}{\ln(s_2/s_1)}
- \left(\frac{\omega R_a}{u_{ea}}\right)^2 \left(H_\rho - M_e^2\right) \frac{\ln(R_{2}/R_{1})}{\ln(s_2/s_1)} = 0
\] (3.1)

where, for example,

\[H_a = \frac{1}{2} (H_1 + H_2)\]

Similarly, the shape parameter equation (2.7) is discretized using logarithmic central two-point differences for accuracy. It becomes:

\[
\frac{\ln(H_2^*/H_1^*)}{\ln(s_2/s_1)} + \frac{s_a}{\theta_a} \left(\frac{C_f}{2} - \tilde{C}_D\right) + \left(2 \frac{H^*}{H_a^*} + 1 - H_a\right) \frac{\ln(u_{e2}/u_{e1})}{\ln(s_2/s_1)}
+ \left(H_\rho - 2 \frac{H^*}{H_a^*}\right) \left(\frac{\omega R_a}{u_{ea}}\right)^2 \frac{\ln(R_{2}/R_{1})}{\ln(s_2/s_1)} = 0
\] (3.2)

where, following the notation introduced in Drela [9],

\[\tilde{C}_D = \frac{2}{H^*} \frac{C_d}{H^*}\]

In order to initialize the variable values at the very first point after the leading edge, the boundary layer is assumed to be similar and thus the values are obtained from theory (see Cebeci and Bradshaw [4]). If the case under consideration involves a blunt leading edge with a stagnation point, then
\[
\frac{R \, ds}{u_e} \frac{d \theta}{u_e} = \frac{\ln(u_{e2}/u_{e1})}{\ln(s_2/s_1)} = 1 \quad \text{and} \quad \frac{R \, ds}{\theta} \frac{d \theta}{\theta} = \frac{\ln(\theta_2/\theta_1)}{\ln(s_2/s_1)} = 0
\]

The values for \(C_f\), \(C_d\), \(H_p\), and \(H^{**}\) are given by the functional dependencies previously presented. For laminar flows, \(C_f\) is given from (2.21) and for turbulent flows from (2.26). \(C_d\) is obtained from (2.22) for laminar flows and from (2.33) for turbulent. Equation (2.25) gives the correlation for \(H_p\) and (2.26) the correlation for \(H^{**}\) for all flows.

There is a third equation to be discretized. What equation this is depends on whether the flow is laminar or turbulent. If the flow is turbulent, the shear stress coefficient lag equation is discretized. The dependent variables for the three equations become \(\theta\), \(\delta^*\), and \(C_t^{0.5}\). This equation is discretized differently to avoid numerical difficulties. Backward Euler is used because (2.37) becomes spatially stiff due to the small quantity \(\delta\). Then the discretized shear stress coefficient equation becomes:

\[
2 \frac{\delta_2}{C_{t_2}^{0.5}} \frac{C_{t_2}^{0.5} - C_{t_3}^{0.5}}{\xi_2 - \xi_1} = K_C \left( C_{eq_2}^{0.5} - C_{t_2}^{0.5} \right)
\]

(3.3)

where \(\delta_2\) is given by

\[
\delta_2 = \theta_2 \left[ 3.15 + \frac{1.72}{H_{k_2} - 1} \right] + \delta^*
\]

(3.4)

and \(C_{eq}\) is given by equation (2.41).
If the flow is laminar, the amplification equation (2.50) is discretized. It becomes:

\[
\frac{\hat{n}_2 - \hat{n}_1}{\xi_2 - \xi_1} = \frac{d\hat{n}}{d\xi} (H_{k_2}) \frac{\beta(H_{k_2}) + 1}{2} \alpha(H_{k_2}) \frac{1}{\theta_2}
\]

(3.4)

where \(\alpha(H_k)\) and \(\beta(H_k)\) are given from (2.48) and (2.49) respectively. The dependent variable introduced by this equation is \(\hat{n}\) which replaces \(C_f^{0.5}\) in the laminar region.

### 3.2 The numerical algorithm

The numerical algorithm solves the nonlinear discretized equations (3.1-3.4) at each station by the Newton method. The algorithm is broken down in concise subroutines in order to make it as simple as possible to follow. Appendix C contains a complete listing of the program.

One thing that can be noted about the algorithm is that the order of certain calculations affects the result. For example, in the term

\[
\frac{s \ C_f}{\theta^2}
\]

found in equation (3.1), near the leading edge \(s\) and \(\theta\) go to zero, \(C_f\) goes to infinity, but the entire term asymptotes to a constant. Hence, it is better to discretize this term as
\[ \frac{1}{2} \left( \frac{s_1 + s_2}{\theta_1 + \theta_2} \right) \frac{1}{2} \left( C_{f_1} + C_{f_2} \right) \]

rather than

since averaging a nearly constant function is accurate, while averaging near a singularity is not accurate.

For the turbulent case, if the values for all the variables at station one are given, equations (3.1), (3.2), and (3.3) are the three relations which determine the three unknowns \( \theta, \delta^*, \) and \( C_t^{0.5} \) at station two. Note that in the laminar case equation (3.4) replaces equation (3.3) and the amplification factor replaces \( C_t^{0.5} \) as an unknown. If a guess for these unknowns is available, then a better guess can be calculated using the Newton method to solve the following linearized system:

\[
\begin{bmatrix}
\Delta \theta_2 \\
\Delta \delta^*_2 \\
\Delta C_t^{0.5}
\end{bmatrix} = - \begin{bmatrix}
I
\end{bmatrix}
\begin{bmatrix}
R
\end{bmatrix}
\]

41
$\mathbf{R}$ is the vector of residuals of equations (3.1-3.3) and $\mathbf{J}$ is the $3 \times 3$ Jacobian matrix of these residuals. Both $\mathbf{R}$ and $\mathbf{J}$ are evaluated using the current guess for the values of $\theta$, $\delta^*$, and $C_t^{0.5}$ at station two. The changes produced by the Newton method are then used to update the current guess. In other words:

$$\theta_2 + \Delta \theta_2 \rightarrow \text{new } \theta_2$$

$$\delta_2^* + \Delta \delta_2^* \rightarrow \text{new } \delta_2^*$$

$$C_t^{0.5}_2 + \Delta C_t^{0.5}_2 \rightarrow \text{new } C_t^{0.5}_2$$

The process is repeated until convergence is achieved. The next downstream station can then be calculated.

The FORTRAN code in appendix C show how the linearization of the equations is done numerically and how the Newton method is applied to solve them. The code is quite readable and fairly easy to understand.
4. TEST CASES

In order to evaluate the accuracy of the integral method presented, three test cases will be considered. The first case involves a rotating disk in potential flow, the second case the boundary layer inside a conical surface due to swirl, and the last case involves a cascade with quasi three-dimensional effects.

It is interesting to note that there is not a great number of test cases available for consideration. Exact solutions to the Navier-Stokes equations in cases where the flow could be considered quasi three-dimensional are few. Also, experimental results involving rotating turbomachinery boundary layer measurements are hard to obtain. Therefore, the number of examples that can be considered and that the method can be compared to is limited.

4.1 The rotating disk

The rotating disk in a fluid at rest is a rare example where an exact solution to the Navier-Stokes equations exists. It is a problem which was considered as early as 1921 by von Karman. Good discussions of the problem are presented in Schlichting [20], (pp. 102-107, 647-652) and White [27], (pp. 164-172).

The problem involves the steady flow which results when a disk rotates at constant angular velocity \( \omega \) about the axis \( r=0 \). Figure 4.1, reproduced from Schlichting, demonstrates the nature of the flow.
Fig. 4.1 Flow in the neighborhood of a disk rotating in a fluid at rest

The viscous drag of the rotating surface sets up a swirling flow toward the disk. All three components of the velocity, u, v, and w, are involved resulting in fully three-dimensional motion. Fortunately, because of radial symmetry, these velocity components are independent of ϕ as is the pressure p. The governing equations can be non-dimensionalized and put in terms of six first order differential equations which can be solved numerically. Both Schlichting and White follow this procedure.

The problem considered here for the quasi three-dimensional flow, however, differs somewhat from the one presented in Schlichting ([20]). Now, an outer potential flow is applied to the
problem. There is streamtube expansion and rotation. Although the
governing equations remain the same as in the case of the disk
rotating in the fluid at rest (see Schlichting), now the boundary
condition far upstream has a velocity \( u \) that is not equal to zero.
For a potential

\[
\Phi = \left( \frac{1}{2} r^2 - z^2 \right) k
\]

the velocity \( u = kr \) and therefore \( u_e = kr \). This boundary condition has
to be satisfied.

When the disk is not rotating \((\omega = 0)\), the flow can be
considered axisymmetric. For a value of \( k = 40 \text{s}^{-1} \) and air being the
fluid, the boundary layer thickness calculated using the 99 percent
definition of free stream velocity is constant and has a value of
0.168cm (Schlichting [20]). The value obtained using the quasi
three-dimensional method is also constant and is 0.163cm. Both
these values show a boundary layer thickness that is quite small
and also compare well with the value White [27] gives (0.165cm).
For different values of \( k \) with no rotation and for laminar flow,
the boundary layer thickness remains constant as the thinning due
to stream acceleration exactly balances the thickening due to shear
diffusion. It also remains constant for a small value of \( k \), a value
of \( \omega \) not equal to zero, and flow that is still laminar. For \( k = 0.1 \)
and a rotational speed of 200rad/s, the integral method yields a
value of \( \delta \) of about 0.418cm which compares well with a value of
\( \delta = 0.42 \text{cm} \) for a rotating disk calculated using the finite difference
method. As the rotational speed increases, the value of $\delta$ decreases.

The boundary layer thickness is no longer constant when the flow becomes turbulent. Now, the value of $\delta$ increases over the radius of the disk. Figure 4.2 demonstrates how the boundary layer thickness varies over the radius of a disk of 1ft. diameter, rotating at $500s^{-1}$.

![Graph: Turbulent $\delta$ vs. $r$ for a rotating disk](image)

**Fig. 4.2** Turbulent $\delta$ vs. $r$ for a rotating disk

As in the laminar case, the faster the disk rotates, the smaller the value of the boundary layer thickness for the same value of $u_e$. For the turbulent case of the rotating disk, there are
no data from another source to compare with the results that the integral method gives. However, the data for the turbulent case presented above should be accurate. As will be demonstrated by the other two test cases and especially by the quasi three-dimensional cascade, the method works quite well when the flow is turbulent.

4.2 Swirling flow in a cone

The boundary layer in the swirling flow in a converging nozzle has been discussed by Taylor [23], Cooke [6], and Weber [25]. The problem considered is one in which a fluid enters tangentially a chamber of radius $R_3$ and emerges through an orifice of radius $R_2$ which is considerably less than $R_3$, after passing down a converging cone whose vertex subtends an angle $2\alpha$. If the entry conditions are suitably designed, the longitudinal component of velocity is small compared with the swirl component except at points close to the orifice. The geometry of the flow is demonstrated in figure 4.3.

In this problem, the main body of the fluid is taken to move with velocity $\omega/r$ where $r$ is the distance from the axis. If the boundary layer is assumed to be thin, then the variation in $p$ through its thickness may be neglected. There is a streamtube reduction which, for the integral method, is assumed to be proportional to the reduction in area through the converging nozzle. Taylor solves the problem for the laminar boundary layer by transforming the Navier-Stokes equations into integral equations.
and using the Karman-Polhausen method with an assumed velocity
distribution in the boundary layer. Cooke also solves the problem
using the Polhausen method, but the different procedure that he
follows gives somewhat different results. Weber provides a solution
for both the laminar and the turbulent boundary layers with the
laminar case being the same as Taylor's.

![Diagram](image)

**Fig. 4.3** Swirling flow in a converging nozzle

It is fairly simple to use the integral method to solve this
case for both laminar and turbulent flow. A long cone is used with
$\alpha$ equal to 45 degrees. The streamtube thickness variation is
proportional to the area of the cone. The normalized results
obtained are compared with the laminar and turbulent cases found in
Taylor for the laminar and Weber for the turbulent in figures 4.4
and 4.5. The solid line shows results calculated using the quasi
three-dimensional integral method and the marks show data from Taylor and Weber.

\[ \frac{\delta}{R_0} \sqrt{\frac{\omega}{v \sin \alpha}} \]

**Fig. 4.4** Laminar boundary layer flow in a converging nozzle

\[ R_1 = \frac{R}{R_0} \]

\[ \frac{\delta}{R_0} \sqrt{\frac{\omega}{v \sin \alpha}} \]

**Fig. 4.5** Turbulent boundary layer flow in a converging nozzle
As can be seen in the previous two figures, the results obtained using the new method agree well with results obtained by Taylor and Weber. In the turbulent case however, the integral method gives higher values than Weber does. These differences and the differences in the laminar case could be explained from the fact that the assumptions made to solve the problem by each researcher affect the results. Overall, the results tend to agree.

4.3 Cascade with quasi three-dimensional effects

The final test case considered involves flow in a cascade with quasi three-dimensional effects. Data obtained in this case is compared with data from Calvert and Herbert [3]. They use an inviscid-viscous interaction method to predict the blade-to-blade performance of axial compressors. They couple the inviscid and the viscous regions using an integral method similar to the one considered here for the viscous region.

Accurate prediction of the boundary layer is important since compressor performance is critically dependent upon blade surface boundary layers. The blade-to-blade flow surface is assumed to be axisymmetric, and the three-dimensional effects which are included are the varying radius and the varying streamtube thickness in the axial direction. This means that still some three-dimensional effects are neglected, but the accuracy obtained compared to the cost of the calculation is very acceptable.
The geometry of the cascade is known and the edge velocity in the boundary layer used for the quasi three-dimensional method is the one given by Calvert for a cascade known as the V1 cascade. The streamtube thickness variation is also given by Calvert and is assumed to be constant along the blade. This is not an entirely accurate assumption, but the lack of any data about this variation does not give us any other choice. The step size used is 0.1% of the chord. The inlet Mach number is 0.8, the chord has a length of 80mm, the pitch to chord/ratio is 0.45, the Reynolds number is 500,000, the flow angle is 54.5 degrees, the approximate outlet number is 8 degrees and the streamtube thickness reduction is 1.228. The Mach number distribution for the suction and pressure sides is shown in figure 4.6.

![Mach number distribution for V1 cascade](image)

**Fig. 4.6** Mach number distribution for V1 cascade
Given the boundary layer edge velocity distribution and the rest of the data, the displacement thickness can be calculated for both the suction and pressure sides. On the suction side, however, there is a problem since separation occurs. When the boundary layer approaches separation, the forward mode of calculation becomes ill-conditioned, as a small change in the inviscid Mach number distribution produces very large changes in shape parameter and displacement thickness values.

The quasi three-dimensional method works very well, as figure 4.7, shows until the point of separation. To avoid the problem, the computation can be switched to the inverse mode where the displacement thickness is given and the edge velocity is calculated. Indeed, if we do that, then we get back the Mach number distribution shown in figure 4.6. To determine where the switch from the forward mode to the inverse mode of calculation takes place, a limiting value for the shape parameter could be established. Once that value has been reached, the switch would occur.

On the pressure side of the blade, the problem of separation does not exist since the blade is operating at optimum incidence. Pressure surface boundary layer measurements were taken only at 95% chord. The value of the displacement thickness over the chord that Calvert obtains is 0.0040mm and the measured value is 0.0052mm. The value calculated by the quasi three-dimensional integral method is 0.0042mm which is in reasonable agreement with the other values. If one considers that the length of the chord is 80 mm, then the error
in absolute measurement is only 0.08mm which is a very small distance.

![Graph showing boundary layer growth in the V1 cascade.](image)

**Fig. 4.7** Boundary layer growth in the V1 cascade.

On the suction side, comparisons with the measured values of the displacement thickness after the point of separation cannot be made. If the method was coupled to an inviscid solver and the inverse mode was used, than the inverse mode of the calculation could be used and the results obtained. Up to the point of separation however, the values obtained using the current method agree very well with Calvert's calculations. A difference is that separation is predicted just a little bit sooner than in Calvert's calculations. However, the results of the current method and of
Calvert's method are within plotting accuracy. Also note, that the flow becomes turbulent at about 20% of the chord.
5. CONCLUSIONS

This thesis has presented an integral method based on the compressible integral boundary layer equations for calculating quasi three-dimensional flows. The momentum and shape parameter equations with rotational, axial, and radial effects were developed from the Prandtl boundary layer equations. The amplification factor equation for the laminar region and the shear lag coefficient equation for the turbulent region of the flow were added to the system of equations. The necessary functional dependencies for laminar and turbulent closure were presented and the equations were discretized.

Three test cases were presented. The first involved a disk with and without rotation. Without rotation, the axisymmetric stagnation flow and results for laminar flow of this kind obtained using the current method were very close to theoretical results. The boundary layer thickness remained constant for laminar flow. For turbulent flow the boundary layer thickness increased. Although results from another source were not available to compare the current method with for the turbulent case, the boundary layer thickness should be accurate.

This claim for accuracy is backed up by the results obtained in the second and third test cases. The second case involved swirling flow in a converging nozzle and had radial, rotational and streamtube thickness variation effects. The agreement with results given by Taylor [23] and Weber [25] was very good.
The final case involved quasi three-dimensional flow in a cascade and had radial and streamtube thickness variation considerations. There was a problem because the flow separated on the suction side at about 65% of the chord and the forward mode of the solver could not be used. Up to that point however, agreement with the results obtained by Calvert and Herbert [3] was very good as was the agreement with measurements on the pressure side. If the quasi three-dimensional integral method was coupled to an inviscid solver, the inverse mode could be used with the displacement thickness specified and the edge velocity the unknown.

The method has been demonstrated to be effective for various flows with some three-dimensional effects. The next step is to couple it to an inviscid solver and use it for more detailed calculations. The method does in fact represent a more economical approach to some flows where a three-dimensional finite difference calculation would be prohibitively expensive.
REFERENCES


APPENDIX A

THE DISPLACEMENT THICKNESS MODEL

The main effect of a boundary layer on the external flow is to displace the streamlines away from the surface in the direction of the surface normal. This appendix shows that the quasi three-dimensional inviscid part of the flow moves a distance equal to the displacement thickness of the boundary layer.

If conservation of mass is to hold for the viscous flow situation shown in figure A.1,

![Figure A.1 Viscous flow boundary layer](image)

the resulting equation will be:

\[
\rho_e v_e R_h = \int_0^{\eta_e} \frac{\partial}{\partial \eta} (\rho v R_h) \, d\eta
\]

\[
= - \int_0^{\eta_e} \frac{\partial}{\partial \xi} (\rho u R_h) \, d\eta
\]
where
\[
\frac{d\xi}{R} = \frac{ds}{R}, \quad \frac{dn}{R}
\]

and \(s\) and \(n\) are the surface tangential and surface normal coordinates respectively.

After dividing and multiplying through by \(R\), and adding and subtracting
\[
\frac{d}{d\xi} \left( \rho e u e R h \right)
\]

the equation becomes
\[
\rho e v e R h = \int_{0}^{\eta e} \left[ \frac{1}{R} \frac{d}{d\xi} \left( \rho e u e - \rho u \right) R h \right] R d\eta - \frac{d}{d\xi} \left( \rho e u e R h \right) \eta e
\]

But,
\[
\delta^* = \int_{0}^{\infty} \left( 1 - \frac{\rho u}{\rho e u e} \right) R d\eta \quad \text{and} \quad d\xi = \frac{ds}{R}
\]

and so the expression can be rewritten as
\[
\rho e v e R h = \frac{d}{ds} \left( \rho e u e \delta^* R h \right) - \frac{d}{ds} \left( \rho e u e R h \right) R \eta e \quad (A.1)
\]
For the equivalent inviscid flow over a displacement surface shown in figure A.2 below,

![Diagram of displacement surface](image)

**Figure A.2** The displacement thickness model

for conservation of mass to hold,

\[
\rho_{e v e R h} = \rho_{e u e} \frac{d\Delta}{ds} R h - \int_{\Delta/\eta}^{\eta e} \frac{\partial}{\partial \eta} (\rho v R h) \, d\eta
\]

If we observe that

\[
\int_{\Delta/\eta}^{\eta e} \frac{\partial}{\partial \eta} (\rho v R h) \, d\eta = - \int_{\Delta/\xi}^{\eta e} \frac{\partial}{\partial \xi} (\rho_{e u e R h}) \, d\eta
\]

and since

\[
\frac{d}{ds} (\rho_{e u e \Delta R h}) = \rho_{e u e R h} \frac{d\Delta}{ds} - \Delta \frac{d}{ds} (\rho_{e u e R h})
\]

we are led to the result that
\[
\rho_e v_e R_h = \frac{d}{ds} \left( \rho_e u_e \Delta R_h \right) - \frac{d}{ds} (\rho_e u_e R_h) \eta_e
\]  \hspace{1cm} (A.2)

The requirement that \( v_e \) is the same for both the viscous and the inviscid flows, and the comparison of equations (A.1) and (A.2) suggests that \( \Delta = \delta^* \) for the quasi three-dimensional case, a result which is the same as the one obtained in the two-dimensional case.
APPENDIX B

DERIVATION OF THE INTEGRAL BOUNDARY LAYER EQUATIONS

The integral boundary layer equations can be derived starting from the well known boundary layer equations for two dimensions first put forth by Prandtl in 1904.

\[
\text{mass: } \frac{\partial (\rho u)}{\partial s} + \frac{\partial (\rho v)}{\partial n} = 0 \quad (B.1)
\]

\[
\text{momentum: } \rho u \frac{\partial u}{\partial s} + \rho v \frac{\partial u}{\partial n} = \rho e \frac{\partial u_e}{\partial s} + \frac{\partial \tau}{\partial n} \quad (B.2)
\]

For the quasi three-dimensional case the non-dimensionalized coordinates \( \xi \) and \( \eta \) are introduced defined by \( ds = R d\xi \) and \( dn = R d\eta \). When the effects due to \( h, R, \) and \( \omega \) are taken into account and the continuity and momentum equations are applied to a cube of dimensions \( R d\xi \times R d\eta \times h \) these equations become:

\[
\frac{\partial}{\partial \xi} (\rho u R h d\eta) + \frac{\partial}{\partial \eta} (\rho v R h d\xi) d\eta = 0
\]

or,

\[
\frac{\partial}{\partial \xi} (\rho u R h) + \frac{\partial}{\partial \eta} (\rho v R h) = 0 \quad (B.3)
\]

and

\[
\rho u \frac{\partial \xi}{\partial \xi} + \rho v \frac{\partial \eta}{\partial \eta} - \rho \omega^2 R \frac{\partial R}{\partial \xi} = \rho e \frac{\partial u_e}{\partial \xi} - \rho e \omega^2 R \frac{\partial R}{\partial \eta} + \frac{\partial \tau}{\partial \eta} \quad (B.4)
\]
In order to derive the integral momentum equation, equation (B.3) is multiplied by \( u \), (B.4) by \( Rh \), the results are then added and integrated across the boundary layer. Then:

\[
\begin{align*}
    u \frac{\partial}{\partial \xi} (puRh) + u \frac{\partial}{\partial \eta} (pvRh) + puRh \frac{\partial u}{\partial \xi} + pvRh \frac{\partial u}{\partial \eta} - Rh \left( \rho_e - \rho \right) \omega^2 R \frac{dR}{d\xi} \\
    - \rho_{eueR} \frac{du_e}{d\xi} - Rh \frac{\partial \tau}{\partial \eta} &= 0
\end{align*}
\]

or, after applying the chain rule and multiplying by \( R \),

\[
\begin{align*}
    \int_0^\delta \left( \frac{\partial}{\partial \xi} (pu^2 Rh) + \frac{\partial}{\partial \eta} (puvRh) + (\rho_e - \rho) \omega^2 R^2 h \frac{dR}{d\xi} \\
    - \rho_{eueRh} \frac{du_e}{d\xi} - Rh \frac{\partial \tau}{\partial \eta} \right) R d\eta &= 0
\end{align*}
\]

However,

\[
\begin{align*}
    \int_0^\delta \left( \frac{\partial}{\partial \xi} (puvRh) \right) R d\eta &= \rho_{euev_eR^2h} = -\int_0^\delta \left( u_e \frac{\partial}{\partial \xi} (puRh) \right) R d\eta,
\end{align*}
\]

and,

\[
\begin{align*}
    \int_0^\delta Rh \frac{\partial \tau}{\partial \eta} R d\eta &= -R^2 h \tau_w
\end{align*}
\]
Using these relations, applying the chain rule again, and noting that the integrand is 0 for $\eta > \delta$:

\[
\int_{0}^{\infty} \frac{\partial}{\partial \xi} \left[ \rho u R \left( u - u_e \right) \right] R \, d\eta + \int_{0}^{\infty} (\rho u - \rho_e u_e) R \frac{du_e}{\partial \xi} R \, d\eta
\]

\[
+ \int_{0}^{\infty} (\rho_e - \rho) \omega^2 R^2 h \frac{dR}{\partial \xi} R \, d\eta + R^2 \tau_w = 0
\]

The displacement thickness $\delta^*$, the momentum thickness $\theta$, the skin friction coefficient $C_f$, and $\delta_p$ are defined as:

\[
\delta^* = \int_{0}^{\infty} \left( 1 - \frac{\rho u}{\rho_e u_e} \right) R \, d\eta, \quad \delta_p = \int_{0}^{\infty} \left( 1 - \frac{\rho}{\rho_e} \right) R \, d\eta,
\]

\[
\theta = \int_{0}^{\infty} \left( 1 - \frac{u}{u_e} \right) \frac{\rho u}{\rho_e u_e} R \, d\eta, \quad C_f = \frac{2 \tau_w}{\rho_e u_e^2}
\]

Substituting these definitions in the equation, it becomes

\[- \frac{d}{\partial \xi} \left( \rho_h (u - u_e)^2 \right) - \rho_e u_e \delta^* \frac{du_e}{\partial \xi} + \rho_e \omega^2 R^2 h \delta_p \frac{dR}{\partial \xi} + R^2 \tau_w = 0\]

Defining the shape parameters $H$ and $H_p$ as

\[H = \frac{\delta^*}{\theta}, \quad H_p = \frac{\delta_p}{\theta},\]

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then dividing through by $R^2 \rho_e u_e^2 h \theta$, noting that $ds = R d\xi$ is the true arc length, and substituting in the equation, we obtain:

\[
\frac{1}{R h} \frac{d(Rh)}{ds} + \frac{1}{\rho_e} \frac{d\rho}{ds} + \frac{2}{u_e} \frac{d u_e}{ds} + \frac{1}{\theta} \frac{d\theta}{ds} = \frac{1}{2} \frac{C_f}{\theta} - H \frac{1}{u_e} \frac{d u_e}{ds} + \left(\frac{\omega R}{u_e}\right)^2 H_p \frac{1}{R} \frac{dR}{ds}
\]

But,

\[
\frac{1}{\rho_e} \frac{d\rho}{ds} = \left(\frac{\omega R}{u_e}\right)^2 \frac{1}{M_e^2} \frac{dR}{R ds} - M_e^2 \frac{1}{u_e} \frac{d u_e}{ds}
\]

and so, finally:

\[
\frac{1}{\theta} \frac{d\theta}{ds} + \frac{1}{R h} \frac{d(Rh)}{ds} = \frac{1}{2} \frac{C_f}{\theta} - H \frac{1}{u_e} \frac{d u_e}{ds} + \left(\frac{\omega R}{u_e}\right)^2 \frac{1}{R} \frac{dR}{ds}
\]

(B.5)

The kinetic energy equation can be derived if (B.3) is multiplied by $(u^2 - u_e^2)$, (B.4) by $2uR h$, the results are added, and then integrated once more over the boundary layer. Then:
\[
\int_0^\infty \left( u^2 - u_e^2 \right) \frac{\partial}{\partial \xi} (\rho u) \, R \, d\eta + \int_0^\infty r u_R h 2u_e \frac{\partial u_e}{\partial \xi} \, R \, d\eta^* + \int_0^\infty (\rho - \rho_e) u_R h 2u_e \frac{\partial u_e}{\partial \eta} \, R \, d\eta^* - \int_0^\infty \left[ \left( u^2 - u_e^2 \right) \rho v_R \right] R \, d\eta^* + \int_0^\infty 2u_R \frac{\partial \tau}{\partial \eta} \, R \, d\eta^* + \int_0^\infty (\rho_e - \rho) 2u^2 R^2 h \frac{dR}{d\xi} \, R \, d\eta^* = 0
\]

Applying the chain rule:

\[
\int_0^\infty \frac{\partial}{\partial \xi} \left[ \rho u_R \left( u^2 - u_e^2 \right) \right] R \, d\eta + \int_0^\infty (\rho - \rho_e) u_R h 2u_e \frac{\partial u_e}{\partial \xi} \, R \, d\eta^* - \left[ 2u^2 h R \right]_0^R + \int_0^\infty 2u_R \tau \frac{\partial u}{\partial \eta} \, R \, d\eta^* - \int_0^\infty (\rho_e - \rho) 2u^2 R^2 h \frac{dR}{d\xi} \, R \, d\eta^* = 0
\]

Defining the kinetic energy thickness \( \theta^* \), the density thickness \( \delta^{**} \) and the dissipation coefficient \( C_D \) as

\[
\delta^{**} = \int_0^\infty \left( 1 - \frac{\rho}{\rho_e} \right) \frac{u}{u_e} \, R \, d\eta, \quad \theta^* = \int_0^\infty \left[ 1 - \left( \frac{u}{u_e} \right)^2 \right] \frac{\rho u}{\rho u_e} \, R \, d\eta
\]

\[
C_D = \frac{1}{\rho_e u_e^2} \int_0^\infty \tau \frac{\partial u}{\partial \eta} \, d\eta
\]

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and the shape parameters

\[ H^* = \frac{\theta^*}{\theta}, \quad H^{**} = \frac{\delta^{**}}{\theta} \]

the kinetic energy equation can be expressed as:

\[
\frac{1}{R:\theta} \frac{d(Rh)}{ds} + \frac{1}{\rho_e} \frac{d\rho_e}{ds} + 3 \frac{du_e}{ds} + \frac{1}{\theta^*} \frac{d\theta^*}{ds} + 2\delta^{**} \frac{1}{\theta^*} \frac{du_e}{ds} - 2\delta^{**} \left(\frac{\omega R}{u_e}\right)^2 \frac{1}{R} \frac{dR}{ds} = \frac{2C_p}{\theta^*}
\]

or, finally:

\[
\frac{1}{\theta^*} \frac{d\theta^*}{ds} + \frac{1}{R} \frac{d(Rh)}{ds} + \left(2 \frac{H^{**}}{H^*} + 3 - M_0^2\right) \frac{1}{u_e} \frac{du_e}{ds} - \left(2 \frac{H^{**}}{H^*} - M_0^2\right) \left(\frac{\omega R}{u_e}\right)^2 \frac{1}{R} \frac{dR}{ds} = \frac{2C_p}{\theta^*} \quad (B.6)
\]

However,

\[ H^* = \frac{\theta^*}{\theta} \]

and so

\[
\frac{1}{\theta^*} \frac{d\theta^*}{ds} = \frac{1}{H^*} \frac{dH^*}{ds} + \frac{1}{H^*} \frac{d\theta}{\theta \; ds}
\]
Using this result and substituting the momentum equation into the kinetic energy equation leads to the shape parameter equation:

\[
\frac{1}{H^*} \frac{dH^*}{ds} = \frac{1}{\theta} \frac{2 \cdot C_D}{H^*} - \frac{1}{\theta} \frac{2 \cdot C_f}{2} - \left( \frac{2}{H^*} \frac{H^{**} + 1 - H}{H^*} \right) \frac{1}{u_e} \frac{du_e}{ds} - \left( H_p - 2 \frac{H^{**}}{H^*} \right) \left( \frac{dR}{u_e} \right)^2 \frac{1}{R} \frac{dR}{ds}
\]

(B.7)

This equation along with the momentum equation (B.5) are the two integral compressible boundary layer equations used in the method presented in this thesis.
APPENDIX C

PROGRAM LISTING

A listing of the main program subroutine that does the integral boundary layer calculations is provided in the pages that follow. There are numerous comments explaining what the certain parts of the program do, and the logical steps taken to reach the solution can be followed fairly easily.
SUBROUTINE BLINT
& (RE,N,X,XTR1,UDM,GHT,IMODE,UEDGE,DSTAR,THETA,CF,NDONE,
& OMEGA,RB,SH)

***************************************************************************
* Compressible, Laminar/Turbulent Integral Boundary Layer Program
* Component of ISES airfoil design/analysis code
* Solution Scheme:
* Momentum and Dissipation Equation Solution
* Falkner-Skan, G-beta parameter closure
* Options currently implemented
* (streamwise quantity prescribed:
  1) Ue
  2) Dstar
  3) Ue* Dstar  (= mass defect )
  4) Mass flow + channel height (IBLT)
* Mark Drela  Sept 1986
* MIT Computational Fluid Dynamics Lab
* Anastasios Ioannidis Jan 1990
***************************************************************************

Input:
------

RE  Reynolds number based on reference quantities
N  number of streamwise points
X(.)  streamwise arc length array
XTR1  forced transition x location
   - free transition will occur if detected earlier
   - set XTR1=999.0 to guarantee laminar flow
UDM(.)  specified quantity array contining:
   - IMODE=1:  Uedge
   - IMODE=2:  Dstar
   - IMODE=3:  Mass defect (Uedge* Dstar)
   - IMODE=4:  Total channel mass flux
HGT(.)  channel height array
   - used only if IMODE=4 ...
    UEDGE = UDM / (HGT - Dstar)
IMODE  mode switch (see above)

New input: (This is the input that is new to quasi 3-D flows
-------

OMEGA  This is the rotational speed of the body
RB(.)  This is the body radius to the point considered
SH(.): This is the streamtube reduction ratio due to the geometry of the body

Output:
--------

UEDGE(.): edge velocity array
DSTAR(.): displacement thickness array
THETA(.): momentum thickness array
CF(.): skin friction coefficient array
NDONE(.): index of last calculated point
  - if calculation was successful, NDONE=N

Certain parameter values in the program may change for certain cases.

The include files 'blq3d.inc' and 'iblq3d.inc' contain variables some or all of which are shared from the various subroutines called by BLINT.

INCLUDE 'blq3d.inc'
INCLUDE 'iblq3d.inc'
DIMENSION X(N), UDM(N), HGT(N), UEDGE(N), DSTAR(N), THETA(N), CF(N)

DIMENSION RB(N), SH(N)

IF(N.GT.IMAX) STOP 'BLINT: Array overflow... N > IMAX'

C---- default quantities:  RSTBL is the stagnation density
                            HSTBL is the stagnation enthalpy
                            GAMBL is Cp/Cv
                            HVISBL is Sutherland's constant
                            AMCRIT is critical A/Ao for transition

by varying the value of HSTBL compressibility effects can be considered or neglected.

RSTBL = 1.0
HSTBL = 250.0
GAMBL = 1.4
GM1BL = GAMBL - 1.0
HVISBL = 0.4*HSTBL
AMCRIT = 4.0

IEND = N
REYBL = RE
KODE = IMODE

ROTVEL = OMEGA

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DO 5 I=1,IEND
   XI(I) = X(I)
   SPEC(I) = UDM(I)
   HEIGHT(I) = HGT(I)
RBODY(I) = RB(I)
STUBEH(I) = SH(I)
5 CONTINUE

C----- various upwinding parameters
C- 0.5 = Trapezoidal (most accurate)
C- 1.0 = Backward Euler (most stable)
C- WC refers to the shear coefficient lag equation
C- WT refers to the momentum equation
C- WH refers to the shape parameter equation
WC2 = 0.5
WT2 = 0.5
WH2 = 1.0
WC1 = 1.0 - WC2
WT1 = 1.0 - WT2
WH1 = 1.0 - WH2

C----- shape parameter for separation criterion
HILMAX = 4.0

C----- set forced transition arc length position
XIFORC = XTR1

C----- initialize similarity station with Thwaites' relations
I = 2
XSI = XI(I)

RBI = RBODY(I)
SHI = STUBEH(I)
IF(KODE.EQ.1) THEN
   UEI = SPEC(I)
   BULE = ALOG(SPEC(I+1)/SPEC(I)) / ALOG(XI(I+1)/XI(I))
   TSQ = 0.45/(UEI*(5.0*BULE+1.0)*RE) * XSI
   THI = SQRT(TSQ)
   DSI = 2.4*THI
ELSE IF(KODE.EQ.2) THEN
   DSI = SPEC(I)
   BDLE = ALOG(SPEC(I+1)/SPEC(I)) / ALOG(XI(I+1)/XI(I))
   BULE = 1.0 - 2.0*BDLE
   THI = DSI/2.4
   TSQ = THI*THI
   UEI = 0.45/(TSQ*(5.0*BULE+1.0)*RE) * XSI
ELSE IF(KODE.EQ.3) THEN
   MDI = SPEC(I)
   BMLE = ALOG(SPEC(I+1)/SPEC(I)) / ALOG(XI(I+1)/XI(I))
   BULE = 2.0*BMLE - 1.0
   UTSQ = 0.45/((5.0*BULE+1.0)*RE) * XSI
THI = UTSQ*2.4 / MDI
DSI = 2.4*THI
UEI = MDI/DSI
ELSE IF(KODE.EQ.4) THEN
   BULE = -ALOG(HEIGHT(I+1)/HEIGHT(I)) / ALOG(XI(I+1)/XI(I))
   UEI = SPEC(I)/HEIGHT(I)
   TSQ = 0.45/(UEI*(5.0*BULE+1.0)*RE) * XSI
   THI = SQRT(TSQ)
   DSI = 2.4*THI
ENDIF
C
C------ initialize amplification ratio
   AMI = 0.
C
C------ initialize Ctau for first turbulent station
   CTI = 0.03
C
   TRAN = .FALSE.
   TURB = .FALSE.
   ITTRAN = IEND
C
C------ march downstream
   NDONE = 0
   DO 1000 I=2, IEND
      IM = I-1
C
      SIMI = I.EQ.2
      WAKE = .FALSE.
C
C------ prescribed quantities
   XSI = XI(I)
   IF(KODE.EQ.1) UEI = SPEC(I)
C
   RBI = RBODY(I)
   SHI = STUBEH(I)
C
C------ check for transition and set appropriate flags and things
   IF((I.GT.2) .AND. (.NOT.TURB)) CALL TRCHEK
C
C------ Newton iteration loop for current station
   DO 100 ITBI=1, 25
C------ assemble 10x3 linearized system for dCtau, dTh, dDs, dUe, dXi
C   at the previous "1" station and the current "2" station
C   (the "1" station coefficients will be ignored)
C
   CALL BLSYS
C
   IF(KODE.EQ.1) THEN
      VS2(4,1) = 0.
      VS2(4,2) = 0.
      VS2(4,3) = 0.
      VS2(4,4) = 1.0
      VSRE2(4) = 0.
   ELSE IF(KODE.EQ.2) THEN
      VS2(4,1) = 0.
      VS2(4,2) = 0.
      VS2(4,3) = 1.0
   ENDIF
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VS2(4,4) = 0.
VSREZ(4) = SPEC(I) - DSI
ELSE IF(KODE.EQ.3) THEN
  VS2(4,1) = 0.
  VS2(4,2) = 0.
  VS2(4,3) = UEI
  VS2(4,4) = DSI
  VSREZ(4) = SPEC(I) - UEI*DSI
ELSE IF(KODE.EQ.4) THEN
  UESPEC = SPEC(I) / (R2*(HEIGHT(I)-DSI))
  VS2(4,1) = 0.
  VS2(4,2) = 0.
  VS2(4,3) = -UESPEC/(HEIGHT(I)-DSI) + UESPEC/R2 * R2U2
  VS2(4,4) = 1.0
  VSREZ(4) = UESPEC - DSI
ENDIF

C----- solve Newton system for current "2" station
CALL SOLVIT(4,4,VS2,VSREZ)

C----- determine max changes and underrelax if necessary
D MAX = AMAX1 ( ABS(VSREZ(2)/THI),
     &    ABS(VSREZ(3)/DSI) )
IF(I.LT.ITRAN) D MAX = AMAX1 ( D MAX, ABS(VSREZ(1)/AMCRIT))
IF(I.GE.ITRAN) D MAX = AMAX1 ( D MAX, ABS(VSREZ(1)/CTI) )

C
RLX = 1.0
IF(DMAX.GT.0.3) RLX = 0.3/DMAX

C----- update as usual
IF(I.LT.ITRAN) AMI = AMI + RLX*VSREZ(1)
IF(I.GE.ITRAN) CTI = CTI + RLX*VSREZ(1)
THI = THI + RLX*VSREZ(2)
DSI = DSI + RLX*VSREZ(3)
UEI = UEI + RLX*VSREZ(4)

C
IF(KODE.EQ.1 .AND. DSI/THI.GT.HLMAX) THEN
  WRITE(6,*) 'BLINT: Separation at x = ',XI(I)
  NDONE = I-1
RETURN
ENDIF
STOP
C
IF(DMAX.LE.1.0E-5) GO TO 110
C
CONTINUE
WRITE(6,*) 'BLINT: Convergence failed at x = ',XI(I)
WRITE(6,*) 'BL problem is probably physically ill-posed'
NDONE = I-1
RETURN
STOP
C
C----- pick up here after the Newton iterations
110    CONTINUE
   NDONE = I
C
C---- store primary variables
UEDGE(I) = UEI
DSTAR(I) = DSI
THETA(I) = THI
CF(I) = CF2
C---- the next two lines can become of use
C
RBODY(I) = RBI
STUBEH(I) = SHI
C
C---- set "1" variables to "2" variables for
C next streamwise station
DO 310 NC=1, NCOM
   COM1(NC) = COM2(NC)
310 CONTINUE
C
C---- turbulent intervals will follow transition interval
IF(TRAN) TURB = .TRUE.
TRAN = .FALSE.
C
1000 CONTINUE
C
RETURN
C---- here subroutine BLINT ends
END
SUBROUTINE TRCHEK
C
C Checks if free or forced transition
C occurred in the current x interval
C
INCLUDE 'blq3d.inc'
INCLUDE 'iblq3d.inc'
C
IF(XIFORC.EQ.999.0) RETURN
C
XI = XI(I-1)
X2 = XI(I )
C
C---- calculate AMPL2 value
CALL DAMPL( HK1, T1, RT1, AX, AXHK1, AXT1 )
AMPL2 = AMPL1 + AX*(X2-X1)
AMI = AMPL2
C
C---- test for free or forced transition
TRFREE = AMI.GE.AMCRT
TRFORC = XIFORC.GT.X1 .AND. XIFORC.LE.X2
C
C---- set transition interval flag
TRAN = TRFORC .OR. TRFREE
C
IF(.NOT.TRAN) THEN
   ITRAN = I+2
RETURN
ENDIF
C
C---- resolve if both forced and free transition
IF(TRFREE .AND. TRFORC) THEN
   TRFORC = XIFORC .LT. XT
   TRFREE = XIFORC .GE. XT
ENDIF
C
IF(TRFORC) THEN
C---- if forced transition, then XT is prescribed
XI = XIFORC
XTA1 = 0.
XTX1 = 0.
XTD1 = 0.
XTU1 = 0.
XTX2 = 0.
XTT2 = 0.
XTD2 = 0.
XTU2 = 0.
ELSE
C---- if free transition, XT is related to BL variables
XT = (AMCRIT-AMPL1)/AX + XI
XTAX = -(AMCRIT-AMPL1)/AX**2
C
XTA1 = -1.0/AX
XTX1 = 1.0
XTT1 = XTAX*(AXHK1*HK1T1 + AXT1)
XTD1 = XTAX*(AXHK1*HK1D1)
XTU1 = XTAX*(AXHK1*HK1U1)
XTX2 = 0.
XTT2 = 0.
XTD2 = 0.
XTU2 = 0.
ENDIF

C
C----- save transition location
ITRAN = I
TFORCE = TRFORC
C
RETURN
C----- end of subroutine TRCHEK
END
SUBROUTINE BLSYS

C Sets up the BL Newton system governing the current interval
C
C | VS1 | dT1 | + | VS2 | dT2 | = | VSREZ |
C | dD1 | | dD2 |
C | dU1 | | dU2 |
C | dX1 | | dX2 |

3x5 5x1 3x5 5x1 3x1

C The system as shown corresponds to a laminar station
C If TRAN, then dS2 replaces dA2
C If TURB, then dS1, dS2 replace dA1, dA2

C................................................................
IMPLICIT REAL(M)
INCLUDE 'iblq3d.inc'
C
C---- set primary BL variables from current values
X2 = XSI
AMPL2 = AMI
S2 = CTI
U2 = UEI
T2 = THI
D2 = DSI
C
RB2 = RBI
SH2 = SHI
C
C---- calculate secondary BL variables and their sensitivities
IF(WAKE) THEN
CALL BLVAR(3)
ELSE IF(TURB.OR.TRAN) THEN
CALL BLVAR(2)
ELSE
CALL BLVAR(1)
ENDIF
C
C---- for the similarity sation, "1" and "2" variables are the same
IF(SIMI) THEN
DO 3 NC=1, NCOM
   COM1(NC) = COM2(NC)
3 CONTINUE
ENDIF
C---- now appropriate finite difference system for current interval
   IF(Tinan) THEN
       CALL TRDIF
   ELSE IF(Simi) THEN
       CALL BLDIFF(0)
   ELSE IF(.NOT. TURB) THEN
       CALL BLDIFF(1)
   ELSE IF(TURB) THEN
       CALL BLDIFF(2)
   ENDIF

C
   IF(Simi) THEN
C---- at similarity station, "l" variables are really "2" variables
   DO 10 K=1, 4
       DO 101 L=1, 5
           VS2(K,L) = VS1(K,L) + VS2(K,L)
           VS1(K,L) = 0.
   101 CONTINUE
   10 CONTINUE
   ENDIF

C
   RETURN

C---- end of subroutine BLSYS
   END
SUBROUTINE BLVAR(ITYP)

C Calculates all secondary "2" variables from the primary "2" variables X2, U2, T2, D2, S2.
C Also calculates the sensitivities of the secondary variables wrt the primary variables.
C
C ITYP = 1 : laminar
C ITYP = 2 : turbulent
C ITYP = 3 : turbulent wake
C
C IMPLICIT REAL(M)
INCLUDE 'iblq3d.inc'
C
C---- shear coefficient constan (proportional to Cebeci & Smith 0.0168)
CTCON = 0.015
C
C---- set edge Mach number ** 2
M2 = U2*U2 / (GM1BL*(STBL-0.5*U2*U2))
TR2 = 1.0 + 0.5*GM1BL*M2
M2U2 = 2.0*M2*TR2/U2
C
C---- set edge static density (isentropic relation)
R2 = RSTBL*TR2**(-1.0/GM1BL)
R2U2 = -R2/TR2 * 0.5*M2U2
C
C---- set shape parameter
H2 = D2/T2
H2D2 = 1.0/T2
H2T2 = -H2/T2
C
C---- set edge static enthalpy and molecular viscosity
HEDGE = HSTBL - 0.5*U2*U2
V2=SQR((HEDGE/HSTBL)**3)*((HSTBL+HVISBL)/(HEDGE+HVISBL)/REYBL
V2U2 = V2*U2*(1.0/(HEDGE+HVISBL)-1.5/HEDGE)
C
C---- set kinematic shape parameter
CALL HKIN( H2, M2, HK2, HK2H2, HK2M2 )
C
IF(ITYP.EQ.3) HK2 = AMAX1(HK2,1.0005)
IF(ITYP.NE.3) HK2 = AMAX1(HK2,1.05)
C
HK2U2 = HK2M2*M2U2
HK2T2 = HK2H2*H2T2
HK2D2 = HK2H2*H2D2
C
C---- set momentum thickness Reynolds number
RT2 = R2*U2*T2/V2
RT2U2 = RT2*(1.0/U2 + R2U2/R2 - V2U2/V2)
RT2T2 = RT2/T2
C density thickness shape parameter (H**)
CALL HCT( HK2, M2, HC2, HC2HK2, HC2M2 )
HC2U2 = HC2HK2*HK2U2 + HC2M2*M2U2
HC2T2 = HC2HK2*HK2T2
HC2D2 = HC2HK2*HK2D2

C set new shape parameter (Hrho)
CALL HRT( HK2, M2, HR2, HR2HK2, HR2M2 )
HR2U2 = HR2HK2*HK2U2 + HR2M2*M2U2
HR2T2 = HR2HK2*HK2T2
HR2D2 = HR2HK2*HK2D2

C set KE thickness shape parameter from H - H** correlations
IF(ITYP.EQ.1) THEN
    CALL HSL( HK2, RT2, M2, HS2, HS2HK2, HS2RT2, HS2M2 )
ELSE
    CALL HST( HK2, RT2, M2, HS2, HS2HK2, HS2RT2, HS2M2 )
ENDIF

HS2U2 = HS2HK2*HK2U2 + HS2RT2*RT2U2 + HS2M2*M2U2
HS2T2 = HS2HK2*HK2T2 + HS2RT2*RT2T2
HS2D2 = HS2HK2*HK2D2

C normalized slip velocity Us
US2 = 0.5*HS2*(3.0 - 4.0*(HK2-1.0)/H2)/3.0
US2HS2 = 0.5*(3.0 - 4.0*(HK2-1.0)/H2)/3.0
US2HK2 = 0.5*HS2*(-4.0)/H2)/3.0
US2H2 = 0.5*HS2*(-4.0*(HK2-1.0)/H2**2)/3.0

C US2U2 = US2HS2*HS2U2 + US2HK2*HK2U2
US2T2 = US2HS2*HS2T2 + US2HK2*HK2T2 + US2H2*H2T2
US2D2 = US2HS2*HS2D2 + US2HK2*HK2D2 + US2H2*H2D2

C equilibrium wake layer shear coefficient (Ctau)EQ ** 1/2
HKB = HK2 - 1.0
USB = 1.0 - US2
CQ2 =
   & Sqrt( CTCNOW2*HS2*HKB**3 / (USB*H2**2*HKB**2) )
CQ2HS2 = CTCNOW2 * HKB**3 / (USB*H2**2*HKB**2) * 0.5/CQ2
CQ2US2 = CTCNOW2*HS2*HKB**3 / (USB*H2**2*HKB**2) / USB * 0.5/CQ2
CQ2HK2 = CTCNOW2*HS2*HKB**2 / (USB*H2**2*HKB**2) * 3.0 * 0.5/CQ2
& - CTCNOW2*HS2*HKB**3 / (USB*H2**2*HKB**2) * 2.0 * 0.5/CQ2
CQ2H2 = -CTCNOW2*HS2*HKB**3 / (USB*H2**2*HKB**2) / H2 * 0.5/CQ2

C CQ2U2 = CQ2HS2*HS2U2 + CQ2HK2*HK2U2 + CQ2US2*US2U2
CQ2T2 = CQ2HS2*HS2T2 + CQ2HK2*HK2T2 + CQ2US2*US2T2
CQ2D2 = CQ2HS2*HS2D2 + CQ2HK2*HK2D2 + CQ2US2*US2D2

C CQ2T2 = CQ2T2 + CQ2H2*H2T2
CQ2D2 = CQ2D2 + CQ2H2*H2D2

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C skin friction coefficient (zero in wake)
IF(ITYP.EQ.3) THEN
    CF2 = 0.
    CF2HK2 = 0.
    CF2RT2 = 0.
    CF2M2 = 0.
ELSE IF(ITYP.EQ.1) THEN
    CALL CFL( HK2, RT2, M2, CF2, CF2HK2, CF2RT2, CF2M2 )
ELSE
    CALL CFT( HK2, RT2, M2, CF2, CF2HK2, CF2RT2, CF2M2 )
ENDIF

CF2U2 = CF2HK2*HK2U2 + CF2RT2*RFT2U2 + CF2M2*M2U2
CF2T2 = CF2HK2*HK2T2 + CF2RT2*RRT2T2
CF2D2 = CF2HK2*HK2D2

C set similarity variables if not defined
IF(SIMI) THEN
    HK1 = HK2
    HK1T1 = HK2T2
    HK1D1 = HK2D2
    HK1U1 = HK2U2
    RT1 = RT2
    RT1T1 = RT2T2
    RT1U1 = RT2U2
    M1 = M2
    M1U1 = M2U2
ENDIF

C define stuff for midpoint CF
HKA = 0.5*(HK1 + HK2)
RTA = 0.5*(RT1 + RT2)
MA = 0.5*(M1 + M2)

C midpoint skin friction coefficient (zero in wake)
IF(ITYP.EQ.3) THEN
    CFA = 0.
    CFAHKA = 0.
    CFARTA = 0.
    CFAMA = 0.
ELSE IF(ITYP.EQ.1) THEN
    CALL CFL( HKA, RTA, MA, CFA, CFAHKA, CFARTA, CFAMA )
ELSE
    CALL CFT( HKA, RTA, MA, CFA, CFAHKA, CFARTA, CFAMA )
ENDIF

C dissipation function
IF(ITYP.EQ.1) THEN
    CALL DIL( HK2, RT2, DI2, DI2HK2, DI2RT2 )
C
DI2U2 = DI2HK2*HK2U2 + DI2RT2*RRT2U2
DI2T2 = DI2HK2*HK2T2 + DI2RT2*RRT2T2
DI2D2 = DI2HK2*HK2D2
DI2C2 = 0.

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ELSE
   CALL DIT( HS2, US2, CF2, S2, DI2, &
            DI2HS2, DI2US2, DI2CF2, DI2S2 )
C
   DI2U2 = DI2HS2*HS2U2 + DI2US2*US2U2 + DI2CF2*CF2U2
   DI2T2 = DI2HS2*HS2T2 + DI2US2*US2T2 + DI2CF2*CF2T2
   DI2D2 = DI2HS2*HS2D2 + DI2US2*US2D2 + DI2CF2*CF2D2
ENDIF
C
C----- BL thickness (Delta) from simplified Green's correlation
   DE2 = (3.15 + 1.72/(HK2-1.0))**T2 + D2
   DE2HK2 = ( - 1.72/(HK2-1.0)**2)**T2
C
   DE2U2 = DE2HK2*HK2U2
   DE2T2 = DE2HK2*HK2T2 + (3.15 + 1.72/(HK2-1.0))
   DE2D2 = DE2HK2*HK2D2 + 1.0
C
RETURN
C----- end of subroutine BLVAR
END
SURBROUTINE TRDIF

C Sets up the Newton system governing the transition interval. Equations governing
C the laminar part X1 < xi < XT and
C the turbulent part XT < xi < X2
C are simply summed.

IMPLICIT REAL(M)
INCLUDE 'ib1q3d.inc'
REAL BL1(4,5), BL2(4,5), BLREZ(4)
& BT1(4,5), BT2(4,5), BTREZ(4)

C---- save variables and sensitivities for future restoration
DO 5 NC=1, NCOM
  C1SAV(NC) = COM1(NC)
  C2SAV(NC) = COM2(NC)
5 CONTINUE

C---- weighting factors for linear interpolation to transition point
WF2 = (XT-X1)/(X2-X1)
WF2XT = 1.0/(X2-X1)

C
WF2A1 = WF2XT*XTA1
WF2X1 = WF2XT*XTX1 + (WF2-1.0)/(X2-X1)
WF2X2 = WF2XT*XTX2 - WF2/(X2-X1)
WF2T1 = WF2XT*XTT1
WF2T2 = WF2XT*XTT2
WF2D1 = WF2XT*XTD1
WF2D2 = WF2XT*XTD2
WF2U1 = WF2XT*XTU1
WF2U2 = WF2XT*XTU2

C
WF1 = 1.0 - WF2
WF1A1 = -WF2A1
WF1X1 = -WF2X1
WF1X2 = -WF2X2
WF1T1 = -WF2T1
WF1T2 = -WF2T2
WF1D1 = -WF2D1
WF1D2 = -WF2D2
WF1U1 = -WF2U1
WF1U2 = -WF2U2

C
C
C**** FIRST, do laminar part between X1 and XT
C
C---- interpolate primary variables to transition point
TT = T1*WF1 + T2*WF2
TTA1 = T1*WF1A1 + T2*WF2A1
TTX1 = T1*WF1X1 + T2*WF2X1
TTX2 = T1*WF1X2 + T2*WF2X2
TTT1 = T1*WF1T1 + T2*WF2T1 + WF1
TTT2 = T1*WF1T2 + T2*WF2T2 + WF2
TTD1 = T1*WF1D1 + T2*WF2D1
TTD2 = T1*WF1D2 + T2*WF2D2

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TTU1 = T1*WF1U1 + T2*WF2U1
TTU2 = T1*WF1U2 + T2*WF2U2

C

DT = D1*WF1 + D2*WF2
DTA1 = D1*WF1A1 + D2*WF2A1
DTX1 = D1*WF1X1 + D2*WF2X1
DXX2 = D1*WF1X2 + D2*WF2X2
DTT1 = D1*WF1T1 + D2*WF2T1
DTT2 = D1*WF1T2 + D2*WF2T2
DTD1 = D1*WF1D1 + D2*WF2D1 + WF1
DTD2 = D1*WF1D2 + D2*WF2D2 + WF2
DTU1 = D1*WF1U1 + D2*WF2U1
DTU2 = D1*WF1U2 + D2*WF2U2

C

UT = U1*WF1 + U2*WF2
UTA1 = U1*WF1A1 + U2*WF2A1
UTX1 = U1*WF1X1 + U2*WF2X1
UTX2 = U1*WF1X2 + U2*WF2X2
UTT1 = U1*WF1T1 + U2*WF2T1
UTT2 = U1*WF1T2 + U2*WF2T2
UTD1 = U1*WF1D1 + U2*WF2D1
UTD2 = U1*WF1D2 + U2*WF2D2
UTU1 = U1*WF1U1 + U2*WF2U1 + WF1
UTU2 = U1*WF1U2 + U2*WF2U2 + WF2

C

C----- set "2" variables to primary "T" variables at XT
X2 = XT
U2 = UT
T2 = TT
D2 = DT

C

RBT = RB1*WF1 + RB2*WF2
SHT = SH1*WF1 + SH2*WF2

C

RB2 = RBT
SH2 = SHT

C

C----- calculate laminar secondary "T" variables
CALL BLVAR(I)

C=

C= at this point, all "2" variables
C= are really "T" variables at XT
C=

C

C----- set up Newton system for dAm, dTh, dDs, dUe, dXi at X1 and XT
CALL BLDIF(I)

C

C----- The current Newton system is in terms of "1" and "T"
C variables, so calculate its equivalent in terms of "1" and
C "2" variables. In other words, convert residual sensitivities
C wrt "T" variables into sensitivities wrt "1" and "2"
C variables. The amplification equation is unnecessary here,
C so the K=1 row is left empty.
DO 10 K=2, 3
   BLREZ(K) = VSRES(K)

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C

BL1(K,1) = VS1(K,1)
& + VS2(K,2)*TTA1
& + VS2(K,3)*DTA1
& + VS2(K,4)*UTA1
& + VS2(K,5)*XTA1

BL1(K,2) = VS1(K,2)
& + VS2(K,2)*TTT1
& + VS2(K,3)*DTT1
& + VS2(K,4)*UTT1
& + VS2(K,5)*XTT1

BL1(K,3) = VS1(K,3)
& + VS2(K,2)*TTD1
& + VS2(K,3)*DTD1
& + VS2(K,4)*UTD1
& + VS2(K,5)*XTD1

BL1(K,4) = VS1(K,4)
& + VS2(K,2)*TTU1
& + VS2(K,3)*DTU1
& + VS2(K,4)*UTU1
& + VS2(K,5)*XTU1

BL1(K,5) = VS1(K,5)
& + VS2(K,2)*TTX1
& + VS2(K,3)*DTX1
& + VS2(K,4)*UTX1
& + VS2(K,5)*XTX1

C

BL2(K,1) = 0.
BL2(K,2) = VS2(K,2)*TTT2
& + VS2(K,3)*DTT2
& + VS2(K,4)*UTT2
& + VS2(K,5)*XTT2

BL2(K,3) = VS2(K,2)*TTD2
& + VS2(K,3)*DTD2
& + VS2(K,4)*UTD2
& + VS2(K,5)*XTD2

BL2(K,4) = VS2(K,2)*TTU2
& + VS2(K,3)*DTU2
& + VS2(K,4)*UTU2
& + VS2(K,5)*XTU2

BL2(K,5) = VS2(K,2)*TTX2
& + VS2(K,3)*DTX2
& + VS2(K,4)*UTX2
& + VS2(K,5)*XTX2

C
10 CONTINUE
C
C
C**** SECOND, set up turbulent part between XT and X2 ****
C
C----- calculate equilibrium shear coefficient CQT at transition
C point
CALL BLVAR(2)
C
C---- set initial shear coefficient value ST at transition point
C
( note that CQ2, CQ2T2, etc. are really "CQT", "CQTTT", etc.)
CTR = 0.70
ST = CTR*CQ2
STTT = CTR*CQ2T2
STDT = CTR*CQ2D2
STUT = CTR*CQ2U2
C
C---- calculate ST sensitivities wrt
C
the actual "1" and "2" variables
ST1 = STTT*TTA1 + STDT*DTA1 + STUT*UTA1
STX1 = STTT*TTX1 + STDT*DTX1 + STUT*UTX1
STX2 = STTT*TTX2 + STDT*DTX2 + STUT*UTX2
STT1 = STTT*TTT1 + STDT*DTT1 + STUT*UTT1
STT2 = STTT*TTT2 + STDT*DTT2 + STUT*UTT2
STD1 = STTT*TTD1 + STDT*DTD1 + STUT*UTD1
STD2 = STTT*TTD2 + STDT*DTD2 + STUT*UTD2
STU1 = STTT*TTU1 + STDT*DTU1 + STUT*UTU1
STU2 = STTT*TTU2 + STDT*DTU2 + STUT*UTU2
C
S2 = ST
C
C---- recalculate turbulent secondary "T" variables using proper ST
CALL BLVAR(2)
C
C---- set "1" variables to "T" variables and rest "2" variables
C
to their save turbulent values
DO 30 NC=1, NCOM
   COM1(NC) = COM2(NC)
   COM2(NC) = C2SAV(NC)
30 CONTINUE
C
C---- set up Newton system for dCt, dTh, dDs, dUe, dXi at XT and X2
CALL BLDIF(2)
C
C---- convert sensitivities wrt "T" variables into sensitivities
C
wrt "1" and "2" variales as done before for the laminar part
DO 40 K=1, 3
   BTREZ(K) = VSREZ(K)
C
   BT1(K,1) = VS1(K,1)*STA1
   & + VS1(K,2)*TTA1
   & + VS1(K,3)*DTA1
   & + VS1(K,4)*UTA1
   & + VS1(K,5)*XTA1
   BT1(K,2) = VS1(K,1)*STT1
   & + VS1(K,2)*TTT1
   & + VS1(K,3)*DTT1
   & + VS1(K,4)*UTT1
   & + VS1(K,5)*XTT1
   BT1(K,3) = VS1(K,1)*STD1
   & + VS1(K,2)*TTD1
   & + VS1(K,3)*DTD1
   & + VS1(K,4)*UTD1
   & + VS1(K,5)*XTD1

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BT1(K,4) = VS1(K,1)*STU1
& + VS1(K,2)*TTU1
& + VS1(K,3)*DTU1
& + VS1(K,4)*UTU1
& + VS1(K,5)*XTU1

BT1(K,5) = VS1(K,1)*STX1
& + VS1(K,2)*TTX1
& + VS1(K,3)*DTX1
& + VS1(K,4)*UTX1
& + VS1(K,5)*XTX1

BT2(K,1) = VS2(K,1)
BT2(K,2) = VS2(K,2)
& + VS1(K,1)*TTT2
& + VS1(K,2)*TTT2
& + VS1(K,3)*DTT2
& + VS1(K,4)*DTT2
& + VS1(K,5)*XTT2

BT2(K,3) = VS2(K,3)
& + VS1(K,1)*TTT2
& + VS1(K,2)*TTT2
& + VS1(K,3)*DTT2
& + VS1(K,4)*DTT2
& + VS1(K,5)*XTT2

BT2(K,4) = VS2(K,4)
& + VS1(K,1)*STU2
& + VS1(K,2)*TTU2
& + VS1(K,3)*DTU2
& + VS1(K,4)*DTU2
& + VS1(K,5)*XTU2

BT2(K,5) = VS2(K,5)
& + VS1(K,1)*STX2
& + VS1(K,2)*TTX2
& + VS1(K,3)*DTX2
& + VS1(K,4)*DTX2
& + VS1(K,5)*XTX2

40 CONTINUE

C--- Add up laminar and turbulent parts to get final system

in terms of honest-to-goodness "1" and "2" variables

VSREZ(1) = BTREZ(1)
VSREZ(2) = BLREZ(2) + BTREZ(2)
VSREZ(3) = BLREZ(3) = BTREZ(3)

DO 60 L=1, 5
  VS1(1,L) = BT1(1,L)
  VS2(1,L) = BT2(1,L)
  VS1(2,L) = BL1(2,L) + BT1(2,L)
  VS2(2,L) = BL2(2,L) + BT2(2,L)
  VS1(3,L) = BL1(3,L) + BT1(3,L)
  VS2(3,L) = BL2(3,L) + BT2(3,L)
60 CONTINUE

91
C---- To be sanitray, restore "1" quantities which got clobbered
C in all of the numerical gymnastics above. The "2" variables
C were already restored for the XT-X2 differencing part.
DO 70 NC=1, NCOM
   COM1(NC) = C1SAV(NC)
 70 CONTINUE
C
RETURN
C---- end of subroutine TRDIF
END
SUBROUTINE BLDIF(ITYP)
C....................................................................................
C Sets up the Newton system coefficients and residuals
C
C ITYP = 0 : similarity station
C ITYP = 1 : laminar station
C ITYP = 2 : turbulent interval
C
C This routine knows nothing about a transitional
C interval which is taken care of by TRDIF.
C....................................................................................
IMPLICIT REAL(M)
INCLUDE 'ib1q3d.inc'
C
C---- shear coefficient lag constant (see M.D.'s thesis)
SCC = 4.2
C
C IF(ITYP.EQ.0) THEN
C---- similarity logarithmic differences (prescribed)
XLOG = 1.
ULOG = BULE
TLOG = 0.5*(1.0-BULE)
HLOG = 0.
DDLOG = 0.
C
RLOG = 1.
RHLOG = 1.
ELSE
C---- usual logarithmic differences
XLOG = ALOG(X2/X1)
ULOG = ALOG(U2/U1)
TLOG = ALOG(T2/T1)
HLOG = ALOG(HS2/HS1)
DDLOG = 1.0
C
RLOG = ALOG(RB2/RB1)
RHLOG = ALOG((RB2*SH2)/(RB1*SH1))
ENDIF
C
DO 55 K=1, 4
VSR2(K) = 0.
DO 551 L=1, 5
VSI(K,L) = 0.
VSI2(K,L) = 0.
551 CONTINUE
55 CONTINUE
C
IF(ITYP.EQ.0 .OR. XIFORC.EQ.99.0) THEN
C**** LE point --> set zero amplification factor
VS2(1,1) = 1.0
VSR2(1) = -AMPL2
C
ELSE IF(ITYP.EQ.1) THEN
C
C**** laminar part --> set amplification equation
CALL DAMPL(HK1, T1, RT1, AX, AXHK1, AXT1)
REZC = AMPL2 - AMPL1 - AX*(X2-X1)
ZAX = -(X2-X1)
C
VS1(1,1) = -1.0
VS1(1,2) = ZAX*AXHK1*HK1T1 - (X2-X1)*AXT1
VS1(1,3) = ZAX*AXHK1*HK1D1
VS1(1,4) = ZAX*AXHK1*HK1U1
VS1(1,5) = AX
VS2(1,1) = 1.0
VS2(1,2) = 0.
VS2(1,3) = 0.
VS2(1,4) = 0.
VS2(1,5) = -AX
VSREZ(1) = -REZC
C
ELSE
C
C**** turbulent part --> set shear lag equation
C
SA = WC1*S1 + WC2*S2
CQA = WC1*CQ1 + WC2*CQ2
DEA = WC1*DE1 + WC2*DE2
C
DXI = X2 - X1
SRAT = (S2 - S1) / SA
C
REZC = SCC*(CQA - SA)*DXI - 2.0*DEA*SRAT
C
ZSRAT = -2.0*DEA
ZCQ1 = WC1*SCC*DXI
ZCQ2 = WC2*SCC*DXI
ZDE1 = -WC1*2.0*SRAT
ZDE2 = -WC2*2.0*SRAT
ZDXI = SCC*(CQA - SA)
C
ZS1 = ZSRAT*(-1.0/SA - WC1*SRAT/SA) - WC1*SCC*DXI
ZS2 = ZSRAT*( 1.0/SA - WC2*SRAT/SA) - WC2*SCC*DXI
C
VS1(1,1) = ZS1
VS1(1,2) = ZCQ1*CQ1T1 + ZDE1*DE1T1
VS1(1,3) = ZCQ1*CQ1D1 + ZDE1*DE1D1
VS1(1,4) = ZCQ1*CQ1U1 + ZDE1*DE1U1
VS1(1,5) = -ZDXI
VS2(1,1) = ZS2
VS2(1,2) = ZCQ2*CQ2T2 + ZDE2*DE2T2
VS2(1,3) = ZCQ2*CQ2D2 + ZDE2*DE2D2
VS2(1,4) = ZCQ2*CQ2U2 + ZDE2*DE2U2
VS2(1,5) = ZDXI
VSREZ(1) = -REZC
C
ENDIF
C
C**** Set up momentum equation
XA = WT1*X1 + WT2*X2
TA = WT1*T1 + WT2*T2
HA = WT1*H1 + WT2*H2
MA = WT1*M1 + WT2*M2
CFA = WT1*CF1 + WT2*CF2
C
RBA = WT1*RB1 + WT2*RB2
HRA = WT1*HR1 + WT2*HR2
UA = WT1*U1 + WT2*U2
C
XOT = XA/TA
BTMP = HA + 2.0 - MA
C
CTMP = (ROTVEL*RBA/UA)**2
C
RE2T = TLOG + BTMP*ULOG - XLOG*0.5*CFA*XOT
& - CTMP*(HRA-MA)*RLOG + RHLOG
ZCFA = -XLOG*0.5*XOT
ZHA = ULOG
ZMA = -ULOG + CTMP*RLOG
ZTA = XLOG * 0.5*CFA*XOT/TA
ZXA = -XLOG * 0.5*CFA /TA
C
ZUA = RLOG * (HRA-MA)*2.0*CTMP/UA
ZHR = -RLOG * CTMP
ZXL = -DDLOG * 0.5*CFA*XOT
ZUL = DDLOG * BTMP
ZTL = DDLOG
C
VS1(2,2) = WT1*(ZCFA*CF1T1 + ZHA*H1T1 + ZTA) - ZTL/T1
& + WT1*(ZHR-H1T1)
VS1(2,3) = WT1*(ZCFA*CF1D1 + ZHA*H1D1 )
& + WT1*(ZHR-H1D1)
VS1(2,4) = WT1*(ZCFA*CF1U1 + ZMA*M1U1 ) - ZUL/U1
& + WT1*(ZHR-H1U1 + ZUA)
VS1(2,5) = WT1*ZXA - ZXL/X1
VS2(2,2) = WT1*(ZCFA*CF2T2 + ZHA*H2T2 + ZTA) + ZTL/T2
& + WT1*(ZHR-H2T2)
VS2(2,3) = WT1*(ZCFA*CF2D2 + ZHA*H2D2 )
& + WT1*(ZHR-H2D2)
VS2(2,4) = WT1*(ZCFA*CF2U2 + ZMA*M2U2 ) + ZUL/U2
& + WT1*(ZHR-H2U2 + ZUA)
VS2(2,5) = WT2*ZXA + ZXL/X2
VSREZ(2) = -RE2T
C
C**** set shape parameter equation
XA = WH1*X1 + WH2*X2
TA = WH1*T1 + WH2*T2
HA = WH1*H1 + WH2*H2
HSA = WH1*HS1 + WH2*HS2
HCA = WH1*HC1 + WH2*HC2
DIA = WH1*DI1 + WH2*DI2
CFA = WH1*CF1 + WH2*CF2
RBA = WH1*RB1 + WH2*RB2  
HRA = WH1*HR1 + WH2*HR2  
UA = WH1*U1 + WH2*U2  

XOT = XA/TA  
BTEMP = 2.0*HCA/HSA + 1.0 - HA  
CTMP = (ROTVEL*RBA/UA)**2  

REZH = HLOG + BTEMP*ULOG + XLOG*(0.5*CFA-DIA)*XOT  
& + (HRA - 2.0*HCA/HSA)*CTMP*RLOG  
ZCFA = XLOG*0.5*XOT  
ZDIA = -XLOG*XOT  
ZHS1 = -WH1*2.0*HCA*ULOG/HSA**2 + WH1*2.0*HCA*CTMP*RLOG/HSA**2  
& - DDLOG/HS1  
ZHS1 = -WH2*2.0*HCA*ULOG/HSA**2 + WH2*2.0*HCA*CTMP*RLOG/HSA**2  
& + DDLOG/HS2  
ZHCA = 2.0*ULOG/HSA - 2.0*RLOG*CTMP/HSA  
ZHA = -ULOG  
ZTA = -XLOG*(0.5*CFA-DIA)*XOT/TA  
ZXA = XLOG*(0.5*CFA-DIA)/TA  

ZHRA = RLOG*CTMP  
ZUA = RLOG*(-2.0)*(HRA - 2.0*HCA/HSA)*CTMP/UA  

ZXL = DDLOG*(0.5*CFA-DIA)*XOT  
ZUL = DDLOG*BTEMP  

VS1(3,1) = WH1*(ZDIA*DI1S1)  
VS1(3,2) = ZHS1*HS1T1 + WH1*(ZCFA*CF1T1 + ZHA*H1T1 + ZTA)  
& + WH1*(ZHRA*HR1T1)  
VS1(3,3) = ZHS1*HS1D1 + WH1*(ZCFA*CF1D1 + ZHA*H1D1 )  
& + WH1*(ZHRA*HR1U1 + ZUA)  
VS1(3,4) = ZHS1*HS1U1 + WH1*(ZCFA*CF1U1) - ZUL/U1  
& + WH1*(ZHRA*HR1U1 + ZUA)  
VS1(3,5) = WH1*(ZXA ) - ZXL/X1  
VS2(3,1) = WH2*(ZDIA*DI2S2)  
VS2(3,2) = ZHS2*HS2T2 + WH2*(ZCFA*CF2T2 + ZHA*H2T2 + ZTA)  
& + WH2*(ZHRA*HR2T2)  
VS2(3,3) = ZHS2*HS2D2 + WH2*(ZCFA*CF2D2 + ZHA*H2D2 )  
& + WH2*(ZHRA*HR2U2 + ZUA)  
VS2(3,4) = ZHS2*HS2U2 + WH2*(ZCFA*CF2U2) + ZUL/U2  
& + WH2*(ZHRA*HR2U2 + ZUA)  
VS2(3,5) = WH2*(ZXA ) + ZXL/X2  

VS1(3,2) = VS1(3,2) + WH1*(ZHCA*HC1T1 + ZDIA*DI1T1)  
VS1(3,3) = VS1(3,3) + WH1*(ZHCA*HC1D1 + ZDIA*DI1D1)  
VS1(3,4) = VS1(3,4) + WH1*(ZHCA*HC1U1 + ZDIA*DI1U1)  
VS2(3,2) = VS2(3,2) + WH2*(ZHCA*HC2T2 + ZDIA*DI2T2)  
VS2(3,3) = VS2(3,3) + WH2*(ZHCA*HC2D2 + ZDIA*DI2D2)  
VS2(3,4) = VS2(3,4) + WH2*(ZHCA*HC2U2 + ZDIA*DI2U2)  

VSREZ(3) = -REZH  

RETURN  
C--- end of subroutine BLDIF  
END
SUBROUTINE DAMPL( HK, TH, RT, AX, AXHK, AXTH )
C
C output: AX spatial amplification rate
C input : HK kinematic shape parameter
C TH momentum thickness
C RT momentum thickness Reynolds number
C
C---- log10(Critical Rth) - H correlation for Falkner-Skan profiles
C HMI = 1.0/(HK-1.0)
C GRCRIT = (1.415*HMI-0.489)*TANH(20.*HMI-12.9)+3.295*HMI+0.440
C
C IF(ALOG10(RT) .LT. GRCRIT) THEN
C
C---- no amplification for Rtheta < Rcrit
AX = 0.
AXHK = 0.
AXTH = 0.
C
ELSE
C---- Amplification envelope slpe correlation for Falkner-Skan
ARG = 2.4*HK - 3.7 + 2.5*TANH(1.5*(HK-3.1))
DADR = 0.01*SQRT(ARG*ARG + 0.25)
DADRHK = 0.01/SQRT(ARG*ARG + 0.25) * ARG
& * (2.4 + 3.75/(COSH(1.5*(HK-3.1)))**2)
C---- convert amplification rate in Rtheta to rate in xi
TFS = (6.54*HK - 14.07 )/HK**2
TFSHK = -(6.54 - 28.14/HK)/HK**2
BUH = (0.058*(HK-4.0)**2/(HK-1.0) - 0.068)/TFS
BUHHK = (0.058*(HK-4.0) / (HK-1.0)
& * (2.0 - (HK-4.0)/(HK-1.0)) )/TFS
& - BUH/TFS * TFSHK
AX = 0.5*(BUH+1.0)*TFS*DADR/TH
AXHK = 0.5 * TFS*DADR/TH * BUHHK
& + 0.5*(BUH+1.0) * DADR/TH * TFSHK
& + 0.5*(BUH+1.0)*TFS / TH * DADRHK
AXTH = -AX/TH
C
ENDIF
C
IF(AX.LE.0.0) THEN
AX = 0.000001
AXHK = 0.
AXTH = 0.
ENDIF
C
RETURN
C---- end of subroutine DAMPL
END
SUBROUTINE HKIN( H, MSQ, HK, HKH, HKMSQ )
REAL MSQ
C
C----- calculate kinematic shape parameter (assuming air)
C (from Whitfield)
HK = (H - 0.29*MSQ)/(1.0 + 0.113*MSQ)
HKH = 1.0/(1.0 + 0.113*MSQ)
HKMSQ = (-.29 - 0.113*HK) / (1.0 + 0.113*MSQ)
C
RETURN
C----- end of subroutine HKIN
END

SUBROUTINE HCT( HK, MSQ, HC, HCHK, HCMSQ )
REAL MSQ
C
C----- density shape parameter (from Whitfield)
HC = MSQ * (0.064/(HK-0.8) + 0.251)
HCHK = MSQ * (-.064/(HK-0.8)**2 )
HCMSQ = 0.064/(HK-0.8) + 0.251
C
RETURN
C----- end of subroutine HCT
END

SUBROUTINE DIL( HK, RT, DI, DIHK, DIRT )
C
C----- Laminar dissipation function ( 2D/H* ) (from Falkner-Skan)
IF(HK.LT.4.0) THEN
DI = ( 0.00205 * (4.0-HK)**5.5 + 0.207 ) / RT
DIHK = ( -.00205*5.5*(4.0-HK)**4.5 ) / RT
ELSE
HKB = HK - 4.0
DEN = 1.0 + 0.02*HKB**2
DI = (-.003 * HKB**2 /DEN + 0.207 ) / RT
DIHK = (-.003*2.0*HKB*(1.0/DEN - 0.02*HKB**2/DEN**2) ) / RT
ENDIF
DIRT = -DI/RT
C
RETURN
C----- end of subroutine DIL
END
SUBROUTINE DIT( HS, US, CF, ST, DI, DIHS, DIUS, DICF, DIST )
C
C---- Turbulent dissipation function (2D/H*)
  DI = ( 0.5*CF*US + ST*ST*(1.0-US) ) * 2.0/HS
  DIHS = - ( 0.5*CF*US + ST*ST*(1.0-US) ) * 2.0/HS**2
  DIUS = ( 0.5*CF + ST*ST ) * 2.0/HS
  DICF = ( 0.5 *US ) * 2.0/HS
  DIST = ( 2.0*ST*(1.0-US) ) * 2.0/HS
C
RETURN
C----- end of subroutine DIT
END

SUBROUTINE HSL( HK, RT, MSQ, HS, HSHK, HSRT, HSMSQ )
C
C---- Laminar HS correlation (from Falkner-Skan)
  IF(HK.LT.4.0) THEN
    HS = 0.076*(HK-4.0)**2/HK + 1.515
    HSHK = 0.076*(1.0-16.0/HK**2)
  ELSE
    HS = 0.020*(HK-4.0)**2/HK + 1.515
    HSHK = 0.020*(1.0-16.0/HK**2)
  ENDIF
C
  HSRT = 0.
  HSMSQ = 0.
C
RETURN
C----- end of subroutine HSL
END

SUBROUTINE HST( HK, RT, MSQ, HS, HSHK, HSRT, HSMSQ )
  REAL MSQ
C
C---- Turbulent HS correlation (from Whitfield profile family)
  GRT = ALOG(RT)
  SRT = SQRT(RT)
  HO = 3.0 + 400.0/RT
  HORT = - 400.0/RT**2
  IF(RT.LT.400.0) THEN
    HO = 4.0
    HORT = 0.0
  ENDIF
C
  IF(HK.LT.HO) THEN
    HEX = (HO-HK)**1.6
    RTMP = 0.165 - 1.6/SRT
    HT = 1.505 + 4.0/RT + RTMP*HEX/HK
    HTHK = RTMP*HEX/HK*(-1.6/(HO-HK) - 1.0/HK)
    HTRT = -4.0/RT**2 + HEX/HK*0.8/SRT/RT
    & + RTMP*HEX/HK*1.6/(HO-HK)*HORT
  ENDIF
ELSE
    HDIF = HK - HO
    RTMP = HK - HO + 4.0/GRT
C
    HTMP = 0.02/HK + 0.007*GRT/RTMP**2
    HT = 1.505 + 4.0/RT + HDIF**2 * HTMP
    HTHK = 2.0*HDIF*HTMP + HDIF**2*(-.02/HK**2 - 0.014*GRT/RTMP**3)
    HTRT = -4.0/RT**2
    & + HDIF**2*0.014/RTMP**2*(1.0 + 2.0*GRT/RTMP*4.0/GRT**2)/RT
    & + 2.0*HDIF*HTMP*(-HORT)
ENDIF
C
C----- Whitfield's minor additional compressibility correction
    FM = 1.0 + 0.014*MSQ
    HT = ( HT + 0.028*MSQ ) / FM
    HTHK = ( HTHK ) / FM
    HTRT = ( HTRT ) / FM
    HTMSQ = 0.028/FM - 0.014*HT/FM
C
C----- fudge HS slightly to make sure HS -> 2 as HK -> 1
    HTF = 0.485/9.0 * (HK-4.0)**2/HK + 1.515
    HTFHK = 0.485/9.0 * (1.0-16.0/HK**2)
    ARG = AMAX1( 10.0*(1.0 - HK), -15.0 )
    HXX = EXP(ARG)
    HXXHK = -10.0*HXX
C
    HS = (1.0-HXX)*HT + HXX*HTF
    HSHK = (1.0-HXX)*HTHK + HXX*HTFHK
    & + ( -HT + HTF )*HXXHK
    HSRT = (1.0-HXX)*HTRT
    HSMSQ = (1.0-HXX)*HTMSQ
C
RETURN
C----- end of subroutine HST
END

SUBROUTINE CFL( HK, RT, MSQ, CF, CFHK, CFRT, CFMSQ )
    REAL MSQ
C
    C----- Laminar skin friction function (Cf) (from Falkner-Skan)
    IF(HK.LT.7.4) THEN
        TMP = (7.4-HK)**2 / (HK-1.0)
        CF = ( 0.03954*TMP -0.134 ) / RT
        CFHK = ( -.03954*TMP * (2.0/(7.4-HK) + 1.0/(HK-1.0)) ) / RT
    ELSE
        TMP = 1.0 - 1.4/(HK-6.0)
        CF = ( 0.044*TMP**2 - 0.134 ) / RT
        CFHK = ( 0.088*TMP*1.4/(HK-6.0)**2 ) / RI
    ENDIF
    CFRT = -CF/RT
    CFMSQ = 0.0
C
RETURN
C----- end of subroutine CFL
END
SUBROUTINE CFT( HK, RT, MSQ, CF, CFHK, CFRT, CFMSQ )
REAL MSQ
C
C---- Turbulent skin friction function (Cf) (Swafford)
FC = SQRT(1.0 + 0.2*MSQ)
GRT = ALOG(RT/FC)
GRT = AMAX1(GRT/3.0)
GEX = -1.74 - 0.31*HK
ARG = 4.0 - HK/0.875
ARG = AMIN1( 10.0, ARG )
ARG = AMAX1(-10.0, ARG )
CFO = 0.3*EXP(-1.33*HK) * (GRT/2.3026)**GEX
CF = ( CFO + 1.1E-4*(TANH(ARG)-1.0) ) / FC
CFHK = (-1.33*CFO - 0.31*ALOG(GRT/2.3026)*CFO
& -1.1E-4/COSH(ARG)**2 / 0.875) / FC
CFRT = GEX*CFO/(FC*GRT) / RT
CFMSQ = GEX*CFO/(FC*GRT) * (-0.1/FC**2) - 0.1*CF/FC**2
C
RETURN
C---- end of subroutine CFT
END

SUBROUTINE HRT( HK, MSQ, HR, HRHK, HRMSQ )
REAL MSQ
C
C---- new shape parameter
HR = MSQ * (0.185*HK + 0.15)
HRHK = MSQ * (0.185 )
HRMSQ = (0.185*HK + 0.15)
C
RETURN
C---- end of subroutine HRT
END
SUBROUTINE SOLVIT(NSIZ, NN, Z, R)
C******************************************************************************
C
C Solves general NxN system in N unknowns
C Assumes system is invertible...
C ...if it isn't, a divide by zero will result.
C
C Z is the coefficient matrix...
C ...destroyed during solution process.
C R is the righthand side...
C ...replaced by the solution vector.
C
C******************************************************************************
C
DIMENSION Z(NSIZ,NSIZ), R(NSIZ)
C
DO 1 NP=1, NN-1
   NP1 = NP+1
C
C---- find max pivot index NX
   NX = NP
   DO 11 N=NP1, NN
      IF(ABS(Z(N,NP))-ABS(Z(NX,NP))) 11,11,111
111
      NX = N
   11 CONTINUE
C
   PIVOT = 1.0/Z(NX,NP)
C
C---- switch pivots
   Z(NX,NP) = Z(NP,NP)
C
C---- switch rows & normalize pivot row
   DO 12 I=NP1, NN
      TEMP = Z(NX,I)*PIVOT
      Z(NX,I) = Z(NP,I)
      Z(NP,I) = TEMP
12 CONTINUE
C
   TEMP = R(NX)*PIVOT
   R(NX) = R(NP)
   R(NP) = TEMP
C
C---- forward eliminate everything
   DO 15 K=NP1, NN
      ZTMP = Z(K,NP)
      DO 151 L=NP1, NN
         Z(K,L) = Z(K,L) - ZTMP*Z(NP,L)
151 CONTINUE
   R(K) = R(K) - ZTMP*R(NP)
15 CONTINUE
C
1 CONTINUE
C
C---- solve for last row
   R(NN) = R(NN)/Z(NN,NN)
C
C---- back substitute everything
   DO 2 NP=NN-1, 1, -1
    NP1 = NP+1
   DO 21 L=NP1, NN
    R(NP) = R(NP) - Z(NP,L)*R(L)
 21    CONTINUE
 2    CONTINUE
C
RETURN
C---- end of subroutine SOLVIT
END