SELF-CONSISTENT SOLUTION OF NONLINEAR PROBLEMS IN UNSTEADY RADIATION GASDYNAMICS

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ABSTRACT

It is shown that the system of equations which governs the unsteady flow of a radiating gas in chemical nonequilibrium is hyperbolic, and standard methods are applied to the solution of situations in radiation gasdynamics (RGD) which are restricted neither in optical depth nor the radiation-convection parameter. Absorption of shock layer radiation by the upstream gas is included in a numerical method of characteristics, and it is proved that only a Mark-type radiation boundary condition is appropriate to the lowest order full-range differential approximation of one-dimensional radiative fields. To illustrate the method, the flow fields generated by planar and cylindrical pistons inserted into ideal gases with arbitrary absorption properties are investigated both with the differential approximation and the full transfer equation. The results obtained are believed to be the first in a realistic aerodynamic situation and show that the differential approximation predicts surface pressures and heat transfer rates very accurately and general flow fields within ten percent for the cases considered. It was found that linearized theories of piston insertions may be in error near the start of motion. In addition, nonmonotonic (entropy layer induced) surface pressure histories have been noted, and it is observed that variable surface emissivity and wall temperature may exert "blowing" or "suction" upon entropy layers. Upstream absorption is shown to be a dominant mechanism in the evolution of unsteady flow fields, and contrary to previous predictions it is shown that the effects of radiation upon pressure and velocity may be comparable to those upon temperature in many cases. A rigorous hypersonic piston analogy has been developed so that the results of the investigation may be interpreted in terms of corresponding steady situations in RGD.

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LIST OF SYMBOLS

a_f	frozen speed of sound, Eq.(4.21)
a_{ij}	coefficient of general compatibility relation, Eq.(5.9)
A	frozen specific heat in the presence of equilibrium radiation, Eq.(4.38a)
$A_k^{(\pm)}$	coefficients in half-range Legendre Polynomial expansion, Eq.(3.35)
$A_{ij}^{(k)}$	three-dimensional coefficient matrix of general quasi-linear system of equations, Eq.(4.10)
A_ℓ^m	coefficients in spherical harmonic expansion, Eq.(3.58)
A_ℓ	coefficients in full-range Legendre Polynomial expansion, Eq.(3.25)
A_{mn}	Einstein spontaneous emission probability per unit time
b_i	inhomogeneous term of general compatibility relation, Eq.(5.9)
b_1, b_2	exponents in absorption coefficient, Eq.(5.200)
B	defined by Eq.(4.38b)
B_ℓ	contribution of emission to l^{th} approximate boundary condition, Eq.(3.99)
B_5	defined by Eq.(5.43b)
B_7	defined by Eq.(5.44b)
B_ν	Planck function, Eq.(2.15)
B_{mn}	Einstein absorption probability per unit time and intensity
B_{nm}	Einstein induced emission probability per unit time and intensity
$B(T)$	black body steradiancy, $\sigma T^4/\pi$

LIST OF SYMBOLS (continued)

$\tilde{\mathcal{B}}$	nonequilibrium source function, Eq.(A.35)
$\hat{\mathcal{B}}$	effective source function, Eq.(5.48d)
$\mathcal{B}_o$	Boltzmann number, Eq. (3.18)
c	speed of light
$\overset{\Rightarrow}{C}$	chemical contribution to characteristic matrix
C_i	nonradiative nonequilibrium variables
C_{ic}	defined by Eq.(7.6a)
C_{nc}	defined by Eq.(7.6b)
C_{P_f}	frozen specific heat, $\frac{\partial h}{\partial T}\Big)_{P,\tilde{C}}$
C_s	defined by Eq.(5.43c)
C_1	inverse wave speed, Eq.(5.147a)
C_γ	defined by Eq.(5.44c)
E	energy per unit mass
E_c	electronic energy per unit mass
$E_n(x)$	Exponential Integral of order n, Eq.(2.68)
f_ν	photon distribution function
$\overset{\Rightarrow}{F}$	gasdynamic contribution to characteristic matrix
$F_j(r_o)$	geometric correction to free stream emission in asymptotic solution, Eq.(5.113)
$\mathcal{F}_j$	defined by Eq.(3.81)
$G(\xi,\xi')$	Green's function
$G_j(r,r_o)$	geometric attenuation in asymptotic solution, Eq.(5.116)
$\mathcal{G}_{(\pm)}$	half-range intensities, Eq.(5.48a)

LIST OF SYMBOLS (continued)

h	Planck's constant, enthalpy per unit mass
$\mathcal{I}_0$	first radiation moment, Eq.(2.22)
$\vec{\mathcal{I}}_n$	$(n+1)^{st}$ tensor moment of the specific intensity
I_ν	specific intensity of radiation of frequency ν
$\mathbb{I}$	perturbation to first radiation moment, Eq.(7.44d)
j_ν	emission coefficient, Eq.(2.3)
J_ν	source function (emission)
k	Boltzmann constant, wave number
K_H	hypersonic similarity parameter, $(M_{f_\infty}\delta)$
ℓ_i	propagation direction cosine
m, n	parameters in kernel substitution, Eq.(3.20)
$m^{(\pm)}$	intensity averaged direction cosine, Eq.(3.39a)
m_a	atom (ion) mass
$\vec{\vec{M}}$	coefficient matrix of general quasi-linear system in two independent variables, Eq.(4.31)
M_{f_∞}	frozen free stream Mach number
$\vec{n}$	unit outward normal to a surface
$\vec{N}$	inhomogeneous terms of general quasi-linear system in two independent variables, Eq.(4.31)
N_n	number of systems in states with principal quantum number, n
p	Laplace transform variable
P	pressure
$\vec{p}$	momentum
$P_\ell(\mu)$	Legendre Polynomial

LIST OF SYMBOLS (continued)

p_1	pressure in conical starting solution, Eq.(5.63a)
$\overset{\Rightarrow}{P}^{(R)}$	radiation stress tensor, Eq.(2.24)
$p^{(1)}, p^{(2)}$	defined in Eq.(4.56)
$\mathcal{P}_\nu(\vec{\Omega}'\cdot\vec{\Omega})$	scattering phase function, Eq.(2.6)
$\mathcal{P}$	pressure perturbation, Eq.(7.44a)
q_1	heat flux in conical starting solution, Eq.(5.63d)
$\vec{q}^{(R)}$	radiant energy flux, Eq.(2.23)
Q	volumetric radiant energy loss, Eq.(4.7)
$Q_{a,i}$	atom (ion) electronic partition functions
$\mathcal{Q}$	heat flux perturbation, Eq.(7.44c)
$\vec{r}$	position vector
r_d	diffuse reflection coefficient
r_R	Reimann invariant, Eq.(7.38)
r_s	specular reflection coefficient
$r_0(t)$	moveable upper boundary in asumptotic solution
$\overset{\Rightarrow}{R}$	radiative contribution to characteristic matrix
$R^{(I)}$	defined by Eq.(5.31)
$R^{(M)}$	defined by Eq.(5.28)
$R^{(q)}$	defined by Eq.(5.30)
$R^{(u)}$	defined by Eq.(5.29)
$R_p(t)$	piston path
s	photon path length
s_R	Reimann invariant, Eq.(7.38)

LIST OF SYMBOLS (continued)

$\mathcal{S}'$	modified source function (emission and scattering), Eq.(2.32b)
t	time
t_i^*	chemical rate parameter, Eq.(3.18)
T	temperature
T_j	ionization temperature, $(h\nu_j)/k$
T_1	temperature in conical starting solution, Eq.(5.63b)
$\mathcal{T}$	temperature perturbation, Eq.(7.44b)
u	velocity in x-direction
$\vec{u}$	velocity vector
u_1	velocity in conical starting solution, Eq.(5.63c)
$u^{(R)}$	radiation energy density
$\hat{u}$	effective velocity, Eq.(5.58)
$\mathcal{U}$	velocity perturbation, Eq.(7.44c)
$\hat{\mathcal{U}}$	defined by Eq.(7.42)
v	velocity in y-direction
$\vec{v}$	left-hand eigenvectors of $\vec{\vec{M}}$
V	volume
w	velocity in z-direction
W_i	mass rate of production of variable C_i
W_ν	rate of production of photons of class $(\nu, d\nu)$, $(\vec{\Omega}, d\vec{\Omega})$
Υ	absorption variable, Eq.(A.32)
$Y_\ell^m(\vec{\Omega})$	spherical harmonics, Eq.(3.57)

LIST OF SYMBOLS (continued)

α_a	mean volumetric absorption coefficient, Eq.(A.8)
α_e	mean volumetric emission coefficient, Eq.(2.39)
α_k	direction cosines of local normal to characteristic manifold, Eq.(4.12)
α_M	inverse Planck mean absorption coefficient, Eq.(A.52)
α_p	Planck mean volumetric emission coefficient
α_R	Rosseland mean volumetric absorption coefficient, Eq.(2.52)
α_T	temperature mean absorption coefficient, Eq.(A.53)
α_ν	volumetric absorption coefficient for radiation of frequency ν
$\alpha_a^{(n)}$	$(n+1)^{st}$ tensor moment of α_a, Eq.(A.12)
$\bar{\alpha}$	absorption coefficient ratio, α_p/α_R
β	sound-light parameter, Eq.(3.18)
β_1	defined by Eq.(5.147b)
β_ν	volumetric extinction coefficient, $(\alpha_\nu + \tilde{\eta}_\nu)$
Γ	specific heat ratio parameter, $(\gamma_{f\infty}-1)/\gamma_{f\infty}$
δ	fineness ratio of slender body
Δ	defined by Eq.(4.38c)
ϵ	emissivity
ζ	conical variable, (r/t)
$\hat{\zeta}$	reduced inner time variable, Eq.(5.48c)
η	optical depth, Eq.(2.65)
η_{q_1}	generalized optical depth, Eq.(3.75)
$\hat{\eta}_1$	reduced conical velocity, Eq.(5.70)

LIST OF SYMBOLS (continued)

$\tilde{\eta}_\nu$	volumetric scattering coefficient
θ	declination
θ_ν	nonequilibrium temperature
κ	radiation temperature slip coefficient, Eq.(3.118)
λ	characteristic direction
μ	direction cosine, Eq.(2.64)
μ_j	discrete ordinate, Eq.(3.103)
$\mu^{(t)}$	flux averaged direction cosine, Eq.(3.39b)
Ξ_n	n^{th} generalized boundary moment, Eq.(3.114)
ρ	density
σ	Stephan-Boltzmann constant
$d\sigma$	elemental area
τ	optical depth, Eq.(2.55)
τ_ℓ	Bouguer number, Eq.(3.18)
$\hat{\tau}$	reduced optical depth, Eq.(5.48b)
ν	frequency
ϕ	azimuth, heat flux potential
$\mathcal{H}$	radiation-convection parameter, Eq.(5.147c)
$\chi(I_{0w})$	free stream emission index, Eq.(5.121)
ψ	stream function, general property
$\psi(\bar{r},t)$	characteristic manifold
ω	radian frequency
ϖ_ℓ	coefficients in Legendre Polynomial expansion of phase function, Eq.(3.4)

LIST OF SYMBOLS (continued)

$\vec{\Omega}$	propagation vector

Subscripts

$(\ )_a$	absorption averaged quantity, Eq.(2.40) ff.
$(\ )_e$	local equilibrium condition
$(\ )_{ij}$	average quantity, Eq.(5.13a)
$(\ )_j,\ j=1,7$	quantity at a point in the characteristics calculation
$(\ )_s$	quantities referred to shock wave
$(\ )_{SL}$	quantity referred to sea level
$(\ )_u$	quantity referred to unsteady flow
$(\ )_w, (\ )_p$	quantities referred to piston surface
$(\ )_o, (\ )_\infty$	quantity referred to an undisturbed state
$(\ )_\nu$	per unit frequency
$(\ )_j^i$	difference, Eq.(5.13b)

Superscripts

$(\pm)$	quantity referred to photons for which, $\vec{\Omega}\cdot\vec{n} \gtrless 0$
$*$	dimensional quantity
$\overline{(\ )}$	normalized quantity for large velocity, small time analysis; normalized variable in Hypersonic Piston Analogy
$(\ )^n$	n^{th} iterate

CHAPTER 1

INTRODUCTION

1.1 The Importance of Thermal Radiation

The design of vehicles for entry into planetary atmospheres from deep space missions is a compromise between aerodynamics and gas-dynamics. It has long been recognized that the surfaces of such vehicles will have to be protected from the potentially severe aero-dynamic heating. Since practical considerations which obviate the use of minimum energy orbital transfer dictate that reentry must occur at greater than parabolic speeds, conditions within the shock layers of these vehicles will be favorable for appreciable excitation of internal degrees of freedom of the gas, for dissociation, and for ionization. Once these processes occur, radiation of the energy which they required becomes possible. It has been pointed out (Allen, 1962) that considerations of radiative heating will dictate the philosophy of future spacecraft design. Although slender vehicles present decelera-tion problems because of their low drag relative to bluff ones, the concurrent weak shock wave may provide a sufficient reduction of radiative heat transfer to justify their use. Through an admittedly crude analysis Allen has attempted an optimization of conical bodies with ablative surfaces. Almost without exception, those who have undertaken the gasdynamic analysis of high speed atmospheric flight have cited these preliminary results. Trustworthy quantitative investigations of slender bodies still do not exist, however, and if the situation is ever to be truly optimized, the details of both viscous and inviscid flows in chemical and radiative nonequilibrium must be better understood. Unless one is willing to make approximations that seem sometimes rather drastic, the complete problem cannot yet be attempted—even numerically. It is hoped that the investigation to be described will lead to a better understanding of the coupling between radiative transfer and fluid mechanics in realistic aerodynamic circumstances.

Emphasis shall be placed upon self-consistent problems as Sherman (1966 b) defines them. These are problems in which the absorption properties and radiant energy source function are not specified à priòri but are obtained from the simultaneous solution of the governing radiative, fluid mechanical, and thermodynamic relations. The most general non-self-consistent problems, in which the radiative source is specified and fluid mechanics is determined to be compatible with it, are usually found in neutron physics. (See e.g. Podney, et.al., 1966). However, any problem in which radiation is uncoupled from fluid mechanics satisfies the definition of non-self-consistency. Part of the emphasis of this investigation rests on the development of new methods for the solution of problems in radiation gasdynamics (RGD), thus a summary of existing schemes and some of the most representative problems to which they have been applied is appropriate. The shortcomings of these methods—many of which were developed without aerodynamics in mind - will be pointed out, and the present analysis will be justified. Some familiarity with gasdynamics and radiative transfer is assumed; therefore, the uninitiated reader may wish to scan Chapter 2 and Section 3.1 before reading the remainder of this introduction.

1.2 Review of Problems in Radiative Transfer

1.2.1 Nonaerodynamic Applications

The theory of radiative transfer in astrophysics is well documented (Kourganoff, 1952, Chandrasekhar, 1960). Although there are numerous variations of the problem, the astrophysicist is fundamentally interested in the prediction of temperature distributions within and energy fluxes emerging from stellar atmospheres (which may be assumed planar because of their enormous radii of curvature). With this information, observations of intensities may be interpreted in terms of surface temperatures. The analysis is quite complicated, even though the situation is one of radiative and mechanical equilibrium with the interior of the star providing a limitless energy source which maintains the emergent flux. Since neutrons of a given energy travel at a constant speed in a manner analogous to that by which photons travel at the speed of light, it is not surprising that the theoretical aspects of the motion of neutrons through absorbing and scattering media closely parallel those of astrophysical radiative transfer (Davison, 1958; Weinberg and Wigner, 1958). Problems of reactor design are complicated by the wide range of material and geometric boundary conditions which the neutron flux must satisfy. Only the mathematical techniques of astrophysics and neutron transport are similar, the individual problems are quite different.

The disciplines of meteorology and chemical engineering must be given credit for the first attempts at analysis of the coupling between radiation and other forms of heat transfer in fluids. Radiation is the primary mode of energy transfer to and from planetary atmospheres. These atmospheres are usually in a state of mechanical equilibrium, and radiation plays a dominant role in the maintenance of atmospheric temperature gradients. Should one disrupt the concentration of absorbers present (say by burning fossil fuels), energy which would have been lost or at least dispersed over large distances, due to an ordinarily weaker absorption, may be trapped in the lower layers of the atmosphere, hence the "greenhouse" effect. The works of Goody (1956, 1960) and of Murgai and Varma (1965a, b) are perhaps the most recent meteorological applications of interest to the aerodynamicist. In these, the effects of radiation upon such natural phenomena as gravitationally induced cellular convection and the rising of hot gas plumes were investigated. An extensive treatment of atmospheric radiation may be found in Chapters 5 - 7 of Brunt's classical work (Brunt, 1952). Chemical engineers have been concerned with similar problems, but their interest lies with fluids which are very viscous and geometries which are bounded (e.g., channels and containers). For instance, radiation is of importance in glass furnaces. Beyond the pioneering efforts of Hottel (see e.g. Hottel and Cohen, 1958), Cess (1964a, b) has analyzed the effects of radiation upon viscous, heat conducting fluids in various simple flow situations. Whereas he makes judicious use of perturbation procedures when radiative transfer is large or small relative to whatever other mode of heat transfer is present (conduction, natural or forced convection, etc.), Viskanta, et al. (Viskanta, 1963, 1964a; Viskanta and Grosh, 1961, 1962; Viskanta and Merriam, 1967) have attempted more general solutions of similar situations as has Greif (1964, 1965a, b). Sparrow and Cess (1966) present the chemical engineer's viewpoint.* A considerable background exists in the non-aerodynamic applications of radiative fluid mechanics. Although the aerodynamicist is a relative newcomer to radiative transfer, there have also been many aerodynamically oriented works.

1.2.2 Radiation in Aerodynamics

When it was realized that radiative energy transport might be of great aerodynamic import, experimental and theoretical investigation of the physical processes which contribute to thermal radiation were embarked upon in earnest. Clearly, phenomena which predominate in stellar and atmospheric radiation are not always those which contribute most to radiation from shock layers; therefore the catalog of information which existed required elaboration. For instance, scattering, diffuse reflection, and absorption by dust and water vapor are of extreme importance in meteorological radiation (Brunt, 1952, Chapter VI), whereas they are insignificant in aerodynamics. One of the most detailed

* The analysis of radiative transfer between sources and surfaces in vacua is of little interest in the present context.

investigations appropriate to shock heated air is that of Kivel and Bailey (1957). The works of Armstrong, et al. (1961) and of Churchill (1966) are representative of more recent theoretical analyses. Experiments conducted to date are summarized by Gruszczynski and Warren (1967). Zeldovich and Raizer (1966) present both the theoretical and experimental aspects of radiation physics and Bond, Watson, and Welch (1965) provide a useful compendium of results. The spectral, or frequency dependent, aspects of radiation lead to cumbersome formulations. One usually ignores the fact that radiation of different frequencies is absorbed differently and employs mean absorption properties over the entire spectrum. This "gray gas" approximation is employed in almost all of the investigations to be mentioned. An improvement of this assumption has been suggested (Finkleman,and Chien, to be published) and will be commented upon in Section 2.6 and in Appendix A.

The formulation of the radiation coupled dynamics of gases was undertaken by Goulard in his now classic papers (Goulard, 1962, 1963a). Sampson (1965b) has placed particular emphasis upon the radiation field of gases in motion while fairly general summaries are presented by Zhigulev et al. (1963), Vincenti and Kruger (1965), Bond, Watson, and Welch (1965), and Pai (1966). Sections 2.1 through 3.2 present the essential concepts.

Aerodynamically oriented forays into radiation have been a good deal less general than astrophysical or chemical engineering investigations. Astrophysicists need not be particularly concerned with the presence of intervening solid boundaries; the aerodynamicist is, however, ultimately concerned with the interaction of complicated boundaries with the radiative field so that heat transfer rates may be predicted. He must also account for the fact that the fluid is moving whereas astrophysicitsts need not. As opposed to chemical engineers, he must consider a fluid in complex motion whose composition and state are not at his disposal. Furthermore, relative to the size of bodies of aerodynamic interest, flow fields are unbounded since radiation is global and affects all of the space occupied by the fluid. It is quite easy to understand why gas-dynamicists who deal with radiation often seek simplifications.

1.2.3 Limiting Cases

The most familiar simplification results when the gas emits much more strongly than it absorbs. The justification for this so-called "optically thin" assumption will be discussed in Section 2.5. To this approximation radiative quantities are isotropic. The only complication over nonradiating representations is the addition of a nonadiabatic term in the energy balance. The energy loss is of the form of Stefan's Law, $2(\epsilon/\ell)\sigma T^4$, where the effective "emissivity per unit length", ϵ/ℓ, depends upon the emission (absorption) properties of the gas. At worst some additional non-linearity is introduced in the formulation. If, in addition, convection is predominant (as determined after identification of the appropriate dimensionless parameters) the radiation may be uncoupled from the now adiabatic gas-dynamics, and the nonradiating temperature and pressure distributions may be used to predict radiative quantities. The earliest attempts at RGD adopted this non-self-consistent approach.

"Thin" analyses of both blunt and slender bodies have been performed. One of the first uncoupled blunt body investigations was that in which Yoshikawa and Wick (1961) compared radiative and convective heat transfer rates. They used an average shock layer temperature and assumed the layer to be an isothermal, planar slab. This analysis and others similar to that of Kennet and Strack (1961) probably laid the foundation for Allen's conclusions. Kennet (1962) used the predictions of a Lighthill constant density solution to arrive at "uncoupled" radiative losses, and Koh (1962) used Newtonian predictions of the nonradiating flow to observe that convective heat transfer coefficients decreased with increasing blunt body nose radius while radiative heat transfer increased. All of the latter made the planar slab assumption, and Wing (1962) corrected them by considering the shock layer to be spherical, concentric with the body.

Uncoupled results must overpredict radiative effects when radiation and convection are comparable. If all surfaces present are cold and black, the shock layer can only lose energy if emission alone is allowed. Thus the temperature levels at which emission occurs will be reduced. Goulard (1961) carried out a perturbation analysis of the radiation coupled flow about a blunt body for small values of the parameter which measures the importance of radiation relative to convection (the Boltzmann number). He obtained quantitative estimates for the reduction in shock layer thickness and radiative heat flux due to cooling of the shock layer. Bird (1960) also considered the coupled flow field under the assumption of a linear variation of velocity along the surface from the stagnation point to the sonic point. From this investigation there came the widely accepted conclusion that coupling of radiation and convection has a large effect upon temperature and density distributions, but relatively little effect upon velocity and pressure as compared with those of nonradiating flows. It shall be shown (Chapter 6) that this is not generally the case. More recently Wilson and Hoshizaki (1964) and Hoshizaki and Wilson (1965) have attempted both viscous and inviscid analyses of optically thin blunt body flows.

The thin assumption is best motivated within the shock layers of slender bodies. Koh and DeSilva (1962) examined the flat plate boundary layer within this approximation, while Chin and Hearne (1964, 1965) have considered both unyawed and yawed cones. Romishevskii (1963) has obtained similarity solutions applicable to slender bodies, and Jischke (1966) has applied an integral method to wedge flows. Olfe (1966) has investigated wedge flows in the thin limit with a perturbation expansion in the density ratio across the strong shock wave. The relative ease with which optically thin analyses may be carried out allows one to gain considerable insight into the qualitative implications of RGD; the applicability of these solutions to realistic situations is another matter.

Unless radiative transfer is in fact small in some sense, the shocked gas must eventually lose all of its energy because there is no absorption mechanism. In addition, the energy emitted from the shock layer must be absorbed in part by the surrounding gas. Implicit in the thin approximation is the fact that energy losses are spread over large distances, but if losses are appreciable or if the gas ahead of the body is strongly absorbing the oncoming flow must be aware of the disturbance and the field will be greatly modified.

At the opposite extreme, considerable complication is avoided if the medium is strongly absorbing and nearly in radiative equilibrium. In this case variations in the temperature field must be small, and since absorption and emission are balanced radiation becomes a localized phenomenon. The heat flux may then be expressed in a manner analogous to Fourier conduction, with the thermal conductivity depending upon the absorption properties of the gas. It is well known (Cess, 1964) that this approximation will also overpredict heat fluxes when applied outside of its region of validity. For instance, Goulard (1963b) has demonstrated that a square root singularity exists in the flux at the surface of a flat plate as predicted in this "optically thick" situation. Obviously (see Section 2.3) the flux can be no larger than the black body value, σT^4, which must be bounded. Just as in molecular transport problems, the effects of boundaries upon the radiative field to this approximation will be confined to small regions near those boundaries.

The close resemblance of radiative transfer to molecular conduction and the limited influence of boundaries first led investigators to apply the thick approximation to the structure of shock waves propagating at constant speed into undisturbed gases. Such a situation might obtain, for instance, in the late stages of motion of a planar piston into a gas at rest. The shock wave is sustained by the agency which does work on the piston, but it has reasonably little knowledge of the existence of either the piston or its source of energy. The effects of molecular transport phenomena on such a situation are understood (see e.g. Landau and Lifschitz, 1959) thus the method of attack is well documented. Sachs (1946) was probably the first to consider optically thick shock structure in any detail. There then followed the analyses of Sen and Guess (1957), who employed standard phase plane analyses, and Marshak (1958), who obtained similarity solutions. All of the latter considered the complete viscous, heat conducting problem and noted the fact that the gas upstream of the shock wave is forewarned of its approach. Scala and Sampson (1964) have examined the effects of both "thin" and "thick" radiative transfer upon shock structure and boundary layers, while Steiger and Khosla (1965) have investigated the possibility of the existence of similarity solutions in both extremes in viscous shear flows. They found, however, that conditions under which similarity was possible were quite unrealistic.

1.2.4. The Full Formulation – Planar Methods

Because of the inherent isotropy of both limiting cases, the geometrical complication of the problems attempted has been minimal. When one is forced to consider the complete radiative process—both absorption and emission—analysis becomes increasingly difficult. In Section 3.1 it will be shown that the general formulation of radiation gasdynamics involves a system of coupled, non-linear, integro-differential equations. The differential portions account for the fluid mechanical conservation of mass, momentum, and energy, while integral terms arise from the global effect of radiation upon the energy balance and from the possibility of the scattering of photons in the radiative transfer equation. The integrals involve the radiative source distribution weighted over space and frequency, hence also the variation of both temperature and pressure. Further complicating the picture is the fact that the radiative properties of intervening boundaries enter the governing equations directly (as is indicated for planar situations by Goulard, 1960, and further mentioned in Sections 2.7 and 2.8).

In fact, only for radiative transfer in planar media have the geometrical aspects of the general formulation been resolved.

The simplest analyses of the complete problem are those which eliminate the complications of geometry and surfaces, but even in the prediction of shock structure integro-differential equations must be solved. Moreover, the kernels of the integral terms are not separable; therefore, successive differentiation will not remove the integrals from the formulation. The most common means of overcoming this difficulty is to approximate the exact kernel by one which is separable. Any such kernel may be approximated arbitrarily closely by an expansion in terms of a set of functions complete over the range of the variable of interest (Murty, 1965); however, the more terms one includes the greater is the number of differentiations which are required, and a differential equation of infinite order results. Usually only one term is retained, and the analysis is rendered more reasonable. More will be said of this in Section 3.2.1.

The fact that successive differentiations raise the order of the governing equations and, therefore, require more boundary conditions is immaterial in shock structure since infinitely many derivatives of all physical quantities must vanish far up- and downstream and there is no radiative input from these regions in the absence of external sources. An example of the kernel substitution approach may be found in the work of J. F. Clarke (1962) who concluded that inviscid radiating gases were capable of sustaining discontinuities only in the derivatives of physical properties and not in the properties themselves. Heaslet and Baldwin (1963a) using the same scheme qualified this conclusion and further demonstrated that discontinuous profiles were possible for a wide range of up- and downstream conditions. They noted also that the upstream flow field for sufficiently strong shocks may be quite optically thick whereas downstream an optically thin layer might obtain. Thus analyses based on either extreme would be in error. These conclusions have since been verified by numerous independent analyses. The influence of molecular transport phenomena upon the situation within the framework of kernel substitution was considered by Cohen and J. H. Clarke (1965). Apart from shock structure, kernel substitution has been widely used in aerodynamic problems. Yoshikawa and Chapman (1962) applied it to the blunt body problem but were unaware of the simplification which it afforded and treated the approximate kernel only as a convenient numerical device. The method has been applied to linearized problems by Lick (1964, 1965), Vincenti and Baldwin (1962), Moore (1964), and Rhyming (1965a, b). The linear models will be commented upon in Section 5.5. When kernel substitution is not appropriate, there are other approximations which may be relied on.

The integral terms may be simplified in nearly "thick" planar situations if the "local temperature" concept is applied. This is accomplished by assuming the temperature distribution to be accurately represented by the first few terms of a Taylor series about the point of interest. Then the integrals become a series of derivatives of the temperature distribution, each weighted by a coefficient proportional to a moment of the kernel, and a purely differential form ensues. Yoshikawa and Chapman (1962) used this in the investigation noted above, and Chisnell (1966) employed it in the study of the stagnation region of a blunt body for the case in which radiation is less important than convection. Both endeavors considered only "cold" walls. Shwartz (1966) has studied boundary layers in this manner. The method is not appropriate when the derivatives of temperature are large, and this precludes application near a shock wave if upstream absorption is included. For this reason, local temperature expansions are of limited value in general.

Integral or "strip" methods, in the sense of Belotserkovskii (1959), have also been tried. In these the governing equations are first expressed in as nearly divergence forms, $\partial P_{ij}/\partial x_j = c_i(\vec{x})$, as possible and are then integrated in one of the less important coordinate directions (across a shock or boundary layer). Suitable profiles with free parameters are assumed for the quantities, $P_{ij}(\vec{x})$, e.g. $P_{ij}(x,y;t) = \sum_{k=0}^{N} a_k(x;t) W_k(y)$ where the $W_k(y)$ are specified a priori. The number of separate strips required depends upon the number of free parameters $a_k, k=0,N$, and suitable matching conditions at the boundaries of the strips are devised.

When the radiation integrals may be written in a reasonably straightforward manner, the integrations may be carried out so that only the appropriate a_k's each weighted by a moment of the kernel with respect to the W_k's, remain. The effect upon the radiative transfer is analogous to that of the local temperature approximation, but in this case the problem is reduced to the solution of a system of equations in one less independent variable than before. Sforza (1963) has formulated the solution of the radiating flat plate boundary layer by such an integral method, and Viskanta and Lall (1965, 1966) are particularly fond of this approach in cylindrical gas masses whose absorption properties are constant.

A summary of methods in planar radiative transfer must mention the Shuster-Schwarzschild or Forward-Reverse approximation (see e.g. Zeldovich and Raizer, 1966, and Section 3.2). This scheme is well known in astrophysics. The assumption that there exist at each point two separate isotropic radiation intensities in the forward and reverse directions along the single axis of variation of properties makes two coupled differential equations of the radiative transfer equation. These equations provide the properties which would ordinarily require integrals in the energy equation. They are, however, coupled to that equation by the appearance of temperature. Adrianov and Polyak (1963) discuss this approximation in non-aerodynamic situations as do Heaslet and Baldwin (1963b). Traugott and Wang (1964) have suggested improvements in the model, and Wang (1963) has applied it to the similarity solution of planar, piston generated flow fields.

Although there are several methods for the solution of planar problems, (kernel substitution, local temperature expansions, strip methods, and a forward-reverse approximation*) many investigators

* The application of the Ambartsumian-Chandrasekhar invariance principles (Chandrasekhar, 1960) has not been mentioned. These rely on the fact that the addition of a small slab of absorbing gas to a planar medium should have negligible effect upon emergent fluxes and internal temperature distributions as the thickness of the slab vanishes. Bellman, et al. (1965a, b) have applied these to non-astrophysical situations, but the possibility of their extension to aerodynamics is doubtful.

disdain the use of such approximations. Some of these problems may, in fact, be solved exactly with iterative schemes, and this was done by Hoshizaki and Wilson (1966) for blunt bodies with mass transfer cooling and by Howe and Viegas (1963) for the detached viscous shock layer into which an absorbing gas was injected. Neither included upstream absorption, and both used "planar" radiation models in nonplanar situations. Hoshizaki and Wilson did, however, include the spectral aspects of radiation. There are numerous uncertainties in the previous approximate methods. The physical implications of the essentially mathematical procedure of kernel substitution are not immediately clear. It is not known how nearly optically thick a flow field must be for local temperatures to provide accurate results. Strip methods are notoriously sensitive to the profiles one assumes even in simple situations. Jischke's "thin" analysis (1966) assumes, for instance, different profiles for a quantity proportional to temperature, T, and for T^4, which is an obvious inconsistency in the radiation. Furthermore the sources of inaccuracy in the forward-reverse approximation are not apparent. Perhaps most important is the fact that there seems to be no systematic extension of these essentially "planar" methods to nonplanar situations. One is forced, therefore, to apply a planar radiative formulation to flow fields which are two or three-dimensional.

1.2.5 Nonplanar Methods

At this point an important distinction must be made between the terms "planar" and "one-dimensional." Most investigators have used these terms interchangeably, and planar radiative transfer is commonly referred to as "one dimensional" in the literature. In fact Sampson (1965) uses such loose terminology. Single dimensionality is, however, a more general situation which encompasses cylindrical and spherical geometries as well as planar ones. The geometrical complication of even cylindrical or spherical radiative transfer makes straightforward, closed-form expression prohibitive except when the variation of the absorption properties of the gas may be neglected. Cuperman, et.al. (1963), Viskanta and Lall (1965), and Davison (1958) give the expressions for spherically symmetric media, and quite recently Heaslet and Warming (1966) have presented the cylindrical case (See Appendix C). The additional difficulties of multi-dimensional problems and the inapplicability of most planar methods to them will be examined in Sections 2.7 and 3.2.1. At present it seems that only three models may be conveniently applied to problems involving multi-dimensional radiative transfer. These are Hottel's zoning technique (Hottel and Cohen, 1958; Einstein, 1963) Monte Carlo schemes, and the various modifications of the so-called moment methods.

The first involves the derivation of "exchange factors" which account for the mutual interaction of the elemental portions of the flow field after division into cells for numerical purposes. These exchange factors include the effects of geometry and thus must be rederived separately for each situation encountered. The method is currently being investigated in aerodynamic situations by this author's colleague, M. Jischke, and will not be discussed further. The second method is statistical in nature and traces selected photons through their interaction with matter. The philosophy of this approach is summarized by Kahn (1954), and Campbell (1967) outlines its most recent application to radiative transfer in addition to citing earlier attempts. The additional complication of fluid mechanics in Monte-Carlo radiative transfer presents a formidable numerical problem and has not yet been attempted. The methods above are essentially numerical devices, and their physical implications are difficult to assess without considerable computational effort. Thus, moment methods, although not inherently as exact, seem to be the most reasonable approach to multidimensional problems.

It is the anisotropy of radiation that leads to the major problems—especially the geometrical difficulties—encountered in radiative transfer. The gross properties of the radiative field (e.g., the heat flux) which are of primary concern are averages (or moments) of the specific intensity of radiation weighted over the range of the angular variables. If these moments could be obtained without regard for the intensity itself, considerable simplification would ensue. The circumstance is analogous to the dynamics of gases; all gross properties (density, temperature, mass velocity, etc.) of a medium may be deduced if the distribution functions of its constituent particles are known, but one need not necessarily know the distribution function to determine these properties quite accurately. Examination of only a few moments of a velocity distribution function allows one to obtain all of the necessary macroscopic information he desires. This philosophy can lead to the governing equations (Navier-Stokes) with which the gas-dynamicist is familiar. Such equations are merely the first members of an infinite hierarchy, and simplifying assumptions were required before the system could be closed. The similarities between radiative transfer and molecular gasdynamics have been pointed out by Eckert (1963) and by Heaslet and Warming (1963b). The analogy cannot be carried very far, however, for reasons to be stated.

In kinetic theory the distribution function may be approximated arbitrarily closely by a series of functions complete over the range of the velocity variable. These turn out in general to be Laguerre Polynomials or functions related to them which are orthogonal over $(0, \infty)$. Analogously one is led to the choice of spherical harmonics which are complete over the range of the angular variable in radiative transfer. Just as the series is truncated in kinetic theory, the series representation of the photon distribution function (which is related to the more fundamental quantity, specific intensity, as will be discussed in Section 2.1) may be truncated. Similarly each of the time and space dependent coefficients of the terms in the series is related to the moments of the specific intensity with respect to the angular variables just as the coefficients of the kinetic theory expansion are related to velocity moments of the distribution function. Successive integration of the governing equation, weighted over the unit sphere, results in a hierarchy of equations. Because of the explicit appearance of an angular variable in the fundamental relation, these equations are coupled one order with the next. When the expansion of the intensity is truncated at a given stage, it is implicitly assumed that all successive terms vanish, and this provides unique relations among higher and lower moments of the intensity thus closing the system of equations. When "n" coefficients are retained, the integral terms (even those which describe scattering) are removed from the system and are replaced by "n-1" additional first order differential equations. This is the essence of the differential approximation (see Section 3.3). The analogy between the specific intensity and a velocity distribution function is no accident, as will be shown in Section 2.1.

The differential approximation is common in neutron transport theory (Davison, 1958; Mark, 1944, 1945). Kourganoff (1952) refers to the planar analog of the general moment approach for problems in radiative equilibrium, and Traugott has applied it to gasdynamics. The approach adopted by Chou (1967), which requires selective averaging over specified regions of solid angle, is very closely related to this formulation. If the validity of the differential approximation is accepted, there is no longer a need to solve integro-differential equations. The complications of geometry are removed, and completely general problems may be formulated once and for all.

On the other hand, some important "macroscopic" information has been hidden. Molecular gasdynamics possesses the advantage of a good deal of macroscopic experience that is not available in radiative transfer. The first few moments of a velocity distribution function are physically very meaningful, and observation of the interaction of fluids with surfaces leads to unambiguous formulation of boundary conditions for the equations normally encountered. Since one may frequently assume the gas to be a continuum, the phenomena of temperature and velocity slip, which may only be predicted on a microscopic scale, may often be neglected. This is not the case in radiative transfer since even the thick formulation possesses a greal deal of slip (see Section 3.4). Moreover, physical understanding of the angular moments of the specific intensity rapidly diminishes as the order of the moment increases. Photons are only remotely like material particles, and macroscopic experience cannot be relied upon to dictate appropriate boundary conditions (i.e., photons may be annihilated in the course of their interaction with material particles; thus the number of particles is not necessarily conserved). The interaction of radiation with surfaces requires special consideration and will be investigated in Section 3.4.

Because of the complication of boundary conditions attention has even in this approximation been brought to bear upon shock structure. Traugott (1962, 1964) has extensively investigated the problem in a viscous, heat conducting gas. It will be shown Section 3.2.3, that in such planar problems the "moment method" is formally equivalent to a kernel substitution. Goody (1956, 1960) has employed a differential approximation, derived from the forward-reverse approach, in his meteorological studies. Cheng (1964, 1965) has attempted linearized two-dimensional problems with this scheme, and Cheng and Vincenti (1966) have applied the method to a two-dimensional blunt body flow by the method of series truncation (Swigart, 1963). Cohen (1967) has used the moment method (with incorrect surface interactions) to sketch the structure of a blunt body flow field at large Mach numbers when radiation is predominant. His investigation exploits the analytical possibilities that the differential approximation offers. Wang (1965a, 1965b) considered slender body problems, and Sherman (1966b) has examined various moment methods and assessed their errors in mainly astrophysical situations.

In summary, moment methods are convenient for the study of nonplanar situations unrestricted in optical depth, and they provide a tractable system of differential equations. However, the effects of the omission of higher order coefficients in the expansion are not as well known as in kinetic theory, where they are certainly very small in most situations. Also, the formulation of the interaction of radiation with surfaces, consistent with the differential approximation, is not yet systematic.

1.2.6 The Complete Problem

In this brief description of the methods which have been applied to the solution of problems in RGD no mention has been made of the numerous perturbation schemes which simplify the integral terms in planar problems since these are most closely related to the limiting thick or thin approximations. In addition, little reference has been made to the spectral aspects of radiation or to the physical processes which contribute to the emission or absorption of radiant energy. The fact that the significant contributions to the study of radiation gasdynamics can be summarized in these few paragraphs is testimony to the need for further investigation.

If completeness in a problem of RGD includes simultaneously (a) no restriction on optical depth, (b) the effects of emitting, absorbing, and reflecting surfaces, (c) allowance for absorption (by the gas upstream) of energy emitted from the shock layer, and (d) the consideration of geometries more realistic than simply planar, then no analysis to date is complete. Chien (1967) has come close by including effects (a) and (c). Although there are many approximate methods of solution available, there are no complete exact solutions at all. "Exact" in this context includes not only the full transfer equation but also the various approximate models, e.g., moment methods. Without exact solutions the error involved in the various approximate schemes cannot be estimated. Indeed perturbation schemes, although useful, yield little insight into cases in which all fluid mechanical and radiative effects are comparable; only exact solutions yield information of that type. Part of the purpose of this investigation is, therefore, to ascertain whether exact solutions of "complete" problems in radiation gasdynamics are possible.

1.3 The Usefulness of Unsteady Gasdynamics

Recently much attention has been focused upon the prediction of steady flow field through the observation of an unsteady flow, beginning from arbitrary initial conditions. The advantage of this approach is that the unsteady equations of inviscid gasdynamics are always hyperbolic whereas steady flows need by locally supersonic in order to be hyperbolic (see e.g., Stanyukovich, 1960; Von Mises, 1958). Thus, by analyzing the development of a flow field until temporal changes are sufficiently small, quite general flow fields, including imbedded subsonic regions, may be determined in a stable computational manner. Because of their elliptic character, such subsonic regions force the Cauchy problem to be ill-posed in the steady state. In RGD the situation in the steady state is even worse, because the global nature of radiation prevents the system of equations from being sufficiently hyperbolic for hyperbolicity to be used to advantage – even if the flow is locally supersonic (see Section 4.1.2).

Methods for the numerical solution of hyperbolic problems have been explored extensively. Burstein (1964) comments critically on the various methods available. The investigation conducted by Bohachevsky and Mates (1966) is representative of what may be referred to as "strict finite difference" methods in which the first step is the choice of a difference scheme. These may be modified and stability improved by judicious choice of the difference formulae. The Lax-Wendroff scheme, which is very closely related to pseudo viscosity methods (see e.g., Lax and Wendroff, 1962; Richtmeyer, 1957), has been most widely applied. Difficulty arises in the prediction of weak solutions, those which contain imbedded discontinuities. Just as in truly viscous situations, in the Lax-Wendroff scheme sharp discontinuities are smoothed, and any shock waves present are spread over several mesh elements (see Burstein, op. cit.). A characteristics approach, while a good deal more complicated for systems of equations in several independent variables (e.g., Holt, 1956; Sauerwein, 1964, 1966), predicts these sharp discontinuities exactly. In this approach the equations are written in characteristic form (see Section 4.2) before a difference scheme is applied. The use of characteristics has in the past allowed the solution of quite general inviscid real gas flows with few complications over other nonisentropic situations (see Sedney, South, and Gerber, 1962; Sussman, 1966). There seems, also, to be less controversy concerning numerical stability among the adherents of characteristics calculations then among those who favor finite difference approaches. It will be shown, however, in Section 7.2 that one of the greatest advantages of multi-dimensional characteristics calculations – that large matrices need not be inverted at each stage – may be lost when radiation is included. Sauerwein (1964) has compared "strict" finite difference schemes with the method of characteristics on ten different points (stability, ease of handling fixed and free boundaries, computation time, programmer effort, etc.) and has concluded that the method of characteristics is potentially the most powerful.

In a general unsteady situation the radiative field is established on a much shorter time scale than is the fluid mechanics since photons travel at the speed of light. Although temporal derivatives appear in the equations which govern radiative properties, they are usually neglected. The radiation then becomes "quasi-steady" and depends upon time only implicity through the temporal variation of temperature and pressure. It is not clear, therefore, whether the same advantages may be gained from unsteady flow fields in RGD that obtain in other real gas flows. This question was probably first raised by Nemchinov (1960), who recognized the hyperbolic nature of even the equations of unsteady RGD and formulated and solved numerous linearized problems within the framework of a forward-reverse, planar formulation of the Lagrangian equations of which Russian investigators are so fond. The work of Nemchinov provided much of the background for this investigation.

Most of the methods which have been applied to problems in RGD stem from astrophysics or chemical engineering. The application of a method of characteristics benefits from its development in an aerodynamic context, but such an approach in RGD raises many questions. This is not the first time that a characteristics approach to RGD has been suggested. Besides Nemchinov's comments, Khosla (1966) briefly mentioned the possibility after the start of the present investigation.

1.4 The Purpose of the Investigation

Detailed general solutions of the coupled integr-differential equations of RGD have not been presented heretofore. For the most part, investigations carried out to date have been limited to restricted ranges of optical depth, to instances in which radiation is in some sense insignificant or predominant, or to planar cases (in which the representation of the radiative field or approximations thereto are greatly simplified). There have been no investigations of nonlinear problems in RGD which have included the effects of absorption of shock layer radiation by the gas upstream in the presence of impermeable surfaces of arbitrary opacity. It is apparent that more general situations must be investigated. Many questions still exist concerning the applicability of planar methods to nonplanar situations, and the ambiguity in the modeling of the interaction of radiation with solid surfaces consistent with the various differential approximations must be resolved.

The explicit purposes of this investigation are, therefore, clear. They are:

1. To develop a method, based upon the theory of characteristics, with which general multi-dimensional problems in RGD, including upstream absorption, may be treated.
2. To examine critically the question of boundary conditions appropriate for use with the differential approximation.
3. To solve, using the above method, some of the planar problems which have as yet not been investigated in their full, nonlinear sense, and
4. To assess the applicability of one-dimensional (not necessarily planar) approximations to radiative transfer in two-dimensional or cylindrically symmetric radiating hypersonic slender body theory.

Before these may be accomplished, one must be aware of the fundamentals of radiative transfer.

CHAPTER 2

FUNDAMENTALS OF RADIATIVE TRANSFER

2.1 A Kinetic Theory for Photons - The Transfer Equation

Because the relationship between photons and material particles has been stressed, the point of view taken by Sampson (1965) and Simon (1963) is adopted and radiative transport is treated in the light of kinetic theory. The reader must be aware of the elements of radiative transfer if he is to appreciate the present investigation, thus the most fundamental concepts are discussed first. The basic discussion may be found elsewhere in more detail, and the experienced reader may wish to move immediately to Section 2.4 in which some new concepts are introduced. All which shall be presented in this chapter is necessary for reference in the subsequent analysis.

Planck's idea that emission and absorption of radiant energy take place in quanta and Einstein's idea that these quanta travel at the speed of light led to the acceptance of the concept of photons. Photons belong to the general class of entities which have the characteristics of both waves and particles, but they are unique among Bosons in that they have zero rest mass and chemical potential. The particulate quality of photons is of greatest concern here, since their interaction with matter relies most on this aspect of their nature. Their wave-like behavior cannot be disregarded since it is important in such phenomena as refraction and polarization, and it enters strongly into the interaction of radiation with obstacles.

Photons of a given energy have associated with them a frequency characteristic of the state of the source from which they were emitted and the process involved in their emission. The frequency and energy of course, are related by $E = h\nu$, where h is Planck's constant. If $\vec{\Omega}$ is the unit vector in the direction of motion of a photon, then its velocity and momentum are:

$$\vec{V} = c\,\vec{\Omega}$$

$$\vec{P} = \left(\frac{E}{c}\right)\vec{\Omega}$$

Following Samson (1965), one may consider the distribution function for photons. Let $f_\nu(\vec{P}, \vec{r}, t)$ be the number of photons per unit volume in the frequency interval $d\nu$ about ν, propagating in direction within an element of solid angle $d\vec{\Omega}$ about the direction $\vec{\Omega}$ at a position in physical space $\vec{r}$ and at a time t. (This differs from the distribution function employed by Simon (1963) by a factor $(c^3/h^3\nu^2)$ since he considers the distribution in photon momentum space). These will be referred to as photons of class $(\nu, d\nu)$, $(\vec{\Omega}, d\vec{\Omega})$. Just as for material particles, a Boltzmann equation may be written for photons. Consideration of the flux of photons through an elemental volume in phase space leads to

$$\frac{\partial f_\nu}{\partial t} + \vec{V}\cdot \text{grad}\, f_\nu = W_\nu(\vec{P}, \vec{r}, t) \qquad (2.1)$$

upon omission of the perturbing influence of external fields. W_ν accounts for all losses or gains to the class of interest due to absorption (i.e., the fact that radiant energy of frequency ν has been transformed into another form of energy or into radiant energy of a different frequency), due to emission, and due to scattering. The distribution of particles in momentum space now involves only an angular distribution since all of them move with the same speed. The behavior of photons is already unlike that of material particles.

The same restrictions apply as for any Boltzmann equation. It is really only one of a BBGKY hierarchy of coupled equations which govern the n-particle distribution functions. It applies, for instance, when the time between encounters with other particles is much greater than the duration of those encounters. A general discussion of the numerous restrictions may be found in Wu's recent text (Wu, 1966) and the specialization to photons is given by Sampson (1965).

Although distribution functions illustrate the material nature of photons, it is more common to deal with the specific intensity, I_ν. This is defined as the amount of radiant energy flowing through a unit area of a surface per unit time which is carried by photons in $(\nu, d\nu)$ travelling in directions within solid angle element $d\vec{\Omega}$ centered about $\vec{n}$, the outward normal to the surface. Since the number of photons of the class mentioned which cross dA in an interval dt is,

$$f_\nu(\vec{\Omega}, \vec{r}, t)\,(c\,dt\,dA)\,\vec{\Omega}\cdot\vec{n}$$

the energy carried by these must be

$$dE_\nu = f_\nu(\vec{\Omega}, \vec{r}, t)\, h\nu c\,(\vec{\Omega}\cdot\vec{n})\, dA\, dt$$

Thus the specific intensity is related to the photon distribution function through

$$I_\nu(\vec{\Omega}, \vec{r}, t) = h\nu c\, f_\nu(\vec{\Omega}, \vec{r}, t) \qquad (2.2)$$

A good deal of the specification of the source function, W_ν, may be carried out without specific regard for the atomic and molecular processes involved in the absorption, emission, or scattering of photons. One must, however, be aware of the consequences of these processes.

Absorption in general raises the energy level of the absorber and may cause rotational, vibrational, or electronic excitation in molecules. More rarely, it may cause dissociation of the molecule. It may also cause electronic transitions in atoms, or it may lead to ionization. On the other hand, if for any reason a molecule, an atom, or combinations of these with electrons emit radiation, energy is lost through processes which are the inverse of those just mentioned. The number of photons in $(\nu, d\nu)$, $(\vec{\Omega}, d\vec{\Omega})$ emitted in the volume dV per unit time due to any molecular or atomic process is indicated by

$$d^4 n_{\nu_e} = \left(\frac{j_\nu}{h\nu}\right) d\nu \, d\vec{\Omega} \, dV \tag{2.3}$$

j_ν is referred to as the emission coefficient.

Assume for now that the gas of atoms and molecules in which the photon gas resides is stationary. If a photon encounters a particle of the background gas, it may be either scattered or absorbed. Scattering is interpreted herein as that interaction in which the direction of motion of the photon changes as a consequence of encounter but in which there is negligible exchange of energy between the photon and the scatterer. This is, for instance, nearly the case in the elastic collision of material particles which have greatly different masses but nearly the same kinetic energy, say an electron and an atom. The result is a large change in momentum of the scattered particle due to the change in direction. Experiment and theory dictate that the loss of photons to the class due to absorption and scattering are respectively

$$d^4 n_{\nu_A} = \rho \kappa_{\nu_T} f_\nu(\vec{\Omega}, \vec{r}, t) \, d\nu \, ds \tag{2.4}$$

$$d^4 n_{\nu_{(S^-)}} = \rho \kappa_{\nu_S} f_\nu(\vec{\Omega}, \vec{r}, t) \, d\nu \, ds \tag{2.5}$$

That is, if $\rho\kappa_{\nu_T}$ and $\rho\kappa_{\nu_S}$ are the volumetric true absorption and scattering coefficients, then the respective losses to the class of interest are proportional to the number of photons of that class which are present in and to the path length, ds, which a photon travels through dV. The symbolism which Cheng (1965a) and Vincenti and Kruger (1965) use renders the notation less ambiguous

$$\alpha_\nu \equiv \rho \kappa_{\nu_T}$$
$$\tilde{\eta}_\nu \equiv \rho \kappa_{\nu_S}$$

In general α_ν and $\tilde{\eta}_\nu$ do not depend upon $\vec{\Omega}$ and the sum $\beta_\nu \equiv \alpha_\nu + \tilde{\eta}_\nu$ is referred to as the volumetric extinction coefficient since it accounts for all losses from the class in question (Goulard, 1962, employs $\rho\beta_\nu$ for the volumetric extinction coefficient). From Eq. (2.5) the number of photons in $(\nu, d\nu)$, $(\vec{\Omega}', d\vec{\Omega}')$ which are scattered into other directions is

$$\tilde{\eta}_\nu(\vec{r},t)\, f_\nu(\vec{\Omega}',\vec{r},t)\, d\nu\, d\vec{\Omega}'\, dV\, ds$$

Of this, a fraction $\mathcal{P}_\nu(\vec{\Omega}'\cdot\vec{\Omega},\vec{r},t)$ is directed into $(\vec{\Omega}, d\vec{\Omega})$. $\mathcal{P}_\nu(\vec{\Omega}'\cdot\vec{\Omega},\vec{r},t)$ is Goulard's phase function, $f(\alpha)$ (Goulard, 1962). Accounting for all of the scattered contributions, the total number of photons scattered into the class $(\nu, d\nu)$, $(\vec{\Omega}, d\vec{\Omega})$ is

$$d^4 n_{\nu_{(S^+)}} = \tilde{\eta}_\nu(\vec{r},t)\left[\int_{\Omega'} \mathcal{P}_\nu(\vec{\Omega}'\cdot\vec{\Omega},\vec{r},t)\, f_\nu(\vec{\Omega}',\vec{r},t)\frac{d\Omega'}{4\pi}\right] d\nu\, ds\, d\vec{\Omega} \tag{2.6}$$

In order that photons not be annihilated in the scattering process, $\mathcal{P}_\nu(\vec{\Omega}'\cdot\vec{\Omega},\vec{r},t)$ must be normalized to unity.

$$\int_{\Omega'} \mathcal{P}_\nu(\vec{\Omega}'\cdot\vec{\Omega},\vec{r},t)\, d\vec{\Omega}' = 1$$

Chandrasekhar (1960, page 7) indicates the following Legendre Polynomial expansion for the phase function*

$$\mathcal{P}_\nu(\vec{\Omega}'\cdot\vec{\Omega},\vec{r},t) = \sum_{l=0}^{\infty} \varpi_{l_\nu} P_l(\vec{\Omega}'\cdot\vec{\Omega}) \tag{2.6a}$$

In an interval dt the net increase of photons of class $(\nu, d\nu)$, $(\vec{\Omega}, d\vec{\Omega})$ due to emission is given by Eq. (2.3), that due to scattering by Eq. (2.6). The net loss due to extinction is the sum of Eq. (2.4) and Eq. (2.5). Since $c\,dt = ds$, the function W_ν may be evaluated with the result

* Since all absorption processes, including frequency rearrangement, have been lumped into the absorption coefficient, albedos (Chandrasekhar, 1960) need not be defined. This means that Rayleigh or Thompson scattering are by definition true scattering processes, whereas Raman scattering, in which there is energy exchange, must be grouped among absorption processes. The entire question is academic, however, since scattering will ultimately be neglected (see Section 3.3).

$$\frac{\partial f_\nu}{\partial t} + \vec{v}\cdot \text{grad}\, f_\nu = \frac{j}{h\nu} + c\tilde{\eta}_\nu \int \mathcal{P}_\nu(\vec{\Omega}'\cdot\vec{\Omega}) f_\nu(\vec{\Omega}') \frac{d\vec{\Omega}'}{4\pi} - \beta_\nu f_\nu \tag{2.7}$$

Multiplication by $h\nu$ and use of Eq. (2.2) results in

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \vec{\Omega}\cdot \text{grad}\, I_\nu = j_\nu - \beta_\nu I_\nu + \tilde{\eta}_\nu \int \mathcal{P}_\nu(\vec{\Omega}'\cdot\vec{\Omega}) I_\nu(\vec{\Omega}') \frac{d\vec{\Omega}'}{4\pi} \tag{2.8}$$

In the above the derivative along the photon path may be written in the following forms

$$\vec{\Omega}\cdot \text{grad} = \frac{\partial}{\partial s} = \ell_i \frac{\partial}{\partial x_i} \tag{2.9}$$

where s is the distance variable along the photon path, and the ℓ_i are direction cosines of the photon world line relative to a Cartesian system of coordinates. These are the governing equations of radiative transfer. Equations (2.7) or (2.8) represent a doubly infinite set of equations, one for each frequency, ν , and each direction, $\vec{\Omega}$. Each is coupled with all others through absorption and scattering. Photons do not interact among themselves, thus all possible contributions are included (at least symbolically) in the equations above.

The difference between the kinetic theory of material particles and that for photons is now more precise. Material particles are neither absorbed nor emitted (except perhaps at very special kinds of surfaces), their interactions occur mainly among themselves, and (if only elastic encounters are considered, as is usually the case) scattering is the major contribution to the source term. In fact, $\left(\frac{c}{\tilde{\eta}_\nu}\right)$ may be interpreted as a collision frequency and the phase function $\mathcal{P}_\nu$ is related to the product of the scattering particle distribution and a differential scattering cross-section which may be obtained from the laws of classical mechanics once the interaction potential is known. Many investigators note the similarity of Eq. (2.7) or (2.8) in the absence of scattering to the Krook kinetic equation. The similarity is, however, purely mathematical since the physical processes which remain have, in general, no counterpart in the kinetic theory of ideal gases. Since the equations above are merely a symbolism for the actual physical processes involved in radiation, further postulates are necessary in order to proceed.

It is of interest to outline what sort of knowledge is required to determine the absorption properties of a gas. The most simple models consider the radiating system to be hydrogenic and to consist of an electron harmonically bound to a massive positively charged nucleus. The energy radiated by this constantly accelerating charge is what is referred to as "radiation". Unfortunately, general atomic and molecular systems are not so simple, and correction factors are introduced to make them fit such a model. The most common of these is the oscillator strength or f-number which is the number (perhaps non-integral) of such oscillators which an atomic or molecular system undergoing a given transition must correspond to if it is to be represented by the simple model. The oscillator strength may be a function of a discrete variable, ν, for line radiation or of a continuous ν for processes involving free electrons. The "strengths" for upward (absorptive) and downward (emissive) transitions are not necessarily the same, just as the relation between absorption and emission is not that of a positive to a negative photograph. The restriction on the f's is that their sum over all pairs of states between which radiation is possible must be the number of excited electrons which may participate in the radiation. (The sum in this context is taken to include integrals over continuous spectra). Often the oscillator strength may be factored into vibrational, rotational, and electronic portions corresponding to the separability of the wave functions of the systems which enter various matrix elements of the dipole moment. One may also introduce a Gaunt factor which accounts for the nonhydrogenicity of many electron systems. In addition, a shape factor is necessary in order to correct the oscillator model for the perturbing influences of interparticle encounters, external fields (that of a passing ion may have an appreciable effect), and the fact that the system is in continual thermal motion. These are the mechanisms of collisional, Stark, and Doppler broadening respectively. Since while the oscillator moves it continually loses energy through radiation, the oscillation is damped. This leads to the natural line width. Broadening is quite important, since the energy content of a given line in the spectrum can be significantly increased if it is broadened. It may have a marked effect upon the various mean coefficients to be defined shortly (Section 2.4.2). All of the terminology mentioned above stems from Einstein's original success in making the Planck formula consistent with the Bohr theory of the atom and has been retained mainly for compatibility with the earliest investigations. If the quantum mechanics of the systems of interest were resolved, then all of the properties mentioned above could be calculated. Unfortunately, such a wealth of information is unavailable, and various approximations must be invoked. Chapin (1962) discusses the processes involved in atomic radiation, and Hunt and Sibulkin (1966a, b, c) present a summary of the calculation of absorption coefficients in molecular systems.

Consider nonequilibrium molecular radiation. Einstein postulated that the probability per system in state "n" that such will make a transition downward (to state "m") while spontaneously emitting radiation in $(\vec{\Omega}, d\vec{\Omega})$ during an interval dt is

$$P_{nm_E} = A_{nm}\, dt\, d\vec{\Omega} \tag{2.10}$$

Furthermore, he assumed that the probability that the transition downward may be induced by the radiative field itself is

$$P_{nm_I} = B_{nm}\, I_{\nu_{nm}}\, dt\, d\vec{\Omega} \tag{2.11}$$

where

$$h\nu_{nm} = E_n - E_m \tag{2.12}$$

Similarly, the probability per system in state "m" for upward transition to state "n" upon absorption of radiation of frequency ν_{nm} in $(\vec{\Omega}, d\vec{\Omega})$ is:

$$P_{nm_A} = B_{mn} I_{\nu_{nm}} \, dt \, d\vec{\Omega} \qquad (2.13)$$

Therefore, the nonscattering terms in Eq. (2.8) are

$$j_\nu - \alpha_\nu I_\nu = \alpha_\nu' (J_\nu - I_\nu) \qquad (2.14a)$$

$$J_\nu = \frac{N_n A_{nm}}{N_m B_{mn}} \left\{ 1 - \frac{N_n B_{nm}}{N_m B_{mn}} \right\}^{-1} \qquad (2.14b)$$

where

$$\alpha_\nu' = h \nu_{nm} B_{mn} N_n \left\{ 1 - \frac{N_n B_{nm}}{N_m B_{mn}} \right\} \qquad (2.14c)$$

In detail J_ν and α_ν' should be sums over all pairs of states which may emit or absorb radiation of frequency ν_{nm}. This will be assumed to have been done. Knowledge of the radiative field of a gas in thermodynamic equilibrium yields still more information about the quantities above, and this is described in the next section.

2.2 Kirchoff's Law and Local Thermodynamic Equilibrium

As was pointed out by this author (Finkleman, 1964) and by Cheng (1965a) there seems to be widespread confusion in the literature concerning the definition of equilibrium. One must be careful to distinguish between mechanical equilibrium and thermodynamic (or thermal) equilibrium. It is known that Kirchoff's Law holds in the latter situation, in which homogeneity and isotropy prevail. The most general statement of this law is that the ratio of the "disposition" of a black body to emit thermal radiation of a given frequency to that to absorb depends only upon temperature when the body is in thermal equilibrium. There has been considerable confusion in the interpretation of this "law".

Agassi (1967) points out that there are three restrictions implicit in this statement. The radiation must be thermal, the radiating body must be black, and the system must be in equilibrium. There are, however, no universally accepted definitions of these terms. By omitting one, two, or all of the restrictions one may interpret the law in eight different ways. Kirchoff's Law, which depended upon no specific mechanism for radiation, attracted Planck because of its utter generality.

Agassi gives a very interesting account of the evolution and interpretation of Planck's well known formula for equilibrium radiation

$$(I_\nu)_e = \frac{2h\nu^3}{c^2}\left(e^{\frac{h\nu}{kT}}-1\right)^{-1} \tag{2.15}$$

This relation and the constants "h" and "h/k" were recognized by Planck well before he had thought of quantization. They arose when he attempted to explain Kirchoff's Law by means of the first and second laws of thermodynamics. It remained for Einstein to assume the existence of induced emission and to correlate Planck's ideas with Bohr's theory of the atom. The original reasoning which Einstein employed may be found in the work cited (Agassi, 1967) among other sources. The relation above is accepted henceforth, and its interpretation and several derivations are left to the reader.

In thermodynamic equilibrium, the scattering terms in Eq. (2.8) are obviously irrelevant because of isotropy. Isotropy and homogeneity then cause the left hand side to vanish as well. Hence an alternate statement of Kirchoff's Law is

$$(J_\nu)_e = \left(\frac{j_\nu}{\alpha'_\nu}\right)_e = I_{\nu_e}(T)$$

From equilibrium statistical mechanics

$$\left(\frac{N_n}{N_m}\right)_e = \frac{g_n}{g_m}\, e^{-\frac{h\nu_{nm}}{kT}} \tag{2.16}$$

Therefore the relationship between the Einstein coefficients is

$$(B_{nm}/B_{mn}) = (g_m/g_n) \tag{2.17a}$$

$$(A_{nm}/B_{nm}) = (2h\nu_{nm}^3/c^2) \tag{2.17b}$$

where the g's are degeneracies of the upper and lower states. Note that these relations depend only upon microscopic properties of the radiating systems, thus they must apply in general. Equation (2.14b) is, therefore

$$J_\nu = \frac{2h\nu_{nm}^3 N_n}{c^2 N_m}\left\{1 - \frac{g_m N_n}{g_n N_m}\right\}^{-1} \tag{2.18}$$

The main difficulty one encounters in nonequilibrium situations lies in the definitions of the macroscopic concepts of temperature and entropy. On the microscopic level this difficulty stems from ignorance of the nonequilibrium distribution of particles among the available states. Suppose a temperature, Θ_ν, is defined as follows

$$\Theta_\nu \equiv \frac{k}{h\nu_{nm}} \log\left(\frac{N_m g_n}{N_n g_m}\right) \tag{2.19}$$

The specification of Θ_ν is not arbitrary, but depends upon the physics of the problem. From Eq. (2.19) it follows that

$$j_\nu - \alpha_\nu I_\nu = \alpha'_{\nu_{nm}}(B_{\nu_{nm}} - I_{\nu_{nm}}) \tag{2.20a}$$

where

$$B_{\nu_{nm}}(\Theta_{\nu_{nm}}) = \frac{2h\nu_{nm}^3}{c^2}\left\{e^{\frac{h\nu_{nm}}{k\Theta_{\nu_{nm}}}} - 1\right\}^{-1} \tag{2.20b}$$

$$\alpha'_{\nu_{nm}} = h\nu_{nm} B_{mn} N_m \left\{1 - e^{-\frac{h\nu_{nm}}{k\Theta_{\nu_{nm}}}}\right\} \tag{2.20c}$$

where the astrophysical symbolism, B_ν, has been introduced for the Planck function. Since it is understood that the effect of induced emission is to decrease absorption, the prime may henceforth be deleted from the volumetric absorption coefficient. It is important that the nonequilibrium temperature, Θ_ν, must depend upon ν. It is close to the translational temperature of the gas for frequencies at which α_ν' is small, but not otherwise (Sampson, 1965).

If the various energy levels are always populated according to a Boltzmann distribution at a temperature characteristic of the local average translational energy of the medium, then Eq. (2.19) implies that $\Theta_\nu = T$. This is the assumption of local thermodynamic equilibrium (LTE). Physically such an assumption is unwarranted unless the populations of levels from which radiation may occur is maintained by some external agency. In many cases the rate equations which govern the number densities of the various species of material particles and their energy level populations are predominated by interactions with other material particles (collision dominance) and the interaction with radiation is insignificant. Thus LTE may exist if the rate at which energy is lost through radiation from atomic or molecular systems is balanced by collisional excitation such that Boltzmann distribution at the local translational temperature is maintained. Collision dominance should not be confused with LTE. Usually the two do occur together, but not always. For instance if the photon free path is sufficiently small (an optically thick gas) then LTE may hold even if the gas is not collision dominated. For a more complete discussion of LTE and its implications, see Chapter 10 of Sampson's monograph (1965). The symbolism B_ν will hereafter be employed for the source function. It is understood, however, that this Planck function will depend upon a suitably defined nonequilibrium temperature, Θ_ν, in the most general situations. This is a matter of convenience only; in fact the source function is not necessarily the Planck function at all.

Thus far the elemental properties of the radiative field source function, absorption and scattering coefficients, specific intensity have been defined. These will now be related to the "observable" quantities of thermal radiation.

2.3 Macroscopic Properties of the Radiative Field

Just as the mass flux, heat flux, and stresses (momentum fluxes) in a material gas are averages of properties of individual particles, corresponding quantities may be defined for a photon gas. These include the spatial density of radiant energy, the radiant heat flux, and the radiation stress tensor.

Since the number of photons in $(\nu, d\nu)$, $(\vec{\Omega}, d\vec{\Omega})$ per unit volume is $f_\nu d\vec{\Omega}$, the net density of energy due to the presence of photons is frequency at any point and time must be

$$U_\nu^{(R)} = \int_{4\pi} h\nu f_\nu(\vec{\Omega}') d\vec{\Omega}' = \frac{1}{c}\int_{4\pi} I_\nu(\vec{\Omega}') d\vec{\Omega}' \qquad (2.21)$$

Note that the first moment of the intensity which appears above

$$I_{o_\nu} \equiv \int_{4\pi} I_\nu(\vec{\Omega}') d\vec{\Omega}' \qquad (2.22)$$

is quite important in RGD.

If $b(\vec{v})$ is some property of an individual photon, the net flux of b per unit frequency from all photons which may cross a unit area with outward normal $\vec{n}$ is

$$\vec{B}\cdot\vec{n} = \left[\int_{\Omega} b(\vec{\Omega}) f_\nu(\vec{\Omega},\vec{r},t)\, c\vec{\Omega}\, d\vec{\Omega}\right]\cdot\vec{n}$$

If $b = h\nu$ the energy- or heat-flux vector follows

$$\vec{q}_\nu = \int_{\Omega} c h\nu \vec{\Omega} f_\nu(\vec{\Omega}) d\vec{\Omega} = \int_{\Omega} I_\nu \vec{\Omega}\, d\vec{\Omega} \qquad (2.23)$$

Similarly if $b = \vec{P}_\nu$ is the momentum of a photon, the radiation stress tensor results

$$\vec{P}_\nu^{(R)} = \int_{\Omega} \left(\frac{h\nu}{c}\vec{\Omega}\right) f_\nu \, c\vec{\Omega} \, d\vec{\Omega} = \frac{1}{c}\int I_\nu \vec{\Omega}\vec{\Omega} \, d\vec{\Omega} \tag{2.24}$$

These may be expressed in a Cartesian coordinate system in which

$$\vec{\Omega} = l_i \vec{e}_i = l_1 \vec{e}_1 + l_2 \vec{e}_2 + l_3 \vec{e}_3 \tag{2.25a}$$

where the $\vec{e}_i$ are unit vectors along the coordinate axes. If ϕ is the azimuth measured from the x_1 axis and θ the declination measured from the x_3 axis then

$$l_1 = \sin\theta \cos\phi \tag{2.25b}$$

$$l_2 = \sin\theta \sin\phi \tag{2.25c}$$

$$l_3 = \cos\theta \tag{2.25d}$$

$$d\vec{\Omega} = \sin\theta \, d\theta \, d\phi \tag{2.25e}$$

In this scheme

$$u_\nu^{(R)} = \int I_\nu \, d\vec{\Omega} \tag{2.21}$$

$$(\vec{q}_\nu)_i = \int I_\nu l_i \, d\vec{\Omega} \tag{2.23a}$$

$$\left(\vec{P}^{(R)}\right)_{ij} = \frac{1}{c}\int I_\nu l_i l_j \, d\vec{\Omega} \tag{2.24a}$$

It follows that the fundamental quantities may be interpreted as moments of the specific intensity with respect to the angular variable or as "momentum" moments of the photon distribution function. The general moment may be written

$$(I_\nu^{(n)})_{ijk\ldots} = \left\{\int (l_i^{\,a} l_j^{\,b} l_k^{\,c} \cdots) \, I_\nu(\vec{\Omega}) \, d\vec{\Omega}\right\}_{a+b+c+\cdots=n}$$

Beyond $(I_\nu)^{(2)}_{ij}$ these have no simple physical interpretation.

It is important in some of the arguments to follow that one distinguish between photons incident upon an elemental area whose unit normal is $\vec{n}$ and those which are emergent from it. The superscript notation $(+)$, $(-)$ is employed to denote quantities referred to those photons moving in directions, $\vec{\Omega}$, such that $\vec{\Omega}\cdot\vec{n} > 0$ or < 0 respectively. One may infer that

$$
\begin{aligned}
I_{o\nu}^{(\pm)}(\vec{r},t) &= \int_{\vec{\Omega}\cdot\vec{n} \gtrless 0} I_\nu^{(\pm)}(\vec{\Omega},\vec{r},t)\, d\vec{\Omega} \\
\vec{n}\cdot\vec{q}_\nu^{(\pm)}(\vec{r},t) &= \pm \int_{\vec{\Omega}\cdot\vec{n} \gtrless 0} I_\nu^{(\pm)}(\vec{\Omega},\vec{r},t)\,\vec{n}\cdot\vec{\Omega}\, d\vec{\Omega}
\end{aligned}
\qquad (2.26)
$$

where $I_\nu = I_\nu^{(\pm)}$ as $\vec{\Omega}\cdot\vec{n} \gtrless 0$. Also

$$
\begin{aligned}
I_{o\nu}(\vec{r},t) &= I_{o\nu}^{(+)}(\vec{r},t) + I_{o\nu}^{(-)}(\vec{r},t) \\
\vec{q}_\nu(\vec{r},t) &= \vec{q}_\nu^{(+)}(\vec{r},t) - \vec{q}_\nu^{(-)}(\vec{r},t)
\end{aligned}
\qquad (2.27)
$$

These quantities and the radiation stress tensor (Eq. (2.24, 2.24a)) are, however, only the contributions due to photons of a single frequency (or energy). If ψ_ν is any property for a given frequency then the net effect of photons of all frequencies is

$$
\psi = \int_0^\infty \psi_\nu(\vec{r},t)\, d\nu \qquad (2.28)
$$

and these integrated influence the fluid dynamics.

Suppose now, that the field were isotropic; since I_ν would no longer depend upon direction Eq. (2.26) would become

$$
\begin{aligned}
I_{o\nu}^{(\pm)} &= 2\pi\, I_\nu(\vec{r},t) \\
\vec{n}\cdot\vec{q}_\nu^{(\pm)} &= \pi\, I_\nu(\vec{r},t)
\end{aligned}
$$

where $\vec{n}$ may be in any direction whatever. Also

$$I_{o_\nu} = 4\pi I_\nu(\vec{r},t)$$

$$\vec{q}_\nu = 0$$

and, the stress tensor would be diagonal

$$\overleftrightarrow{P}_\nu^{(R)} = \frac{4\pi}{3c} I_\nu(\vec{r},t) \overleftrightarrow{1}$$

If, furthermore, the situation were one of thermal equilibrium, $I_\nu = B_\nu(T)$, and

$$I_o = \int_0^\infty I_\nu d\nu = 4\pi B(T) = c u^{(R)}$$

$$\overleftrightarrow{P}^{(R)} = \frac{1}{3c} I_o(\vec{r},t) \overleftrightarrow{1} \tag{2.29}$$

where

$$B(T) = \int_0^\infty \frac{2h\nu^3}{c^2}\left(e^{\frac{h\nu}{kT}}-1\right)^{-1} d\nu = \frac{\sigma T^4}{\pi} \tag{2.30}$$

and

$$\sigma = (2\pi^4 k^4/15h^3c^2)$$

is the Stefan-Boltzmann constant. Note that, for any isotropic situation

$$\overleftrightarrow{P}^{(R)} = \frac{1}{3} u^{(R)} \overleftrightarrow{1} \tag{2.31}$$

as may also be deduced from Lambert's law. The flux emergent from a surface in an isothermal "black" body in either direction is incidentally

$$\vec{q}^{(+)} = \vec{q}^{(-)} = \sigma T^4$$

which is the Stephen-Boltzmann law.

All of the properties of the radiative field which will be necessary to the formulation of RGD are now available.

2.4 Formal Solution of the Transfer Equation - Asymptotic Situations and Semi-Gray Radiative Transfer

2.4.1 The Solution of the Transfer Equation

The source function includes scattering when Eq. (2.8) is written in the following form

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \frac{\partial I_\nu}{\partial s} = \beta_\nu \left(S_\nu - I_\nu \right) \tag{2.32a}$$

where

$$S_\nu = \frac{1}{\beta_\nu}\left\{ \left[\tilde{\eta}_\nu \int_{\Omega'} P_\nu (\vec{\Omega}'\cdot\vec{\Omega}) I_\nu(\vec{\Omega}') \frac{d\vec{\Omega}'}{4\pi} + \alpha_\nu B_\nu \right] \right\} \tag{2.32b}$$

The fact that S_ν depends upon I_ν and that the solution of the differential equation may lead only to an equivalent integral equation is not important. The solution of Eq. (2.32a) is readily deduced (see e.g., Zeldovich and Raizer, 1966, pg. 132)

$$I_\nu(s,t) = I_\nu(s_o, t-\tfrac{s-s_o}{c}) e^{-\int_{s_o}^{s} \beta_\nu (s'', t-\frac{s-s''}{c}) ds''} + \int_{s_o}^{s} \beta_\nu(s') S_\nu(s', t-\tfrac{s-s'}{c}) e^{-\int_{s_o}^{s'} \beta_\nu (s'', t-\frac{s-s''}{c}) ds''} ds' \tag{2.33}$$

where some boundary is located at a distance s_o along the ray of interest. It is convenient to express Eq. (2.63) in a more compact form by choosing the origin of coordinates arbitrarily and defining

$$\tau_\nu (s', t) = \int_{s_o}^{s'} \beta_\nu (s'', t - \tfrac{s-s''}{c}) \, ds'' \tag{2.34}$$

Then

$$I_\nu(\tau_\nu) = I_\nu(\tau_{\nu_o}) e^{-(\tau_\nu - \tau_{\nu_o})} + \int_{\tau_{\nu_o}}^{\tau_\nu} S_\nu(\tau_\nu') e^{\tau_\nu' - \tau_\nu} d\tau_\nu' \tag{2.35}$$

This expression belies several important points. First, the source function, S_ν is not known in general. It must be deduced from energy balances and master equations which in turn depend upon the radiative field through I_ν. Second, the relationship between the photon path length, s, and the coordinate distances of interest is not apparent. This relation emphasizes the anisotropy of the specific intensity. Nevertheless the problem has been simplified somewhat.

The spectral intensity is insufficient in RGD, since all frequencies must be allowed. If the integrated specific intensity is defined by

$$I(\vec{\Omega},\vec{r},t) = \int_0^\infty I_\nu(\vec{\Omega},\vec{r},t)\,d\nu \tag{2.36}$$

then the transfer Eq. (2.32) becomes

$$\frac{1}{c}\frac{\partial I}{\partial t} + \frac{\partial I}{\partial s} = \alpha_e B(\theta) - \beta_a I + \tilde{\eta}_0 S_1 \tag{2.37}$$

where

$$B(\theta) = \int_0^\infty B_\nu(\theta_\nu)\,d\nu \qquad [\theta = \theta(\vec{r},t)] \tag{2.38a}$$

$$S_1(\vec{\Omega},\vec{r},t) = \int_0^\infty d\nu \int_{4\pi} \beta_\nu(\vec{\Omega}'\cdot\vec{\Omega})\, I_\nu(\vec{\Omega}')\,\frac{d\vec{\Omega}'}{4\pi} \tag{2.38b}$$

Three mean properties of the medium must be introduced after integration. The mean emission coefficient is

$$\alpha_e = \int_0^\infty \alpha_\nu B_\nu\,d\nu \Big/ B(\theta) \tag{2.39}$$

α_e reduces to the Planck mean emission coefficient (Zeldovich and Raizer, pg. 166) in LTE. The remaining mean properties are dependent upon the unknown intensity.

$$\beta_a(\vec{\Omega},\vec{r},t) = \int_0^\infty \beta_\nu(\vec{r},t)\, I_\nu(\vec{\Omega},\vec{r},t)\, d\nu / I(\vec{\Omega},\vec{r},t) \quad = \alpha_a + \tilde{\eta}_a \tag{2.40a}$$

$$\alpha_a(\vec{\Omega},\vec{r},t) = \int_0^\infty \alpha_\nu(\vec{r},t)\, I_\nu(\vec{\Omega},\vec{r},t)\, d\nu / I(\vec{\Omega},\vec{r},t) \tag{2.40b}$$

$$\tilde{\eta}_a = \int_0^\infty \tilde{\eta}_\nu(\vec{r},t)\, I_\nu(\vec{\Omega},\vec{r},t)\, d\nu \,/ I(\vec{\Omega},\vec{r},t) \tag{2.40c}$$

The existence of such mean coefficients has been recognized by Kourganoff (1952, pg. 27) and by Viskanta (1964b). Both seem to have ignored the inherent anisotropy of $\alpha_a, \tilde{\eta}_a$ even in a nonscattering medium as was pointed out to the author by Dr. D. H. Sampson (1967).

Several of the approximate treatments of α_a in a nonscattering gas (Finkleman and Chien, to be published) are presented in Appendix A. The terminology "Semi-Gray Radiative Transfer" has been applied to problems investigated with Eq. (2.37). This is in contrast to the common assumption of a completely gray gas in which α_ν is in fact independent of frequency so that $\alpha_e = \alpha_a = \alpha_\nu$. The general solution of Eq. (2.37) is

$$I(s,t) = I(\tau_{a_0})\, e^{-(\tau_a - \tau_{a_0})} + \int_{\tau_{a_0}}^{\tau_a} \frac{\alpha_e}{\beta_a} S(\tau_a')\, e^{-(\tau_a - \tau_a')} d\tau_a' \tag{2.41}$$

$$\tau_a = \int_0^s \beta_a(s'', t - \tfrac{s-s''}{c})\, ds'' \qquad S = \int_0^\infty S_\nu\, d\nu \tag{2.42}$$

when this is compared with Eq. (2.35) the only effect that semi-gray transfer has upon an otherwise gray problem is to modify the effective source function. Reference may be made to Appendix A for further comments on the absorption coefficient α_a. Although the concept of an absorption mean is not new, the recognition of its implications in subsequent remarks are original.

The terms "large" and "small" which have been used require qualification in accord with appropriate normalization. In order to examine the asymptotic situations, (Section 1.2.3) it is supposed that there is some reference level of absorption, α_o^*, physical length scale, ℓ^*, and reference material velocity, u_o^*. In terms of the normalized quantities (asterisks denote dimensional quantities)

$$\alpha_j = \alpha_j^*/\alpha_o^* \qquad [j = \nu, a, e]$$
$$t = t^* u_o^*/\ell^*$$
$$s = s^*/\ell^*$$

Eq. (2.35) becomes

$$I_\nu(s,t) = I_\nu(\tau_{\nu_o}) e^{-\tau_\ell(\tau_\nu - \tau_{\nu_o})} + \tau_\ell \int_{\tau_\ell \tau_{\nu_o}}^{\tau_\ell \tau_\nu} S_\nu(\tau_\nu') e^{-\tau_\ell(\tau_\nu - \tau_\nu')} d\tau_\nu' \tag{2.43a}$$

where

$$\tau_\nu = \int_0^s \beta_\nu [s', t - \beta(s-s')] ds' \qquad \left[\beta = \frac{u_o^*}{c^*}, \tau_\ell = \alpha_o^* \ell^*\right] \tag{2.43b}$$

A corresponding form may be found for Eq. (2.41). Since the expression is homogeneous in the dimensions of the specific intensity and the source function, their normalization is immaterial. The magnitude of the parameter τ_ℓ will determine the simplifications allowed. For simplicity the boundary term is ignored for the moment either because $I_\nu(\tau_{\nu_o}) \ll I_\nu(\tau_\nu)$ or $\tau_\ell(\tau_\nu - \tau_{\nu_o}) \gg 1$. However, it can be important, and its implications will be discussed shortly.

2.4.2 The Optically "Thin" and "Thick" Limits

Suppose that the physical length scale of interest is much smaller than the reference photon free path, $1/\alpha_o^*$, i.e., $\tau_\ell \ll 1$. Then

$$e^{-\tau_\ell(\tau_\nu - \tau_\nu')} = 1 - \tau_\ell(\tau_\nu - \tau_\nu') + O(\tau_\ell^2)$$

hence

$$I_\nu(\tau_\nu) = \int_{\tau_\ell \tau_{\nu_o}}^{\tau_\ell \tau_\nu} \tau_\ell S_\nu(\tau_\nu') d\tau_\nu' + O(\tau_\ell^2)$$

The transfer equation is, to lowest order

$$\beta \frac{\partial I_\nu}{\partial t} + \frac{\partial I_\nu}{\partial s} = \tau_\ell \beta_\nu B_\nu$$

The scattering term is neglected on the basis that it is $O(I_\nu)$, so that $S_\nu = B_\nu + O(\tau_\ell)$. Equations (2.9), (2.22), and (2.23), in a nonscattering gas lead to

$$\beta \frac{\partial I_{o\nu}}{\partial t} + div\, \vec{q}_\nu = \tau_\ell\,(4\pi\alpha_\nu B_\nu) + O(\tau_\ell^2) \tag{2.44a}$$

Upon integration over all frequencies, the divergence of the net radiative heat flux is

$$div\, \vec{q} = 4\pi\tau_\ell \alpha_e B - \beta \frac{\partial I_o}{\partial t} + O(\tau_\ell^2) \tag{2.44b}$$

which for a steady flow (or for $\beta \ll 1$), in LTE reduces to

$$div\, \vec{q} = \tau_\ell \cdot 4\alpha_p \sigma T^4 + O(\tau_\ell^2)\ , \quad [\alpha_e \to \alpha_p] \tag{2.45}$$

This is the most common form of the optically thin limit. The analogy with the Stefan -Boltzmann law is clear, and the appropriate mean emission coefficient is α_p.

Except very near to a surface, the boundary condition term may certainly be omitted if $\tau_\ell \gg 1$. Because the origin of coordinates is arbitrary the lower limit of integration, Eq. (2.43a), may be allowed to approach $-\infty$. Since the photon free path is much larger than the length scale of interest, it may be assumed that all physical properties, in particular the temperature, vary little over an optical path $\sim 1/\alpha_o^*$. Because of the dominance of the absorption emission process scattering ignored and $B_\nu(\tau_\nu')$ may be expanded in a Taylor series about τ_ν with the result

$$I_\nu(\tau_\nu) = \tau_\ell \int_{-\infty}^{\tau_\ell \tau_\nu} \left\{ B_\nu(\tau_\nu) + [\tau_\nu' - \tau_\nu] \frac{\partial B_\nu}{\partial \tau_\nu}\Big|_{\tau_\nu} + \frac{[\tau_\nu' - \tau_\nu]^2}{\underline{|2}} \frac{\partial^2 B_\nu}{\partial \tau_\nu^2}\Big|_{\tau_\nu} + \cdots \right\} e^{-\tau_\ell(\tau_\nu - \tau_\nu')} d\tau_\nu'$$

or

$$I_\nu(\tau_\nu) = B_\nu(\tau_\nu) - \frac{dB_\nu}{d\tau_\nu}(\tau_\nu) \cdot \frac{1}{\tau_\ell} + \frac{d^2 B}{d\tau_\nu^2}(\tau_\nu) \cdot \frac{1}{\tau_\ell^2} - \cdots \tag{2.46}$$

The definitions (Eq. (2.34)) yield

$$\frac{d}{d\tau_\nu} = \frac{1}{\alpha_\nu(s,t)} \left\{ \beta \frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right\} \tag{2.47}$$

so that Eq. (2.46) is in fact a particular solution of Eq. (2.32). In LTE, Eq. (2.46), (2.21), (2.23), and (2.24) lead to

$$u_\nu^{(R)} = \frac{4\pi}{c} B_\nu + O\left(\frac{1}{\tau_\ell}\right) \tag{2.48}$$

$$\vec{q}_\nu = -\frac{4\pi}{3\tau_\ell \alpha_\nu} \left(\frac{dB_\nu}{dT}\right) \text{grad}(T) + O\left(\frac{1}{\tau_\ell^2}\right) \tag{2.49}$$

$$\overrightarrow{\overrightarrow{P}}_\nu^{(R)} = \frac{u_\nu^{(R)}}{3} \overrightarrow{\overrightarrow{I}} + O\left(\frac{1}{\tau_\ell}\right) \tag{2.50}$$

where Eq. (2.49) is a consequence of the fact that the average value of any direction cosine vanishes and of Eq. (2.9). The radiative field is nearly isotropic since Eq. (2.50) holds only if that is the case, and this justifies the complete emission scatterings. In LTE the net heat flux is

$$\vec{q} = -\frac{4\sigma}{3\tau_\ell \alpha_R} \text{grad}(T^4) + O\left(\frac{1}{\tau_\ell^2}\right) \tag{2.51}$$

where the Rosseland mean absorption coefficient (Zeldovich and Raizer, 1966, pg. 153) is defined by

$$\frac{1}{\alpha_R} = \int_0^\infty \frac{1}{\alpha_\nu} \frac{dB_\nu}{dT} d\nu \Big/ \int_0^\infty \frac{dB_\nu}{dT} d\nu \qquad (2.52)$$

The condition $\tau_\ell \gg 1$ is equivalent to $\frac{1}{B_\nu} \frac{dB_\nu}{d\tau_\nu} \ll 1$, which is the manner in which Vicenti and Kruger (1965) choose to derive the expressions. The similarity with Fourier conduction is obvious. It has been recognized that the "thick" limit is singular (Olstad, 1965). Sufficiently close to a boundary variations may not necessarily be small over distances less than a radiation free path. The singular nature of this approximation will be interpreted in a new light in Section 4.2.

There are at least two different ways in which mean absorption properties of the gas may be defined. The choice of an effective α in the gray approximation thus presents a problem. The general case is neither "thin" nor "thick", thus neither α_p nor α_R is universally appropriate. Recognition of this fact prompted the investigation of which is summarized in Appendix A. If it is assumed that α_a does not depend upon $\vec{\Omega}$ (the "quasi-isotropic" assumption) then to a first approximation in a nonscattering gas

$$\alpha_a(\vec{r},t) = \alpha_p(\vec{r},t) \cdot \left(\frac{T}{T_\infty}\right)^4 \left\{ 1 - 2 \int_{(T/T_\infty)}^{1} \frac{\alpha_p}{\alpha_R} \left(\frac{T}{T_\infty}\right)^4 d\left(\frac{T}{T_\infty}\right)^4 \right\}^{-\frac{1}{2}} \qquad (2.53)$$

which yields the correct thin and thick cases. Its effect on intermediate situation has not yet been assessed, but it should be better than either α_p or α_R. However, Eq. (2.53) fails to include the effects of boundaries or the possibility of appreciable anisotropy.

In this section it has been shown that the familiar limiting results apply in a gas which may scatter photons as well as emit and absorb them. Futhermore, the extension of these results to unsteady cases has been indicated. These facts are well known in astrophysics (see Sampson, 1965b), but they are less widely recognized by gasdynamicists.

2.5 Useful Expression for Radiative Transfer and Exact Formula for the Planar Case

In this section expressions useful in a future context (e.g., Appendix C) are derived. The detail with which Goulard (1962) treats the problem can hardly be improved upon. Referring to Fig. 1, Eq. (2.35) may be written

$$I_\nu(M) = I_\nu(Q)\, e^{-\tau_{\nu,MQ}} + \int_{P=Q}^{P=M} S_\nu(P)\, e^{-\tau_{\nu,MP}}\, d\tau_{\nu,MP} \tag{2.54}$$

where, analogous to Eq. (2.34)

$$\tau_{\nu,MP} = \int_M^P \beta_\nu\left(s'', t - \frac{s-s''}{c}\right) ds'' \tag{2.55}$$

It may be shown that

$$I_\nu(Q) = \epsilon_{\theta,\nu}\, B_\nu(Q) + \int_{2\pi} \rho_{\theta',\nu}\, I_\nu(\vec{\Omega}')\, f_{\theta',\theta,\nu}\, \frac{\vec{\Omega}'\cdot\vec{n}}{\pi}\, d\vec{\Omega}' \tag{2.56}$$

where the first term accounts for emission from the surface at Q, $\epsilon_{\theta,\nu}$ being the directional emissivity. $f_{\theta',\theta,\nu}$ is the true reflection coefficient and is analogous to the phase function previously defined. It represents that fraction of the incident energy in directions $\theta' = \cos^{-1}(\vec{\Omega}'\cdot\vec{n})$ reflected into the directions $\theta = \cos^{-1}(\vec{\Omega}\cdot\vec{n})$ and is normalized to unity. $\rho_{\theta',\nu}$ is the directional reflectivity which may have both specular and diffuse parts. In general $\epsilon_{\theta,\nu}$; $f_{\theta',\theta,\nu}$; $\rho_{\theta',\nu}$ are properties of the surface and of the radiative field in question. Similar expressions may be derived in analogy with Eq. (2.41).

It will subsequently be clear that the most important radiative contribution to RGD is the divergence of the heat flux. Equation (2.32), the definitions (2.23) and (2.9), and the fact that $\mathcal{P}_\nu$ is normalized to unity lead to

$$\frac{1}{c}\frac{\partial I_{0_\nu}}{\partial t} + \operatorname{div}\vec{q}_\nu = \alpha_\nu\left(4\pi B_\nu - I_{0_\nu}\right) \tag{2.57}$$

so that the fact that scattered energy is not "stored" (scattered photons are not annihilated) appears consistent in the formulation. Figure 1 requires that the incremental solid angle subtend by $d\sigma_{\perp}$ at M and the incremental radiating volume be

$$d\vec{\Omega} = \frac{d\sigma_{\perp}}{\overline{MP}^2}$$

$$dV(P) = d\sigma_{\perp}\, ds$$

whence

$$d\tau_{\nu,MP}\, d\vec{\Omega} = \beta_{\nu}\, ds \cdot \frac{d\sigma_{\perp}}{\overline{MP}^2} = \frac{\beta_{\nu}}{\overline{MP}^2}\, dV(P)$$

Similarly

$$d\sigma_A = \frac{\overline{MQ}^2\, d\vec{\Omega}}{\vec{\Omega}\cdot\vec{n}}$$

is the elemental area on the boundary surface which subtends a solid angle increment $d\vec{\Omega}$ at M. It is, then, a simple matter to write

$$\frac{1}{c}\frac{\partial I_{o\nu}(M)}{\partial t} + div\left(\vec{q}_{\nu}(M)\right) =$$

$$= 4\pi\alpha_{\nu}\, B_{\nu}(M) - \alpha_{\nu}(M)\int_V \beta_{\nu} S_{\nu}\, \frac{e^{-\tau_{\nu,MP}}\, dV(P)}{\overline{MP}^2} - \alpha_{\nu}(M)\int_A I_{\nu}(Q)\, e^{-\tau_{\nu,MQ}}\, \frac{\vec{\Omega}\cdot\vec{n}\, d\sigma_A(Q)}{\overline{MQ}^2} \qquad (2.58)$$

Although the formulation of the surface interaction is no longer so clear the analogous result from analysis of Eq. (2.37) is

$$\frac{1}{c}\frac{\partial I_o(M)}{\partial t} + div\left(\vec{q}(M)\right) =$$

$$= 4\pi\alpha_e\, B(M) - \int_V \alpha_a(M)\alpha_b(P) S(P)\, \frac{e^{-\tau_{q,MP}}\, d\bar{V}(P)}{\overline{MP}^2} - \int_A \alpha_a(M) I(Q)\, \frac{\vec{\Omega}\cdot\vec{n}\, e^{-\tau_{q,MQ}}\, d\sigma_A(Q)}{\overline{MQ}^2} \qquad (2.59)$$

or, simply

$$\frac{1}{c}\frac{\partial I_0(M)}{\partial t} + \operatorname{div}(\vec{q}(M)) = 4\pi\alpha_e B(M) - \int \alpha_a(M)\, I\, d\vec{\Omega} \tag{2.60}$$

In this case α_a must be retained in the integration since it receives contributions from the entire volume occupied by the gas and from the whole surface. Alternately, an angular mean of α_a may be defined

$$\alpha_a^{(0)} = \int \alpha_a(\vec{\Omega})\, I(\vec{\Omega})\, d\vec{\Omega} \,/\, I_0 \tag{2.61}$$

In this case $\alpha_a(M)$ in Eq. (2.59) may be replaced by $\alpha_a^{(0)}(M)$ which may be carried outside of the integration if the integrand is carefully redefined with Eq. (2.41) and Eq. (2.21) and (2.28). The transition from solid angles to areas and volumes may seem pointless at first glance since Eq. (2.58) and (2.59) are equivalent to Eq. (2.57) and (2.60). However, the volume and area forms permit rather straightforward formulation of cylindrical and spherical transfer problems, as will be shown in Appendix C.

In an entirely equivalent manner,

$$\vec{n}_1\cdot\vec{q}_\nu = \int_{4\pi} I_\nu\,(\vec{n}_1\cdot\vec{\Omega})\,d\vec{\Omega} = \int_A I_\nu(Q)\,e^{-\tau_{\nu,MQ}}\frac{(\vec{\Omega}\cdot\vec{n}_1)(\vec{\Omega}\cdot\vec{n})\,d\sigma_A(Q)}{\overline{MQ}^2} + \int_V \beta_\nu(P) S_\nu(P)\, e^{-\tau_{\nu,MP}}\frac{(\vec{\Omega}\cdot\vec{n}_1)\,dV(P)}{\overline{MP}^2} \tag{2.62}$$

$$\vec{n}_1\cdot\vec{q} = \int_A I(Q)\,e^{-\tau_{a,MQ}}\frac{(\vec{\Omega}\cdot\vec{n}_1)(\vec{\Omega}\cdot\vec{n})\,d\sigma_A(Q)}{\overline{MQ}^2} + \int_V \alpha_e(P) S(P)\, e^{-\tau_{a,MP}}\frac{(\vec{\Omega}\cdot\vec{n}_1)\,dV(P)}{\overline{MP}^2} \tag{2.63}$$

where $\vec{n}_1 \cdot \vec{q}$ is interpreted as the component of $\vec{q}$ in the direction $\vec{n}_1$. In all integrals $\vec{n}$ is the outward normal from the surface A at point Q, and all integrations are carried out over the entire volume occupied by the gas and over all portions of bounding surfaces which are "seen" by point M.

It may be stated without qualification that in all but planar geometries and a few cases in which the absorption coefficient α_ν is a constant a general reduction of the integrals which appear is not possible (see e.g., Heaslet and Warming, 1966). The reduction of the planar expressions is carried out in all standard texts. A presentation closely related to RGD (Vincenti and Baldwin, 1962) follows as a prelude to the general problem.

Consider Fig. 2 where the coordinate x_3 is referred to simply as x. Suppose that at some point x_W there is a surface of infinite extent in the x_1, x_2 plane and that no properties depend upon x_1 or x_2, hence that the radiative field cannot depend upon the azimuth, ϕ. The distance along a ray in a direction θ is related to distance along the x-axis through

$$ds = \frac{dx}{\mu} \qquad [\mu = \cos\theta] \tag{2.64}$$

It follows that, if

$$\eta_\nu = \int_0^x \alpha_\nu\left(x'', t - \frac{x - x''}{\mu c}\right) dx'' \tag{2.65}$$

where the origin is taken at the surface, then in Eq. (2.54)

$$\tau_{\nu, MP} = \frac{1}{\mu}\{\eta_\nu(P) - \eta_\nu(M)\}$$

Thus

$$I_\nu^{+}(\mu, x, t) = \left\{ I_\nu^{+}(\mu, 0, t) + \int_0^{\eta_\nu} S_\nu(\eta_\nu') e^{\frac{\eta_\nu'}{\mu}} \frac{d\eta_\nu'}{\mu} \right\} e^{-\frac{\eta_\nu}{\mu}}$$

$$I_\nu^{(-)}(\mu, x, t) = -e^{-\frac{\eta_\nu}{\mu}} \int_{\eta_\nu}^{\infty} S_\nu(\eta_\nu') e^{\frac{\eta_\nu'}{\mu}} \frac{d\eta_\nu'}{\mu}$$

where $I_\nu = I_\nu^{(\pm)}$ according as $\vec{\Omega}\cdot\vec{n} = \mu \gtrless 0$. The heat flux in the x-direction is

$$q_\nu = \int_{4\pi} I_\nu \mu \, d\vec{\Omega} = 2\pi \int_{-1}^{1} I_\nu \mu \, d\mu$$

$$= 2\pi \int_0^1 \mu I_\nu^+(\mu,0,t) e^{-\frac{\eta_\nu}{\mu}} d\mu + 2\pi \int_0^{\eta_\nu} S_\nu(\eta_\nu') \int_0^1 d\mu \, e^{-\frac{\eta_\nu-\eta_\nu'}{\mu}} d\eta_\nu' + 2\pi \int_{\eta_\nu}^{\infty} S_\nu(\eta_\nu') \int_{-1}^{0} (-1) \, e^{\frac{\eta_\nu-\eta_\nu'}{\mu}} d\mu \, d\eta_\nu'$$

or

$$q_\nu(x,t) = 2\pi \int_0^1 \mu I_\nu^+(\mu,0,t) e^{-\frac{\eta_\nu}{\mu}} d\mu + 2\pi \int_0^{\eta_\nu} S_\nu(\eta_\nu') E_2(\eta_\nu - \eta_\nu') d\eta_\nu' + -2\pi \int_{\eta_\nu}^{\infty} S_\nu(\eta_\nu') E_2(\eta_\nu' - \eta_\nu) d\eta_\nu' \quad (2.66)$$

also, it may be shown that

$$I_{o_\nu} = 2\pi \int_0^1 I_\nu^+(\mu,0,t) e^{-\frac{\eta_\nu}{\mu}} d\mu + 2\pi \int_0^{\eta_\nu} S_\nu(\eta_\nu') E_1(\eta_\nu - \eta_\nu') d\eta_\nu' + 2\pi \int_{\eta_\nu}^{\infty} S_\nu(\eta_\nu') E_1(\eta_\nu' - \eta_\nu) d\eta_\nu' \quad (2.67)$$

where $E_1(x)$ and $E_2(x)$ are the exponential integrals of order unity and two which are members of the general class of exponential integrals.

$$E_n(x) \equiv \int_0^1 e^{-\left(\frac{x}{\mu}\right)} \mu^{n-2} d\mu \quad (2.68)$$

These functions are discussed at length by Kourganoff (1963) and extensive tables may be found in NBS AMS No. 55 (Abramowitz and Stegun, 1964). The quantity $I_\nu^+(\mu,0,t)$ contains emission from the surface as well as specular and diffuse reflection. The emission is isotropic; if the wall is in LTE:

$$I_{\nu_E}^+(\mu,0,t) = \epsilon_{\nu,0} B_\nu(T_w)$$

The specular contribution is easily determined

$$I^+_{\nu_s}(\mu,0,t) = r_{s,\theta,\nu} I^-_\nu(-\mu,0,t) = r_{s,\nu}\int_0^\infty S_\nu(\eta_\nu')e^{-\frac{\eta_\nu'}{\mu}}d\eta_\nu'$$

If the heat flux into the wall is examined, and the diffuse reflection coefficient is independent of direction,

$$I^+_{\nu_d}(\mu,0,t) = 2r_{d,\nu}\int_0^\infty S_\nu(\eta_\nu')E_2(\eta_\nu')d\eta_\nu'$$

Thus, if $\epsilon_{\nu,\theta}$ and $r_{s,\theta,\nu}$ are independent of θ, the angular integration yields

$$q_\nu = 2\pi\Big\{\epsilon_\nu B_\nu(T_w)E_3(\eta_\nu) + r_{s,\nu}\int_0^\infty S_\nu(\eta_\nu')E_2(\eta_\nu+\eta_\nu')d\eta_\nu' + \\ + 2r_{d,\nu}\int_0^\infty S_\nu(\eta_\nu')E_2(\eta_\nu')d\eta_\nu' E_3(\eta_\nu) + \\ + \int_0^\infty S_\nu(\eta_\nu')\,\mathrm{sgn}(\eta_\nu-\eta_\nu')E_2\{|\eta_\nu-\eta_\nu'|\}d\eta_\nu'\Big\} \quad (2.69)$$

Similarly Eq. (2.67) and (2.57) result in

$$\frac{1}{c}\frac{\partial I_{0\nu}}{\partial t} + \mathrm{div}\,\vec{q}_\nu = -2\pi\Big\{\Big[\epsilon_\nu B_\nu(T_w) + 2r_{d,\nu}\int_0^\infty S_\nu(\eta_\nu')E_2(\eta_\nu')d\eta_\nu'\Big]E_2(\eta_\nu) + \\ + r_{s,\nu}\int_0^\infty S_\nu(\eta_\nu')E_1(\eta_\nu+\eta_\nu')d\eta_\nu' + \int_0^\infty S_\nu(\eta_\nu')E_1\{|\eta_\nu-\eta_\nu'|\}d\eta_\nu' - 2B_\nu(\eta_\nu)\Big\} \quad (2.70)$$

Analogous expressions follow for the semi-gray approximation in a quasi-isotropic gas. The only modification necessary is that all η optical variables involve α_a and the source function appears always in the weighted form $\frac{\alpha_e}{\alpha_a}S(\eta_a)$. If quasi-isotropy does not hold, exponential integrals do not necessarily appear. Of course, the emissivity and reflectivities in the modified form are not necessarily the same as those in Eq. (2.69) and (2.70). Examination of thermal equilibrium reveals the corrollary of Kirchoffs law.

$$\epsilon_\nu = 1 - r_{d,\nu} - r_{s,\nu} \quad (2.71)$$

which is, the most common form in which the law is stated.

In this brief summary it has been shown that photons may be treated as material particles up to a point. The equation which governs radiative transfer in an absorbing, emitting and scattering medium at rest has been written, and the implications of Kirchoffs law have been examined. In addition some of the physics which is involved in the calculation of absorption coefficients has been indicated. The concept of local thermodynamic equilibrium has been introduced in the framework of general, nonequilibrium radiation. The macroscopic properties of the radiative field have been defined. Furthermore the full transfer equation has been solved (or at least reduced to an integral equation), and it has been used to obtain the limiting optically "thin" and "thick" approximations and to justify a semi-gray model for radiative transfer. This model should be better then a gray one and significantly easier to apply in practice then the full nongray formulation. Finally the general, multidimensional expressions for the heat flux and its divergence have been indicated, and the planar forms were written explicitly. All of the definitions and derivations stated in this chapter will now be adapted to the problems at hand.

CHAPTER 3

THE FORMULATION OF RADIATION GAS DYNAMICS

3.1 Radiative Contributions to Gas Dynamics

The derivation of the equations which govern non-radiating real gas dynamics is well documented and will not be pursued herein. By the same procedure that the conservation relations are obtained from the Boltzmann equations of material particles, analogous relations may be obtained for the photon gas. Simon (1963) treats matter and radiation consistently within the assumptions of special relativity (see Sampson, 1965). Sampson's "simplified method" of obtaining the equations of RGD involves decoupling the radiative and material Boltzmann equations and then adding the right and left hand sides of the equations for photon momentum and energy transport to the corresponding members of the respective equations for material particles. This is valid if, and only if, the gas is collision dominated, which is very often the case; but if it is not, the approach must be reëxamined. The radiative transfer equation applies generally only in inertial systems while α_ν is computed in the non-inertial system which is moving at the local mass velocity. Sampson (1965) indicates that errors involved in neglecting this fact are usually insignificant (of order $[\sqrt{RT^*_\infty}/c^*]^2$ which shall ultimately be neglected).

The governing equations of RGD as derived by Simon (1963) are as follows:

$$\frac{\partial \rho^*}{\partial t} + \mathrm{div}(\rho^* \vec{u}^*) = 0 \tag{3.1}$$

$$\rho^* \frac{D\vec{u}^*}{Dt^*} + \operatorname{div}\left(\vec{\vec{P}}^* + \vec{\vec{P}}^{*(R)}\right) + \frac{1}{c^{*2}}\left\{\frac{\partial \vec{q}^{*(R)}}{\partial t} + \frac{\partial}{\partial t}\left(\vec{\vec{P}}^{*(R)}\cdot\vec{u}^* + u^{*(R)}\vec{u}^*\right) + \vec{u}^*\cdot\operatorname{grad}\vec{q}^{*(R)} + \right.$$
$$\left. + \vec{q}^{*(R)}\operatorname{div}\vec{u}^* + \vec{u}^*\operatorname{div}(\vec{q}^{*(R)}) + \vec{q}^{*(R)}\cdot\operatorname{grad}\vec{u}^*\right\} \tag{3.2}$$

$$\rho^* \frac{Dh^*}{Dt^*} - \frac{DP^*}{Dt^*} + u^{*(R)}\operatorname{div}\vec{u}^* = \vec{u}^*\cdot\left(\operatorname{div}\vec{\vec{P}}^{*(R)}\right) - \operatorname{div}\left(\vec{\vec{P}}^{*(R)}\cdot\vec{u}^*\right) + \frac{\vec{u}^*}{c^{*2}}\cdot\left\{\frac{\partial \vec{q}^{*(R)}}{\partial t} + \right.$$
$$+ \vec{u}^*\operatorname{div}\vec{q}^{*(R)} + \vec{q}^{*(R)}\cdot\operatorname{grad}\vec{u}^* + \vec{u}^*\cdot\operatorname{grad}\vec{q}^{*(R)}$$
$$\left. + \vec{q}^{*(R)}\operatorname{div}\vec{u}^* + \frac{\partial}{\partial t^*}\left[\vec{\vec{P}}^{*(R)}\cdot\vec{u}^* + u^{*(R)}\vec{u}^*\right]\right\}$$
$$- \operatorname{div}\vec{q}^{*(R)} - \frac{2}{c^{*2}}\frac{\partial}{\partial t^*}\left(\vec{q}^{*(R)}\cdot\vec{u}^*\right) \tag{3.3}$$

The possibility of N non-equilibrium chemical processes is included symbolically in the form:

$$\frac{DC_i}{Dt^*} = -\frac{W_i^*}{\rho^*} \qquad (i = 1, N), \quad \left[W_i^* = W_i^*\left(P^*, T^*, \vec{C}, I_\nu^*\right)\right] \tag{3.4}$$

where the W_i^* are the rates of progress of reactions involving the chemical parameters C_i. Also

$$\frac{1}{c^*}\frac{\partial I_\nu^*}{\partial t^*} + \vec{\Omega}\cdot\operatorname{grad} I_\nu^* = \beta_\nu^*\left(S_\nu^* - I_\nu^*\right) \tag{3.5}$$

hence:

$$\frac{\partial u^{*(R)}}{\partial t^*} + \operatorname{div}\left(\vec{q}^{*(R)}\right) = \int_0^\infty d\nu \int_{4\pi} d\Omega\, \beta_\nu^*\left(S_\nu^* - I_\nu^*\right) \tag{3.6}$$

$$\frac{1}{c^{*2}}\frac{\partial \vec{q}^{*(R)}}{\partial t^*} + \operatorname{div}\left(\vec{\vec{P}}^{*(R)}\right) = \int_0^\infty \frac{1}{c^*}\, d\nu \int_{4\pi} d\Omega\, \vec{\Omega}\, \beta_\nu^*\left(S_\nu^* - I_\nu^*\right) \tag{3.7}$$

etc.

To these must be added the state and thermodynamic relations:

$$\rho^* = \rho^*(P^*, T^*, \vec{c}) \tag{3.8}$$

$$h^* = h^*(P^*, T^*, \vec{c}) \tag{3.9}$$

Since radiation depends most strongly upon temperature, T must be included in the specification. Although the choice of P^* or ρ^* for the remaining variable of state is arbitrary, pressure has been chosen in this work. Only the properties of the material gas are included in the enthalpy, h^*, pressure, P^*, etc.; since the radiative contributions have been grouped separately in the equations. The integro-differential nature of the equations is clear. For instance, the divergence of the radiative heat flux required for the energy Eq. (3.3) is expressed as an integral involving the temperature distribution through Eq. (3.6). Eq. (3.5) is integro-differential as well since S_ν^* involves integrals of I_ν^* in the scattering terms. In general, Eq. (3.4) will involve integrals in the radiative interaction which must be allowed in the w_i^* rate terms. Such integrals must also incorporate the nonequilibrium parameters. (This is clear if some of the c_i are mass fractions since the absorption coefficients must depend on concentration.) Therefore Eq. (3.3), (3.4), and (3.5) may all be integro-differential equations.

Since present interest lies in the study of the unsteady flow fields generated by bodies moving into gases initially at rest, a normalization which depends only upon the undisturbed state has been chosen. Thus Mach numbers do not enter the governing equations but do appear in the boundary conditions. Molecular transport phenomena are neglected, but no restriction is yet made upon the caloric or thermal properties of the medium. The appropriately normalized variables are:

$$
\begin{aligned}
&P = P^*/P_o^* && I_\nu = I_\nu^*/B_{\nu_o}^* \\
&\vec{u} = \vec{u}^*/u_o^* && S_\nu = S_\nu^*/B_{\nu_o}^* \\
&T = T^*/T_o^* && \beta_\nu = \beta_\nu^*/\beta_{\nu_o} \\
&\rho = \rho^*/\rho_o^* && \vec{q}^{(R)} = \vec{q}^{*(R)}/\sigma T_o^{*4} \\
&h = h^*/C_{p_{f_o}}^* T_o^* && \overset{\Rightarrow}{P}^{(R)} = \overset{\Rightarrow}{P}^{*(R)}/\left(\frac{\sigma T_o^{*4}}{C^*}\right) \\
&W_i = W_i^* \cdot \rho_o^* t_{i_o}^* && u^{(R)} = u^{*(R)}/\left(\frac{\sigma T_o^{*4}}{C^*}\right)
\end{aligned}
\tag{3.10}
$$

$$t = t^* u_o^*/\ell^* \ , \quad \vec{x} = \vec{x}^*/\ell^*$$

The reference velocity is taken to be the isothermal speed of sound so that ratios of specific heats enter the formulation as infrequently as possible. The other quantities employed above are, $t_{i_o}^*$, the reference chemical relaxation time for process "i", and the frozen specific heat of the reference state $C_{p_{f_o}}^* = \left.\frac{\partial h^*}{\partial T^*}\right)_{p_o^*, \tilde{c}_o}$. The normalized quantities $u^{(R)}$ and I_o , $\vec{q}^{(R)}$ and $\vec{I}_1$, $\overset{\Rightarrow}{P}^{(R)}$ and $\overset{\Rightarrow}{I}_2$ are used interchangeably. In cases in which confusion will not arise the superscript, R , will be deleted as well.

The normalized forms of Eq. (3.1) through (3.9) are

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{u}) = 0 \tag{3.11}$$

$$\rho \frac{D\vec{u}}{Dt} + \operatorname{grad} P + \left(\frac{\beta}{B_o}\right) \operatorname{div}\left(\overset{\Rightarrow}{P}^{(R)}\right) + \left(\frac{\beta^2}{B_o}\right) \left\{ \frac{\partial \vec{q}^{(R)}}{\partial t} + \frac{\partial}{\partial t}\left(\overset{\Rightarrow}{P}^{(R)} \cdot \vec{u} + u^{(R)} \vec{u}\right) + \vec{u} \cdot \operatorname{grad} \vec{q}^{(R)} + \vec{q}^{(R)} \operatorname{div} \vec{u} + \vec{u} \operatorname{div} \vec{q}^{(R)} + \vec{q}^{(R)} \cdot \operatorname{grad} \vec{u} \right\} = 0 \tag{3.12}$$

$$
\begin{aligned}
\frac{\rho}{\Gamma} \frac{Dh}{Dt} - \frac{DP}{Dt} + \left(\frac{\beta}{B_o}\right) \frac{Du^{(R)}}{Dt} = {} & \beta \vec{u} \cdot \operatorname{div}\left(\overset{\Rightarrow}{P}^{(R)}\right) - \beta \operatorname{div}\left(\overset{\Rightarrow}{P}^{(R)} \cdot \vec{u}\right) + \\
& + \frac{\beta^2 \vec{u}}{B_o} \cdot \left\{ \frac{\partial \vec{q}}{\partial t} + \vec{u} \operatorname{div} \vec{q}^{(R)} + \vec{q}^{(R)} \operatorname{div} \vec{u} + \vec{u} \cdot \operatorname{grad} \vec{q}^{(R)} + \vec{q}^{(R)} \cdot \operatorname{grad} \vec{u} \right\} + \\
& + \frac{\beta}{B_o} \frac{\partial}{\partial t} \left\{ \overset{\Rightarrow}{P}^{(R)} \cdot \vec{u} + u^{(R)} \vec{u} \right\} - \frac{1}{B_o} \operatorname{div} \vec{q}^{(R)}
\end{aligned}
\tag{3.13}
$$

$$\frac{DC_i}{Dt} = - \frac{1}{\tau_i} \left(\frac{W_i}{\rho}\right) \tag{3.14}$$

$$\beta \frac{\partial I_\nu}{\partial t} + \vec{\Omega} \cdot \text{grad}\, I_\nu = \tau_\ell \beta_\nu \, (S_\nu - I_\nu) \tag{3.15}$$

$$\beta \frac{\partial u^{(R)}}{\partial t} + \text{div}\, \vec{q}^{(R)} = \tau_\ell \int_0^\infty d\nu \int_{4\pi} d\vec{\Omega}\, \beta_\nu (S_\nu - I_\nu) \tag{3.16}$$

$$\beta \frac{\partial \vec{q}^{(R)}}{\partial t} + \text{div}\, \overrightarrow{\overrightarrow{P}}^{(R)} = \tau_\ell \int_0^\infty d\nu \int_{4\pi} d\vec{\Omega}\, \vec{\Omega} \beta_\nu (S_\nu - I_\nu) \tag{3.17}$$

The dimensionless parameters which appear have been discussed by Goulard (1963a). They are

$$\begin{aligned} \tau_\ell &= \beta_0^* \ell^* \\ \beta &= u_0^*/c^* \\ B_0 &= \rho_0^* u_0^{*3}/\sigma T_0^{*4} \\ \tau_i &= t_{i_0}^* u_0^*/\ell^* \\ \Gamma &= (\gamma_{f_0} - 1)/\gamma_{f_0} \end{aligned} \tag{3.18}$$

τ_ℓ, the Bouguer number is the ratio of the reference physical length scale to the extinction length of the radiation; β is a measure of the propagation velocity of acoustic waves relative to that of light; B_0, the Boltzmann number, indicates the importance of convective relative to radiative energy fluxes, and the $(1/\tau_i)$ are rate parameters which compare the time that fluid elements reside in the region of physical interest to the appropriate relaxation time. Goulard has grouped these into intrinsic parameters (which affect only the formulation of the radiation) and extrinsic parameters (which affect only the interaction of radiation with fluid mechanics). It has been noted (Section 2.4.2) that τ_ℓ is an intrinsic parameter. The most important extrinsic parameters are the combination τ_ℓ/B_0 and τ_i, while β may be considered both intrinsic and extrinsic. The distinction is purely academic.

It is common to neglect all terms $O(\beta)$ and higher. This is not necessarily the case in the current formulation since in stellar structure or during the initial stages of intense blasts, temperatures may be sufficiently large, $T^* \sim 10^7 \,^\circ K$, (see Bethe, 1965) that $\beta \sim 10^{-3}$, $\frac{\beta}{B_0} \sim 1$ and must be retained.

Contributions of order β^2 are almost always negligible and are henceforth omitted.

The W_i functions disguise many processes about which little is known. They may contain parameters which are quite significant and may depend upon the radiative field. Often such dependence is ignored (e.g., Scala and Sampson, 1963) for the purpose of the prediction of qualitative trends in the limiting cases. Such assumptions are no longer sufficient since definitive works are now needed. In fact, the scaling chosen may not be at all appropriate when specific chemical processes are decided upon (Penner, Thomas and Adomeit, 1963). The solution of the system of Eq. (3.11) through (3.14) presents formidable difficulties unless certain approximations are invoked. Several of these are examined in the following sections.

3.2 Approximate Methods in Planar Problems

3.2.1 The Forward-Reverse Method and Kernel Substitution

For simplicity, the medium is assumed to be semi-gray and non-scattering unless it is stated otherwise. The non-gray and gray analogs are obvious in all cases, and the normalization remains that of Section 3.1.

The most widely applied method in planar problems is the kernel substitution referred to in Section 1.2.4. In the steady state of the non-scattering formulation, the integral expression for div($\vec{q}$), Eq. (3.16), presents the greatest difficulty. Equation (2.70) confirms that the integrals cannot be removed from the expressions by successive differentiation since the E_n functions are not separable.

Murty (1965) proposed the formal expansion

$$E_2(x) = \sum_{i=1}^{N} m_i e^{-n_i t} \qquad (3.19)$$

Of course, once any one of the E_n's is known the others follow from the recursion relations. The manner in which the m_i and n_i parameters are chosen is at best ambiguous. As $N \rightarrow \infty$ the choice is made least arbitrary since matching, say N moments of the exact $E_2(x)$ and the values its first N derivatives at a given point with those obtained from Eq. (3.19) provides $2N$ equations for the m_i and n_i . If N is finite, only a few of the moments or derivatives may be chosen. (Baldwin, 1962, indicates how the choice may be optimized.) It will be shown (Section 3.4) that the mathematical procedure does not always yield physically reasonable expressions. Furthermore, no number of terms will correctly reproduce the logarithmic singularity of $E_1(x) = -\frac{dE_2}{dx}$ at the origin.

Consider the one term expression

$$E_2(x) = m e^{-nx} \tag{3.20}$$

Then, in the steady state of a non-scattering gas in LTE

$$q = 2\left\{\left[\epsilon T_w^4 + 2r\tau_\ell \int_0^\infty \frac{\alpha_p}{\alpha_R} T^4(\eta') m e^{-n\eta'\tau_\ell} d\eta'\right] \frac{m e^{-n\eta\tau_\ell}}{n} + r\tau_\ell \int_0^\infty \frac{\alpha_p}{\alpha_R} T^4(\eta') m e^{-n(\eta+\eta')\tau_\ell} d\eta' + \tau_\ell \int_0^\infty \frac{\alpha_p}{\alpha_R} T^4(\eta') \operatorname{sgn}(\eta-\eta') m e^{-n|\eta-\eta'|\tau_\ell} d\eta'\right\} \tag{3.21}$$

The integral may now be eliminated by successive differentiation of Eq. (3.21) with the result:

$$\frac{\partial^2 q}{\partial \eta^2} - n^2 \tau_\ell^2 q = 4m \frac{\partial}{\partial \eta}\left(\tau_\ell \frac{\alpha_p}{\alpha_R} T^4\right) \tag{3.22}$$

Since if $\tau_\ell \gg 1$ this reduces to

$$q = -\frac{4m}{n^2 \tau_\ell \alpha_R} \frac{\partial T^4}{\partial x} + O\left(\frac{1}{\tau_\ell^2}\right) \tag{3.23}$$

Equation (2.51) requires that

$$m = \frac{1}{3} n^2 \tag{3.24}$$

if the correct thick limit is to ensue. Therefore the parameters m and n are not independent. Note that the limit of Eq. (3.22) as is a singular perturbation since the highest order derivative is lost. A "boundary layer" of thickness $O(\sqrt{\tau_e})$ may be expected; however the situation is much more complicated than is indicated (see Olstad, 1965). It shall soon be pointed out that the use of Eq. (3.21) alone is in violation of the physics of the problem and that neither of the parameters m or n may be chosen arbitrarily. In Section 3.2.3 it will be shown that different "choices" of m and n imply drastically different physical restrictions upon the radiative field.

3.2.2 Moment Methods

The analysis of problems in astrophysical radiative equilibrium forms the foundation for most approximate methods in radiative transfer. Since stellar atmospheres have negligible curvature, the basic formulations are all planar. Furthermore in all such problems there is no incoming photon flux at the edge of the atmosphere (which is taken as the origin of coordinates with x increasing inward). This is indicated by the condition $I_\nu^-(-\mu, 0, t) = 0$. For a non-scattering atmosphere in LTE radiative equilibrium exists when absorption and emission are balanced; hence, the right hand side of Eq. (3.16) vanishes and, in the steady state, the heat flux is constant. Then Eq. (2.37) and the definitions Eq. (2.22) and (2.23) may be combined into an integro-differential equation for the integrated intensity I. Since the integral involved is that in Eq. (2.22) over the angular variable θ, Eddington (1926) was led to assume the expansion

$$I(\mu, x, t) = \sum_{l=0}^{N} A_l(x,t) P_l(\mu) \tag{3.25}$$

This expansion is exact in the least square sense as $N \rightarrow \infty$ because the Legendre polynomials $P_l(\mu)$ are complete over (-1,1) corresponding to θ in $(0, \pi)$. This approach is useful in RGD as well as in astrophysics.

There are two philosophies in the use of Eq. (3.23). The first, which is preferred by Traugott (1962), involves taking moments of Eq. (2.37) with respect to μ. These entail expressions of the form

$$I_n(x,t) = 2\pi \int_{-1}^{1} \mu^n I(\mu,x,t)\, d\mu \tag{3.26}$$

Since μ appears explicitly in Eq. (2.37), the equation for a general moment I_n will involve I_{n+1} as well, thus the system is not closed. Another complication arises in the most general case, since α_a depends upon μ. The successive moments of Eq. (2.37) require angular averages of α_a

$$I_n \alpha_a^{(n)} = 2\pi \int_{-1}^{1} \mu^n \alpha_a(\mu,x,t)\, I(\mu,x,t)\, d\mu \tag{3.27}$$

For the moment the isotropic form of α_a, Eq. (2.53), is assumed so that

$$\alpha_a^{(n)} = \alpha_a \quad \text{for all } n \tag{3.28}$$

One may express μ^n as a linear combination of the "n + 1" Legendre polynomials of order "n" and lower, therefore the I_n-moments, Eq. (3.26), may be expressed as linear combinations of the A_l coefficients in Eq. (3.25). If the expansion is truncated by setting $A_l = 0, l > N$ (the P_N - approximation), relationships between the I_n, $n > N$ and those for $n \leq N$ result. This condition closes the system of equations obtained from Eq. (2.37). This approach is most convenient only in the planar case, and the alternate procedure will be employed in general..

If Eq. (3..25) is inserted directly in Eq. (2.27) before moments are taken, the recursion relations among the Legendre polynomials:

$$\mu P_l(\mu) = \{ l P_{l-1}(\mu) + (l+1) P_{l+1}(\mu) \} / (2l+1) \tag{3.29}$$

may be used to obtain the following system of equations for the coefficients,

$$\beta \frac{\partial A_l}{\partial t} + \frac{(l+1)}{(2l+3)} \frac{\partial A_{l+1}}{\partial x} + \frac{l}{(2l-1)} \frac{\partial A_{l-1}}{\partial x} = \tag{3.30}$$

$$= \tau_l \left\{ \frac{\tilde{\eta}\, \bar{\omega}_l A_l}{(2l+1)} - (\alpha_a + \tilde{\eta}) A_l + \alpha_e B\, \delta_{0l} \right\}$$

Now the truncation, $A_\ell = 0$, $\ell > N$ closes the system of equations directly. For instance from Eq. (2.22) through (2.24) and (2.28)

$$\begin{aligned} I_0(x,t) &= 2\pi\int_{-1}^{1} I(\mu,x,t)\,d\mu = 4\pi A_0(x,t) \\ q(x,t) &= 2\pi\int_{-1}^{1} \mu\, I(\mu,x,t)\,d\mu = \frac{4\pi}{3} A_1(x,t) \\ \overrightarrow{\overrightarrow{P}}^{(R)}(x,t) &= 2\pi\int_{-1}^{1} \mu^2 I(\mu,x,t)\,d\mu \cdot \overrightarrow{\overrightarrow{1}} = \frac{4\pi}{3}\left(A_0 + \frac{2}{5}A_2\right)\overrightarrow{\overrightarrow{1}} \end{aligned} \tag{3.31}$$

In general

$$A_\ell(x,t) = \frac{2\ell+1}{2}\int_{-1}^{1} P_\ell(\mu)\, I(\mu,x,t)\,d\mu \tag{3.32}$$

so that $A_\ell = 0$, $\ell = K$, will specify a linear relation between all of the I_ℓ for $\ell \le K$. The astrophysical equivalent of Eq. (3.29) is obtained by omitting the unsteady term and noting that the statement of radiative equilibrium is:

$$4\pi \alpha_a A_0 = \alpha_e B \tag{3.33}$$

Because the moments of I which are involved in Eq. (3.31) are taken over the complete range of the angular variable, the formulation consisting of Eq. (3.25),(3.30) and (3.31) is referred to as the planar, full-range moment method. Caution is required since the consistent formulation of boundary conditions has not been investigated.

Because boundary conditions are often specified in terms of $I^{(+)}$ or $I^{(-)}$, the half range intensities, a formulation in terms of half range moments is also instructive in some cases. Since the $P_\ell(2\mu-1)$ are complete in $(0,1)$ and $P_\ell(2\mu+1)$ are in (-1,0), an expansion of the form below is applicable.

$$I^{(\pm)} = \sum_{\ell=0}^{\infty} A_\ell^{(\pm)}(x,t)\, P_\ell(2\mu \mp 1) \tag{3.34a}$$

where:

$$A_k^{(+)}(x,t) = (2k+1)\int_0^1 P_k(2\mu-1)\, I^{(+)}(\mu,x,t)\, d\mu \qquad (3.35a)$$

$$A_k^{(-)}(x,t) = (2k+1)\int_{-1}^0 P_k(2\mu+1)\, I^{(-)}(\mu,x,t)\, d\mu \qquad (3.35b)$$

The operations of the full range problem result in:

$$\beta \frac{\partial A_\ell^{(\pm)}}{\partial t} + \frac{1}{2}\left\{\frac{\ell+1}{2\ell+3}\frac{\partial A_{\ell+1}^{(\pm)}}{\partial x} + \frac{\ell}{2\ell-1}\frac{\partial A_{\ell-1}^{(\pm)}}{\partial x} \pm \frac{\partial A_\ell^{\pm}}{\partial x}\right\} = \mathcal{I}_\ell\left\{\alpha_e B \delta_{0\ell} - \alpha_a A_\ell^{\pm}\right\} \qquad (3.36)$$

in a nonscattering gas.

Sherman (1967) and Traugott and Wang (1964) are in agreement that half range moments are best suited when boundary conditions are applied at the interface between a radiating and a nonradiating medium. For instance the condition $I^{(-)}(\mu,0,t) = 0$ is simply $A_k^{(-)}(0,t) = 0$ for all k. The specification of the $A_k^{(+)}(0,t)$ is not clear, however. In fact their specification in astrophysical problems would provide the "Law of Darkening" (Kourganoff, 1952) which is one of the more important quantities to be determined. Krook (1955) has discussed the astrophysical counterparts of both full-and half-range moments.

There is yet another moment method which although not as rigorous as the previous two is nevertheless just as widely applied. The original idea is due to Schuster and Schwarzschild, but their formulation is only one member of a general class of forward-reverse formulations. The parametric nature of these approximations has to this author's knowledge not been recognized previously. The intensity itself is sacrificed in favor of its lowest moments through Eq. (2.26) and (2.27).

$$I_0^{(\pm)} = \int_{\vec{\Omega}\cdot\vec{n}\geq 0} I(\vec{\Omega},x,t)\, d\vec{\Omega} = 2\pi\int_{\mu\geq 0} I^{(\pm)}(\mu)\, d\mu \equiv 2\pi m^{(\pm)} I^{(\pm)} \qquad (3.37)$$

$$q^{(\pm)} = \int_{\vec{\Omega}\cdot\vec{n}\geq 0} \vec{\Omega}\, I(\vec{\Omega},x,t)\, d\vec{\Omega} = 2\pi\int_{\mu\geq 0} \mu\, I^{(\pm)}_\mu\, d\mu = \pi\mu^{(\pm)} I^{(\pm)} \qquad (3.38)$$

where

$$m^{(\pm)} \equiv \int_{\mu \gtrless 0} I^{(\pm)}(\mu)\, d\mu \;/\; I^{(\pm)} \tag{3.39a}$$

$$\mu^{(\pm)} \equiv 2\int_{\mu \gtrless 0} \mu\, I^{(\pm)}(\mu)\, d\mu \;/\; I^{(\pm)} \tag{3.39b}$$

Integration of Eq. (2.37) over the half-ranges results in:

$$\beta \frac{\partial}{\partial t}\left(m^{(\pm)} I^{(\pm)}\right) + \frac{\partial}{\partial x}\left(\mu^{(\pm)} I^{(\pm)}\right) = \tau_\ell \left\{ 2\pi \alpha_e B - 2\alpha_a m^{(\pm)} I^{(\pm)} \right\} \tag{3.40}$$

If $\mu^{(+)} = \mu^{(-)} \equiv \bar{\mu}$, $m^{(+)} = m^{(-)} \equiv \bar{m}$, and $\bar{\mu}$ and $\bar{m}$ are constants:

$$\beta \frac{\partial I_0}{\partial t} + \frac{\partial q}{\partial x} = \tau_\ell \left\{ 4\pi \alpha_e B - \alpha_a I_0 \right\} \tag{3.41}$$

$$\beta \frac{\partial q}{\partial t} + \frac{1}{4}\left(\frac{\bar{\mu}}{\bar{m}}\right)^2 \frac{\partial I_0}{\partial x} = -\tau_\ell \alpha_a q \tag{3.42}$$

By definitions, Eq. (2.26), (2.27), and (2.28), Eq. (3.41)is an exact relation. Equation (3.42) is, however, only approximately correct. In fact Eq. (3.17) indicates that the approximation is:

$$\overline{\overline{P}}^{(R)} = \frac{1}{4}\left(\frac{\bar{\mu}}{\bar{m}}\right)^2 I_0 \;\overline{\overline{1}} \tag{3.43}$$

Since Eq. (3.41) and (3.42) form a complete system for I_0 and q, a two-parameter-family of approximations has been obtained. If the optically thick limit must be reproduced when $\tau_\ell \gg 1$, then:

$$\left(\frac{\bar{\mu}}{\bar{m}}\right)^2 = \frac{4}{3} \tag{3.44}$$

Equation (3.43) reduces to the isotropic relation, Eq. (2.29) in this case, and there is only a one-parameter-family of approximations. Although the remaining parameter does not appear in the equations, it will enter the boundary conditions. Historically, equations (3.41) and (3.42) with assumption, Eq. (3.44), constitute the Milne-Eddington approximation (Eddington, 1926, p. 100). The original Eddington approximation in an astrophysical context consisted not only of the assumption stated, but also of an approximate boundary condition, $I_0(0,t) - 2q(0,t) = 0$, but both of these cannot be found consistently from the forward-reverse formulation (Section 3.4).

There are, therefore, at least three approximate methods for the study of planar radiative transfer. The first two, full- and half-range moment methods, possess the advantage of being mathematically systematic; the last, the forward-reverse approximation, while not nearly as systematic is perhaps physically more clear. Criteria for the suitability of a given approximation to the integro-differential equations (specifically to the integral formulation of div q) must now be developed. One which has been noted is that the optically thick and thin limits must always be reproduced. This requires a second criterion, namely that that the first of the moment equations obtained from Eq. (2.37), Eq. (3.41), must always be satisfied. Having established these criteria, the desirability of the various orders of approximation in all three methods above may be examined.

3.2.3 The Suitability of Planar Moment Methods, and Their Relation with Kernel Substitution

Kernel substitution will be examined first in order to bring its physical interpretation to light. It has been noted that the use of Eq. (3.21) alone when assumption, Eq. (3.20), is involved is not appropriate. This is clear if it is noted that the exact heat flux, Eq. (2.69) or its divergence, Eq. (2.70), specifies an exact I_0 through Eq. (3.41). If assumption, Eq. (3.20), is invoked in both q and I_0 an inconsistency arises which can be resolved if, and only if, $m = 1$. Then one is forced to choose $n = \sqrt{3}$. There are many reasons for employing Eq. (3.20) in the form:

$$E_2(x) = e^{-\sqrt{3}x} \tag{3.45}$$

TABLE I

$A_\ell = 0$ for $\ell >$	$\frac{1}{\tau_\ell}(\beta I_{0_t} + q_x)$	$(\frac{\beta}{\tau_\ell}\frac{\partial}{\partial t} + \alpha_a^{(0)}) I_0$	$\frac{1}{\tau_\ell} I_{0_x}$	$\frac{1}{\tau_\ell}(\beta q_t + P_x^{(R)})$	$\frac{1}{\tau_\ell}(\beta P_t^{(R)} + I_{3x})$	$\frac{1}{\tau_\ell}(\beta I_{3_t} + I_{4_x})$	I
1	$4\pi\alpha_e B - \alpha_a^{(0)} I_0$	——	$-3(\frac{\beta}{\tau_\ell}\frac{\partial}{\partial t} + \alpha_a^{(1)}) q$	——	——	——	$\frac{1}{4\pi}(I_0 + 3\mu q)$
2	↓	$\frac{5}{3}(\frac{\beta}{\tau_\ell}\frac{\partial}{\partial t} + \alpha_a^{(2)}) P^{(R)} + \frac{4}{9}(4\pi\alpha_e B)$	——	$-\alpha_a^{(1)} q$	——	——	$\frac{1}{4\pi}\{\frac{3}{4}(3+5\mu^2) I_0 + 3\mu q + \frac{15}{8}(3\mu^2-1) P^{(R)}\}$
3	↓	——	$\frac{10\beta}{\tau_\ell}(\frac{7}{6} I_{3_t} - q_t) + 10(\frac{7}{6}\alpha_a^{(3)} I_3 - \alpha_a^{(1)} q)$	↓	$\frac{4\pi}{3}\alpha_e B - \alpha_a^{(2)} P^{(R)}$	——	
4	↓	$(\frac{\beta}{\tau_\ell}\frac{\partial}{\partial t})(\frac{21}{5} I_4 - \frac{14}{3} P^{(R)}) + (\frac{21}{5}\alpha_a^{(4)} I_4 - \frac{14}{3}\alpha_a^{(2)} P^{(R)}) + \frac{386}{225}(4\pi\alpha_e B)$	——	↓	↓	$-\alpha_a^{(3)} I_3$	
5	↓	——	$\frac{\beta}{\tau_\ell}\frac{\partial}{\partial t}\{\frac{I_4}{1155} + 103 I_3 + 21 q\} + \{\frac{\alpha_a^{(4)} I_4}{1155} + 103\alpha_a^{(3)} I_3 + 21\alpha_a^{(1)} q\}$	↓	↓	↓	
6	↓			↓	↓	↓	

$q = I_1 \; , \; P^{(R)} = I_2$

Although this relation corresponds with the exact result $E_2(0) = 1$, the recursion relations require that $E_n(0) = 3^{(1-\frac{n}{2})}, n > 2$, which are wrong. Rhyming (1965a) has justified Eq. (3.45) when radiation is weak ($B_0 >> 1$) by forcing the approximate and exact solutions to be the same near a shock wave. However, his exact solution near the shock was obtained with a Neumann series of doubtful convergence.

The consequences of the full range moment method to various approximations are displayed in Table I. It is evident that none of the even approximations ($A_\ell = 0, \ell > N$-even) yield the correct optically thin limit. For instance,

$$div \vec{q} = \tau_\ell \left(\frac{5}{9}\right) 4\pi d_e B \qquad [N=2] \qquad (3.46a)$$

$$div \vec{q} = \tau_\ell \left(\frac{161}{225}\right) 4\pi d_e B \qquad [N=4] \qquad (3.46b)$$

Thus all even orders of approximation may be eliminated except for expository purposes (Section 4.2). This line of reasoning seems not to have been pursued before, and the justification for neglecting even orders usually relies upon vague analogies with neutron transport theory. Since Eq. (3.41) is always satisfied, the odd orders of approximation are all acceptable. Of course all orders display the correct optically thick behavior. The P_1 equations are identical with the forward-reverse equations according to Eq. (3.41), thus the physical implications of the moment methods become more clear.

When half-range moments are examined, both of the $A_k^{(\pm)}$ must vanish for $k > N$ in order for the approximation to be consistent. The tabular form is no longer appropriate, but the first two approximations are:

$$A_n^{(\pm)} = 0, \; n > 0$$

$$2\pi I^{(\pm)} = 2 q^{(\pm)} = I_0^{(\pm)} = 2\pi A_0^{(\pm)} \qquad (3.47a,b)$$

$$\beta \frac{\partial g^{(\pm)}}{\partial t} \pm \frac{1}{2}\frac{\partial g^{(\pm)}}{\partial x} = \tau_e \{\pi \alpha_e B(T) - \alpha_a^{(0)} g^{(\pm)}\} \qquad (3.47c,d)$$

$A_n = 0, n > 1$

$$\frac{(1 \mp 2\mu)}{2\pi}(I_0^{(\pm)} - 3g^{(\pm)}) = I^{(\pm)} \qquad (3.48a,b)$$

$$p^{(n)(\pm)} = g^{(\pm)} - \frac{1}{6} I_0^{(\pm)} \qquad (3.48c,d)$$

$$\beta \frac{\partial I_0^{(\pm)}}{\partial t} \pm \frac{\partial g^{(\pm)}}{\partial x} = \tau_e \{2\pi \alpha_e B - \alpha_a^{(0)} I_0^{(\pm)}\} \qquad (3.48e,f,)$$

$$\beta \frac{\partial g^{(\pm)}}{\partial t} \pm \frac{\partial g^{(\pm)}}{\partial x} = \tau_e \{\pi \alpha_e B - \alpha_a^{(1)} g^{(\pm)}\} \qquad (3.48g,h)$$

Equations (3.47c,d) in light of Eq. (3.47a,b) are in fact Eq. (3.41) as are Eq. (3.48e,f). Thus both are acceptable on that count. Of course, the correct optically thin limit is always a consequence of Eq. (3.41). However, Eq. (3.47a,b) are equivalent to Eq. (3.37) and (3.38) if $\bar{m} = 1$ and $\bar{\mu} = 1$ which is in fact the case if Eq. (3.47a,b) hold. The reason for the terminology forward-reverse is that $I^{(+)}$ and $I^{(-)}$ are assumed to be isotropic "streams" of radiation in the respective unit hemishperes about a point. Since the first half-range moment approximation corresponds with the forward-reverse approximation if $(\bar{\mu}/\bar{m}) = 1$, it is obvious from Eq. (3.44) that the wrong optically thick limit follows. The next order of approximation, Eq. (3.48), is difficult to analyze since $I_0^{(\pm)}$ may be eliminated completely from the last moment equation which one retains. It may be shown through the first moment procedure which was outlined that in all even orders of the full-range moments and in all odd orders of those over the half range one of the moment equations may always be reduced to a relation for the operation $(\beta \frac{\partial}{\partial t} + \tau_e \alpha_a^{(0)})$ on I_0 or $I_0^{(\pm)}$ respectively in terms of all of the other moments but not of their spatial derivatives (see Table I). This has serious effects upon the characteristics approach to be outlined in Chapter 4 and is in fact the reason for the erroneous "thin" behavior. The half-range moment method introduces two equations for each order, thus its application to the same order as a full-range formulation will be twice as cumbersome.

Sherman (1967) has compared the full- and half-range methods in various simple situations. He has found close correspondence between the P_1 full-range and P_0 half-range methods, but his conclusions cannot necessarily be accepted since the situations he examined were non-self-consistent, hence unrealistic in RGD. The investigation conducted by Nemchinov (1960) employed the P_0 half-range formulation given by Eq. (3.47c,d) as $\beta \to 0$.

It is noted that the insertion of the kernel substitution Eq. (3.45) into Eq. (2.69) and the determination of I_0 through Eq. (3.41) yields identically the P_1-approximation of the planar moment method. Therefore the simple mathematical artifice corresponds to the Eddington assumption that the isotropic relationship exists between radiation pressure and energy density. Zeldovich and Raizer (1966) refer to this as the diffusion approximation, since the P_1-approximation contains the dimensional equation:

$$q^{(R)} = -\frac{c}{3\alpha_R}\frac{\partial u^{(R)}}{\partial X} \tag{3.49}$$

which may be interpreted as the photon gas analog of Fick's Law. Fortunately the analogy is not complete, and its effect upon the equations of RGD is quite different from that of a mere molecular transport process. From previous comments concerning Eq. (3.37) and (3.38), it is clear that the case $\bar{\mu} = \bar{m} = 1$ of the forward-reverse approximation corresponds to the kernel substitution, Eq. (3.20), with $m = 1$, $n = 2$.

It is instructive to consider the case in which the temperature field is uniform and any walls present are at the same temperature as the gas. Since there can be no heat flux

$$\epsilon + r_s + 2r_d E_3(0) - 1 = 0 \tag{3.50}$$

With Eq. (2.71), Eq. (3.50) may be reduced to:

$$r_d \{ 2E_3(0) - 1 \} = 0 \tag{3.51}$$

However, according to the one-term kernel substitution:

$$E_3(0) = (m/n) \tag{3.52}$$

Therefore, only if $m/n = 1/2$ can diffuse reflection be considered. Since Eq. (3.24) must hold the result is $n = 3/2$, $m = 3/4$ Therefore, diffuse reflection may be included only at the expense of either the optically thick limit ($m = 1$, $n = 2$) or the satisfaction of Eq. (3.41) (if $m = 3/4, n = 3/2$). It must be pointed out as well that at large distances from regions of radiative disturbance, all radiative properties behave asymptotically as $E_n(x)$, which for all n is $\sim e^{-x}/x$. To this there corresponds me^{-nx}, therefore the nearer n is to unity the better the approximation will be in the far field. Clearly an approximation which is consistent with diffuse reflection will decay in almost the correct manner in the far field. The physical implications of this phenomenon may be deduced from Chandrasekhar's (1960) comments concerning the polarization of a beam of photons upon interaction with a surface. Crudely, any kernel substitution corresponds to the assumption of different isotropic intensities in different regions of the unit sphere about each point in space. Since the effect of diffuse reflection is to smooth anisotropies in the incoming beams, the photons isotropically reflected into the "right" half sphere (assuming as always that the surface is on the left) will not match the isotropic distribution(s) already assumed to exist therein unless the kernel substitution leads to the exact relation $E_3(0) = 1/2$. The ultimate choice, Eq. (3.45), does not satisfy this relation. Because of the formal equivalence of kernel substitution, forward-reverse approximations, and moment methods to lowest order, diffuse reflection must be omitted in all if an approximation which satisfies Eq. (3.41) is to yield both "thick" and "thin" limits consistently and decay in approximately the right manner far from regions of radiative disturbance. Higher order methods are not necessarily so restricted. The difficulties associated with them (see e.g. Murty, 1965) do not seem to justify the additional refinement, however. Since the implications of the omission of diffuse reflection upon a physically realistic situation cannot yet be determined, when the differential approximations are employed in the subsequent analysis it will be understood that diffuse reflection is ignored.

3.3 Non-Planar Methods

The extension of the methods above to more general situations is by no means straightforward. Heaslet and Warming (1966) indicate the extension of the forward-reverse method to cylindrical and spherical cases. The treatment is very much like that presented previously except that angular variations become more important. The transfer equation (2.37) may be written

$$\beta \frac{\partial I}{\partial t} + \sin^{(2-\nu)}(\phi)\left\{\mu \frac{\partial I}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I}{\partial \mu}\right\} = \tau_e \left\{4\pi \alpha_e B - \alpha_a I\right\} \qquad (3.53)$$

where $\nu = 1, 2$ for the cylindrical and spherical cases respectively. Here $\mu = \cos\theta$ expresses the same relation between s and r as that between s and x in Eq. (2.64). In the cylindrical case the angular variable ϕ is necessary to account for the inclination of the photon path with the cylindrical axis. Since $\ell_1 = \mu \sin^{(2-\nu)}(\phi)$ is the single direction cosine of the problem, Eq. (2.22), (2.23), (2.24) become:

$$I_o = \int_{4\pi} I(\vec{\Omega})\, d\vec{\Omega} \qquad (3.54a)$$

$$q = \int_{4\pi} \mu \sin^{(2-\nu)}(\phi)\, I(\vec{\Omega})\, d\vec{\Omega} \qquad (3.54b)$$

$$P^{(R)} = \int_{4\pi} \{\mu \sin^{(2-\nu)}(\phi)\}^2\, I(\vec{\Omega})\, d\vec{\Omega} \qquad (3.54c)$$

The heat flux is in the radial direction,and Eq. (3.41) follows for all cases. The second moment equation, that obtained by averaging Eq. (3.53) over $\vec{\Omega}$ after multiplying by ℓ_1, is, however, different in each geometry.

$$\beta \frac{\partial q}{\partial t} + \frac{\partial P}{\partial r} + \frac{1}{r}(3P - I_o) = -\tau_e \alpha_a^{(1)} q \qquad [\nu = 2] \qquad (3.55a)$$

$$\rho \frac{\partial q}{\partial t} + \frac{\partial P}{\partial r} + \frac{1}{r}\left\{2P - \int_0^{2\pi} d\theta \int_0^{\pi} I(\theta,\phi)\sin^3\phi \, d\phi\right\} = -\tau_\ell \alpha_a^{(1)} q, \; \gamma = 1 \quad (3.55b)$$

If the Eddington assumption, Eq. (2.50), is applied it follows that

$$\rho \frac{\partial q}{\partial t} + \frac{1}{3}\frac{\partial I_0}{\partial r} = -\tau_\ell \alpha_a^{(1)} q \quad (3.56)$$

for both cylindrical and spherical geometries in analogy with P_1 - planar methods. The derivation above still does not represent a systematic approach.

The planar full-range moment method is the specialization of a more general multi-dimension approach. Since the spherical harmonics

$$Y_\ell^m(\vec{\Omega}) = \left(\frac{\ell - m}{\ell + m}\right) e^{im\phi} P_\ell^{|m|}(\mu) \quad (3.57)$$

form a complete orthonormal set over $0 \leq \phi \leq 2\pi, \; -1 \leq \mu < 1$, these are the functions which must be used in an angular expansion. In analogy with Eq. (3.25),

$$I(\vec{\Omega}, \vec{r}, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_\ell^m(\vec{r}, t) Y_\ell^m(\vec{\Omega}) \quad (3.58)$$

The derivation of the equations is best outlined by Davison (1958, pp 157-163) in the context of neutron transport theory. Indeed the method appears in its most general form in problems of neutron diffusion. The expansion was first applied in a manner consistent with RGD by Cheng (1964,a,b, 1965a,b). The formulation is easily extended to semi-gray radiative transfer.

If the phase function $\mathcal{P}_\nu$ is independent of frequency, the frequency integrated equations, Eq. (2.37), are:

$$\frac{1}{c}\frac{\partial I}{\partial t} + \vec{\Omega} \cdot \mathrm{grad}\, I = \tau_\ell \left\{\alpha_e B - \alpha_a I + \tilde{\eta}_a \int_{\Omega'} \mathcal{P}(\vec{\Omega}' \cdot \vec{\Omega}) I(\vec{\Omega}') \frac{d\vec{\Omega}'}{4\pi}\right\} \quad (3.59)$$

Application of the theorem:

$$P_\ell(\vec{\Omega}'\cdot\vec{\Omega}) = \sum_{m=-\ell}^{\ell} Y_\ell^m(\vec{\Omega})\, Y_\ell^{m*}(\vec{\Omega}') \tag{3.60}$$

and the expansion:

$$\mathcal{P}(\vec{\Omega}'\cdot\vec{\Omega}) = \sum_{\ell=0}^{\infty} \overline{\omega}_\ell(\vec{r},t)\, P_\ell(\vec{\Omega}'\cdot\vec{\Omega}) \tag{3.61}$$

in addition to Eq. (3.58)and the various recursion relations, results in

$$\begin{aligned}
&\beta\frac{\partial A_\ell^m}{\partial \tau} + \frac{\sqrt{(\ell+m+2)(\ell+m+1)}}{2(2\ell+3)}\left\{\frac{\partial A_{\ell+1}^{m+1}}{\partial x_1} + i\frac{\partial A_{\ell+1}^{m+1}}{\partial x_2}\right\} + \frac{\sqrt{(\ell-m)(\ell-m-1)}}{2(2\ell-1)}\left\{\frac{\partial A_{\ell-1}^{m+1}}{\partial x_1} + i\frac{\partial A_{\ell-1}^{m+1}}{\partial x_2}\right\} + \\
&+ \frac{\sqrt{(\ell-m+2)(\ell-m+1)}}{2(2\ell+3)}\left\{\frac{\partial A_{\ell+1}^{m-1}}{\partial x_1} - i\frac{\partial A_{\ell+1}^{m-1}}{\partial x_2}\right\} - \frac{\sqrt{(\ell+m)(\ell+m-1)}}{2(2\ell-1)}\left\{\frac{\partial A_{\ell-1}^{m-1}}{\partial x_1} - i\frac{\partial A_{\ell-1}^{m-1}}{\partial x_2}\right\} + \\
&+ \frac{\sqrt{(\ell+m+1)(\ell-m+1)}}{2\ell+3}\frac{\partial A_{\ell+1}^m}{\partial x_3} + \frac{\sqrt{(\ell+m)(\ell-m)}}{2\ell-1}\frac{\partial A_{\ell-1}^m}{\partial x_2} = \tau_\ell\left\{\frac{\tilde{\eta}_a^{(\ell)}\overline{\omega}_\ell A_\ell^m}{2\ell+1} - \beta_a^{(\ell)} A_\ell^m + d_\ell B\delta_{0m}\delta_{0\ell}\right\}
\end{aligned} \tag{3.62}$$

It is easily shown that $\overline{\omega}_0 = 1$ because $\mathcal{P}_a$ is normalized to unity. Furthermore $\overline{\omega}_1 = 3\overline{\mu'}$, where $\overline{\mu'}$, the average angle of scattering, vanishes if scattering is isotropic. The quantities $\tilde{\eta}_a^{(\ell)}$ and $\beta_a^{(\ell)}$ are defined in a manner similar to Eq. (3.27) and will be discussed shortly. Equations (2.25b,c,d) may be expressed compactly in the forms:

$$\ell_1 = \sin\theta\cos\phi = \frac{1}{\sqrt{2}}\left(Y_1^1 - Y_1^{-1}\right) \tag{3.63a}$$

$$\ell_2 = \sin\theta\sin\phi = \frac{1}{\sqrt{2}}\left(Y_1^1 + Y_1^{-1}\right) \tag{3.63b}$$

$$\ell_3 = \cos\theta = Y_1^0 \tag{3.63c}$$

Although the equations above are restricted to a cartesian coordinate system, once the coefficients are grouped to form moments there will be no complex quantities, and the equations may be expressed in universal notation. Consider, for instance, that:

$$I_0 = 4\pi A_0^0 \tag{3.64a}$$

$$q_1 = i\frac{2\sqrt{2\pi}}{3}(A_1^{-1} - A_1^1) \tag{3.64b}$$

$$q_2 = -i\frac{2\sqrt{2\pi}}{3}(A_1^{-1} + A_1^1) \tag{3.64c}$$

$$q_3 = \frac{4\pi}{3}A_1^0 \tag{3.64d}$$

$$P_{11}^{(R)} = 4\pi\left\{\frac{1}{3}A_0^0 + \frac{A_2^2 + A_2^{-2}}{5\sqrt{6}}\right\} \tag{3.64e}$$

$$P_{22}^{(R)} = 4\pi\left\{\frac{1}{3}A_0^0 - \frac{A_2^2 + A_2^{-2}}{5\sqrt{6}}\right\} \tag{3.64f}$$

$$P_{33}^{(R)} = 4\pi\left\{\frac{1}{3}A_0^0 + \frac{2}{15}A_2^0\right\} \tag{3.64g}$$

$$P_{12}^{(R)} = P_{21}^{(R)} = -4\pi i\,(A_2^2 - A_2^{-2})/5\sqrt{6} \tag{3.64h}$$

$$P_{13}^{(R)} = P_{31}^{(R)} = -4\pi\,(A_2^1 + A_2^{-1})/5\sqrt{3} \tag{3.64i}$$

$$P_{23}^{(R)} = P_{32}^{(R)} = -4\pi i\,(A_2^1 - A_2^{-1})/5\sqrt{3} \tag{3.64j}$$

Note that Eq. (3.64a and d) correspond exactly with the planar formulation, Eq. (3.31a and b). It follows immediately that the multi-dimensional analog of Eq. (3.41) results from Eq. (3.62) for $\ell = 0$, $m = 0$. With the aid of Eq. (3.64a, b, c, d)

$$\beta\frac{\partial I_0}{\partial t} + \operatorname{div}\vec{q} = \tau_\ell\left\{4\pi\alpha_\ell B - \beta_a^{(0)} I_0 + \overline{\omega}_0 \tilde{\eta}_a^{(0)} I_0\right\} \tag{3.65}$$

The generalized P_1 approximation may be developed by setting $A_\ell^m = 0,\ \ell > 1$. The immediate consequence of this is that the radiation stress term is reduced to the frequency integrated form of the isotropic relation Eq. (2.50)

$$P_{ij}^{(R)} = \frac{1}{3} I_0 \delta_{ij} \tag{3.66}$$

Truncation at $\ell = 1$ allows equations (3.62) for $\ell = 1,\ m = 1, 0, 1$ to be combined in the form:

$$\beta \frac{\partial \vec{q}}{\partial t} + \frac{1}{3}\,\mathrm{grad}\, I_0 = -\tau_\ell \left\{ \beta_a^{(1)} + \tilde{\eta}_a^{(1)} [1 - \bar{\mu}] \right\} \vec{q} \tag{3.67}$$

which corresponds to the P_1 planar approximation, to the planar forward-reverse equations (3.42), with $(\bar{\mu}/\bar{m})^2 = (4/3)$ and to the planar, cylindrical, and spherical Eddington approximations, Eq. (3.56). As is noted in Appendix A and Eq. (3.27) for any function $\psi_a(\vec{\Omega}, \vec{r}, t)$ the following angular averages may be defined.

$$I_0 \psi_a^{(0)}(\vec{r}, t) = \int_{4\pi} \psi_a(\vec{\Omega}, \vec{r}, t)\, I(\vec{\Omega}, \vec{r}, t)\, d\vec{\Omega} \tag{3.68a}$$

$$\vec{q}\, \psi_a^{(1)}(\vec{r}, t) = \int_{4\pi} \psi_a(\vec{\Omega}, \vec{r}, t)\, \vec{\Omega}\, I(\vec{\Omega}, \vec{r}, t)\, d\vec{\Omega} \tag{3.68b}$$

where ψ_a may correspond to α_a, β_a, or $\tilde{\eta}_a$. In general, the "tensor" moments of ψ_a will be tensors as well (e.g. the averages of ψ_a with respect to each direction cosine in Eq. (3.68b) are not necessarily the same). The notation will not be further complicated with this fact. Analogous to the entry at the top of Table I, the intensity is, to this approximation

$$I = \frac{1}{4\pi} \left(I_0 + 3 \vec{\Omega} \cdot \vec{q} \right) \tag{3.69}$$

Equations (3.65) and (3.67) constitute the first approximation of a purely differential formulation of the equations of radiative transfer. Although the scheme is mathematically systematic, it is obvious that extension to higher orders will prove cumbersome. In view of the comments of Section 3.2.3, even orders of approximation need not be considered. In the future the general full-range method will be referred to as "the" differential approximation. It will shortly be pointed out that the formulation is in fact not "purely" differential at all.

As long as one distinguishes among the various angular moments of α_a, the effect upon the governing equations is the same as if scattering were included. For instance, I_0 and $\mathcal{g}$ are weighted by different effective absorption coefficients. The reader is referred to Sampson(1965) for the justification of the omission of scattering in a given situation. For complete generality scattering should be allowed, but since so little physical information about scattering in aerodynamic situations is available it may be omitted.

3.4 A Critical Analysis of Boundary Conditions Appropriate for Moment Methods

Throughout the discussion of approximate methods in radiative transfer, care was taken not to consider boundary conditions. Since the integro-differential formulation contained all of the information necessary to determine, say, the heat flux once a temperature distribution was specified, somewhere in the formulation the relative simplicity of the interaction of radiation with surfaces has been lost. The reader may recall the introductory comments offerred in Chapter I. The problem is related to the fact that boundary conditions are specified for the intensity itself and not for its individual moments. Any expansion of the intensity in the governing equations must also be applied to the boundary conditions, and different conditions should be specified for the various moments in different orders of expansion. It is well known that inconsistencies often arise in the choice of boundary conditions once one truncates the appropriate expansion (Davison, 1958, p. 129; Chandrasekhar,1944; Marshak, 1947). Furthermore, since the governing equations are never satisfied exactly, the boundary conditions will never be precise either. The important point is that even though the boundary conditions are inexact, they must remain consistent with the approximate equations. The boundary conditions appropriate for the various approximate methods in RGD have not been investigated exhaustively previously. Cheng (1964a,b; 1965a,b) gave them only passing comment. A modification of the so-called "Mark" boundary condition will be shown to be the only formulation of the interaction of radiation with absorbing and specularly reflecting surfaces which is consistent with a P_1 differential approximation.

3.4.1 The Method of Solidification

Consider the P_1 equations in a planar, cylindrical, or spherical medium as $\beta \to 0$. Suppose that there are no boundaries present but that the temperature and pressure are nevertheless non-uniform. The equations which govern the radiative field may be written in the form:

$$\operatorname{div} \vec{q} = \tau_e \{ 4\pi \alpha_e B - \alpha_a^{(0)} I_0 \} \tag{3.70}$$

$$\operatorname{grad} I_0 = -3\tau_e \alpha_a^{(1)} \vec{q} \tag{3.71}$$

A heat flux potential similar to that used by Cohen (1965) and Traugott (1966) may be defined as follows:

$$\vec{q} = \frac{1}{\alpha_a^{(1)}} \operatorname{grad} \phi \tag{3.72}$$

Eq. (3.71) then requires that

$$I_0 = -3\tau_e \phi \tag{3.73}$$

and both of these lead to

$$\operatorname{div}\left(\frac{1}{\alpha_a^{(1)}} \operatorname{grad} \phi\right) - 3\tau_e^2 \alpha_a^{(0)} \phi = 4\pi \tau_e \alpha_e B \tag{3.74}$$

In a one-dimensional situation the appropriate optical depth is

$$\eta_{a_1} = \int_{R(t)}^{r} \sqrt{\alpha_a^{(0)}(\xi,t)\,\alpha_a^{(1)}(\xi,t)}\; d\xi \tag{3.75}$$

where a parameter $R(t)$ has been introduced for reasons which will soon become clear. Equation (3.74) may now be written

$$\frac{\partial^2 \phi}{\partial \bar{\eta}_{a_1}^2} - \phi = \frac{4\pi}{3\tau_e}\left(\frac{\alpha_e}{\alpha_a^{(0)}} B\right) - \frac{g}{\tau_e\sqrt{3}}\,\frac{\alpha_a^{(1)}}{\alpha_a^{(0)}}\,\frac{\partial}{\partial \bar{\eta}_{a_1}}\left[\log\left(r^j\sqrt{\frac{\alpha_a^{(0)}}{\alpha_a^{(1)}}}\right)\right] \tag{3.76}$$

where

$$\bar{\eta}_{a_1} = \sqrt{3}\,\tau_e\,\eta_{a_1} \tag{3.77}$$

and

$$r(\eta) = R(t) + \int_{-\bar{\eta}_{W a_1}}^{\bar{\eta}_{a_1}} \frac{d\bar{\eta}'}{\tau_e\sqrt{3\alpha_a^{(1)}\alpha_a^{(0)}}} \tag{3.78a}$$

$$\eta_{W_{a1}} = \int_0^{R(t)} \sqrt{\alpha_a^{(0)}(\xi)\,\alpha_a^{(1)}(\xi)}\; d\xi \tag{3.78b}$$

Henceforth the subscript $a1$ will be suppressed.

Under conditions previously stated, Eq. (3.76) may be reformulated. If symmetry is invoked in all cases, it may be assumed that

$$g \to 0 \quad \text{as} \quad r \to 0, \infty$$

Since data is specified on $\partial\phi/\partial\eta$, the Green's function appropriate for the solution of Eq. (3.76) is

$$G(\bar{\eta},\bar{\eta}') = \begin{cases} \cosh(\bar{\eta}'+\bar{\eta}_W)\, e^{-(\bar{\eta}+\bar{\eta}_W)}, & \bar{\eta} > \bar{\eta}' \\ \cosh(\bar{\eta}+\bar{\eta}_W)\, e^{-(\bar{\eta}'+\bar{\eta}_W)}, & \bar{\eta} < \bar{\eta}' \end{cases} \tag{3.79}$$

so that the solution may be written as follows*

$$\phi(\bar{\eta}) = -\frac{1}{3\tau_\ell} \int_{-\bar{\eta}_W}^{\infty} G(\bar{\eta},\bar{\eta}')\mathcal{F}_j(\bar{\eta}')\,d\bar{\eta}' \tag{3.80}$$

where

$$\mathcal{F}_j(\bar{\eta}') = 4\pi \frac{\alpha_e}{\alpha_a^{(0)}} B - \frac{\alpha_a^{(1)}}{\alpha_a^{(0)}} \hat{\tau} \sqrt{3} \frac{\partial}{\partial \bar{\eta}} \left\{ \log\left(r^j \sqrt{\frac{\alpha_a^{(0)}}{\alpha_a^{(1)}}} \right) \right\} \tag{3.81}$$

Suppose that the gas contained within $0 \leq r \leq R(t)$ (or an isolated slab of gas $(-\frac{R}{2} \leq x \leq \frac{R}{2})$ in the planar case) is separated from its sorroundings by a fictitious membrane. Assume further that by some means the absorption properties of the isolated mass are controllable and that the gas therein is maintained isothermal at temperature T_W.

If the properties of the solitary gas are manipulated so that all incident radiation is absorbed, $\alpha_a^{(0)}/\alpha_a^{(1)} \to 1$, $\alpha_p \to \alpha_a^{(0)} \to \alpha_a^{(1)} \to \infty$. There follow

$$I_o(\eta) = 2\pi B(T_W) e^{-\sqrt{3}\tau_\ell \eta} + \frac{\sqrt{3}}{2}\tau_\ell \int_0^{\infty} e^{-\sqrt{3}\tau_\ell |\eta-\eta'|} \mathcal{F}_j(\eta')\,d\eta' \tag{3.82}$$

$$q(\eta) = \frac{2\pi}{\sqrt{3}} B(T_W) e^{-\sqrt{3}\tau_\ell \eta} + \frac{1}{2}\tau_\ell \int_0^{\infty} \operatorname{sgn}(\eta-\eta') e^{-\sqrt{3}\tau_\ell |\eta-\eta'|} \mathcal{F}_j(\eta')\,d\eta' \tag{3.83}$$

Since $\eta_W \to \infty$, and the body of gas is isothermal, a black surface of radius $R(t)$ has been simulated. At the surface, the following relationship between I_o and q exists.

$$I_{o_{wall}} + \sqrt{3}\, q_{wall} = 4\pi B(T_W) \tag{3.84}$$

This is Mark's Boundary Condition of Neutron Transport Theory. Mark assumed the isolated medium to be black and applied matching conditions at the surface. It has never been demonstrated that his condition is in fact consistent with general non-gray radiative transfer to a P_1-approximation. The current modification of Mark's ideas may be

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* The solution, Eq. (3.62) with Green's function, Eq. (3.61) and is in fact the equivalent of kernel substitution in the spherical case when $\alpha_p = \alpha_a$ = constant. (See Appendix C.)

extended to non-black interactions as well.

Suppose that the space previously isolated is void. One may object to the use of an η-plane in this case since the entire isolated region degenerates to a single point. However, since $\alpha_p \to \alpha_a^{(0)} \to \alpha_a^{(1)} \to 0$, $q = 0$ everywhere therein so that the distinction is immaterial. The effect of this assumption is merely to let $\eta_W \to 0$, and the result is

$$I_0(\eta) = \sqrt{3}\int_0^{\eta} \cosh(\sqrt{3}\zeta_e\eta')e^{-\sqrt{3}\zeta_e\eta}\,\zeta_e\,\mathcal{F}_j(\eta')d\eta' + \sqrt{3}\int_{\eta}^{\infty} \cosh(\sqrt{3}\zeta_e\eta)e^{-\sqrt{3}\zeta_e\eta'}\,\zeta_e\,\mathcal{F}_j(\eta')d\eta' \quad (3.85)$$

$$q(\eta) = \int_0^{\eta} \cosh(\sqrt{3}\zeta_e\eta')e^{-\sqrt{3}\zeta_e\eta}\,\zeta_e\,\mathcal{F}_j(\eta')d\eta' - \int_{\eta}^{\infty} \sinh(\sqrt{3}\zeta_e\eta)e^{-\sqrt{3}\zeta_e\eta'}\,\zeta_e\,\mathcal{F}_j(\eta')d\eta' \quad (3.86)$$

Obviously $q(\eta = 0) = 0$. Since all radiation incident on the surface along a given ray emerges at an antipotal point unattenuated, the effect upon the field is as though it were normally incident and specularly reflected. Similarly, radiation along rays which do not pass through the origin may be identified with reflection as well, so that a specularly reflecting surface has been constructed.

Lighthill has suggested (Lighthill, 1960) that, analogous to accomodation coefficients in kinetic theory, one may use the emissivity to write

$$\begin{Bmatrix} I_{0_W} \\ q_W \end{Bmatrix} = \epsilon \begin{Bmatrix} I_{0_W} \\ q_W \end{Bmatrix}_{Black} + (1-\epsilon)\begin{Bmatrix} I_{0_W} \\ q_W \end{Bmatrix}_{Reflecting} \quad (3.87)$$

Of course, "black" corresponds to a "diffuse" in kinetic theory; i.e. a reemission of "particles" with energies characteristic of the temperature of the surface.

This implies the following expressions:

$$I_{0_W} + \left(\frac{2-\epsilon}{\epsilon}\right)\sqrt{3}\, q_W = 4\pi B(T_W) \quad (3.88)$$

where

$$I_{o_W} = 2\pi\,\epsilon B(T_W) + \left(1 - \frac{\epsilon}{2}\right)\sqrt{3}\,\tau_e \int^{\infty} e^{-\sqrt{3}\,\tau_e \eta'}\, \mathcal{J}_i(\eta')\,d\eta' \qquad (3.89)$$

Because this artifice amounts to the solidification of the isolated volume, the procedure has been called the Method of Solidification (MOS).
In general, the semi-gray forms of Eq. (2.69) and (2.70) for the planar case when $r_d = 0$ lead to

$$I_{o_W} + \left(\frac{2-\epsilon}{\epsilon}\right) C\, q_W = 2\pi B(T_W)\{E_2(0) + C E_3(0)\} + \qquad (3.90)$$
$$+ 2(2-\epsilon)\int_{R_p(t)}^{\infty} \tau_e \pi B(\xi)\{E_1[\tau_e \eta(\xi)] - C E_2[\tau_e \eta(\xi)]\}\, \alpha_e(\xi)\, d\xi$$

where C may be a function of time at most. It is necessary to retain the notation $E_2(0)$, $E_3(0)$ because a consistent kernel substitution will not yield the exact values 1 and $1/2$ respectively. Thus if $m \to 1$, $n \to \sqrt{3}$ in Eq. (3.20), the relations above reproduce Eq. (3.88) and (3.89). In particular if the constants m and n are left unspecified, then the only consistent results follow if $C = n$. Furthermore, the correct emission, $4\pi B(T_W)$, results if, and only if, $m = 1$.

So far the entire medium outside of the surface has been affected by radiation. What if another impermeable membrane were inserted at some distance, $R_s(t)$, from the surface? Such a situation is of importance in application to shock layers. Suppose the gas in $R_s \le r < \infty$ is non-absorbing, i.e. $\alpha_p = \alpha_a^{(0)} - \alpha_a^{(1)} = 0$. It follows that

$$I_{o_{Shock}} - q_{Shock}\sqrt{3} = 0 \qquad (3.91)$$

Equation (3.69) still holds, of course, but

$$I_{o_{Wall}}\Big)_{N.E.} = 2\pi\,\epsilon B(T_W) + \sqrt{3}\left(1 - \frac{\epsilon}{2}\right)\int_0^{\eta_s} e^{-\sqrt{3}\,\tau_e \eta'}\, \tau_e \mathcal{J}_i(\eta')\,d\eta' \qquad (3.92)$$

Since if $\alpha_p \to 0$ the gas does not emit, this case entails neither upstream absorption nor emission. If, however $\alpha_p \sim \alpha_a^{(0)} \sim \alpha_a^{(1)} \ll 1$ (i.e., small but non-vanishing) then it may be assumed that the temperature level of the upstream medium is unaffected by absorption of radiation from the shock layer. The geometric complication represented by $(\frac{1}{r})$ terms is negligible in such an approximation, hence:

$$I_{0_{Shock}} - q_{Shock}\sqrt{3} = 4 \tag{3.93}$$

in addition to Eq. (3.88) and

$$I_{0_{Wall}}\Big)_{N.A.} = I_{0_{Wall}}\Big)_{N.E.} + 2(2-\epsilon)e^{-\sqrt{3}\tau_\ell \eta_{Shock}} \tag{3.94}$$

These appear to be the only boundary conditions consistent with the approximation.

The implications of Eq. (3.88) - (3.94) are quite important. First, this method of deriving boundary conditions yields not only relationship between I_0 and q, but also an explicit expression for either one in terms of the temperature (and pressure) fields. That the entire field at the time of interest is involved is disconcerting but validly reflects the physics of the situation. The solution of quasi-steady problems (those unsteady problems in which $\beta \to 0$) must always involve iteration in the sense that relationships between I_0 and q exist at the boundary and far from it (perhaps at a shock wave through Eq. (3.93)). One must choose the correct value of, say, I_0 at one boundary in order for proper behavior to ensue at the other. The integral expressions provide a means whereby reasonable guesses may be made.

It is clear that the geometry of the medium does not affect the surface interaction. This is consistent with Davison's comments (Davison, 1958) that the interaction is a purely local phenomenon and should not depend upon geometry. It shall be demonstrated (Section 4.2) that it does not depend upon the time scale either. The effects of upstream emission and absorption can be appreciable. With neither process present Eq. (3.91) indicates that, for comparable temperature distributions, $(\mathcal{I}_o)_{wall}$ will be smallest. Since $\mathcal{I}_o$ is always greater than zero, it is possible that the heat flux into the wall will be drastically underpredicted because the gas upstream offers no resistance to the heat flux from the shock layer. If the free stream is allowed to emit, the resistance of the true situation is more nearly simulated since the upstream gas radiates back as a black body. The full formulation includes not only this effect but also the fact that the free stream radiates at a higher temperature than the undisturbed value. The energy flux incident upon the shock wave from the undisturbed gas is attenuated through the shock layer but must always affect the surface interaction. The boundary condition term in an exact planar formulation would in fact be

$$\mathcal{I}^{(+)}_{wall} E_2(\tau_e \eta) + \mathcal{I}^{(-)}_{shock} E_2\{\tau_e(\eta_{shock} - \eta)\}$$

Therefore the upstream emission term is less important the "thicker" the shock layer and the cooler the free stream. However, there is always a "boundary layer" near the interface in which the term in question must be important. It has not been previously observed that the lack of upstream absorption forces the solutions one might obtain to be invalid over "large" distances along slender bodies (or "times" in piston problems). No matter how small the energy lost by the shock layer is relative to that remaining, large amounts of energy may be lost over large periods of time, and the resistance offered by the upstream gas must eventually be significant no matter how strong the shock wave may be initially. This is the mechanism by which equilibrium is attained. Similar remarks have been made by Thomas (1965) and Olfe (1967). Upstream emission cannot be ignored at all in situations such as the piston problem mentioned earlier since the shock layer is not much warmer than the upstream gas. The interpretation is elucidated further if the relationship of the MOS to a method of images is pointed out. In reality a distribution of radiative sources inside the body has been found which will allow the satisfaction of given boundary conditions. These source (sink) distributions will obviously be different if any portion of the medium is altered (say by omission of free stream emission).

Despite the findings of this section, the relationship of the general Mark-type boundary condition to others which may be formulated is not clear. In the next section several other approximate boundary conditions shall be derived and the relationships among them and inconsistencies inherent in them will be indicated.

3.4.2 The Ambiguity in Full-Range Boundary Conditions

Suppose that some expansion scheme has been imposed upon I, the specific intensity, in order to simplify the formulation. As has been noted, the boundary conditions must be expanded as well. Consider the planar, Legendre polynomial expansion, Eq. (3.25), for simplicity, since geometry has been shown to be unimportant in any event. Then:

$$I(\mu,0,t) = \sum_{\ell=0}^{\infty} A_\ell(0,t)\,P_\ell(\mu) \tag{3.95}$$

The expansion scheme dictates that

$$\frac{2}{2\ell+1}\,A_\ell(x,t) = \int_{-1}^{1} I(\mu,x,t)\,P_\ell(\mu)\,d\mu \tag{3.96}$$

Thus, if the boundary condition is of the form

$$I(\mu,0,t) = B(T)\,, \quad -1<\mu<0 \tag{3.97}$$

there follows

$$\frac{2}{2\ell+1}\,A_\ell(0,t) = B_\ell + \sum_{m=0}^{\infty} A_m(0,t)\int_0^1 P_\ell(\mu)\,P_m(\mu)\,d\mu \tag{3.98}$$

where

$$B_\ell = B(T)\int_0^1 P_\ell(\mu)\,d\mu \tag{3.99}$$

From Eq. (3.98)

$$2A_0 = B(T) + A_0 + \frac{1}{2}A_1 - \frac{1}{8}A_3 + \cdots \tag{3.100a}$$

$$\frac{2}{3} A_1 = -\frac{1}{2} B(T) + \frac{1}{2} A_0 + \frac{1}{3} A_1 + \frac{1}{8} A_2 + \cdots \qquad (3.100b)$$

$$\frac{2}{5} A_2 = \frac{1}{8} A_1 + \frac{1}{5} A_2 + \frac{1}{8} A_3 + \cdots \qquad (3.100c)$$

If the expansion is truncated at lowest order, $A_2 = 0$, either

$$A_0 = \frac{1}{2} A_1 + B \qquad (3.101a)$$

or

$$\frac{2}{3} A_1 = A_0 - B \qquad (3.101b)$$

In terms of more familiar quantities these are

$$I_0 - \frac{3}{2} q = 4\pi B \qquad (3.102a)$$

$$I_0 - 2q = 4\pi B \qquad (3.102b)$$

Thus there are two conditions to choose between if non trivial solutions are desired. Of these two, the latter is most commonly us ed, but there is no particular reason for choosin either. Expression Eq. (3.93) falls almost midway between Eq. (3.102a) and (3.102b). Davison (1958) points out the source of inconsistency in the methods. Suppose that the expansion is truncated at P_{2k} (wherein all even orders of approximation are excluded), then if $\alpha_p = \alpha_a$ = const there are ($N+1$) non-trivial coefficients in the boundary conditions ($N=2k-1$). If solutions of the form $A_n \sim g_n e^{\frac{1}{\nu_n}\alpha x}$ are sought, then if k is even, there exist ($N+1$) roots, ν_n. If, however, k is odd then there are only $M < [(N+1)P_N(0)]^2$ roots, which is N, and one of the boundary conditions cannot be satisfied. The Mark-type boundary condition previously derived avoids any arbitrariness because the only data used (i.e. those invoked in the specification of Green's functions) were homogeneous. If the expansion is truncated at P_{2k}

a Mark-condition is, in fact, equivalent to choosing $\frac{1}{2}(N+1)$ values, μ_j, at which

$$I^{(+)}(\mu_j, 0, t) = \epsilon B(T)\cdot 4\pi + (1-\epsilon) I^{(-)}(-\mu_j, 0, t) \qquad (3.103)$$

if the μ_j are the roots of $P_{N+1}(\mu)=0$. (Since N is even, the roots are two-fold; thus there are $N+1$ equations.) This is also equivalent to evaluating the angular integrals in the full formulation by Gaussian Quadrature. Marshak's boundary condition as Cheng (1965a) uses it is derived from a general rule which requires the choice of $\frac{1}{2}(N+1)$ functions, $\chi_j(\mu)$, in the half range $0 \leq \mu \leq 1$ (or $-1 \leq \mu \leq 0$ as appropriate) and forcing $I^{(+)} - 4\pi\epsilon B(T) - (1-\epsilon) I^{(-)}$ to be orthogonal to them in their range of definition. If these functions are Legendre polynomials in the planar case or the corresponding spherical harmonics for planar surfaces in a multi-dimensional field, this insures conservation of the associated moments at the boundary. There is, however, no rule for the choice of the weight function $\chi_j(\mu)$. If, however, Eq. (3.25) is forced into

$$\int_0^1 \{I^{(+)}(\mu,0,t) - 4\pi\epsilon B - (1-\epsilon) I^{(-)}(-\mu,0,t)\} P_{2k-1}(\mu)\, d\mu = 0;\ k = 1, \tfrac{N+1}{2} \qquad (3.104)$$

then, the P_1 approximation yields

$$I_0 + \left(\frac{2-\epsilon}{\epsilon}\right) 2q = 4\pi B(T_w) \qquad (3.105)$$

which is Eq. (3.88) if $\sqrt{3}\, q$ is replaced by $2\, q$. This is Marshak's boundary condition (1947). Although the half-range schemes are potentially much less general than the differential approximation, half-range boundary conditions can provide some interesting full-range effects. They will be shown to yield useful results in regard to Marshak's boundary condition, Eq. (3.105).

3.4.3 The Relationship Among Full – and Half-Range Boundary Conditions

Suppose that boundary conditions are specified in the general manner

$$I^{(+)}(\mu,0,t) = 4\pi\epsilon B(T) + (1-\epsilon) I^{(-)}(-\mu, 0, t) \qquad (3.106)$$

then the half-range equations (3.34), (3.35a, b) result in

$$\frac{1}{2k+1} A_k^{(+)}(0,t) = 4\pi\epsilon B(T)\int_0^1 P_k(2\mu-1)\,d\mu + (1-\epsilon)\sum_{l=0}^{\infty} A_l^{(-)}(0,t)\int_0^1 P_l(2\mu+1)P_k(2\mu-1)\,d\mu \tag{3.107}$$

The first few of these are:

$$A_0^{(+)}(0,t) = 4\pi\epsilon B + (1-\epsilon)\{A_0^{(-)} + 2A_1^{(-)} + \cdots\} \tag{3.108a}$$

$$\frac{1}{3} A_1^{(+)}(0,t) = (1-\epsilon)\{\tfrac{1}{3}A_1^{(-)} + \cdots\} \tag{3.108b}$$

To lowest order, P_0 half-range, Eq. (2.27) states that

$$q = \pi(A_0^{(+)} - A_0^{(-)}) \tag{3.109a}$$

$$I_0 = 2\pi(A_0^{(+)} + A_0^{(-)}) \tag{3.109b}$$

whence Eq. (3.105) is recovered. However, this approximation yields an incorrect thick limit. Thus the factor of 2 on q is not consistent with the correct "thick" and "thin" limits. From the remarks made after Eq. (3.90), Marshak's condition is consistent only with the approximation $E_2(x) = e^{-2x}$.

Consider further the family of forward-reverse approximations defined by Eq. (3.36) - (3.42). Equations (3.37) and (3.38) yield

$$q = \frac{1}{\pi\bar{\mu}}(I^{(+)} - I^{(-)}) \tag{3.110a}$$

$$I_0 = \frac{1}{2\pi\bar{m}}(I^{(+)} + I^{(-)}) \tag{3.110b}$$

From Eq. (3.106) there follows

$$I_0 + \left(\frac{2-\epsilon}{\epsilon}\right)\left(\frac{2\bar{m}}{\bar{\mu}}\right)q = 4\pi B(T)\cdot\bar{m} \tag{3.111}$$

which corresponds to Eq. (3.88) if $\frac{2\bar{m}}{\bar{\mu}} = \sqrt{3}$ and $\bar{m} = 1$. But this yields $\bar{\mu} = 2/\sqrt{3}$ which is greater than unity, hence impossible. Since $\bar{\mu} \le 1$, $\bar{m}$ will be > 1, and this cannot be allowed since it would imply emission at a rate greater than that of a black body.

Helliwell (1966) has formulated the half-range equations from a full-range expansion equivalent to the P_1-approximation. With this artifice he has obtained the correct thick limit. His approach cannot be used to formulate boundary conditions of the type Eq. (3.106) since while the half-range moments can be found from a full-range expansion, the half-range intensities cannot.

One of the reasons that Gaussian quadrature is most suitable for the evaluation of the moment integrals is that the Gaussian ordinates are symmetric about the midpoint of the interval. Furthermore, the end points are not necessary. Since odd orders of approximation (those which require an even number of points) do not require the use of values at the midpoint ($\mu=0$) where I is discontinuous, this is another reason for the omission of even orders. According to Krook (1955) the planar Legendre polynomial expansion Eq. (3.25) is equivalent to

$$I_n(x,t) = 2\pi\int_{-1}^{1}\mu^n I(\mu,x,t)\,d\mu = 2\pi\sum_{j=1}^{N}(\mu_j)^n a_j\{I(\mu_j,x,t)+(-1)^n I(-\mu_j,x,t)\} \quad (3.112)$$

where the a_j are Gaussian weights (see Chandrasekhar, 1960), and the μ_j are Gaussian coordinates, the zeros of the Legendre polynomials. If one eliminates $I(-\mu_j, x, t)$ from Eq. (3.106) and then employs the closing condition which relates $A_\ell;\ \ell=n,n+1,\dots$ to $A_0, A_1, \dots, A_{n-1}$, it follows that

$$\begin{vmatrix} \Xi_0(0,t) & 1 & \cdots & 1 \\ \Xi_1(0,t) & \mu_1 & & \mu_n \\ \vdots & \vdots & & \vdots \\ \Xi_{n-1}(0,t) & (\mu_1)^{n-1} & & (\mu_n)^{n-1} \\ \Xi_\ell(0,t) & (\mu_1)^\ell & \cdots & (\mu_n)^\ell \end{vmatrix} = 0 \qquad [\ell \geq n] \quad (3.113)$$

where

$$\Xi_n(x,t) = \frac{1}{1+(-1)^n-\epsilon}\left\{I_n(0,t) - 2\pi \sum_{j=1}^{\ell} (\mu_j)^{\ell} a_j \epsilon B\right\} \qquad (3.114)$$

To lowest order, $\ell = 1$, this reproduces Eq. (3.88), and indicates what form the extensions to higher orders may take. Krook's idea has never been displayed in the general form above. The P_1-approximation is equivalent to assuming that $I^{(+)}$ and $I^{(-)}$ are two discrete streams of radiation incident from angles $\cos^{-1}(\pm\frac{1}{\sqrt{3}})$ respectively, and each is proportional to $(I_0 \pm q\sqrt{3})$. The successive P_N approximations may be thought of as representing $I^{(\pm)}$ by N discrete streams each (Chandrasekhar, 1960). In the multi-dimensional case, each stream has both a declination, θ, and an azimuth, ϕ; hence in the spherical harmonic expansion there are $(2\ell+1)$ individual streams in $I^{(\pm)}$ to a P_ℓ approximation. Chou's (1967) method involves the choice of optimum regions of the unit sphere for averaging. His approach is not general, although it has been demonstrated that it is more accurate than the differential approximation in a few cases.

Some comments concerning the MOS in more general situations are now offered without proof. Extension of the method to higher orders of approximation in one-dimensional cases will require the solution of differential equations of higher and higher order. If these were carried out the result would be:

1. n-1 linear combinations of the n moments related to a function which would depend upon the temperature of the boundary, and

2. A single condition which would relate the value of one of the moments at the boundary to integrals of the temperature and pressure distributions.

Thus the differential approximation is not purely differential at all. The integrals have been removed from the governing equations at the expense of their appearance in the boundary conditions, but this is a considerable simplification. The application of the MOS to multi-dimensional situations should be straightforward but cumbersome. Particular difficulties will arise if the bounding surfaces are concave outward so that they can "see" themselves. It is hoped that such extension will not be necessary, since it has been amply demonstrated that in at least three different geometries the boundary conditions are the same. The problem could easily have been solved in terms of retarded coordinates (just as the transfer equation was solved to obtain Eq. (2.33)). In this manner it could be verified that Eq. (3.88) and its analogs Eq. (3.92) and (3.94) hold even when $\beta \neq 0$. This shall be justified in another context in Section 4.2.

Kourganoff (1957) has studied the effects of several boundary conditions upon the P_1 -planar approximation in astrophysics. He has shown that for stellar structure the errors in the Eddington approximation

$$p^{(R)} = \frac{1}{3} I_0 \; ; \quad I_0(0,t) = 2 q(0,t) \tag{3.115}$$

compensate each other at large optical depths so that the discrepancies from the exact solution are approximately 3 percent. However, at the interface the approximation is 15 percent in error. The boundary condition, Eq. (3.88), is exact at small optical depths, but diverges by a consistent 8 percent for $\eta \gtrsim 2$. Even though the specific intensity is hopelessly anisotropic at the interface, Mark's boundary condition gives good results thereat. Since aerodynamic interest lies partially in the prediction of heating rates, it seems that a Mark-type boundary condition is best for this application. In fact if emission from all depths is considered, it may be shown that at the edge of the atmosphere (Eddington, 1926, p.333).

$$I_0 = 2 q \left(1 + \frac{3}{2} \cos\theta\right) \tag{3.116}$$

so that

$$I_0(0,t) = \frac{7}{4} q(0,t) \tag{3.117}$$

which differs from Eq. (3.88) by only a fraction of one percent.

3.4.4 A Consistent Derivation of Optically Thick "Slip" Boundary Conditions

The analogy between "optically thick" radiative transfer and molecular conduction has been pointed out previously (Chapter 1). Probstein (1963) and others (e.g. Tien and Greif, 1964) realized that the phenomenon of molecular slip had an analog in radiative transfer and sought expressions of the form

$$T(0) - T_W = K \, \mathrm{grad}\, T \cdot \vec{n} \tag{3.118}$$

where $T(0)$ is the temperature of the gas at the wall and K, the slip coefficient, was to be determined from matching with the black body result in the "thin" limit before the optically thick catastrophe could occur. It shall be shown that an expression with a definite value of K follows from "general" boundary conditions of the form of Eq. (3.88) and (3.105) which are written symbolically as

$$I_0 + \left(\frac{2-\epsilon}{\epsilon}\right) C_1 q = 4\pi B(T_W) \tag{3.119}$$

for a gas in LTE. Recall Eq. (3.65) and (3.67) in a non-scattering, quasi-osotropic, semi-gray gas.

$$\beta \frac{\partial I_0}{\partial t} + \mathrm{div}\,\vec{q} = \tau_\ell \left(4\alpha_p T^4 - \alpha_a^{(0)} I_0\right) \tag{3.120}$$

$$\beta \frac{\partial \vec{q}}{\partial t} + \mathrm{grad}\, I_0 = -3\tau_\ell \alpha_a^{(1)} \vec{q} \; ; \; \left[\alpha_a^{(0)} = \alpha_a^{(1)} \equiv \alpha_a\right] \tag{3.121}$$

In the thick limit
$$I_0 = 4 \frac{\alpha_p}{\alpha_a} T^4 + O\left(\frac{1}{\tau_\ell}\right) \tag{3.122}$$

and

$$\vec{q} = -\frac{4}{3\tau_\ell \alpha_R} \mathrm{grad}\, T^4 \tag{3.123}$$

Thus Eq. (3.119) in light of Eq. (3.122) and (3.123) is

$$\left(4\frac{\alpha_p}{\alpha_a}T(0)^4 - 4T_W^4\right) = \left(\frac{2-\epsilon}{\epsilon}\right)c_1 \cdot \frac{4}{3\tau_\ell \alpha_R}\, \mathrm{grad}\,T \cdot \vec{n} \qquad (3.124)$$

Since gradients must be small and $T(0)$ is not too different from T_W

$$\frac{\alpha_p}{\alpha_a} \simeq 1 \qquad (3.125)$$

$$\frac{T(0)}{T_W} = 1 + \frac{T'}{T_W} + 0\left(\frac{1}{\tau_\ell^2}\right) \qquad (3.126)$$

so that

$$4T_W^4 \cdot 4\frac{T'}{T_W} = \left(\frac{2-\epsilon}{\epsilon}\right)c_1 \cdot \frac{4T_W^4}{3\alpha_R\tau_\ell}\, \mathrm{grad}\left(1+4\frac{T'}{T_W}+\cdots\right)\cdot\vec{n}$$

or

$$T' = \left(\frac{2-\epsilon}{\epsilon}\right)\frac{c_1}{3\tau_\ell\alpha_R}\, \mathrm{grad}\,T' \cdot \vec{n} \qquad (3.127)$$

and by Eq. (3.126)

$$T(0) - T_W \cong \left(\frac{2-\epsilon}{\epsilon}\right)\frac{c_1}{3\tau_\ell \alpha_R}\, \mathrm{grad}\,T \cdot \vec{n} \qquad (3.128)$$

Equation (3.128) is the general form of Probstein's "slip" boundary condition. In fact, for a black wall and Marshak's boundary condition ($c_1 = 2$), Eq. (3.128) is identically Probstein's result! It is, therefore, not surprising that his results agree with those of the differential approximation to which Lick applied Marshak's boundary condition! We claim, however, that the most appropriate slip coefficient is

$$K = \left(\frac{2-\epsilon}{\epsilon}\right)\frac{1}{\sqrt{3}\,\alpha_R} \qquad (3.129)$$

Equation (3.129) is again in direct analogy with the molecular result if ϵ is associated with the thermal accomodation coefficient (See Eq. (3.87)). There is no arbitrary matching required. The non-linear condition, Eq. (3.124),was derived by Deissler (1964) in a very elegant manner but unfortunately only for the case of a constant α_ν. His formula to $O(\frac{1}{\tau_0^2})$ corresponds to Marshak's boundary condition, $c_1 = 2$. Slip is always present when molecular transport phenomena are ignored, thus no "special" treatment is necessary if one is consistent. The previous comments do, however, clarify the concept of radiation slip.

CHAPTER 4

THE METHOD OF CHARACTERISTICS IN RADIATION GASDYNAMICS

4.1 The Characteristic Surfaces of the Unsteady RGD

The governing equations which will be discussed are those consistent with a P_1 differential approximation to $o(\beta^2)$ in a semi-gray medium which is not necessarily in LTE.

$$\rho_P)_{T,\bar{c}} \frac{DP}{Dt} + \rho_T)_{P,\bar{c}} \frac{DT}{Dt} + \rho \, div \vec{u} = \sum_i \rho_{c_i}|_{P,T,c_j} \left(\frac{W_i}{t_i^* \rho}\right) \tag{4.1}$$

$$\rho \frac{D\vec{u}}{Dt} + grad\, P = \beta \left(\frac{\tau_\ell}{B_0}\right) \alpha_a^{(1)} \vec{q} \tag{4.2a, b, c}$$

$$\left\{\frac{\rho h_P|_{T,\bar{c}}}{\Gamma} - 1\right\} \frac{DP}{Dt} + \frac{\rho h_T|_{P,\bar{c}}}{\Gamma} \frac{DT}{Dt} + \frac{4}{3}\left(\frac{\beta}{B_0}\right) I_0 \, div \vec{u} = 3 \frac{\tau_\ell}{B_0} \beta \vec{u} \cdot \alpha_a^{(1)} \vec{q} + \frac{\tau_\ell}{B_0} Q \tag{4.3}$$

$$\beta \frac{\partial I_0}{\partial t} + div \vec{q} = -\tau_\ell Q \tag{4.4}$$

$$\beta \frac{\partial \vec{q}}{\partial t} + \frac{1}{3} grad\, I_0 = -\tau_\ell \alpha_a^{(1)} \vec{q} \tag{4.5a, b, c}$$

$$\frac{Dc_i}{Dt} = -\frac{W_i}{t_i^* \rho} \tag{4.6}$$

where

$$Q = \alpha_a^{(0)} I_0 - 4\pi \alpha_e B \tag{4.7}$$

The equations of state and thermodynamics have been used in the form

$$\rho = \rho(P, T, \vec{c}) \tag{4.8}$$

$$h = h(P, T, \vec{c}) \tag{4.9}$$

The mathematical character of the above will be examined according to a procedure similar to that outlined by Von Mises (1958).

Consider the quasi-linear system of equations

$$\sum_{j=1}^{m} \left\{ \sum_{k=1}^{n} A_{ij}^{(k)} \frac{\partial u_j}{\partial y_k} \right\} + C_i(\vec{u}, \vec{y}) = \vec{0} \tag{4.10}$$

which are "m" equations for the determination of "m" unknowns u_i, $i = 1$ to m in "n" independent variables y_k, $k = 1$ to n. The $A_{ij}^{(k)}(\vec{u}, \vec{y})$ are three dimensional matrices whose general elements are the coefficients of the derivatives of u_j with respect to y_k in the i^th governing equation. Consider further a surface, S, described by

$$\psi(\vec{y}) = 0$$

in which the coordinates $\xi_0, \xi_1, \cdots, \xi_{n-1}$ are defined such that $\xi_0 = 0$ is the surface S. The ξ_i, $i = 1, n-1$ are coordinates in S. In the new coordinate system Eq. (4.10) is

$$\sum_{k=1}^{n} \sum_{j=1}^{m} \left\{ \sum_{\ell=1}^{n-1} A_{ij}^{(k)} \frac{\partial \xi_\ell}{\partial y_k} \frac{\partial u_j}{\partial \xi_\ell} \right\} + C_i = -\sum_{j=1}^{m} \sum_{k=1}^{n} \left(A_{ij}^{(k)} \frac{\partial \psi}{\partial y_k} \right) \frac{\partial u_j}{\partial \xi_0} \tag{4.11}$$

Given initial data upon all of the u_i, $i=1,m$ on the surface S, a solution may be initiated locally in Taylor series if the normal derivatives of the u_i on $\xi_0 = 0$, i.e., the $\partial u_j / \partial \xi_0$, can be obtained from Eq. (4.11). Once this has been accomplished the Cauchy-Kowalewski theorem (Garabedian, 1964, pg. 6 ff.) assures a unique solution. The required derivatives cannot be obtained if

$$\left| \sum_{k=1}^{n} A_{ij}^{(k)} \alpha_k \right|_{\psi=0} = 0 \tag{4.12a}$$

where

$$\alpha_k \equiv \frac{\partial \psi}{\partial y_k} \tag{4.12b}$$

A manifold whose description satisfies Eq. (4.12) is, therefore, exceptional for the specification of initial data and is called "characteristic". A manifold is characteristic at a point if the direction cosines of the normal to its tangent plane satisfy Eq. (4.12). Equation (4.12) is a homogeneous polynomial of order "m" in the direction cosines, α_k. If it yields a relation from which real values of any one of the α_k may be found for any real values of all of the others, the direction defined by that α_k is time-like, and the surface to which that direction is normal is space-like. If there are "m" real and distinct roots of this type the system of equations is totally hyperbolic. On the other hand, if the $A_{ij}^{(k)}$ are symmetric for each k or if any linear combination of them is definite the system is termed symmetric hyperbolic (see Jeffrey and Taniuti, 1964). Maxwell's equations are of the latter type. Finally, if some of the roots of Eq. (4.12) are real and some are imaginary, the problem is ultrahyperbolic, and if <u>all</u> are complex, it is elliptic. To be hyperbolic in the weakest sense a system need have only one space-like characteristic manifold at every point. These are merely convenient names and bear little relation to the ease of solution of the respective cases.

A hyperbolic system may be simplified if one makes use of its characteristic surfaces. Since derivatives normal to these surfaces are indeterminate, the governing equations may be written upon them in one less independent variable than the original system requires. The systematic method of forming combinations of the equations which allow this involves first multiplying the system (Eq. (4.10)) by some unspecified vector $v_i(\vec{u},\vec{y})$ so that

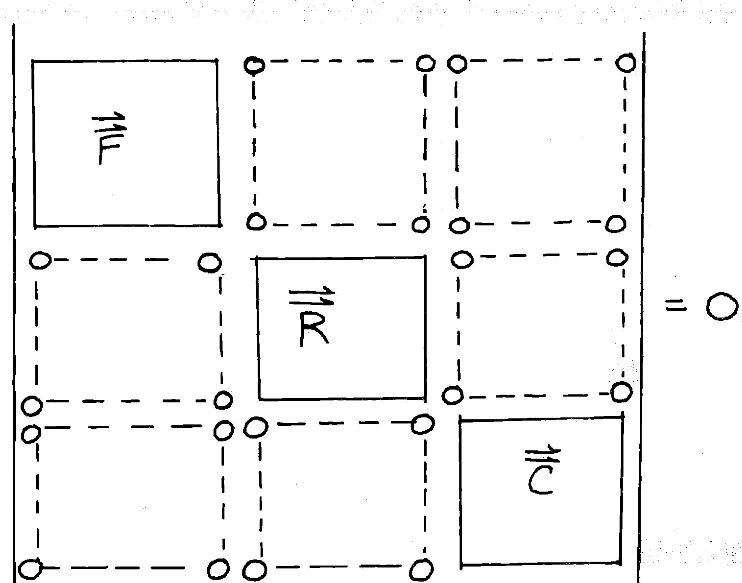
$$\sum_j \sum_k \sum_i \left(v_i A_{ij}^{(k)} \frac{\partial \xi_0}{\partial y_k}\right)\frac{\partial u_j}{\partial \xi_0} = -\sum_i \left\{ v_i c_i + \sum_k \sum_j \sum_l v_i A_{ij}^{(k)} \frac{\partial \xi_l}{\partial y_k}\frac{\partial u_j}{\partial \xi_l} \right\} \quad (4.13a)$$

The normal derivative $\partial u_j / \partial \xi_0$ will be indeterminate on the characteristic manifold, thus tenable equations follow only if the components of $\vec{v}$ are such that

$$\vec{v} \cdot \sum_k \vec{\vec{A}}^{(k)} \frac{\partial \xi_0}{\partial y_k} = \vec{0} \quad (4.13b)$$

The right hand side of Eq. (4.13a) then provides the simplified equations in $(n-1)$ independent variables. These are referred to as compatibility relations, and one is assured that there will be as many of them as there are dependent variables only if the directions α_k are all distinct. The geometry of the transformation $\vec{y}$ to $\vec{\xi}$ is very difficult, but it has been undertaken with some success (Holt, 1956; Butler, 1960; Chu, 1964, 1967; Sauerwein, 1964, 1966).

It is obvious that Eqs. (4.4), (4.5), and (4.6) do not enter the specification of the fluid mechanical characteristic surfaces. That is, the radiation and the fluid mechanics are not coupled through differentiated terms, but rather through inhomogeneous terms, such as . For three-dimensional unsteady flow Eq. (4.12) is



(4.14)

where $\vec{\vec{F}}$, $\vec{\vec{R}}$, and $\vec{\vec{C}}$ are the fluid mechanical, radiative, and chemical submatrices given by

$$\vec{\vec{F}} = \begin{bmatrix} \rho_p|_{T,\bar{c}}\,\mathcal{D}\psi & \rho\psi_x & \rho\psi_y & \rho\psi_z & \rho_T|_{p,\bar{c}}\,\mathcal{D}\psi \\ \frac{1}{\rho}\psi_x & \mathcal{D}\psi & 0 & 0 & 0 \\ \frac{1}{\rho}\psi_y & 0 & \mathcal{D}\psi & 0 & 0 \\ \frac{1}{\rho}\psi_z & 0 & 0 & \mathcal{D}\psi & 0 \\ \left\{\frac{\rho h_p|_{T,\bar{c}}}{\Gamma}-1\right\} & \frac{4}{3}\left(\frac{\beta}{B_0}\right)I_0\psi_x & \frac{4}{3}\left(\frac{\beta}{B_0}\right)I_0\psi_y & \frac{4}{3}\left(\frac{\beta}{B_0}\right)I_0\psi_z & \frac{\rho h_T|_{p,\bar{c}}}{\Gamma}\mathcal{D}\psi \end{bmatrix} \qquad (4.15)$$

$$\vec{\vec{R}} = \begin{bmatrix} \beta\psi_t & \psi_x & \psi_y & \psi_z \\ \frac{1}{3}\psi_x & \beta\psi_t & 0 & 0 \\ \frac{1}{3}\psi_y & 0 & \beta\psi_t & 0 \\ \frac{1}{3}\psi_z & 0 & 0 & \beta\psi_t \end{bmatrix} \qquad (4.16)$$

$$\vec{\vec{C}} = \mathrm{diag}\,(\mathcal{D}\psi) \qquad (4.17)$$

$\mathcal{D}$ is the Eulerian derivative

$$\mathcal{D} = \frac{\partial}{\partial t} + \vec{u}\cdot\mathrm{grad} \qquad (4.18)$$

The part of Eq. (4.14) due to fluid mechanics yields the following

$$(\mathcal{D}\psi)^2 = a_f^2 (\mathrm{grad}\,\psi)\cdot(\mathrm{grad}\,\psi) \tag{4.19}$$

$$(\mathcal{D}\psi)^2 = 0 \tag{4.20}$$

where

$$a_f^2 = \left.\frac{\partial p}{\partial \rho}\right)_{s,\vec{c}} = \frac{h_T\big|_{p,\vec{c}} - \frac{\rho_T|_{p,\bar{c}}}{\rho^2}\cdot\frac{4}{3}\Gamma\left(\frac{\beta}{B_0}\right)I_0}{h_T\big|_{p,\bar{c}}\,\rho_p\big|_{T,\bar{c}} - h_p\big)_{T,\bar{c}}\,\rho_T\big|_{p,\bar{c}} + \frac{\Gamma\rho_T|_{p,\bar{c}}}{\rho}} + O(\beta^2) \tag{4.21}$$

is the frozen propagation velocity in the presence of equilibrium radiation. (In the above and in all that follows, subscript notation is employed for partial differentiation.) That Eq. (4.21) is correct may be verified through the second law of thermodynamics if one is careful to identify the radiative contributions to energy and pressure. The quantity a_f is the same as the propagation speed formed by Sachs (1946) and cited by Sen and Guess (1957) and Marshak (1958). The determination of characteristic surfaces will always specify the propagation speeds under the conditions of interest since characteristic surfaces may alternately be interpreted as wave-fronts (Staniukovich, 1960). This seems quite the easiest way to approach the situation since all of the requisite thermodynamics is already in the governing equations. The application of standard methods (see e.g., Garabedian, 1964) to Eq. (4.19) and (4.20) yields the equations of the generalized Mach cones and particle world lines.

$$\{(\vec{r}-\vec{r}_0) - \vec{u}(t-t_0)\}\cdot\{(\vec{r}-\vec{r}_0) - \vec{u}(t-t_0)\} = a_f^2 (t-t_0)^2 \tag{4.22}$$

and

$$d\vec{r} = \vec{u}\,dt \tag{4.23}$$

The particle paths are, in fact, $N+2$ fold degenerate if there are N chemical processes since $\det(\bar{\bar{C}}) = 0$ requires that $(\mathcal{D}\psi)^N = 0$.

Equation (4.22) verifies the time-like nature of the variable t. These results and their consequences in real, nonradiating gases are well known, and more detail may be found in the works of Sauerwein (1964) and Sussman (1966). Of greatest interest in the present context are the consequences of $det(\overleftrightarrow{R})=0$. If moments of the transfer equation had not been taken, then $det\,\overleftrightarrow{R}=0$ would, have been merely

$$\mathcal{D}_\beta(\psi) \equiv \left(\beta\frac{\partial}{\partial t} + \vec{\Omega}\cdot grad\right)\psi = 0 \tag{4.24}$$

which yields the nondegenerate photon path line,

$$d\vec{r} = \frac{\vec{\Omega}}{\beta}\,dt \tag{4.25}$$

When only moments of the specific intensity are considered, the situation changes drastically. Consider the infinite, unclosed system of moment equations

$$\beta\frac{\partial I_0}{\partial t} + div\,\vec{I}_1 = \tau_\ell\{4\pi\alpha_e B - \alpha_a^{(0)} I_0\} \tag{4.26a}$$

$$\beta\frac{\partial \vec{I}_1}{\partial t} + div\,\overleftrightarrow{I}_2 = -\tau_\ell\alpha_a^{(1)}\vec{I}_1 \tag{4.26b}$$

$$\beta\frac{\partial \overleftrightarrow{I}_2}{\partial t} + div\,\overleftrightarrow{I}_3 = \tau_\ell\left(\frac{4\pi}{3}\right)\alpha_e B\,\overleftrightarrow{I} - \alpha_a^{(2)}\overleftrightarrow{I}_2 \tag{4.26c}$$

where the moments of I are defined according to Eq. (3.68). The determinant $\overleftrightarrow{R}$ for this system is of the form

$$\overline{\overline{R}} = \begin{bmatrix} \beta\psi_t & \psi_x & \psi_y & \psi_z & 0 & \cdots \\ 0 & \beta\psi_t & 0 & 0 & \psi_x & \\ 0 & 0 & \beta\psi_t & 0 & 0 & \\ 0 & 0 & 0 & \beta\psi_t & 0 & \\ \vdots & \vdots & \vdots & 0 & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \tag{4.27}$$

Even though the system is unclosed, $\overline{\overline{R}}$ must be a square matrix of infinite dimensions. All nonzero elements appear above the major diagonal, thus $\det(\overline{\overline{R}})=0$ yields the infinitely degenerate characteristic

$$\lim_{M\to\infty} (\beta\psi_t)^M = 0 \tag{4.28}$$

which is a cylindrical surface $\psi = \psi(\vec{r})$. The effect is that of instantaneous propagation even though the speed of light, $1/\beta$, is finite. This characteristic has no obvious physical interpretation but will be referred to as the pseudo photon path. Obviously the confluence of all of the cylinders must be the photon path line. In fact, the system is parabolic since there is only one compatibility relation, the transfer equation, which applies along the path line.

The implications of the truncation of a spherical haromic expansion are now apparent. Since the closing condition relates higher order moments to those of lower order, it transfers nonzero elements from the right of the diagonal to the left in the "N^{th}" row if $\overline{\overline{I}}_N$ is the last moment retained. This provides only cone-like characteristic surfaces in odd orders of approximation and "$N-1$" cones in addition to a pseudo photon path in even orders. In fact, it is truncation which makes the system of moment equations hyperbolic. Since the closing condition relates the highest order moment to all of the lower ones the characteristic directions are solutions of an N^{th} or $(N+1)^{st}$ degree polynomial because that number of elements will be moved to the opposite side of the diagonal. For instance, Eq. (4.14) leads to Eq. (4.28) once and to

$$3\beta^2 (\psi_t)^2 = (\text{grad}\,\psi)\cdot(\text{grad}\,\psi) \qquad (4.29)$$

the solution of which is

$$(t-t_0)^2 = (3\beta^2)(\vec{r}-\vec{r}_0)\cdot(\vec{r}-\vec{r}_0) \qquad (4.30)$$

These shall be called "light cones." As opposed to Mach cones, they have straight generatrices. Their projection on physical space is a sphere of radius $(t-t_0)/\beta\sqrt{3}$ whose center is fixed whereas the projection of a Mach conoid on physical space is a sphere whose center moves along the projected path line. The characteristic surfaces of two-dimensional unsteady flow are indicated in Fig. 3. As the elements to the left of the diagonal move downward in $\bar{\bar{R}}$ the semi-angle of the light cone is more nearly $(\frac{1}{\beta})$ rather than a fraction of it. This will be demonstrated in Section 4.2. Since the time axis lies within the light cone, disturbances must eventually extend over all of space.

The difficulty with steady flow is now obvious, since Eq. (4.28) then defines no time-like directions. The system of equations becomes ultra-hyperbolic (Jeffery and Taniuti, 1964, p. 37). Ultra-hyperbolic systems allow discontinuities not only in space-time but also in ordinary space, thus the nature of the problem is drastically altered. Nevertheless, in unsteady situations the characteristic surfaces are always real since as $\beta \to 0$ a solution of Eq. (4.29) is $\psi - t = \text{const}$. The situation will be clarified in one-dimensional unsteady flow. In Appendix B the implications of radiative transfer in two (or three)-dimensional steady flow are briefly investigated.

4.2 One-Dimensional Unsteady RGD

4.2.1 Verification of Hyperbolicity

The essence of hyperbolic methods which involve the use of characteristics is best illustrated by problems in only two independent variables. The Von Mises approach is particularly instructive in these cases. The appropriate form of Eq. (4.10) is

$$\frac{\partial \vec{V}}{\partial r} + \vec{\vec{M}} \cdot \frac{\partial \vec{V}}{\partial t} = \vec{N}(\vec{V}, r, t) \tag{4.31}$$

Premultiplying Eq. (4.31) by the unspecified vector, $\vec{v}$, one sees that if

$$\vec{v} \cdot (\vec{\vec{M}} - \lambda \vec{\vec{I}}) = \vec{0} \tag{4.32}$$

Equation (4.30) is

$$\vec{v} \cdot \left(\frac{\partial \vec{V}}{\partial r} + \lambda \frac{\partial \vec{V}}{\partial t} \right) = \vec{v} \cdot \vec{N} \tag{4.33}$$

which are ordinary differential equations in the directions

$$\frac{dr}{dt} = \frac{1}{\lambda} \tag{4.34}$$

This is an equivalent definition of characteristic. Since differentiation only along the line, Eq. (4.34) is involved, one can never move away from it if data is specified on it <u>alone.</u> The directions, λ, are the eigenvalues of $\vec{\vec{M}}$ and once the left hand eigenvectors of $\vec{\vec{M}}$ (the $\vec{v}$) are found the compatibility relations are specified by Eq.(4.33). This is the advantage of dealing with two independent variables.

The specializations of the previous classes of hyperbolicity follow directly. A system of equations is hyperbolic if all of the directions, λ, are real and if there are as many compatibility relations of the form (4.32) as there are dependent variables. This is assured if all of the λ are distinct, in which case the system is completely hyperbolic. If there are "m" distinct eigenvectors even though some of the λ are degenerate then the definition of a symmetric hyperbolic system is satisfied. However, if there are insufficient eigenvectors even though the λ which exist are real, the system is parabolic. The one-dimensional analogs of Eq. (4.1)-(4.6) may be written in the form (4.31) if the following are identified. In all equations $j=0$, 1, or 2 for planar, cylindrical, or spherical geometries.

$$\vec{V} = \{\rho, u, T, I_0, g, c_1, \cdots, c_N\} \tag{4.35}$$

$$\overset{\Rightarrow}{M} = \begin{bmatrix} -\frac{1}{u\Delta}\left\{\frac{B\rho_T|_{p,\vec{c}}}{\rho\rho_p|_{T,\vec{c}}} + \frac{A}{\Gamma}\right\} & \frac{A}{u^2\Gamma\Delta}\,\frac{\rho}{\rho_p|_{T,\vec{c}}} & 0 & 0 & 0 & 0_1 & \cdots & \cdots & 0_N \\ \frac{1}{\rho u^2\Delta}\left\{\frac{B\rho_T|_{p,\vec{c}}}{\rho\rho_p|_{T,\vec{c}}} + \frac{A}{\Gamma}\right\} & \frac{-1}{u\Delta}\left\{\frac{B\rho_T|_{p,\vec{c}}}{\rho\rho_p|_{T,\vec{c}}} + A/\Gamma\right\} & 0 & 0 & 0 & 0_1 & \cdots & \cdots & 0_N \\ -\frac{B}{\rho u^3\rho_p|_{T,\vec{c}}\Delta} & \frac{B}{u^2\rho_p|_{T,\vec{c}}\Delta} & \frac{1}{u} & 0 & 0 & 0_1 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 3\beta & 0_1 & \cdots & \cdots & \cdots \\ \vdots & \vdots & 0 & \beta & 0 & 0_1 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & 0 & 0 & (1/u)_1 & 0_2 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & 0 & 0_1 & (1/u)_2 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0_1 & \cdots & 0_N \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & (1/u)_N \end{bmatrix} \tag{4.36}$$

$$\vec{N} = \begin{bmatrix} \left\{ \frac{j}{r}\left(\frac{A}{\Gamma\Delta}\right)\frac{\rho}{\rho_p|_{T,\vec{c}}} + \frac{\tau_\ell}{uB_0\cdot\Delta}Q\frac{\rho_T|_{p,\vec{c}}}{\rho\rho_p|_{T,\vec{c}}} + \frac{\beta\tau_\ell}{B_0\cdot\Delta}\alpha_a^{(1)}q\left[\frac{A}{\Gamma u^2\rho_p|_{T,\vec{c}}} - 3\frac{\rho_T|_{p,\vec{c}}}{\rho\rho_p|_{T,\vec{c}}}\right] + \right. \\ \left. + \frac{1}{u\Delta}\sum_i\left(\frac{W_i}{t_i^*}\right)\left[\frac{\rho_T|_{p,\vec{c}}}{\rho\rho_p|_{T,\vec{c}}}\left(h_{c_i}|_{p,T,c_j} - \frac{4}{3}\left(\frac{\beta}{B_0}\right)\frac{I_0\rho_{c_i}|_{p,T,c_j}}{\rho^2}\right) - \frac{A}{\Gamma}\,\frac{\rho_{c_i}|_{p,T,c_j}}{\rho}\right]\right\} \\ \left\{\frac{1}{\rho u^2\Delta}\right\}\left\{-\frac{j}{r}\left(\frac{A}{\Gamma}\right)\frac{\rho u}{\rho_p|_{T,\vec{c}}} - \frac{\tau_\ell}{B_0}Q\frac{\rho_T|_{p,\vec{c}}}{\rho\rho_p|_{T,\vec{c}}} + \frac{\beta\tau_\ell}{B_0}u\alpha_a^{(1)}q\left[3\frac{\rho_T|_{p,\vec{c}}}{\rho\rho_p|_{T,\vec{c}}} - \frac{A}{\Gamma} - \frac{B\rho_T|_{p,\vec{c}}}{\rho\rho_p|_{T,\vec{c}}}\right] - \right. \\ \left. - \sum_i\left(\frac{W_i}{t_i^*}\right)\left[\frac{\rho_T|_{p,\vec{c}}}{\rho\rho_p|_{T,\vec{c}}}\left(h_{c_i}|_{p,T,c_j} - \frac{4}{3}\frac{\beta}{B_0}\frac{I_0\rho_{c_i}|_{p,T,c_j}}{\rho^2}\right) - \frac{A}{\Gamma}\frac{\rho_{c_i}|_{p,T,c_j}}{\rho}\right]\right\} \\ \left\{\frac{1}{\rho u\Delta}\right\}\left\{\frac{j}{r}\frac{B\rho u}{\rho_p|_{T,\vec{c}}} - \frac{\tau_\ell}{B_0}Q\left(1 - \frac{1}{u^2\rho_p|_{T,\vec{c}}}\right) + \frac{\beta\tau_\ell}{B_0}u\alpha_a^{(1)}\vec{q}\left[\frac{B}{u^2\rho_p|_{T,\vec{c}}} + 3\left(1 - \frac{1}{u^2\rho_p|_{T,\vec{c}}}\right)\right] - \right. \\ \left. - \sum_i\left(\frac{W_i}{t_i^*}\right)\left[\frac{B\rho_{c_i}|_{p,T,c_j}}{\rho} + \left(1 - \frac{1}{u^2\rho_p|_{T,\vec{c}}}\right)\left(h_{c_i}|_{p,T,c_j} - \frac{4}{3}\frac{\beta}{B_0}I_0\frac{\rho_{c_i}|_{p,T,c_j}}{\rho^2}\right)\right]\right\} \\ -3\tau_\ell\alpha_a^{(1)}q \\ -\left(\tau_\ell Q + j\frac{g}{r}\right) \\ -\frac{W_1}{t_1^*\rho u} \\ \vdots \\ -\frac{W_N}{t_N^*\rho u} \end{bmatrix} \tag{4.37}$$

where

$$A = h_T)_{p,\vec{c}} - \frac{4}{3}\Gamma\left(\frac{\beta}{B_0}\right)\frac{I_0}{\rho^2}\rho_T)_{p,\vec{c}} \tag{4.38a}$$

$$B = 1 + \frac{4}{3}\left(\frac{\beta}{B_0}\right)\frac{I_0}{\rho}\rho_p)_{T,\vec{c}} - \rho\frac{h_p)_{T,\vec{c}}}{\Gamma} \tag{4.38b}$$

$$\Delta = -\frac{B}{\rho}\frac{\rho_T)_{p,\vec{c}}}{\rho_p)_{T,\vec{c}}} - \frac{A}{\Gamma}\left(1 - \frac{1}{u^2\rho_p)_{T,\vec{c}}}\right) \tag{4.38c}$$

It is also useful to note that

$$\Delta = -\frac{A}{\Gamma\rho_p|_{T,\vec{c}}}\;\frac{(u^2 - a_f^2)}{u^2 a_f^2} \tag{4.38d}$$

where a_f^2 is given by Eq. (4.21). With the information, the following characteristic directions and eigenvectors are found.

$$\lambda^{(1),(2)} = 1/(u \pm a_f) \tag{4.39a}$$

$$\vec{v}^{(1),(2)} = \{1, \pm\rho a_f, 0, O(\beta^3), O(\beta^2), 0, \ldots, 0\} \tag{4.39b}$$

$$\lambda^{(3)} = 1/u \tag{4.40a}$$

$$\vec{v}^{(3)} = \{-\frac{\Gamma B}{\rho A}, 0, 1, O(\beta^3), O(\beta^2), 0, \cdots; 0\} \tag{4.40b}$$

$$\lambda^{(4),(5)} = \pm\beta\sqrt{3} \tag{4.41a}$$

$$\vec{v}^{(4),(5)} = \{0, 0, 0, 1, \mp\sqrt{3}, 0, \ldots, 0\} \tag{4.41b}$$

$$\lambda^{(6)} \to \lambda^{(5+N)} = 1/u \tag{4.42a}$$

$$\vec{v}^{(6)} \to \vec{v}^{(5+N)} = \{0, \ldots\ldots, 0_{K-1}, 1, 0_{K+1}, \ldots, 0_N\},\; K = 6, 5+N \tag{4.42b}$$

If $\beta \to 0$, $\lambda^{(4)} = \lambda^{(5)}$ becomes degenerated, but $\vec{\upsilon}^{(4)}$ and $\vec{\upsilon}^{(5)}$ remain distinct so that the hyperbolic nature of the problem is retained. The gasdynamic relations including terms of $o(\beta)$ will be of no further interest in this investigation; however, those which govern the radiation contain significant information. Equations (4.41a, b) and (4.37) reveal that

$$\frac{1}{\sqrt{3}} \frac{d^{\pm}}{dr} \{ I_o \pm q\sqrt{3} \} = \pm \tau_e \{ 4\pi \alpha_e B - (\alpha_a^{(0)} I_o \pm \alpha_a^{(1)} q\sqrt{3}) \} \mp j \frac{q}{r} \tag{4.43a}$$

on

$$\frac{dr}{dt} = \pm \frac{1}{\beta\sqrt{3}} \tag{4.43b}$$

Since the characteristic of negative slope carries information from upstream, far from any disturbance Eq. (4.43a) indicates that Eq. (3.93) holds even in the unsteady case. Therefore, the boundary conditions derived in Chapter 3 hold always.

Examination of the next approximation will reveal additional discrepancies inherent in "even" truncation of moment approximations. The governing equations are listed in the second row of Table I. They are

$$\beta \frac{\partial I_o}{\partial t} + \frac{\partial q}{\partial x} = -\tau_e \{ \alpha_a^{(0)} I_o - 4\pi \alpha_e B \} \tag{4.44}$$

$$\beta \frac{\partial q}{\partial t} + \frac{\partial p^{(R)}}{\partial x} = -\tau_e \alpha_a^{(1)} q \tag{4.45}$$

$$\beta \frac{\partial p^{(R)}}{\partial t} + \frac{3}{5} \frac{\partial q}{\partial x} = -\tau_e \{ \alpha_a^{(2)} p^{(R)} - \frac{4\pi}{3} \alpha_e B \} \tag{4.46}$$

The characteristic directions and compatibility relations associated with these are

$$\frac{d^{\pm}}{dx} \{ p^{(R)} \pm \sqrt{\tfrac{3}{5}}\, q \} = -\sqrt{\tfrac{5}{3}} \{ [\alpha_a^{(2)} p^{(R)} \pm \sqrt{\tfrac{3}{5}}\, \alpha_a^{(1)} q] - \frac{4\pi}{3} \alpha_e B \} \tag{4.47a}$$

on

$$\frac{dx}{dt} = \pm \frac{1}{\beta} \sqrt{\tfrac{3}{5}} \tag{4.47b}$$

$$\beta \frac{d}{dt}\left[I_0 - \frac{5}{3}p^{(R)}\right] = -\tau_e\left\{\left[\alpha_a^{(1)} I_0 - \frac{5}{3}\alpha_a^{(2)} p^{(R)}\right] - \frac{16\pi}{9}\alpha_e B\right\} \quad (4.48a)$$

on

$$\chi = const. \quad (4.48b)$$

The limit $\beta \to 0$ is singular since it implies that the right hand side of Eq. (4.48a) must vanish. The system remains hyperbolic however, since I_0 may be eliminated completely in favor of $p^{(R)}, B$. Far from regions of disturbance the right hand sides of these equations must vanish. Since the field at these points is nearly isotropic $\alpha_a^{(0)} \to \alpha_a^{(1)} \to \alpha_a^{(2)} = \alpha_a \to \alpha_e$ so that

$$I_0 - \frac{5}{3}p^{(R)} = \frac{16\pi}{9}B \quad (4.49)$$

$$p^{(R)} - \sqrt{\frac{3}{5}}\,q = \frac{4\pi}{3}B \quad (4.50)$$

These two imply that

$$I_0 - \sqrt{\frac{5}{3}}\,q = 4\pi B \quad (4.51)$$

a form which is familiar. In the far field, isotropy ensues as $q \to 0$. To next order the characteristic directions are $\lambda = \pm\beta\sqrt{5 \pm \sqrt{40/3}}$ where the negative sign within the outer radical corresponds to the "light cone". All even orders possess a characteristic which carries information about the moments from far in the past, whereas other characteristics possess only a slight delay. When $\beta \to 0$ there are bits of information which cannot be accepted. The effects must eventually decay, hence the singular behavior is serious only on scales of $O(\beta)$.

Even though β is not entirely negligible, it is nevertheless small $\lesssim O(10^{-2})$. Therefore the time increments of a numerical scheme would have to be $O(\beta)$ and several time increments would be required before any effect would appear in fluid mechanics. Although $O(\beta)$ will henceforth be neglected, it is again stressed in the study of stellar structure or the initial stages of nuclear blasts $O(\beta)$ and $O(\beta/B_0)$ may not be entirely negligible.

Another consequence of the omission of $O(\beta)$ is best illustrated by the planar forms of Eq. (4.26).

$$\frac{\partial I_1}{\partial x} = \tau_e \left\{ 4\pi \alpha_e B - \alpha_a^{(0)} I_0 \right\} \tag{4.52a}$$

$$\frac{\partial I_2}{\partial x} = -\tau_e \cdot \alpha_a^{(1)} I_1 \tag{4.52b}$$

$$\frac{\partial I_3}{\partial x} = \tau_e \left\{ \frac{4\pi}{3} \alpha_e B - \alpha_a^{(2)} I_2 \right\} \tag{4.52c}$$

In the moment space these may be written in the form

$$\frac{\partial I_1}{\partial I_2} = \frac{\alpha_a^{(0)} I_0 - 4\pi \alpha_e B}{\alpha_a^{(1)} I_1} \tag{4.53a}$$

$$\frac{\partial I_2}{\partial I_3} = \frac{-\alpha_a^{(1)} I_1}{\frac{4\pi}{3} \alpha_e B - \alpha_a^{(2)} I_2} \tag{4.53b}$$

As an undisturbed state is approached, the numerators and denominators of these expressions vanish simultaneously, so that regions of radiative equilibrium are singular in the sense that to approach them a saddle point must be traversed. Therefore, arbitrarily small errors in I_0 at the wall will lead to arbitrarily large divergence in the far field. This is not to be confused with Hadamard's definition of an ill-posed problem which requires large changes in the solution everywhere for infinitesimal changes in data. It is noted that if all moments of the specific intensity were retained, the confluence of the saddles would be a regular point since the full transfer equation is regular as equilibrium is attained. Truncation of the moments forces one always to deal with saddle points. Sherman (1967) has recognized the "stiff" nature of these equations but his techniques for handling the saddle are inapplicable in the present treatment.

The domain of dependence of the solution at a given point in space-time now includes all of the past and the present. This is absolutely the largest domain of depencence allowable in any physical problem. The compatibility relations are

$$\frac{d^{\pm}p}{dt} \pm \rho a_f \frac{d^{\pm}u}{dt} =$$

$$= -\frac{\tau_\ell}{B_0} Q \frac{\rho_T|_{p,\bar{c}}}{\rho \rho_p|_{T,\bar{c}}} \left(\frac{\Gamma a_f^2 \rho_p|_{T,\bar{c}}}{A}\right) - j \frac{\rho u}{r \rho_p|_{T,\bar{c}}} \left(a_f^2 \rho_p|_{T,\bar{c}}\right) -$$

$$- \frac{\Gamma a_f^2 \rho_p|_{T,\bar{c}}}{A} \sum_i \left(\frac{W_i}{t_i^*}\right) \left\{ \frac{\rho_T|_{p,\bar{c}}}{\rho \rho_p|_{T,\bar{c}}} h_{c_i}\Big|_{p,T,c_j} - \frac{A \rho_{c_i}|_{p,T,c_j}}{\Gamma \rho} \right\} \tag{4.54a}$$

on

$$\frac{dr}{dt} = u \pm a_f \tag{4.54b}$$

$$\frac{dT}{dt} - \frac{\Gamma B}{\rho A} \frac{dp}{dt} = \frac{\tau_\ell}{B_0} \left(\frac{\Gamma}{\rho A}\right) Q + \frac{1}{\rho} \sum_i \left(\frac{W_i}{t_i^*}\right) \frac{\Gamma}{A} h_{c_i}\Big|_{p,T,c_j} \tag{4.54c}$$

and

$$\frac{dc_i}{dt} = -\frac{W_i}{t_i^* \rho} \tag{4.54d}$$

on

$$\frac{dr}{dt} = u \tag{4.54e}$$

and

$$\frac{dq}{dr} = -\left\{ \tau_\ell Q + j \frac{q}{r} \right\} \tag{4.55a}$$

$$\frac{dI_0}{dr} = -3 \tau_\ell \alpha_a^{(1)} q \tag{4.55b}$$

on

$$t = \text{const.} \tag{4.55c}$$

Of course Eq. (4.55a) is merely the energy equation considered previously.

The singular nature of the optically thick limit is illustrated by the characteristics approach. When the result $\alpha_a^{(0)} I_0 \simeq 4\pi \alpha_e B$ to $O(1/\tau_\ell)$ is applied, one of the compatibility relations is lost, and still another is required because $\mathrm{div}\,\vec{q}$ must be found. The problem then becomes truly parabolic, and a method of characteristics will never be suitable. No matter how small Q, Eq. (4.7), may be, its retention in the system through Eq. (4.55a) serves as a bridge between τ and $\vec{g}$ and renders the problem hyperbolic. Therefore $O(1/\tau_s^2)$ need always be retained in unsteady problems (and the propagation of shock waves falls into this category). The thin limit is not singular because two dependent variables and two compatibility relations are lost simultaneously. These results have never been examined in the manner above.

4.2.2 The Existence of Shock Waves – Generalized Rankine Hugoniot Relations

It is well known (see e.g., Garabedian, 1964) that hyperbolic systems of equations admit solutions with strong discontinuities, that is discontinuities in the variables themselves rather than just in derivatives. When such discontinuities occur, the derivatives of the dependent variables are locally ill-defined; hence the governing equations cannot be satisfied. Consider the following quasi-linear system of equations written in divergence form

$$\frac{\partial}{\partial t}\left\{\vec{P}^{(1)}(x,t,\vec{V})\right\} + \frac{\partial}{\partial x}\left\{\vec{P}^{(2)}(x,t,\vec{V})\right\} + \vec{n}(x,t,\vec{V}) = 0 \tag{4.56}$$

If the vector $\vec{V}$ is defined in some domain, D, a solution is weak if for any vector $\vec{v}$ which vanishes outside of some region of D and which has continuous first derivatives everywhere:

$$\iint_S \left\{ \vec{P}^{(1)} \cdot \frac{\partial \vec{v}}{\partial t} + \vec{P}^{(2)} \cdot \frac{\partial \vec{v}}{\partial x} - \vec{n}\cdot\vec{v} \right\} dt\, dx = 0 \tag{4.57}$$

If the line across which the discontinuity may exist is examined in detail and it is recognized that $\vec{v}$ is arbitrary, one finds that the line is determined by

$$\left.\frac{dx}{dt}\right|_i = \frac{[P_i^{(2)}]}{[P_i^{(1)}]} \tag{4.58}$$

where brackets denote the discontinuity in the variable of interest.

$$[\psi] = \psi_1 - \psi_2$$

If the equations were linear, these lines would in fact be characteristics. That the line of discontinuity is not a characteristic is a manifestation of nonlinearity. Equation (4.58) represent, of course, as many equations as there are elements of the dependent variable vector, $\vec{V}$; and these equations relate the discontinuities in all of the variables to the "slope" of the discontinuity. (For further details see Jeffrey and Taniuti, 1964, Ch. III; Courant and Hilbert, 1962, Vol. II, pp. 486-490).

Fortunately, the equations of gasdynamics, with or without radiation, may be cast in the form Eq. (4.56). This is fundamentally so because the equations may be derived from a control volume considerations which require the application of the divergence theorem in order for a system of purely differential equations to result. Since the Divergence (Gauss) Theorem may be applied only in regions in which the dependent variables have continuous first derivatives, it is obvious that the governing equations cannot apply near strong discontinuities. As was mentioned earlier, radiation and fluid mechanics are not coupled by derivatives, therefore the ordinary arguments for the existence of shock waves still hold. If $O(\beta)$ is included it follows that

$$\rho_1(u_1 - u_s) = \rho_2(u_2 - u_s) \tag{4.59}$$

$$\left(P_1 + \frac{1}{3}\frac{\beta}{B_0}I_{0_1}\right) + \rho_1 u_1(u_1 - u_s) = \left(P_2 + \frac{1}{3}\frac{\beta}{B_0}I_{0_2}\right) + \rho_2 u_2(u_2 - u_s) \tag{4.60}$$

$$\frac{\Gamma}{B_0}q_1 + \rho_1 H_1(u_1 - u_s) + \Gamma P_1 u_s + \Gamma\left(\frac{\beta}{B_0}\right)I_{0_1}\left(\frac{4}{3}u_1 - u_s\right) =$$

$$= \frac{\Gamma}{B_0}q_2 + \rho_2 H_2(u_2 - u_s) + \Gamma P_2 u_s + \Gamma\left(\frac{\beta}{B_0}\right)I_{0_2}\left(\frac{4}{3}u_2 - u_s\right) \tag{4.61a}$$

where

$$H = h + \frac{\Gamma}{2}u^2 . \tag{4.61b}$$

Also,

$$\begin{bmatrix} \beta u_s & -1 \\ -1 & 3\beta u_s \end{bmatrix} \begin{Bmatrix} I_{0_1} \\ q_1 \end{Bmatrix} = \begin{bmatrix} \beta u_s & -1 \\ -1 & 3\beta u_s \end{bmatrix} \begin{Bmatrix} I_{0_2} \\ q_2 \end{Bmatrix} \tag{4.62, 4.63}$$

where $U_s = \frac{dx}{dt}\Big|_s$ is the slope of the line across which the discontinuity occurs. This line is hereafter referred to as the shock wave and its slope, U_s, as the shock velocity. Equations (4.59), (4.60) and (4.61) were derived by Sachs (1946) and Marshak (1958) through quite different arguments. Equations (4.62) and (4.63) imply, however, that I_0 and g are continuous across allnonrelativistic shocks $U_s \neq \frac{1}{3\sqrt{3}}$. Therefore radiation pressure can do no work upon the discontinuity, and the energy density must be the same in both sides. This is readily justified since in the P_1 approximation the intensity I is a linear combination of I_0 and g. Physically, the specific intensity must be continuous across all nonrelativistic shock waves because the mean free path for photons is always much larger than that of material particles and the shock is a discontinuity on the latter scale. The heat flux must be continuous since there is no mechanism whereby energy may be dissipated in the (mathematical) discontinuity. Thus I_0 must be continuous as well. These arguments would not hold in a viscous, heat conducting gas, but as long as transport phenomena are neglected the ordinary Rankine-Hugoniot relations hold across a shock wave in a real, radiating gas if effects of order β/β_0 are neglected. If that order is important, the energy density of radiation is convected across the shock by the fluid at an effective velocity, $(\frac{4}{3}u - U_s)$.

It is easily demonstrated that the nonequilibrium chemistry must be frozen across the shock wave because the continuity and rate equations lead to

$$U_s = \frac{[\rho u c_i]}{[\rho c_i]} \qquad (4.64)$$

while continuity alone yields Eq. (4.59), which may also be written as

$$U_s = \frac{[\rho u]}{[\rho]} \qquad (4.65)$$

It is clear that all of the c_i must be continuous across the shock. Again, if transport phenomena were allowed, the existence of diffusive fluxes would obviate the approach, since the velocities of the individual species would not be the same as the mass velocity of the gas. These arguments hold for two-dimensional steady flow as well. Tirumalesa (1967) has transcribed the mathematical formulation. He fails to notice, however, that frozen flow must hold across any inviscid shock.

In summary, across a shock wave propagating at velocity u_s the following hold

$$\rho_1 (u_1 - u_s) = \rho_2 (u_2 - u_s) \tag{4.66}$$

$$P_1 + \rho_1 u_1 (u_1 - u_s) = P_2 + \rho_2 u_2 (u_2 - u_s) \tag{4.67}$$

$$\rho_1 H_1 (u_1 - u_s) + \Gamma P_1 u_s = \rho_2 H_2 (u_2 - u_s) + \Gamma P_2 u_s \tag{4.68}$$

$$I_{o_1} = I_{o_2} \tag{4.69}$$

$$g_1 = g_2 \tag{4.70}$$

$$(c_i)_1 = (c_i)_2 \qquad i = 1, N \tag{4.71}$$

Any assumptions other than frozen flow (Eq. (4.63)) across a transparent shock (Eq. (4.69), (4.70)) are inconsistent with the governing equations. Note that the shock wave does not propagate into an undisturbed medium. Since external sources of radiation are not allowed, the only way in which Eq. (4.69) and (4.70) may be satisfied is through a loss of energy through the shock.

The mathematical analysis is quite rigorous, but one must not forget the physical assumptions involved in practical problems. Although the restriction has not yet been made, it must often be assumed that LTE exists. In light of introductory remarks, almost every condition for the existence of LTE is violated within the shock wave. Collision dominance is obviously impossible. However, this local failure is insufficient to invalidate the solutions which one might obtain. Indeed, even the Navier-Stokes equations do not apply within the shock. Since concern lies only with what happens across the shock wave and not within it, sufficient justification exists for the assumption of LTE everywhere. Ultimate justification rests, however, with the investigation of non-LTE shock structure.

In this chapter it has been proved that the equations of RGD are hyperbolic and that methods based upon the use of characteristics may be applied to the insolution. The singular nature of the optically thick limit in unsteady RGD has been interpreted in terms of the hyperbolic-parabolic transition of the governing equations. It has been demonstrated that the one-dimensional formulation of RGD remains hyperbolic as $\beta \to 0$, but that saddle points simultaneously appear in regions of radiative equilibrium. The existence of shock waves in RGD has been verified, and consistent shock relations have been derived. All of this has been done with the use only of the hyperbolic nature of the equations. These observations must now be applied to the solution of realistic problems in unsteady RGD.

CHAPTER 5

THE INVESTIGATION OF A SPECIFIC PROBLEM

5.1 The Piston Problem in One Spatial Dimension

In order to determine the practical implications of the hyperbolic nature of unsteady radiation gasdynamics, the case of a planar, cylindrical, or spherical piston, impulsively started, and moving in an arbitrary manner into a gas which was initially at rest will be examined. The piston velocity is taken to be $\frac{dR_p}{dt} = U_p(t)$ which is normalized by the free stream isothermal speed of sound according to Eq. (3.10). The radiative aspects of the problem are emphasized through the omission of chemical nonequilibrium and consideration of a perfect gas. In this manner one is led to assume that LTE holds throughout, but the gas is allowed to be semi-gray. The piston problem is a model for various realistic flow situations. However, the greater the complication of the gas model, the more restricted is the class of analogies. At least one steady flow analog will be developed subsequently.

Nonlinear piston problems have been investigated in ideal gases (Taylor, 1946), in real gases (Chu, 1957), and even in radiating gases (Wang, 1963). Wang's investigation of the inverse problem is perhaps the most general to date. However, his analysis rests upon the omission of upstream absorption, the presence of only cold, black walls, and the assumption that the shock wave must be strong for radiation to be important. It will be shown (Chapter 6) that none of these are required in general. Above all, his search for similarity solutions forced the consideration of nonuniform upstream gases (just as did Romishevskii's, 1963,optically thin solutions). Marshak (1958) has pointed out that the stages of flow development which one observes through similarity solutions depend upon the approximations he is forced to make and that different similarity treatments of the same situation lead to results which differ even qualitatively. There is certainly sufficient justification for the examination of general piston problems in radiating gases – even though Wang has claimed that the similarity approach is the only way in which the nonlinear problem may be solved.

In this chapter the numerical procedures required for the solution problem described above will be formulated. Chemical equilibrium could easily be incorporated but it shall not be attempted at this time. It is admitted that the exclusion of nonequilibrium phenomena precludes direct prediction of real situations in all but a restricted range of the parameters of the problem $(B_0, \Gamma, \tau_\ell, \epsilon, T_w, U_p)$. In particular, any conclusions concerning upstream absorption of shock layer radiation must be made with the reservation that precursor ionization may play a significant role in the disposition of this energy.

The formulation of the problem requires Eqs. (4.1)-(4.9) in their one-dimensional form

$$P_t + uP_r - \frac{P}{T}(T_t + uT_r) + P(u_r + j\frac{u}{r}) = 0 \qquad (5.1a)$$

$$u_t + uu_r + \frac{T}{P}P_r = 0 \qquad (5.2a)$$

$$\frac{P}{\Gamma T}(T_t + uT_r) - (P_t + uP_r) = \frac{\tau_\ell}{B_0}Q \qquad (5.3a)$$

$$q_r = -\{\tau_\ell Q + j\frac{q}{r}\} \qquad (5.4a)$$

$$I_{0_r} = -3\tau_\ell \alpha_a^{(1)} q \qquad (5.5a)$$

where $$Q = \alpha_a^{(0)} I_0 - 4\alpha_p T^4$$

These are discarded in favor of the equivalent compatibility relations (Eqs. (4.54)-(4.55)) and the following thermodynamic and state equations for a calorically perfect gas.

$$\frac{d^+P}{dt} + \left(\frac{Pa}{T}\right)\frac{d^+u}{dt} = \frac{1}{(c_p/\Gamma - 1)}\left\{\frac{\tau_\ell}{B_0}Q - j\frac{c_p}{\Gamma}\frac{Pu}{r}\right\} \text{ on } \frac{dr}{dt} = u + a \quad (\Gamma_+) \quad (5.1b)$$

$$\frac{d^-P}{dt} - \left(\frac{Pa}{T}\right)\frac{d^-u}{dt} = \frac{1}{(c_p/\Gamma - 1)}\left\{\frac{\tau_\ell}{B_0}Q - j\frac{c_p}{\Gamma}\frac{Pu}{r}\right\} \text{ on } \frac{dr}{dt} = u - a \quad (\Gamma_-) \quad (5.2b)$$

$$\frac{d}{dt}\left\{\ln\left[\frac{P}{T^{(c_p/\Gamma)}}\right]\right\} = -\frac{\tau_\ell}{B_0}(Q/P) \text{ on } \frac{dr}{dt} = u \quad (\Gamma_0) \quad (5.3b)$$

$$\frac{dq}{dr} = -\left\{\tau_\ell Q + j\frac{q}{r}\right\} \quad (5.4b)$$

$$\frac{dI_0}{dr} = -3\tau_\ell \alpha_0^{(1)} q \quad (5.5b)$$

$$\rho = \frac{P}{T} \quad (5.6)$$

$$h = h_0 + \int_{T_0}^{T} c_p(T')dT' \quad (5.7)$$

where

$$a^2 = \left(\frac{c_p T}{c_p - \Gamma}\right) \quad (5.8)$$

The types of data which are necessary to completely specify the problem may be determined once the characteristics and the information which they convey have been recognized (Jeffrey and Taniuti, 1964, Ch.II) Eq. (5.4b) and (5.5b) indicate that initial data on radiation is irrelevant. However, special consideration must be given to times or order β. It follows that initial data is called for on the fluid mechanics P, u, T, boundary data on u or P, and boundary data on I_0 and q. Since the fluid must move with the piston, the surface is a particle path and data must be specified on a characteristic. This is why initial as well as boundary data on u are necessary. The fact that boundary data on I_0 and q are necessary is misleading, since all one knows is the relationship between the intensities $I^{(+)}$ and $I^{(-)}$ at the piston and far from it. The relationship is given in terms of $\mathcal{I}_o$ and $\mathcal{g}$ in Eq. (3.88) and (3.89); therefore a two-point boundary value problem must be solved. Unless the correct value of $\mathcal{I}_o$ (or $\mathcal{g}$) is chosen at the wall, the radiative saddle point in the gas "far" upstream cannot be traversed. Even though arbitrarily small errors in $\mathcal{I}_o$ at the wall will cause an arbitrarily large divergence from the true solution, the problem is not ill-posed because the smaller the error in initial data, the farther the computation may be carried into the flow field before divergence occurs. By Hadamard's definition, the solutions of ill-posed problems change by arbitrarily large amounts everywhere if arbitrarily small changes are made in initial data. Equations of the type which govern $\mathcal{I}_o$ and $\mathcal{g}$ are often classified as "stiff". The appearance of a two point boundary value problem in a characteristics scheme is apalling since the use of characteristics requires that one "march" away from data lines. To march from the free stream downward would require knowledge of the asymptotic behavior of derivatives since an undisturbed state is not unique to any particular piston motion; therefore iteration is required.

All nonlinear hyperbolic problems have in common the fact that their characteristic surfaces are unknown until the solution is obtained. The radiative equations are linear as a consequence of their foundation in Maxwell's equations, thus the radiative characteristics may always be drawn a priori. The gasdynamic characteristics must be determined simultaneously with the dependent variables, thus the system of equations is enlarged. The various numerical approximations which may be made in practice are considered next.

5.2 Numerical Approximations and the Inclusion of Upstream Absorption

5.2.1 Averaging Schemes

The compatibility relations and the equations which determine the characteristic curves are indicated symbolically by the fact that

$$a_{ij}\frac{du_j}{dt} = b_i\,(\vec{u}, r, t) \quad ; \quad i, j = 1, m \tag{5.9}$$

on the curve

$$\frac{dr}{dt} = \lambda_j (\vec{u}, r, t) \tag{5.10}$$

Suppose that Eq. (5.9) are integrated along the appropriate lines (Eq. (5.10)) from the points (r_j, t_{n-1}) from which characteristics emanate up to the point (r_1, t_n) at which properties are desired. These relations are of the form

$$\int_{t_{n-1}}^{t_n} a_{ij}\, du_j = \int_{t_{n-1}}^{t_n} b_i\, dt \tag{5.11}$$

$$r_j - r_1 = \int_{t_{n-1}}^{t_n} \lambda_j\, dt \tag{5.12}$$

An as yet unspecified quadrature formula is applied, and these are written as

$$(a_{ij})_{1j} (\Delta u_j)_j^1 = (b_i)_{1j} \Delta t \tag{5.13a}$$

$$(\Delta r)_j^1 = (\lambda_j)_{1j} \Delta t \tag{5.13b}$$

where $(\ )_{1,j}$ denotes some average along line (5.12) between its end points and

$$(\Delta u_j)_j^1 = u_j(r_1, t_n) - u_j(r_j, t_{n-1}) \tag{5.14}$$

The time increment of the difference scheme is Δt. Equations (5.13) and (5.14) now constitute $2m$ algebraic equations in $2m$ unknowns, the dependent variables $u_j(r_1, t_n)$ and m coordinates, which may be those of the points (r_j, t_{n-1}) if r_1 is specified or those of $(m-1)$ of those points and the point (r_1, t_n) if one of the r_j is specified. Inquiry is made first into the choice of averaging scheme.

Such a choice was made when a quadrature formula was needed in order to evaluate the radiation integrals. Here there is the disadvantage that only the values of the variables at the end points may be used.

Podney, et. al. (1966) have assumed that all quantities vary linearly along the characteristics. Consider the coefficient $\left(\frac{Pa}{T}\right)$ which by Eq. (5.8) is $\frac{P}{\sqrt{T}}(1-r)^{-\frac{1}{2}}$. If both P and T are assumed to vary linearly along whatever line is of interest, then

$$\left[\frac{Pa}{T}\right]_{1,j} = \left(\frac{1}{\Delta t}\right)\int_{t_{n-1}}^{t_n}\left\{\frac{P_1 + \frac{(\Delta P)_j^1}{\Delta t}(t-t_1)}{\sqrt{T_1 + \frac{(\Delta T)_j^1}{\Delta t}(t-t_1)}}\right\}dt$$

or

$$\left[\frac{Pa}{T}\right]_{1,j} = \left(\frac{Pa}{T}\right)_1\left\{\frac{2}{3}\left(1+2\frac{P_j}{P_1}\right)-2\frac{T_j}{T_1}\right\}\left(\frac{T_1}{(\Delta T)_j^1}\right)^2 +$$

$$+\left(\frac{Pa}{T}\right)_j\left\{\frac{2}{3}\left(1+2\frac{P_1}{P_j}\right)-2\frac{T_1}{T_j}\right\}\left(\frac{T_j}{(\Delta T)_j^1}\right)^2 \tag{5.15}$$

Division by the potentially small quantity $(\Delta T)_j^1$ is required. Similarly, for a quantity of the form (b/a), it follows that

$$\left[\frac{b}{a}\right]_{1,j} = \left(\frac{b_1-b_j}{a_1-a_j}\right) + \left\{b_j - a_j\frac{(b_1-b_j)}{(a_1-a_j)}\right\}\frac{\log\left(\frac{a_j}{a_1}\right)}{(a_1-a_j)} \tag{5.16}$$

which requires the quotients and differences of small quantities. Such formulae are undesirable for numerical analysis, and they present difficulty in the formulation of iterative schemes. A simpler formula in which the entire coefficient is assumed to vary linearly over the range (t_{n-1}, t_n) is,

$$[\psi]_{1,j} = \frac{1}{2}\{\psi_1 + \psi_j\} \tag{5.17}$$

In fact, for the quantity $\left(\frac{Pu}{r}\right)$, say, the difference between the formulae analogous to Eq. (5.15) or (5.16) and (5.17) is

$$\int_{t_{n-1}}^{t_n} \left(\frac{Pu}{r}\right) dt = \frac{1}{2}\left\{\left(\frac{Pu}{r}\right)_j + \left(\frac{Pu}{r}\right)_1\right\}\Delta t + \\ + O\left\{\frac{1}{3}(u_j + u_1)\left(\frac{1}{r_j} + \frac{1}{r_1}\right)(\Delta t)^2\right\} + \\ + O\left\{\frac{(\Delta P)_j'}{P_j}\Delta t,\ \frac{(\Delta u)_j'}{u_j}\Delta t\right\} \tag{5.18}$$

so that if is small enough the two are equivalent. Still a third method requires averages of all quantities in a given term. For instance

$$\left[\frac{Pu}{r}\right]_{1,j} = \frac{[P]_{1,j}\,[u]_{1,j}}{[r]_{1,j}} \tag{5.19}$$

where the brackets on the right hand side are evaluated according to Eq. (5.17) (since Eq. (5.17) and the method of Eqs. (5.15) and (5.16) are equivalent for the average of a single quantity). This method, which was used by Sussman (1966), cannot be justified in the sense of a quadrature formula applied to Eq. (5.11) and (5.12). As Chou and Karpp (1965) and Makino and Shear (1961) have shown, the errors associated with the last two methods are of the same order, $O(\Delta t)^2$. Their analysis is outlined in Appendix D*. This is as far as one can proceed without specifying the overall numerical procedure to be used.

5.2.2 The Computation of Properties at Field and Boundary Points

The method of characteristics as it is most often applied (see Hoskin, 1964) requires a characteristic mesh. That is, known points are allowed to determine the locations of points in the "future". If initial data is specified, the envelope of points determined by a characteristic mesh becomes increasingly skewed and the flow field last known

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* The advantage of the schemes indicated in Eq. (5.16) and (5.19) appears to be that no difficulty arises near the origin, , in cylindrical or spherical cases. None of the situations investigated exhibited anomalous behavior near the origin even though Eq. (5.17) was used because of compensating radiative and geometric effects in the compatibility relations. This is a minor point, and those who pursue similar problems in the future may choose whatever scheme they prefer.

consists of points scattered in space-time. Such a scheme is ill-suited for the radiation problem since it is imperative that a flow field be determined in all of space at a given time if the radiation is to be included in a self-consistent manner. Therefore the only choice available is the constant-time increment scheme developed by Hartree some fifteen years ago (Hartree, 1953, 1958). Recently Chou, Huang, and Karpp (1967) compared both methods in a simple isentropic flow. They found negligible difference between them. The basic computation scheme is outlined in Fig. 4 and is described below.

According to the scheme of Eq. (5.13) and (5.14), Eqs. (5.1b) – (5.5b) may be written in the following difference forms to .

$$(P_1 - P_3) - \left(\frac{Pa}{T}\right)_{13} (u_1 - u_3) = R^{(M)}_{13} \Delta t \tag{5.20}$$

$$(P_1 - P_4) + \left(\frac{Pa}{T}\right)_{14} (u_1 - u_4) = R^{(M)}_{14} \Delta t \tag{5.21}$$

$$\ln \left\{ \left(\frac{P_1}{P_2}\right) \frac{(T_2)^{c_{p_2}/R}}{(T_1)^{c_{p_1}/R}} \right\} = R^{(u)}_{12} \frac{\Delta t}{\left(\frac{c_p}{R}\right)_{12}} \tag{5.22}$$

$$q_1 - q_5 = - R^{(q)}_{15} \Delta r \tag{5.23}$$

$$I_{o_1} - I_{o_5} = - R^{(I)}_{15} \Delta r \tag{5.24}$$

where points 2, 3, and 4 are determined according to

$$r_3 - r_1 = -(u - a)_{13} \Delta t \tag{5.25}$$

$$r_2 - r_1 = -(u)_{12} \Delta t \tag{5.26}$$

$$r_4 - r_1 = -(u + a)_{14} \Delta t \tag{5.27}$$

and

$$R^{(M)} = \frac{1}{\left(\frac{c_p}{R} - 1\right)} \left\{ \frac{\tau_\ell}{B_0} Q - j \frac{c_p}{R} \left(\frac{Pu}{r}\right) \right\} \tag{5.28}$$

$$R^{(u)} = -\left(\frac{\tau_{\ell}}{B_0}\right)\left(\frac{Q}{P}\right) \tag{5.29}$$

$$R^{(g)} = \tau_{\ell}\, Q + j\left(\frac{g}{r}\right) \tag{5.30}$$

$$R^{(I)} = 3\tau_{\ell} d_a^{(1)} g \tag{5.31}$$

The spatial increment Δr in Eq. (5.23) and (5.24) and the time increment Δt are only weakly dependent upon each other, and the radiative equations need not necessarily be written in this manner. Equation (5.22) has been chosen over any other form of the path line relation because it requires no averaging on the left hand side and eliminates a possible source of error in the all-important temperature distribution. Equations (5.20) and (5.21) may immediately be solved for P_1, and U_1.

$$U_1 = \frac{U_3\left(\frac{Pa}{T}\right)_{13} + U_4\left(\frac{Pa}{T}\right)_{14} + \left[R_{14}^{(M)} - R_{13}^{(M)}\right]\Delta t + P_4 - P_3}{\left(\frac{Pa}{T}\right)_{13} + \left(\frac{Pa}{T}\right)_{14}} \tag{5.32}$$

$$P_1 = P_3 + \left(\frac{Pa}{T}\right)_{13}(U_1 - U_3) + R_{13}^{(M)}\Delta t \tag{5.33}$$

whence

$$T_1 = (T_2)^{\frac{C_{P_2}}{C_{P_1}}}\left\{\frac{P_1}{P_2}\, e^{-R_{12}^{(u)}\Delta t}\right\}^{R/C_{P_1}} \tag{5.34}$$

In the framework of the system of equations above, one is forced to assume properties at point 1 in order to initiate the calculation. Fewest assumptions need be made if the following general procedure is employed. If only a velocity and temperature are assumed at point 1, enough information exists to determine the locations of points 2, 3, and 4 to a first approximation by "searching" the data known at time t_{n-1}. The most obvious step once properties at these points are known is to assume that all of the average coefficients and inhomogeneous terms are simply the values at the respective "known" points. Then Eq. (5.32), (5.33) and (5.34) yield P_1, u_1, and T_1 independent of I_0 or g at 1 which are found from the properties at point 5 with Eq. (5.23) and (5.24) under the same assumption. Equations (5.32), (5.33), (5.34), (5.23), (5.24) may be written more clearly in the form

$$u_1^{(n)} = \frac{u_3^{(n-1)}\left(\frac{\rho a}{T}\right)_{13}^{(n-1)} + u_4^{(n-1)}\left(\frac{\rho a}{T}\right)_{14}^{(n-1)} + \left\{R_{14}^{(m)} - R_{13}^{(m)}\right\}^{(n-1)}\Delta t + P_4^{(n-1)} - P_3^{(n-1)}}{\left(\frac{\rho a}{T}\right)_{13}^{(n-1)} + \left(\frac{\rho a}{T}\right)_{14}^{(n-1)}} \tag{5.36}$$

$$P_1^{(n)} = P_3^{(n-1)} + \left(\frac{\rho a}{T}\right)_{13}^{(n-1)}\left\{u_1^{(n)} - u_3^{(n-1)}\right\} + R_{13}^{(M),(n-1)}\Delta t \tag{5.37}$$

$$T_1^{(n)} = \left\{T_2^{(n-1)}\right\}^{\left(\frac{C_{P_2}}{C_{P_1}}\right)^{(n-1)}}\left\{\left(\frac{P_1^{(n)}}{P_2^{(n-1)}}\right) e^{-R_{12}^{(u_1),(n-1)}\Delta t}\right\}^{\left(\frac{r}{C_{P_1}}\right)^{(n-1)}} \tag{5.38}$$

$$g_1^{(n)} = g_5 - R_{15}^{(g),(n-1)}\Delta t \tag{5.39}$$

$$I_{0_1}^{(n)} = I_{0_5} - R_{15}^{(I),(n-1)}\Delta t \tag{5.40}$$

where the superscript $(n-1)$ denotes quantities which depend upon a previous iteration. The superscript is necessary for the properties at points 2, 3, 4 as well, because these points move about in space as iteration progresses. In a characteristic mesh calculation, the points at the extremes of the domain of dependence at point 1 are fixed while point 1 and the $(m-2)$ points from which characteristics emanate move. Courant, Isaacon, Reës (1952) have demonstrated the convergence of the procedure outlined above. The nonlinearity is built up on each iteration, and the first step is a local linearization. At a point on the surface (Fig. 4b), the velocity is specified, but there is no Γ_+ characteristic available. Therefore, the relations along $\Gamma^{(-)}$ and $\Gamma^{(o)}$ are sufficient to determine the pressure and temperature through Eq. (5.37) and (5.38). However there is no point from "below" at which I_0 and q are known. All that is really available is the relation Eq. (3.88) between I_0 and q, thus Eq. (3.89), (3.92), or (3.94) become important. More will be said of this subsequently.

5.2.3 Computation of the Shock Point: The Inclusion of Upstream Absorption

The analysis of a point on the shock wave is by far the most important consequence of a characteristics treatment of one-dimensional RGD. From Fig. 4c and Eq. (4.66) – (4.71) it follows that fifteen simultaneous equations must be solved for P_1, u_1, T_1, I_{o1}, q_1, P_7, u_7, T_7, I_{o7}, u_7, and r_2, r_3, r_4, r_6, u_s. Equations (4.69) and (4.70) eliminate two of the unknowns trivially. The location of point 1 follows from the integration of

$$\left.\frac{dr}{dt}\right)_s = u_s(t) \tag{5.41}$$

which leads to

$$R_s(t_n) - R_s(t_{n-1}) = [u_s]_{t_n, t_{n-1}} \Delta t \tag{5.42}$$

As before, it is assumed that $[u_s]_{t_n,t_{n-1}} = u_s(t_n)$ to a first approximation. The relations along $\Gamma^{(-)}$, Γ^{o}, and $\Gamma^{(+)}$ from points 3, 2, and 4 immediately determine P_1, u_1, and T_1 if r_2, r_3, and r_4 are located according to the previous scheme. The Rankine-Hugoniot relations may be solved by inserting Eq. (5.6) and (5.7) into Eq. (4.66), (4.67) and (4.68). Equations (4.66) may be used to eliminate ρ_7 from Eq. (4.67) and (4.68) in favor of u_7 and u_s. Then Eq. (4.69) yields T_7 in terms of u_7 and u_s, whence Eq. (4.68) may be solved for $u_7(u_s, P_1, u_1, T_1)$ or $u_s(u_7, P_1, u_1, T_1)$. The results are

$$u_s(u_7, P_1, u_1, T_1) = \frac{1}{2}\left\{B_s + \sqrt{B_s^2 + 4C_s}\right\} \tag{5.43a}$$

where

$$B_5 = \frac{(1-\frac{\Gamma}{2})u_7^2 - \Gamma u_1 u_7 + (\frac{3}{2}\Gamma - 1)u_1^2}{(1-\Gamma)(u_7 - u_1)} \tag{5.43b}$$

$$C_5 = \frac{u_7[(T_1 + u_1^2) + (\frac{\Gamma}{2} - 1)u_1 u_7] - u_1[T_1 + \frac{\Gamma}{2}u_1^2]}{(1-\Gamma)(u_7 - u_1)} \tag{5.43c}$$

or

$$u_7 = \frac{1}{2}\left\{B_7 + \sqrt{B_7^2 + 4C_7}\right\} \tag{5.44a}$$

where

$$B_7 = \frac{(1-\Gamma)u_5^2 + \Gamma u_1 u_5 - (T_1 + u_1^2)}{(1-\Gamma/2)(u_5 - u_1)} \tag{5.44b}$$

$$C_7 = u_1\left\{\frac{u_5[u_5 - (\frac{3}{2}\Gamma - 1)u_1] + (T_1 + \frac{\Gamma}{2}u_1^2)}{(1-\Gamma/2)(u_5 - u_1)}\right\} \tag{5.44c}$$

Initial guesses for u_5, u_1, and T_1 were necessary to locate 1, hence 2, 3, and 4. Thus Eq. (5.44a-c) yield a consistent assumption for u_7 and T_7 which in turn allow a first estimate of the location of point 6. The analog of Eq. (5.21) may then be written along the $\Gamma^{(-)}$-characteristic 6-7.

$$P_7^{(n)} = P_6^{(n-1)} - \left(\frac{Pa}{T}\right)_{67}^{(n-1)}\left\{u_7^{(n)} - u_6^{(n-1)}\right\} + R_{67}^{(M),(n-1)}\Delta t \tag{5.45}$$

But Eq. (4.67) with Eq. (5.43) inserted in it is another equation for $P_7^{(n)}$ and $u_7^{(n)}$ which may be combined with Eq. (5.45) to yield the following nonlinear algebraic equation for $u_7^{(n)}$.

$$P_6^{(n-1)} - R_{67}^{(m),(n-1)} \Delta t - \left(\frac{P}{T}\right)_1^{(n-1)} \left\{T_1 + u_1^2\right\}^{(n-1)} + \left(\frac{Pa}{T}\right)_{67}^{(n-1)} u_6^{(n-1)} +$$

$$+ u_7^{(n)} \left\{\frac{P_1 u_1}{T_1} - \left(\frac{Pa}{T}\right)_{67}\right\}^{(n-1)} + u_s\left(u_7^{(n)}\right) \cdot \left(\frac{P_1}{T_1}\right)^{(n-1)} \left\{u_1^{(n-1)} - u_7^{(n)}\right\} = 0 \qquad (5.46)$$

where $u_s(u_7^{(n)})$ is given by Eq. (5.43). A new value of $R_s(t_n) = r_1$ may be determined with an averaged shock velocity and iteration may commence. I_0 and q still follow from Eq. (5.37) and (5.38). Similar procedures are used, for instance, in the determination of the flow field near the trailing shock wave of a slender body of finite length. A nonuniform upstream flow field may be handled with little more difficulty than a uniform one since even if P_1, u_1, T_1, are known, the nonlinear algebraic equation (5.46) must always be solved. The most striking advantage of the use characteristics in RGD is the ease with which upstream absorption may be included*.

5.2.4 The Overall Iterative Solution

The specific manner in which the radiative properties I_0 and q are to be found may now be considered. It has been decided that fluid dynamics determines P, u, and T at each point independent of the current iteration upon I_0 and q thereat**. This suggests two possible

* Note that $c_p = 1$ above hence the flow is chemically frozen everywhere. The inclusion of chemical equilibrium results in the loss of Eq. (4.71); hence discontinuities in the c_i are allowed. Equations (4.66), (4.67) and (4.68) must then be solved iteratively with an appropriate Mollier Diagram (i.e., $h = h(P,T)$ is known only implicitly). Although the formulation is not as "simple" as that above, the inclusion of chemical equilibrium should present no conceptual difficulties.

** This is not the only assumption possible, since I_{0_1} and q_1 may be found immediately if only a temperature is assumed at point 1 (T_s would be most appropriate). These may then be used in the averages of Eq. (5.36) and (5.37). Similarly Eq. (5.36) and (5.37) provide enough information that only T need be averaged in the solution of Eq. (5.38). The possible schemes are infinite in number, and they cannot all be mentioned. Any reasonable iterative scheme should converge.

iterative schemes. In the first (Method 1) one would assume a value for at the wall (say, with the aid of Eq. (3.89), (3.92), or (3.94) and the temperature distribution of the previous time increment). Then $\mathcal{I}_o$ and q would be determined at each point along with P, u and T to a first approximation by integrating upward. The various coefficients and inhomogeneous terms in the compatibility relations would be averaged, new points in the past located, and the equations iterated at each point individually in the standard manner. Of course, the saddle point could not be approached arbitrarily closely; thus when divergence in $\mathcal{I}_o$ or q appeared the calculation would be reinitiated at the boundary, this time with revised temperature and pressure distributions with which the integrals might be computed more accurately. The process would then be iterated, a whole flow field being required for each iteration.

The second (Method 2) makes best use of the $\mathcal{I}_o$ and q independence. Because of this independence one might calculate P, u, and T to a first approximation at each point in space. The temperature distribution could then be inserted in the integral boundary condition, and in the planar case $\mathcal{I}_o$ and q could be determined consistent at all points with P, u, T while leaving those properties unchanged. In spherical or cylindrical cases this procedure would be cumbersome, because a heat flux distribution would have to be assumed for insertion in the integrals (of course if $\alpha_\nu = const.$ the problem is simpler). However, in the planar case the independence allows solution of the integro-differential equation by the procedure above. This is why the symbol Q has been used in all of the relations. Instead of using Eq. (4.7) as the definition of Q one could just as easily use Eq. (2.70). Each iteration would yield a complete flow field to a consistent order of approximation whereas Method 1 could not be truncated arbitrarily.

Both methods are severely limited by the finite accuracy of numerical quadrature. In Method 1, the integral boundary condition cannot be computed accurately enough for the radiative field to approach equilibrium arbitrarily close. Therefore another method must be found to guess $\mathcal{I}_o$ at wall. In Method 2 errors compound in the heat flux because divq has been obtained from partial integration of Eq. (2.70). Thus, near regions of radiative equilibrium quantities of like order must be subtracted. Equation (2.70) could be integrated numerically to obtain g, but then triple and quadruple quadratures would be necessary and accuracy would be lost through interpolation and round off. The procedure would be very time consuming as well. Nevertheless, both Methods 1 and 2 have been investigated. Before solutions may be attempted, initiating procedures must be decided upon and, for Method 1, some approximate scheme for traversing the saddle point must be formulated.

5.3 Starting Procedures

5.3.1 The Existence of Entropy Layers

When radiation is important in the establishment of a flow field a "frozen flow" does not necessarily apply for "small" times as it does in the case of chemical nonequilibrium (Sussman, 1966; Sedney, South, and Gerber, 1962). Gradual acceleration of the piston in order to force the shock to form spontaneously is not practical. Even though the effect of initial conditions must decay exponentially along characteristics, the growth of "entropy layers" near the piston and the shock may be affected by starting solutions. These entropy layers are the manifestation of the history of the flow and in an analogous steady flow are due to vorticity production from finite shock radii of curvature and whatever nonequilibrium processes (chemical or radiative) are of interest. When radiation is considered, the entire concept of entropy layers must be re-evaluated because the thermodynamic system which consists of only the gas is open and energy may flow from the gas to the wall if the emissivity does not vanish. The system made up of only shocked gas is always an open one, since even without upstream absorption entropy leaks through the shock. Just as suction or blowing can change the properties of a viscous boundary layer, emission and absorption at the wall may thicken or bleed off an entropy layer. The only place that classical intuition may be trusted is at the shock, which is locally adiabatic. The only requirement on the flow across the shock is that entropy must increase. In addition the shock must propagate supersonically relative to the gas immediately ahead of it, and the isentropic Mach number immediately downstream must be subsonic. Because of the complicated nonadiabatic nature of the field, overall phenomena cannot be predicted intuitively. For instance if the piston were cold, black, and moving with constant velocity, the temperature of the gas at the wall would be very small (see e.g., Yoshikawa and Chapman, 1962). However, the velocity of the gas at the surface must be that of the piston; therefore, the flow somewhere behind the shock might be locally supersonic. However, the regions of influence of disturbances are not affected, since all disturbances are communicated everywhere instantaneously as $\beta \to 0$. This implies that Mach numbers are no longer meaningful; they account for the ratio of kinetic energy to thermal energy of random motion but completely ignore the energy which is always transit - the radiation. Wang (1965) fails to allow for this possibility in some of his conclusions.

Although the radiative field forms almost instantaneously beneath the fluid mechanics, it is conjectured that fluid mechanical shock formation predominates initially. At least for moderate wall temperatures, this assumption is justified because the shock is necessary to produce the elevated temperatures that make radiation possible. This is not the case in situations which involve large energy inputs (blast waves), but it should be true for piston problems. In addition, no matter how fast the piston moves, in a time of $O(\beta)$ only a small amount of energy has been introduced into the flow through work done on the piston. It is assumed, therefore, that the number of photons (none at all if external sources are excluded) is frozen at the initial instant $t = 0$. With this assumption the effect of initiating procedures upon the growth of entropy layers may be investigated.

Consider a time of $O(\beta)$ after the instant of shock formation. Within the frame of inner and outer expansions the inner region in which $t^* = t/\beta$ is of order unity may be developed by expanding all properties in asymptotic series. LTE should not hold on a scale during which few collisions occur, but it is accepted nevertheless. The gas is assumed a continuum as well. If β is sufficiently small, fluid mechanical properties cannot vary on this scale, and to first order P, U, and T are the same functions of the spatial coordinate which they were initially. The radiation is governed by the linear system (Eq. (4.43)) which is reproduced below in a somewhat modified form for a quasi-isotropic gas.

$$\left(\frac{\partial}{\partial \hat{\jmath}} \mp \frac{\partial}{\partial \hat{\tau}}\right) G_{\pm}(\hat{\tau},\hat{\jmath}) + G_{\pm}(\hat{\tau},\hat{\jmath}) = \hat{B}(\hat{\tau}) - j\,\frac{q}{\tau_\ell\, r\, \alpha_a} \tag{5.47}$$

where α_a, α_p, and T are their "initial" values as functions of r or x and

$$G_{\pm} = I_0 \pm q\sqrt{3} \tag{5.48a}$$

$$\hat{\tau} = \sqrt{3}\,\tau_\ell \int_0^r \alpha_a(\xi)\,d\xi \tag{5.48b}$$

$$\hat{\jmath} = \tau_\ell\, \alpha_a(r)\, t^* \tag{5.48c}$$

$$\hat{B} = 4\tau_\ell \frac{\alpha_p}{\alpha_a} T(\hat{\tau}) \tag{5.48d}$$

$$t^* = t/\beta \sim O(1) \tag{5.48e}$$

In the planar case (Fig. 5a)

$$G_+(\hat{\tau},\hat{\jmath}) = e^{-\hat{\jmath}} G_+(\hat{\tau}-\hat{\jmath},0) + e^{-\hat{\tau}} \int_{\hat{\tau}-\hat{\jmath}}^{\hat{\tau}} \hat{B}(\eta')e^{\eta'}d\eta' \tag{5.49a}$$

$$G_-(\hat{\tau},\hat{\jmath}) = e^{-\hat{\jmath}} G_-(\hat{\tau}+\hat{\jmath},0) + e^{\hat{\tau}} \int_{\hat{\tau}}^{\hat{\tau}+\hat{\jmath}} \hat{B}(\eta')e^{-\eta'}d\eta' \qquad \hat{\tau}-\hat{\jmath} > 0 \tag{5.49b}$$

$$G_+(\hat{\tau},\hat{\jmath}) = e^{-\hat{\jmath}} G_+(0,\hat{\jmath}-\hat{\tau}) + e^{-\hat{\tau}} \int_{\hat{\jmath}-\hat{\tau}}^{\hat{\tau}} \hat{B}(\eta')e^{\eta'}d\eta' \tag{5.50a}$$

$$G_-(\hat{\tau},\hat{\jmath}) = e^{-\hat{\jmath}} G_-(0,\hat{\tau}+\hat{\jmath}) + e^{\hat{\tau}} \int_{\hat{\tau}}^{\hat{\tau}+\hat{\jmath}} \hat{B}(\eta')e^{-\eta'}d\eta' \qquad \hat{\tau}-\hat{\jmath} < 0 \tag{5.50b}$$

where, since the lower boundary is at $R_p = \beta u_p t^* + O(\beta^2)$

$$G_+(0,\hat{\xi}) = 4\epsilon T_W^4 + (1-\epsilon)\, G_-(0,\hat{\xi}) \tag{5.51a}$$

$$G_-(0,\hat{\xi}) = e^{-\hat{\xi}} G_-(\hat{\xi},0) + \int_0^{\hat{\xi}} \hat{B}(\eta')e^{-\eta'}d\eta' \tag{5.51b}$$

The ordering of the series is in doubt (at least as far as the fluid mechanics in concerned) within distances of $O(\beta)$ of the piston and the shock and in these regions entropy layers begin to grow. In regions near the shock or piston in which $r_i^* = \frac{r_i - R_i(t)}{\beta} \sim O(1)$, where $i = s$ for shock or p for piston Eq. (5.1a)-(5.9a) may be written as follows

$$\left(\frac{P}{T}\right)_{t^*} + (u-\dot{R}_i)\left(\frac{P}{T}\right)_{r_i^*} + \left(\frac{P}{T}\right)\left\{u_{r_i^*} + j\frac{u}{r^* + \frac{R_i}{\beta}}\right\} = 0 \tag{5.52}$$

$$u_{t^*} + (u-\dot{R}_i)\, u_{r_i^*} = -\left(\frac{T}{P}\right) P_{r_i^*} \tag{5.53}$$

$$-\left\{P_{t^*} + (u-\dot{R}_i)P_{r_i^*}\right\} + \frac{P}{\mu T}\left\{T_{t^*} + (u-\dot{R}_i)T_{r^*}\right\} = \beta \frac{\tau_\ell}{B_o} Q \tag{5.54}$$

$$\beta\left\{I_{o_{t^*}} - \dot{R}_i I_{o_{r_i^*}}\right\} + \left\{q_{r_i^*} + j\frac{q}{r_i^* + R_i/\beta}\right\} = \beta\tau_\ell Q \tag{5.55}$$

$$\beta\left\{q_{t^*} - \dot{R}_i q_{r_i^*}\right\} + \frac{1}{3} I_{o_{r_i^*}} = -\beta\tau_\ell \alpha_a q \tag{5.56}$$

However, to lowest order

$$R_i = R_i(t_o) + \dot{R}_i(t_o)\cdot t + \cdots$$

$$= R_{o_i} + \beta t^* \dot{R}_{i_o} + O(\beta^2) \tag{5.57}$$

so that if

$$\hat{u}_i = u - \dot{R}(t^*) \qquad (5.58)$$

then

$$\frac{\partial \hat{u}}{\partial t^*} = \frac{\partial u}{\partial t^*} + O(\beta) \qquad (5.59)$$

This allows Eq. (5.52)-(5.56) to be written to lowest order as follows.

$$\left(\frac{P}{T}\right)_{t^*} + \hat{u}\left(\frac{P}{T}\right)_{r^*} + \frac{P}{T}u_{r^*} = O(\beta) \qquad (5.60)$$

$$\hat{u}_{t^*} + \hat{u}\,\hat{u}_{r^*} = -\frac{T}{P}P_{r^*} + O(\beta) \qquad (5.61)$$

$$\frac{P}{T^r} = \frac{P}{T^r}(r_i^* \to \pm\infty, t^*) + O(\beta) \qquad (5.62)$$

$$q_{r_i^*} = O(\beta) \qquad (5.62a)$$

$$I_{o r_i^*} = O(\beta) \qquad (5.62b)$$

But these are the ordinary equations of isentropic planar gasdynamics. Furthermore, the radiative quantities do not vary appreciably in these regions (which are below $\tau - \xi = 0$). In Fig. 5b the entropy levels of flow field in Regions I, above the shock, and II, between the shock and the piston, are specified completely by initial and boundary data. Regions IIIa, IIIb, and IV are isentropic but at different entropy levels in general. If solutions were required, the equation would first be integrated from $r_s^* \to \infty$, where data from Region I are known, down to $r_s^* = 0$. The Rankine Hugoniot relations apply to P, T, and $\hat{u}$, and $\dot{R}_s$ at this point, and an eigenvalue shock velocity would be determined such that the values of P, T, and $\hat{u}$ in Region II were approached as $r_s^* \to -\infty$. This would fix the entropy level in Region II, and that level would apply in Region IV as well. Region IV would then be determined by P and T in Region II $(r_p^* \to \infty)$ and by the fact that $\hat{U}(r_p^* \to 0, t^*) = 0$. Therefore Regions II, IIIa, IIIb, and IV would be coupled. If the current initiating assumption holds, the eigen shock velocity is the frozen valve, thus the entropy level of the entire initial solution is that of the nonradiating flow, and this specifies the entropy levels of the initial numerical solution which applies as $t^* \to \infty$. Therefore, the application of the appropriate nonradiating starting solution will not affect the growth of entropy layers within the limits of the present physical model.

5.3.2 Quasi-Steady Starting Solutions

5.3.2.1 Moderate Piston Velocities

In order to begin the numerics, initial flow fields — including radiation, if possible — are required. These will be applied at some time $0 \ll t_i \ll 1$. Upstream absorption may be neglected since the shock layer is extremely thin, and matters are further simplified by consideration of only a gray gas. First, frozen solutions are attempted in the usual manner (Sussman, 1966) by searching for a conical flow. If piston velocities are "moderate", $u_p \sim O(1)$, $T_W \sim O(1)$, $B_0 \gtrsim O(1)$, and the piston does not accelerate rapidly, $\dot{R}_p \gg \ddot{R}_p t$, the following series solution is appropriate.

$$P(\xi,t) = P_1(\xi) + tP_2(\xi) + \cdots \tag{5.63a}$$

$$T(\xi,t) = T_1(\xi) + t\,T_2(\xi) + \cdots \tag{5.63b}$$

$$u(\xi,t) = u_1(\xi) + t\,u_2(\xi) + \cdots \tag{5.63c}$$

$$q(\xi,t) = t\,q_1(\xi) + \cdots \tag{5.63d}$$

$$I_0(\xi,t) = 4 + t\sqrt{3}\,q_1(\xi_{Shock}) + t^2 I_{02} + \cdots \tag{5.63e}$$

The last relation is a consequence of Eq. (3.93) and the conical variable is

$$\xi = \left(\frac{r}{t}\right) \tag{5.64}$$

The starting solution will be nonself-consistent to this approximation. Equations (5.1a)-(5.9a) may be reduced to the following form to $O(\beta)$

$$\frac{dT_1}{d\xi} = -j\,\frac{\hat{\eta}_1\left\{\frac{\hat{\eta}_1}{\xi}+1\right\}}{D(\hat{\eta}_1,\xi)} \qquad (5.65)$$

$$\frac{d\hat{\eta}_1}{d\xi} = \frac{1}{D}\left\{-\frac{\hat{\eta}_1^2}{T_1}\left(\frac{1}{\Gamma}-1\right) + \frac{1}{\Gamma}\left(j\frac{\hat{\eta}_1}{\xi}+j+1\right)\right\} \qquad (5.66)$$

$$\frac{dq_1}{d\xi} = 4\tau_1\alpha_P\left\{T_1^4-1\right\} - j\,\frac{q_1}{\xi} \qquad (5.67)$$

$$\frac{dI_{01}}{d\xi} \sim o(t) \qquad (5.68)$$

where

$$D(\hat{\eta}_1,\xi) = \frac{\hat{\eta}_1^2}{T_1}\left\{\frac{1}{\Gamma}-1\right\} - \frac{1}{\Gamma} \qquad (5.69)$$

$$\hat{\eta}_1 = U_1 - \xi \qquad (5.70)$$

These are subject to

$$\hat{\eta}_1\,(\xi = U_p) = 0 \qquad (5.71)$$

and to the Rankine Hugoniot Equations (4.58)-(4.62) which, in the absence of upstream absorption take the form

$$U_s = \frac{1}{2}\left\{\left(\frac{1-\frac{\Gamma}{2}}{1-\Gamma}\right)U_{1(-)} + \left[\left(\frac{1-\frac{\Gamma}{2}}{1-\Gamma}\right)^2 U_{1(-)}^2 + \frac{4}{1-\Gamma}\right]^{1/2}\right\} \qquad (5.72)$$

$$P_1 = 1 + U_{1(-)}U_s \qquad (5.73)$$

$$T_1 = P_1\left(1 - \frac{U_{1(-)}}{U_s}\right) \qquad (5.74)$$

at $\xi = u_{1(-)}$. The planar problem may be solved trivially, but the cylindrical and spherical cases require numerical solution. A value of $u_{1(-)}$ must be "guessed" at $\xi = u_{1(-)}$ such that integration backwards to $\xi = u_p$ will lead to satisfaction of the condition $\hat{\eta}_1 = 0$. This is the same problem which was investigated by Taylor (1946). In light of forthcoming remarks concerning iteration schemes the solution may be easily carried out with linear influence coefficients. The numerical analysis is straight forward, and representative starting data are indicated in Figs. 15g, h, i, j.

Since the radiation is uncoupled from the fluid mechanics, Eq. (5.67) is readily solved subject to Eq. (3.88) if

$$T_w(t) = 1 + \dot{T}_w(0) \cdot t + O(t^2) \tag{5.75}$$

Equations (3.88) and (5.63d, e) indicate the following two point boundary condition

$$(q_1)_{Shock} + \left(\frac{2-\epsilon}{\epsilon}\right)(q_1)_{Wall} = \frac{16}{\sqrt{3}}\,\dot{T}_w \tag{5.76}$$

The heat flux is merely a quadrature

$$q_1(\xi) = \frac{1}{\xi^j}\left\{\int_{u_p}^{\xi} \xi^j f(\xi)\,d\xi + \frac{\epsilon\left[\frac{16}{\sqrt{3}}\dot{T}_w - u_s^{-j}\int_{u_p}^{u_s} \xi^j f(\xi)\,d\xi\right]}{\epsilon u_s^{-j} + (2-\epsilon)\,u_p^{-j}}\right\} \tag{5.77}$$

where

$$f(\xi) = 4\tau_p \alpha_p \{T_1^4 - 1\} \tag{5.78}$$

In the planar case, $j = 0$

$$q_1 = 4\tau_p \alpha_p \{T_1^4 - 1\}\left\{(\xi - u_s) + \left(1 - \frac{\epsilon}{2}\right)(u_s - u_p)\right\} + \frac{8}{\sqrt{3}}\,\epsilon\,\dot{T}_w \tag{5.79}$$

where T_2 is found from Eq. (5.74) with $u_{(-)} = u_p$.

The solutions obtained by the procedures above are more nearly correct the smaller the time at which they are applied. However, there is a limit beyond which the initiating time cannot be decreased, since if $t \sim O(\beta)$ the situation is clearly different. This is in contrast to flows with chemical nonequilibrium where the uncoupled solution must be correct as $t \to 0$. Any scheme will eventually predict a very small heat flux for small t, and intuitively this is correct.

5.3.2.2 Large Piston Velocities: The Assumption of Impulsive Radiation

The previous solution certainly does not hold if the piston moves rapidly. Suppose that $u_p \equiv \frac{1}{\epsilon_2} >> 1$ and that for $t \sim O(\epsilon_2)$ the piston path is approximated by its tangent. For simplicity the gas is taken to be gray and the absorption coefficient is described by

$$\alpha = p^{b_1} T^{b_2} \tag{5.80}$$

Consider the scaling

$$t = \bar{t} f(\epsilon_2) \ , \quad 0 < f(\epsilon_2) \ll 1 \tag{5.81}$$

Since $R_p < r < R_s$ and $u_s/u_p \sim O(1)$ as a result of the Rankine-Hugonoit equations, it must be assumed that

$$r = \bar{r} \frac{f(\epsilon_2)}{\epsilon_2} \tag{5.82}$$

so that, for small times

$$\bar{t} \leq \bar{r} \leq \beta_s \bar{t} \tag{5.83}$$

where

$$\beta_s = \frac{u_s}{u_p} \sim O(1) \tag{5.84}$$

The Rankine-Hugoniot relations also require that

$$P - 1 = \frac{\bar{P}}{\epsilon_2^2} \quad , \quad \bar{P} \sim O(1) \tag{5.85}$$

$$T - \left(\frac{r}{2-r}\right) = \frac{\bar{T}}{\epsilon_2^2} \quad , \quad \bar{T} \sim O(1) \tag{5.86}$$

since

$$U = \bar{u}/\epsilon_2 \quad , \quad \bar{u} \sim O(1)$$

This states merely that P and T behave as Mach number squared. Suppose that

$$I_0 - 4 = \bar{I}_0 \cdot h(\epsilon_2) \tag{5.87}$$

$$q = \bar{q} \cdot g(\epsilon_2) \tag{5.88}$$

then Eq. (5.4) and (5.7) in conjunction with Eq. (5.80), (5.81), (5.82), (5.85), and (5.86) lead to

$$\left(\frac{h}{f}\right)\left(\frac{\epsilon_2 g}{h}\right)\left(\bar{q}_{\bar{r}} + j\,\frac{\bar{q}}{\bar{r}}\right) \sim 4\,\bar{P}^{b_1}\,\bar{T}^{b_2+4} \Big/ \epsilon_2^{\,2(b_1+b_2)+8} \tag{5.89}$$

$$\left(\frac{\epsilon_2 h}{f}\right) \bar{I}_{0_{\bar{r}}} \sim \frac{g}{\epsilon^{2(b_1+b_2)}}\; \bar{P}^{b_1}\,\bar{T}^{b_2}\,\bar{q} \tag{5.90}$$

hence

$$g(\epsilon_2) = f(\epsilon_2) \Big/ \epsilon_2^{\,2(b_1+b_2)+9} \tag{5.91}$$

$$h(\epsilon_2) = f^2(\epsilon_2) \Big/ \epsilon_2^{\,4(b_1+b_2)+10} \tag{5.92}$$

The energy equation indicates that

$$\left(\frac{\bar{P}}{\epsilon_2^2}\right) \frac{(\bar{T}_{\bar{t}} + \bar{u}\,\bar{T}_{\bar{r}})}{\epsilon_2^2 f(\epsilon_2)} \sim \frac{\epsilon_2\, g(\epsilon_2)}{B_0\, f(\epsilon_2)} \left(\bar{q}_{\bar{r}} + j\,\frac{\bar{q}}{\bar{r}}\right) \tag{5.93}$$

Therefore, the scaling is completely determined

$$g = B_0/\epsilon_2^3 \tag{5.94}$$

$$f = B_0\,\epsilon_2^{2(b_1+b_2)+6} \tag{5.95}$$

$$h = B_0^2\,\epsilon_2^2 \tag{5.96}$$

These results are summarized in the following exact analogs of Eqs. (5.1a)-(5.9a)

$$\left(\bar{T}+\frac{\epsilon_2^2\Gamma}{2-\Gamma}\right)(\bar{P}_{\bar{t}}+\bar{u}\bar{P}_{\bar{r}})-(\bar{P}+\epsilon_2^2)(\bar{T}_{\bar{t}}+\bar{u}\bar{T}_{\bar{r}})+(\bar{P}+\epsilon_2^2)(\bar{T}+\epsilon_2^2)\left(\bar{u}_{\bar{r}}+j\frac{\bar{u}}{\bar{r}}\right)=0 \tag{5.97}$$

$$(\bar{P}+\epsilon_2^2)(\bar{u}_{\bar{t}}+\bar{u}\bar{u}_{\bar{r}}) = -\left(\bar{T}+\epsilon_2^2\frac{\Gamma}{2-\Gamma}\right)\bar{P}_{\bar{r}} \tag{5.98}$$

$$-\left(\bar{T}+\epsilon_2^2\frac{\Gamma}{2-\Gamma}\right)(\bar{P}_{\bar{t}}+\bar{u}\bar{P}_{\bar{r}})+\frac{(\bar{P}+\epsilon_2^2)}{\Gamma}(\bar{T}_{\bar{t}}+\bar{u}\bar{T}_{\bar{r}})=-\left(\bar{T}+\epsilon_2^2\frac{\Gamma}{2-\Gamma}\right)\left(\bar{q}_{\bar{r}}+j\frac{\bar{q}}{\bar{r}}\right) \tag{5.99}$$

$$\bar{q}_{\bar{r}}+j\frac{\bar{q}}{\bar{r}} = \tau_e(\bar{P}+\epsilon_2^2)^{b_1}\left(\bar{T}+\epsilon_2^2\frac{\Gamma}{2-\Gamma}\right)^{b_2}\left\{4(\bar{T}+\epsilon_2^2)^4-4\epsilon_2^8-B_0^2\epsilon_2^{10}\bar{I}_0\right\} \tag{5.100}$$

$$\bar{I}_{0_{\bar{r}}} = -3\tau_e(\bar{P}+\epsilon_2^2)^{b_1}\left(\bar{T}+\epsilon_2^2\frac{\Gamma}{2-\Gamma}\right)^{b_2}\bar{q} \tag{5.101}$$

Subject at $\bar{r}=\beta_s\bar{t}$ to

$$\bar{T} = \left(\frac{1-\Gamma}{1-\frac{\Gamma}{2}}\right)\left[\beta_s^2-\frac{\epsilon_2^2}{1-\Gamma}\right]\left[\frac{\Gamma}{2-\Gamma}+\frac{\epsilon_2^2}{\beta_s^2(1-\Gamma)}\right]+\frac{\epsilon_2^4}{\beta_s^2(1-\Gamma/2)} \tag{5.102a}$$

$$\bar{P} = \left(\frac{1-\Gamma}{1-\Gamma/2}\right)\left(\beta_s^2-\frac{\epsilon_2^2}{1-\Gamma}\right) \tag{5.102b}$$

$$\bar{u} = \left(\frac{1-\Gamma}{1-\frac{\Gamma}{2}}\right)\left(\beta_s-\frac{\epsilon_2^2}{\beta_s(1-\Gamma)}\right) \tag{5.102c}$$

$$B_0\,\epsilon_2^5\,\bar{I}_0 = \bar{q}\sqrt{3}+\frac{4\epsilon_2^3}{B_0} \tag{5.102d}$$

and at $\bar{r} = \bar{t}$

$$\bar{u} = 1 \tag{5.103a}$$

$$B_o \epsilon_2^5 \bar{I}_o + \left(\frac{2-\epsilon}{\epsilon}\right)\sqrt{3}\, \bar{q} = \frac{4\epsilon_2^2}{B_o}(T_w^4 - 1) \tag{5.103b}$$

where

$$r = B_o \cdot \epsilon_2^{5+2(b_1+b_2)} \bar{r} \tag{5.103c}$$

$$t = B_o \epsilon_2^{6+2(b_1+b_2)} \bar{t} \tag{5.103d}$$

To lowest order Eq. (5.100) is

$$div\, \bar{q} = \tau_e \bar{p}^{b_1} \bar{T}^{b_2} \cdot 4\bar{T}^4 + O(\epsilon_2^{2(b_1+b_2+4)}) \tag{5.104}$$

which is merely the optically "thin" result, Eq. (2.45). In fact, reabsorption is $O(\epsilon^{10})$! Furthermore, the energy lost to the wall is of order ϵ_2^2 at most if $T_w \lesssim O(1)$. Romishevskii (1963) has sought similarity solutions for the system Eq. (5.97)-(5.101) to lowest order. Such solutions are possible for the boundary conditions above if and only if $\Gamma = 0,\ (\gamma = 1)$. Wang (1963) has verified these statements. In fact, if the previous conical flow is assumed then, near the piston $(J - u_p \sim \epsilon_2)\frac{dT_1}{T_1} \sim \frac{d\hat{q}_1}{\hat{q}_1}$, so that all of the energy of a fluid element would be radiated away as it approached the surface. The "thin" equations apply here even though the normalized absorption coefficient is "large" $[\alpha \sim O(\epsilon_2^{-2(b_1+b_2)}) \gg 1]$. The flow is not really "thin" but is more appropriately emission dominated.

The optically thin equations cannot be solved as trivially as the conical Eqs. (5.65)-(5.67) could. Therefore, it is not worthwhile to pursue the present approach if $u_p \gg 1$. The predictions of the conical equations were poor in some cases; therefore because of the labor involved for all but moderate piston velocities it was best to assume that radiation was turned on impulsively at the initiating instant, t_i. This is accomplished by forcing α_ν to vanish for all ν at times $t < t_i$ so that initial distributions of I_o and q are immaterial. In accordance with the previous section, "impulsive radiation" will not affect the growth of entropy layers as long as $t_i \ll 1$ and the initial solution is the correct nonradiating one rather than one arbitrarily chosen. Unfortunately these statements must be examined more closely in nonplanar geometries.

Since a specific radiative behavior will be imposed upon the flow initially, the time required for the field to "forget" must be predicted. Use of a nonradiating solution at time t_i produces a shock layer with too much energy. Once radiation is begun this energy, which is $\sim \rho E (U_s - U_p) t_i$, is radiated away. The time required is

$$t_R \sim \frac{\langle P \rangle (U_s - U_p)}{\Gamma B_0 [\langle T^4 \rangle - 1]} \cdot t_i \qquad (5.105)$$

where the brackets denote some average levels of pressure and temperature within the shock layer. Naturally, it must take longer to discard the excess energy the longer one waits to turn the radiation on. Also, if is small, the free stream is hot and will resist the energy loss from the shock. It is expected that the flow field will be rather unpredictable for $t < t_R$. Despite all of these arguments the effect of initiating time upon the resulting solutions must be investigated. Before this can be done the manner in which the saddle point is to be dealt with must be decided upon.

5.4 Passage Through the Saddle Point

It was pointed out in Chapter 4 that the approach to radiative equilibrium is singular if $t \gg \beta$. Equations (4.55) and (4.56) clearly indicate that this behavior is not a result of any approximations other than the omission of terms of $O(\beta)$. Truncation of the system of moment equations aggravates the phenomenon. Since the problem is self consistent, the method outlined by Curtiss and Hirshfelder (1952) is not very useful.* A "movable upper boundary" technique, similar to that used by Scala and Gordon (1967), has been adopted.

* See Appendix G

Suppose that a value of I_o at the piston surface, I_{ow}, has been "guessed" and that integration has proceeded upward to some point at which an undisturbed situation has nearly been attained. A parameter is provided for perturbation if it is assumed that $\beta \ll t \sim o(\epsilon_3) \ll 1$. Since the far field will be affected mainly by radiation and not by convection, Eq. (5.1a)-(5.5a) indicate that if

$$(I_o - 4) \sim q \sim O(\epsilon_3)$$

then

$$O(P-1) \sim O(T-1) \sim O(\epsilon_3^2)$$

$$O(u) \lesssim O(\epsilon_3^2)$$

A point, $r_o(t)$, may always be chosen so that these conditions hold as long as I_{ow} is not too much in error. It may be verified that a consistent perturbation results if

$$I_o = 4 + \epsilon_3 I_o' + \cdots \tag{5.106a}$$

$$q = \epsilon_3 q' + \cdots \tag{5.106b}$$

$$P = 1 + \epsilon_3^2 P' + \cdots \tag{5.106c}$$

$$T = 1 + \epsilon_3^2 T' + \cdots \tag{5.106d}$$

$$u = \epsilon_3^3 (B_o/\tau_\ell) u' + \cdots \tag{5.106e}$$

Equations (5.4a)-(5.5a) then lead to

$$q_r' + j\frac{q'}{r} = -\tau_\ell I_o' + O(\epsilon_3) \tag{5.107}$$

$$I_{o\,r}' = -3\tau_\ell q' + O(\epsilon_3) \tag{5.108}$$

hence

$$I_{o\,rr}' + \frac{j}{r} I_{o\,r}' - 3\tau_\ell^2 I_o' = 0 \tag{5.109}$$

which is of standard form. However, this equation admits solutions which are exponentially divergent. For instance, if $j=0$ the solution is, naturally

$$I_o' = A_1 e^{-\sqrt{3}\tau_e x} + B_1 e^{\sqrt{3}\tau_e x} \tag{5.110a}$$

$$\sqrt{3}\, q' = A_1 e^{-\sqrt{3}\tau_e x} - B_1 e^{\sqrt{3}\tau_e x} \tag{5.110b}$$

Since integration upward from the boundary predicts both $I_o(r_o)$ and $q(r_o)$, both A_1 and B_1 may exist. Unless the condition

$$I_o'(r_o) = q'(r_o)\cdot\sqrt{3} + O(\epsilon_3) \tag{5.111}$$

is satisfied B_1 will not vanish, and the solution must diverge. This is an alternate statement of the boundary condition, Eq. (3.93). It reflects the fact that once the integration has reached a station which is effectively undisturbed, the gas from there onward radiates as a black body. Condition (5.111) may be generalized for cylindrical and spherical geometries as follows

$$(I_o - 4)F_j(r_o) - \sqrt{3}\, q(r_o) - O(\epsilon_3^2) \tag{5.112}$$

$$F_j(r_o) = \begin{cases} 1 & j=0 \\ K_1(\sqrt{3}\tau_e r_o)/K_o(\sqrt{3}\tau_e r_o) & j=1 \\ \left(1 + \frac{1}{\sqrt{3}\tau_e r_o}\right) & j=2 \end{cases} \tag{5.113a, b, c}$$

where the K_n are modified Bessel functions. The additional geometric factors are a consequence of the fact that nowhere has it been assumed that $r_o \gg 1$ *.

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* If upstream absorption is neglected, F_j is unity for all j and r_o is the shock location. If upstream emission is ignored the 4 in $(I_o - 4)$ must be deleted. Unless upstream emission is deleted radiative equilibrium is not attained in the free stream and there is, in fact, no saddle.

Analogously

$$I_0' = \left\{ \frac{I_0(r_0,t) - 4}{\epsilon_3} \right\} G_j(r,r_0) \qquad (5.114)$$

$$g' = \left\{ \frac{g(r_0,t)}{\epsilon_3} \right\} G_j(r,r_0) \qquad (5.115)$$

where

$$G_j(r,r_0) = \begin{cases} e^{-\sqrt{3}\tau_\ell (r-r_0)} & j=0 \\ K_0(\sqrt{3}\tau_\ell r)/K_0(\sqrt{3}\tau_\ell r_0) & j=1 \\ \left(\frac{r_0}{r}\right) e^{-\sqrt{3}\tau_\ell (r-r_0)} & j=2 \end{cases} \qquad (5.116a,b,c)$$

At this point a further assumption is necessary, since if $t \sim O(1)$ there is no small parameter available. Consider, however, that for $5/3 > \gamma > 9/7$ $.4 > \Gamma > .2$ while as $\gamma \to 1$, $\Gamma \to 0$*. Therefore, one may make use of the fact that Γ is reasonably small (Moore, 1964, 1966; Vicenti and Baldwin, 1962). When $\epsilon_3 = \Gamma$ the orders of approximation of P and u are larger than before

$$P_{(\Gamma \ll 1)} = 1 + \epsilon_3 P' + \cdots \qquad (5.117a)$$

$$u_{(\Gamma \ll 1)} = \epsilon_3^2 \left(\frac{B_0}{\tau_\ell}\right) u' + \cdots \qquad (5.117b)$$

In general

$$P' = \left[\frac{\Gamma}{1-\Gamma}\right]^k \int_0^{(t/\epsilon_3^k)} I_0'(r, \epsilon_3^k t') \, dt' \qquad (5.118)$$

$$u' = \left[\frac{\Gamma}{1-\Gamma}\right]^k \int_0^{(t/\epsilon_3^k)} d\xi \int_0^{(\xi/\epsilon_3^k)} dt' \frac{\partial I_0'}{\partial r}(r, \epsilon_3^k t') \qquad (5.119)$$

---

*If $\Gamma \to 0$ the radiative and gasdynamic fields are in fact always uncoupled (see Section 7.2).

where $k=1$ if $t \sim O(\epsilon_3)$, $\Gamma \sim O(1)$ and $k=0$ if $\Gamma \sim O(\epsilon_3)$, $t \sim O(1)$.

As these relations stand, the point $r_0(t)$ at which Eq. (5.112) is satisfied for a given value of ϵ_3 must be found and the integrals must be evaluated in Eq. (5.118) and (5.119). This is not necessary, since if $r_0(t)$ is known only approximately, an ϵ may be chosen to match. With negligible additional error, it may be assumed that

$$\frac{u(r)}{u(r_0)} = \frac{P(r)-1}{P(r_0)-1} \approx \frac{T(r)-1}{T(r_0)-1} \approx \frac{I_0(r)-4}{I_0(r_0)-4} \approx \frac{g(r)}{g(r_0)} \approx G_j(r, r_0) \;,\; r > r_0 \tag{5.120}$$

The relations above are not really necessary since the quantities which they yield are most always small. Obviously the numerical integration could be carried to infinity if I_{0_W} were known, but that certainly is not the pragmatic approach. Condition (5.112) indicates when the solution may be terminated safely. Furthermore it proves that boundary conditions must be chosen consistent with the governing equations since if they were not divergence from radiative equilibrium in the far field would be inevitable.

Although the accuracy with which the integral expressions Eq. (3.89), (3.92), or (3.94) may be found is limited, they may provide good initial estimates for I_{0_W}. Conditions (5.112) provide a single parameter whose behavior will indicate how good a given value of I_{0W} may be. One must solve the equation

$$\chi = [I_0(r_0) - 4]\, F_j\{r_0(I_{0_W})\} - \sqrt{3}\, g(r_0) = O(\epsilon_3^2) \tag{5.121}$$

where now the explicit dependence upon I_{0_W} may be indicated. Naturally, iteration is required, and each iteration involves an entire flow field. The most obvious approach is to assume that χ is linear in I_{0W} so that the method of linear influence coefficients (implicit Newton-Rhapson iteration) may be employed. New values of I_{0W} could be obtained from

$$(I_{0W})^{(n+1)} = (I_{0W})^{(n)} - \frac{\chi(I_{0W}^{(n)})}{\left(\frac{\partial \chi}{\partial I_{0W}}\right)^{(n)}} \tag{5.122}$$

where the derivative is approximated by a difference quotient. Unfortunately, χ was found to be a highly nonlinear function of I_{ow}; therefore, the method of Xerikos and Anderson (1962) was applied. The scheme was devised in conjunction with the sonic singularity in the integral approach to blunt body perfect gas flows, but the techniques are appropriate to many situations with saddle points.

The method involves bracketing the correct value of I_{ow} and then computing a new guess from Eq. (5.122). Since χ is in general nonlinear, these guesses may be greater than the high bracket or less than the low one. If this be the case, the next guess is taken as half the sum of the high and low brackets. The process is continued until a sufficiently small portion of the $\chi(I_{ow})$ curve has been isolated wherein the variation is indeed linear. In the problem at hand, the first guess was taken either from the integral or an approximation to it drawn from the value of I_{ow} at a previous time. The second guess was made arbitrarily larger or smaller than the first (depending upon whether $I_{ow}^{(1)}$ was too small or large) so that a bracket could be found. Once the bracket was obtained, the next two guesses were, typically, forced to halve. This led to $\chi \sim O(10^{-3})$ after which influence coefficients were appropriate, and accuracy was improved by an order of magnitude upon each successive iteration. Normally four to five iterations were sufficient not only to achieve reasonable asymptotic behavior but also to cause the flow field within the shock layer and the shock slope to be reasonably insensitive to the value of I_{ow}. In no case were more than ten iterations required.

Physical intuition determines whether a guess at I_{ow} is too large or too small. Recall that I_o and q are continuous across the shock wave but that temperature, hence $div(\vec{q})$, is not. Since there are no external sources of radiation, $div(\vec{q})$ must change sign near the upstream side of the shock. Suppose, without loss of generality, that the shock layer is everywhere hotter than the free stream. The larger I_{ow} is taken, the algebraically smaller will q_w be. The growth of I_o is determined by the initial value of the heat flux; therefore, if I_o is too large at the wall it will be even worse at the shock so that $div(\vec{q})$ might not change sign. It is proposed that the integration be carried out beyond the shock until either the divergence of the heat flux or the flux itself change sign from that which they had immediately upstream of the shock. If χ at that point is positive, I_{ow} was chosen too large, and conversely. Analogous arguments hold when the wall is cold and the shock layer may be cooler than the undisturbed gas (upstream "cooling" was observed by Chien, 1967).

There are several different ways in which χ may be written. As Eq. (5.121) is stated it seems that quantities which are in fact of the same order will be subtracted and this can lead to a loss of accuracy. Several different forms were investigated. Two possibilities are $\chi_1 = \chi / g(\tau_o)$ or $\chi_2 = \dfrac{\chi}{I_o(\tau_o) - q}$. No one form converged more rapidly or gave more accurate results than any other, therefore Eq. (5.121) was retained. As stated, an asymptotic solution is theoretically redundant because one should always be able to guess a better value of I_{ow} and thus carry the solution a bit further. The limits of the computer are rapidly approached in this manner, since I_{ow} must be accurate within $10^{-5}\%$ if χ is to be $\lesssim 10^{-4}$. Solutions which joined with the asymptotic forms, Eq. (5.120), were compared with others in which I_{ow} was iterated into double precision. It was confirmed that the saving in computer time and storage more than balanced the loss of accuracy (which is $O(\epsilon_3^3)$). Thus all of the techniques necessary for one to attempt the numerical investigation of radiating piston problems have been developed. Before the detailed investigation is outlined, the methods currently available for the solution of these problems will be reviewed, and their steady flow analogs will be developed.

5.5 The Linearized Analysis

The predictions of linearized theories in gasdynamics are generally quite accurate and have revealed many interesting phenomena. The linearized analysis of ordinary gasdynamics (OGD) is quite simple since only the two velocity variables need be considered, and they may be combined into a single velocity potential. Pressures follow from Bernoulli's equation and temperatures, if they are necessarily, from the constancy of total enthalpy. When nonequilibrium chemistry is included, the analysis becomes much more complicated since at least one more dependent variable need be accounted for. Instead of the second order Prandtl-Glauert equations of OGD, a third order equation governs the velocity potential (see e.g., Sussman, 1966; Vincenti and Kruger, 1965). Although the planar case has previously been investigated (Lick, 1964; Cheng, 1964a,b; Moore, 1966; Vincenti and Baldwin, 1962; Baldwin, 1962) in this section it will be shown that the equations of RGD (to the P_1- approximation) lead to a linearized problem which is governed by a fifth order partial differential equation. The refinement offered is the consideration of a semi-grey gas and the trivial inclusion of cylindrical and spherical geometries. Rhyming (1965) has considered weakly nongrey situations in diatomic gases, and Baldwin (1962) has indicated how a general nongrey planar theory might be carried out.

Suppose that the piston is inserted (or retracted) in a manner such that displacement from its initial position and deviation of its wall temperature from unity are never large for $t \sim O(1)$.

$$R_p(t) = \delta_1 f(t) + R_0 \qquad 0 < \delta_1 \ll 1 \tag{5.123}$$
$$T_W(t) = 1 + \delta_1 T_W'(t)$$

All quantities are linearized about their undisturbed values as follows

$$\begin{aligned} P &= 1 + \delta_1 P' + \cdots & \alpha_a &= 1 + \delta_1 \alpha_a'(P', T') + \cdots \\ u &= \delta_1 u' + \cdots & \alpha_p &= 1 + \delta_1 \alpha_p'(P', T') + \cdots \\ T &= 1 + \delta_1 T' + \cdots & I_0 &= 4 + \delta_1 I_0' + \cdots \end{aligned} \tag{5.124}$$
$$q = \delta_1 q' + \cdots$$
$$\alpha_R = \frac{1}{\bar{\alpha}} + \delta_1 \alpha_R' + \cdots, \quad \bar{\alpha} = \alpha_{p_\infty}^* / \alpha_{R_\infty}^*$$

These are consistent with the governing equations (5.1b)-5.9b) which now become, to lowest order

$$P_t' - T_t' + u_r' + j\frac{u'}{r'} = 0 \tag{5.125}$$

$$u_t' + P_r' = 0 \tag{5.126}$$

$$\frac{1}{\Gamma} T_t' - P_t' = \frac{\tau_e}{\mathcal{B}_0} Q' \tag{5.127}$$

$$q_r' + j\frac{q'}{r} = -\tau_e Q' \tag{5.128}$$

$$I_{0\,r}' = -3\tau_e q' \tag{5.129}$$

where the parameters $\mathcal{B}$, τ_e, and Γ are assumed to be at most of order unity. From Eq. (2.53) it follows that

$$\alpha_a' = \alpha_p' + 4(1-\bar{\alpha})T' \tag{5.130}$$

In general $\bar{\alpha} > 1$ (in fact Scala and Sampson, 1963, are quite fond of $\bar{\alpha} = 83$, Goody, 1960, suggests that for meteorological applications $\bar{\alpha} \sim 100$). Furthermore

$$Q' = I_0' - 16T' + 4(\alpha_a' - \alpha_p')$$

hence, with Eq. (5.130)

$$Q' = I_0' - 16\bar{\alpha}T' \tag{5.131}$$

Therefore, the essential effects of a nongrey gas appear through the difference between the Planck and Rosseland means. The stream function, ψ, and heat flux potential, ϕ are introduced through

$$u' = \psi_r \tag{5.132}$$

$$q' = \phi_r \tag{5.133}$$

so that Eq. (5.126) and (5.129) lead to

$$p' = -\psi_t \tag{5.134}$$

$$I_0' = -3\tau_e \phi \tag{5.135}$$

if the field is initially undisturbed. The remaining equations are

$$\nabla^2\psi - \psi_{tt} = T'_t \tag{5.136}$$

$$\nabla^2\psi - \frac{1}{\gamma}\psi_{tt} = -\frac{\Gamma\tau_\ell}{B_0}(3\tau_\ell\phi + 16\bar{\alpha}T') \tag{5.137}$$

$$\nabla^2\phi - 3\tau_\ell^2\phi = 16\bar{\alpha}\tau_\ell T' \tag{5.138}$$

where $\nabla^2 = \frac{1}{r^j}\frac{\partial}{\partial r}\left(r^j\frac{\partial}{\partial r}\right)$ as usual. Note that the operations upon ψ in Eq. (5.136) and (5.137) are isothermal and isentropic Prandtl-Glauert operators respectively. (The isothermal wave speed is unity and the isentropic one is $\sqrt{\gamma}$ in the current normalization). In accordance with the accepted notation these operators are

$$W_T(\psi) = \left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\psi \tag{5.139}$$

$$W_S(\psi) = \left(\nabla^2 - \frac{1}{\gamma}\frac{\partial^2}{\partial t^2}\right)\psi \tag{5.140}$$

Although Eq. (5.136)-(5.138) are sufficient, they may be combined in a single equation. Application of the operation ∇^2 to Eq. (5.137) and the use Eq. (5.138) allows elimination of ϕ. Differentiation of the result with respect to time, permits removal of the temperature perturbation through Eq. (5.136) with the result

$$\frac{\partial}{\partial t}(\nabla^2 W_S) - 3\tau_\ell^2\frac{\partial W_S}{\partial t} + \left(\frac{16\Gamma\tau_\ell}{B_0}\right)\bar{\alpha}\,\nabla^2 W_T = 0 \tag{5.141}$$

If Traugott's equations (1965) had been used the factor $\bar{\alpha}$ would have appeared in the second term rather than the last one. These equations are subject to the boundary conditions

$$\psi_r = \dot{f}(t) \tag{5.142}$$

$$-3\tau_1\varphi + \left(\frac{2-\epsilon}{\epsilon}\right)\sqrt{3}\varphi_r = 16\dot{T}_w(t) \quad , \quad at \quad r = R_o \tag{5.143}$$

and

$$\psi, \varphi \text{ finite}; \quad \psi_r, \varphi_r \to 0 \quad as \quad r \to \infty \tag{5.144}$$

where $R_o = 1$ in all but the planar case, in which $R_o = 0$ without loss of generality.

As an aside, it is mentioned that the linearized analysis of meteorological problems $(\bar{\alpha} \gg 1)$ is considerably simplified when quasi-isotropic, semi-grey radiative transfer is employed. Equation (5.138) requires that $T' \sim O(\delta_1)$ so that $\bar{\alpha}T' \sim O(1)$. Then Eq. (5.136) forces the gasdynamic field to be determined only of isothermal propagation. Therefore ψ may be readily determined whence Eq. (5.137) allows φ to be found in terms of ψ. The analysis is straightforward in all respects.

The linear model includes none of the interaction of the nonlinear problem. For instance, the characteristics Eq. (4.44b), (4.45c) become straight lines, therefore individual "pulses" may never coalesce to form a shock wave. They must always coalesce in reality, thus the omission of their curvature leads to nonuniformity in the far field. In Section 4.2.2 it was indicated that strong discontinuities are still allowed across the leading Mach wave. Therefore, one must allow for the possibility of discontinuity relations such as Moore (1966) employs and Baldwin (1962) justifies rigorously. Linearized solutions are in general not uniformly valid for large times or distances, and more sophisticated perturbation procedures will be required.

The system (Eq. (5.136)-(5.138)) as incorporated in Eq. (5.141) has been analyzed on many occasions. If one assumes that

$$\psi = \psi_o \cdot \left\{ \begin{matrix} e^{ikx} \\ H_o^{(1)}(kr) \\ \frac{1}{r} e^{ikr} \end{matrix} \right\} e^{-i\omega t} \qquad \begin{matrix} j=0 \\ j=1 \\ j=2 \end{matrix} \tag{5.145}$$

where $H_0^{(1)}$ is the Hankel function of the first kind, the dispersion relation obtained by Vincenti and Baldwin follows. In their notation

$$C_1^4\left(1-2i\frac{\beta_1\mathcal{K}}{\gamma}\right)+C_1^2\left\{1-\beta_1^2-2i\beta_1\mathcal{K}\right\}-\beta_1^2=0 \tag{5.146}$$

where

$$C_1=i\left(\frac{k}{\omega}\right) \tag{5.147a}$$

$$\beta_1=\sqrt{3}\,\gamma\left(\frac{\tau_e}{\omega}\right) \tag{5.147b}$$

$$\mathcal{K}=\frac{8}{\sqrt{3}}\left(\frac{\Gamma\mathcal{I}}{B_0}\right) \tag{5.147c}$$

All comments made by Vincenti and Baldwin are appropriate without modification. They apply directly to pistons which oscillate, $f(t)\sim e^{-i\omega t}$, so that only a single Fourier component may participate. Vincenti and Baldwin were forced to assume that $\Gamma \ll 1$ in order to perform their analysis. Cheng's wavy wall analysis (1965a, b) proceeds in the same manner. The effect of the steady flow may easily be inferred from the unsteady analysis if a Gallilean transformation is applied.

Baldwin has investigated the equations by a Laplace transform method as have Lick (1964) and Moore (1966). Unfortunately they have investigated only the case of an impulsively started and instantaneously stopped black wall, $f(t)=H(t)=\begin{cases}0, t<0\\ 1, t>0\end{cases}$. This case is particularly simple because all Fourier components participate equally. It does, however, have a discontinuous solution. The phenomena due to a pulsed wall temperature for a stationary surface have been investigated by Solan and Cohen (1966). The case of a general $f(t)$ has not yet been investigated since, even though the Laplace transform solution is easily obtained, its inversion to the physical plane is almost impossible. This is illustrated if Eq. (5.136)-(5.138) are considered rather than Eq. (5.141) since the boundary conditions are obviously better suited to those.

If the ordinary Laplace transform is performed upon Eq. (5.136)-(5.138), T' may be eliminated. The result is

$$g(p)\nabla^2\bar{\psi} - p^2\{g(p)+\gamma\Gamma p\}\bar{\psi} - \frac{3\tau_e^2}{B_0}\Gamma p\bar{\phi} = 0 \tag{5.148}$$

$$g(p)\nabla^2\bar{\phi} - p\cdot 3\tau_e^2\bar{\phi} - 16\Gamma\tau_e\alpha\left(\frac{1}{\gamma}p^2\right)\bar{\psi} = 0 \tag{5.149}$$

$$\overline{T'} = \frac{\Gamma}{g(p)}\left\{\left(\frac{1}{\gamma}\right)p^2\bar{\psi} - \frac{3\tau_e^2}{B_0}\bar{\phi}\right\} \tag{5.150}$$

where

$$g(p) = p + \frac{16\Gamma\tau_e\alpha}{B_0} \tag{5.151}$$

and

$$\bar{\psi}(r,p) = \int_0^\infty \psi(r,t)e^{-pt}dt \tag{5.152}$$

for instance. These are subject to

$$\bar{\psi}_r = p\bar{f}(p) \tag{5.153}$$

$$-3\tau_e\bar{\phi} + \left(\frac{2-\epsilon}{\epsilon}\right)\sqrt{3}\,\bar{\phi}_r = 16\,\overline{T'_w}(p)\,, \quad at\ r=R_0 \tag{5.154}$$

$$\bar{\psi},\bar{\phi}\ \text{finite},\quad \bar{\psi}_r,\bar{\phi}_r \to 0 \quad as \quad r\to\infty$$

Since the effect of geometry appears only through K_0 Bessel functions or an additional $(1/r)$ decay, attention is confined to the planar case. It is understood that $\tau_e = 1$ because the planar case lacks a physical length scale. Obviously Eq. (5.148) and (5.149) admit solutions of the form

$$\overline{\psi} = \sum_i \overline{\psi}_i e^{-\kappa_i x} \tag{5.155}$$

$$\overline{\phi} = \sum_i \overline{\phi}_i e^{-\kappa_i x} \tag{5.156}$$

then

$$\begin{bmatrix} \left\{ \kappa^2 - p^2 \left[1 + \frac{\gamma \Gamma p}{g} \right] \right\} & \frac{3\tau_e^2}{B_0} \frac{\Gamma p}{g} \\ -\frac{16 \Gamma \tau_e}{\gamma} \alpha \left(\frac{p^2}{g} \right) & \left\{ \kappa^2 - \frac{3\tau_e^2 p}{g(P)} \right\} \end{bmatrix} \begin{Bmatrix} \overline{\psi}_i \\ \overline{\phi}_i \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{5.157}$$

so that nontrivial solutions exist if and only if the determinant of the coefficient matrix vanishes.

Of the four roots above, at least two will yield solutions which do not behave properly as $x \to \infty$. Then Eq. (5.157) relates the $\overline{\psi}_i$ to the $\overline{\phi}_{i,\, i=1,2}$ after which Eq. (5.153) and (5.154) allow solution for either $\overline{\psi}_1$ and $\overline{\psi}_2$ or $\overline{\phi}_1$ and $\overline{\phi}_2$. The expressions for the κ_i's, $\overline{\phi}_i$'s, and $\overline{\psi}_i$'s themselves are cumbersome and without additional simplifying assumptions one cannot hope to handle the problem analytically. Since almost every simplification has been pursued extensively and because the purpose of this investigation is not the exhaustive analysis of linearized problems, the matter will not be pursued further.

It would, of course, be interesting to compare the linearized and nonlinear solutions of the same problem just as Sussman did when considering chemical nonequilibrium. In the present investigation the flow field does not terminate at the wavehead as his did; therefore only asymptotic solutions are available without considerable labor, and general comparison is inappropriate at the present time. It has been argued that large time solutions to the linearized problem are not valid unless special precautions are taken.

Also, large time solutions to nonlinear problems are not worthwhile because they closely parallel the study of inviscid radiating shock structure which is well understood. It was discovered that the nonlinear solution of situations which would allow linearization possessed noticeble radiative fields for small times only if the Boltzmann number were small. Examination of $g(p)$, Eq. (5.151) reveals that small time (large P) approximate linear solutions are possible in all cases except $B_0 \ll 1$. Similarly, any approximation based upon $B_0 \ll 1$ will not be valid for small times. The small time behavior is easily obtained if $B_0 \gtrsim O(1)$ since the radiative and fluid mechanical fields are almost uncoupled. If and $\frac{\Gamma p^3}{B_0 g^2} \ll 1$ (i.e., $t \ll 1$)

$$\bar{\psi} \simeq -\frac{1}{\gamma} F(p) \sqrt{\frac{p}{p+A_B}}\, e^{-\sqrt{p(p+A_B)}\,(x/\sqrt{\gamma})}, \quad A_B = \frac{16\Gamma^2 \tau_0 \bar{T}}{B_0} \tag{5.158}$$

$$\bar{\phi} \simeq -\frac{16}{3\tau_e} \frac{\epsilon \overline{T_w'(p)}}{\left\{\epsilon + (2-\epsilon)\sqrt{\frac{p}{g(p)}}\right\}}\, e^{-\sqrt{3}\,\tau_e x \sqrt{p/g(p)}} \tag{5.159}$$

Note that since only one root remains in each of $\bar{\phi}$ and $\bar{\psi}$, $\bar{\phi}$ must vanish if the wall temperature is unperturbed. That is, the fluid mechanics has not had time to influence the radiation whereas the radiation has some influence upon the fluid mechanics. This casts some doubt upon the "initiating" assumption which claimed that just the opposite must happen. The anomoly is easily resolved, because if $p^3/B_0 g^2$ is neglected relative to K^4 then A_B should neglect relative to P, in which case an ordinary uncoupled flow results. The inversion of Eq. (5.158) will be indicated so that there is at least some comparison available.

$$\psi \simeq -\frac{1}{\gamma} e^{-\frac{A_B}{2} x \sqrt{\gamma}} \int_0^{t - x/\sqrt{\gamma}} f\left(t-\tau-\frac{x}{\sqrt{\gamma}}\right) e^{-\frac{A_B}{2}\tau}\, I_0\left\{\frac{1}{2} A_B \sqrt{\tau\left(\tau + \frac{2x}{\sqrt{\gamma}}\right)}\right\} d\tau \tag{5.160}$$

There is no upstream tail to this approximation because pulsed wall temperatures are not considered in this analysis, the inversion of

Eq. (5.159) is not of interest. From Eq. (5.157) it may readily be inferred that for very small or very large P, the wave head travels at the isentropic propagation speed. At intermediate times it slows to nearly the isothermal speed and then accelerates to the isentropic speed when $t \gg 1$.

Because closed form linear analyses of general piston paths are prohibitively difficult, the numerical investigation of nonlinear piston motions is clearly necessary. Furthermore, piston histories which allow linearization produce hardly enough radiative to cause noticeable effects in practice. Even though piston insertions bear a close relation to shock tube phenomena, it is instructive to think in terms of a realistic steady situation. In the next section the appropriate analogy is derived.

5.6 A Hypersonic Piston Analogy

Piston analogies of ideal and real gas problems have been developed in considerable detail (Hayes and Probstein, 1959; Sussman, 1966), but an analysis of this type in a radiating gas has not yet been approached rigorously. Khosla (1966) has developed the method in linearized situations and his analogy compares quite favorably with Cheng's two dimensional solutions (Cheng, 1965a,b). It is hoped that the following analysis will prove at least as useful for nonlinear problems.

Since the previous normalization Eq. (3.10), is inappropriate for steady situations, one must begin with the dimensional equations (Eq. (3.1)-(3.4)) for a general gas.

$$\rho^*_p)_{T,\bar{c}} \frac{DP^*}{Dt^*} + \rho^*_T|_{P,\bar{c}} \frac{DT^*}{Dt^*} + \rho^* \operatorname{div} \vec{u}^* = \sum_i \rho^*_{c_i}|_{P,T,c_j} \left(\frac{W_i^*}{\rho}\right) \tag{5.161}$$

$$\rho^* \frac{D\vec{u}^*}{Dt^*} + \operatorname{grad}(P^*) = 0 \tag{5.162}$$

$$\{\rho^* h^*_p|_{T,\bar{c}} - 1\} \frac{DP^*}{Dt^*} + \rho^* h^*_T|_{P,\bar{c}} \frac{DT^*}{Dt^*} = -\operatorname{div} \vec{q}^* + \sum_i h^*_{c_i}|_{P,T,c_j} W_i^* \tag{5.163}$$

$$\operatorname{div} \vec{q}^{\,*} = 4\alpha_e^{*} B^{*} - \alpha_a^{(0)*} I_0^{*} \tag{5.164}$$

$$\operatorname{grad} I_0^{*} = -3\alpha_a^{(1)*} \vec{q}^{\,*} \tag{5.165}$$

$$\rho^{*} \frac{Dc_i}{Dt^{*}} = - W_i^{*}(\rho^{*}, T^{*}, \vec{c}) \tag{5.166}$$

where

$$\frac{D}{Dt^{*}} = u^{*}\frac{\partial}{\partial x} + v^{*}\frac{\partial}{\partial r}$$

Consider a slender body $B(x,r) = r - \delta R_p(x) = 0$.

The conditions under which a steady analog of one-dimensional unsteady flow exists will be determined by the assumptions which must be made to force the steady equations into the unsteady form. Since the shock layer is thin, the following may immediately be assumed.

$$t = x^{*}/\ell^{*} \tag{5.167}$$

$$r = r^{*}/\delta\ell^{*} \tag{5.168}$$

$$u = u^{*}/u_\infty^{*} \tag{5.169a}$$

$$v = v^{*}/\delta \cdot u_\infty^{*} \tag{5.169b}$$

The parameter δ may be, for instance, the fineness ratio of the slender body of interest, $0 < \delta << 1$. Normalization of the continuity equation introduces no factors of order $o(\delta)$. Therefore if

$$\rho = \rho^{*}/\rho_\infty^{*} \tag{5.170a}$$

$$p = p^{*}/p_\infty^{*} \tag{5.170b}$$

$$T = T^*/T_\infty^* \tag{5.170c}$$

$$W_i = W_i^*/\rho_\infty^* t_{i_\infty}^* \tag{5.170d}$$

the continuity equation is

$$\rho_P)_{T,c} \frac{DP}{Dt} + \rho_T)_{P,c} \frac{DT}{Dt} + \rho \, \mathrm{div}\, \vec{u} = \sum_i \rho_{c_i})_{P,T,c_j} \frac{W_i}{\rho t_i^*} \tag{5.171}$$

where

$$t_i^* = u_\infty^* t_{i_\infty}^* / \ell$$

The normalization of most of the variables,is irrelevant so far, since the expression is homogeneous in all but the dimensions of velocity and length. Since the oncoming axial flow predominates,

$$u = 1 + \delta \bar{u} \tag{5.172}$$

at most. Then Eq. (5.171) corresponds with the unsteady continuity equation if transverse motion is identified with that of the piston. To this approximation $\frac{D}{Dt} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial r}$.

The momentum equations (5.162) and (5.165) are

$$u u_t + v u_r = -\frac{1}{\gamma M_\infty^2} \left(\frac{P_t}{\rho}\right) \tag{5.173}$$

$$u v_t + v v_r = -\frac{1}{\gamma (M_\infty \delta)^2} \left(\frac{P_r}{\rho}\right) \tag{5.174}$$

where a perfect gas at infinity has been assumed. Under assumption Eq. (5.172), (5.174) will correspond with the unsteady momentum equation if

$$P = \gamma K_H^2 \bar{P} \tag{5.175}$$

where $K_H = M_{f_\infty} \delta$ must be of order unity. This implies, however, that the information obtained from Eq. (5.173) will yield higher order corrections to the axial field and it is neglected for now. If h, I_o, α_a, α_p and q_1, q_2 are normalized as follows

$$h = h^* / C_{p_{f_\infty}}^* T_\infty^* \tag{5.176a}$$

$$I_o = I_o^* / \sigma T_\infty^{*4} \tag{5.176b}$$

$$\alpha_{a,p} = \alpha_{a,p}^* / \alpha_{p_\infty}^* \tag{5.176c}$$

$$q_{1,2} = q_{1,2}^* / \sigma T_\infty^{*4} \tag{5.176d}$$

the energy equation is

$$\left\{ \frac{\rho h_p|_{T,c}}{P} - 1 \right\} \frac{DP}{Dt} + \frac{\rho h_T|_{p,\tilde{c}}}{P} \frac{DT}{Dt} = \frac{\tau_\ell \delta}{K_H \sqrt{\delta} (B_o)_\infty} \left\{ \alpha_a^{(\omega)} I_o - 4 \alpha_p B \right\} + \sum_i h_{c_i}|_{P,T,c_j} \left(\frac{w_i}{t_i^*} \right) \tag{5.177a}$$

where

$$(B_o)_\infty = \frac{\rho_\infty^* (R T_\infty^*)^{3/2}}{\sigma T_\infty^{*4}} \tag{5.177b}$$

Clearly it must be assumed that

$$T = \delta K_H^2 \bar{T} \tag{5.178a}$$

$$I_o = \delta^4 K_H^8 \bar{I_o} \tag{5.178b}$$

Thus Eq. (5.177) corresponds with its unsteady analog if

$$\frac{\tau_\ell \delta}{K_H \sqrt{\delta} (B_o)_s} = \frac{(\tau_\ell)_u}{(B_o)_u} \tag{5.178c}$$

where the subscripts s, u denote the corresponding parameters of the steady and unsteady problems. The unsteady situation is assumed to possess a characteristics time, t_c^*, at which the motion is terminated, corresponding to the finite length of a body in steady flow. The unsteady Bouguer number is, therefore

$$(\tau_\ell)_u = t_c^* \sqrt{RT_\infty^*}\, \alpha_{p_\infty}^* \tag{5.178d}$$

The radiation governing equations must now be examined. Normalization is no longer arbitrary, therefore Eq. (5.165), (5.166) and (5.167) become

$$\delta(q_1)_t + (q_2)_r + j\frac{q_2}{r} = -\delta\tau_{\ell_s}\left(\alpha_a^{(0)} I_0 - 4\alpha_e B\right) \tag{5.179}$$

$$I_{0_t} = -3\tau_{\ell_s}\alpha_a^{(1)} q_1 \tag{5.180}$$

$$I_{0_r} = -3(\delta\tau_{\ell_s})\alpha_a^{(1)} q_2 \tag{5.181}$$

Equation (5.181) indicates that steady-unsteady correspondence ensues if

$$\delta\tau_{\ell_s} = \tau_{\ell_u} \tag{5.182}$$

and

$$q_2 = \bar{q}_2 \cdot \gamma^4 K^8 \tag{5.183a}$$

Then Eq. (5.180) indicates that

$$q_1 = \bar{q}_1 \cdot \delta \cdot \gamma^4 K^8 \tag{5.183b}$$

and yields information of a higher order. With the above, Eq. (5.179) becomes

$$(\bar{q}_2)_r + j\frac{\bar{q}_2}{r} = -(\delta\tau_{\ell_s})\left\{\alpha_a^{(0)}\bar{I}_0 - 4\alpha_e \bar{B}\right\} + o(\delta^2) \tag{5.184}$$

Once continuity has been forced into steady-unsteady correspondence, the rate equations, (5.168), correspond as well if $(t_i^*)_s = (t_i^*)_u$. The normalization of the nonequilibrium variables depends upon the specific temperature and pressure dependence of the W_i and, by virtue of Eq. (5.175) and (5.178a), is not arbitrary. According to Eq. (5.172), the x-momentum equation becomes

$$\bar{u}_t + v\bar{u}_r = -\frac{\bar{p}_t}{\bar{\rho}} \tag{5.185a}$$

which is a single partial differential equation of first order whose solution if found from

$$\frac{d\bar{u}}{dr} = -\frac{\bar{p}_t(r,t)}{\bar{\rho}\, v(r,t)} \tag{5.185b}$$

on

$$\frac{dr}{dt} = v(r,t) \tag{5.185c}$$

where all of the terms on the right hand sides are known so that integration may immediately be carried out along the characteristic. The physical interpretation of Eq. (5.185b) is that the rate at which first order pressure $\bar{p}$ does work on each slab of fluid is balanced by the convection of first order transverse moment $\bar{\rho}v$ in the axial direction.

In summary

$$\bar{\rho}_p\big)_{T,\bar{c}} \frac{D\bar{p}}{Dt} + \bar{\rho}_T\big|_{p,\bar{c}} \frac{D\bar{T}}{Dt} + \bar{\rho}\left(\bar{v}_r + j\frac{\bar{v}}{r}\right) = \sum_i \bar{\rho}_{c_i}\big|_{p,T,c_j}\left(\frac{W_i}{\bar{\rho}\, t_i^*}\right) \tag{5.186}$$

$$\frac{D\bar{v}}{Dt} = -\bar{p}_r/\bar{\rho} \tag{5.187}$$

$$\left\{\frac{\bar{\rho}\bar{h}_p\big|_{T,\bar{c}}}{r} - 1\right\}\frac{D\bar{p}}{Dt} + \frac{\rho\bar{h}_T\big|_{p,\bar{c}}}{r}\frac{D\bar{T}}{Dt} = \frac{(\tau_{e_s}\delta)}{K_H\sqrt{\delta}\,\theta_{0_s}}\left(\alpha_a^{(0)}\bar{I}_0 - 4\alpha_e\bar{B}\right) + \sum_i \bar{h}_{c_i}\big|_{p,T,c_j}\left(\frac{W_i}{t_i^*}\right) \tag{5.188}$$

$$(\bar{q}_2)_r = -\left\{(\tau_{e_s}\delta)\left[\alpha_a^{(0)}\bar{I}_0 - 4\alpha_e\bar{B}\right] + j\frac{\bar{q}_2}{r}\right\} \tag{5.189}$$

$$\overline{I}_{D_r} = -3\,(\delta \tau_{\ell_s})\,\alpha_0^{(1)}\,\overline{q}_2 \qquad (5.190)$$

$$\bar{\rho}\,\frac{Dc_i}{Dt} = -\frac{w_i}{t_i^*} \qquad (5.191)$$

where, for correspondence

$$(\delta \cdot \tau_{\ell_s}) = \tau_{\ell_u} \qquad (5.192a)$$

$$\frac{1}{K_H \sqrt{\gamma}\, Bo_s} = \frac{1}{Bo_u} \qquad (5.192b)$$

$$(t_i^*)_s = (t_i^*)_u \qquad (5.192c)$$

In addition

$$(\tau_\ell)_s = \alpha_{p_\infty}^* \ell^* \qquad (5.192d)$$

$$(\tau_\ell)_u = t_c^* \sqrt{RT_\infty^*}\ \alpha_{p_\infty}^* \qquad (5.192e)$$

$$Bo = \rho_\infty^* (RT_\infty^*)^{3/2} / \sigma T_\infty^{*4} \qquad (5.192f)$$

$$(t_i^*)_s = u_\infty^* t_{i\infty}^* / \ell \qquad (5.192g)$$

$$(t_i^*)_u = (t_{i\infty}^* / t_c^*) \qquad (5.192h)$$

$$K_H = M_{f_\infty}\,\delta \qquad (5.192i)$$

Equation (5.192a) indicates that the optical depth should have been based upon shock layer thickness instead of body length since traverse radiation is most significant. The normalized variables illustrate that the usual hypersonic small disturbance assumptions have appeared naturally from the search for steady-unsteady correspondence.

$$t = x^*/\ell^* \tag{5.193a}$$

$$r = r^*/\delta\ell^* \tag{5.193b}$$

$$u = u^*/u_\infty^* = 1 + \delta^2 \bar{u} \tag{5.193c}$$

$$v = v^*/u_\infty^* \delta = \bar{v} \tag{5.193d}$$

$$p = p^*/p_\infty^* = \gamma K_H^2 \bar{p} = \frac{\rho_\infty^* u_\infty^{*2} \delta^2}{p_\infty^*} \bar{p} \tag{5.193e}$$

$$T = T^*/T_\infty^* = \gamma K_H^2 \bar{T} \tag{5.193f}$$

$$I_0 = I_0^*/\sigma T_\infty^{*4} = \gamma^4 K_H^8 \bar{I_0} \tag{5.193g}$$

$$q_1 = q_1^*/\sigma T_\infty^{*4} = \delta \cdot \gamma^4 K_H^8 \bar{q}_1 \tag{5.193h}$$

$$q_2 = q_2^*/\sigma T_\infty^{*4} = \gamma^4 K_H^8 \bar{q}_2 \tag{5.193i}$$

In addition higher order information is furnished by Eq. (5.183), (5.185), and (5.180) which takes the form

$$\bar{q}_1 = -\frac{1}{3(\xi_s \delta)\alpha_a^{(1)}} \frac{\partial \bar{I_0}}{\partial t} \tag{5.194}$$

These allow the determination of the axial velocity and heat flux perturbations immediately after one obtains the other variables. The velocity boundary condition on the surface is

$$\frac{DB_o}{Dt} = 0 \quad \text{on} \quad B(r,t) = 0$$

which becomes,

$$\bar{v} = \dot{R}_p(t)\{1 + O(\delta^2)\} \tag{5.195}$$

Obviously the system (5.186)-(5.191) above must allow the same kind of discontinuity (shock wave) that the corresponding unsteady equations did. This could be verified through an expansion of the two-dimensional steady Rankine-Hugoniot equations, but in light of the comments in Section 4.2.2 this is not necessary. The radiative boundary conditions do not depend upon the $x^* \rightarrow t$ relation, hence they are still Eq. (3.88), and (3.89), (3.92) or (3.94). The correspondence is, therefore, complete.

This scheme must be modified in situations without physical length or time scales (cones and wedges). It is best to shrink the axial extent of the field rather than to expand the traverse one in these cases. The appropriate normalization of the independent variables is

$$t = x^*\delta / \ell^* \tag{5.196a}$$

$$r = r^* / \ell^* \tag{5.196b}$$

That of the dependent variables remains the same. The equations and boundary conditions are invariant if $\delta\tau_\ell$ is replaced by unity and

$$\delta(t_i^*)_s = (t_i^*)_u \tag{5.197}$$

which indicates that the corresponding unsteady flow field due to a piston in linear motion is very nearly in chemical equilibrium. This is said with reservation because in general the normalizations of the W_i are not known, and they might introduce factors which would change the conclusions.

It is interesting to note that the definition of $\mathcal{B}_o$, Eq. (5.192f), is consistent with both steady and unsteady cases so that the two have comparable values of that parameter.

Several comments on the mathematical and physical assumptions implicit in the analysis are in order. Initially the system of equations was seventh order, ultra-hyperbolic, while after reduction the formal representation is hyperbolic and of fifth order. The order of the system has not been reduced, however. Two equations were merely uncoupled from all of the others, and this was possible because data upon q_1 and u were chosen so that they could be satisfied within the current ordering. Even though the mathematical structure of the problem has changed, it is not a singular perturbation.

Implicit in all of the relations derived is the assumption that the effect of adjacent slabs of gas one upon all of the others is of higher order than the outward effect of radiation upon the flow field. Axial upstream absorption and emission have been omitted in favor of the corresponding sidewise effects. This is justified by the overwhelming importance of axial convection and could also have been accomplished with the full transfer equation. In that instance the omission of $\partial I/\partial x$ relative to $\partial I/\partial r$ is just as easily justified as was $q_{1_x} \ll q_{1_r}$. Then the heat flux and its divergence might be, for instance, the planar expressions Eq. (2.69) and (2.70) in the r-direction. This was done by Khosla (1966).

Although Eq. (5.185b), (5.185c), and (5.194) allows the determination of the axial heat flux and velocity perturbations, this is not so simple in practice. To accurately determine $\overline{P}_t$ and $\overline{I}_{ot}$ Eqs. (5.186)-(5.191) and their boundary conditions must be differentiated with respect to t. The solution for the temporal derivatives could be carried out in the same manner as that of the original system, and the later solution would enter through in homogeneous terms and coefficients. Even though the system would be linear, the labor involved seems hardly worthwhile. There is a strong analogy with the method of parametric differentiation (Rubbert, 1966) since t is only a parameter as far as the radiative field of lowest order is concerned.

Equation (5.194) confirms that the analogy is invalid near discontinuities in the piston path or its temperature history since if $\partial \bar{I}_0 / \partial t$ is large the governing assumptions are violated. In particular this will occur at the impulsive start of motion. Since the dominant mode of axial heat transfer has been assumed to be convection it must always follow that

$$\rho^* u^* \frac{\partial h^*}{\partial x^*} \gg - \frac{\partial q_1^*}{\partial x^*} \tag{5.198}$$

The Rosseland formula, Eq. (2.51), overpredicts worst, therefore, a conservative result will be given by

$$(q_1^*)_{max} = - \frac{16}{3} \frac{\sigma T^{*3}}{\alpha_R^*} \frac{\partial T^*}{\partial x^*}$$

If S_1 is distance along the surface of a slender body measured from the nose

$$\rho^* u^* \partial h^* / \partial x^* \sim \rho_\infty^* U_\infty^* C_{p_{f\infty}}^* \frac{\Delta T^*}{S_1^*}$$

$$- \frac{\partial q_1^*}{\partial x^*} \sim \frac{16}{3} \frac{\sigma T^{*3}}{\alpha_R^*} \frac{\Delta T^*}{S_1^*}$$

so that Eq. (5.198) becomes

$$\alpha_R^* S_1^* \gg \frac{16}{3} \frac{\Gamma}{K_H (B_0)_S} \delta \tag{5.199}$$

Thus, the more important radiation is the farther from the nose the analogy holds. A consistent analogy must always include self-absorption since it is certain that the shock layer is not necessarily optically thin. The unique result the approach outlined above is that it is not restricted to non self-absorbing shock layers as most analyses are.

5.7 The Choice of an Absorption Coefficient Model and Radiation Similarity Parameters

Since the model proposed omits some important physical processes, it is best to choose a description of the absorption coefficient consistent with this omission. The calculation or measurement of absorption coefficients is quite difficult, and one cannot at present hope to be able to include all possible phenomena in a given gas. At "low" temperatures (1 ev and less) molecular radiation dominates the spectrum, thus attention should be confined to models which favor molecular processes. To this author's knowledge only the work of Armstrong, et.al. (1961) includes tabulation of both Planck and Rosseland mean absorption coefficients for air at reasonably low temperatures. All others present K_ν as it depends upon frequency (or wave-number) under given conditions. From the definition of the Rosseland mean, Eq. (2.52), it appears that the greatest uncertainties arise from frequencies in the wings of broadened lines. This may be particularly true for low frequencies. At "low" temperatures the peak of the weight function dB_ν/dT moves toward lower frequencies as well. However, most tabulations neglect low frequencies; therefore the calculation of Rosseland means may be quite difficult. Since most tabulations of the Rosseland coefficient are, furthermore, suitable only for high temperatures (> 1 ev) it seems that the distinction between Planck and Rosseland means is better motivated in future investigations, in which the temperature range of interest may be expanded. There are in addition many uncertainties in semi-grey radiative transfer, particularly in the quasi-isotropic approximation, that would be better resolved in more simple situations than those which shall be considered. Therefore, even though a semi-grey analysis has been maintained up to this point, it must be abandoned in favor of a grey one for the sake of simplicity. All of the computer programs to be described allow for the inclusion of semi-greyness, but henceforth only the Planck mean absorption coefficient is considered. At least qualitatively, the Planck and Rosseland means display the same behavior as temperature and pressure vary.

The trends of the Planck mean are well understood. It increases almost linearly with density, but its behavior with temperature in nonmonotonic, increasing when $T < 1$ ev but decreasing rapidly when $T > 10$ ev because processes involving free electrons (e.g., Bremsstrahlung) predominate. Many investigators have sought formulae which approximate this behavior. Fortunately the store of data is growing quite rapidly*. Nevertheless only a crude approximation can be expected, and the results obtained from this investigation must be regarded as qualitative rather than quantitative. Thid does not in any way detract from the method or the numerical analysis since any absorption coefficient model may be used.

Among others, Traugott (1964) and Shwartz (1965) have investigated the choice of an appropriate absorption coefficient model. Both agree that the data of Kivel and Bailey (1957) are representative of most investigations. Traugott incorporates an extensive correlation of data including that of Brownlee (1963), Nardone et.al (1963), and Meyerott et.al. (1960). Since Traugott's formula validly reflects the essential phenomena in the low temperature regime, in the future the general form

$$\alpha_p = P^{b_1} T^{b_2} \tag{5.200}$$

shall be used. Both Traugott and Shwartz arrive at

$$b_1 = 1 \tag{5.201a}$$

$$b_2 = 4 \tag{5.201b}$$

for $T < 1$ ev in air. At higher temperatures, when the gas in nearly completely ionized it may be shown that $b_1 \sim 2$, $b_2 \sim -11/2$ (Pai, 1966, pg. 79; Zeldovich and Raizer, 1966, pg. 260). Gruszczynski and Warren (1967) indicate that under conditions of superorbital entry short wave length radiation is predominant. Furthermore, Biberman et.al. (1964) have shown that the majority of total gas radiance over wide ranges of conditions comes from atomic line radiation (in the vacuum uv, 500-1600 A°). Therefore an atomic model is adopted in most cases and $\gamma = 5/3$ $(\Gamma = 0.4)$ is employed.** Because of the manner in which the Boltzmann number was defined, the parameters τ_1 and B_0 are not independent once an absorption coefficient

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* See Appendix F for a concise discussion of the labor involved in the computation of absorption coefficients.

** For expository purposes such a model is qualitatively correct. In choosing the absorption coefficient model above, one weights line radiation over that in the continuum, nevertheless the empirical formula is obtained from experiments in air, which is an essentially diatomic gas at the temperatures of interest.

model has been chosen. Furthermore, they depend only upon properties of the undisturbed gas. According to Eq. (5.192f) and (5.201a, b)

$$B_0 / B_{0_{S.L.}} = \frac{(P^*/P^*_{SL})}{(T^*/T^*_{SL})^{1/2}} \sim 10^3 \frac{P_{(\text{in atm})}}{T^4_{(\text{in } {}^\circ K^4)}} \tag{5.202a}$$

$$\alpha_p / \alpha_{p_{S.L.}} = \left(\frac{P^*}{P^*_{SL}}\right)\left(\frac{T^*}{T^*_{SL}}\right)^4 \tag{5.202b}$$

where the sea level values are

$$B_{0_{SL}} \simeq 3.12 \times 10^4 \tag{5.203a}$$

$$\alpha_{p_{SL}} \simeq 5.59 \times 10^{-5}\ m^{-1} \tag{5.203b}$$

Thus Goulard's radiation-convection parameter is

$$\frac{\tau_\ell}{B_0} = \frac{\tau_{\ell_{S.L.}}}{B_{0_{S.L.}}} \left(\frac{T^*}{T^*_{SL}}\right)^{15/2} \tag{5.204}$$

which may be reasonably large at high altitudes. The variation of $(B_0/B_{0_{SL}})$ and $\alpha_p/\alpha_{p_{SL}}$ with altitude are presented in Fig. 6.

The parameters governed by surface properties are the wall emissivity, ϵ, its temperature, T_w, and the piston velocity, u_p. Because the temperature downstream of the shock is $\sim O(u_p^2)$ and attention is restricted to $T <$ 1 ev, the motion is confined to $u_p \lesssim 3$. If the wall were very cold or very hot this value might be higher or lower, but it is most instructive to examine $T_w \sim O(1)$. In fact, if T_w were not $\sim O(1)$ then there would be another temperature scale, T_W, in the problem. The only obvious effect which emissivity might have is that the heat flux into the wall must diminish as ϵ does. The limiting process $\epsilon \to 0$ is, however, another singular perturbation since in that case no matter how cold the wall may be there will be no heat flux into it and the temperature slip will always be significant.

Viskanta and Grosh (1962) have examined the effect of emissivity in a simple situation, but their conclusions do not necessarily apply to cases of interest herein. For completeness one should examine several emissivities.

The region in which physically realizable situations may be modeled by the present equations is indicated schematically in Fig. 7. The $(P/B_o, T_w)$ and (U_p, T_w) planes and the appropriate surface of constant wall emissivity delineates the volume of interest. If equilibrium thermodynamics were included, another decade in P/B_o could be allowed without straining the gas model.

The purpose of the present investigation is not only the study of specific instances in a given gas but also the prediction of the effects of radiation (and the worth of the numerical program) over a wide range of conditions. The specific parameters discussed apply only for air, but reentry problems are not necessarily confined to the earth. In fact, one of Jupter's satellites has an atmosphere of methane, whose absorption coefficients in the temperature range considered are a good deal larger than those of air (see Main and Bauer, 1967). A complete investigation would involve the study of all permutations of small, moderate, and large Boltzmann numbers; small and moderate wall temperatures; wall emissivities near zero and unity, and both small and moderate piston velocities. It would be interesting also to investigate the effects which radiation may have upon aerodynamics through the modification of pressure distributions on slender bodies. Such an investigation would require at least forty separate cases, and this is beyond consideration. Therefore, parameters were chosen first to simulate situations which might be encountered in the atmosphere, but second to produce situations in which the effects of radiation are if not predominant then at least pronounced. The problems to be presented accomplish these goals. Before the cases studied are described, some comments on the numerical experience acquired are of interest.

5.8 Numerical Requirements

5.8.1 Stability

The numerical stability of hyperbolic problems in only two independent variables is seldom questioned. The Courant-Friedrichs-Lewy (CFL) stability criterion for hyperbolic systems in any number of dependent variable requires that the domain of dependence of a point according to the difference equations must contain the domain of dependence of the differential equation. It is well known that in all simplical networks* the CFL condition is sufficient for stability. Of course the only network available in a problem in two independent variables is simplical since it must use the ends of a line segment (two points in a one-dimensional data surface) to determine each new point. However, a characteristic mesh might be only "marginally" stable whereas a constant time increment scheme is unconditionally so. This happens because in the Martree scheme in a nonradiating situation the properties at points from which characteristics emanate are found by interpolation which includes points from outside the true domain of dependence. When radiation is included, stability in space-time is assured since the difference and differential equations possess exactly the same domain of dependence. This conclusion follows for multidimensional problems as well. Since errors are of order $(\Delta r)^2$, spatial increments in the radiative equations must not be too large. It is easily shown that errors oscillate with exponentially increasing amplitude unless $\Delta r < \left|\frac{\partial^2 \xi/\partial r^2}{2\,\partial \xi/\partial r}\right|\left(\frac{1}{\kappa_e}\right)$.

Once the numerical stability of the system is assured, the manner in which time increments may be chosen must be determined. This is dictated by the requirement that the domain of dependence of all points at the current time, t_n, must include only points at the previous time, t_{n-1}. Consider Fig. 8 in which properties at a point arbitrarily close to either the piston surface or the shock wave are desired. Let that point be at a distance $\epsilon_q (R_s - R_p)$, $0 < \epsilon_q < 1$ from either "boundary". Then, to a first approximation either

$$\Delta t \cdot (u_1 - q)_{n-1} = \{R_s - \epsilon_q (R_s - R_p)\}_n - R_{s_{n-1}}$$

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* Simplical networks are those that determine a new point by using L + 1 points of an L-dimensional simplex in an initial data surface of dimension L.

or

$$\Delta t \cdot (U_p + q(R_p))_{n-1} = R_{p_n} + \varepsilon (R_s - R_p)_n$$

where the subscripts $(-)$ or p denote properties just downstream of the shock and at the piston face respectively. If all properties are linear over the increment, Δt, and the piston does not accelerate too rapidly

$$\Delta t \le \min \begin{cases} \dfrac{\varepsilon_q (R_s - R_p)_{n-1}}{q_{p_{n-1}} - \varepsilon_q (U_s - U_p)_{n-1}} \\[2ex] \dfrac{\varepsilon_q (R_s - R_p)_{n-1}}{U_{s_{n-1}} - (U_{(-)} - q_{(-)})_{n-1} + \varepsilon_q (U_s - U_p)_{n-1}} \end{cases} \qquad (5.205)$$

Usually, the first estimate (that which allows an arbitrarily close approach to the piston) is smallest. Nevertheless, if the body accelerates or decelerates this will not always be the case.

As the shock layer grows optically thick there must be regions near the piston or the shock in which rapid variations may occur over short distances (entropy layers). Therefore, it is best to cause points within the shock layer to accumulate near the piston and the shock. If there are N points within the shock layer $r_1 = R_p$ and $r_N = R_s$, one procedure which allows this is the following

$$\frac{r_i - R_p}{R_s - R_p} = \frac{1}{2} \left\{ 1 + (2\varepsilon_q - 1) \cos\left[\pi \frac{(i-2)}{(N-3)}\right] \right\} \qquad 2 \le i \le N-1 \qquad (5.206)$$

In this manner the second and $(N-1)^{st}$ points are $\varepsilon_q (R_s - R_p)$ from piston and shock respectively and the other points become farther apart as one moves inward or outward toward $i \le N/2$. An obvious choice for N is that integer nearest to $1 + 1/\varepsilon_q$. In the calculation to be cited points were chosen symmetrically about the shock wave until "N/2" upstream points had been determined. Thereafter the spatial increment between points was increased by a factor, D, until the largest increment allowed by spatial stability was approached. Beyond that station all spatial increments were equal. These procedures are incorporated in the numerical programs described in Appendix E.

5.8.2 Accuracy and Numerical Experience

Chou, Karpp and Huang (1967) were able to determine the accuracy of their characteristics scheme through comparison with an exact analytic solution. Since there are no solutions available for comparison in RGD, some other method must be devised. If energy is conserved by the numerical procedure within a reasonable tolerance, then one may have confidence in the solutions obtained. Since the increase in energy of the flow in a given time is the difference between the work done upon the piston and the energy lost through radiation, energy is conserved if

$$\int_{R_p(t_2)}^{R_{Max}(t_2)} E(r',t)(r')^j dr' - \int_{R_p(t_1)}^{R_{Max}(t_1)} E(r',t)(r')^j dr' = \Gamma \int_{t_1}^{t_2} P_{Surface}(t')\, u_p(t')\, R_p^j(t')\, dt' - \frac{\Gamma}{B_o} \int_{t_1}^{t_2} \{ q_{Max}(t') R_{Max}^j(t') - q_{wall}(t') R_p^j(t') \}\, dt' \tag{5.207}$$

where $j = 0, 1, 2$ for planar, cylindrical or spherical cases and for a perfect gas

$$E = \frac{P}{T} \{ (1-\Gamma)T + \frac{\Gamma}{2} u^2 \} \tag{5.208}$$

If upstream absorption is included, $R_{Max} \to \infty$ and $q_{Max} = 0$, otherwise $R_{Max} = R_{Shock}$ and q_{Max} is q_{Shock}. The heat flux is taken positive outward from the piston.

This criterion was applied in several planar and cylindrical cases. Without exception energy was conserved with 1 percent in planar and within

1.5 percent in cylindrical cases over approximately fifty time increments. This is certainly acceptable, since the radiative transfer equation is never satisfied exactly. Errors of this type could be blamed upon either the quadratures or the linear interpolation which was used.

The program required a good deal of "knowhow" in the choice of parameters. Generally, the integral boundary condition was used as the initial guess only when the program was begun. After that it was much easier to strip the final, iterated value of I_{o_w} of all terms dependent upon T_w, ϵ, or η_{shock} (if upstream absorption were neglected). The information remaining was the integral, and this was used as the first estimate for the integral of the next time increment. The parameters ACUR1 and ACUR2 (see Appendix E) help establish the high and low brackets for I_{o_w} once an initial guess is made. Generally these may be taken to be 0.2 or less, the higher values being necessary for more rapidly moving pistons. The parameter XD1 is a measure of the largest allowable spatial increment, and XD2 is the factor by which the increment outside of the shock is increased with each step. Both of these must vary inversely as τ_ℓ since smaller increments are necessary in "thick" situations. The dummy variable KLUP allows one to monitor intermediate results interspersed judiciously throughout the program. Caution forces allowance for possible difficulties for combinations of parameters not yet investigated.

In general, three iterations upon the compatibility relations were allowed in program A (see Appendix E) although up to five were used sometimes. Of course an error criterion could have been built into the individual subroutines, but this was not deemed necessary. In the initial stages of flow development the procedure described above for guessing I_{o_w} converged on the first or at most the second iteration, but as the shock layer grew more optically thick the far field behavior became more and more sensitive to I_{o_w}. On the average four or five iterations were required upon the entire flow field (these shall be referred to as Grand Iterations), but at times eight were necessary. At no time were more than ten iterations required, and never did the procedure fail to converge. If the shock layer were very optically thick the boundary condition, I_{o_w}, would have to be guessed into double precision in order to obtain convergence to the degree called for thus far. Fortunately this can be avoided since very optically thick behavior is reasonably well understood and may be amenable to perturbation analyses in order to determine the structure of the optically thin layers near shock and surface. Also, radiation plays a relatively minor role in the compatibility relations since $div\, q$ is small; therefore the flow field becomes insensitive to small errors in I_{o_w} just as rapidly as the heat flux distribution becomes sensitive to them. This behavior has been confirmed in practice.

The smaller the initiating time the more rapidly will the field forget discrepancies in the initial data. However, the allowable time increments are also smaller the smaller t_i is. Therefore about the same amount of computer time is required for the solution to settle out regardless of what t_i might be. The initial time is, in fact, the parameter, h, which is characteristic of the allowable step site even though ϵ_4 plays just as important a role. The reason for this is that ϵ_4 must be continually decreased if "entropy layers" are to be monitored. If the same value of ϵ_4 were used throughout, the points in the shock layer would grow farther apart as time progressed. Usually ϵ_4 was taken to be 0.25 (five points in the shock layer) at the start of motion, but values as small as 1/15 (17 points in the shock layer) were necessary at times in order to monitor steep gradients. The size of the time increment depends upon the parameters chosen, but usually four or five time increments could be covered in one minute of computer time. If upstream absorption is neglected, the time required decreases by roughly an order of magnitude. Another crude estimate is that about forty-five minutes of 7094 computation are necessary to travel to a time $\sim .25$ if $t_i \lesssim 10^{-2}$.

The location of the point $r_o(t)$ at which the solution is terminated is an indication of the extent of upstream absorption. This station always remained well ahead of the shock, and as the shock layer grew it occurred fewer and fewer shock detachment distances upstream. Generally thirty to fifty points were necessary in the flow field. Only the last two or three came from the asymptotic solution. Thus that solution serves only to complete the data set thereby making interpolation in the far field less ambiguous. The reader will note in Appendix E that one-hundred points were allowed in the flow field. In a very optically thin situation more points then this might be required because of the slow radiative decay. Unfortunately, the storage capacity of the MIT IBM system which existed when this investigation was terminated allowed the use of no more than 100 points. Therefore the study of very thin situations or very large times (when the shock layer would be so thick physically that many points would be required) was not possible. The current MIT IBM Model 360-65 might handle as many as two hundred points but not many more.

Unfortunately, program B (the full transfer equations) is not nearly as rapid as program A. Quadrature is required in order to obtain the optical depth, η, which is needed for the quadratures involved in RAD and QTAU. Since the quadrature must be performed at each of points 1, 2, 3, and 4 in each Grand Iteration, it is understandable that up to five minutes of 7094 time were required for the three grand iterations necessary to complete a single time increment.[1] Only a few time increments in a single problem were computed. This is sufficient, however, to have proved that "exact" solutions of "complete" problems in RGD are possible, and to obtain estimates of the computational effort involved. If the physics of radiation and its interaction with other nonequilibrium phenomena were completely resolved, the inclusion of chemical nonequilibrium would involve very few operations in addition to those which are now necessary. Therefore, estimates arrived at are representative of more general calculations as well. Further comments are reserved for the next chapter, in which specific results and difficulties will be discussed.

CHAPTER 6

RESULTS AND DISCUSSION

The computer program allows investigation of one-dimensional piston motions with or without shock waves, and without restriction on surface properties, piston paths, or the upstream state of the gas. Of course, exhaustive investigation of all possible situations was not feasible. The cases considered are outlined in Table II. Indicated are the parameters of the problems and the extent of the investigation as measured by the "times" at which computation was terminated. In all cases the wall was at the free stream temperature and all are planar, include upstream absorption, and use the Planck mean absorption coefficient given by Eq. (5.200) and (5.201) unless it is stated otherwise.

TABLE II

Piston Path $R_p(t)$	Bo	τ_f	ϵ_w	t_{max}	Comments
0.5t	0.5	1.00	1.00	0.050	α=constant
1.5t	0.5	1.00	1.00	0.284	α=constant without upstream absorption
0.1t	0.1	1.00	0.50	0.060	
t(2.0-t)	10.0	0.02	1.00	0.163	
t(2.0-t)	1.0	1.00	1.00	0.443	
1.5t	2.0	1.00	0.25	0.313	cylindrical
1.5t	2.0	1.00	1.00	0.356	cylindrical

These situations were chosen because they employed varied combinations of the radiative parameters, thereby insuring efficient operation of the program in general. Most important, regimes which are unapproachable with perturbation procedures were investigated. A restricted comparison with the linearized theory of Section 5.5 was attempted, and with the aid of programs B and C (see Appendix E) the error inherent in the use of the differential approximation has been determined. This is believed to be the first such comparison in a self-consistent gasdynamic problem.

No attempt was made to investigate the effects of various approximations upon a single situation. For instance, no case was investigated both with and without upstream absorption. It became clear that the effects of upstream absorption could be seen even without comparison with the more restricted solutions. Close examination of Table II reveals that large and small values of all parameters have been considered so that qualitative trends may be predicted. The sole nonplanar case studied is of such importance that a complete subsection is allotted to it.

6.1 Planar Investigations

The starting solutions of Section 5.3.2 predicted radiative properties quite well in general, but due to their lack of self-consistency were quite poor otherwise. Virtually no approximation could be greatly in error at small times since as the shock layer thickness vanishes so does the volume of radiating gas. Furthermore, if a tangent wedge solution is applied to any piston near the initiation of motion, then the flow field is uniform, and there are no driving gradients within the shock layer. This is true only in planar instances.

The effects of inconsistency in initial solutions are illustrated in Figures 9a-9e. Although the effects of radiation upon velocity and pressure are less than those upon temperature, the former two best demonstrate transient behavior. In general, the gas should be compressed within the shock layer as the piston is approached. Even though a uniform profile is applied initially, the opposite situation (expansion from shock to piston) ensues. In spite of the discontinuity in temperature at the shock wave, most of the energy liberated from the shock layer initially is given up by the gas near the piston, and a rarefaction moves from the piston toward the shock. Finally, the optically thin region near the shock radiates both toward the piston and the upstream gas, and the behavior which was anticipated occurs. Even though velocity perturbations upstream of the shock wave are small, slight nonmonotonic behavior occurs during the transient with a boundary layer-like region near the shock wave, and this is the first indication of the formation of an entropy layer. These phenomena are a consequence of the starting procedure and do not necessarily represent physical occurences. Nevertheless, they provide insight into the effects of radiation upon nonlinear wave propagation. If the end of the transient is taken to be that time at which continual compression from shock to piston occurs, then the estimates provided by (5.105) are very good.

The case presented in Figures 9a-9e was the first to be studied and was used primarily to determine the influence of initiating time upon the solutions; therefore, it was not carried very far temporally. The dashed lines in Figure 9a indicate pressure profiles for a starting time of 0.001 as opposed to 0.005 which the solid lines represent. Transient effects upon pressure are significant; however, those upon velocity within the shock layer and temperature are scarcely noticeable. The heat flux distributions display the almost linear relation with distance that is expected from nearly optically thin isothermal shock layers. Even though radiative effects are enhanced by the choice Bo = 0.5, the flux is always small because the shock is weak. It is observed that values of the physical variables near the piston surface are rather insensitive to the initiating procedure. Even though the surface pressure reaches the correct asymptote within a time the order of the initiating time (Figure 9d), the heat flux at the wall adjusts almost instantaneously. The shock layer attempts to rid itself of excess energy as rapidly as possible, but the wall can accept this energy only at a rate which is fixed by its temperature and emissivity. This accounts for the rarefaction previously mentioned and is the major reason that it takes longer to forget the later the radiation is turned on. The classical exothermic weakening of the shock wave is displayed in Figure 9e. The analysis shows that the radiation induced transient gasdynamics rapidly decays and that the initiating assumption of Section 5.3.2 is justified in planar flow fields.

Figures 10a-10e illustrate the effects of the omission of upstream absorption. Since a large piston velocity, U_p=1.5, was chosen, the shock wave is reasonably strong, and radiative effects are more pronounced than they were in the previous case. The pressure transient, Figure 10a, decays very rapidly because unlimited amounts of energy may be lost to the nonabsorbing free stream. These pressure distributions and the velocity and temperature histories of Figures 10b and 10c indicate a tendency toward invariance as functions of the conical variable $\frac{x-x_p(t)}{x_s-x_p} = \frac{x}{(U_s-U_p)t} - \frac{U_p}{(U_s-U_p)}$; hence a rapid approach to radiative equilibrium. In fact, this is the only instance in which a steady state absorption-emission balance was even close. Since thin solutions decay most rapidly, situations which ignore upstream absorption should attain equilibrium more rapidly than those which include all effects.

The evolution of the heat flux at the wall and the surface pressure is given in Figure 10d. The effect of initiating time is slight, and results have been extrapolated to t_i=0. The nonmonotonic pressure distribution is well-documented in nonisentropic flows and has been observed in flow fields in which shock waves are sufficiently curved to produce marked entropy layers. Traugott (1960) comments on non-monotonic pressure distributions on blunted cones, and Sussman (1966) has observed them in flow fields which incorporated chemical non-equilibrium. To this author's knowledge, the phenomenon has not heretofore been reported in radiating flows. Since the behavior is intimately associated with nonlinearity no linear theory could predict it. In addition, most nonlinear analyses to date have considered only cold, black walls so that entropy layers were "sucked" away before this interesting phenomenon could occur. It has been demonstrated that the rise in surface pressure does not depend upon the inclusion of upstream absorption, hence a nonlinear hypersonic small disturbance theory should predict it even in a less than "complete" approximation. In fact Lee (1965) has made similar qualitative observations in flows with chemical nonequilibrium with a small disturbance approach.

Comparison of Figures 9e and 10e reveals that, consistent with a rapid approach to equilibrium, the omission of upstream absorption leads to a very rapid shock wave decay. The investigation of each of the cases mentioned thus far involved twenty minutes of IBM 7094 computation, even though the latter was examined ten times as "long". Lack of upstream absorption led to more rapid computation not only because fewer points were necessary but also because convergence of the numerical iteration was greatly enhanced. Aside from the physical information derived, these cases may be regarded as a critical examination of the applicability of the numerical techniques to situations both with and without upstream absorption. However, the capabilities of the program were not determined without difficulty, and some limitations were noted. The most severe of these is discussed in Section 6.2.

Since the most interesting aspects of radiative transfer are not illustrated by situations which allow linearization, very little computer time was allotted to the comparison with linear theories. Furthermore, the pulsed pistons which are readily attacked analytically are impossible to reproduce numerically. At least a few cycles of a high frequency piston oscillation could have been examined, but this would have precluded the existence of a shock wave, and the radiative field would have been very weak. The best choice was, therefore, a piston moving with a small, constant velocity. Unfortunately, radiation was insignificant over the interval of interest in all cases except those with very small Boltzmann numbers; thus, only the situation in which small time approximations to the linear theory are doubtful (see Section 5.5) could be investigated.

A wedge-like piston, $U_p=0.1$, was initiated at $t_i=0.0001$. Representative flow field perturbations, as computed by the nonlinear program, are presented in Figures 11a and 11b. The heat flux is again that of a thin radiative field. The shock wave, which initially traveled at $U_s=1.359$, was considerably stronger than the Mach wave which linear theory would have, $a=\sqrt{\gamma}=1.276$. The shock wave decayed negligibly during the interval of interest. Eq. (5.160) yields the following linearized predictions for the surface pressure and velocity perturbations:

$$\frac{p_s'(t)}{p_s'(0)} = e^{-\varkappa t} I_{(0)}(\varkappa t) \;; \quad \varkappa = \frac{8\Gamma^2\tau_e}{B_0} \tag{6.1}$$

$$\frac{u'(x,t)}{U_p} = \varkappa e^{-\varkappa x/\sqrt{\gamma}} \int_0^{t-x/\sqrt{\gamma}} e^{-\varkappa\tau} \left\{ I_{(0)}\left[\varkappa\sqrt{\tau(\tau+2x/\sqrt{\gamma})}\right] - 2\tau I_{(1)}\left[\varkappa\sqrt{\tau(\tau+2x/\sqrt{\gamma})}\right] \right\} d\tau + \frac{1}{\sqrt{\gamma}} e^{-\varkappa t} I_0\left\{\varkappa\sqrt{t^2 - x^2/\gamma}\right\} \tag{6.2}$$

The perturbations to surface pressure and velocity downstream of the wavehead are compared with the results of the nonlinear calculation in Figures 11c and 11d. The nonlinear theory predicts a much more rapid decay, and even for such small times four or five percent errors are evident. It is difficult to say whether discrepancies arise because of the use of Eq. (6.1) and (6.2) and the approximations inherent in them or because of discrepancies in the linear theory itself. The parameter $\varkappa = 8\Gamma^2/B_0$ is, however, not very large because Γ^2/B_0 is still of order unity, thus the assumptions are not badly violated. Even though definitive comparison cannot be achieved, it is clear that large errors may arise as the motion progresses; hence, nonlinear interactions may play a much more important role in RGD than in other nonequilibrium situations.

Because of excessive computer requirements, program B was used sparingly. Nevertheless, the order of error introduced by the differential approximation to radiative transfer has been predicted. The acceptable error depends upon the properties which are of interest. If prediction of the effect of radiation upon aerodynamics is important, then I_0 is crucial since it is the only radiation variable that enters the divergence of the heat flux which in turn determines the fluid mechanics. Program C was used to determine the error in I_0. Since the differential approximation should be at its worst in nearly thin situations (Olfe, 1967), the flow fields of the three previous investigations (Figures 9, 10, and 11) were employed. Flow fields were chosen at representative times, and the exact values of I_0 were compared with the values predicted by the approximate method (Figure 12a). When upstream absorption is neglected I_0 is underpredicted, the worst underprediction occurring "far" from boundaries. On the other hand, when upstream absorption is included I_0 is overpredicted. Overprediction is to be expected because the exact formulation has an algebraic decay in addition to the exponential (Section 3.2.3). The overprediction is least near the piston and is small everywhere. The largest error in I_0 is less than eight percent, and this lack of appreciable error is attributed to the use of the consistent boundary conditions, Eq. (3.88). This is in agreement with Kourganoff's (1957) astrophysical investigation and with the results of Heaslet and Warming (1965). Since is nearly correct at the surface, the divergence of the heat flux, hence the flux itself, should be accurate as well. A program C comparison confirms this as well. Therefore, all variables are predicted accurately at the wall, and both aerodynamics and heat transfer as given by the differential approximation are quite good. The comparison is not complete because the error may be cumulative in time, and this is the next point which must be considered.

In order to determine the cumulative nature of the error, the first case mentioned (Figures 9), which included upstream absorption, was investigated with program B. The calculation was initiated with a solution based upon the differential approximation and was allowed to progress until the degree of error growth could be determined. The results of the last time increment are shown in Figure 12b. The velocity field and shock slope are underpredicted by the differential approximation whereas the temperature and pressure fields are over-predicted. The errors in Figure 12b are smaller than would be expected based upon the approximations inherent in the differential approach. The investigation indicates that error growth is extremely slow, the levels shown in the figure having changed very little over several time increments.

One may inquire into the excellence of the differential approximation. The main assumption is near isotropy, Eq. (3.66). Following Zeldovich and Raizer (1966), it is observed that the quantity $|q|/I_0$ is an excellent index of anisotropy. If the radiation were unidirectional, maximum anisotropy, $q^* = c^* U^{(R)*}$, since the radiant energy density would be convected with the speed of light. If complete isotropy prevailed, the heat flux would vanish. Therefore the radiative field progresses from isotropy to complete anisotropy as $0 \leq |q|/I_0 \leq 1$. Examination of all numerical results proves that in no case is the anisotropy index greater than 0.25. It can only be concluded that one-dimensional piston generated radiative fields are appreciably anisotropic only near surfaces and that the boundary condition Eq. (3.88) accounts for that anisotropy as well. In fact, Eq. (3.88) is the exact relationship between $\mathcal{I}_0$ and $\mathcal{g}$ along the ray $\mu = 0$. Thus the gasdynamicist is justified in relying upon the differential approximation. Its inherent simplicity more than counterbalances the attendant errors.

The flow fields generated by retracted and inserted pistons may be combined according to a piston analogy to simulate a two-dimensional airfoil at angle of attack. Beside the fact that the piston analogy fails near discontinuities in slope, such discontinuities may not be allowed because the formulation of the difference equations of the numerical procedure requires that there be no appreciable variations over each time increment. Therefore, the upper surface of an airfoil (retracted piston) must begin and the lower surface (inserted piston) must end with zero slope. A combination of piston paths which accomplishes this is

$$R_p)_{bottom} = U_0 t(1-\frac{t}{2}) \qquad (6.3a)$$

$$R_p)_{top} = -\frac{U_0}{2}t^2 \qquad (6.3b)$$

which corresponds to a biconvex airfoil whose angle of attack is four times its thickness ratio. The upper surface may also correspond to a boat-tail while the lower one to the ogival nose of a body at zero angle of attack. The value $U_0=2.0$ was chosen since it generated a reasonably strong shock wave but did not cloud radiative cooling with a very rapid deceleration. The specific heat ratio $\gamma=1.2$ is indicative of a hypersonic environment, and Eq. (5.202) and (5.203) predict that realistic aerodynamic situations in air at moderate altitudes require $\tau_\ell \sim 10^{-2}$ and $B_0 \sim 10$.

The study was begun on the lower surface with $\tau_\ell = 0.02$ and $B_0 = 10$. Unfortunately, the program was not written so that a nonradiating calculation could be performed for comparison; therefore, shock-expansion theory (Mahony, 1955) was used to predict the corresponding idealized flow. It was found that the difference between solutions with the parameters above and those of shock-expansion theory were of the order of the error inherent in the latter procedure. The upper surface is, therefore, not worthy of investigation since the gradual acceleration would not require significant radiation, and simpler theories would predict pressures rather well.

Figures 13a and 13b show the evolution of pressure and velocity distributions for the parameters noted above. Even though the shock has a finite radius of curvature from the outset, the solution was initiated at $t_i=0.001$ with a tangent wedge. There is no indication of the formation of entropy layers, and the deviation from nonradiating values is probably no more than a few percent. The deceleration over this period of time is insufficient to have such marked effects; therefore, radiation is certainly not insignificant.

Radiative phenomena are enhanced by setting $\tau_\ell = Bo = 1$. Although this is no longer a realistic situation in air, it might be appropriate to gases rich in hydrocarbons.* Comparison of Figures 13a and 13b with 14a and 14b proves the importance of the parameters τ_ℓ and Bo. A noticeable entropy layer begins to develop near the shock wave very early in the motion when radiation is strong, and the entire character of the profiles is altered. Temperature distributions are presented in Figure 14c, but the entropy layer is much more prominent in the velocity and pressure distributions. Thus, the commonly accepted conclusion that radiation affects temperature substantially more than it does velocity or pressure is perhaps premature.

A significant upstream flow field can be detected in Figure 14c. This is expanded upon in Figure 14d in which the upstream temperature profile is presented explicitly. Since upstream velocity disturbances are small and the temperature is not too far from unity, continuity requires that the pressure and temperature profiles behave qualitatively the same, hence the figure is representative of both variables. The most convenient distance for normalization is the shock layer thickness, but its use can be misleading. The upstream extent of the disturbance may appear to decrease with time, but in reality the shock layer grows thick much more rapidly so that the upstream influence of radiation will always grow. Figures 14c and 14d prove that upstream absorption is prevalent for the order of three free stream optical path lengths. Note that at moderate times temperature levels upstream of the shock wave may be greater than those at points within the shock layer. The flow evolves slowly initially with both piston deceleration and radiation contributing to the weakening of the shock wave. The

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* The computations of Main and Bauer (1967) in hydrocarbon-air mixtures indicate that Planck and Rosseland mean absorption coefficients may be as much as five orders of magnitude higher in a mixture which is half methylene (CH_2) and half air by volume than in air alone.

upstream gas absorbs energy at first, but when the piston has undergone considerable deceleration the upstream gas begins to radiate back toward the shock layer and helps sustain the shock wave. This is indicated by the slight decrease in upstream temperature when $t \sim 0.35$ (see Figure 14c). The effects of radiation are quite strong even though the shock is only of moderate strength, and this refutes Wang's statements (Section 5.1). It is clear that upstream absorption cannot be neglected in general. Indeed in this case it is a dominant mechanism.

Nowhere is the evolution of the radiative field better described than in the heat flux distributions of Figure 14e. For small times the field is the linear one of a thin shock layer with a slow decay upstream. As time progresses an optically thin layer grows near the shock while the heat flux becomes "thick" elsewhere. A slight thin behavior appears near the piston as well. Thus the radiative field is characteristically optically thick, and the shock wave and piston can be "seen" only by the gas very close to them. This is in contrast to the case with less important radiation $(\frac{\tau_\ell}{B_0} \ll 1)$. The flux profiles at two times for both sets of parameters are compared in Fig. 14f. Long after the case with $\frac{\tau_\ell}{B_0} \sim O(1)$ has ceased to look thin that for which $\frac{\tau_\ell}{B_0} \ll 1$ still remains so, and the heat flux levels at both piston and shock are drastically different.

In Figures 14g, 14h, and 14i nonmonotonic (entropy layer induced) behavior of surface properties is investigated. The surface pressure distribution for both sets of parameters is presented in Figure 14g. As is the case when almost any nonequilibrium process is considered, the variation of surface pressure is reasonably insensitive to significant changes in most parameters. The greatest deviations, approximately twelve percent, occur near the "nose", but farther back along the surface the shock layer is sufficiently thick that disturbances reflected from it are rapidly attenuated and an essentially isentropic expansion ensues. The pressure distribution is monotonic only because the piston is decelerating, but it is clear that there may be significant alterations of aerodynamic moment coefficients if the situation is taken to represent an airfoil. Since the trend in reentry vehicles is toward slender bodies, this can have important consequences in aerodynamic stability. The variation with time of the temperature of the gas at the wall is shown in Figure 14h. When radiation is more important, the temperature decreases very rapidly initially and tends asymptotically to the wall temperature as the field grows increasingly optically thick. The behavior of the heat flux into the wall, Figure 14i, is quite different, since as the shock wave decays and the piston decelerates the rate of loss of energy to the wall must decrease. As the temperature of the gas at the wall approaches the wall temperature and the shock layer grows thick, the gas at the wall cannot radiate toward it and the fluxes from elsewhere are attenuated significantly before they reach the wall. Thus it is verified that radiative heating of the surfaces of slender bodies is most severe near the nose. The effect of Boltzmann and Bouguer numbers upon shock decay is shown in Fig. 14j, while the complete history of the motion is given in Fig. 14k. Obviously the decay is more rapid the larger the radiation-convection parameter, τ_ℓ/B_o .

The reasons for termination of solutions prior to the attainment of radiative equilibrium are apparent, since all of the significant effects had appeared by $t \sim 0.35$. The last curves in Figures 14c and 14e prove that the qualitative aspects of the flow had ceased to change. More important, large velocity gradients had begun to appear. This required the insertion of more and more points in the flow field with a concurrent decrease in step size and increase of computation time. This forewarns of the fact that molecular transport phenomena would become important so that the present inviscid analysis would be inappropriate. The investigation of this case alone involved approximately 2.5 hours of 7094 computation.

6.2 Specific Difficulties

Although most difficulties mentioned have been resolved, that associated with the study of large piston velocities is serious. Despite the fact that the study of $U_p \lesssim 3$ may be justified on various physical grounds (Section 5.7) the analysis must eventually be extended. The study of pistons with large velocities leads rapidly to numerical difficulties. The non-self-consistent initial temperature distributions of Section 5.3.2 predict large values of I_0 at the wall; hence large heat fluxes exist as well. Consider that if $U_p \sim 3.0$, the temperature downstream of the shock is ~ 10, hence $\alpha_p T^4 \sim 10^9$! The initial guess for I_0 at the wall is extremely critical because the field is emission dominated and the numerical approximation to the piston path is discontinuous, the driving force being applied in discrete pulses at each step in time. Therefore the field may radiate so strongly while it waits for the next input that it may lose more energy than is available. One obvious cure for such behavior is to shrink the time increments. This radiation limitation is, in fact, the most stringent stability criterion, since one would expect the allowable time increments to be larger when the shock and piston move rapidly. Physical shrinking of time increments is not the proper approach.

When quantities of order 10^9 appear, the normalization scheme is obviously inappropriate. The difficulty can be traced to the fact that there are two disparate length scales: the upstream and downstream absorption lengths. Neither is universally appropriate, and since the shock will continually weaken the levels of temperature and pressure downstream of the shock, which are appropriate for normalization, are not known. Suppose, however, that these are assumed to be the nonradiating values. The Rankine-Hugoniot relations in the form (5.102) require that

$$u_s \sim \left(\frac{\gamma+1}{2}\right) U_p \tag{6.4a}$$

$$T_{(-)} \sim \left(\frac{\gamma-1}{2}\right) U_p^2 \tag{6.4b}$$

$$P_{(-)} \sim \left(\frac{\gamma+1}{2}\right) U_p^2 \tag{6.4c}$$

if $U_p \gg 1$. Therefore, the governing equations may be renormalized by the quantities above.

$$\bar{u} = u/\sqrt{T_{(-)}} \qquad (6.4d)$$
$$\bar{T} = T/T_{(-)} \qquad (6.4e)$$
$$\bar{P} = P/T_{(-)} \qquad (6.4f)$$
$$\bar{t} = t\sqrt{T_{(-)}}\,\alpha_p(P_{(-)},T_{(-)}) \qquad (6.4g)$$
$$\bar{X} = X\,\alpha_p(P_{(-)},T_{(-)}) \qquad (6.4h)$$

If the parameters τ_ℓ and Bo are based upon properties downstream of the nonradiating shock wave the form of the equations is unchanged. Upstream of the shock the emission condition (5.112) requires that

$$\bar{I}_o - \bar{q}\sqrt{3} = 4\left(\frac{1}{T_{(-)}}\right)^4 \sim 4\left(\frac{\kappa T_\infty^*}{\frac{\gamma-1}{2}u_p^{*2}}\right)^4 << 1 \qquad (6.5)$$

so that unless the ratio of specific heats is near unity the upstream gas hardly emits or absorbs. The same is true at the wall if $T_w \sim O(1)$. However, nonemitting, nonabsorbing upstream states are singular for large times. Thus either upstream absorption is neglected so that nonuniformity may ensue as the motion proceeds, or it is retained at the expense of keeping track of exceedingly small quantities. This suggests perturbation procedures, but their consideration must be withheld at this stage since even the qualitative trends are not clear.

6.3 A Cylindrical Investigation

Cylindrical situations may lead to geometrical difficulties since near the cylindrical axis the (j/r) term in the compatibility relations Eqs. (5.20), (5.21), and (5.23) may outweigh the other terms. Examination of (5.28) and (5.30) reveals that the potentially dangerous terms may be balanced by radiative effects, and this was in fact the case. The nonplanar investigations were begun just as the planar ones were, with the study of the effect of the initiating assumption. The results are shown in Figures 15a-15l.

Since the difficulty mentioned above was expected at the outset, larger initiating times ($t_i \sim 0.025$) were used than were appropriate for planar cases. Although Figure 15e indicates an inflection in the shock wave, this behavior may be shown to be consistent with previous remarks (Section 5.5). It has been pointed out that the wavehead first moves rapidly (at the isentropic propagation speed), then more slowly (at nearly the isothermal speed), and finally at the larger speed; hence an inflection. This leads to consideration of two possible long time behavior patterns for a general wedge or cone-like situation.

The pattern exhibited by the last planar case discussed, the ogive, presages one possibility. The field is optically thin initially so that energy is given up by the shock layer and is absorbed upstream. As the shock wave moves frather ahead of the piston, the radiative field grows optically thick so that eventually the shock cannot "see" the radiative disturbance generated at the piston face. Eventually there is no radiative input to the upstream gas from the piston, and the shock layer may regain the energy which it lost by engulfing the preheated gas. In such a case the shock at large times from the start of motion must move at nearly the nonradiating shock speed. Therefore there must be an inflection somewhere. If radiation were strong, it is possible that so much energy would be lost in the initial stages that sufficiently large upstream disturbances would be induced for compression from the free stream to the piston to be possible without a shock wave. It has been noted in Chapter 1 that the analysis of Heaslet and Baldwin (1963a) allows for such behavior. Which of the two situations will obtain depends upon the evolution of the flow field at intermediate times. If the radiation were turned on at a fairly large t_i the radiative field would be optically thick from the outset. There would be a large, almost instantaneous loss of energy which would then be rapidly reabsorbed. Thus the early inflection in Figure 15c is justified. This situation would not occur physically unless the motion were initiated (in a shock tube, say) with a weakly absorbing gas and at some later stage large amounts of a strong absorber were introduced. The possibility is interesting.

Figure 15a indicates extreme nonmonotonic behavior in surface pressure. After the initial dip, all of the curves coalesce within a timeof order t_i. Figure 15b shows quite different trends in the gas temperature at the wall. The distrib utions obtained for different initiating times do not coalesce for quite a long time. The reason for this is seen in Figure 15d which presents the history of the heat flux to the wall. The non-self-consistent starting solution underpredicts the heat flux, thus all solutions begin with essentially zero flux no matter when the radiation is begun. The greater the excess energy in the shock layer the greater is the instantaneous loss. Because of fixed wall temperature and emissivity (ϵ =0.25 in this case) the maximum heat flux is rapidly achieved, and the gas near the wall must cool at the same rate independent of the excess energy which remains after the initial loss.

As the time at which radiation is turned on, t_i, is decreased, a near vanishing heat flux is more nearly correct. This is illustrated by the fact that the shock slope has no inflection for $t_i < 0.025$. The overshoot in surface pressure and heat flux disappear, and the correct behavior may be deduced. The results, extrapolated to zero initiating time, are presented in Figures 15e and 15f. Solutions were carried out both for black and nearly reflecting piston surfaces in order to insure that what was observed was due mainly to cylindrical geometry. Naturally the heat flux to the wall is considerably less when emissivity is small than when it is nearly unity. Furthermore, nonmonotonic behavior in pressure is very pronounced and the gas temperature at the wall is nonmonotonic as well. Figure 15f illustrates the "suction" exerted upon entropy layers the nearer the wall is to being a perfect absorber. Nonmonotonic behavior is much more pronounced when the flux into the wall is highly constrained ($\epsilon \ll 1$).

It is noted that surface pressure eventually rises above the nonradiating conical flow value whereas the shock continually decays. A similar phenomenon occurs on blunted cones (Traugott, 1960) wherein the surface pressure rises above the pressure at the junction of the conical frustrum and the blunt nose. This behavior is independent of emissivity. As the shock layer grows thick more energy is reflected from the wall if the emissivity is not unity. Eventually radiative cooling takes place exclusively near the shock; energy reflected from the wall is absorbed very close to it, and the gas located there grows warmer. Note that at small times, when the heat flux is always small, emissivity has little effect upon the flow field.

The phenomena which have been noted lead one to distrust the initiating assumption in nonplanar flows. Pressure, velocity, and temperature profiles at selected intervals for both emissivities are presented in Figures 15g, 15h, and 15i. The conical starting solution (which is indicated on each of the figures by t=0) requires continual compression from shock to piston; therefore, the pressure distribution no longer can reflect the transient as it did in the planar case. Velocity is rather insensitive as well, thus the temperature distribution is now the key to the explanation of the difficulties. Since the conical flow is isentropic, the temperature must behave qualitatively exactly as the pressure does. Thus the starting solution in Figure 15i possesses highest temperatures near the piston. This is contrary to what the radiative field would prefer, and large heat fluxes are required. Temperature gradients did not exist in the tangent wedge planar starting solutions, and those solutions always maintained a near zero flux into the wall after radiation was begun. The interaction between radiation and convection is, therefore, much more important in nonplanar than in planar situations. From the outset the radiation will resist the formation of adverse temperature gradients. Nevertheless, if the shock layer is thin enough there can be no appreciable energy loss, and very close to the "nose" the conical flow is still appropriate.

The pressure distributions of Figure 15g do not respond to changes in emissivity. It has been shown that the effect of a small emissivity is to force the majority of the energy loss to occur at the shock wave; thus the entropy layer near the shock should be more pronounced when $\epsilon=0.25$ than when $\epsilon=1.0$. This is demonstrated in Figure 15i. The development of the temperature profiles was monitored very closely, and the inflected profile is correct. (The appearance of such profiles precludes any "integral" approach.) The heat flux distributions given in Figure 15j are qualitatively the same as those observed in Figure 14e. The (j/r) term in the continuity and heat flux equations resists optically thick behavior, and gradients near the shock are less steep than in the planar case. The flux distributions are displaced when emissivity is increased, but the same net energy losses occur. The loss is redistributed between the wall and the upstream gas, hence the differences in the magnitude and extent of upstream absorption in Figure 15k. It must be realized that if energy is lost to the wall ($\epsilon \simeq 1$) it may not be recovered whereas it is still available if it is lost to the upstream gas ($\epsilon \ll 1$). This may have serious consequences in the subsequent motion. The extent of upstream absorption is relatively less than that of the planar case, but dimensional considerations alone dictate that the precursor must extend the order of a free stream absorption length (Cohen, 1967). Since the energy loss from the shock layer is insensitive to emissivity, it is not surprising that the shock layer thickness, Figure 15ℓ, is insensitive as well.

The insensitivity of the pressure and velocity fields induced by cylindrical pistons to variations in wall emissivity is predicted by Eq. (3.70)

$$I_{o_w} = 2\epsilon T_w^4 + \left(1-\frac{\epsilon}{2}\right)\sqrt{3}\int_0^\infty e^{-\sqrt{3}\tau_\ell \eta'}\,\tau_\ell\left\{4\frac{\alpha_e}{\alpha_a}T^4 - j\frac{g}{r\tau_\ell \alpha_a}\right\}d\eta' \qquad (6.6)$$

As the emissivity decreases the second term is weighted more heavily than the first. In addition the heat flux is constrained for small values of the optical variable, η, where the weight $e^{-\sqrt{3}\tau_\ell \eta}$ is largest, so that the integral itself will be larger. Thus geometrical effects may compensate for changes in emissivity. Since only I_0 affects the fluid dynamics, it is clear that nonplanar fields will be less sensitive to emissivity (and wall temperature) variations than planar fields will be.

It is conjectured that the rise of surface pressure above the nonradiating value (Figure 15e) and the decay of the shock velocity to nearly the piston velocity are harbingers of the loss of the shock wave. The imminent loss can be traced to upstream absorption. Furthermore, upstream absorption delays the approach to radiative equilibrium sufficiently that it is not computationally feasible to pursue it. The inclusion of semi-grey radiative transfer might improve the situation since in general $\alpha_P > \alpha_R$ and the approach to equilibrium might be more rapid if a smaller absorption coefficient (hence greater cooling) were required. The cone and wedge-like flow fields observed may not be interpreted as those over the corresponding portions of finite bodies because the characteristic length scale, $1/\alpha_\infty$, forces the field to be aware of its finite termination from the very start of motion. The piston analogy reveals that although the previous investigation was terminated at the corresponding distance along a slender cone is the order of 10^3 meters in air and this is quite far indeed.

The distributions of I_0 have not been discussed in detail in any of the investigations since they have no clear physical meaning. The character of the velocity field induced upstream has been neglected as well. Representative values of these variables are presented as they came from the computer in Section E.3. It is obvious that the upstream velocity perturbation is not important.

6.4 Conclusions

The suitability of a characteristics approach to general problems in unsteady RGD has been demonstrated through the investigation of one-dimensional piston problems in ideal, radiating gases. The following general conclusions have been deduced as well.

1. Only the Mark-type boundary condition is consistent with a P_1 differential approximation if upstream absorption of shock layer radiation is allowed. If any other data are applied, divergence in the far field is inevitable.

2. The singular nature of the optically thick limit is related to the transition of the character of the governing equations from hyperbolic to parabolic. Furthermore, recognition of the hyperbolic nature of the equations of unsteady RGD has proved that only transparent shock waves across which nonequilibrium chemical processes must be frozen are allowed in general.

3. The manner in which the radiative field is initiated cannot affect the entropy levels of the subsequent flow fields. The coupling between radiation and convection in the early stages of motion is, however, more important in nonplanar fields than in planar ones.

4. The nonlinear interaction of disturbances may be more important in RGD than in other nonequilibrium flows. Thus linear theories may not be as appropriate as they are in other situations.

5. The differential approximation to radiative transfer is quite adequate for most investigations of slender body flows in RGD. What is believed to be the first self-consistent comparison of the full transfer equation with the differential approximation in RGD indicates that surface properties are predicted quite well by the approximate method, and although errors as high as ten percent occur in the velocity field error growth is slow. The differential approximation is most accurate when upstream absorption is included.

6. Nonmonotonic behavior of surface properties may occur in radiating flows. Variable surface emissivity and temperature may exert "blowing" or "suction" upon entropy layers and thus inhibit or enhance nonmonotonic (entropy layer induced) effects.

7. The pressure distributions along the surfaces of slender bodies are insensitive to large variations in Boltzmann and Bouguer numbers and to changes in wall emissivity. The observable motion, as indicated by the shape of the shock wave, is insensitive to variations in wall emissivity.

8. Upstream absorption of shock layer radiation may be a dominant mechanism even in slender body flows with reasonably weak shock waves. The extent of the upstream precursor is the order of a free stream radiation absorption length and is less in cylindrical than in planar geometries.

9. Contrary to previous observations, the effects of radiation upon pressure and velocity may be comparable to those upon temperature

The investigation presented required approximately twelve hours of IBM 7094 computation of which five were required for debugging. The inclusion of chemical processes should accentuate the phenomena which have been observed. It is hoped that the present approach will aid in the refinement of the design of slender reentry vehicles.

CHAPTER 7

EXTENSIONS AND FUTURE DEVELOPMENTS

Previous results indicate the utility of a characteristics approach to unsteady RGD. The possibilities have hardly been explored, and future investigation can take either of two courses. Either one-dimensional problems may be pursued in great detail with improved gas models and numerical techniques, or the "simple" analysis may be extended to multi-dimensional cases. Both types of extensions will be discussed below in addition to new approaches suggested by the difficulties which have been encountered.

7.1 Improvement of the Physical Model

The first steps in a more realistic description of one-dimensional situations should include equilibrium chemistry. The most severe complications arise in the solution of the Rankine Hugoniot equations. The most accurate determination of the changes across shock waves in gases in chemical equilibrium requires iteration with the aid of a (numerical) Mollier diagram. If curve fits to Mollier charts are excluded, the shock relations become a set of nonlinear algebraic and functional equations, and although their solution is not difficult in principle it requires reasonably long computation times. A model which involves a mixture of atoms, ions, and electrons is particularly suitable because the radiative processes admissible in such a system are well understood and because the equilibrium thermodynamics is reasonable. The use of a semi-gray gas with such a model is still justifiable. It may be assumed that little difficulty will be encountered in the pursuit of these analyses, and a more sophisticated approach will now be considered.

Recently Ferrari and Clarke (1964) and Clarke and Ferrari (1965) have suggested a model for nonequilibrium radiation in an ionized gas. Their study was prompted by observations in shock tubes and in reentry phenomena of rather large electron precursors. The situation is not yet resolved, but it has been proposed that the electron precursor may be due to ambi-polar diffusion, to photoionization, or to both. Lederman and Wilson (1967) claim to have resolved some of the experimental uncertainties, and they conclude that precursor ionization is caused by radiation from the shocked gas. Furthermore, they observe that the precursor can extend for several meters ahead of the shock (roughly $\sim 1/\alpha_{P_\infty}$) and that the farthest extent of the precursor moves at nearly the shock velocity. Both of these observations agree with the present ideal gas results (Sections 5.9.2, 6.3).

Space does not permit the presentation of the details of the Clarke and Ferrari analysis. Their nonequilibrium model may, however, be incorporated in a semi-gray analysis. The authors mention that one might attempt a differential analog of their nonequilibrium model, but they do not indicate how this might be done. The procedure will be described in the following pages.

The radiative heat flux is divided into contributions from atomic lines, from their wings (the part to which broadening contributes), and from the continuum. Clarke and Ferrari then argue that some suitably chosen frequency averaged absorption coefficient is associated with each contribution and that the extent of the ionization precursor region will be of the same order as the reciprocal of the continuum average. They conclude further that the line contribution is almost always optically thick (hence, a minor effect), and that the contribution of the wings is extremely optically thin in the temperature range $T \lesssim 1 ev$. Only the continuum radiation, which is contributed to by bound-free processes, enters the production or destruction of ions. This is becaus e a photon of greater than a certain minimum frequency is required to free an electron from a given shell. All three contributions enter the energy balance, however. The "thick" line contribution may be neglected if temperature gradients are moderate. Therefore, radiative properties for $\nu < \nu_j$, the frequency which corresponds to the ionization potential, are hardly of consequence.

The following dimensional transfer and rate equations may be derived if induced radiative recombination is neglected.

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \vec{\Omega}\cdot \mathrm{grad}\, I_\nu = \rho \kappa_\nu (1-C)\left\{\left(\frac{C^2}{1-C}\right)\left(\frac{1-C_e}{C_e^2}\right) B_\nu(T) - I_\nu\right\} \tag{7.1}$$

$$\rho\left\{\frac{\partial C}{\partial t} + \vec{u}\cdot \mathrm{grad}\, C\right\} = -m_a \,\mathrm{div}\int_{\nu_j}^{\infty} \frac{\vec{q}_\nu}{h\nu}\, d\nu + \frac{\rho C}{\tau_c}\left\{\frac{(1-C)}{(1-C_e)} C_e^2 - C^2\right\} \tag{7.2}$$

$$C \equiv \frac{\rho_i}{\rho} = \frac{n_i}{(n_a + n_i)} \tag{7.3}$$

The quantity C is the mass fraction of ions and it is assumed that

$$T_j = \left(\frac{h\nu_j}{k}\right) \gg T$$

Furthermore, κ_ν is assumed to have been corrected for induced emission. The last term in Eq. (7.2) is the familiar "Freeman Rate" expression (Clarke and McChesney, 1964), and the collisional relaxation time is

$$\tau_c = \frac{c_{nc} T}{\rho^2}\left\{1 - 2\left(\frac{T}{T_j}\right) + O\left(\frac{T}{T_j}\right)^2\right\}^{-1} \tag{7.4}$$

The local equilibrium ionization fraction is given by the Saha equation

$$\frac{C_e^2}{1-C_e} = \frac{c_{nc}}{c_{ic}}\left(\frac{T^{3/2}}{\rho}\right) e^{-\left(\frac{T_j}{T}\right)} \tag{7.5}$$

wherein the relevant parameters are

$$\frac{1}{c_{ic}} = \left(\frac{8\pi k}{m_e}\right)^{\frac{1}{2}}\left(\frac{e^4}{m_a \epsilon_j^2}\right) \tag{7.6a}$$

$$\frac{1}{c_{nc}} = \left(\frac{h^3}{2\pi k}\right)\left(\frac{Q_a}{Q_i}\right)\left(\frac{e^4}{m_a^2 m_e^2 \epsilon_j^2}\right) \tag{7.6b}$$

where $\epsilon_j = h\nu_j$ is the ionization potential for the lowest electronic state, m_e and $m_a \simeq m_i$ are electron and atom (ion) masses, and Q_a and Q_i are the electronic partition functions of atoms and ions. Note that if the collisional relaxation time is "small", $\alpha \simeq \alpha_e$ and the source function is given by the Planck Function. These equations will now be incorporated in a semi-gray model.

Suppose Eq. (7.1) is integrated over the frequency ranges $(0, \nu_j)$, (ν_j, ∞). The following two-group quantities are designated.

$$I^{(1)} = \int_0^{\nu_j} I_\nu \, d\nu \tag{7.9a}$$

$$I^{(2)} = \int_{\nu_j}^{\infty} I_\nu \, d\nu \tag{7.9b}$$

so that

$$I = I^{(1)} + I^{(2)} \tag{7.10}$$

The quantities I_0 and q are approximated analogously. If the reasoning of Chapters 2 and 3 is applied and one assumes quasi-isotropy in α_a, then:

$$\frac{1}{c}\frac{\partial I_0^{(2)}}{\partial t} + \operatorname{div} \vec{q}^{(2)} = (1-c)\left\{\left(\frac{c^2}{1-c}\right)\left(\frac{1-c_e}{c_e^2}\right)4\pi \alpha_{p,2} B^{(2)} - \alpha_{a,2} I_0^{(2)}\right\} \tag{7.11}$$

$$\frac{1}{c}\frac{\partial \vec{q}^{(2)}}{\partial t} + \frac{1}{3}\operatorname{grad} I_0^{(2)} = -\alpha_{a,2}(1-c)\vec{q}^{(2)} \tag{7.12}$$

and similarly for the quantities in the lower range. The subscripts on the absorption coefficients denote the respective group averages. Clarke and Ferrari examine the quantities $\int_{\nu_j}^{\infty} (h\nu)^{-m} I_\nu \, d\nu$ and conclude that (their Corollary II):

$$\int_{\nu_j}^{\infty} \frac{\vec{q}_\nu}{h\nu} \, d\nu \simeq \frac{\vec{q}^{(2)}}{h\nu_j} \tag{7.13}$$

whence

$$\rho\left\{\frac{\partial c}{\partial t} + \vec{u}\cdot\operatorname{grad} c\right\} = -m_a \operatorname{div}\left(\frac{\vec{q}^{(2)}}{h\nu_j}\right) + \frac{\rho c}{\tau_c}\left\{\left(\frac{1-c}{1-c_e}\right)c_e^2 - c^2\right\} \tag{7.14}$$

In addition, the two-group volumetric energy gain is:

$$\operatorname{div} \vec{q} = (1-c)\left\{\left(\frac{c^2}{1-c}\right)\left(\frac{1-c_e}{c_e^2}\right)4\pi\left[\alpha_{p,1} B^{(1)} + \alpha_{p,2} B^{(2)}\right] - \left[\alpha_{a,1} I_0^{(1)} + \alpha_{a,2} I_0^{(2)}\right]\right\} \tag{7.15}$$

If the lines are weak and $\frac{T_j}{T} \gg 1$, one may safely assume that for $\nu \gtrsim \nu_j$

$$B_\nu(T) \simeq \frac{2h\nu^3}{c^2} e^{-\frac{h\nu}{kT}} \tag{7.16}$$

Therefore:

$$B^{(2)}(T) = \left(\frac{2k\nu_j^3}{c^2}\right) T e^{-\frac{T_j}{T}} \tag{7.17}$$

The Clarke and Ferrari assumption

$$\kappa_\nu = \kappa \left(\frac{\nu_j}{\nu}\right)^2 \tag{7.18}$$

leads to

$$\alpha_{P,2} = \rho\kappa \left(\frac{T_j}{T}\right)^2 \left\{ \frac{\frac{T_j}{T} + 1}{\left(\frac{T_j}{T}\right)^3 + 3\left(\frac{T_j}{T}\right)^2 + 6\left(\frac{T_j}{T}\right) + 6} \right\} \tag{7.19}$$

Similarly

$$\alpha_{R,2} = \rho\kappa \left(\frac{T_j}{T}\right)^2 \left\{ \frac{\left(\frac{T_j}{T}\right)^4 + 4\left(\frac{T_j}{T}\right)^3 + \frac{4!}{2!}\left(\frac{T_j}{T}\right)^2 + 4!\left(\frac{T_j}{T}\right) + 4!}{\left(\frac{T_j}{T}\right)^6 + 6\left(\frac{T_j}{T}\right)^5 + \frac{6!}{4!}\left(\frac{T_j}{T}\right)^4 + \frac{6!}{3!}\left(\frac{T_j}{T}\right)^3 + \frac{6!}{2!}\left(\frac{T_j}{T}\right)^2 + 6!\left(\frac{T_j}{T}\right) + 6!} \right\} \tag{7.21}$$

$$= \alpha_{P,2} + O\left(T/T_j\right)$$

The fact that the gas is nearly gray for $\nu \gtrsim \nu_j$ *follows for all power laws of the form, Eq. (7.18). If line radiation is neglected and the gas is assumed to be grey, Eq. (7.11), (7.12), (7.14), and (7.15) complete the system. The thermodynamic and state relations are:

$$\rho = \frac{m_a P}{k(1+c)T} \tag{7.22}$$

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* In Appendix A it is shown that the Rosseland mean is not necessarily appropriate.

$$h = \frac{5}{2}\left(\frac{k}{m_a}\right)T + E_c + c\left(\frac{k}{m_a}\right)T_j + O\left(\frac{T}{T_j}\right) \tag{7.23}$$

where E_c is the electronic energy of atoms and ions

$$E_c = \sum_{k_a, k_i} \epsilon_k \frac{e^{-\left(\frac{\epsilon_k}{RT}\right)}}{Q_{a,i}} \tag{7.24}$$

The indices a, i include all electronic levels of atoms and ions.

If all variables are normalized according to the Scheme of Chapter 3, the rate equation becomes

$$\rho\left\{\frac{\partial c}{\partial t} + \vec{u}\cdot \operatorname{grad} c\right\} = -\left(\frac{T_\infty}{T_j}\right)\frac{\operatorname{div}\vec{q}^{(2)}}{B_0} + \frac{1}{t_1^*}\left(\frac{\rho c}{\tau_c}\right)\left\{\left(\frac{1-c}{1-c_e}\right)c_e^2 - c^2\right\} \tag{7.25}$$

where

$$t_1^* = \frac{(C_{no})_\infty T_\infty \sqrt{RT_\infty}}{\rho_\infty^2 \ell} \tag{7.26}$$

Also

$$\rho = \frac{P}{T(1+c)} \tag{7.27}$$

$$h = T + \frac{2}{5}E_c + \frac{2}{5}c\left(\frac{T_j}{T_\infty}\right) \tag{7.28}$$

where $\left(c_{p_f}\right)_\infty = \frac{5}{2}\left(\frac{k}{m_a}\right)$

The transfer equations remain unchanged. In the limit $\beta \to 0$ the significant one dimensional compatibility relations, Eq. (4.44) - (4.46) are:

$$\frac{d^\pm p}{dt} \pm \rho a_f \frac{d^\pm u}{dt} = \left(\frac{\tau_\ell}{B_0}\right)\frac{2}{5}q^{(2)}\frac{(1+c)a_f^2}{\left(1+\frac{2}{5}\frac{dE_c}{dT}\right)} - j\,\frac{\rho u}{r}a_f^2 - \frac{\frac{2}{5}a_f^2}{T\left(1+\frac{2}{5}\frac{dE_c}{dT}\right)}\cdot W\cdot\frac{2}{5}\left(\frac{T_j}{T_0}\right) \tag{7.29}$$

$$\frac{dT}{dt} - \frac{1}{\rho(\frac{5}{2}+\frac{dE_c}{dT})}\frac{dP}{dt} = \frac{\tau_1}{B_o}\frac{\{Q^{(1)}+Q^{(2)}\}}{(\frac{5}{2}+\frac{dE_c}{dT})} + \frac{2}{5}\left(\frac{W}{\rho}\right)\frac{(T_j/T_0)}{(\frac{5}{2}+\frac{dE_c}{dT})} \tag{7.30}$$

where

$$a_f^2 = \frac{5}{3}T\cdot(1+c)\left\{\frac{1+\frac{2}{3}\frac{dE_c}{dT}}{1+\frac{2}{3}(E_c-c)}\right\} \tag{7.31}$$

and W is the negative of the righthand side of Eq. (7.25)

$$W = \left(\frac{T_\infty}{T_j}\right)\frac{\operatorname{div}\vec{g}^{(2)}}{B_o} + \frac{1}{t_1^*}\frac{\rho c}{\tau_c}\left\{\left(\frac{1-c}{1-c_e}\right)c_e^2 - c^2\right\} \tag{7.32}$$

In accordance with previous assumptions, Eq. (7.32) shows that unless $B_o \ll 1$ the effect of radiation upon ionization is of higher order in (T_∞/T_j) than is the effect of ionization upon radiation. Therefore, line radiation may be included according to the scheme suggested previously. As long as the gas is assumed grey, even for $\nu < \nu_j$ no difficulties arise, and a two-group model may be handled with reasonable ease. In fact if α_a is evaluated in a region of chemical equilibrium and radiative near-equilibrium then Eq. (2.53) still holds. Even the most sophisticated attempts at a non-gray analysis (Smith and Hassan, 1966; Stone and Gaustad, 1961) use no more than a two-group model in the ensuing computations.

Clarke and Ferrari had little success in predicting ionization precursors with their model. This may be due to their rather drastic simplifications, e.g. $c \ll 1$, and the model outlined above should predict more realistic effects. Since it has been demonstrated that the MOC may be readily applied to radiating flows, the investigation of a flow with a chemical and radiative nonequilibrium is quite feasible, even in the two-group approximation.

7.2 Extension to Multi-Dimensional Radiative Transfer and Other Proposals

With the general approach outlined in Section 4.1.1 it has been shown that the fluid dynamic variables ($P, u, v, w, T, c_1, \ldots, c_N$) lead to ordinary Mach conoids and particle path lines. Note, however, that the compatibility relations from which P, u, v, w are found apply along generatrices of the Mach conoid*. There are infinitely many such lines, and depending upon which three are chosen different answers may result. If more than three are chosen the network becomes non-simplical and stability problems may arise. Chu(1967) has proposed a novel scheme whereby many lines may be used, and he has employed an error minimization technique which is quite satisfying. In this way one may arrive at unambiguous solutions. Time scales of order β are excluded because they would require numerical time steps $O(\beta)$ as well. As $\beta \to 0$ in a P_N , N-odd , approximation, the characteristic surface for the radiative properties is merely t = const (see equation (4.29)). On that surface the three equations of the differential approximation apply:

$$div\, \vec{q} = \tau_e \{4\pi \alpha_e B - \alpha_a^{(0)} I_0\} \tag{7.33}$$

$$grad\, I_0 = -3\tau_e \alpha_a^{(1)} \vec{q} \tag{7.34}$$

Thus there are still as many compatibility relations as there are dependent variables.

In a given surface of constant time, each point is affected by every other point; therefore only an implicit scheme is justifiable. The geometry for the solution of a two-dimensional unsteady problem is indicated in Fig. 16. A body moving along the x-axis with constant velocity is shown in the figure. Several Mach conoids are shown as is a constant time surface, but the shock wave has been omitted. An iterative scheme similar to that of Chapter 5 is now proposed. The surface t = const is divided with an appropriately chosen grid (not necessarily evenly spaced), and

* The temperature would follow from the relation along the path line.

to a first approximation, P, u, v, T may be found at each node independent of I_0, q_1, and q_2. With the properties of the gas now specified, the linear system, Eq. (7.33), (7.34) may be solved implicitly for I_0, q_1, and q_2 at each point (say with a relaxation method) after which another iteration could be made on P, u, v, and T. The method employs "hyperbolic" techniques through the use of characteristic surfaces for P, u, v and T, and "elliptic" techniques for the radiation. Although the scheme is simple in principle, there are certainly many hidden difficulties. Obviously a part of the advantage of unsteady gas dynamics is lost since large matrices must be inverted at each stage. In addition, numerical stability has not yet been investigated. It seems, however, that there should be no problem with P, u, v, or T since the CFL criterion is certainly satisfied. Rather, spatial instabilities in I_0, q_1, and q_2 will probably be the most serious difficulty.

Since integration over all of space is required, one may justifiably question the use of the differential approximation. In fact, the multi-dimensional technique for radiation appears very much like Hottel's zoning. However, the use of the approximate formulae avoids the geometric difficulties which one would most certainly encounter otherwise. The situation in more than one spatial dimension and time is very much different than that which was investigated in detail previously, but if one-dimensional experience is at all indicative of computational trends, it must be concluded that the solution of even a two-dimensional problem must await a new generation of high speed computers.

In addition to elaboration of the gas model and extension of the scheme to more spatial variables, the physical difficulties which have arisen must be examined. The need for the study of the region near initiation of motion, $\beta \ll t \ll 1$, is obvious. Furthermore, if pistons with discontinuous paths are to be investigated, the regions near these discontinuities must be examined analytically. A perfectly reflecting wall is in a sense singular, thus it might be of considerable interest to formulate a perturbation scheme based on a small emissivity. The phenomena associated with large piston velocities suggest a perturbation approach to the inclusion of upstream absorption. The appropriate parameter for expansion would be $(R T_\infty)/U_p^2$. Such a perturbation might well be singular for large times.

The fact that $\Gamma = \frac{\gamma_R - 1}{\gamma_R}$ is small presents some interesting possibilities. Moore (1966) and Mahony (1955) have considered the implications of γ near unity. In all of fluid dynamics this limit leads to great simplification. Equations (3.1) - (3.7) clearly show that, no matter what the nonequilibrium processes might be:

$$\frac{Dh}{Dt} = 0 \tag{7.35}$$

if $\Gamma \equiv 0$. Thus temperature is constant along a particle path. Since all particle paths originate in the undisturbed gas, it follows that $T = 1$ everywhere. The radiative equations are immediately uncoupled from the fluid mechanics. For instance, in the planar case it follows that

$$I_0(x,t) = 4 + 2\epsilon\,(T_W^4 - 1)\exp\left\{-\sqrt{3}\,\tau_e \int_{R_p(t)}^{x} \alpha_a^{(0)}(\xi,t)\,d\xi\right\} \tag{7.36}$$

$$q(x,t) = \frac{2}{\sqrt{3}}\,\epsilon\,(T_W^4 - 1)\exp\left\{-\sqrt{3}\,\tau_e \int_{R_p(t)}^{\infty} \alpha_a^{(1)}(\xi,t)\,d\xi\right\} \tag{7.37}$$

The fluid dynamics may then be handled readily in the framework of Reimann invariants which in this case are

$$\left.\begin{matrix} r_R \\ s_R \end{matrix}\right\} = \ln(P) \pm u \tag{7.38}$$

They are constant respectively on the isothermal Mach lines

$$\frac{dr}{dt} = u \pm 1 \tag{7.39}$$

The system may be inverted for the solution of x and t in terms of s_R and r_R.

$$\frac{\partial^2 t}{\partial s \partial r} + \frac{1}{4}\left(\frac{\partial t}{\partial s} + \frac{\partial t}{\partial r}\right) = 0 \tag{7.40}$$

This equation may be solved by the standard technique of Reimann-Green's Functions (Garabedian, 1964). Equation (7.40) may be reduced to

$$\hat{U}_{sr} - \frac{1}{16}\hat{U} = 0 \tag{7.41}$$

where

$$\hat{\mathcal{U}} = t\sqrt{P} \tag{7.42}$$

Just as hypergeometric functions are the fundamental solutions for ideal gas dynamics, the fundamental solution of Eq. (7.41) is

$$R_m(r,s;\xi,\eta) = J_{(0)}\left\{\frac{i}{2}\sqrt{(r-\xi)(s-\eta)}\right\} \tag{7.43}$$

where the modified Bessel functions, $K_{(0)}$ or $I_{(0)}$, are chosen depending upon the types of data specified. Therefore, given sufficient boundary data, the entire solution of the non-linear problem may be written to lowest order in Γ in closed form. These observations are independent of what specific nonequilibrium processes are present.

Consider next a local linearization about the non-linear, radiating solution. Let

$$\mathbb{P} = \frac{\partial P}{\partial \Gamma} \tag{7.44a}$$

$$\mathbb{T} = \frac{\partial T}{\partial \Gamma} \tag{7.44b}$$

$$\mathcal{U} = \frac{\partial u}{\partial \Gamma} \tag{7.44c}$$

$$\mathbb{I} = \frac{\partial I_0}{\partial \Gamma} \tag{7.44d}$$

$$\mathbb{q} = \frac{\partial q}{\partial \Gamma} \tag{7.44e}$$

so that, e.g.

$$P = P_0(x,t;\Gamma=0) + \mathbb{P}\,\Gamma + O(\Gamma^2) \tag{7.45}$$

A characteristics solution for $\mathbb{P}$, $\mathbb{T}$, $\mathcal{U}$, $\mathbb{I}$, and $\mathbb{q}$ is easily formulated. It may be shown that

$$\frac{d\mathbb{T}}{dt} = F_3(x,t;\mathbb{T},\mathbb{I}) \quad \text{on} \quad \frac{dx}{dt} = u_0(x,t) \tag{7.46}$$

$$\frac{d}{dt}\left[\ln(\mathbb{P}) \pm \mathcal{U}\right] = \frac{1}{P_0}\left\{F_2 - P_0 F_3 \pm P_0 F_1\right\} \text{ on } \frac{dx}{dt} = u_0(x,t) \pm 1 \qquad (7.47)$$

$$\frac{d}{dx}(q) = F_4(x, t; \mathbb{P}, \mathbb{T}, \mathbb{I}) \qquad (7.48)$$

$$\frac{d}{dx}(\mathbb{I}) = F_5(x, t; \mathbb{P}, \mathbb{T}, q) \qquad (7.49)$$

where:

$$F_1 = -\mathcal{U}(u_{0_x}) - \frac{(P_0)_x}{P_0}\left(\mathbb{T} - \frac{\mathbb{P}}{P_0}\right) \qquad (7.50)$$

$$F_2 = -\mathbb{P}(u_0)_x - \mathcal{U}(P_0)_x \qquad (7.51)$$

$$F_3 = -P_0 \cdot (u_0)_x - \frac{\tau_\ell}{B_0}\alpha_p(P_0, T_0)\left\{4 I_0 + 16\mathbb{T} - \mathbb{I}\right\} \qquad (7.52)$$

$$F_4 = \tau_\ell(4 - I_0)\left\{\alpha_{p_0}\mathbb{P} + \frac{\partial \alpha_p}{\partial T}\Big|_0 \mathbb{T}\right\} + \tau_\ell \alpha_{p_0}\left\{16\mathbb{T} - \mathbb{I}\right\} \qquad (7.53)$$

$$F_5 = -3\tau_\ell\left\{q_0\left[\alpha_{p_0}\mathbb{P} + \frac{\partial \alpha_p}{\partial T}\Big|_0 \mathbb{T}\right] + \alpha_{p_0} q\right\} \qquad (7.54)$$

Since the characteristics are known, the solution is somewhat simpler than that outlined previously. The expansion must nevertheless be non-uniform, since the characteristics of the "uncoupled" solution will appear at each stage. However, the shock wave will be altered each time. In detail, the exact compatibility relations and characteristic equations must be perturbed according to the standard PLK technique, and all quantities—including x and t —must be considered functions of the characteristic variables. The resolution of the difficulties will be worthwhile.

Many areas for investigation have been opened merely by recognition of the hyperbolic nature of unsteady RGD. The methods which have been proposed, while complicated, are far simpler than most which have been suggested. Furthermore, the use of characteristics provides insight into the non-uniform nature of various perturbation procedures.

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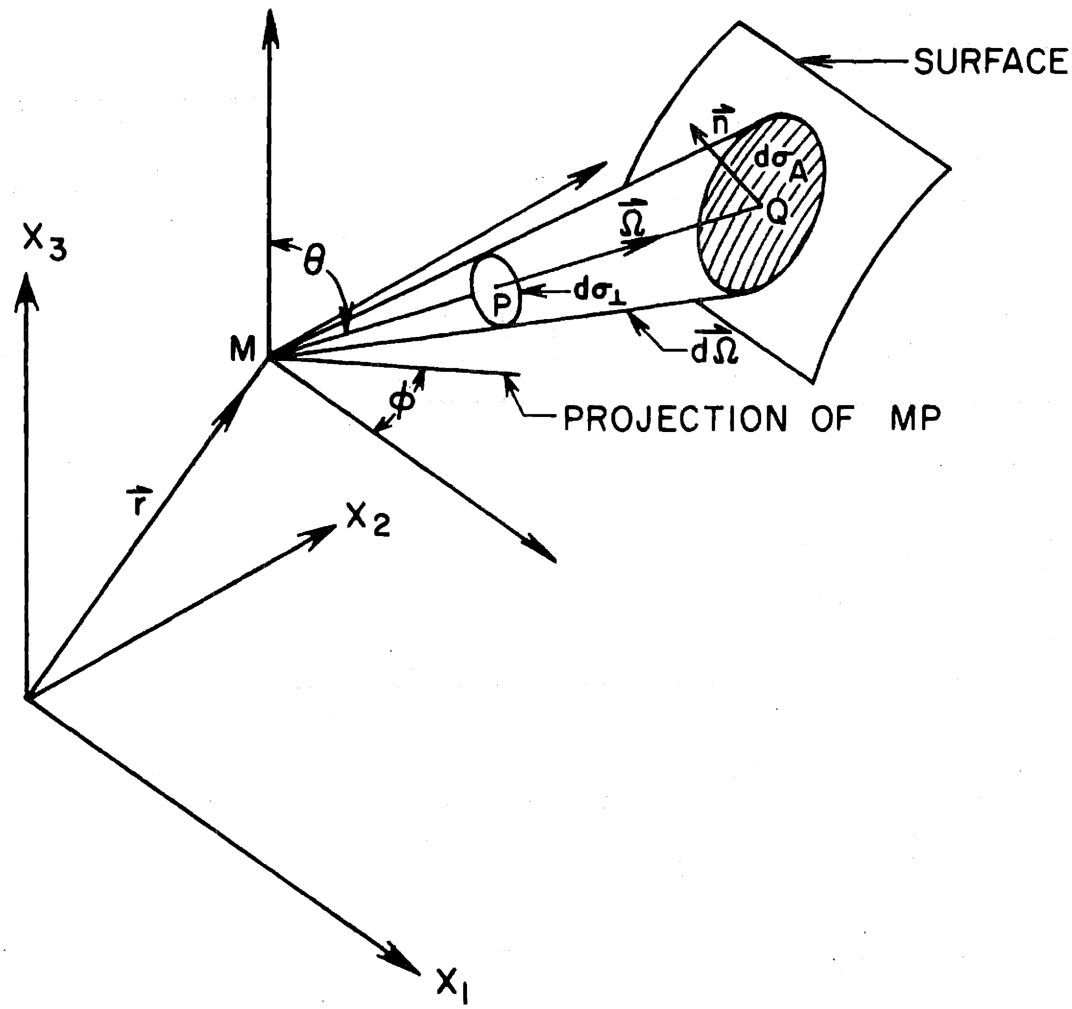

Figure 1. Geometry of general radiative transfer

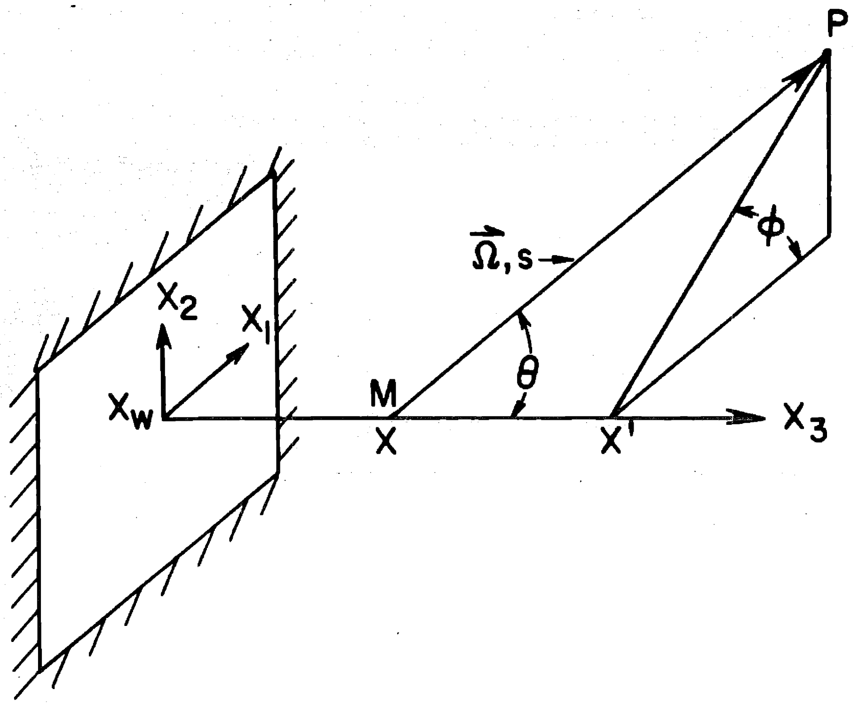

Figure 2. Geometry of planar radiative transfer

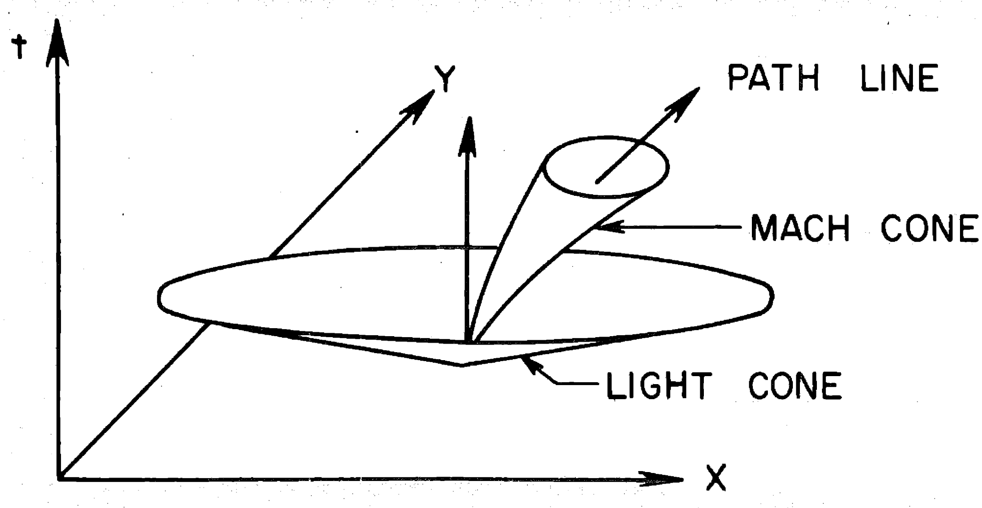

Figure 3. Characteristic surfaces of radiation gasdynamics

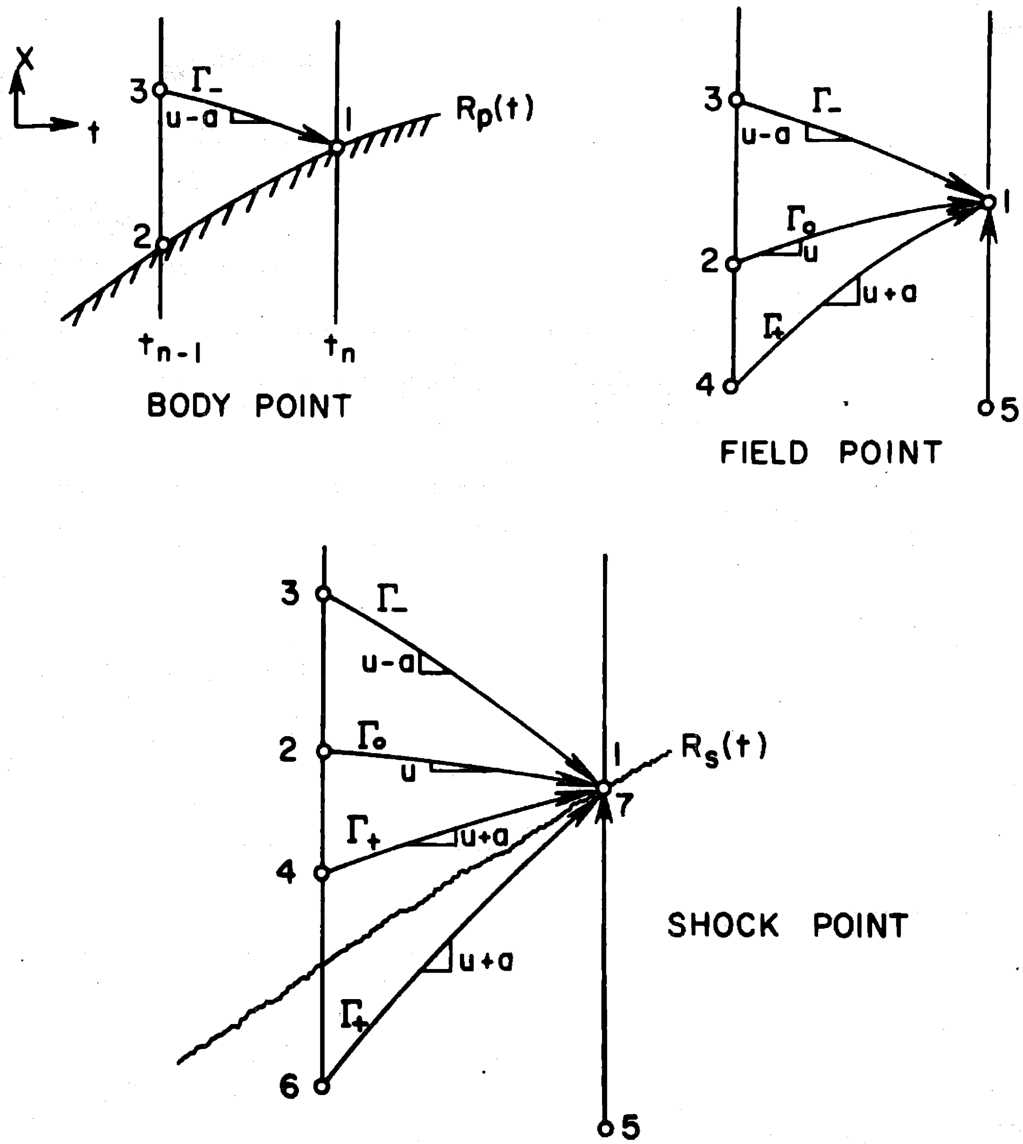

Figure 4. Geometry of the characteristics solution

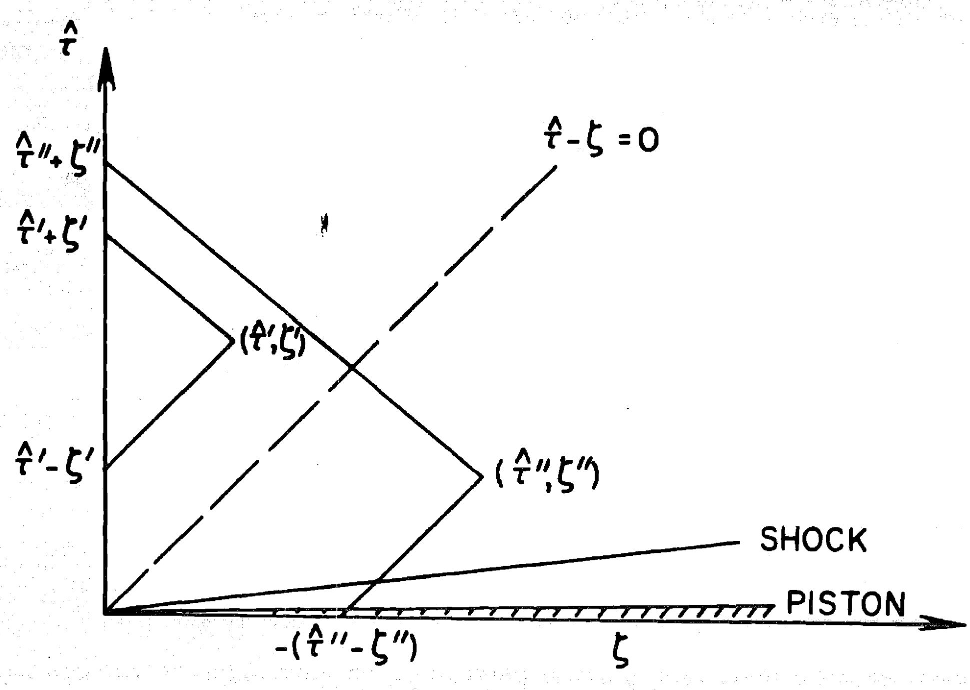

Figure 5a. Solution of radiative field for $t \sim 0(\beta)$

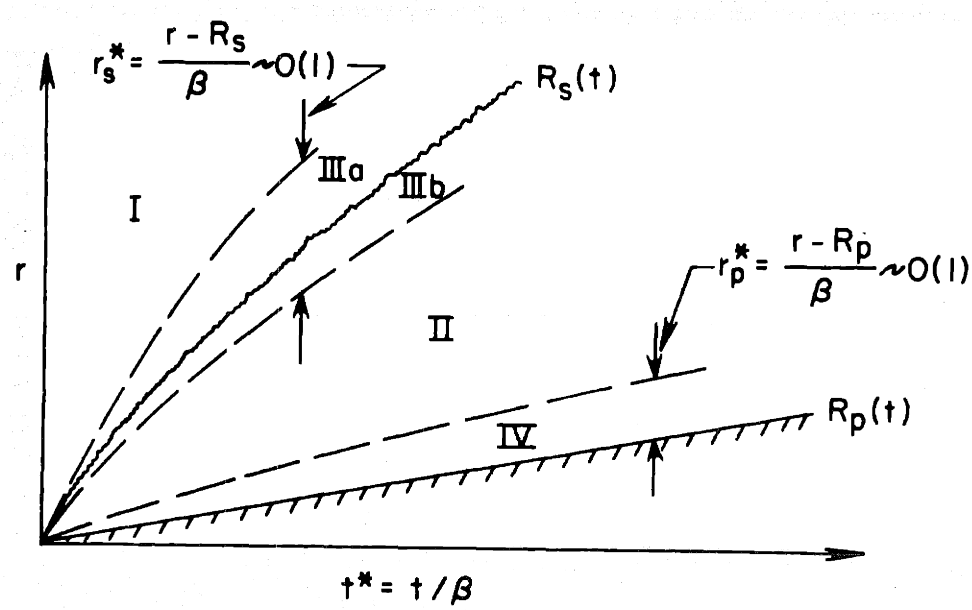

Figure 5b. Schematic solution of flow field for $t \sim 0(\beta)$

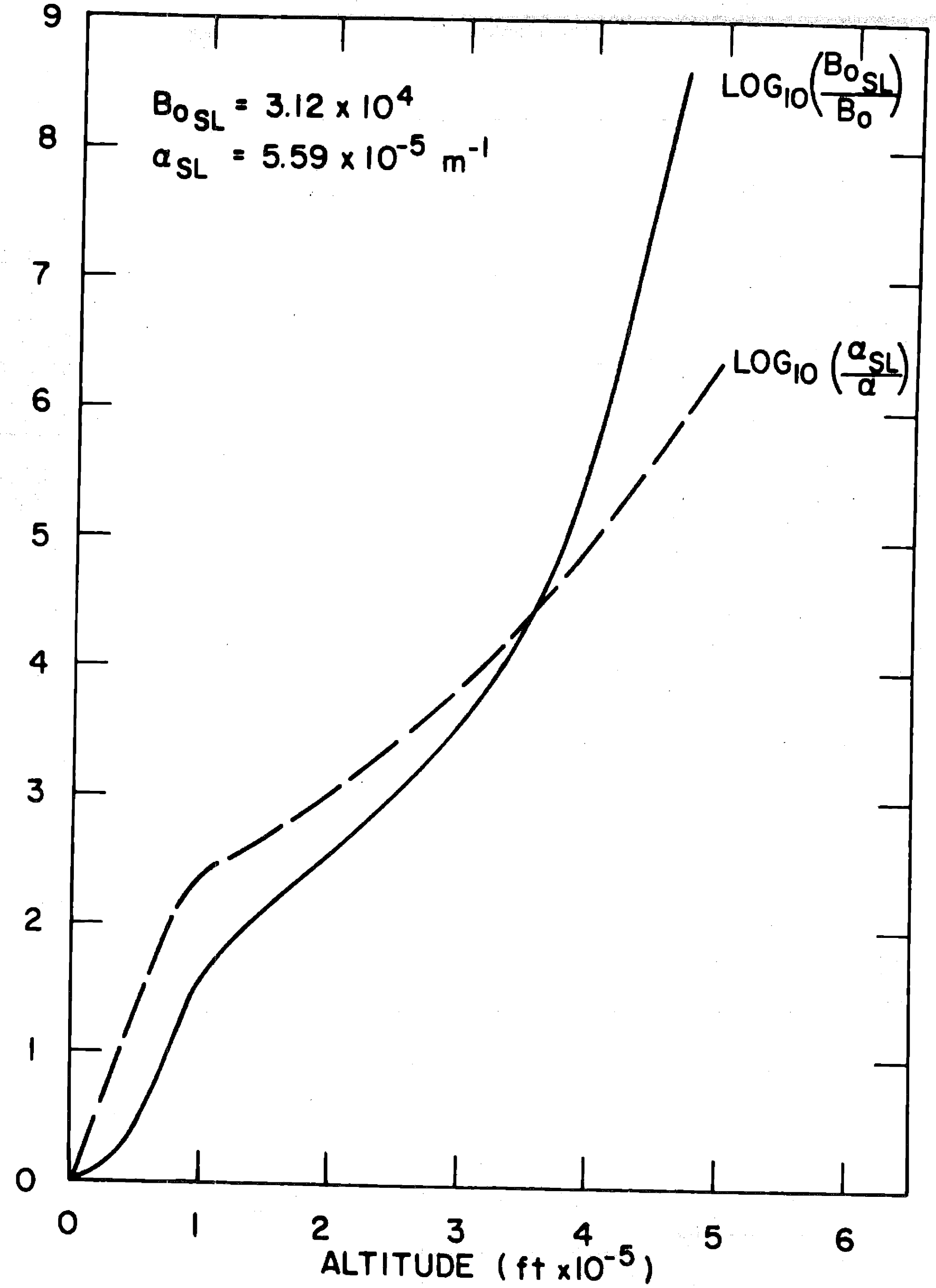

Figure 6. Radiative properties of air

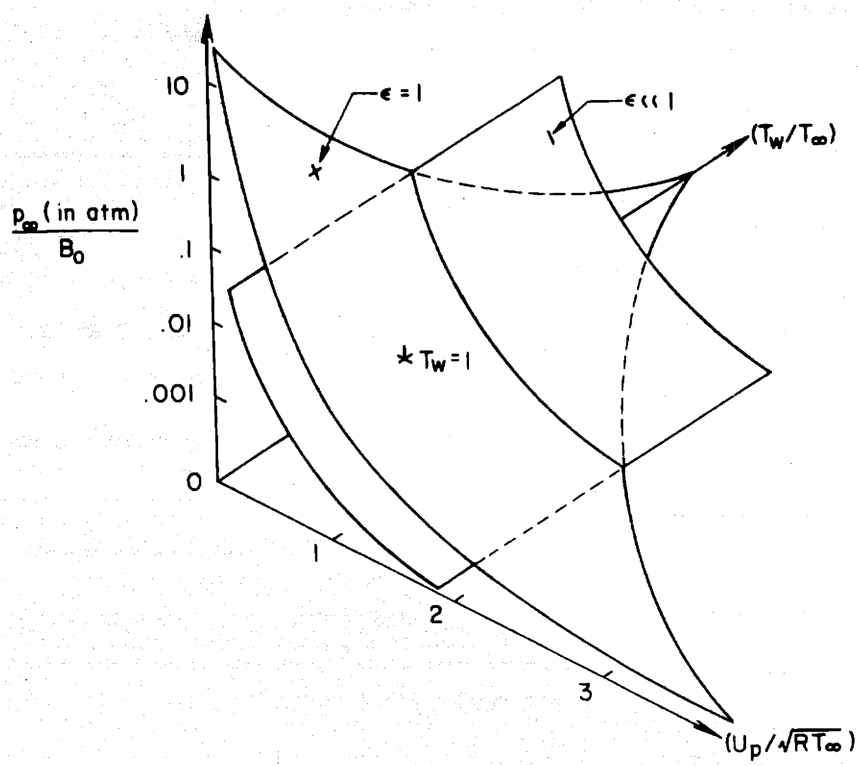

Figure 7. Region of validity of the investigation

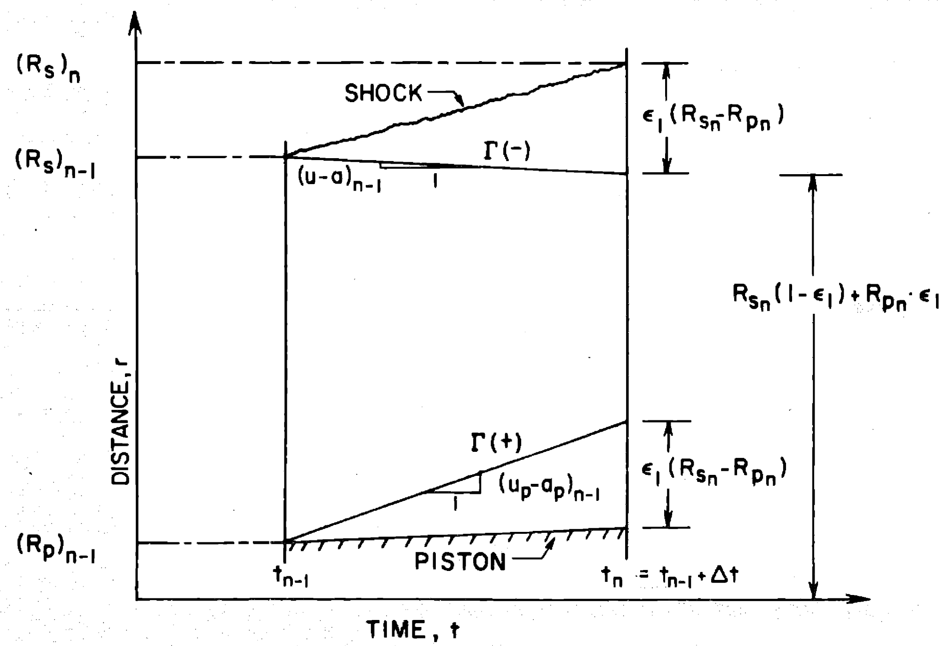

Figure 8. Determination of allowable step size

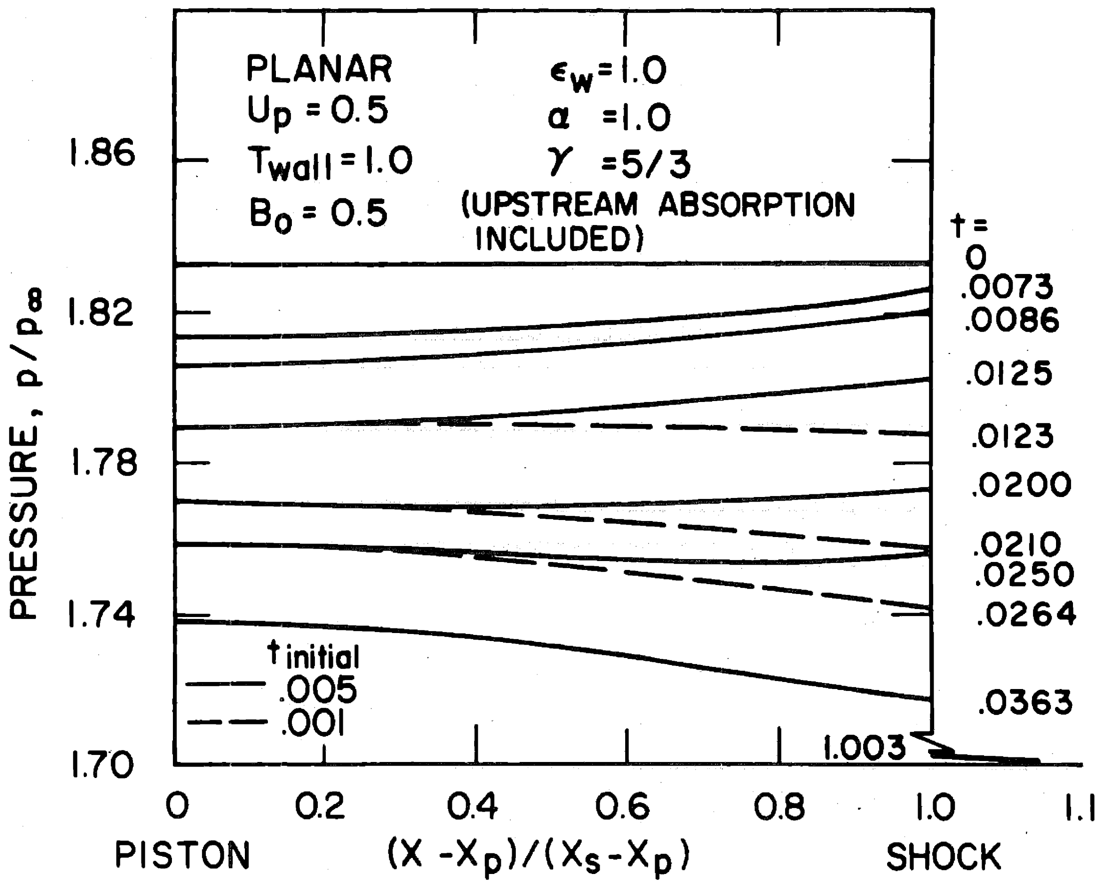

Figure 9a. Effect of initiating time upon pressure field induced by a planar piston in uniform motion

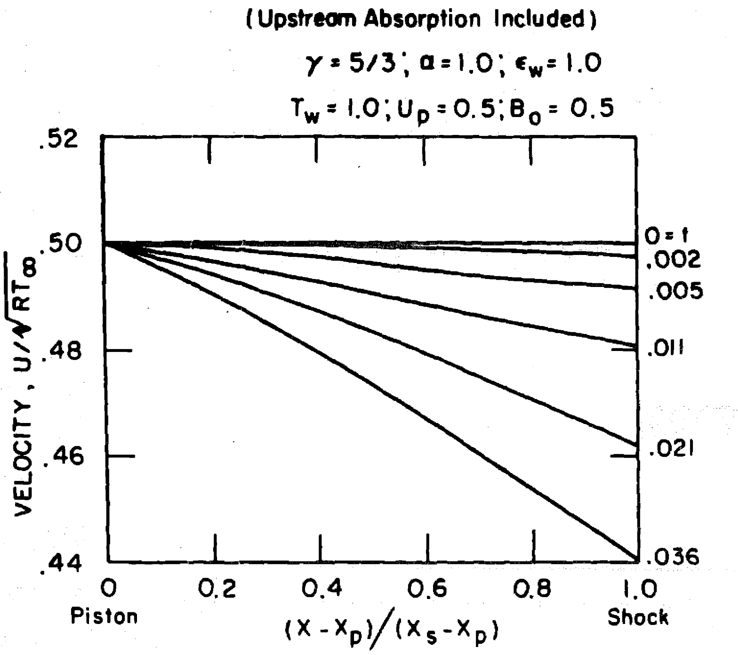

Figure 9b. Velocity field induced by a planar piston in uniform motion

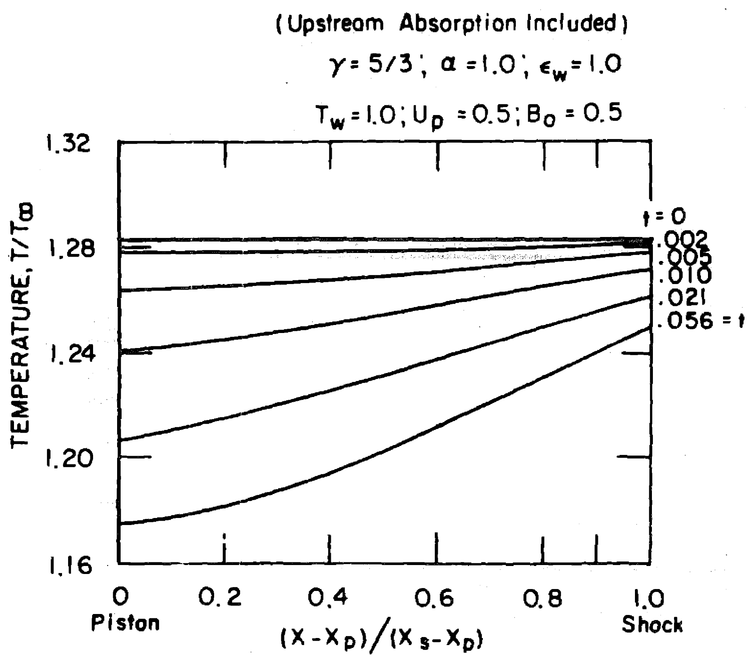

Figure 9c. Temperature field induced by a planar piston in uniform motion

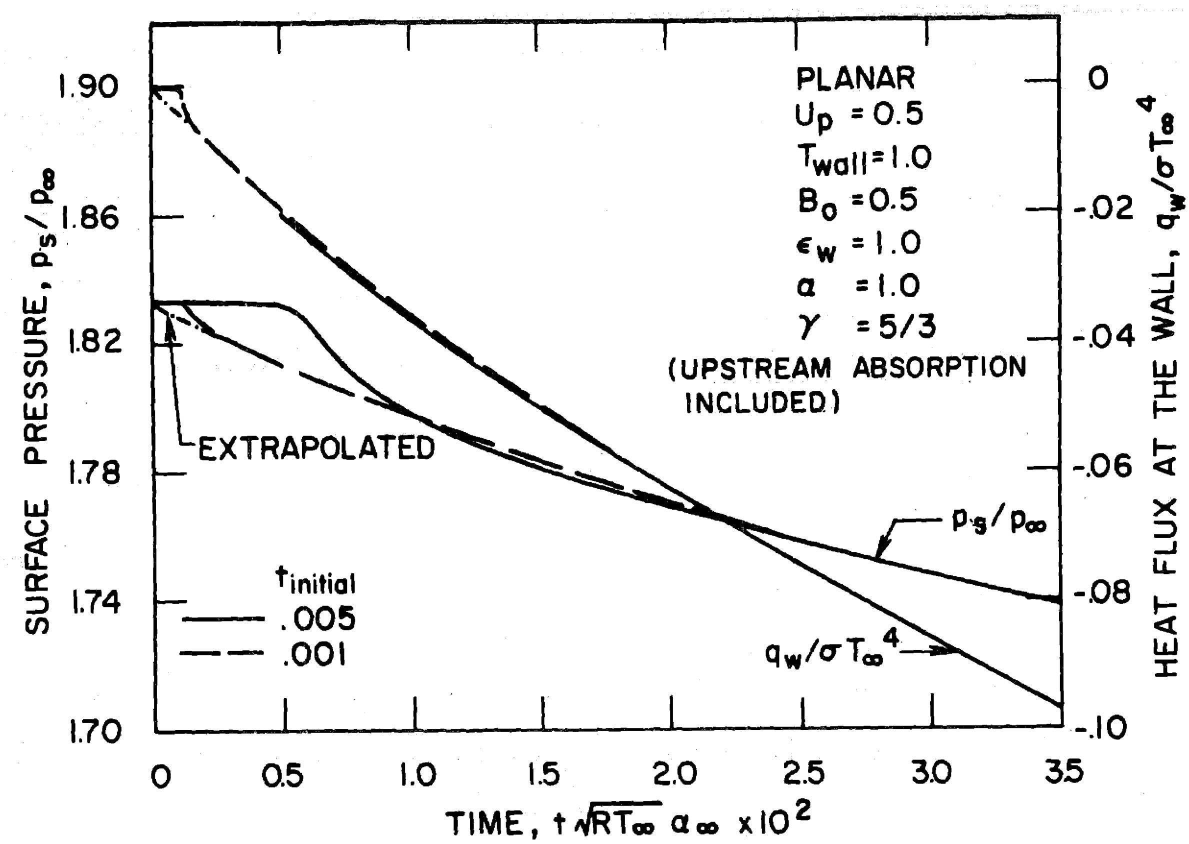

Figure 9d. Effect of initiating time upon heat flux and pressure at the surface of a planar piston in uniform motion

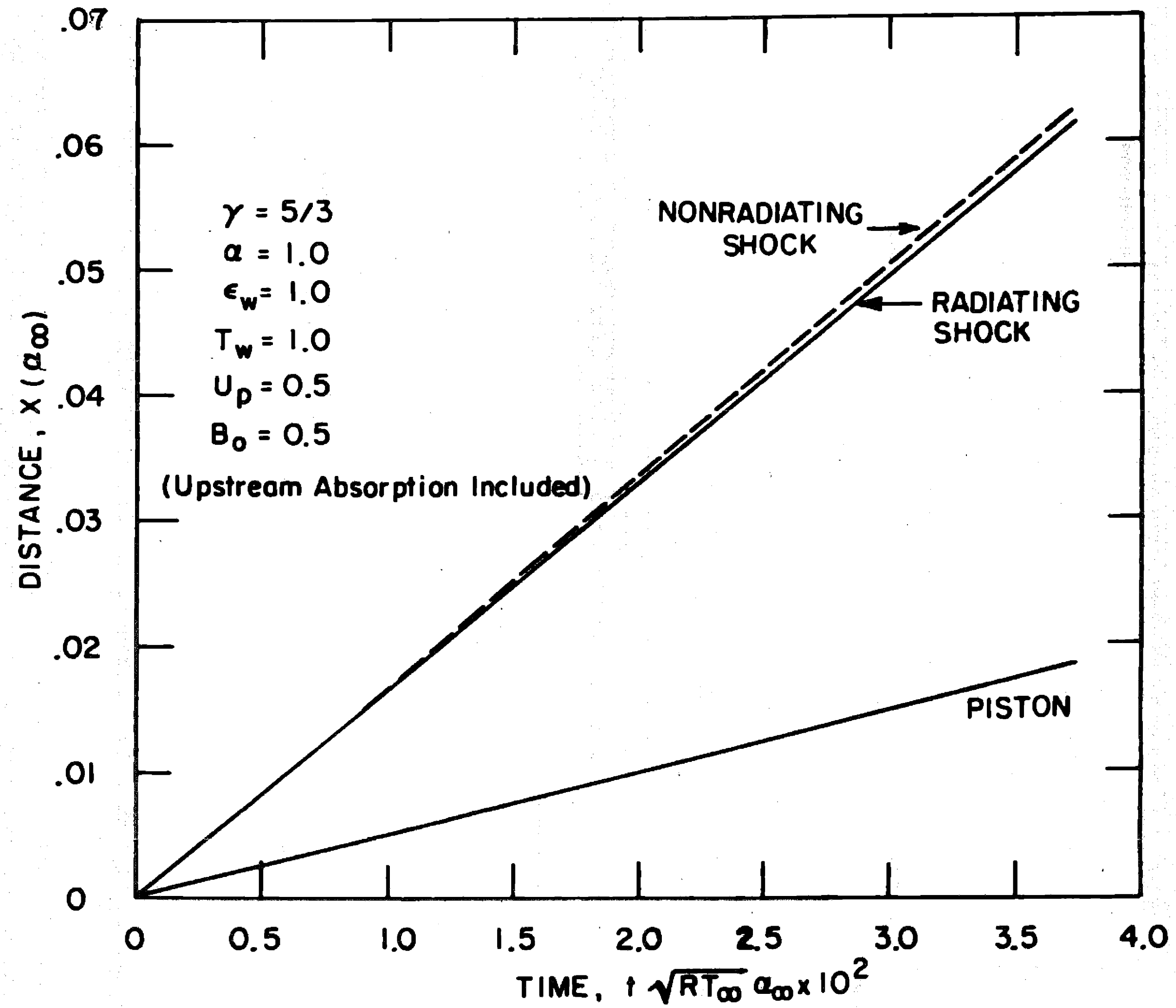

Figure 9e. Shock wave generated by a planar piston in uniform motion

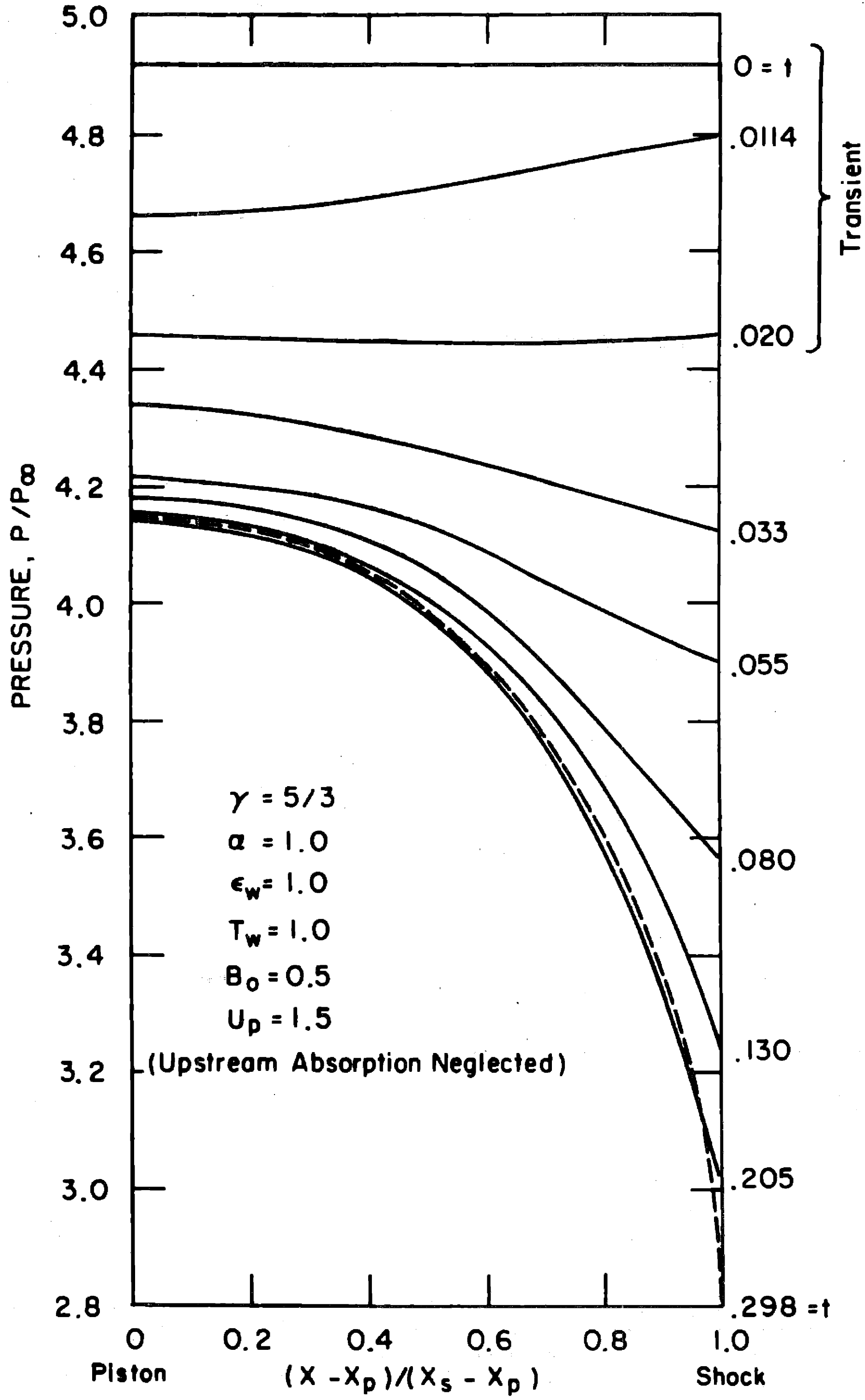

Figure 10a. Pressure field induced by a planar piston in uniform motion

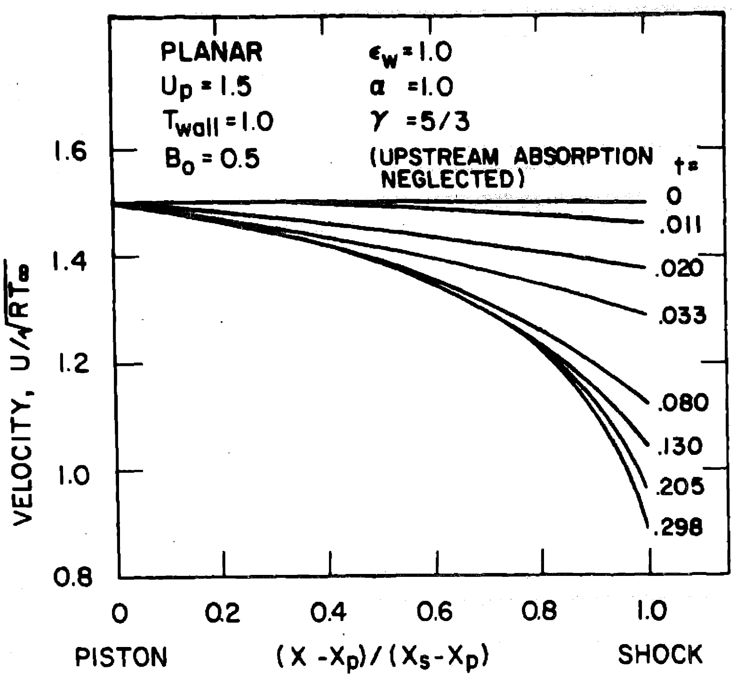

Figure 10b. Velocity field induced by a planar piston in uniform motion

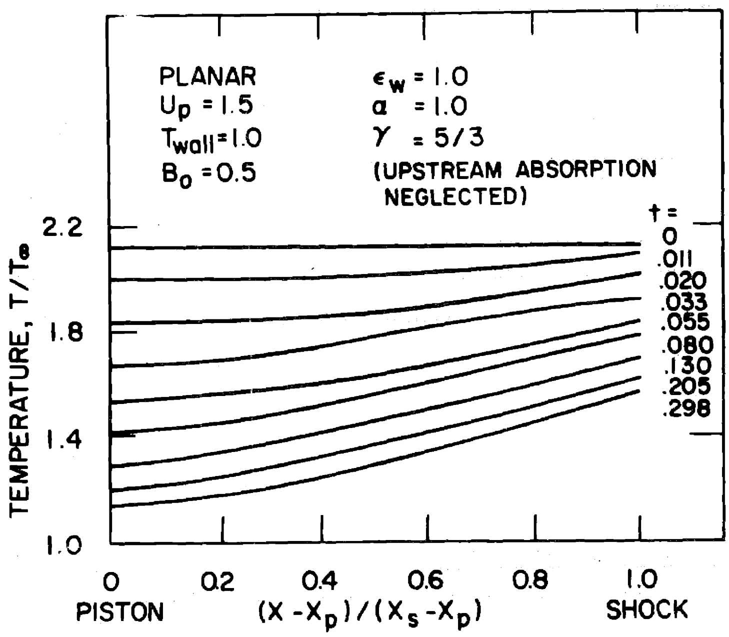

Figure 10c. Temperature field induced by a planar piston in uniform motion

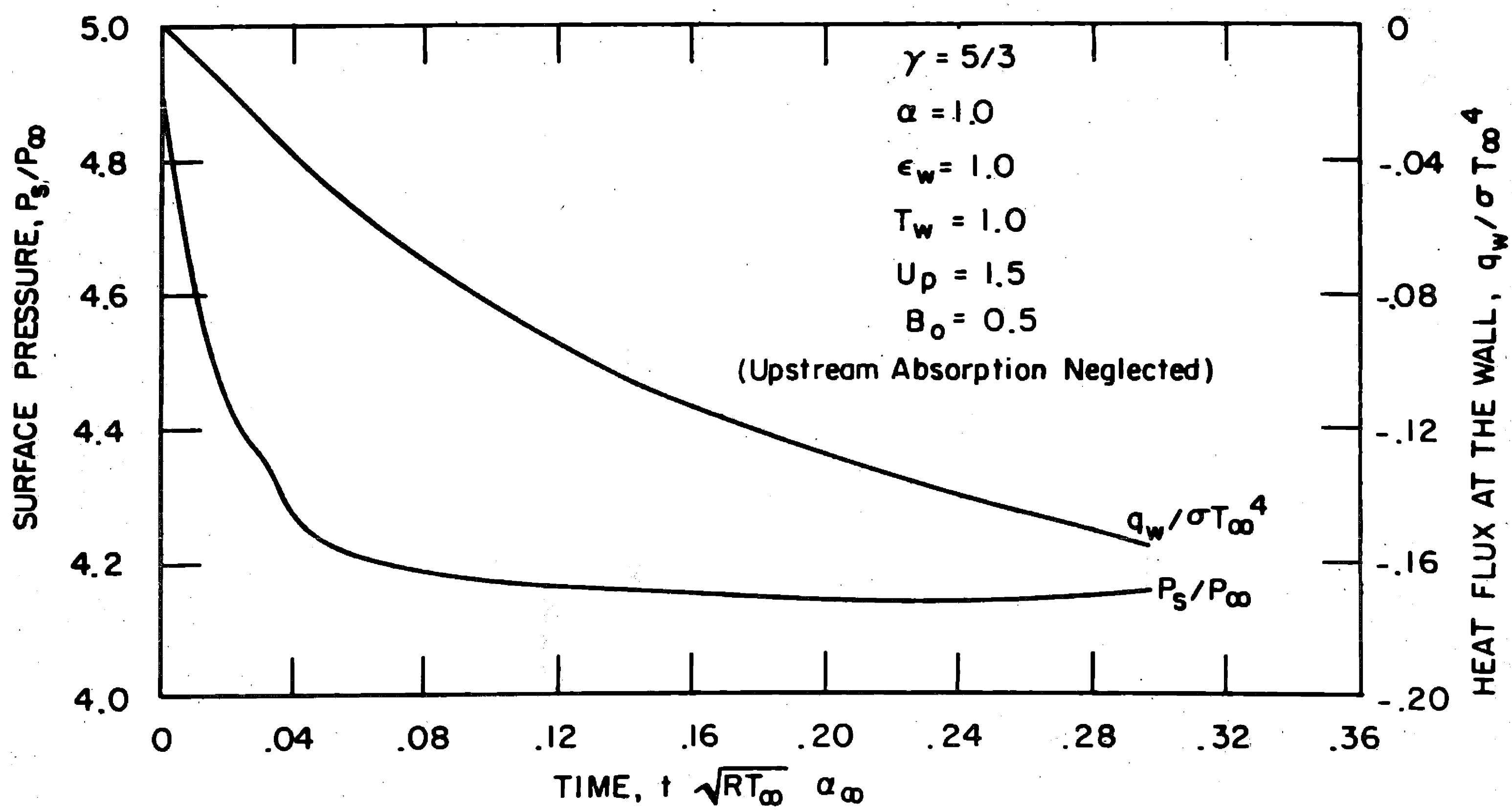

Figure 10d. Heat flux and pressure at the surface of a planar piston in uniform motion

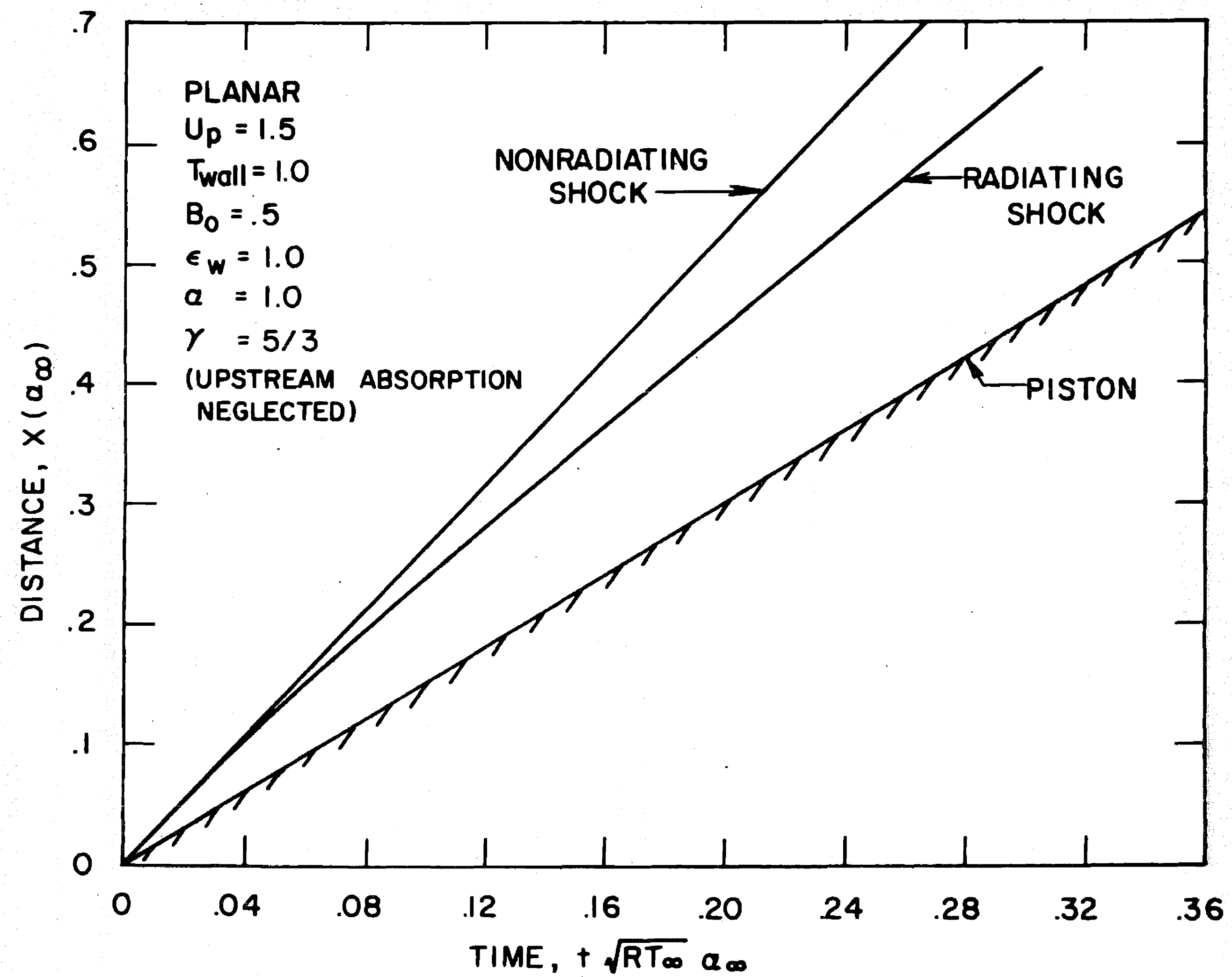

Figure 10e. Shock wave generated by a planar piston in uniform motion (2° semi-angle wedge, 350 k ft., 3×10^4 ft/sec)

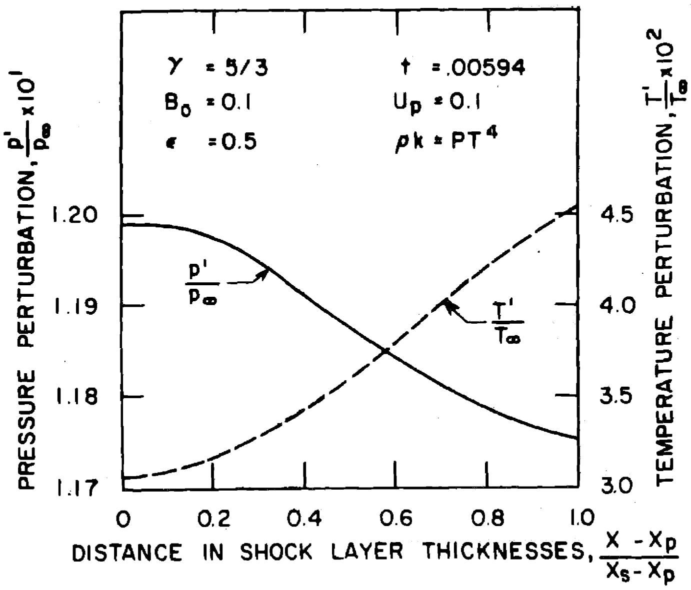

Figure 11a. Representative pressure and temperature perturbations in a situation which allows linearization

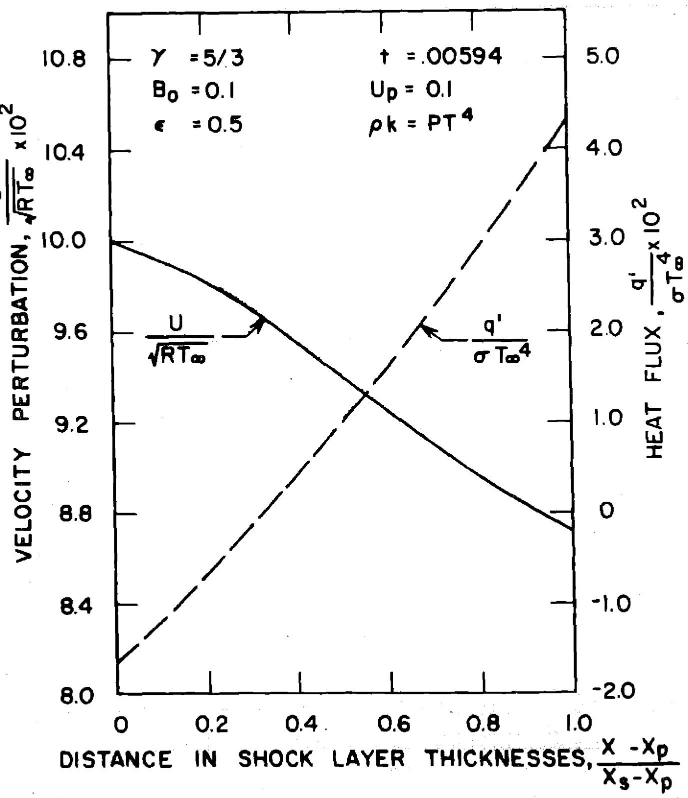

Figure 11b. Representative heat flux and velocity perturbations in a situation which allows linearization

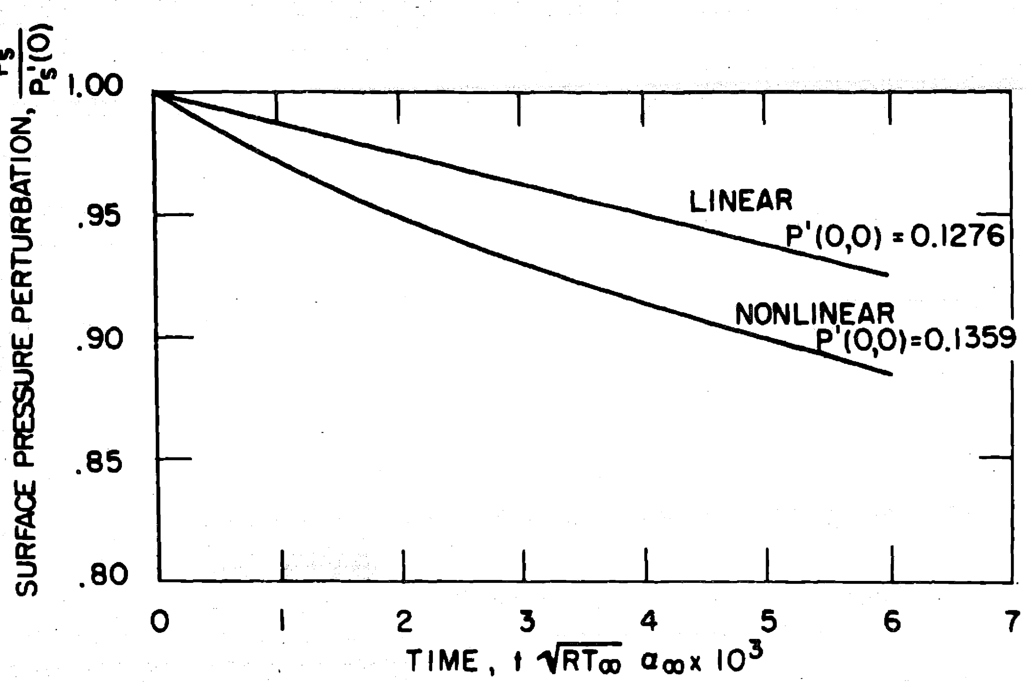

Figure 11c. Comparison of surface pressure perturbations predicted by linear and nonlinear theories

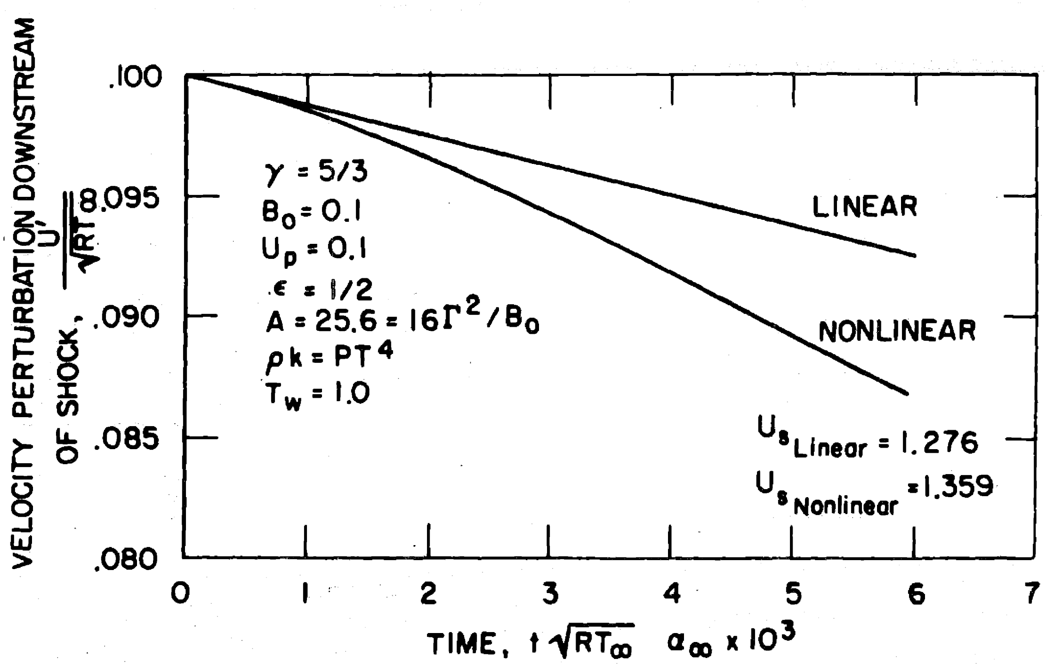

Figure 11d. Comparison of velocity perturbations predicted by linear and nonlinear theories

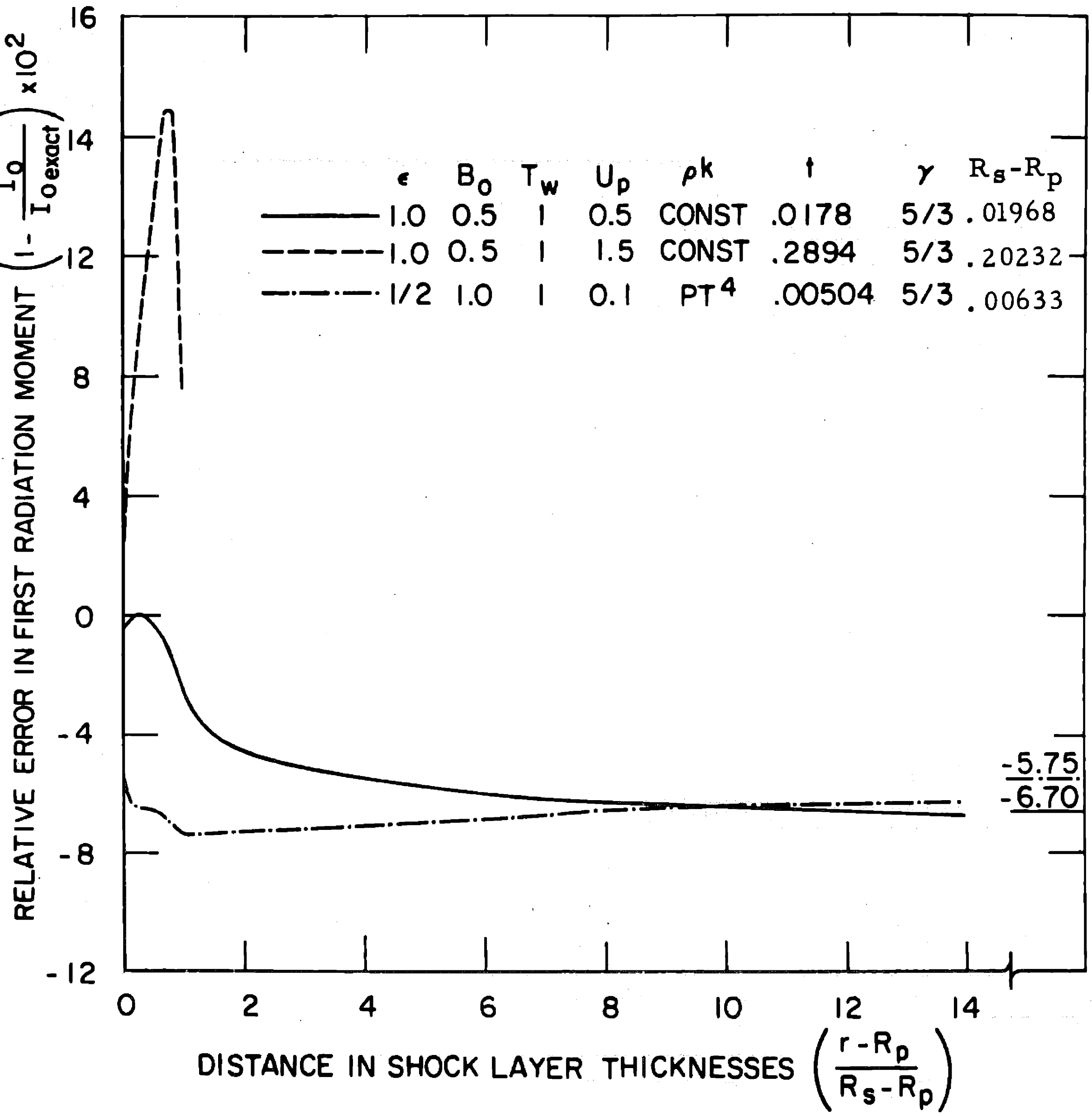

Figure 12a. Comparison of first radiative moment as predicted by the differential approximation and the full transfer equation

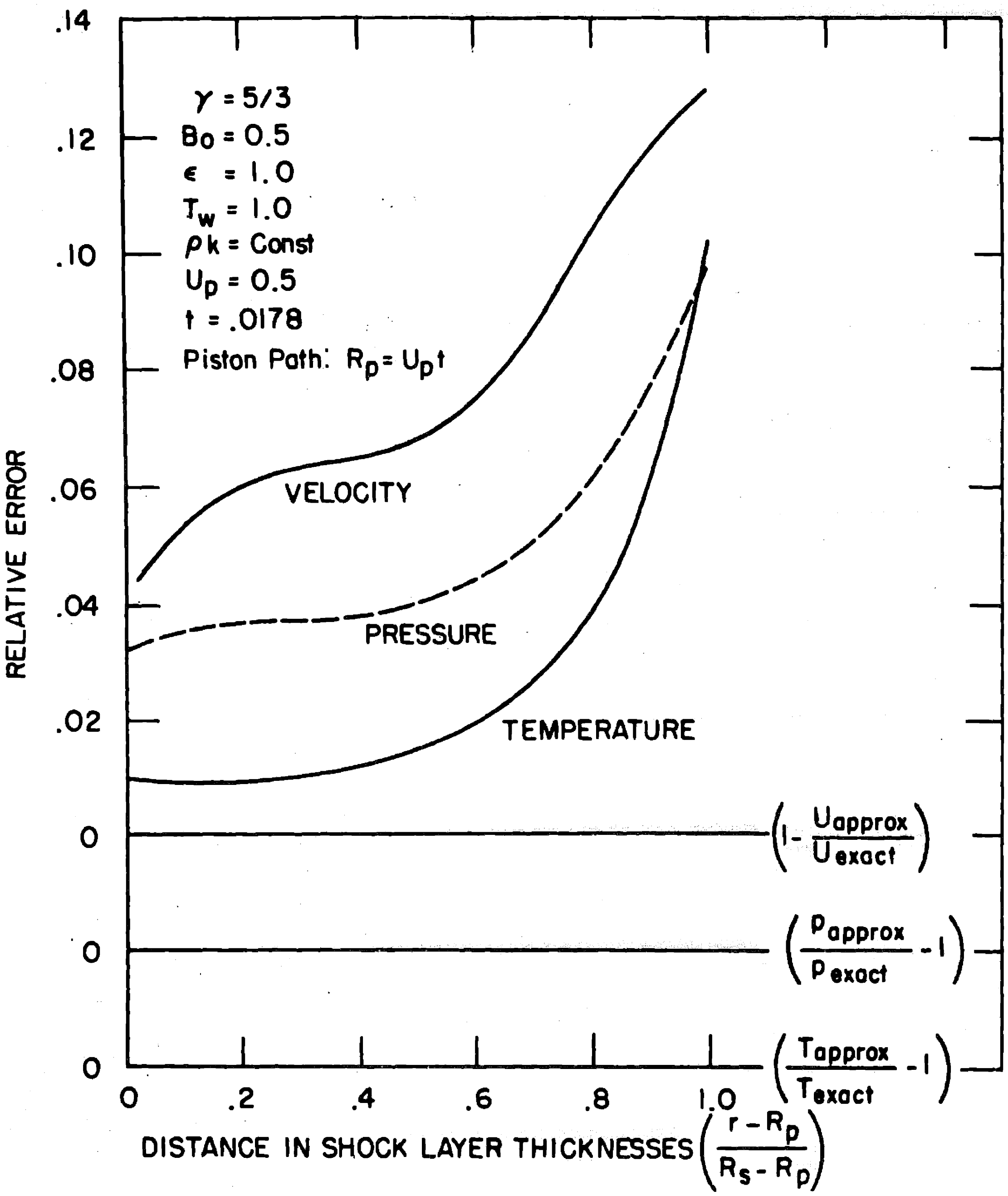

Figure 12b. Cumulative error in a flow field predicted by the differential approximation

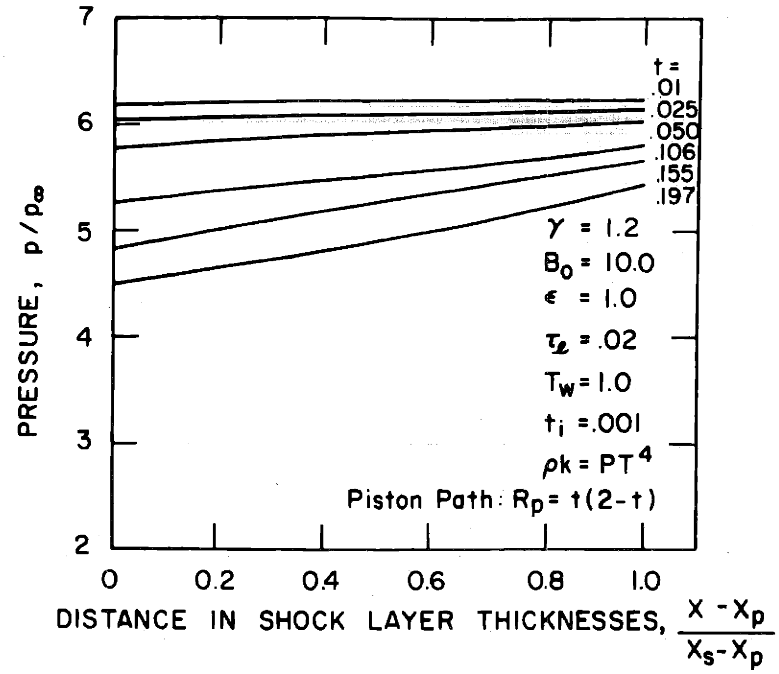

Figure 13a. Pressure field induced by a decelerating planar piston

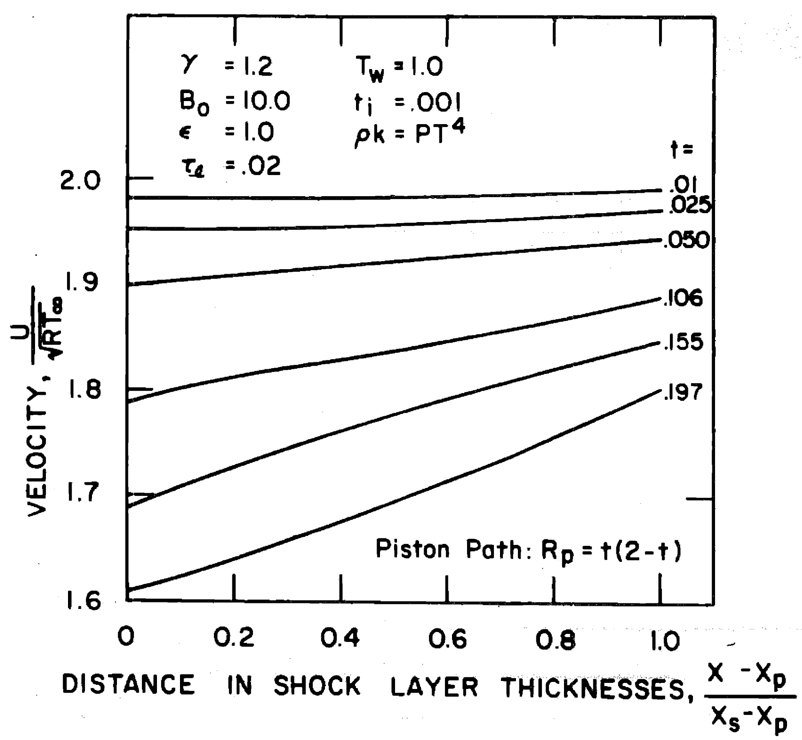

Figure 13b. Velocity field induced by a decelerating planar piston

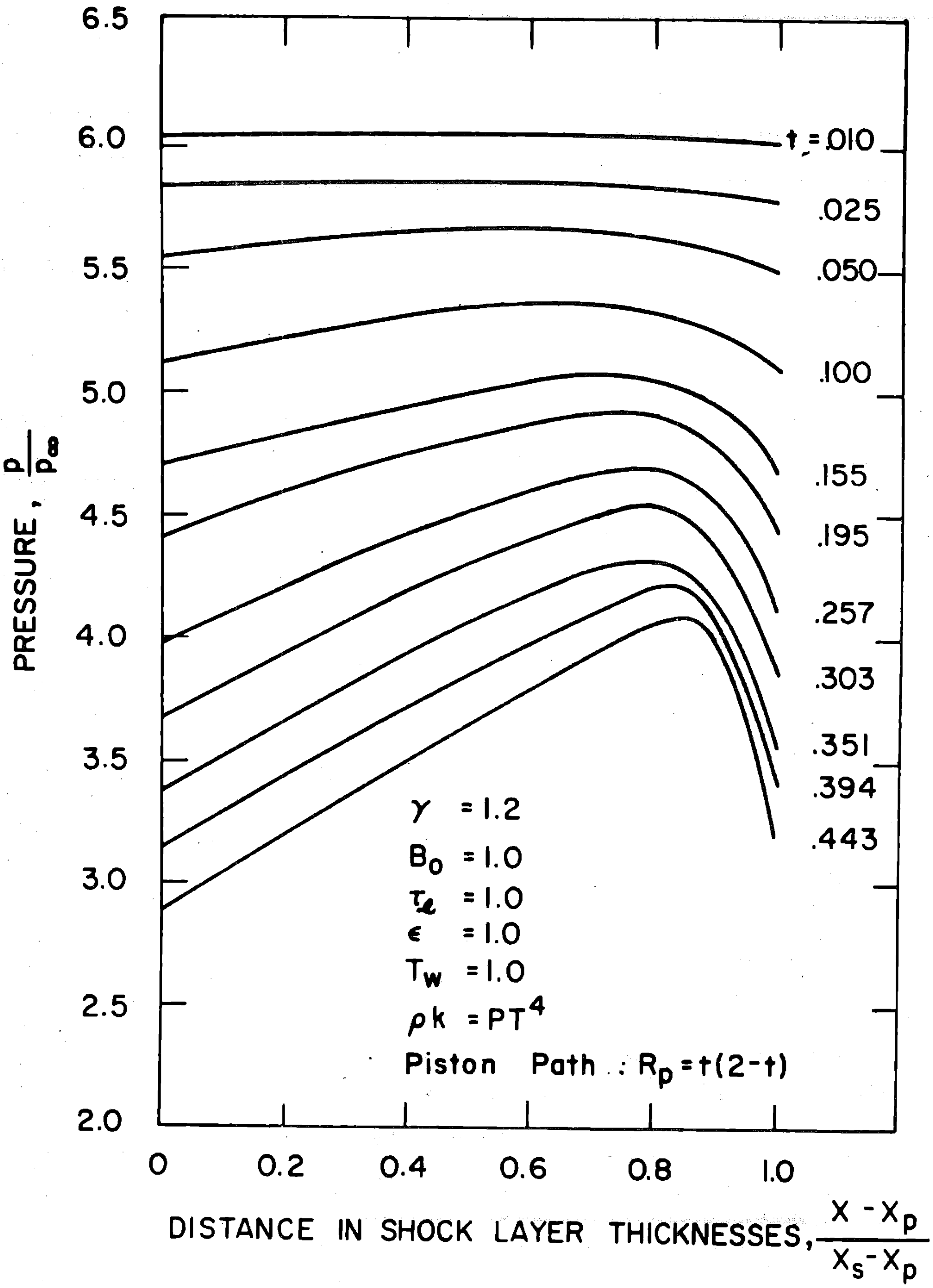

Figure 14a. Pressure field induced by a decelerating planar piston

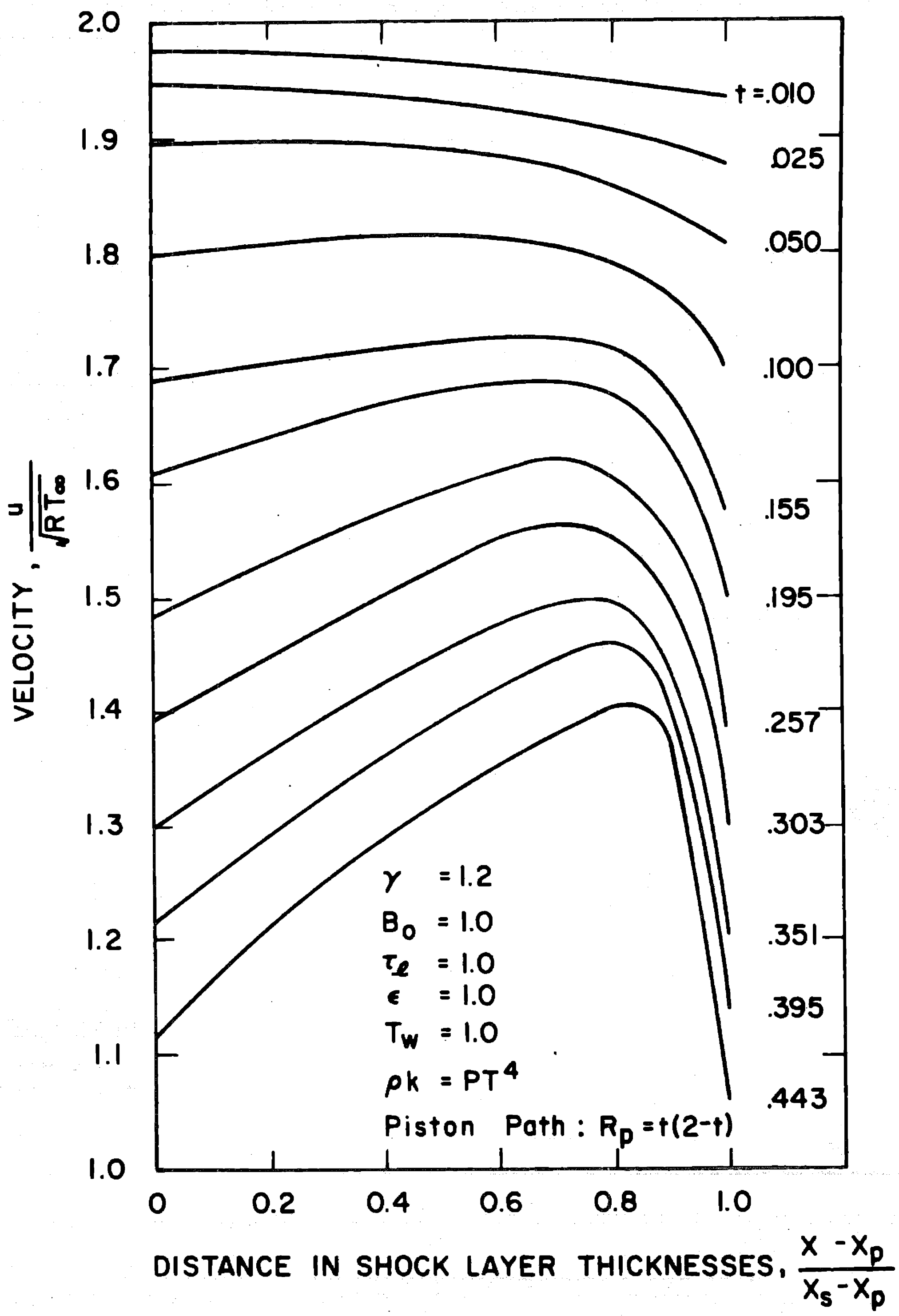

Figure 14b. Velocity field induced by a decelerating planar piston

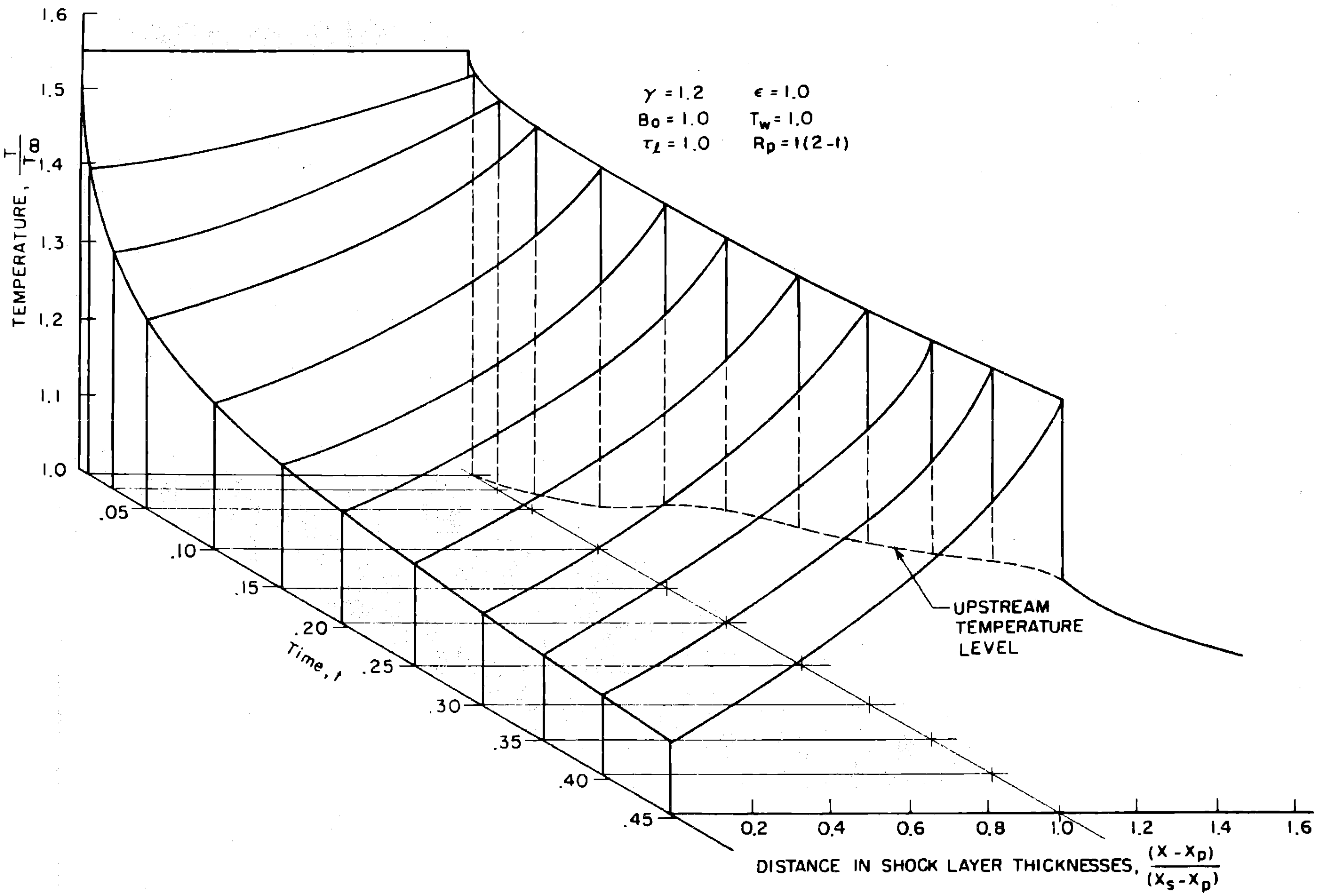

Figure 14c. Temperature field induced by a decelerating planar piston

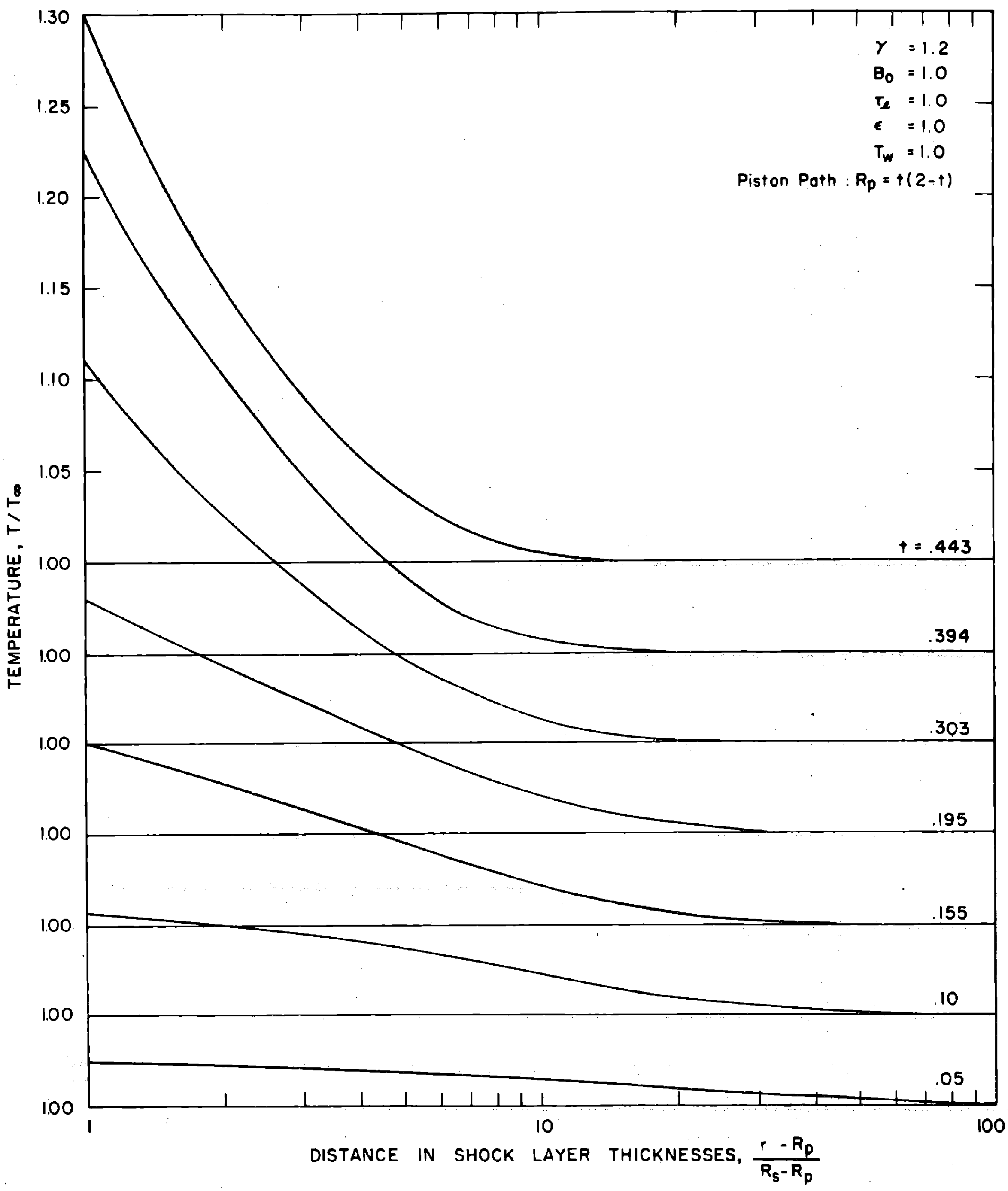

Figure 14d. Evolution of flow field induced upstream of the shock wave generated by a decelerating planar piston

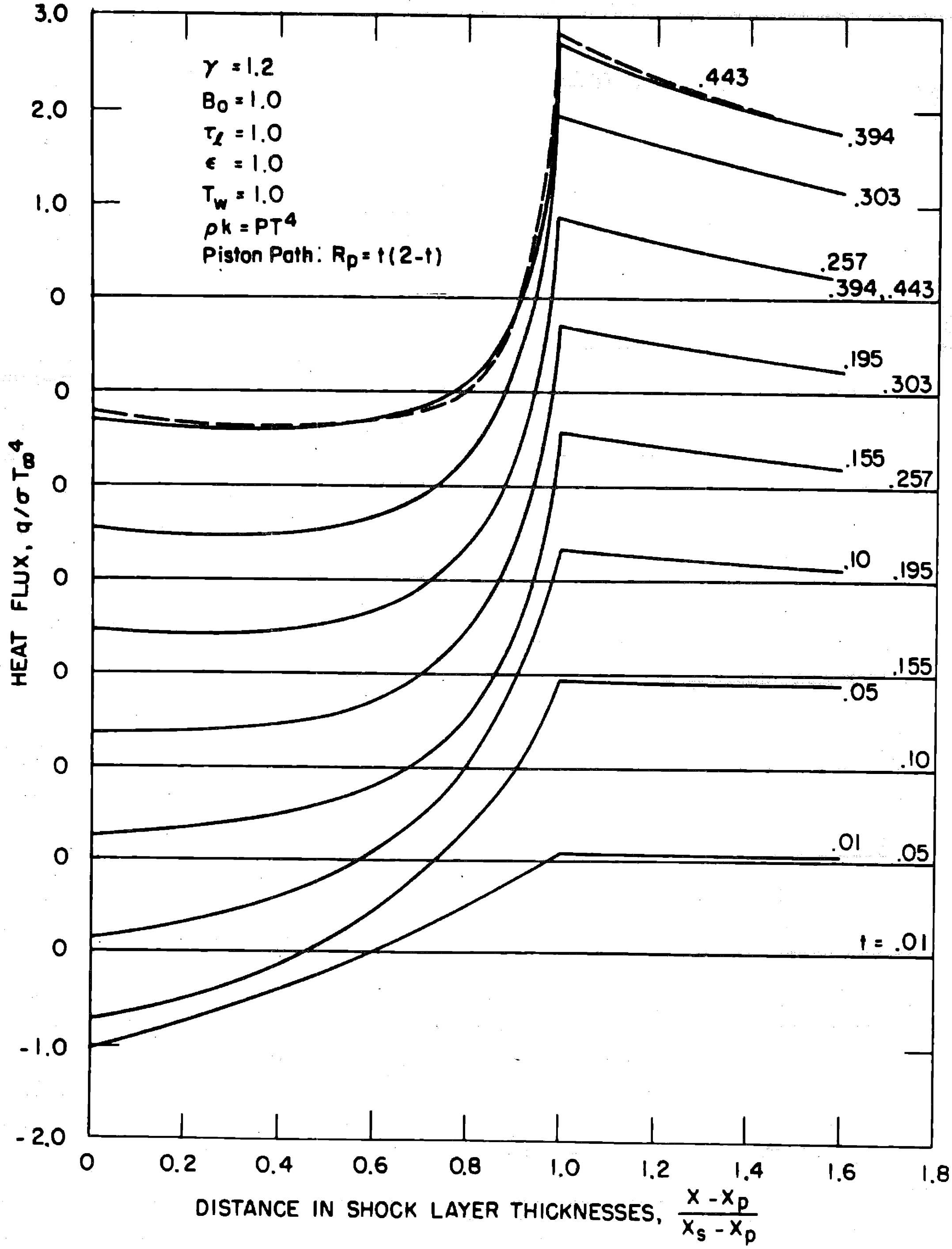

Figure 14e. Radiative field induced by a decelerating planar piston

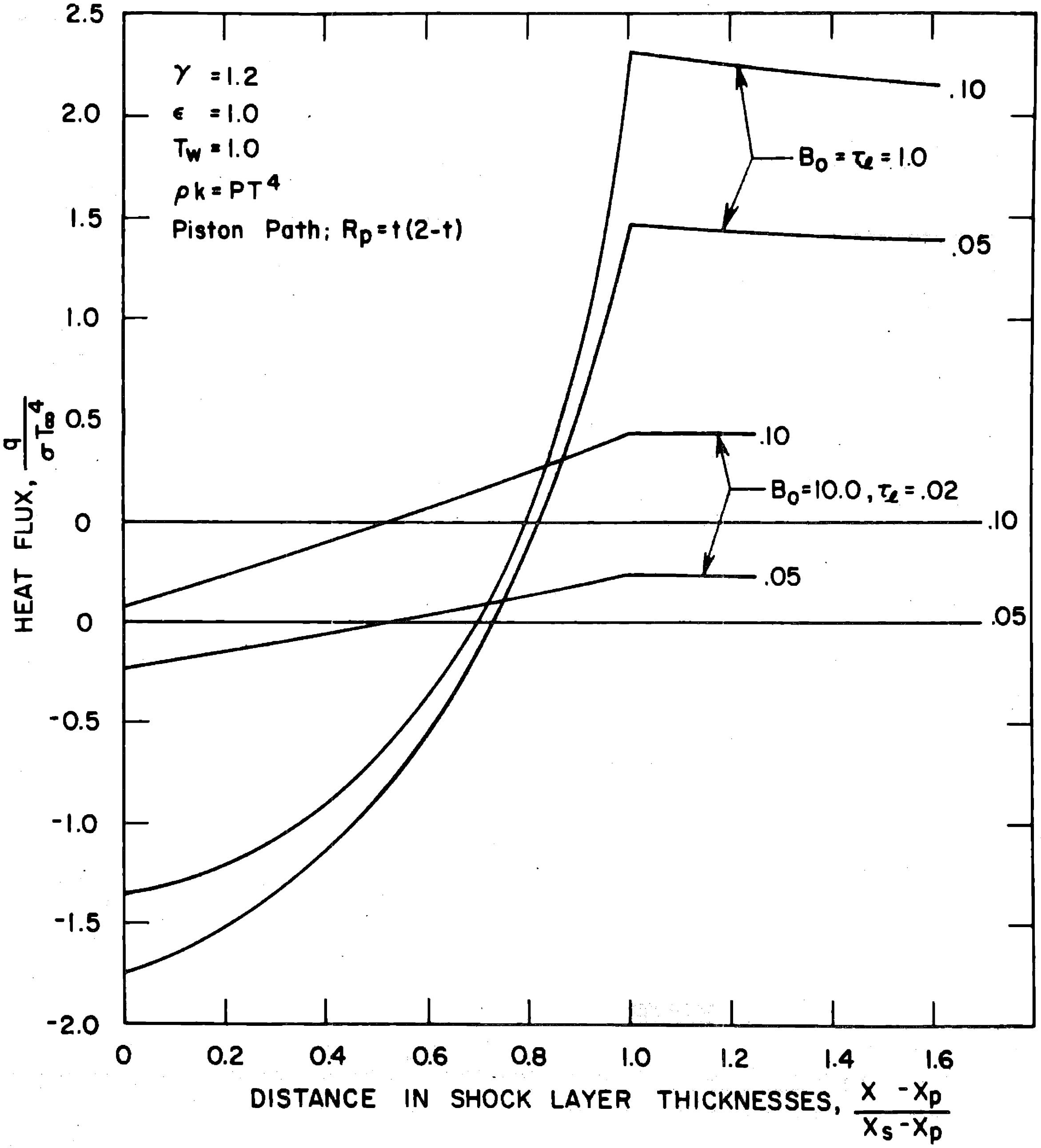

Figure 14f. Effect of Boltzmann and Bouguer numbers upon radiative field induced by a decelerating planar piston

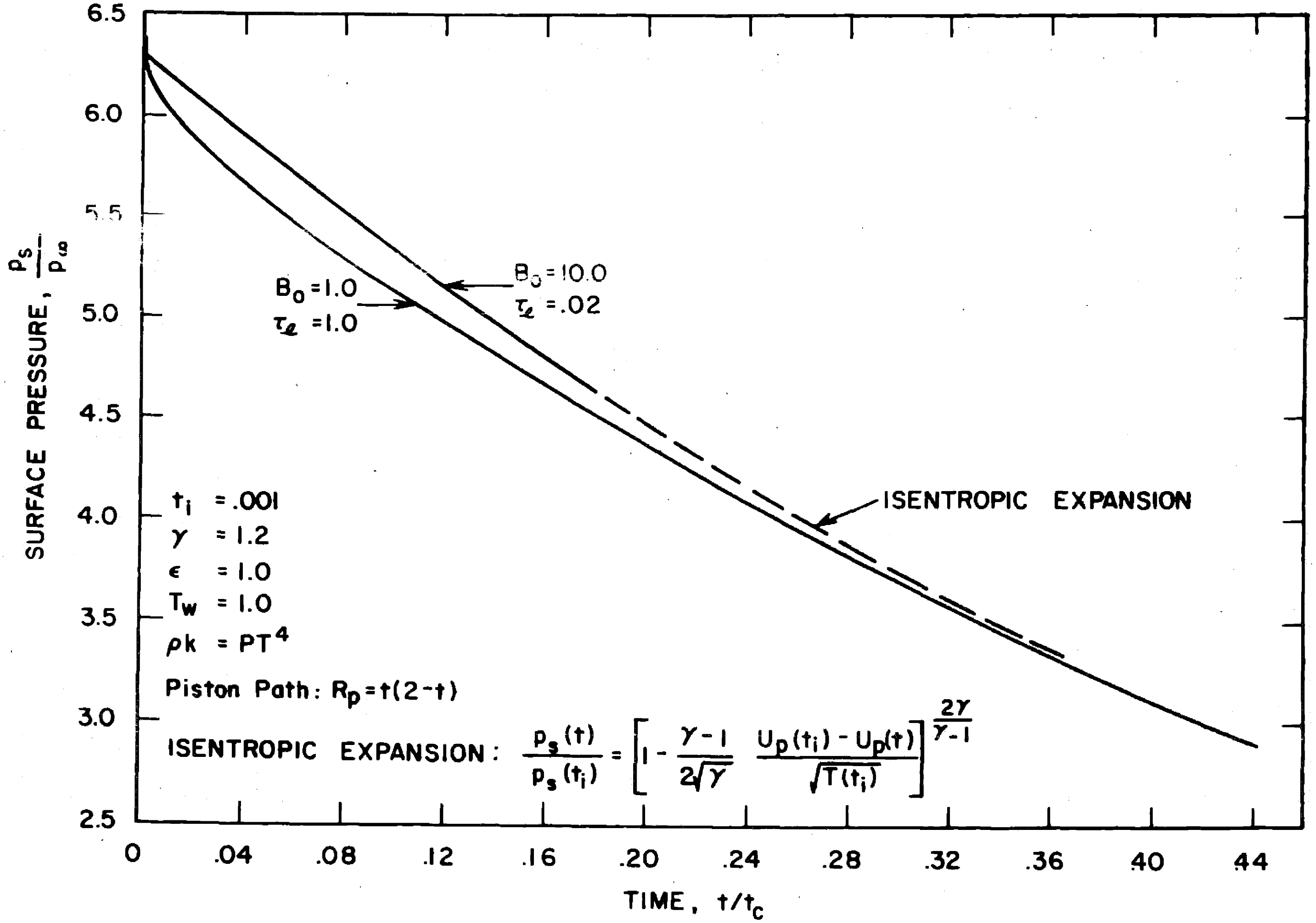

Figure 14g. Effect of Boltzmann and Bouguer numbers upon pressure at the surface of a decelerating planar piston

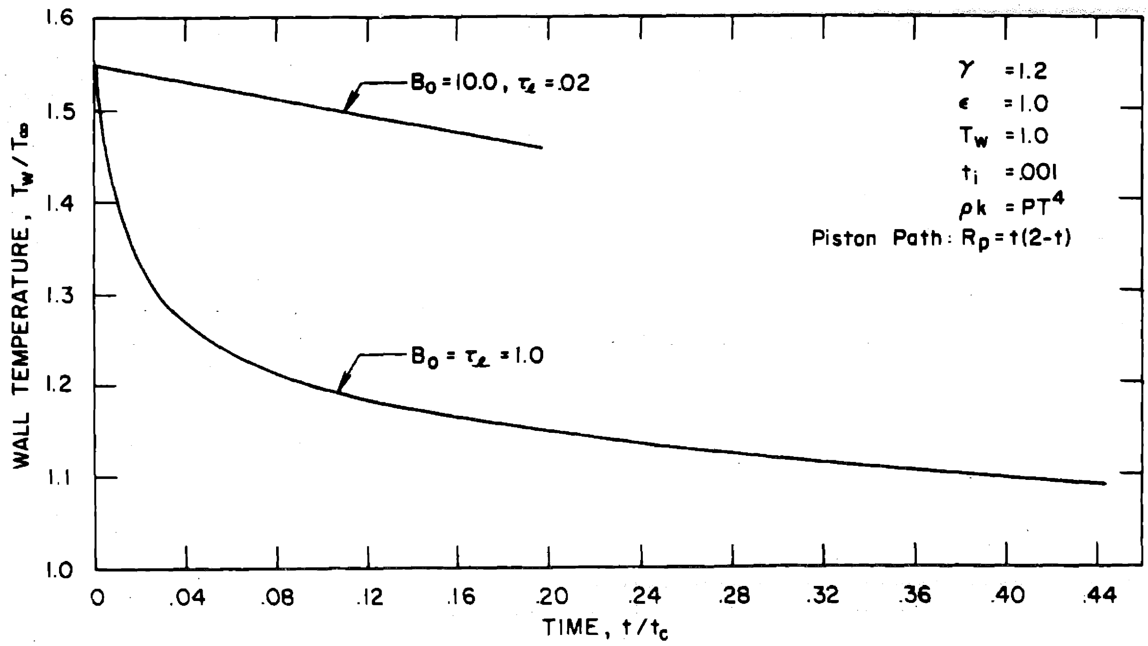

Figure 14h. Effect of Boltzmann and Bouguer numbers upon temperature slip at the surface of a decelerating planar piston

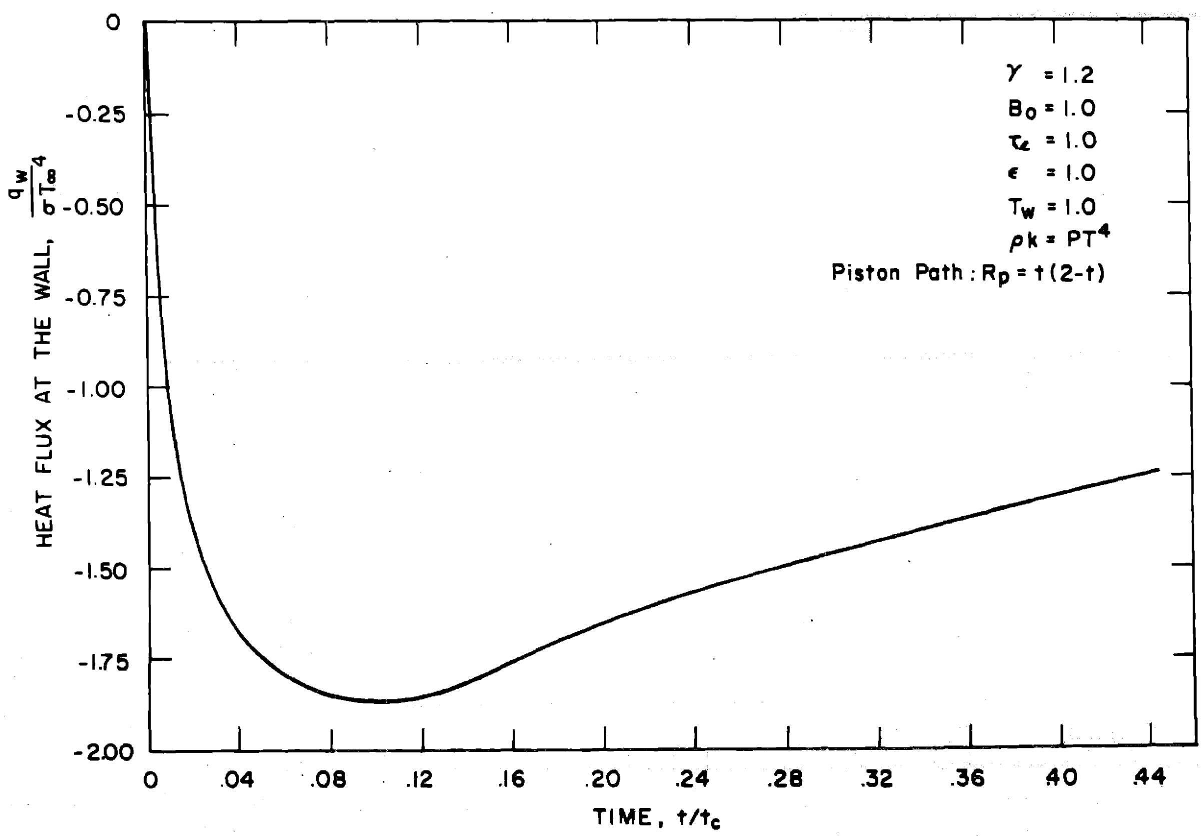

Figure 14i. Heat flux at the surface of a decelerating planar piston

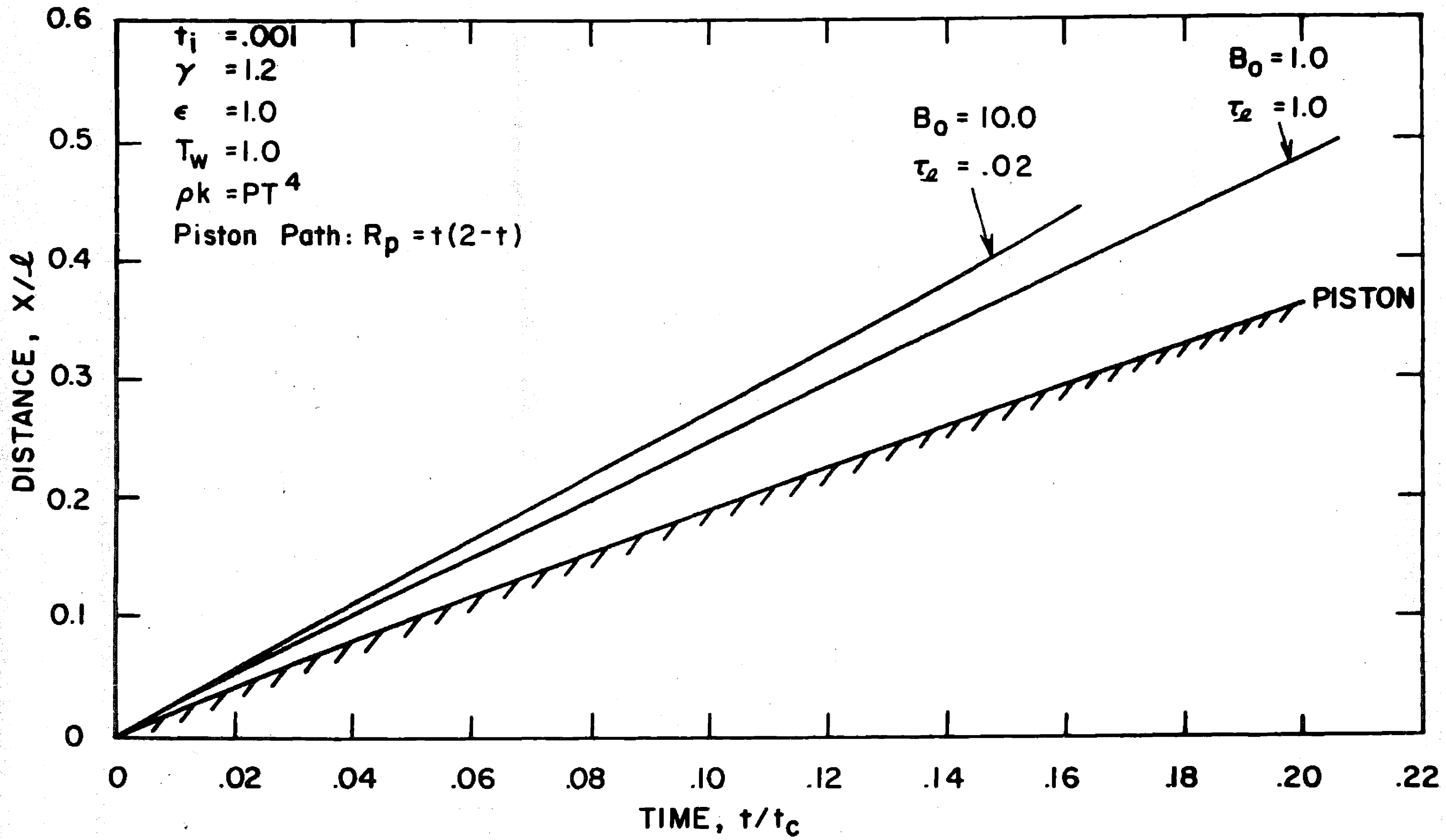

Figure 14j. Effect of Boltzmann and Bouguer numbers upon shock wave generated by a decelerating planar piston

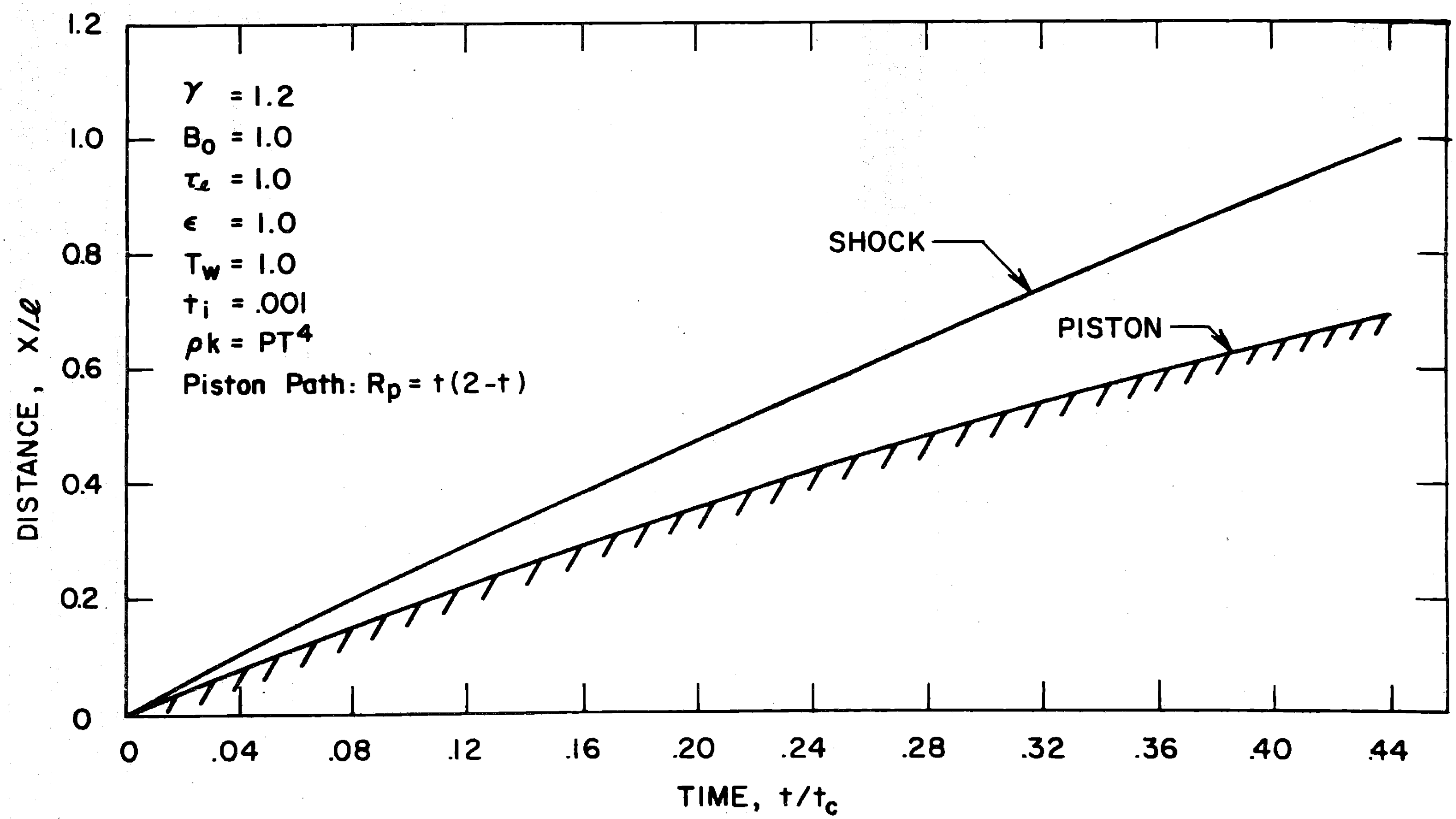

Figure 14k. Evolution of shock wave generated by a decelerating planar piston

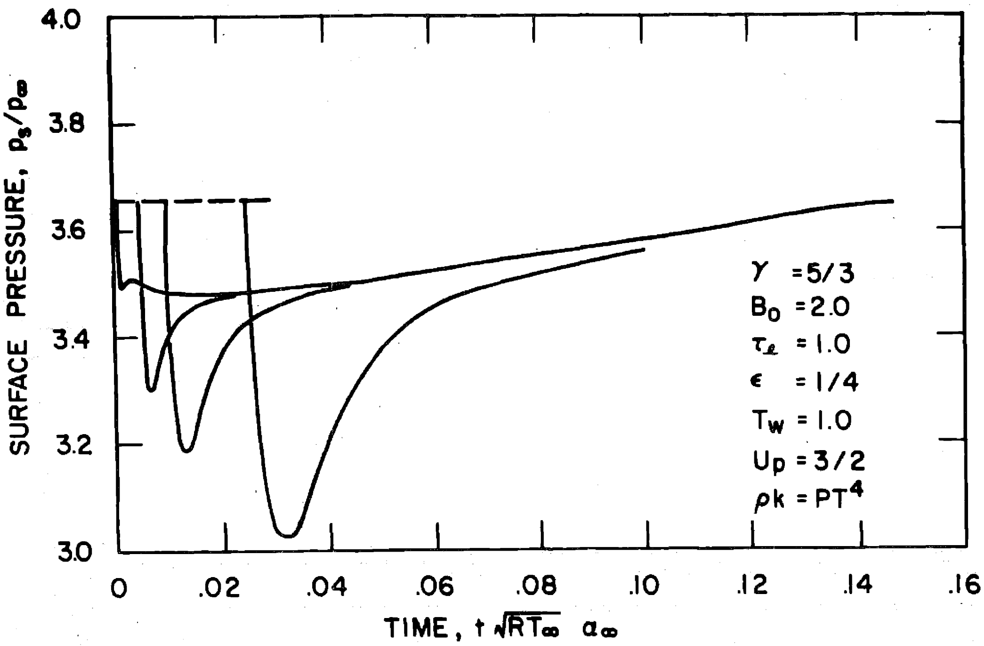

Figure 15a. Effect of initiating time upon pressure at the surface of a cylindrical piston in uniform motion

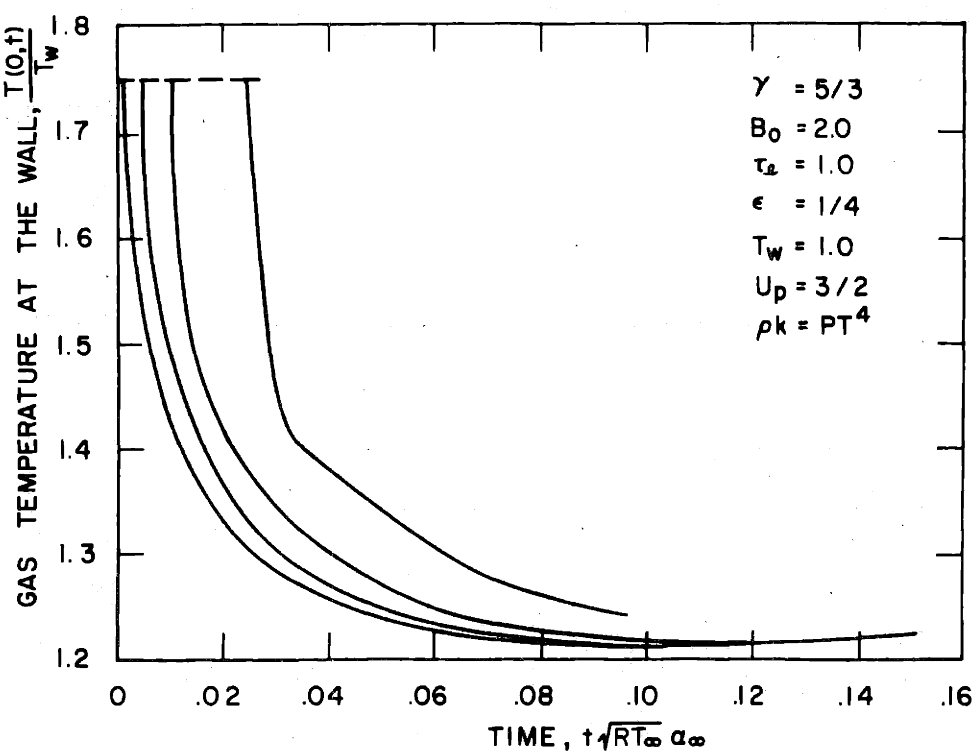

Figure 15b. Effect of initiating time upon temperature slip at the surface of a cylindrical piston in uniform motion

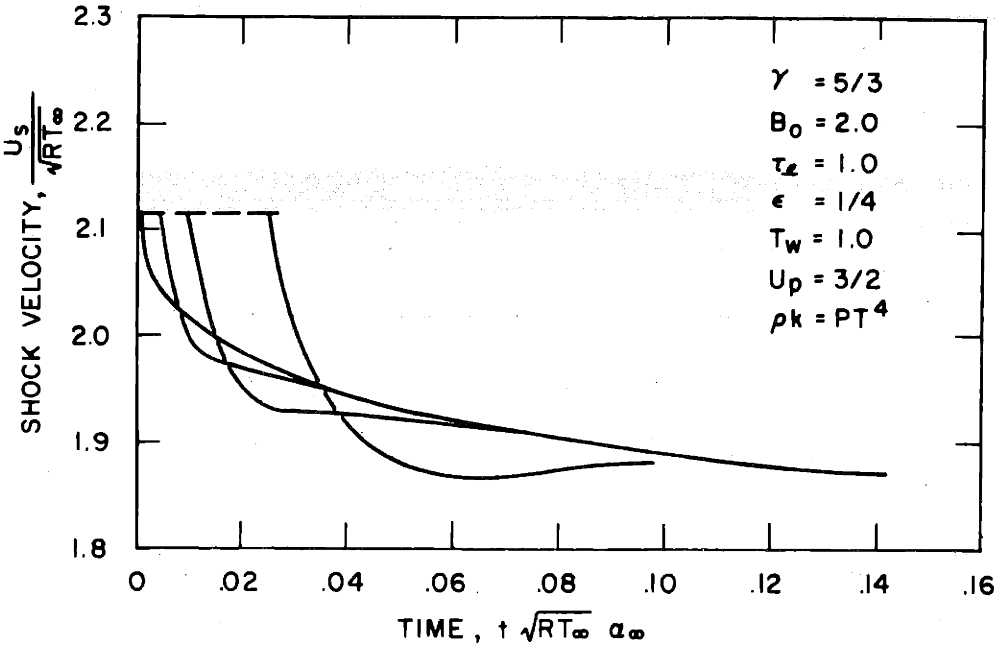

Figure 15c. Effect of initiating time upon velocity of the shock wave generated by a cylindrical piston in uniform motion

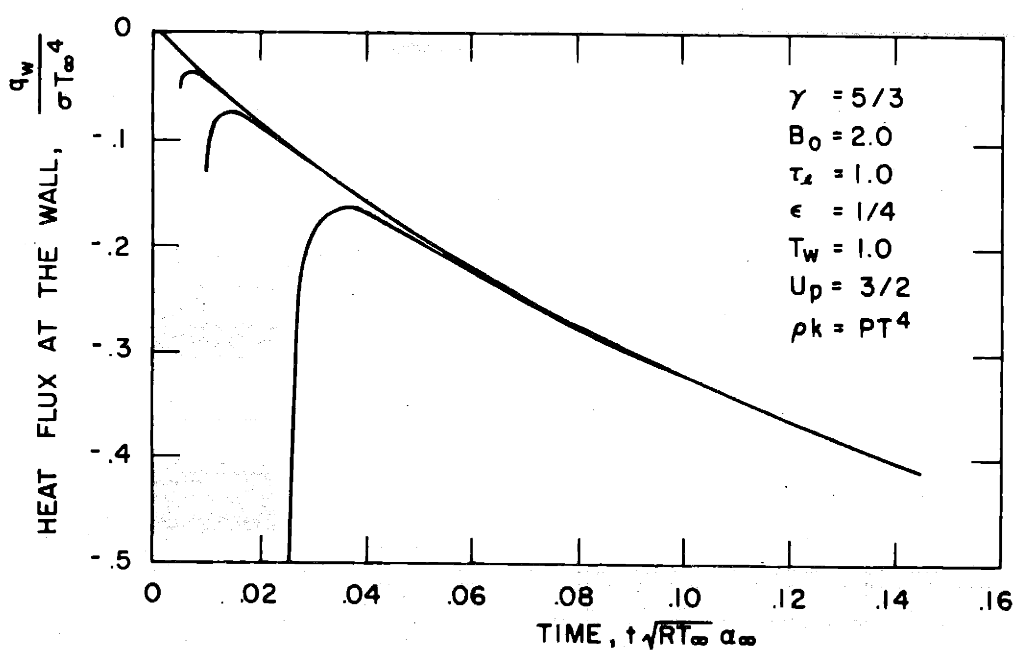

Figure 15d. Effect of initiating time upon heat flux at the wall of a cylindrical piston in uniform motion

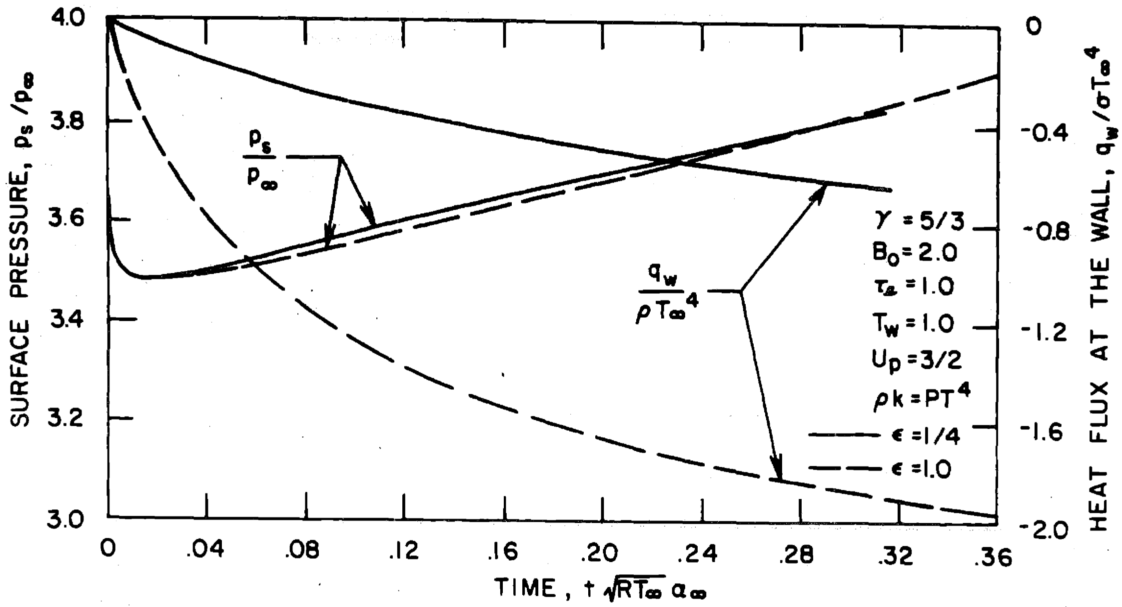

Figure 15e. Effect of emissivity upon pressure and heat flux at the surface of a cylindrical piston in uniform motion

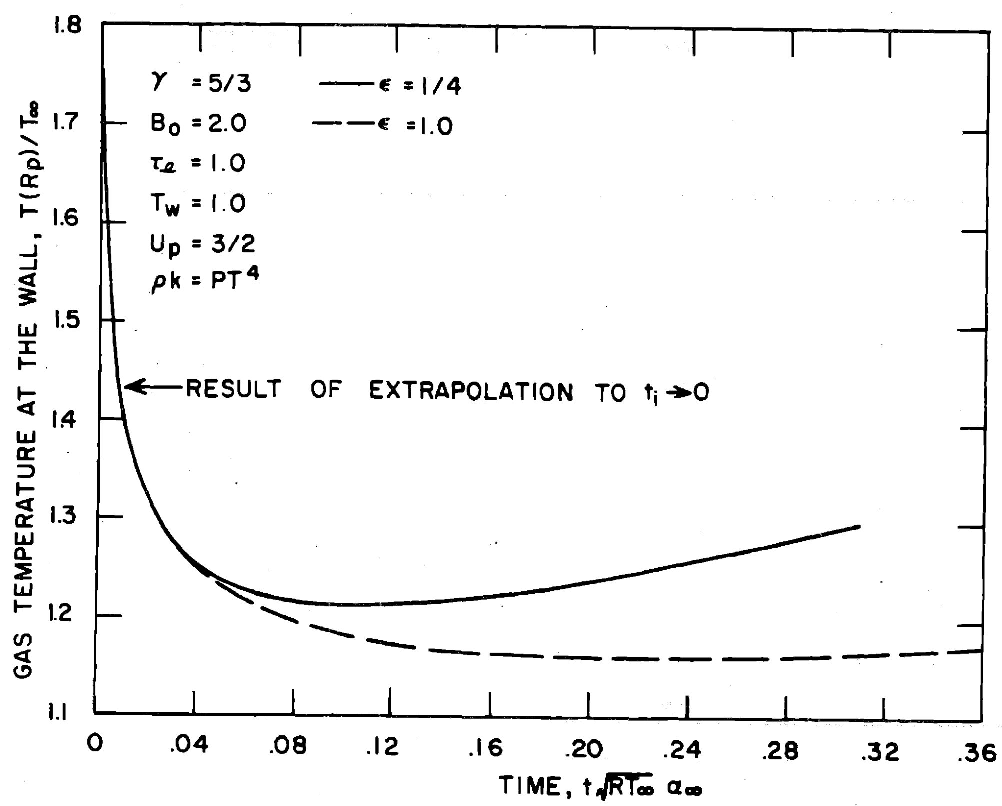

Figure 15f. Effect of emissivity upon temperature slip at the surface of a cylindrical piston in uniform motion

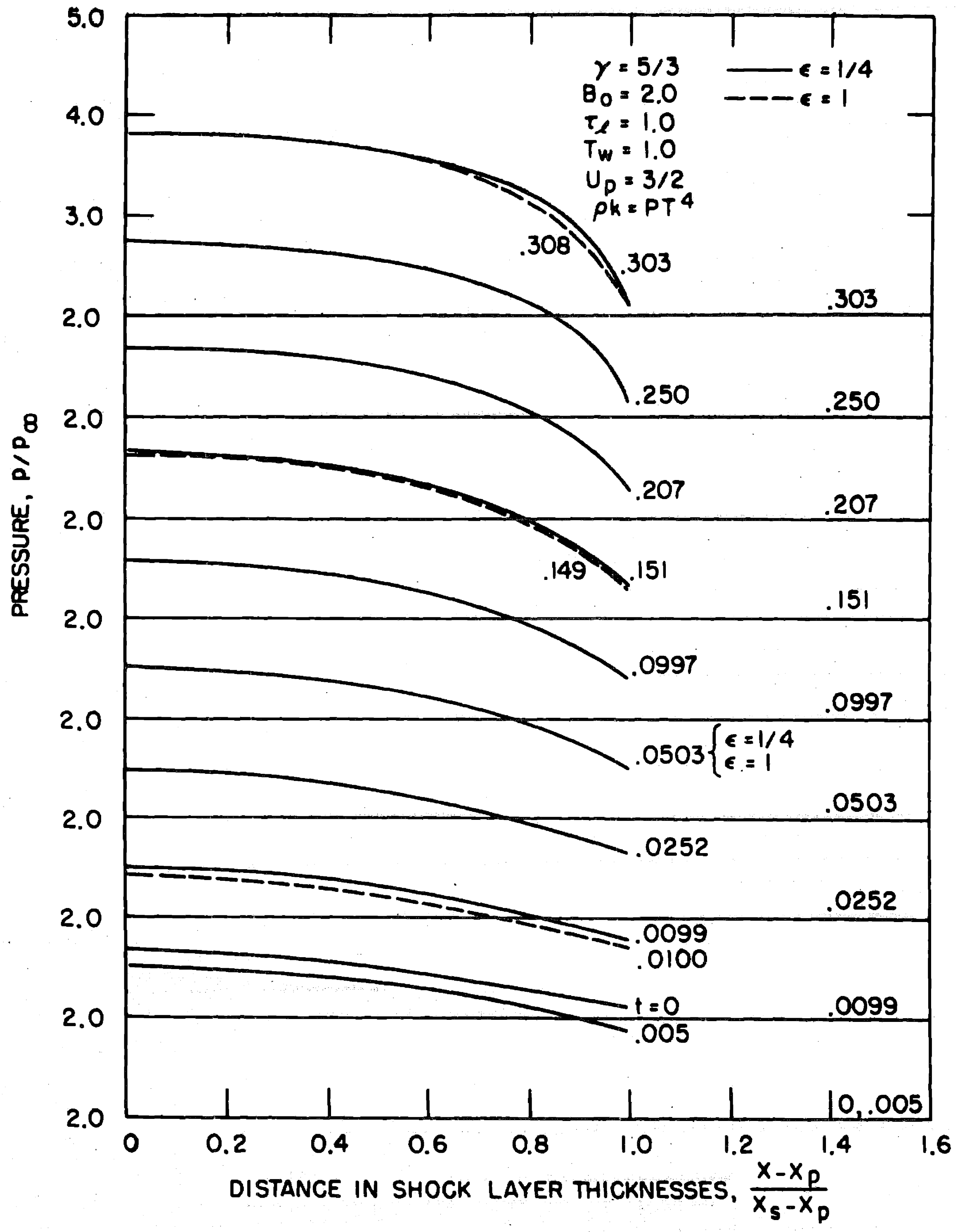

Figure 15g. Pressure field induced by a cylindrical piston in uniform motion

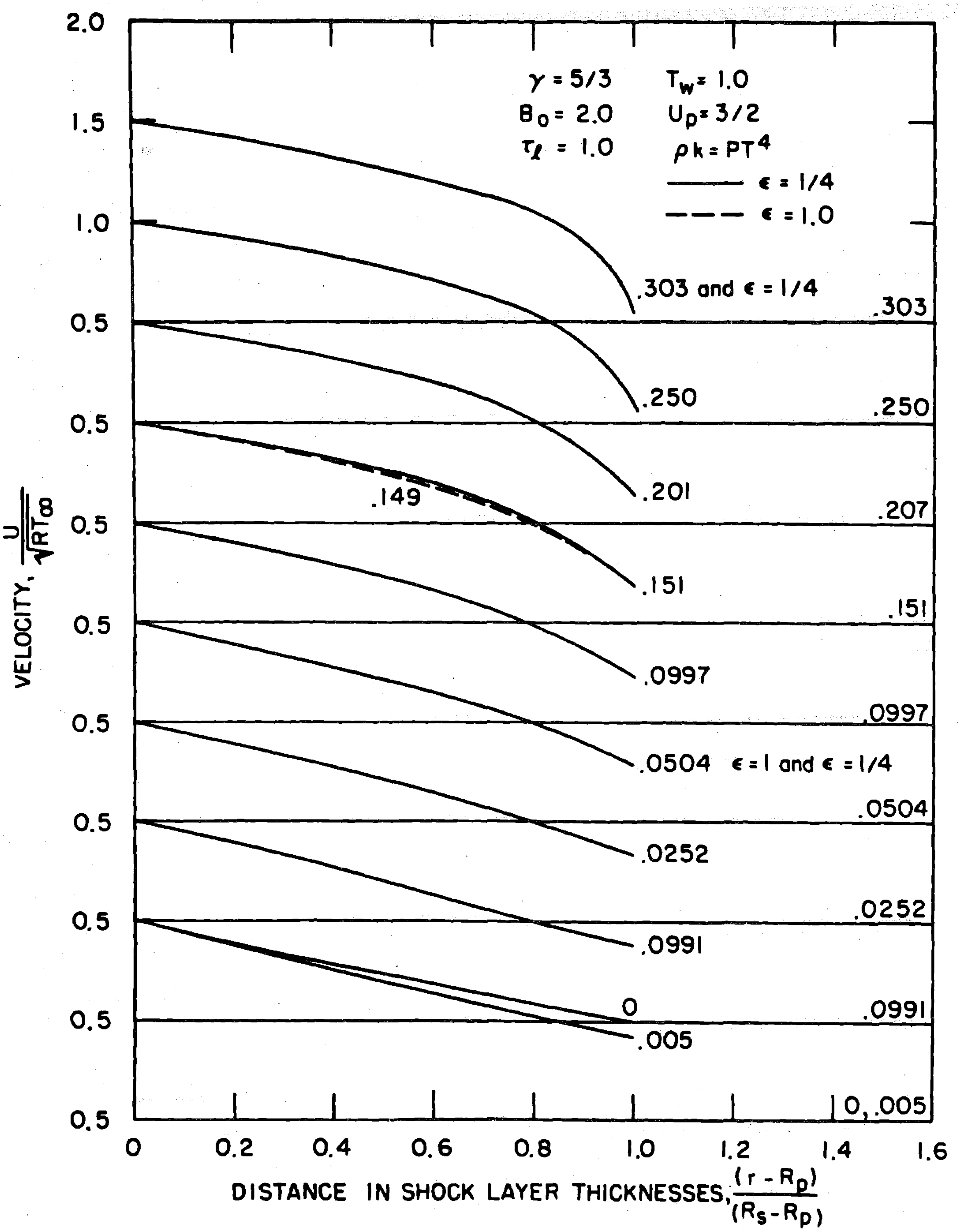

Figure 15h. Velocity field induced by a cylindrical piston in uniform motion

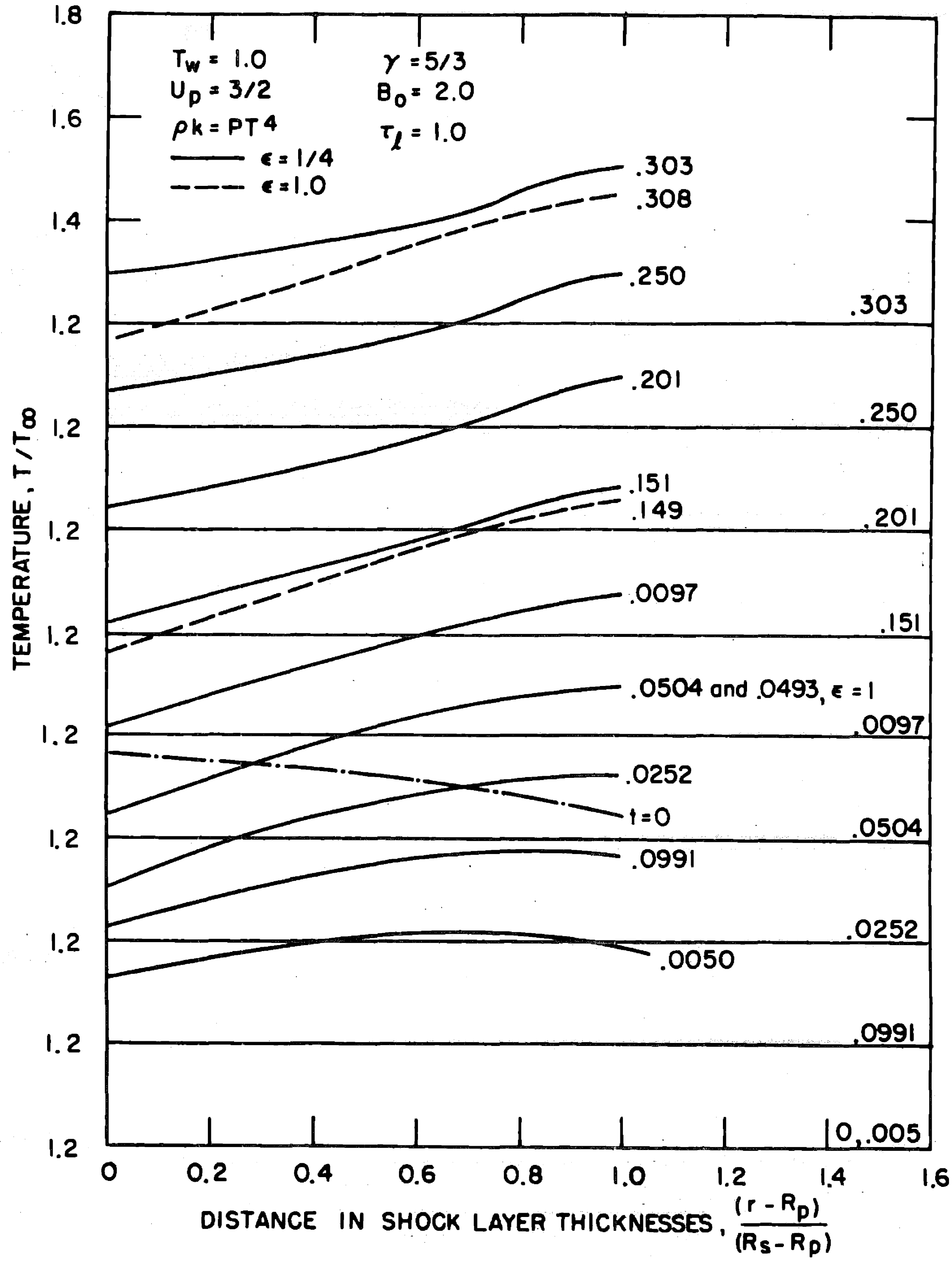

Figure 15i. Temperature field induced by a cylindrical piston in uniform motion

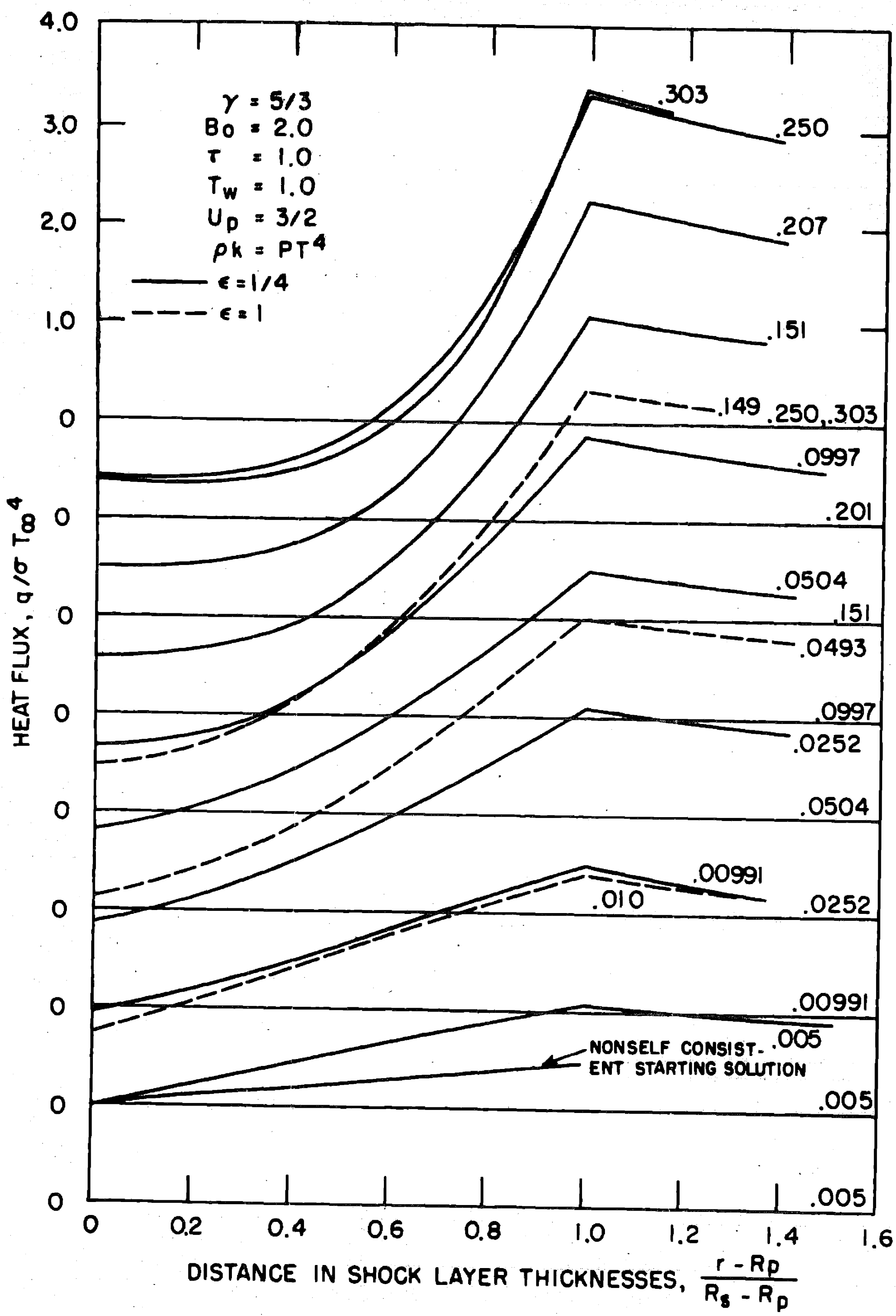

Figure 15j. Effect of emissivity upon radiative field induced by a cylindrical piston in uniform motion

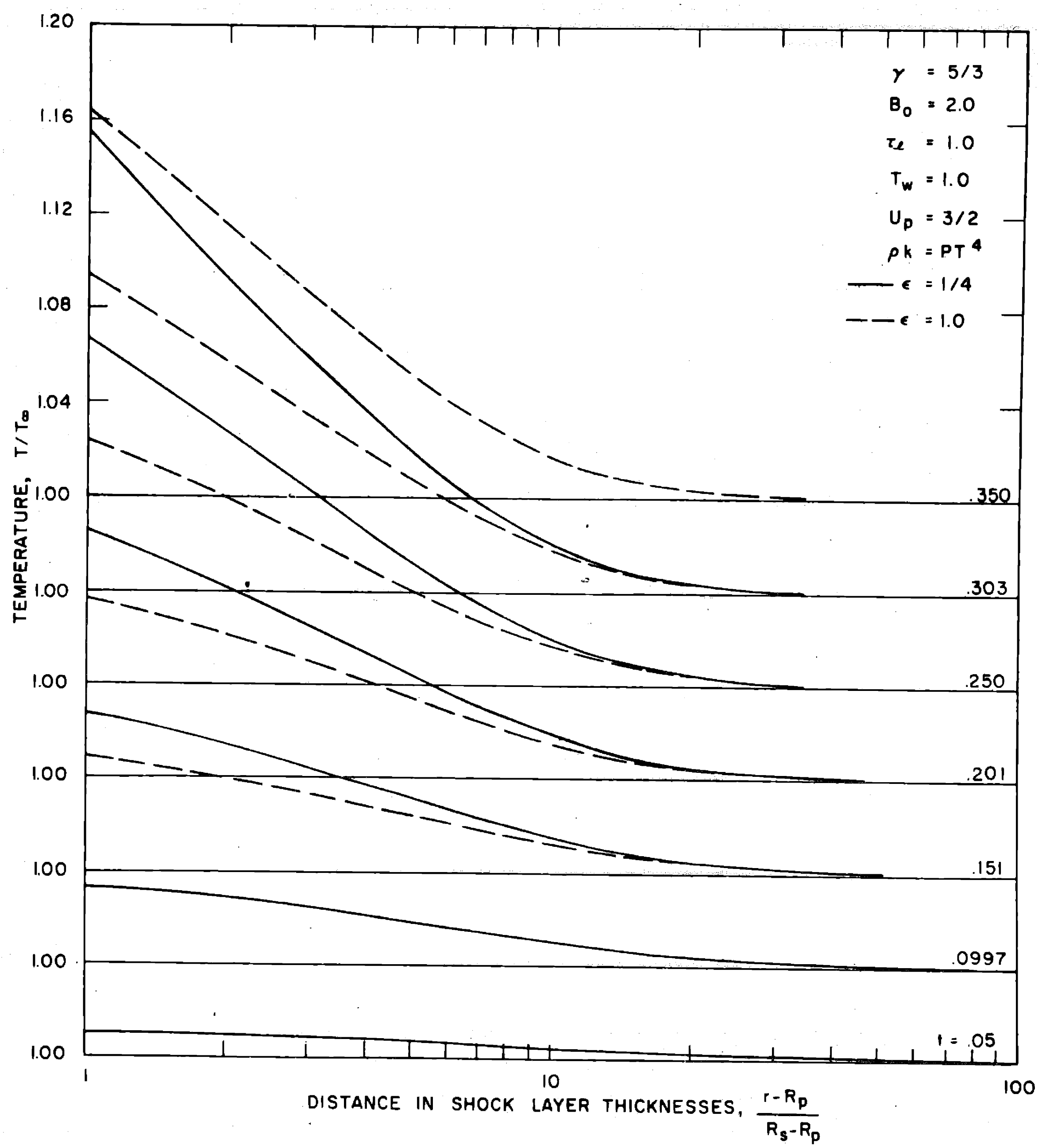

Figure 15k. Evolution of the flow field induced upstream of the shock wave generated by a cylindrical piston in uniform motion

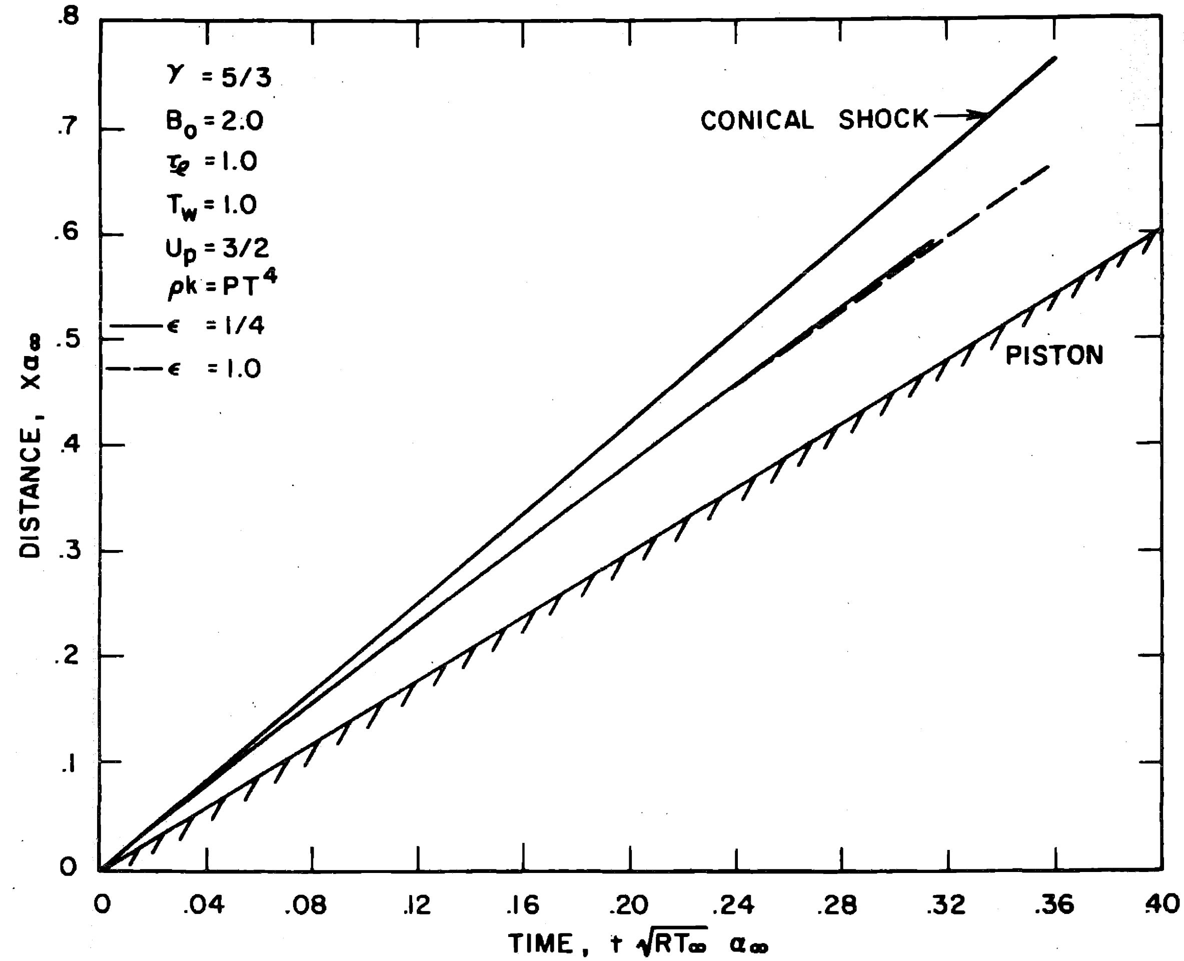

Figure 151. Effect of emissivity upon shock wave generated by a cylindrical piston in uniform motion

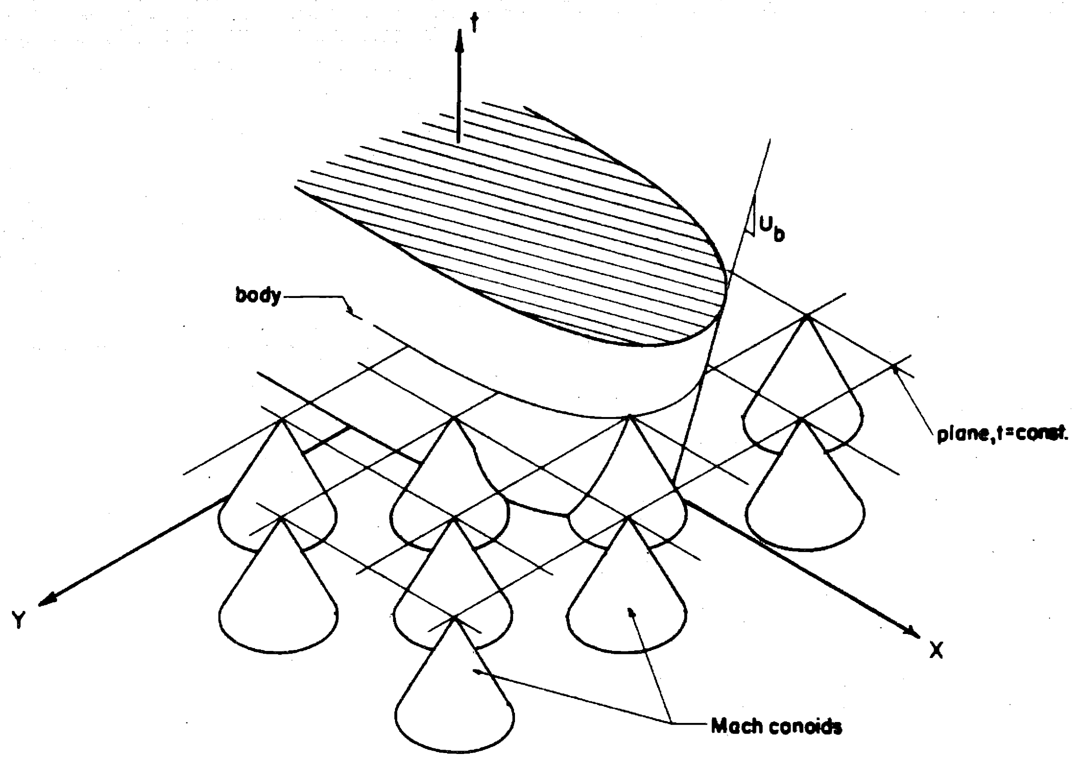

Figure 16. Geometry of the multi-dimensional characteristics solution

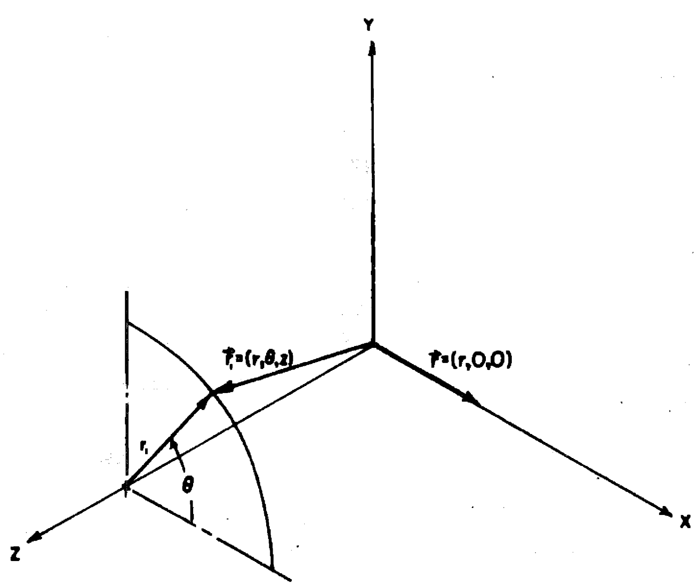

Figure 17. Geometry of cylindrically symmetric radiative transfer

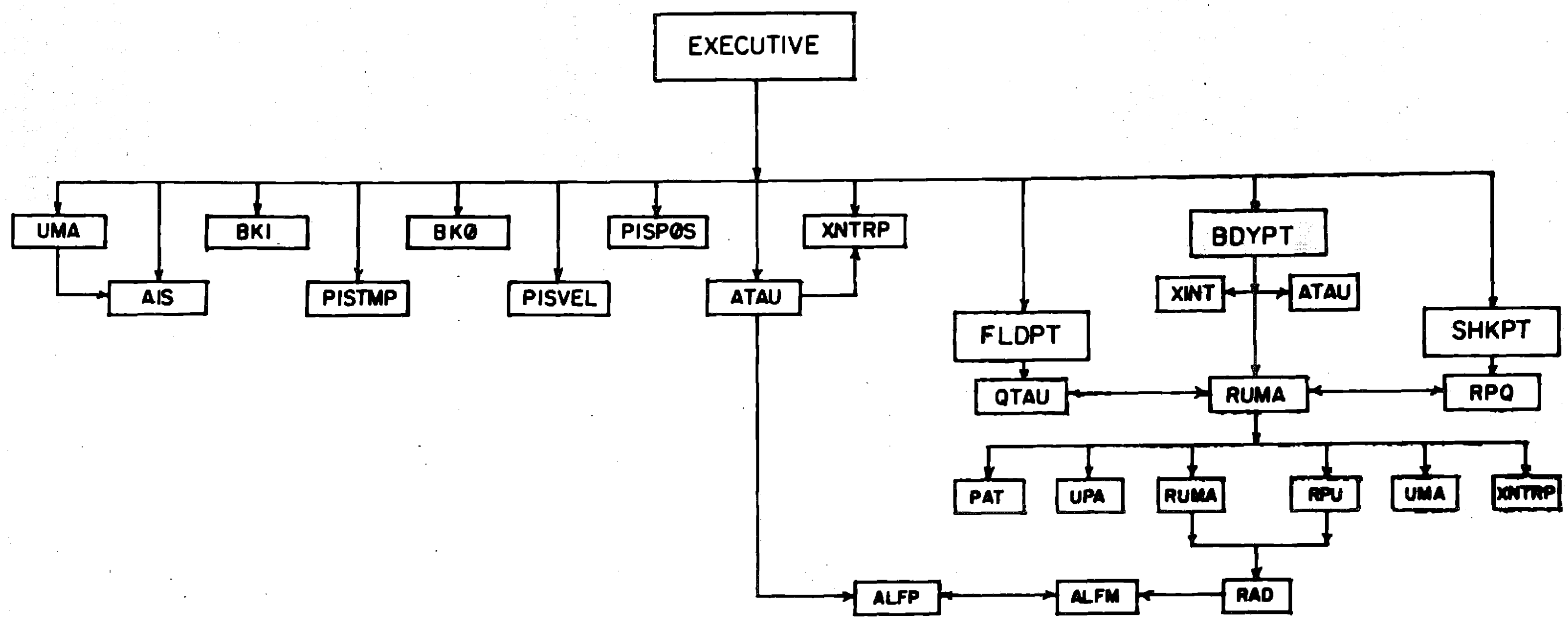

Figure 18. Flow chart of major programs

APPENDIX A

SEMI-GREY RADIATIVE TRANSFER

The assumption of gray radiative transfer is inadequate in most situations since a single mean absorption (emission) coefficient is conceptually correct only at the optically thick or thin extremes. If the gas is non-grey, several mean absorption coefficients appear when the transfer equation is integrated over all frequencies. In this note a mean absorption coefficient is defined in addition to an emission coefficient, and both means are employed in the formulation of radiant heat exchange. It is demonstrated that when local thermodynamic equilibrium (LTE) is not invoked several mean absorption properties must be defined. The resulting approximations are meant for specific application in radiation gas dynamics and has been termed "semi-grey" radiative transfer.

A.1 Non-Grey Absorption in Local Thermodynamic Equilibrium

In a non-scattering gas which is unencumbered by material boundaries or unsteady effects the assumption of local thermodynamic equilibrium leads to the following transfer equation for the specific intensity I_ν,

$$\vec{\Omega}\cdot \mathrm{grad}\,(I_\nu) = \alpha_\nu \,(B_\nu - I_\nu) \qquad \text{(A.1)}$$

where

$$B_\nu = \frac{2h\nu^3}{c^2}\left\{e^{\frac{h\nu}{kT}} - 1\right\}^{-1} \qquad \text{(A.2)}$$

$$\alpha_\nu = \rho k_\nu (1 - e^{\cdots})$$ (A.3)

and $\vec{\Omega}$ is the unit vector in the direction of propagation. When Eq. (A.1) is integrated over frequency, both mean "emission" and "absorption" coefficients must be defined in the integrated formulation.

$$\vec{\Omega} \cdot grad(I) = \alpha_p B(T) - \alpha_a I$$ (A.4)

where

$$I(\vec{\Omega}, \vec{r}) = \int_0^\infty I_\nu \, d\nu$$ (A.5)

$$B(T) = \int_0^\infty B_\nu \, d\nu$$ (A.6)

$$\alpha_p(\vec{r}) = \int_0^\infty \alpha_\nu B_\nu \, d\nu \Big/ B(T)$$ (A.7)

$$\alpha_a(\vec{\Omega}, \vec{r}) = \int_0^\infty \alpha_\nu I_\nu \, d\nu \Big/ I$$ (A.8)

Viskanta (1964) refers to α_a and α_p as mean volumetric absorption and emission (Planck) coefficients. Unlike the emission mean, the absorption mean depends upon the solution of a specific problem; e.g., upon an entire flow field. In addition, it is anisotropic in general (Sampson, 1967).

All properties are normalized as follows (asterisks denote dimensional quantities)

$$\vec{q} = \vec{q}^* / \sigma T_\infty^{*4} \qquad \vec{r} = \vec{r}^* / \ell \qquad T = T^* / T_\infty^*$$

$$\alpha = \alpha^* / \alpha_{p_\infty} \qquad \vec{I}_n = \vec{I}_n^* / \sigma T_\infty^{*4}$$

where ℓ is an appropriate length scale, $(\,)_\infty$ denotes some reference state in an undisturbed gas, and the corresponding Bouguer number is $\tau_\ell = \alpha_{p_\infty} \ell$. The moment equations which hield the macroscopic properties of the radiative field are:

$$div\, \vec{q} = \tau_\ell \{ 4\alpha_p T^4 - \alpha_a^{(0)} I_0 \}$$ (A.9)

$$\mathrm{div}(\overleftrightarrow{I}_2) = -\tau_\ell \alpha_a^{(1)} \vec{q} \tag{A.10}$$

$$\vdots \qquad\qquad \vdots$$

$$\mathrm{div}(\overleftrightarrow{I}_n) = \tau_\ell \{ C_n \cdot 4\alpha_p T^4 - \alpha_a^{(n-1)} \overleftrightarrow{I}_{n-1} \} \tag{A.11}$$

where the C_n result from appropriate angular integration. Since is not isotropic, the angular moments of the absorption coefficient as well as those of the specific intensity must be defined.

$$\alpha_a^{(n)}(\vec{r}) \overleftrightarrow{I}_n = \int_{4\pi} \Omega^n \alpha_a(\vec{\Omega},\vec{r}) I(\vec{\Omega},\vec{r}) d\vec{\Omega} \tag{A.12}$$

$$\overleftrightarrow{I}_n = \int_{4\pi} \vec{\Omega}^n I(\vec{\Omega},\vec{r}) d\vec{\Omega} \tag{A.13}$$

where it is understood that $\vec{\Omega}^n$ is an n[th] order tensor. The moments are tensors as well, but the notation will not be complicated with that fact. The notation $\vec{q}$ is used interchangeably with $\vec{I}_1$ which is the radiative heat flux. Similarly $\overleftrightarrow{P}^{(R)} = \frac{1}{c} \overleftrightarrow{I}_2$ is the radiation stress tensor and $I_0 = c\,u^{(R)}$ is related to the radiative energy density. It is clear that the correct optically thin limit must always follow independent of α_a, thus examination of the thick situation may clarify the meaning of the absorption mean.

If the absorption length is much smaller than the dimension of the physical region of interest, the following Taylor series is a particular solution of Eq. (A.1) (Goulard, 1963a)

$$I_\nu = B_\nu - \frac{1}{\tau_\ell \alpha_\nu} \frac{dB_\nu}{dT} \vec{\Omega} \cdot \mathrm{grad}\, T + \frac{1}{\tau_\ell \alpha_\nu} \vec{\Omega} \cdot \mathrm{grad} \left\{ \frac{1}{\tau_\ell \alpha_\nu} \frac{dB_\nu}{dT} \vec{\Omega} \cdot \mathrm{grad}\, T \right\} + \cdots \tag{A.14}$$

Hence the first two spectral moments of the specific intensity are

$$I_{0_\nu} = 4\pi B_\nu + \frac{4\pi}{3\tau_\ell^2} \left\{ \frac{1}{\alpha_\nu^2} \frac{d^2 B_\nu}{dT^2} - \frac{1}{\alpha_\nu^3} \frac{\partial \alpha_\nu}{\partial T} \frac{dB_\nu}{dT} \right\} (\mathrm{grad}\, T)^2 - \frac{4\pi}{3\tau_\ell^2 \alpha_\nu^3} \frac{\partial \alpha_\nu}{\partial P} \frac{dB_\nu}{dT} \mathrm{grad}\, T \cdot \mathrm{grad}\, P + \frac{4\pi}{3\tau_\ell^2 \alpha_\nu^2} \nabla^2 T + \cdots \tag{A.15}$$

$$\vec{q}_\nu = -\frac{4\pi}{3\tau_\ell \alpha_\nu} \frac{dB_\nu}{dT} \mathrm{grad}\, T + \cdots \tag{A.16}$$

From Eq. (A.12) it follows that

$$\alpha_a^{(0)} = \alpha_p + \frac{4}{3\tau_f^2 B}\left\{\left(\frac{1}{\alpha_f} - \frac{\alpha_p}{\alpha_b^2}\right)\sigma_1 T^2 - \left(\frac{1}{\alpha_g} - \frac{\alpha_p}{\alpha_c^2}\right)\sigma_2 T^3\right\}(\mathrm{grad}\,T)^2 - \frac{4\sigma_2 T^3}{3\tau_f^2 B}\left\{\frac{1}{\alpha_h} - \frac{\alpha_p}{\alpha_d^2}\right\}\mathrm{grad}\,T\cdot\mathrm{grad}\,P \tag{A.17}$$
$$+ \frac{4\sigma_2 T^3}{3\tau_f^2 B}\left\{\frac{1}{\alpha_R} - \frac{\alpha_p}{\alpha_e^2}\right\}\nabla^2 T + O\left(\frac{1}{\tau_f^3}\right)$$

$$\alpha_a^{(1)} = \alpha_R + O\left(\frac{1}{\tau_f^2}\right) \tag{A.18}$$

where the Rosseland mean absorption coefficient is

$$\frac{1}{\alpha_R} = \int_0^\infty \frac{1}{\alpha_\nu}\frac{dB_\nu}{dT}\,d\nu \Big/ \left(\frac{\sigma_2 T^3}{\pi}\right), \quad \frac{\sigma_2 T^3}{\pi} = \int_0^\infty \frac{dB_\nu}{dT}\,d\nu \tag{A.19}$$

and the following properties of the medium may be identified as well.

$$\frac{1}{\alpha_b^2} = \int_0^\infty \frac{1}{\alpha_\nu^2}\frac{d^2B_\nu}{dT^2}\,d\nu \Big/ \left(\frac{\sigma_1 T^2}{\pi}\right), \quad \frac{\sigma_1 T^2}{\pi} = \int_0^\infty \frac{d^2B_\nu}{dT^2}\,d\nu \tag{A.20}$$

$$\frac{1}{\alpha_c^2} = \int_0^\infty \frac{1}{\alpha_\nu^3}\frac{\partial\alpha_\nu}{\partial T}\frac{dB_\nu}{dT}\,d\nu \Big/ \left(\frac{\sigma_2 T^3}{\pi}\right) \tag{A.21}$$

$$\frac{1}{\alpha_d^2} = \int_0^\infty \frac{1}{\alpha_\nu^3}\frac{\partial\alpha_\nu}{\partial P}\frac{dB_\nu}{dT}\,d\nu \Big/ \left(\frac{\sigma_2 T^3}{\pi}\right) \tag{A.22}$$

$$\frac{1}{\alpha_e^2} = \int_0^\infty \frac{1}{\alpha_\nu^2}\frac{dB_\nu}{dT}\,d\nu \Big/ \left(\frac{\sigma_2 T^3}{\pi}\right) \tag{A.23}$$

$$\frac{1}{\alpha_f} = \int_0^\infty \frac{1}{\alpha_\nu}\frac{d^2B_\nu}{dT^2} \Big/ \left(\frac{\sigma_1 T^2}{\pi}\right) \tag{A.24}$$

$$\frac{1}{\alpha_g} = \int_0^\infty \frac{1}{\alpha_\nu^2}\frac{\partial\alpha_\nu}{\partial T}\frac{dB_\nu}{dT} \Big/ \left(\frac{\sigma_2 T^3}{\pi}\right) \tag{A.25}$$

$$\frac{1}{\alpha_h} = \int_0^\infty \frac{1}{\alpha_\nu^2}\frac{\partial\alpha_\nu}{\partial P}\frac{dB_\nu}{dT} \Big/ \left(\frac{\sigma_2 T^3}{\pi}\right) \tag{A.26}$$

where $\alpha_\nu(P,T)$ is a known property of the gas. If the moment equations, Eq. (A.9) - (A.11) are truncated in the usual manner (see Section 3.2.2), the result consistent with a P_1 differential approximation is, to lowest order in Bouguer number

$$\mathrm{div}\,\vec{q} = \tau_f\,\alpha_p\{4T^4 - I_0\} \tag{A.27}$$

$$grad(I_0) = -3\tau_\ell \alpha_R \vec{q} \tag{A.28}$$

which are identically Traugott's equations (Traugott, 1966). They must be in error far from regions of radiative disturbance since as gradients vanish, α_a does in fact become isotropic so that $\alpha_a = \alpha_a^{(0)} = \alpha_a^{(1)} = \cdots = \alpha_P$. Nevertheless the implication is that the various moments of α_a might not all be the same.

Examination of the optically thick solution of Eq. (A.4) leads to

$$I = \frac{\alpha_P}{\alpha_a} B(T) - \frac{1}{\tau_\ell \alpha_a} \vec{\Omega}\cdot grad\left(\frac{\alpha_P}{\alpha_a} B\right) + O\left(\frac{1}{\tau_\ell^2}\right) \tag{A.29}$$

whereas the exact frequency integrated expression from Eq. (A.14) is

$$I = B(T) - \frac{1}{\tau_\ell \alpha_R} \vec{\Omega}\cdot grad\,(B) + O\left(\frac{1}{\tau_\ell^2}\right) \tag{A.30}$$

Compatibility of these expressions requires that

$$\frac{1}{\alpha_a} \vec{\Omega}\cdot grad\left(\frac{\alpha_P}{\alpha_a} B\right) - \frac{1}{\alpha_R} \vec{\Omega}\cdot grad\,(B) \simeq B(T)\left\{\frac{\alpha_P}{\alpha_a} - 1\right\} \tag{A.31}$$

If

$$\Upsilon(\vec{\Omega}, \vec{r}) = \frac{\alpha_P(\vec{r})}{\alpha_a(\vec{\Omega}, \vec{r})} B(T) \tag{A.32}$$

then Eq. (A.31) may be recast in the form

$$\Upsilon \vec{\Omega}\cdot grad(\Upsilon) - \left(\frac{\alpha_P}{\alpha_R}\right) B\, \vec{\Omega}\cdot grad(B) = -\alpha_P B\{\Upsilon - B\} \tag{A.33}$$

Eq. (A.33) is a nonlinear Abel equation (Murphy, 1960) and has closed form solutions only for certain special functional forms of $\frac{\alpha_P}{\alpha_R} B$. It is an indication of the manner in which α_a depends upon the familiar properties α_P and α_R and upon the flow field. In general, Eq. (A.33) can be solved only numerically.

Suppose that α_a is independent of $\vec{\Omega}$ (quasi-isotropy). Equation (A.33) may be averaged over the unit sphere and easily solved with the result

$$\Upsilon(\vec{r}) = \left\{1 - 2\int_T^1 \frac{\alpha_P}{\alpha_R}[P(T'),T']\, T'^4 dT'^4\right\}^{1/2} \tag{A.34}$$

The Steiltjes representation is necessary mathematically because the temperature will not have continuous derivatives in general. (Henceforth the Steiltjes form is implied in all cases.) Eq. (A.34) satisfies the condition $I_\nu \to B_\nu$ as $T \to 1$. If α_P/α_R depends only upon temperature, which is a reasonable approximation, then α_a may be determined from Eq. (A.34) once and for all. If Y were expanded in a series of spherical harmonics, the approximation indicated above would occupy the same position in the truncated system that the Milne-Eddington approximation does in the differential formulation of radiative transfer.

The optically thick limit has long been known to be singular, and the consequences of carrying out a matching procedure in a singular limit are not clear. The anisotropy of α_a is evident when boundaries are present since in that case a solution such as Eq. (A.34) could not be found because the values of Y on the boundary would be necessary. Since those values would depend upon the direction in which the boundary was "seen", it is proposed that a first approximation might be

$$\Upsilon^2(\vec{\Omega},\vec{r}) = \Upsilon_o^2(\vec{\Omega}) - 2\int_T^{T_o(\vec{\Omega})} \frac{\alpha_P}{\alpha_R}[P(T'),T']\, T'^4 dT'^4 \tag{A.35}$$

Consistent with Eq. (A.33) the unknown constant might be determined from

$$\int_{4\pi} \{\Upsilon(\vec{\Omega}) - B\}\, d\Omega = 0 \tag{A.36}$$

The analysis has not been restricted to specific geometries thus far. However, a procedure for correcting Eq. (A.34) in the planar case has been considered. In analogy with Olfe's (1967) recent modification of the differential approximation, even in the thick case terms dependent upon radiative interaction with boundaries must be retained. When there is a black boundary at $x = 0$, to lowest order, there result:

$$q_\nu = -2\pi\left\{\frac{2}{3\tau_\nu \alpha_\nu}\frac{\partial B_\nu}{\partial x} - B_\nu(T_w)E_3(\tau_\nu \eta_\nu) + \cdots\right\}, \quad \eta_\nu = \int_0^x \alpha_\nu(\xi)d\xi \tag{A.37}$$

$$I_{o_\nu} = 2\pi \{ 2 B_\nu(T) + B_\nu(T_w) E_2(\tau_o \eta_\nu) + \cdots \} \qquad \text{(A.38)}$$

It is assumed that the gas is nearly grey, without ambiguity $\alpha_\nu = \alpha_R$ in the above. The manipulations carried out previously then yield the following integral equation for a quasi-isotropic Y.

$$Y^2 = 1 - 2\int_x^\infty \left\{ \frac{\alpha_P}{\alpha_R} T^{-4} \frac{\partial T^4}{\partial x} + \frac{3}{2} \alpha_P T^{-4} T_w^4 \left[E_3\left\{ \tau_o \int_0^{x'} \frac{\alpha_P T^4(\xi)}{Y(\xi)} d\xi \right\} - E_3\left\{ \tau_o \int_0^{x'} \alpha_R(\xi) d\xi \right\} \right] \right\} dx' \qquad \text{(A.39)}$$

It is suggested that Eq. (A.34) may be inserted in the argument of the first exponential integral, so that a second approximation to Y, including boundary effects, may be found.

Contrary to the approach which may be taken in astrophysics (Sampson, 1965a) consideration has nowhere been restricted to a specific situation or an assumed temperature distribution. No general conclusions can be drawn from such comparisons in RGD because the nature of the flow field can seldom be predicted in advance. The implications of semi-grey radiative transfer upon the general formulation of RGD have been examined in Chapter 2. For instance, even in the planar case the geometrical aspects of the radiative field cannot be resolved and exponential integrals do not appear in general. It is stressed that neither (A.34) nor (A.39) may offer improvement in all cases.

If the gas were truly grey, $\alpha_P = \alpha_R$, and complete greyness, $\alpha_a = \alpha_P = \alpha_R$, is in fact a particular solution of Eq. (A.32). If

$$\frac{\alpha_P}{\alpha_R} = \left(\frac{m+1}{n^2}\right) T^{8m} \qquad \text{(A.40)}$$

then a solution of Eq. (A.33) is, to a quasi-isotropic approximation.

$$\alpha_a = \sqrt{(m+1)\alpha_P \alpha_R} \qquad \text{(A.41)}$$

It often happens that $0 < m < 1$ (Scala and Sampson, 1964); thus

$$\alpha_R < \alpha_a < \alpha_P \qquad \text{(A.42)}$$

if $\frac{T^{4m}}{n} > 1$. Eq. (A.41) does not lead to the correct undisturbed result $\alpha_a \to \alpha_p$. If m = 0, Traugott's equations may be reproduced if I_o is redefined within a multiplicative constant. When derived in this manner, the approximation is not restricted to near thick situations. The use of a square root mean, Eq. (A.41), is often acceptable (Goody, 1960). It is further noted that Eq. (A.4) is similar to Stewart's non-grey transfer equation (Stewart, 1964).

In planar situations of pure radiative transfer in gray gases B(T) is sought as a function of the optical variable, η . It is suggested that existing solutions may be reinterpreted by considering B(T) to be $\frac{\alpha_p}{\alpha_a} B(T)$ and carrying out the transformation to the physical plane according to

$$\chi = \int_0^{\eta} \frac{d\eta'}{\alpha_a(\eta')} \tag{A.43}$$

In radiative equilibrium absorption and emission are balanced so that

$$\vec{q} \simeq -\frac{4}{3\tau_p \alpha_R} \, grad\,(T^4) = const. \tag{A.44}$$

The same result follows for quasi-isotropic, semi-grey radiation as a consequence of the matching procedure. Thus the same temperature distributions would be predicted by Eq. (A.34) and by Traugott's equations even though the former assumed isotropy in α_a . The behavior of I_o would, of course, be different.

A.2 Semi-Grey Transfer in Non-LTE Gases

The inclusion of the effects of non-greyness in an essentially grey system of equations was aided substantially by consideration of gases in LTE. It will not be shown that the concept may be extended to non-LTE situations. For expository purposes the model of Clarke and Ferrari (1965) was chosen. If induced radiative recombination is neglected in a gas of atoms, ions, and electrons, the transfer equation is

$$\vec{\Omega} \cdot grad\,(I_\nu) = \tau_\ell \alpha_\nu (1-c) \left\{ \left(\frac{c^2}{1-c}\right)\left(\frac{1-c_e}{c_e^2}\right) B_\nu(T) - I_\nu \right\} \tag{A.45}$$

where C is the mass fraction of ions, and the subscript "e" denotes the

Saha equilibrium state at local conditions. Before the concept of α_a may be examined, the modification to the mean absorption properties of the gas must be examined.

Since C does not depend upon frequency the Planck mean is unaffected by the nonequilibrium nature of the radiative field. If the optical depth is

$$\eta_\nu = \int_0^r (1-C)\alpha_\nu(\xi)\,d\xi \tag{A.46}$$

then the familiar approximations lead to

$$I_\nu = \tilde{B}_\nu - \frac{1}{\zeta \alpha_\nu (1-C)} \vec{\Omega}\cdot \mathrm{grad}(\tilde{B}_\nu) + \cdots \tag{A.47}$$

where

$$\tilde{B}_\nu = \left(\frac{C^2}{1-C}\right)\left(\frac{1-C_e}{C_e^2}\right) B_\nu(T) \tag{A.48}$$

Since the source function, $\tilde{B}_\nu$, is not a function of temperature alone

$$\begin{aligned}\vec{\Omega}\cdot \mathrm{grad}(\tilde{B}_\nu) = {} & \frac{C^2}{(1-C)}\frac{\partial}{\partial T}\left[\left(\frac{1-C_e}{C_e^2}\right)B_\nu\right]\vec{\Omega}\cdot \mathrm{grad}(T) + \left(\frac{1-C_e}{C_e^2}\right)\frac{C(2-C)}{(1-C)^2}B_\nu\,\vec{\Omega}\cdot \mathrm{grad}(C) + \\ & + \left(\frac{C^2}{1-C}\right)B_\nu \frac{\partial}{\partial P}\left[\frac{1-C_e}{C_e}\right]\vec{\Omega}\cdot \mathrm{grad}\,P\end{aligned} \tag{A.49}$$

Thus it follows that

$$\begin{aligned}\vec{q} = -\frac{4\pi\tilde{B}}{3\zeta_\ell(1-C)}\Big\{ & \frac{1}{\alpha_T}\mathrm{grad}\,T + \frac{1}{\alpha_M}\frac{(2-C)}{(1-C)}\mathrm{grad}(\ln C) \\ & + \frac{1}{\alpha_M}\frac{\partial}{\partial P}\left[\ln\left(\frac{1-C_e}{C_e^2}\right)\right]\mathrm{grad}(P)\Big\} + o\left(\frac{1}{\zeta_\ell^2}\right)\end{aligned} \tag{A.50}$$

where

$$\tilde{B} = \int_0^\infty \tilde{B}_\nu\,d\nu \tag{A.51}$$

$$\frac{1}{\alpha_M} = \int \left(\frac{1}{\alpha_\nu}\right) B_\nu\,d\nu \Big/ B \tag{A.52}$$

$$\frac{1}{\alpha_T} = \frac{1}{\alpha_M}\frac{\partial}{\partial T}\left\{\ln\left(\frac{1-C_e}{C_e^2}\right)\right\} + \frac{\sigma_2}{T\cdot\alpha_R} \tag{A.53}$$

The detailed model incorporates a heat flux dependent reaction rate, thus the gradient of the atom mass fraction depends upon the heat flux (or at least upon part of it). This bears a striking resemblance to the Hall effect in MHD. Eq. (A.50) proves that concentration and pressure gradients as well as temperature gradients may produce heat fluxes. Since the Saha equation for $\ln\left(\frac{1-C_e}{C_e^2}\right)$ is proportional to (T_j/T), where T_j is the temperature which corresponds to the ionization potential, an approximation consistent with the assumptions $\left(\frac{T_j}{T} >> 1\right)$ must omit the Rosseland mean completely. In a non-LTE situation optically thick radiation no longer is directly related to molecular conduction, and even the limiting case may involve more than a single mean absorption coefficient.

If Eq. (A.45) is integrated over all frequencies and the procedure outlined previously is applied it may be shown that

$$\frac{\alpha_p}{\alpha_a}\tilde{B} - \left(\frac{1}{\tau_\ell \alpha_a}\right)\vec{\Omega}\cdot grad\left(\frac{\alpha_p}{\alpha_a}\tilde{B}\right) = \tilde{B} - \frac{\tilde{B}}{\tau_\ell}\left\{\frac{1}{\alpha_T}\vec{\Omega}\cdot grad\,T + \frac{1}{\alpha_M}\left[\frac{(2-c)}{(1-c)}\vec{\Omega}\cdot grad(\ln c) + \frac{\partial}{\partial P}\left\{\ln\left(\frac{1-C_e}{C_e^2}\right)\right\}\vec{\Omega}\cdot grad\,P\right]\right. \tag{A.54}$$

Furthermore, if quasi-isotropy is assumed and the reference state is one of chemical as well as radiative equilibrium and

$$\tilde{\Upsilon} = \frac{\alpha_p}{\alpha_a}\tilde{B} \tag{A.55}$$

then

$$\tilde{\Upsilon}^2 = 1 - 2\int_{|\vec{r}|}^{\infty}\frac{\tilde{B}^2}{1-c}\left\{\frac{\alpha_p}{\alpha_T}grad(T) + \frac{\alpha_p}{\alpha_M}\left[\left(\frac{2-c}{1-c}\right)grad(\ln c) + \frac{\partial}{\partial P}\left\{\ln\left(\frac{1-C_e}{C_e^2}\right)\right\}grad\,P\right]\right\}\cdot d\vec{r} \tag{A.56}$$

Eq. (A.56) can never be evaluated once and for all just as Eq. (A.34) cannot unless α_p/α_R depends only upon temperature. In both cases, only iterative solutions of general problems are possible. Thus Eq. (A.56) lacks the potential simplicity of Eq. (A.34). In chemical equilibrium Eq. (A.56) and (A.34) are identical.

A.3 Conclusions

Considerable caution is necessary in the physical interpretation of the absorption coefficient, α_a. The demonstration of the firm physical foundation of Traugott's equations is but one of the many implications that it may have. The formulation is quite general, but relies upon the assumption if isotropy in α_a. Even though only iterative schemes may be used in general, this approach is simpler than the inclusion of the numerous detailed radiative processes which participate in non-grey radiative transfer. Either experiment or exact solutions in non-grey gases are necessary before the usefulness of the quasi-isotropic assumption may be determined. In any event the systematic specification of mean absorption coefficients is a good deal more complicated than has heretofore been recognized*.

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* An independent analysis similar to the one presented herein has recently come to our attention (Patch, 1967). Although the basic concepts are indeed very closely related, the detailed formulations are quite different as are the purposes for which the mean coefficients are intended.

APPENDIX B

TWO-DIMENSIONAL STEADY FLOW: A POSSIBLE APPROACH BY THE METHOD OF GARABEDIAN AND LIBERSTEIN

Consider the appropriately normalized equations of two-dimensional, steady RGD. For simplicity only a perfect gas is employed, but similar results would follow if the general case of Chapter III were considered. The equations are:

$$\rho(u_x+v_y) + u\rho_x + v\rho_y - \left(\frac{\rho u}{T}\right)T_x - \left(\frac{\rho v}{T}\right)T_y = 0 \qquad \text{(B.1)}$$

$$u u_x + v u_y = -\left(\frac{T}{\rho}\right)\rho_x \qquad \text{(B.2)}$$

$$u v_x + v v_y = -\left(\frac{T}{\rho}\right)\rho_y \qquad \text{(B.3)}$$

$$(uT_x+vT_y)+\Gamma u(uu_x+vu_y)+\Gamma v(uv_x+vv_y) = \Gamma\left(\frac{\tau_e}{B_0}\right)\left(\frac{T}{\rho}\right)Q \qquad \text{(B.4)}$$

$$(q_1)_x + (q_2)_y = -\tau_e Q \qquad \text{(B.5)}$$

$$(I_0)_x = -3\tau_e \alpha_a q_1 \qquad \text{(B.6)}$$

$$(I_0)_y = -3\tau_e \alpha_a q_2 \qquad \text{(B.7)}$$

where

$$Q = \alpha_a I_0 - 4\alpha_p T^4$$
$$\Gamma = (\gamma_\infty - 1)/\gamma_\infty \qquad \text{(B.8)}$$

Following Cohen (1965), in a quasi-isotropic gas one may assume that

$$\vec{q} = \frac{1}{\alpha_a} \operatorname{grad} \phi \tag{B.9}$$

Then Eq. (B.5) may be written in the form

$$\nabla^2 \phi - [\operatorname{grad}(\ln \alpha_a)] \cdot \operatorname{grad} \phi = -\alpha_a \tau_e \phi \tag{B.10}$$

It follows from Eq. (B.6) and (B.7) that

$$I_0 = -3 \tau_e \phi \tag{B.11}$$

if all arbitrary constants are absorbed into the definition of .
Eq. (B.1) - (B.7) may be cast in the following form

$$\frac{\partial}{\partial y}\begin{Bmatrix} p \\ u \\ v \\ T \\ \phi_2 \\ \phi_1 \\ \phi \end{Bmatrix} + \begin{bmatrix} \frac{uv}{\Delta_1}(1-\Pi) & \frac{pv}{\Delta_1} & -\frac{pu}{\Delta_1} & 0 & 0 & 0 & 0 \\ T/pv & u/v & 0 & 0 & \vdots & \vdots & \vdots \\ -\frac{Tu}{p\Delta_1}(1-\Pi) & -T/\Delta_1 & \frac{uv(1-\Pi)}{\Delta_1} & 0 & \vdots & \vdots & \vdots \\ \frac{\Pi T^2}{p\Delta_1}\left(\frac{u}{v}\right) & \frac{\Pi T}{\Delta_1} v & -\frac{\Pi T}{\Delta_1} u & \frac{u}{v} & 0 & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & 0 \\ \vdots & & & \vdots & -1 & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial x}\begin{Bmatrix} p \\ u \\ v \\ T \\ \phi_2 \\ \phi_1 \\ \phi \end{Bmatrix} = \tag{B.12}$$

$$= \begin{bmatrix} \frac{v\tau_e}{\Delta_1 \cdot B_0} Q \\ 0 \\ -\frac{\tau_e}{\Delta \cdot B_0}\left(\frac{T}{p}\right) Q \\ \frac{\tau_e}{\Delta \cdot B_0} \frac{T(v^2-\Pi)}{pv} Q \\ -\tau_e Q + (\ln \alpha_a)_x \phi_1 + (\ln \alpha_a)_y \phi_2 \\ 0 \\ \phi_2 \end{bmatrix}$$

where

$$\phi_1 = \phi_x \tag{B.13}$$

$$\phi_2 = \phi_y \tag{B.14}$$

$$\Delta_1 = T - v^2(1-r) = (1-r)(a^2 - v^2) \tag{B.15}$$

The characteristics of this system are

$$\frac{dx}{dy} = 0,\ \frac{u}{v},\ \frac{u}{v},\ \frac{-\frac{uv}{a^2} \pm \sqrt{M^2-1}}{1 - v^2/a^2},\ \pm i \tag{B.16}$$

where $M^2 = \dfrac{u^2+v^2}{a^2}$

Therefore the problem is not fully hyperbolic even if $M > 1$. The zero characteristic is redundant since it is merely a consequence of the comutative nature of the differentiations performed (see von Mises, 1958). The corresponding compatibility relations for the fluid mechanics are

$$\left\{ u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \right\} P + \rho \left\{ u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \right\} \left(\frac{u^2+v^2}{2} \right) = 0 \qquad \text{on} \quad \frac{dx}{dy} = \frac{u}{v} \tag{B.17}$$

$$\left\{ u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \right\} T + r \left\{ u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \right\} \left(\frac{u^2+v^2}{2} \right) = \frac{\tau_\ell}{\rho B_0} Q \qquad \text{on} \quad \frac{dx}{dy} = \frac{u}{v} \tag{B.18}$$

$$\pm\sqrt{M^2-1} \left\{ [T-(1-r)v^2]\frac{\partial}{\partial y} + [uv(r-1) \pm T\sqrt{M^2-1}]\frac{\partial}{\partial x} \right\} P + \rho \left\{ [T-(1-r)v^2]\frac{\partial}{\partial y} + [uv(r-1) \pm T\sqrt{M^2-1}]\frac{\partial}{\partial x} \right\} \left(\frac{u^2+v^2}{2} \right) =$$
$$= -\frac{\tau_\ell}{B_0} Q \cdot (u \mp \sqrt{M^2-1}\, v) \qquad \text{on} \quad \frac{dx}{dy} = \frac{-\frac{uv}{a^2} \pm \sqrt{M^2-1}}{1 - v^2/a^2} \tag{B.19}$$

Bernoulli's equation, Eq. (B.17), is unchanged by radiation because radiation pressure has been ignored. The Mach and path line relations are modified by emission locally and by the integrated effect of the component of the heat flux along them since

$$\varphi = \int_0^s \vec{q} \cdot \alpha_n \, d\vec{s}$$

along any line the distance along which is s. It may be shown that there is no analog to Crocco's theorem. The compatibility relations which govern the radiative properties are complex. It is proposed that the problem be continued analytically into the complex domain just as

Garabedian (1957) and Garabedian and Liberstein (1958) have done. The method would best be applied to subsonic regions, since then most of the characteristics would be imaginary. The difficulty introduced by subsonic regions in the real domain is that they force the problem to be ill-posed with respect to initial data. In the complex domain such problems are well posed (see especially Garabedian, 1964, Chap. 16). The difficulties are traded, since the continuation of the initial data from the real to the complex domain may well be unstable (Lin, 1957). Because of the mathematical uncertainties associated with the approach, it has not been pursued. Nevertheless, the method may prove worthwhile, and its advantages must be weighed against those of the unsteady approach to a steady flow.

APPENDIX C

CYLINDRICAL AND SPHERICAL RADIATIVE TRANSFER WHEN ρk_ν IS CONSTANT

Consider Eq. (2.59) in the steady state when α_ν is a constant and there are no boundaries present. In the notation of Figure 1,

$$\operatorname{div} \vec{q} - 4\alpha \tau_e T^4 = -\int_V \tau_e^2 \alpha^2 \frac{4\pi B(P) e^{-\tau_e \tau_{,MP}}}{\overline{MP}^2} dV(P) \qquad \text{(C.1)}$$

The geometry of Figure 17 in the cylindrical case is considered first. Heaslet and Warming (1966) note that

$$\tau_{,MP} = \alpha |\vec{r} - \vec{r}_1|$$

$$|\overline{MP}|^2 = |\vec{r} - \vec{r}_1|^2$$

Clearly,

$$dV(P) = r_1 \, dr_1 \, d\theta \, dz \qquad \text{(C.2)}$$

$$|\vec{r} - \vec{r}_1|^2 = u^2 + z^2 \qquad \text{(C.3)}$$

where

$$u^2 = r^2 + r_1^2 - 2 r r_1 \cos\theta \qquad \text{(C.4)}$$

If $|\vec{r}-\vec{r}_1|$ is denoted by $\hat{t}$, then

$$dz = \frac{\hat{t}\,d\hat{t}}{\sqrt{\hat{t}^2-u^2}} \tag{C.5}$$

whence

$$I_0 = \alpha\int_0^\infty 4\pi B(r_1)\cdot r_1 dr_1 \int_0^\pi d\theta \int_u^\infty \frac{e^{-\alpha\hat{t}}}{\hat{t}\sqrt{\hat{t}^2-u^2}}\, d\hat{t} \tag{C.6}$$

It can be shown that

$$\int_u^\infty \frac{e^{-\alpha\hat{t}}}{(\hat{t})^n\sqrt{\hat{t}^2-u^2}}\, d\hat{t} = \frac{\alpha^n}{\lfloor (n-1)} \int_1^\infty (y-1)^{(n-1)} K_0(\alpha u y)\, dy \tag{C.7}$$

Thus

$$\sqrt{\alpha r}\; I_0(r) = \int_0^\infty 4\sigma_e \alpha T^4(r_1)\sqrt{\alpha r_1}\; K(\alpha r, \alpha r_1)\, dr_1 \tag{C.8}$$

where the symmetrized kernel is

$$K(\alpha r, \alpha r_1) = \frac{\alpha\sqrt{r r_1}}{\pi}\int_0^\pi d\theta \int_1^\infty K_0(\alpha u_0 y)\, dy \tag{C.9}$$

One of the standard texts (Watson, 1944) may be employed to show that

$$\int_0^\pi K_0\{\alpha u_0(\theta) y\}\, d\theta = \pi \begin{cases} K_0(\alpha r y)\, I_{(0)}(\alpha r_1 y) & r > r_1 \\ K_0(\alpha r_1 y)\, I_{(0)}(\alpha r y) & r < r_1 \end{cases} \tag{C.10}$$

where $I_{(0)}$ above is the modified Bessel Function. Thus:

$$K(\alpha r, \alpha r_1) = \alpha\sqrt{r r_1} \begin{cases} \int_1^\infty K_0(\alpha r y)\, I_0(\alpha r_1 y)\, dy & r > r_1 \\ \int_1^\infty K_0(\alpha r_1 y)\, I_0(\alpha r y)\, dy & r < r_1 \end{cases} \tag{C.11}$$

The analogous expression to a P_1 differential approximation may be derived as follows. In the cylindrical or spherical cases the heat flux potential may be applied

$$q = \frac{1}{\alpha}\frac{\partial \phi}{\partial r} \tag{C.12}$$

$$I_0 = -3\phi \tag{C.13}$$

to yield

$$\frac{\partial}{\partial \xi_1}\left(\xi_1^j \frac{\partial \phi}{\partial \xi_1}\right) - \xi_1^j \phi = \frac{4}{3} T^4 \xi_1^j \tag{C.14}$$

where

$$\xi_1 = r_1 \alpha \sqrt{3} \tag{C.15}$$

The solution of Eq. (C.14) is

$$\phi(\xi_1) = -\int_0^\infty \xi_1'^{\,j}\, G(\xi_1, \xi_1') \cdot \frac{4}{3} T^4(\xi_1')\, d\xi_1' \tag{C.16}$$

where $G(\xi_1, \xi_1')$ satisfies

$$\begin{gathered}\frac{\partial}{\partial \xi_1}\left(\xi_1^j \frac{\partial G}{\partial \xi_1}\right) - \xi_1^j G = -\delta(\xi_1 - \xi_1') \\ \left[G(\xi_1, \xi_1')\right]_{\xi = \xi_1'^{(-)}}^{\xi = \xi_1'^{(+)}} = 0 \\ \left[\xi_1^j \frac{\partial G}{\partial \xi_1}\right]_{\xi = \xi_1'^{(-)}}^{\xi = \xi_1'^{(+)}} = -1\end{gathered} \tag{C.17}$$

and $G(\xi_1, \xi_1')$ must decay exponentially as $\xi_1 \to \infty$ and be regular at $\xi_1 = 0$. In the cylindrical case

$$G(\xi_1, \xi_1') = \begin{cases} I_0(\xi_1') K_0(\xi_1) & \xi_1 > \xi_1' \\ I_0(\xi_1) K_0(\xi_1') & \xi_1 < \xi_1' \end{cases} \tag{C.18}$$

It follows that

$$I_0(\xi_1) = \int_0^\infty \xi_1' G(\xi_1, \xi_1') \cdot \frac{4}{3} T^4(\xi_1')\, d\xi_1' \tag{C.19}$$

This is equivalent to Eq.(C.8) if

$$G(r,r') = \begin{cases} \int_1^\infty K_0(\alpha r y) I_{(0)}(\alpha r' y)\, dy & r > r' \\ \int_1^\infty K_0(\alpha r' y) I_{(0)}(\alpha r y)\, dy & r < r' \end{cases} \tag{C.20}$$

As long as αr is not too small, it may be assumed that for $y \gtrsim 1$

$$K_0(\alpha r y) \sim \frac{e^{-\alpha r y}}{\sqrt{2\pi \alpha r y}} \tag{C.21}$$

$$I_{(0)}(\alpha r y) \sim \frac{e^{+\alpha r y}}{\sqrt{2\pi \alpha r y}} \tag{C.22}$$

(see e.g. Abramowitz and Stegun, 1964). Thus

$$G(r,r') \simeq \frac{\alpha}{2\pi} E_1(\alpha |r - r'|) \tag{C.23}$$

If

$$E_1(x) \simeq \sqrt{3}\, e^{-\sqrt{3} x} \tag{C.24}$$

then Eq. (C.21) and (C.22) lead to (C.18) exactly! Therefore the relationship between kernel substitution and the differential approximation holds in cylindrical as well as planar geometries if α_μ is a constant. Since the Bouguer number is implicit in α , the comments below Eq. (C.20) indicate why the differential approximation can be expected to diverge most from the exact formulation in the optically thin situation.

The formulation of the spherical problem may be found in detail in the works of Cuperman, et. al. (1964) and in Davison (1958, pg. 96). In this case Eq. (C.3) becomes

$$dV(P) = r'^2 dr' d\varphi\, d\mu \tag{C.25}$$

where

$$\mu = \cos\theta = \frac{\vec{r} \cdot \vec{r}'}{r r'} \tag{C.26}$$

Therefore

$$\frac{d\mu}{|\vec{r} - \vec{r}'|} = \frac{d|\vec{r} - \vec{r}'|}{r r'} \tag{C.27}$$

It follows that

$$rI_o = \alpha \int_0^{\infty} 2r' T^4(r') \left\{ E_1(\alpha|r-r'|) - E_1(\alpha[r+r']) \right\} dr' \tag{C.28}$$

Application of the differential approximation in the same manner as before (see e.g. Eq. (3.66)) yields,

$$\begin{aligned} rI_o = {} & \sqrt{3} \int_0^{\alpha r} \cosh(\sqrt{3}\alpha r') e^{-\sqrt{3}\alpha r} 4\alpha T^4(r') dr' + \\ & + \sqrt{3} \int_{\alpha r}^{\infty} \cosh(\sqrt{3}\alpha r) e^{-\sqrt{3}\alpha r'} \cdot 4\alpha T^4(r') dr' \end{aligned} \tag{C.29}$$

which is identical with (C.28) under assumption (C.24). The equivalence of kernel substitution and the differential approximation is thus further extended. The differential approximation is, however, quite general and applies even if α_ν is not constant. The integration involved in the exact formulation cannot be attempted unless α_ν is a constant. Because of the simplicity of the differential approach, no other approximation can be justified for the initial investigation of nonplanar RGD.

APPENDIX D

ERRORS ASSOCIATED WITH THE FINITE DIFFERENCE ANALYSIS

In this appendix it shall be shown that the errors involved in all of the methods of solution indicated in Section 5.2 are $O(\Delta t)^2$. For expository purposes, consider Eq. (5.2b) which applies along the $\Gamma^{(-)}$ Mach line

$$dP - (\rho a)\, du = \frac{1}{(\frac{C_p}{R}-1)} \left\{ \frac{\tau_\ell}{B_0} \left[\alpha_0^{(1)} I_0 - 4\alpha_p T^4\right] - j \frac{C_p}{R} \frac{Pu}{r} \right\} dt \qquad \text{(D.1)}$$

In the notation of Figure 4, Eq.(5.20) may be reproduced

$$(P_1 - P_3) - \left(\frac{Pa}{T}\right)_{13} (u_1 - u_3) = \frac{1}{(\frac{C_p}{R}-1)_{13}} \left\{ \frac{\tau_\ell}{B_0} \left[\alpha_0^{(1)} I_0 - 4\alpha_p T^4\right] - j \frac{C_p}{R} \frac{Pu}{r} \right\}_{13}$$

A point is inserted between 1 and 3 at a distance $\frac{1}{n}(\Delta s)_{13}$, $n > 1$ from 1. Only one of the inhomogeneous terms on the right hand side of Eq. (D.2) need be retained, since the effects of all are the same. All properties may be expanded in Taylor series about the intermediate point, 3' as follows

$$u_3 = u_{3'} + \left(\frac{du}{ds}\right)_{3'} (n-1)\left(\frac{\Delta s}{n}\right) + \frac{1}{\lfloor 2} \left(\frac{d^2u}{ds^2}\right)_{3'} (n-1)^2 \left(\frac{\Delta s}{n}\right)^2 + \cdots \qquad \text{(D.3a)}$$

$$u_1 = u_{3'} - \left(\frac{du}{ds}\right)_{3'} \left(\frac{\Delta s}{n}\right) + \frac{1}{\lfloor 2} \left(\frac{d^2u}{ds^2}\right)_{3'} \left(\frac{\Delta s}{n}\right)^2 + \cdots \qquad \text{(D.3b)}$$

Then

$$u_{3'} = \overline{u}_{13} + \frac{(-1)}{4}(n^2 - 2n + 2) \left(\frac{d^2u}{ds^2}\right)_{3'} \left(\frac{\Delta s}{n}\right)^2 + O\left(\frac{\Delta s}{n}\right)^4 \qquad \text{(D.4a)}$$

$$\left(\frac{du}{ds}\right)_{3'} = \frac{u_1 - u_3}{\Delta s} - \frac{1}{\lfloor 3} \frac{(n^3 - 3n^2 + 3n - 2)}{(n)^3} \left(\frac{d^3u}{ds^3}\right) (\Delta s)^2 + O(\Delta s)^4 \qquad \text{(D.4b)}$$

where

$$\overline{u}_{13} = \frac{1}{2}(u_1 + u_3) \qquad \text{(D.4c)}$$

Performing the same operations on P and inserting the result in Eq. (D.1) written at point 3', one finds that

$$\frac{(\Delta u)_3^1}{\Delta s} - \frac{\bar{P}_{13}\,\bar{q}_{13}}{\bar{T}_{13}} \frac{(\Delta P)_3'}{\Delta s} + \frac{j}{\left(\frac{c_p}{R}-1\right)_{13}} \overline{\left(\frac{c_p}{R}\right)}_{13} \frac{\bar{P}_{13}\,\bar{u}_{13}}{\bar{F}_{13}} \frac{\Delta t}{\Delta s} = o(\Delta s)^2 \tag{D.5}$$

Suppose the entire coefficients and inhomogeneous terms in Eq. (D.1) were averaged. Then the Taylor series yield

$$\left(\frac{Pq}{T}\right)_3 = \left(\frac{Pq}{T}\right)_{3'} + \left\{\frac{d}{ds}\left(\frac{Pq}{T}\right)\right\}_{3'} \left(\frac{n-1}{n}\right)\Delta s + \cdots \tag{D.6a}$$

$$\left(\frac{Pq}{T}\right)_1 = \left(\frac{Pq}{T}\right)_{3'} - \left\{\frac{d}{ds}\left(\frac{Pq}{T}\right)\right\}_{3'} \frac{\Delta s}{n} + \cdots \tag{D.6b}$$

so that

$$\left(\frac{Pq}{T}\right)_{3'} = \overline{\left(\frac{Pq}{T}\right)}_{13} - \frac{1}{\lfloor 2} \frac{(n^2-2n+2)}{n^2} \left\{\frac{d^2}{ds^2}\left(\frac{Pq}{T}\right)\right\}_{3'} (\Delta s)^2 + \cdots \tag{D.7}$$

It is clear that this will lead to the same error, $o(\Delta s)^2$, as the previous procedure did. Since $ds^2 = dx^2 + dt^2$ the result is proved. In fact, even the method of Podney, et. al. (1966) is equivalent to the previous two in this sense. If h is some parameter which determines the size of the allowable time increments, these are said to contain errors of $o(h^2)$. Once h has been decided upon, solutions may be corrected by extrapolation to $h = 0$ (see Chou, Karpp, and Huang, 1967).

APPENDIX E

DESCRIPTION OF COMPUTER PROGRAMS

E.1 The Calculation of Flow Fields According to the Differential Approximation -- Program A

Since it is the most general tool employed in the investigation, Program A will be described in detail. The entire Fortran II listing is included in section E.3 and Method I of Section 5.4.2 is employed. Just as in the schemes of Sauerwein (1964) and Sussman (1966) an Executive program calls upon three subprograms which calculate individually points on the boundary (BDYPT), within the field (FLDPT), and on the shock (SHKPT). The overall flow chart is presented in Figure 18. Table E.I lists each of the programs and those subprograms which they must have access to. Generality was sought, sometimes at the expense of expediency. This philosophy is certainly justified. Rather than endure recompilation of the numerous programs in order to make minor corrections, some check procedures are retained which, in light of experience with the program, are no longer necessary. These anachronisms will be pointed out as they appear. The description begins with the simplest subprograms and builds toward the Executive. Mnemonics are used throughout.

Function subprograms

The functions PISPØS(AT), PISVEL (AT), and PISTMP (AT) represent piston position, velocity, and wall temperature respectively as functions of time. Their specification is arbitrary. Chemical non-equilibrium may be included if the external function AIS(T) is considered the frozen speed of sound. That which was employed computes a according to Eq. (5.8) in a perfect gas

$$AIS(T) = \sqrt{\frac{T}{1-\Pi}} \qquad (E.1)$$

The absorption properties of the gas are external inputs as well. The Planck mean absorption coefficient is referred to as ALFP(P, T), and Eq. (5.200) was used with b_1 and b_2 arbitrary. The absorption mean was programmed as well, even though it has not yet been used. If

$$\frac{\alpha_P}{\alpha_R} = N(M+1)T^{BM} \qquad \text{(E.2)}$$

then the quasi-isotropic absorption mean, Eq. (2.53), is

$$\alpha_a = \alpha_p \frac{T^4}{\sqrt{1+N+NT^{BM}}} \qquad \text{(E.3)}$$

The function ALFM(P, T), which calls upon ALFP(P,T), calculates the above. In the cylindrical case the modified Bessel Functions $K_0(x)$ and $K_1(x)$ are necessary. The numerical approximations given by Abramowitz and Stegun (1964, pp378, 379) were employed. The coefficients $(u+a_f)$, $(u-a_f)$, and ρa_f are necessary in the compatibility relations. In order to allow for a general state relation, these have been included as UPA(T), UMA(T), and PAT(P,T) (where the latter name comes from the fact that in a perfect gas $\rho a_f = \frac{pa}{T}$).

The right hand sides of the compatibility relations have also been kept separate from the body of the program. It was originally planned to use spatial increments exclusively. If one insists upon $d^{\pm}/dr$ instead of $d^{\pm}/dt$ on the Mach lines, the right hand sides differ by factors of $1/(u \pm a)$. Even though time increments are used, the individual identities of the inhomogeneous terms were retained. The parameter $\mathcal{Q}$ defined by Eq. (4.7) enters most of these inhomogeneous terms; therefore the following function was defined.

$$RAD(P,T,XI\emptyset) = \frac{\tau_0}{B_0} Q = \frac{\tau_0}{B_0}\{\alpha_a^{(0)} I_0 - 4\alpha_p T^4\} \qquad \text{(E.3)}$$

Then the functions RUPA(R, P, U, T, XIØ), RUMA(R, P, U, T, XIØ) are identical and correspond with Eq.(5.28).

$$R^{(m)}(r,p,u,T,I_0) = \frac{1}{(\frac{c_p}{r}-1)}\{RAD(P,T,XI\emptyset) - j\frac{c_p}{r}\frac{pu}{T}\} \qquad \text{(E.4)}$$

RPU(R, P, U, T, XIØ) is the inhomogeneous term in the compatibility relation along the path line and represents the difference form of Eq. (5.29)

$$R^{(u)}(r, p, u, T, I_0) = -\frac{RAD}{p} \tag{E.5}$$

On the line of constant time the inhomogeneous terms in Eq.(p.23) and (5.24) are designated QTAU(P, T, Q) and RPQ(R, P, T, XIØ, Q). They represent, respectively

$$QTAU(p, T, q) = 3\tau_e \, \alpha_a^{(1)}(p, T) \, q \tag{E.6}$$

$$RPQ(r, p, T, I_0, q) = B_0 \cdot RAD + j \, q/r \tag{E.7}$$

Finally it is necessary that the optical depth be calculated. This is accomplished with the function

$$ATAU(R) = \eta_a(r) = \int_{R_p(t)}^{r} \alpha_a(\xi, t) \, d\xi \tag{E.8}$$

in which a Gaussian Quadrature whose order may be controlled externally is employed. Spacing of the points in the shock layer is accomplished by SPACE(CLØSE, CØUNT, CNTM) according to Eq. (5.206). These eighteen external functions allow for any eventuality, and they keep the program of reasonable length.

Subroutines

The subroutine XNTRP is designed to interpolate within a known data set for the properties P, U, T, I_0, q at any point. The data is divided among three matrices the rows of which contain all of the variables. There is one row for each point at which data is known, and the numbering begins at the piston with unity. The point downstream of the shock wave is denoted by IS, NP, or ISW and the last point known is designated IMAX, NMAX, or IMAXW. Data from a previous time increment is stored within the matrix XØLD(I, J) where I=1, IMAX and J=1, 7 denotes r, P, U, T, I_0, q, U_s respectively. For instance, XØLD(IS+1, 3) is the velocity immediately upstream of the shock. Data computed at the current time is entered into XNEW(I, J), I=1, NMAX with the downstream side of the shock at I=NP. The matrix within which interpolation is carried out is designated WØRK(I, J), I=1, IMAXW with shock at ISW. WORK is loaded with XOLD, XNEW, or some modified form of either or both depending upon the purpose for which interpolation is carried out. The program accounts for the fact that the data have a discontinuity at the shock, and the interpolated properties are entered as VAR(J), J=1, 7 which is in common among the programs. The subroutine which was programmed uses linear interpolation. Lagrangian interpolation was programmed as well, but there was negligible difference in the results while computation time increased noticeably.

The integral in the boundary conditions Eq.(3.89), (3.92), or (3.94) is calculated by XINT. The program uses Gaussian Quadrature of order NW with ordinates and weights XW(I) and W(I), I=1, NW. XINT calls upon ATAU at each point. Since data is not known past some R_{max} the various properties are assumed to be their undisturbed values from R_{max} to infinity, and the additional contribution is added to the integrated quantity when appropriate.

Considerable time was spent formulating a quick and accurate method of determining points from which characteristics reach point 1. At first a very sophisticated scheme which always used the current average slope was tried. The program required iteration, and although it was quite good and led to more rapid convergence of the compatibility relations, there were a few situations in which it did not function properly. In that event the program would send an error message back to the calling program giving the generic designation of the point which was sought (ISTØP). The sophisticated search program was abandoned, but the major subroutines are still capable of handling it.

The searching procedure decided upon is the standard one in most respects. Initially a guess is made for properties at point 1 and the characteristics are projected backward from it with slopes based upon those properties. On the next iteration the properties at these points ((PRØP(I, J), J=1, 7) I=2, 3, 4) and the new ones at point 1 (PRØP(1, J), J=1, 7) provide average slopes for the lines projected backward. Thus the slopes are always one iteration behind the properties. In this manner points 2, 3, and 4 would execute damped oscillations in space. This is opposed to the characteristic mesh in which point 1 itself would move each time. When one deals with problems in ordinary gasdynamics it is quite safe to determine points 3 and 4 and then to choose point 2, from which a path line emanates, midway between them. This is inappropriate when there exist regions in which gradients may be large. Indeed this approxima tion was found to be greatly in error near the shock as an optically thick situation was approached. Therefore the subroutine SEARCH(K, L, UMA1, UPA1, U1) determines the locations of all points , K=2, 3, 4, in the L^{th} iteration of the compatibility relations based upon the $(L-1)^{st}$ iterates of U-a, U+a, and U at point 1. The matrix PROP(I, J) in which properties at points 1, 2, 3, 4, 5 (and 6 and 7 at shock waves) are stored while they are needed is in common among all of the subprograms.

Now that the building blocks are available, the discussion may turn to major subprograms which calculate properties at boundary, field, and shock points. Since the programs themselves are generously interspersed with comments, they will only be described schematically. The subroutine BDYPT(MTIME, NIT, XXI) calculates P, U, T, q on the piston based upon a guessed value for I_0. In the first major iteration of the program guesses at properties at point 1 are based upon the last known body point. They are taken as the most current body point on successive iterations. Similarly the matrix WØRK(I, J) in the evaluation of the integrals is loaded with old data, scaled linearly to fit within the expanded shock layer, on the first overall iteration and with the data of the previous iteration on successive ones. The arguments MTIME, NIT, and XXI are explained within the program itself. One function they serve it to allow the solution of non-self-consistent problems in which case XXI is the prescribed value of I_{0_w}. The subprogram solves the equations

$$P_1 = P_3 + \left(\frac{\rho a}{T}\right)_{13} (U_p - U_3) + R_{13}^{(M)} \Delta t \tag{E.9}$$

$$T_1 = T_2 \left\{ \frac{P_1}{P_2} e^{-R_{12}^{(M)} \Delta t} \right\}^n \tag{E.10}$$

$$I_{0_w} = 2\epsilon T_w^4(t) + \left(1 - \frac{\epsilon}{2}\right)\sqrt{3} \int_0^{R_{max}} e^{-\sqrt{3}\tau_\ell \eta'} \mathcal{F}_j(\eta') d\eta' + QSH \tag{E.11}$$

$$q_w = \frac{\epsilon (4T_w^4 - I_0)}{\sqrt{3}(2-\epsilon)} \tag{E.12}$$

where, for a quasi-isotropic, semi-grey gas

$$\mathcal{F}_j = 4\tau_\ell \frac{\alpha_p}{\alpha_a} T^4 - j \frac{q}{r\alpha_a} \tag{E.13}$$

and $R^{(M)}$, $R^{(u)}$ are defined by Eq. (5.28) and (5.29). $R_{max} \to \infty$ and $QSH = 0$ if upstream absorption is included, but if upstream absorption is neglected $R_{max} = R_{shock}$ and $QSH = 0$ if upstream emission is disallowed or $QSH = 2(1-\epsilon)e^{-\eta_3 I_0 \eta_3}$ if it is included. Point 2 is the boundary point at the previous time, and point 3 is found with the SEARCH subroutine.

The subroutine FLDPT(MTIME, KASM, X1, X2) calculates properties at points whose domain of dependence includes neither the shock wave nor the boundary. It solves Eq.(E.9) and (E.10) after first solving

$$u_1 = \frac{u_3 \left(\frac{pq}{T}\right)_{13} + u_4 \left(\frac{pq}{T}\right)_{14} + \left\{R^{(M)}_{14} - R^{(M)}_{13}\right\}\Delta t + p_4 - p_3}{\left(\frac{pq}{T}\right)_{13} + \left(\frac{pq}{T}\right)_{14}} \tag{E.14}$$

In addition I_0 and q are found from

$$q_1 = q_5 - R^{(q)}_{15}\Delta t \tag{E.15}$$

$$I_{0_1} = I_{0_5} - R^{(I)}_{15}\Delta t \tag{E.16}$$

where $R^{(q)}$ and $R^{(I)}$ are defined by Eq.(5.30) and (5.31). The argument KASM allows for the solution of non-self-consistent problems in which Eq. (E.15) and (E.16) are omitted, the values X1 and X2 being the locally prescribed values of I_0 and q. Similar to FLDPT, the subroutine SHKPT(RSH, MTIME) first determines properties at point 1. Then, as outlined in Section 5.2.3 it solves Eq. (5.46) by a Newton-Rhapson scheme. The shock location is corrected at each stage in the standard manner, and provisions for omission of upstream absorption are obvious. Each of the three major subroutines is allowed LM iterations on the compatibility relations.

The longest program of all, the Executive, needs the least clarification. After reading in a starting solution and other parameters, it begins by choosing the time increment, Δt, according to Eq. (5.205). Provision has been made for the computation of flow fields at the exact times AA and AB which might be of interest. The body point subroutine is called, and the field points, spaced according to Eq. (5.206) are calculated by the field point subroutine. The shock point is calculated, and then further field points are called upon. The program allows the computation of flow fields without shock waves. In that case the leading characteristics are

are monitored instead of the shock wave. The iteration and monitoring scheme of Section 5.4 is then used. The variable XM(IDØ) corresponds to χ_n in Eq.(5.121). Once staisfactory convergence is obtained, the asymptotic solution, Eq.(5.120), is joined to that already computed. The new data may be printed or punched for input or both, and it is then entered as old data for the next time increment.

E.2 Calculation of Planar Flow Fields with the Full Transfer Equation -- Programs B and C

This program is more streamlined than the one described above. It employs Method II outlined in Section 5.2.4, namely, simultaneous iteration of all points in the flow field. Only those programs which are different from those above will be mentioned. First the exponential integrals are necessary. Again Abramowitz and Stegun (1964, pg 231) give numerical approximations, and these were programmed as E1(X), E2(X), and E3(X). The radiation integrals are computed by RAD(R, P, T, K, I1) and QTAU(R, P, T, I2) of which only the first, which represents -div(q), is necessary in the computation of the flow field. The argument K is necessary because the interpolation matrices must be loaded with old data when the integrals are computed at points 2, 3, and 4 but with new data at point 1. This is done by ALØAD(K, I4). If one allows for the fact that data is known only up to $r=R_{max}$ beyond which the field is undisturbed, the following formulae hold.

$$
\begin{aligned}
RAD(R,P,T,K,I1) = {} & 2\frac{\tau_g}{\theta_0}\alpha_a\Big\{\Big[\epsilon T_W^4 + 2r_d\int_{R_p}^{R_{max}}\tau_e\alpha_p T^4(\xi)E_2\{\tau_e\eta_a(\xi)\}d\xi\Big]E_2(\tau_e\eta_a) + \\
& + r_s\int_{R_p}^{R_{max}}\tau_e\alpha_p T^4(\xi)E_1[\tau_e\{\eta_a(x)+\eta_a(\xi)\}]\,d\xi \; + \\
& + \int_{R_p}^{R_{max}}\tau_e\alpha_p T^4(\xi)E_1[\tau_e|\eta_a(x)-\eta_a(\xi)|]\,d\xi\Big\} + \frac{4\tau_e}{\theta_0}\alpha_p(x,t)T^4(x,t) + \\
& - 2\frac{\tau_g}{\theta_0}\alpha_a\Big\{2r_d E_2[\tau_e\eta_a(x)]E_3[\tau_e\eta_a(R_{max})] + r_s E_2[\tau_e\{\eta_a(x)+\eta_a(R_{max})\}] \\
& + E_2\{\tau_e[\eta_a(R_{max})-\eta_a(x)]\}\Big\}
\end{aligned}
\qquad (E.17)
$$

$$
\begin{aligned}
QTAU(R,P,T,I2) = 2 \Big\{ & [\epsilon T_w^4 + 2 r_d \int_{R_p}^{R_{max}} [\tau_e \alpha_p T^4 E_2\{\tau_e \eta_a(\xi)\} d\xi] E_3[\tau_e \eta_a(x)] + r_s \int_{R_p}^{R_{max}} \tau_e \alpha_p T^4 E_2[\tau_e \{\eta_a(x) + \eta_a(\xi)\}] d\xi \\
& + \int_{R_p}^{R_{max}} \mathrm{sgn}\{\eta_a(x) - \eta_a(\xi)\} \tau_e \alpha_p T^4(\xi) E_2[\tau_e |\eta_a(x) - \eta_a(\xi)|] d\xi \Big\} + \\
& + 2\{ r_d E_3[\tau_e \eta_a(x)] E_3[\tau_e \eta_a(R_{max})] + r_s E_3[\tau_e \{\eta_a(x) + \eta_a(R_{max})\}] \\
& - E_3[\tau_e \{\eta_a(R_{max}) - \eta_a(x)\}] \}
\end{aligned}
\qquad \text{(E.18)}
$$

Because of the extra arguments, the functions RUMA(AR, P, U, T, K, I) and RPU(AR, P, U, T, K, I) had to be modified from the previous functions of the same names.

The searching procedure is as before, but now slopes at points 2, 3, and 4 must be stored. Therefore the search routine is called SEARCH(K, L, UMA1, UPA1, U1, UM3, UP4, XU2). The major subroutines are essentially unchanged as is the Executive program. The program computes new points until either the temperature is nearly unity or until the shock is reached depending upon whether upstream absorption is included. After a sufficient number of iterations has been made, the properties q and I_0 are computed.

In addition a short program (Program C) was written so that given temperature and pressure distributions one could use ALØAD, RAD, QTAU, XNTRP, ALFP, and ALFM to determine the consistent heat flux and first moment distributions.

Many alternate procedures are available. For instance one could, by least squares, have faired a polynomial to the function $\alpha_p T^4$ as it depends upon η_a and then evaluated the integrals explicitly as Viskanta and Merriam did. Although this procedure might be rapid, it would become difficult to handle many points accurately. Convergence of the iterative scheme is by no means assured. The only justification is that difficulties were never encountered in practice. Convergence was enhanced by playing a neat trick in ALØAD. On each iteration the new data matrix was manipulated in such a manner that at point "M" at the current time it consisted of data of the current iteration for points i = 1 through "M-1" and of data from the last iteration for points "M+1" through NMAX. In this manner the integral terms lag behind the current iteration less and less the more points which are computed. Only at the wall are these terms a full iteration behind. Nevertheless the solution of the full transfer equation taxed computer storage although it was quite successful.

Table E.I

PROGRAM	REQUIRES ACCESS TO	
	FUNCTIONS	SUBROUTINES
EXECUTIVE	ATAU(PLACE) AIS(T) UMA(U,T) SPACE(CLOSE,COUNT,CNTM) PISPOS(TIME) PISVEL(TIME) BKØ(RØ) BKI(RI)	BDYPT(MTIME,IDO,XIW(IDO),MWITH) XNTRP FLDPT(MTIME) SHKPT(RSH,MTIME,MWITH)
BDYPT(MTIME,IDO,XIW(IDO),MWITH)	RUMA(R,P,U,T,XIØ) RPU(P,U,T,XIØ) PAT(P,T) ATAU(PLACE) UMA(U,T)	SEARCH(K,L,UMAI,UPAI,UI) XINT
FLDPT(MTIME)	PAT(P,T) RUMA(R,P,U,T,XIØ) RPU(P,U,T,XIØ) QTAU(P,T,Q) RPQ(R,P,T,XIØ,Q)	SEARCH(K,L,UMAI,UPAI,UI)

Table E.I (Continued)

PROGRAM	REQUIRES ACCESS TO	
	FUNCTIONS	SUBROUTINES
FLDPT(MTIME)	UMA(U,T) UPA(U,T)	
SHKPT(RSH,MTIME,MWITH)	PAT(P,T) RUMA(R,P,U,T,XIØ) RPU(P,U,T,XIØ) QTAU(P,T,Q) RPQ(R,P,T,XIØ,Q) UMA(U,T) UPA(U,T)	SEARCH(K,L,UMAI,UPAI,UI)
XINT	ALFP(P,T) ATAU(PLACE)	XNTRP
SEARCH(K,L,UMAI,UPAI,UI)	UPA(U,T) UMA(U,T)	XNTRP
ATAU(PLACE)	ALFM(P,T)	XNTRP

Table E.I (Concluded)

PROGRAM	REQUIRES ACCESS TO	
	FUNCTIONS	SUBROUTINES
RUMA(R,P,U,T,XIØ)	RAD (P,T,XIØ)	NONE ↓
RPU(P,U,T,XIØ)	RAD (P,T,XIØ)	
RPQ(R,P,T,XIØ,Q)	RAD (P,T,XIØ)	
QTAU(P,T,Q)	ALFM(P,T) ALOAD(K,I) ATAU(PLACE)	
ALFM(P,T)	ALFP(P,T)	
RAD(P,T,XIØ)	ALFM(P,T),ALFP(P,T), ALOAD(K,I) ATAU(PLACE)	
UPA(U,T), UMA(U,T)	AIS(T)	
PAT(P,T)	AIS(T)	
XNTRP	NONE ↓	
PISPØS(AT)		
PISVEL(AT)		
PISTMP(AT)		
AIS(T)		
BKØ (RNAUT)		
BKI (RNAUT)		
ALFP(P,T)		
SPACE(CLOSE,COUNT, CNTM)		
ALOAD(K,I)		

Small letters denote programs necessary only for solution of the fulltransfer equation.

E.3 Computer Program and Representative Output

DIFFERENTIAL APPROXIMATION

```
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),YOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      COMMON EMIS,TW,AA,AB,AC,AD,RS,KISS,MWITH,KLUP,MWTH2
       DIMENSION XIW(100),XM(100)
C
C     ****************************************************************
C
C     EXECUTIVE PROGRAM -- ITERATION WITH THE AID OF INFLUENCE
C     COEFFICIENTS.
C
C     ****************************************************************
C
 1010 FORMAT(I3)
 1015 FORMAT(2E12.8)
 1030 FORMAT(5F12.8)
 1050 FORMAT(4F12.8/2I3)
 1070 FORMAT(4F12.5)
 1090 FORMAT(2I3/(6F12.8/F12.8))
 1105 FORMAT(1H ,6E12.5)
 1110 FORMAT(1H ,19HCANNOT FIND POINT 3,2X,4E14.8)
 3001 FORMAT(I3/(2F12.8))
C
C     READ IN INITIAL DATA.
C
    2 READ 1090,IS,IMAX,((XOLD(I,J),J=1,7),I=1,IMAX)
C
C     READ IN THE ORDER OF THE GAUSSIAN QUADRATURE AND THE ASSOCIATED
C     WEIGHTS AND ORDINATES.
C
      READ 3001,NW,(W(I),XW(I),I=1,NW)
C
C     A1, B1, AND B2 ARE CONSTANTS NEEDED FOR THE PLANCK MEAN ABSORP-
C     TION COEFFICIENT.     XMA AND XNA APPEAR IN THE ABSORPTION
C     AVERAGED MEAN.
C
      READ 1030,A1,B1,B2,XMA,XNA
C
C      THE FIRST THREE OF THESE ARE ACCURACY SPECIFICATIONS IN THE
C     SEARCH SUBROUTINE.  XE IS THE FRACTION OF THE PISTON-SHOCK
C     SEPARATION AT WHICH THE FIRST POINT ABOVE THE PISTON IS LOCATED.
C     IBMAX IS THE MAXIMUM NUMBER OF ITERATIONS ALLOWED, AND NTOT IS
C     AN OPTIONAL ARGUMENT WHICH ALLOWS ONE TO ITERATE ON THE INTEGRAL
C     BOUNDARY CONDITION EACH TIME IF IT IS SET EQUAL TO UNITY.
C
      READ 1050,EPS,EPS1,EPS2,XE,IBMAX,NTOT
C
C     WALL EMISSIVITY (NOT ALLOWED TO VARY IN TIME) AND TIME.
C
      READ 1015,EMIS,TIME
C
C     THE MAXIMUM NUMBER OF ITERATIONS ALLOWED ON THE COMPATIBILITY
C     RELATIONS
C
      READ 1010,LM
C
C     GAM=(GAMMA-1)/GAMMA, WHERE GAMMA IS THE RATIO OF SPECIFIC HEATS.
C     IF GNU IS 0, 1, OR 2, WE ARE INVOLVED IN A PLANAR, CYLINDRICAL,
C     OR SPHERICAL CALCULATION, RESPECTIVELY.
C
      READ 1070,GAM,GNU,TAU,BO
```

```
C
C     OVERALL ACCURACY REQUIREMENTS.
C
      READ 1015,EPS4,EPS5
      SLEPS=EPS4
C
C     IF MTIME IS UNITY, THE RADIATION IS TURNED OFF DURING THE
C     PREVIOUS TIME INCREMENT.  IF IT IS TWO OR GREATER , IT IS ON.
C
      READ 1010,MTIME
      EPSL=EPS5                                                         00470
C
C     IF IDO IS UNITY, THE INTEGRAL IS USED TO PROVIDE AN INITIAL
C     GUESS.  OTHERWISE, ABC IS AN ESTIMATE OF THE INTEGRAL PART OF THE
C     BOUNDARY CONDITION.
C
      READ 1098,IDO,ABC
 1098 FORMAT(I3,F12.8)
C
C     XD1*XD2 IS THE MAXIMUM SPATIAL INCREMENT.  A STABILITY CRITERION.
C
      READ 1070,XD1,XD2,AA,AB
C
C     THE PROGRAM PRINTS EVERY (KPRM+1)TH AND PUNCHES EVERY (KPNM+1)TH
C     TIME INCREMENT.  IF KPRNT=KPRM+1, THE RESULT OF THE FIRST TIME
C     INCREMENT WILL BE DISPLAYED.
C
      READ 1010,KPRNT,KPNCH,KPRM,KPNM
C
C     IF MWITH IS LESS THAN OR EQUAL TO UNITY UPSTREAM ABSORPTION IS
C     INCLUDED.
C
      READ 1010,MWITH
C
C     IF MWTH2=1 AND MWITH IS GREATER THAN UNITY, RHO-KAPPA
C     VANISHES UPSTREAM.  IF MWTH2 AND MWITH ARE BOTH GREATER THAN
C     UNITY, UPSTREAM ABSORPTION IS NEGLECTED BUT THE FREE STREAM
C     STILL EMITS.
C
      READ 1010,MWTH2
C
C     INCREMENTS FOR XHIGH AND XLOW RELATIVE TO FIRST GUESSES IN
C     ITERATION PROCEDURES.
C
      READ 1015,ACUR1,ACUR2
C
C     FRACTION OF DELTA TIME WHICH IS USED.
C
      READ 1015,FDEL,AD
C
C     PRINTS FLOW FIELD WITHIN THE SHOCK ON EACH ITERATION IF KLUP=1.
C     HANDLES PISTON RETRACTIONS IF KSHOK IS GREATER THAN UNITY.
C     THE PROGRAM THEN TRACES THE PROGRESS OF THE LEADING CHARACTERISTIC
C     INSTEAD OF THE SHOCK.
C
      READ 1010,KLUP,KSHOK
C
C     COMPUTE DELTA TIMES BASED UPON POINTS NEAREST TO SHOCK AND BODY.
C
   22 DELN=XE*(XOLD(IS,1)-XOLD(1,1))                                    00490
      DU=XE*(XOLD(IS,7)-XOLD(1,3))                                      00500
      DEL1=DELN/(AIS(XOLD(1,4))-DU)
      DEL2=DELN/(-UMA(XOLD(IS,3),XOLD(IS,4))+XOLD(IS,7)+DU)
```

```
C
C     CHOOSE THE SMALLEST DELTA TIME.
C
      DELT=FDEL*MIN1F(DEL1,DEL2)
      IF(TIME-AA) 1,7,7
    1 IF(ABSF(TIME/AA-1.)-EPS) 4,4,9
    9 IF(TIME+DELT-AA) 4,3,3
    3 TIME=AA
      GO TO 5
    7 IF(TIME+DELT-AB) 4,8,8
    8 TIME=AB
      GO TO 5
    4 TIME=TIME+DELT
C
C     PISTON POSITION, VELOCITY, AND WALL TEMPERATURE ARE EXTERNAL
C     FUNCTIONS OF TIME.
C
    5 RP=PISPOS(TIME)
      UP=PISVEL(TIME)
      TW=PISTMP(TIME)
C
C     THE SHOCK POSITION IS EXTRAPOLATED LINEARLY INITIALLY.
C
      RS=XOLD(IS,1)+XOLD(IS,7)*DELT
      APE=XOLD(IS,7)
      DO 17 I=1,IMAX
      WORK(I,1)=RP+(XOLD(I,1)-XOLD(1,1))*(RS-RP)/(XOLD(IS,1)-XOLD(1,1))
      DO 17 J=2,7
      WORK(I,J)=XOLD(I,J)
   17 CONTINUE
      ISW=IS
      RSINT=RS
      IMAXW=IMAX
      RMAX=WORK(IMAX,1)
      IF(IDO-1) 18,18,19
   19 IF(MWITH-1) 12,12,13
   13 IF(MWTH2-1) 12,12,1738
   12 QSH=0.
      GO TO 14
 1738 ARGA=1.7320508*TAU*ATAU(RSINT)
      QSH=2.*(2.-EMIS)*EXPF(-ARGA)
   14 XIB=2.*EMIS*(TW**4.)+QSH
      XIW(3)=XIB+(1.-.5*EMIS)*ABC
      XIW(1)=(1.-ACUR1)*XIW(3)
      XIW(2)=(1.+ACUR1)*XIW(3)
      XM(1)=-.1
      XM(2)=.1
      XHIGH=XIW(2)
      XLOW=XIW(1)
      IDO=3
   18 RSH=RS
C
C     BODY POINT SUBROUTINE.
C     IF IDO IS UNITY, THE INTEGRAL BOUNDARY CONDITION IS USED.  IF
C     IT IS GREATER THAN UNITY, THE GUESS COMPUTED AT THE END OF THE
C     PROGRAM IS USED INSTEAD.
C
    6 KISS=1
      CALL BDYPT(MTIME,IDO,XIW(IDO))
      IF(ISTOP-1) 10,10,20
   10 DO 50 I=1,7
      XNEW(1,I)=PROP(1,I)
   50 CONTINUE
```

```
      IF(IDO-1) 955,955,91
  955 XIW(1)=PROP(1,5)
      GO TO 91
C
C     ERROR MESSAGE SENT FROM SUBROUTINE WHICH SEARCHES FOR THE
C     CHARACTERISTICS.
C
   20 PRINT 1110,GR(1),GR(2),RK(1),RK(2)
      GO TO 2
C
C     THE NUMBER OF POINTS BETWEEN BODY AND SHOCK IS BASED UPON THE
C     FRACTION OF THE SHOCK-PISTON SEPARATION AT WHICH THE FIRST POINT
C     IS LOCATED.
C
   91 XNNP=1.+1./XE
      NP=XNNP
C
C     THE MATRIX WITH WHICH THE INTERPOLATION ROUTINE OPERATES IS
C     NOW RESET TO CONTAIN ONLY OLD DATA.
C
      DO 21 I=1,IMAX
      DO 21 J=1,7
      WORK(I,J)=XOLD(I,J)
   21 CONTINUE
      ISW=IS
      IMAXW=IMAX
      NPM=NP-1
      XNP=NPM
      DO 301 I=2,NPM
      XI=I
C
C     POINTS BETWEEN PISTON AND SHOCK ARE CLUSTERED NEAR TO THE BODY
C     AND SHOCK AND SPACED FARTHER APART IN THE SPACE BETWEEN.
C
      PROP(1,1)=RP+(RS-RP)*SPACE(XE,XI,XNP)
C
C     INITIAL GUESSES AT PROPERTIES ARE MADE ALONG RAYS PROJECTED
C     FROM DATA AT THE PREVIOUS TIME INCREMENT.   THESE GUESSES ARE
C     NECESSARY IN ORDER TO FIND THE CHARACTERISTICS.
C
      XR=(XOLD(IS,1)-XOLD(1,1))*(PROP(1,1)-RP)/(RS-RP)+XOLD(1,1)
      CALL XNTRP
      DO 40 K=2,7
      PROP(1,K)=VAR(K)
   40 CONTINUE
C
C     POINT 5 IS LOCATED IMMEDIATELY ALONG THE TIME=CONSTANT
C     CHARACTERISTIC.
C
      DO 41 J=1,7
      PROP(5,J)=XNEW(I-1,J)
   41 CONTINUE
C
C     FIELD POINT SUBROUTINE
C
   42 KISS=1
      CALL FLDPT(MTIME,1,0.,0.)
      IF(ISTOP-1) 100,100,102
  100 DO 301 K=1,7
      XNEW(I,K)=PROP(1,K)
  301 CONTINUE
      GO TO 150
  102 PRINT 1200,ISTOP
```

```
      PRINT 1201,GR(1),GR(2),RK(1),RK(2)
 1200 FORMAT(19H CANNOT FIND POINT ,I3)
 1201 FORMAT(1H ,4E12.5)
      GO TO 2
C
C     SHOCK POINT SUBROUTINE
C
  150 KISS=2
      IF(KSHOK-1) 154,154,156
  154 CALL SHKPT(RSH,MTIME)
      RS=RSH
      GO TO 153
  156 PROP(1,1)=RS
      PROP(5,1)=XNEW(NP-1,1)
      DO 159 J=2,7
      PROP(1,J)=XNEW(NP-1,J)
      PROP(5,J)=XNEW(NP-1,J)
  159 CONTINUE
      CALL FLDPT(MTIME,1,0.,0.)
      PROP(1,7)=UPA(PROP(1,3),PROP(1,4))
      DO 157 J=1,7
      PROP(7,J)=PROP(1,J)
  157 CONTINUE
      US=PROP(1,7)
      RS=.5*(XOLD(IS,7)+US)*DELT+XOLD(IS,1)
  153 IF(ISTOP-1) 152,152,151
  151 PRINT 1200,ISTOP
      PRINT 1201,GR(1),GR(2),RK(1),RK(2)
      GO TO 2
  152 NPP=NP+1
      DO 160 J=1,7
      XNEW(NP,J)=PROP(7,J)
      XNEW(NPP,J)=PROP(1,J)
  160 CONTINUE
      IF(KLUP-1) 161,161,162
  161 PRINT 1105,((XNEW(I,J),J=1,6),I=1,NPP)
  162 CONTINUE
C
C     IF UPSTREAM ABSORPTION IS ACCOUNTED FOR (MWITH=1) THE COMPUTATION
C     IS CONTINUED.  IF UPSTREAM ABSORPTION IS NOT INCLUDED, IO IS
C     COMPARED TO ITS UNDISTURBED VALUE.  IF IT IS NOT SUFFICIENTLY
C     CLOSE THERETO, A NEW ITERATION IS BEGUN.
C
      IF(MWITH-1) 164,164,166
  166 IF(MWTH2-1) 2012,2012,2013
 2012 AB1A=0.
      GO TO 2014
 2013 AB1A=4.
 2014 XM(IDO)=PROP(1,5)-AB1A-PROP(1,6)*1.7320508
      IF(ABSF(XM(IDO)/4.)-EPS4) 168,168,335
  168 NMAX=NP
      GO TO 736
  164 NP2=NP+2
      DO 650 I=NP2,100
  200 DO 210 J=1,6
      PROP(1,J)=XNEW(I-1,J)
      PROP(5,J)=XNEW(I-1,J)
  210 CONTINUE
C
C     POINTS BEYOND THE SHOCK ARE SPACED SUCCESSIVELY FARTHER APART.
C
      IF(I-NP2) 242,242,244
  242 PROP(1,1)=PROP(1,1)+XNEW(NP,1)-XNEW(NP-1,1)
```

```
      GO TO 670
  244 IF(XNEW(I-1,1)-XNEW(I-2,1)-XD1) 245,245,246
  245 PROP(1,1)=PROP(1,1)+XD2*(XNEW(I-1,1)-XNEW(I-2,1))
      GO TO 670
  246 PROP(1,1)=PROP(1,1)+XD2*XD1
  670 KISS=2
      CALL FLDPT(MTIME,1,0.,0.)
      IF(ISTOP-1) 101,101,103
  101 DO 220 K=1,7
      XNEW(I,K)=PROP(1,K)
  220 CONTINUE
      IF(KLUP-1) 203,203,207
  203 PRINT 1105,(XNEW(I,J),J=1,6)
  207 IKILL=I
      XNU=GNU+1.
      NU=XNU
      RO=PROP(1,1)*TAU*1.7320508
      GO TO (810,820,830),NU
  810 XF=1.
      GO TO 840
  820 XF=BK1(RO)/BKO(RO)
      GO TO 840
  830 XF=1.+1./RO
C
C     IF XM VANISHES, THE ASYMPTOTIC SITUATION HAS ARRIVED.
C
  840 XM(IDO)=(PROP(1,5)-4.)*XF-PROP(1,6)*1.7320508
      IF(I-99) 841,841,335
  841 IF(ABSF(XM(IDO)/4.)-EPS4) 700,700,330
C
C     ONE KNOWS THAT HE IS ON THE WRONG BRANCH OF THE FAMILY OF
C     SOLUTIONS IF EITHER IO-4. OR Q CHANGE SIGN FROM THAT WHICH THEY
C     HAD AT THE SHOCK.
C
  330 ERTES=RPQ(XNEW(I,1),XNEW(I,2),XNEW(I,4),XNEW(I,5),XNEW(I,6))
      IF(XNEW(IS,6)) 900,900,910
  900 IF(ERTES) 920,920,335
  920 IF(XNEW(I,6)) 650,335,335
  910 IF(ERTES) 335,310,310
  310 IF(XNEW(I,6)) 335,335,650
  650 CONTINUE
      GO TO 335
  103 PRINT 1200,ISTOP
      PRINT 1201,GR(1),GR(2),RK(1),RK(2)
      GO TO 2
C
C     PROCEDURE FOR OBTAINING A NEW GUESS FOR IO AT THE WALL IS BASED
C     UPON FINDING TWO INITIAL VALUES,ONE TOO HIGH AND ONE TOO LOW,
C     AND THEN EITHER HALVING THE DIFFERENCE OR EMPLOYING INFLUENCE
C     COEFFICIENTS, WHICHEVER GIVES THE BEST RESULT.
C     IF XM IS GREATER THAN ZERO, XIW WAS CHOSEN TOO LARGE
C     AND CONVERSELY.
C
  335 IF(IDO-1) 340,340,350
C
C     A SECOND GUESS MUST BE MADE INTUITIVELY.  IF THE FIRST GUESS
C     IS LOW, THE SECOND GUESS WILL BE HIGHER, AND CONVERSELY.
C
  340 IF(XM(1)) 343,342,342
  342 EPSL=-EPSL
  343 XIW(2)=XIW(1)*(1.+EPSL)
      XLOW=(1.-ACUR2)*XIW(1)
      XHIGH=(1.+ACUR2)*XIW(1)
```

```
      IDO=2
      GO TO 6
  350 IF(XM(IDO)) 2000,2000,2001
 2000 XLOW=XIW(IDO)
      GO TO 2015
 2001 XHIGH=XIW(IDO)
C
C     A GUESS IS MADE WITH A LINEAR INFLUENCE COEFFICIENT.
C
 2015 AA=XM(IDO)-XM(IDO-1)
      XIW(IDO+1)=XIW(IDO)-(XIW(IDO)-XIW(IDO-1))*XM(IDO)/AA
      IF(XIW(IDO+1)-XLOW) 2020,2020,2025
C
C     IF THE INFLUENCE COEFFICIENT GUESS IS LESS THAN THE LOWEST
C     ESTIMATE OR GREATER THAN THE HIGHEST ESTIMATE, THE HALVING
C     PROCEDURE OVERRIDES IT.
C
 2025 IF(XIW(IDO+1)-XHIGH) 2030,2020,2020
 2020 XIW(IDO+1)=.5*(XHIGH+XLOW)
 2030 IDO=IDO+1
      IF(KLUP-1) 2102,2102,2104
 2102 PRINT 4987,XLOW,XHIGH,XIW(IDO)
 4987 FORMAT(6H XLOW=,E13.7,7H XHIGH=,E13.7,11H NEW GUESS=,E13.7///)
 2104 IF(IDO-IBMAX) 6,6,690
  690 EPS4=10.*EPS4
      PRINT 1987,EPS4
 1987 FORMAT(10H ACCURACY=,E12.5)
      IF(EPS4-.01) 694,694,695
  694 XIW(1)=XLOW
      XIW(2)=XHIGH
      XM(1)=-.1
      XM(2)=.1
      XIW(3)=XIW(IDO)
      IDO=3
      GO TO 6
  695 PRINT 1988
 1988 FORMAT(18H ACCURACY EXCEEDED)
      GO TO 2
C
C      IF THE ASYMPTOTIC SOLUTION HAS BEEN REACHED ARBITRARILY CLOSELY
C     THE SOLUTION IS FAIRED THERETO FOR THE REMAINDER OF THE
C      COMPUTATION OF THE FLOW FIELD.
C
  700 IF(XNEW(IS,6)) 930,930,932
  930 IF(XNEW(IKILL,5)-4.) 934,934,730
  934 IF(XNEW(IKILL,6)) 750,750,730
  932 IF(XNEW(IKILL,5)-4.) 730,740,740
  740 IF(XNEW(IKILL,6)) 730,750,750
  730 I1=IKILL
      GO TO 745
  750 I1=IKILL+1
  745 DO 735 II=I1,100
      IF(II-NP2) 766,766,767
  766 XNEW(II,1)=XNEW(II-1,1)+XNEW(NP,1)-XNEW(NP-1,1)
      GO TO 768
  767 XNEW(II,1)=XNEW(II-1,1)+2.*(XNEW(II-1,1)-XNEW(II-2))
  768 R1=1.7320508*TAU*XNEW(II,1)
      R2=1.7320508*TAU*XNEW(II-1,1)
      GO TO (760,765,770),NU
  760 XXF=EXPF(-R1+R2)
      GO TO 775
  765 XXF=BKO(R1)/BKO(R2)
      GO TO 775
```

```
  770 XXF=R2*EXPF(-R1+R2)/R1
  775 XNEW(II,5)=4.+(XNEW(II-1,5)-4.)*XXF                                    02400
      XNEW(II,6)=XNEW(II-1,6)*XXF                                            02410
      XNEW(II,2)=1.+(XNEW(II-1,2)-1.)*XXF
      XNEW(II,3)=XNEW(II-1,3)*XXF
      XNEW(II,4)=1.+(XNEW(II-1,4)-1.)*XXF
      IA=II
      IF(XNEW(II,6)-EPS2) 710,710,735                                        02450
  735 CONTINUE                                                               02460
  710 NMAX=IA
C
C     THE FLOW FIELD JUST COMPUTED IS ENTERED AS OLD DATA.
C
  736 DO 720 I=1,NMAX
      DO 720 J=1,7
      XOLD(I,J)=XNEW(I,J)                                                    02610
  720 CONTINUE                                                               02620
      XNU=GNU+1.
      NU=XNU
      IF(KPRNT-KPRM) 1723,1723,1703
 1703 PRINT 1702,XNEW(1,1),XNEW(NP,1)
 1702 FORMAT(1H1,7X,16HPISTON POSITION=,E12.5,17H  SHOCK POSITION=,E12.5
     1//)
      KPRNT=1
      DO 790 I=1,NMAX
      XNEW(I,1)=(XNEW(I,1)-RP)/(RS-RP)
  790 CONTINUE
      PRINT 1708,TW,TAU,BO,NU
 1708 FORMAT(7X,6HTWALL=,F8.4,6X,4HTAU=,F6.3,6X,3HBO=,F6.3,6X,4HGNU=,I3
     1//)
      PRINT 4985,EMIS
 4985 FORMAT(23X,11HEMISSIVITY=,F6.3//)
      PRINT 1705,TIME
 1705 FORMAT(20X,11HTHE TIME IS,E12.5//)                                     02510
      PRINT 1710,XNEW(NP,7)
 1710 FORMAT(18X,15HSHOCK VELOCITY=,E12.5//7X,1HR,11X,1HP,9X,1HU,11X,1HT
     1,10X,2HIO,8X,1HQ/)
      PRINT 1720,((XNEW(I,J),J=1,6),I=1,NMAX)
 1720 FORMAT(1H ,2F11.5,E11.4,2F11.5,E11.4)
      PRINT 1722
 1722 FORMAT(1H1)
 1723 KPRNT=KPRNT+1
      EPSL=XNEW(1,5)/XIW(1)-1.                                               02570
      MTIME=2                                                                02580
      EPS4=SLEPS
      IS=NP                                                                  02630
      IMAX=NMAX                                                              02640
      IF(KPNCH-KPNM) 1724,1724,1726
 1726 PUNCH 1090,IS,IMAX,((XOLD(I,J),J=1,7),I=1,IMAX)
      KPNCH=1
 1724 KPNCH=KPNCH+1
C
C     RELEVANT PARAMETERS ARE INITIALIZED, AND ABC IS COMPUTED
C     BASED UPON THE SOLUTION JUST FOUND.
C
      DO 9887 I=1,IMAX
      DO 9887 J=1,7
      WORK(I,J)=XOLD(I,J)
 9887 CONTINUE
      RSINT=XOLD(IS,1)
      ISW=IS
      RMAX=XOLD(IMAX,1)
      IMAXW=IMAX
```

```
      XE=XE*AD
      IF(MWITH-1) 1732,1732,1730
 1730 IF(MWTH2-1) 1732,1732,1731
 1732 QSH=0.
      GO TO 1735
 1731 ARGA=1.7320508*TAU*ATAU(RSINT)
      QSH=2.*(2.-EMIS)*EXPF(-ARGA)
 1735 ABC1=XOLD(1,5)-2.*EMIS*(TW**4.)-QSH
      ABC=ABC1/(1.-.5*EMIS)
      IF(ABSF(TIME/AB-1.)-EPS) 9999,22,22
 9999 CONTINUE
      CALL EXIT
      END
```

```
      SUBROUTINE BDYPT(MTIME,NIT,XXI)
C
C     THIS SUBROUTINE INTEGRATES THE COMPATIBILITY RELATIONS
C     REQUIRED TO COMPUTE PROPERTIES IN THE GAS NEAR THE WALL.  IT
C     ALSO INVOLVES MAKING A GUESS AT THE VALUE OF THE FIRST MOMENT, IO.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      COMMON EMIS,TW,AA,AB,AC,AD,RS,KISS,MWITH,KLUP,MWTH2
      L=1
C
C     INTERPOLATION MATRIX IS LOADED WITH OLD DATA.
C
    2 DO 5 I=1,IMAX
      DO 5 J=1,7
      WORK(I,J)=XOLD(I,J)
    5 CONTINUE
      ISW=IS
      IMAXW=IMAX
C
C     INITIAL GUESS AT P AND T ARE BASED UPON EITHER OLD DATA OR THE
C     LAST VALUES COMPUTED.
C
      IF(NTOT-1) 6,6,7
    6 IF(L-1) 106,106,9
  106 PROP(1,2)=XOLD(1,2)
      PROP(1,4)=XOLD(1,4)
      GO TO 8
    7 PROP(1,2)=XNEW(1,2)
      PROP(1,4)=XNEW(1,4)
    8 PROP(1,1)=RP
      PROP(1,3)=UP
C
C     THE POINTS FROM WHICH CHARACTERISTICS EMANATE TOWARD POINT 1
C     ARE FOUND AS ARE THE PROPERTIES THEREAT.
C
    9 CALL SEARCH(3,L,UMA1,UPA1,U1)
      IF(ISTOP-1) 20,20,10
   10 ISTOP=3
      RETURN
   20 IF(KLUP-1) 21,21,22
   21 PRINT 5000,(PROP(3,J),J=1,6)
 5000 FORMAT(1H ,6E12.5)
   22 PAT3=PAT(PROP(3,2),PROP(3,4))
C
C     IF MTIME=1 THE RADIATION IS TURNED OFF DURING THE PREVIOUS
C     TIME INCREMENT, IF MTIME=2 IT IS ON.  IT IS NECESSARY TO TURN IT
C     OFF ONLY WITHIN THE NON-RADIATING STARTING SOLUTION.
C
      IF(MTIME-1) 15,15,25
   15 RUMA3=0.
      RPU2=0.
      GO TO 29
   25 RUMA3=RUMA(PROP(3,1),PROP(3,2),PROP(3,3),PROP(3,4),PROP(3,5))
      RPU2=RPU(XOLD(1,2),XOLD(1,3),XOLD(1,4),XOLD(1,5))
C
C     IN THE FIRST ITERATION COEFFICIENTS AND INHOMOGENEOUS TERMS ARE
C     ASSUMED EQUAL TO THOSE ON THE APPROPRIATE CHARACTERISTICS.
C     ALL COEFFICIENTS AND INHOMOGENEOUS TERMS IN THE COMPATIBILITY RE-
C     LATIONS ARE EXTERNAL FUNCTIONS SO THAT THEY MAY BE CHANGED
```

```
C       WITHOUT DISTURBING THE COMPILATION OF THE EXECUTIVE PROGRAM
C       AND SEVERAL  OF THE SUBROUTINES.
C
   29 IF(L-1) 30,30,40
   30 PAT1=PAT3
      RUMA1=RUMA3 .
      RPU1=RPU2
C
C       IN SUCCESSIVE ITERATIONS THESE COEFFICIENTS AND INHOMOGENEOUS
C       TERMS ARE AVERAGES OF THOSE AT POINT 1 (WHICH ARE COMPUTED
C       LATER WITHIN THIS SUBROUTINE) AND THOSE AT POINTS 2, 3, OR 4 AS
C       APPROPRIATE.
C
   40 PAT13=0.5*(PAT1+PAT3)
      RUM13=0.5*(RUMA1+RUMA3)
      RPU12=0.5*(RPU1+RPU2)
C
C       THE COMPATIBILITY RELATIONS.
C
      PROP(1,2)=PROP(3,2)+PAT13*(UP-PROP(3,3))+RUM13*DELT
      POWER=RPU12*DELT*GAM
      PP=PROP(1,2)/XOLD(1,2)
      IF(KLUP-1) 42,42,44
   42 PRINT 4000,PP
 4000 FORMAT(1H ,3HPP=,E12.5)
   44 P1P2=PP**GAM
      PROP(1,4)=XOLD(1,4)*EXPF(-POWER)*P1P2
C
C       THE INTERPOLATION ROUTINE OPERATES WITH A MATRIX CALLED WORK.
C       IT IS NECESSARY TO SET IN THIS MATRIX THE DATA APPROPRIATE FOR
C       THE COMPUTATION OF THE INTEGRAL BOUNDARY CONDITION AT THIS POINT.
C
      IF(NTOT-1) 50,50,100
C
C       IN THE FIRST ITERATION THE PREVIOUS FLOW FIELD IS RE-SCALED AND
C       USED WITH THE EXECPTION OF THE BODY POINT (WHICH HAS JUST BEEN
C       COMPUTED).
C
   50 DO 60 I=1,IMAX
      WORK(I,1)=RP+(XOLD(I,1)-XOLD(1,1))*(RS-RP)/(XOLD(IS,1)-XOLD(1,1))
   60 CONTINUE
      DO 70 J=2,4
      WORK(1,J)=PROP(1,J)
   70 CONTINUE
      ISW=IS
      IMAXW=IMAX
      IF(L-1) 80,80,90
   80 WORK(1,5)=XOLD(1,5)
      WORK(1,6)=XOLD(1,6)
      WORK(1,7)=0.
      GO TO 95
   90 WORK(1,5)=PROP(1,5)
      WORK(1,6)=PROP(1,6)
   95 RMAX=WORK(IMAXW,1)
      RSINT=RS
      GO TO 150
C
C       IN SUCCESSIVE ITERATIONS THE PREVIOUS FLOW FIELD IS EMPLOYED
C       WITH THE EXCEPTION OF THE FIRST POINT.
C
  100 WORK(1,1)=RP
      ISW=NP
      IMAXW=NMAX
```

```
      DO 110 J=2,4
      WORK(1,J)=PROP(1,J)
  110 CONTINUE
      IF(L-1) 112,112,114
  112 WORK(1,5)=XNEW(1,5)
      WORK(1,6)=XNEW(1,6)
      GO TO 116
  114 WORK(1,5)=PROP(1,5)
      WORK(1,6)=PROP(1,6)
  116 DO 120 I=2,NMAX
      DO 120 J=1,7
      WORK(I,J)=XNEW(I,J)
  120 CONTINUE
      RMAX=XNEW(NMAX,1)
      RSINT=XNEW(NP,1)
C
C     IF NIT IS GREATER THAN OR EQUAL TO 2 THE INFLUENCE COEFFICIENT
C     ITERATION IS USED, AND IO AT THE WALL IS GUESSED.  IF NIT IS UNITY
C     THE ITERATION IS PERFORMED WITH THE AID OF THE INTEGRAL.
C     THE FORM OF THE BOUNDARY CONDITION IS DIFFERENT IF UPSTREAM
C     ABSORPTION IS NOT ALLOWED THAN IF IT IS.
C
  150 IF(NIT-1) 152,152,153
  152 CALL XINT
      H1=G
      IF(MWITH-1) 155,155,164
  164 IF(MWTH2-1) 155,155,156
  155 QSH=0.
      GO TO 162
  156 ARGA=1.7320508*TAU*ATAU(RSINT)
      QSH=2.*(2.-EMIS)*EXPF(-ARGA)
  162 PROP(1,5)=2.*EMIS*(TW**4.)+(1.-.5*EMIS)*H1*1.7320508+QSH
      GO TO 157
  153 PROP(1,5)=XXI
  157 PROP(1,6)=EMIS*(4.*(TW**4.)-PROP(1,5))/(1.7320508*(2.-EMIS))
  166 L=L+1
      IF(KLUP-1) 172,172,174
  172 PRINT 5000,(PROP(1,J),J=1,6)
C
C     RETURN TO THE EXECUTIVE PROGRAM AFTER A SUFFICIENT NUMBER OF
C     ITERATIONS HAVE BEEN PERFORMED.
C
  174 IF(L-LM) 160,160,170
C
C     COEFFICIENTS AND INHOMOGENEOUS TERMS AT POINT 1.
C
  160 PAT1=PAT(PROP(1,2),PROP(1,4))
      RUMA1=RUMA(PROP(1,1),PROP(1,2),PROP(1,3),PROP(1,4),PROP(1,5))
      RPU1=RPU(PROP(1,2),PROP(1,3),PROP(1,4),PROP(1,5))
      UMA1=UMA(PROP(1,3),PROP(1,4))
      GO TO 2
  170 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE FLDPT(MTIME,KASM,X1,X2)
C
C     THIS SUBROUTINE INTEGRATES THE COMPATIBILITY RELATIONS
C     REQUIRED TO COMPUTE PROPERTIES AT A FIELD POINT.
C     THE LOGIC IS COMPLETELY ANALOGOUS TO THAT FOR A BODY POINT
C     EXCEPT THAT THREE ADDITIONAL COMPATIBILITY RELATIONS ARE NEEDED.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
     3EMIS,TW,AA,AB,AC,AD,RS
      L=1
   10 CALL SEARCH(3,L,UMA1,UPA1,U1)
C
C     IF A GIVEN POINT FROM WHICH A CHARACTERISTIC EMENATES CANNOT BE
C     FOUND WITHIN THE SPECIFIED NUMBER OF ITERATIONS, ISTOP IS SET
C     EQUAL TO THE INDICATOR OF THAT POINT, AND THE EXECUTIVE PROGRAM
C     IS INFORMED THAT AN ERROR HAS OCCURRED.
C
      IF(ISTOP-1) 11,11,120
   11 CALL SEARCH(2,L,UMA1,UPA1,U1)
      IF(ISTOP-1) 12,12,130
   12 CALL SEARCH(4,L,UMA1,UPA1,U1)
      IF(ISTOP-1) 13,13,140
   13 PAT3=PAT(PROP(3,2),PROP(3,4))
      PAT4=PAT(PROP(4,2),PROP(4,4))
      IF(MTIME-1) 15,15,25
   15 RUPA4=0.
      RUMA3=0.
      RPU2=0.
      GO TO 16
   25 RUPA4=RUPA(PROP(4,1),PROP(4,2),PROP(4,3),PROP(4,4),PROP(4,5))
      RUMA3=RUMA(PROP(3,1),PROP(3,2),PROP(3,3),PROP(3,4),PROP(3,5))
      RPU2=RPU(PROP(2,2),PROP(2,3),PROP(2,4),PROP(2,5))
   16 QT5=QTAU(PROP(5,2),PROP(5,4),PROP(5,6))
      RQ5=RPQ(PROP(5,1),PROP(5,2),PROP(5,4),PROP(5,5),PROP(5,6))
      IF(L-1) 80,80,90
   80 PAT1=PAT4
      PT1=PAT3
      RUPA1=RUPA4
      RUMA1=RUMA3
      RPU1=RPU2
      QT1=QT5
      RQ1=RQ5
   90 PAT14=0.5*(PAT1+PAT4)
      PAT13=0.5*(PT1+PAT3)
      RUP14=0.5*(RUPA1+RUPA4)
      RUM13=0.5*(RUMA1+RUMA3)
      RPU12=0.5*(RPU1+RPU2)
      QT15=0.5*(QT1+QT5)
      RQ15=0.5*(RQ1+RQ5)
      PROP(1,3)=(PROP(4,3)*PAT14+PROP(3,3)*PAT13+
     1(RUP14-RUM13)*DELT+PROP(4,2)-PROP(3,2))/(PAT13+PAT14)
      PROP(1,2)=PROP(3,2)+PAT13*(PROP(1,3)-PROP(3,3))+RUM13*DELT
      PROP(1,4)=PROP(2,4)*EXPF(-RPU12*DELT*GAM)*(PROP(1,2)/PROP(2,2))
     1**GAM
      IF(KASM-1) 94,94,96
   94 PROP(1,5)=PROP(5,5)-QT15*(PROP(1,1)-PROP(5,1))
      PROP(1,6)=PROP(5,6)-RQ15*(PROP(1,1)-PROP(5,1))
      GO TO 98
   96 PROP(1,5)=X1
```

```
      PROP(1,6)=X2
   98 L=L+1
      IF(L-LM) 100,100,110
  100 PAT1=PAT(PROP(1,2),PROP(1,4))
      RUPA1=RUPA(PROP(1,1),PROP(1,2),PROP(1,3),PROP(1,4),PROP(1,5))
      RUMA1=RUMA(PROP(1,1),PROP(1,2),PROP(1,3),PROP(1,4),PROP(1,5))
      RPU1=RPU(PROP(1,2),PROP(1,3),PROP(1,4),PROP(1,5))
      QT1=QTAU(PROP(1,2),PROP(1,4),PROP(1,6))
      RQ1=RPQ(PROP(1,1),PROP(1,2),PROP(1,4),PROP(1,5),PROP(1,6))
      PT1=PAT1
      UMA1=UMA(PROP(1,3),PROP(1,4))
      UPA1=UPA(PROP(1,3),PROP(1,4))
      U1=PROP(1,3)
      GO TO 10
  110 CONTINUE
      RETURN
  120 ISTOP=3
      RETURN
  130 ISTOP=2
      RETURN
  140 ISTOP=4
      RETURN
      END
```

```
      SUBROUTINE SHKPT(RSH,MTIME)
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      COMMON EMIS,TW,AA,AB,AC,AD,RS,KISS,MWITH,KLUP
C
C     THIS SUBROUTINE COMPUTES SIMULTANEOUSLY THE PROPERTIES  ON
C     BOTH SIDES OF THE SHOCK.   IT BOTH INTEGRATES THE REQUIRED COM-
C     PATIBILITY RELATIONS AND SOLVES THE RANKINE-HUGONIOT RELATIONS.
C     THE LOGIC IS ANALOGOUS TO THAT FOR A BODY POINT.
C
      L=1
      DO 10 J=2,7
      PROP(5,J)=XNEW(NP-1,J)
      PROP(7,J)=XOLD(IS,J)
   10 CONTINUE
      IF(MWITH-1) 2,2,4
    2 DO 3 J=2,7
      PROP(1,J)=XOLD(IS+1,J)
    3 CONTINUE
      GO TO 12
    4 PROP(1,2)=1.
      PROP(1,3)=0.
      PROP(1,4)=1.
   12 PROP(5,1)=XNEW(NP-1,1)
   15 PROP(7,1)=RSH
      PROP(1,1)=RSH
C
C     PROPERTIES AT POINT 1, UPSTREAM OF THE SHOCK, ARE FOUND AS AT
C     A FIELD POINT.
C
      CALL SEARCH(3,L,UMA1,UPA1,U1)
      IF(ISTOP-1) 30,30,20
   20 ISTOP=3
      RETURN
   30 CALL SEARCH(2,L,UMA1,UPA1,U1)
      IF(ISTOP-1) 50,50,40
   40 ISTOP=2
      RETURN
   50 CALL SEARCH(4,L,UMA1,UPA1,U1)
      IF(ISTOP-1) 70,70,60
   60 ISTOP=4
      RETURN
   70 CALL SEARCH(6,L,UMA1,UPA7,U1)
      IF(ISTOP-1) 100,100,90
   90 ISTOP=6
      RETURN
  100 PAT4=PAT(PROP(4,2),PROP(4,4))
      PAT3=PAT(PROP(3,2),PROP(3,4))
      IF(MTIME-1) 102,102,104
  102 RUPA4=0.
      RUPA6=0.
      RUMA3=0.
      RPU2=0.
      GO TO 106
  104 RUPA4=RUPA(PROP(4,1),PROP(4,2),PROP(4,3),PROP(4,4),PROP(4,5))
      RUPA6=RUPA(PROP(6,1),PROP(3,2),PROP(6,3),PROP(6,4),PROP(6,5))
      RUMA3=RUMA(PROP(3,1),PROP(3,2),PROP(3,3),PROP(3,4),PROP(3,5))
      RPU2=RPU(PROP(2,2),PROP(2,3),PROP(2,4),PROP(2,5))
  106 QT5=QTAU(PROP(5,2),PROP(5,4),PROP(5,6))
      RQ5=RPQ(PROP(5,1),PROP(5,2),PROP(5,4),PROP(5,5),PROP(5,6))
```

```
      IF(L-1) 120,120,130
  120 PAT1=PAT4
      PT1=PAT3
      RUPA1=RUPA4
      RUMA1=RUMA3
      RPU1=RPU2
      QT7=QT5
      RQ7=RQ5
  130 PAT14=0.5*(PAT1+PAT4)
      PAT13=0.5*(PT1+PAT3)
      RUP14=0.5*(RUPA1+RUPA4)
      RUM13=0.5*(RUMA1+RUMA3)
      RPU12=0.5*(RPU1+RPU2)
      QT57=0.5*(QT5+QT7)
      RQ57=0.5*(RQ5+RQ7)
      IF(MWITH-1) 132,132,134
  132 PROP(1,3)=(PROP(4,3)*PAT14+PROP(3,3)*PAT13+(RUP14-RUM13)*DELT+
     1PROP(4,2)-PROP(3,2))/(PAT13+PAT14)
  136 PROP(1,2)=PROP(3,2)+PAT13*(PROP(1,3)-PROP(?,3))+RUM13*DELT
      PROP(1,4)=PROP(2,4)*EXPF(-RPU12*DELT*GAM)*(PROP(1,2)/PROP(2,2))
     1**GAM
      GO TO 138
C
C     IF UPSTREAM ABSORPTION IS NOT INCLUDED, POINT 1 IS UNDISTURBED.
C
  134 PROP(1,3)=0.
      PROP(1,2)=1.
      PROP(1,4)=1.
  138 PROP(7,5)=PROP(5,5)-QT57*(PROP(7,1)-PROP(5,1))
      PROP(7,6)=PROP(5,6)-RQ57*(PROP(7,1)-PROP(5,1))
      PAT6=PAT(PROP(6,2),PROP(6,4))
      IF(L-1) 140,140,150
  140 PAT7=PAT6
      RUPA7=RUPA6
  150 PAT67=0.5*(PAT6+PAT7)
      RUP67=0.5*(RUPA6+RUPA7)
C
C     THE PROPERTIES AT POINT 1 ALONG WITH THE COMPATIBILITY
C     RELATION ON 6-7 AND PROPERTIES AT POINT 6 ARE EMPLOYED TO SOLVE
C     THE RANKINE HUGONIOT RELATIONS BY A NEWTON-RHAPSON SCHEME.
C
      P1=PROP(1,2)
      U1=PROP(1,3)
      T1=PROP(1,4)
      P6=PROP(6,2)
      U6=PROP(6,3)
      NTIME=1
      IF(L-1) 152,152,160
  152 U7=XOLD(IS,3)
  160 AS=(1.-GAM)*(U7-U1)
      BS=((1.-0.5*GAM)*U7*U7-GAM*U1*U7+(1.5*GAM-1.)*U1*U1)/AS
      CS=(U7*(T1+U1*U1+(0.5*GAM-1.)*U1*U7)-U1*(T1+0.5*GAM*U1*U1))/AS
      SS=SQRTF(BS*BS+4.*CS)
      US=0.5*(BS+SS)
      FUN=U7*(PAT67-P1*(U1-US)/T1)-(P6+PAT67*U6+RUP67*DELT)+P1*(T1+U1*U1
     1-U1*US)/T1
      IF(ABSF(FUN)-EPS) 190,190,180
  180 BSU=((2.-GAM)*U7-GAM*U1-(1.-GAM)*BS)/AS
      CSU=(T1+U1*U1+(GAM-2.)*U1*U7-(1.-GAM)*CS)/AS
      USU7=0.5*((BS/SS+1.)*BSU+2.*CSU/SS)
      FUNP=PAT67+P1*(US-U1)/T1+P1*(U7-U1)*USU7/T1
C
C     NEWTON-RHAPSON GUESS.
```

```
C
      U7=U7-FUN/FUNP
      NTIME=NTIME+1
      IF(NTIME-IBMAX) 160,160,170
  170 PRINT 1012
 1012 FORMAT(24H CANNOT FIND SHOCK SPEED)
      PRINT 1011,NTIME,U7
 1011 FORMAT(1H ,6HAFTER ,I3,18H ITERATIONS, U7 IS,E12.5)
      ISTOP=7
      RETURN
  190 PROP(7,7)=US
      IF(KLUP-1) 192,192,194
  192 PRINT 9875,US
 9875 FORMAT(1H ,3HUS=,E12.5)
  194 PROP(7,3)=U7
      PROP(7,2)=P1*(T1+U1*U1-U1*US-(U1-US)*U7)/T1
      PROP(7,4)=T1*PROP(7,2)*(U7-US)/((U1-US)*P1)
      PROP(1,5)=PROP(7,5)
      PROP(1,6)=PROP(7,6)
      PROP(1,7)=PROP(7,7)
      L=L+1
      IF(L-LM) 200,200,600
  200 PAT1=PAT(PROP(1,2),PROP(1,4))
      PAT7=PAT(PROP(7,2),PROP(7,4))
      PT1=PAT1
      RUPA1=RUPA(PROP(1,1),PROP(1,2),PROP(1,3),PROP(1,4),PROP(1,5))
      RUPA7=RUPA(PROP(7,1),PROP(7,2),PROP(7,3),PROP(7,4),PROP(7,5))
      RUMA1=RUMA(PROP(1,1),PROP(1,2),PROP(1,3),PROP(1,4),PROP(1,5))
      RPU1=RPU(PROP(1,2),PROP(1,3),PROP(1,4),PROP(1,5))
      QT7=QTAU(PROP(7,2),PROP(7,4),PROP(7,6))
      RQ7=RPQ(PROP(7,1),PROP(7,2),PROP(7,4),PROP(7,5),PROP(7,6))
      UMA1=UMA(PROP(1,3),PROP(1,4))
      UPA1=UPA(PROP(1,3),PROP(1,4))
      U1=PROP(1,3)
      UPA7=UPA(PROP(7,3),PROP(7,4))
      RSH=0.5*(XOLD(IS,7)+US)*DELT+XOLD(IS,1)
      GO TO 15
  600 CONTINUE
      RETURN
      END
```

```
      FUNCTION ATAU(PLACE)
C
C     THIS FUNCTION COMPUTES THE INTEGRAL FROM R-PISTON TO R
C     OF RHO-KAPPA DR.  RHO-KAPPA IS ALLOWED TO BE AN ARBITRARY FUNCTION
C     OF TEMPERATURE AND PRESSURE.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      COMMON EMIS,TW,AA,AB,AC,AD,RS
      T1=0.
      T2=0.
      IF(PLACE-RSINT) 10,10,20
   10 DO 15 I=1,NW
      UR=0.5*(PLACE-RP)*XW(I)+0.5*(PLACE+RP)
      XR=UR
      CALL XNTRP
      T1=W(I)*ALFM(VAR(2),VAR(4))+T1
   15 CONTINUE
      ATAU=.5*(PLACE-RP)*T1
      RETURN
   20 DO 30 I=1,NW
      UR=0.5*(RSINT-RP)*XW(I)+0.5*(RSINT+RP)
      XR=UR
      CALL XNTRP
      T1=W(I)*ALFM(VAR(2),VAR(4))+T1
   30 CONTINUE
      TAU1=.5*(RSINT-RP)*T1
      DO 40 I=1,NW
      UR=0.5*(PLACE-RSINT)*XW(I)+0.5*(PLACE+RSINT)
      XR=UR
      CALL XNTRP
      T2=W(I)*ALFM(VAR(2),VAR(4))+T2
   40 CONTINUE
      ATAU=TAU1+.5*(PLACE-RSINT)*T2
      RETURN
      END
```

```
      SUBROUTINE XINT
C     THIS SUBROUTINE COMPUTES THE INTEGRAL PART OF THE BOUNDARY
C     CONDITION.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),           000
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,CAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      COMMON EMIS,TW,AA,AB,AC,AD,RS,KISS,MWITH,KLUP,MWTH2
      G=0.
      DO 10 I=1,NW
      R=.5*(RSINT-RP)*XW(I)+.5*(RSINT+RP)
      XR=R
      CALL XNTRP
      RAH=4.*TAU*ALFP(VAR(2),VAR(4))*VAR(4)**4.-GNU*VAR(6)/R
      TA=ATAU(R)
      ARG=1.7320508*TAU*TA
      EX=EXPF(-ARG)
      G=RAH*EX*W(I)+G
   10 CONTINUE
      PART1=.5*(RSINT-RP)*G
      IF(MWITH-1) 100,100,200
  100 G=0.
      DO 30 I=1,NW
      R=.5*(RMAX-RSINT)*W(I)+.5*(RMAX+RSINT)
      XR=R
      CALL XNTRP
      RAH=4.*TAU*ALFP(VAR(2),VAR(4))*VAR(4)**4.-GNU*VAR(6)/R
      TA=ATAU(R)
      ARG=1.7320508*TAU*TA
      EX=EXPF(-ARG)
      G=RAH*EX*W(I)+G
   30 CONTINUE
      PART2=.5*(RMAX-RSINT)*G
      ARG12=1.7320508*TAU*ATAU(RMAX)
      ADD=2.3094011*EXPF(-ARG12)
      GO TO 300
  200 ADD=0.
      PART2=0.
  300 G=PART1+PART2+ADD
      RETURN
      END
```

```
      SUBROUTINE XNTRP
C
C     THIS SUBROUTINE INTERPOLATES LINEARLY AMONG THE PROPERTIES IN
C     AN IMAXW BY 7 MATRIX WITH A DISCONTINUITY BETWEEN ISW AND ISW+1.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      COMMON EMIS,TW,AA,AB,AC,AD,RS
      IF(XR-WORK(1,1)) 90,5,5
    5 VAR(1)=XR
      IF(XR-WORK(ISW,1)) 10,20,20
   10 IMW=ISW
      I1=1
      GO TO 30
   20 IF(XR-WORK(IMAXW,1)) 25,70,70
   25 IMW=IMAXW
      I1=ISW+1
   30 DO 40 I=I1,IMW
      IF(XR-WORK(I,1)) 50,50,40
   40 CONTINUE
   50 P2=WORK(I,1)-WORK(I-1,1)
      P3=XR-WORK(I-1,1)
      DO 60 J=2,7
      VAR(J)=WORK(I-1,J)+P3*(WORK(I,J)-WORK(I-1,J))/P2
   60 CONTINUE
      RETURN
   70 VAR(2)=1.
      VAR(3)=0.
      VAR(4)=1.
      VAR(5)=4.
      VAR(6)=0.
      RETURN
   90 PRINT 1000,XR
 1000 FORMAT(38H THE POINT SOUGHT IS INSIDE THE PISTON,E12.5)
      ISTOP=5
      RETURN
      END
```

```
      SUBROUTINE SEARCH(K,L,UMA1,UPA1,U1)
C
C     THIS ROUTINE SEARCHES AMONG A SET OF DATA FOR A POINT FROM WHICH A
C     CHARACTERISTIC INTERSECTS THE POINT OF INTEREST.  K IS THE
C     DESIGNATION OF THE POINT SOUGHT (2=STREAMLINE,3=MINUS MACH LINE,
C     4=PLUS MACH LINE,6=PLUS MACH LINE TO THE POINT ON THE DOWNSTREAM
C     SIDE OF THE SHOCK).  L IS THE NUMBER OF ITERATIONS WHICH HAVE
C     BEEN MADE ON THE COMPATIBILITY RELATIONS AT THE POINT OF INTER-
C     EST.   UMA1,UPA1, AND U1 ARE VELOCITY MINUS OR PLUS THE SPEED
C     OF SOUND AND THE VELOCITY RESPECTIVELY AT POINT 1.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK1(3),PROP(7,7),GR1(3),RK(100),GR(100)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK1,GR1
      COMMON EMIS,TW,AA,AB,AC,AD,RS,KISS
    1 IF(K-6) 2,3,3
    2 GO TO (10,20,30,10),K
   10 IF(L-1) 100,100,102
  100 S4=UPA(PROP(1,3),PROP(1,4))
      GO TO 104
  102 S4=.5*(UPA1+UPA(PROP(4,3),PROP(4,4)))
  104 RK(1)=PROP(1,1)-S4*DELT
      IF(KISS-1) 42,42,14
   14 IF(RK(1)-XOLD(IS,1)) 16,16,40
   16 RK(1)=XOLD(IS,1)*(1.+EPS1)
      GO TO 40
   20 IF(L-1) 200,200,202
  200 S2=PROP(1,3)
      GO TO 204
  202 S2=.5*(U1+PROP(2,3))
  204 RK(1)=PROP(1,1)-S2*DELT
      GO TO 42
   30 IF(L-1) 300,300,302
  300 S3=UMA(PROP(1,3),PROP(1,4))
      GO TO 304
  302 S3=.5*(UMA1+UMA(PROP(3,3),PROP(3,4)))
  304 RK(1)=PROP(1,1)-S3*DELT
      GO TO 42
    3 IF(L-1) 400,400,402
  400 S6=UPA(PROP(7,3),PROP(7,4))
      GO TO 404
  402 S6=.5*(UPA1+UPA(PROP(6,3),PROP(6,4)))
  404 RK(1)=PROP(7,1)-S6*DELT
   42 IF(RK(1)-XOLD(1,1)) 43,43,40
   43 RK(1)=XOLD(1,1)*1.0001
   40 XR=RK(1)
      CALL XNTRP
      DO 50 J=1,7
      PROP(K,J)=VAR(J)
   50 CONTINUE
      RETURN
      END
```

```
C
C     ****************************************************************
C
      FUNCTION PISPOS(AT)
C
C     PISTON POSITION AS A FUNCTION OF TIME.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      COMMON EMIS,TW,AA,AB,AC,AD,RS
      PISPOS=XOLD(1,3)*AT
      RETURN
      END
C
C     ****************************************************************
C
      FUNCTION PISVEL(AT)
C
C     PISTON VELOCITY AS A FUNCTION OF TIME.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      COMMON EMIS,TW,AA,AB,AC,AD,RS
      PISVEL=XOLD(1,3)
      RETURN
      END
C
C     ****************************************************************
C
      FUNCTION PISTMP(AT)
C
C     PISTON TEMPERATURE AS A FUNCTION OF TIME.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      COMMON EMIS,TW,AA,AB,AC,AD,RS
      PISTMP=1.
      RETURN
      END
```

```
C
C     ****************************************************************
C
      FUNCTION AIS(T)
C
C     THE ISENTROPIC SPEED OF SOUND FOR CONSTANT RATIO OF SPECIFIC
C     HEATS.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      AIS=SQRTF(T/(1.-GAM))
      RETURN
      END
C
C     ****************************************************************
C
      FUNCTION UPA(U,T)
C
C     VELOCITY PLUS THE SPEED OF SOUND AS A FUNCTION OF VELOCITY AND
C     TEMPERATURE.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      UPA=U+AIS(T)
      RETURN
      END
C
C     ****************************************************************
C
      FUNCTION UMA(U,T)
C
C     VELOCITY MINUS THE SPEED OF SOUND AS A FUNCTION OF VELOCITY AND
C     TEMPERATURE
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      UMA=U-AIS(T)
      RETURN
      END
```

```
C
C     ******************************************************************
C
      FUNCTION RUMA(AR,P,U,T,XIO)
C
C     THE RIGHT HAND SIDL OF THE COMPATIBILITY RELATIONS ALONG BOTH
C     PLUS AND MINUS MACH LINES AS A FUNCTION OF POSITION, PRESSURE,
C     VELOCITY, TEMPERATURE, ANC THE FIRST RADIATION MOMENT.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      RUMA=(RAD(P,T,XIO)-GNU*P*U/(AR*GAM))/(1./GAM-1.)
      RETURN
      END
C
C     ******************************************************************
C
      FUNCTION RUPA(AR,P,U,T,XIO)
C
C     THE RIGHT HAND SIDE OF TF= COMPATIBILITY RELATIONS ALONG BOTH
C     PLUS AND MINUS MACH LINES AS A FUNCTION OF POSITION, PRESSURE,
C     VELOCITY, TEMPERATURE, ANC THE FIRST RADIATION MOMENT.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      RUPA=(RAD(P,T,XIO)-GNU*P*U/(AR*GAM))/(1./GAM-1.)
      RETURN
      END
C
C     ******************************************************************
C
      FUNCTION RPU(P,U,T,XIO)
C
C     THE RIGHT HAND SIDE OF THE COMPATIBILITY RELATION ALONG THE
C     STREAMLINE AS A FUNCTION OF PRESSURE, VELOCITY, TEMPERATURE,
C     AND THE FIRST RADIATION MOMENT.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      RPU=-RAD(P,T,XIO)/P
      RETURN
      END
```

```
C
C     ******************************************************************
C
      FUNCTION ALFP(P,T)
C
C     THE PLANCK MEAN ABSORPTION COEFFICIENT AS A FUNCTION OF PRESSURE
C     AND TEMPERATURE.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      ALFP=A1*(P**B1)*(T**B2)
      RETURN
      END
C
C     ******************************************************************
C
      FUNCTION ALFM(P,T)
C
C     THE ABSORPTION MEAN COEFFICIENT AS A FUNCTION OF PRESSURE AND
C     TEMPERATURE.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      ALFM=ALFP(P,T)*T**4./SQRTF(1.-XNA+XNA*(T**(8.*(XMA+1.))))
      RETURN
      END
C
C     ******************************************************************
C
      FUNCTION PAT(P,T)
C
C     COEFFICIENT -- RHO-A INVOLVED IN COMPATIBILITY RELATIONS ALONG
C     PLUS AND MINUS MACH LINES.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,AI,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      PAT=P*AIS(T)/T
      RETURN
      END
```

```
C
C     *****************************************************************
C
      FUNCTION RAD(P,T,XIO)
C
C     DEL-DOT THE HEAT FLUX.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      RAD=TAU*(ALFM(P,T)*XIO-4.*ALFP(P,T)*T**4.)/BO
      RETURN
      END
C
C     *****************************************************************
C
      FUNCTION RPQ(AR,P,T,XIO,Q)
C
C     RIGHT HAND SIDE OF THE COMPATIBILITY RELATION FOR THE HEAT FLUX
C     ALONG THE TIME=CONSTANT CHARACTERISTIC.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      RPQ=RAD(P,T,XIO)*BO+GNU*Q/AR
      RETURN
      END
C
C     *****************************************************************
C
      FUNCTION QTAU(P,T,Q)
C
C     RIGHT HAND SIDE OF THE COMPATIBILITY RELATION FOR THE FIRST
C     RADIATION MOMENT WHICH APPLIES ALONG THE TIME=CONSTANT CHARACTER-
C     ISTIC.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      QTAU=3.*Q*ALFM(P,T)*TAU
      RETURN
      END
```

```
C
C     ******************************************************************
C
      FUNCTION BKO(RNAUT)
C
C     BESSEL FUNCTION -- K-NAUT
C
      IF(RNAUT-2.) 10,10,20
   10 T=RNAUT/3.75
      T2=T*T
      X1=1.+T2*(3.5156229+T2*(3.0899424+T2*(1.2067492+T2*(.2659732+
     1T2*(.0360768+.0045813*T2)))))
      X=RNAUT/2.
      X2=X*X
      BKO=-LOGF(X)*X1-.57721566+X2*(.42278420+X2*(.23069756+X2*(
     10.03488590+X2*(.00262698+X2*(.00010750+.00000740*X2)))))
      RETURN
   20 X=2./RNAUT
      X1=1.25331414+X*(-.07832358+X*(.02189568+X*(-.01062446+X*
     1(.00587872+X*(-.00251540+X*.00053208)))))
      BKO=X1*EXPF(-RNAUT)/SQRTF(RNAUT)
      RETURN
      END
C
C     ******************************************************************
C
      FUNCTION BK1(RNAUT)
C
C     BESSEL FUNCTION K-ONE
C
      IF(RNAUT-2.) 10,10,20
   10 T=RNAUT/3.75
      T2=T*T
      AX1=0.5+T2*(.87890594+T2*(.51498869+T2*(.15084934+T2*(.02658733+
     1T2*(.00301532+.00032411*T2)))))
      X1=RNAUT*AX1
      X=RNAUT/2.
      X2=X*X
      AX2=1.+X2*(.15443144+X2*(-.67278579+X2*(-.18156897+X2*
     1(-.01919402+X2*(-.00110404-.00004686*X2)))))
      BK1=LOGF(X)*X1+AX2/RNAUT
      RETURN
   20 X=2./RNAUT
      AX3=1.25331414+X*(.23498619+X*(-.03655620+X*(.01504268-X*
     1(.00780353+X*(.00325614-.00068245*X)))))
      BK1=AX3*EXPF(-RNAUT)/SQRTF(RNAUT)
      RETURN
      END
C
C     ******************************************************************
C
      FUNCTION SPACE(CLOSE,COUNT,CNTM)
C
C     THE FUNCTION WHICH SPACES THE POINTS BETWEEN PISTON AND SHOCK.
C
      PI=3.141593
      SPACE=.5*(1.+(2.*CLOSE-1.)*COSF(PI*(COUNT-2.)/(CNTM-2.)))
      RETURN
      END
```

FULL TRANSFER EQUATION

```
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP1,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      COMMON EMIS,TW,AA,AB,AC,AD,RS,KISS,MWITH,KLUP,RDIFF
      COMMON MWTH2
C
C     ******************************************************************
C
C     EXECUTIVE PROGRAM -- FULL TRANSFER EQUATION
C
C     ******************************************************************
C
 1010 FORMAT(I3)
 1015 FORMAT(2F12.8)
 1030 FORMAT(5F12.8)
 1050 FORMAT(4F12.8/2I3)
 1070 FORMAT(4F12.8)
 1090 FORMAT(2I3/(6F12.8/F12.8))
 1105 FORMAT(1H ,6E12.5)
 3001 FORMAT(I3/(2F12.8))
 1596 FORMAT(F12.8)
C
C     READ IN INITIAL DATA
C
    2 READ 1090,IS,IMAX,((XOLD(I,J),J=1,7),I=1,IMAX)
C
C     READ IN THE ORDER OF THE GAUSSIAN QUADRATURE AND THE ASSOCIATED
C     WEIGHTS AND ORDINATES
C
      READ 3001,NW,(W(I),XW(I),I=1,NW)
C
C     A1,B1, AND B2 ARE CONSTANTS NEEDED FOR THE PLANCK MEAN ABSORP-
C     TION COEFFICIENT.  XMA AND XNA APPEAR IN THE ABSORPTION
C     AVERAGED MEAN.
C
      READ 1030,A1,B1,B2,XMA,XNA
C
C     THE FIRST THREE OF THESE ARE ACCURACY SPECIFICATIONS IN THE
C     SEARCH SUBROUTINE.  XE IS THE FRACTION OF THE PISTON SHOCK
C     SEPARATION AT WHICH THE FIRST  POINT ABOVE THE PISTON IS LOCATED.
C     IBMAX IS THE MAXIMUM NUMBER OF ITERATIONS ALLOWED, AND NTOT IS
C     THE INDEX OF THE ITERATION CURRENTLY UNDERWAY.
C
      READ 1050,EPS,EPS1,EPS2,XE,IBMAX,NTOT
C
C     WALL EMISSIVITY (NOT ALLOWED TO VARY IN TIME) AND TIME
C
      READ 1015,EMIS,TIME
C
C     THE MAXIMUM NUMBER OF ITERATIONS ALLOWED
C
      READ 1010,LM
C
C     GAM=(GAMMA-1)/GAMMA, WHERE GAMMA IS THE RATIO OF SPECIFIC HEATS.
C     IF GNU IS 0, 1, OR 2, WE ARE INVOLVED IN A PLANAR, CYLINDRICAL,
C     SPHERICAL CALCULATION, RESPECTIVELY.  (GNU IS ALWAYS UNITY IN
C     THIS PROGRAM.)
C
      READ 1070,GAM,GNU,TAU,BO
C
C     ARBITRARY ACCURACY REQUIREMENTS INSERTED FOR FLEXIBILITY.
```

```
C
      READ 1015,EPS4,EPS5
C     IF MTIME IS UNITY, THE RADIATION IS TURNED OFF DURING THE
C     PREVIOUS TIME INCREMENT.
C
      READ 1010,MTIME
C
C     AA AND AB ARE TWO CHARACTERISTIC TIMES OF THE PROBLEM
C
      READ 1070,XD1,XD2,AA,AB
C
C     THE PROGRAM PRINTS EVERY KPRMTH TIME INCREMENT AND PUNCHES
C     EVERY KPNMTH TIME INCREMENT.
C
      READ 1010,KPRNT,KPNCH,KPRM,KPNM
C
C     IF MWITH IS LESS THAN OR EQUAL TO UNITY UPSTREAM ABSORPTION IS
C     INCLUDED
C
      READ 1010,MWITH
C
C     IF MWTH2=1 AND MWITH IS GREATER THAN UNITY, RHO-KAPPA
C     VANISHES UPSTREAM.  IF MWTH2 AND MWITH ARE BOTH GREATER THAN
C     UNITY, UPSTREAM ABSORPTION IS NEGLECTED BUT THE FREE STREAM
C     STILL EMITS.
C
      READ 1010,MWTH2
C
C     FDEL IS THE FRACTION OF THE COMPUTED TIME INCREMENT WHICH IS USED
C
      READ 1015,FDEL,AD
      READ 1596,RDIFF
C
C     IF KLUP IS UNITY INTERMEDIATE RESULTS ARE PRINTED, IF KSHOK
C     IS GREATER THAN UNITY A PISTON RETRACTION CAN BE HANDLED.
C
      READ 1010,KLUP,KSHOK
C     COMPUTE DELTA TIMES BASED UPON POINTS NEAREST TO SHOCK AND BODY
C
   22 DELN=XE*(XOLD(IS,1)-XOLD(1,1))
      DU=XE*(XOLD(IS,7)-XOLD(1,3))
      DEL1=DELN/(AIS(XOLD(1,4))-DU)
      DEL2=DELN/(-UMA(XOLD(IS,3),XOLD(IS,4))+XOLD(IS,7)+DU)
C
C     CHOOSE THE SMALLEST DELTA TIME
C
      DELT=FDEL*MINIF(DEL1,DEL2)
      IF(TIME-AA) 1,7,7
    1 IF(ABSF(TIME/AA-1.)-EPS) 4,4,9
    9 IF(TIME+DELT-AA) 4,3,3
    3 TIME=AA
      GO TO 5
    7 IF(TIME+DELT-AB) 4,8,8
    8 TIME=AB
      GO TO 5
    4 TIME=TIME+DELT
C
C     PSITON POSITION, VELOCITY, AND WALL TEMPERATURE ARE EXTERNAL
C     FUNCTIONS OF TIME
C
    5 RP=PISPOS(TIME)
```

```
      UP=PISVEL(TIME)
      TW=PISTMP(TIME)
      RP1=RP
C
C     THE SHOCK POSITION IS EXTRAPOLATED LINEARLY INITIALLY.
C
      RS=XOLD(IS,1)+XOLD(IS,7)*DELT
      RSH=RS
      KISS=1
C
C     BODY POINT SUBROUTINE
C
    6 CALL BDYPT(MTIME)
      DO 50 I=1,4
      XNEW(1,I)=PROP(1,I)
   50 CONTINUE
      XNNP=1.+1./XE
      NP=XNNP
      NPM=NP-1
      XNP=NPM
      DO 301 I=2,NPM
      XI=I
C
C     POINTS BETWEEN PISTON AND SHOCK ARE CLUSTERED NEAR TO THE
C     PISTON AND SHOCK AND ARE SPACED FARTHER APART IN BETWEEN
C
      PROP(1,1)=RP+(RS-RP)*SPACE(XE,XI,XNP)
      IF(NTOT-1) 19,19,23
   19 DO 21 I1I=1,IMAX
      DO 21 J=1,7
      WORK(I1I,J)=XOLD(I1I,J)
   21 CONTINUE
      XR=(XOLD(IS,1)-XOLD(1,1))*(PROP(1,1)-RP)/(RS-RP)+XOLD(1,1)
      CALL XNTRP
      DO 39 K=2,7
      PROP(1,K)=VAR(K)
   39 CONTINUE
C
C     FIELD POINT SUBROUTINE
C
   23 KISS=1
      CALL FLDPT(MTIME,I)
      DO 40 J=1,4
      XNEW(I,J)=PROP(1,J)
   40 CONTINUE
      IF(KLUP-1) 41,41,301
   41 PRINT 1105,(XNEW(I,J),J=1,5)
  301 CONTINUE
      IF(KSHOK-1) 154,154,156
  154 KISS=2
      CALL SHKPT(RSH,MTIME)
      RS=RSH
      IF(ISTOP-7) 153,2,2
  156 PROP(1,1)=RS
      IF(NTOT-1) 159,159,155
  159 DO 158 J=2,7
      PROP(1,J)=XNEW(NP-1,J)
  158 CONTINUE
  155 CALL FLDPT(MTIME,NP)
      PROP(1,7)=UPA(PROP(1,3),PROP(1,4))
      DO 157 J=1,7
      PROP(7,J)=PROP(1,J)
  157 CONTINUE
```

```
      RS=XOLD(IS,1)+.5*(XOLD(IS,7)+PROP(1,7))*DELT
  153 NPP=NP+1
      DO 160 J=1,7
      XNEW(NP,J)=PROP(7,J)
      XNEW(NPP,J)=PROP(1,J)
  160 CONTINUE
      IF(KLUP-1) 161,161,162
  161 PRINT 1105,(XNEW(NP,J),J=1,6),(XNEW(NPP,J),J=1,6)
  162 IF(MWITH-1) 164,164,166
  166 NMAX=NP
      GO TO 335
  164 NP2=NP+2
      DO 650 I=NP2,100
      IF(I-NP2) 242,242,244
  242 PROP(1,1)=PROP(1,1)+XNEW(NP,1)-XNEW(NP-1,1)
      GO TO 670
  244 IF(XNEW(I-1,1)-XNEW(I-2,1)-XD1) 245,245,246
  245 PROP(1,1)=PROP(1,1)+XD2*(XNEW(I-1,1)-XNEW(I-2,1))
      GO TO 670
  246 PROP(1,1)=PROP(1,1)+XD2*XD1
      IF(NTOT-1) 665,665,670
  665 DO 666 J=2,7
      PROP(1,J)=XNEW(I-1,J)
  666 CONTINUE
  670 IF(I-NMAX) 671,671,672
  672 XNEW(I,1)=PROP(1,1)
      XNEW(I,2)=1.
      XNEW(I,3)=0.
      XNEW(I,4)=1.
      NMAX=I
  671 CALL FLDPT(MTIME,I)
      DO 220 J=1,4
      XNEW(I,J)=PROP(1,J)
  220 CONTINUE
      IG=I
      IF(KLUP-1) 203,203,207
  203 PRINT 1105,(XNEW(I,J),J=1,6)
  207 IF(ABSF(XNEW(I,4)-1.)-EPS4) 335,335,650
  650 CONTINUE
  335 NTOT=NTOT+1
      NMAX=IG
      IF(KLUP-1) 1335,1335,1336
 1335 PRINT 1010,NMAX
      PRINT 1010,NP
      PRINT 1105,((XNEW(I,J),J=1,6),I=1,NMAX)
 1336 IF(NTOT-LM) 6,6,720
  720 DO 336 K=1,NMAX
      XNEW(K,6)=QTAU(XNEW(K,1),XNEW(K,2),XNEW(K,4),1)
      QQ=BO*RAD(XNEW(K,1),XNEW(K,2),XNEW(K,4),1,K)/TAU
      XNEW(K,5)=(4.*ALFP(XNEW(K,2),XNEW(K,4))*XNEW(K,4)**4.+QQ)
     1/ALFM(XNEW(K,2),XNEW(K,4))
      IF(KLUP-1) 338,338,336
  338 PRINT 1105,(XNEW(K,J),J=1,6)
  336 CONTINUE
      DO 735 I=1,NMAX
      DO 735 J=1,7
      XOLD(I,J)=XNEW(I,J)
  735 CONTINUE
      IF(KPRNT-KPRM) 1723,1723,1703
 1703 PRINT 1702,XNEW(1,1),XNEW(NP,1)
 1702 FORMAT(1H1,7X,16HPISTON POSITION=,E12.5,17H   SHOCK POSITION=,E12.5
     1//)
      KPRNT=1
```

```
      DO 790 I=1,NMAX
      XNEW(I,1)=(XNEW(I,1)-RP)/(RS-RP)
  790 CONTINUE
      PRINT 1708,IW,TAU,BO
 1708 FORMAT(7X,6HTWALL=,F8.4,6X,4HTAU=,F6.3,6X,3HBO=,F6.3,6X,7HGNU= 1 /
     1/)
      PRINT 4985,EMIS,RDIFF
 4985 FORMAT(23X,11HEMISSIVITY=,F6.3/22X,13HREFLECTIVITY=,F6.3//)
      PRINT 1705,TIME
 1705 FORMAT(20X,11HTHE TIME IS,E12.5//)
      PRINT 1710,XNEW(NP,7)
 1710 FORMAT(18X,15HSHOCK VELOCITY=,E12.5//7X,1HR,11X,1HP,9X,1HU,11X,1HT
     1,10X,2HIO,8X,1HQ/)
      PRINT 1720,((XNEW(I,J),J=1,6),I=1,NMAX)
 1720 FORMAT(1H ,2F11.5,E11.4,2F11.5,E11.4)
      PRINT 1722
 1722 FORMAT(1H1)
 1723 KPRNT=KPRNT+1
      MTIME=2
      IS=NP
      IMAX=NMAX
      IF(KPNCH-KPNM) 1724,1724,1726
 1726 PUNCH 1090,IS,IMAX,((XOLD(I,J),J=1,7),I=1,IMAX)
      KPNCH=1
 1724 KPNCH=KPNCH+1
      NTOT=1
      XE=XE*AD
      IF(ABSF(TIME/AB-1.)-EPS) 9999,22,22
 9999 CONTINUE
      CALL EXIT
      END
```

```
      SUBROUTINE BDYPT(MTIME)
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      COMMON EMIS,TW,AA,AB,AC,AD,RS,KISS,MWITH,KLUP,RDIFF
      IF(NTOT-1) 100,100,110
  110 PAT1=PAT(XNEW(1,2),XNEW(1,4))
      RUMA1=RUMA(XNEW(1,1),XNEW(1,2),XNEW(1,3),XNEW(1,4),1,1)
      RPU1=RPU(XNEW(1,1),XNEW(1,2),XNEW(1,3),XNEW(1,4),1,1)
      UMA1=UMA(XNEW(1,3),XNEW(1,4))
  100 DO 5 I=1,IMAX
      DO 5 J=1,7
      WORK(I,J)=XOLD(I,J)
    5 CONTINUE
      ISW=IS
      IMAXW=IMAX
      IF(NTOT-1) 6,6,7
    6 PROP(1,2)=XOLD(1,2)
      PROP(1,4)=XOLD(1,4)
      GO TO 8
    7 PROP(1,2)=XNEW(1,2)
      PROP(1,4)=XNEW(1,4)
    8 PROP(1,1)=RP
      PROP(1,3)=UP
      CALL SEARCH(3,NTOT,UMA1,UPA1,U1,UMA3,UPA4,U2)
      IF(KLUP-1) 21,21,22
   21 PRINT 5000,(PROP(3,J),J=1,6)
 5000 FORMAT(2H ,6E12.5)
   22 PAT3=PAT(PROP(3,2),PROP(3,4))
      UMA3=UMA(PROP(3,3),PROP(3,4))
      IF(MTIME-1) 15,15,25
   15 RUMA3=0.
      RPU2=0.
      GO TO 29
   25 RUMA3=RUMA(PROP(3,1),PROP(3,2),PROP(3,3),PROP(3,4),3,1)
      RPU2=RPU(XOLD(1,1),XOLD(1,2),XOLD(1,3),XOLD(1,4),2,1)
   29 IF(NTOT-1) 30,30,40
   30 PAT1=PAT3
      RUMA1=RUMA3
      RPU1=RPU2
   40 PAT13=.5*(PAT1+PAT3)
      RUM13=.5*(RUMA1+RUMA3)
      RPU12=.5*(RPU1+RPU2)
      PROP(1,2)=PROP(3,2)+PAT13*(UP-PROP(3,3))+RUM13*DELT
      POWER=RFJ12*DELT*GAM
      PP=PROP(1,2)/XOLD(1,2)
      IF(KLUP-1) 42,42,44
   42 PRINT 4000,PP
 4000 FORMAT(1H ,3HPP=,E12.5)
   44 P1P2=PP**GAM
      PROP(1,4)=XOLD(1,4)*EXPF(-POWER)*P1P2
      RETURN
      END
```

```
      SUBROUTINE FLDPT(MTIME,I3)
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      DIMENSION UMA3(100),UPA4(100),U2(100)
      COMMON YR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      COMMON EMIS,TW,AA,AB,AC,AD,RS,KISS,MWITH,KLUP,RDIFF
      IF(NTOT-1) 1,1,200
  200 PAT1=PAT(XNEW(I3,2),XNEW(I3,4))
      PT1=PAT1
      RUMA1=RUMA(XNEW(I3,1),XNEW(I3,2),XNEW(I3,3),XNEW(I3,4),1,I3)
      RUPA1=RUMA1
      RPU1=RPU(XNEW(I3,1),XNEW(I3,2),XNEW(I3,3),XNEW(I3,4),1,I3)
      UMA1=UMA(XNEW(I3,3),XNEW(I3,4))
      UPA1=UPA(XNEW(I3,3),XNEW(I3,4))
      U1=XNEW(I3,3)
    1 DO 12 I=1,IMAX
      DO 12 J=1,7
      WORK(I,J)=XOLD(I,J)
   12 CONTINUE
      ISW=IS
      IMAXW=IMAX
      CALL SEARCH(3,NTOT,UMA1,UPA1,U1,UMA3(I3),UPA4(I3),U2(I3))
      CALL SEARCH(2,NTOT,UMA1,UPA1,U1,UMA3(I3),UPA4(I3),U2(I3))
      CALL SEARCH(4,NTOT,UMA1,UPA1,U1,UMA3(I3),UPA4(I3),U2(I3))
      PAT3=PAT(PROP(3,2),PROP(3,4))
      PAT4=PAT(PROP(4,2),PROP(4,4))
      UMA3(I3)=UMA(PROP(3,3),PROP(3,4))
      UPA4(I3)=UPA(PROP(4,3),PROP(4,4))
      U2(I3)=PROP(2,3)
      IF(MTIME-1) 15,15,25
   15 RUPA4=0.
      RUMA3=0.
      RPU2=0.
      GO TO 16
   25 RUPA4=RUMA(PROP(4,1),PROP(4,2),PROP(4,3),PROP(4,4),4,I3)
      RUMA3=RUMA(PROP(3,1),PROP(3,2),PROP(3,3),PROP(3,4),3,I3)
      RPU2=RPU(PROP(2,1),PROP(2,2),PROP(2,3),PROP(2,4),2,I3)
   16 IF(NTOT-1) 80,80,90
   80 PAT1=PAT4
      PT1=PAT3
      RUPA1=RUPA4
      RUMA1=RUMA3
      RPU1=RPU2
   90 PAT14=.5*(PAT1+PAT4)
      PAT13=.5*(PT1+PAT3)
      RUP14=.5*(RUPA1+RUPA4)
      RUM13=.5*(RUMA1+RUMA3)
      RPU12=.5*(RPU1+RPU2)
      PROP(1,3)=(PROP(4,3)*PAT14+PROP(3,3)*PAT13+
     1(RUP14-RUM13)*DELT+PROP(4,2)-PROP(3,2))/(PAT13+PAT14)
      PROP(1,2)=PROP(3,2)+PAT13*(PROP(1,3)-PROP(3,3))+RUM13*DELT
      PROP(1,4)=PROP(2,4)*EXPF(-RPU12*DELT*GAM1*(PROP(1,2)/PROP(2,2))
     1**GAM
      RETURN
      END
```

```
      SUBROUTINE SHKPT(RSH,MTIME)
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      DIMENSION UMA3(4),UPA4(4),U2(4)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      COMMON EMIS,TW,AA,AB,AC,AD,RS,KISS,MWITH,KLUP,RDIFF
      IF(NTOT-1) 1,1,200
  200 PAT1=PAT(XNEW(NP+1,2),XNEW(NP+1,4))
      PAT7=PAT(XNEW(NP,1),XNEW(NP,4))
      PT1=PAT1
      RUPA7=RUMA(XNEW(NP,1),XNEW(NP,2),XNEW(NP,3),XNEW(NP,4),1,NP)
      RUMA1=RUMA(XNEW(NP+1,1),XNEW(NP+1,2),XNEW(NP+1,3),XNEW(NP+1,4),
     11,NP+1)
      RUPA1=RUMA1
      RPU1=RPU(XNEW(NP+1,1),XNEW(NP+1,2),XNEW(NP+1,3),XNEW(NP+1,4),1,
     1NP+1)
      UPA1=UPA(XNEW(NP+1,3),XNEW(NP+1,4))
      UMA1=UMA(XNEW(NP+1,3),XNEW(NP+1,4))
      U1=XNEW(NP+1,3)
      UPA7=UPA(XNEW(NP,3),XNEW(NP,4))
      DO 20 J=1,7
      PROP(1,J)=XNEW(NP+1,J)
      PROP(7,J)=XNEW(NP,J)
   20 CONTINUE
      GO TO 40
    1 DO 10 J=2,7
      PROP(1,J)=XOLD(IS+1,J)
      PROP(7,J)=XOLD(IS,J)
   10 CONTINUE
   40 IF(MWITH-1) 60,60,62
   62 PROP(1,2)=1.
      PROP(1,3)=0.
      PROP(1,4)=1.
   60 PROP(1,1)=RSH
      PROP(7,1)=RSH
      DO 61 I=1,IMAX
      DO 61 J=1,7
      WORK(I,J)=XOLD(I,J)
   61 CONTINUE
      ISW=IS
      IMAXW=IMAX
      CALL SEARCH(3,NTOT,UMA1,UPA1,U1,UMA3(1),UPA4(1),U2(1))
      CALL SEARCH(2,NTOT,UMA1,UPA1,U1,UMA3(1),UPA4(1),U2(1))
      CALL SEARCH(4,NTOT,UMA1,UPA1,U1,UMA3(1),UPA4(1),U2(1))
      CALL SEARCH(6,NTOT,UMA1,UPA7,U1,UMA3(1),UPA4(2),U2(1))
      PAT4=PAT(PROP(4,2),PROP(4,4))
      PAT3=PAT(PROP(3,2),PROP(3,4))
      UMA3(1)=UMA(PROP(3,3),PROP(3,4))
      UPA4(1)=UPA(PROP(4,3),PROF(4,4))
      U2(1)=PROP(2,3)
      UPA4(2)=UPA(PROP(6,3),PROP(6,4))
      IF(MTIME-1) 102,102,104
  102 RUPA4=0.
      RUPA6=0.
      RUMA3=0.
      RPU2=0.
      GO TO 106
  104 RUPA4=RUMA(PROP(4,1),PROP(4,2),PROP(4,3),PROP(4,4),4,NP)
      RUPA6=RUMA(PROP(6,1),PROP(5,2),PROP(6,3),PPOP(6,4),6,NP)
      RUMA3=RUMA(PROP(3,1),PROP(3,2),PROP(3,3),PROP(3,4),3,NP)
      RPU2=RPU(PROP(2,1),PROP(2,2),PROP(2,3),PROP(2,4),2,NP)
```

```
106 IF(NTOT-1) 120,120,130
120 PAT1=PAT4
    PT1=PAT3
    RUPA1=RUPA4
    RUMA1=RUMA3
    RPU1=RPU2
130 PAT14=.5*(PAT1+PAT4)
    PAT13=.5*(PT1+PAT3)
    RUP14=.5*(RUPA1+RUPA4)
    RUM13=.5*(RUMA1+RUMA3)
    RPU12=.5*(RPU1+RPU2)
    IF(MWITH-1) 132,132,134
132 PROP(1,3)=(PROP(4,3)*PAT14+PROP(3,3)*PAT13+(RUP14-RUM13)*DELT+
   1PROP(4,2)-PROP(3,2))/(PAT13+PAT14)
    PROP(1,2)=PROP(3,2)+PAT13*(PROP(1,3)-PROP(3,3))+RUM13*DELT
    PROP(1,4)=PROP(2,4)*EXPF(-RPU12*DELT*GAM)*(PROP(1,2)/PROP(2,2))
   1**GAM
    GO TO 138
134 PROP(1,3)=0.
    PROP(1,2)=1.
    PROP(1,4)=1.
138 IF(NTOT-1) 140,140,150
140 PAT7=PAT6
    RUPA7=RUPA6
150 PAT67=.5*(PAT6+PAT7)
    RUP67=.5*(RUPA6+RUPA7)
    P1=PROP(1,2)
    U1=PROP(1,3)
    T1=PROP(1,4)
    P6=PROP(6,2)
    U6=PROP(6,3)
    NTIME=1
    IF(NTOT-1) 152,152,160
152 U7=XOLD(IS,3)
160 AS=(1.-GAM)*(U7-U1)
    BS=((1.-.5*GAM)*U7*U7-GAM*U1*U7+(1.5*GAM-1.)*U1*U1)/AS
    CS=(U7*(T1+U1*U1+(.5*GAM-1.)*U1*U7)-U1*(T1+.5*GAM*U1*U1))/AS
    SS=SQRTF(BS*BS+4.*CS)
    US=.5*(BS+SS)
    FUN=U7*(PAT67-P1*(U1-US)/T1)-(P6+PAT67*U6+RUP67*DELT)+P1*(T1+U1*U1
   1-U1*US)/T1
    IF(ABSF(FUN)-EPS) 190,190,180
180 BSU=((2.-GAM)*U7-GAM*U1-(1.-GAM)*BS)/AS
    CSU=(T1+U1*U1+(GAM-2.)*U1*U7-(1.-GAM)*CS)/AS
    USU7=.5*((BS/SS+1.)*BSU+2.*CSU/SS)
    FUNP=PAT67+P1*(US-U1)/T1+P1*(U7-U1)*USU7/T1
    U7=U7-FUN/FUNP
    NTIME=NTIME+1
    IF(NTIME-IBMAX) 160,160,170
170 PRINT 1012
1012 FORMAT(24H CANNOT FIND SHOCK SPEED)
    PRINT 1011,NTIME,U7
1011 FORMAT(1H 6HAFTER ,I3,18H ITERATIONS, U7 IS,E12.5)
    ISTOP=7
    RETURN
190 PROP(7,7)=US
    IF(KLUP-1) 192,192,194
192 PRINT 9875,US
9875 FORMAT(1H ,3HUS=,E12.5)
194 PROP(7,3)=U7
    PROP(7,2)=P1*(T1+U1*U1-U1*US-(U1-US)*U7)/T1
    PROP(7,4)=T1*PROP(7,2)*(U7-US)/((U1-US)*P1)
    PROP(1,7)=PROP(7,7)
    RSH=.5*(XOLD(IS,7)+US)*DELT+XOLD(IS,1)
    RETURN
    END
```

```
      FUNCTION RAD(R,P,T,K,I1)
C
C     DEL-DOT THE HEAT FLUX AS A RESULT OF HAVING SOLVED THE FULL
C     PLANAR TRANSFER EQUATION.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),GR(3),PROP(7,7)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      COMMON EMIS,TW,AA,AB,AC,AD,RS,KISS,MWITH,KLUP,RDIFF
      COMMON MWTH2
      CALL ALOAD(K,I1)
      ARG=TAU*ATAU(R)
      IF(KLUP-1) 2,2,4
    2 PRINT 2020,ARG
 2020 FORMAT(6H RARG=,E12.5)
    4 FB1=0.
      FBB1=0.
      FB2=0.
      FBB2=0.
      G1=0.
      G1A=0.
      G2=0.
      G2A=0.
      DO 10 I=1,NW
      PR=0.5*(RSINT-RP)*XW(I)+0.5*(RSINT+RP)
      ARG1=TAU*ATAU(PR)
      XR=PR
      CALL XNTRP
      RAH=TAU*ALFP(VAR(2),VAR(4))*VAR(4)**4.
      FB1=RAH*E2(ARG1)*W(I)+FB1
      FBB1=RAH*E1(ARG+ARG1)*W(I)+FBB1
   10 CONTINUE
      GLUB=.5*(RSINT-RP)
      FB1=FB1*GLUB
      FBB1=FBB1*GLUB
      IF(ABSF((RMAX-RSINT)/RMAX)-.1) 151,151,15
   15 DO 20 I=1,NW
      PR=.5*(RMAX-RSINT)*XW(I)+.5*(RMAX+RSINT)
      ARG1=TAU*ATAU(PR)
      XR=PR
      CALL XNTRP
      RAH=TAU*ALFP(VAR(2),VAR(4))*VAR(4)**4.
      FB2=RAH*E2(ARG1)*W(I)+FB2
      FBB2=RAH*E1(ARG+ARG1)*W(I)+FBB2
   20 CONTINUE
      FLUB=.5*(RMAX-RSINT)
      FB2=FB2*FLUB
      FBB2=FBB2*FLUB
      GO TO 152
  151 FB2=0.
      FBB2=0.
  152 FBC1=EMIS*TW**4.+2.*RDIFF*(FB1+FB2)
      FBC2=(1.-RDIFF-EMIS)*(FBB1+FBB2)
      IF(KLUP-1) 153,153,154
  153 PRINT 1000,FBC1,FBC2
 1000 FORMAT(7H RFBC1=,E12.5,2X,6HRFBC2=,E12.5)
  154 IF(R-RSINT) 110,110,130
  110 IF(ABSF(R/RP-1.)-.0001) 112,112,114
  114 DO 120 I=1,NW
      PR=0.5*(R-RP)*XW(I)+0.5*(R+RP)
      ARG1=TAU*ATAU(PR)
```

```
      XR=PR
      CALL XNTRP
      RAH=TAU*ALFP(VAR(2),VAR(4))*VAR(4)**4.
      G1=RAH*E1(ARG-ARG1)*W(I)+G1
  120 CONTINUE
      GLB=.5*(R-RP)
      G1=G1*GLB
      GO TO 160
  112 G1=0.
      GO TO 160
  130 DO 140 I=1,NW
      PR=0.5*(RSINT-RP)*XW(I)+0.5*(RSINT+RP)
      ARG1=TAU*ATAU(PR)
      XR=PR
      CALL XNTRP
      RAH=TAU*ALFP(VAR(2),VAR(4))*VAR(4)**4.
      G1=RAH*E1(ARG-ARG1)*W(I)+G1
  140 CONTINUE
      GLB=.5*(RSINT-RP)
      G1=G1*GLB
      DO 150 I=1,NW
      PR=0.5*(R-RSINT)*XW(I)+0.5*(R+RSINT)
      ARG1=TAU*ATAU(PR)
      XR=PR
      CALL XNTRP
      RAH=TAU*ALFP(VAR(2),VAR(4))*VAR(4)**4.
      G1A=RAH*E1(ARG-ARG1)*W(I)+G1A
  150 CONTINUE
      GLB=.5*(R-RSINT)
      G1A=G1A*GLB
      G1=G1+G1A
  160 IF(KLUP-1) 203,203,205
  203 PRINT 1020,G1
 1020 FORMAT(5H RG1=,E12.5)
  205 IF(R-RSINT) 200,200,230
  200 DO 210 I=1,NW
      PR=0.5*(RSINT-R)*XW(I)+0.5*(RSINT+R)
      ARG1=TAU*ATAU(PR)
      XR=PR
      CALL XNTRP
      RAH=TAU*ALFP(VAR(2),VAR(4))*VAR(4)**4.
      G2=RAH*E1(ARG1-ARG)*W(I)+G2
  210 CONTINUE
      GL2=.5*(RSINT-R)
      G2=G2*GL2
      IF(ABSF((RMAX-RSINT)/RMAX)-.1) 260,260,215
  215 DO 220 I=1,NW
      PR=.5*(RMAX-RSINT)*XW(I)+.5*(RMAX+RSINT)
      ARG1=TAU*ATAU(PR)
      CALL XNTRP
      RAH=TAU*ALFP(VAR(2),VAR(4))*VAR(4)**4.
      G2A=RAH*E1(ARG1-ARG)*W(I)+G2A
  220 CONTINUE
      G22=.5*(RMAX-RSINT)
      G2A=G2A*G22
      G2=G2+G2A
      GO TO 260
  230 DO 235 I=1,NW
      PR=.5*(RMAX-R)*XW(I)+.5*(RMAX+R)
      ARG1=TAU*ATAU(PR)
      XR=PR
      CALL XNTRP
      RAH=TAU*ALFP(VAR(2),VAR(4))*VAR(4)**4.
```

```
      G2=RAH*E1(ARG1-ARG)*W(I)+G2
  235 CONTINUE
      G23=.5*(RMAX-R)
      G2=G2*G23
  260 FFLD=G1+G2
      IF(KLUP-1) 263,263,261
  263 PRINT 1030,FFLD
 1030 FORMAT(7H RFFLD=,E12.5)
  261 ARG3=TAU*ATAU(RMAX)
      IF(MWTH2-1) 270,270,280
  270 FLEFT=0.
      GO TO 290
  280 FLEFT=2.*RDIFF*E3(ARG3)*E2(ARG)+(1.-RDIFF-EMIS)*E2(ARG3+ARG)+E2(AR
     1G3-ARG)
  290 Q=2.*ALFM(P,T)*(E2(ARG)*FBC1+FBC2+FFLD+FLEFT)-4.*ALFP(P,T)*T**4.
      RAD=Q*TAU/BO
      RETURN
      END
```

```
      FUNCTION QTAU(R,P,T,I2)
C
C     HEAT FLUX
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),GR(3),PROP(7,7)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      COMMON EMIS,TW,AA,AB,AC,AD,RS,KISS,MWITH,KLUP,RDIFF
      COMMON MWTH2
      CALL ALOAD(1,I2)
      ARG=TAU*ATAU(R)
      IF(KLUP-1) 2,2,4
    2 PRINT 2020,ARG
 2020 FORMAT(6H QARG=,E12.5)
    4 FB1=0.
      FBB1=0.
      FB2=0.
      FBB2=0.
      G1=0.
      G1A=0.
      G2=0.
      G2A=0.
      DO 10 I=1,NW
      PR=0.5*(RSINT-RP)*XW(I)+0.5*(RSINT+RP)
      ARG1=TAU*ATAU(PR)
      XR=PR
      CALL XNTRP
      RAH=TAU*ALFP(VAR(2),VAR(4))*VAR(4)**4.
      FB1=RAH*E2(ARG1)*W(I)+FB1
      FBB1=RAH*E2(ARG+ARG1)*W(I)+FBB1
   10 CONTINUE
      GLUB=.5*(RSINT-RP)
      FB1=FB1*GLUB
      FBB1=FBB1*GLUB
      IF(ABSF((RMAX-RSINT)/RMAX)-.1) 151,151,15
   15 DO 20 I=1,NW
      PR=.5*(RMAX-RSINT)*XW(I)+.5*(RMAX+RSINT)
      ARG1=TAU*ATAU(PR)
      XR=PR
      CALL XNTRP
      RAH=TAU*ALFP(VAR(2),VAR(4))*VAR(4)**4.
      FB2=RAH*E2(ARG1)*W(I)+FB2
      FBB2=RAH*E2(ARG+ARG1)*W(I)+FBB2
   20 CONTINUE
      FLUB=.5*(RMAX-RSINT)
      FB2=FB2*FLUB
      FBB2=FBB2*FLUB
      GO TO 152
  151 FB2=0.
      FBB2=0.
  152 FBC1=EMIS*TW**4.+2.*RDIFF*(FB1+FB2)
      FBC2=(1.-RDIFF-EMIS)*(FBB1+FBB2)
      IF(KLUP-1) 153,153,154
  153 PRINT 1000,FBC1,FBC2
 1000 FORMAT(7H QFBC1=,E12.5,2X,6HQFBC2=,E12.5)
  154 IF(R-RSINT) 110,110,130
  110 DO 120 I=1,NW
      PR=0.5*(R-RP)*XW(I)+0.5*(R+RP)
      ARG1=TAU*ATAU(PR)
      XR=PR
      CALL XNTRP
```

```
      RAH=TAU*ALFP(VAR(2),VAR(4))*VAR(4)**4.
      G1=RAH*E2(ARG-ARG1)*W(I)+G1
  120 CONTINUE
      GLB=.5*(R-RP)
      G1=G1*GLB
      GO TO 160
  130 DO 140 I=1,NW
      PR=0.5*(RSINT-RP)*XW(I)+0.5*(RSINT+RP)
      ARG1=TAU*ATAU(PR)
      XR=PR
      CALL XNTRP
      RAH=TAU*ALFP(VAR(2),VAR(4))*VAR(4)**4.
      G1=RAH*E2(ARG-ARG1)*W(I)+G1
  140 CONTINUE
      GLB=.5*(RSINT-RP)
      G1=G1*GLB
      DO 150 I=1,NW
      PR=0.5*(R-RSINT)*XW(I)+0.5*(R+RSINT)
      ARG1=TAU*ATAU(PR)
      XR=PR
      CALL XNTRP
      RAH=TAU*ALFP(VAR(2),VAR(4))*VAR(4)**4.
      G1A=RAH*E2(ARG-ARG1)*W(I)+G1A
  150 CONTINUE
      GLB=.5*(R-RSINT)
      G1A=G1A*GLB
      G1=G1+G1A
  160 IF(KLUP-1) 203,203,205
  203 PRINT 1020,G1
 1020 FORMAT(5H QG1=,E12.5)
  205 IF(R-RSINT) 200,200,230
  200 DO 210 I=1,NW
      PR=0.5*(RSINT-R)*XW(I)+0.5*(RSINT+R)
      ARG1=TAU*ATAU(PR)
      XR=PR
      CALL XNTRP
      RAH=TAU*ALFP(VAR(2),VAR(4))*VAR(4)**4.
      G2=RAH*E2(ARG1-ARG)*W(I)+G2
  210 CONTINUE
      GL2=.5*(RSINT-R)
      G2=G2*GL2
      IF(ABSF((RMAX-RSINT)/RMAX)-.1) 260,260,215
  215 DO 220 I=1,NW
      PR=.5*(RMAX-RSINT)*XW(I)+.5*(RMAX+RSINT)
      ARG1=TAU*ATAU(PR)
      CALL XNTRP
      RAH=TAU*ALFP(VAR(2),VAR(4))*VAR(4)**4.
      G2A=RAH*E2(ARG1-ARG)*W(I)+G2A
  220 CONTINUE
      G22=.5*(RMAX-RSINT)
      G2A=G2A*G22
      G2=G2+G2A
      GO TO 260
  230 DO 235 I=1,NW
      PR=.5*(RMAX-R)*XW(I)+.5*(RMAX+R)
      ARG1=TAU*ATAU(PR)
      XR=PR
      CALL XNTRP
      RAH=TAU*ALFP(VAR(2),VAR(4))*VAR(4)**4.
      G2=RAH*E2(ARG1-ARG)*W(I)+G2
  235 CONTINUE
      G23=.5*(RMAX-R)
      G2=G2*G23
```

```
  260 FFLD=G1-G2
      IF(KLUP-1) 263,263,261
  263 PRINT 1030,FFLD
 1030 FORMAT(7H QFFLD=,E12.5)
  261 ARG3=TAU*ATAU(RMAX)
      IF(MWTH2-1) 270,270,280
  270 FLEFT=0.
      GO TO 290
  280 FLEFT=2.*RDIFF*E3(ARG3)*E3(ARG)+(1.-RDIFF-EMIS)*E3(ARG3+ARG)-E3(AR
     1G3-ARG)
  290 QTAU=2.*(FBC1*E3(ARG)+FBC2+FFLD)+FLEFT*2.
      RETURN
      END
```

```
      SUBROUTINE SEARCH(K,L,UMA1,UPA1,U1,UM3,UP4,XU2)
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK1(3),PROP(7,7),GR1(3),RK(100),GR(100)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINI,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK1,GR1
      COMMON EMIS,TW,AA,AB,AC,AD,RS,KISS
    1 IF(K-6) 2,3,3
    2 GO TO (10,20,30,10),K
   10 IF(L-1) 100,100,102
  100 S4=UPA(PROP(1,3),PROP(1,4))
      GO TO 104
  102 S4=.5*(UPA1+UP4)
  104 RK(1)=PROP(1,1)-S4*DELT
      IF(KISS-1) 42,42,14
   14 IF(RK(1)-XOLD(IS,1)) 16,16,40
   16 RK(1)=XOLD(IS,1)*(1.+EPS1)
      GO TO 40
   20 IF(L-1) 200,200,202
  200 S2=PROP(1,3)
      GO TO 204
  202 S2=.5*(U1+XU2)
  204 RK(1)=PROP(1,1)-S2*DELT
      GO TO 42
   30 IF(L-1) 300,300,302
  300 S3=UMA(PROP(1,3),PROP(1,4))
      GO TO 304
  302 S3=.5*(UMA1+UM3)
  304 RK(1)=PROP(1,1)-S3*DELT
      GO TO 42
    3 IF(L-1) 400,400,402
  400 S6=UPA(PROP(7,3),PROP(7,4))
      GO TO 404
  402 S6=.5*(UPA1+UP4)
  404 RK(1)=PROP(7,1)-S6*DELT
   42 IF(RK(1)-XOLD(1,1))43,43,40
   43 RK(1)=XOLD(1,1)*1.0001
   40 XR=RK(1)
      CALL XNTRP
      DO 50 J=1,7
      PROP(K,J)=VAR(J)
   50 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE ALOAD(K,I4)
C
C     THIS SUBROUTINE LOADS THE DATA MATRIX FROM WHICH THE INTEGRALS
C     ARE COMPUTED.
C
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),GR(3),PROP(7,7)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NMAX,PROP,RK,GR
      COMMON EMIS,TW,AA,AB,AC,AD,RS,KISS,MWITH,KLUP,RDIFF
      IF(K-1) 500,500,450
  450 DO 460 I=1,IMAX
      DO 460 J=1,7
      WORK(I,J)=XOLD(I,J)
  460 CONTINUE
      RMAX=XOLD(IMAX,1)
      RSINT=XOLD(IS,1)
      RP=XOLD(1,1)
      IMAXW=IMAX
      ISW=IS
      RETURN
  500 DO 510 I=1,NMAX
      DO 510 J=1,7
      WORK(I,J)=XNEW(I,J)
  510 CONTINUE
      RMAX=XNEW(NMAX,1)
      RSINT=XNEW(NP,1)
      RP=XNEW(1,1)
      ISW=NP
      IMAXW=NMAX
      RETURN
      END
```

```
      FUNCTION E1(X)
C
C     EXPONENTIAL INTEGRAL OF FIRST ORDER
C
      A0=-.57721566
      A1=.99999193
      A2=-.24991055
      A3=.05519968
      A4=-.00976004
      A5=.00107857
      B1=8.57332874
      B2=18.05901697
      B3=8.634760893
      B4=.2677737343
      C1=9.573322345
      C2=25.63295615
      C3=21.09965308
      C4=3.95849692
      EXXE=0.1E-29
      IF(X-EXXE) 100,100,200
  100 E1=69.0309
      RETURN
  200 IF(X-1.) 1,1,2
    1 E1=-LOGF(X)+A0+X*(A1+X*(A2+X*(A3+X*(A4+X*A5))))
      RETURN
    2 TOP=B4+X*(B3+X*(B2+X*(B1+X)))
      BOT=C4+X*(C3+X*(C2+X*(C1+X)))
      E1=EXPF(-X)*TOP/(BOT*X)
      RETURN
      END
C
C      ****************************************************************
C
      FUNCTION E2(X)
C
C     EXPONENTIAL INTEGRAL OF SECOND ORDER
C
      IF(X-.0001) 2,2,1
    1 E2=EXPF(-X)-X*E1(X)
      GO TO 3
    2 E2=1.
    3 CONTINUE
      RETURN
      END
C
C      ****************************************************************
C
      FUNCTION E3(X)
C
C     EXPONENTIAL INTEGRAL OF THIRD ORDER
C
      E3=.5*(EXPF(-X)-X*E2(X))
      RETURN
      END
```

```
      FUNCTION RUMA(AR,P,U,T,K,I)
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NAMX,PROP,RK,GR
      COMMON EMIS,TW,AA,AB,AC,AD,RS,KISS,MWITH,KLUP,RDIFF
      RUMA=RAD(AR,P,T,K,I)/(1./GAM-1.)
      RETURN
      END
C
C      ***************************************************************
C
      FUNCTION RPU(AR,P,U,T,K,I)
      DIMENSION WORK(100,7),VAR(7),W(30),XW(30),XOLD(100,7),
     1XNEW(100,7),RK(3),PROP(7,7),GR(3)
      COMMON XR,WORK,ISW,IMAXW,VAR,RSINT,RMAX,RP,XMA,XNA,A1,
     1B1,B2,GNU,TAU,G,W,XW,NW,GAM,BO,IS,IMAX,XOLD,NTOT,DELT,
     2XNEW,IBMAX,EPS,EPS1,EPS2,ISTOP,LM,UP,NP,NAMX,PROP,RK,GR
      COMMON EMIS,TW,AA,AB,AC,AD,RS,KISS,MWITH,KLUP,RDIFF
      RPU=-RAD(AR,P,T,K,I)/P
      RETURN
      END
```

CYLINDRICAL PISTON
$\gamma = 5/3$

PISTON POSITION= .37627E-01 SHOCK POSITION= .51135E-01

TWALL= 1.0000 TAU= 1.000 RO= 2.000 GNU= 2

EMISSIVITY= .250

THE TIME IS .25085E-01

SHOCK VELOCITY= .19335E 01

R	P	U	T	IO	Q
0.	3.43557	.1500E 01	1.37745	5.29897	-.1071E 00
.24875	3.38702	.1338E 01	1.41656	5.28377	.3219E 00
.50000	3.20937	.1176E 01	1.46360	5.19760	.8557E 00
.75125	2.92197	.9988E 00	1.49337	5.02537	.1467E 01
1.00000	2.55286	.8019E 00	1.49416	4.78763	.2036E 01
1.00000	1.00265	.1344E-03	1.00265	4.78763	.2036E 01
1.24875	1.00259	.1260E-03	1.00259	4.76749	.1908E 01
1.62188	1.00250	.1148E-03	1.00250	4.73953	.1742E 01
2.18157	1.00237	.1012E-03	1.00238	4.70185	.1539E 01
3.02111	1.00222	.8577E-04	1.00222	4.65282	.1307E 01
4.28042	1.00202	.6949E-04	1.00202	4.59163	.1060E 01
6.16938	1.00178	.5382E-04	1.00178	4.51878	.8182E 00
9.00283	1.00151	.3972E-04	1.00151	4.43630	.5983E 00
13.25300	1.00122	.2800E-04	1.00122	4.34782	.4130E 00
19.62824	1.00093	.2147E-04	1.00093	4.25838	.2667E 00
29.19112	1.00067	.1484E-04	1.00067	4.17409	.1579E 00
43.53543	1.00042	.8845E-05	1.00042	4.10161	.8234E-01
65.05190	1.00022	.4842E-05	1.00022	4.04688	.3457E-01
108.08483	1.00006	.1414E-05	1.00006	4.01369	.1009E-01
194.15070	1.00001	.1440E-06	1.00001	4.00139	.1028E-02
366.28242	1.00000	.1895E-08	1.00000	4.00002	.1353E-04

PISTON POSITION= .76113E-01 SHOCK POSITION= .10059E 00

TWALL= 1.0000 TAU= 1.000 BC= 2.000 GNU= 2

EMISSIVITY= .250

THE TIME IS .50742E-01

SHOCK VELOCITY= .19215E 01

R	P	U	T	IE	C
0.	3.51057	.1500E 01	1.26578	6.36336	-.1949E 00
.24726	3.46520	.1362E 01	1.33981	6.36578	.1347E 00
.50000	3.30484	.1214E 01	1.40851	6.26684	.7121E 00
.75274	3.00666	.1032E 01	1.46681	5.98187	.1560E 01
1.00000	2.51434	.7838E 00	1.48926	5.50563	.2494E 01
1.00000	1.01027	.7741E-03	1.01027	5.50563	.2494E 01
1.24726	1.00997	.7254E-03	1.00997	5.45944	.2344E 01
1.61816	1.00954	.6604E-03	1.00954	5.39522	.2148E 01
2.17450	1.00897	.5804E-03	1.00897	5.30852	.1904E 01
3.00902	1.00822	.4883E-03	1.00822	5.19574	.1620E 01
4.26079	1.00729	.3904E-03	1.00729	5.05551	.1310E 01
6.13845	1.00620	.2955E-03	1.00620	4.89019	.9998E 00
8.95494	1.00499	.2139E-03	1.00499	4.70685	.7126E 00
13.17967	1.00375	.1511E-03	1.00375	4.51755	.4674E 00
19.51677	1.00258	.9920E-04	1.00258	4.33830	.2749E 00
29.02241	1.00154	.5558E-04	1.00154	4.18663	.1376E 00
43.28088	1.00077	.2701E-04	1.00077	4.07706	.5168E-01
64.66859	1.00031	.1161E-04	1.00031	4.01575	.9084E-02
107.44400	1.00004	.1503E-05	1.00004	4.00204	.1176E-02
192.99482	1.00000	.3034E-07	1.00000	4.00004	.2374E-04

PISTON POSITION= .11275E 00 SHOCK POSITION= .14730E 00

TWALL= 1.0000 TAU= 1.000 PC= 2.000 GNU= 2

EMISSIVITY= .250

THE TIME IS .75165E-01

SHOCK VELOCITY= .19037E 01

R	P	U	T	IO	C
0.	3.54635	.1500E 01	1.22931	7.18616	-.2628E 00
.24776	3.50047	.1370E 01	1.31015	7.21152	.1570E-01
.50000	3.34246	.1226E 01	1.38105	7.11663	.5957E 00
.75224	3.02878	.1040E 01	1.44787	6.75740	.1556E 01
1.00000	2.45746	.7574E 00	1.48126	6.08102	.2714E 01
1.00000	1.02073	.1921E-02	1.02072	6.08102	.2714E 01
1.24776	1.02001	.1792E-02	1.01999	6.00627	.2549E 01
1.61940	1.01900	.1625E-02	1.01899	5.90270	.2332E 01
2.17886	1.01765	.1420E-02	1.01765	5.76365	.2061E 01
3.01304	1.01591	.1183E-02	1.01591	5.58431	.1742E 01
4.26732	1.01379	.9325E-03	1.01379	5.36428	.1394E 01
8.14874	1.01134	.6894E-03	1.01134	5.11011	.1043E 01
8.97087	1.00871	.4754E-03	1.00871	4.83687	.7180E 00
13.20407	1.00612	.3035E-03	1.00612	4.56764	.4440E 00
19.55386	1.00381	.1755E-03	1.00381	4.33005	.2358E 00
29.07855	1.00201	.8753E-04	1.00201	4.14940	.9721E-01
43.38558	1.00089	.3810E-04	1.00089	4.03977	.2258E-01
71.93964	1.00013	.5499E-05	1.00013	4.00574	.3259E-02
129.08777	1.00000	.1370E-06	1.00000	4.00014	.6120E-04

PISTON POSITION= .15427E 00 SHOCK POSITION= .19974E 00

TWALL= 1.0000 TAU= 1.000 BO= 2.000 GNU= 2

EMISSIVITY= .250

THE TIME IS .10285E 00

SHOCK VELOCITY= .18855E 01

R	P	U	T	IO	Q
0.	3.58950	.1500E 01	1.21561	7.97307	-.3277E 00
.24875	3.54351	.1377E 01	1.29495	8.03337	-.9836E-01
.50000	3.38763	.1237E 01	1.36408	7.95946	.4685E 00
.75125	3.05837	.1047E 01	1.43522	7.54164	.1515E 01
1.00000	2.39647	.7272E 00	1.47510	6.65920	.2870E 01
1.00000	1.03544	.3924E-02	1.03536	6.65920	.2870E 01
1.24875	1.03396	.3638E-02	1.03390	6.54734	.2690E 01
1.62188	1.03195	.3275E-02	1.03191	6.39342	.2452E 01
2.18157	1.02929	.2832E-02	1.02926	6.18882	.2154E 01
3.02111	1.02591	.2326E-02	1.02589	5.92871	.1804E 01
4.28042	1.02187	.1794E-02	1.02186	5.61600	.1420E 01
6.16939	1.01735	.1286E-02	1.01734	5.26493	.1035E 01
9.00283	1.01269	.8451E-03	1.01269	4.90221	.6835E 00
13.25300	1.00834	.5010E-03	1.00833	4.56402	.3942E 00
19.62825	1.00473	.2604E-03	1.00473	4.28795	.1851E 00
29.19112	1.00220	.1140E-03	1.00220	4.10091	.5888E-01
43.53544	1.00077	.4409E-04	1.00077	4.00838	.5494E-02
72.22407	1.00006	.3666E-05	1.00006	4.00070	.4567E-03
129.60132	1.00000	.3037E-07	1.00000	4.00001	.3784E-05

PISTCN PCSITICN= .22597E 00 SHOCK POSITION= .28943E OC

TWALL= 1.COOO TAU= 1.COO HO= :.COO GNU= 2

EMISSIVITY= .250

THE TIME IS .1506:E CO

SHOCK VELOCITY= .18735E 01

R	P	U	T	IO	Q
C.	3.66030	.1500E 01	..22059	9.16267	-.4258E 00
.16518	3.64110	.1429E 01	1.26770	9.27137	-.3681E 00
.26324	3.61416	.1385E 01	1.29384	9.32925	-.2696E CO
.5CCOC	3.45339	.1258E 01	1.35388	9.33334	.2308E CO
.73676	3.12152	.1079E 01	1.41613	8.94048	.1228E 01
.83482	2.93683	.9789E 00	1.45409	8.57132	.1859E 01
1.CCCCO	2.34065	.6923E 00	1.48248	7.63849	.3102E C1
1.CCCOO	1.06684	.9294E-02	1.06645	7.63849	.3102E C1
1.16518	1.06445	.8775E-02	1.06411	7.50768	.2959E 01
1.41294	1.06114	.8085E-02	1.06085	7.32483	.2764E 01
1.78459	1.05666	.7198E-02	1.05643	7.07678	.2505E C1
2.34205	1.05085	.6123E-02	1.05068	6.75357	.2180E C1
3.17826	1.04369	.4912E-02	1.04357	6.35387	.1797E 01
4.43256	1.03543	.3665E-02	1.03536	5.89C94	.1379E 01
6.31402	1.02665	.2504E-02	1.02661	5.39624	.9649E CO
9.13620	1.01816	.1540E-02	1.01814	4.91706	.5957E CO
13.36947	1.01085	.8325E-03	1.01084	4.50655	.3070E 00
19.71939	1.00541	.3824E-03	1.00541	4.20751	.1166E 00
29.24426	1.00207	.1421E-03	1.C0207	4.03562	.2338E-C1
48.29399	1.C0021	.1408E-04	1.C0021	4.00353	.2315E-02
86.39347	1.CC000	.1637E-06	1.CC000	4.00004	.2692E-C4

PISTON POSITION= .30154E 00 SHOCK POSITION= .38351E 00

TWALL= 1.0000 TAU= 1.000 RC= 2.000 GNU= 2

EMISSIVITY= .250

THE TIME IS .20103E 00

SHOCK VELOCITY= .18603E 01

R	P	U	T	IO	Q
0.	3.69888	.1500E 01	1.24097	10.12454	-.5051E 00
.16435	3.68353	.1432E 01	1.27774	10.31269	-.4987E 00
.26266	3.66113	.1391E 01	1.29899	10.42652	-.4327E 00
.50000	3.52436	.1272E 01	1.35123	10.56665	-.9792E-02
.73734	3.19995	.1099E 01	1.41282	10.23165	.9965E 00
.83565	3.00456	.9935E 00	1.45727	9.80239	.1724E 01
1.00000	2.27178	.6496E 00	1.49125	8.60051	.3238E 01
1.00000	1.10688	.1801E-01	1.10560	8.60051	.3238E 01
1.16435	1.10201	.1681E-01	1.10089	8.39184	.3072E 01
1.41088	1.09539	.1525E-01	1.09447	8.10552	.2846E 01
1.78767	1.08659	.1329E-01	1.08596	7.72682	.2550E 01
2.33536	1.07577	.1100E-01	1.07525	7.24953	.2183E 01
3.16739	1.06291	.8529E-02	1.06257	6.68365	.1758E 01
4.41543	1.04885	.6097E-02	1.04865	6.06112	.1305E 01
6.28750	1.03478	.3955E-02	1.03467	5.43513	.8698E 00
9.09561	1.02210	.2204E-02	1.02204	4.87047	.4990E 00
13.30776	1.01200	.1151E-02	1.01197	4.42643	.2269E 00
19.62599	1.00510	.4969E-03	1.00509	4.13852	.6490E-01
28.77597	1.00165	.1727E-03	1.00165	4.01259	.7986E-02
47.07592	1.00010	.1036E-04	1.00010	4.00076	.4792E-03
83.67581	1.00000	.4404E-07	1.00000	4.00000	.2036E-05

PISTON POSITION= .37527E 00 SHOCK POSITION= .47462E 00

TWALL= 1.0000 TAU= 1.000 BC= 2.000 GNU= 2

EMISSIVITY= .250

THE TIME IS .25018E 00

SHOCK VELOCITY= .18463E 01

R	P	U	T	IO	Q
0.	3.75972	.1500E 01	1.26801	10.91260	-.5701E 00
.16239	3.74683	.1435E 01	1.29502	11.19587	-.5872E 00
.26127	3.72616	.1396E 01	1.31219	11.37584	-.5442E 00
.50000	3.59847	.1283E 01	1.35756	11.68064	-.1904E 00
.73873	3.27584	.1114E 01	1.41559	11.44745	.8033E 00
.83761	3.06921	.1003E 01	1.46405	10.97035	.1617E 01
1.00000	2.19381	.6025E 00	1.49804	9.53839	.3327E 01
1.00000	1.15056	.2975E-01	1.14747	9.53339	.3327E 01
1.16239	1.14218	.2743E-01	1.13953	9.23150	.3139E 01
1.40598	1.13106	.2449E-01	1.12890	8.81975	.2884E 01
1.77135	1.11683	.2090E-01	1.11520	8.29088	.2554E 01
2.31942	1.09963	.1685E-01	1.09849	7.64834	.2152E 01
3.14152	1.08021	.1265E-01	1.07950	6.91915	.1696E 01
4.37467	1.06001	.8711E-02	1.05960	6.15582	.1223E 01
6.22440	1.04085	.5414E-02	1.04064	5.42876	.7831E 00
8.99899	1.02455	.2976E-02	1.02445	4.80998	.4232E 00
13.16088	1.01239	.1406E-02	1.01234	4.35283	.1747E 00
19.40371	1.00477	.5393E-03	1.00475	4.07261	.4694E-01
31.88937	1.00045	.5119E-04	1.00045	4.00689	.4456E-02
56.86068	1.00000	.5383E-06	1.00000	4.00007	.4686E-04

PISTON POSITION= .44655E 00 SHOCK POSITION= .56206E 00

TWALL= 1.0000 TAU= 1.000 BC= 2.000 GNU= 2

EMISSIVITY= .250

THE TIME IS .29770E 00

SHOCK VELOCITY= .18336E 01

R	P	U	T	IO	Q
0.	3.81685	.1500E 01	1.29208	11.57661	-.6249E 00
.16077	3.80683	.1438E 01	1.31351	11.96577	-.6489E 00
.26013	3.78817	.1399E 01	1.32842	12.21879	-.6212E 00
.50006	3.67108	.1292E 01	1.36910	12.70965	-.3251E 00
.73987	3.35387	.1127E 01	1.42240	12.61501	.6439E 00
.83923	3.13648	.1011E 01	1.47272	12.10075	.1535E 01
1.00000	2.1152[illegible]	.5554E 00	1.50312	10.46657	.3384E 01
1.00000	1.19549	.4404E-01	1.18940	10.46657	.3384E 01
1.16077	1.18271	.4014E-01	1.17756	10.04004	.3173E 01
1.40193	1.16604	.3525E-01	1.16194	9.48170	.2891E 01
1.76368	1.14531	.2948E-01	1.14230	8.78667	.2532E 01
2.30629	1.12111	.2317E-01	1.11910	7.97353	.2102E 01
3.12020	1.09487	.1690E-01	1.09366	7.08946	.1625E 01
4.341[illegible]8	1.06873	.1127E-01	1.06807	6.20580	.1142E 01
6.17239	1.04505	.6757E-02	1.04472	5.40357	.7053E 00
8.91936	1.02583	.3562E-02	1.02569	4.75409	.3605E 00
13.03981	1.01220	.1595E-02	1.01214	4.30089	.1335E 00
19.22049	1.00419	.5740E-03	1.00416	4.04619	.2941E-01
31.58185	1.00029	.3937E-04	1.00029	4.00317	.2017E-02
56.30457	1.00000	.2162E-06	1.00000	4.00002	.1108E-04

APPENDIX F

THE CALCULATION OF ABSORPTION COEFFICIENTS

Mayer (1964) outlines very concisely how one goes about the calculation of absorption coefficients. The absorption coefficient may be written in the form

$$\alpha_\nu = \sum_i N_i \sigma_i(\nu) \qquad \text{(F.1)}$$

where the N_i are the number densities of particles capable of absorbing photons of frequency ν in the i^{th} absorption process, the cross-section for which is $\sigma_i(\nu)$. For line (band) radiation

$$\sigma(\nu) = \frac{\pi e^2}{mc} f_{u\ell}(\nu)\, b(\nu) \qquad \text{(F.2)}$$

is related to the Einstein coefficients (f-numbers) for downward transitions and to the shape factor, $b(\nu)$, whereas for continuum radiation

$$\sigma = \int_{\nu_j}^{\infty} \left(\frac{\pi e^2}{mc}\right) \frac{df}{d\nu}\, d\nu \qquad \text{(F.3)}$$

where ν_j might correspond to the ionization potential of an atom for instance. For a medium in LTE the N_i would first have to be determined. This is a problem in quantum mechanics which requires determination of energy levels and wave functions. Unfortunately, one is ignorant of most inter-particle potentials and at best a good approximation can be found. Next the absorption lines and edges need be found, and this involves the specification of the allowed transitions among the uncertain energy levels. In this step it must be decided which terms contribute to the sum above. The third step requires calculation of the oscillator strength (or its derivative with respect to frequency for processes which involve free electrons). This requires the approximate wave functions and usually is taken to include only dipole matrix elements. Approximate wave functions are usually quite bad in the regions which are important in the overlap integrals, hence a major source of error. Last, and most difficult, the shape factor is necessary, and to find it the quantum many-body problem must be solved. When all of this has been accomplished for a given system, the absorption coefficient is known for a single combination of temperature and pressure. To determine the variations of mean coefficients, the entire process must be repeated at many temperatures and pressures. Since most gas dynamicists are not yet proficient enough in quantum mechanics to attempt such calculations, they must rely on the work of others. For more details reference may be made to Chapter V of Zeldovich and Raizer (1966) and to Hunt and Sibulkin (1966a, b, c).

APPENDIX G

THE STIFF NATURE OF THE EQUATIONS

Curtiss and Hirschfelder (1952) indicate that if an equation of the form

$$\frac{dy}{dx} = a(x,y)\left\{ y - G(x) \right\} \tag{G.1}$$

is to be solved numerically, divergence may result as $y \to G$. They prove that the implicit forward difference scheme

$$\frac{y_n - y_{n-1}}{\Delta x} = \frac{y_n - G_n}{a(x_n, y_n)} \tag{G.2}$$

or

$$y_n = \left\{ G_n - \frac{a(x_n,y_n)}{\Delta x} y_{n-1} \right\} / \left\{ 1 - \frac{a(x_n,y_n)}{\Delta x} \right\} \tag{G.3}$$

converges to the correct "equilibrium" solution regardless of the initial conditions. If $|a(x,y)/\Delta x| \ll 1$ the equation is "stiff."

If the P_1 steady equations

$$\frac{\partial q}{\partial r} + j\frac{q}{r} = \tau_e \left\{ 4\kappa_p T^4 - \alpha_a^{(0)} I_0 \right\} \tag{G.4}$$

$$\partial I_0 / \partial r = -3\tau_e \alpha_a^{(1)} \vec{q} \tag{G.5}$$

are operated upon in this manner, one finds that

$$q_n = \left\{ q_{n-1} + (4\kappa_p T^4)_n \Delta r - \alpha_{a\,n}^{(0)} \Delta r\, I_{0\,n-1} \right\} / \Delta_3 \tag{G.6}$$

$$I_{0\,n} = \left\{ \left(1 + j\frac{\Delta r}{r_n}\right) I_{0\,n-1} - 3\alpha_{a\,n}^{(1)} \Delta r \left[(4\kappa_p T^4)_n \Delta r + q_{n-1} \right] \right\} / \Delta_3 \tag{G.7}$$

$$\Delta_3 = 1 + j\frac{\Delta r}{r_n} - 3\alpha_{a\,n}^{(0)} \alpha_{a\,n}^{(1)} (\Delta r)^2 \tag{G.8}$$

On the other hand, the iterative characteristics scheme yields

$$g_n = \left\{ g_{n-1}\left[1 - \frac{j}{2}\frac{\Delta r}{r_{n-1}} + \frac{3}{4}\alpha_{a_n}^{(0)}\alpha_{a_{n-1}}^{(1)}(\Delta r)^2\right] + \Delta r\left[2(\alpha_p T^4)_n + 2(\alpha_p T^4)_{n-1} - \frac{1}{2}(\alpha_{a_{n-1}}^{(0)} + \alpha_{a_n}^{(0)})I_{0_{n-1}}\right]\right\} / \Delta_3' \tag{G.9}$$

$$I_{0_n} = \left\{ I_{0_{n-1}}\left[1 + \frac{j}{2}\frac{\Delta r}{r_{n-1}} + \frac{3}{4}\alpha_{a_n}^{(1)}\alpha_{a_{n-1}}^{(0)}(\Delta r)^2\right] - \frac{3}{2}\alpha_{a_n}^{(1)}\Delta r\left[g_{n-1}\left\{\frac{\alpha_{a_{n-1}}^{(1)}}{\alpha_{a_n}^{(1)}}\left(1 + \frac{j}{2}\frac{\Delta r}{r_n}\right) + \left(1 - \frac{j}{2}\frac{\Delta r}{r_n}\right)\right\} + 2(\alpha_p T^4)_n + (2\alpha_p T^4)_{n-1}\right]\right\} / \Delta_3' \tag{G.10}$$

$$\Delta_3' = 1 + j/2\left(\frac{\Delta r}{r_n}\right) - 3/4\, \alpha_{a_n}^{(0)}\alpha_{a_n}^{(1)}(\Delta r)^2 \tag{G.11}$$

which are identically Eq. (G.6)-(G.8) to orders $(\frac{\Delta r}{r_n})$, $(\Delta r)^2$ as equilibrium is attained.

Sherman (1967) uses the Curtiss-Hirschfelder approach in the solution of shock structure problems. Although the method prevents divergence from radiative equilibrium, it gives no insight into the correction of the boundary condition at a surface. In the terminology of Section 5.4, even though both $I_0(r_0)$ and $g(r_0)$ are known, the numerical approximation will force radiative equilibrium in the far field. This may lead to the development of "hot" or "cold" spots so that the flux distributions may be corrected. Clearly the approximation is not as suitable as the one which was used in Chap. 5 when self-consistent situations are to be examined. (Sherman's modification of the Curtiss-Hirschfelder approximation is equivalent to the use of Eq. (5.121) to lowest order in $\Delta r/r$.)

The procedure is best suited to the alleviation of the optically thick I_{0_W} sensitivity (in which case $\frac{q(x,y)}{\Delta x} \doteq 1/\tau_0 \ll 1$) or to the solution of non-self-consistent problems.

BIOGRAPHY

Although David Finkleman was born in Brooklyn, New York, he attended public schools in Washington, D. C., graduating from McKinley Technical High School in June 1958. He was employed from June 1958 until September 1963 by the David Taylor Model Basin (currently the Naval Ship Research Center), Aerodynamics Laboratory, in conjunction with the cooperative option in aerospace engineering at the Virginia Polytechnic Institute. While at DTMB, first in the Transonic Division and later in the Gasdynamics Division, he participated in the conduct and analysis of wind tunnel tests of naval aircraft and missiles. Among his publications were the DTMB memoranda:

> "Design of a High Enthalpy, Low Density, Plasma Facility", September, 1962.
>
> "Tables and Charts for the Compressible Flow of Calorically Imperfect Air", June, 1963.

the former of which was presented at the September, 1962 meeting of the Supersonic Tunnel Association and appeared as:

> "Preliminary Design Considerations for a High Enthalpy Low Density, Plasma Tunnel", AIAA Student J., Vol. 1, No. 2, December, 1963, pp. 35-45.

At VPI he was the recipient of awards from the National Defense Transportation Association and the Society of American Military Engineers.

Having been awarded an NASA Traineeship and a General Dynamics Graduate Fellowship, he entered MIT in September, 1963, and received an S.M. in Aeronautics and Astronautics in June, 1964 with thesis entitled,

> "An Experimental Investigation of Thermal Diffusion in Binary Gas Mixtures"

From June-September, 1964 he was employed by the MIT Aerophysics Laboratory and participated in the planning of wake studies. His interest in wakes was continued during the summer of 1965 when at Mithras Inc. he participated in theoretical investigations which culminated in the following.

"An Improved Transformation of the Boundary Layer Equations Applicable to the Near Wake and a Revised Method for the Calculation of Near Wake Properties", Mithras Memo A-110, July, 1965.

"Calculation of Turbulent Near Wake Length, Neck Radius, and Core Enthalpy with Base Bleed", Mithras Memo A-119, August, 1965.

These were incorporated in:

"Wind Tunnel Measurements of Turbulent Wake Cooling with Base Injection", Mithras Rpt. MC 64-85-R3, AFOSR 66-1929, August, 1966.

Since September, 1964, the author has been a doctoral candidate at MIT and since September, 1966 a research assistant at the Aerophysics Laboratory. Having taken ROTC as an undergraduate, he is currently a First Lieutenant in the U. S. Air Force and has been assigned to the Frank J. Seiler Research Laboratory, United States Air Force Academy. Mr. Finkleman is a member of the AIAA, the Society of American Military Engineers, Sigma Xi, Sigma Gamma Tau, Tau Beta Pi, and the American Association for the Advancement of Science.

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