Remote Sounding of the Upper Atmosphere by Microwave Measurements

by

William Benjamin Lenoir

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REMOTE SOUNDING OF THE UPPER ATMOSPHERE BY MICROWAVE MEASUREMENTS

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ABSTRACT

A unified theory of radiative transfer in a medium with polarization dependent parameters is developed. This theory is applied to the emission and propagation of microwaves in the upper atmosphere. The complex propagation matrix for the Zeeman-broadened, molecular oxygen resonances at 5 mm is developed. General characteristics of the brightness temperature coherency matrix spectrum are presented and discussed.

Satellite-borne microwave experiments to remotely sound the temperature profile of the upper atmosphere on a global basis are suggested. A seven frequency experiment to sound the atmosphere from 10-80 km is proposed with bandwidths ranging from 1.5 mc for the upper heights to 200 mc for the lower heights. The heights of the seven weighting function peaks are 12, 18, 27, 40, 50, 60, 73 km, with the latter two depending somewhat on magnetic latitude. The effects of different temperature structures, receiver polarization, and receiver position (latitude and longitude) are investigated.

Experimental results from a balloon-borne radiometer system are presented. A conclusion drawn from these results is that the Van-Vleck-Weisskopf collision-broadened line-shape does not adequately describe the true line-shape. Laboratory and atmospheric experiments are suggested to clarify this uncertainty.

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Chapter I  Introduction

An interesting problem is the question of the possibility of remotely sounding the height dependence of atmospheric parameters, in particular, temperature, on a global scale. This thesis is a study of the possibility of doing this with a microwave radiometer system in an earth-orbiting satellite.

There are several methods now employed to measure the height dependence of the temperature. Balloon-borne radiosondes permit the temperature to be measured to heights of about 30 km. These measurements are localized to the vicinity of the launch site. Very few such measurements have been made in the southern hemisphere.

The Meteorological Rocket Network (MRN) has, for several years, had a program which measures the temperature from 30-55 km. These measurements are even more localized than the balloon measurements, being confined to the vicinity of five launch sites.

The infra-red bands of CO$_2$ afford a means of remotely probing the temperature profile. Several satellite experiments have been conducted or are being planned. The major advantage of these would be the global coverage. Measurements would be made over the oceans, jungles, polar ice caps, and large land areas. One disadvantage is that, because of the moderately weak intensity of these bands, heights up to only 30 km are accessible.

None of the existing methods have the possibility of measuring the temperature profile to a high altitude (70-80 km) on a global basis. The microwave spectrum of the O$_2$ molecule, however, shows promise of being able to do this.
Many constituents of the atmosphere have molecular resonances in the centimeter and millimeter wavelength range. The $\text{O}_2$ molecule is a particularly interesting one of these because of its abundance ($\text{N}_2$ has no microwave spectrum) and constant mixing ratio up to about 100 km. In the vicinity of 5 mm (60 gc), the $\text{O}_2$ molecule has many resonance lines. Several of these are quite intense and should be able to be used to remotely sound the temperature profile to high altitudes. Thus, it seems possible to expect to remotely sound the temperature profile to about 80 km with a satellite-borne, swept-frequency radiometer or a multi-channel radiometer. Such an experiment in a polar orbit would permit truly global coverage.

For heights above 50 km, the terrestrial magnetic field causes the $\text{O}_2$ microwave absorption to be polarization dependent (Zeeman effect). Because of this, it will be necessary to develop a matrix theory of radiative transfer in a medium with polarization dependent parameters. No such rigorous treatment of the problem exists presently. After the theory of the propagation of partially-polarized electromagnetic waves is developed, it will be applied to the specific case at hand, i.e. $\text{O}_2$ microwave absorption above 50 km.

General features of calculated spectra can then be presented and discussed. The effects of position (latitude and longitude), polarization, and resonance line changes can then be investigated. Finally, experiments to sound the upper atmosphere can be designed. The effects of the various parameters can be investigated to see to what extent they will modify the experiment.

Experimental results presented in Appendix 3 indicate that the true line-shape may not be too well known. Until this is resolved, (by ac-
curate laboratory experiments) it will be a disadvantage to the microwave experiments. The line-shape uncertainties affect the infra-red experiments less because of the nature of them. In the infra-red experiments, the receiver band-width encompasses several lines; whereas, in the microwave experiments, the receiver band-width encompasses a small part of one line. Hence, the exact line-shape is more important to the microwave experiments.
Chapter II  Radiative Transfer in a Medium with Polarization Dependent Properties

1. Introduction

In many cases of practical interest, the emission and absorption properties of a medium depend on the polarization of the radiation. The resonant absorption of the oxygen molecule at 5mm. is an excellent example. In these cases, for a general treatment, it does not suffice to treat the radiative transfer in the framework of the scalar equation of radiative transfer. The following is the development of a general method of handling such radiation. A spatially- and angularly-incoherent TEM wave travelling in the +z direction is assumed.

2. Maxwell's Equations

Let \( \vec{E}(\vec{r}, \nu) \) and \( \vec{H}(\vec{r}, \nu) \) be the electric field and magnetic field vectors, respectively, in the frequency domain. Let \( \vec{\alpha} \) and \( \vec{\beta} \) be normalized spatial vectors (not necessarily real) which represent two opposite polarizations. Then \( \vec{E}(\vec{r}, \nu) \) and \( \vec{H}(\vec{r}, \nu) \) can be expressed in terms of their \( \vec{\alpha} \) and \( \vec{\beta} \) components.

\[
\vec{E}(\vec{r}, \nu) = E_\alpha(\vec{r}, \nu) \vec{\alpha} + E_\beta(\vec{r}, \nu) \vec{\beta} \]  \hspace{1cm} (2.1a)

\[
\vec{H}(\vec{r}, \nu) = H_\alpha(\vec{r}, \nu) \vec{\alpha} + H_\beta(\vec{r}, \nu) \vec{\beta} \]  \hspace{1cm} (2.1b)

\( \vec{\alpha} \) and \( \vec{\beta} \) form a polarization basis (i.e., they are orthonormal and span the two dimensional vector space of polarizations). Eqs. 2.1 can be written as column vectors:

\[
\vec{E}(\vec{r}, \nu) = \begin{pmatrix} E_\alpha(\vec{r}, \nu) \\ E_\beta(\vec{r}, \nu) \end{pmatrix} \]  \hspace{1cm} (2.2a)
\[
\vec{H}(\vec{r}, \gamma) = \begin{pmatrix}
H_\alpha(\vec{r}, \gamma) \\
H_\beta(\vec{r}, \gamma)
\end{pmatrix}
\] (2.2b)

where the basis $\vec{\alpha}, \vec{\beta}$ is assumed.

Allowing the medium to have polarization dependent properties the constituency relations become:

\[
\vec{B}(\vec{r}, \gamma) = \bar{\mathcal{M}}(\gamma) \vec{H}(\vec{r}, \gamma) \quad (2.3a)
\]

\[
\vec{D}(\vec{r}, \gamma) = \bar{\mathcal{E}}(\gamma) \vec{E}(\vec{r}, \gamma) \quad (2.3b)
\]

\[
\vec{J}(\vec{r}, \gamma) = \bar{\mathcal{G}}(\gamma) \vec{E}(\vec{r}, \gamma) \quad (2.3c)
\]

where the double bar under a symbol indicates the two dimensional matrix of a linear operator on vectors written in the basis $\vec{\alpha}, \vec{\beta}$. $\bar{\mathcal{M}}(\gamma), \bar{\mathcal{E}}(\gamma)$ and $\bar{\mathcal{G}}(\gamma)$ are complex matrices, in general. They can be written:

\[
\bar{\mathcal{M}}(\gamma) = \bar{\mathcal{M}}_1(\gamma) + i \bar{\mathcal{M}}_2(\gamma) \quad (2.4a)
\]

\[
\bar{\mathcal{E}}(\gamma) = \bar{\mathcal{E}}_1(\gamma) + i \bar{\mathcal{E}}_2(\gamma) \quad (2.4b)
\]

\[
\bar{\mathcal{G}}(\gamma) = \bar{\mathcal{G}}_1(\gamma) + i \bar{\mathcal{G}}_2(\gamma) \quad (2.4c)
\]

where the matrices on the right of the equal signs are all hermitian (i.e., they are equal to their transpose (super 't'), complex-conjugates (super '*')). It is interesting to note that this decomposition (unique) is similar to the real and imaginary part decomposition of scalars.

Maxwell's Equations are:

\[
\nabla \times \vec{E}(\vec{r}, \gamma) = -i 2\pi \gamma \bar{\mathcal{M}}(\gamma) \vec{H}(\vec{r}, \gamma) \quad (2.5a)
\]

\[
\nabla \times \vec{H}(\vec{r}, \gamma) = \left[ \bar{\mathcal{G}}(\gamma) + i 2\pi \gamma \bar{\mathcal{E}}(\gamma) \right] \vec{E}(\vec{r}, \gamma) \quad (2.5b)
\]
\[ \nabla \cdot \vec{E}(\vec{r}, \nu) = 0 \quad (2.5c) \]
\[ \nabla \cdot \vec{H}(\vec{r}, \nu) = 0 \quad (2.5d) \]

where 2.5c assumes no free space charge.

Of immediate interest (because this is the case for atmospheric O_2 in a magnetic field) is the case:
\[ \bar{\sigma}_z(\nu) = \bar{\varepsilon}_z(\nu) = \bar{\mu}_z(\nu) = 0 \quad (2.6a) \]
\[ \bar{\mu}_z(\nu) = \bar{\mu}_0 \frac{I}{z} \quad (2.6b) \]
\[ \bar{\varepsilon}_z(\nu) = \bar{\varepsilon}_0 \frac{I}{z} \quad (2.6c) \]

where \( \frac{I}{z} \) is the unit matrix
\[ \frac{I}{z} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.7) \]

For propagation in the \( z \) direction, the wave equations from 2.5a, 2.5b, and 2.6 are:
\[ \frac{d^2}{dz^2} \vec{E}(\vec{r}, \nu) = - (2\pi \nu)^2 \bar{\mu}_0 \bar{\varepsilon}_0 \begin{pmatrix} I + i \bar{\varepsilon}_z'(\nu) \end{pmatrix} \vec{E}(\vec{r}, \nu) \quad (2.8a) \]
\[ \frac{d^2}{dz^2} \vec{H}(\vec{r}, \nu) = - (2\pi \nu)^2 \bar{\mu}_0 \bar{\varepsilon}_0 \begin{pmatrix} I + i \bar{\varepsilon}_z'(\nu) \end{pmatrix} \vec{H}(\vec{r}, \nu) \quad (2.8b) \]

with:
\[ \bar{\varepsilon}_z'(\nu) = \frac{1}{\bar{\varepsilon}_0} \bar{\varepsilon}_z(\nu) \]

letting:
\[ \bar{G}(\nu)^2 = - (2\pi \nu)^2 \bar{\mu}_0 \bar{\varepsilon}_0 \begin{pmatrix} I + i \bar{\varepsilon}_z'(\nu) \end{pmatrix} \quad (2.9) \]

The general solution to 2.8a is:
\[ \vec{E}(\vec{r}, \nu) = e^{\bar{G}(\nu) \frac{z}{z}} \vec{E}_+ (\vec{r}_y, \nu) + e^{- \bar{G}(\nu) \frac{z}{z}} \vec{E}_- (\vec{r}_y, \nu) \quad (2.10) \]

(See Appendix 1 for the definition of the exponential of a matrix.)

It is seen that \( \bar{G}(\nu) \) is the matrix equivalent of the complex propaga-
tion constant, $\gamma$; hence, will be called the complex propagation matrix. $\vec{E}_+^r$ and $\vec{E}_-$ are arbitrary vectors in the $x$, $y$ plane.

$\mathbf{G}(\nu)$ has a decomposition as in 2.4

$$\mathbf{G}(\nu) = \mathbf{A}(\nu) + i \mathbf{B}(\nu)$$  \hspace{1cm} (2.11)

with $\mathbf{A}(\nu)$ and $\mathbf{B}(\nu)$ both hermitian. $\mathbf{A}(\nu)$ is the attenuation matrix and $\mathbf{B}(\nu)$ is the propagation matrix. This is equivalent to the statement:

$$\gamma = \alpha + i \beta.$$

The solution to 2.8b consistent with 2.10 is:

$$\vec{H}(\vec{r}, \nu) = \nu (\nu) \left[ e^{\frac{-\mathbf{C}(\nu) \vec{r}}{\mathbf{E}_{+}(\vec{r}_{xy}, \nu)}} - e^{\frac{-\mathbf{C}(\nu) \vec{r}}{\mathbf{E}_{-}(\vec{r}_{xy}, \nu)}} \right]$$  \hspace{1cm} (2.12a)

$$\nu (\nu)^2 = \frac{\varepsilon_0}{\mu_0} \left[ \frac{I}{2} + i \varepsilon_2 \nu \right]$$  \hspace{1cm} (2.12b)

Just as in the scalar case the + subscript indicates a wave travelling in the +z direction and the - subscript, one travelling in the -z direction.

$\mathbf{G}(\nu)^2$ can be easily solved for $\mathbf{G}(\nu)$, hence $\mathbf{A}(\nu)$ and $\mathbf{B}(\nu)$, if a small loss assumption can be made. The small loss assumption assumes that the fractional power lost by the radiation fields to the medium over a wavelength's distance is very small. (i.e., If $\mathbf{G}(\nu) = \mathbf{A}(\nu) + i \mathbf{B}(\nu)$, then trace $\mathbf{A}(\nu) \ll$ trace $\mathbf{B}(\nu)$ is the condition to be fulfilled.) This amounts to fulfilling the condition trace (anti-hermitian part $\mathbf{G}(\nu)^2$) $\ll$ trace (hermitian part $\mathbf{G}(\nu)^2$). Thus, the condition is trace $\varepsilon_2 \nu \ll$ trace $I = 2$, for the case at hand.

The equations to be solved for $\mathbf{A}(\nu)$ and $\mathbf{B}(\nu)$ are (from 2.9):

$$\mathbf{A}(\nu)^2 - \mathbf{B}(\nu)^2 = - (2 \pi \nu^2 \mathbf{M_0} \varepsilon_0 \mathbf{I})$$  \hspace{1cm} (2.13a)
\[
\mathbf{A}(\nu) \mathbf{B}(\nu) + \mathbf{B}(\nu) \mathbf{A}(\nu) = -(2\pi\nu)^2 \mathcal{M}_0 \varepsilon_0 \varepsilon_\nu'(\nu) \quad (2.13b)
\]

These can be solved to yield \( \mathbf{A}(\nu) \) and \( \mathbf{B}(\nu) \) for the small loss case.

\[
\mathbf{A}(\nu) = \pi \nu \sqrt{\mathcal{M}_0 \varepsilon_0} \varepsilon_\nu'(\nu) \quad (2.14a)
\]

\[
\mathbf{B}(\nu) = 2\pi \nu \sqrt{\mathcal{M}_0 \varepsilon_0} \quad (2.14b)
\]

also:

\[
\gamma_0(\nu) = \sqrt{\frac{\varepsilon_0}{\mathcal{M}_0}} \quad (2.14c)
\]

Hence, only the attenuation is polarization dependent. Since only the +z directed wave is of interest, let \( E_z = 0 \), then:

\[
\mathbf{\varepsilon}(\mathbf{r}, \nu) = \mathbf{\varepsilon}(-\varepsilon(\nu) z) \mathbf{\varepsilon}^* (\mathbf{r}, \nu) \quad (2.15a)
\]

\[
\mathbf{\varepsilon}(\mathbf{r}, \nu) = \sqrt{\frac{\varepsilon_0}{\mathcal{M}_0}} \mathbf{\varepsilon}(-\varepsilon(\nu) z) \mathbf{\varepsilon}^* (\mathbf{r}, \nu) \quad (2.15b)
\]

The small loss assumption is not necessary for 2.15a but is implied by the specific form of 2.15b.

3. The Coherency Spectrum Matrix

Let \( \mathbf{E}_T(\mathbf{r}, \nu) \) be the Fourier Transform of a sample, \( T \) in duration, of the time domain electric field. Then:

\[
\mathbf{E}_T(\mathbf{r}, \nu) = \begin{pmatrix} \mathcal{E}_u(\mathbf{r}, \nu) \\ \mathcal{E}_f(\mathbf{r}, \nu) \end{pmatrix} \quad (2.16)
\]

A "coherency spectrum matrix" can now be defined as:

\[
\overline{\mathbf{G}}(\nu) = \lim_{T \to \infty} \frac{\mathbf{E}_T(\mathbf{r}, \nu) \cdot \mathbf{E}_T(\mathbf{r}, \nu)^*}{T} \quad (2.17)
\]

in which the bar over the relation denotes ensemble average. 2.1
This $\mathcal{J}(\nu)$ is related to the coherency matrix of Wolf$^{2.2}$ but does not require a narrow band assumption. Substitution of 2.16 into 2.17 yields:

$$
\mathcal{J}(\nu) = \left( \begin{array}{cc}
\left( \lim_{T \to \infty} \frac{|E_{a,T}(\vec{r},\nu)|^2}{T} \right) & \left( \lim_{T \to \infty} \frac{E_{a,T}(\vec{r},\nu) E_{b,T}^*(\vec{r},\nu)}{T} \right) \\
\left( \lim_{T \to \infty} \frac{E_{a,T}(\vec{r},\nu) E_{b,T}(\vec{r},\nu)}{T} \right) & \left( \lim_{T \to \infty} \frac{|E_{b,T}(\vec{r},\nu)|^2}{T} \right)
\end{array} \right)
$$

Using the relation for power spectral density:

$$
\mathcal{F}_{ij}(\nu) = \left( \lim_{T \to \infty} \frac{E_{i,T}(\nu) E_{j,T}^*(\nu)}{T} \right) \quad (2.19)
$$

Eq. 2.18 becomes:

$$
\mathcal{J}(\nu) = \begin{pmatrix}
\mathcal{F}_{aa}(\nu) & \mathcal{F}_{ab}(\nu) \\
\mathcal{F}_{ba}(\nu) & \mathcal{F}_{bb}(\nu)
\end{pmatrix} \quad (2.20)
$$

which is more general since $\mathcal{F}_{ij}(\nu)$ is defined for random noise-like fields whereas $E(\vec{r},\nu)$ is not.

Alternately, a "Poynting Vector Spectrum Matrix" could have been defined:

$$
\mathcal{S}(\nu) = \left( \lim_{T \to \infty} \frac{E_T(\vec{r},\nu) \times H_T^*(\vec{r},\nu)}{T} \right) \quad (2.21a)
$$

in which each element of the above S matrix is a vector. Since $E_T$ and $H_T$ are assumed in the x, y plane only, then the only vector components of 2.21a will be the $\vec{E}_z$ terms. This z-component is related to $\mathcal{J}(\nu)$ through:

$$
S_z(\nu) = \sqrt{\frac{E_o}{M_o}} \mathcal{J}(\nu) \quad (2.21b)
$$

if the wave admittance is as in 2.14c. Otherwise, $S_z(\nu)$ is not so simply related to $\mathcal{J}(\nu)$ and $\mathcal{J}(\nu)$ is not directly the power coherency.
spectrum matrix.

Examination of 2.20, or equivalently 2.18, shows at once that \( \mathbb{J}(\nu) \) is hermitian (self-adjoint), i.e. \( \mathbb{J}(\nu) = \mathbb{J}(\nu)^{t*} \). It is also obvious that \( \mathcal{F}_{\alpha\alpha}(\nu) \) and \( \mathcal{F}_{\rho\rho}(\nu) \) are real and non-negative. \( \mathcal{F}_{\alpha\alpha}(\nu) \) and \( \mathcal{F}_{\rho\rho}(\nu) \) are the power spectral densities in polarization \( \hat{\alpha} \) and polarization \( \hat{\rho} \), respectively. Hence, the trace of \( \mathbb{J}(\nu) \)

\[
\text{tr} \mathbb{J}(\nu) = \mathcal{F}_{\alpha\alpha}(\nu) + \mathcal{F}_{\rho\rho}(\nu) \geq 0 \tag{2.22}
\]

is the total spectral density of the radiation.

The off diagonal terms, \( \mathcal{F}_{\alpha\rho}(\nu) \) and \( \mathcal{F}_{\rho\alpha}(\nu) \), measure the degree of coherence between the radiation with polarization \( \hat{\alpha} \) and that with polarization \( \hat{\rho} \). By Schwartz' inequality, it is seen that the determinant of \( \mathbb{J}(\nu) \)

\[
\text{det} \mathbb{J}(\nu) = \mathcal{F}_{\alpha\alpha}(\nu) \mathcal{F}_{\rho\rho}(\nu) - |\mathcal{F}_{\alpha\rho}(\nu)|^2 \tag{2.23}
\]

is real and non-negative.

The analysis, so far, has assumed \( \hat{\alpha}, \hat{\rho} \) to be the polarization basis. The change from one polarization basis, \( \hat{\alpha} \) and \( \hat{\rho} \), to another, \( \hat{x} \) and \( \hat{y} \), is effected through an unitary transformation, \( \mathcal{U} \).

\[
\begin{pmatrix}
E_x(\nu) \\
E_y(\nu)
\end{pmatrix} = \mathcal{U} \begin{pmatrix}
E_\alpha(\nu) \\
E_\rho(\nu)
\end{pmatrix} \tag{2.24}
\]

This changes only the polarization basis in which the radiation is described. It does not change the radiation itself in any way.

From 2.18 and 2.24, the coherency spectrum matrix is transformed to the basis \( \hat{x}, \hat{y} \) through:

\[
\mathbb{J}_{x,y}(\nu) = \mathcal{U} \mathbb{J}_{x,\rho}(\nu) \mathcal{U}^{t*} \tag{2.25}
\]
from which it is obvious that \( \mathbf{J}_{x,y}(\nu) \) is hermitian, since \( \mathbf{J}_{\alpha,\beta}(\nu) \) is.

From the properties of unitary transformations, it can be shown (see Appendix 1) that the trace, determinant, and eigen values, are invariant under a transformation such as 2.25. So that, \( \text{tr} \mathbf{J}(\nu) \), \( \det \mathbf{J}(\nu) \), \( \lambda_1(\nu) \), \( \lambda_2(\nu) \) are the trace, determinant and eigen values, respectively, of \( \mathbf{J}(\nu) \) in any polarization basis.

Then there exists an unitary transformation, \( U_\nu(\nu) \) to a polarization basis, \( \hat{m}, \hat{n} \) in which \( \mathbf{J}(\nu) \) is diagonal.

\[
\mathbf{J}_{m,n}(\nu) = U_\nu^\dagger(\nu) \mathbf{J}_{\alpha,\beta}(\nu) U_\nu(\nu) = \begin{pmatrix} \mathbf{J}_m(\nu) & 0 \\ 0 & \mathbf{J}_n(\nu) \end{pmatrix}
\]

(2.26)

with \( \mathbf{J}_m(\nu) = \lambda_1(\nu) \) and \( \mathbf{J}_n(\nu) = \lambda_2(\nu) \)

Assuming, with no loss of generality, that \( \mathbf{J}_m(\nu) \geq \mathbf{J}_n(\nu) \) then:

\[
\mathbf{J}_{m,n}(\nu) = \mathbf{J}_n(\nu) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \mathbf{J}_m(\nu) - \mathbf{J}_n(\nu) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

(2.27)

where both matrices on the right are valid coherency spectrum matrices.

The first is seen to be that of a randomly polarized (unpolarized) wave; while the second is that of a totally polarized wave with the polarization \( \hat{m} \). Such a decomposition exists uniquely (though a function of \( \nu \), in general) for all coherency spectrum matrices. The fractional polarization spectrum can now be defined as the power spectral density of the polarized part divided by the total power spectral density.

\[
\rho(\nu) = \frac{\mathbf{J}_m(\nu) - \mathbf{J}_n(\nu)}{\mathbf{J}_m(\nu) + \mathbf{J}_n(\nu)}
\]

(2.28)

\( \rho(\nu) \) is independent of the polarization basis and is given more generally by:
\[
\rho(\nu) = \sqrt{1 - \frac{4 \det \mathcal{J}(\nu)}{[\text{tr} \mathcal{J}(\nu)]^2}}
\]

(2.29)

4. The Brightness Temperature Coherency Spectrum Matrix

It is often convenient to use the concept of brightness temperature rather than intensity or spectral flux density. For the scalar description, the brightness temperature (in the microwave region where \(\nu \ll kT\)) is defined by Rayleigh-Jeans Law:

\[
\mathcal{I}(\nu) = \frac{2k\nu^2}{c^2} \mathcal{T}_B(\nu)
\]

(2.30)

where \(\mathcal{I}\) is the intensity, \(\mathcal{T}_B\) the brightness temperature, \(k\) Boltzmann's constant, and \(c\) the velocity of light.

It will also be convenient to define a brightness temperature coherency spectrum matrix in a similar fashion. Since \(\frac{2k\nu^2}{c^2}\) is constant (for a given \(\nu\)) and \(\mathcal{J}(\nu)\) is the intensity coherency spectrum matrix (except for a multiplicative constant), then define:

\[
\mathcal{T}_B(\nu) = \sqrt{\frac{\varepsilon_0}{\hbar \nu}} \frac{c^2}{2k\nu^2} \mathcal{J}(\nu)
\]

(2.31)

as the brightness temperature coherency spectrum matrix. Thus, in the polarization basis, \(\alpha, \beta\):

\[
\mathcal{T}_B(\nu) = 
\begin{pmatrix}
T_{\alpha\alpha}(\nu) & T_{\alpha\beta_r}(\nu) + iT_{\alpha\beta_\ell}(\nu) \\
T_{\alpha\beta_r}(\nu) - iT_{\alpha\beta_\ell}(\nu) & T_{\beta\beta}(\nu)
\end{pmatrix}
\]

(2.32)

where \(T_{\alpha\alpha}(\nu)\) and \(T_{\beta\beta}(\nu)\) are the brightness temperatures (in a scalar sense) of the radiation with polarization \(\alpha\) and polarization \(\beta\), respectively. The general properties of \(\mathcal{J}(\nu)\) apply also to \(\mathcal{T}_B(\nu)\).

In particular, the fraction polarization spectrum, \(\rho(\nu)\), is
unchanged.

\[ \rho(\nu) = \sqrt{1 - \frac{4 \det T_\nu(\nu)}{[\text{tr} T_\nu(\nu)]^2}} \quad (2.33) \]

5. **Matrix Equation of Radiative Transfer**

Equation 2.15a represents the general solution for a \( \nu \) travelling wave under the restrictions already stated.

\[ \overrightarrow{E}(\overrightarrow{r}, \nu) = e^{-\frac{G(\nu)z}{\nu}} \overrightarrow{E}_r(\overrightarrow{r}_{x,y}, \nu) \quad (2.15a) \]

This certainly obeys the wave equation 2.8a; however, it must obey a less general equation as well, in particular:

\[ \frac{d}{dz} \overrightarrow{E}(\overrightarrow{r}, \nu) = -\frac{G(\nu)}{\nu} \overrightarrow{E}(\overrightarrow{r}, \nu) \quad (2.34) \]

hence:

\[ \frac{d}{dz} \overrightarrow{E}(\overrightarrow{r}, \nu) = -\frac{G(\nu)}{\nu} \overrightarrow{E}(\overrightarrow{r}, \nu) \quad (2.35) \]

To find the differential equation for \( J(\nu) \), consider \( \frac{d}{dz} J(\nu) \). From 2.17

\[ \frac{d}{dz} J(\nu) = \lim_{T \to \infty} \frac{d}{dz} \left[ \overrightarrow{E}_r(\overrightarrow{r}, \nu) \cdot \overrightarrow{E}_r^{*}(\overrightarrow{r}, \nu) \right] \quad (2.36) \]

and

\[ \frac{d}{dz} J(\nu) = \lim_{T \to \infty} \left( \overrightarrow{E}_r(\overrightarrow{r}, \nu) \cdot \overrightarrow{E}_r^{*}(\overrightarrow{r}, \nu) + \overrightarrow{E}_r(\overrightarrow{r}, \nu) \cdot \frac{d}{dz} \overrightarrow{E}_r^{*}(\overrightarrow{r}, \nu) \right) \quad (2.37) \]

Taking the transpose, complex conjugate of 2.35 yields:

\[ \frac{d}{dz} \overrightarrow{E}_r^{*}(\overrightarrow{r}, \nu) = -\overrightarrow{E}_r^{*}(\overrightarrow{r}, \nu) \frac{G^{*}(\nu)}{\nu} \quad (2.38) \]
Thus, with 2.17, 2.35, and 2.38, equation 2.37 becomes:

\[
\frac{d}{dz} \mathcal{J}(\nu) + \mathcal{G}(\nu) \mathcal{J}(\nu) + \mathcal{J}(\nu) \mathcal{G}(\nu) \xi = 0
\]  

(2.39)

Equation 2.39 is the matrix equation of radiative transfer ignoring emission from the medium. The equivalent equation in the brightness temperature notation is:

\[
\frac{d}{dz} \mathcal{T}_B(\nu) + \mathcal{G}(\nu) \mathcal{T}_B(\nu) + \mathcal{T}_B(\nu) \mathcal{G}(\nu) \xi = 0
\]  

(2.40)

Consider the propagation through a slab of finite thickness, \( \Delta z \), having a complex propagation matrix \( \mathcal{G}(\nu) \). If the BTCSM (brightness temperature coherency spectrum matrix) incident on the slab is known to be \( \mathcal{T}_B(\nu) \) then 2.40 allows \( \mathcal{T}_B(\nu) \) (the emergent BTCSM) to be found while the electric field quantities remain unknown (and, in fact, will often be random variables).

![Figure 2.1](image)

If \( \vec{E}_{\text{in}}(x,y,\nu) \) were known, 2.15a would yield \( \vec{E}_{\text{out}}(x,y,\nu) \) as:

\[
\vec{E}_{\text{out}}(x,y,\nu) = e^{-\mathcal{G}(\nu)\Delta z} \vec{E}_{\text{in}}(x,y,\nu)
\]  

(2.41)

and \( \mathcal{J}_{\text{out}}(\nu) \) would be:

-20-
\[
\bar{J}_{out}(\nu) = \lim_{T \to \infty} \frac{E_{\text{out}, T}(\vec{r}, \nu) \cdot E_{\text{out}, T}(\vec{r}, \nu)}{T}
\]  

(2.42)

which, with 2.41, gives:

\[
\bar{J}_{out}(\nu) = e^{-G(\nu) \Delta z} \bar{J}_{in}(\nu) e^{-G^{*}(\nu) \Delta z}
\]  

(2.43)

or, equivalently:

\[
\bar{T}_{B_0}(\nu) = e^{-G(\nu) \Delta z} \bar{T}_{B_0}(\nu) e^{-G^{*}(\nu) \Delta z}
\]  

(2.44)

It is easily verified that this is indeed the solution to 2.40.

Emission from the medium has been ignored, so far. To complete

the analysis, it must be included. One method of including it would

be to add a source current density, \( \vec{J}_S(\vec{r}, \nu) \), (a random variable,

not a deterministic variable) to the right hand side of 2.5b. Then

\( \vec{J}_S(\vec{r}, \nu) \) should be included in the equations for \( \bar{T}_B(\vec{r}, \nu) \) and \( \bar{E}(\vec{r}, \nu) \). Thermodynamic considerations would determine the statistics of

\( \vec{J}_S(\vec{r}, \nu) \).

It is equally as valid, and easier, to insert the emission directly

into 2.40 since the corresponding \( \vec{J}_S(\vec{r}, \nu) \) would be random anyway.

Let \( S_e(\nu) \) be the emission spectrum matrix so that 2.40 becomes:

\[
\frac{d}{d\Delta z} \bar{T}_B(\nu) + \bar{G}(\nu) \bar{T}_B(\nu) + \bar{T}_B(\nu) \bar{G}(\nu)^* = S_e(\nu)
\]  

(2.45)

\( S_e(\nu) \) must be hermitian since the left side of 2.45 is.

\( S_e(\nu) \) is found through definition of an emission temperature spec-

trum matrix, \( T_e(\nu) \). Define \( T_e(\nu) \) as the BTCSM necessary to fulfill

the condition that if the input BTCSM, \( T_{B_i}(\nu) \), is equal to \( T_e(\nu) \) then

the output BTCSM, \( T_{B_o}(\nu) \), is also equal to \( T_e(\nu) \) independent of \( \Delta z \).
(See Fig. 2.1 for geometry.) (Note that $T_e(\nu)$ is a property of the medium, not the radiation.)

This definition of $T_e(\nu)$ requires:

$$S_e(\nu) = G(\nu) T_e(\nu) + T_e(\nu) G^t(\nu)$$  \hspace{1cm} (2.46)

so that the emission spectrum matrix depends on the emission temperature spectrum matrix and the complex propagation matrix of the medium. This is the matrix equivalent to the scalar relation$^{2.4}$

$$j = \alpha \ T_e$$ \hspace{1cm} (2.47)

in which $j$ is the emission coefficient, $\alpha$, the power absorption coefficient, and $T_e$, the scalar emission temperature.

Equations 2.45 and 2.46 are combined to give the matrix equation of radiative transfer:

$$\frac{d}{dz} T_{B_0}(\nu) + G(\nu) T_{B_0}(\nu) + T_{B_0}(\nu) G^t(\nu)$$

$$= G(\nu) T_e(\nu) + T_e(\nu) G^t(\nu)$$  \hspace{1cm} (2.48)

The solution to 2.48 appropriate to Figure 2.1 with $T_e(\nu)$ as an added property of the medium is:

$$T_{B_0}(\nu) = e^{-G(\nu)\Delta z} T_{B_0}(\nu) e^{-G^t(\nu)\Delta z} + [T_e(\nu) - e^{-G(\nu)\Delta z} T_e(\nu) e^{-G^t(\nu)\Delta z}]$$ \hspace{1cm} (2.49)

The first term on the right represents the part of $T_{B_0}(\nu)$ due to the $T_{B_i}(\nu)$ that is incident on the slab. The second term (in brackets) represents the emission in the slab and its subsequent propagation through the remainder of the slab.

If the medium is in local thermodynamic equilibrium, then:

$$T_e(\nu) = \frac{T_K}{T_e} = t_K I$$  \hspace{1cm} (2.50)

where $T_K$ is the kinetic temperature matrix, $t_K$ is the scalar kinetic temperature,
and the unit matrix.

If local thermodynamic equilibrium obtains, then 2.48 becomes:

\[ \frac{d}{dz} T_b(\nu) + \frac{G(\nu) T_b(\nu) + G(\nu) G^t(\nu)}{G^t(\nu)} = t_K \left[ \frac{G(\nu) + G(\nu) G^t(\nu)}{G^t(\nu)} \right] \]  
(2.51)

which, using 2.11 becomes:

\[ \frac{d}{dz} T_b(\nu) + A(\nu) T_b(\nu) + T_b(\nu) A(\nu) = 2t_K A(\nu) \]  
(2.52a)

If small loss obtains, then the special form of \( \) allows 2.52a to be written:

\[ \frac{d}{dz} T_b(\nu) + A(\nu) T_b(\nu) + T_b(\nu) A(\nu) = 2t_K A(\nu) \]  
(2.52b)

The solutions to 2.52a and 2.52b appropriate to figure 2.1 with the slab at a temperature of \( t_K \) are:

\[ T_{b_0}(\nu) = e^{-\frac{E(\nu) \Delta \nu}{T_{b_0}(\nu)}} \frac{T_{b_0}(\nu)}{T_{b_0}(\nu)} e^{-\frac{E(\nu) \Delta \nu}{T_{b_0}(\nu)}} + t_K \left[ \frac{1}{T_{b_0}(\nu)} - e^{-\frac{E(\nu) \Delta \nu}{T_{b_0}(\nu)}} \right] \]  
(2.53a)

\[ T_{b_0}(\nu) = e^{-\frac{A(\nu) \Delta \nu}{T_{b_0}(\nu)}} T_{b_0}(\nu) e^{-\frac{A(\nu) \Delta \nu}{T_{b_0}(\nu)}} + t_K \left[ \frac{1}{T_{b_0}(\nu)} - e^{-\frac{A(\nu) \Delta \nu}{T_{b_0}(\nu)}} \right] \]  
(2.53b)

6. Finite Band-Width Considerations

Equation 2.49 integrated over a finite frequency band will yield the brightness temperature coherency matrix for that center frequency, \( \nu_c \), and that band-width, \( \Delta \nu \).

\[ \nu_c + \frac{\Delta \nu}{2} \]

\[ T_{b_0}(\nu_c, \Delta \nu) = \frac{1}{\Delta \nu} \int \frac{T_{b_0}(\nu)}{\nu_c - \Delta \nu} d\nu \]  
(2.54)

This \( T_{b_0}(\nu_c, \Delta \nu) \) would describe the radiation appropriate to equation 2.49 after it had been passed through an appropriate band pass filter.
The fractional polarization of $T_{B_0}(\nu, \Delta \nu)$ is given by:

$$
\rho(\nu_c, \Delta \nu) = \sqrt{1 - \frac{\int \text{det} \frac{\partial T_{B_0}(\nu_c, \Delta \nu)}{\partial T_{B_0}(\nu_c, \Delta \nu)} \, d\nu}{[\text{Tr} \frac{\partial T_{B_0}(\nu_c, \Delta \nu)}{\partial T_{B_0}(\nu_c, \Delta \nu)}]^2}}
$$

(2.55)

Note that this is not equal to $\int \rho \, d\nu$

In general, equation 2.54 represents quite a formidable integration, but is necessary when $T_{B_0}(\nu)$ varies with $\nu$ over the interval of integration. Special cases do exist in which equation 2.54 can be greatly simplified. Physically, these cases are of large importance as they encompass the narrow band cases. (Unfortunately, these do not include the particular case of interest described in Chapters IV and V.)

If $g(\nu)$ exhibits essentially no $\nu$ dependence over the interval $(\nu_c - \frac{\Delta \nu}{2}, \nu_c + \frac{\Delta \nu}{2})$ then 2.54 assumes the same form as 2.49 with all temperature coherency spectrum matrices replaced by their integrals over the band-width in question. That is:

$$
\frac{1}{\Delta \nu} \int_{\nu_c - \frac{\Delta \nu}{2}}^{\nu_c + \frac{\Delta \nu}{2}} T_{B}(\nu) \, d\nu
$$

(2.56)

For cases in which $T_{B_0}(\nu)$ and $T_e(\nu)$ are essentially independent of $\nu$ over the band-width as well, this reduces further to:

$$
T_{B}(\nu) \rightarrow T_B(\nu_c)
$$

(2.57)

7. Assumptions and Restrictions in Retrospect

1. Spatial and angular incoherence was assumed.

2. $g(\nu)$ was independent of $z$. 

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This could be relaxed to permit slight \( z \) dependence. The governing rule would be that the \( z \) dependence \( g(v) \) be small enough so that:

\[
\frac{d}{d\tilde{z}} e^{-\frac{G(v)}{\tilde{z}}} \approx -\frac{G(v)}{\tilde{z}} e^{-\frac{G(v)}{\tilde{z}}}
\]

otherwise, the relations derived will not hold.

3. Small loss was implied.

In particular:

\[
B(v) \approx 2\pi v \sqrt{\mu_0 \varepsilon_0} \frac{I}{\tilde{z}}
\]

and

\[
\gamma_0(v) \approx \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{I}{\tilde{z}}
\]

The former relation is not critical; however, the latter is quite important because, if it does not hold (at least as \( \gamma_0(v) = (\text{real constant}) \frac{I}{\tilde{z}} \)) then 2.21b would not be satisfied and \( tr \gamma(v) \) would not be the intensity of radiation. In this case, a simple plane wave solution to Maxwell's Equations will probably not exist.
Chapter III  The Complex Propagation Matrix for Molecular Oxygen Absorption Near 5μm.

1. Introduction

The $O_2$ molecule in the ground state has no electric dipole moment, but does have a magnetic dipole moment resulting from the unpaired spins of two electrons. This magnetic moment permits microwave transitions between the fine-structure levels of the molecular rotational states.\textsuperscript{3,1}

The electron spin quantum number, $S$, is equal to 1 for the $O_2$ molecule. The total angular momentum quantum number, $J$, is given by:

$$J = N + 1$$
$$J = N$$

or:

$$J = N - 1$$

where $N$ is the rotational quantum number. $N$ must be odd because of the exclusion principle. The selection rules permit the transitions:

$$J = N \longrightarrow J = N + 1$$
called the $N^+$ transition, and:

$$J = N \longrightarrow J = N - 1$$
called the $N^-$ transition.

If there is no external magnetic field, then the radiation will be isotropic and unpolarized. However, the introduction of an external magnetic field will greatly complicate the picture, introducing radiation that is neither isotropic nor unpolarized.

2. Zeeman Splitting

The application of an external magnetic field will cause a splitting of the transition lines. This arises because the magnetic moment
associated with the $O_2$ molecule couples with the external field to split
the energy associated with a given $J$ into $2J + 1$ levels corresponding to
$M = -J, \ldots, 0, \ldots J$, where $M$ is the quantum number associated with the
projection of the $O_2$ magnetic moment along the direction of the external
field. This energy perturbation is given by: $^3_1$

$$\Delta W = -1.001 M \mu H \frac{J(J+1) + S(S+1) - N(N+1)}{J(J+1)}$$  \hspace{1cm} (3.2)

in which:

$\mu$ = Bohr magneton

$H$ = external field strength

$S = 1$

$\Delta W$ = change in energy of level $J, N, M$.

The selection rules permit transitions of $^4_1$ in $J$ to be accompanied
by changes in $M$ of:

$$\Delta M = 0, ^1_1$$  \hspace{1cm} (3.4)

Thus, there are six types of transitions of interest:

$\Delta J = ^4_1$, each of which may have $\Delta M = 0, ^1_1$

The $\Delta M = 0$ components are commonly called $\pi$ components; while the
$\Delta M = ^1_1$ components are called $\sigma$ components. The transition frequencies
for the case of no external magnetic field are well known. $^3_2$

(See appendix 4.) The change in this frequency, $\Delta \nu$, caused by an ex-
ternal magnetic field is given in table 1.

**Table 1.** Frequency Change, $\Delta \nu$

- $\Delta M = M_{\text{final}} - M_{\text{initial}}$
- $K = 2.8026$ for $\Delta \nu$ in m/s
\[
\begin{array}{c|c|c}
\Delta M = +1 & \frac{\mathcal{K} h}{N+1} \left( 1 + M \frac{N-1}{N} \right) & -\frac{\mathcal{K} h}{N} \left( 1 + M \frac{N+2}{N+1} \right) \\
\Delta M = 0 & \frac{\mathcal{K} h}{N+1} M \frac{N-1}{N} & -\frac{\mathcal{K} h}{N} M \frac{N+2}{N+1} \\
\Delta M = -1 & \frac{\mathcal{K} h}{N+1} \left( -1 + M \frac{N-1}{N} \right) & -\frac{\mathcal{K} h}{N} \left( -1 + M \frac{N+2}{N+1} \right) \\
\end{array}
\]

3. Polarization and Matrix Elements

The radiation from the emitting processes is that of a magnetic dipole with appropriate dipole moment. These dipole moments are found from the quantum mechanic matrix elements between the two states of interest. The appropriate transitions are denoted by:

\[
(J, M) \rightarrow (J', M')
\]

where:

\[
\Delta M = M' - M = 0, \pm 1
\]

\[
\Delta J = J' - J = \pm 1
\]

\[
J = N
\]

as noted earlier.

The matrix elements, \( \mathcal{M}(N, M, \Delta J, \Delta M) \) of such transitions are given by: 3.3

\[
\begin{align*}
\mathcal{M}(N, M, +1, \pm 1) &= i \mathcal{C}_+ (N) \frac{1}{2} \sqrt{(N\pm M+1)(N\pm M+2)} \left( \hat{l}_x \pm i \hat{l}_y \right) \\
\mathcal{M}(N, M, +1, 0) &= \mathcal{C}_+ (N) \sqrt{(N+1)^2 - M^2} \hat{l}_z \\
\mathcal{M}(N, M, -1, \pm 1) &= \pm \mathcal{C}_- (N) \frac{1}{2} \sqrt{(N\mp M)(N\mp M-1)} \left( \hat{l}_x \pm i \hat{l}_y \right) \\
\mathcal{M}(N, M, -1, 0) &= \mathcal{C}_- (N) \sqrt{N^2 - M^2} \hat{l}_z
\end{align*}
\]
in which $\vec{l}_x$, $\vec{l}_y$, $\vec{l}_z$, are real unit vectors in the $x$, $y$, and $z$ directions, respectively, with $z$ being the direction of the perturbing magnetic field, and $C_{ul}(N)$ is independent of $M$.

The radiation fields for a radiator having a magnetic dipole moment as in 3.3a or 3.3c are:

\[
\vec{E}_{z}^{r}(\nu) = C \frac{M_{z}}{\sqrt{\varepsilon}} \frac{e^{i\beta R}}{R} \left[ i \vec{l}_y - i \cos \psi \vec{l}_\phi \right]
\]

\[
\vec{H}_{z}^{r}(\nu) = C \frac{M_{z}}{\sqrt{\varepsilon}} \frac{e^{i\beta R}}{R} \left[ i \cos \psi \vec{l}_y + \vec{l}_\phi \right]
\]

with $R$, $\psi$, $\phi$ being spherical coordinates as in Fig. 3.1 and $C = \frac{j}{4\pi} \left( \frac{2\pi}{c} \right)^{2}$

\[
\text{Figure 3.1}
\]

Similarly, for 3.3b or 3.3d it follows that:

\[
\vec{E}_{0}^{r}(\nu) = i C \frac{M_{z}}{\sqrt{\varepsilon}} \frac{\sin \psi e^{-i\beta R}}{R} \vec{l}_\phi
\]

\[
\vec{H}_{0}^{r}(\nu) = -i C \frac{M_{z}}{\sqrt{\varepsilon}} \frac{\sin \psi e^{-i\beta R}}{R} \vec{l}_y
\]
The Coherency Matrices in the basis \( T_1, T_2 \) associated with 3.4a and 3.5a are:

\[
J_{\pm} = \frac{|c|^2 |M_{\pm}|^2}{R^2} \begin{pmatrix} 1 & \mp i \cos \psi \\ \mp i \cos \psi & \cos^2 \psi \end{pmatrix} \quad (3.6a)
\]

\[
J_0 = \frac{|c|^2 |M_0|^2}{R^2} \begin{pmatrix} 0 & 0 \\ 0 & \sin^2 \psi \end{pmatrix} \quad (3.6b)
\]

Let \( \rho_{\pm} \) and \( \rho_0 \) be the matrix parts of 3.6a and 3.6b with the angular dependence.

\[
\rho_{\pm} = \begin{pmatrix} 1 & \mp i \cos \psi \\ \mp i \cos \psi & \cos^2 \psi \end{pmatrix} \quad (3.7a)
\]

\[
\rho_0 = \begin{pmatrix} 0 & 0 \\ 0 & \sin^2 \psi \end{pmatrix} \quad (3.7b)
\]

In the case of an unperturbed (no external field) transition, the radiation must be isotropic and unpolarized. If \( \mathbf{A} \) is the quantum mechanic matrix element between these unperturbed states, (these \( |\mathbf{A}|^2 \)'s are well known 3.1, 3.2) then the coherency matrix for this transition is:

\[
J = \frac{|c|^2 |\mathbf{A}|^2}{R^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.8)
\]

-30-
Hence, the $C_{\Delta \lambda} (N)$ in 3.3 are found by requiring:

$$\bar{J} = \sum_{M = -1}^{J} \left( J_{+1} + J_{0} + J_{-1} \right) \quad (3.9)$$

as the $H_{\text{ext}} = 0$ limit would require.

The matrix elements $|x_{\Delta} (N,M_{\Delta}, \Delta J, \Delta M)|^{2}$ are given in table 2.

**Table 2. Magnitude - Squared of Matrix Elements**

<table>
<thead>
<tr>
<th>$\Delta M$</th>
<th>$\Delta J = +1$</th>
<th>$\Delta J = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta M = +1$</td>
<td>$\frac{3N(N+M+1)(N+M+2)}{2(N+1)^{2}(2N+1)} M_{0}^{2}$</td>
<td>$\frac{3(N+1)(N-M)(N-M-1)}{2N^{2}(2N+1)} M_{0}^{2}$</td>
</tr>
<tr>
<td>$\Delta M = 0$</td>
<td>$\frac{3N[(N+1)^{2}-M^{2}]}{(N+1)^{2}(2N+1)} M_{0}^{2}$</td>
<td>$\frac{3(N+1)(N^{2}-M^{2})}{N^{2}(2N+1)} M_{0}^{2}$</td>
</tr>
<tr>
<td>$\Delta M = -1$</td>
<td>$\frac{3N(N-M+1)(N-M+2)}{2(N+1)^{2}(2N+1)} M_{0}^{2}$</td>
<td>$\frac{3(N+1)(N+M)(N+M-1)}{2N^{2}(2N+1)} M_{0}^{2}$</td>
</tr>
</tbody>
</table>

4. Attenuation Matrix

Using the Van Vleck-Weisskopf line shape for a collision broadened line together with the standard Doppler line shape, (See appendix 2 for further detail.) the attenuation matrix for one line $(N, \Delta J)$ and one type of radiation $(\Delta M)$ is:

$$\tilde{A}_{\Delta \lambda \Delta \lambda}^{\langle \nu \rangle} = \left[ C \frac{P_{\nu}^{2}}{T_{\text{d}}} e^{-\frac{E_{\nu}}{T}} \sum_{M = -1}^{J} |x_{\Delta} (N,M_{\Delta}, \Delta J, \Delta M)|^{2} F(\nu, \nu_{0}, \Delta \nu_{x}, \Delta \nu_{0}) \right] R_{\Delta M} \quad (3.10)$$
in which:

\[ P = \text{total pressure in mm. of Hg} \]

\[ T = \text{temperature in } ^{0}\text{K} \]

\[ \nu = \text{frequency in } \text{gc/s} \]

\[ E_{N} = \text{energy of } N^{th} \text{ rotation level in } ^{0}\text{K} \]

\[ \zeta = .305, \text{ a constant for units of } \text{km}^{-1} \]

\[ F(\nu, \nu_{0}, \Delta \nu_{c}, \Delta \nu_{d}) = \text{line shape (See appendix 2)} \]

The total attenuation matrix for a given line is:

\[ A_{N,\Delta \nu}^{(v)} = A_{N,\Delta \nu,+1}^{(v)} + A_{N,\Delta \nu,0}^{(v)} + A_{N,\Delta \nu,-1}^{(v)} \]  \hspace{1cm} (3.11)

since each of the absorption types ( \( \Delta M' \)'s) is independent of the others, permitting a straight addition.

The overall attenuation matrix is found by summing over all possible lines.

\[ A^{(v)} = \sum_{N=1}^{\infty} \left[ A_{N,+1}^{(v)} + A_{N,-1}^{(v)} \right] \]  \hspace{1cm} (3.12)

Finally, the complex propagation matrix is:

\[ G^{(v)} = A^{(v)} + i \frac{2\pi}{\nu\sqrt{\varepsilon_{0}M_{0}}} I \]  \hspace{1cm} (3.13)

with \( A^{(v)} \) as in 3.10, 3.11, 3.12.

Equation 3.13 has assumed that a small loss assumption is valid.

For the range of values of \( P, T \) and \( \nu \) of interest in the terrestrial atmosphere, this is certainly valid inasmuch as:

\[ \left| \text{tr} A^{(v)} \right|_{\text{max}} \left( \frac{4\pi \nu \sqrt{\varepsilon_{0}M_{0}}}{4\pi \nu \sqrt{\varepsilon_{0}M_{0}}} \right) \lesssim 10^{-5} \]  \hspace{1cm} (3.14)
The polarization basis has been chosen so that the 1-1 element of
the matrices (A, α, η, etc.) will be appropriate to linear polar-
zation with the H plane in the φ direction (E plane in the ψ direc-
tion). Similarly, the 2-2 element is appropriate to the opposite lin-
ear polarization; i.e., H plane in the ψ direction. In other words,
if ψ = 90° (the magnetic equator), then A is diagonal and the 1-1 ele-
ment will be for the σ Zeeman components and the 2-2 element will be
for the τ Zeeman component. But this separation is evident only for
the magnetic equator. Note that the above angles (ψ, φ) are mea-
sured from the magnetic polar axis, not the geographic polar axis as
will be the case in later chapters.
1. **Method of Calculation.**

The appropriate equation of propagation is 2.52b.

\[ \frac{d}{dz} T_B(v) + A(v) T_B(v) + \int T_B(v) A(v) = z \tau \kappa A(v) \]  

(2.52b)

The terrestrial atmosphere was approximated as a series of constant-temperature, constant-pressure layers, each 1 km thick. One hundred such layers from the ground to a height of 100 km were used. (Above 100 km, there is insufficient oxygen to be of any importance.) See Fig. 4.1

![Diagram](image)

**Figure 4.1**

The solution to 2.52b for one such layer is given by 2.53b:

\[ T_B(v) = e^{-A(v)\Delta z} T_B(v) e^{-A(v)\Delta z} + z \tau \kappa \left[ 1 - e^{-2A(v)\Delta z} \right] \]  

(2.53b)

with:

- \( T_B(v) \) being incident from below

- \( A(v) \) being a constant matrix appropriate to the oxygen for the layer
\( t_k \) being the kinetic temperature of the layer and:
\( T_{\infty}(\nu) \) being emergent from the top of the layer.

The ground-atmosphere interface is not of importance here as only frequencies in the center of the complex are considered. For these frequencies, the attenuation from the ground to 100 km. is so great that emission below about 10 km. does not contribute to the final \( T_\infty \).

Since the ground radiation is unimportant, the \( T_{\infty}(\nu) \) emergent at 100 km. is found by continual application of 2.53b for each of the layers between 0 and 100 km.

The temperature and pressure for each layer were obtained from a model atmosphere. Five different model atmospheres were used:

1. 1962 U.S. Standard Atmosphere\(^{4.1}\) (called MODATS II)
2. Typical Low Latitude (0° N) Summer\(^{4.2}\)
3. Typical Low Latitude (0° N) Winter\(^{4.2}\)
4. Typical High Latitude (70° N) Summer\(^{4.2}\)
5. Typical High Latitude (70° N) Winter\(^{4.2}\)

The temperature profiles of the model atmospheres are as shown in Fig. 4.2. The pressures were found by assuming hydrostatic equilibrium and a base pressure of 760mm. Hg.

A somewhat different analysis emphasizing the contribution to \( T_{\infty}(\nu) \) of the individual layers was actually used. The emission from each layer is carried through all layers above it to give its (the layer's) contribution to the total \( T_{\infty}(\nu) \). These contributions are then summed for \( T_{\infty}(\nu) \). In accordance with this, a weighting function matrix can be defined as:

\[
\overline{W}(\lambda, \nu) = \frac{P(\lambda, \nu)\int \frac{1}{\lambda} - e^{-2A(\nu, \lambda)}\lambda^4 h}{P(\lambda, \nu)} \quad (4.1)
\]
with:

\[ P(\bar{h},\nu) = e^{-\frac{A(\nu,100)}{\Delta \bar{h}}} e^{-\frac{A(\nu,99)}{\Delta \bar{h}}} \ldots e^{-\frac{A(\nu,2)}{\Delta \bar{h}}}. \]  \hspace{1cm} (4.2)

The emergent \( T_E(\nu) \) is given by:

\[ T_E(\nu) = \sum_{\bar{h}=2}^{100} WF(\bar{h},\nu) \chi(\bar{h}). \]  \hspace{1cm} (4.3)

Equation 4.3 and the weighting function concept are very useful as they emphasize the thermal-height structure.

Most calculations were made using MODATS II as the model atmosphere. The other four model atmospheres were used mainly to see the effect of some different temperature structures on the weighting functions for specific experiments. These are discussed in the next chapter. The general calculations presented here were all made with MODATS II.

The polarization basis in which all calculations were made and results plotted was a fairly natural choice. Linear polarizations were chosen as these are, by far, the most common receiver polarizations. The directions were related to geographic directions rather than geomagnetic directions as these are most easily defined in a satellite. Fig. 4.3 shows the geographic coordinate system which was used.
The unit vectors of interest for a +R travelling wave would be \( \overrightarrow{L_R} \) which points in the direction of increasing \( \Theta \) (along a longitude circle), and \( \overrightarrow{L_\phi} \), which points in the direction of increasing \( \phi \) (along a latitude circle).

In the polarization basis chosen, the 1-1 elements of the matrices represent linear polarization with the H field in the \( ^+\phi \) direction while the 2-2 elements represent linear polarization with the H field in the \( ^+\Theta \) direction.

The model of the magnetic field which was used was a dipole of peak strength 0.62 gauss at the magnetic pole. The orientation \( ^4 \Theta \) was as in Fig. 4.4.

\[
\begin{align*}
\Theta_o &= 14^\circ \\
\phi_o &= -101^\circ
\end{align*}
\]

![Figure 4.4 Magnetic Dipole Orientation](image)

It should be recalled that the relations derived in Chapter III were in terms of \( \Psi \), the angle between the magnetic field vector and \( \overrightarrow{L_R} \). Calculations made to demonstrate the magnetic latitude behavior were all made on the \( \phi = \phi_o = -101^\circ = 101^\circ W \) meridian. On this meridian, the magnetic field (of the dipole model) has \( \overrightarrow{L_R} \) and \( \overrightarrow{L_\phi} \) components only and the \( \Psi \) and \( \Theta \) directions coincide. This makes the calculations easier and, more important, allows the results to be presented in an
easily understandable form.

Along this meridian, the type of polarization from a $\Pi$ component is $\Theta$ linear, independent of the magnetic latitude. It does, however, have a $\sin^2 \psi$ amplitude dependence on latitude. The type of polarization from a $\sigma$ component depends on $\psi$. For $\psi = 90^\circ$, (magnetic equator) the polarization is $\phi$ linear. For $\psi = 0^\circ$, (magnetic pole) the polarization is circular with the sense depending on which $\sigma$ component is being considered. For $0^\circ < \psi < 90^\circ$, the polarization is elliptical with major axis in the $\phi$ direction and minor axis in the $\Theta$ direction (the sense again depending on which $\sigma$ component is being considered).

Because of the polarizations of the $\Pi$ and $\sigma$ components along this meridian, the total radiation will have $\text{Re} \left[ T_{\theta r}(\psi) \right] = 0$ for all latitudes. This says that there is no linear coherence between the 1-1 element and the 2-2 element, which is seen from the preceding paragraph. The $\text{Im} \left[ T_{\theta r}(\psi) \right]$ can, however, be nonzero, being a measure of the circular coherence between the two elements. In fact, $\text{Im} \left[ T_{\theta r}(\psi) \right]$ is seen to be identically zero only for $\psi = 90^\circ$, the magnetic equator.

To simplify computations (greatly reduce the required computer time) a further simplification of the magnetic dipole field was made. The magnitude of the field at 65 km was used for all heights. (i.e., the $R$ dependence of the field was ignored.) The direction of the field is independent of height so this was unaffected. The total variation of the magnitude of the field over the height ranges of importance was less than 2%. This was felt to be negligibly small. (Anamolies and variations from a true dipole field would probably be larger than this.)
2. **General Calculation Results.**

As discussed in the previous section, the calculations presented here were all made along the meridian of the magnetic pole \( (\phi = -101^\circ) \). Hence, the relation between the magnetic latitude and the geographic coordinates is:

<table>
<thead>
<tr>
<th>Magnetic Latitude</th>
<th>0</th>
<th>90° (magnetic pole)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0^\circ ) (magnetic equator)</td>
<td>104°</td>
<td>-101°</td>
</tr>
<tr>
<td>30°</td>
<td>74°</td>
<td>-101°</td>
</tr>
<tr>
<td>45°</td>
<td>59°</td>
<td>-101°</td>
</tr>
<tr>
<td>60°</td>
<td>44°</td>
<td>-101°</td>
</tr>
</tbody>
</table>

**Figure 4.5** is the computed \( T_\alpha (\nu) \) as would be seen by a satellite over the magnetic equator for frequencies near the 5+ resonance. The small ripples are caused by the individual Zeeman split resonances. \( T_{\alpha_{\nu}} (\nu) \) results from emission and absorption by the \( \Delta M = \pm 1 \) processes (\( \sigma \) components); while \( T_{\alpha_{zz}} (\nu) \) results from the \( \Delta M = 0 \) processes (\( \pi \) components). \( T_{\alpha_{zz}} (\nu) = 0 \) for all \( \nu \).

**Figure 4.6** shows \( T_\alpha (\nu) \) for the same line but over the magnetic pole. The \( \pi \) components are not visible here so the absorption and emission mechanisms are with right-circular or left-circular polarization, so \( T_{\alpha_{\nu}} (\nu) = T_{\alpha_{zz}} (\nu) \) and \( \text{Im} \left[ T_{\alpha_{zz}} (\nu) \right] \neq 0 \).

The fine structure is more evident now (the magnetic field is twice as strong as over the magnetic equator) and the overall splitting is twice as wide.

**Figure 4.7** shows \( T_\alpha (\nu) \) for a magnetic latitude of \( 45^\circ \). Figures 4.8-4.9 show some of the weighting functions as \( \nu \) changes across the resonance for the above cases. Figures 4.10-4.11 show the heights
**BRIGHTNESS TEMPERATURE MATRIX SPECTRUM**

- **λ**: 59.5910 μm
- \( \nu = \nu_0 + \Delta \nu \)
- \( T_{B,11} = T_{B,12} \)
- Left scale
- In \( T_{B,12} \)
- Right scale

Graph showing temperaturer versus wave frequency (Δν) in mc/s.
\[
\gamma = \gamma_0 + \Delta \gamma
\]

\[
WF_{\mu} = WF_{\infty}
\]

**FIGURE 4.9**
of the peak and half-peaks (the heights at which the weighting function has one-half of its maximum value) of the weighting functions as functions of $\nu$.

Figure 4.10 (magnetic equator) shows clearly the effect of the Zeeman splitting. At frequencies nearest resonance $WF_{22}$ peaks at a very high altitude (77-78 km.) while $WF_{11}$ peaks lower (69 km.). This is because the $T$ components are most intense near the main resonance, while the $\sigma$ components are most intense at frequencies displaced form the main resonance. (The most intense $\sigma$ components fall at $\pm 0.7$ mc from the $V_f$.) This is seen as $\nu$ increases away from $V_f$.

The roles of the $WF$'s have interchanged at $\Delta V=0.7$ mc where the $\sigma$ components are most intense. Further increase in $\Delta V$ brings both $WF$'s down deeper into the atmosphere (warmer temperatures at these heights) as neither are very intense at high altitudes. As the pressure increases (height decreasing) the lines are pressure-broadened to a greater extent causing an increase in the intensity of the lines. When the $WF$ peaks reach about 50 km polarization is no longer important since the pressure-broadening is much larger than the Zeeman splitting.

Figure 4.11 (magnetic pole) exhibits a strange phenomenon, a double peaked $WF$. This is reasonable insomuch as the mechanisms of absorption and emission are right or left circularly polarized over the pole. For $\Delta V^2 1.0-1.4$ mc the $\Delta M=1$ components are quite intense while the $\Delta M=-1$ components are quite weak; thus the radiation with right circular polarization comes predominantly from high (70-80 km) in the atmosphere, while the radiation with left circular
WEIGHTING FUNCTION PEAKS  5+ Line  Magnetic Equator MODATS II

Height

km

\[ \Delta V \text{ mc} \]

--- height of \( W_F \) peak

--- --- height of \( W_F = \frac{1}{2} \) peak

FIGURE 4.10
$WF_{i} = WF_{p} + WF_{e}$

- - - height of $WF$ peak
- - - height of $WF = t$ peak

$V = V_0 + \Delta V$

FIGURE 4.11
comes from lower (45-55 km). Both of these contribute to the linear polarization terms causing the double peaked shape.

Figures 4.12-4.18 show $T_B(v)$ for several other lines at both magnetic polar and equatorial positions.

The significant features of the 1+ lines are few Zeeman components and their extreme narrowness.

The 7- lines are similar to the 5+ lines except what the Zeeman structure is more evident. The WF's associated with the 7- lines will depend more strongly on frequency (within the Zeeman complex) than the 5+ lines do.

The 15+ line is less intense than the 5+ or 7- lines, hence it will penetrate deeper into the atmosphere causing warmer $T_{B11}$ and $T_{B22}$ within the Zeeman complex. The larger number of Zeeman components and their close spacing (compared to the pressure-broadening) prohibits any fine structure from being detectable.

The 19+ and 21- lines are less intense than the 15+ line and penetrate still deeper into the atmosphere. The small range of values for the $T_B$'s is because the height of the WF peaks is now about the height of the stratopause where the temperature inverts. Thus, significant changes in height may have very small changes in the $T_B$'s.

For penetration below 50 km a scalar treatment will suffice as polarization is no longer important. Figures 4.17 and 4.18 indicate that it is just barely detectable for the 19+ and 21- lines.
BRIGHTNESS TEMPERATURE MATRIX SPECTRUM

\[ V_0 = 5 \times 26.4 \, \text{K} \]

\[ T_{B_{ll}} = T_{B_{22}} \]

\[ \text{left scale} \]

\[ \text{right scale} \]

\[ \Delta V_{mc} \]
BRIGHTNESS TEMPERATURE MATRIX SPECTRUM

15+ Line  Magnetic Equator  MODATS II

$V_0 = 62.99 \text{GHz}$

$T_{B_{11}}$, $T_{B_{22}}$

$\Delta V_{mc}$

FIGURE 4.16
BRIGHTNESS TEMPERATURE MATRIX SPECTRUM

21-Line
Magnetic Equator
MODATS II

$\gamma_0 = 54.6728 \pi$
BRIGHTNESS TEMPERATURE MATRIX SPECTRUM

$V_0 = 64.1272 \, \mu\text{K}$

$T_{B11} = T_{B22}$

left scale

$\text{Im}(T_{B12})$

right scale

$\Delta \nu_{mc}$

FIGURE 4.18
Chapter V  Microwave Experiments to Sound the Upper Atmosphere

The height variation of the terrestrial atmospheric temperature is conveniently divided into three sections over the range of interest to this paper: (see Figure 4.2)

1. Troposphere; ground to about 15 km
2. Stratosphere; 15 km to about 50 km
3. Mesosphere; 50 km to about 90 km

These three sections are separated by regions in which the temperature inverts (i.e., changes the sign of the lapse rate); the tropopause at about 15 km (a relative temperature minimum) and the stratopause at about 50 km (a relative temperature maximum). These divisions are approximate, depending on latitude, season, and other factors.

Chapter IV has indicated that the absorption due to molecular oxygen very near a transition frequency will afford a means of remotely probing the mesosphere from about 75 km down.

The instrument used to take these measurements would be a satellite-borne radiometer, most likely a standard Dicke superheterodyne-comparison radiometer. If $T_n$ is the effective noise temperature of the receiver; BW, the pre-detection band-width; and $\bar{T}$ the post-detection integration time, then the RMS deviation of the receiver output expressed in temperature units would be:

$$\Delta T_{RMS} = \frac{2}{\sqrt{BW \cdot \bar{T}}}$$  \hspace{1cm} (5.1)

$T_n$'s of $7 - 10 \times 10^3$ °K are presently realizable, with the $7 \times 10^3$ °K figure probably a little optimistic. For a satellite in a polar orbit similar to that of a Nimbus satellite, an effective ground speed of 6 mi/sec is experienced. The desired spatial resolution will, therefore,
limit the length of the integration time, \( \tau \).

1. Mesospheric Soundings

For a sounding having a weighting function that peaks as high in the atmosphere as possible, the most intense lines should be used. The relative intensity dependence of \( N \) is:

\[
\sim (2N+1) e^{-\frac{2.07}{2} N(N+1)}
\]  \hspace{1cm} (5.2)

The \( N_{\text{max}} \) for peak intensity for \( T \)'s appropriate to the mesosphere appear in Table 5.1

Table 5.1  \( N_{\text{max}} \)

<table>
<thead>
<tr>
<th>( T )</th>
<th>( N_{\text{max}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>200°</td>
<td>5, 7</td>
</tr>
<tr>
<td>250°</td>
<td>7</td>
</tr>
<tr>
<td>300°</td>
<td>7, 9</td>
</tr>
</tbody>
</table>

A line with larger \( N \) has an advantage over one with lower \( N \). The higher lines (larger \( N \)) have more Zeeman components with approximately the same total splitting; hence, each component is closer to its neighbor. This minimizes the relative transparency (deeper probing) of frequencies between Zeeman components. Figure 4.11 shows this effect. This effect will tend to broaden the final weighting function when a finite band-width encompassing peaks and valleys is permitted. For this reason, the 7+ line is considered as a means of probing the mesosphere above 70 km rather than the 5+ line. The 9+ line would be even better from the fine-structure point of view, but it is somewhat beyond the intensity maximum for the cold mesospheric temperatures.

The brightness temperature for a finite band-width is:
\[
T_B(\nu_0, \Delta \nu) = \frac{1}{\Delta \nu} \int_{\nu_0 - \Delta \nu}^{\nu_0 + \Delta \nu} T_B(\nu) \, d\nu
\]  

(2.54)

Similarly, the weighting function matrix is given by:

\[
WF(\nu_0, \Delta \nu) = \frac{1}{\Delta \nu} \int_{\nu_0 - \Delta \nu}^{\nu_0 + \Delta \nu} WF(\nu, \nu_0) \, d\nu
\]  

(5.3)

From which \( T_B \) can be found by:

\[
T_B(\nu_0, \Delta \nu) = \int_0^{100} WF(\nu, \nu_0, \Delta \nu) \, T_B(\nu) \, d\nu
\]  

(5.4)

A weighting function such as 5.3 will be diagonal if \( \nu_0 \) is a transition frequency and they are computed on the meridian of the magnetic pole (previous calculations were made here). This is because Re[\( T_{Bz}(\nu) \)] = 0 on this meridian and Im[\( T_{Bz}(\nu) \)] is an odd function around a transition frequency. This will make the weighting functions for finite band-width easy to discuss as there are only the two main diagonal terms. In this case, the fractional polarization of 5.4 is easily found as:

\[
\rho = \left| \frac{T_B_{ll} - T_B_{zz}}{T_B_{ll} + T_B_{zz}} \right|
\]  

(5.5)

Chapter IV has indicated that the weighting function and brightness temperatures are quite different over the magnetic pole from what they are over the magnetic equator. First, consider the 7+ Equatorial line.

For the 7+ Equatorial line, the frequency averaged weighting functions are shown in Figure 5.1 for band-widths, BW, of 0.5, 1.0, 1.5, 2.0,
FIGURE 5.1b
2.5 mc. The $T_g (\nu)$ for this line is shown in Figure 5.2. The significance of the 1.5 mc BW is seen in Figure 5.2. The overall Zeeman splitting is on the order of 1.5 mc and this BW will include all the components of the line but will not include too much of the deeply penetrating wings. A BW of 1.5 mc is seen to have the least fractional polarization so that receiver polarization would matter very little.

The effect of different atmosphere structures is explored in Figures 5.3 - 5.5. Figure 5.3 shows the weighting functions for a BW of 1.5 mc for the MODATS II, Low Latitude Summer, and Low Latitude Winter Atmospheres. The colder atmospheres are seen to have weighting functions that peak higher than in the warmer atmospheres. This is due to the $T^{-3}$ dependence of $A(\nu)$. Figures 5.4 and 5.5 show the $T_g (\nu)$ for these atmospheres.

For the 7+ Polar line, the frequency averaged weighting functions are shown in Figure 5.6 for BW = 0.5, 1.0, 1.5, 2.0, 2.5, 300 mc. $T_g (\nu)$ is shown in Figure 5.7. Increasing BW widens the weighting function, which for a polar observation can be written as a scalar weighting function times the unit matrix. The explanation for this is found in Figure 4.11 where the double-peaked weighting functions appear.

Different atmospheres are used to compute the weighting functions and brightness temperatures of Figures 5.8-5.10. Figure 5.8 shows the behavior of the weighting function for BW = 1.5 mc for the MODATS II, High Latitude Summer, and High Latitude Winter Atmospheres; while Figures 5.9 and 5.10 show $T_g (\nu)$ for the latter two atmospheres.

A BW of 1.5 mc seems quite reasonable over the magnetic pole as well as over the magnetic equator. However, over the pole, the weighting function, while truly polarization independent, peaks somewhat low-
FIGURE 5.3b
$WF_{11} = WF_{22}$  7+ Line  Magnetic Pole  MODATS II

FIGURE 5.6
BRIGHTNESS TEMPERATURE MATRIX SPECTRUM

7+ Line Magnetic Pole MODATS II

\[ \gamma_0 = 6.0 \times 10^6 \text{ cm} \]
\[ \gamma = \gamma_0 + \Delta \gamma \]

\[ T_{B11}, T_{B22} \]

left scale

\[ \text{right scale} \]

\[ \Delta \gamma \text{ max} \]
$WF_{11}=WF_{22}$  7+ Line  Magnetic Pole  $BW = 1.5$ mc

Atmosphere:
- MODATS II
- HLS
- HILW

FIGURE 5.8
er (by about 7 - 10 km) and is wider (by about 5 - 7 km). Figure 5.1 illustrates the dependence of the weighting function peak heights and widths on the magnetic latitude.

Table 5.2 summarizes the proposed 7+ line experiment.

<table>
<thead>
<tr>
<th>Table 5.2</th>
<th>7+ Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>V₀ = 60.4348 gc</td>
<td></td>
</tr>
<tr>
<td>BW = 1.5 mc</td>
<td></td>
</tr>
<tr>
<td>MODATS II Atmosphere</td>
<td></td>
</tr>
<tr>
<td>h₀ = height of weighting function peak</td>
<td></td>
</tr>
<tr>
<td>Δh = full width between half peak values</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Equator</th>
<th>Pole</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>11 polarization</td>
<td>22 polarization</td>
</tr>
<tr>
<td>h₀</td>
<td>72.5 km</td>
<td>74.0 km</td>
</tr>
<tr>
<td>Δh</td>
<td>20 km</td>
<td>20 km</td>
</tr>
</tbody>
</table>

If a measurement as summarized in Table 5.2 were made with a radiometer having Tₓ = 10⁴ °K and τ = 10 sec. (60 mi. on ground), the ΔTₓ would be about 5⁰K. If the plan was to space 6 - 8 radiometer frequencies and band-widths so that the h₀'s were spaced from about 15 km to 75 km and to try an inversion method (deducing the thermal profile from Tₓ measurements, see Appendix 5) on the resulting 6 - 8 measurements, a ΔTₓ = 5⁰ K would probably be too high to achieve a convergent iteration. Generally, a ΔTₓ of 1% of the measured Tₓ is desired.5.3, 5.4

If Tₓ = 7 x 10³ °K and τ = 30 sec. (180 mi. on ground), then ΔTₓ = 2⁰ K which is about 1% of the Tₓ reading. This measurement
could probably be part of several measurements which are to be inverted. Even the $T_N = 10^{40}$K and $\tau = 10$ sec measurement should have meteorological value as an average of the mid-upper mesospheric temperature.

For a narrow band-width experiment, the required frequency stability and accuracy will be a critical requirement on the equipment. Generally speaking, the frequency stability and accuracy should be on the order of 1/10 the band-width.

Since the Zeeman effect and polarization are most important above 50 km, another frequency and band-width can be similarly chosen to give a sounding of the temperature near 60 km. The 17+ line is ideal for this. Using a BW of 1.5 mc (for the same reason as with the 7+ line) the frequency averaged weighting functions are shown in Figure 5.12. The $T_{\text{sec}}(v)$ of these lines are shown in Figures 5.13 and 5.14. The magnetic latitude dependence is similar to that of the 7+ line. The effect of the other atmospheres is very small as the temperature variations of them at 55-60 km are small.

Table 5.3 summarizes the proposed 17+ line experiment.

<table>
<thead>
<tr>
<th>Table 5.3</th>
<th>17+ Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_0 = 63.5685$ gc</td>
<td></td>
</tr>
<tr>
<td>BW = 1.5 mc</td>
<td></td>
</tr>
<tr>
<td>MODATS II Atmosphere</td>
<td></td>
</tr>
<tr>
<td>$h_0 = \text{height of weighting function peak}$</td>
<td></td>
</tr>
<tr>
<td>$\Delta h = \text{full width between half peak values}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Equator</td>
</tr>
<tr>
<td>--------</td>
<td>--------------------------</td>
</tr>
<tr>
<td></td>
<td>1-1 polarization</td>
</tr>
<tr>
<td>( h_0 )</td>
<td>60 km</td>
</tr>
<tr>
<td>( \Delta h )</td>
<td>21 km</td>
</tr>
</tbody>
</table>

For this experiment, a receiver with \( T_N = 10^4 \, ^\circ K \) and \( \tau = 10 \) sec has \( \Delta T_{RMS} = 5^\circ K \); while a receiver with \( T_N = 7 \times 10^3 \, ^\circ K \) and \( \tau = 30 \) sec has \( \Delta T_{RMS} = 2^\circ K \). The comments about the 7+ experiment would apply here as well.

2. Stratospheric and Upper-Tropospheric Soundings

For heights of 50 km and below, it is best to take \( \gamma \) to be on the wing of a line or in a valley between two lines. It is still possible to have weighting functions peak at these heights for BW's centered on transition frequencies, but these would be very large \( N \)'s (e.g., The 23+ line with BW = 1.5 mc has a weighting function which peaks at 50 km but it is about 30 km wide.) and they would have very large \( \Delta h \)'s.

A typical spectrum of \( h_0 \) and the half peak heights, for frequencies away from the transition frequency, is shown in Figure 5.15. As this figure shows, an experiment with \( \gamma = 60.80 \, gc \) and BW = 250 mc would yield \( h_0 = 18 \) km and \( \Delta h = 8 \) km.

For the widest BW's, the valleys between the transition frequencies are the best positions for the \( \gamma \)'s. (See Appendix 4 for a list of the transition frequencies.) The maximum possible \( h_0 \) for a "valley" experiment is about 27 km (the valley between the closest two transition frequencies). For \( h_0 \)'s above this, it is necessary to position \( \gamma \) on the wing of a line.
Experiments covering the range of heights from about 10-50 km can be planned. (10 km is used as a lowest height since probing deeper would require knowledge of the emission coefficient of the surface. Weighting functions peaking above 10 km will attenuate ground radiation to the extent that it is not visible from above the atmosphere.)

Table 5.4 describes a five frequency experiment. The effects of different atmospheres are studied and seen to be very small.

Table 5.4

Five Frequency Experiment

Atmospheres:
   HLS - high latitude summer
   HLW - high latitude winter
   LLS - low latitude summer
   LLW - low latitude winter

<table>
<thead>
<tr>
<th></th>
<th>$\nu_0 = 64.47$ gc</th>
<th>$\nu_0 = 60.82$ gc</th>
<th>$\nu_0 = 58.388$ gc</th>
<th>$\nu_0 = 60.4409$ gc</th>
<th>$\nu_0 = 60.4365$ gc</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta h$</td>
<td>$\Delta h$</td>
<td>$\Delta h$</td>
<td>$\Delta h$</td>
<td>$\Delta h$</td>
</tr>
<tr>
<td></td>
<td>km.</td>
<td>km.</td>
<td>km.</td>
<td>km.</td>
<td>km.</td>
</tr>
<tr>
<td>MODATS II</td>
<td>12</td>
<td>10</td>
<td>18</td>
<td>7</td>
<td>27</td>
</tr>
<tr>
<td>HLS</td>
<td>12</td>
<td>11</td>
<td>18</td>
<td>9</td>
<td>27</td>
</tr>
<tr>
<td>HLW</td>
<td>12</td>
<td>12</td>
<td>18</td>
<td>9</td>
<td>27</td>
</tr>
<tr>
<td>LLS</td>
<td>12</td>
<td>11</td>
<td>18</td>
<td>7</td>
<td>27</td>
</tr>
<tr>
<td>LLW</td>
<td>12</td>
<td>11</td>
<td>18</td>
<td>7</td>
<td>27</td>
</tr>
</tbody>
</table>

The weighting functions for the five experiments described in Table 5.4 are shown in Figure 5.16. The more or less even spread and only slight overlap is an obvious feature of the functions. Inasmuch as they do not overlap a great deal, any inversion attempt will be helped toward rapid convergence since the temperature from a given height
falls predominantly in one weighting function and not in several overlapping ones.

3. **Atmospheric Soundings, 10-75 km**

The seven frequency experiment described in Table 5.5 shows soundings which cover the 10-75 km range. If the $\Delta T$'s are sufficiently small, an inversion of these measurements (See appendix 5) should yield a good temperature-height profile for this height range.

**Table 5.5**

Seven Frequency Experiment

**MODATS II Atmosphere**

$\Delta T_1 = \Delta T_{RMS}$ for $T_n = 10^4 \, ^\circ K$, $\tau = 10 \, sec.$

$\Delta T_2 = \Delta T_{RMS}$ for $T_n = 7,000 \, ^\circ K$, $\tau = 30 \, sec.$

<table>
<thead>
<tr>
<th>$\gamma_0$ gc</th>
<th>$BW_{mc}$</th>
<th>$h_0$ km</th>
<th>$\Delta h$ km</th>
<th>$\frac{\Delta T_1}{\circ K}$</th>
<th>$\frac{\Delta T_2}{\circ K}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>64.47</td>
<td>200</td>
<td>12</td>
<td>11</td>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>60.82</td>
<td>200</td>
<td>18</td>
<td>7</td>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>58.388</td>
<td>30</td>
<td>27</td>
<td>9</td>
<td>1.2</td>
<td>0.5</td>
</tr>
<tr>
<td>60.4409</td>
<td>2.5</td>
<td>40</td>
<td>12</td>
<td>4.0</td>
<td>1.5</td>
</tr>
<tr>
<td>60.4365</td>
<td>1.0</td>
<td>50</td>
<td>20</td>
<td>6.0</td>
<td>2.5</td>
</tr>
<tr>
<td>63.5685</td>
<td>1.5</td>
<td>60</td>
<td>21</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>54</td>
<td>26</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>60.4348</td>
<td>1.5</td>
<td>73</td>
<td>20</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>66</td>
<td>26</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

4. **Balloon Soundings**

Appendix 3 summarizes the results of a program of balloon measurements of the 9+ transition. The radiometer that made these mea-
falls predominantly in one weighting function and not in several overlapping ones.

3. **Atmospheric Soundings, 10-75 km**

The seven frequency experiment described in Table 5.5 shows soundings which cover the 10-75 km range. If the $\Delta T$'s are sufficiently small, an inversion of these measurements (See appendix 5) should yield a good temperature-height profile for this height range.

<table>
<thead>
<tr>
<th>$\gamma_{gc}$</th>
<th>$\delta$</th>
<th>$h_0$</th>
<th>$\Delta h$</th>
<th>$\Delta T_1$</th>
<th>$\Delta T_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>64.47</td>
<td>200</td>
<td>12</td>
<td>11</td>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>60.82</td>
<td>200</td>
<td>18</td>
<td>7</td>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>58.388</td>
<td>30</td>
<td>27</td>
<td>9</td>
<td>1.2</td>
<td>0.5</td>
</tr>
<tr>
<td>60.4409</td>
<td>2.5</td>
<td>40</td>
<td>12</td>
<td>4.0</td>
<td>1.5</td>
</tr>
<tr>
<td>60.4365</td>
<td>1.0</td>
<td>50</td>
<td>20</td>
<td>6.0</td>
<td>2.5</td>
</tr>
<tr>
<td>63.5685</td>
<td>1.5</td>
<td>60</td>
<td>21</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>60.4348</td>
<td>1.5</td>
<td>73</td>
<td>20</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>66</td>
<td>26</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

4. **Balloon Soundings**

Appendix 3 summarizes the results of a program of balloon measurements of the 9+ transition. The radiometer that made these mea-
surements can easily be adapted (with a few mechanical changes) to check the proposed experiment which has its weighting function peak at 18 km. Since the balloon must rise through the atmosphere, it will eventually measure and at termination it must descend through the atmosphere it has measured, adequate knowledge of the true temperature profile will exist to check the theory. In particular, more information on a better line shape may be obtained (see Appendix 3).

The first three frequencies in Table 5.5 could be incorporated in a balloon-borne radiometer prior to a satellite experiment. The balloon would have to rise to about 35 km so that the last weighting function (peaking at 27 km) would be small at the height of the balloon.
Chapter VI Concluding Remarks and Suggestions

The previous chapter has indicated that measurements of the atmospheric temperature averaged over a height range are possible for heights up to about 30 km. For example, the 7+ line experiment (Table 5.2) by itself would yield a measure of the average temperature in the 60-80 km range. Uncertainties caused by latitude variations and polarization variations can be resolved, assuming that the latitude and receiver polarization sense are known. Thus, even with $T_N = 10^4$ K and $\tau = 10$ sec, a $\Delta T_{RMS} = 5^0$K is obtained. While this is probably not a sufficiently small $\Delta T$ for this measurement to be part of an inversion experiment; it, nonetheless, should prove to be a useful measurement in its own right. Especially since this measurement is for a range of altitudes about which little is known on a global scale.

As pointed out above, if the latitude and polarization sense of the receiver are known, then these effects can be taken into account in the data analysis. The line-shape and weighting function characteristics depend on magnetic latitude. (See Figure 5.11 ) However, from a meteorological viewpoint, the variation of the temperature with geographic latitude would be more interesting. Knowledge of the geographic latitude and longitude are sufficient to determine the magnetic latitude. The height range of the weighting function depends on geographic longitude as well as geographic latitude. This will be somewhat troublesome but can be tolerated. Over the geographic equator, the weighting function dependence on geographic longitude is small. This is easily seen in Figure 5.11, realizing that the geographic equator has a magnetic latitude between 14°N and 14°S.
The preceding discussion has assumed a well-defined and well-known magnetic field (a dipole, in this case). If this is not true, then this uncertainty will introduce variations on the height dependence of the weighting function. Small magnetic anomalies will have negligible effects. Larger anomalies (magnitude of the perturbation not very small compared to the magnitude of the unperturbed field) however, will have the effect of perturbing the weighting function with respect to its height dependence. Therefore, the magnetic latitude variation of the weighting function can be accounted for only to the degree of accuracy with which the magnetic field is known. At these altitudes, very little is known about the magnetic field variations.\textsuperscript{6.1} (These heights are too high for balloon-borne magnetometers and too low for satellite-borne magnetometers.)

The 17+ line experiment has features similar to the 7+ line experiment but the height range is about 10 km lower. The same comments as for the 7+ line would apply here as well.

If the integration time, $\tau$, is increased to 30 sec (about 180 miles on ground) and $T_N$ is decreased to 7,000 °K, Table 5.5 shows that the seven frequency experiment, sounding the atmosphere from 10-90 km, could well produce invertible data (using the rule of thumb that accuracy of 1\% is required for invertible data). The increase of $\tau$ should prove to be of no difficulty. The decrease of $T_N$, however, may present microwave hardware problems. Nonetheless, it is not unreasonable to expect that in the near future (two to five years), a radiometer with $T_N = 7,000$ °K will present no problem whatsoever. (A local company claims to have built a radiometer for satellite operation at 5 mm with
$T_N = 3,500^\circ\text{K}$. They employ a Klystron local oscillator, and for their purposes, the frequency stability and accuracy are not critical.)

There are two areas in which more work needs to be done before an experiment such as proposed in Table 5.5 should be attempted. More theoretical work on inversion methods, and especially, the effect of uncertainties, needs to be accomplished. Experimental work on the exact line-shape and appropriate parameters is also in order.

The results presented in Appendix 3 indicate that the Van Vleck-Weisskopf line-shape may not sufficiently describe these $O_2$ lines. Laboratory measurements of the line shape and intensity at a variety of pressures and temperatures are called for. The difficulty of resolving the high pressure lines might be averted by investigating the 1-line at 2.5 mm. However, working at 2.5 mm is much more difficult than working at 5 mm would be. Laboratory measurements might be coupled with further balloon experiments, especially downward looking experiments. Until the line-shape is more accurately known, the hope of achieving a useful inversion of experimental results seems dim.

Appendix 5 describes the general considerations of most inversion methods. Two things concerning the inversion methods need further theoretical investigation; namely, the choice of a set of functions to expand the atmospheric temperature in, and investigation of how measurement noise propagates through the inversion procedure. Very little has been done with respect to the latter point, but for an inversion method to be practical, this must be better understood.
Appendix 1, Matrix Relations

Strictly speaking, the polarization space is a two dimensional vector space, \( V \), over the field of complex numbers, \( \mathbb{C} \), with an inner product. The matrices to be considered will be \( 2 \times 2 \) with complex elements. These are the matrices representing linear transformations relative to a particular ordered basis. The inner product of the two vectors \( \vec{a}, \vec{b} \) in the vector space is \( \vec{a} \cdot \vec{b}^* \) where the dot indicates a conventional "dot product".

I. Definitions:

1. **Hermitian.**

   A matrix \( \mathbf{H} \) is hermitian iff:
   \[
   \mathbf{H} = \mathbf{H}^* \tag{A1.1}
   \]

2. **Unitary.**

   A matrix \( \mathbf{U} \) is unitary iff it is invertible and:
   \[
   \mathbf{U}^{-1} = \mathbf{U}^* \tag{A1.2}
   \]

3. **Determinant.**

   The determinant of a matrix \( \mathbf{A} \), \( \det \mathbf{A} \), is:
   \[
   \det \mathbf{A} = a_{11}a_{22} - a_{12}a_{21} \tag{A1.3}
   \]

   where
   \[
   \mathbf{A} = \begin{pmatrix}
   a_{11} & a_{12} \\
   a_{21} & a_{22}
   \end{pmatrix} \tag{A1.4}
   \]

   with \( a_{ij} \in \mathbb{C} \); \( i,j = 1,2 \)

4. **Trace.**

   The trace of a matrix \( \mathbf{A} \), \( \text{tr} \mathbf{A} \), is:
   \[
   \text{tr} \mathbf{A} = a_{11} + a_{22} \tag{A1.5}
   \]

5. **Eigen Values** (or characteristic values).

   The eigen values, \( \lambda_1, \lambda_2 \), of a matrix, \( \mathbf{A} \), are the scalars
\( \xi \in \mathbb{C} \) that satisfy the equation:
\[
\det \left( A - \lambda I \right) = 0 \tag{Al.6}
\]

Eq. Al.6 is a second order polynomial in \( \lambda \) (\( n^{\text{th}} \) order if \( A \) is \( n \times n \)); hence, there will be two \( \lambda' \)'s: \( \lambda, \lambda' \); satisfying (Al.6). (It is possible that \( \lambda = \lambda' \).

6. **Similar Matrices.**

Two matrices \( A \) and \( B \) are similar iff an invertible matrix, \( P \), exists such that:
\[
A = P^{-1} B P \tag{Al.7}
\]

7. **Unitarily Equivalent Matrices.**

Two matrices \( A \) and \( B \) are unitarily equivalent iff an unitary matrix, \( U \), exists such that:
\[
A = U^{-1} B U \tag{Al.8}
\]

8. **Exponential of a Matrix.**

The exponential of a matrix, \( A \), is defined by:
\[
e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \tag{Al.9}
\]

with
\[e^1 = I\tag{Al.10}\]

9. **Basis.**

A basis is a pair of vectors \( \vec{\alpha}, \vec{\beta} \in V \) such that:
\[
\vec{\alpha} \cdot \vec{\alpha}^* = 1 \tag{Al.11a}
\]
\[
\vec{\beta} \cdot \vec{\beta}^* = 1 \tag{Al.11b}
\]
\[
\vec{\alpha} \cdot \vec{\beta}^* = 0 \tag{Al.11c}
\]

Then any \( \vec{\gamma} \in V \) can be expressed as:
\[
\vec{\gamma} = a \vec{\alpha} + b \vec{\beta} \tag{Al.12}
\]

with \( a, b \in \mathbb{C} \).

II. **Properties**

1. **Change of Basis**
If \( \alpha', \beta' \) are a basis and \( \chi, \gamma \) are a different basis, then there is an unique unitary transformation, \( U \), such that:

\[
\begin{pmatrix} \chi \\ \gamma \end{pmatrix} = U \begin{pmatrix} \alpha \\ \beta \end{pmatrix}
\]  

(Al.13)

Proof: Since \( \alpha', \beta' \) is a basis and both \( \chi, \gamma \) are \( \in \mathcal{V} \) then there are \( a_{ij} \)'s of \( \mathcal{C} \) such that:

\[
\begin{align*}
\chi &= a_{u} \chi' + a_{z} \beta' \\
\gamma &= a_{z} \chi' + a_{\bar{z}} \beta'
\end{align*}
\]  

(Al.14a)

(Al.14b)

The basis relations, (Al.11), require:

\[
\begin{align*}
|a_{u}|^2 + |a_{z}|^2 &= 1 \\
|a_{\bar{z}}|^2 + |a_{z}|^2 &= 1 \\
\begin{bmatrix}
a_{u} & a_{\bar{z}}
\end{bmatrix} \begin{bmatrix} a_{u} & a_{\bar{z}} \\
a_{z} & a_{z} \end{bmatrix} &= 0
\end{align*}
\]  

(Al.15a)

(Al.15b)

(Al.15c)

which gives:

\[
A \dot{A^\text{t}^*} = I
\]  

(A1.16)

with

\[
A = \begin{pmatrix} a_{u} & a_{\bar{z}} \\ a_{z} & a_{z} \end{pmatrix}
\]

hence, \( A \) is unitary. q.e.d.

If \( T_{\chi, \rho} \) is the matrix of a linear transformation in the basis \( \chi, \rho \), then \( A T_{\chi, \rho} A^\text{t}^* \) is the matrix of the linear transformation in the basis \( \chi, \gamma \).

Proof: Let

\[
E_{0, \chi, \rho} = T_{\chi, \rho} E_i \alpha, \beta
\]  

(A1.17)

with \( E_{0, \chi, \rho}, E_i \alpha, \beta \) written in basis \( \chi', \beta' \) then:

\[
E_{0, \chi, \gamma} = A E_{0, \chi, \rho}
\]  

(A1.18)

or:

\[
E_{0, \chi, \rho} = A^\text{t}^* E_{0, \chi, \gamma}
\]  

(A1.19)

and

\[
E_i \alpha, \beta = A^\text{t}^* E_i \chi, \gamma
\]  

(A1.20)
hence, (A.17) becomes:
\[ \mathcal{A}^* E_{\alpha,\beta} \mathcal{A}^+ = \mathcal{T}_{\beta,\alpha} \mathcal{A}^{t\star} E_{\alpha,\beta} \]
and
\[ \bar{E}_{\alpha,\beta} = \mathcal{A} \mathcal{T}_{\beta,\alpha} \mathcal{A}^{t\star} E_{\alpha,\beta} \] (A.21)
which yields:
\[ \mathcal{T}_{\alpha,\beta} = \mathcal{A} \mathcal{T}_{\alpha,\beta} \mathcal{A}^{t\star} \] (A.22)
q.e.d.

The det, tr, and \( \lambda \)'s of a transformation, \( T \), are invariant under an unitary transformation such as A.22.

Proof:
\[ \det \mathcal{T}_{\alpha,\beta} = \det \left( \mathcal{A} \mathcal{T}_{\beta,\alpha} \mathcal{A}^{t\star} \right) \]
\[ = \left( \det \mathcal{A} \right) \left( \det \mathcal{T}_{\beta,\alpha} \right) \left( \det \mathcal{A}^{t\star} \right) \]
\[ = \left( \det \mathcal{A} \right) \left( \det \mathcal{A}^{t\star} \right) \left( \det \mathcal{T}_{\beta,\alpha} \right) \]
\[ \det \mathcal{T}_{\alpha,\beta} = \det \mathcal{T}_{\beta,\alpha} \] (A.23)
also
\[ \text{tr} \mathcal{T}_{\alpha,\beta} = \text{tr} \left( \mathcal{A} \mathcal{T}_{\beta,\alpha} \mathcal{A}^{t\star} \right) \]
which becomes, with the use of \( \text{tr} \left( CE \right) = \text{tr} \left( EC \right) \)
\[ \text{tr} \mathcal{T}_{\alpha,\beta} = \text{tr} \left( \mathcal{A}^{t\star} \mathcal{A} \mathcal{T}_{\beta,\alpha} \right) \]
or
\[ \text{tr} \mathcal{T}_{\alpha,\beta} = \text{tr} \mathcal{T}_{\beta,\alpha} \] (A.24)
Finally:
\[ \det \left( \mathcal{T}_{\alpha,\beta} - \lambda \mathcal{T} \right) = 0 \]
which gives:
\[ \lambda = \frac{\text{tr} \mathcal{T}_{\alpha,\beta}}{2} \left[ 1 \pm \sqrt{1 - \frac{4 \det \mathcal{T}_{\alpha,\beta}}{\left( \text{tr} \mathcal{T}_{\alpha,\beta} \right)^2}} \right] \] (A.25)
Since the right hand side of A.25 is independent of basis, it follows that \( \lambda \)'s are, also.
q.e.d.

2. Hermitian Matrices.
If $\Gamma_{x,\mu}$ is hermitian, then $\Gamma_{x,y}$ is hermitian for all bases $\vec{x}, \vec{y}$.

Proof:

$$\Gamma_{x,y} = A \Gamma_{x,\mu} A^* \quad (A.22)$$

$$\Gamma_{x,y}^{t*} = (A^{t*})^{t*} \Gamma_{x,\mu}^{t*} A^{t*}$$

which follows from:

$$(A \otimes c)^{t*} = c^{t*} B^{t*} A^{t*}$$

and hence:

$$\Gamma_{x,y}^{t*} = A \Gamma_{x,\mu} A$$

$$\Gamma_{x,y}^{t*} = \Gamma_{x,y} \quad (A.26)$$

q.e.d.

If A is hermitian with $\det A > 0$ and $\text{tr} A > 0$, then the eigenvalues of A are real and non-negative.

Proof: Since both $\det A$ and $\text{tr} A > 0$ then $\lambda_i > 0$ and $\lambda_{ii} > 0$

$$(\text{tr} A)^2 = (\lambda_i + \lambda_{ii})^2 = \lambda_i^2 + 2 \lambda_i \lambda_{ii} + \lambda_{ii}^2$$

$$(\text{tr} A)^2 - 4 \lambda_i \lambda_{ii} = (\lambda_i - \lambda_{ii})^2 > 0$$

$$(\text{tr} A)^2 > 4 \lambda_i \lambda_{ii} > 4 \left[ \lambda_i \lambda_{ii} - (\lambda_i)^2 \right] = 4 \det A$$

hence:

$$\frac{4 \det A}{(\text{tr} A)^2} < 1 \quad (A.27)$$

and $\lambda_i, \lambda_{ii}$ real and $> 0$.

q.e.d.

3. Exponentials of Matrices.

a. Diagonal Matrices

If D is diagonal with

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \quad (A.28)$$

then

$$e^D = \begin{pmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{pmatrix} \quad (A.29)$$
Proof: \[ e^D = \sum_{n=0}^{\infty} \frac{D^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} d, n & d \\ \vdots & \ddots \\ 0 & \cdots & d^n \\ \end{pmatrix} \]

\[ = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{d, n}{n!} \\ \vdots \\ \sum_{n=0}^{\infty} \frac{d^n}{n!} \\ \end{pmatrix} \begin{pmatrix} e^d \\ 0 \\ 0 \\ e^d \\ \end{pmatrix} \]

q.e.d.

b. Hermitian Matrices

Let \( U \) be the unitary matrix that diagonalizes \( H \), an hermitian matrix,

\[ D = U H U^t \]

or

\[ H = U^t D U \]

and

\[ H^n = U^t D^n U \]

so that

\[ e^H = U^t e^D U \]

(c. Differentiation)

The derivative of an exponential matrix is given by:

\[ \frac{d}{dx} e^{Ax} = A e^{Ax} = e^{Ax} A \]

\[ \text{with } A \text{ independent of } x. \]

Proof:

\[ \frac{d}{dx} e^{Ax} = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{A^n x^n}{n!} \]

\[ = \sum_{n=0}^{\infty} \frac{n A^n x^{n-1}}{n!} \]

\[ = A \sum_{k=0}^{\infty} \frac{A^n x^n}{n!} \]
\[
\frac{d}{dx} e^{A^n x^n} = \left( \sum_{n=0}^{\infty} \frac{A^n x^n}{n!} \right) A^n
\]

q.e.d.

d. Commutation Considerations.

If \( AB = BA \) then \( e^A e^B = e^B e^A \) and \( e^A e^B = e^{A+B} \).

Proof:
\[
e^{A} e^{B} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{A^n B^m}{n! m!} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{B^m A^n}{m! n!}
\]

so that:
\[
e^A e^B = e^B e^A
\]

For the last part, (i.e. \( e^A e^B = e^{A+B} \)) consider:
\[
e^{A+B} = \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!} = I + (A+B) + \frac{1}{2} (A^2 + 2AB + B^2) + \frac{1}{6} (A^3 + 3A^2 B + 3AB^2 + B^3) + \ldots
\]
since \( AB = BA \) this becomes:
\[
e^{A+B} = I + (A+B) + \frac{1}{2} (A^2 + 2AB + B^2) + \frac{1}{6} (A^3 + 3A^2 B + 3AB^2 + B^3) + \ldots
\]

Now consider:
\[
e^{\frac{A}{x}} e^{\frac{B}{x}} = \sum_{n=0}^{\infty} \frac{A^n}{n! x^n} \sum_{m=0}^{\infty} \frac{B^m}{m! x^m}
\]
\[
= \frac{1}{x} + \frac{A}{x} + \frac{1}{2} \frac{A^2}{x^2} + \frac{1}{6} \frac{A^3}{x^3} + \ldots
\]
\[
+ \frac{B}{x} + \frac{AB}{x^2} + \frac{1}{2} \frac{A^2 B}{x^3} + \frac{1}{6} \frac{A^3 B}{x^4} + \ldots
\]
\[
+ \frac{1}{2} \frac{B^2}{x} + \frac{1}{2} \frac{AB^2}{x^2} + \frac{1}{4} \frac{A^2 B^2}{x^3} + \ldots
\]
\[
+ \frac{1}{6} \frac{B^3}{x} + \frac{1}{2} \frac{A B^3}{x^2} + \ldots
\]

Diagonal summing of this shows it to be identical with the expression for \( e^{A+B} \).
Hence: \( e^A e^B = e^B e^A = e^{A+B} \) if \( AB = BA \)

q.e.d.

If \( AB \neq BA \) then it is true, in general, that \( e^A e^B \neq e^B e^A \)

and neither are equal to \( e^{A+B} \).

It is seen that care must be taken in dealing with exponential matrices as the scalar commutation properties are no longer valid. This should certainly be expected since even the simple product of two matrices does not follow the scalar property of multiplication commutation.
Appendix 2. **Line Shape**

1. **Derivation**

The problem of the exact line-shape when both collision-broadening and Doppler-broadening processes are important is not new. In astrophysics, this line-shape is known as the Voigt profile. Some tabulations of this do exist, but suffer from large approximation errors as well as inadequately covering the range of interest here.

The form factor for a purely collision-broadened resonance line consists of two terms: a resonant term and a non-resonant term.

\[
F_c(\nu, \nu_0, \Delta \nu_c) = F_{c,\text{res}}(\nu, \nu_0, \Delta \nu_c) + F_{c,\text{nonres}}(\nu, \nu_0, \Delta \nu_c) \quad (A2.1)
\]

with

\[
F_{c,\text{res}}(\nu, \nu_0, \Delta \nu_c) = \frac{1}{\Delta \nu_c} \left( \frac{\nu - \nu_0}{\Delta \nu_c} \right)^2 + 1
\]

and

\[
F_{c,\text{nonres}}(\nu, \nu_0, \Delta \nu_c) = \frac{1}{\Delta \nu_c} \left( \frac{\nu + \nu_0}{\Delta \nu_c} \right)^2 + 1
\]

\[
\Delta \nu_c = \text{collision half-width}
\]

\[
\nu_0 = \text{resonant frequency}
\]

\[\nu = \text{frequency}\]

The form factor for a purely Doppler-broadened resonance line is:

\[
F_0(\nu, \nu_0, \Delta \nu_d) = \sqrt{\frac{\nu_0^2}{\pi}} \frac{1}{\Delta \nu_d} - \ln 2 \left( \frac{\nu - \nu_0}{\Delta \nu_d} \right)^2 \quad (A2.2)
\]

with

\[
\Delta \nu_d = \text{Doppler half-width}.
\]

The thermal motion of the molecules causes the resonant frequencies to be Doppler-shifted. The appropriate form for the line-shape for one such Doppler-broadened transition is given by:

\[
F(\nu, \nu_0, \Delta \nu_c, \Delta \nu_d) = \int_{-\infty}^{\infty} F_c(\nu, \nu_0, \Delta \nu_c) \mathcal{P}(\Omega, \nu_0, \Delta \nu_d) \, d\Omega \quad (A2.3)
\]

where \(\mathcal{P}(\Omega, \nu_0, \Delta \nu_d)\) is the probability that the resonant frequency has
been Doppler-shifted to $\Omega$, and is given by:

$$\mathcal{P}(\Omega, \nu_0, \Delta \nu_d) = F_d(\nu_0, \Omega, \Delta \nu_d)$$  \hspace{1cm} (A2.4)

Hence, the true line-shape is seen to be the convolution of $F_d$ with $F_c$.

Equation A2.3 can be divided into two integrals; one for the resonant collision line-shape term, the other for the non-resonant collision line-shape term.

$$F(\nu, \nu_0, \Delta \nu_c, \Delta \nu_d) = F_{RES}(\nu, \nu_0, \Delta \nu_c, \Delta \nu_d) + F_{NONRES}(\nu, \nu_0, \Delta \nu_c, \Delta \nu_d)$$  \hspace{1cm} (A2.5a)

with

$$F_{RES}(\nu, \nu_0, \Delta \nu_c, \Delta \nu_d) = \int_{-\infty}^{\infty} F_c(\nu, \nu_0, \Delta \nu_c) F_d(\nu_0, \Omega, \Delta \nu_d) d\Omega$$  \hspace{1cm} (A2.5b)

and

$$F_{NONRES}(\nu, \nu_0, \Delta \nu_c, \Delta \nu_d) = \int_{-\infty}^{\infty} F_c(\nu, \nu_0, \Delta \nu_c) F_d(\nu_0, \Omega, \Delta \nu_d) d\Omega$$  \hspace{1cm} (A2.5c)

In the frequency range of interest, the $F_c$ term is essentially a delta-function compared to $F_d$ nonres; hence:

$$F_{NONRES}(\nu, \nu_0, \Delta \nu_c, \Delta \nu_d) \approx F_{NONRES}(\nu, \nu_0, \Delta \nu_c)$$  \hspace{1cm} (A2.6)

The calculations described below refer to Equation A2.5b.

2. Method of Calculation

There are two variables ($f$, $R$) that are more natural to the considerations at hand than the four previously mentioned ones ($\nu$, $\nu_0$, $\Delta \nu_c$, $\Delta \nu_d$). Let:

$$R = \frac{\Delta \nu_d}{\Delta \nu_c}$$  \hspace{1cm} (A2.7a)

$$f = \frac{\nu - \nu_0}{\Delta \nu_c} \quad \text{if} \quad R \leq 1$$  \hspace{1cm} (A2.7b)

$$f = \frac{\nu - \nu_0}{\Delta \nu_d} \quad \text{if} \quad R > 1$$  \hspace{1cm} (A2.7c)

also, let:

$$D(x) = \sqrt{\frac{\hbar}{2\pi}} e^{-\frac{\hbar}{\pi} x^2}$$  \hspace{1cm} (A2.8a)

$$C(x) = \frac{1}{1 + x^2}$$  \hspace{1cm} (A2.8b)

Then, if $R \leq 1$

$$F_{RES}(\nu, \nu_0, \Delta \nu_c, \Delta \nu_d) = \frac{1}{\Delta \nu_c} \int_{-\infty}^{\infty} C(f - R \xi) D(\xi) d\xi$$  \hspace{1cm} (A2.9a)
and if $R > 1$

$$F_{RES} (Y, Y_o, \Delta Y_c, \Delta Y_d) = \frac{1}{\Delta Y_d} \int_{-\infty}^{\infty} R \left[ R(F-x) \right] D(x) dx \quad (A2.9b)$$

The calculations were divided into two types: those with $R \leq 1$ and those with $R > 1$. These computations were all made on the IBM 7094 at the MIT Computation Center.

For $R \leq 1$, the procedure was as follows:

1. Choose equal $x$ intervals such that $x_{\text{min}} = -5.2$, $x_{\text{max}} = 5.2$, and $x_i - x_{i-1} = 0.02$

2. Approximate $D(x)$ and $C(x)$ by:

$$D(x) = D \left( \frac{x_i + x_{i+1}}{2} \right)$$

$$C(F-Rx) = C \left[ F - R \left( \frac{x_i + x_{i+1}}{2} \right) \right]$$

$$D(x) = 0 \quad ; \quad x < x_{\text{min}} \quad \text{or} \quad x > x_{\text{max}}$$

3. Multiply and sum

$$S_c (F, R) = \sum_{i = \text{min}}^{\text{max}-1} D \left( \frac{x_i + x_{i+1}}{2} \right) C \left[ F - R \left( \frac{x_i + x_{i+1}}{2} \right) \right] \Delta x_i \quad (A2.10)$$

then

$$F_{RES} (Y, Y_o, \Delta Y_c, \Delta Y_d) = \frac{1}{\Delta Y_c} S_c (F, R) \quad (A2.11)$$

For $R > 1$, the procedure was as follows:

1. Choose equal intervals of $x$ such that $x_{\text{min}} = -5.2$, $x_{\text{max}} = 5.2$, and $x_i - x_{i-1} = \frac{0.02}{R}$

   (This insures proper coverage of $C$.)

2. Approximate $D(x)$ and $C(x)$ as before

3. Multiply and sum

$$S_d (F, R) = \sum_{i = \text{min}}^{\text{max}-1} D \left( \frac{x_i + x_{i+1}}{2} \right) C \left[ R \left( F - \frac{x_i + x_{i+1}}{2} \right) \right] \Delta x_i \quad (A2.12)$$

now

$$F_{RES} (Y, Y_o, \Delta Y_c, \Delta Y_d) = \frac{1}{\Delta Y_o} S_d (F, R) \quad (A2.13)$$

$S_c$ and $S_d$ are shown in figure A2.1 as functions of $F$ with $R$ as a parameter. Using the 1962 U.S. Standard Atmosphere as a basis, $R$ vs. height is shown in Fig. A2.2; and the effective line width (full
width at points where true line-shape is half its maximum value) vs. 
height is shown in Fig. A2.3

An array of $S_C(F, R)$ and $S_D(F, R)$ values was computed and compu-
ter cards punched accordingly. A program was written which returns 
$F(v, v_o, \delta v_c, \delta v_d)$ when given $v, v_o, \delta v_c, \delta v_d$ as arguments and access 
to the array mentioned above. It searches the array for the appro-
priate value, doing a two dimensional linear interpolation to arrive 
at the result. A Lorentz shape (collision-broadening line-shape) 
matching the boundary value and the boundary slope is used for $F$'s 
too large to be found in the array.

The values of $F$ and $R$ for which the array was computed were cho-
zen to make the maximum interpolation error less than 0.3% over the 
ranges:

- $R = \text{or less than 10}$ all $F$
- $10 < R < R$ $F < \text{less than 10}$

which excludes only a very small, relatively unimportant range of the 
parameters.

The approximation errors in computing $S_C$ and $S_D$ are quite small. 
The truncation error is totally negligible (less than $10^{-6}$) due to the 
rapid decay of the Gaussian. The errors due to the piece-wise constant 
approximations to the functions are less than $10^{-4}$ for all values in 
the data array (i.e. \[ \left| \frac{S_{\text{true}}(F, R) - S_{\text{calc.}}(F, R)}{S_{\text{true}}(F, R)} \right| < 10^{-4} \]).

Hence, the overall error results from the interpolation error. 
This is always less than 0.3%.
$R$ vs. height 1962 U.S. Standard Atmosphere

$R = \frac{\Delta V_e}{\Delta V_c}^{10}$
Line Widths vs. Height  
1962 U.S. Standard Atmosphere

**FIGURE A 2.3**

Line Width  $\text{mc/s}$
3. Half-Widths

The Doppler half-width is given by:\[3.2.6\]
\[\Delta \nu_d = 3.581 \times 10^{-7} \sqrt{\frac{T}{M}} \nu_o\] (A2.14)
where: \(T = \) kinetic temperature, deg K
\(M = \) molecular weight.

The exact relation for \(Dv_c\) is not so well known,\[3.2.7 - 3.2.12\] The relation used in this paper was:
\[\Delta \nu_c = \left[ \alpha_{o_2} F_{o_2} + \alpha_{n_2} F_{n_2} \right] \rho \left( \frac{300}{T} \right)^x\] (A2.15)
where \(F_{o_2}, F_{n_2} = \) fractional part of air that is \(O_2, N_2\)
\(\alpha_{o_2}, \alpha_{n_2} = \) half-width per mm Hg at \(T = 300^oK\) for \(O_2-O_2, O_2-N_2\) collisions, respectively
\(\rho = \) pressure (total) in mm Hg
\(x = \) fraction between 0.5 and 1.0 (0.85 was used in calculations.)

For the range of interest, (heights less than 100 km,) the \(F_{o_2}, F_{n_2}\) terms are constants. \(\alpha_{o_2}\) is also a constant, but \(\alpha_{n_2}\) is taken as having a slight pressure dependence. The form for A2.15 that was used in the computations was:
\[\Delta \nu_c = \left( 0.4095 + \beta \right) \rho \left( \frac{300}{T} \right)^{0.85}\] (A2.16)
with \(\beta = 0.38025\) \(p = \) or greater than 250 mm
\(\beta = 1.14075\) \(p = \) or less than 19 mm
\(\beta\) linear with \(p\) between these values.

A2.14 gives \(Dv_c\) in m/s.

Actually, the pressure dependence of \(Dv_c/p\) is probably due (at least in part) to the increasing importance of three-body collisions and not to a change in the \(O_2-N_2\) collision cross-section. However, A2.16 is unreasonable with respect to the published laboratory measurements,
which are limited. Furthermore, a very large percentage of the radiation of interest in this report originates at heights with $p$ less than 19 mm. Appendix 3, however, indicates that the Van Vleck-Weisskopf line-shape may not adequately describe the true collision line-shape. Nonetheless, for calculation purposes, this shape was used.
Appendix 3  Balloon Experiment Results

In order to check the theory and to obtain more information regarding the line-shape, intensity, and pressure half-width, a series of balloon experiments was begun in 1963. An altitude of 30 km was chosen because, at that height, the resonance lines are distinguishable and that height is not difficult to achieve with helium filled balloons.

1. Theory

Using MODATS II as the atmosphere, the brightness temperature versus frequency at 30 km was calculated for the 9+ line for several zenith angles. Figure A3.1 shows these spectra. Frequencies of 20, 60, and 200 mc away from \( \sqrt{9+} = 61.1506 \) gc represent, approximately, peak, middle, and wing for this line. The brightness temperature versus height for these frequencies and a zenith angle of 60° is shown in Figure A3.2.

The weighting functions for an experiment such as this with the antennas directed upward (less than 90° from zenith) are quite different from those of a "downward" looking experiment (such as the previously mentioned satellite experiments). For the balloon experiments, all weighting functions have a maximum value at the observing height. Figure A3.3 shows the six weighting functions for the six channels of the balloon-borne radiometer at float (30 km). Any differential height selectivity is now a second-order effect.

2. Radiometer

A block diagram of the radiometer designed and built (not by the author) for this experiment is shown in Figure A3.4. The audio and d-c
BRIGHTNESS TEMPERATURE vs. FREQUENCY

9+ line Height = 30 km.

1962 U.S. Standard Atmosphere
\( V_o = 61,1506 \ \text{ge/s} \)

\( \Delta V \) \text{mc/s}

- zenith angle = 60°
- zenith angle = 75°
BRIGHTNESS TEMPERATURE vs. HEIGHT

9+ line

Zenith Angle = 60°

1962 U.S. Standard Atmosphere

V0 = 61.1506 km/s

ΔV = ± 20 mc/s
ΔV = ± 60 mc/s
ΔV = ± 200 mc/s

FIGURE A 3.2
BALLOON WEIGHTING FUNCTIONS

\[ Y(x) = (\gamma_b + \Delta \gamma) \]

\[ \gamma_b = 61.1506 \] γc

\[ \Delta \gamma = 20 \] mc

\[ \Delta \gamma = 60 \] mc

\[ \Delta \gamma = 200 \] mc

\[ \Delta \gamma = 20 \] mc

\[ \Delta \gamma = 60 \] mc

\[ \Delta \gamma = 200 \] mc

height = 30 km

FIGURE A 3.3
portion is time-shared by the three intermediate frequency (IF) amplifiers. The four input ports (60° antenna, 347°K calibration, 77°K calibration, 75° antenna; in order) are switched by a mechanical switch. Each input is applied for one minute. After a complete cycle of inputs, the IF is changed and the input cycle repeated, etc. Thus, a complete cycle requires 12 minutes.

The local oscillator is a Varian Klystron in an oil bath for cooling. The Klystron and Klystron power supplies require about 30 watts of power. An absorption cavity is used to lock the local oscillator onto the correct frequency.

The integration time is 10 sec. and bandwidths are 10, 10, and 15 mc for the 20, 60, and 200 mc IF's, respectively. The measured $\Delta T_{\text{RMS}}$'s for these channels are about 1°K, 2°K, and 2°K; yielding $T_N$'s of about 5,000°K, 10,000°K, and 12,000°K.

3. Balloon Flights

Four sets of balloon flights have been made from the NCAR Balloon Base, Palestine, Texas. The flight profile for all is the same:

1. Ascent to 30 km in about two hours
2. Float at 30 km for about four hours
3. Descent (by valve) to 5 km in about two hours
4. Parachute to ground.

The dates and nomenclature (balloon base) of the flights are:

3P - Aug. 1963
32P - 6 Feb. 1964
33P - 9 Feb. 1964
55P - 23 July 1964
57P - 28 July 1964
Flights 3, 32, and 57 yielded no data due to radiometer malfunctions; while flights 33, 55, and 88 yielded partial data. Flight 89 was the only flight yielding complete data.

Prior to 55P, the colder calibration load (770K) was a heated load at about 3100K. The telemetry was operating only for 88P and 89P.

4. Results

Figures A3.5-A3.9 show the results of the four flights from which data was obtained. The calculated curves are based on a temperature profile that is derived as follows:

0-30 km - measured by thermometer on balloon
30-40 km - transition to match two ends
40- MODATS II

The central transition region of this atmosphere affects the calculation for height above about 27-28 km only. The 40 km and up region has no effect on the calculations.

The general characteristics of the results are consistent from flight to flight. In particular, the $T_B$ for the 60 mc and 200 mc channels (both 60° antenna and 75° antenna) follows the kinetic temperature profile for 2-4 km longer than the theoretical curves. The $T_B$ then falls off consistent with theory but displaced in height an amount appropriate to the height of departure from the kinetic temperature. The 20 mc channel (both antennas) does not exhibit this phenomenon. Its $T_B$ departs from the kinetic temperature at the predicted height or at a height lower than predicted.
MEASURED BRIGHTNESS TEMPERATURE vs. HEIGHT

Flight 55P
Palestine, Texas
23 July 1964

Calculated Curves

IF freq. zenith angle

- 20 mc/s  60°
X 20 mc/s  75°
O 60 mc/s  60°
● 60 mc/s  75°
▲ 200 mc/s  60°
△ 200 mc/s  75°

(60 mc/s channel not operable for the latter part of ascent)

FIGURE A 3.6
At float height, the general results are similar. The 20 mc measured $T_B$'s are below the theoretical curves by 10-20°K while the 60 mc and 200 mc measured $T_B$'s are above the theoretical curves by 10-20°K.

Because of the shape of the weighting functions, (all monotonic decreasing with increasing height) the temperature of the layers closest to the balloon are the most important for all six measurements. Attempting to infer the temperature of the nearest layer from the 20 mc measurements only, makes the 60 mc and 200 mc measurements worse. No reasonable temperature profile is consistent with the measured values within the Van Vleck-Weisskopf line-shape theory. (A temperature-profile varying from $-14,000°K$ to $+8,000°K$ between 30 and 35 km could give the measured results, but is hardly reasonable.)

Likewise, attempting to fit the Van Vleck-Weisskopf line-shape to the measurements by varying the line-width parameter, $\Delta V$, is similarly unsuccessful. The 20 mc results are in conflict with the 60 mc and 200 mc results.

In view of the preceding, more measurements (especially, controlled laboratory experiments) are indicated. The answer is, perhaps, that a different line-shape actually applies; or, possibly, that $\frac{\Delta V}{P}$ has more pressure dependence than indicated by measurements to date.
## Appendix 4. Microwave O$_2$ Transition Frequencies

<table>
<thead>
<tr>
<th>N</th>
<th>$\nu_{N^+}$</th>
<th>$\nu_{N^-}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>56.2648$^*$</td>
<td>118.7505$^+$</td>
</tr>
<tr>
<td>3</td>
<td>58.4466$^*$</td>
<td>62.4863$^*$</td>
</tr>
<tr>
<td>5</td>
<td>59.5910$^*$</td>
<td>60.3061$^*$</td>
</tr>
<tr>
<td>7</td>
<td>60.4348$^*$</td>
<td>59.1642$^*$</td>
</tr>
<tr>
<td>9</td>
<td>61.1506$^*$</td>
<td>58.3239$^*$</td>
</tr>
<tr>
<td>11</td>
<td>61.8002$^*$</td>
<td>57.6125$^+$</td>
</tr>
<tr>
<td>13</td>
<td>62.4112$^*$</td>
<td>56.9682$^+$</td>
</tr>
<tr>
<td>15</td>
<td>62.9980$^+$</td>
<td>56.3634$^+$</td>
</tr>
<tr>
<td>17</td>
<td>63.5685$^*$</td>
<td>55.7839$^+$</td>
</tr>
<tr>
<td>19$^-$</td>
<td>64.1272$^+$</td>
<td>55.2214$^+$</td>
</tr>
<tr>
<td>21</td>
<td>64.6779$^+$</td>
<td>54.6728$^+$</td>
</tr>
<tr>
<td>23</td>
<td>65.2241$^*$</td>
<td>54.1294$^+$</td>
</tr>
<tr>
<td>25</td>
<td>65.7626$^+$</td>
<td>53.5960$^+$</td>
</tr>
<tr>
<td>27</td>
<td>66.2978$^+$</td>
<td>53.0695$^+$</td>
</tr>
<tr>
<td>29</td>
<td>66.8313$^+$</td>
<td>52.5458$^+$</td>
</tr>
</tbody>
</table>

* Measured$^{a4.1}$ (agree with calculations)

+ Calculated$^{a4.2}$
Appendix 5. Inversion of Radiative Transfer Equation

1. Scalar Equation.

The equation to be inverted is:

\[ T_{\theta_i}(\gamma_i, \Delta \gamma_i) = \int_0^H \mathcal{W}(H, \gamma_i, \Delta \gamma_i) \, d\gamma \]  \hspace{1cm} (A5.1)

with:

\[ \mathcal{W}(H, \gamma_i, \Delta \gamma_i) = \int \alpha(\gamma, t(H), \rho(H)) e^{\frac{\gamma_i + \Delta \gamma_i}{2}} \, d\gamma \]  \hspace{1cm} (A5.2)

\[ \alpha(\gamma, t(H), \rho(H)) = \text{scalar power absorption coefficient} \]

Since one is limited to a finite number (usually 4-10) of \( T_{\theta} \) measurements, the problem becomes that of finding a suitable approximation to \( t(h) \) that would cause the measured \( T_{\theta} \)'s a5.1-a5.3 Equation A5.1 assumes ground radiation to be negligible.

Suppose \( t(h) \) is to be inferred from \( N \) independent \( T_{\theta} \) measurements. Equation A5.1 then represents \( N \) equations (for \( i = 1, 2, \ldots N \)). Most inversion methods are iterative procedures which approximate \( t(h) \) by a finite sum of functions.

\[ t(H) = \sum_{j=1}^{N} \alpha_j f_j(H) \]  \hspace{1cm} (A5.3)

where the \( \alpha_j \)'s are real constants that are to be determined by the inversion and the \( f_j(h) \)'s are predetermined functions of \( h \). Typical \( f_j(h) \)'s that have been such use are:

1. Powers of \( h \) (i.e., \( 1, h, h^2, h^{N-1} \))
2. Unity for a range of \( h \), zero elsewhere
3. Sines and cosines
4. Various polynomials (e.g., Legendre, Tschebyschëff, etc.)
Since $WF(h, \gamma_i, \Delta \gamma_i)$ depends on $t(h)$ equations A5.1 are not separable for an easy solution for the $a_j$'s. The procedure, generally, is as follows:

1. Form an initial "guess" for $t_0(h)$. (This may be a "guess" for the $a_j$'s or, more likely, $t_0(h)$ may come from a standard atmosphere.)

2. Calculate $WF_0(h, \gamma_i, \Delta \gamma_i)$ from this $t_0(h)$ and the corresponding $p_0(h)$.

3. Assume $t_1(h)$ to be of the form:

$$t_2(h) = \sum_{j=1}^{N} A_j \frac{1}{r} f_j(h)$$  \hspace{1cm} (A5.4)

4. Substitute these into eqs. A5.1 to obtain a set of $N$ linear equations containing the $N$ unknowns $a_j$.

$$T_{\theta_i}(\gamma_i, \Delta \gamma_i) = \sum_{j=1}^{N} A_j \int_{0}^{h} f_j(h) WF_0(h, \gamma_i, \Delta \gamma_i) d\frac{h}{r}$$  \hspace{1cm} (A5.5)

$$i = 1, 2, \ldots, N$$

Writing A5.5 as a matrix equation

$$T_{\theta} = C_0 \cdot a'$$  \hspace{1cm} (A5.6)

with

$$T_{\theta} = \begin{pmatrix} T_{\theta_1} \\ T_{\theta_2} \\ \vdots \\ T_{\theta_N} \end{pmatrix}, \quad a' = \begin{pmatrix} a_1' \\ a_2' \\ \vdots \\ a_N' \end{pmatrix}$$

and

$$C_0 = N \times N \text{ matrix with elements}$$

$$C_{ij} = \int_{0}^{h} f_j(h) WF_0(h, \gamma_i, \Delta \gamma_i) d\frac{h}{r}$$

$$i, j = 1, 2, \ldots, N.$$
5. If $C_o$ is invertible, then
\[
\alpha' = C_o^{-1} T_b \]
(A5.7)

The first solution for $t(h)$ is now:
\[
t_k(h) = \alpha' \left[ T_b \right] \cdot F(h) \]
(A5.8)

where $F(h)$ is a column vector
\[
F(h) = \begin{pmatrix}
F_r(h) \\
F_z(h) \\
\vdots \\
F_\nu(h)
\end{pmatrix}
\]

The procedure is now repeated so that, if $K$ is the iteration number, A5.6 becomes:
\[
T_b = C_k^{-1} \alpha^k \]
(A5.9)

with
\[
C_{k-1,i,j} = \int_0^h f_j(h) W F_{k-1}(h, y_i, \Delta y_i) \, dh
\]
and $WF_{K-1}(h, y_i, \Delta y_i)$ being determined with $t_{K-1}(h)$.

Hence,
\[
t_k(h) = \alpha^k \left[ T_b \right] \cdot F(h) \]
(A5.10)

or
\[
t_k(h) = \left[ T_b \right]^{-1} F(h)
\]

Convergence occurs when $\left| t_K(h) - t_{K-1}(h) \right|$ falls within a previously specified criterion. (e.g., RMS $\left\{ t_K(h) - t_{K-1}(h) \right\} < \varepsilon$; $\left| t_K(h) - t_{K-1}(h) \right|_{\text{max}} < \delta$; etc.)

For any given inversion method, several questions need to be answered:
1. Does the iterative procedure converge?

2. If so, does it converge to the "best fit" of the expansion?

3. If so, is the "best fit" good enough? (i.e., Is the particular set of $F_j(h)$'s sufficient?)

4. How do uncertainties in the measurements affect the results?

Factors which will certainly affect the convergence and accuracy characteristics of an inversion are:

1. The noise inherent in the $T_\theta$ measurements
2. The accuracy with which the relation for $\omega$ is known
3. The particular set of $F_j(h)$'s that was chosen.


The inversion of the matrix equation of radiative transfer is much more difficult. At the present time, this has not been attempted (by anyone). However, for experiments such as suggested in this paper, a matrix inversion is unnecessary since all measurements are scalars (the $T_\theta$ of one polarization). The scalars, however, now depend on latitude, orientation (receiver polarization), and the terrestrial magnetic field vector; as well as the other dependent factors ($t$, $P$, etc.). The above factors must all be added to the list of considerations helping to determine whether an inversion method leads to a convergent answer and, if so, how accurate an answer results.


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BIOGRAPHY

The author was born in Miami, Florida on March 14, 1939. He entered Massachusetts Institute of Technology as a freshman in September, 1957. In June of 1961, he received an S.B. degree from M.I.T. and in June of the following year, he received his M.S. degree from the Institute.

In February of 1962, he was appointed a teaching assistant at M.I.T. and in June, 1964, an instructor. He has taught graduate and undergraduate subjects in the Electrical Engineering Department since February, 1962.

The author is a member of the Eta Kappa Nu and Sigma Xi professional honor societies and is also a member of the American Geophysical Union.

He is married to the former Elizabeth Frost and is the father of a son, William Benjamin, Jr.