ON THE SOLUTION OF FRICTIONAL CONTACT PROBLEMS

by

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Bachelor of Science in Mechanical Engineering, Purdue University (1988)

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Abstract

The problem of bodies in frictional contact is often encountered in industry. But the theoretical and numerical analysis of frictional contact problems is a difficult task. The nonlinear and irreversible nature of the contact problem requires a reliable and stable numerical algorithm for achieving iteration convergence and solution accuracy. The objective of this thesis is to gain insight into the solution of frictional contact problems with the finite element method so that more powerful and general procedures for solving contact problems can be designed. The thesis essentially consists of two parts.

The first part of the thesis consists of the finite element analysis of a problem which has been solved analytically by Dundurs and Comninou. The problem consists of an unbounded body with a semi-infinite cut. The body is first compressed normal to the cut and then a concentrated force is applied at the tip of the cut. The problem contains many interesting physical phenomena which allows us to check the reliability and accuracy of our finite element solutions and thus test the validity of the contact algorithm in the finite element program ADINA.

The second part of the thesis consists of the analysis of two problems. The purpose of considering the first problem is to find an efficient load stepping scheme for the finite element analysis of a typical elastic-plastic plate forming process. A numerical experiment was performed to find an efficient load step by investigating the relationship between the load step size and the contact segment size. The results were then generalized for problems with similar geometric configurations and nonlinear material properties. The insight obtained here can be used to establish automatic-time-stepping schemes for finite element solutions. The second contact problem solved was the insertion of a rubber seal into a rigid frame. The purpose was to solve for the vertical force required to insert the rubber seal into the rigid frame and also to find the final deformed shape of the rubber seal. This problem was also used to identify whether the experiences gained on load step and contact segment size selections with the elastic-plastic plate problem could also be
applied in the solution of this rubber seal insertion problem.

Thesis Supervisor: Dr. Klaus-Jürgen Bathe
Title: Professor of Mechanical Engineering
To my wife Jolene

For her love and support
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Chapter 1

Introduction

The problem of bodies in frictional contact is one that engineers frequently encounter. Much research has been devoted to finding a unified theory and general solution method for contact problems. Analytical solutions to many problems and numerical procedures for solving many classes of problems have already been published[2, 12, 13, 15, 16, 17, 21, 23, 24, 25]. However, the solution of these problems, either analytically or numerically, is not straightforward. Analytical solutions often involve abstract mathematical concepts that obscure the physics of the problem. Furthermore, analytical solutions are usually possible for non-frictional contact problems only and there are very few closed form solutions of contact problems with friction. Shibuya et al. analyzed an elastic contact problem with friction for a half-space indented by a flat annular stamp but he assumed full adhesion between the half-space and the annular stamp[28]. During the period 1981-1983, Dundurs and Comninou published a series of three papers analyzing a contact problem with Coulomb friction in the context of elasticity for an unbounded body with a semi-infinite cut[7, 8, 9]. Although the problem corresponds to no practical application,
it addresses a number of issues which provide some insight into modeling more complex frictional contact problems. Some of the interesting issues addressed are the distinction between weak and strong friction, effects of loading history, transition from separation to slip and from slip to stick, residual shear tractions left by unloading, and the presence of a slip zone for the case of weak friction[11]. These analytical solutions can play an important role in the validation of numerical algorithms used for solving contact problems.

With the advent of more powerful computers the onus has been placed on the numerical method for the solution of contact problems which are intrinsically nonlinear. Tremendous progress has been made in the solution of nonlinear problems with the finite element method. The numerical method most commonly used for solving nonlinear problems is the incremental Newton method in which the static and kinematic variables are updated at successive load steps until the load path is completely traced out. However, the selection of an appropriate load step is crucial for iteration convergence and considerable judgement may be required of the analyst. Among other aspects, the degree of geometric and material nonlinearity determines the size of the load step required for convergence. A very small load step is usually used to avoid gross overlap between the two bodies since this will usually cause fatal termination of the solution method. Too large a load step will not result in convergence and too small a load step will consume enormous computer time and memory. To advance the capabilities for solving contact problems, a number of adaptive incremental loading procedures have been proposed[14, 27]. Although the analysis may be more straightforward, the analyst still needs to use his/her
judgement to input certain solution parameters. The procedures shall then automatically adjust the size of the load step if convergence is not reached or if divergence is detected, hence the initial input by the user will affect the rate of convergence of the iterative scheme. Since most of these procedures are heuristic in nature, they usually work well for a certain class of problems but not for others. Therefore, until a general and always effective incremental solution algorithm is found there is need for more research to obtain insight into the solution of contact problems, insight that should help in designing more powerful and general procedures.

To obtain these insights, we have organized our studies into two parts:

1. The first part of this study, discussed in Chapter 2, consists of the finite element analysis of the problem discussed by Dundurs and Comninou. The problem consists of an unbounded body with a semi-infinite cut. First, the body is compressed normal to the cut and then a concentrated force is applied at the tip of the cut. The concentrated force is then unloaded and reloaded again. The finite element solutions for the surface tractions and the relative displacements of the contact surfaces are then compared with the corresponding analytical solutions obtained by Dundurs and Comninou. The detailed behavior of the contact surfaces given by the finite element solution are then checked with the behavior of the contact surfaces given by the analytical solution to ensure that the physical phenomena discussed by Dundurs and Comninou are also exhibited by the finite element solution.

2. The second part of the study consists of the analysis of two problems:
(a) The purpose of analyzing the first problem, discussed in Chapter 3, is to find an efficient load stepping scheme for the finite element analysis of a typical elastic-plastic plate forming process and study the physical and numerical phenomena of the contact problem and solutions. A numerical experiment is performed to find a load step/segment size ratio that will allow the iterative scheme to converge efficiently. The results are then generalized by investigating how the geometric dimensions and material properties affect the rate of convergence of the iterative scheme.

(b) The aim of analyzing the second problem, discussed in Chapter 4, is to find the vertical force required to insert a rubber seal into a rigid frame and also to find the final deformed shape of the rubber seal. The problem was analyzed for four different cases of coefficients of friction and was also used to identify whether the experiences gained on load step and contact segment size selections with the elastic-plastic plate problem could also be applied in the solution of this rubber seal insertion problem.

In our numerical studies, the general purpose finite element program ADINA was used[1]. The contact surface pairs (contactor and target surfaces) are modeled with groups of two noded segments with linear displacement interpolations between adjacent nodes. The contact constraints are imposed by the classical Lagrange multiplier method. Overlap occurs when the contactor nodes penetrate the target segments but target nodes are permitted to penetrate contactor segments. Distributed tractions on the contact
segments are used for deciding whether a node is sticking, sliding or releasing.
Chapter 2

Elastic Contact Problem with Friction

2.1 Introduction

In this chapter we analyze the elastic frictional contact problem discussed by Dundurs and Comninou in a series of three papers[7, 8, 9]. Although the problem does not correspond to a practical application, it allows the close study of the detailed behavior of the contact surfaces during loading, unloading and reloading.

The problem consists of an unbounded elastic body with a semi-infinite cut. The body is first compressed normal to the cut and then a concentrated force is applied at the tip of the cut (see Figure 2-1). When the angle between the concentrated force and the cut, $\alpha$, is given by $0 < \alpha < \pi/2$, separation of the contact surfaces near the tip of the cut occurs. The part of the contact surfaces immediately adjacent to the separation zone slips and the contact surfaces beyond the separation and slip zones stick. When $\pi/2 \leq \alpha < \pi$ there is no separation of the contact surfaces and the part of the contact
Figure 2-1: Geometric model of the unbounded body with a semi-infinite cut surfaces near the tip of the cut will slip and the contact surfaces beyond the slip zone are sticking.

Unloading and reloading phenomena in the presence of weak or strong friction was also discussed by Dundurs and Comninou. The case $f_k < \tan \alpha$ is defined as weak friction while the case $f_k > \tan \alpha$ is defined as strong friction where $f_k$ is the coefficient of kinetic friction. Since the ADINA finite element program does not include the use of a coefficient of kinetic friction, we will be using $f_k = f_s$ where $f_s$ is the coefficient of static friction. The extents of the separation, slip and stick zones and the distribution of the surface tractions differ for weak and strong friction during unloading and reloading of the concentrated force. There is no distinction between weak or strong friction during the initial loading phase of the concentrated force.

The solution presented by Dundurs and Comninou is based on the use of dislocation
theory[5, 19] and the singular integral equations obtained are solved for the surface tractions and the relative displacements between the contact surfaces of the cut[22, 23]. The problem is formulated by superimposing the elasticity solution with the dislocation theory solution and imposing the condition that the solution is bounded at the transition points[10], ie. from separation to slip and from slip to stick. The solution is singular at the point \( x = 0 \) since a solution that is bounded at both ends of the singular integral equation does not exist[22]. Although not all the physical boundary conditions are satisfied at the tip of the cut, it is conjectured that the analytical solution would still provide a good reference for the finite element solution.

The problem presents a few salient features to test the validity of a finite element formulation. There is a singularity at the tip of the cut where the concentrated force is applied and the boundary conditions vary along the contact surfaces of the cut. The problem was analyzed by Böhm using the finite element method but the mesh he used was too coarse[6]. Thus his results were not conclusive.

The finite element model of the problem used in this study and the loading history is shown in Figure 2-2. The size of the model is chosen such that the effects of the constraints on the boundary surfaces\(^1\) are minimized. Three boundary surfaces are constrained by rollers and the remaining boundary surface which is parallel to the cut is then displaced such that the normal tractions at the cut are equal to the desired pressure, \( p^\infty \).

A concentrated force is applied at the tip of the cut. The force is increased until it

---

\(^1\)To avoid confusion we use the term ‘boundary surfaces’ to describe the surrounding surfaces of the body and the term ‘contact surfaces’ to describe the surfaces of the cut.
$P = 2$

$p = 1$

$\alpha = 45^\circ$

$f = 0.2$

$E = 3000$

$\nu = 0.25$

Figure 2-2: Finite element model of the unbounded body with a semi-infinite cut and the loading history of the concentrated load $P$. 

22
reaches $P_1$, then it is unloaded to $P_2$ and finally it is reloaded to $P_3$. The shear and normal tractions, the gap and tangential slip between the upper and lower contact surfaces are then compared with the corresponding analytical solutions for both cases of weak and strong friction.

2.2 Overview of Analytical Solution

This section reiterates some of the main findings by Dundurs and Comninou and summarizes the nomenclature before presenting our numerical solutions. The definitions of the variables and the analytical solutions for the surface tractions and the relative displacements are given in the Appendix.

2.2.1 Initial Loading Phase for the Case with Separation

The extents of the separation and slip zones, $a$ and $b$ respectively, during loading are proportional to the magnitude of the concentrated force, $P$, and inversely proportional to the applied pressure, $p^\infty$. The normal tractions, $N(x)$, are zero in the separation zone and approach $-p^\infty$ asymptotically beyond the separation zone. The shear tractions, $S(x)$, are also zero in the separation zone, $0 < x < a$, but increase monotonically over the slip zone, $a < x < b$, and decay quickly to zero in the stick zone, $b < x < \infty$. When the concentrated force reaches $P_1$, the shear tractions are $S_1(x)$ and the extents of the separation and slip zones are $a_1$ and $b_1$ respectively. The distribution of the shear tractions, $S_1(x)$, are shown in Figure 2-3.

The analytical solutions for the extents of the separation and slip zones, the nor-
mal and shear tractions, the gap and tangential slip between the upper and lower contact surfaces of the cut after initial loading are given in the Appendix (See equations (A.1) to (A.6)).

2.2.2 Unloading and Reloading with Weak Friction

Upon complete loading, i.e. \( \dot{P} = 0 \), the upper and lower contact surfaces stopped moving relative to each other. Hence, we can consider all the contact surfaces as either separated, \( 0 < x < a_1 \), or sticking, \( a_1 < x < \infty \).

Upon unloading to \( P_2 \), the separation zone decreases from \( a_1 \) to \( a_2 \) and the gap closes between \( a_1 \) and \( a_2 \). For the case of weak friction, the upper and lower contact surfaces that have just come into contact between \( a_1 \) and \( a_2 \) are slipping relative to each other. At the same time, slipping is also extending beyond \( a_1 \) to \( b_2 \). Hence during unloading, a slip zone, \( a_2 < x < b_2 \), is expanding about the point \( a_1 \) into the separation and stick zones (See Figure 2-3). If the concentrated force is completely unloaded, i.e. \( P_2 = 0 \), residual shear tractions of constant magnitude, \( f_k p^\infty \), remain between \( 0 < x < b_1 / 2 \).

When the unloading process ends, i.e. \( \dot{P} = 0 \) and \( P_2 \geq 0 \), the contact surfaces can be regarded as either separated, \( 0 < x < a_2 \), or sticking, \( a_2 < x < \infty \), since there are no relative motions between the upper and lower contact surfaces. When reloading to \( P_3 \), the separation zone increases from \( a_2 \) to \( a_3 \). The part of the contact surfaces immediately adjacent to the separation zone will begin slipping again. The slip zone now extends from \( a_3 \) to \( b_3 \). When \( P_3 > P_1 \), the results obtained are the same as those obtained by direct loading. It should be noted that when \( P_3 = P_1 \), the size of the slip zone suddenly jumps
Figure 2-3: Shear tractions after initial loading phase, unloading and reloading with weak friction.

Figure 2-4: Relative positions of separation and slip zones after loading, unloading and reloading with weak friction.
from \(b_2\) to \(b_1\). The sudden change in size of the slip zone is due to the total eradication of the residual shear tractions between \(b_2\) and \(b_3\). The relative positions of \(a_1, b_1, a_2, b_2, a_3, b_3\) after reloading are shown in Figure 2-4.

The analytical results for the extents of the separation and slip zones, the normal and shear tractions and the tangential slip after unloading and reloading and the residual shear tractions after complete unloading for the case of weak friction are given by equations (A.7) to (A.21) in the Appendix.

2.2.3 Unloading and Reloading with Strong Friction

Similar to the case of weak friction, we can regard the contact surfaces as either separated or sticking at the end of the initial loading process. During unloading to \(P_2\), the separation zone decreases from \(a_1\) to \(a_2\) and the gap closes between \(a_1\) and \(a_2\). But for the case of strong friction, the contact surfaces that have just come into contact between \(a_1\) and \(a_2\) are sticking. Therefore, for the case of strong friction the stick zone follows immediately after the contracting separation zone. When the concentrated force is unloaded to \(P_2\), the shear and normal tractions remain at zero between \(0 < x < a_2\). Although there is no slipping during unloading, part of the residual shear tractions remain constant at \(p^\alpha \tan \alpha\) between \(0 < x < a_1\) upon complete unloading, ie. \(P_2 = 0\) (See Figure 2-5).

When the unloading process ends, ie. \(\dot{P} = 0\) and \(P_2 \geq 0\), the contact surfaces are either separated, \(0 < x < a_2\), or sticking, \(a_2 < x < \infty\). Upon reloading to \(P_3\), the separation zone expands from \(a_2\) to \(a_3\) and the contact surfaces beyond the separation zone are still sticking. The shear distributions are the same for reloading and unloading for the
Figure 2-5: Shear tractions after loading, unloading and reloading with strong friction.
Figure 2-6: Normal and shear tractions for the case of loading without separation case of strong friction. If the concentrated force is reloaded to $P_3 = P_1$, the shear tractions will be the same as those obtained by direct loading, i.e. all residual shear tractions have been eradicated. Loading beyond $P_1$ gives the same results as direct loading.

The analytical results for the extents of the separation and slip zones, the shear tractions and the tangential slip after unloading and reloading and the residual shear tractions after complete unloading for the case of strong friction are given by equations (A.22) to (A.33) in the Appendix.

2.2.4 Loading for the Case without Separation

When $\pi/2 \leq \alpha \leq \pi$, there is no separation between the upper and lower contact surfaces of the cut. The part of the contact surfaces immediately adjacent to the tip of the
cut slips, $0 < x < b$, and the rest of the contact surfaces stick, $b < x < \infty$. The normal tractions are singular at the tip of the cut and approach $-p^\infty$ asymptotically from $-\infty$ (See Figure 2-6). The shear tractions are also singular at the tip of the cut and decay to zero with a discontinuity in the slope of the curve at $x = b$. The analytical solutions for the extent of the slip zone, the normal and shear tractions, and the tangential slip for the case of loading without separation are given by equations (A.34) to (A.38) in the Appendix.

2.3 Finite Element Solution

The problem was solved numerically using the finite element program ADINA [1]. The finite element mesh used in this analysis is shown in Figure 2-2. The mesh consists of 1180 8-node plane strain elements with a total number of 7078 degrees of freedom. At the contact surfaces, 7-node plane strain elements were used because ADINA assumes a linear displacement variation between adjacent nodes of the contact surfaces. The model is initially loaded by prescribing a displacement to the free boundary surface with no friction between the contact surfaces of the cut to obtain the desired pressure, $p^\infty$. The program is then restarted with coefficient of kinetic friction, $f_k$, between the contact surfaces of the cut. The concentrated force is increased until $P_1$, unloaded to $P_2$ and then reloaded to $P_3$. A total of 6 load cases were studied (See Table 2.1). The load cases 2 and 4 were also analyzed using nonlinear elastic truss elements to model the cut. This analysis using nonlinear elastic truss elements serves as a check for the contact algorithm. The finite element solutions were also checked to identify whether the contact surfaces
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<th>$E$</th>
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<td>0.4</td>
<td>3000</td>
<td>0.25</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>No Separation</td>
<td>10</td>
<td>5</td>
<td>135$^\circ$</td>
<td>0.2</td>
<td>3000</td>
<td>0.25</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2.1: Load cases

are slipping, ie. $|S(x)| = f_s|N(x)|$, or sticking, ie. $|S(x)| < f_s|N(x)|$(Recall that we are using $f_s = f_k$).

2.3.1 Loading, Unloading and Reloading with Weak Friction

The results for load cases 1 to 4 are shown in Figures 2-7 to 2-16. The extents of the separation and slip zones obtained by the finite element method are smaller than that obtained by Dundurs and Comninou in their analysis. The plots for the tangential slip and the gap between the upper and lower contact surfaces explain the differences between the finite element solutions and the analytical solutions. The analytical solutions for the tangential slip and the gap at the tip of the cut are non-zero. However, the finite element solutions for the tangential slip and the gap are zero at the tip of the cut since the node at the tip of the cut is shared by the upper contact surface(contactor) and the lower contact surface(target). The reason that the analytical solutions for the tangential slip and the gap are non-zero at the tip of the cut can be found in the derivation of the analytical solution. In satisfying the condition that the solution is bounded at the transition points, the solution to the singular Cauchy integral equation is singular at $x = 0$ since a solution
Figure 2-7: Normal and shear tractions after initial loading
Figure 2-8: Residual-shear tractions after complete unloading and shear tractions after reloading
Figure 2-9: Tangential slip and gap between upper and lower contact surfaces of cut after initial loading

P = 2
P(INF) = 1
ALPHA = 45 DEG
FRICITION COEFFICIENT = 0.2
YOUNG'S MODULUS = 3000.0
V = 0.25
A = 0.00005
B = 0.00032
Figure 2-10: Tangential slip between upper and lower contact surfaces of cut after unloading and reloading
Figure 2-11: Normal and shear tractions after initial loading with the concentrated load applied horizontally
Figure 2-12: Tangential slip and gap between upper and lower contact surfaces of cut after initial loading with the concentrated load applied horizontally.
Figure 2-13: Normal and shear tractions after initial loading with the concentrated load applied vertically
Figure 2-14: Tangential slip and gap between upper and lower contact surfaces of cut after initial loading with the concentrated load applied vertically
Figure 2-15: Normal and shear tractions after initial loading with frictionless contact
Figure 2-16: Tangential slip and gap between upper and lower contact surfaces of cut after initial loading with frictionless contact.
that is bounded at both ends of the integral does not exist. Thus, the finite element solution differs from the analytical solution near the tip of the cut. Away from the tip of the cut, the finite element solution converges to the analytical solution. Figure 2-13 clearly shows the differences between the finite element solutions and the analytical solutions for the normal and shear tractions. While the analytical solutions for the normal and shear tractions remain at a finite value other than zero at the tip of the cut, the finite element solutions for the tractions are zero at the tip of the cut.

Also, for the case of frictionless contact, the finite element solution for the tangential slip between the upper and lower contact surfaces decreases to zero as $x$ increases because the boundary is finite and constrained while the analytical solution for the tangential slip remains constant (See Figure 2-16).

Load cases 2 and 4 were also analyzed using nonlinear elastic truss elements to model the cut (See Figure 2-17). The plots for the axial truss forces and the relative nodal displacements of the truss elements for load cases 2 and 4 are shown in Figures 2-18 and 2-19. Comparing these plots with the corresponding plots in Figures 2-20 and 2-21 which were obtained with the ADINA contact algorithm, we can see that the point where the separation zone ends is exactly the same. Thus, we have validated our numerical results. The results clearly show that the unloading and reloading characteristics predicted by Dundurs and Comninou are also present in the finite element solution. Furthermore, the finite element solutions for the relative locations of the transition points and the residual-shear tractions left by unloading agree with the analytical solutions.
Figure 2-17: Modeling of cut with nonlinear elastic truss elements
Figure 2-18: Axial truss forces and nodal displacements after initial loading with concentrated load applied horizontally (Truss element model of case 2)
Figure 2-19: Axial truss forces and nodal displacements after initial loading (Truss element model of case 4)
Figure 2-20: Normal nodal contact forces and displacements after initial loading with concentrated load applied horizontally (Case 2)
Figure 2-21: Normal nodal contact forces and displacements after initial loading (Case 4)
2.3.2 Loading, Unloading and Reloading with Strong Friction

The results for the case of loading, unloading and reloading with strong friction are shown in Figures 2-22 to 2-26. The same discrepancies in the extents of the separation and stick zones between the finite element solutions and the analytical solutions are also present. The reasons for the discrepancies are the same as those given in Section 2.3.1. The residual shear tractions after complete unloading are $p^\infty \tan \alpha$ as predicted by Dundurs and Comninou. Thus, the contact surfaces are sticking since $p^\infty \tan \alpha < p^\infty f_k$. After reloading the concentrated force to $P_3 = P_1$, the surface tractions, tangential slip and gap are the same as those reached by direct loading (See Figures 2-22, 2-24 and 2-26).

2.3.3 Loading for the Case without Separation

The results for the case of loading without separation are shown in Figures 2-27 and 2-28. In the finite element solutions, the normal and shear tractions are zero at $x = 0$ while the surface tractions in the analytical solutions are singular. However, the finite element solutions for the normal and shear tractions are almost equal to the analytical solutions except when $x$ is very small, i.e. near the tip of the cut. The finite element solution for the extent of the slip zone is also slightly smaller than that of the analytical solution.

The normal tractions approach $-p^\infty$ from $-\infty$ continuously and the shear tractions approach zero from $-\infty$ with a discontinuity in the slope of the curve at $x = b$, the transition point from slip to stick. The shear tractions have an infinite slope at the transition from slip to stick which is in agreement with the asymptotic analysis of such a transition[10].

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Figure 2-22: Normal and shear tractions after initial loading with strong friction
Figure 2-23: Residual-shear tractions after complete unloading and shear tractions after reloading with strong friction
Figure 2-24: Tangential slip and gap between upper and lower contact surfaces of cut after initial loading with strong friction
Figure 2-25: Tangential slip and gap between upper and lower contact surfaces of cut after unloading and reloading with strong friction
Figure 2-26: Shear tractions and tangential slip after reloading to $P_3 = P_1$ with strong friction
Figure 2-27: Normal and shear tractions after loading for the case without separation
Figure 2-28: Tangential slip and gap between upper and lower contact surfaces of cut after initial loading for the case without separation
2.4 Summary

Despite the differences in the sizes of the separation and slip zones, the finite element solution exhibits exactly the unloading and reloading characteristics predicted by Dundurs and Comninou. The distinction between weak and strong friction, the expanding slip zone during unloading for the case of weak friction, the absence of a slip zone during unloading for the case of strong friction, the differences in the residual shear tractions between weak and strong friction, and the relative positions of the separation, slip and stick zones are all clearly evident in the finite element solution.

The results obtained with the nonlinear truss element model of the cut were consistent with those obtained with the ADINA contact algorithm. The six load cases were selected to analyze the problem at the geometrical limits of the problem, ie. $\alpha = 0^\circ$, $\alpha = 90^\circ$ and $f_k = 0$, to test the reliability of the ADINA contact algorithm. The results obtained by the finite element method were consistent and agree very well with the analytical solutions obtained by Dundurs and Comninou. The finite element solution is therefore reliable and in fact makes more physical sense than the analytical solution because the mathematical model does not define the physical problem realistically at the tip of the cut.
Chapter 3

Forming of Elastic-Plastic Plate

3.1 Introduction

The forming of a plate is a common process in the metal-working industry. The problem involves continuously changing boundary conditions and is difficult if not impossible to solve analytically. Therefore, the problem has to be solved numerically. However, the selection of an appropriate load step is crucial for convergence of the iterative scheme and considerable judgement may be required of the analyst. The size of the load step that would result in rapid convergence of the iterative scheme depends on the degree of nonlinearity of the problem. Since the degree of nonlinearity varies during the loading process, a numerical algorithm with an adaptive load step would be the most efficient way of solving contact problems. Presently, a reliable and general adaptive incremental loading procedure is still not available although a number of heuristic algorithms have been proposed.

Therefore, until a reliable and general adaptive incremental loading procedure is avail-
able, there is still need for more research to obtain insight into the solution of contact problems in order that more general and reliable algorithms can be developed. Thus the purpose of this numerical experiment is to determine an incremental loading procedure such that the iterative scheme will converge efficiently.

The problem used for this numerical experiment consists of an elastic-plastic plate which is pressed into a rigid die (See Figure 3-1). This problem was suggested by G. Müller and H. Lenz[18]. The finite element model used in our analysis is shown in Figure 3-2. The vertical displacement is prescribed at the center of the plate and the displacement is incremented until the plate reaches the desired depth. Since the displacement is prescribed, the applied load is unknown. Hence, we investigate the relationship between the incremental displacement and the contact segment size since the segment size will affect the relative positions of the target and contactor nodes, the die and plate respectively, when the displacement is being incremented. Thus we seek an incremental displacement/segment size ratio that would allow the iterative scheme to converge efficiently for this type of problems. The problem is solved for a range of geometric dimensions and material properties so that the results can be generalized for similar problems. The solution times were then analyzed to determine how the rate of convergence of the iterative scheme is affected by the geometric dimensions and material properties. Finally, we tested the incremental displacement/segment size ratio identified as effective by using it to analyze a different problem.

In our finite element analysis, the full Newton iteration method was used for the equilibrium iterations because of its better convergence performance for such highly nonlinear
Figure 3-1: Geometric model of the forming of an elastic-plastic plate

Figure 3-2: Finite element model of the forming of an elastic-plastic plate
systems. The automatic restart routine in ADINA was used when convergence was not reached. A maximum of four automatic restarts was permitted since the use of too many automatic restarts would be inefficient. A maximum of 50 iterations was allowed for each load step since it is anticipated that many iterations would be required for regions of the loading which are highly nonlinear.

3.2 Results and Discussions of Finite Element Analysis

This study essentially consists of three parts. The first part is to study how the geometric dimensions and material properties affect the rate of convergence of the iterative scheme. The second part is to establish an incremental displacement/segment size ratio for which the iterative scheme would converge efficiently. In the last part, the incremental displacement/segment size ratio identified as effective is tested on a different problem to establish the general applicability of the results to similar types of problems.

To compare how each geometric dimension or material property affects the solution time\textsuperscript{1}, a total of twelve cases were analyzed (See Table 3-1). The displacement was prescribed incrementally and a total of 360 steps was used to prescribe the displacement completely. The incremental displacement for each step ranges from 0.1mm to 0.25mm. A varying incremental displacement instead of a constant incremental displacement was used to ensure convergence for each of the twelve cases. Also, the same finite element mesh was used for all twelve cases except for case 12. For case 12, a finer mesh was used for the plate because of the small corner radius of the die.

\textsuperscript{1}The solution times were obtained on a Convex C1 minicomputer
<table>
<thead>
<tr>
<th>Run</th>
<th># of incr. displ.</th>
<th>Time (secs)</th>
<th>Plate (nodes)</th>
<th>Die (nodes)</th>
<th>Variable</th>
<th>Total # of ite.</th>
<th># of ite. per incr. displ.</th>
<th>Time per ite. (secs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>360</td>
<td>6309.86</td>
<td>109</td>
<td>39,40</td>
<td>-</td>
<td>1555</td>
<td>4.32</td>
<td>4.06</td>
</tr>
<tr>
<td>2</td>
<td>360</td>
<td>4494.00</td>
<td>109</td>
<td>40,41</td>
<td>f = 0.0</td>
<td>958</td>
<td>2.66</td>
<td>4.69</td>
</tr>
<tr>
<td>3</td>
<td>360</td>
<td>6239.49</td>
<td>109</td>
<td>40,41</td>
<td>f = 0.05</td>
<td>1609</td>
<td>4.47</td>
<td>3.88</td>
</tr>
<tr>
<td>4</td>
<td>360</td>
<td>6322.69</td>
<td>109</td>
<td>39,40</td>
<td>f = 0.7</td>
<td>1606</td>
<td>4.46</td>
<td>3.94</td>
</tr>
<tr>
<td>5</td>
<td>360</td>
<td>5901.55</td>
<td>109</td>
<td>39,40</td>
<td>E=168000</td>
<td>1532</td>
<td>4.26</td>
<td>3.85</td>
</tr>
<tr>
<td>6</td>
<td>360</td>
<td>6417.66</td>
<td>109</td>
<td>39,40</td>
<td>E=252000</td>
<td>1611</td>
<td>4.48</td>
<td>3.98</td>
</tr>
<tr>
<td>7</td>
<td>360</td>
<td>5670.83</td>
<td>111</td>
<td>38,39</td>
<td>t = 2.0</td>
<td>1502</td>
<td>4.17</td>
<td>3.78</td>
</tr>
<tr>
<td>8</td>
<td>360</td>
<td>7180.10</td>
<td>106</td>
<td>40,41,42</td>
<td>t = 8.0</td>
<td>1703</td>
<td>4.73</td>
<td>4.22</td>
</tr>
<tr>
<td>9</td>
<td>360</td>
<td>5974.24</td>
<td>117</td>
<td>36,37</td>
<td>R = 25.0</td>
<td>1495</td>
<td>4.15</td>
<td>4.00</td>
</tr>
<tr>
<td>10</td>
<td>360</td>
<td>5901.60</td>
<td>111</td>
<td>38,39</td>
<td>R = 15.0</td>
<td>1528</td>
<td>4.24</td>
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<tr>
<td>11</td>
<td>360</td>
<td>6794.44</td>
<td>108</td>
<td>41,42</td>
<td>R = 5.0</td>
<td>1647</td>
<td>4.58</td>
<td>4.13</td>
</tr>
<tr>
<td>12</td>
<td>378</td>
<td>13,565.76</td>
<td>78</td>
<td>25,26</td>
<td>R = 2.0</td>
<td>2075</td>
<td>5.49</td>
<td>6.54</td>
</tr>
</tbody>
</table>

Table 3.1: Computation times for various geometric dimensions and material properties
The geometric and material variables are the thickness of the plate, the corner radius of the die, the Young's modulus of the plate and the coefficient of friction. From Table 3.1, we notice that the final points of contact between the plate and the die vary very little among the twelve cases. Figures 3-3 and 3-4 show that the normalized computation times vary almost linearly over a wide range of Young's moduli, friction coefficients, plate thicknesses and corner radii of the die.

It is also obvious that the coefficient of friction does not affect the solution time at all except for the special case of no friction. Since the displacement was prescribed in the same way for all the twelve cases, the solution time is a good indicator of how each geometric dimension or material property affects the rate of convergence of the iterative scheme.

The original and deformed shapes of the elastic-plastic plate for case 1 are shown in Figure 3-5. The plots for the vertical and horizontal reactions at the center of the plate versus the prescribed displacement are shown in Figure 3-6. The vertical reaction at the center of the plate is solely due to the reaction at the top node, node 6 in Figure 3-5, where the displacement is prescribed. Since the middle and bottom nodes, nodes 129 and 7 respectively, were not constrained vertically, the vertical reactions are zero at these two nodes. But nodes 6, 129 and 7 were all constrained in the horizontal direction. The Force-YY versus displacement curve shown in Figure 3-6 was obtained by summing the horizontal reactions at nodes 6, 129 and 7. Since the horizontal reactions at nodes 6 and 7 are almost equal in magnitude and acting in opposite directions, the total horizontal reaction is primarily due to the horizontal reaction at node 129(See Figure 3-7).
Figure 3-3: Normalized computation times for various Young's moduli and friction coefficients
Figure 3-4: Normalized computation times for various corner radii and plate thicknesses
Figure 3-5: Elastic-plastic plate before and after deformation
Figure 3-6: Reactions at the center of the plate during deformation
Figure 3-7: Horizontal reactions at nodes 6, 129 and 7 at displacement 40mm
magnitude of the total horizontal reaction is much smaller than the magnitude of the horizontal reactions at nodes 6 and 7. The reaction plots for the other cases have the same general profile.

Convergence difficulties were encountered at the regions where the discontinuities occur in the reaction curves in Figure 3-6. The discontinuities are due to the stick-slip behavior[26] at the contact points. Figures 3-8 to 3-10 show the plastic zones for consecutive time steps where a discontinuity is observed on the reaction plots. Note that the plastic zones of consecutive time steps differ significantly. The sudden unloading due to the stick-slip behavior causes some parts of the plate to change from plastic to elastic which results into sudden stiffening of the plate. The discontinuity in the plastic zone shown in Figure 3-8 when the displacement is 6.75mm is a result of the coarse mesh being used. The same discontinuity is also present in the effective stress band plot shown in Figure 3-11. On the contrary, the plastic zones shown in Figures 3-12 to 3-14 look almost similar for consecutive time steps for the smooth parts of the reaction plots.

Next, we solved another ten cases of the problem using different incremental displacements and two finite element meshes with different contact segment sizes(See Table 3.2). The incremental displacements were from 0.125mm to 1.0mm and the two segment sizes used were 0.5mm and 1.0mm. For comparison purposes, the problem was also solved with varying incremental displacements.

From Figure 3-15, the incremental displacement/segment size ratios with the shortest computation times were 0.5 for segment size 0.5(case 15 in Table 3.2) and 0.25 for segment size 1.0(case 20 in Table 3.2). Therefore, we can conclude that an incremental
Figure 3-8: Plastic zones at displacements 6.5mm and 6.75mm are shown in black
Figure 3-9: Plastic zones at displacements 24.75mm and 25.0mm are shown in black
Figure 3-10: Plastic zones at displacements 35.5mm and 35.75mm are shown in black
Figure 3-11: Effective stresses in plate at displacement 6.75mm
Figure 3-12: Plastic zones at displacements 4.5mm and 4.75mm are shown in black
Figure 3-13: Plastic zones at displacements 23.5mm and 23.75mm are shown in black
Figure 3-14: Plastic zones at displacements 33.5mm and 33.75mm are shown in black
<table>
<thead>
<tr>
<th>Run</th>
<th># of incr. displ.</th>
<th>Time (secs)</th>
<th>Total # of ite.</th>
<th>Time per ite.</th>
<th># of eqns</th>
<th># of contact segments</th>
<th>Size of contact segment</th>
<th>incr. displ.</th>
<th>subinc. max. (ave)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>360 (ref.)</td>
<td>6309.86</td>
<td>1555</td>
<td>4.32</td>
<td>4.06</td>
<td>567</td>
<td>64</td>
<td>0.5</td>
<td>0.048</td>
</tr>
<tr>
<td>13</td>
<td>193 (-38.9%)</td>
<td>3855.50</td>
<td>972</td>
<td>5.03</td>
<td>3.97</td>
<td>567</td>
<td>64</td>
<td>0.5</td>
<td>0.10 (.1)</td>
</tr>
<tr>
<td>14</td>
<td>320 (-6.4%)</td>
<td>5907.47</td>
<td>1414</td>
<td>4.42</td>
<td>4.18</td>
<td>567</td>
<td>64</td>
<td>0.5</td>
<td>0.125 (.101)</td>
</tr>
<tr>
<td>15</td>
<td>160 (-32.9%)</td>
<td>4234.76</td>
<td>930</td>
<td>5.81</td>
<td>4.55</td>
<td>567</td>
<td>64</td>
<td>0.5</td>
<td>0.25 (3)</td>
</tr>
<tr>
<td>16</td>
<td>107 (+11.5%)</td>
<td>7036.33</td>
<td>938</td>
<td>8.77</td>
<td>7.50</td>
<td>567</td>
<td>64</td>
<td>0.5</td>
<td>0.374 (1.19)</td>
</tr>
<tr>
<td>17</td>
<td>80 (+57.0%)</td>
<td>9907.21</td>
<td>930</td>
<td>11.63</td>
<td>10.65</td>
<td>567</td>
<td>64</td>
<td>0.5</td>
<td>0.50 (2.45)</td>
</tr>
<tr>
<td>18</td>
<td>193 (-63.6%)</td>
<td>2298.43</td>
<td>894</td>
<td>4.63</td>
<td>2.57</td>
<td>375</td>
<td>32</td>
<td>1.0</td>
<td>0.10 (1.02)</td>
</tr>
<tr>
<td>19</td>
<td>320 (-46.4%)</td>
<td>3381.91</td>
<td>1374</td>
<td>4.29</td>
<td>2.46</td>
<td>375</td>
<td>32</td>
<td>1.0</td>
<td>0.125 (1.00)</td>
</tr>
<tr>
<td>20</td>
<td>160 (-68.8%)</td>
<td>1964.16</td>
<td>763</td>
<td>4.77</td>
<td>2.57</td>
<td>375</td>
<td>32</td>
<td>1.0</td>
<td>0.25 (1.01)</td>
</tr>
<tr>
<td>21</td>
<td>80 (-30.9%)</td>
<td>4360.17</td>
<td>628</td>
<td>7.85</td>
<td>6.94</td>
<td>375</td>
<td>32</td>
<td>1.0</td>
<td>0.5 (1.55)</td>
</tr>
<tr>
<td>22</td>
<td>40 (+25.5%)</td>
<td>7918.60</td>
<td>634</td>
<td>15.85</td>
<td>12.49</td>
<td>375</td>
<td>32</td>
<td>1.0</td>
<td>1.0 (3.05)</td>
</tr>
</tbody>
</table>

Table 3.2: Computation times for various segment sizes and incremental displacements
Figure 3-15: Computation times versus incremental displacement/segment size ratios
<table>
<thead>
<tr>
<th></th>
<th># of incr. displ.</th>
<th>Time (secs)</th>
<th>Total # of ite.</th>
<th># of ite. per incr. displ.</th>
<th>Time per ite.</th>
</tr>
</thead>
<tbody>
<tr>
<td>plate</td>
<td>360</td>
<td>7180.10</td>
<td>1703</td>
<td>4.73</td>
<td>4.22</td>
</tr>
<tr>
<td>seal</td>
<td>400</td>
<td>17054.77</td>
<td>4110</td>
<td>10.28</td>
<td>4.15</td>
</tr>
</tbody>
</table>

Table 3.3: Comparison of solution data between seal insertion and plate forming problem displacement/segment size ratio of 0.25 to 0.5 would be an optimum ratio to use for achieving rapid convergence of the iterative scheme.

A totally different problem was then used to test the results. The test problem used is the insertion of an elastic-plastic seal as shown in Figure 3-16. An incremental displacement to segment size ratio of 0.25 was used. Since this problem has a thickness/length ratio of 0.15, we can compare the solution time with case 8 of Table 3.1 which has a thickness/length ratio of 0.1455. The comparison of the solution times is shown in Table 3.3. The solution time of 17054.77 seconds for the seal insertion problem might seem very large compared to the solution time of 7180.10 seconds for the plate forming problem. However, the total displacement to length ratio of the seal insertion problem is 2.75 times larger than that of the plate forming problem. Also, the seal insertion problem is harder to solve than the plate forming problem because it involves some unloading towards the end of the prescribed displacement when the contact area moves from the straight part of the seal to the curved ends of the seal. When the seal changes from plastic to elastic due to the unloading towards the end of the prescribed displacement, the system will stiffen and divergence can occur[3, 20]. Therefore, the solution time of 17,054.77 seconds for the seal insertion problem is acceptable.
Figure 3-16: Insertion of elastic-plastic seal
3.3 Summary

Although the results we have obtained are not based on any theoretical concepts, the results provide insight into why convergence problems are sometimes encountered during the iteration process. It must be noted though, that the results can only be applied to problems with similar geometric configurations and material properties.

Our results showed that the computation time is a linear function of the geometric dimensions and material properties. Thus, we can use the incremental displacement/segment size ratio of 0.25 to 0.5 for a wide range of problems and expect rapid convergence.

It was also observed in our numerical experiments that if convergence is not reached within 15 iterations in each load step then it is unlikely that it will ever be reached. In fact, convergence is usually reached within 4 to 8 iterations. Therefore, it is unnecessary to use a large iteration limit for each load step since this will result in excessive computation time. It is not efficient to use more than 4 automatic restarts since this will also result into excessive computational effort. We can use these results to establish automatic-time-stepping schemes for finite element solutions.

One important point to note is that in our analysis we have prescribed the displacement at the center of the plate while a vertical force is usually applied in the actual forming process. In the actual forming process, the vertical force is being increased gradually until the plate reaches the required depth. The plate is thus forced to deform without unloading of the vertical force. Hence, the force-displacement curve may be
considerably smoother in an actual forming process.
Chapter 4

Insertion of Rubber Seal

4.1 Introduction

The problem analyzed in this chapter consists of a rubber seal being inserted into a rigid frame. The aim is to determine the force required to insert the seal into the frame and also to find the deformed shape of the seal. The problem was also used to identify whether the experiences gained on load step and contact segment size selections with the elastic-plastic plate problem are also applicable to the solution of this rubber seal insertion problem. This problem has the same basic geometric configuration as the plate forming problem discussed in Chapter 3. The two differences are the material, which is now incompressible rubber, and the curved ends of the seal. The material model used is the Mooney-Rivlin model with $C_1 = 80, C_2 = 20$ and $\kappa = 10^6$. The geometric and finite element models are shown in Figures 4-1 and 4-2 respectively. A finite element mesh of 50 9-node plane strain elements was used. Displacement-pressure formulated elements were used since the displacement-based elements have difficulties in the analysis of problems
Figure 4-1: Geometric model of insertion of seal problem

Figure 4-2: Finite element model of insertion of seal problem
with incompressible material[4].

A displacement is prescribed at the center of the seal via a rigid truss until the seal reaches a depth of one inch. The vertical load and horizontal reaction at the center of the seal are plotted against the prescribed displacement. The problem was solved for four different cases of friction coefficients.

4.2 Results and Discussions of Finite Element Analysis

The original and final deformed shapes of the seal are shown in Figure 4-3 and the intermediate shapes at displacements of 0.40 inch and 0.80 inch are shown in Figure 4-4. The vertical load and horizontal reaction for friction coefficients 0.0, 0.2, 0.5 and 0.7 are shown in Figures 4-5, 4-6, 4-7 and 4-8 respectively. The vertical load is solely due to the vertical reaction at the node which lies on the top of the rigid truss where the displacement is prescribed. The total horizontal reactions are obtained by summing the horizontal reactions at the nodes which lie on the centerline of the plate.

The stick-slip phenomena, characterized by the discontinuities in the vertical load and horizontal reaction versus displacement curves, are more pronounced for larger coefficient of friction. Despite these discontinuities, the vertical load and horizontal reaction versus displacement curves for all the four friction coefficients have the same general profile. The vertical load reaches a maximum when the displacement reaches about 0.4 to 0.6 inch. After the vertical load reaches the maximum, the slope of the vertical load versus displacement curve changes suddenly when the displacement reaches about 0.8 inch. This is clearly shown in Figure 4-5 where the reaction plots are much smoother. The change in
slope is due to the contact area having moved from the straight part to the curved part of the seal (See Figure 4-4). Thus the rate of relative movement between the seal's contact surface and the frame's contact surface has been changed by the orientation of the seal's contact segments at the curved ends. When the displacement reaches about 0.95 inch, the slope of the vertical reaction curve changes again although not as sudden this time. The change in slope is due to the contact region having moved from the top half to the bottom half of the curved sides of the frame and this causes a sudden unloading of the seal.

A plot of the pressure distribution in the rubber seal is shown in Figure 4-9. The distribution of the pressure in the rubber seal is fairly continuously except for some small pressure jumps across the coarser meshes. This implies that the mesh we have used is fine enough for this problem and the results that we have obtained would have minimal errors due to discretization.

In our analysis, an incremental displacement/segment size ratio of 4.0 was used. This differs greatly from the incremental displacement/segment size ratio of 0.25 to 0.5 which was found to be the most efficient ratio to use for the solution of the elastic-plastic plate forming problem discussed in the previous chapter. Therefore, the experiences gained on the load step and segment size selections with the elastic-plastic plate problem are not directly applicable to the solution of this rubber seal insertion problem. This is not unexpected since the properties of elastic-plastic material and incompressible rubber material are very different.
Figure 4-3: Original and final deformed shape of seal
Figure 4-4: Deformed shape of seal at displacements 0.40 inch and 0.80 inch
Figure 4-5: Vertical and horizontal reactions at center of seal for friction coefficient 0.0
Figure 4-6: Vertical and horizontal reactions at center of seal for friction coefficient 0.2
Figure 4-7: Vertical and horizontal reactions at center of seal for friction coefficient 0.5
Figure 4-8: Vertical and horizontal reactions at center of seal for friction coefficient 0.7
Figure 4-9: Distribution of pressure in rubber seal
4.3 Summary

The highly discontinuous vertical load and horizontal reaction versus displacement curves are due to the stick-slip behavior typical in frictional contact problems with large deformations. However, in the actual insertion process a force is usually applied at the center of the plate instead of prescribing the displacement. Therefore, the effect of the stick-slip phenomenon will not be so predominant.

The solution times for the cases with friction coefficients 0.0, 0.2, 0.5 and 0.7 are 1142.42, 2050.00, 1778.50 and 1714.38 seconds respectively. The large difference in the solution times between the frictionless and frictional cases is due to the different algorithms used by ADINA for solving frictionless and frictional contact problems. The ADINA automatic-time-stepping routine was also used to ensure convergence during the equilibrium iterations.

The relatively continuous pressure distribution in the rubber seal indicates that the mesh that we have used for solving this problem is fine enough and the results due to discretization errors are minimal.

We have also found that the experiences gained on load step and contact segment size selections with the elastic-plastic plate forming problem are not directly applicable in the solution of this rubber seal insertion problem.
Chapter 5

Conclusions

We have presented the finite element solutions of three frictional contact problems in this thesis. Each problem provides insight into the solution of frictional contact problems and the physical phenomena governing the behavior of deformable bodies in contact. The first problem presented in Chapter 2 involves a body with a semi-infinite cut. The body is first compressed normal to the cut and a concentrated force is then applied at the tip of the cut. We compared our finite element solutions for the surface tractions and the relative displacements with the corresponding analytical solutions derived by Dundurs and Comninou. The finite element solutions agree well with the analytical solutions but there are some differences between the analytical and finite element solutions largely because the mathematical solution of Dundurs and Comninou does not satisfy all physical boundary conditions at the tip of the cut. However, we believe that this problem analyzed by Dundurs and Comninou is still a good reference for checking and validating finite element contact algorithms because the detailed behavior of the contact surfaces during unloading and reloading in the presence of weak or strong friction provides a rigorous
check on whether the contact algorithm has been properly formulated. The problem contributes to the understanding of complex issues such as the effects of the loading history, distinction between weak and strong friction and residual shear tractions left by unloading.

The purpose of analyzing the second problem was to find an efficient loading scheme for the finite element solution of a typical elastic-plastic plate forming problem. An incremental displacement/segment size ratio that would result in efficient convergence of the iterative scheme was obtained by investigating the relationship between the incremental displacement and contact segment size. The results were generalized to similar problems by studying how the convergence rate of the iterative scheme was affected by the geometrical dimensions and material properties. We found that an incremental displacement/segment size ratio of 0.25 to 0.5 would result in efficient convergence for similar problems. Our numerical experiment also showed that a maximum of 15 iterations is all that is required for convergence at each time step and no more than 4 automatic restarts should be used when convergence is not reached. The use of too many automatic restarts and large iteration limits would result in excessive computational effort.

The last problem that we analyzed was the insertion of a rubber seal into a rigid frame. The purpose was to solve for the vertical load and horizontal reaction at the center of the seal and also to find the deformed shape of the seal. It was found that the discontinuities in the reaction versus displacement curves were more pronounced when the friction coefficient between the rubber seal and frame is larger. The discontinuities were due to the stick-slip phenomena at the contact area. Despite the discontinuities, the
reaction versus displacement curves have the same general profile for different friction coefficients. The problem was also used to identify whether the experiences gained on load step and segment size selections with the elastic-plastic plate forming problem are applicable in the solution of the rubber seal insertion problem. But it was found that the effective incremental displacement/segment size ratio used in the solution of the rubber seal insertion problem is much larger than the ratio identified as effective for the solution of the elastic-plastic plate forming problem. Thus, the experiences gained on effective load step and segment size selections with the elastic-plastic plate forming problem cannot be directly applicable to the solution of the rubber seal insertion problem.
Appendix

A.1 Nomenclature

\( a_i \) = extent of separation zone
\( b_i \) = extent of slip zone
\( N_i(x) \) = Normal tractions at faces of cut
\( S_i(x) \) = Shear tractions at faces of cut
\( g_i(x) \) = \( u_y(x,0^+) - u_y(x,0^-) \) = gap between faces of cut
\( h_i(x) \) = \( u_x(x,0^+) - u_x(x,0^-) \) = tangential slip between faces of cut

A.2 Initial Loading Phase for Case without Separation

The analytical solutions for the initial loading phase\[7\] are:

\[
\begin{align*}
  a_1 & = \frac{\kappa - 1}{\pi(\kappa + 1)} \frac{P_1 \cos \alpha}{p^\infty} \\
  b_1 & = a_1 \left( 1 + \frac{\tan \alpha}{f_k} \right) \\
  N_1(x) & = -p^\infty \left( 1 - \frac{a_1}{x} \right)^{1/2} \quad a_1 < x < \infty \\
  S_1(x) & = -f_k p^\infty \left\{ \left( 1 - \frac{a_1}{x} \right)^{1/2} - \left( 1 - \frac{b_1}{x} \right)^{1/2} \right\} \quad a_1 < x < b_1 \\
  & \quad b_1 < x < \infty
\end{align*}
\]
\[ g_1(x) = \frac{\kappa + 1}{4\mu} p^\infty a_1 \left\{ \frac{\pi}{2} + \sin^{-1} \left( 1 - \frac{2x}{a_1} \right) - 2 \left[ \frac{x}{a_1} \left( 1 - \frac{x}{a_1} \right) \right]^{1/2} \right\} \\
0 < x < a_1 \]  
(A.5)

\[ h_1(x) = -\frac{\kappa + 1}{4\mu} f_k p^\infty b_1 \left\{ \frac{\pi}{2} \left( 1 - \frac{a_1}{b_1} \right) + \sin^{-1} \left( 1 - \frac{2x}{b_1} \right) - 2 \left[ \frac{x}{b_1} \left( 1 - \frac{x}{b_1} \right) \right]^{1/2} \right\} + \frac{\kappa + 1}{4\mu} f_k p^\infty a_1 \right\} \\
0 < x < a_1 \]  
(A.6)

\[ a_2 = \frac{\kappa - 1}{\pi(k+1)} \frac{P_2 \cos \alpha}{p^\infty} \]  
(A.7)

\[ b_2 = \frac{1}{2} (a_1 + a_2) + \frac{1}{2} (a_1 - a_2) \frac{\tan \alpha}{f_k} \]  
(A.8)

\[ N_2(x) = -p^\infty \left( 1 - \frac{a_2}{x} \right)^{1/2} \]  
(a_2 < x < \infty)  
(A.9)

\[ S_2(x) = f_k p^\infty \left\{ \left( 1 - \frac{a_2}{x} \right)^{1/2} H(x - a_2) - 2 \left( 1 - \frac{b_2}{x} \right)^{1/2} H(x - b_2) \right\} \]  
\left\{ \left( 1 - \frac{b_1}{x} \right)^{1/2} H(x - b_1) \right\} \]  
0 < x < \infty  
(A.10)

\[ h_2^\ast(x) = \frac{\kappa + 1}{4\mu} f_k p^\infty \left\{ \pi \left( b_2 - a_1 \right) - a_1 \left\{ \sin^{-1} \left( 1 - \frac{2x}{a_1} \right) - \right\} \right\} + 2b_2 \left\{ \sin^{-1} \left( 1 - \frac{2x}{b_2} \right) \right\} \\
-2 \left[ \frac{x}{b_2} \left( 1 - \frac{x}{b_2} \right) \right]^{1/2} \right\} \]  
(a_2 < x < a_1)  
(A.11)

\[ h_2^\ast(x) = \frac{\kappa + 1}{4\mu} f_k p^\infty \left\{ \pi b_2 + 2b_2 \left\{ \sin^{-1} \left( 1 - \frac{2x}{b_2} \right) \right\} \\
-2 \left[ \frac{x}{b_2} \left( 1 - \frac{x}{b_2} \right) \right]^{1/2} \right\} \]  
(a_1 < x < b_2)  
(A.12)
\[ h_2 = h_1 + h_2^c \]  \hspace{1cm} (A.13)

Upon complete unloading of the concentrated force, i.e. \( P_2 = 0 \),

\[ a_2 = 0 \]  \hspace{1cm} (A.14)

\[ b_2 = \frac{1}{2} a_1 \left( 1 + \frac{\tan \alpha}{f_k} \right) = \frac{1}{2} b_1 \]  \hspace{1cm} (A.15)

and the residual shear tractions are:

\[ S_2(x) = f_k p^\infty \left\{ 1 - 2 \left( 1 - \frac{b_2}{x} \right)^{1/2} H(x - b_2) + \left( 1 - \frac{b_1}{x} \right)^{1/2} H(x - b_1) \right\} 0 < x < \infty \]  \hspace{1cm} (A.16)

The analytical solutions for the reloading phase[8] are:

\[ a_3 = \frac{\kappa - 1}{\pi (\kappa + 1)} \frac{P_3 \cos \alpha}{p^\infty} \]  \hspace{1cm} (A.17)

\[ b_3 = \frac{1}{2} \left[ a_3 \left( 1 + \frac{\tan \alpha}{f_k} \right) - a_2 \left( \frac{\tan \alpha}{f_k} - 1 \right) \right] \]  \hspace{1cm} (A.18)

\[ S_3(x) = f_k p^\infty \left\{ - \left( 1 - \frac{a_3}{x} \right)^{1/2} H(x - a_3) + 2 \left( 1 - \frac{b_3}{x} \right)^{1/2} H(x - b_3) + 2 \left( 1 - \frac{b_2}{x} \right)^{1/2} H(x - b_2) + \left( 1 - \frac{b_1}{x} \right)^{1/2} H(x - b_1) \right\} 0 < x < \infty \] \hspace{1cm} (A.19)

\[ h_3^c(x) = -\frac{\kappa + 1}{2\mu} f_k p^\infty b_3 \left\{ \frac{\pi}{2} + \sin^{-1} \left( 1 - \frac{2x}{b_3} \right) - 2 \frac{x}{b_3} \left( 1 - \frac{x}{b_3} \right)^{1/2} \right\} + \frac{\kappa + 1}{4\mu} f_k p^\infty a_2 \left\{ \frac{\pi}{2} + \sin^{-1} \left( 1 - \frac{2x}{a_2} \right) - 2 \frac{x}{a_2} \left( 1 - \frac{x}{a_2} \right)^{1/2} \right\} \]

\[ H(a_2 - x) + \frac{\kappa + 1}{4\mu} f_k p^\infty a_3 \left\{ \frac{\pi}{2} + \sin^{-1} \left( 1 - \frac{2x}{a_3} \right) \right\} \]

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\[ -2 \left[ \frac{x}{a_3} \left( 1 - \frac{x}{a_3} \right) \right]^{1/2} \right \} H(a_3 - x) \quad 0 < x < b_3 \]  
\[ h_3(x) = h_2 + h_3^c \]  

Note that the above equations for reloading are for part 1 of the reloading phase, \( P_3 < P_1 \), since loading beyond \( P_1 \) gives similar results as direct loading.

### A.4 Unloading and Reloading with Strong Friction

The analytical solutions for the unloading phase[8] are:

\[ a_2 = \frac{\kappa - 1}{\pi (\kappa + 1)} \frac{P_2 \cos \alpha}{p^\infty} \]  
\[ b_2 = a_2 \]  
\[ S_2(x) = p^\infty \left\{ \tan \alpha \left( 1 - \frac{a_2}{x} \right)^{1/2} H(x - a_2) - (f_k + \tan \alpha) \left( 1 - \frac{a_1}{x} \right)^{1/2} \right\} H(x - a_1) + f_k \left( 1 - \frac{b_1}{x} \right)^{1/2} H(x - b_1) \quad 0 < x < \infty \]  
\[ h_2^c = -\frac{\kappa + 1}{4\mu} p^\infty a_2 \tan \alpha \left\{ \frac{\pi}{2} + \sin^{-1} \left( 1 - \frac{2x}{a_2} \right) - 2 \left[ \frac{x}{a_2} \left( 1 - \frac{x}{a_2} \right) \right]^{1/2} \right\} \]  
\[ H(a_2 - x) + \frac{\kappa + 1}{4\mu} p^\infty a_1 \tan \alpha \left\{ \frac{\pi}{2} + \sin^{-1} \left( 1 - \frac{2x}{a_1} \right) \right\} \]  
\[ -2 \left[ \frac{x}{a_1} \left( 1 - \frac{x}{a_1} \right) \right]^{1/2} \right \} \quad 0 < x < a_1 \]  
\[ h_2(x) = h_1 + h_2^c \]  

Upon complete unloading, ie \( P_2 = 0 \),

\[ a_2 = 0 \]  

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\[ S_2(x) = p^\infty f_k \left\{ \frac{tan\alpha}{f_k} - \left( 1 + \frac{tan\alpha}{f_k} \right) \left( 1 - \frac{a_1}{x} \right)^{1/2} H(x - a_1) \right. \\
+ \left. \left( 1 - \frac{b_1}{x} \right)^{1/2} H(x - b_1) \right\} \quad 0 < x < \infty \quad (A.28) \]

The analytical solutions for the reloading phase[8] are:

\[ a_3 = \frac{\kappa - 1}{\pi (\kappa + 1)} \frac{P_3 cos\alpha}{p^\infty} \quad (A.29) \]

\[ b_3 = a_3 \quad (A.30) \]

\[ S_3(x) = p^\infty \left\{ tan\alpha \left( 1 - \frac{a_3}{x} \right)^{1/2} H(x - a_3) - (f_k + tan\alpha) \left( 1 - \frac{a_1}{x} \right)^{1/2} \right. \\
H(x - a_1) + f_k \left( 1 - \frac{b_1}{x} \right)^{1/2} H(x - b_1) \right\} \quad 0 < x < \infty \quad (A.31) \]

\[ h_3^c = \frac{\kappa + 1}{4\mu} p^\infty a_2 tan\alpha \left\{ \frac{\pi}{2} - 2 \left[ \frac{x}{a_2} \left( 1 - \frac{x}{a_2} \right) \right]^{1/2} + sin^{-1} \left( 1 - \frac{2x}{a_2} \right) \right\} \]

\[ H(a_2 - x) - \frac{\kappa + 1}{4\mu} p^\infty a_3 tan\alpha \left\{ \frac{\pi}{2} - 2 \left[ \frac{x}{a_3} \left( 1 - \frac{x}{a_3} \right) \right]^{1/2} + sin^{-1} \left( 1 - \frac{2x}{a_3} \right) \right\} \quad 0 < x < a_3 \quad (A.32) \]

\[ h_3(x) = h_2 + h_3^c \quad (A.33) \]

Note that the shear tractions given by (A.31) is for \( P_3 < P_1 \) since loading beyond \( P_1 \) gives similar results as direct loading.

### A.5 Loading for the Case without Separation

The analytical solutions for the initial loading phase without separation[7] are:

\[ a_1 = 0 \quad (A.34) \]
\[ b_1 = \frac{\kappa - 1}{\pi (\kappa + 1)} \frac{P \cos \alpha}{p^\infty} \left( 1 + \frac{\tan \alpha}{f_k} \right) \]  \hspace{1cm} (A.35)

\[ N_1(x) = -p^\infty \left\{ 1 - \frac{f_k}{f_k + \tan \alpha 2|x|} \right\} \quad -\infty < x < \infty \]  \hspace{1cm} (A.36)

\[ S_1(x) = f_k p^\infty \frac{f_k}{f_k + \tan \alpha 2|x|} - p^\infty \left\{ \begin{array}{ll}
1 & 0 < x < b_1 \\
1 - \left(1 - \frac{b_1}{x}\right)^{1/2} & b_1 < x < \infty
\end{array} \right. \]  \hspace{1cm} (A.37)

\[ h_1(x) = -\frac{\kappa + 1}{4\mu} f_k p^\infty b_1 \left\{ \frac{\pi}{2} + \sin^1 \left(1 - \frac{2x}{b_1}\right) \right\} \]  \hspace{1cm} 0 < x < b_1 \]  \hspace{1cm} (A.38)
References


