Equilibrium and Stability Studies of Strongly Shaped Tokamaks

by

S. Pekka Hakkarainen


Submitted to the Department of Nuclear Engineering in partial fulfillment of the requirements for the degree of

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Applied Plasma Physics

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Abstract

This thesis is concerned with the cross-sectional shaping of Tokamak experiments in thermonuclear fusion research. The work includes a study, based on a simple constant pressure model for the plasma region, to optimize the cross-sectional shape given practical experimental constraints. The results of this study are then tested using a more sophisticated plasma model, where the plasma region is close to what is thought to exist in the laboratory. In order to make experimental studies concentrating on the effects of plasma shaping easier, we also present a method for fast determination of the shape of a laboratory plasma from magnetic measurements. The work is motivated by a search for improved Tokamak performance to achieve plasma conditions where self-sustaining fusion reactions become possible.

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Omistan tämän työn
Äidille, Isälle ja Susan’ille
Täydestä sydämestäni
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S.P.H., January 1988
1. Introduction

Tokamaks as thermonuclear fusion devices have been the subject of intense theoretical and experimental interest ever since the concept was introduced in the Soviet Union in the late 1960's and successful experiments performed with the T-3 Tokamak\textsuperscript{[1]}. A tokamak consists of a toroidal plasma column held in place by a magnetic field that is produced by currents in a variety of equilibrium coil systems in the vicinity of the plasma as well as the current flowing in the plasma itself (See Figure (1.1)). This type of equilibrium is susceptible to many instabilities, which limit such plasma parameters as the pressure and electric current. The most severe of these instabilities occur in present experiments on a time scale of a few microseconds and are referred to as disruptions. Any successful experiment must be designed so as to be stable against disruptive instabilities. Theoretically, disruptions can be related to fluid like instabilities known as magnetohydrodynamic (MHD) modes — the plasma behaves collectively as a fluid, not as a collection of independent charged particles.

Classification of Instabilities

In the ideal MHD model, which neglects plasma resistivity, disruptive instabilities are produced by a combination of plasma pressure and plasma current flowing parallel to the magnetic field and are referred to as kink modes. They exhibit relatively long wavelengths perpendicular to the magnetic field and are, therefore, characteristic globally inside the plasma. When the plasma surface is displaced during such an instability, the kink mode is said to be external, meaning that the magnetic fields in the vacuum region outside the plasma are also perturbed. Otherwise, when only the plasma region changes during the instability, the kink mode is classified to be internal. The external
kink mode is theoretically the most persistent instability for a tokamak, and becomes unstable at sufficiently high values of the plasma current, regardless of the details of the plasma pressure and current distribution. The corresponding limit on the average plasma current gives an absolute upper bound for the current in an experiment[^2]. While internal modes often set stricter limits on the total plasma pressure and current than the external kink mode, these instabilities can, to some extent, be stabilized by modifying the pressure and current distributions themselves.

Tokamak performance can also be limited by another class of MHD modes, *viz.* those that are driven by the component of the plasma current flowing perpendicular to the magnetic field. Through the ideal MHD momentum equation,

\[ \nabla p = J \times B, \]  

(1.1)

these modes can be analysed in terms of the plasma pressure gradient and are, therefore, customarily referred to as pressure driven modes. Such perturbations have very short wavelengths perpendicular to the magnetic field and are localized to regions where the pressure gradient exceeds some threshold value. The pressure driven modes do not generally lead to disruptions due to their local character — rather they set a limitation on the plasma pressure, but not the current, that can be contained in a given experiment.

Pressure driven modes are usually further classified into interchange and ballooning modes. The former resemble a Rayleigh–Taylor instability becoming unstable when the magnetic field lines are convex toward the plasma thus allowing the plasma to push the field lines in the same direction that they tend to bend towards under their own tension. In tokamaks, the magnetic geometry is arranged in such a way as to achieve average favourable curvature and the interchange instability only appears as a localized instability at the magnetic axis, leading to a necessary condition for stability known as the Mercier criterion[^3]. The ballooning modes are internal modes which are like the interchange instability, except that the perturbation varies along the magnetic field, taking advantage of the fact that in a tokamak the magnetic field curvature is alternating between favourable and unfavourable along a field line. This leads to an overall "ballooning" like structure, and these modes set definite limits on the total
achievable plasma pressure. Such modes can be stabilized most effectively by letting the plasma pressure drop below the instability threshold value.

Previous Numerical Work

MHD modes have interested theorists for quite some time. Due to the complicated magnetic geometry of a tokamak, a lot of this work has had to be done numerically. Purely analytic models have been developed for extremely thin bicycle-tire-like tori with very specific plasma pressure and current distributions — analytic extensions of these models to more conventional “fat” experimental machines have been complicated and remain inadequate. Of course, all experimental machines, such as JT-60 in Japan, JET in Europe, TFTR at Princeton, D-III-D in San Diego, Alcator-C at MIT and
the future experiments CIT and Alcator-C-MOD, have been supported by extensive numerical studies to ensure and improve on MHD stability, but due to the very large number of variables in this type of investigation, these studies have always been limited in scope while attempting to address a large number of issues involved. In particular, such studies have performed a limited scan over those variables determining the cross-sectional shape of the plasma while focusing on the problem of optimizing over those parameters that determine the details of the plasma pressure and current distribution in any given machine.

In the early days of tokamak experiments, stability studies were limited to circular cross-section plasmas, as almost all the experimental devices were circular in shape. The Alcator-C\textsuperscript{[4]} experiment discovered that a circular cross-section tokamak enabled us to reach a Lawson parameter\textsuperscript{[5]} of $n\tau_E > 0.8 \times 10^{20} \, m^{-3} sec$, ($n$ is the electron density and $\tau_E$ the energy confinement time), which is required for thermalized break-even at a higher temperature than was obtained in this experiment. A break-even condition is reached when the plasma pressure is high enough so that the fusion power produced equals the power spent on maintaining the plasma equilibrium. However, it was discovered that circular plasmas are not sufficient for reaching ignition, where the charged $\alpha$-particles produced in a fusion reaction, eg.

$$^2D + ^3T \rightarrow ^4\alpha(3.5\text{MeV}) + ^1n(14\text{MeV}),$$

(1.2)

can alone account for sustaining the operation and the neutrons can be used for generating output power. Other plasma configurations were investigated leading to the discovery that elongated cross-sections allowed for a larger total plasma current and larger total pressure without exceeding the thresholds of instability for kink and ballooning modes. A parametric study on the effect of changing the detailed shapes of the pressure and current profiles was performed for the "Dee-shaped" JET cross-section and some other similar ones, and the results were expressed in the form of a "scaling law", known after its discoverer as the Troyon Scaling law\textsuperscript{[6]}:

$$\beta = C_T \frac{I}{aB} \%.$$  

(1.3)
Here $\beta$ is the ratio of the average plasma pressure to the average magnetic energy, and thus measures the efficiency of plasma confinement by the magnetic field. $I$ is the plasma current in $MA$, $a$ the minor radius of the tokamak in meters and $B$ the magnetic field at the magnetic axis inside the plasma in Tesla (See Figure (1.2)). $C_T$ is known as the Troyon constant, and its value is $3.5 - 4$ depending on the details of the geometry and the plasma regions. These details might not seem so interesting if it were not for the fact that the values of $\beta$ for circular plasmas are not quite sufficient for self-sustained fusion operation, i.e. operation beyond the ignition limit. However, improvements by orders of magnitude are not required, either. It then becomes a task for the physicist to find out how to arrange the geometry and plasma properties of the experiment so as to reach this ignition condition in a stable fashion.

The ignition condition can be expressed in terms of plasma $\beta$ in the following simple manner. The fusion power output from the $\alpha$-particles in a $D - T$ reaction is given by

$$P_\alpha = \frac{n_i^2}{4} \langle \sigma v \rangle Q_\alpha,$$

(1.4)
where \( n_i \) is the ion density (\( D \) and \( T \)) in the plasma, \(<\sigma v>\) is the fusion cross-section averaged over the velocity distribution of the ions and \( Q_\alpha \) is the energy of the \( \alpha \)-particles, taken to be 14 \( MeV \). The density of the plasma is here taken to be a constant. Assuming that the power losses from the plasma can be estimated by

\[
P_L = \frac{3n_i T}{\tau_E}, \tag{1.5}
\]

where the temperature \( T \) is taken to be a constant and \( \tau_E \) is the energy confinement time of the system, we obtain the ignition condition by equating the fusion power produced in the \( \alpha \)-particles to the power losses,

\[
P_\alpha = P_L. \tag{1.6}
\]

The plasma \( \beta \) in this model is given by

\[
\beta = \frac{4\mu_0 n_i T}{B_0^2}, \tag{1.7}
\]

where \( B_0 \) is the applied magnetic field strength. Eliminating \( n_i \) in favour of \( \beta \) we obtain

\[
\frac{B_0^2\beta}{4\mu_0 \tau_E} = \frac{Q_\alpha B_0^4}{64\mu_0^2} \beta^2 \left( \frac{<\sigma v>}{T^2} \right)_M, \tag{1.8}
\]

where the power output has been maximized by maximizing \(<\sigma v>/T^2\) as a function of temperature. This yields, after rearranging,

\[
\beta \tau_E = \frac{16\mu_0 Q_\alpha}{B_0^2} \left( \frac{<\sigma v>}{T^2} \right)_M^{-1}, \tag{1.9}
\]

in other words, for a given confinement time and construction of the experiment (magnetic field and temperature), a definite plasma \( \beta \) is required for ignition to be reached. A more complete derivation of the ignition condition and related analysis is given by Freidberg and Wesson\(^7\). In fusion machines that produce net electric power, it is projected that values of \( \beta \) in excess of 10% are needed. The results of the circular cross-section Tokamaks Alcator–C and TFTR\(^8\) show that values of \( \beta \) about 0.5 – 2% can be reached without shaping the plasma.
At first sight it would seem from equation (1.3) that we can improve on the performance ($\beta$) by simply increasing the plasma current. This indeed is true, even in practice, but only up to a certain point. Associated with equation (1.3), one also needs to know how kink modes limit the total plasma current. These limits have traditionally been expressed in terms of the so-called safety factor, which is defined as

$$q_\psi = \frac{RB_\phi}{2\pi} \int \frac{dl}{R^2 B_p}, \quad (1.10)$$

where $R$ is the major radius of the tokamak, $B_\phi$ is the toroidal and $B_p$ the poloidal magnetic field and the integral is taken around the curve tracing the plasma cross-section. $q_\psi$ is the ratio of how many times the field lines twist around in the toroidal direction over the number of times in the poloidal direction, i.e. a measure of how the magnetic field twists around the plasma surface. It has been established that, at zero plasma pressure, the kink limit can be expressed in terms of a simple limit on $q_\psi$, such as $q_\psi > 2$ for circular and slightly elongated cross-sections$^6$. Due to the fact that $B_p$ is directly proportional to the plasma current through Ampère's law, this limit then effectively sets a maximum value for the plasma current (at zero pressure) for stable operation. However, it has since been discovered that the limit on $q_\psi$ varies from plasma cross-section to another, being equal to 3 or 4 for some elongated cross-sections$^6$. Moreover, the limit does not appear to coincide with an integer value when the plasma pressure is increased from zero. Effectively this last observation suggests, that more than one phenomenon is responsible for setting the limits of tokamak operation.

Mathematical analysis of ballooning modes is greatly simplified when use is made of the fact that the modes are localized inside the plasma, i.e. they have short wavelengths perpendicular to the magnetic field. The ballooning equations can then be solved numerically for any tokamak shape, and the results are usually presented in terms of an ($s, \alpha$) diagram. Here, $s$ denotes the magnetic shear and defined by

$$s = 2\psi \frac{d q_\psi}{q_\psi} \frac{d \psi}{d \psi}, \quad (1.11)$$

$\psi$ being the toroidal magnetic flux function, and $\alpha$ is a measure of the local pressure gradient and is defined by

$$\alpha = -q_\psi^2 R_0 \beta'. \quad (1.12)$$
Figure 1.3 \((s, \alpha)\) stability diagram for ballooning modes.

An example of such a diagram is shown in Figure (1.3), which shows that at a given magnetic shear the plasma is stable to small pressure gradients and becomes unstable when \(\alpha\) exceeds a threshold value. This is the ballooning instability that we have discussed. Following the work of Coppi\(^{[10]}\), however, we also find the surprising result that when the pressure gradient is increased even further, the ballooning mode is restabilized. This "second region of stability" is the basis for the PBX–experiment at Princeton University.

Both kink and ballooning modes contribute to the value of the coefficient \(C_T\) in equation (1.3). Which limit dominates over the other is largely a matter of what the cross-sectional shape of the plasma is like and how the details of the pressure and current distributions in the plasma region are arranged. In the so-called high–\(\beta\) ordering, where use is made of the assumption that the plasma minor radius is very much smaller than the major radius, ballooning modes do not, in fact, set a limit on \(\beta\) itself, but rather on the quantity \(\beta q_*^2\), where \(q_*\) is defined as

\[
q_* = \frac{2A_p B_\phi}{R \mu_0 I},
\]  

(1.13)
with $A_p$ being the cross-sectional area of the plasma. In effect, $q_*^{-1}$ measures the total plasma current $I$ in any given discharge. From the viewpoint of the ballooning mode analysis, it then seems that $\beta$ can be increased by simply decreasing the toroidal component of the magnetic field — the ballooning threshold will not change in terms of $\beta q_*^2$, which is independent of $B_\phi$, and $\beta$ will increase. In relaxing the assumption of large aspect ratio ($= R/a$), a dependence on $q_*$ itself is introduced. This dependence is important in regions where the magnetic shear is low, i.e. $s \ll 1$. In a tokamak, $s$ is small near the magnetic axis, where the ballooning analysis yields a stability limit against local interchange modes — the Mercier criterion. It is essentially through this criterion on the on-axis value of $q_\psi$ of the plasma that the ballooning modes indirectly set a limit on the overall plasma $\beta$ for circular tokamaks of finite aspect ratio. Yamazaki and his coworkers discovered\(^{[11]}\), however, that when making plasma cross-sections that are more favourable to ballooning stability, such as bean and crescent shaped plasmas with the tips pointing toward the middle of the torus, no separate limit is observed in terms of $q_*$. Thus, for these shapes ballooning modes contribute to the value of the coefficient $C_T$ in Troyon Scaling law, equation (1.3), but not to the limit on the overall plasma current. Furthermore, it has also turned out that external kink modes set a more severe limit on the achievable plasma $\beta$ than ballooning modes do.

In the high-$\beta$ ordering, the equation determining stability against kink modes turns out to be a function of two variables, $\beta/\epsilon n^2$ and $nq_*$, where $\epsilon$ is the inverse aspect ratio ($= a/R$) and $n$ the so-called toroidal mode number giving the number of wavelengths for the perturbations going the long way round the torus. From this observation, it is clear that the most severe limits are obtained for the long wavelength $n = 1$ modes — we leave aside the axisymmetric $n = 0$ modes for the time being, since the analysis, results and methods of stabilization are quite different for these modes. In arbitrary ordering, a separate dependence on $q_*$ is introduced, which has led to the introduction of the so-called infernal modes by Manickam and coworkers at Princeton University\(^{[12]}\). The current understanding is, however, that in stabilizing these modes by lowering the total $\beta$, the $n = 1$ modes are the most persistent ones and the last ones to become
completely stable.

Some very successful experiments have been designed with the stability calculations based on the Ideal MHD model, and in particular on stability against ideal ballooning and kink modes. The results from the PBX-experiment indicate that a bean shaped tokamak can reach plasma $\beta$-values of about 5%, while plasmas stable to 6% $\beta$ have been produced with the D-III-D experiment. Both of these experiments exceed the result obtained with the Dee-shaped JET tokamak. The design specifications for the Compact Ignition Tokamak (CIT) set the plasma $\beta$ at 7–8%.

The overall picture then appears to be the following: In order to approach the ignition condition and improve stability over the original circular cross-section devices, we need configurations that allow stable operation at higher values of $\beta$. Since it is not clear that much experimental control can be exercised over the details of the pressure and current distributions — in fact, there are many claims of the inevitability of "profile consistency", or the fact that these distributions will in practice follow some strict empirical rules no matter what the experimentalist does — it seems that shaping the plasma cross-section offers much more room for improvement. It has also been discovered, that by coincidence both ballooning and kink modes may favour the same type of cross-sectional shape. It is then natural to inquire what such an optimal shape would be like. It is the purpose of this work to provide a partial answer to this generally very ambitious problem.

This Work

For the reasons discussed above, in Chapter 2 we limit ourselves to investigating the pressure driven external kink modes. Furthermore, pressure and current distributions will be fixed in order to concentrate solely on the effects of cross-sectional shaping of the plasma. We first investigate the model that takes the plasma pressure to be a constant with an infinitely sharp drop to zero at the plasma boundary. In this model, all the current flows on the surface of the plasma. This sharp boundary model does not suffer from the computational problem of having a complicated plasma equilibrium which
takes a long time to evaluate numerically — the shape optimization can be performed quickly and, within the assumptions of the model, reliably. However, due to the nature of the model, only external modes can be excited, while internal modes will have to be left for those models that compute the whole plasma equilibrium with more realistic pressure and current distributions. Due to axial symmetry in a tokamak, it is possible to analyse all toroidal mode numbers \( (n) \) separately. In the past, only \( n = 1 \) modes have been investigated numerically, but in this work we perform a simultaneous shape optimization with respect to both the \( n = 1 \) and the axisymmetric \( (n = 0) \) modes. It is also ensured that all optimal shapes are stable to \( n = 2 \) modes.

Due to the simplicity of the sharp boundary model, the numerical values obtained will also be subject to debate. Generally, it has turned out that this model predicts values for critical plasma \( \beta \) for stability which are larger by factors of the order of about three than those obtained from the full diffuse equilibrium calculations.

Nevertheless, the advantages of investigating this model are many. Firstly, due to its simplicity, the model can be exhaustively studied. This allows us to define the problem very sharply — in particular, since practical optimization problems usually are accompanied by some physical constraints, it will become clear what the proper constraints as well as definitions of optimized quantities will be. When these constraints are properly defined, the optimal shape against external kink modes is found to be a “peapod”, or a shape that has relatively pointy ends and a circular outer edge combined with some indentation on the inside of the torus. Such a shape has critical \( \beta \) values larger by a factor of 2.5 than a circular shape and a current limit which is considerably higher than for a circle. Secondly, some general consequences of the definitions and constraints can be discussed and practical conclusion may be drawn. These include such issues as “honest” definitions of the plasma \( \beta \) and designs of the toroidal field coil system. Thirdly, despite the fact the numerical values obtained may be overly optimistic, it is nevertheless possible to qualitatively compare different cross-sectional shapes and gain an understanding of the overall driving forces in the system. Finally, it seems unlikely that the topological properties of the stable region of operation would change when going
from the sharp boundary plasma to a diffuse plasma model. Given this assumption, some qualitatively correct conclusions can be drawn from the sharp boundary model concerning stability limits for external kink modes. In particular, it will be possible to look for a more useful description of the limit on the total plasma current than has been used in the past. It will be shown that for all possible cross-sectional shapes the kink limits are better described in terms of $q_*$ rather than $q_\psi$.

In Chapter 3 we next test the results obtained with the sharp boundary model using a more realistic prescription of the plasma region. In this chapter, we are able to investigate the impact of our "honest" definition of the plasma $\beta$ when determining optimal cross-sectional shapes for tokamak operation while retaining pressure and current distributions that are of more than academic interest. As hinted to above, however, this investigation is necessarily of limited scope due to the large number of independent parameters involved. Nevertheless, it is shown that if parameters measuring the efficiency of an experiment to contain a given amount of plasma (such as $\beta$) are defined in such a way that all the externally created energy in the system is accounted for, shapes similar to those obtained with the sharp boundary model are also favourable here. In other words, the sharp boundary model is seen to address the correct physical issues. A comparison is made between the optimal "peapod" shape from the sharp boundary model and the more conventional bean and Dee-shaped plasmas of existing experiments. As a result of this study, it is seen that the "peapod" shape exhibits more robust stability properties than a bean or Dee-shaped plasma, having a higher current limit and a wider region of stability. The ballooning analyses of these cross-sectional shapes indicate that the Troyon coefficients for the three shapes are close to equal, while the kink limits allow higher plasma currents for the "peapod". The result is a higher stable value of $\beta$ for a "peapod" than for the other two shapes.

Finally, in Chapter 4 we turn to a different but related question, that of determining the plasma shape in an experiment. In a typical situation, magnetic measurements are available at a surface outside the plasma region, such as the wall of the vacuum chamber, but little accurate detail is usually known about quantities inside the plasma.
This problem turns out to be ill-posed in a mathematical sense, and leads to interesting conclusions about how to best extract information concerning the shape of the plasma boundary. It is shown that in most cases, the plasma shape can be determined very accurately and systematically, which is important for the experimentalist if the plasma performance is to be studied as a function of its cross-sectional shape. The resulting numerical code is fast and reliable, and is based on an application of the vector Green's theorem and a Fourier analysis of $\psi$, the poloidal magnetic flux, and $B_p$, the poloidal magnetic field. The Fourier analysis is performed in terms of an angle-like variable $\nu$, which is used to parametrize a series of surfaces in the vacuum region between the wall and the plasma. It is shown that the results of calculating $\psi$ and $B_p$ on the plasma surface are as accurate as the measurements given at the vacuum chamber wall. Furthermore, an optimal number of Fourier harmonics is shown to exist for each of the calculation surfaces. This allows us to maximize the accuracy of the calculation.
1.1 References


2. Sharp Boundary Model of Plasma

Summary of Previous work

Global kink modes have interested plasma theorists for quite some time, because some of the earliest plasma experiments, the Z-pinches, were found to be very unstable to plasma perturbations that had "kink" like characteristics. Despite the fact that considerable doubt exists in the community concerning the accuracy and reliability of the sharp boundary model, it has been investigated by a number of authors\textsuperscript{[1-9]}, primarily because of the simplicity of its equilibrium.

It is frequently found that the stability limits given by the sharp boundary model are considerably higher than those given by diffuse plasma models\textsuperscript{[1][2]}. This can be explained in terms of the model itself: internal plasma instabilities driven by the local current and pressure gradients are excluded from the model by construction. The fact that this model predicts plasma $\beta$'s for marginal stability, which are higher than those obtained using more realistic diffuse plasma models, can be thought to be a result of the fact that in a diffuse model the plasma has to be stable to both global modes and those local modes that are driven by gradients in the pressure and current profiles. The sharp boundary model treats only the global modes, so that as well as excluding ballooning and Mercier\textsuperscript{[10]} modes, such low $n$ (toroidal mode number) modes as internal kink modes and the infernal modes of Manickam\textsuperscript{[11]} are also not included. For this reason, when numerical codes for analyzing these plasma modes together with the global modes became available, the sharp boundary model was soon forgotten.
The sharp boundary model has one advantage, however, over the more complicated ones: its equilibrium is simple and analytic and its stability calculation can be reduced to an evaluation of integrals over the plasma surface only, making the calculation computationally very fast. Moreover, the most general plasma perturbations can be included in the calculation by analyzing only the normal component of the perturbation vector on the plasma surface. It has also been found, that even though the values of critical $\beta$ for stability invariably are overly optimistic, their trends are consistent for each cross-sectional shape of the plasma. For these reasons, the sharp boundary model is convenient as a starting point for calculations aiming at optimizing the plasma shape against global kink modes, which is the problem we are concerned with in this chapter.

Most calculations in the past have involved assumptions of large aspect ratio and have accordingly made use of expansions in $\epsilon$, the inverse aspect ratio. The calculations have typically been done in one of two regimes, the conventional tokamak regime[3][4][5] which assumes $\beta \sim \epsilon^2$ or the high $\beta$ tokamak regime[6][7][8], where the $\beta$ scaling is taken to be $\beta \sim \epsilon$. Freidberg and Grossmann[9] developed a procedure for calculating the plasma stability in a very elegant way for an arbitrary shaped tokamak plasma with no assumptions about the magnitude of the aspect ratio, and, consequently, no a priori model for the magnitude of the plasma $\beta$. They used Green's theorem to allow them to express the perturbed magnetic energy of the plasma and vacuum regions in terms of the perturbation vector at the plasma surface and were thus able to express the total potential energy of the system as a function of the normal component of the surface perturbation. They treated the $n = 1$ modes and investigated the stability of five cross-sectional shapes, the circle, ellipse, triangle, square and the doublet shape, using a coordinate mesh on the plasma surface corresponding to equal arc lengths. It is an extension of this treatment that will be discussed in this chapter. A more convenient parametrization of the plasma surface will enable us to efficiently optimize the shape in terms of the parameters describing it, viz. quantities such as elongation, triangularity, indentation, etc.
About This Work

The stability analysis is carried out using Fourier analysis methods for expressing the normal component of the plasma displacement on the surface of the plasma. The assumption of axial symmetry results in the decoupling of the toroidal harmonics, in other words perturbations periodic in the direction the long way round the torus may be analysed one by one. As has already been pointed out, theoretically the $n = 1$ mode is the most unstable one in the large aspect ratio limit, if we for the moment do not consider the axisymmetric $n = 0$ modes. However, in order to provide a check in the case of an arbitrary aspect ratio, both the $n = 1$ and $n = 2$ modes will be analysed in this work. It is found that the $n = 2$ mode is stable to higher values of plasma $\beta$ than the $n = 1$ mode, confirming the theoretical prediction obtained from the large aspect ratio limit.

As far as the axisymmetric perturbations are concerned, it is possible to analyse them very effectively in the sharp boundary model. Whereas the Universal Minimizing Principle of Ideal MHD\cite{12} shows that for the non-axisymmetric modes the most unstable perturbations are incompressible,

$$\nabla \cdot \xi = 0,$$

(2.1)

where $\xi$ is the perturbation vector, this need not be the case for the axisymmetric ones. It will be shown, that in the sharp boundary model, the most unstable $n = 0$ modes always have a constant compressibility,

$$\nabla \cdot \xi = K,$$

(2.2)

with $K$ a constant. Theoretical considerations will not allow us to determine the value of the constant, but our numerical simulations have shown that, in fact, $|K| \leq 10^{-9}$ in all the cases that we have investigated. Nevertheless, the fully compressible formulation is retained in the analysis of the axisymmetric modes.

Finally, the sharp boundary model is sometimes also called the skin current model, implying that all the equilibrium current density is concentrated on the plasma surface.
That a skin current distribution implies constant pressure inside the plasma (the sharp boundary model) is immediately clear from the Ideal MHD momentum equation

\[ \nabla p = J \times B. \tag{2.3} \]

However, a constant pressure inside the plasma does not necessarily imply a current distribution that is concentrated at the plasma surface. For this reason, a more appropriate name for the model would be the skin current model, but both terms are used interchangeably in literature.

### 2.1 Equilibrium and Coordinate System

#### Coordinates

We consider the sharp boundary model with a uniform plasma pressure and all the current flowing on the surface of the plasma. Following previous analysis, the toroidal component of the magnetic field both outside and inside the plasma will satisfy

\[ B_{\phi} R = \text{const} \tag{2.4} \]

the constant being in general different inside and outside the plasma. We find it convenient to write the radial coordinate \( R \) on the plasma surface as follows

\[ R = R_0 (1 + \epsilon x) \tag{2.5} \]

where \( R_0 \) and \( \epsilon \) are constants being roughly equal to the major radius and inverse aspect ratio respectively – their exact relationship to the major radius and inverse aspect ratio will be given in Section (2.3). Similarly, the vertical portion of the plasma boundary will be given by
\[ Z = \epsilon R_0 y \]  

(2.6)

If we now take the functions \( x \) and \( y \) to be parametrized in an angle like variable \( 0 \geq v < 2\pi \), say, then the boundary of the plasma will be the locus of the points \((R, Z)\) as \( v \) moves from 0 to \( 2\pi \).

**Equilibrium Jump Condition**

Just inside the plasma boundary, the toroidal field is given by

\[ B_\phi = \frac{B_i}{1 + \epsilon x(v)}, \]  

(2.7)

and just outside by

\[ \hat{B}_\phi = \frac{B_0}{1 + \epsilon x(v)}, \]  

(2.8)

where \( B_i, B_0 \) are constants, giving the magnitudes of the fields at the point \( R = R_0 \).

In the notation used here, \(^\wedge\) (hat) refers to field quantities in the vacuum region outside the plasma. To complete the equilibrium specification, we need to give the magnitude of the poloidal component (component perpendicular to the toroidal direction) of the magnetic field. The Ideal MHD momentum equation and Ampère’s law are

\[ \nabla p = \mathbf{J} \times \mathbf{B}, \]  

(2.9)

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \]  

(2.10)

**Eliminating** \( \mathbf{J} \) **gives**

\[ \mu_0 \nabla p = (\nabla \times \mathbf{B}) \times \mathbf{B}, \]  

(2.11)

or
\[ \nabla \mu_0 p = -\nabla \left( \frac{B^2}{2} \right) + (B \cdot \nabla) B. \] (2.12)

This can be written in tensor form as follows

\[ \text{div} \left[ \left( \mu_0 p + \frac{B^2}{2} \right) \vec{I} - BB \right] = 0 \] (2.13)

where \( \vec{I} \) is the unit tensor, \( \delta_{ij} \).

Integrating this equation over an infinitesimal penny-shaped volume on the plasma surface, we find, using the divergence theorem

\[ \oint_{\partial V} dS \cdot \left[ \left( \mu_0 p + \frac{B^2}{2} \right) \vec{I} - BB \right] = 0 \] (2.14)

where \( \partial V \) is the surface of the penny. Letting the thickness of the penny shrink to zero, and taking the normal (to the face of the penny) component of Eq. (2.14) we find

\[ [\mu_0 p + \frac{B^2}{2}] = [B_n^2], \] (2.15)

where \( B_n = n \cdot B \), and the brackets \([Q]\) denote the jump in the quantity \( Q \), i.e.

\[ [Q] = (Q)_1 - (Q)_2, \] (2.16)

with 1 denoting \( Q \) evaluated just outside the plasma, and 2 just inside. Similarly, we derive from

\[ \nabla \cdot B = 0, \] (2.17)

the result

\[ [B_n] = 0, \] (2.18)

and since there is no field component perpendicular to the toroidal direction inside the plasma, \( B_n = 0 \) on both sides of the plasma-vacuum boundary. Then Eq. (2.15) gives the component of the magnetic field parallel to the plasma-vacuum boundary and perpendicular to the toroidal direction, denoted here by \( \hat{B}_p \):
\[
\frac{\hat{B}_p^2}{B_0^2} = \frac{2\mu_0 p}{B_0^2} - \frac{1 - B_i^2/B_0^2}{(1 + \varepsilon \omega)^2}
\]  

(2.19)

In order to specify the equilibrium completely for any given cross-sectional shape and plasma size, we need to provide the constants \(p\) and \(B_i^2/B_0^2\). We note from Eq. (2.19) that \(R_0\), the major radius, has been eliminated and that the value of the toroidal component of the vacuum magnetic field, \(B_0\), is just a scaling factor.

Three special cases of Eq. (2.19) can be immediately identified. The first corresponds to zero plasma pressure, \(p = 0\). The second is the case where all the plasma is confined by the poloidal field, i.e. \(B_0 = B_i\). In this case the last term is identically zero, and can here be referred to as the neutral paramagnetic–diamagnetic case. The final case is the so called equilibrium limit where the poloidal field becomes zero at the point of the plasma surface closest to the axis of symmetry of the tokamak.

**Normalization of Equations**

Due to the fact that the toroidal component of the magnetic field, \(B_0\), has become a mere scaling factor, we find it convenient to define some dimensionless quantities that will completely specify the equilibrium. Let us then define

\[
b_p = \frac{\hat{B}_p}{B_0},
\]

(2.20)

\[
\lambda^2 = \frac{2\mu_0 p}{B_0^2},
\]

(2.21)

\[
k = \frac{1 - B_i^2/B_0^2}{\lambda^2 (1 + \varepsilon x_m)^2},
\]

(2.22)

where

\[
x_m = \min_{0 \leq v \leq 2\pi} x(v)
\]

(2.23)

and in general \(x_m < 0\). Then Eq. (2.19) may be written
\[ b_p^2 = \lambda^2 \left[ 1 - k \left( \frac{1 + \varepsilon x_m}{1 + \varepsilon x(v)} \right)^2 \right]. \]  

(2.24)

It is now clear that the three cases discussed above correspond to \( \lambda = 0 \) (with \( \lambda^2 k \) remaining finite) for the zero pressure case, \( k = 0 \) for the neutral paramagnetic–diamagnetic case and \( k = 1 \) for equilibrium limit. We shall also have to define precisely how \( k = 1 \) yields an equilibrium “limit”.

Following the definition of Chapter 1, we define a dimensionless measure of the plasma current as follows:

\[ q_* = \frac{2A_p B_V}{\mu_0 R_V I_p}, \]  

(2.25)

where the quantities \( R_V \) and \( B_V \) refer to the major radius and the toroidal field, respectively, at a conveniently chosen point which is held constant when the plasma shape is varied. \( A_p \) is the plasma cross-sectional area, defined by

\[
A_p = \int dZ \, dR \\
= -2\varepsilon^2 R_0^2 \int_0^\pi y \dot{z} \, dv, 
\]  

(2.26)

where \( \dot{z} \equiv dx/dv \). The plasma current, \( I_p \), can be calculated from Ampère’s law

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \]  

(2.27)

which gives, using Stokes’ theorem

\[ \oint \mathbf{B} \cdot d\ell = \mu_0 I_p, \]  

(2.28)

where \( I_p = \int \mathbf{J} \cdot d\mathbf{S} \), the integral being evaluated over the plasma cross-section. Using previous notation, this gives

\[ \mu_0 I_p = B_0 \int_0^{2\pi} b_\nu \left( \dot{r}^2 + \dot{z}^2 \right)^{1/2} \, dv, \]  

(2.29)
where we have used \( \dot{\equiv} d/dv \). This can be rewritten

\[
\mu_0 I_p = \epsilon R_0 B_0 \int_0^{2\pi} b_k \Delta dv,
\]  

(2.30)

where

\[
\Delta \equiv (\dot{x}^2 + \dot{y}^2)^{1/2}.
\]  

(2.31)

Substitution into the formula for \( q_* \) yields

\[
q_* = \frac{2A_p B_V}{R_V \epsilon R_0 B_0} \left\{ \int_0^{2\pi} \lambda \left[ 1 - k \left( \frac{1 + \epsilon x_m}{1 + \epsilon x} \right)^2 \right]^{1/2} \Delta dv \right\}^{-1},
\]  

(2.32)

where \( R_0 \) and \( R_V \) (and consequently \( B_0 \) and \( B_V \)) are not in general equal, although for most practical cross-sections they are close to equal. It is found in Section (2.3) convenient to hold the position of the centre of the vacuum chamber, rather than the centre of the plasma, constant, and the relationship between \( R_0 \) and \( R_V \) denoting this location will be given there.

An important dimensionless figure of merit is the plasma \( \beta \), which is defined as the total plasma pressure divided by the total magnetic energy, or symbolically

\[
\beta = \frac{\int p \, dV}{\int \frac{B^2}{2\mu_0} \, dV},
\]  

(2.33)

where the integrals are taken over the volume of the system. In most present studies the integrals are evaluated over the plasma volume only, but in optimization studies of the kind that we are concerned with in this thesis, it is important that we account for all the work that we need to expend in order to find the appropriate "optimum" for the quantity of interest. In the present context, we need to account for all the magnetic energy produced by the equilibrium coil system before we can claim that a particular configuration produces the best possible confinement scheme. These issues will be discussed further in Section (2.3) where the optimization problem will be defined.
fully. Suffice it to say here that, in principle, the above definition of plasma $\beta$ involves evaluation of the total magnetic energy in the system. The subsequent optimization w.r.t. parameters defining the plasma cross-section will be done for this definition of plasma $\beta$.

**Equilibrium Limit**

We are now in a position to discuss how $k = 1$ corresponds to an equilibrium limit. Treating $\lambda$ and $k$ as independent parameters, we note that the product $\beta_T q_*^2$ is independent of $\lambda$. Hence, the toroidal plasma $\beta$ is defined as

$$\beta_T = \frac{\int pdV}{\int \frac{B^2 dV}{2\mu_0}} = \lambda^2 \frac{V_p}{\int \left( \frac{B_\phi}{B_0} \right)^2 dV}, \quad (2.34)$$

and so

$$\beta_T q_*^2 = \frac{V_p}{\int \left( \frac{B_\phi}{B_0} \right)^2 dV} \frac{4A^2 p B^2}{\epsilon^2 R^2 p R^2 B^2} \left\{ \int_0^{2\pi} \left[ 1 - k \left( \frac{1 + \epsilon x_\mu}{1 + \epsilon x} \right)^2 \right]^{1/2} \Delta d\nu \right\}^{-2}. \quad (2.35)$$

Furthermore, since the only $k$-dependence is in the last integral, which clearly is a monotonically decreasing function of $k$, the largest value of $\beta q_*^2$ is obtained for $k = 1$. It is in this sense that $k = 1$ is termed the equilibrium limit, viz. it maximizes the product $\beta_T q_*^2$, or equivalently gives the highest equilibrium plasma $\beta_T$ for any given plasma current. It is also relatively straightforward to see that the same conclusion applies to $\beta q_*^2$ with the total plasma $\beta$, but this will be discussed in detail below in the context of the optimization problem, since we have not yet defined how to evaluate the poloidal field energy of the system. (See section (2.3), Eq. (2.134) and Appendix E.)

While it is conceivable that the largest critical plasma $\beta$ for stability may not correspond to the equilibrium limit, in practice this has not been found to be the case. Indeed, it is found that the critical plasma $\beta$ for any given cross-sectional shape is a
monotonically (strictly) increasing function of \( k \). This is illustrated in Figure (2.1) where the stable region of a circular and bean shaped plasma are shown in a plot of plasma \( \beta \) against \( 1/q_* \) (plasma current). The equilibrium limit and critical \( \beta \) for stability are indicated. Also shown is a hypothetical case where the highest critical \( \beta \) would be obtained for some value of \( k \neq 1 \). Since this is not observed for "reasonable" cross-sectional shapes in practice, we will in this work concentrate on optimizing the plasma shape at the equilibrium limit \( k = 1 \).

### 2.2 Evaluation of \( \delta W \)

The stability calculation is based on an evaluation of \( \delta W \). Due to the assumed axial symmetry of the torus, the toroidal direction becomes ignorable and a Fourier analysis of the plasma displacement vector \( \xi \) in the toroidal direction will be convenient:

\[
\xi = e^{-in\phi} \xi(R, Z). \tag{2.36}
\]

When formulated in this manner the different toroidal harmonics, denoted by the harmonic number \( n \), can be analysed independently.

The Ideal MHD Universal Minimizing Principle\(^{12}\) shows that the vector \( \xi \) minimizing \( \delta W \) satisfies, for any \( n \):

\[
(B \cdot \nabla) \nabla \cdot \xi = 0. \tag{2.37}
\]

In our model, since inside the plasma the magnetic field has only a toroidal component, and since the variation of \( \nabla \cdot \xi \) in the toroidal direction has the form \( e^{-in\phi} \), we can rewrite the above in the form

\[
-\frac{inB_\phi}{R} \nabla \cdot \xi = 0. \tag{2.38}
\]
It is then clear, that whenever the toroidal mode number \( n \neq 0 \), the minimizing perturbations are incompressible:

\[
\nabla \cdot \xi = 0. \tag{2.39}
\]

However, when \( n = 0 \), compressible perturbations must in general be allowed. Appendix D shows that in the sharp boundary model the minimizing perturbation satisfies

\[
\nabla \cdot \xi = K, \tag{2.40}
\]

where \( K \) is a constant. The value of this constant can be determined in terms of the normal component of the perturbation vector evaluated on the plasma surface. It will be shown below that all the other contributions to \( \delta W \) can also be expressed in terms of the normal component of the perturbation vector,

\[
\xi_n = n \cdot \xi, \tag{2.41}
\]

where \( n \) is the unit normal at the plasma surface. We then find it convenient to Fourier analyse \( \xi_n \) in terms of the parameter \( v \) which parametrizes the plasma surface. We write

\[
\xi_n(R(v), Z(v)) = e^{-i n \phi} \sum_m \xi_m e^{i m v}, \tag{2.42}
\]

where the notation \((R(v), Z(v))\) denotes a point on the plasma surface, as defined before:

\[
R(v) = R_0 (1 + \epsilon x(v)), \tag{2.43}
\]

\[
Z(v) = R_0 \epsilon y(v). \tag{2.44}
\]

It can be shown that the plasma potential energy is given by

\[
\delta W = \delta W_p + \delta W_V + \delta W_S, \tag{2.45}
\]

where the plasma contribution is
\[ \delta W_p = \frac{1}{2} \int_{V_p} dr \left\{ \frac{|Q|^2}{\mu_0} + \gamma p |\nabla \cdot \xi|^2 - \xi^* \cdot J \times B + \left( \xi^* \cdot \nabla p \right) \nabla \cdot \xi^* \right\}, \tag{2.46} \]

the vacuum contribution is

\[ \delta W_V = \frac{1}{2} \int_{V_v} dr \frac{|\hat{Q}|^2}{\mu_0}, \tag{2.47} \]

and the surface term is given by

\[ \delta W_S = \frac{1}{2\mu_0} \int_{\partial V_p} |\xi|^2 n \cdot \left[ \nabla \frac{B^2}{2} \right] dS, \tag{2.48} \]

where \( n \) is the outward normal to the plasma surface. \( V_p \) denotes the plasma region, \( V_v \) the vacuum region and \( \partial V_p \) the plasma surface. In the above formulae,

\[ Q = \nabla \times (\xi \times B), \tag{2.49} \]

the perturbed magnetic field in the plasma, \( \hat{Q} \) is the perturbed magnetic field in the vacuum region and \( \gamma \) is the ratio of the principal specific heats of the plasma

\[ \gamma = \frac{c_p}{c_v}. \tag{2.50} \]

The details of the derivation of the above formulae are given by Freidberg \[13].

A. Surface contribution to \( \delta W \)

The surface term in \( \delta W \) is the most straightforward to evaluate, so we do that first. The surface element \( dS \) can be written

\[ dS = R d\phi d\ell_p, \tag{2.51} \]

where \( d\ell_p \) is the poloidal arc length along the plasma boundary. This can be further expressed as
\[ dS = R_0^2 \epsilon (1 + \epsilon x) (\hat{x}^2 + y^2)^{1/2} d\nu d\phi. \] (2.52)

Using the fact that

\[ \nabla \frac{B^2}{2} = B \times (\nabla \times B) + (B \cdot \nabla) B, \] (2.53)

we find

\[ n \cdot \nabla \frac{B^2}{2} = n \cdot (B \times (\nabla \times B)) + (B \cdot \nabla) (n \cdot B) - B \cdot (B \cdot \nabla) n. \] (2.54)

But inside the plasma as well as in the vacuum \((\nabla \times B) = 0\), and just outside the plasma boundary \(n \cdot B = 0\), since the boundary is a flux surface, so that

\[ n \cdot [\nabla \frac{B^2}{2}] = -[B \cdot (B \cdot \nabla) n]. \] (2.55)

Substituting for

\[ B = B_T e_\phi + B_p t, \] (2.56)

and for

\[ n = \frac{\dot{y} e_R - \dot{x} e_Z}{\Delta}, \] (2.57)\[ t = \frac{\dot{x} e_R + \dot{y} e_Z}{\Delta}, \] (2.58)

we obtain after a short calculation (see Appendix A):

\[ n \cdot [\nabla \frac{B^2}{2}] = -\frac{\dot{B}_p^2}{\epsilon R_0 \Delta^3} (\dot{x} \dot{y} - y \dot{x}) - \frac{B_0^2 - B_p^2}{R_0 \Delta} \frac{\dot{y}}{(1 + \epsilon x)^3}. \] (2.59)

The surface term may then be written

\[ \frac{\delta W_s}{2 \pi R_0} = -\frac{B_0^2 \pi}{\mu_0} \sum_{m,p} \xi_m H_{mp} \xi_p^* \] (2.60)
\[
H_{mp} = \frac{1}{2\pi} \int_0^{2\pi} \nu \left[ \frac{b_p^2}{\Delta^2} (\dot{x} \dot{y} - \dot{y} \dot{x}) + \epsilon \left( 1 - \frac{B_1^2}{B_0^2} \right) \frac{\dot{y}}{(1 + \epsilon x)^3} \right] \times (1 + \epsilon x)^{(m-p)\nu}.
\]

(2.61)

Since both \(\delta W_p\) and \(\delta W_v\) are positive definite, the sole destabilizing contribution comes from the surface term. It would then at first glance seem attractive to minimize the destabilizing contribution from the surface term in order to find the optimum cross-sectional shape for the plasma. It may be demonstrated, however, that the surface term is minimized by an indefinitely thin vertical slit (the limit of an ellipse whole elongation becomes infinite) centered at the major radius. This is not the answer that we obtain from the full problem where the vacuum and plasma contributions are retained. For this reason we need not analyse the surface term any further on its own, except to note that the first term in the square brackets represents the destabilizing curvature of the poloidal magnetic field. Sharp corners in the flux surface, corresponding to a separatrix, are allowed only when the poloidal field, \(b_p\), vanishes at that point to compensate for the fact that either \(\ddot{x}\) or \(\ddot{y}\) becomes infinitely large. The second term represents curvature of the toroidal field component and is destabilizing on the outside (for a diamagnetic plasma) and stabilizing on the inside. This term also causes the mode to “balloon” on the outside of the torus, so that these external global modes may be described as “ballooning” kink modes.

B. Vacuum Contribution

As noted above the vacuum potential energy is given by

\[
\delta W_v = \frac{1}{2\mu_0} \int_{V_v} |\hat{Q}|^2 dV,
\]

(2.62)

where the integral is taken over the entire vacuum region extending to infinity, and \(\hat{Q} = \hat{B}\) is the perturbed magnetic field in the vacuum region. The minimizing perturbations
must satisfy $\nabla \cdot \mathbf{B}_1 = 0$ so we may minimize $\delta W_V$ with respect to this constraint by introducing a Lagrange multiplier $\hat{\psi}$:

$$\delta W_V \equiv \mathcal{L} = \frac{1}{\mu_0} \int_{V_V} dV \left( \frac{1}{2} |\mathbf{B}_1|^2 + \hat{\psi} \nabla \cdot \mathbf{B}_1 \right). \tag{2.63}$$

Varying $\mathcal{L}$ w.r.t. $\mathbf{B}_1$ yields

$$\delta \mathcal{L} = \frac{1}{\mu_0} \int dV \left( \dot{\mathbf{B}}_1 \cdot \delta \mathbf{B}_1 + \hat{\psi} \nabla \cdot \delta \mathbf{B}_1 \right)$$

$$= \frac{1}{\mu_0} \int dV \delta \dot{\mathbf{B}}_1 \cdot \left( \mathbf{B}_1 - \nabla \hat{\psi} \right) + \frac{1}{\mu_0} \oint dS \cdot \hat{\psi} \delta \mathbf{B}_1. \tag{2.64}$$

The surface integral vanishes since $dS \cdot \delta \mathbf{B}_1 = 0$ (the plasma surface is a flux surface) and so the minimizing perturbations $\mathbf{B}_1$ satisfy

$$\dot{\mathbf{B}}_1 = \nabla \hat{\psi}, \tag{2.65}$$

where $\hat{\psi}$ can be thought of as the magnetic potential. Since $\nabla \cdot \dot{\mathbf{B}}_1 = 0$, $\hat{\psi}$ satisfies

$$\nabla^2 \hat{\psi} = 0. \tag{2.66}$$

Then we may write

$$\delta W_V = \frac{1}{2\mu_0} \int dV \nabla \hat{\psi} \cdot \nabla \hat{\psi}^*$$

$$= \frac{1}{2\mu_0} \int dV \left[ \nabla \cdot \left( \hat{\psi}^* \nabla \hat{\psi} \right) - \hat{\psi}^* \nabla^2 \hat{\psi} \right]$$

$$= \frac{1}{2\mu_0} \oint dS \cdot \hat{\psi}^* \nabla \hat{\psi}, \tag{2.67}$$

where the surface integral is over the surface of the plasma. The contribution from infinity vanishes since we require $\hat{\psi}$ to be regular there. We now expand $\hat{\psi}$ and $\nabla \hat{\psi}$ in Fourier series as follows.
\[ \epsilon R_0 \Delta (1 + \epsilon x) \mathbf{n} \cdot \nabla \hat{\psi} = -iB_0 e^{in\phi} \sum \hat{a}_m e^{im\nu}, \]  
(2.68) \\
\[ \hat{\psi} = iB_0 e^{-in\phi} \sum \hat{b}_m e^{-im\nu}. \]  
(2.69)

Then we have

\[ \frac{\delta W_V}{2\pi R_0} = \frac{\pi B_0}{\mu_0} \sum \hat{a}_m \hat{b}_m^*. \]  
(2.70)

### Boundary Conditions

The coefficients \( \hat{a}_m \) and \( \hat{b}_m \) can be related to the coefficients \( \xi_m \) of the normal displacement through boundary conditions at the plasma surface and through the solution of Laplace’s equation for the magnetic potential in the vacuum region. As demonstrated in Appendix B, the boundary condition at the plasma surface is given by

\[ [\mathbf{n} \cdot \nabla \hat{\psi}]_S = [\mathbf{n} \cdot \nabla \times (\xi \times \mathbf{B})]_S. \]  
(2.71)

A short calculation yields, repeated in Appendix B,

\[ [\mathbf{n} \cdot \nabla \hat{\psi}]_S = -i \frac{nB_0 \xi}{R_0 (1 + \epsilon x)^2} + \frac{1}{\epsilon R_0^2 \Delta (1 + \epsilon x)} \frac{\partial}{\partial \nu} \xi R \hat{B}_p. \]  
(2.72)

Substituting the Fourier expansions for \( \xi \) and \( [\mathbf{n} \cdot \nabla \hat{\psi}]_S \) we get

\[ -iB_0 \sum \hat{a}_m e^{im\nu} = \sum_m \left[ -i \frac{nB_0 \Delta \epsilon}{1 + \epsilon x} \xi_m e^{im\nu} + \frac{1}{R_0} \frac{\partial}{\partial \nu} \xi_m e^{im\nu} R \hat{B}_p \right], \]  
(2.73)

or

\[ \hat{a}_p = \sum_m \left[ \frac{n\epsilon}{2\pi} \int_0^{2\pi} \frac{\Delta}{1 + \epsilon x} \cos (m - p) \nu \, dv - \frac{m}{2\pi} \int_0^{2\pi} \left( \frac{\hat{B}_p}{B_0} \right) (1 + \epsilon x) \cos (m - p) \nu \, dv \right] \xi_m. \]  
(2.74)
This can be written

\[ \hat{a}_p = \hat{G}_{pm} \xi_m, \]  

(2.75)

with

\[ \hat{G}_{pm} = G_{pm} - K_{pm}, \]  

(2.76)

where \[ \dagger \]

\[ G_{pm} = \frac{m \epsilon}{2 \pi} \int_0^{2\pi} \frac{\Delta \cos (m - p) v \, dv}{1 + \varepsilon x}, \]  

(2.77)

\[ K_{pm} = \frac{m}{2 \pi} \int_0^{2\pi} \left( \frac{\hat{B}_p}{\hat{B}_0} \right) (1 + \varepsilon x) \cos (m - p) v \, dv. \]  

(2.78)

Solution for Flux Function \( \hat{\psi} \)

The coefficients \( \hat{b}_m \) (See Eq. (2.69)) may be related to the \( \hat{a}_m \) through the solution of Laplace’s equation:

\[ \nabla^2 \hat{\psi} = 0. \]  

(2.79)

Introducing the infinite space Green’s function \( G(x, x') \) given by

\[ G(x, x') = \frac{1}{4\pi|x - x'|}, \]  

(2.80)

and satisfying

\[ \nabla^2 G = \delta(x - x'), \]  

(2.81)

we have the identity

\[ \dagger \] These formulae are slightly different from those given by Freldberg and Grossmann, since a minor algebraic error has been corrected.
\[ \nabla \cdot \left( G \nabla \hat{\psi} - \hat{\psi} \nabla G \right) = G \nabla^2 \hat{\psi} - \hat{\psi} \nabla^2 G. \quad (2.82) \]

Integrating both sides over the vacuum region gives

\[
\int dS' \cdot \left( G \nabla \hat{\psi} - \hat{\psi} \nabla G \right) = - \int \hat{\psi} \nabla^2 G \, dV' = - \int \hat{\psi} \delta (x - x') \, dV'. \quad (2.83)
\]

When the observation point \( x \) moves to the plasma boundary \( S' \), the value of the integral on the right hand side is \(-\frac{1}{2} \hat{\psi}(x)\), and if \( n' \) is the outward unit normal at the plasma surface, we get the following integral equation for \( \hat{\psi} \) and \( n \cdot \nabla \hat{\psi} \):

\[
\frac{1}{2} \hat{\psi}(x) = - \int dS' \left( \hat{\psi}(x') \, n' \cdot \nabla' G - G n' \cdot \nabla' \hat{\psi}(x') \right). \quad (2.84)
\]

Substitution of the Fourier expansions for \( \hat{\psi} \) and \( n \cdot \nabla \hat{\psi} \) yields a relationship between \( \hat{a}_m \) and \( \hat{b}_m \):

\[
\hat{b}_p - \sum_m A_{pm} \hat{b}_m = \sum_m B_{pm} \hat{a}_m, \quad (2.85)
\]

where the matrices \( A \) and \( B \) are given by

\[
A_{pm} = \frac{1}{4\pi^2} \int_0^{2\pi} d\nu \int_0^{2\pi} d\nu' \epsilon R_0 \Delta (1 + \epsilon x') \, n' \cdot \nabla' G_n \cos (mv' - pv), \quad (2.86)
\]

\[
B_{pm} = \frac{1}{4\pi^2} \int_0^{2\pi} d\nu \int_0^{2\pi} d\nu' \, G_n \cos (mv' - pv), \quad (2.87)
\]

and \( G_n \) is the reduced Green's function

\[
G_n = -4\pi R_0 \int_0^{2\pi} G(x, x') \cos n (\phi - \phi') \, d\phi'. \quad (2.88)
\]

Fast Fourier transform methods may be successfully used for fast evaluations of the matrices \( A \) and \( B \) and the logarithmic singularities in \( G_n \) and \( n' \cdot \nabla' G_n \) when \( v \to v' \) can
be removed and integrated analytically. These procedures are outlined in Appendix C. It then follows that the vacuum energy is given by

$$\frac{\delta W_V}{2\pi R_0} = \frac{B_0^2 \pi}{\mu_0} \sum_{m,p} \xi^*_m W^V_{mp} \xi_p,$$  \hspace{1cm} (2.89)

where

$$W^V = \hat{G}^T \cdot (I - A)^{-1} \cdot B \cdot \hat{G},$$  \hspace{1cm} (2.90)

a symmetric matrix since $\delta W_V$ is symmetric. The matrices $\hat{G}$, $A$ and $B$ are defined in Eqs. (2.76)–(2.78) and (2.86)–(2.87).

C. Plasma contribution

As noted above, the plasma displacement vector is incompressible for modes with non-zero toroidal mode number, but for axisymmetric modes the displacement in general is compressible. For this reason, the $n = 0$ and $n \neq 0$ cases have to be treated separately. We first calculate the plasma energy for the case $n \neq 0$, because it closely parallels the vacuum energy calculation.

$n \neq 0$ modes

As in the vacuum calculation, the minimizing perturbations $B_1$ satisfy $\nabla \times B_1 = 0$ so that all the perturbed current flows on the plasma surface. We may then write

$$B_1 = \nabla \psi,$$  \hspace{1cm} (2.91)

with the magnetic potential $\psi$ satisfying Laplace’s equation. The plasma energy is then given by

$$\delta W_p = \frac{1}{2\mu_0} \oint \psi^* \nabla \psi \cdot dS,$$  \hspace{1cm} (2.92)

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where we have required $\psi$ to be regular everywhere inside the plasma. Expanding in Fourier series for $\psi$ and $\partial\psi/\partial n$ as follows

\begin{equation}
\epsilon R_0 \Delta (1 + \varepsilon x) \mathbf{n} \cdot \nabla \psi = -i B_0 e^{-i n \phi} \sum a_m e^{i m \nu}, \tag{2.93}
\end{equation}

\begin{equation}
\psi = -i B_0 e^{-i n \phi} \sum b_m e^{i m \nu}, \tag{2.94}
\end{equation}

we have

\begin{equation}
\frac{\delta W_p}{2\pi R_0} = \frac{\pi B_0^2}{\mu_0} \sum a_m b_m^*. \tag{2.95}
\end{equation}

The $a_m$ may be related to the $\xi_m$ through the boundary condition on the plasma surface:

\begin{equation}
[n \cdot \nabla \psi]_s = [n \cdot \nabla \times (\xi_n \mathbf{n} \times \mathbf{B})]_s, \tag{2.96}
\end{equation}

which yields, after a short calculation (see Appendix B)

\begin{equation}
[n \cdot \nabla \psi]_s = -i \frac{n B_i \xi}{R_0 (1 + \varepsilon x)^2}. \tag{2.97}
\end{equation}

This gives the relationship

\begin{equation}
a_m = \frac{B_i}{B_0} G_{mp} \xi_p, \tag{2.98}
\end{equation}

where $G$ is the same matrix that we had in the vacuum calculation, given by Eq. (2.77). A relationship between the $a_m$ and $b_m$ is obtained through the solution of Laplace's equation. We solve that in terms of the Green's function

\begin{equation}
\frac{1}{2} \psi (x) = \int dS' [\psi (x') \mathbf{n'} \cdot \nabla' G - G \mathbf{n'} \cdot \nabla' \psi (x')], \tag{2.99}
\end{equation}

which yields, similarly to the vacuum calculation,

\begin{equation}
b_p + \sum_m A_{pm} b_m = \sum_m B_{pm} a_m, \tag{2.100}
\end{equation}

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with $A$ and $B$ the same matrices that we had before, given by Eqs. (2.86) and (2.87). The plasma energy is now given by

$$\frac{\delta W_p}{2\pi R_0} = \frac{\pi B_0}{\mu_0} \xi_m W^p_{mp} \xi^*_p,$$  \hspace{1cm} (2.101)

where the matrix $W^p$ is

$$W^p = G^T \cdot (I + A)^{-1} \cdot B \cdot G,$$  \hspace{1cm} (2.102)

a symmetric matrix.

$n = 0$ modes

For the $n = 0$ modes the full plasma energy is given by

$$\delta W_p = \frac{1}{2} \int_{V_p} dr \left\{ \frac{|Q|^2}{\mu_0} + \gamma p_0 |\nabla \cdot \xi|^2 \right\},$$  \hspace{1cm} (2.103)

where we have allowed the displacement to be compressible in general. $\gamma$ is the ratio of the principal specific heats of the plasma and is in this work taken to be $\gamma \geq 1$. It is shown in Appendix D that for axisymmetric modes, the plasma displacement is arranged in such a way that the perturbed magnetic field vanishes inside the plasma and $\delta W_p$ only contains a contribution from the plasma compressibility. Furthermore, the compressibility is shown to be a constant, which enables us to write

$$\frac{\delta W_p}{2\pi R_0} = \frac{\pi B^2_0}{\mu_0} 2 \frac{\gamma \lambda^2 |a_m \xi_m|^2 \epsilon^2 R_0^3}{V_p},$$  \hspace{1cm} (2.104)

where $V_p$ is the plasma volume, $\lambda^2 = 2\mu_0 p_0 / B^2_0$ is the normalized plasma pressure, $\xi_m$ is the $m$'th Fourier component of the displacement vector normal to the surface of the plasma and the quantities $a_m$ are defined by

$$a_m = \int_0^{2\pi} dv (1 + \epsilon x) \Delta e^{imv}. $$  \hspace{1cm} (2.105)

For the details of the derivation see Appendix D.
D. Energy Principle

We are now in a position to state the Energy Principle for the sharp boundary model. By combining the expressions in the previous sections, we have for non-axisymmetric modes \( (n \neq 0) \),

\[
\frac{\delta W^{(n \neq 0)}}{2\pi R_0} = \frac{\pi B_0^2}{\mu_0} \xi_m W_{mp} \xi_p, \tag{2.106}
\]

where the matrix \( W \) is given by

\[
W = -H + \frac{B_i}{B_0^2} G^{(n)T} \left( I + A^{(n)} \right)^{-1} B^{(n)} G^{(n)} + \hat{G}^{(n)T} \left( I - A^{(n)} \right)^{-1} B^{(n)} \hat{G}^{(n)}. \tag{2.107}
\]

In the above, we have indicated the \( n \)-dependence of a matrix by a superscript \( (n) \). For the axisymmetric modes \( (n = 0) \) the expression is

\[
\frac{\delta W^{(n = 0)}}{2\pi R_0} = \frac{\pi B_0^2}{\mu_0} \xi_m W^0_{mp} \xi_p, \tag{2.108}
\]

where the matrix \( W^0 \) is

\[
W^0 = -H + D + K^T \cdot (I - A^0)^{-1} \cdot B^0 \cdot K, \tag{2.109}
\]

and the matrix \( D \) is defined as

\[
D_{mp} = \frac{1}{2} \gamma \lambda^2 \epsilon^2 R_0^3 \frac{a_m}{V_p} a_p, \tag{2.110}
\]

with

\[
a_m = \int_0^{2\pi} dv (1 + \epsilon x) \Delta e^{imv}. \tag{2.111}
\]

Both \( W_{mp} \) and \( W^0_{mp} \) are symmetric matrices, since they have been derived from an Energy Principle that is originally symmetric in form. Notice further that since \( H \propto \lambda^2, D \propto \lambda^2 \) and \( K \propto \lambda \), we have \( W^0 \propto \lambda^2 \). Effectively, this means that if a particular cross-sectional shape is stable to axisymmetric perturbations, it will be stable to them at any plasma \( \beta \), i.e. at any value of \( \lambda^2 \) (see the next section). This means that the cross-sectional optimization is not possible with respect to \( n = 0 \) modes alone but some non-axisymmetric perturbations have to be included if a critical value of plasma \( \beta \) is to
be found. It is worth noting that the stability to axisymmetric modes for more realistic plasma configurations is also found to be essentially independent of plasma $\beta^{[13]}$.

2.3 Statement of The Optimization Problem

A definition of the optimization problem can now be sharply stated. In order to achieve this we must clearly define the quantity that is to be maximized, viz. plasma $\beta$, and all constraints that naturally accompany these kinds of optimization problems. For any given cross-sectional shape with the plasma at the equilibrium limit ($k = 1$), we can calculate the critical value of $\lambda = \lambda_c$ that just makes $\delta W$ vanish. The corresponding critical plasma $\beta$ ($\beta = \beta_c$) will then be expressed as a function of $\lambda_c$ and the appropriate geometric factors. It is the maximization of $\beta_c$ with respect to geometric parameters defining the plasma surface that we mean when using the term optimization procedure. The corresponding $\beta_c$ will be called the optimal $\beta_c$ and the cross-sectional shape the optimal shape.

Modes Included in The Problem

As well as keeping the $m = 0$ component in the Fourier analysis in the poloidal direction, 30 positive and negative $m$'s were also retained. An exact calculation would, of course, require all $m$-numbers to be included for $-\infty < m < \infty$, but from a computational point of view it was found that for the cross-sectional shapes analysed in this work the range $-30 \leq m \leq 30$ is sufficient. This ensures that the magnitude of the last harmonic for $n \cdot \xi_\perp$ is smaller by about five orders of magnitude than the largest component.

Due to axial symmetry in the system, the toroidal modes can be analysed one by one. Since theoretical considerations at large aspect ratio indicate that the $n = 1$ mode should have the lowest critical $\beta$ among all the non-zero mode numbers, we perform this
work for three toroidal mode numbers, \( n = 0, 1, 2 \). Furthermore, the \( n = 2 \) calculation is done \textit{a posteriori} as a check to make sure that the critical \( \beta \) for that mode is indeed higher than for the \( n = 1 \) mode. This is found to be the case for all the cross-sectional shapes considered.

The balance between the \( n = 0 \) and \( n = 1 \) modes is slightly more complicated. As noted above, no optimization with the pure \( n = 0 \) mode is possible, since \( \delta W^0 \) is directly proportional to plasma \( \beta \). When a shape is stable to axisymmetric modes, it is so at any plasma \( \beta \) and therefore there are two types of plasma shapes, those that are \( n = 0 \) stable and those that are \( n = 0 \) unstable. With the pure \( n = 0 \) mode, there is no way of choosing between the various stable ones. Clearly, some combination of \( n = 0 \) and \( n = 1 \) modes is required in the calculation. If the optimal cross-section for the pure \( n = 1 \) modes is stable to axisymmetric modes, we need look no further. However, if it is unstable, as is frequently the case, some modification of the cross-section is needed to make it stable to \( n = 0 \) modes. This modification results in the lowering of the critical \( \beta \).

Let us assume that we have a set of surfaces close to the optimal shape with pure \( n = 1 \) modes, such that the optimal shape is unstable to the \( n = 0 \) mode. Let the critical \( \beta \) for the pure \( n = 1 \) mode with the optimal shape be denoted by \( \beta_c^{(1)} \). Let us also write the perturbation vector as follows

\[
\xi = \alpha e^{i\phi} \xi_1 + (1 - \alpha^2)^{1/2} \xi_0, \tag{2.112}
\]

where the first term corresponds to the \( n = 1 \) mode and the second to the \( n = 0 \) mode, as indicated by their \( \phi \)-dependences. Then our assumption about the critical \( \beta \) at \( \alpha = 1 \) (pure \( n = 1 \) mode) is

\[
-\frac{\partial \beta_c^{(1)}}{\partial \alpha} \bigg|_{\alpha=1} < 0. \tag{2.113}
\]

Also, it follows from the fact that for shapes stable to the \( n = 0 \) mode the \( \beta \)-limit is the maximum equilibrium value, that

\[
\beta_c^{(1)} \leq \beta_c^{(0)}, \tag{2.114}
\]
where $\beta_c^{(0)}$ is the maximum $\beta$ at $\alpha = 0$. By the Mean Value Theorem applied to $\partial \beta_c / \partial \alpha$, there exists an $\alpha_c$ such that
\[ \frac{\partial \beta_c}{\partial \alpha} \bigg|_{\alpha=\alpha_c} = 0. \] (2.115)

Since the $n = 0$ and $n = 1$ modes decouple, we can write
\[ \delta W = \alpha^2 \xi_1 \cdot W^{(n=1)} \cdot \xi_1 + (1 - \alpha^2) \xi_0 \cdot W^{(n=0)} \cdot \xi_0 \] (2.116)
where the matrices $W^{(n=1)}$ and $W^{(n=0)}$ depend only on geometrical quantities. From section 2.2 we also know that
\[ \xi_1 \cdot W^{(n=1)} \cdot \xi_1 = A_1 \lambda^2 + A_2 \lambda + A_3, \] (2.117)
\[ \xi_0 \cdot W^{(n=0)} \cdot \xi_0 = B_1 \lambda^2, \] (2.118)
with $A_1$, $A_2$, $A_3$ and $B_1$ constants. By construction, at the marginal stability point $\alpha = \alpha_c$ and $\beta_c = \beta_c^{\alpha_c}$,
\[ \alpha^2 \Lambda_1 + (1 - \alpha^2) \Lambda_2 = 0, \] (2.119)
where
\[ \Lambda_1 = A_1 \lambda^2 + A_2 \lambda + A_3, \] (2.120)
\[ \Lambda_2 = B_1 \lambda^2. \] (2.121)

Keeping the shape fixed and varying $\alpha$, we have at marginal stability
\[ 2 \delta \alpha (\Lambda_1 - \Lambda_2) + \alpha^2 \delta \lambda (2 \lambda A_1 + A_2 - 2 \lambda B_1) = 0, \] (2.122)
where we have used the fact that $A_1$, $A_2$, $A_3$ and $B_1$ are only functions of the shape. If $2 \lambda A_1 + A_2 - 2 \lambda B_1 = 0$, we have immediately $\Lambda_1 = \Lambda_2$ and by Eq. (2.119) they both vanish. If $2 \lambda A_1 + A_2 - 2 \lambda B_1 \neq 0$,
\[ \delta \lambda = -2 \delta \alpha \frac{\Lambda_1}{\alpha (1 - \alpha^2) (2 \lambda A_1 + A_2 - 2 \lambda B_1)}, \] (2.123)

† See Fig. (2.12)
where we have used the fact that

\[ \Lambda_2 = -\frac{\alpha^2}{1 - \alpha^2} \Lambda_1. \]  \hspace{1cm} (2.124)

Since \( \beta \) is a monotonic function of \( \lambda \), in other words

\[ \delta \beta \propto \delta \lambda, \]  \hspace{1cm} (2.125)

we have \( \delta \lambda = 0 \) at the point \( \alpha = \alpha_c \). It then follows that

\[ \Lambda_1 = \Lambda_2 = 0, \]  \hspace{1cm} (2.126)

so that at \( \alpha = \alpha_c \) (and \( \beta = \beta_{c}^{\text{c}} \)), the corresponding cross-sectional shape is marginally stable to both the \( n = 0 \) and the \( n = 1 \) mode. This is the marginal stability point that we seek in this work.

**Optimized Quantity and Constraints**

The most obvious definitions of the quantity to be maximized (plasma \( \beta \)) and the constraint for the optimization (fixing the aspect ratio) do not yield a practical answer. That a constraint akin to fixing the aspect ratio of the machine is necessary is obvious, since the critical \( \beta_c \) is a monotonically decreasing function of aspect ratio — no finite maximum exists. To properly study the problem requires an optimization for each fixed value of a parameter suitably chosen to represent the size of the machine. Designers of experiments and reactors may then choose based on constraints outside our model how small they can afford to make the aspect ratio to maximize \( \beta_c \).

The most obvious definition of the plasma \( \beta \) in our model is \( \beta = 2\mu_0 p_0 / B_\phi^2 = \lambda^2 \), and that for the machine size is the conventional aspect ratio \( A = R_c / a \), where \( R_c \) is the major radius at the geometrical centre of the plasma and \( a \) the minor radius along the midplane. Let us then assume, for the sake of argument, that we fix the aspect ratio to some relatively large value. It is well known that the minimum overall curvature of a closed line of given length is obtained for the circle. Following this idea, the minimum destabilizing contribution is obtained for a cross-sectional shape that has
circular sections on the outside and inside of the torus, because the overall surface curvature is minimized. The optimal shape corresponds to a spherical shell tokamak with very small sections cut away at the top and bottom so that the field lines can pass from the outside to the inside. The field lines go round the tokamak both inside and outside, and they are sharply connected at the tips (See Figure (2.2)). The resulting extremely high surface curvature at the tips is not destabilizing, because the destabilizing term containing a contribution from this curvature is multiplied by $B_p^2$, a quantity which pointwise vanishes at the same location when operating at the equilibrium limit. If one tries to numerically maximize the plasma performance using these simple definitions of plasma $\beta$ and aspect ratio, such a tendency towards a spherical shell tokamak is precisely what one will observe. Even at finite aspect ratio the optimal shape will have the above characteristics and will be a fat-in-the-middle spherical shell. It is clear that such an answer is not desirable from a practical viewpoint. Firstly, the plasma volume becomes very small compared to the total surface area, and secondly, questions about the shapes and sizes of the toroidal field coils and equilibrium coils required to produce such a plasma pose serious problems. In particular, when the toroidal field coils have a straight leg on the inside of the tokamak, a large amount of magnetic energy is produced that is in no way accounted for by this definition.

These problems naturally lead to better definitions of both the plasma $\beta$ and machine size. For the sake of simplicity, we will in this work assume that the toroidal field coils are rectangular in shape, but their elongation is in no way restricted. While the shape of these coils is, of course, arbitrary, and different shapes will certainly produce optima that differ from the results of this study, the rectangular shape corresponds closely to those shapes that are in use in present experiments. Moreover, the rectangular shape illustrates very well the properties of this optimization problem. These properties are discussed in more detail in section 2.4.

The definition of plasma $\beta$ must account for all the magnetic energy that is supplied through the equilibrium coil system and the toroidal field coils for the optimization to
be meaningful. The toroidal field energy is defined as

\[
W_{\phi} = \frac{1}{2\mu_0} \int_{V_{VC}} \hat{B}_\phi^2 dV, \tag{2.127}
\]

where \(\hat{B}_\phi\) is the toroidal component of the vacuum field and the integral is taken over the volume \((V_{VC})\) inside the toroidal coil set (the vacuum chamber).

Since an explicit set of equilibrium coils is not included in our model, the total poloidal magnetic energy associated with that coil set is estimated by calculating the total field energy due to the currents flowing in the plasma, in other words the self-inductance of the plasma:

\[
W_\theta = \frac{\mu_0}{8\pi} \int \frac{\mathbf{J}_\phi(x) \cdot \mathbf{J}_\phi(x') dx dx'}{|x - x'|}. \tag{2.128}
\]

The exclusion of this term from the definition of \(\beta\) does not substantially alter the answer, but it is included for the sake of completeness. Appendix E outlines how \(W_\theta\) is calculated in practice.

The size of the machine is set by fixing the total volume \(V_{VC}\) and total cross-sectional area \(A_V\) of the vacuum chamber, in other words the region inside the toroidal field coil set. One of these corresponds to just a length normalization in the system, since the major radius,

\[
R_V = \frac{V_{VC}}{2\pi A_V}, \tag{2.129}
\]

scales completely out of the analysis. The other then fixes the chamber aspect ratio

\[
\epsilon_V = \frac{1}{R_V} \sqrt{\frac{A_V}{\pi}}, \tag{2.130}
\]

where the factor \(\sqrt{\pi}\) is included for compatibility when comparing with circular shaped plasmas.

The toroidal vacuum field strength is also a simple scaling factor in the problem, and all magnetic fields are normalized to the field given by

\[
B_V^2 = \frac{2\mu_0 W_{\phi}}{V_{VC}}. \tag{2.131}
\]
The quantity to be maximized is then given by

$$\beta = \frac{W_p}{W_\phi + W_\theta},$$  \hspace{1cm} (2.132)

where

$$W_p = \int_{V_p} p\,dV,$$  \hspace{1cm} (2.133)

the total plasma energy contained in the system. With the result of Appendix E, it can be seen that $\beta$ is a monotonic function of $\lambda^2$ and has the form

$$\beta = \frac{\lambda^2}{a^2 + b^2\lambda^2},$$  \hspace{1cm} (2.134)

where $a$ and $b$ depend only on geometrical quantities.

**Solution Procedure**

The optimization procedure is then as follows. We fix the size of the machine ($\epsilon_V$), and find the critical $\beta$ for any given cross-sectional shape at the equilibrium limit ($k = 1$) by varying $\lambda$ so that $\delta W = 0$. The procedure is repeated for different shapes in a systematic fashion (using non-linear maximization procedures) so that the optimal cross-section and the corresponding $\beta$ are found. The parameter $\alpha$ is also varied during the optimization, so that the maximization of $\beta$ is achieved in such a way that the corresponding shape is simultaneously stable to $n = 0$ and $n = 1$ modes.

**Coordinate System**

We require the plasma to be up-down symmetric and find it convenient to parametrize the plasma surface in terms of the parameter $v$, $0 \leq v < 2\pi$, as discussed above. It follows that the horizontal distance from the axis of symmetry, $R(v)$, is an even function of $v$ while the vertical distance from the midplane, $Z(v)$, is an odd function. We set

$$Z(v) = \epsilon R_0 \sin v,$$  \hspace{1cm} (2.135)
and allow arbitrary shapes by writing

$$R(v) = R_0 + \epsilon R_0 \sum_n c_n \cos n v,$$

(2.136)

where in principle the sum is infinite. In practice, we find it sufficient to keep 4 coefficients \(\{c_n\}\) in the sum. By maximizing the plasma \(\beta\) with respect to the shape of the cross-section we then mean maximizing with respect to this set of coefficients \(\{c_n\}\).

It is straightforward to relate the quantities \(\epsilon \equiv a/R_0\) and \(R_0\) to the quantities defining the machine size, \(\epsilon V\) and \(R_V\), and the result is

$$\epsilon = \frac{(\pi/A)^{1/2} \epsilon V}{1 - \epsilon V \left(\frac{\pi}{A}\right)^{3/2} \frac{V}{\pi}}$$

(2.137)

and

$$R_0 = \frac{R_V}{1 + \epsilon V A},$$

(2.138)

where

$$A = 2y_{\max}(x_{\max} - x_{\min}),$$

(2.139)

$$V = \frac{1}{2} A (x_{\max} + x_{\min}).$$

(2.140)

We have written in general

$$R = R_0 (1 + \epsilon x),$$

(2.141)

$$Z = \epsilon R_0 y,$$

(2.142)

and the subscripts max and min refer to, respectively, the maximum and minimum values of the functions \(x(v)\) and \(y(v)\). For circular and elliptical plasma surfaces, the quantities \(R_0\) and \(R_V\) are equal, and \(2\epsilon R_0\) is equal to the distance across the plasma on the midplane. However, for more complicated plasma shapes, \(R_0\) takes values (slightly) different from \(R_V\) and consequently this simple interpretation of \(\epsilon\) is lost. Nevertheless, for reasonable plasma shapes \(\epsilon\) and \(R_0\) remain close to the conventional inverse aspect ratio and major radius, respectively.
2.4 Results and Discussion

While the results presented here are directly a consequence of the shape that we have chosen for the vacuum chamber, or equivalently the toroidal field coil set, surrounding the plasma, the rectangular shape illustrates very well how the different competing forces determine the optimal shape. We shall first give physical explanations to some of the properties of the optimal shapes. These explanations, albeit plausible in the given context, do not, however, by themselves constitute any proof. The proof is obtained by means of numerical analysis and numerical optimization over the set of Fourier coefficients \( \{c_n\} \) determining the cross-sectional shape. The physical picture will lead us to two constraints on the set \( \{c_n\} \), which allow us to eliminate two of the coefficients in the problem and perform the optimization over a space of 2 fewer dimensions than in the original problem. The justification for this procedure must lie in the fact that the full problem\( ^\dagger \) produces the same answer as the reduced one\( ^\ddagger \). This is indeed the case, and the results presented here can be thought of as results from either of the two methods. In practice, they are the results from the reduced problem, since a reduction by two of the number of dimensions in the optimization makes the procedure computationally much faster and more reliable. The results were checked using the full problem formulation for a subset of the cases investigated.

\[\begin{align*}
\dagger & \text{ From here on we refer to the formulation where all coefficients } \{c_n\} \text{ are included in the optimization as the full problem.} \\
\ddagger & \text{ The formulation which reduces the number of Fourier coefficients by two by means of the constraints discussed below is referred to as the reduced problem.}
\end{align*}\]
Physical Picture

The major destabilizing term, as noted in section 2.2, is that arising from the overall surface curvature of the plasma in $\delta W_s$. The destabilizing contribution is larger on the outside of the torus than on the inside, so that the tendency for a good cross-sectional shape is to have a very "circular" looking boundary on the outside of the torus. On the inside of the torus, the shape is not circular in general, because the magnitude of the poloidal field is much smaller there and allows for bigger variation in the boundary shape. At the equilibrium limit, in particular, the poloidal field vanishes at the point closest to the axis of symmetry of the tokamak, and therefore shapes with very high surface curvature at those points become permissible from the viewpoint of stability.

The main contribution to increasing $\beta$ comes from the plasma potential energy. The plasma tries to fill out as much of the available volume as possible while still maintaining a sufficiently low overall curvature so as to keep the shape stable. Since sharp corners in the shape are allowed at points where the poloidal field vanishes, such corners develop whenever it is advantageous to increasing $\beta$. This corresponds to a separatrix moving onto the plasma surface and thus represents the equilibrium limit. Of course, even in more realistic plasma models, as well as in experiments, sharp corners in the shape of the flux surface are allowed at a separatrix without jeopardizing stability. We may, therefore, think of the equilibrium limit in the sharp boundary model as representing the plasma in an experiment with a divertor, where the plasma is defined to extend all the way to the flux surface containing the separatrix. Then our optimization procedure will yield the optimal shape for the surface containing the separatrix. Viewed this way, it is natural to expect the optimal shape to have sharp corners — for, if it does not, we may increase $\beta$ by just sharpening out the ends of the plasma because the total volume increases. This does not alter the stability properties significantly, because the poloidal field vanishes at the sharp corner anyway. Figure (2.3) illustrates how the diagonal element of the $H$ matrix (divided by $\lambda^2$; see section 2.2), representing the destabilizing term in $\delta W$, behaves as the corner at the plasma tip is made sharper. The fact that the diagonal element does not become infinite for cross-sections with a sharp tip, demonstrates the
point made here. For, if $H_{nn}/\lambda^2$ becomes very large, the destabilizing term in $\delta W_e$ also becomes large suggesting that the corresponding shape has a very low value of $\beta_c$ for stability. In the case illustrated here, which is for our optimal shapes to be discussed below, $H_{nn}/\lambda^2$ in fact decreases when the tip is made sharper, suggesting that not only do we gain by volume utilization in making $\beta_c$ larger, but also by a possible reduction in the destabilizing factor. Analytically, we would expect the destabilizing contribution to remain finite, since at the equilibrium limit ($k = 1$) the poloidal field exactly vanishes at the point closest to the axis of symmetry, while the radius of curvature varies in this procedure and becomes zero for shapes that have an infinitely pointy tip. The product of the two must, therefore, vanish at that point.

The idea of volume utilization influences the resulting optimal shape in another way, too. Along the midplane, the plasma tries to expand as far as possible toward the vacuum chamber walls. It turns out that for a rectangular vacuum chamber no intermediate optimum exists: the best result is obtained when the plasma lies right next to the wall along the midplane. Other shapes for the vacuum chamber may produce different results, but a rectangular shape demonstrates well the tendency that the plasma has: to fill the allowed space.

We have now accounted for two features of the optimal shape. These features allow us to state the two constraints that reduce the dimension of the optimization space by two, which was referred to above. What remains to be optimized is the total elongation of the plasma as well as details in the boundary shape between the midplane and the plasma tips. How many Fourier coefficients are needed to represent the remaining characteristics of the shape is a matter of convergence of the answer. In practice, two coefficients were used to account for the elongation and the details of the boundary shape in addition to the two already determined using the constraints discussed above, making it a total of four coefficients $\{c_n\}$. Convergence was tested by performing the optimization with 10 coefficients $\{c_n\}$ in the full problem and the agreement between the answers was seen to be very good.
Fourier Coefficients and Geometric Quantities

It is now helpful to see how the Fourier coefficients \( \{c_n\} \) are related to the various geometric quantities. Figure (2.4) shows a typical crescent shaped cross-section where various points have been denoted by numbers 1 to 4. Point 1 is the outer edge of the plasma along the midplane and point 3 the corresponding inner edge. Point 2 denotes the top of the plasma column and point 4 is the point closest to the axis of symmetry. In our notation using the parameter \( \nu \), points 1, 2 and 3 correspond to \( \nu = 0, \pi/2 \) and \( \pi \) respectively, while point 4 satisfies \( R(\nu) = 0 \). Figure (2.4) also defines four lengths associated with the plasma and the vacuum chamber, viz. \( a, b, d \) and \( i \). The elongation of the vacuum chamber is given by

\[
\kappa \nu = \frac{b}{a}. \tag{2.143}
\]

Similarly, we may define the plasma triangularity as

\[
\delta = \frac{d}{a}, \tag{2.144}
\]

and the indentation by

\[
i = \frac{i}{a}. \tag{2.145}
\]

The coordinates of points 1 to 3 are given by

\[
R = R_0 + \epsilon R_0 \sum c_n \quad Z = 0 \quad \text{(point 1)},
\]
\[
R = R_0 + \epsilon R_0 \sum (-)^n c_{2n} \quad Z = \epsilon R_0 \quad \text{(point 2)},
\]
\[
R = R_0 + \epsilon R_0 \sum (-)^n c_n \quad Z = 0 \quad \text{(point 3)}.
\]

When we assume that the plasma shape has sharp tips, the points 2 and 4 coincide, and we can easily calculate \( \kappa \nu \) and \( \delta \) in terms of the \( \{c_n\} \). We find

\[
b = \epsilon R_0. \tag{2.146}
\]

The quantity \( a \) is defined as the width of the vacuum chamber. When the point closest to the axis of symmetry is not on the midplane (as shown in the figure), in other words
when $\delta \geq 1$, and when the points 2 and 4 coincide corresponding to a sharp tip in the shape, $a$ is equal to half the distance along the $R$-direction between points 1 and 3, and is given by

$$2a = \epsilon R_0 \sum (c_n - (-)^nc_{2n}) = \epsilon R_0 \left[ \sum c_{2n-1} + 2 \sum c_{4n-2} \right]. \tag{2.147}$$

When points 2 and 4 do not coincide, the expression for $a$ will be different. This case will be discussed below. When $\delta \leq 1$, we find

$$2a = \epsilon R_0 \sum (c_n - (-)^nc_n) = \epsilon R_0 \sum c_{2n-1}. \tag{2.148}$$

Similarly, we find

$$d = \frac{1}{2} \epsilon R_0 \sum (c_n - (-)^nc_n) - \epsilon R_0 \sum (-)^nc_{2n}$$

$$= \epsilon R_0 \sum (c_{2n} - (-)^nc_{2n})$$

$$= 2\epsilon R_0 \sum c_{4n-2}. \tag{2.149}$$

Then $\kappa_V$ and $\delta$ can be expressed as follows

$$\kappa_V = \frac{2}{s_1 + 2s_2}, \tag{2.150}$$

$$\delta = \frac{4s_2}{s_1 + 2s_2},$$

where

$$s_1 = \sum c_{2n-1},$$

$$s_2 = \sum c_{4n-2}. \tag{2.151}$$

When $\delta \leq 1$, the expressions are

$$\kappa_V = \frac{1}{s_1}, \tag{2.152}$$

$$\delta = \frac{2s_2}{s_1}.$$

We also note that the indentation is not independent of the triangularity in this notation. Specifically, from the figure,

$$2a - i = 2(2a - d), \tag{2.153}$$

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and we obtain, when \( \delta \geq 1 \),

\[
\iota = 2(\delta - 1).
\]  \hspace{1cm} (2.154)

This relationship can, in principle, be extended to \( \delta < 1 \), but in that regime it becomes meaningless to talk about indentations, since they would become negative.

Thus far we have analysed the case where points 2 and 4 coincide, corresponding to an infinitely sharp tip on the plasma boundary. Since in general we expect to analyse also shapes which are pointy, but not infinitely sharp, we find it convenient to define a measure of the radius of curvature of the boundary at \( \nu = \pi/2 \). The radius of curvature, \( R_c \), at a point along the boundary is given by

\[
\frac{1}{R_c} = |t \cdot \nabla t|,
\]  \hspace{1cm} (2.155)

where \( t \) is a unit vector tangent to the surface and is given by

\[
t = \frac{\dot{R}e_R + \dot{Z}e_Z}{(\dot{R}^2 + \dot{Z}^2)^{1/2}},
\]  \hspace{1cm} (2.156)

where dot \( \cdot \) denotes differentiation with respect to \( \nu \). A short calculation yields

\[
R_c = \left| \frac{(\dot{Z}^2 + \ddot{R}^2)^{3/2}}{\dot{Z} \dddot{R} - \dddot{Z} \dot{R}} \right|,
\]  \hspace{1cm} (2.157)

at any point along the plasma surface. In particular, at \( \nu = \pi/2 \) we have \( \dot{Z} = 0 \) and \( \dddot{Z} = \epsilon R_0 \), and so

\[
R_c = \epsilon R_0 s_3^2,
\]  \hspace{1cm} (2.158)

where

\[
s_3 = \sum_n (-)^{n+1} (2n - 1)c_{2n-1}.
\]  \hspace{1cm} (2.159)

The constraint of an infinitely sharp tip corresponds to \( s_3 = 0 \). Note that \( R_c = 0 \) is possible only at \( \nu = \pi/2 \) since in our representation \( \dot{Z} = 0 \) only at \( \nu = \pi/2 \). However, due to the arbitrary character of the parameter \( \nu \), the sharp tip is free to develop anywhere in real space, but is constrained to be located at the top of the plasma column. This is
a consequence of the fact that the $Z$–direction of the boundary has been parametrized using a very simple function (sin $\nu$). For this reason, our optimization must be thought of as an optimization over up–down symmetric plasma shapes with a separatrices at the top and bottom of the plasma columns. This is not a real restriction, however, since if the optimal shape did not satisfy this constraint, a solution of the full problem with 10 Fourier coefficients would be able to find such a shape. It would not be able to exactly reproduce an infinitely sharp tip, but 10 coefficients is plenty to produce a very sharp tip anywhere along the plasma boundary with a small non-zero radius of curvature at the tip.

Our physical picture discussed at the beginning of this section suggests that the optimal shape should satisfy $s_3 = 0$. However, the case of an infinitely sharp tip on the plasma boundary is difficult to analyse using a finite grid size. For this reason, we in practice choose a small but non–zero radius of curvature for the plasma tip. For this case, the expressions for the elongation and triangularity given above are not exact, since the expression for $a$ is only approximately valid. However, we shall retain these same expressions here for the sake of simplicity. The exact $\kappa_V$ and $\delta$ corresponding to the optimal shape can, of course, be calculated a posteriori.

It is convenient to normalize the radius of curvature of the plasma tip using a length scale that remains constant. One such length scale is the square root of the cross–sectional area of the vacuum chamber, giving us a normalized radius of curvature

$$\rho_c = \frac{R_c}{\epsilon R_0 / \sqrt{\kappa_V}} = \sqrt{\kappa_V} s_3^2. \tag{2.160}$$

In summary, if $\kappa_V$, $\delta$ and $\rho_c$ are specified, we have the following constraints for the coefficients $\{c_n\}$, when $\delta \geq 1$,

$$s_1 + 2s_2 = \frac{2}{\kappa_V},$$

$$2s_2 = \frac{\delta}{\kappa_V},$$

$$s_3^2 = \frac{\rho_c}{\sqrt{\kappa_V}}. \tag{2.161}$$

We note that the coefficients $\{c_{4n}\}$ are not included in these constraints and remain free. In practice, we usually keep four coefficients $\{c_1, c_2, c_3, c_4\}$ in the calculation. Three of
these four can be related to the quantities $\kappa_V$, $\delta$ and $\rho_e$ as follows. In the case of 4 Fourier coefficients,

$$s_1 = c_1 + c_3,$$

$$s_2 = c_2,$$

$$s_3 = c_1 - 3c_3,$$

and we obtain

$$c_1 = \frac{3}{4} \frac{2 - \delta}{\kappa_V} + \frac{1}{4} \frac{\sqrt{\rho_e}}{\kappa_V^{1/4}},$$

$$c_2 = \frac{\delta}{2\kappa_V},$$

$$c_3 = \frac{12 - \delta}{4 \kappa_V} - \frac{1}{4} \frac{\sqrt{\rho_e}}{\kappa_V^{1/4}}.$$  \hfill (2.162)

When $\delta < 1$, the expressions are given by

$$c_1 = \frac{3}{4\kappa_V} + \frac{1}{4} \frac{\sqrt{\rho_e}}{\kappa_V^{1/4}},$$

$$c_2 = \frac{\delta}{2\kappa_V},$$

$$c_3 = \frac{1}{4\kappa_V} - \frac{1}{4} \frac{\sqrt{\rho_e}}{\kappa_V^{1/4}}.$$  \hfill (2.163)

In both cases the coefficient $c_4$ remains independent of the geometrical quantities $\kappa_V$, $\delta$ and $\rho_e$. It is related to the details of the surface shape between the extrema $v = 0$, $v = \pi/2$, $v = \pi$ and $v = 3\pi/2$.

**Results of Numerical Computation**

The full problem is concerned with finding the optimal shape in the unrestricted 4-dimensional space $\{\kappa_V, \delta, \rho_e, c_4\}$. The physical picture above would suggest the use of the constraints

$$\delta = 1,$$

$$\rho_e = 0,$$  \hfill (2.165)

in order to reduce the problem to the 2-dimensional one. In practice, however, we choose

$$\delta = 1,$$

$$\rho_e = 0.03,$$  \hfill (2.166)
and optimize over the 2-dimensional space \( \{ \kappa_V, \epsilon_4 \} \). The results of this optimization are verified using the full problem formulation with 10 coefficients in the Fourier series for \( R \), and the two are found to be in good agreement. Due to computational limitations, however, this verification has been performed only for \( \epsilon_V = 0.5 \) with \( \alpha = 1 \), corresponding to a pure \( n = 1 \) mode. Specifically, with the full problem formulation we obtain an optimal peapod shape with sharp tips and the plasma filling the vacuum chamber along the midplane. The corresponding critical \( \beta_V, \beta_V = 0.221 \) with peapod elongation \( \kappa_V = 4.6 \). The reduced formulation yields \( \beta_V = 0.217 \) with \( \kappa_V = 4.4 \) with the pure \( n = 1 \) mode. As a result of this investigation, we use the reduced formulation to give the general results presented below. Since these results are obtained by optimizing over a subspace of cross-sectional shapes, the critical \( \beta_V \)-values obtained represent lower bounds for the true critical \( \beta_V \)’s.

We next investigate the results of simultaneous optimization with respect to both \( n = 0 \) and \( n = 1 \) modes. This optimization is achieved by manually varying the \( \alpha \)-parameter of section (2.3). Figure (2.5) shows the critical \( \beta_V \) as a function of \( \epsilon_V \), the machine size parameter. Also shown is the curve representing critical \( \beta_V \)’s for circular cross-section plasmas of the same size. It should be emphasized again that for a given value of \( \epsilon_V \), the circular shape has a much smaller value of the conventional aspect ratio \( (A = R/a) \) than the optimal shape, which is quite elongated. The optimal shape for \( \epsilon_V = 0.5 \) is shown in Figure (2.6). For comparison, a circular plasma with \( \epsilon_V = 0.5 \) is shown in Figure (2.7), demonstrating the difference in the aspect ratios of the two plasmas. Due to its appearance, we have called the optimal shape a “peapod”†.

Figure (2.8) shows the value of the optimized \( q_* \), the normalized plasma current parameter, as a function of \( \epsilon_V \), corresponding to the optimal shape and \( \beta_c \). The corresponding \( q_* \) for the circular shape is also given. It is seen that shaping the plasma cross-section allows higher plasma currents, or lower \( q_* \)’s, before the external kink becomes unstable. Figure (2.9) shows how the optimal elongation varies with \( \epsilon_V \). It is

† The fusion community has in the past demonstrated its vegetarian tendencies, for example in naming guiding centre orbits. This is just another garden produce added to the list.
concluded that to a good approximation, \( q_* \) is independent of \( \epsilon_V \) for the optimal shapes, while \( \kappa_V \) increases linearly as the inverse aspect ratio is increased. The result is that when \( \epsilon_V \) is increased, more elongated peapods become stable to both \( n = 0 \) and \( n = 1 \) modes. In fact, it will be shown below that the axisymmetric mode is the one that sets a limit on the plasma elongation.

We have also investigated how the critical \( \beta_V \) varies with the elongation of a peapod, when the machine size parameter \( \epsilon_V \) is fixed. The calculation is performed for \( \epsilon_V = 0.5 \) and \( \alpha = 1 \), denoting pure \( n = 1 \) modes. This resulting \( \beta_c \) is shown in Figure (2.10) as a function of elongation, and the maximum is seen to be a relatively flat one — small changes in the elongation do not change the critical \( \beta_V \) very much. When the simultaneous optimization with respect to \( n = 0 \) and \( n = 1 \) modes is performed, the maximum in \( \beta_V \) occurs at a lower value of \( \kappa_V \), and this maximum has a slightly lower value than given here for the pure \( n = 1 \) mode. However, we wish to avoid the added complication of the manual variation \( \alpha \), and present these results for the case \( \alpha = 1 \).

The variation of \( q_* \) corresponding to the above \( \beta_V \) as a function of \( \kappa_V \) (with \( \epsilon_V = 0.5 \)) is shown in Figure (2.11). Note that \( q_* \) is a weak function of elongation for peapod shapes. It can be concluded that while the critical \( \beta_V \) corresponds to a \( q_* \) which can be widely different for different cross-sectional shapes, for shapes close to the optimal shape the stability limit corresponds to a nearly constant and low value of \( q_* \), and therefore to a high value of the plasma current.

The stability diagram for the optimal peapod at \( \epsilon_V = 0.5 \) is shown in Figure (2.12), where the plasma \( \beta_V \) is plotted against \( 1/q_* \), the plasma current. The stable region is bounded from above by the equilibrium limit \( (k = 1) \) and from the right by the kink stability limit. The stable region can be thought to be generated by a set of equilibrium curves corresponding to different values of the equilibrium parameter \( k \in (-\infty, 1] \). The equilibrium curve corresponding to \( k = -\infty \) is the \( \beta_V = 0 \) axis, since this value corresponds to \( p_0 = 0 \) in the equilibrium equation. The stability limits for each \( k = \text{const.} \) are then obtained by increasing \( p_0 \) until \( \delta W = 0 \) for that equilibrium. The maximum \( \beta_c \) is obtained at the equilibrium limit \( k = 1 \). It is the locus of these
\( \beta_c(k = 1) \) for different \( \epsilon_V \) that give \( \beta_c \) as a function of \( \epsilon_V \) as shown in Figure (2.5). It can be seen from Figure (2.12) that if \( q_* \) is in the interval \((q_{*1}, q_{*2})\), and \( \beta_V \) is increased at a constant current, we first encounter an unstable region before getting to a region stable to \( n = 1 \) kink modes. This leads to the conclusion that for a peapod plasma the increase in the plasma pressure has to be programmed so it is suitably coupled with the increase of the plasma current. For circular and elliptical shapes the stable region is different in shape and such "seconds regions" of stability do not exist. The equilibria always become more unstable when \( \beta_V \) is increased at a constant current.

Figure (2.12) also demonstrates that in the sharp boundary model there is no simple description of the stability boundary against the external kink modes. Certainly, \( q_\psi \), the MHD safety factor, cannot be used for describing this boundary since it varies considerably along the boundary and is equal to \( \infty \) at the equilibrium limit, which corresponds to the highest \( \beta \)-value. However, \( q_* \) remains relatively constant along the stability boundary, suggesting that a \( q_* \)-limit, or a direct limit on the plasma current, should be used when defining the region of validity for the Troyon scaling law. Admittedly, however, the sharp boundary model does not produce such a scaling law. This is because the plasma \( \beta \) at a given current is ultimately limited only by the equilibrium limit. A proper treatment of the plasma region will take into account ballooning modes as well as internal kink and infernal modes, giving rise to another stability limit. This limit is a limit on plasma \( \beta \) at a given plasma current, and is seen before the equilibrium limit is reached. Nevertheless, the role of the external kink mode giving rise to a limit on the total plasma current remains the same no matter how the plasma region is treated.

A few comments about the interplay between the \( n = 0 \) and \( n = 1 \) modes are appropriate here. It is found that at large values of \( \epsilon_V \) (\( \geq 0.7 \)) the optimal peapod shape found by including only the \( n = 1 \) modes in the calculation is also stable to the axisymmetric modes. However, at lower values of \( \epsilon_V \) the shape found by optimizing against the pure \( n = 1 \) modes is found slightly unstable to the \( n = 0 \) mode. The shape stable to both \( n = 0 \) and \( n = 1 \) modes was then found by manually varying the parameter \( \alpha \) that controls the ratio between the amplitudes of the \( n = 1 \) and \( n = 0 \)
components in a general mixed mode displacement (see section (2.3)). It is discovered that a decrease in the elongation of the shape is needed in order to make it stable to both modes at the same time. The behaviour of $\beta_c$ as $\alpha$ is varied is given in Figure (2.13) for $\epsilon \nu = 0.3$. The minimum corresponds to the shape marginally stable to both modes and thus gives the true critical $\beta_c$. This minimum moves toward the right (toward purely $n = 1$ modes) as $\epsilon \nu$ is increased, but the magnitude of $\beta_c$ at the minimum is only slightly lower than with the pure $n = 1$ mode.

Figure (2.14) shows the variation of the corresponding elongation with $\alpha$. It can be seen that $\kappa \nu$ is a strong function of $\alpha$ and, we may therefore conclude, that the axisymmetric mode plays the main role in determining how elongated the plasma is. Since the minimum in $\beta_c$ as a function of $\alpha$ moves toward the pure $n = 1$ modes with an increasing $\epsilon \nu$, the result is, as stated above, that higher elongations become stable to both $n = 0$ and $n = 1$ modes when the aspect ratio becomes tighter.

Finally, $q_*$ is shown as a function of $\alpha$ for $\epsilon \nu = 0.3$ in Figure (2.15). It can again be seen that the current corresponding to $\beta_c$ at a given $\alpha$ varies very little as the mode mixture is changed from the pure $n = 1$ modes towards the axisymmetric mode. We have then seen that $q_*$ at the critical $\beta \nu$ for stability is very nearly constant for a large variety of plasma elongations whenever the shape otherwise resembles a peapod, in other words has a fat belly and tip pointing towards the axis of symmetry of the tokamak.
2.5 Conclusion

We have formulated the linear MHD stability problem for an arbitrary aspect ratio sharp boundary model tokamak plasma and investigated low \( n (=0,1,2) \) stability limits for such a configuration. The cross-sectional shape of a tokamak can be optimized in this model, and a simultaneous optimization with respect to \( n = 0 \) and \( n = 1 \) modes has been performed. It is also ensured that the shape is completely stable to \( n = 2 \) modes. We find that the optimal shape is a "peapod" with very high values of plasma \( \beta \) for stability. The improvement over a circular cross-section plasma of the same size (ie. the same volume and cross-sectional area of the vacuum chamber) is approximately by a factor of 2.5 for plasma \( \beta \) and the allowed plasma current is likewise much higher than for a circle. The sharp boundary model also allows us to address some issues concerning the "honest" definition of the plasma \( \beta \) and engineering constraints for the description of the vacuum chamber shape. These constraints are very general, and are useful not only in the context of the sharp boundary model, but also when more realistic plasma configurations are investigated. Given the good performance of a peapod plasma using the sharp boundary model for the plasma, it is natural to inquire how this type of plasma shape behaves in a configuration where the plasma pressure and current profiles are more realistic. This problem will be addressed in the next chapter.

2.6 Acknowledgments

I am deeply indebted to Professor Jeffrey Freidberg for his intuition concerning the sharp boundary model and acknowledge with pleasure his influence in all phases of this work. I am also grateful to Professor Hans Goedbloed for valuable comments during his stay at M.I.T. and to Scott Haney for continuous discussions. I would also like to thank Vonya Perham for helping with the typing of this chapter.
2.7 References


Figure 2.1 Typical stability diagrams for (a) circular plasma, (b) bean shaped plasma, and (c) hypothetical case where highest $\beta$ does not correspond to the equilibrium limit.
Figure 2.2 A large aspect ratio spherical shell tokamak with pointy tips.
Figure 2.3 Variation of $H_{nn}$, the diagonal element of the matrix representing $\delta W_{ss}$, with sharpness of the plasma tip. $R_c = 0$ represents a vanishing radius of curvature at the tip.
Figure 2.4 Typical crescent shaped cross-section demonstrating the relationship between geometrical quantities and the Fourier coefficients (see text).
Figure 2.5 $\beta_c$ as a function of $\epsilon_V$. 
Figure 2.6 Optimal shape stable to $n = 0$ and $n = 1$ modes for $\varepsilon_V = 0.5$

Figure 2.7 Circular plasma with $\varepsilon_V = 0.5$
Figure 2.8 $q_*$ as a function of $\varepsilon_V$ corresponding to maximum $\beta_V$. 

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Figure 2.9 Optimal elongation as a function of $\epsilon_V$. 
Figure 2.10 $\beta_c$ as a function of elongation for $\epsilon_V = 0.5$, $\alpha = 1$ (pure $n = 1$ mode).
Figure 2.11 $q_\ast$ corresponding to maximum $\beta_V$ as a function of elongation for $\epsilon_V = 0.5, \alpha = 1$. 
Figure 2.12 Stability diagram for optimal peapod with $\epsilon_Y = 0.5$. 

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Figure 2.13 $\beta_c$ as a function of $\alpha$ for $\epsilon_V = 0.3$. 
Figure 2.14 $\kappa_V$ corresponding to $\beta_c$ as a function of $\alpha$ for $\varepsilon_V = 0.3$. 
Figure 2.15 $q_*$ corresponding to $\beta_c$ as a function of $\alpha$ for $\epsilon_Y = 0.3$. 
Appendix A  Calculation of the Surface Term

The surface term is given by

\[ \delta W_* = \frac{1}{2\mu_0} \int_{\partial V_p} |\xi_\perp \cdot n|^2 n \cdot \left[ \nabla \frac{B^2}{2} \right] dS. \]  \hspace{1cm} (2A.1)

We have already shown in the text that

\[ n \cdot \left[ \nabla \frac{B^2}{2} \right] = - \left[ B \cdot (B \cdot \nabla)n \right]. \]  \hspace{1cm} (2A.2)

Now,

\[ n = \frac{\dot{\xi} e_R - \dot{\xi} e_Z}{\epsilon R_0 \Delta}, \]  \hspace{1cm} (2A.3)

where \( \Delta = (\dot{x}^2 + \dot{y}^2)^{1/2} \) and \( \dot{\gamma} \equiv \frac{d}{dv} \). We have also written

\[ R = R_0(1 + \epsilon x), \]  \hspace{1cm} (2A.4)
\[ Z = \epsilon R_0 y \]  \hspace{1cm} (2A.5)

Now, just inside the plasma surface

\[ B = B_i e_\phi, \]  \hspace{1cm} (2A.6)

and just outside the surface

\[ B = B_o e_\phi + B_p t, \]  \hspace{1cm} (2A.7)

where

\[ t = \frac{\dot{\xi} e_R + \dot{\xi} e_Z}{\epsilon R_0 \Delta}. \]  \hspace{1cm} (2A.8)

We also have, in general

\[ B \cdot \nabla = \frac{B_\phi}{R} \frac{\partial}{\partial \phi} + \frac{B_p}{\epsilon R_0 \Delta} \frac{\partial}{\partial v}, \]  \hspace{1cm} (2A.9)

so that

\[ B \cdot (B \cdot \nabla)n = (B_\phi e_\phi + B_p t) \left[ \frac{B_\phi}{R} \frac{\partial}{\partial \phi} + \frac{B_p}{\epsilon R_0 \Delta} \frac{\partial}{\partial v} \right] \frac{\dot{\xi} e_R - \dot{\xi} e_Z}{\Delta} \]

\[ = \frac{B_\phi^2}{R} \frac{\dot{\gamma}}{\Delta} + \frac{B_p^2}{\epsilon R_0 \Delta^2} \frac{\dot{\gamma} \dot{y} - \dot{y} \dot{\gamma}}{\Delta}. \]  \hspace{1cm} (2A.10)
This gives immediately
\[ n \cdot \left[ \nabla \frac{B^2}{2} \right] = -\frac{B^2_p}{\epsilon R_0 \Delta^3} (\dot{x} y - \dot{y} \dot{x}) - \frac{B^2_0 - B^2_i}{R_0 \Delta} \frac{\dot{y}}{(1 + \epsilon x)^3}. \] (2A.11)

Substituting the Fourier expansion for the normal component of the displacement vector on the surface,
\[ n \cdot \xi_\perp = \sum_m \hat{\xi}_m e^{imv}, \]
\[ n \cdot \xi_\parallel^* = \sum_p \hat{\xi}_p^* e^{-ipv}, \]
and using the fact that
\[ dS = \epsilon R_0^2 (1 + \epsilon x) \Delta \, dv \, d\phi, \] (2A.12)
we get
\[ \frac{\delta W_s}{2 \pi R_0} = -\frac{\pi B^2_0}{\mu_0} \sum_{m,p} \hat{\xi}_m H_{mp} \hat{\xi}_p^*, \] (2A.13)
where
\[ H_{mp} = \frac{1}{2\pi} \int_0^{2\pi} dv \, (1 + \epsilon x) \cos(m - p)v \left\{ \frac{B^2_p}{B^2_0 \Delta^2} (\dot{x} y - \dot{y} \dot{x}) + \frac{\epsilon (1 - B^2_i/B^2_0) \dot{y}}{(1 + \epsilon x)^3} \right\}. \] (2A.14)
Appendix B  Simplification of Boundary Conditions  
at Plasma Surface

It is shown in Appendix G that the boundary conditions in terms of the displacement
vector $\xi$ and the magnetic field strength $B$ at the plasma surface are given by
\begin{align}
(n_0 \cdot \nabla \psi)_S &= n_0 \cdot \nabla \times (\xi \times B_0)|_S, \quad (2B.1) \\
(n_0 \cdot \nabla \dot{\psi})_S &= n_0 \cdot \nabla \times (\xi \times \dot{B}_0)|_S, \quad (2B.2)
\end{align}

where quantities with subscript 0 refer to the unperturbed equilibrium. The vector
$\xi$ has, in general, three components, but it is immediately clear that the component
parallel to $B$ has no contribution to the boundary conditions. Writing,
\begin{equation}
\xi = \xi n_0 + \zeta n_0 \times b + \xi_b b, \quad (2B.3)
\end{equation}

where $b$ is a unit vector parallel to the magnetic field, we note that $\xi_b$ does not appear
in the boundary condition. Noting further that $n_0 \cdot b = 0$ so that $(n_0 \times b) \times b = -n_0$, and that for an axisymmetric tokamak
\begin{equation}
n_0 \cdot \nabla \times n_0 = 0, \quad (2B.4)
\end{equation}

we find that $\zeta$, too, disappears from the boundary conditions. Eqs. (2B.1) and (2B.2)
then become
\begin{align}
(n_0 \cdot \nabla \psi)_S &= n_0 \cdot \nabla \times (\xi n_0 \times B_0)|_S, \quad (2B.5) \\
(n_0 \cdot \nabla \dot{\psi})_S &= n_0 \cdot \nabla \times (\xi n_0 \times \dot{B}_0)|_S, \quad (2B.6)
\end{align}

and the normal component of the perturbed magnetic field is related only to the normal
component of the displacement vector.

We then write $n \equiv n_0$ and use
\begin{equation}
n = \frac{\dot{y}e_R - \dot{x}e_Z}{\Delta}, \quad (2B.7)
\end{equation}
where \( \Delta^2 = \dot{y}^2 + \dot{z}^2 \) and \( \dot{y} \equiv dy/dv, \dot{z} \equiv dz/dv \), with

\[
B = B_T e_\phi + B_p t, \tag{2B.8}
\]

where \( t \) is the tangential vector at the plasma surface perpendicular to the \( \phi \)-direction,

\[
t = \frac{\dot{z} e_R + \dot{y} e_Z}{\Delta}. \tag{2B.9}
\]

We then obtain

\[
n \times B = B_T t - B_p e_\phi. \tag{2B.10}
\]

Using the facts that for any vector \( a \),

\[
(\nabla \times a)_z = \frac{1}{R} \frac{\partial}{\partial R} Ra_\phi - \frac{1}{R} \frac{\partial a_R}{\partial \phi},
\]

\[
(\nabla \times a)_R = - \frac{\partial a_\phi}{\partial Z} + \frac{1}{R} \frac{\partial a_Z}{\partial \phi},
\]

we get

\[
n \cdot \nabla \times (-\xi B_p e_\phi) = \frac{\dot{y}}{\Delta} \frac{\partial}{\partial y} \xi B_p + \frac{\dot{z}}{\Delta} \frac{1}{R} \frac{\partial}{\partial x} R \xi B_p
\]

\[
= \frac{1}{\epsilon R_0^2 (1 + \epsilon x) \Delta} \frac{\partial}{\partial v} \xi R B_p. \tag{2B.11}
\]

Similarly,

\[
n \cdot \nabla \times (\xi B_T t) = \frac{\dot{y}}{R \Delta} \frac{(-i \xi) B_T \dot{y}}{\Delta} + \frac{\dot{z}}{R \Delta} \frac{(-i \xi) B_T \dot{z}}{\Delta}
\]

\[
= -i \frac{n B_T \xi}{R_0 (1 + \epsilon x)}. \tag{2B.12}
\]

Finally, the boundary conditions become

\[
(n \cdot \nabla \psi)_S = -i \frac{n B_t \xi}{R_0 (1 + \epsilon x)^2}, \tag{2B.13}
\]

\[
(n \cdot \nabla \dot{\psi})_S = -i \frac{n B_0 \xi}{R_0 (1 + \epsilon x)^2} + \frac{1}{\epsilon R_0^2 (1 + \epsilon x) \Delta} \frac{\partial}{\partial v} (R B_p \xi), \tag{2B.14}
\]

where the notation of the text has been used so that \( B_t \) and \( B_0 \) are the toroidal magnetic fields at \( R = R_0 \) inside and outside the plasma, respectively, and \( B_p \) is the poloidal magnetic field at the plasma surface.
Appendix C  Analytic Evaluation of the Reduced Green’s Function

The matrices $A$ and $B$ are defined by

$$A_{pm} = \frac{1}{4\pi^2} \int_0^{2\pi} dv \int_0^{2\pi} dv' C_A(v, v') \cos(mv' - pv), \quad (2C.1)$$

$$B_{pm} = \frac{1}{4\pi^2} \int_0^{2\pi} dv \int_0^{2\pi} dv' C_B(v, v') \cos(mv' - pv), \quad (2C.2)$$

where

$$C_A(v, v') = \varepsilon R_0 (1 + \varepsilon x(v')) n' \cdot \nabla' G_n(v, v'), \quad (2C.3)$$

$$C_B(v, v') = G_n(v, v'). \quad (2C.4)$$

In the above, $n$ denotes the toroidal mode number $(0, 1, 2, \ldots)$ and the Reduced Green’s Function $G_n(v, v')$ is defined by

$$G_n(v, v') = -4\pi R_0 \int_0^{2\pi} d\phi' G(x, x') \cos n(\phi - \phi'), \quad (2C.5)$$

where the $G(x, x')$ is the infinite space Green’s Function

$$G(x, x') = -\frac{1}{4\pi|x - x'|}. \quad (2C.6)$$

Two analytic steps can be taken to speed up the numerical evaluation of the matrix elements $A_{pm}$ and $B_{pm}$. First, the $\phi'$-integration in the Reduced Green’s Function can be performed analytically and thus the integrands $C_A$ and $C_B$ are expressed explicitly as functions of $v$ and $v'$. Second, the resulting functions $C_A$ and $C_B$ have integrable logarithmic singularities along the line $v = v'$, and these singularities can be analytically removed by adding and subtracting a convenient function which can then be integrated analytically. The resulting expressions are smooth functions, and Fast Fourier Transform techniques can be used on a relatively coarse grid $(128 \times 128)$ in the $(v, v')$-space in order to achieve fast and accurate evaluations of the matrix elements. These steps will now be discussed in some detail.
\( \phi' \)-integration

Using the expression

\[
|\mathbf{x} - \mathbf{x}'| = [R^2 + R'^2 - 2RR' \cos(\phi - \phi') + (Z - Z')^2]^{1/2}, \tag{2C.7}
\]

where \( R = R_0 + \varepsilon x(v) \), \( R' = R_0 + \varepsilon x(v') \), \( Z = Z(v) \), \( Z' = Z(v') \), we can rewrite the Reduced Green's Function in the form

\[
G_n(v, v') = \frac{R_0 k^{1/2}}{(RR')^{1/2}} L_n(v, v'), \tag{2C.8}
\]

where

\[
L_n(v, v') = \int_0^{2\pi} \frac{d\psi \cos n\psi}{(1 + k^2 - 2k \cos \psi)^{1/2}}. \tag{2C.9}
\]

In the above, we have written

\[
k = \alpha - (\alpha^2 - 1)^{1/2}, \tag{2C.10}
\]

with

\[
\alpha = 1 + \frac{e^2}{2} \frac{(x(v) - x(v'))^2 + (y(v) - y(v'))^2}{(1 + \varepsilon x(v))(1 + \varepsilon x(v'))}. \tag{2C.11}
\]

\( G_n \) and \( \mathbf{n'} \cdot \nabla' G_n \) can now be evaluated for any \( n \) in terms of the complete elliptic integrals of the first and second kinds (see Appendix J). In general, using the fact that

\[
\mathbf{n'} = \frac{\mathbf{y}' e_R - \mathbf{z}' e_Z}{\Delta'}, \tag{2C.12}
\]

where \( \Delta' = ((\mathbf{z'})^2 + (\mathbf{y'})^2)^{1/2} \), and \( \mathbf{z'} \equiv dx(v')/dv' \) and \( \mathbf{y'} \equiv dy(v')/dv' \), we obtain

\[
\mathbf{n'} \cdot \nabla' G_n = \frac{1}{\varepsilon R_0 \Delta'} \left[ \mathbf{y}' \frac{\partial}{\partial x(v')} - \mathbf{z}' \frac{\partial}{\partial y(v')} \right] G_n. \tag{2C.13}
\]

Using Eq. (2C.8), we have

\[
\mathbf{n'} \cdot \nabla' G_n = \frac{1}{\varepsilon \Delta'} \left[ k^{1/2} L_n(k) \left( \frac{\mathbf{y}' \partial}{\partial x(v')} - \mathbf{z}' \frac{\partial}{\partial y(v')} \right) \right] \frac{1}{(RR')^{1/2}}
\]

\[
+ \frac{1}{(RR')^{1/2}} \left( \mathbf{y}' \frac{\partial k}{\partial x(v')} - \mathbf{z}' \frac{\partial k}{\partial y(v')} \right) \frac{d(k^{1/2} L_n)}{dk}. \tag{2C.14}
\]
A short straightforward calculation shows that

\[
\dot{y}' \frac{\partial k}{\partial x(v')} - \dot{x}' \frac{\partial k}{\partial y(v')} = \frac{k}{k - \alpha} \left[ \varepsilon^2 \left\{ \dot{x}' \left[ x(v') - x(v) \right] - \dot{x}' \left[ y(v') - y(v) \right] \right\} \right. \\
\left. \frac{(1 + \varepsilon(x(v))) (1 + \varepsilon(x(v')))}{(1 + \varepsilon(x(v'))) (1 + \varepsilon(x(v)))} + \frac{(1 - \alpha) \varepsilon \dot{y}'}{(1 + \varepsilon(x(v'))) (1 + \varepsilon(x(v)))} \right].
\]

(2C.15)

Using the identities

\[
\frac{k}{k - \alpha} = \frac{2k^2}{k^2 - 1},
\]

(2C.16)

\[
1 - \alpha = -\frac{(1 - k)^2}{2k},
\]

(2C.17)

and the facts that

\[
\frac{\partial R'}{\partial y(v')} = 0,
\]

(2C.18)

\[
\frac{\partial R'}{\partial x(v')} = \varepsilon R_0,
\]

(2C.19)

\[
\frac{\partial}{\partial x(v')} \frac{1}{(RR')^{1/2}} = -\frac{\varepsilon R_0}{2R^{1/2}R'^{3/2}}
\]

(2C.20)

we obtain, after some rearranging

\[
C_A(v, v') = -\frac{\varepsilon \dot{y}'}{(1 + \varepsilon(x(v)))^{1/2} (1 + \varepsilon(x(v')))^{1/2}} \frac{k^{3/2}}{1 + k} \left[ L_n - (1 - k) \frac{dL_n}{dk} \right] \\
+ \frac{\varepsilon^2 \left\{ \dot{x}' \left[ x(v') - x(v) \right] - \dot{x}' \left[ y(v') - y(v) \right] \right\} k^{3/2}}{(1 + \varepsilon(x(v)))^{3/2} (1 + \varepsilon(x(v')))^{1/2}} \frac{k^{3/2}}{k^2 - 1} \left[ L_n + 2k \frac{dL_n}{dk} \right].
\]

(2C.21)

From Eqs. (2C.4) and (2C.8) it follows immediately that

\[
C_B(v, v') = \frac{4k^{1/2} L_n}{(1 + \varepsilon(x(v)))^{1/2} (1 + \varepsilon(x(v')))^{1/2}}.
\]

(2C.22)
Explicit expressions for \( n = 0, 1, 2 \).

To proceed further we need to fix the value of \( n \). In this work, we consider \( n = 0, 1, 2 \) modes, and give explicit expressions for \( G_0, G_1 \) and \( G_2 \). For \( n = 0 \) modes, we have

\[
L_0 = \int_0^{2\pi} \frac{d\psi}{(1 + k^2 - 2k \cos \psi)^{1/2}}. \tag{2C.23}
\]

Substituting \( m = 2\sqrt{k}/1 + k \) and \( \psi = 2x \), we obtain, using \( \cos 2x = 2\cos^2 x - 1 \),

\[
L_0 = \frac{4}{1 + k} \int_0^{\pi/2} \frac{dx}{(1 - m^2 \cos^2 x)^{1/2}},
\]

\[
= 4K(m)
\]

\[
= 4K(k),
\]

where we have used a result from Appendix J. Then we have immediately,

\[
G_0 = \frac{4R_0k^{1/2}K(k)}{(RR')^{1/2}}. \tag{2C.24}
\]

This gives

\[
C_B^{(n=0)}(v, v') = \frac{4k^{1/2}K(k)}{(1 + \epsilon x(v))^{1/2}(1 + \epsilon x(v'))^{1/2}}. \tag{2C.25}
\]

Using the identity

\[
\frac{dK}{dk} = \frac{E(k)}{k(1 - k^2)} - \frac{K(k)}{k}, \tag{2C.26}
\]

(see Appendix J) we obtain the result

\[
C_A^{(n=0)}(v, v') = -\frac{4\epsilon y'k^{1/2}}{(1 + \epsilon x(v))^{1/2}(1 + \epsilon x(v'))^{1/2}} \left[ \frac{K(k)}{1 + k} - \frac{E(k)}{(1 + k)^2} \right]
\]

\[
+ \frac{4\epsilon^2 k^{3/2}\{y'[x'(v') - x(v)] - x'[y'(v') - y(v)]\}}{(1 + \epsilon x(v))^{3/2}(1 + \epsilon x(v'))^{1/2}(1 - k^2)} \left[ K(k) - \frac{2E(k)}{1 - k^2} \right]. \tag{2C.27}
\]

For \( n = 1 \) modes, we have

\[
L_1 = \int_0^{2\pi} \frac{d\psi \cos \psi}{(1 + k^2 - 2k \cos \psi)^{1/2}}. \tag{2C.28}
\]

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Substituting \( \psi = 2x \) we obtain

\[
L_1 = 4 \int_0^{\pi/2} \frac{dx \cos 2x}{(1 + k^2 - 2k \cos 2x)^{1/2}}. \tag{2C.29}
\]

Using the identity

\[
\cos 2x = 2 \cos^2 x - 1, \tag{2C.30}
\]

we get

\[
L_1 = 4 \int_0^{\pi/2} \frac{dx (2 \cos^2 x - 1)}{(1 + k)(1 - m^2 \cos^2 x)^{1/2}}, \tag{2C.31}
\]

where \( m = 2\sqrt{k}/(1 + k) \). Using the results of Appendix J, we have after a short calculation

\[
L_1 = \frac{4(K(k) - E(k))}{k}, \tag{2C.32}
\]

and so

\[
C_B^{(n=1)}(v, v') = \frac{4R_0}{(RR')^{1/2}} \frac{K(k) - E(k)}{k^{1/2}}. \tag{2C.33}
\]

We next note that

\[
\frac{dL_1}{dk} = \frac{4}{k^2} \left( \frac{E(k)}{1 - k^2} - K(k) \right). \tag{2C.34}
\]

Another short calculation yields then

\[
C_A^{(n=1)}(v, v') = -\frac{4eij'}{(1 + \epsilon x(v))^1/2 (1 + \epsilon x(v'))^{1/2} k^{1/2}} \left[ K(k) \frac{K(k) - (1 + k + k^2)E(k)}{(k + k^2)^2} \right]
\]

\[
+ \frac{4e^2 k^{1/2} [y'(v') - y(v)] - x'(v') - y'(v') - y(v)]}{(1 + \epsilon x(v))^{3/2} (1 + \epsilon x(v'))^{1/2} (1 - k^2)} \left[ K(k) - \frac{1 + k^2}{1 - k^2} E(k) \right]. \tag{2C.35}
\]

For \( n = 2 \) modes, similar calculations are performed. We have

\[
L_2 = \int_0^{2\pi} \frac{d\psi \cos 2\psi}{(1 + k^2 - 2k \cos \psi)^{1/2}}. \tag{2C.36}
\]

Substituting \( \psi = 2x \) and using \( m = 2\sqrt{k}/(1 + k) \), we obtain

\[
L_2 = 4 \int_0^{\pi/2} \frac{dx \cos 4x}{(1 + k^2 - 2k \cos 2x)^{1/2}}. \tag{2C.37}
\]

Using the identities

\[
\cos 2x = 2 \cos^2 x - 1, \tag{2C.38}
\]
\[ \cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1, \quad (2C.39) \]

we have, after a lengthy calculation

\[ L_2 = \frac{4}{3k^2} \left[ (2 + k^2)K(k) - (2 + 2k^2)E(k) \right], \quad (2C.40) \]

where techniques similar to those employed in Appendix J have been used to obtain the final answer. A shorter calculation shows that

\[ \frac{dL_2}{dk} = \frac{4}{3k^3} \left[ \frac{2 - k^2 + 2k^4}{1 - k^2} E(k) - (2 - k^2)K(k) \right]. \quad (2C.41) \]

These results enable us to calculate the final expressions for \( C_A \) and \( C_B \). Another lengthy calculation yields

\[
C_A^{(n=2)}(v, v') = -\frac{16\epsilon y'k^{1/2}}{(1 + \epsilon x(v))^{1/2}(1 + \epsilon x(v'))^{1/2}} \left[ \frac{(4 - 2k - k^2 + 2k^3)K(k)}{3k^2(1 + k)} \right.
- \left. \frac{(4 + 2k - k^2 + 2k^3 + 4k^4)E(k)}{3k^2(1 + k)^2} \right]
+ \frac{16\epsilon^2 \{ y'[x(v') - x(v)] - x'[y(v') - y(v)] \}}{(1 + \epsilon x(v))^{3/2}(1 + \epsilon x(v'))^{1/2}k^{1/2}(1 - k^2)} \left[ (2 - k^2)K(k) - \frac{2 - 2k^2 + 2k^4}{1 - k^2} E(k) \right],
\]

(2C.42)

and Eqs. (2C.4), (2C.8) and (2C.40) immediately give

\[ C_B^{(n=2)}(v, v') = \frac{16}{3(1 + \epsilon x(v))^{1/2}(1 + \epsilon x(v'))^{1/2}k^{3/2}} \left[ (2 + k^2)K(k) - (2 + 2k^2)E(k) \right]. \quad (2C.43) \]

**Analytic Evaluation of Logarithmic Singularities**

It remains to analytically remove the logarithmic singularities of the integrands \( C_A^{(n)} \) and \( C_B^{(n)} \). These result when \( v \to v' \), as \( \alpha \to 1 \) and so \( k \to 1 \). Then\(^{[15]}\)

\[ \lim_{k \to 1} K(k) = 2 \ln 2 - \frac{1}{2} \lim_{k \to 1} \ln (1 - k^2), \quad (2C.44) \]

and an integrable singularity results from the complete elliptic integral of the first kind. The calculation will be performed in some detail for the \( n = 0 \) mode and the results
are stated for the \( n = 1 \) and \( n = 2 \) modes, since those calculations do not qualitatively differ from the \( n = 0 \) calculation.

Let \( v' = v + \delta \). Then

\[
x(v') = x + \delta \dot{x} + \frac{1}{2} \delta^2 \ddot{x} + \ldots,
\]

\[
y(v') = y + \delta \dot{y} + \frac{1}{2} \delta^2 \ddot{y} + \ldots,
\]

where the functions on the right hand side are evaluated at \( v \). Writing \( \alpha = 1 + \delta \alpha \), we have

\[
\delta \alpha = \frac{e^2[(x(v') - x(v))^2 + (y(v') - y(v))^2]}{2(1 + \varepsilon x(v))(1 + \varepsilon x(v'))}
\]

\[
= \frac{e^2 \Delta^2 \delta^2}{2(1 + \varepsilon x(v))^2}.
\]

Then a short calculation yields

\[
1 - k = \sqrt{2\delta \alpha},
\]

and so

\[
1 - k^2 = 2\sqrt{2\delta \alpha}.
\]

Another short calculation yields

\[
\lim_{k \to 1} K(k) = \ln 2 - \frac{1}{4} \ln \frac{e^2 \Delta^2}{(1 + \varepsilon x(v))^2} - \frac{1}{2} \ln \frac{|v - v'|}{2},
\]

where we have written, in general, \( |\delta| = |v - v'| \). We also note that

\[
\lim_{k \to 1} E(k) = 1.
\]

Writing

\[
\dot{y}' = \dot{y} + \delta \ddot{y} + \ldots,
\]

\[
\dot{x}' = \dot{x} + \delta \ddot{x} + \ldots,
\]

we obtain, after another short calculation,

\[
\dot{y}'[x(v') - x(v)] - \dot{x}'[y(v') - y(v)] = \frac{1}{2} \delta^2 (\dot{x} \ddot{y} - \dot{y} \ddot{x}).
\]
Substituting these results into Eq. (2C.27) yields, after some rearranging,

$$
\lim_{v' \to v} C_A^{(n=0)}(v, v') = -\frac{2\epsilon \dot{y}}{(1 + \epsilon x(v))} \left[ \ln 2 - \frac{1}{4} \ln \frac{\epsilon^2 \Delta^2}{(1 + \epsilon x(v))^2} - \frac{1}{2} \right] - \frac{\dot{x} \dot{y} - \dot{y} \ddot{x}}{\Delta^2} + \frac{\epsilon \dot{y}}{(1 + \epsilon x(v))} \ln \frac{|v - v'|}{2}.
$$

(2C.52)

Similar calculations yield, using Eq. (2C.25),

$$
\lim_{v' \to v} C_B^{(n=0)}(v, v') = \frac{1}{(1 + \epsilon x(v))} \left[ \ln 16 - \ln \frac{\epsilon^2 \Delta^2}{(1 + \epsilon x(v))^2} - 2 \ln \frac{|v - v'|}{2} \right].
$$

(2C.53)

We then define

$$
\tilde{C}_A^{(n=0)}(v, v') = C_A^{(n=0)}(v, v') - \frac{\epsilon \dot{y}}{(1 + \epsilon x(v))} \ln \left| \sin \frac{|v - v'|}{2} \right|,
$$

(2C.54)

$$
\tilde{C}_B^{(n=0)}(v, v') = C_B^{(n=0)}(v, v') + \frac{2}{(1 + \epsilon x(v))} \ln \left| \sin \frac{|v - v'|}{2} \right|,
$$

(2C.55)

so that

$$
\tilde{C}_A^{(n=0)}(v, v) = -\frac{2\epsilon \dot{y}}{(1 + \epsilon x(v))} \left[ \ln 2 - \frac{1}{4} \ln \frac{\epsilon^2 \Delta^2}{(1 + \epsilon x(v))^2} - \frac{1}{2} \right] - \frac{\dot{x} \dot{y} - \dot{y} \ddot{x}}{\Delta^2}
$$

(2C.56)

and

$$
\tilde{C}_B^{(n=0)}(v, v) = \frac{1}{(1 + \epsilon x(v))} \left[ \ln 16 - \ln \frac{\epsilon^2 \Delta^2}{(1 + \epsilon x(v))^2} \right],
$$

(2C.57)

both finite. It then follows that the matrix elements $A_{pm}$ and $B_{pm}$ are given by

$$
A_{pm}^{(n=0)} = \frac{1}{4\pi^2} \int_0^{2\pi} dv \int_0^{2\pi} dv' \tilde{C}_A^{(n=0)}(v, v') \cos(mv' - pv) + \hat{A}_{pm},
$$

(2C.58)

$$
B_{pm}^{(n=0)} = \frac{1}{4\pi^2} \int_0^{2\pi} dv \int_0^{2\pi} dv' \tilde{C}_B^{(n=0)}(v, v') \cos(mv' - pv) + \hat{B}_{pm},
$$

(2C.59)

where

$$
\hat{A}_{pm} = -\frac{1}{4\pi^2} \int_0^{2\pi} dv \int_0^{2\pi} dv' \frac{\epsilon \dot{y}}{(1 + \epsilon x(v))} \ln \left| \sin \frac{|v - v'|}{2} \right| \cos(mv' - pv),
$$

(2C.60)

$$
\hat{B}_{pm} = \frac{1}{4\pi^2} \int_0^{2\pi} dv \int_0^{2\pi} dv' \frac{2}{(1 + \epsilon x(v))} \ln \left| \sin \frac{|v - v'|}{2} \right| \cos(mv' - pv).
$$

(2C.61)
Using the identity
\[
\cos(mv' - pv) = \cos m(v - v') \cos (m - p)v - \sin m(v - v') \sin (m - p)v
\]  
(2C.62)
and results calculated in Appendix F, we obtain
\[
\hat{A}_{pm} = \frac{S_m}{2\pi} \int_0^{2\pi} dv \frac{\epsilon y}{(1 + \epsilon x(v))} \cos (m - p)v,
\]  
(2C.63)
\[
\hat{B}_{pm} = -\frac{S_m}{\pi} \int_0^{2\pi} dv \frac{\cos (m - p)v}{(1 + \epsilon x(v))},
\]  
(2C.64)
where
\[
S_m = \begin{cases} 
\ln 2, & m = 0, \\
\frac{1}{2|m|}, & m \neq 0.
\end{cases}
\]
These results enable us to evaluate the matrix elements quickly and reliably using Fast Fourier Transforms, because the resulting integrands for both the one and two-dimensional integrals are smooth functions.

For the \(n = 1\) modes the corresponding results are
\[
\hat{A}_{pm}^{(n=1)} = \frac{1}{4\pi^2} \int_0^{2\pi} dv \int_0^{2\pi} dv' \tilde{C}_A^{(n=1)}(v, v') \cos (mv' - pv) + \hat{A}_{pm},
\]  
(2C.65)
\[
\hat{B}_{pm}^{(n=1)} = \frac{1}{4\pi^2} \int_0^{2\pi} dv \int_0^{2\pi} dv' \tilde{C}_A^{(n=1)}(v, v') \cos (mv' - pv) + \hat{B}_{pm},
\]  
(2C.66)
where
\[
\tilde{C}_A^{(n=1)}(v, v') = C_A^{(n=1)}(v, v') - \frac{\epsilon y}{(1 + \epsilon x(v))} \ln \left| \frac{\sin \frac{v - v'}{2}}{2} \right|,
\]  
(2C.67)
\[
\tilde{C}_B^{(n=1)}(v, v') = C_B^{(n=1)}(v, v') + \frac{2}{(1 + \epsilon x(v))} \ln \left| \frac{\sin \frac{v - v'}{2}}{2} \right|,
\]  
(2C.68)
and the limiting values are
\[
\tilde{C}_A^{(n=1)}(v, v) = -\frac{2\epsilon y}{(1 + \epsilon x(v))} \left[ \ln 2 - \frac{1}{4} \ln \frac{\epsilon^2 \Delta^2}{(1 + \epsilon x(v))^2} - \frac{3}{2} \right] - \frac{\dot{x} y - \dot{y} \ddot{x}}{\Delta^2},
\]  
(2C.69)
\[
\tilde{C}_B^{(n=1)}(v, v) = \frac{1}{(1 + \epsilon x(v))} \left[ \ln 16 - \ln \frac{\epsilon^2 \Delta^2}{(1 + \epsilon x(v))^2} - 4 \right].
\]  
(2C.70)
The \(\hat{A}_{pm}\) and \(\hat{B}_{pm}\) are exactly the same as before.
Finally, for the $n = 2$ modes the results are

\[
A_{pm}^{(n=2)} = \frac{1}{4\pi^2} \int_0^{2\pi} dv \int_0^{2\pi} dv' \tilde{C}_A^{(n=2)}(v, v')\cos(mv' - pv) + 4\hat{A}_{pm}, \quad (2C.71)
\]

\[
B_{pm}^{(n=2)} = \frac{1}{4\pi^2} \int_0^{2\pi} dv \int_0^{2\pi} dv' \tilde{C}_A^{(n=2)}(v, v')\cos(mv' - pv) + \frac{4}{3}\hat{B}_{pm}, \quad (2C.72)
\]

where

\[
\tilde{C}_A^{(n=2)}(v, v') = C_A^{(n=2)}(v, v') - \frac{4\varepsilon\dot{y}}{(1 + \varepsilon x(v))} \ln \left| \sin \frac{v - v'}{2} \right|, \quad (2C.73)
\]

\[
\tilde{C}_B^{(n=2)}(v, v') = C_B^{(n=2)}(v, v') + \frac{8}{3(1 + \varepsilon x(v))} \ln \left| \sin \frac{v - v'}{2} \right|, \quad (2C.74)
\]

and the limiting values are

\[
\tilde{C}_A^{(n=2)}(v, v) = -\frac{8\varepsilon\dot{y}}{(1 + \varepsilon x(v))} \left[ \ln 2 - \frac{1}{4} \ln \frac{\varepsilon^2\Delta^2}{(1 + \varepsilon x(v))^2} - \frac{11}{6} \right] - 4\frac{x\ddot{y} - \dot{y}\ddot{x}}{\Delta^2}, \quad (2C.75)
\]

\[
\tilde{C}_B^{(n=2)}(v, v) = \frac{16}{(1 + \varepsilon x(v))} \left[ \ln 2 - \frac{1}{4} \ln \frac{\varepsilon^2\Delta^2}{(1 + \varepsilon x(v))^2} - \frac{4}{3} \right]. \quad (2C.76)
\]

Again, the $\hat{A}_{pm}$ and $\hat{B}_{pm}$ are exactly the same as before.
Appendix D  Plasma Contribution for Axisymmetric Modes

The standard form for $\delta W_p$ is given by

$$\delta W_p = \frac{1}{2} \int dr \left\{ \frac{|Q|^2}{\mu_0} - \xi^* \cdot J \times Q + \gamma p |\nabla \cdot \xi|^2 + (\xi \cdot \nabla p) \nabla \cdot \xi \right\}, \quad (2D.1)$$

where

$$\mu_0 J = \nabla \times B, \quad (2D.2)$$
$$Q = \nabla \times (\xi \times B), \quad (2D.3)$$

$p$ is the plasma pressure and $\xi$ the plasma displacement vector. In the sharp boundary model of the plasma $\nabla p = 0$, so that the last term vanishes. We next calculate $Q$. Expanding the cross products,

$$Q = -B (\nabla \cdot \xi) + (B \cdot \nabla) \xi - (\xi \cdot \nabla) B, \quad (2D.4)$$

where we have used $\nabla \cdot B = 0$. The last term yields

$$(\xi \cdot \nabla) B = b(\xi \cdot \nabla) B, \quad (2D.5)$$

where $B = |B|$, $b$ is the unit vector in the direction of $B$ and use has been made of the fact that $(\xi \cdot \nabla) b = 0$. This is because in the sharp boundary model $b = e_\phi$ inside the plasma region, and $e_\phi$ has a non–zero derivative only in the direction of $e_\phi$ itself. We then obtain from

$$J \times B = 0, \quad (2D.6)$$

the relation

$$\nabla B = (b \cdot \nabla) B, \quad (2D.7)$$

and finally

$$(\xi \cdot \nabla) B = B(\xi \cdot \kappa), \quad (2D.8)$$
where $\kappa = (b \cdot \nabla)b$. We also know from the fact that $b = e_\phi$ that the vector $(b \cdot \nabla)x$ for any vector $x$ is parallel to $b$. Therefore, we have

$$
(B \cdot \nabla)\xi_\perp = B \cdot (b \cdot \nabla)\xi_\perp
$$

$$
= -B \xi_\perp(b \cdot \nabla)b
$$

$$
= -B \xi_\perp \cdot \kappa. \tag{2D.9}
$$

Here we have used the fact that the axisymmetric mode is independent of $\phi$, i.e. $(b \cdot \nabla)\xi = 0$, $\xi$ being the magnitude of the displacement vector. Then we have

$$
Q = -(\nabla \cdot \xi_\perp + 2\xi_\perp \cdot \kappa)B, \tag{2D.10}
$$

and

$$
J \times Q = 0.
$$

The plasma contribution to $\delta W$ then becomes

$$
\delta W_p = \frac{1}{2} \int \text{d}r \left\{ \gamma p_0 |\nabla \cdot \xi_\perp|^2 + \frac{B^2}{\mu_0} |\nabla \cdot \xi_\perp + 2\xi_\perp \cdot \kappa|^2 \right\}. \tag{2D.11}
$$

**Simplification of $\delta W_p$**

Let us now vary $\delta W_p$ by setting $\xi_\perp \rightarrow \xi_\perp + \delta \xi_\perp$. A short calculation yields

$$
\delta(\delta W_p) = \int_{\partial V_p} \text{d}S \left[ \gamma p_0 \nabla \cdot \xi_\perp + \frac{B^2}{\mu_0} (\nabla \cdot \xi_\perp + 2\xi_\perp \cdot \kappa) \right] \mathbf{n} \cdot \delta \xi_\perp^* 
$$

$$
- \int_{V_p} \text{d}r \nabla \left\{ \gamma p_0 \nabla \cdot \xi_\perp + \frac{B^2}{\mu_0} (\nabla \cdot \xi_\perp + 2\xi_\perp \cdot \kappa) \right\} - 2\kappa \frac{B^2}{\mu_0} (\nabla \cdot \xi_\perp + 2\xi_\perp \cdot \kappa), \tag{2D.12}
$$

where $V_p$ is the plasma volume and $\partial V_p$ the plasma surface. Setting $\delta(\delta W_p) = 0$ gives

$$
\nabla \left[ \gamma p_0 \nabla \cdot \xi_\perp + \frac{B^2}{\mu_0} (\nabla \cdot \xi_\perp + 2\xi_\perp \cdot \kappa) \right] = 2\kappa \frac{B^2}{\mu_0} (\nabla \cdot \xi_\perp + 2\xi_\perp \cdot \kappa). \tag{2D.13}
$$

We note that $\kappa = -e_\phi/R$, and we immediately obtain

$$
\gamma p_0 \nabla \cdot \xi_\perp + \frac{B^2}{\mu_0} (\nabla \cdot \xi_\perp + 2\xi_\perp \cdot \kappa) = f(R), \tag{2D.14}
$$

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and
\[
\frac{df}{dR} = -\frac{2}{R} [f - \gamma p_0 \nabla \cdot \xi_\perp].
\] (2D.15)

It then follows that \( \nabla \cdot \xi_\perp \) and \( \xi_\perp \cdot \kappa \) are functions of \( R \) only and independent of \( Z \).

Also, a short calculation shows that
\[
\nabla \cdot \xi_\perp + 2 \xi_\perp \cdot \kappa = -\frac{\mu_0 R \, df}{2B^2 \, dR}.
\] (2D.16)

Substitution into \( \delta W_p \) yields after a short calculation, using \( d\mathbf{r} = RdRdZd\phi \) and integrating in the \( \phi \)-direction,
\[
\delta W_p = \pi \int dZ \int dRR \left[ \frac{1}{\gamma p_0} \left( f^2 + Rff' + \frac{R^2}{4} f'^2 \right) + \frac{\mu_0 R^2}{4B^2} f'^2 \right].
\]

We can integrate in the \( Z \)-direction and introduce the variable \( v \) for parametrizing the plasma surface. This yields
\[
\delta W_p = \frac{\pi}{\gamma p_0} \int_0^{2\pi} dv \, \dot{R} \dot{Z} \left[ \frac{1}{4} R^2 \left( 1 + \frac{k R^2}{R_0^2} \right) f'^2 + Rff' + f^2 \right],
\] (2D.17)

where
\[
k = \frac{\gamma p_0 \mu_0}{B_0^2},
\] (2D.18)

and prime ('') denotes differentiation with respect to \( R \), so that
\[
f' = \frac{df}{dR} = \frac{\dot{f}}{\dot{R}},
\] (2D.19)

with \( \dot{\cdot} = d/dv \).

**Constraint for \( f \)**

However, \( f \) is not a completely free function but satisfies a certain constraint. We obtain this constraint by integrating the normal component of the displacement vector over the plasma surface:
\[
\int dS \, n \cdot \xi_\perp = \int_0^{2\pi} d\phi \int_0^{2\pi} dv \, Ra_0 \Delta n \cdot \xi_\perp,
\] (2D.20)
\[
= 2\pi a_0 R_0 \int_0^{2\pi} dv \, (1 + \epsilon x) \Delta n \cdot \xi_\perp,
\]
where $\Delta = (\dot{x}^2 + \dot{y}^2)^{1/2}$. We use the Fourier expansion of the normal component of the displacement vector,

$$n \cdot \xi_\perp = \sum_m \hat{\xi}_m e^{im\nu},$$  \hspace{1cm} (2D.21)

and obtain

$$\oint dS \, n \cdot \xi_\perp = 2\pi a_0 R_0 a \cdot \hat{\xi},$$  \hspace{1cm} (2D.22)

where $\hat{\xi}$ represents the (infinite) vector of the Fourier components of $n \cdot \xi_\perp$ and the $m$'th component of $a$ is given by

$$a_m = \int_0^{2\pi} dv \, (1 + \epsilon x) \Delta e^{im\nu}.$$  \hspace{1cm} (2D.23)

Noting that $\xi_\perp \cdot \kappa = -\xi_R / R$, we obtain from Eqs. (2D.14) and (2D.15),

$$\xi_R = \frac{R}{2\gamma p_0} \left[ f + \frac{R}{2} \left( 1 + k \frac{R^2}{R_0^2} \right) \frac{df}{dR} \right].$$  \hspace{1cm} (2D.24)

We also obtain, after a short calculation

$$\frac{\partial \xi_Z}{\partial Z} \equiv \nabla \cdot \xi_\perp - \frac{1}{R} \frac{d}{dR} R \xi_R
\begin{equation}
= -\frac{1}{\gamma p_0} \frac{1}{R} \frac{d}{dR} \left[ \frac{1}{4} R^3 \left( 1 + k \frac{R^2}{R_0^2} \right) \frac{df}{dR} \right].
\end{equation}$$  \hspace{1cm} (2D.25)

Then

$$\xi_Z = -\frac{Z}{\gamma p_0 R} \frac{da}{dR} + h(R),$$  \hspace{1cm} (2D.26)

where

$$a = \frac{1}{4} R^3 \left( 1 + k \frac{R^2}{R_0^2} \right) \frac{df}{dR},$$  \hspace{1cm} (2D.27)

and $h(R)$ is an arbitrary function of $R$. We now note that

$$n \cdot \xi_\perp = \frac{\dot{Z} \xi_R - \dot{R} \xi_Z}{\epsilon R_0 \Delta}.$$  \hspace{1cm} (2D.28)

Another short calculation yields

$$(1 + \epsilon x) \Delta n \cdot \xi_\perp = \frac{1}{\gamma p_0 \epsilon R_0^2} \left[ \frac{d}{dv} (Za) + \frac{d}{dv} \left( \frac{1}{2} Z R^2 f \right) - \frac{Z}{2} \frac{d}{dv} \left( \frac{1}{2} R^2 f \right) + \frac{dH}{dv} \right],$$  \hspace{1cm} (2D.29)

where $H = -\gamma p_0 \int R dR h(R)$. Integrating both sides over $v$ we obtain

$$a \cdot \hat{\xi} = -\frac{1}{\gamma p_0 \epsilon R_0^2} \int_0^{2\pi} \frac{Z}{dv} \left( \frac{1}{2} R^2 f \right) dv.$$  \hspace{1cm} (2D.30)

Eq. (2D.30) represents the constraint for $f$ that we look for.
Final Expression for $\delta W_p$

This constraint can be incorporated into the expression for $\delta W_p$, Eq. (2D.17), by introducing a Lagrange multiplier $\Lambda$,

$$
\delta W_p = \frac{\pi}{\gamma p_0} \int_0^{2\pi} dv \left\{ R \dot{R} Z \left[ \frac{1}{4} R^2 \left( 1 + k \frac{R^2}{R_0^2} \right) \left( \frac{df}{dR} \right)^2 + R f \frac{df}{dR} + f^2 \right] 
- \Lambda \left[ Z \frac{d}{dv} \left( \frac{1}{2} R^2 f \right) + \frac{\gamma p_0 \epsilon R_0^2 a \cdot \dot{\hat{\xi}}}{2\pi} \right] \right\}.
$$

(2D.31)

The Euler–Lagrange equation becomes

$$
\frac{d}{dv} \left[ \frac{R^3 \dot{Z}}{2 \dot{R}} \left( 1 + k \frac{R^2}{R_0^2} \right) \frac{df}{dv} \right] + \frac{d}{dv} \left( R^2 Z f \right) - \frac{1}{2} \frac{d}{dv} \left( \Lambda R^2 Z \right) = 2 R \dot{R} \dot{Z} f - \Lambda R \ddot{R},
$$

(2D.32)

or

$$
2 \frac{d}{dv} (Za) + \frac{d}{dv} R^2 Z (f - \frac{1}{2} \Lambda) = 2 R \dot{R} \dot{Z} (f - \frac{1}{2} \Lambda).
$$

(2D.33)

The solution that satisfies periodicity and evenness conditions is

$$
f = \frac{1}{2} \Lambda,
$$

(2D.34)

which gives $a = 0$. We have now demonstrated that $f$ is a constant, and it follows that $\nabla \cdot \hat{\xi} = f/\gamma p_0$ is also a constant. The value of this constant can be obtained as follows. The constraint on $f$, Eq. (2D.30), gives

$$
f \int_0^{2\pi} dv \ Z \dot{R} \dot{\hat{\xi}} = -\gamma p_0 \epsilon R_0^2 a \cdot \dot{\hat{\xi}},
$$

(2D.35)

or

$$
f = -\frac{2\pi \gamma p_0 \epsilon R_0^2 a \cdot \dot{\hat{\xi}}}{V_p},
$$

(2D.36)

where $V_p$ is the plasma volume. Substituting this into Eq. (2D.17), we obtain the expression for $\delta W_p$ in terms of the normal component of the displacement vector on the plasma surface

$$
\frac{\delta W_p}{2\pi R_0} = \frac{\pi B_0^2}{\mu_0} \frac{1}{2} \gamma \lambda^2 (a \cdot \hat{\xi})^2 \frac{\epsilon^2 R_0^3}{V_p},
$$

(2D.37)

where we have used the fact that $f' = 0$ and have written

$$
\lambda^2 = \frac{2\mu_0 p_0}{B_0^2}.
$$

(2D.38)
Appendix E  Calculation of the Poloidal Field Energy

The poloidal field energy of the system is approximated by evaluating the self-inductance of the plasma and equating this to the total poloidal field energy. This is done, because the model does not include poloidal field coils in the vacuum region. Therefore, the energy due to the poloidal fields in the system is given by

$$W_p = \frac{1}{2\mu_0} \int |\nabla \times A|^2 dV,$$  \hspace{1cm} (2E.1)

where the integration is over all space, and $A$ is the vector potential. Using the fact that

$$\nabla \cdot (A \times (\nabla \times A)) = |\nabla \times A|^2 - A \cdot \nabla \times (\nabla \times A)$$
$$= |\nabla \times A|^2 - A \cdot \nabla \times B,$$  \hspace{1cm} (2E.2)
$$= |\nabla \times A|^2 - \mu_0 A \cdot J$$

we obtain

$$W_p = \frac{1}{2} \int dr' A \cdot J.$$  \hspace{1cm} (2E.3)

Using the Biot–Savart law,

$$A = \frac{\mu_0}{4\pi} \int \frac{J(r')dr'}{|r-r'|},$$  \hspace{1cm} (2E.4)

we have

$$W_p = \frac{\mu_0}{8\pi} \int dr dr' \frac{J(r) \cdot J(r')}{|r-r'|}.$$  \hspace{1cm} (2E.5)

The infinitesimal volume elements can be written

$$dr = RdRdZd\phi$$  \hspace{1cm} (2E.6)

$$dr' = R'dR'dZ'd\phi',$$  \hspace{1cm} (2E.7)

and

$$J(r) \cdot J(r') = J_\phi(R, Z) J_\phi(R', Z') \cos(\phi - \phi'),$$  \hspace{1cm} (2E.8)
so that we get

$$W_p = \frac{\mu_0}{8\pi} \int RdRdZd\phi \int R'dR'dZ' J_\phi(R, Z) J_\phi(R', Z') \int_0^{2\pi} d\phi' \frac{\cos(\phi - \phi')}{|r - r'|}. \quad (2E.9)$$

Now,

$$|r - r'| = [R^2 + R'^2 - 2RR' \cos(\phi - \phi') + (Z - Z')^2]^{1/2},$$

and the $\phi'$ integration can be done analytically. The result is

$$\int_0^{2\pi} d\phi' \frac{\cos(\phi - \phi')}{|r - r'|} = \frac{4}{(RR')^{1/2}} \frac{K(k) - E(k)}{k^{1/2}}, \quad (2E.10)$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, respectively, defined by

$$K(k) = \int_0^{\pi/2} \frac{dx}{(1 - k^2 \cos^2 x)^{1/2}}, \quad (2E.11)$$

$$E(k) = \int_0^{\pi/2} dx (1 - k^2 \cos^2 x)^{1/2}, \quad (2E.12)$$

where the argument $k$ is given by

$$k = \alpha - \sqrt{\alpha^2 - 1}, \quad (2E.13)$$

with

$$\alpha = 1 + \frac{e^2 (x - x')^2 + (y - y')^2}{2(1 + \varepsilon x)(1 + \varepsilon x')}, \quad (2E.14)$$

the prime (') referring to the primed integration variable. Using the facts that in the skin current model

$$J_\phi(R, Z) = \lambda B_o G(v) \delta(Z - Z_o(R)), \quad (2E.15)$$

$$J_\phi(R', Z') = \lambda B_o G(v') \delta(Z' - Z_o(R')), \quad (2E.16)$$

where $\lambda = 2\mu_0 p_0 / B_o^2$, $Z_o$ is the point on the plasma surface corresponding to $R(v)$ and

$$G(v) = \left(1 - k \frac{(1 + \varepsilon x_M)^2}{(1 + \varepsilon x)^2}\right), \quad (2E.17)$$
we get after a short calculation
\[ W_p = \frac{\lambda^2 B_0^2}{\mu_0} \int_0^{2\pi} dv \int_0^{2\pi} dv' C(v, v'), \]  
(2E.18)
where
\[ C(v, v') = e^2 R_0^2 \Delta(v) \Delta(v') (RR')^{1/2} G(v)G(v') \frac{K - E}{k_1^{1/2}}. \]  
(2E.19)

The logarithmic singularity in the integrand when \( v' \to v \) can be removed analytically as follows. We have
\[ \lim_{k^2 \to 1} K = 2 \ln 2 - \frac{1}{2} \ln (1 - k^2), \]  
(2E.20)
and
\[ (1 - k^2) \to \frac{2\varepsilon \Delta(v)}{1 + \varepsilon x(v)} |v - v'|. \]  
(2E.21)

Introducing
\[ \tilde{C}(v, v') = C(v, v') + \frac{1}{2} e^2 R_0^2 \Delta^2(v) R(v) G^2(v) \ln |\sin \frac{(v - v')}{2}|, \]  
(2E.22)
we have after a short calculation
\[ \lim_{k^2 \to 1} \tilde{C} = e^2 R_0^2 \Delta^2(v) R(v) G^2(v) \left[ \ln 2 - \frac{1}{2} \ln \frac{\varepsilon \Delta(v)}{1 + \varepsilon x(v)} - 1 \right]. \]  
(2E.23)

Then
\[ W_p = \frac{\lambda^2 B_0^2}{\mu_0} \int_0^{2\pi} dv \int_0^{2\pi} dv' \tilde{C}(v, v') \]  
(2E.24)
\[ - \frac{\lambda^2 B_0^2}{\mu_0} e^2 R_0^2 \int_0^{2\pi} dv \Delta^2(v) R(v) G^2(v) \int_0^{2\pi} dv' \ln |\sin \frac{(v - v')}{2}|. \]

In Appendix F we show that
\[ \int_0^{\pi/2} dx \ln \sin x = -\frac{\pi}{2} \ln 2, \]  
(2E.25)
which allows us to evaluate the \( v' \) integral. After a change of variables \((v - v')/2 = x\), and noticing that the integral is performed over a period, we get
\[ \int_0^{2\pi} dv' \ln |\sin \frac{(v - v')}{2}| = -2\pi \ln 2. \]  
(2E.26)

Then finally
\[ W_p = \frac{\lambda^2 B_0^2}{\mu_0} \left[ \int_0^{2\pi} dv \int_0^{2\pi} dv' \tilde{C}(v, v') + \pi \ln 2e^2 R_0^2 \int_0^{2\pi} dv \Delta^2(v) R(v) G^2(v) \right]. \]  
(2E.27)

These integrals are straightforward to evaluate numerically.
Appendix F  Evaluation of Some Integrals

In this appendix we consider integrals of the following types

\[ I_p = \frac{1}{2\pi} \int_0^{2\pi} d\theta \ln \left| \sin \frac{\theta}{2} \right| \cos p\theta, \quad \text{(2F.1)} \]

\[ J_p = \frac{1}{2\pi} \int_0^{2\pi} d\theta \ln \left| \sin \frac{\theta}{2} \right| \sin p\theta. \quad \text{(2F.2)} \]

For \( p > 0 \) \( I_p \) can easily be evaluated as follows. We integrate by parts and obtain

\[ I_p = \frac{1}{2\pi} \left\{ \left[ \frac{\sin p\theta}{p} \ln \sin \frac{\theta}{2} \right]^{2\pi}_0 - \int_0^{2\pi} \frac{\sin p\theta \cos \theta/2}{2p} \frac{d\theta}{\sin \theta/2} \right\}. \quad \text{(2F.3)} \]

The first term vanishes and the second term is evaluated by substituting \( z = e^{i\theta} \). We get, after some rearranging

\[ I_p = -\frac{1}{8\pi i p} \oint_{|z|=1} \frac{dz}{z} (z^p - z^{-p}) \frac{(z+1)}{(z-1)}. \quad \text{(2F.4)} \]

The residue at \( z = 1 \) is easily seen to be zero. The other pole is at \( z = 0 \), and only the \( z^{-p} \) term gives a contribution. Furthermore,

\[ -(1+z)(1-z)^{-1} = -(1+z)(1+z+z^2+\ldots), \quad \text{(2F.5)} \]

so that the residue at \( z = 0 \) is equal to \(-2\). Then

\[ I_p = \frac{1}{8\pi i p} 2\pi i (-2) = -\frac{1}{2p}. \quad \text{(2F.6)} \]

For \( p < 0 \) we can immediately see that \( I_p = I_{-p} \) and we have the result

\[ I_p = -\frac{1}{2|p|}, \quad p \neq 0. \quad \text{(2F.7)} \]

\( J_p \) for \( p \neq 0 \) can be evaluated as follows. We write

\[ J_p = \frac{1}{2\pi} \left[ \int_0^{\pi} d\theta \ln \left| \sin \frac{\theta}{2} \right| \sin p\theta + \int_0^{\pi} d\phi \ln \left| \sin \left( \frac{\pi - \phi}{2} \right) \right| \sin(2\pi n - p\phi) \right] \]

\[ = \frac{1}{2\pi} \left[ \int_0^{\pi} d\theta \ln \left| \sin \frac{\theta}{2} \right| \sin p\theta - \int_0^{\pi} d\phi \ln \left| \sin \frac{\phi}{2} \right| \sin p\phi \right] \quad \text{(2F.8)} \]

\[ = 0. \]
We have, of course immediately \( J_0 = 0 \), and so we have the result

\[
J_p = 0. \quad (2F.9)
\]

It remains to evaluate

\[
I_0 = \frac{1}{2\pi} \int_0^{2\pi} d\theta \ln \left| \frac{\sin \frac{\theta}{2}}{2} \right| = \frac{2}{\pi} \int_0^{\pi/2} dx \ln \sin x. \quad (2F.10)
\]

We write

\[
\sin x = \frac{e^{ix} - e^{-ix}}{2i} = \frac{e^{ix}(1 - e^{-2ix})}{2i},
\]

and then obtain

\[
I_0 = \frac{2}{\pi} \int_0^{\pi/2} dx \left[ i x + \ln(1 - e^{-2ix}) - \ln 2 - \ln i \right]
\]

\[
= \frac{2}{\pi} \left[ \frac{i \pi^2}{8} + \int_0^{\pi/2} dx \ln(1 - e^{-2ix}) - \frac{\pi}{2} \ln 2 - i \pi^2/4 \right] \quad (2F.11)
\]

\[
= -\ln 2 - \frac{i \pi}{4} + \frac{2}{\pi} \hat{f},
\]

where

\[
\hat{f} = \int_0^{\pi/2} dx \ln(1 - e^{-2ix}). \quad (2F.12)
\]

Now write \( u = e^{-2ix} \) so that \( dx = i \, du/2u \). Also, when \( x = 0 \), \( u = 1 \) and when \( x = \pi/2 \), \( u = -1 \). Then we have

\[
\hat{f} = -\frac{i}{2} \int_{-1}^1 \frac{du}{u} \ln(1 - u) \quad (2F.13)
\]

\[
= -\frac{i}{2} \left[ \int_0^1 \frac{dv}{v} \ln(1 + v) + \int_0^1 \frac{du}{u} \ln(1 - u) \right].
\]

Expanding

\[
\ln(1 - u) = -\sum_{n=1}^{\infty} \frac{u^n}{n},
\]

\[
\ln(1 + v) = \sum_{n=1}^{\infty} \frac{(-)^{n-1}v^n}{n},
\]

we get

\[
\int_0^1 \frac{du}{u} \ln(1 - u) \quad (2F.14)
\]
and
\[ \int_0^1 \frac{dv \ln(1 + v)}{v} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}. \] 
(2F.15)

Then
\[ \hat{f} = \frac{i}{2} \left[ \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \right] \]
\[ = i \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \]
\[ = i \sum_{n=1}^{\infty} \left( \frac{1}{n^2} - \frac{1}{(2n)^2} \right) \]
\[ = \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}. \] 
(2F.16)

The sum can be easily performed by considering the integral
\[ I = \int \frac{\pi \cot \pi z}{z^2} \, dz \] 
(2F.17)

where the contour of integration is a circle with radius \( R \to \infty \). The poles of the integrand are all natural numbers. Furthermore, the residue at \( z = n \neq 0 \) is
\[ \left. \frac{\pi \cos \pi z}{z^2 \pi \cos \pi z} \right|_{z=n} = \frac{1}{n^2}. \] 
(2F.18)

The residue at \( z = 0 \) is the coefficient of the \( 1/z \) term in
\[ \frac{\pi}{z^2} \frac{1 - \frac{(\pi z)^2}{2} + \frac{(\pi z)^4}{4} - \cdots}{\pi z - \frac{(\pi z)^3}{3} + \frac{(\pi z)^6}{6} - \cdots} = \frac{1}{z^3} \left( 1 - \frac{(\pi z)^2}{2} + \cdots \right) \left( 1 + \frac{(\pi z)^2}{3} + \cdots \right), \] 
(2F.19)

and it can easily be seen that the residue is equal to \( -\pi^2/2 + \pi^2/6 = -\pi^2/3 \). But the value of the integral is zero since the integrand becomes exponentially small along a circle of radius \( R \to \infty \). Then
\[ \sum_{|n|=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3}, \] 
(2F.20)
or
\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \] 
(2F.21)
We then have the result

\[
\hat{I} = i \frac{3 \pi^2}{4} = i \frac{\pi^2}{8},
\]

(2F.22)

and finally

\[
I_0 = -\ln 2.
\]

(2F.23)
Appendix G  Calculation of The Unit Vector Normal to a Surface

In this appendix we calculate the unit vector normal to a surface that has been linearly perturbed. The unit vector is given accurate to first order in small quantities in terms of the leading order unit normal and the perturbation vector.

Let $\xi(r)$ be a perturbation to a surface described by the function $r(u,v)$ where $u$ and $v$ are some parameters describing the surface. In order to simplify the algebra involved in the calculation, we consider, without loss in generality, only locally orthogonal coordinates $u$ and $v$. The calculation can easily be generalized to non-orthogonal coordinates.

When $\xi = 0$, the unit normal to the surface at a point $r$ is given by

$$n_0 = K \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}.$$  \hspace{1cm} (2G.1)

Here $K$ is a normalization constant given by

$$\frac{1}{K^2} = \frac{1}{2} \left( \frac{\partial r}{\partial u} \right)^2 \frac{1}{2} \left( \frac{\partial r}{\partial v} \right)^2,$$  \hspace{1cm} (2G.2)

where we have taken advantage of orthogonality and have set

$$\frac{\partial r}{\partial u} \cdot \frac{\partial r}{\partial v} = 0.$$  \hspace{1cm} (2G.3)

When $\xi \neq 0$, but $\xi \equiv |\xi| \ll |r|$, we have

$$n = L \left( \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} + \frac{\partial r}{\partial u} \times \frac{\partial \xi}{\partial u} + \frac{\partial \xi}{\partial u} \times \frac{\partial r}{\partial v} \right) + O(\xi^2),$$  \hspace{1cm} (2G.4)

where the normalization constant $L$ is given by

$$\frac{1}{L^2} = \left| \frac{\partial r}{\partial u} + \frac{\partial \xi}{\partial u} \right|^2 \left| \frac{\partial r}{\partial v} + \frac{\partial \xi}{\partial v} \right|^2 - \left( \frac{\partial r}{\partial u} + \frac{\partial \xi}{\partial u} \right) \cdot \left( \frac{\partial r}{\partial v} + \frac{\partial \xi}{\partial v} \right).$$  \hspace{1cm} (2G.5)
We expand the expression on the right hand side and obtain

\[ \frac{1}{L^2} = \left| \frac{\partial r}{\partial u} \right|^2 \left| \frac{\partial r}{\partial v} \right|^2 + 2 \left| \frac{\partial r}{\partial v} \right|^2 \left( \frac{\partial r}{\partial u} \cdot \frac{\partial \xi}{\partial u} \right) + 2 \left| \frac{\partial r}{\partial u} \right|^2 \left( \frac{\partial r}{\partial v} \cdot \frac{\partial \xi}{\partial v} \right) + O \xi^2, \]  

(2G.6)

where again we have used the fact that \( \frac{\partial r}{\partial u} \cdot \frac{\partial r}{\partial v} = 0 \). This gives the expression for \( L \)

\[ L = \frac{1}{\left| \frac{\partial r}{\partial u} \right| \left| \frac{\partial r}{\partial v} \right|} \left( 1 - \frac{\partial r}{\partial u} \cdot \frac{\partial \xi}{\partial u} - \frac{\partial r}{\partial v} \cdot \frac{\partial \xi}{\partial v} \right) + O \xi^2. \]  

(2G.7)

The expression for the unit normal becomes

\[ n = \frac{1}{\left| \frac{\partial r}{\partial u} \right| \left| \frac{\partial r}{\partial v} \right|} \left( 1 - \frac{\partial r}{\partial u} \cdot \frac{\partial \xi}{\partial u} - \frac{\partial r}{\partial v} \cdot \frac{\partial \xi}{\partial v} \right) \left( \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} + \frac{\partial r}{\partial u} \times \frac{\partial \xi}{\partial v} + \frac{\partial \xi}{\partial u} \times \frac{\partial \xi}{\partial v} \right) + O \xi^2. \]  

(2G.8)

Using the fact that \( \xi = \xi(r) \), we have

\[ \frac{\partial \xi}{\partial u} = \frac{\partial r}{\partial u} \cdot \nabla \xi, \]  

(2G.9)

\[ \frac{\partial \xi}{\partial v} = \frac{\partial r}{\partial v} \cdot \nabla \xi, \]  

(2G.10)

and then

\[ \frac{\partial r}{\partial u} \cdot \frac{\partial \xi}{\partial u} = \frac{\partial r}{\partial u} \cdot (\nabla \xi) \cdot \frac{\partial r}{\partial u}, \]

\[ \frac{\partial r}{\partial v} \cdot \frac{\partial \xi}{\partial v} = \frac{\partial r}{\partial v} \cdot (\nabla \xi) \cdot \frac{\partial r}{\partial v}. \]

Using the corollary in Appendix H and noting that \( \frac{\partial r}{\partial u} \cdot \frac{\partial r}{\partial v} = 0 \), we have

\[ \frac{\partial r}{\partial u} \cdot \frac{\partial \xi}{\partial u} \frac{\partial r}{\partial v} \cdot \frac{\partial \xi}{\partial v} - \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \cdot (\nabla \xi) \cdot \frac{\partial r}{\partial u} \times \frac{\partial \xi}{\partial v} + \left( \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right)^2 \nabla \cdot \xi \]

\[ = -n_0 \cdot (\nabla \xi) \cdot n_0 + \nabla \cdot \xi. \]  

(2G.11)
Using the theorem of the same appendix, we have

\[
\frac{\partial r}{\partial u} \times \frac{\partial \xi}{\partial v} + \frac{\partial \xi}{\partial u} \times \frac{\partial r}{\partial v} = \frac{\partial r}{\partial u} \times \left( \frac{\partial r}{\partial v} \cdot \nabla \xi \right) + \left( \frac{\partial r}{\partial u} \cdot \nabla \xi \right) \times \frac{\partial r}{\partial v} \\
= \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \nabla \cdot \xi - (\nabla \xi) \cdot \left( \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right) \\
= \left[ \frac{\partial r}{\partial u} \right] \left[ \frac{\partial r}{\partial v} \right] \left[ n_0 (\nabla \cdot \xi - (\nabla \xi) \cdot n_0) \right].
\]

(2G.12)

We have finally

\[
n = n_0 + n_0 [n_0 \cdot \nabla \xi \cdot n_0 - \nabla \cdot \xi] + n_0 \nabla \cdot \xi - (\nabla \xi) \cdot n_0 + O\xi^2 \\
= n_0 + n_1 + O\xi^2,
\]

(2G.13)

where

\[
n_1 = -(\nabla \xi) \cdot n_0 + n_0 [n_0 \cdot \nabla \xi \cdot n_0].
\]

(2G.14)

The boundary condition at any flux surface is given by

\[
n \cdot \hat{B} = 0,
\]

(2G.15)

which gives to leading order \( n_0 \cdot \hat{B}_0 = 0 \). Using this result and a Taylor series expansion for \( \hat{B}_0 \), we get to first order

\[
n_0 \cdot \hat{B}_1 \bigg|_s = -n_1 \cdot \hat{B}_0 \bigg|_s - n_0 (\xi \cdot \nabla) \hat{B}_0 \bigg|_s \\
= n_0 \cdot (\hat{B}_0 \cdot \nabla) \xi \bigg|_s - n_0 (\xi \cdot \nabla) \hat{B}_0 \bigg|_s \\
= n_0 \cdot \nabla \times (\xi \times \hat{B}_0) \bigg|_s,
\]

(2G.16)

where all quantities are evaluated at the unperturbed surface. This gives the boundary condition for a perturbed flux surface whether it is in the plasma or vacuum region, i.e. whether or not a real plasma displacement can be meaningfully defined.
Appendix H  Two Tensor Identities

Theorem. Let $T$ be a rank two tensor and $a$ and $b$ be vectors. Then

$$a \times (b \cdot T) + (a \cdot T) \times b = (\text{tr} \, T)a \times b - T \cdot (a \times b).$$  \hspace{1cm} (2H.1)

Proof

If $a = \alpha b$ for some scalar $\alpha$, then

$$a \times (b \cdot T) + (a \cdot T) \times b = \alpha[b \times (b \cdot T) + (b \cdot T) \times b] = 0.$$  \hspace{1cm} (2H.2)

Also, $a \times b = 0$ and the statement of the theorem is trivially satisfied.

If $a \times b \neq 0$, we write

$$c = a \times b.$$  \hspace{1cm} (2H.3)

Then the $i$'th component of the vector on the left hand side of Eq. (2H.1) is

$$[a \times (b \cdot T) + (a \cdot T) \times b]_i = \epsilon_{ijk}a_jb_lT_{lk} + \epsilon_{ijk}a_ib_jT_{lk}$$

$$= \epsilon_{ijk}a_jb_lT_{lk} - \epsilon_{ijk}a_ib_jT_{lk}$$

$$= \epsilon_{ijk}T_{lk}(a_jb_l - a_ib_j).$$

But,

$$c_m = \epsilon_{mpk}a_pb_k,$$

and

$$c_m\epsilon_{mj} = \epsilon_{mpk}\epsilon_{mj}a_pb_k$$

$$= (\delta_{pj}\delta_{kl} - \delta_{pl}\delta_{kj})a_pb_k$$

$$= a_jb_l - a_ib_j.$$
Then
\[ [a \times (b \cdot T) + (a \cdot T) \times b]_i = \epsilon_{ijk} T_{lk} \epsilon_{mji} c_m \]
\[ = (\delta_{im} \delta_{kl} - \delta_{il} \delta_{mk}) T_{lk} c_m \]
\[ = c_i T_{kk} - T_{ik} c_k \]
\[ = [c(tr T) - T \cdot c]_i , \]
proving the theorem.

Corollary.

\[
(a \times b) \cdot T \cdot (a \times b)
\]
\[= tr T[a^2 b^2 - (a \cdot b)^2] - a^2 (b \cdot T \cdot b) - b^2 (a \cdot T \cdot a) + (a \cdot b)(a \cdot T \cdot b + b \cdot T \cdot a).
\]

Proof

\[
(a \times b) \cdot T \cdot (a \times b)
\]
\[= (a \times b) \cdot [(tr T)(a \times b - a \times (b \cdot T - (a \cdot T) \times b]
\[= tr T(a \times b)^2 - (a \times b) \cdot (a \times (b \cdot T)) - (a \times b) \cdot ((a \cdot T) \times b)
\[= tr T(a \times b)^2 - a^2 b \cdot (b \cdot T) + (a \cdot b)(a \cdot (b \cdot T)) - a \cdot (a \cdot T)b^2 + (a \cdot b)(b \cdot (a \cdot T))
\[= tr T[a^2 b^2 - (a \cdot b)^2] - a^2 (b \cdot T \cdot b) - b^2 (a \cdot T \cdot a) + (a \cdot b)(a \cdot T \cdot b + b \cdot T \cdot a).
\]
Appendix J Some Relations Between Elliptic Integrals

In this appendix we calculate some functional relations between elliptic integrals of the first and second kinds. The standard forms of the integrals are defined by

\[ K(k) = \int_0^{\pi/2} \frac{dx}{(1 - k^2 \sin^2 x)^{1/2}}, \]  
(2J.1)

\[ E(k) = \int_0^{\pi/2} dx(1 - k^2 \sin^2 x)^{1/2}. \]  
(2J.2)

The first functional relationships that we calculate give expressions for elliptic integrals of arguments \(2\sqrt{k}/(1 + k)\) in terms of \(K(k)\) and \(E(k)\). We have

\[ K\left(\frac{2\sqrt{k}}{1 + k}\right) = \int_0^{\pi/2} d\psi \left(1 - \frac{4k}{(1 + k)^2 \sin^2 \psi}\right)^{-1/2}. \]  
(2J.3)

Substituting

\[ \sin \psi = \frac{(1 + k) \sin x}{1 + k \sin^2 x}, \]  
(2J.4)

we get

\[ d\psi = \frac{dx(1 + k)}{(1 - k^2 \sin^2 x)^{1/2}} \left(\frac{1 - k \sin^2 x}{1 + k \sin^2 x}\right), \]  
(2J.5)

\[ \left(1 - \frac{4k}{(1 + k)^2 \sin^2 \psi}\right)^{-1/2} = \frac{1 - k \sin^2 x}{1 + k \sin^2 x}, \]  
(2J.6)

and we have the result

\[ K\left(\frac{2\sqrt{k}}{1 + k}\right) = (1 + k)K(k). \]  
(2J.7)

Similarly,

\[ E\left(\frac{2\sqrt{k}}{1 + k}\right) = \int_0^{\pi/2} d\psi \left(1 - \frac{4k}{(1 + k)^2 \sin^2 \psi}\right)^{1/2}. \]  
(2J.8)

Using the same substitution as before, we get

\[ E\left(\frac{2\sqrt{k}}{1 + k}\right) = \int_0^{\pi/2} \frac{dx(1 + k)}{(1 - k^2 \sin^2 x)^{1/2}} \left(\frac{1 - k \sin^2 x}{1 + k \sin^2 x}\right)^2. \]  
(2J.9)
Writing \( z = \sin x \), we have \( \frac{dz}{\sqrt{1-z^2}} = dx \) and
\[
E\left(\frac{2\sqrt{k}}{1+k}\right) = (1+k) \int_{0}^{1} \frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}} \left(\frac{1-kz^2}{1+kz^2}\right)^2.
\]
(2J.10)

After a straightforward calculation we get
\[
\left(\frac{1-kz^2}{1+kz^2}\right)^2 = 1 + 4 \left[ \frac{a^2}{(z-a)(z+a)} + \frac{a^4}{(z-a)^2(z+a)^2} \right],
\]
(2J.11)

where \( a = i/\sqrt{k} \). It is easy to show, for example by expanding the right hand side below, that
\[
\left(\frac{1-kz^2}{1+kz^2}\right)^2 = 1 + a \left[ \frac{1}{z-a} - \frac{1}{z+a} \right] + a^2 \left[ \frac{1}{(z-a)^2} + \frac{1}{(z+a)^2} \right].
\]
(2J.12)

Let us now write
\[
G(z) = (k^2z^2 - 1)(z^2 - 1),
\]
(2J.13)

where \( G \) satisfies \( G(0) = 1 \) and \( G(1) = 0 \). We also write
\[
I_n(c) = \int_{0}^{1} \frac{(z-c)^n}{\sqrt{G(z)}}.
\]
(2J.14)

Furthermore,
\[
\frac{d}{dz} [2(z-c)^{n+1} \sqrt{G(z)}] = 2(n+1)(z-c)^n \sqrt{G(z)} + (z-c)^{n+1} \frac{G'(z)}{\sqrt{G(z)}}.
\]
(2J.15)

We may write, in general
\[
G(z) = b_0(z-c)^4 + b_1(z-c)^3 + b_2(z-c)^2 + b_3(z-c) + b_4,
\]
(2J.16)

so that
\[
G'(z) = 4b_0(z-c)^3 + 3b_1(z-c)^2 + 2b_2(z-c) + b_3.
\]
(2J.17)

Integrating Eq. (2J.15) between 0 and 1 and writing for the first term on the right hand side \( \sqrt{G(z)} = G(z)/\sqrt{G(z)} \), we get for some set \( \{b_i\} \)
\[
-2(-c)^{n+1} = (2n+6)b_0I_{n+4} + (2n+5)b_1I_{n+3} + (2n+4)b_2I_{n+2} + (2n+3)b_3I_{n+1} + (2n+2)b_4I_n.
\]
(2J.18)
Then we have

\[ E \left( \frac{2\sqrt{k}}{1+k} \right) = (1+k)[I_0 + aI_{-1}(a) - aI_{-1}(-a) + a^2I_{-2}(a) + a^2I_{-2}(-a)], \] (2J.19)

where no argument is necessary for \( I_0 \) since the value of the integral is the same for any finite argument. Let us now determine the \( \{b_i\} \). We have

\[ G(z) = k^2z^4 - (k^2 + 1)z^2 + 1. \]

Expanding \((z - c)^n\) for \( n = 0, 1, 2, 3, 4 \) we get after some algebra

\[ b_0 = k^2, \] (2J.20)
\[ b_1 = 4ck^2, \] (2J.21)
\[ b_2 = -1 - k^2 + 6c^2k^2, \] (2J.22)
\[ b_3 = 4c^3k^2 - 2c(1 + k^2), \] (2J.23)
\[ b_4 = 1 + c^4k^2 - c^2(1 + k^2). \] (2J.24)

Then Eq. (2J.18) yields, with \( n = -2 \)

\[ \frac{2}{c} = 2k^2I_2 + 4ck^2I_1 - [4c^3k^2 - 2c(1 + k^2)]I_{-1} - 2[1 + c^4k^2 - c^2(1 + k^2)]I_{-2}. \] (2J.25)

Writing \( c = a \) and \( c = -a \) successively, and adding the resulting two equations we get, noting that \( a^2 = -1/k \) and \( a^4 = 1/k^2 \),

\[ 0 = 2k^2[I_2(a) + I_2(-a)] + 4k^2a[I_1(a) - I_1(-a)] + 2(1+k)^2[aI_{-1}(a) - aI_{-1}(-a) + a^2I_{-2}(a) + a^2I_{-2}(-a)], \] (2J.26)

and then, eliminating the \( I_{-1} \) and \( I_{-2} \), we get

\[ E \left( \frac{2\sqrt{k}}{1+k} \right) = (1+k) \left[ I_0 - \frac{k^2}{(1+k)^2} \{I_2(a) + I_2(-a) + 2aI_1(a) - 2aI_1(-a)\} \right]. \] (2J.27)

We note that

\[ I_0 = \int_0^1 \frac{dz}{\sqrt{G(z)}} = K(k), \] (2J.28)
\[ \int_0^1 \frac{dz}{\sqrt{1 - k^2 z^2}} = E(k). \]  
(2J.29)

We also have

\[ I_2(a) + I_2(-a) = 2 \int_0^1 \frac{dz(z^2 - \frac{1}{k})}{\sqrt{G(z)}} \]
\[ = \frac{2}{k^2} \int_0^1 \frac{dz(k^2 z^2 - 1 + 1 - k)}{\sqrt{G(z)}} \]
\[ = \frac{2E(k)}{k^2} + \frac{2(1 - k)K(k)}{k^2}. \]  
(2J.30)

Also,

\[ 2aI_1(a) - 2aI_1(-a) = 2 \int_0^1 \frac{dz}{\sqrt{G(z)}} [a(z - a) - a(z + a)] = \frac{4K(k)}{k}. \]  
(2J.31)

This gives the final result, after one line of algebra

\[ E\left(\frac{2\sqrt{k}}{1 + k}\right) = \frac{2E(k) - (1 - k^2)K(k)}{1 + k}. \]  
(2J.32)

The other relationships needed in this work are the expressions in terms of \( k \), \( K(k) \) and \( E(k) \) for the derivatives of the elliptic integrals \( dK/dk \) and \( dE/dk \). The latter is trivial to calculate and we obtain immediately

\[ \frac{dE}{dk} = \int_0^{\pi/2} \frac{dx(-k \sin^2 x)}{(1 - k^2 \sin^2 x)^{1/2}} = \frac{E(k) - K(k)}{k}. \]  
(2J.33)

It remains to calculate \( dK/dk \). We have

\[ \frac{dK}{dk} = \int_0^{\pi/2} \frac{dx k \sin^2 x}{(1 - k^2 \sin^2 x)^{3/2}}. \]  
(2J.34)

Writing \( z = \sin x \), we have

\[ \frac{dK}{dk} = \int_0^1 \frac{dz}{\sqrt{G(z)}} \frac{k z^2}{(1 - k^2 z^2)}. \]  
(2J.35)

It is straightforward to show that

\[ \frac{k z^2}{1 - k^2 z^2} = \frac{1}{k} \left[ -1 - \frac{1}{2k} \left( \frac{1}{z - \frac{1}{k}} - \frac{1}{z + \frac{1}{k}} \right) \right]. \]  
(2J.36)
Then we have
\[ \frac{dK}{dk} = \frac{1}{k} \left[ -I_0 - \frac{1}{2k} \left( I_{-1} \left( \frac{1}{k} \right) - I_{-1} \left( -\frac{1}{k} \right) \right) \right]. \] (2J.37)

Substituting \( c = \pm 1/k \) into Eqs. (2J.20)–(2J.24) we have
\[ b_0 = k^2, \]
\[ b_1 = \pm 4k, \]
\[ b_2 = 5 - k^2, \]
\[ b_3 = \pm \frac{2}{k} (1 - k^2), \]
\[ b_4 = 0. \] (2J.38)

Eq. (2J.18) with \( n = -2 \) yields for \( c = \pm 1/k \)
\[ -2k = 2k^2 I_2 + 4k I_1 - \frac{2}{k} (1 - k^2) I_{-1}, \] (2J.39)
\[ 2k = 2k^2 I_2 - 4k I_1 + \frac{2}{k} (1 - k^2) I_{-1}, \] (2J.40)
respectively. We obtain an expression for \( I_{-1}(1/k) - I_{-1}(-1/k) \) by subtracting the two equations, and have, after some rearranging
\[ \frac{1}{2k} \left[ I_{-1} \left( \frac{1}{k} \right) - I_{-1} \left( -\frac{1}{k} \right) \right] = \frac{k^2}{2(1 - k^2)} \left[ I_2 \left( \frac{1}{k} \right) + I_2 \left( -\frac{1}{k} \right) \right] + \frac{k}{(1 - k^2)} \left[ I_1 \left( \frac{1}{k} \right) - I_1 \left( -\frac{1}{k} \right) \right]. \] (2J.41)

But we have
\[ I_2 \left( \frac{1}{k} \right) + I_2 \left( -\frac{1}{k} \right) = 2 \int_0^1 \frac{dz}{\sqrt{G(z)}} \left( z^2 + \frac{1}{k^2} \right) \]
\[ = \frac{2}{k^2} \int_0^1 \frac{dz}{\sqrt{G(z)}} (k^2 - 1 + 2) \]
\[ = \frac{2}{k^2} (2K(k) - E(k)), \]
and
\[ I_1 \left( \frac{1}{k} \right) - I_1 \left( -\frac{1}{k} \right) = - \int_0^1 \frac{dz}{\sqrt{G(z)}} \]
\[ = - \frac{2K(k)}{k}. \]

Substituting into Eq. (2J.41), we have the final result, after one line of algebra,
\[ \frac{dK}{dk} = \frac{E(k)}{k(1 - k^2)} - \frac{K(k)}{k}. \] (2J.42)

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3. Diffuse Plasma Model

The encouraging results obtained in the previous section using the sharp boundary model of the plasma lead us to investigate the stability of the peapod cross-section (which was found optimal there) with more realistic plasma pressure and current distributions. This constitutes a rather more ambitious undertaking, because we need to solve the Grad-Shafranov equation\(^1\),

\[ \Delta^* \psi = -\mu_0 R J_\psi(R, \psi), \tag{3.1} \]

for the poloidal flux function \(\psi\) for each equilibrium that we consider, rather than just using the simple jump condition, equation (2.19), of the previous chapter. The calculation of stability properties for such an equilibrium is also computationally quite involved, because we now need to evaluate the plasma contribution to \(\delta W\) by integrating over the entire plasma volume — the integral cannot in general be transformed to a simple surface integral in the manner of the previous chapter, and therefore is not as quick to evaluate numerically. Consequently, the stability study of any given equilibrium becomes computationally a much more demanding endeavour and cross-sectional optimization with a diffuse plasma model would not be a practical problem to solve in its full scope. This point is further emphasized by the fact that the Grad-Shafranov equation contains two free functions of the poloidal flux \(\psi\) — functions that are in no way restricted in their form in the ideal MHD model. These functions are the plasma pressure profile, \(p(\psi)\), and the toroidal field profile \(F(\psi) = R B_T\), which together form the toroidal current density \(J_\psi\):

\[ \mu_0 J_\psi = R \mu_0 p'(\psi) + \frac{F F'(\psi)}{R}, \tag{3.2} \]

where prime (\('\)) denotes differentiation with respect to \(\psi\). The so-called profile optimization, in other words the adjustment of these free functions so as to improve stability
properties of the equilibrium, further complicates the issue — in the sharp boundary model this problem did not arise because the pressure and current profiles are fixed there. As a result, some restrictions on the scope of the investigation need to be made.

In this chapter, we examine two features of the peapod shape suggested by the sharp boundary model. Firstly, the stability of a number of peapod equilibria will be investigated with respect to the \( n = 1 \) modes taking conducting walls to be at infinity. Since the aim here is not to provide a comprehensive picture of the stability characteristics of peapods but rather to examine their most striking qualities using a diffuse plasma model. We keep all other aspects as simple as possible, and do not consider the \( n = 0 \) or \( n = 2 \) modes in this chapter. Also we do not add another dimension to the problem by allowing conducting walls to be close to the plasma surface. Because conducting walls are a stabilizing influence, the results obtained with the wall at infinity will provide the most pessimistic estimates of the stability limit for plasma \( \beta \) for the modes and profiles in question. The second feature to be studied here is related to the observation that peapods have the best values of critical \( \beta \) in the sharp boundary model when \( \beta \) is defined according to Eq. (2.132). We compare the performance of a peapod plasma with bean-shaped and Dee-shaped plasmas using the same definition of \( \beta \) that we had in Chapter 2.

### 3.1 Procedure

When examining the stability properties of a peapod shaped plasma equilibrium, the following procedure is used. The plasma pressure profile, \( p(\psi) \) is chosen in a manner consistent with the observation that pressure profiles broader than the current profile, \( J_\psi(\psi, R) \), are favourable to low toroidal mode number \( (n) \) stability while peaked pressure profiles produce unstable infernal modes in the sense of Manickam\(^2\). The current profile is determined in the following manner: We specify the MHD safety factor \( (q(\psi)) \) profile together with the appropriate values of the safety factor at the plasma edge as well as the centre. The central value will be taken to be \( q = 1.04 \), while the edge value is
varied to produce different equilibria. The Grad–Shafranov equation is solved using an equilibrium solver from the PEST family\textsuperscript{[3],[4]}, which takes as an input the shape of the plasma boundary as well as the functional forms and values at the plasma edge and magnetic axis of the pressure profile and the safety factor. Equilibrium solvers that accept as input the toroidal field profile, $F(\psi)$, rather than the safety factor profile, also exist, but they are not as robust and reliable as the version that we have used here. In particular, they experience convergence problems whenever the plasma shape has sharp points along the boundary. The toroidal current distribution corresponding to the given pressure and safety factor profile may then be calculated once the solution to the Grad–Shafranov equation is known. A complete description of the solution procedure can be found in Johnson et al.\textsuperscript{[5]}. This procedure does not always yield current distributions that vanish at the plasma edge, but a procedure has been developed which minimizes the edge current within some given family of safety factor profiles. This aspect will be further discussed in the next section. Nevertheless, reasonable toroidal current profiles are obtained and the stability of such equilibria is analysed using the PEST-2 stability code\textsuperscript{[6]}. The boundaries of the stable region are explored by varying the peak pressure and the edge value of the safety factor, while the pressure profile is kept constant as a function of $\psi$ and the safety factor profile adjusted so that the toroidal current density at the edge is minimized. No other type of profile optimization is performed in this study. Included in the study of peapod shapes will be a limited study on the effects of elongation of the peapod shape on stability limits.

The main effect to be studied is the impact of our definition of plasma $\beta$ on the stability boundaries for different cross-sectional shapes. We have defined the plasma $\beta$ as an average measure of the efficiency of plasma confinement in the entire vacuum chamber rather than just in the plasma region, as is conventionally done. Therefore, we define $\beta$ as

$$< \beta_V > = \frac{\int_V p \, dV}{\int_V \frac{B_\phi^2}{2\mu_0} \, dV},$$

(3.3)

where the integration volume, $V$, is the whole vacuum chamber, and the $B_\phi$ is the
toroidal component of the vacuum field. This definition is justified on the grounds that in order to produce the equilibrium, a certain amount of energy has to be spent to create the confining magnetic field. This field is created not only inside the plasma but also everywhere inside the toroidal field coil system, which we approximate by a rectangular vacuum chamber surrounding the plasma (see Chapter 2). The minimum amount of energy to be spent is given by the denominator in our expression of $\langle \beta_V \rangle$, so this definition suitably reflects the true confinement efficiency. We then apply this definition to the peapod, a bean-shaped and to a Dee-shaped plasma. The impact of this definition of plasma $\beta$ on how one chooses high performance plasma configurations will be illustrated by the comparison of results for critical plasma $\beta$'s between the three cross-sectional shapes.

A few words about the classifications of MHD modes are appropriate here. Ideal modes are in this work classified in two different ways. The first classification refers to the toroidal nature of the mode in terms of the toroidal harmonic number $(n)$. As mentioned above, in this work we concentrate on the $n = 1$ modes and do not investigate axisymmetric $(n = 0)$ or (global) $n \geq 2$ kink modes. However, we carry out the $n \to \infty$ ballooning stability analysis, although no pressure profile optimization against ballooning modes is performed. The second classification refers to the radial structure of the displacements and gives no information about their toroidal nature. Internal modes are strictly confined inside the plasma and the plasma surface remains unperturbed. External modes have only external components, in other words their action is concentrated at the plasma surface. Infernal modes have both internal and external components, when the mode is Fourier analysed in terms of a poloidal angle like variable. In this manner, we talk about, for example, the $(n = 1, m = 2)$ external kink mode or the $n = 1$ infernal mode. No $m$ number is usually assigned to the infernal mode, because it contains, by construction, several $m$ harmonics.

The procedure can then be summarized as follows. Functional forms for the pressure and safety factor profile are chosen with the pressure profile relatively broad so that $n = 1$ infernal modes are not excited. The stability boundaries for a given plasma shape are
mapped in a $<\beta_V>$ vs. $1/q_*$ diagram by investigating the stability of equilibria with differing values of pressure at the axis and safety factor at the edge. This procedure is repeated for peapods of various elongations as well as a bean–shaped and a Dee–shaped plasma. In this manner, quantitative comparisons between the different shapes can be made and the corresponding critical $\beta$ values found.

3.2 Equilibria

Our immediate concern is to obtain solutions to the Grad–Shafranov equation [Eq. (3.1)] in a systematic fashion corresponding to various plasma $\beta$'s and currents. These solutions are calculated using an equilibrium code from the PEST family, specifically the version obtained from Grumman Corporation. Some changes needed to be made to the code in order to obtain reasonable solutions for a peapod shaped plasma, and these changes will be discussed first.

As mentioned above, the obvious choice for specifying the two free functions $p(\psi)$ and $F(\psi)$, would be to directly give them as functions of flux and use the given form in Eq. (3.2). However, two difficulties arise.

The first difficulty has to do with scaling the equilibrium. The axis value of the pressure function and the total plasma current are required as input parameters for the equilibrium iteration when the applied toroidal magnetic field is specified at a given major radius $R = R_0$. The equilibrium code then adjusts the relative magnitude of a normalized $F(\psi)$ at the magnetic axis so that the correct plasma current is obtained while keeping the normalized $F = 1$ at the plasma edge. However, it is not easy, a priori, to choose all these values consistent with the requirement that $q = 1.04$ at the magnetic axis, and usually some iteration is required. In terms of computation time this iteration is quite expensive. While in theory the rescaling of an equilibrium should be possible by changing the value of the applied magnetic field, in practice such a change would result in either a change of the value of $F(\psi)$ at the plasma edge, or a change in the required total plasma current. The former is not desirable, since the stability code
requires \( F = 1 \) at the plasma edge\(^7\), and the latter again requires iterations.

The second difficulty is an even more serious one. The equilibrium code using \( p(\psi) \)
and \( F(\psi) \) as input functions expands its internal grid for the plasma cross-section in
such a way that one flux surface outside the plasma boundary is included. This would be
fine, except that the algorithm experiences convergence problems whenever this outside
flux surface contains sharp corners, i.e. is part of or close to the separatrix surface. As a
consequence, the code using \( p(\psi) \) and \( F(\psi) \) as input functions does not work for peapod
shapes or any shapes with sharp corners except occasionally in the very low plasma \( \beta \)
regime.

These difficulties can be overcome using another formulation of the problem. The
flux averaged safety factor is given by

\[
q(\psi) = \frac{F(\psi)}{2\pi} \oint \frac{dl}{R^2 B_p},
\]

where the integral is taken around a flux surface and the magnitude of the poloidal
magnetic field is given by

\[
RB_p = |\nabla \psi|.
\]

A successful formulation has been developed by Johnson et al.\(^8\) requiring \( q(\psi) \) instead
of \( F(\psi) \) as an input function. This formulation does not require the use of an additional
flux surface outside the plasma region and no convergence difficulties are encountered
until plasma \( \beta \) becomes very large and we begin to approach the equilibrium limit.
Furthermore, since the safety factor profile is specified as an input function, there is no
difficulty setting \( q_0 = 1.04 \), where \( q_0 \) is the value of the safety factor at the magnetic
axis.

There is a disadvantage, however, when using the above procedure. The toroidal
current \( J_\phi(R, \psi) \) does not in general vanish at the plasma edge, when the safety factor
profile is specified instead of the toroidal field profile. With the latter, this is achieved
easily by requiring that the input functions satisfy

\[
p'(\psi) = FF'(\psi) = 0
\]
at the plasma edge. However, when the safety factor profile is used instead, it is not clear how the profile should be chosen to make the toroidal current vanish at the plasma edge, in other words to force the corresponding $F'(\psi) = 0$ there. Moreover, within any given family of profiles it is not even possible to achieve a vanishing plasma current at the edge. Nevertheless, it is possible to minimize the edge current with respect to parameters specifying the safety factor profile, and this is what we have done.

We write

$$F(\psi) = q(\psi)g(\psi),$$

(3.7)

where

$$g(\psi) = 2\pi \left[ \int \frac{d\ell}{R^2 B_\ell} \right]^{-1}.$$  

(3.8)

Differentiating with respect to $\psi$ we obtain

$$F'(\psi) = g(\psi)q'(\psi) + g'(\psi)q(\psi).$$

(3.9)

The parametrization of the safety factor profile can be chosen as

$$q(\psi) = q(\psi; q_0, q_e, \alpha),$$

(3.10)

where $q_0$ and $q_e$ are the axis and edge values, respectively, and are kept fixed for any given equilibrium iteration. $\alpha$ is a parameter controlling the width, or flatness, of the profile. We may then attempt to set

$$F'(\psi_e) = 0,$$

(3.11)

where $\psi_e$ is the value of the flux at the edge. This is done by calculating a value of $\alpha$ in such a way that at iteration $n$

$$F'_n(\psi_e) = g_{n-1}(\psi_e)q'_n(\psi_e; \alpha) + g'_{n-1}(\psi_e)q_n(\psi_e; \alpha) = 0.$$  

(3.12)

This has turned out to be a convergent process for all the equilibria that we have considered. Furthermore, for plasma shapes that have only moderate elongations $\kappa \leq 2$ and triangularities $\delta \leq 0.4$, current profiles with very small edge current densities are
produced for the family of safety factor profiles considered. For the peapod shape, too, the resulting current densities at the edge are quite satisfactory and are typically 15–20 % of the peak value at the magnetic axis.

The family of safety factor profiles considered in this work is given by

\[ q(\psi) = \frac{q_0}{(1 - a\tilde{\psi})^\alpha}, \] (3.13)

where \( q_0 \) is the value at the magnetic axis and \( \tilde{\psi} \) is defined as

\[ \tilde{\psi} = \frac{\psi - \psi_0}{\psi_e - \psi_0}. \] (3.14)

Here \( \psi_0 \) is the value of the flux at the magnetic axis and \( \psi_e \) that at the plasma edge, so that \( \tilde{\psi} = 0 \) at the axis and \( \tilde{\psi} = 1 \) at the edge. The value of \( \alpha \) is chosen so that the safety factor equals the prescribed \( q_e \) at \( \tilde{\psi} = 1 \):

\[ q_e = \frac{q_0}{(1 - a)^\alpha}, \] (3.15)

while \( \alpha \) is chosen so that

\[ \frac{q'(\psi_e; \alpha)}{q(\psi_e; \alpha)} = -\frac{g'(\psi_e)}{g(\psi_e)}. \] (3.16)

The value of \( b \) is fixed to be \( b = 1.2 \), after some investigation with the peapod shape. The final equilibrium does not critically depend on the value of \( b \) chosen as long as \( b \) satisfies \( b > 1 \), which gives \( q'(\psi_0) = 0 \). However, increasing the value of \( b \) substantially beyond unity flattens the safety factor profile and thus broadens the corresponding current profile. This makes it more difficult to decrease the value of the edge current density and for this reason values of \( b \) much larger than unity were not considered in this work. It should also be noted that \( \alpha \) can take both positive or negative values in our formulation, and the corresponding value of \( a \) is positive or negative, respectively. In practice, however, all the resulting values of \( \alpha \) have been positive, and it is for this reason that we have written the functional form of \( q(\tilde{\psi}) \) as given by Eq. (3.13). Typical safety factor and flux averaged toroidal current profiles for a peapod shaped plasma are shown in Figs. (3.1) and (3.2).
The pressure profile is chosen to be broad in comparison with the current profile. This is achieved by writing
\[ p(\psi) = p_0 \left[ \frac{e^{\nu(1-\psi)} - \nu(1 - \tilde{\psi}) - 1}{e^\nu - \nu - 1} \right] e^{\lambda \tilde{\psi}}. \] (3.17)

The above profile satisfies \( p(\psi_0) = p_0, \ p(\psi_e) = 0 \) and \( p'(\psi_e) = 0 \). The factor \( e^{\lambda \tilde{\psi}} \) is included to achieve broadening of the profile. Without this factor the broadest profile would be given by the limit \( \nu \to -\infty \) and would be
\[ p(\psi) = p_0 (1 - \tilde{\psi}). \] (3.18)

In the circular cross-section case this would correspond to a nearly parabolic pressure profile as a function of minor radius.

If hollow pressure profiles are not considered, we obtain a constraint for \( \lambda \). The derivative of the pressure at the axis is given by
\[ p'(\psi_0) = p_0 \left[ \lambda - \frac{\nu(e^\nu - 1)}{e^\nu - \nu - 1} \right]. \] (3.19)

For monotonic profiles we require then \( p'(\psi_0) \leq 0 \) or
\[ \lambda \leq \frac{\nu(e^\nu - 1)}{e^\nu - \nu - 1}. \] (3.20)

It is straightforward to convince oneself that the above constraint also ensures that the profile is monotonic throughout the plasma region for all values of \( \nu \). Since the infernal mode becomes unstable for peaked pressure profiles, we set
\[ \lambda = \frac{\nu(e^\nu - 1)}{e^\nu - \nu - 1} \] (3.21)
in this work. This still leaves \( \nu \) as a free parameter to control the width of the pressure profile. The most peaked profile that can be obtained in this family is given by \( \nu = 0 \), in the sense that the total pressure \( \int p d\tilde{\psi} \) is minimized by choosing \( \nu = 0 \). This can be easily seen by calculating the total pressure and expanding with \( \nu \) a small parameter. Therefore, the most peaked profile is given by
\[ \frac{p(\psi)}{p_0} = e^{2\tilde{\psi}}(1 - \tilde{\psi})^2, \] (3.22)
where we have used the fact that $\lambda \rightarrow 2$ as $\nu \rightarrow 0$. The broadest possible pressure profile is the limit of a constant pressure across the plasma region at the limit $\nu \rightarrow \infty$. In this work we choose $\nu = 1.5$ and the corresponding profile is shown in Fig. (3.3) as a function of $\bar{\psi}$.

In addition to incorporating the above descriptions for the safety factor and pressure profiles in the equilibrium code, the parametrization of the plasma boundary needed modifications. The boundary as defined in the code is modified exactly as discussed in the previous chapter, so that the horizontal and vertical positions of a point along the boundary are given in terms of a parameter $\nu$ as follows:

$$R = R_0 + a \sum_{n=1}^{4} c_n \cos n\nu,$$

(3.23)

$$Z = a \sin \nu.$$  

(3.24)

Here, $0 \leq \nu < 2\pi$ and the $\{c_n\}$ are related to the elongation, triangularity, indentation and sharpness ($\kappa, \delta, \epsilon, h$) as given by Eqs. (2.163)–(2.164).

A final point to be discussed before presenting examples of low and high-$\beta$ equilibria deals with achieving convergence for the equilibrium calculation. It is crucial to have a sufficiently accurate initial guess for the position of the magnetic axis and the shapes of the equilibrium flux surfaces for the calculation to successfully converge. These requirements are the strictest for a peapod equilibrium and can be relaxed considerably for a Dee-shaped plasma, for which even a poor initial guess sometimes suffices. When one equilibrium has been found for a given plasma cross-section, equilibria corresponding to different values of the peak pressure and safety factor at the edge can be routinely calculated by taking as an initial guess the final flux coordinate grid from the known equilibrium. It is then the construction of the first equilibrium that requires most of the attention.

To achieve convergence in the equilibrium calculation we construct an initial grid for the plasma region in such a way that the triangularity of the flux surfaces approaches zero at the magnetic axis and that the shift of the magnetic axis as well as the elongation
of the flux surfaces there can be set arbitrarily. For this reason we define

\[
\sigma(\psi_L) = \sigma_0(1 - \psi_L), \quad (3.25) \\
a(\psi_L) = a_b \psi_L, \quad (3.26) \\
\kappa(\psi_L) = \kappa_0 + (\kappa_b - \kappa_0)\psi_L^2, \quad (3.27) \\
\delta(\psi_L) = \delta_b \psi_L, \quad (3.28) \\
\iota(\psi_L) = \iota_b \psi_L, \quad (3.29) \\
h(\psi_L) = h_b \psi_L. \quad (3.30)
\]

In the above formulae, quantities with subscript \( b \) refer to the values at the plasma boundary while \( \sigma_0 \) and \( \kappa_0 \) are the shift and elongation of the flux surfaces at the magnetic axis, respectively. \( \psi_L \) is a monotonic function of the flux and varies between 0 and 1. At the magnetic axis \( \psi_L = 0 \) and at the plasma boundary \( \psi_L = 1 \). Surfaces of constant \( \psi_L \) are also flux surfaces but \( d\psi_L/d\psi \) is not, in general, constant in the plasma region. For this reason, \( \psi_L \) has been given the name flux label. The equilibrium code calculates, among other things, the relationship between this flux label and the true magnetic flux. The initial grid is then constructed by writing

\[
R(\psi_L, \nu) = R_0 + \sigma(\psi_L) + a(\psi_L) \sum_{1}^{4} c_n(\psi_L) \cos n\nu, \quad (3.31) \\
Z(\psi_L, \nu) = a(\psi_L) \sin \nu. \quad (3.32)
\]

The coefficients \( c_n(\psi_L) \) are calculated as before but with the quantities \( \kappa(\psi_L), \delta(\psi_L), \iota(\psi_L) \) and \( h(\psi_L) \) replacing \( \kappa_b, \delta_b, \iota_b \) and \( h_b \).

For a Dee-shaped plasma it is usually sufficient to have an accurate estimate of just the shift of the magnetic axis in order to achieve convergence. The rest of the initial grid can then be constructed by just projecting the boundary shape along straight lines toward this axis. Unlike our grid above, this approach would maintain constant elongation, triangularity etc. throughout the plasma for the initial grid. This simple prescription is insufficient for peapod shaped plasmas. We also need a reasonably accurate estimate of the elongation of the flux surfaces at the axis as well as making sure
that the triangularity vanishes there. It is also found that a slight overestimate for the
elongation at the axis is better than an underestimate. Experience shows that the above
prescription for the initial guess results in convergence in practically all the cases as long
as \( \kappa_0 \) satisfies

\[
\kappa_T \leq \kappa_0 \leq 1.2\kappa_T,
\]

where \( \kappa_T \) is the true elongation of the flux surfaces at the magnetic axis. When this con-
dition is satisfied, the requirements for the shift of the magnetic axis can be relaxed, and
convergence is achieved for a wide range of \( \sigma_0 \), both overestimates and underestimates
of the actual shift.

We are now in a position to examine typical equilibria produced in the manner
discussed above. For all the cases below, we have chosen the aspect ratio of the vacuum
chamber enclosing the plasma to be \( \varepsilon_V = \sqrt{A_V/R} = 0.5 \), where \( A_V \) is the cross-
sectional area and \( R \) the major radius at the geometrical center of the vacuum chamber.
Fig. (3.4) shows poloidal flux contours for typical low-\( \beta \) and high-\( \beta \) peapod equilibria
while Fig. (3.5) gives the corresponding toroidal current contours. Note that the toroidal
current is reversed near the inside edge of the tokamak, when plasma \( \beta \) becomes suffi-
ciently high. This current reversal has just begun to take place for this high-\( \beta \) case of
the two equilibria shown here. These equilibria are characterized by poloidal plasma \( \beta \)'s
of \( \beta_p = 0.15 \) and \( \beta_p = 1.5 \), respectively, where the poloidal \( \beta \) is defined by

\[
\beta_p = \frac{2\mu_0 \int V_p \, p \, dV}{\bar{B}_p^2 V_p},
\]

where \( V_p \) is the plasma volume, and the average poloidal field is defined by

\[
\bar{B}_p = \frac{\mu_0 I_p}{L},
\]

\( L \) being the length of the circumference of the plasma, and \( I_p \) the plasma current.
Values of \( \beta_p \approx 1 \) typically correspond to the marginal paramagnetic–diamagnetic case.
The elongation of the peapod at the plasma boundary is \( \kappa_V = \kappa = 2.5 \), where \( \kappa \) is
the elongation defined as the ratio of the height of the plasma and the width along the
midplane. \( \kappa_V \) is the elongation of the rectangular vacuum chamber just enclosing the plasma. The flux surface averaged current profile for the high-\( \beta \) peapod equilibrium is shown in Fig. (3.6). The vacuum flux contours for the same case are shown in Fig. (3.7). It seems that there is a relatively large region of bad curvature on the outside of the plasma, so that conducting walls may be needed to stabilize the axisymmetric modes. This issue is not investigated any further in this work, however.

Bean and Dee–shaped equilibria have also been produced for comparison with the peapod shapes. Fig. (3.8) shows the poloidal flux contours for a typical bean shaped equilibrium with \( \beta_p = 0.6 \) and the toroidal current contours for the same case are shown in Fig. (3.9). The corresponding flux and current contours for a Dee–shaped plasma at \( \beta_p = 1.2 \) are shown in Figs. (3.10) and (3.11). The elongation of the rectangular vacuum chamber enclosing the bean shaped plasma is \( \kappa_V = 2.5 \) while the corresponding elongation for the Dee–shaped plasma is \( \kappa_V = 2.0 \). These correspond to elongations that are commonly used in present experiments.

### 3.3 Stability

The plasma pressure profile was chosen to be broad in order to achieve stability against low \( n \) infernal modes. For this reason our stability analysis emphasizes low \( n \) internal, infernal and external modes and no attempt has been made to optimize the profiles to stabilize ballooning modes. Nevertheless, stability against both \( n = 1 \) and \( n = \infty \) ideal MHD modes is checked using numerical procedures developed at the Plasma Physics Laboratory of Princeton University.

The stability of a given equilibrium against high-\( n \) interchange or Mercier modes as well as the high-\( n \) pressure driven or ballooning modes is investigated using the PEST 2.1 ballooning code\(^8\), which includes finite \( n \) corrections to the theoretical \( n \to \infty \) ballooning limit. Included in the analysis is also a variation of the arbitrary integration
function $\chi_0(\psi)$ in the eikonal function $S$ defined by\cite{9}

$$
S(\phi, \psi, \chi) = n \left[ -\phi + \int_{\chi_0}^{\chi} \frac{J B_{\phi}}{R} d\chi' \right].
$$

(3.36)

In the ballooning mode formalism, this integration function is usually thought to represent the location in the angular coordinate $\chi$ of the peak of the instability. Since the mode usually peaks on the outside of the torus where the magnetic curvature is destabilizing, it is sufficient to vary the function $\chi_0$ between $-\pi/2$ and $\pi/2$.

Low $n$ current and pressure driven instabilities are calculated using the PEST 2.4 stability code which is capable of investigating $n = 1, 2, 3, 4$ modes for edge safety factors up to approximately $q_e = 16, 7, 5, 4$, respectively. Since for many of our equilibria $q_e > 7$, only the $n = 1$ modes have been included in the analysis in this section. The restrictions on the size of the $q_e$ value have to do with the resolution that is available in the grid. From the analysis of a large aspect ratio high-$\beta$ tokamak\cite{10} we know that modes with poloidal mode number $m$ can become unstable when the mode rational surface $q = m/n$ exists somewhere inside or slightly outside the edge of the plasma. For that reason at least $n q_e$ poloidal mode numbers should be retained in the analysis. In practice, $n q_e + 10$ mode numbers are kept so that convergence of the Fourier series representation is achieved, and in order that external modes for which the mode rational surface is outside the plasma, are analysed correctly. The available grid size then requires $n q_e \leq 16$. All the $n = 1$ calculations are performed assuming that the stabilizing influence of a conducting wall is not present. In this case, several modes can become unstable.

The most serious instability is the so-called external kink instability which can lead to a major disruption in an experiment. This instability is driven by a toroidal plasma current that has become very large in magnitude. The stability limit is often reported to be $q_e = 2$. This seems to be true for near circular plasmas at low plasma $\beta$, but for peapod plasmas the external kink has been seen to be unstable at higher values of $q_e$. This point will be further discussed below.

Manickam et al.\cite{2} have reported that an Ideal MHD instability which they call an infernal mode becomes unstable when the plasma pressure gradient becomes large in regions where the local shear is small. The infernal mode has both internal and external
components in the displacement vector and often the \( n = 1 \) infernal mode sets the \( \beta \) limit for a plasma when the current is held constant. Although it is conceivable that these modes can be stabilized by increasing the magnetic shear in regions where the pressure gradient is at its largest, no such attempts have been made in our limited study.

When the safety factor is almost constant and equal to a rational number \( q = m_1/n_1 \) over a substantial portion of the plasma, the internal \( n = n_1, \ m = m_1 \) kink mode can become unstable depending on the details of the flux surface geometry. As demonstrated by Ramos\(^{[11]} \), the stability criterion is identical to high-\( n \) interchange mode boundary given by the Mercier criterion\(^{[12]} \). For modes localized at the magnetic axis, one requires for stability

\[
1 \leq q_0^2 \left[ 1 - \frac{3(\kappa_0^2 - 1)(\kappa_0^2 - 2\tau_0)}{(\kappa_0^2 + 1)(3\kappa_0^2 + 1)} - \frac{4\beta_{p0}(\kappa_0 - 1)^2}{\kappa_0(\kappa_0 + 1)(3\kappa_0^2 + 1)} \right],
\]

(3.37)

in regions where formally \( q = q_0 + O(\epsilon^2) \), \( \epsilon \) being a small parameter denoting the local inverse aspect ratio. In the above, \( \kappa_0 \) is the elongation of the flux surfaces and \( \tau_0 \) is their triangularity near the magnetic axis. \( \beta_{p0} \) is the poloidal plasma \( \beta \) at the axis and it is related to the shift \( \sigma_0 \) of the magnetic axis:

\[
2\sigma_0 = \frac{2(\kappa_0^2 + 1)\beta_{p0} + \kappa_0^2 - 2\tau_0}{3\kappa_0^2 + 1}.
\]

(3.38)

With increasing plasma currents, the safety factor and current profiles flatten near the magnetic axis and these cause the triangularity of the flux surfaces near the axis to decrease and their elongation to increase. As a consequence, these internal kink modes, as well as the high-\( n \) interchange modes become unstable near the axis even at very low plasma \( \beta \) and in fact set a limit on the total plasma current often before the external kink mode becomes unstable. While it is clear that these modes can be stabilized by changing the current (safety factor) profile near the axis, no profile optimization has been performed in this study, and the current limit is taken as that current which results in marginal stability against these internal modes. A further increase in the total current is required before the external kink mode becomes unstable, and therefore the external kink does not contribute to the current limits in this work.
3.4 Numerical Results and Discussion

Peapod equilibria of elongations $\kappa_V = 2.5$, $2.8$ and $3.1$ have been tested against stability to $n = 1$ modes. These results are compared with the corresponding results for a Dee–shaped and a bean–shaped plasma of elongations $\kappa_V = 2.0$ and $\kappa_V = 2.5$, respectively, while maintaining the same pressure profile as a function of flux and the same prescription for calculating the safety factor and current profiles. These results can be conveniently summarized in a diagram where the volume averaged toroidal $\beta$ is plotted against $1/q_*$, a measure of the plasma current. The volume averaged toroidal $\beta$ is defined by

$$< \beta_V > = \frac{\int_V p \, dV}{\int_V \frac{B_\phi^2}{2\mu_0} \, dV},$$

(3.39)

with the integrals taken over the vacuum chamber volume. $q_*$ measures the average plasma current density and is defined by

$$q_* = \frac{2A_p B}{R\mu_0 I_p},$$

(3.40)

where $A_p$ is the cross–sectional area of the plasma, $B$ the magnetic field at $R$, the geometrical center of the vacuum chamber, and $I_p$ the total plasma current. The results for the three peapod, the bean and the Dee–shaped plasma equilibria are shown in Figs. (3.12) a–e, where the stable region against $n = 1$ modes is shown with some unstable points included. These unstable points set limits on both the plasma $\beta$ and the plasma current from above.

The most striking feature about the stability diagrams is the fact that while in the past theoretical studies have shown the bean shaped plasmas to be able to maintain stable equilibria with very high plasma $\beta$'s, about $10 - 11\%$, these values are in fact reduced by almost a factor of two when $\beta$ is defined according to Eq. (3.39). Here we pay a penalty for all the magnetic energy that needs to be generated in the vacuum.
chamber, and not only inside the plasma region. In fact, with this definition the highest stable $\beta$ value, $\beta_V = 6.7\%$, was obtained for a peapod shape of elongation $\kappa_V = 2.5$ and not the bean, which reached $\beta_V = 6.6\%$. The conventional volume averaged $\beta$ for this bean equilibrium is $\beta = 10.1\%$, while that for the peapod is $\beta = 8.1\%$. In all the cases that we have investigated, the $n = 1$ infernal mode becomes unstable when plasma pressure is increased while holding the plasma current constant. We therefore conclude, that the infernal mode sets the $\beta$-limit with respect to $n = 1$ modes. A plot of a typical mode structure for the infernal mode is shown in Fig. (3.13).

Another result which is immediately clear from the diagrams is the fact that the current limit is highest for the peapod shapes of elongations $\kappa_V = 2.5$ and 2.8 and considerably lower for the Dee-shaped and bean-shaped equilibria. Note that $1/q_*$ measures average current density $(I_p/A_p)$ and therefore our results indicate that the peapod shape is stable to much higher average current densities than the other shapes. The mode limiting the current in all cases is the $m = 1$, $n = 1$ internal kink mode, with the Mercier criterion becoming violated near the axis for safety factors $q_0 < 1.05$ as the central current density increases. A typical mode structure for this mode is shown in Fig. (3.14). The differences for the various cross-sectional shapes seem to amount to the following. The on-axis triangularity of the Dee-shaped plasma is never very high, so the Mercier criterion is violated for rather modest total plasma currents, which cause the on-axis triangularity to further decrease. For the bean-shaped plasma the situation is slightly different. The total cross-sectional area of the plasma is smaller than for the other shapes. Therefore, as the total current increases, the current profile flattens on the axis much faster than for the other shapes, because the safety factor on axis, and thus the peak value of the current density, are held fixed. This seems to increase the on-axis elongation for the bean-shaped plasma faster than for the peapod equilibria, and the Mercier criterion is again violated at lower total currents. The peapod shape then represents a convenient compromise between a Dee-shaped and bean-shaped plasma and allows stable operation at a larger average plasma current.

The external kink mode, which has only one external component and not a mixture
of poloidal components as seen with the infernal mode, has also been found for some equilibria to be unstable, most notably the $\kappa = 2.8$ peapod at $1/q_* = 1.06$. Two things are apparent here. These equilibria correspond to edge safety factors of $q_e \approx 4.0$, while the external kink mode does not become unstable for near circular plasmas until $q_e < 2^{[14][18]}$. However, when expressed in terms of the average current density $1/q_*$, the limit for the peapod shape is higher than for a circular shape, for which the MHD safety factor $q_e$ and $q_*$ coincide. We therefore conclude, that while a particular $q_e$–limit may be relevant in a context where the plasma shape is fixed or nearly fixed, this limit does not appropriately reflect the operational limits of experiments with widely varying cross-sectional shapes. Therefore, we choose to express the current limit in this work directly in terms of quantities that have meaning to the experimentalist, such as $1/q_*$ which is made up of quantities that can easily be measured. For the $\kappa_V = 2.5$ and 2.8 peapod shapes the normalized current limit is given approximately by $q_* = 1.25$, while the corresponding limits for the bean and Dee–shaped plasmas are $q_* = 1.4$ and 1.5, respectively.

In comparison with the Dee–shaped equilibria, the peapods of elongations $\kappa_V = 2.5$ and 2.8 exhibit both higher $\beta$–limits and higher current limits. While the highest stable $\beta$ value for the bean–shaped equilibrium is comparable to the highest values obtained for these two peapods, the current window for the bean–shaped equilibrium is very narrow. The fact that no unstable points are shown above some stable equilibrium points for both the bean and peapod shapes is a consequence of the fact that the equilibrium calculations did not converge for higher values of $\beta$ without an accompanying increase in the plasma current. We therefore conclude that these points represent operation near the equilibrium limit for the corresponding cross–sectional shape with the given pressure and safety factor profiles.

The peapod with elongation $\kappa_V = 3.1$ does not seem as good as the less elongated peapod equilibria. From this study we conclude that peapods should not have elongation higher than about 2.8 in order to maximize performance. However, the stable regions for the $\kappa_V = 2.5$ and $\kappa_V = 2.8$ peapods are very similar in terms of both highest
stable current density and highest plasma beta. This confirms the observation from the calculations with the sharp boundary model, that the highest stable $\beta$ value is a very weak function of the peapod elongation near its maximum.

Finally, since the pressure profile was not chosen with stability to ballooning modes in mind, not surprisingly some of the equilibria which are stable to $n = 1$ modes are unstable to ballooning modes. Nevertheless, stability to ballooning modes seems reasonably good. We have found that equilibria with

$$\beta > 4 \frac{I_p}{aB},$$

are unstable to ballooning modes while equilibria with $\beta$-values less than this threshold are stable to ballooning modes. The lines of

$$\beta = 4 \frac{I_p}{aB},$$

are shown in each of the stability diagrams. In the above, $a$ is the minor radius in meters of the plasma along the midplane, $I_p$ is the plasma current in $MA$, $B$ the magnetic field in Tesla and $\beta$ is given in per cent. In all the cases analyzed, the ballooning mode was centered along the midplane of the cross-section so that the most unstable mode was obtained for $\chi_0 = 0$, $\chi_0$ being the arbitrary integration function representing the angle along which the mode is centered. The ballooning studies have indicated that substantial improvement in plasma $\beta$ is possible without violating stability to ballooning modes. This can be achieved by broadening the pressure profile further, since the profile chosen for this study has its highest gradient in a region where the flux averaged local shear does not have its maximum value. The mode becomes unstable over a relatively narrow region representing less than one quarter of the plasma region in all cases, and is located at the flux surface inside the plasma region where the pressure gradient has its maximum value.
3.5 Conclusions

We have studied the stability of peapod, bean and Dee–shaped plasma equilibria with respect to \( n = 1 \) modes with relatively broad pressure and current profiles. Our results show that the peapod equilibria have the most robust stability properties with a wide region of stability when the plasma current is varied. The achieved plasma \( \beta \) values are also quite reasonable, and show that the peapod shape is more desirable than a bean or Dee–shaped plasma. Our study supports the conclusions of Chapter 2, where based on the sharp boundary model it was shown that the peapod shape shows better stability properties against low \( n \) number ideal MHD modes than other shapes. The optimal elongation for a peapod has not been conclusively found, but it has been shown that this elongation is somewhat less than 3, although probably higher than 2.

Although vertical stability has not been investigated in this work, the shapes of the vacuum flux contours suggest that in order to stabilize vertical instabilities in peapods, we may need to employ conducting walls near the plasma. This aspect as well as more extensive studies regarding pressure profile optimization with respect to ballooning modes would be good subjects of a future investigation. The present ballooning analyses indicate that the Troyon coefficients for the three shapes are close to equal, while the kink limits allow higher plasma currents for the peapod. The result is a higher stable value of \( \beta \) for a peapod than for the other two shapes.
3.6 Acknowledgments

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3.7 References


Figure 3.1 A typical safety factor profile for a peapod equilibrium.

Figure 3.2 The corresponding flux averaged toroidal current profile.
Figure 3.3 Pressure profile as a function of $\bar{\psi}$ with $\nu = 1.5$. 
Figure 3.4a Poloidal flux contours for a peapod with $\beta_p = 0.15$. 
Figure 3.4b Poloidal flux contours for a peapod with $\beta_p = 1.5$. 
Figure 3.5a Toroidal current contours for a peapod with $\beta_p = 0.15$. 
Figure 3.5b Toroidal current contours for a peapod with $\beta_p = 1.5$. 
Figure 3.6 Flux surface averaged current profile for a peapod with $\beta_p = 1.5$.

Figure 3.7 Vacuum flux contours for a peapod with $\beta_p = 1.5$. 
Figure 3.8 Poloidal flux contours for a bean-shaped plasma with $\beta_p = 0.6$. 
Figure 3.9 Toroidal current contours for a bean-shaped plasma with $\beta_p = 0.6$. 
Figure 3.10 Poloidal flux contours for a Dee–shaped plasma with $\beta_p = 1.2$. 
Figure 3.11 Toroidal current contours for a Dee-shaped plasma with $\beta_p = 1.2$. 
Figure 3.12a Stability diagram for a peapod with $\kappa_V = 2.5$. $\bullet$ denotes stable and o unstable. Also shown is the approximate stability limit for ballooning modes.
Figure 3.12b Stability diagram for a peapod with $\kappa V = 2.8$. $\bullet$ denotes stable and $\circ$ unstable. Also shown is the approximate stability limit for ballooning modes.
Figure 3.12c Stability diagram for a peapod with $\kappa_N = 3.1$. $\bullet$ denotes stable and $\circ$ unstable. Also shown is the approximate stability limit for ballooning modes.
Figure 3.12d Stability diagram for a bean-shaped plasma with \( \kappa V = 2.5 \). ● denotes stable and ○ unstable. Also shown is the approximate stability limit for ballooning modes.
Figure 3.12e Stability diagram for a Dee–shaped plasma with $\kappa_\nu = 2.0$. $\bullet$ denotes stable and $\circ$ unstable. Also shown is the approximate stability limit for ballooning modes.
Figure 3.13 Typical mode structure of an infernal mode showing internal and external components. Also shown is the safety factor profile.
Figure 3.14 Typical mode structure of an $n = 1, m = 1$ internal kink mode showing the single component on-axis mode. Also shown is the safety factor profile.
4. Determination of Plasma Shape in Experiments

In this chapter we present a model for determining the plasma shape in an experimental situation from magnetic measurements that are available some distance outside the plasma region. An accurate determination of the shape is necessary when the experiment is designed to study the performance of the plasma when the shape is varied.

4.1 Introduction

This section outlines the experimental problem and gives a summary of existing methods for obtaining reasonably accurate solutions. We also introduce the new solution procedure and present a basis for the maximization of accuracy in the solution.

A. Experimental Problem

Magnetic diagnostic techniques have developed rapidly in recent years in conjunction with the need to analyze sophisticated plasma fusion experiments. One area of interest involves the diagnosing of strongly non-circular plasmas, perhaps including a divertor generated separatrix. A corresponding classic problem facing the experimentalist is that of determining the shape of the plasma boundary, given an appropriate set of externally measured magnetic probe data.

This problem is ill-posed mathematically, and herein lies the difficulty. Specifically, the aim of the problem is to determine the shape of the flux surfaces $\psi (R, Z) = \text{const}.$ where the flux function satisfies the vacuum form of the Grad-Shafranov equation$^{[1]}$, 

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\[ \Delta^* \psi = 0 \]  
(4.1)

\[ \Delta^* \psi \equiv R^2 \nabla \cdot (\nabla \psi / R^2) \]  
(4.2)

The boundary conditions assume that \( \psi \) and its normal derivative \( n \cdot \nabla \psi \) (i.e. tangential magnetic field) are specified on a known surface exterior to the plasma. This is an accurate approximation to the experimental situation in which tangential magnetic field probes and flux loops are located on the vacuum chamber. Since the operator \( \Delta^* \) is elliptic, the mathematical problem of interest requires the solution of an elliptic partial differential equation, subject to Cauchy boundary-conditions. As is well known this is an ill-posed problem. In principle, small changes in the boundary data lead to large changes in the solution a short distance away, a mathematically unstable situation.

B. Existing Mathematical Procedures

The Cauchy problem for elliptic equations has been extensively studied and is relatively well understood\[^2\]. A number of practical methods have been devised to overcome the ill-posedness. These methods are usually based on an expansion of the flux function in terms of solutions to the vacuum Grad-Shafranov equation; that is

\[ \psi (R, Z) = \sum c_i \chi_i (R, Z) \]  
(4.3)

with the basis functions \( \chi_i \) satisfying

\[ \Delta^* \chi_i = 0. \]  
(4.4)

Stabilizing functionals are also often employed in the procedure for inverting or quasi-inverting (minimization in the least squares sense of some cost function involving the measurement data) the Grad-Shafranov operator.

One class of solution procedures makes use of an expansion for \( \psi \) in terms of orthogonal toroidal functions. Such procedures were used for the Tuman-3 tokamak\[^8\]
and the ISX-B⁴ experiment. In these cases, no information about the plasma current distribution is required. The coefficients cᵢ are determined to directly give a best fit to both ψ and n · Vψ on the boundary. One then uses the general radial and angular dependence of the expansion functions to determine ψ away from the boundary. This procedure works reasonably well in practice, although it suffers from several drawbacks. The main difficulty is that the procedure is quite sensitive to the ill-posedness, as no information is provided about the plasma current. The situation is particularly difficult because the expansion functions are in general not natural to the probe geometry, thus implying the need for many terms in the expansion. As will shown below, the errors associated with ill-posedness increase rapidly with the number of expansion functions.

Another class of solution procedures eliminates the ill-posedness by assuming a certain model for the plasma current, viz. a filament model. Such a procedure was first proposed by Wootton⁵, who makes use of an expansion in the inverse aspect ratio in his analysis. A general procedure where the filament current magnitudes were obtained by minimizing a least squares type cost function assuming fixed locations for the filaments was used in ISX-B⁶. Stabilizing functionals were used in determining the plasma shape for Doublet III⁷ while the filament model for the plasma was retained. Methods of this type work reasonably well in practice but still suffer from two drawbacks. First, the weight factors employed in the stabilizing functionals seem arbitrary, an important point since the results obtained depend quite strongly on the particular weight factor used. Second, for high accuracy, a good approximation is required for the plasma current distribution. This is a difficult task unless one is willing to solve the full Grad-Shafranov equation, a costly procedure in terms of computer time for between shot analysis. Note that in principle the shape of the plasma is completely determined by the boundary data, and no information about the plasma current distribution is required. The purpose of introducing the plasma properties is solely to overcome the ill-posedness.

A procedure based on using a current sheet on a control surface to represent the plasma current is now routinely used at JET⁸. Here again, ψ is expanded in terms of two-dimensional basis functions and the observed fields are matched with the calculated
ones in the least squares sense. A good overview of all the existing methods has been given by Braams\textsuperscript{[9]}. 

C. New Procedure

We have developed a fast [270 ms CPU on a DEC VAX-780] and accurate method to solve for the poloidal flux function in the vacuum region. The method requires no knowledge of the plasma current distribution. The inherent ill-posedness of the problem is overcome by a formulation based on a non-standard application of Green’s theorem. Our results show that in realistic practical applications the accuracy of the calculated solution on the plasma surface is approximately the same as the accuracy of the input data on the measurement surface.

The procedure assumes that $\psi$ and $n \cdot \nabla \psi$ are specified on a given surface exterior to the plasma, e.g. the vacuum chamber. The Green’s function formulation allows us to calculate the angular dependence of $\psi$ and $n \cdot \nabla \psi$ on a sequence of conveniently chosen surfaces filling the region interior to the measurement surface. The advantages of this procedure are twofold. First, the Green’s function method requires no explicit knowledge of the radial dependence of the solution or any associated expansion functions. This is particularly important because the measurement geometry is in general quite complicated, and simple, natural radial expansion functions do not exist. Second, since the solution on each of the interior surfaces involves the determination of the angular dependence of a periodic function, the obvious choice of expansion functions corresponds to a Fourier series. Furthermore, as is shown in the analysis, there is a natural choice for the angle coordinate on the measurement surface and each of the interior surfaces. This ultimately results in the need for relatively few Fourier harmonics to obtain a given accuracy.

The speed of the method (in practice) is due to the fact that all computations related to the Green’s functions can be carried out ahead of time once the experimental geometry has been determined, and the results stored on disk. The data storage requirements are quite modest: in addition to the locations of the equilibrium coil currents one needs to
typically store 10 complex matrices of size usually no greater than $30 \times 30$. The actual computational requirements to solve for $\psi$ are a one dimensional Fourier analysis of the measurement data, approximately 10 matrix multiplications and the one dimensional Fourier reconstruction to obtain the solution in real space, a very rapid procedure indeed.

D. Accuracy

In the practical implementation of the procedure, two steps are taken which greatly enhance the accuracy of the solutions. First, it is assumed that the values of the net plasma current, and the currents in the equilibrium coils, the ohmic transformer, and any other active and passive poloidal field coils are known and can be reasonably well approximated by a sequence of circular current filaments. The flux and tangential magnetic field due to these known current filaments are subtracted from the measurement data prior to the calculation. Once the solution procedure is completed the identical contributions are added back to form the final solution. Note that small errors in the measurements of the coil currents are not important as precisely the same contributions are first subtracted and then added to the solution. The benefit of this step is that once the initial subtraction has been carried out, the resulting input data is a much smoother function; that is, the presence of large localized coil currents near the measurement probes leads to a high harmonic content in the measurement data. Analytically subtracting these known contributions eliminates most if not all of the high harmonic content. This is a critical point, since, as discussed below, the basic ill-posedness of the problem increases considerably as the number of harmonics is increased.

The second step improving accuracy is the recognition that there is an optimum number of harmonics (typically 4–5 for our application) that should be maintained in the analysis. Specifically, even if the number of measurement probes is increased by a large factor, indicating high harmonic accuracy in the input data, one should still carry out the procedure with a much lower, "optimum" number of harmonics to obtain the most accurate $\psi$. The existence of an optimum is the result of the competition between poor resolution with too few harmonics versus poor accuracy due to ill-posedness with
too many harmonics. The relationship between ill-posedness and harmonic content can be investigated through a simple example.

Consider a right circular cylinder of radius \( r = a \) and the function \( \phi \) satisfying Laplace’s equation within the cylinder:

\[
\nabla^2 \phi = 0
\]

(4.5)

In analogy with the diagnostic problem assume the boundary conditions are given by

\[
\phi (a, \theta) = (\phi_m + \epsilon \phi_1) \cos m\theta
\]

(4.6)

\[
\frac{\partial \phi}{\partial r} (a, \theta) = \frac{m}{a} (\phi_m - \epsilon \phi_1) \cos m\theta
\]

Here, \( \phi_m \) is the correct amplitude of the \( m' \)th harmonic and \( \epsilon \phi_1 \) is the error, perhaps due to detector accuracy or calibration procedures. The error amplitude \( \epsilon \phi_1 \) is assumed independent of \( m \) and is scaled as the product of the small number \( \epsilon \) with the fundamental \( m = 1 \) harmonic amplitude \( \phi_1 \). This is a good approximation to the error situation in an actual experiment. When \( \epsilon = 0 \), the solution is

\[
\phi = \phi_m \left( \frac{r}{a} \right)^m \cos m\theta
\]

(4.7)

which is well behaved and regular for \( 0 < r < a \). For \( \epsilon \) small but non-zero the solution inside the cylinder becomes

\[
\phi = \phi_m \left( \frac{r}{a} \right)^m \cos m\theta + \epsilon \phi_1 \left( \frac{a}{r} \right)^m \cos m\theta
\]

(4.8)

If the “plasma surface” corresponds to some interior surface \( r = r_0 < a \), then the ratio of the error in the “calculated” solution [Eq. (4.8)] to the “exact” solution [Eq. (4.7)] on the surface \( r = r_0 \) is given by

\[
\frac{\phi(\text{error})}{\phi(\text{exact})} = \epsilon \left( \frac{\phi_1}{\phi_m} \right) \left( \frac{a}{r_0} \right)^{2m}
\]

(4.9)
The quantity $\phi_1/\phi_m$ is in general an increasing function of $m$ for large $m$. Hence, the error in the solution increases at least exponentially with harmonic number $m$ for fixed $\epsilon$ and $(r_0/a)$. It is for this reason that it is important for the input data be as smooth as possible. The analysis also explains why accuracy degrades when the number of harmonics becomes too large thus implying the existence of an optimum number of harmonics. These points are discussed in detail in Sec. 4.4.

In summary, the purpose of this study is to provide a reliable formulation of the diagnostic problem, leading to a rapid and accurate determination of the flux function despite the ill-posedness. The nature of the ill-posedness is investigated in detail for realistic experimental situations, allowing a reliable determination of the accuracy of the procedure. A discussion is also presented of an issue of experimental importance viz. how many magnetic field probes and flux loops are required to provide sufficiently accurate input data for a correspondingly accurate determination of the plasma surface.

4.2 Formulation

A. Statement of the Problem

Consider the experimental diagnostic problem that arises in many axisymmetric toroidal fusion devices. Surrounding the plasma is a vacuum chamber upon which is mounted a series of magnetic probes and flux loops which measure the poloidal flux $\psi$, and the component of poloidal magnetic field tangent to the vacuum chamber, $B_t$. Experimentally, one would like to use this data to calculate the vacuum flux surfaces, including the location of the separatrix if one is present.

The situation is illustrated in Figure 4.1. The surface $S_1$ represents the vacuum chamber upon which $\psi$ and $B_t$ are specified. In general $S_1$ is not a flux surface because of the short resistive diffusion time of the vacuum chamber.
The region of interest lies between the vacuum chamber and the last flux surface carrying plasma current denoted by $\psi_a$. In systems with a divertor, $\psi_a$ corresponds to the separatrix, as shown in Figure 4.1. In systems with a limiter, $\psi_a$ represents the flux surface just intersecting the limiter. A critical point is that the region of interest corresponds to a vacuum region: $\nabla \times B_p = \nabla \cdot B_p = 0$ where $B_p$ is the poloidal magnetic field.

The analysis presented here describes a procedure for calculating $\psi$ and $B_i$ on any arbitrary interior surface, such as $S_2$, lying in the vacuum region. Once $\psi$ is known, it is then a simple matter to calculate $B_n$, the magnetic field normal to $S_2$.

It is important to realize that the procedure can also be used to calculate $\psi$ and $B_i$ on arbitrary surfaces such as $S_3$ and $S_4$ that intersect or lie within the last surface carrying current. This follows by recognizing that the plasma current density $J_\phi$ consistent with the observed $\psi$ and $B_i$ on $S_1$ is not unique. For example, in a circular cross section plasma, $J_\phi$ can be replaced by an equivalent set of multipoles on axis, and the field external to plasma remains unchanged. In systems with more general cross sections one can assume that a more complicated, but nonetheless equivalent current density distribution can be found which lies entirely within the last observation surface ($S_4$ in Figure 4.1). We emphasize that it is not necessary to explicitly calculate such a $J_\phi$, but only to recognize that one exists. The region between the equivalent $J_\phi$ and the vacuum chamber is now entirely comprised of vacuum, thus allowing application of the procedure over a wider region of space. Clearly, however, the resulting solutions are valid only in the true vacuum region lying between the vacuum chamber and the surface $\psi = \psi_a$. It is the application of the procedure to such surfaces as $S_3$ and $S_4$ that ultimately allows the determination of the flux surface $\psi = \psi_a$.

It should again be noted that the surfaces $S_j$, $(j \geq 2)$ used in the analysis are arbitrary and are chosen for mathematical convenience. In general they will not be flux surfaces since $\psi$ is not known prior to the calculation. By applying the procedure to a sufficient number of $S_j$, $\psi$ and $B_p$ can be determined over the entire region of interest.

The mathematical formulation of the procedure is based on a somewhat non-
standard application of the vector Green’s theorem. The goal is to derive and then simultaneously solve a set of coupled integral equations for $\psi$ and $\mathbf{n} \cdot \nabla \psi$ in the vacuum region.

The analysis begins with the specification of the poloidal magnetic field which in an axisymmetric toroidal geometry can be written as

$$\mathbf{B}_p = \frac{\nabla \psi \times \mathbf{e}_\phi}{R}$$  \hspace{1cm} (4.10)

Here $\psi$ is the unknown flux function to be determined and $(R, \phi, Z)$ are standard cylindrical coordinates. (See Figure 4.1.) Since the region of interest is a vacuum, $\psi$ must satisfy

$$\Delta^* \psi \equiv R^2 \nabla \cdot \left( \frac{\nabla \psi}{R^2} \right) = 0$$  \hspace{1cm} (4.11)

The boundary conditions are chosen to represent the experimental situation: the poloidal flux and tangential field are specified on the surface $S_1$. In terms of $\psi$ this is equivalent to

$$\frac{\psi}{S_1} = \psi_b$$ \hspace{1cm} (4.12a)

$$\frac{n \cdot \nabla \psi}{R} \bigg|_{S_1} = B_t$$ \hspace{1cm} (4.12b)

where $\mathbf{n}$ is the outward normal to $S_1$ and $\psi_b, B_t$ are given input data which are functions of poloidal angle. The mathematical statement of the problem under consideration is the requirement that Eq. (4.11) be solved subject to Eq. (4.12).

Note that $\Delta^*$ is an elliptic operator while Eq. (4.12) corresponds to Cauchy type boundary conditions. As is well known from the theory of partial differential equations, the problem just formulated is ill-conditioned; that is, the correct amount of boundary data is provided, but it is distributed improperly for an elliptic problem. The procedure presented here provides an effective method for solving the ill-conditioned problem by making use of a non-standard application of the vector Green’s theorem.
B. Green's Theorem for the Vector Potential

The first step in the procedure is to express the solution of $\Delta^* \psi = 0$ in terms of the vector Green's theorem using the infinite space Green's Function. In applying the theorem, note that the region of interest is the vacuum region bounded by the vacuum chamber $S_1$ and any arbitrarily chosen interior surface $S_2$. One starts with the general three dimensional vector Green's theorem for the vector potential $A$ given by\(^{[10]}\) (see Appendix A)

$$\sigma A = \sum_{j=1}^{2} \int_{S_j} \left[ (n' \cdot A') \nabla' \hat{G} + (n' \times A') \times \nabla' \hat{G} + \hat{G} (n' \times \nabla' \times A') \right] dS' \quad (4.13)$$

Here primed coordinates denote integration variables, unprimed variables denote observation point, $n$ is the outward pointing normal to the integration surface and $\hat{G} = -(1/4\pi)/|r' - r|$. The quantity $\sigma$ depends upon the exact location of the observation point with respect to the region of interest and is given by

$$\sigma = \begin{cases} 
1 & (R, Z) \text{ between } (S_1, S_2) \\
1/2 & (R, Z) \text{ on } S_1 \text{ or } S_2 \\
0 & (R, Z) \text{ outside } (S_1, S_2) 
\end{cases} \quad (4.14)$$

The next step is to simplify Eq. (4.13) by making use of the assumption of axisymmetry and the fact that only $A_\phi \equiv \psi / R \neq 0$. A short calculation yields

$$\sigma \psi = \sum_{j=1}^{2} \int_{S_j} \frac{1}{R'^3} \left[ \psi' (n' \cdot \nabla' \hat{H}) - \hat{H} (n' \cdot \nabla' \psi') \right] dS' \quad (4.15)$$

where the Green's function for the vector potential has the form

$$\hat{H} (R', \phi', Z'; R, \phi, Z) = -\frac{R'R \cos (\phi' - \phi)}{4\pi |r' - r|} \quad (4.16)$$

and

$$|r' - r| = [R'^2 + R^2 - 2RR' \cos (\phi' - \phi) + (Z' - Z)^2]^{1/2} \quad (4.17)$$
The two basic equations used in the solution procedure are obtained by applying Eq. (4.15) to the cases where the observation point first lies on the surface \( S_2 \) and then on the surface \( S_1 \). This gives

\[
\frac{1}{2} \psi_2 - \int_{S_2} \frac{1}{R_2^2} \left[ \psi_2' \left( \mathbf{n}' \cdot \nabla' \hat{H}_{22} \right) - \hat{H}_{22} (\mathbf{n}' \cdot \nabla' \psi_2') \right] dS_2' = \int_{S_1} \frac{1}{R_1^2} \left[ \psi_1' \left( \mathbf{n}' \cdot \nabla' \hat{H}_{12} \right) - \hat{H}_{12} (\mathbf{n}' \cdot \nabla' \psi_1') \right] dS_1' \tag{4.18}
\]

\[
\frac{1}{2} \psi_1 - \int_{S_1} \frac{1}{R_1^2} \left[ \psi_1' \left( \mathbf{n}' \cdot \nabla' \hat{H}_{11} \right) - \hat{H}_{11} (\mathbf{n}' \cdot \nabla' \psi_1') \right] dS_1' = \int_{S_2} \frac{1}{R_2^2} \left[ \psi_2' \left( \mathbf{n}' \cdot \nabla' \hat{H}_{21} \right) - \hat{H}_{21} (\mathbf{n}' \cdot \nabla' \psi_2') \right] dS_2' \tag{4.19}
\]

Here, the subscripts refer to the surface under consideration. Equations (4.18) and (4.19) should be viewed as two coupled integral equations for the unknowns \( \psi_2 \) and \( \mathbf{n} \cdot \nabla \psi_2 \) in terms of the known boundary data \( \psi_1 \) and \( \mathbf{n} \cdot \nabla \psi_1 \).

### 4.3 Solution Procedure

Several steps are required to solve the coupled integral equations. First, a coordinate system and surface parametrization must be introduced. Second, these are substituted into the integral equations which then reduce considerably. In particular the explicit \( \phi \) dependence of the Green’s function \( \hat{H} \) is removed by an analytic integration. Next, the reduced equations are solved by Fourier analysis, resulting in a set of linear algebraic equations in standard form for numerical computation. Finally, several practical experimental and numerical issues are addressed. These steps are discussed below.
A. Coordinate System

The analysis begins with the parametrization of the vacuum chamber surface $S_1$ and the arbitrary observation surface $S_2$ in terms of an angle like variable $v$ whose range is $0 < v \leq 2\pi$. Each surface is written as

\[ R_j = R_j(v) \]

\[ Z_j = Z_j(v) \]  

(4.20)

The choice of $v$ is arbitrary. For instance $v$ can represent normalized arclength, or the familiar poloidal angle $\theta$ defined by $R_j = R_0 + r_j(\theta) \cos \theta$, $Z_j = r_j(\theta) \sin \theta$. In Sec. 4.4 it is shown that there exist "natural" choices for $v$ for both the measurement and observation surfaces. For the present analysis, the angle $v$ is treated as arbitrary; the particular choice, based on mathematical and/or numerical convenience, is deferred until the end of the calculation. Since the shape of the vacuum chamber and the observation surface are assumed to be given, $R_j(v), Z_j(v)$ are hereafter considered as known quantities.

The normal vector to each surface can be expressed in terms of $R_j$ and $Z_j$ as follows:

\[ \hat{n}_j(v) = \left( \dot{Z}_j e_R - \dot{R}_j e_Z \right) / Q_j \]

(4.21)

where

\[ Q_j = \left( \dot{R}_j^2 + \dot{Z}_j^2 \right)^{1/2} \]

(4.22)

and $\dot{R}_j, \dot{Z}_j$ denote $dR_j/dv, dZ_j/dv$. Note that the outward normal $n$ is related to $\hat{n}$ by: $n_1 = \hat{n}_1$ and $n_2 = -\hat{n}_2$.

Next, observe that the incremental arc length along each surface can be written as

\[ d\ell_j = Q_j dv \]  

(4.23)
This implies that the differential area element on each surface has the form

$$dS_j = R_j d\phi dl_j = R_j Q_j d\phi dv$$  (4.24)

Finally, it is convenient to introduce the normal derivative as follows

$$Q_j \hat{n} \cdot \nabla = \dot{Z}_j \frac{\partial}{\partial R_j} - \dot{R}_j \frac{\partial}{\partial Z_j} \equiv \frac{\partial}{\partial n}$$  (4.25)

**B. Simplification of the Basic Equations**

Upon introducing the coordinate system just discussed, one finds that the basic equations are significantly simplified. The main reduction results from the fact that, because of axisymmetry, the $\phi$ dependence of the Green's function can be analytically integrated. A short calculations yields

$$\frac{1}{2} \psi_2 + \int_{S_2} \left( \frac{\psi_1}{R_2} \frac{\partial H_{22}}{\partial n'_2} - \frac{H_{22}}{R_2} \frac{\partial \psi_2'}{\partial n'_2} \right) dv' - \int_{S_1} \left( \frac{\psi_1}{R_1} \frac{\partial H_{12}}{\partial n'_1} - \frac{H_{12}}{R_1} \frac{\partial \psi_1'}{\partial n'_1} \right) dv' = 0$$  (4.26)

$$\frac{1}{2} \psi_1 - \int_{S_1} \left( \frac{\psi_1}{R_1} \frac{\partial H_{11}}{\partial n'_1} - \frac{H_{11}}{R_1} \frac{\partial \psi_1'}{\partial n'_1} \right) dv' + \int_{S_2} \left( \frac{\psi_2}{R_2} \frac{\partial H_{21}}{\partial n'_2} - \frac{H_{21}}{R_2} \frac{\partial \psi_2'}{\partial n'_2} \right) dv' = 0$$  (4.27)

Here, the reduced Green's function is given by

$$H_{ij} = \int_{0}^{2\pi} \tilde{H}_{ij} d\phi' = -\frac{1}{4\pi} \int_{0}^{2\pi} \frac{R_i R_j \cos (\phi' - \phi) d\phi'}{|r'_i - r_j|}$$  (4.28)

and

$$|r'_i - r_j| = \left[ R_i^2 + R_j^2 - 2 R_i R_j \cos (\phi' - \phi) + (Z_i' - Z_j)^2 \right]^{1/2}$$  (4.29)

Upon evaluating the integral, one obtains

$$H_{ij} = -\frac{(R_i R_j)^{1/2}}{2\pi} \left[ \frac{(2 - k^2) K - 2E}{k} \right]$$  (4.30)

where $K(k)$ and $E(k)$ are the complete elliptic functions and
\[ k^2 = \frac{4R_i' R_j}{(R_i' + R_j)^2 + (Z_i' - Z_j)^2} \]  \hspace{1cm} (4.31)

Also required in the calculation is the normal derivative of \( H_{ij} \) which is easily calculated and is given by

\[ \frac{1}{R_i'} \frac{\partial H_{ij}}{\partial n_i'} = \frac{1}{2\pi} \left( \frac{R_j}{R_i'} \right)^{1/2} \left\{ \Lambda_{ij} \left[ \frac{(2-k^2)E - 2(1-k^2)K}{k} \right] + \Gamma_{ij}k(E-K) \right\} \]  \hspace{1cm} (4.32)

with

\[ \Lambda_{ij} = \frac{\dot{Z}_i' (R_i' - R_j) - \dot{R}_i' (Z_i' - Z_j)}{(R_i' - R_j)^2 + (Z_i' - Z_j)^2} \]  \hspace{1cm} (4.33)

\[ \Gamma_{ij} = \frac{\dot{Z}_i'}{2R_i'} \]

C. Fourier Analysis

The solution to the basic equations [Eqs. (4.26) and (4.27)] can now be found by standard Fourier analysis. The unknowns \( \psi_2 \) and \( (1/R_2) (\partial \psi_2/\partial n_2) \) are expanded on the observation surface \( S_2 \) as

\[ \psi_2 = \sum_{-L}^{L} a_{\ell} e^{i\nu} \]  \hspace{1cm} (4.34)

\[ \frac{1}{R_2} \frac{\partial \psi_2}{\partial n_2} = \sum_{-L}^{L} b_{\ell} e^{i\nu} \]  \hspace{1cm} (4.35)

In principle the sum over \( \ell \) extends over the range \(-\infty < \ell < \infty\). In practice, numerical considerations require that the series be truncated at a finite value \( |\ell| = L \). The aim of the Solution Procedure is to calculate the unknown Fourier coefficients \( a_{\ell}, b_{\ell} \) in terms of the input data.
In a similar manner, the input quantities \( \psi_1 \) and \( (1/R_1)(\partial\psi_1/\partial n_1) \) are expanded on the vacuum chamber surface \( S_1 \) as

\[
\psi_1 = \sum_{-M}^{M} \hat{a}_m e^{im\nu}
\]

\[
\frac{1}{R_1} \frac{\partial \psi_1}{\partial n_1} = \sum_{-M}^{M} \hat{b}_m e^{im\nu}
\]

(4.36) (4.37)

Note that the coefficients \( \hat{a}_m \) and \( \hat{b}_m \) are known quantities, easily derivable from the input data which consists of: (1) the vacuum chamber surface \( R_1 (\nu) \), \( Z_1 (\nu) \); (2) the poloidal flux on \( S_1 \), \( \psi_b (\nu) \) and (3) the tangential field on \( S_1 \), \( B_t (\nu) \). Also the input Fourier series is in general truncated using a different number of terms from the output series. To avoid the problems associated with ill-posedness, the usual situation is characterized by \( L \leq M \). Substituting into Eq. (4.12) yields

\[
\hat{a}_m = \frac{1}{2\pi} \int_{0}^{2\pi} \psi_b e^{-im\nu} d\nu
\]

(4.38)

\[
\hat{b}_m = \frac{1}{2\pi} \int_{0}^{2\pi} Q_1 B_t e^{-im\nu} d\nu
\]

(4.39)

The Fourier analysis is completed by substituting these expansions into Eqs. (4.26) and (4.27), leading to a standard problem in linear algebra

\[
W \cdot y = V \cdot x
\]

(4.40)

The terms are defined as follows. The vector \( y \) is of length \( 4L + 2 \) and consists of the output coefficients \( a_\xi \) and \( b_\xi \):

\[
y = \begin{pmatrix} a \\ \vdots \\ b \end{pmatrix}
\]

(4.41)

The vector \( x \) is of length \( 2M + 2 \) and consists of the input coefficients \( \hat{a}_\xi \) and \( \hat{b}_\xi \):
\[
x = \begin{pmatrix}
\hat{a} \\
\vdots \\
\hat{b}
\end{pmatrix}
\] (4.42)

The matrix \(W\) has dimensions \((4L + 2) \times (4L + 2)\) and can be written as

\[
W = \begin{pmatrix}
I + A & -C \\
\vdots & \\
-B & D
\end{pmatrix},
\] (4.43)

where \(I\) is the identity matrix and the elements of \(A, B, C,\) and \(D\) are given by

\[
A_{\ell\ell'} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} \left( \frac{1}{R_2'} \frac{\partial H_{12}}{\partial \eta_2'} \right) e^{i\ell' \varphi' - iv \varphi} dv d\varphi
\] (4.44)

\[
B_{\ell\ell'} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} \left( \frac{1}{R_2'} \frac{\partial H_{21}}{\partial \eta_2'} \right) e^{i\ell' \varphi' - iv \varphi} dv d\varphi
\] (4.45)

\[
C_{\ell\ell'} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} H_{22} e^{i\ell' \varphi' - iv \varphi} dv d\varphi
\] (4.46)

\[
D_{\ell\ell'} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} H_{21} e^{i\ell' \varphi' - iv \varphi} dv d\varphi
\] (4.47)

The ranges of \(\ell\) and \(\ell'\) are \(-L \leq \ell \leq L\) and \(-L \leq \ell' \leq L\).

The matrix \(V\) has dimensions \((4L + 2) \times (4M + 2)\) and has the form

\[
V = \begin{pmatrix}
P & -T \\
\vdots & \\
I - S & U
\end{pmatrix}
\] (4.48)

The elements of \(P, S, T,\) and \(U\) are given by

\[
P_{\ell m} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} \left( \frac{1}{R_1'} \frac{\partial H_{12}}{\partial \eta_1'} \right) e^{im \varphi' - iv \varphi} dv d\varphi'
\] (4.49)

\[
S_{\ell m} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} \left( \frac{1}{R_1'} \frac{\partial H_{11}}{\partial \eta_1'} \right) e^{im \varphi' - iv \varphi} dv d\varphi'
\] (4.50)
\[ T_{lm} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} H_{12} e^{im\psi' - it\psi} d\psi' d\psi \]  
(4.51)

\[ U_{lm} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} H_{11} e^{im\psi' - it\psi} d\psi' d\psi \]  
(4.52)

The ranges of \( \ell \) and \( m \) are \(-L \leq \ell \leq L \) and \(-M \leq m \leq M \).

The output vector \( y \) is easily found numerically from Eq. (4.40) as follows.

\[ y = M \cdot x \]  
(4.53)

\[ M = (W)^{-1} \cdot V \]

Observe that the elements of \( W \) and \( V \) depend only upon the shape of the vacuum chamber surface \( S_1 \) and the observation surface \( S_2 \). Hence, for a given experimental application in which a vacuum chamber and a sequence of \( N_S \) observation surfaces are specified, the matrix \( M \) need only be computed one time (for each of the \( N_S \) surfaces) and stored. Analysis of a given set of probe data then requires the evaluation of two one dimensional Fourier series for \( \hat{a}_m \) and \( \hat{b}_m \) and \( N_S \) matrix multiplications. Typically \( N_S \approx 10 \) and the dimensions of a given matrix \( M \) are of the order \((30 \times 30)\). The implication is that the numerical procedure should be very fast indeed.

The implementation of the Solution Procedure is largely a straightforward numerical problem. Still, there are several numerical subtleties and these, along with a concise summary of the relevant relations are given in Sec. 4.3.E and F. Before proceeding however, we present an important practical generalization of the procedure in Sec. 4.3.D.

D. Generalization when \( \psi \) and \( B_t \) are Measured on Different Surfaces

In many practical situations the poloidal flux \( \psi \), and the tangential magnetic field \( B_t \) are not measured at the same poloidal angle or on the same surface. For example, in a typical application the flux probes are located outside the vacuum chamber while the magnetic field probes are on the inside. A further complication is that the vacuum
chamber often carries significant currents, particularly during startup. These issues are addressed here and result in straightforward extensions to the Solution Procedure.

The situation of interest is illustrated in Figure 4.2a. The vacuum chamber has a finite thickness. \( B_t \) and \( \psi_b \) are assumed measured on the inner and outer surfaces \( S_1 \) and \( S_0 \) respectively. The goal of the analysis is to obtain an analytic solution for \( \psi \) in the region lying between \( S_0 \) and \( S_1 \). In particular, once \( \psi \) is known on \( S_1 \), the problem reduces to one in which the Solution Procedure can be directly applied.

To obtain analytic solutions, two important assumptions are made. First, the distance \( c \) between the surfaces \( S_0 \) and \( S_1 \) is assumed small compared to the average minor radius \( b \) of the vacuum chamber: \( c \ll b \). This essentially reduces the analysis to a local 1-D problem. Second, the quantities \( B_t \) and \( \psi_b \) are assumed to change slowly in time with respect to the magnetic diffusion time of the vacuum chamber. This allows a simple expansion solution for the diffusion equation.

The solution for \( \psi \) is obtained as follows. The flux function within the vacuum chamber wall satisfies \( \Delta^* \psi = -\mu_0 RJ_\phi \). In the limit \( c \ll b \), then \( \Delta^* \approx \partial^2 / \partial \rho^2 \) where \( \rho \) is physical distance measured perpendicular to the vacuum chamber. The origin is chosen such that the surface \( S_1 \) corresponds to \( \rho = 0 \), while \( S_0 \) corresponds to \( \rho = c \). Consequently, \( \psi \) satisfies

\[
\frac{\partial^2 \psi}{\partial \rho^2} \approx -\mu_0 RJ_\phi
\]  

(4.54)

The quantity \( J_\phi \) is found from Faraday's law using the relations \( \mathbf{E} = \gamma \mathbf{J} \) and \( \mathbf{B} = \nabla \times (\psi \mathbf{e}_\phi / R) \) where \( \gamma \) is the resistivity of the wall. A simple calculation gives

\[
RJ_\phi = -\frac{1}{\gamma} \frac{\partial \psi}{\partial t}
\]  

(4.55)

In deriving Eq. (4.55), a free integration function has been set to zero, corresponding to the requirement that \( J_\phi \) be non-zero only if the flux is changing; that is, there is no externally driven current in the vacuum chamber.

Combining Eqs. (4.54) and (4.55) yields the familiar diffusion equation

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\[
\frac{\partial \psi}{\partial t} = \frac{\eta}{\mu_0} \frac{\partial^2 \psi}{\partial \rho^2}
\]

(4.56)

The boundary conditions require

\[
\psi (c, v, t) = \psi_b (v, t)
\]

(4.57)

\[
\frac{\partial \psi}{\partial \rho} (0, v, t) = R_1 (v) B_t (v, t)
\]

(4.58)

where \( \mathbf{n} \cdot \nabla \psi \approx \partial \psi / \partial \rho \).

If \( \psi_b \) and \( B_t \) vary slowly in time, initial transients occurring during a wall diffusion time can be ignored. (The wall diffusion time is given by \( \tau_W = \mu_0 c^2 / \eta \).) The solution for \( \psi \) is then obtained by expanding

\[
\psi (\rho, v, t) = \psi_b (v, t) + \psi_1 (\rho, v, t) + \ldots
\]

(4.59)

with \( \psi_1 \ll \psi_b \). The flux \( \psi_1 \) satisfies

\[
\frac{\partial^2 \psi_1}{\partial \rho^2} = \frac{\mu_0}{\eta} \frac{\partial \psi_b}{\partial t}
\]

(4.60)

\[
\psi_1 (c, v, t) = 0
\]

\[
\frac{\partial \psi_1}{\partial \rho} (0, v, t) = R_1 B_t
\]

The solution for \( \psi \) is easily found and is given by

\[
\psi = \psi_b - \frac{\mu_0}{2\eta} \frac{(c^2 - \rho^2)}{2} \frac{\partial \psi_b}{\partial t} - R_1 B_t (c - \rho)
\]

(4.61)

The first correction term represents the effect of current diffusion in the wall while the second correction term represents the small change in flux from \( \rho = c \) to \( \rho = 0 \) because of the finite wall thickness.

From Eq. (4.61) it follows that \( \psi \) and \( \mathbf{n} \cdot \nabla \psi \) on the inner surface \( \rho = 0 \) are given by
\begin{equation}
\left. n \cdot \nabla \psi \right|_{S_1} = R_1 B_t \tag{4.62}
\end{equation}

\begin{equation}
\psi \left|_{S_1} = \psi_b - \frac{\mu_0 c^2}{2 \eta} \frac{\partial \psi_b}{\partial t} - R_1 B_t c \right. \tag{4.63}
\end{equation}

Equations (4.62) and (4.63) represent an equivalent set of boundary conditions on
$S_1$ permitting a direct application of the Solution Procedure. The only difference is that
$\psi (S_1)$ appearing in Eq. (4.12a) should be replaced by the more complete form given by
Eq. (4.63).

As a practical matter, when $c/b \ll 1$, the corrections to the boundary conditions
due to the chamber thickness $c$, and the wall diffusion current $J_\phi$ are each of the order
$c/b$. Even so, it is important to include these corrections because small errors in the
input can lead to large errors in the output, a consequence of the ill-posedness.

The last issue to be addressed involves the situation in which the $\psi$ and $B_t$ probes
are located at different poloidal angles as shown in Figure 4.2b. A mathematical represen-
tation of the problem is illustrated in Figure 4.3a. Shown here are curves of $R_1$ and
$Z_1$ as a function of $v$. Also shown are the locations of the non-overlapping flux loops
and magnetic field probes.

The issue is easily resolved by introducing two separate parametrizations of $R_1(v)$
and $Z_1(v)$ on $S_1$, one for $\psi$ and the other for $B_t$. As a specific example, assume that an
equally spaced grid in the angle $v$ is defined consisting of $N_p$ points where $N_p$ represents
the number of $B_t$ probes. For simplicity assume there are also $N_p$ flux loops. On the
surface $S_{1\psi}$, $R_{1\psi} (v_n)$ and $Z_{1\psi} (v_n)$ are chosen to correspond to the location of the flux
loops. Similarly on the surface $S_{1B}$, $R_{1B} (v_n)$ and $Z_{1B} (v_n)$ represent the location of the
$B_t$ probes. The surfaces $R_{1\psi}$, $Z_{1\psi}$, $R_{1B}$, $Z_{1B}$ are plotted vs $v$ in Figure 4.3b. Observe
that even though the $\psi$ and $B_t$ probes are located on the same physical surface $S_1$, they
appear on different surfaces in $v$-space.

The net result is that when evaluating $\hat{a}_m$, $\hat{b}_m$ from Eqs. (4.38)-(4.39), and $P_{\ell m}$,
$S_{\ell m}$, $T_{\ell m}$, $U_{\ell m}$ from Eqs. (4.49)-(4.52), one must use the appropriate representation of
$S_1$ (i.e. $S_{1\psi}$ or $S_{1B}$).
E. Numerical Issues

As stated previously, the implementation of the Solution Procedure is a relatively straightforward numerical problem. There are, however, several subtleties worth discussing involving (1) the consequences of symmetry, (2) the choice of natural angular coordinates, (3) the existence of logarithmic singularities in certain Green’s functions, and (4) the subtraction of the external coil currents from the Solution Procedure.

Consider first the effects of symmetry. If the experiment of interest has the flux loops, magnetic probes and trial surfaces possessing up-down symmetry then all of the matrices are purely real: A, B, C, D, P, S, T and U. This is true even when the plasma itself does not possess such symmetry, as for instance in a system with a single null poloidal divertor. On the other hand, the input Fourier coefficients \( \hat{a}_m \) and \( \hat{b}_m \) are purely real only if the plasma and corresponding input probe data have up-down symmetry.

The second issue involves the natural choice of the angle \( v \) on each surface of interest. Since the numerical procedure makes extensive use of fast Fourier transform techniques, it is convenient for the information to be specified on an equi-spaced mesh in \( v \). Thus, if there are \( N_g \) grid points describing each surface of interest, a mesh \( v \) is established as follows:

\[
v_n = \frac{2\pi n}{N_g}, \quad 0 \leq n \leq N_g - 1. \tag{4.64}
\]

When calculating the Fourier coefficients of the input data, \( \hat{a}_n \) and \( \hat{b}_n \), we use a similar equi-spaced mesh in \( v \) with \( N_g \) replaced by \( N_p \), the number of flux probes and magnetic loops, so that

\[
R_1(v_n) = \hat{R}_n \tag{4.65}
\]

\[
Z_1(v_n) = \hat{Z}_n
\]

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where \((\hat{R}_n, \hat{Z}_n)\) are the actual coordinates of the \(n'\)th probe. Since \(N_p\) is usually not very large, good accuracy in evaluating the elements of \(W\) and \(V\) requires \(N_p \gg N_p\). We usually arrange the probes on the measurement surface in such a way that the \(N_p\) locations lie on a subset of the \(N_g\) grid points so as to ensure that the two parametrizations of the measurement surface are numerically identical at the probe locations.

In general, one can also easily find natural coordinates for the interior surfaces, although these depend upon the specific shape of the surfaces. For example, in the application considered here the interior surfaces are chosen as a set of similar ellipses. Consequently, the natural parametrization of the \(j\)th interior surface can be written as

\[
R_j(v_n) = R_0 + a_j \cos v_n
\]

\[
Z_j(v_n) = b_j \sin v_n
\]

(4.66)

where \((a_j, b_j)\) are the width and height of the surface under consideration.

The third issue concerns the evaluation of \(A, C, S, \) and \(U\), the matrices in which the observation surface coincides with the integration surface. Each of the corresponding Green's functions possesses an integrable logarithmic singularity when \(v' = v\). While this represents acceptable analytic behavior it leads to problems of numerical accuracy. This practical difficulty is avoided by adding and subtracting an appropriate function to each Green's function, thereby allowing the logarithmic singularity to be integrated analytically. The modified Green's functions are given as follows.

\[
\tilde{H}_{11}(v', v) = H_{11} - \frac{R_1}{4\pi} \ln \sin^2 \left(\frac{v' - v}{2}\right)
\]

\[
\tilde{H}_{22}(v', v) = H_{22} - \frac{R_2}{4\pi} \ln \sin^2 \left(\frac{v' - v}{2}\right)
\]

\[
\frac{1}{R_1^l} \frac{\partial \tilde{H}_{11}(v', v)}{\partial n_1^l} = \frac{1}{R_1^l} \frac{\partial H_{11}}{\partial n_1^l} - \frac{\hat{Z}_1}{8\pi R_1} \ln \sin^2 \left(\frac{v' - v}{2}\right)
\]

(4.67)
\[
\frac{1}{R'_2} \frac{\partial \tilde{H}_{22}}{\partial n'_2}(v', v) = \frac{1}{R'_2} \frac{\partial \tilde{H}_{22}}{\partial n'_2} - \frac{\dot{Z}_2}{8\pi R_2} \ln \sin^2 \left( \frac{v' - v}{2} \right)
\]

When the observation and integration points coincide (i.e., \( v' = v \)) the logarithmic singularities cancel and each of the modified Green's functions remains finite:

\[
\tilde{H}_{11}(v, v) = \frac{R_1}{4\pi} \left( \ln \frac{\dot{R}_1^2 + \dot{Z}_1^2}{16R_1^2} + 4 \right)
\]

\[
\tilde{H}_{22}(v, v) = \frac{R_2}{4\pi} \left( \ln \frac{\dot{R}_2^2 + \dot{Z}_2^2}{16R_2^2} + 4 \right)
\]

\[
\frac{1}{R'_1} \frac{\partial \tilde{H}_{11}}{\partial n'_1}(v, v) = \frac{1}{4\pi} \left[ \frac{\dot{R}_1 \ddot{Z}_1 - \dot{Z}_1 \ddot{R}_1}{\dot{R}_1^2 + \dot{Z}_1^2} + \frac{\dot{Z}_1}{2R_1} \left( 2 + \ln \frac{\dot{R}_1^2 + \dot{Z}_1^2}{16R_1^2} \right) \right]
\]

\[
\frac{1}{R'_2} \frac{\partial \tilde{H}_{22}}{\partial n'_2}(v, v) = \frac{1}{4\pi} \left[ \frac{\dot{R}_2 \ddot{Z}_2 - \dot{Z}_2 \ddot{R}_2}{\dot{R}_2^2 + \dot{Z}_2^2} + \frac{\dot{Z}_2}{2R_2} \left( 2 + \ln \frac{\dot{R}_2^2 + \dot{Z}_2^2}{16R_2^2} \right) \right]
\]

Using these relations, the matrices \( A, C, S, \) and \( U \) are given by

\[
A_{\ell'} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} \left( \frac{1}{R'_2} \frac{\partial \tilde{H}_{22}}{\partial n'_2} \right) e^{i\ell'v' - i\ell'v} dv' dv + \frac{F_{\ell'}}{2\pi} \int_0^{2\pi} \frac{\dot{Z}_2}{R_2} e^{i(\ell' - \ell)v} dv
\]

\[
S_{\ell m} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} \left( \frac{1}{R'_1} \frac{\partial \tilde{H}_{11}}{\partial n'_1} \right) e^{imv' - i\ell'v} dv' dv + \frac{F_m}{2\pi} \int_0^{2\pi} \frac{\dot{Z}_1}{R_1} e^{i(m - \ell)v} dv
\]

\[
C_{\ell'} = -\frac{\ell'}{\pi} \int_0^{2\pi} \int_0^{2\pi} \tilde{H}_{22} e^{i\ell'v' - i\ell'v} dv' dv - \frac{\ell' F_{\ell'}}{\pi} \int_0^{2\pi} R_2 e^{i(\ell' - \ell)v} dv
\]

\[
U_{\ell m} = -\frac{\ell'}{\pi} \int_0^{2\pi} \int_0^{2\pi} \tilde{H}_{11} e^{imv' - i\ell'v} dv' dv - \frac{m F_m}{\pi} \int_0^{2\pi} R_1 e^{i(m - \ell)v} dv
\]

where

\[
F_{\ell'} = \frac{1}{4\pi} \int_0^{2\pi} \ln \sin^2 \left( \frac{v' - v}{2} \right) e^{i\ell'(v' - v)} dv' = \begin{cases} -\ln 2 & \ell' = 0 \\ -\frac{1}{2|\ell'|} & \ell' \neq 0 \end{cases}
\]

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In this form the matrices can be evaluated in a straightforward manner. Finally, note that no numerical problems arise in the evaluation of $B$, $D$, $P$ and $T$ since the observation and integration points lie on different surfaces and hence do not overlap.

The fourth and final numerical issue involves the subtraction of the exterior coil currents prior to the application of the Solution Procedure. This task is readily accomplished by writing the total flux $\hat{\psi}$ as follows:

$$\hat{\psi} (R, Z, t) = \psi (R, Z, t) + \sum_j \mu_0 I_j (t) \Psi_j (R, Z)$$

(4.71)

Here

$$\Psi_j (R, Z) = \frac{(RR_j)^{1/2}}{2\pi} \left[ \frac{(2 - k^2) K (k) - 2E (k)}{k} \right]$$

(4.72)

$$k^2 = \frac{4RR_j}{(R + R_j)^2 + (Z - Z_j)^2}$$

and $(R_j, Z_j)$ are the coordinates of the $j$'th filament. The sum over $j$ includes the currents in the exterior equilibrium field coils and the ohmic transformer. It also includes the net plasma current which is assumed located at any arbitrary interior point, for example $R = R_0, Z = 0$. If there are current carrying conductors internal to the vacuum chamber, their currents must be included in the sum. For each of these currents, the corresponding $I_j (t)$ is assumed known. Equation (4.72) indicates that the contribution to the flux from each current is approximated as that due to a thin circular filament located at $(R_j, Z_j)$.

The task of the Solution Procedure is to accurately calculate the residual contribution to the flux, $\psi$. Since $\psi$ still satisfies $\Delta^* \psi = 0$, all the previous analysis remains valid. The only difference is that the external current contributions must be subtracted from the boundary data. Consequently, if $\psi_b$ and $B_b$ are the experimentally measured data, the appropriate boundary conditions for the Solution Procedure become
\[ n \cdot \nabla \psi \bigg|_{S_1} = R_1 B_t - \sum_j \mu_0 I_j n \cdot \nabla \psi_j \]

\[ \psi \bigg|_{S_1} = \psi_b - \frac{\mu_0 c^2}{2 \eta} \frac{\partial \psi_b}{\partial t} - R_1 B_t c \]

\[ - \sum_j \left[ \mu_0 I_j \psi_j - \frac{\mu_0^2 c^2}{2 \eta} \frac{dI_j}{dt} \psi_j - \mu_0 c I_j n \cdot \nabla \psi_j \right]. \tag{4.73} \]

Clearly, once \( \psi \) has been determined, the external coil contributions must be added back to obtain the full solution \( \dot{\psi} \).

F. Summary

For convenience a summary is presented of the basic relations required for the solution procedure.

Coupled integral equations:

\[ \frac{1}{2} \psi_2 + \int_{S_3} \left( \frac{\psi_2'}{R_2'} \frac{\partial H_{22}}{\partial n_2'} - \frac{H_{22}}{R_2'} \frac{\partial \psi_2'}{\partial n_2'} \right) dv' - \int_{S_1} \left( \frac{\psi_1'}{R_1'} \frac{\partial H_{12}}{\partial n_1'} - \frac{H_{12}}{R_1'} \frac{\partial \psi_1'}{\partial n_1'} \right) dv' = 0 \tag{4.74} \]

\[ \frac{1}{2} \psi_1 - \int_{S_1} \left( \frac{\psi_1'}{R_1'} \frac{\partial H_{11}}{\partial n_1'} - \frac{H_{11}}{R_1'} \frac{\partial \psi_1'}{\partial n_1'} \right) dv' + \int_{S_2} \left( \frac{\psi_2'}{R_2'} \frac{\partial H_{21}}{\partial n_2'} - \frac{H_{21}}{R_2'} \frac{\partial \psi_2'}{\partial n_2'} \right) dv' = 0 \tag{4.75} \]

Green's Functions:

\[ H_{ij} = -\frac{(R_i'R_j')^{1/2}}{2\pi} \left[ \frac{(2 - k^2) K - 2E}{k} \right] \]  

\[ \frac{1}{R_i'} \frac{\partial H_{ij}}{\partial n_i'} = \frac{1}{2\pi} \left( \frac{R_j'}{R_i'} \right)^{1/2} \left\{ \Lambda_{ij} \left[ \frac{(2 - k^2) E - 2(1 - k^2) K}{k} \right] + \Gamma_{ij} k (E - K) \right\} \tag{4.77} \]

\[ k^2 = \frac{4R_i'R_j}{(R_i' + R_j')^2 + (Z_i' - Z_j')^2} \tag{4.78} \]
\[
\Lambda_{ij} = \frac{\dot{Z}_i'(R_i' - R_j) - \dot{R}_i'(Z_i' - Z_j)}{(R_i' - R_j)^2 + (Z_i' - Z_j)^2}
\]

\[
\Gamma_{ij} = \frac{\dot{Z}_i'}{2R_i'}
\]

Fourier analysis:

\[
\psi_2 = \sum_{-L}^{L} a_{m} e^{i\nu}
\]

\[
\frac{1}{R_2} \frac{\partial \psi_2}{\partial n_2} = \sum_{-L}^{L} b_{m} e^{i\nu}
\]

\[
\psi_1 = \sum_{-M}^{M} \hat{a}_{m} e^{i\nu}
\]

\[
\frac{1}{R_1} \frac{\partial \psi_1}{\partial n_1} = \sum_{-M}^{M} \hat{b}_{m} e^{i\nu}
\]

Input Fourier coefficients:

\[
\hat{a}_{m} = \frac{1}{2\pi} \int_{0}^{2\pi} \left[ \psi_b - \frac{\mu_0 c^2}{2\eta} \frac{\partial \psi_b}{\partial t} - R_1 B_t c \\
+ \sum_{j} \left( \mu_0 I_j H_{1j} - \frac{\mu_0 c^2}{2\eta} \frac{d I_j}{d t} H_{1j} - \frac{\mu_0 c I_j}{Q_1} \frac{\partial H_{1j}}{\partial n_1} \right) \right] e^{-i\nu \sigma} d\nu
\]

Output Fourier coefficients:

\[
\begin{pmatrix}
\hat{a} \\
\hat{b}
\end{pmatrix} = \begin{pmatrix}
I + A & -C \\
- B & D
\end{pmatrix}^{-1} \begin{pmatrix}
P \\
I - S & U
\end{pmatrix} \begin{pmatrix}
P \\
- T
\end{pmatrix} \begin{pmatrix}
\hat{a} \\
\hat{b}
\end{pmatrix}
\]

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Matrix elements:

\[ B_{\ell\ell'} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} \left( \frac{1}{R_2'} \frac{\partial H_{21}}{\partial \eta_2'} \right) e^{i \ell' \psi' - i \ell \psi} d\psi d\psi' \]  \hspace{1cm} (4.87)

\[ P_{\ell m} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} \left( \frac{1}{R_1'} \frac{\partial H_{12}}{\partial \eta_1'} \right) e^{im\psi' - i \ell \psi} d\psi d\psi' \]  \hspace{1cm} (4.88)

\[ D_{\ell \ell'} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} H_{21} e^{i \ell' \psi' - i \ell \psi} d\psi d\psi' \]  \hspace{1cm} (4.89)

\[ T_{\ell m} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} H_{12} e^{im\psi' - i \ell \psi} d\psi d\psi' \]  \hspace{1cm} (4.90)

\[ A_{\ell\ell'} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} \left( \frac{1}{R_2'} \frac{\partial \tilde{H}_{22}}{\partial \eta_2'} \right) e^{i \ell' \psi' - i \ell \psi} d\psi d\psi' + \frac{F_{\ell'}}{2\pi} \int_0^{2\pi} \frac{\dot{Z}_2}{R_2} e^{i(\ell' - \ell) \psi} d\psi \]  \hspace{1cm} (4.91)

\[ S_{\ell m} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} \left( \frac{1}{R_1'} \frac{\partial \tilde{H}_{11}}{\partial \eta_1'} \right) e^{im\psi' - i \ell \psi} d\psi d\psi' + \frac{F_m}{2\pi} \int_0^{2\pi} \frac{\dot{Z}_1}{R_1} e^{i(m - \ell) \psi} d\psi \]  \hspace{1cm} (4.92)

\[ C_{\ell\ell'} = -\frac{\ell'}{\pi} \int_0^{2\pi} \int_0^{2\pi} \tilde{H}_{22} e^{i \ell' \psi' - i \ell \psi} d\psi d\psi' - \frac{\ell' F_{\ell'}}{\pi} \int_0^{2\pi} R_2 e^{i(\ell' - \ell) \psi} d\psi \]  \hspace{1cm} (4.93)

\[ U_{\ell m} = -\frac{m}{\pi} \int_0^{2\pi} \int_0^{2\pi} \tilde{H}_{11} e^{im\psi' - i \ell \psi} d\psi d\psi' - \frac{m F_m}{\pi} \int_0^{2\pi} R_1 e^{i(m - \ell) \psi} d\psi \]  \hspace{1cm} (4.94)

and

\[ F_{\ell'} = \begin{cases} -\ln 2 & \ell' = 0 \\ -\frac{1}{2|\ell'|} & \ell' \neq 0 \end{cases} \]  \hspace{1cm} (4.95)

Modified Green's functions for arbitrary \( \psi' \) and \( \psi \):

\[ \tilde{H}_{11}(\psi', \psi) = H_{11} - R_1 f \]  \hspace{1cm} (4.96)

\[ \tilde{H}_{22}(\psi', \psi) = H_{22} - R_2 f \]  \hspace{1cm} (4.97)

\[ \frac{1}{R_1'} \frac{\partial \tilde{H}_{11}}{\partial \eta_1'}(\psi', \psi) = \frac{1}{R_1'} \frac{\partial H_{11}}{\partial \eta_1'} - \frac{\dot{Z}_1}{2R_1} f \]  \hspace{1cm} (4.98)
\[
\frac{1}{R_2'} \frac{\partial \tilde{H}_{22}}{\partial n_2'} (v', v) = \frac{1}{R_2'} \frac{\partial H_{22}}{\partial n_2'} - \frac{\ddot{Z}_2}{2R_2} f
\] (4.99)

and

\[
f (v', v) = \frac{1}{2\pi} \ln \sin^2 \left( \frac{v' - v}{2} \right)
\] (4.100)

Modified Green's functions for \(v' = v\):

\[
\tilde{H}_{11} (v, v) = \frac{R_1}{4\pi} (g_1 + 4)
\] (4.101)

\[
\tilde{H}_{22} (v, v) = \frac{R_2}{4\pi} (g_2 + 4)
\] (4.102)

\[
\frac{1}{R_1'} \frac{\partial \tilde{H}_{11}}{\partial n_1'} (v, v) = \frac{1}{4\pi} \left[ h_1 + \frac{\ddot{Z}_1}{2R_1} (2 + g_1) \right]
\] (4.103)

\[
\frac{1}{R_2'} \frac{\partial \tilde{H}_{22}}{\partial n_2'} (v, v) = \frac{1}{4\pi} \left[ h_2 + \frac{\ddot{Z}_2}{2R_2} (2 + g_2) \right]
\] (4.104)

and

\[
g_i (v) = \ln \frac{\dot{R}_i^2 + \dot{Z}_i^2}{16R_i^2}
\] (4.105)

\[
h_i (v) = \frac{\dot{R}_i \dot{Z}_i - \dot{Z}_i \dot{R}_i}{\dot{R}_i^2 + \dot{Z}_i^2}
\] (4.106)

Full solution on interior surface \(k\):

\[
\dot{\psi} = \psi - \sum_j \mu_0 I_j \tilde{H}_{kj}
\] (4.107)

\[
B_i = \frac{1}{R_k Q_k} \frac{\partial \psi}{\partial n_k} - \sum_j \frac{\mu_0 I_j}{R_k Q_k} \frac{\partial H_{kj}}{\partial n_k}
\] (4.108)

This completes the formulation of the Solution Procedure.
4.4 Results and Discussion

A Test Case

A numerical code has been written to test the accuracy of the Green's Function Solution Procedure. Comparisons with analytic large aspect ratio, circular cross-section plasmas and vacuum chambers have been made. In all such comparisons, the agreement between the analytic and the numerically computed flux function is excellent, with the magnitude of maximum local deviation less than 1 part in 10^5.

As a more realistic test, we consider a model that provides an accurate representation of the poloidal field in the Alcator C-Mod experiment, viz. a finite aspect ratio \((R_0/a = 3)\), elongated \((\kappa = 1.7)\) plasma with a separatrix generated by divertor coils. In order to have an analytic model with which to compare, we choose a set of circular current filaments each carrying current \(I_j\) to represent the plasma, the equilibrium field coils and the ohmic transformer. The analytic flux and field at any arbitrary point \((R_i, Z_i)\) can then be written as

\[
\psi(R_i, Z_i) = -\sum_j \mu_0 I_j H_{ij}, \tag{4.109}
\]

\[
B_t(R_i, Z_i) = -\sum_j \frac{\mu_0 I_j}{R_i Q_i} \frac{\partial H_{ij}}{\partial n_i}, \tag{4.110}
\]

where the reduced Green's function \(H_{ij}\) is given by equation (4.30) and the modulus of the Elliptic functions appearing in \(H_{ij}\) is

\[
k^2 = \frac{4R_i R_j}{(R_i + R_j)^2 + (Z_i - Z_j)^2}, \tag{4.111}
\]

\((R_j, Z_j)\) being the location of the \(j\)th current filament. The current filament geometry and the corresponding analytic flux contours for typical Alcator C-Mod parameters are
illustrated in Figure 4.4. This represents quite a realistic model upon which we shall test the solution procedure†.

B Computational Surfaces and Error Definition

In applying the procedure we must first identify the measurement surface $S_1$ and the interior surfaces $S_k$, $k \geq 2$. For simplicity we choose the measurement surface to be an ellipse of elongation $\kappa = 2.255$ approximating the Alcator C-Mod vacuum chamber. An elliptic shape was chosen for numerical convenience, but it does not constitute a critical assumption, since the exact vacuum chamber shape can easily be incorporated. For the interior surfaces we choose a set of 9 ellipses of various elongations and radii filling the "vacuum" region, the innermost one having a surface area 0.241 times that of the measurement surface. The computational geometry is illustrated in Figure 4.5, superimposed over the current filaments that represent the plasma, the equilibrium field coils and the ohmic transformer.

The angle $v$ used for parametrizing each surface, and the corresponding $[R_j(v), Z_j(v)]$, are chosen in accordance with equation (4.66). Also, no attempt has been made to improve accuracy by optimizing the spacing of the probes on the measurement surfaces, which are located at equal $v$ intervals as shown in Figure 4.5.

Consider now the evaluation of the error. Two different definitions are introduced, a local error as a function of the poloidal angle $v$, and a Root Mean Squared (RMS) error characterizing the average error on each interior surface. Furthermore, separate errors are required for the flux and the tangential magnetic field. The local errors on any given surface are defined as

$$\epsilon_\psi(v) = \frac{2\pi}{R_0 \mu_0 I_p} |\psi_n(v) - \psi_e(v)|,$$

$$\epsilon_B(v) = \frac{2\pi}{\mu_0 I_p} \left| \frac{1}{R} \frac{\partial \psi_n}{\partial n}(v) - \frac{1}{R} \frac{\partial \psi_e}{\partial n}(v) \right|,$$  

(4.112)

† D. Humphreys and S. Wolfe at MIT Plasma Fusion Center helped us obtain the filament model.
where the subscript \( n \) refers to the numerically computed solution and \( a \) to the exact analytic result. Note that the errors are normalized to the plasma current \( I_p \), which determines the characteristic size of \( \psi \) and \( B_t \). In this sense the magnitudes of \( \epsilon_{\psi} \) and \( \epsilon_B \) provide a reasonable, absolute measure of the accuracy of the solutions.

As a relative measure of the accuracy, it is useful to compare \( \epsilon_{\psi} \) and \( \epsilon_B \) with the errors in the boundary data, \( \hat{\epsilon}_{\psi} \) and \( \hat{\epsilon}_B \), which are also defined in accordance with equation (4.112), and are calculated using the difference between the "exact" measured boundary data and the reconstructed data from the truncated Fourier series. The latter serve as an input to the Solution Procedure. Clearly, one can at best expect \( \epsilon_{\psi} \approx \hat{\epsilon}_{\psi} \) and \( \epsilon_B \approx \hat{\epsilon}_B \). When \( \epsilon_{\psi} \gg \hat{\epsilon}_{\psi} \) or \( \epsilon_B \gg \hat{\epsilon}_B \), we have an indication of the fact that large errors have developed as a result of ill-posedness.

It is also useful to define RMS errors for each surface to facilitate simultaneous comparisons with many different cases. These are defined in the usual manner:

\[
< \epsilon_{\psi} > = \frac{1}{N_g} \left[ \sum_{k=1}^{N_g} \epsilon_{\psi}^2(u_k) \right]^{1/2},
\]

\[
< \epsilon_B > = \frac{1}{N_g} \left[ \sum_{k=1}^{N_g} \epsilon_B^2(u_k) \right]^{1/2},
\]

where \( u_k = 2(k - 1)\pi/N_g \) and \( N_g \) is the number of grid points.

C Numerical Computation for the Test Case

As a first test of the Solution Procedure we have numerically computed \( \psi \) and \( \partial \psi / \partial n \) for the analytic model just described. In carrying out the calculation we assume that there are \( N_p = 25 \) magnetic probes and flux loops on the measurement surface. The number of grid points is taken to be \( N_g = 100 \). Also, each of the Fourier series appearing in the analysis is truncated to \( L = M = 5 \) harmonics. The measurement data for \( \hat{\psi} \) and \( n \cdot \nabla \hat{\psi} \) on \( S_1 \) is taken exactly from the analytic solution. It is also assumed for simplicity that the magnetic probes and flux loops are overlapping so that the thickness of the vacuum chamber vanishes, \( c = 0 \). For the results presented in this section, all the
external currents as well as the net plasma current represented by a filament at $R = R_0$, $Z = 0$ have been initially subtracted from the solution and then added back at the end.

The first quantity of interest is the accuracy of the input data. Illustrated in Figures 4.6a and 4.6b are curves of $\psi_a(\nu)$, $\psi_n(\nu)$ and $\dot{\epsilon}_\psi(\nu)$ on $S_1$. Similar sets of curves for the tangential field at the surface are shown in Figures 4.6c and 4.6d. The Fourier series approximation to the input data is quite accurate for $L = M = 5$ harmonics and $N_p = 25$ probes, with the RMS errors for the input data being $< \dot{\epsilon}_\psi > = 0.02\%$ and $< \dot{\epsilon}_B > = 0.2\%$.

Consider now the results of applying the Solution Procedure. Illustrated in Figures 4.7a and 4.7b are the flux contours from the numerically computed solution and the analytic solution respectively. To the resolution of the printer, they are identical. A more quantitative comparison between the measurement surface and the computation surfaces is shown in Figure 4.8, where we have plotted curves of $< \epsilon_\psi >$ and $< \epsilon_B >$ as functions of $x_j$, which is defined as the ratio between the cross-sectional areas enclosed by the computation surface $S_j$ and the measurement surface $S_1$:

$$x_j = \frac{A_j}{A_1}. \quad (4.115)$$

Note that the RMS errors are quite small, comparable to the errors in the input data; there is no degradation in accuracy due to ill-posedness. The accuracy, in fact, increases when moving towards the plasma centre. This is because, typically, fewer harmonics are required to accurately represent the flux and magnetic field near the plasma centre than at the vacuum chamber wall. We conclude, therefore, that in general it is not appropriate to use the same number of harmonics for each computation surface — a smaller number should be used for inner ones than for those close to the measurement surface.

**D Effect of Subtracting External Currents**

It has been previously stated that the accuracy of the Solution Procedure is significantly increased by first subtracting the contribution to the flux and field from all the known external currents and then performing the calculation on the residual flux.
function. At the end, one adds back the contributions from the external currents to obtain the full flux function. This point is illustrated in Figures 4.9–4.11.

The calculated flux and the corresponding analytic flux are shown in Figure 4.9, when the contribution from external currents was retained in the expressions for the input data. Computationally, these results are obtained by setting each \( I_j = 0 \) in equation (4.73). The case illustrated here was computed using \( N_p = 25 \) and \( L = M = 5 \) corresponding to the same resolution in the input data as in the previous section. However, finite differences in the computed and analytic flux contours can now easily be seen, implying considerably increased errors.

The increase in the errors is due to a combination of poor convergence of the Fourier series for the input data and the natural ill-posedness of the problem. Figure 4.10 demonstrates the poor convergence, where the exact and truncated Fourier series representations of the input data are shown for \( \psi \) and \( \partial \psi / \partial n \) on surface \( S_1 \). The convergence deteriorates because of the influence of the nearby external currents, which add significantly to the high harmonic content in the input data. Furthermore, the accuracy of the truncated Fourier series representation is reduced relative to that of the previous section for the same number of harmonics. Indeed, the RMS errors increase by a factor of 10 to \( < \varepsilon_\psi > = 0.2\% \) and \( < \varepsilon_B > = 3\% \) for the input data.

The effect of the ill-posedness is shown in Figure 4.11. Illustrated are the RMS errors as functions of computation surface \( S_J \). As expected, the magnitude of the errors is considerably higher when the external currents are not subtracted from the calculation, a result of the poor convergence using the same number of harmonics. The increase in the RMS errors as \( x_J \) decreases (i.e. moving away from the measurement surface \( S_1 \)) is an indication of errors in the higher harmonics in the input data. This leads to an increasing error away from the measurement surface as a result of ill-posedness.
E  Effect of Varying the Number of Probes

Having shown that the Solution Procedure is capable of producing quite accurate answers for the flux function in the vacuum region, we now investigate the effect of increasing the number of probes \( N_p \) while holding the number of harmonics \( M = L \) constant. In accordance to the results of the previous sections, we subtract the contributions from the known external currents and perform the computation on the residual flux function. Intuitively, we expect the critical parameter to be \( N_p/M \), corresponding to the number of measurements available for resolving the highest harmonic used in the computation. Illustrated in Figure 4.12a are curves of \( <\epsilon_\psi> \) as function of \( N_p/M \) for several values of \( M \), the number of harmonics, for a typical computation surface \( S_j \). The maximum value of \( M \) used in this study is chosen sufficiently low so that the effects of ill-posedness are not important.

We observe that the accuracy of the solution increases (the error decreases) as the number of harmonics \( M \) is increased. This is to be expected before the onset of divergence due to ill-posedness. The interesting feature in Figure 4.12a is the fact that for all values of \( M \) the accuracy of the solution saturates when \( N_p/M \) exceeds the value 4. Intuitively, once a sufficient number of measurements is available for resolving the highest harmonic, no additional increases in accuracy result from further increase to \( N_p/M \). From an experimental point of view, this is an important conclusion: there is no need to increase the number of probes beyond \( 5M \) where \( M \) represents the highest significant harmonic content of the measured data as a function of \( v \) with the known external currents subtracted out. For this reason we set \( N_p/M = 5 \) for the cases studied below.

F  Effect of Varying the Number of Harmonics

The formulation of the Solution Procedure has not assumed that the number of harmonics used for the input data \( M \) equals the number used for the output functions \( L \). It is then natural to investigate whether \( M = L \) is always the best choice, and if not, how the two can be best determined. In order to do this, we carry out the Solution
Procedure for different values of $L$ at fixed $M$ and $N_p/M = 5$. The optimal $L_o$ at a given $M$ is then chosen as the one that corresponds to the smallest value of $< \epsilon_\psi >$ for the surface $S_j$ in question. A plot of $< \epsilon_\psi >$ versus $L$ is shown in Figure 4.12b for surface $S_8$. Observe the existence of the optimum $L_o$ representing the competition between convergence and ill-posedness.

Consider next the variation of $L_o$ with $M$ for surfaces $S_4$ and $S_8$ as shown in Figures 4.13a and 4.13b, respectively. It is evident that some saturation in the optimal $L_o$ is observed for surface $S_8$. This can be attributed to the ill-posedness of the problem as discussed before, and a full analysis to determine the optimal $L_o$ will be given after the next section. The conclusion from this section is that in general the choice $L = M$ appears to give the highest accuracy over almost the entire parameter range of interest. Furthermore, accuracy continues to improve as the value of $M$ is increased. This is a consequence of the fact that the “exact” boundary data from the analytic solution is used as an input to the numerical solution. Numerical noise in the data is so low that modern computers provide accurate answers, even with 15 harmonics present. We shall see in the next section, that when random errors are deliberately introduced into the measurement data (representing the actual experimental situation), there is an optimum $M_o$ for each surface; values of $L \geq 8$ invariably lead to large errors due to ill-posedness, even when the random errors are relatively small, approximately 1%. Usually it is not necessary to have a value of $M$ larger than the largest value of $L$.

G Effect of Random Errors in Measurements

As just stated, in order to more effectively simulate a realistic experimental situation, we introduce random errors into the given boundary data. These might correspond to errors in the measured values of the magnetic fields or currents due to uncertainties in the diagnostic equipment, e.g. orientation of the probes, calibration procedures etc. Specifically, we assume that the locations of the equilibrium field coils, the ohmic transformer and the diagnostic probes are accurately known, while the quantities measured during the experiment, the magnetic flux and fields as well as the currents in the plasma
and external coils, contain random errors. These are introduced in our model by changing the values of the "measured" flux and field on the boundary \( S_1 \) (obtained from the analytic filament model) by a maximum of \( \varepsilon_M \) so that each measurement, \( \psi_j \) say, is changed to \( \psi_j(1 + \varepsilon_M \tau_j) \), where \( \tau_j \) is a uniformly distributed random number in the interval \([-1, 1]\). Similarly, currents in the equilibrium coils and the ohmic transformer \( I_j \) as well as the total plasma current \( I_p \) are changed to \( I_j(1 + \varepsilon_M \tau_j) \) and \( I_p(1 + \varepsilon_M \tau_j) \) respectively. Several values of \( \varepsilon_M \) have been studied up to \( \varepsilon_M = 0.05 \). The results given below correspond to the case \( \varepsilon_M = 0.01 \), although qualitatively, the same conclusions apply to all values of \( \varepsilon_M \) investigated.

Figures 4.14a and 4.14b illustrate the numerically computed and analytic flux contours respectively, when 1% random errors are present in all measurements. The calculation is performed with \( N_p = 25 \) and \( M = L = 3 \). The two flux plots are very close to identical. Figure 4.15a shows the RMS error \( < \psi > \) as a function of the measurement surface \( S_j \). It can be seen that for \( x_j > 0.4 \) the errors in the calculated values are approximately the same as the errors in the input data, which is the best accuracy that we can expect to obtain. For \( x_j < 0.4 \) the effects of ill-posedness become apparent. Figure 4.15b shows, for comparison with the boundary values given before, the calculated and analytic flux as a function of \( v \) for the innermost surface in the calculation, \( S_9 \). The approximation is seen to be reasonable.

We next investigate the effect of varying \( L \), the number of output harmonics, for given values of \( M \), the number of input harmonics. The optimal \( L_o \) is chosen as before to correspond to the smallest values of \( < \psi > \) for the given surface. The optimal \( L_o \) is shown in Figures 4.16a and 4.16b as a function of \( M \) for the same two surfaces \( S_4 \) and \( S_8 \) as in the previous section.

There are several important points to note. First, on a given surface, \( S_8 \) say, the optimal \( L_o \) equals \( M \) for sufficiently small values of \( M \leq 3 \). For \( M \geq 4 \), there is a rapid saturation in accuracy. Even with very high resolution of the input data, characterized by \( M = 13 \), \( N_p/M = 5 \), the optimal number of output harmonics remains fixed at \( L_o = 3 \), a dramatic demonstration of the consequences of ill-posedness. Clearly, from
an experimental point of view, the optimal choice of parameters for surface $S_b$ is $L_o = 3$, $M_o = 3$, $N_p/M_o = 5$.

The second point to note is that on any surface $S_j$, there is a similar optimal set of parameters $L_o = M_o$, $N_p/M_o = 5$ with, however, the critical $L_o$ varying from surface to surface. Surfaces close to the measurement surface are characterized by higher values of $L_o$. This is a consequence of the fact that the effects of ill-posedness increase, not only with harmonic number, but also with distance from the measurement surface.

Finally, observe that the formulation of the Solution Procedure is such that the number of output harmonics can be varied from surface to surface. Therefore, in a maximally efficient experimental implementation, one chooses $M_o$ from a surface relatively close to the measurement surface. For instance, on $S_3$ the saturation in accuracy occurs at $M_o = 6$. To resolve the highest harmonic we require $N_p/M_o = 5$. Therefore, $N_p = 30$ represents the optimal number of measurement probes. The value of $L_o$ is chosen as $L_o = M_o = 6$ for surfaces close to the measurement surface $S_1$. As one moves inward, away from $S_1$, $M$ remains fixed at $M = M_o = 6$ (there is no reason not to use every piece of available data) but the value of $L_o$ is gradually decreased because of ill-posedness so that, on surfaces $S_8$ and $S_9$, $L_o = 3$. This simple procedure leads to the optimal choice of experimental parameters.

**H Error Analysis**

The fact that an optimal $L_o$ exists is a result of the competition between ill-posedness in the formulation and poor resolution due to truncation. The former would require a small number of harmonics, since the highest ones diverge fastest, while the latter would require a large number of harmonics to maintain accuracy in the Fourier series representation. Indeed, the optimal $L_o$ is expected and observed to be a function of the error level in the measurements as well as a function of the distance between the calculation and measurement surfaces.

A simple model based on the right circular cylinder of Section 4.1 demonstrates both of these effects as well as predicting the saturation of $L_o$ with $M$ for large $M$. 

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Consider the problem
\[ \nabla^2 \phi = 0, \quad (4.116) \]
with boundary conditions
\[ \phi(r = a) = \sum_{n=0}^{M} (\phi_n + \varepsilon_n) \cos n\theta, \quad (4.117) \]
\[ \frac{\partial \phi}{\partial r}(r = a) = \frac{1}{a} \sum_{n=0}^{M} n(\phi_n - \varepsilon_n) \cos n\theta. \quad (4.118) \]
The analytic solution corresponding to these boundary conditions is given by
\[ \phi(r, \theta) = \sum_{n=0}^{M} \left[ \phi_n \left( \frac{r}{a} \right)^n + \varepsilon_n \left( \frac{a}{r} \right)^n \right] \cos n\theta. \quad (4.119) \]
The coefficients \( \phi_n \) represent the "exact" answer, while the coefficients \( \varepsilon_n \) represent the "error" in the boundary data which renders the solution singular at the center of the cylinder. The correct solution with no errors that is regular at the origin, is of course, written as
\[ \phi_R = \sum_{n=0}^{M} \phi_n \left( \frac{r}{a} \right)^n \cos n\theta. \quad (4.120) \]
Assume now, in analogy with the Solution Procedure, that the solution given by equation (4.119) is truncated using \( L \leq M \) harmonics. This solution, denoted by \( \phi_T \), has the form
\[ \phi_T = \sum_{n=0}^{L} \left[ \phi_n \left( \frac{r}{a} \right)^n + \varepsilon_n \left( \frac{a}{r} \right)^n \right] \cos n\theta. \quad (4.121) \]
The difference between \( \phi_T \) and \( \phi_R \) is a measure of the error between the calculated and the exact solutions. Writing \( \Delta \phi = \phi_T - \phi_R \) we obtain
\[ \Delta \phi(r, \theta) = \sum_{n=0}^{L} \varepsilon_n \left( \frac{a}{r} \right)^n \cos n\theta - \sum_{L+1}^{M} \phi_n \left( \frac{r}{a} \right)^n \cos n\theta. \quad (4.122) \]
Note that the last term in equation (4.122) is non-zero for \( L \leq M - 1 \) and vanishes for \( L = M \). Defining the RMS error as usual by
\[ < \varepsilon^2(r) > = \left[ \frac{1}{2\pi} \int_0^{2\pi} d\theta (\Delta \phi)^2 \right] / \phi_M^2, \quad (4.123) \]
where $\phi_M = \max_\theta |\phi(a, \theta)|$, yields

$$2 < \epsilon^2 > = \sum_{n=0}^{L} \left( \frac{\epsilon_n}{\phi_M} \right)^2 \left( \frac{a}{r} \right)^{2n} + \sum_{L+1}^{M} \left( \frac{\phi_n}{\phi_M} \right)^2 \left( \frac{r}{a} \right)^{2n}.$$  \hspace{1cm} (4.124)

Equation (4.124) can be qualitatively generalized to non-circular cross-sections by replacing $r^2/a^2$ by $x_j = A_j/A_1$, the ratio of the cross-sectional areas of the calculation and measurement surfaces. The first term in equation (4.124) represents the errors due to ill-posedness while the second term represents the errors resulting from the truncation of the Fourier series.

To proceed further we need knowledge of the $n$-dependences of $\epsilon_n$ and $\phi_n$. As a simple model that is also quite realistic in comparison with the experimental situation we assume

$$\frac{\epsilon_n}{\phi_M} = \frac{\epsilon_M}{\sqrt{3}},$$  \hspace{1cm} (4.125)

$$\frac{\phi_n}{\phi_M} = Ae^{-\alpha n}.$$  \hspace{1cm} (4.126)

Equation (4.125) indicates that the error in the data is independent of $n$, and the standard deviation of a uniform random distribution has been used to denote the magnitude of the error. In practice, typical experimental errors in the measurements lead to a slowly decreasing dependence of $\epsilon_n$ on $n$. Assuming that the errors in the measurements are random, we have

$$\epsilon_n = \frac{1}{2\pi} \int_0^{2\pi} \phi(a, \theta) \epsilon_M r_i \cos n \theta d\theta,$$  \hspace{1cm} (4.127)

where $r_i$ is a random number in the interval $[-1, 1]$. Since $\phi \epsilon_M r_i$ is of bounded variation, we can use Riemann's Lemma$^{[11]}$ to estimate the magnitude of $\epsilon_n$:

$$\epsilon_n \sim \frac{\phi_M \epsilon_m r_i}{n}.$$  \hspace{1cm} (4.128)

Taking the standard deviation of a uniform random distribution as a measure of the magnitude of $r_i$, we finally obtain

$$\frac{\epsilon_n}{\phi_M} \sim \frac{\epsilon_M}{\sqrt{3n}}.$$  \hspace{1cm} (4.129)
However, the exponential dependence associated with \((a/r)^{2n}\) dominates the behaviour so that treating \(\epsilon_n\) as a constant is a reasonable approximation and greatly simplifies the analysis.

The assumption \(\phi_n/\phi_M = Ae^{-\alpha n}\) is an accurate approximation to typical Alcator C-MOD data using an elliptically shaped measurement surface. For the test case the parameter \(\alpha \approx 0.5\), and the amplitude \(A \approx 1\). Under more general conditions, one still expects \(\phi_n\) to be a rapidly decreasing function of \(n\) for large \(n\), and the choice of an exponential decay with a free fitting parameter \(\alpha\) leads to a considerable simplification of the analysis.

Equations (4.125) and (4.126) are substituted into equation (4.124) and the summations carried out. The result is

\[
2 < \epsilon^2 > = \frac{\epsilon_M^2}{3x_j^L} \left[ \frac{1 - x_j^{L+1}}{1 - x_j} \right] + A^2(\lambda x_j)^{L+1} \left[ \frac{1 - (\lambda x_j)^{M-L}}{1 - \lambda x_j} \right], \tag{4.130}
\]

where \(\lambda = e^{-2\alpha}\). The RMS error \(<\epsilon^2>^{1/2}\) is sketched in Figure 4.17a. Note that there are two possibilities. In case (a) the error has a minimum for a value \(L_o < M\). A straightforward calculation yields

\[
L_o = \ln \left[ \frac{\epsilon_M^2}{3A^2} \frac{(1 - \lambda x_j) \ln x_j}{(1 - x_j) \lambda x_j \ln \lambda x_j} \right] / \ln (\lambda x_j^2). \tag{4.131}
\]

For case (b) the minimum in \(<\epsilon^2>\) occurs for a value \(L > M\), violating the condition for validity \(L \leq M\). In this case, the minimum allowable value of \(<\epsilon^2>\) occurs on the boundary and

\[
L_o = M. \tag{4.132}
\]

A plot of \(L_o\) versus \(M\) is illustrated in Figure 4.17b. Observe the similarities between Figures 4.17b and 4.16. The transition value of \(M\) is given by \(M_o = L_o\), defined by equation (4.131). The optimal experimental design for a given surface corresponds to \(L = M = L_o\) and \(N_p/M = 5\). This leads to the minimum error with the fewest number of probes. A comparison between the observed \(L_o\) and that calculated using equation (4.131) is given for \(\alpha = 0.5\), \(\epsilon_M = 0.01\) in Table 4.1. The agreement is seen to be good.
for a large range of values of \( x_j \). The variation of \( L_o \) with \( \varepsilon_M \) is illustrated in Table 4.2 for the case \( x_j = 0.441, \alpha = 0.5 \). As expected, larger \( \varepsilon_M \) require smaller values of \( L \) to minimize the effects of ill-posedness. The simple model thus explains all the qualitative features obtained from the full numerical studies.

4.5 Conclusion

We have formulated a Green’s function method for reconstructing the poloidal flux surfaces in the vacuum region of a tokamak based on magnetic measurements on a boundary surface at the vacuum vessel wall. The method is very fast to use in a practical situation, and the accuracy is comparable to the accuracy of the input data. We have also addressed the issue of the number of probes required for sufficiently accurate input data, and based on our analysis can conclude that in most cases no more than 30 probes are necessary.

4.6 Acknowledgments

We are indebted to Prof. Ian Hutchinson for useful discussions and suggestions throughout the course of this work and are grateful to Dr. Bas Braams of Princeton Plasma Physics Laboratory for many valuable comments. We also acknowledge the help and comments given to us by Dr. Robert Granetz, who critically reviewed our analysis in detail. We also thank Vonya Perham for helping us type parts of the manuscript and Gerasimos Tinios for writing a substantial portion of the computer code.
4.7 References


Figure 4.1 A typical experimental geometry.
Figure 4.2a Vacuum chamber has finite thickness, c.

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Figure 4.17b The optimal number of output harmonics $L_0$ versus the number of input harmonics $M$ for $x_j = 0.591$ with $\varepsilon_M = 1\%$. 

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<table>
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<th>$x_j$</th>
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<th>Observed $L_o$</th>
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<td>0.241</td>
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</tr>
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</tr>
<tr>
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<td>5</td>
</tr>
<tr>
<td>0.441</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
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</tr>
<tr>
<td>0.591</td>
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<td>6</td>
</tr>
<tr>
<td>0.686</td>
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<td>7</td>
</tr>
<tr>
<td>0.784</td>
<td>6</td>
<td>7</td>
</tr>
</tbody>
</table>

**Table 4.1** The optimal $L_o$ as calculated from equation (4.131) (expected $L_o$) and the corresponding observed $L_o$ for $\varepsilon_M = 1\%$ are shown for each calculation surface.

<table>
<thead>
<tr>
<th>$\varepsilon_M$ %</th>
<th>Expected $L_o$</th>
<th>Observed $L_o$</th>
</tr>
</thead>
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</tr>
<tr>
<td>5.0</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

**Table 4.2** The optimal $L_o$ as calculated from equation (4.131) (expected $L_o$) and the corresponding observed $L_o$ for $x_j = 0.441$ are shown for some values of $\varepsilon_M$. 

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Appendix A  Vector Green’s Theorem

This section contains a derivation of the Vector Green’s Theorem, a related but less well known form of the usual Green’s Theorem. This theorem is used for determining the interior flux surfaces, and its specific application is given in section 4.3.

The derivation begins with the vector identity

\[ \nabla \cdot (A \times \nabla \times G - G \times \nabla \times A) = G \cdot \nabla \times \nabla \times A - A \cdot \nabla \times \nabla \times G. \quad (4A.1) \]

Assume that \( A \) is the vector potential and the region of interest corresponds to a pure vacuum; for instance the region between the vacuum chamber and the last plasma flux surface to carry current. In such a region \( A \) satisfies

\[ \nabla \times \nabla \times A = 0. \quad (4A.2) \]

Assume that \( G \) is chosen as follows:

\[ G = \frac{C}{r}, \quad (4A.3) \]

where

\[ r = \left[ (x' - x)^2 + (y' - y)^2 + (z' - z)^2 \right]^{1/2}, \quad (4A.4) \]

and

\[ C = C_1e_x + C_2e_y + C_3e_z \quad (4A.5) \]

is an arbitrary constant vector.
One now integrates over some vacuum region and uses the divergence theorem. The result is

\[ \int A' \cdot \nabla' \times \nabla' \times G \, dx' = -\sum_j \int n' \cdot (A' \times \nabla' \times G - G \times \nabla' \times A') \, dS'_j. \quad (4A.6) \]

In Eq. (4A.6) prime denotes the integration variable and the unprimed quantities in G denote the observation point. The notation \( \sum' \) denotes the fact that the observation point \((x, y, z)\) is excluded from the domain of integration and is surrounded by a sphere (or hemisphere) if it lies inside (or on) the vacuum region. A contribution from the surface of this sphere (or hemisphere) is included in the right hand side.

The next step is to simplify the individual terms appearing in Eq. (4A.6). The following relations are helpful in this connection. First, since \( C \) is a constant vector

\[ \nabla \times G = -C \times \nabla \frac{1}{r}. \quad (4A.7) \]

Second, since \( \nabla^2(1/r) = 0 \) for \( r \neq 0 \)

\[ \nabla \times \nabla \times G = \nabla \left( C \cdot \nabla \frac{1}{r} \right). \quad (4A.8) \]

Using Eq. (4A.7) one can easily show that

\[ n \cdot A \times \nabla \times G = C \cdot (n \times A) \times \nabla \frac{1}{r}, \quad (4A.9) \]

\[ n \cdot G \times \nabla \times A = -\frac{C \cdot n \times \nabla \times A}{r}. \quad (4A.10) \]

If one now uses the gauge condition \( \nabla \cdot A = 0 \) then Eq. (4A.6) can be rewritten as

\[ C \cdot \sum_j \int \left[ \left( n' \cdot A' \right) \nabla' \frac{1}{r} + (n' \times A') \times \nabla' \frac{1}{r} + \frac{n' \times \nabla' \times A'}{r} \right] \, dS'_j = 0. \quad (4A.11) \]

Since \( C \) is an arbitrary vector,
\[ \sum_{j=1}^{3} \int \left[ \left( n' \cdot A' \right) \nabla' \frac{1}{r} + \left( n' \times A' \right) \times \nabla' \frac{1}{r} + \frac{n' \times \nabla' \times A'}{r} \right] dS'_j = 0. \] (4A.12)

The last step in the derivation is to evaluate the contribution over the observation point analytically in the limit where the sphere (or hemisphere) surrounding it shrinks to zero. In a small sphere (or hemisphere) surrounding the observation point

\[ \nabla \frac{1}{r} = -\frac{e_r}{r^2}, \] (4A.13)

\[ n = -e_r. \]

The contribution to Eq. (4A.12) from the observation point is given by

\[ I_o \equiv \int \left[ \left( \frac{e_r \cdot A'}{r^2} \right) e_r + \left( \frac{e_r \times A'}{r^2} \right) e_r - \frac{e_r \times \nabla' \times A'}{r} \right] dS'_o, \] (4A.14)

and \( I_o \) is easily evaluated by noting that

\[ e_r \times (e_r \times A') = (e_r \cdot A') e_r - A', \] (4A.15)

and that in the limit \( r \to 0 \)

\[ \int Q dS'_o = \lim_{r \to 0} \left( \sigma 4\pi r^2 Q \right). \] (4A.16)

This allows us to write the Vector Green's Theorem in the form

\[ \sigma A(x) = -\frac{1}{4\pi} \sum_j \int \left[ \left( n' \cdot A' \right) \nabla' \frac{1}{r} + \left( n' \times A' \right) \times \nabla' \frac{1}{r} + \frac{n' \times \nabla' \times A'}{r} \right] dS'_j. \] (4A.17)

In equations (4A.16) and (4A.17)
\[ \sigma = \begin{cases} 
1 & \text{interior observation point,} \\
1/2 & \text{surface observation point,} \\
0 & \text{exterior observation point,}
\end{cases} \quad (4A.18) \]

and \( \sum_j \) denotes summation over only the boundary surfaces.