UNIQUE CONTINUATION THEOREMS
for the DIRAC OPERATOR
and the LAPLACE OPERATOR

by

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Unique Continuation Theorems
for the Dirac Operator
and the Laplace Operator

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ABSTRACT

Let \( u \) be a solution of the following differential operator \( L \) in some non-empty connected open subset in \( \mathbb{R}^n \):

\[
L = \Delta + \sum a_j \partial / \partial x_j + b, \quad \text{where}
\]

\( a_j \in L^r_{\text{loc}}(\mathbb{R}^n), \ b \in L^s_{\text{loc}}(\mathbb{R}^n) \) for some suitable \( r, s \). \( L \) is said to satisfy the strong unique continuation property if solutions to \( Lu = 0 \) that vanish at a point to infinite order, vanish identically. One way of proving this property is to prove Carleman type inequalities. The first inequality we are going to consider is the following.

\[
\|e^{t \phi} \nabla f\|_{L^2(U \setminus \{0\}, dx)} \leq C \|e^{t \phi} \Delta f\|_{L^p(U \setminus \{0\}, dx)} \quad \text{for all} \ f \in C^\infty_0(U \setminus \{0\}; C^m),
\]

where \( \phi \) is a suitable weight function, which is a radially decreasing convex function. And \( U \) is an open neighborhood of the origin. The second type of inequality we are going to consider is about the Laplacian.

\[
\|e^{t \phi} u\|_{L^q(U \setminus \{0\}, dx)} \leq C \|e^{t \phi} \Delta f\|_{L^p(U \setminus \{0\}, dx)} \quad \text{for all} \ f \in C^\infty_0(U \setminus \{0\}, C^m)
\]

We are proving this for one special \( p \).
Introduction

We are going to consider unique continuation theorems for the following differential operator $L$:

$$L = \Delta + \sum a_j \partial/\partial x_j + b$$  \hspace{1cm} (1)

or

$$L = D + V$$  \hspace{1cm} (1')

Here $\Delta$ is the Laplace operator and $D$ is the Dirac operator, $a_j \in L^r_{\text{loc}}(R^n), b \in L^s_{\text{loc}}(R^n)$ for some suitable $r, s$. $V$ is a $m \times m$ matrix-valued potential with entries acting on functions with values in $C^m$.

$L$ is said to satisfy the strong unique continuation property if solutions to $Lu = 0$ that vanish at a point to infinite order, vanish identically.

On the other hand, $L$ is said to satisfy the unique continuation property if solutions to $Lu = 0$ that vanish in a non-empty, open subset of a connected set, vanish identically.

Also $L$ is said to satisfy the weak unique continuation property if solutions to $Lu = 0$ that are compactly supported vanish identically. One way of proving this property is to prove Carleman type estimates.

When $n = 2$, $a_j = 0$, for all $j$, and $b$ is bounded, Carleman [8] proved a unique continuation theorem, and all subsequent work follows his basic idea.

The first inequality we are going to consider is the following.

$$\|e^{t\phi}f\|_{L^2(U, dx)} \leq C\|e^{t\phi}Df\|_{L^p(U, dx)} \hspace{1cm} \text{for all } f \in C_0^\infty(U; C^m)$$  \hspace{1cm} (2)

for $U$ an open neighborhood of the origin, where $\phi$ is a suitable weight function.

For instance, weight function $\phi(x_n)$ implies weak unique continuation. If $\phi$ peaks on a hypersurface, we get unique continuation. On the other hand, if $\phi$ peaks at a single point we get strong unique continuation.

Then (2) implies an inequality concerns to the gradient.

$$\|e^{t\phi}f\|_{L^2(U, dx)} \leq C\|e^{t\phi}\Delta f\|_{L^p(U, dx)} \hspace{1cm} \text{for all } f \in C_0^\infty(U; C^m)$$  \hspace{1cm} (2')

The second type of inequality we are going to consider is about the Laplacian.

$$\|e^{t\phi}u\|_{L^2(U, dx)} \leq C\|e^{t\phi}\Delta u\|_{L^p(U, dx)}$$  \hspace{1cm} (3)

The above inequality (2) implies strong unique continuation result for $L = D + V$.

Let $\Omega$ be a connected, open subset of $R^n$.

If $V \in L^r(\Omega; M(m, C))$ and $u$ satisfies $(D + V)u = 0$ in $\Omega$, $Du \in L^2(\Omega; C^m)$.

If $u$ vanishes in a non-empty open subset of $\Omega$, then $u \equiv 0$ in $\Omega$. On the other hand the two inequalities (2'), (3) imply strong unique continuation result for the Laplacian, which put together first and zero order perturbations.

Our main result is the following.
Theorem
Let \(0 \in U \subset \mathbb{R}^n\) be an connected open set, \(u \in H^{2, p}_{loc}\) for some \(p\), \(a_j \in L^p_{loc}(\mathbb{R}^n)\), for all \(j\), \(b \in L^p_{loc}(\mathbb{R}^n)\), \(r = (3n - 2)/2\), \(s > n/2\) and \(u\) satisfies the equation
\[
\Delta u(x) + \sum a_j(x) \frac{\partial u}{\partial x_j} + b(x) u(x) = 0
\]
Then if \(u\) vanishes to infinite order at the origin, \(u\) must be zero on \(U\).

Historically, (2') was proved by Hörmander [1] with a \(C^\infty\) weight function, peaking on a smooth hypersurface. This gives unique continuation not strong unique continuation. Our main contribution is to extend his result to strong unique continuation. This question is explicitly posed in [1](p. ).

Later, Jerison [2] proved the best possible Carleman-type inequalities for (2) and obtained unique continuation property for \(L = D + V, V \in L^\gamma_{loc}(\mathbb{R}^n)\) for \(\gamma = \frac{3n-2}{3}\). Also Jerison and Kenig [3] proved strong unique continuation for solution of (1) with \(a_j = 0, b \in L^{n/2}\). On the other hand, when \(p, q\) are not dual exponents, Barcelo, Ruiz, Kenig, Sogge [4] proved weak unique continuation with exactly the hypothesis on the coefficients in our theorem.

Now we want to make some comments on the weight function.
First, for the weak unique continuation, it is enough to have \(\phi\) as a function of one variable e.g. \(\phi(x) = \phi(x_n) = x_n + x_n^2/2\), \(x = (x', x_n) \in \mathbb{R}^n\) and the solution \(u\) (\(Lu = 0\)) vanishes on the upper half space i.e. on \(x_n > 0\). Then \(\phi(x_n)\) has peaks on the hyperplane \(x_n = 0\).
For the unique continuation, it is enough to have a radial function i.e. \(\phi(r)\) for \(0 < a < b\) and \(u\) has support on a compact interval i.e. \(u \in C^\infty_0([a, b])\). Then \(\phi\) has peaks on some sphere.
Finally, for the strong unique continuation, \(\phi\) is also a radial function and support of \(u\) is non-compact i.e. \(u \in C^\infty_0(0 < r < b)\) and as \(r \to 0\), \(\phi\) is allowed to blow up at a single point.

**Statements of Results**

**Theorem 1**
Let \(n \geq 3\), let \(\phi(x)\) be a smooth real valued convex radial function, which does not vanish identically.

There is a constant \(C\) depending only on \(n\) such that for all \(t \in \mathbb{R}\), and for all \(h \in C^\infty_0((\mathbb{R}^n) \times S)\)
\[
\ell \|e^{4t}h\|_{L^2((\mathbb{R}^n) \times S, dydw)} \leq C \|e^{4t}Dh\|_{L^2((\mathbb{R}^n) \times S, dydw)}
\]

**Corollary 1**
Let \(U \ni 0\) be a connected, open subset of \(\mathbb{R}^n\).
Suppose we have a solution of a Schrödinger operator \((D + V)u = 0\) in \(U\), \(V \in L^\infty\) and \(\int_{|x| < r} |u(x)|^2 dx = 0(e^{4t})\) for any \(N\), then \(u \equiv 0\) on \(U\).

**Theorem 2**
Let \(n \geq 3\), \(\phi(x)\) same as before.
p = (6n-4)/(3n+2), i.e., 1/p - 1/2 = 1/γ, with γ = (3n - 2)/2
There is a constant C depending only on n such that for all t ∈ R

\[ \| e^{t\phi} f \|_{L^2((-\infty,0) \times S, dydw)} \leq C \| e^{t\phi} Df \|_{L^p((-\infty,0) \times S, dydw)} \text{ for all } f \in C_0^\infty((-\infty,0) \times S) \]  

(2')

**Corollary 2**
Let Ω be a connected, open subset of \( R^n \), \( n \geq 3 \).
If \( V \in L^1(\Omega; M(m, C)) \) and u satisfies \( Du \in L^2(\Omega; C^m) \), \( (D + V)u = 0 \) in \( \Omega \).
If u vanishes in a non-empty open subset of \( \Omega \), then u is identically zero in \( \Omega \).

**Corollary 2'**
Let the conditions be the same as above. The fact that \( D^2 = -\Delta \) implies the following inequality.

\[ \| e^{t\phi} \nabla f \|_{L^3((-\infty,0) \times S, dydw)} \leq C \| e^{t\phi} \Delta f \|_{L^p((-\infty,0) \times S, dydw)} \]  

(2'')

**Theorem 3**
Let \( n \geq 3 \) and \( \phi(x) \) as before.
Then there is a constant C depending only on n such that

\[ \| e^{t\phi} u \|_{L^q((-\infty,0) \times S, dydw)} \leq C \| e^{t\phi} \Delta u \|_{L^p((-\infty,0) \times S, dydw)} \text{ for all } f \in C_0^\infty(R^n \setminus 0) \]

and for \((1/q, 1/p)\) in the open triangle ABC (fig.1) with vertices A(1/2, 1/2),
\( B(n/(2n - 2), 1/q_b), C((n^2 + 2n - 4)/(2n - 2), 1/q_c) \) where

\[ \frac{n}{2(n - 1)} - \frac{1}{q_b} = \frac{n^2 + 2n - 4}{2(n - 1)} - \frac{1}{q_c} = \frac{2}{n}. \]

**Corollary 3**
Let \( 0 \in U \subset R^n \) be an open set, \( u \in H^{2,p}_{loc}(U) \), \( p = (6n-4)/(3n+2) \), \( u \) satisfies the equation

\[ \Delta u(x) + v(x) \nabla u(x) + w(x) u(x) = 0 \]  

(3')

where \( v \in L^r_{loc}(R^n) \), \( w \in L^s_{loc}(R^n) \), for \( r = (3n-2)/2, s > n/2 \) and

\[ \int_{\xi < |x| < r} |u(x)|^p dx = 0(r^N) \text{ for any } N, \]

\[ \int_{\xi < |x| < r} |\nabla u(x)|^p dx = 0(r^N) \text{ for any } N, \text{ then } u \equiv 0 \text{ in } U. \]
We will first prove these inequalities on $I \times S$, $I$ is some fixed interval in $\mathbb{R}^n$. Later on we can extend the results to $(-\infty, 0) \times S$ using partition of unity.

The key ingredient is a theorem of C. Sogge [5] giving $L^p$ to $L^q$ estimates for the projection operator onto spherical harmonics of a given degree.

Jerison [3] used a discrete restriction theorem for the Fourier transform and we will follow same methods in this paper.

And for the choice of weight function we will adopt the same one which Baouendi and Alinhac [5] used first, and Hörmander [1] used later for the convexity reason.

First, we want to set up some notations and elementary results, following Jerison [2].

**Polar coordinates**

Let $S$ denote the unit sphere in $\mathbb{R}^n$.

For $y \in R$, and $w \in S, x = e^w w$ gives polar coordinates on $\mathbb{R}^n$, i.e., $y = \log|x|$ and $w = x/|x|$.

The operator

$L = \sum_{j<k} \alpha_{jk} (x_j \partial / \partial x_k - x_k \partial / \partial x_j)$ acts only in the $w$-variables $[L, \partial / \partial y] = 0$.

We will view $L$ as an operator on the sphere $S$. Let

$$\hat{\alpha} = \sum_{j=1}^n \alpha_j x_j / |x|, \quad \text{then}$$

$$\hat{\alpha} D = e^{-w} (\partial / \partial y - L);$$

and since $\hat{\alpha}^2 = -1,$

$$e^w D = \hat{\alpha} (\partial / \partial y - L)$$

(4)
Note that $\hat{\alpha}$ is unitary and $L^* = L$. If we recall that

$$\Delta = e^{-2\nu}(\partial^2/\partial y^2 + (n - 2)\partial/\partial y + \Delta_S),$$

where $\Delta_S$ denotes the Laplace-Beltrami operator of the sphere. It follows from $D^* = D$, $D^2 = -\Delta$ that

$$L(L + n - 2) = -\Delta_S$$

(6)

Since $\phi(x) \sim -\log|x| = -y$, we want to give a rigorous definition i.e., Define $\psi \in C^\infty(R)$ implicitly as a new weight function given by Hörmander as $y = -\psi(y) + e^{-\psi(y)}$.

In general if $\psi \in C^\infty(R)$, then (4) implies that in polar coordinates $x = e^y u$,

$$e^{t\psi(y)}e^y De^{-t\psi(y)}h = \hat{\alpha}A_{t}h$$

(7)

where $A_t = \partial/\partial y - (t\psi'(y) + L)$.

**Proof of Theorem 1**

To prove the inequality, it suffices to compute $A_t^*A_t$ and show $A_t^*A_t \geq ct\psi''(y)$. Since this means,

$$(A_t^*A_t h, h) \geq (t\psi''(y)h, h) = t||\sqrt{\psi''(y)h}||_{L^2(dx)}^2$$

and

$$(A_t^*A_t h, h) = ||A_t h||_{L^2(dx)}^2.$$  

Combining these two implies that theorem 1 holds. But

$$A_t^*A_t = (-\partial/\partial y - n - (t\psi'(y) + L))(\partial/\partial y - (t\psi'(y) + L))$$

$$= -(\partial/\partial y + n/2)^2 + (t\psi'(y) + L + n/2)^2 + t\psi''(y) \geq t\psi''(y).$$

We had the relation $y = -\psi(y) + e^{-\psi(y)}$. From this, we get

$$\psi'(y) = -1/1 + e^{-\psi(y)} < 0,$$

which means $\psi(y)$ is a decreasing function of $y$. We also find

$$\psi''(y) = e^2 e^{-\psi(y)}/(1 + e^{-\psi(y)})^3 \geq e^y.$$
So the claim is true. Since $\psi(y)$ goes to $\infty$ as $y$ goes to $-\infty$, $\psi''(y)$ vanishes as $y$ goes to $-\infty$.

Now we want to show how such an inequality implies unique continuation property.

**Proof of Corollary 1**

First, introduce a cut off function $\eta \in C_0^\infty(R)$ such that $\eta = 1$ for $(-\infty, -T_0)$ for sufficiently large $T_0 \in R$, and $\eta = 0$ for $y > 0$.

Then by limiting process, (2) is equivalent to

$$t||e^{t\psi}(\eta u)||^2_{L^2((-\infty,0) \times S, ds)} \leq C ||e^{t\psi}D(\eta u)||^2_{L^2((-\infty,0) \times S, ds)}$$

since

$$D(\eta u) = (D\eta)u + \eta(Du) = (D\eta)u + \eta(-Vu),$$

this means that the right hand side

$$||e^{t\psi}D(\eta u)||^2_{L^2((-\infty,0) \times S, ds)} \leq C' ||e^{t\psi}u||^2_{L^2((-\infty,0) \times S, ds)}$$

since $||V||_\infty \leq B$.

This implies that as $t$ goes to $\infty$ $u$ is identically zero on the set where $\eta$ is 1.

Now we want to prove theorem 2.

**proof**

Since $A_t = e^{t\psi(y)}e^{y}De^{-t\psi(v)}$, where $A_t = \partial/\partial y - (t\psi(y) + L)$,

it suffices to prove the following inequality first and show later (2') by duality.

$$||f||_{L^2(e^{y}dydw, R-x S)} \leq C ||A_t f||_{L^2(e^{y}dydw, R-x S)} \quad \text{for } f \in C_0^\infty(U)$$

Let $\pi_k$ denote the projection of $L^2(S; C^m)$ onto $E_k$. We can rewrite

$$A_t f = \sum_k (\partial/\partial y - (t\psi(y) + k))\pi_k f.$$ 

If we have an operator of type $\partial/\partial y - ay + b$ for $a$, $b$ constant coefficients, and $a > 0$ then we can find a left inverse operator for $\partial/\partial y - ay + b$ easily.

So first consider an operator

$$\Omega = d/dy - y.$$ 

Jeron [2] exhibited the following exact formula for the symbol of a left inverse of $\Omega$:

There is a unique operator $B$ on $R$ satisfying $B\Omega = I$ and $B(e^{-y^2/2}) = 0$ given by

$$Bf(y) = (1/2\pi) \int F_0(y, \eta)e^{i\eta \cdot \hat{f}(\eta)}d\eta,$$

where

$$F_0(y, \eta) = \sqrt{2} \int_0^\infty e^{-y^2 - 2yv} dv - \int_0^\infty e^{-x^2 - 2x(y-in)} ds. \quad \text{(8)}$$
Now if we have an operator $\partial/\partial y - ay + b$ then

$$\sigma(y, \eta; a, b) = \frac{1}{\sqrt{a}} F_0(\sqrt{ay} \frac{b}{\sqrt{a}}, \frac{\eta}{\sqrt{a}})$$

is the symbol of the left inverse of $\partial/\partial y - ay + b$

Then by the method of freezing coefficient, we get an approximate symbol for the inverse of $\partial/\partial y - (t\psi(y) + k)$. Namely

$$F(y, \eta) = \sigma(y, \eta; t\psi''(y), -t\psi'(y) + t\psi''(y)y - k).$$

Also the following symbol estimate is true.

$$| (\partial/\partial y)^i (\partial/\partial \eta)^j F_0(y, \eta) | \leq C_j, (1 + |y + i\eta|)^{-1-j-l} j, l = 0, 1, \ldots \tag{9}$$

Now we want to have similar estimates for our symbol $F(y, \eta)$ using (9).

Claim

$$| (\partial/\partial y)^i (\partial/\partial \eta)^j F(y, \eta) | \leq C_j (\sqrt{a} + |t\psi(y) + k - i\eta|)^{-1-j-l}(a + |t\psi'(y) + k|)^l \tag{10}$$

Let's denote $A_j = \partial^i y'/\partial y^i, B_j = \partial^i \eta'/\partial y^i$. Then,

$$|(\partial/\partial \eta)^j (\partial/\partial y)^l F(y, \eta)| = |(\partial/\partial \eta)^j (\partial/\partial y)^l 1/\sqrt{a} F_0(y', \eta')|$$

$$= \sum_{k=0}^l (1/\sqrt{a})^j (\partial/\partial \eta)^j ((\partial/\partial y)^l 1/\sqrt{a}) [((\partial/\partial y)^l - k) F_0(y', \eta')],$$

since $\eta' = \eta/\sqrt{a}$.

Now we have to compute $(\partial/\partial y)^k (1/\sqrt{a})$, and $(\partial/\partial y)^l - k F_0(y', \eta')$. First,

$$\partial^k/\partial y^k F_0(y', \eta') = \sum_{j+i+s+t=k} C_{i,j} A_{j}^1 \ldots A_{j}^{s+t} B_{i}^1 \ldots B_{i}^{s+t} (\partial/\partial y)^j (\partial/\partial \eta)^l F_0$$

for

$$1 + 2 + \ldots + q_s = s, \quad 1 + 2 + \ldots + r_t = t \quad \text{and} \quad \tau_1 + \ldots + r_s = j, \quad l_1 + \ldots + l_t = i$$

Now from the relation $y = -\psi(y) + e^{-\psi(y)}$, $\epsilon > 0$ we get

$$\psi'(y) = \frac{1}{1 + \epsilon e^{-\psi(y)}} \quad \psi''(y) = \frac{\epsilon^2 e^{-\psi(y)}}{(1 + \epsilon e^{-\psi(y)})^3}$$

($^*$)

$$|\psi^{(k)}(y)| \leq C_k e^{-\psi(y)} \quad k \geq 2$$

Now since

$$a = t \psi''(y) \sim te^{-\psi(y)}, \quad 1/\sqrt{a} \sim t^{-1/2} e^{\frac{1}{2} \psi(y)}.$$ Therefore,

$$7$$
\[(**)
\]

\[|\partial / \partial y|^{k} \frac{1}{\sqrt{a}} \leq C_{k} t^{-1/2} e^{\epsilon^2 \psi(y)} \leq C' e^{k} \frac{1}{\sqrt{a}}.\]

We can use this estimate to get one for \(\partial / \partial y\).

\[(a) \partial y' / \partial y \sim \sum_{i=0}^{k-1} \partial / \partial y(1/\sqrt{a})(t^i \psi^{k-1-i}(y)) + \partial / \partial y(1/\sqrt{a})(t^i \psi(y) + \epsilon)(y) + \partial / \partial y(1/\sqrt{a})(t^i \psi(y) + \epsilon)^e\psi(y)) \leq C_{k} e^{(1/\sqrt{a})(t^i \psi(y) + \epsilon)} \leq C_{k} e^{(1/\sqrt{a})(t^i \psi(y) + \epsilon)} \leq C_{k} e^{(1/\sqrt{a})(t^i \psi(y) + k)}\]

Using (a) we can deduce that each

\[|A_{j}| \leq C(\sqrt{a} + \frac{1}{\sqrt{a}})|t^i \psi(y) + \epsilon)| \text{ for all } j\]

which implies

\[|A_{j}|^{n_{A_{j}}} \leq C(\sqrt{a} + \frac{1}{\sqrt{a}})|t^i \psi(y) + \epsilon)|^{n_{A_{j}}} \leq C(\sqrt{a} + \frac{1}{\sqrt{a}})|t^i \psi(y) + \epsilon)|^{n_{A_{j}}} \text{ for } i = n_{A_{j}} + \ldots + \eta_{n_{A_{j}}}.

|B_{j}^{l_{B_{j}}} ... B_{j}^{l_{B_{j}}} \leq C(\sqrt{a} + \frac{1}{\sqrt{a}})|t^i \psi(y) + \epsilon)|^{l_{B_{j}}} = C(\sqrt{a} + \frac{1}{\sqrt{a}})|t^i \psi(y) + \epsilon)|^{l_{B_{j}}} \text{ for } j = l_{B_{j}} + \ldots + l_{B_{j}},

the last inequality holds since

\[(\gamma) \frac{\partial \eta'}{\partial y} = \frac{\partial}{\partial y} (1/\sqrt{a}) \eta \leq \frac{c}{\sqrt{a}} \eta \text{ by } (**).

If we combine all the terms,

\[|\partial / \partial \eta|^{j}(\partial / \partial y) |t^i \psi(y, \eta')| = \sum_{k=0}^{l} (1/\sqrt{a})^{j}((\partial / \partial \eta)^{j}((\partial / \partial y)^{k}(1/\sqrt{a}))((\partial / \partial y)^{l-k} F_{0}(y', \eta'))

= \sum_{k=0}^{l} (1/\sqrt{a})^{j}((\partial / \partial y)^{k}(1/\sqrt{a}))((\partial / \partial \eta)^{j}((\partial / \partial y)^{l-k} F_{0}(y', \eta'))

= \sum_{k=0}^{l} C(1/\sqrt{a})^{j+1}((\partial / \partial \eta)^{j}((\partial / \partial y)^{l-k} F_{0}(y', \eta')).}
From (**) and (β) the above is bounded by

\[ C(1/\sqrt{a})^{j+1} \sum_{0 \leq k \leq a + \beta \leq 1 - k} C_{a,\beta} \left| \frac{1}{\sqrt{a}} (t \psi(y) + k) \right|^2 \left| \frac{1}{\sqrt{a}} \eta \right|^\beta \left| (\partial/\partial y)^{\alpha} (\partial/\partial \eta)^{\beta} F(y', \eta') \right| \]

Using (9) and other estimates above this is bounded by

\[ C_j(1/\sqrt{a})^{j+1} \sum_{\alpha, \beta \leq 1 - k, 0 \leq k \leq 1} C_{a,\beta} \left| \frac{1}{\sqrt{a}} (t \psi(y) + k) \right|^2 \left| \frac{1}{\sqrt{a}} \eta \right|^\beta (1 + \frac{1}{\sqrt{a}} |t \psi(y) + k - i \eta|)^{-1-j-\alpha-\beta} \]

\[ \leq C_j(1/\sqrt{a} + |t \psi(y) + k - i \eta|)^{-1-j-l}(a + |t \psi(y) + k|)^l \]

by cancellations.

The main tool in the proof is the spherical restriction theorem of C. Sogge.

**Theorem**

Let \( \xi_k \) denote the projection operator from \( L^2(S) \) to the space of spherical harmonics of degree \( k \). Then there is a constant \( c \) such that

\[ ||\xi_k g||_{L^p(S)} \leq c k^{1-2/n} ||g||_{L^p(S)} \quad \text{where} \]

\[ p = 2n/(n + 2), p' = 2n/(n - 2). \]

Formula (6) implies that

\[ (L + (n - 2)/2)^2 = -\Delta + (n - 2)^2/4. \]

Hence

\[ T = \text{sgn}(L + (n - 2)/2) = (L + (n - 2)/2)(-\Delta + (n - 2)^2/4)^{-1/2} \]

is a classical pseudodifferential operator on \( S \). Thus \( T \) is bounded from \( L^q(S; C^m) \) to \( L^{q'}(S; C^m) \) for all \( q, 1 < q < \infty \). Moreover,

\[ \pi_k = \frac{1}{2}(1 + T) \xi_k, \quad k = 0, 1, 2, \ldots \]

\[ \pi_k = \frac{1}{2}(1 - T) \xi_k, \quad k = 1 - n, -n, -n - 1, \ldots \]

Therefore, Sogge's theorem implies that

\[ ||\pi_k g||_{L^p(S; C^m)} \leq C k^{1-2/n} ||g||_{L^p(S; C^m)}. \]

Define \( \pi_{M,N} \) by

\[ \pi_k \pi_{M,N} g = \{ \pi_k g \text{ if } M \leq k \leq N, \ 0 \text{ otherwise.} \} \]
The triangle inequality implies

\[ \|π_M NG\|_{L^p(\mathcal{S};C^m)} \leq CN^{1-2/n}(N-M+1)\|g\|_{L^p(\mathcal{S};C^m)} \]

Next use a device due to P. Tomas:

\[ \|π_M NG\|_{L^2}^2 = \int_S \langle π_M NG, g \rangle \leq \|π_M NG\|_{L^p} \|g\|_{L^p} \leq CN^{1-2/n}(N-M+1)\|g\|_{L^p}^2. \]

We conclude that

\[ \|π_M NG\|_{L^2(\mathcal{S};C^m)} \leq CN^{1/p'}(N-M+1)^{1/2}\|g\|_{L^p(\mathcal{S};C^m)} \]

and by duality

\[ \|π_M NG\|_{L^{p'}(\mathcal{S};C^m)} \leq CN^{1/p'}(N-M+1)^{1/2}\|g\|_{L^2(\mathcal{S};C^m)} \]

If we interpolate with the trivial estimate

\[ \|π_M NG\|_{L^q(\mathcal{S};C^m)} \leq \|g\|_{L^2(\mathcal{S};C^m)} \]

we find that

\[ \|π_M NG\|_{L^q(\mathcal{S};C^m)} \leq C(N^{n/2}1)^{1/2-1/q/2}\|g\|_{L^2(\mathcal{S};C^m)} \]

for \(2 \leq q \leq p' = 2n/n - 2\).

Let \(N\) be the integer satisfying \(2^{N-1} \leq 10 \epsilon^{1/2} t^{1/2} \leq 2^N\).

Consider a partition of unity \(\{\phi_\beta\}_{\beta=0}^N\) of the positive real axis satisfying

\[ \sum_{\beta=0}^N \phi_\beta(r) = 1 \quad \text{all } r > 0 \]

\[ \text{supp } \phi_\beta \subset \{r : 2^{\beta-2} \leq r \leq 2^\beta\}, \beta = 1, 2, ... N - 1. \]

\[ \text{supp } \phi_0 \subset \{r : r \leq 1\}, \quad \text{supp } \phi_N \subset \{r : r \geq s/400\}, \]

\[ |(\partial/\partial r)^l \phi_\beta(r)| \leq C_l 2^{-\beta l}, \quad l = 0, 1, ... \]
Define \( F_t^\beta(y, \eta, k) = \phi_\beta(\frac{1}{\sqrt{a}} |t\psi'(y) + k - i\eta|) F_t(y, \eta, k) \) Then \( F_t \) satisfies
\[
|\partial/\partial \eta|^j (\partial/\partial y)^l F_t(y, \eta, k) | \leq C_{j,l}(\sqrt{a} + \|t\psi'(y) + k - i\eta\|)^{-1-j-l}(a + \|t\psi'(y) + k\|)^l
\]
(13)
\[
|\partial/\partial \eta|^j (F_t(y, \eta, k) - F_t(y, \eta, k + 1)) | \leq C_j(\sqrt{a} + \|t\psi'(y) + k - i\eta\|)^{-2-j}
\]
From (13) and the property of \( |\partial/\partial r|^l \phi_\beta(r) | \leq 2^{-\beta l} \), we deduce the following inequalities hold uniformly for \( y \in I = I_t = (-l, -l + 1) \).
\[
|\partial/\partial \eta|^j F_t^\beta(y, \eta, k) | \leq C_j(2^\beta \sqrt{a})^{-1-j}
\]
(14)
\[
|\partial/\partial \eta|^j (F_t^\beta(y, \eta, k) - F_t^\beta(y, \eta, k + 1)) | \leq C_j(2^\beta \sqrt{a})^{-2-j}
\]
Define \( (F_t^\beta f)(y, w) = \sum_k \frac{1}{2\pi} \int F_t^\beta(y, \eta, k) \pi_k \tilde{f}(\eta, \cdot)(w) e^{i\eta \cdot d\eta} \). Then \( F_t = \sum_{\beta=0}^N F_t^\beta \). We begin by estimating \( F_t^N \).
In the case \( \beta = N \), we need different estimates. By the choice of \( N \) such that \( 2^N \sim 10e^{i\sqrt{a}} \), we have the following.
Since \( F_t^N \) is supported where
\[
|t\psi'(y) + k - i\eta| \geq 2^N \sqrt{a} \sim 10t,
\]
we have
\[
|t\psi'(y) + k - i\eta| \geq c(1 + |\eta| + |k|) \text{ uniformly for } y < 0.
\]
Hence
\[
|\partial/\partial \eta|^j (\partial/\partial y)^m F_t^N(y, \eta, k) | \leq C_{j,m}(1 + |\eta| + |k|)^{-1-j-l} j = 0, 1, ...
\]
(15)
\( F_t^N \) is controlled by standard pseudodifferential operators.
In fact,
\[
e^{yD} F_t^N f = \sum_k \frac{1}{2\pi} \int r_t(y, \eta, k) \pi_k \tilde{f}(\eta, \cdot)(w) e^{i\eta \cdot d\eta}, \text{ where}
\]
\[
r_t(y, \eta, k) = \hat{\alpha}((i\eta - k) F_t^N + \partial/\partial y F_t^N).
\]
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It follows from (15) that
\[
|(\partial/\partial \eta)^j(\partial/\partial y)^m r_i(y, \eta, k)| \leq C_{j,m}(1 + |k| + |\eta|)^{-j-m}
\] (16).

In other words \(r_i\) is a classical symbol in the \((y, \eta)\) variables and a bounded multiplier in the \(k\) variable, and thus
\[
||e^y D F_i^N f||_{L^2(I \times S, dx)} \leq C ||f||_{L^2(I \times S, dx)} \quad I = I_l.
\]
Now by the Sobolev inequality, we get
\[
||f||_{L^p(I \times S, e^{\eta y} dy dw)} \leq ||r D f||_{L^2(I \times S, e^{\eta y} dy dw)}, \quad \text{for all } f \in C_0^\infty(I \times S),
\]
and \(p' = 2n/(n-2)\). Here \(r \sim e^{-y}\) in \(I_l\). So we have
\[
||F_i^N f||_{L^q(I \times S, dx)} \leq C ||e^{-y} D F_i^N f||_{L^2(I \times S, dx)}
\]
\[
\leq C' ||e^y D F_i^N f||_{L^2(I \times S, dx)}
\]
\[
\leq C'' ||f||_{L^2(I \times S, dx)} \quad \text{for all } f \in C_0^\infty(I \times S; C^m)
\]
for \(1 \leq q \leq p'\). In particular this holds for \(q = (6n-4)/(3n-6)\).

Let \(M = [-t \psi'(y) - 2^\beta \sqrt{\alpha}] , M' = [M + 22^\beta \sqrt{\alpha}] + 1\). Denote
\[
T_i^\beta(y, \eta) g(w) = \sum K F_i^\beta(y, \eta, k) \pi_k g(w)
\]
Here \(F_i^\beta(y, \eta, k) = 0\) unless \(M \leq k \leq M'\). A summation by parts gives
\[
T_i^\beta(y, \eta) = \sum_{M}^{M'} (F_i^\beta(y, \eta, k) - F_i^\beta(y, \eta, k + 1)) \pi_{M,k} \quad \text{for } M \leq k \leq M'
\]
Now (11) implies
\[
||\pi_{M,k} g||_{L^\infty(S, C^m)} \leq C(t^{-2\beta}(2^\beta \sqrt{\alpha})^{n/2})^{1/2-1/q} ||g||_{L^2(S, C^m)} \quad \text{for } \beta \leq N - 1
\]
Therefore, by (14)
\[
||(|\partial/\partial \eta)^jT_i^\beta(y, \eta) \pi_{M,k} g||_{L^\infty(S, C^m)} \leq C_j(2^\beta \sqrt{\alpha})^{-1-j}(t^{-2\beta}(2^\beta \sqrt{\alpha})^{n/2})^{1/2-1/q} ||g||_{L^2(S, C^m)}
\]
uniformly for \(y \in I\).
Define
\[
K_i^\beta(y, z) = \frac{1}{2\pi} \int T_i^\beta(y, \eta) e^{i\eta} d\eta
\]
\[
= \frac{1}{2\pi} \int (\partial/\partial \eta)^jT_i^\beta(y, \eta) e^{i\eta} d\eta
\]
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and since the length of the interval in $\eta$ where $T_\delta$ is non-zero is less than $2 \times 2^\delta \sqrt{\alpha}$,

$$
\|K_\delta(y, z)g\|_{L^q(S; C^m)} \leq C_j |2^\delta \sqrt{\alpha}z|^{-j} (t^2 \frac{\alpha}{2} (2^\delta \sqrt{\alpha})^{n/2})^{1/2 - 1/q} ||g||_{L^2(S; C^m)}
$$

Furthermore,

$$
\|K_\delta(y, z)g\|_{L^q(S; C^m)} \leq C(1 + |2^\delta \sqrt{\alpha}z|)^{10} (t^2 \frac{\alpha}{2} (2^\delta \sqrt{\alpha})^{n/2})^{1/2 - 1/q} ||g||_{L^2(S; C^m)}
$$

Note that

$$
F_\delta f(y, w) = \int K_\delta(y, y - y')f(y', \cdot)(w)dy'
$$

**Lemma**

Let $H(y, y')$ be a bounded operator from $L^p(S)$ to $L^q(S)$ of operator norm $\leq h(y - y')$ for each $y, y' \in R$. Suppose that $h \in L^r(R)$ for $1/r + 1/p = 1 + 1/q$. Then

$$
Tf(y, w) = \int H(y, y')f(y', \cdot)(w)dy' 	ext{ satisfies}
$$

$$
\|Tf\|_{L^q(R \times S)} \leq \|h\|_{L^r(R)} \|f\|_{L^p(R \times S)}.
$$

Note that

$$
||(1 + |2^\delta \sqrt{\alpha}z|^{10})||_{L^r(S)} \leq C(2^\delta \sqrt{\alpha})^{-1/r}
$$

we see that the Lemma implies that for $\beta \leq N - 1$,

$$
\|F_\delta f\|_{L^q(I, x, S, d\omega)} \leq C(2^\delta \sqrt{\alpha})^{-1/r} (t^{3n-4/4}2^{\beta/n/2} \sqrt{\psi''(y)}^{n/2})^{1/2 - 1/q} ||g||_{L^2(I, x, S, d\omega)}
$$

$$
= C2^{-(n-2)\beta/\gamma} t^{\frac{1}{2} - \frac{1}{4}} (\sqrt{\psi''(y)})^{-1/r} (\sqrt{\psi''(y)})^{n/2 \times 1/\gamma} ||g||_{L^2(I, x, S, d\omega)}
$$

$$
= C2^{-(n-2)\beta/\gamma} (\psi''(y))^{-1/2 + n/4\gamma} ||g||_{L^2(I, x, S, d\omega)}
$$

with $1/2 - 1/q = 1/\gamma$, $1/r + 1/2 = 1 + 1/q$, $\gamma = (3n - 2)/2$, $r = (3n - 2)/(3n - 4)$ and $m = -1/2 + n/4\gamma = -2n + 4/2(3n - 2) > -1/3$.

So the above is bounded by

$$
C2^{-(n-2)\beta/\gamma} (e^\nu)^m ||g||_{L^2(I, x, S, d\omega)}
$$

If we sum the series in $\beta$ and add the final term $\beta = N$, we get

$$
\sum_\beta F_\delta f \|_{L^q(I, x, S, d\omega)} \beta \leq C e^{i\beta/3} \|f\|_{L^2(I, x, S, d\omega)}
$$

(17)
Since we have an estimate only for the main term, we will work on the remainder term from now on. As a reminder, we defined

\[ F_t f(y) = \sum_k \int F_t(y, \eta, k) e^{i\nu \cdot \pi_k} \hat{f}(\eta) d\eta \]

\[ = \int \int F_t(y, \eta, k) e^{i(y-y')^n} f(y', \cdot) dy'd\eta \pi_k \quad \text{and} \]

\[ B_t(y', \eta, k) = \partial/\partial y' - a(y)y' + b(y) \]

is the operator whose inverse has symbol \( F_t(y, \eta, k) \) i.e.

\[ F_t B_t f(y) = \sum_k \int \int F_t(y, \eta, k) e^{i(y-y')^n} (\partial/\partial y' - a(y)y' + b(y)) f(y') dy'd\eta \]

\[ = \int \delta(y-y') e^{i(y-y')^n} f(y') dy' = f(y) \]

Then

\[ R_t f(y) = F_t A_t f(y) - Id f(y) = F_t A_t f(y) - F_t B_t f(y) \]

\[ = \sum_k \int \int F_t(y, \eta, k) \{(\partial/\partial y') - (\xi'(y') + k)\} - (\partial/\partial y' - a(y)y' + b(y)) \}

\[ \times e^{i(y-y')^n} f(y', \cdot)(\omega) dy'd\eta \pi_k \]

\[ = \int \int F_t(y, \eta, k) t(y-y')^2 g(y, y') e^{i(y-y')^n} f(y', \cdot)(\omega) dy'd\eta \pi_k \quad \text{for} \]

\[ g(y, y') = \int_0^1 (1 - s) \psi''(y)(y + s(y' - y)) ds \]

Since we want similar type of inequality, i.e.

\[ \| R_t f \|_{L^2(I \times S, d\omega)} \leq C_t f \|_{L^2(I \times S, d\omega)} \quad \text{in} \quad I_t = (-l, -l + 1) \]

we will adopt same techniques as in the main step by introducing \( \{\phi_\beta\}_{\beta=0}^M \) as partition of unity for \( R^+ \) and \( T^\phi_t, K^\phi_t \) having the same definitions as before. Then

\[ R^t \phi f(y) = \sum \int \int F_t^\phi(y, \eta, k) t(y-y')^2 \phi_{\nu} e^{i\nu \cdot \pi_k} g(y, y') f(y', \cdot)(\omega) dy'd\eta \pi_k \]

The kernel \( K^\phi_t \) was defined as

\[ K^\phi_t(y, z) = \frac{1}{2\pi} \int T^\phi_t(y, \eta) e^{i\nu \cdot \pi_k} d\eta. \]
Using this definition, we define

\[ \tilde{K}_1^\beta(y, y') = \frac{1}{2\pi} \int T_1^\beta(y, \eta) \eta(y - y')^2 e^{i(y - y')\eta} g(y, y') d\eta. \]

We know already when \( z = y - y' \),

\[ \|K_1^\beta(y, z)f\|_{L^1(S; \mathbb{C})} \leq C_j |2^\beta \sqrt{\alpha}z|^{-j} (t^{n-2/2} (2^\beta \sqrt{\alpha})^{n/2})^{1/2 - 1/4} \|f\|_{L^2(S; \mathbb{C})}. \]

From this we get

\[ \|\tilde{K}_1^\beta(y, y')f\|_{L^1(S; \mathbb{C})} \leq C_j |2^\beta \sqrt{\alpha}z|^{-j} |t| |z|^2 \|g\|_\infty (t^{n-2/2} (2^\beta \sqrt{\alpha})^{n/2})^{1/2 - 1/4} \|f\|_{L^2(S; \mathbb{C})}. \]

\[ = C_j |t(2^\beta \sqrt{\alpha})^{-2}\|g\|_\infty |2^\beta \sqrt{\alpha}z|^{-j+2} (t^{n-2/2} (2^\beta \sqrt{\alpha})^{n/2})^{1/2 - 1/4} \|f\|_{L^2(S; \mathbb{C})}. \]

Note that

\[ R^\beta f(y, w) = \int \tilde{K}_1^\beta(y, y')f(y', .)(w)dy'. \]

As we know \( a = t\psi''(y) \sim te^{-d} \) uniformly for \( y \in I_t \), and \( \|g\|_\infty \leq e^{-d} \) since \( y, y' \in I_t \). As a result, \( \|t(2^\beta \sqrt{\alpha})^{-2}\|g\|_\infty \leq C' \). Then using the same method it gives us

\[ \|R^\beta f\|_{L^1(I_t \times S; \mathbb{C})} \leq C' t^{-\beta(n-2/\gamma)} e^{d/3} \|f\|_{L^2(I_t \times S; \mathbb{C})} \quad \text{when} \quad \gamma = (3n - 2)/2 \quad (17') \]

Now consider the case \( \beta = N \). From the definition,

\[ (R^N_t f)(y, w) = \sum_k \frac{1}{2\pi} \int \int F_t^N(y, \eta, k) \pi_k f(y', .)(w)(y - y')^2 g(y, y') e^{i\eta(y - y')} dy' d\eta \]

\[ = \sum_k \frac{1}{2\pi} \int \int \tilde{F}_t^N(y, y', \eta, k) \pi_k f(y', .)(w) e^{i\eta(y - y')} dy' dw \]

when \( \tilde{F}_t^N(y, y', \eta, k) = (y - y')^2 g(y, y') F_t^N \)

Now we can claim this operator is controlled by standard pseudodifferential operators. In fact,

\[ e^{\nu D} \tilde{F}_t^N f = \sum_k \frac{1}{2\pi} \int \tilde{f}_t(y, y', \eta, k) \pi_k f(\eta, .)(w) e^{i\nu \eta} d\eta, \]

where \( \tilde{f}_t(y, y', \eta, k) = (y - y')^2 g(y, y') r_t(y, \eta, k) \)
We already have the estimate for $r_i(y, \eta, k)$ i.e.

$$\left| (\partial/\partial \eta)^j (\partial/\partial y)^m r_i(y, \eta, k) \right| \leq C_{j, m} (1 + |\eta| + |k|)^{-j-m}.$$ 

Since $y, y' \in I_i$,

$$(y - y')^2 g(y, y') \sim (y - y')^2 e^{-d}$$

and $|(\partial/\partial y)^{m_1} (\partial/\partial y')^{m_2} (y - y')^2 g(y, y')| \leq B$. So we get

$$\left| (\partial/\partial \eta)^j (\partial/\partial y)^{m_1} (\partial/\partial y')^{m_2} \tilde{r}_i(y, y', \eta, k) \right| \leq C_{j, m_1, m_2} (1 + |k| + |\eta|)^{-j-m_2}.$$ 

In other words $\tilde{r}_i$ is a classical symbol in the $(y, y', \eta)$ variables and a bounded multiplier in the $k$ variable. Now following the same method as in the main step and we can deduce

$$\| \tilde{F}_i \|_{L^2(I_i \times S, dx)} \leq C \| f \|_{L^2(I_i \times S, dx)}.$$ 

If we sum the series in $\beta$ and add the final term $\beta = N$, we get

$$\| R_i f \|_{L^2(I_i \times S, dx)} \leq C e^{i/2} \| f \|_{L^2(I_i \times S, dx)} \quad (15'')$$

Since we have an estimate on the unit annulus i.e. $I_i \times S$, we want to extend it to the whole ball, in this case $R^- \times S$.

In Theorem 1, we proved

$$\| f \|_{L^2(I \times S, d^{n+1}ydydw)} \leq C \| A_i f \|_{L^2(I \times S, e^{n+1}ydydw)}$$

for $f \in C_c^{\infty}(I \times S)$.

when $I = I_j$, this means

$$t \int_{I_j \times S} |f|^2 e^{(n+1)\eta dydw} \leq C \| A_i f \|_{L^2(I_j \times S, e^{n+1}ydydw)}.$$ 

Since $e^{\eta} \sim e^{-ij}$ in $I_j$, this is equivalent to

$$\sqrt{t} (\int_{I_j \times S} |f|^2 e^{-ij e^{n+1}ydydw})^{1/2} \leq C \| A_i f \|_{L^2(I_j \times S, e^{n+1}ydydw)}.$$ 

i.e.

$$\| f \|_{L^2(I_j \times S, e^{n+1}ydydw)} \leq C e^{ij/2} \| A_i f \|_{L^2(I_j \times S, e^{n+1}ydydw)}.$$ 

So (10) implies

$$\| R_i f \|_{L^2(I_j \times S, e^{n+1}ydydw)} \leq C e^{ij/8} \| A_i f \|_{L^2(I_j \times S, e^{n+1}ydydw)}.$$
Now from the relation \( f(y) = F_t A_t f(y) - R_t f(y) \), we get

\[
\|f\|_{L^q(I \times S, e^{\nu y} dy dw)} \leq \|F_t A_t f\|_{L^q(I \times S, e^{\nu y} dy dw)} + \|R_t F\|_{L^q(I \times S, e^{\nu y} dy dw)} \\
\leq e^{c j/3}\|A_t f\|_{L^q(I \times S, e^{\nu y} dy dw)} + e^{5 c j/6}\|A_t f\|_{L^q(I \times S, e^{\nu y} dy dw)} \\
\leq C' e^{5 c j/6}\|A_t f\|_{L^2(I \times S, e^{\nu y} dy dw)} \quad \text{which is}
\]

\[
\int_{I \times S} |f| e^{\nu y} dy dw \leq e^{c j/6} \left( \int_{I \times S} |A_t f|^2 e^{-c j} e^{\nu y} dy dw \right)^{1/2}.
\]

i.e.

\[
\int_{I \times S} |f|^2 e^{4 c y + \nu y} dy dw \leq (\int_{I \times S} |A_t f|^2 e^{(c+n) y} dy dw)^{1/2}
\]

We get from this

\[
\|f\|_{L^q(I \times S, e^{(c+n)+\nu} y dy dw)} \leq C'\|A_t f\|_{L^2(I \times S, e^{(n+c)+\nu} y dy dw)}.
\]

Now choose \( \{\psi_{jk}\}_{j \in N, k=1,2} \) be partition of unity such that

\[
\psi_{j1} \in C_0^\infty(-j, -j + 3/4), \quad \psi_{j2} \in C_0^\infty(-j + 2/4, -j + 5/4).
\]

Then using \( f = \sum \psi_j f \), (we will just call \( \{\psi_{jk}\} \) as \( \{\psi_j\} \)) and

\[
\|A_t(\psi_j f)\|_{L^2(I \times S)} = \|\psi_j^* f + \psi_j A_t f\|_{L^2(I \times S)} \\
\leq C\|\psi_j^* f\|_{L^2(I \times S)} + C\|\psi_j A_t f\|_{L^2(I \times S)}
\]

We come to the final estimate.

\[
\|f\|_{L^q(d(1+n)/2+n) y dy dw, R \times S)} = \|\sum_j \psi_j f\|_{L^q(d(1+n)/2+n) y dy dw, I \times S)} \\
\leq (1) C_0 \sum_j \|\psi_j f\|_{L^q(d(1+n)/2+n) y dy dw, I \times S)} \\
\leq C_1 \sum_j \|A_t(\psi_j f)\|_{L^2(d(1+n)+y dy dw, I \times S)} \\
\leq C_2 \sum \|\psi_j^* f\|_{L^2(d(1+n)+y dy dw, I \times S)} + C_2 \sum \|\psi_j A_t f\|_{L^2(d(1+n)+y dy dw, I \times S)} \\
\leq (2) C'\|f\|_{L^2(d(1+n)+y dy dw, R \times S)} + C'\|A_t f\|_{L^2(d(1+n)+y dy dw, R \times S)} \\
\leq (3) C''\|A_t f\|_{L^2(e^{y} y dy dw, R \times S)}
\]

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Inequality (1) and (2) hold since for each $x \in R$, only finitely many $\psi_j$'s overlap. (3) comes from $L^2$ estimates. So we get
\[
\|f\|_{L^2(e^{(1+\epsilon/3+n)}\nu_{dydw},R-xS)} \leq C \|A_tf\|_{L^2(e^{n\nu_{dydw},R-xS})}
\]
which is equivalent to
\[
\|e^{t\psi}f\|_{L^2(e^{(1+\epsilon/3+n)}\nu_{dydw},R-xS)} \leq C \|e^{t\psi}Df\|_{L^2(e^{n\nu_{dydw},R-xS})}
\]  
(18)
Now if we choose small $\epsilon$, then $\epsilon < e^{4\epsilon/3}$. So the above is equivalent to
\[
\|e^{t\psi}f\|_{L^p(R-xS,e^{n\nu_{dydw}})} \leq C \|e^{t\psi}Df\|_{L^p(R-xS,e^{n\nu_{dydw}})}
\]  
(18')
Now we want to find a dual version of this i.e.
\[
\|e^{t\psi}f\|_{L^2(R-xS,ds)} \leq C \|e^{t\psi}Df\|_{L^p(R-xS,ds)} \quad \text{for } p = q', f \in C_0^\infty(R^- \times S)
\]
To prove this we will follow a new policy i.e. instead of finding an approximate inverse for $A_t$, we want to find a left inverse operator $B_t$ satisfying $f(y) = B_tA_tf(y)$. If we assume $B_t$ has kernel $K_t$, then the above is equivalent to
\[
\pi_kf(y) = \int K_t(y,s)(\partial/\partial s - t\psi(s) + k)\pi_kf(s)ds
\]
\[
= \int(-\partial/\partial s - t\psi'(y) + k)K_t(y,s)\pi_kf(s)ds
\]
Let
\[
K_{1,t}(y,s) = H(y - s)e^{(\psi(y) - \psi(s)) + k(y-s)} \text{ then}
\]
\[
B_tf(y,w) = \sum_k \int_{-\infty}^y e^{(\psi(y) - \psi(s)) + k(y-s)}\pi_kf(s,..)(w)ds
\]
This integral converges only for $k < -t\psi'(y)$.
On the other hand, the fact that $f \in C_0^\infty(R^- \times S)$ and
\[
(-\partial/\partial s - t\psi(s) - k)e^{(\psi(y) - \psi(s)) + k(y-s)} = 0 \text{ tells us}
\]
\[
K_{2,t}(y,s) = (H(y - s) - 1)e^{(\psi(y) - \psi(s)) + k(y-s)} \text{ is another kernel.}
\]
From this we have another expression which is
\[
B_2f(y,w) = -\sum_k \int_{y}^{\infty}g(s)e^{(\psi(y) - \psi(s)) + k(y-s)}\pi_kf(s,..)(w)ds
\]
where the integral converges for $k > -t\psi'(y)$, where $g(s)$ is a cut off function we introduce for later use such that $g \in C^\infty$ which has value 1 for $s < 0$, and equals 0 for $s > 1$. Then $g(s)f(s) = f(s)$ for $s < 0$. Now we can write
\[
\psi(s) - \psi(y) = (s-y)\psi'(y) + \frac{(s-y)^2}{2}h(y,s), \text{ where}
\]
\[ h(y, s) = \int_0^1 (1 - \xi)\psi''(y + \xi(s - y))d\xi \geq 0 \]

After substituting \( f(s) = \int \tilde{f}(\eta)e^{i\eta s}d\eta \), and \( y - s = s' \), we get

\[
B_1 f(y, w) = \sum_k \int \sigma_{1,i}(y, \eta, k)\tilde{f}(\eta, \cdot)(\omega)e^{i\eta \omega}d\eta, \text{ where}
\]

\[
\sigma_{1,i}(y, \eta, k) = \int_{-\infty}^{-\infty} e^{i(\psi(y) - \psi(s)) + k(u - s) + i(u - v)\eta}ds \quad (-t\psi'(y) > k)
\]

Also

\[
B_2 f(y, w) = \sum_k \int \sigma_{2,i}(y, \eta, k)\tilde{f}(\eta, \cdot)(\omega)e^{i\eta \omega}d\eta, \text{ where}
\]

\[
\sigma_{2,i}(y, \eta, k) = -\int_{-\infty}^{-\infty} g(s)e^{i(\psi(y) - \psi(s)) + k(u - s) + i(u - v)\eta}ds \quad (k > -t\psi'(y)).
\]

Since \( \sigma_{1,i}(y, \eta, k) \) are defined only for \(-t\psi'(y) > k\), we want to extend this for \( k > -t\psi'(y) \). Also we want to extend \( \sigma_{2,i}(y, \eta, k) \) for \( k < -t\psi'(y) \) too. For that we will follow Stein's method ([7],p182).

Lemma

There exist a continuous function \( \phi \) defined on \([1, \infty)\) which is rapidly decreasing at \( \infty \), that is \( \phi(\lambda) = 0(\lambda^{-N}) \), as \( \lambda \to \infty \), for every \( N \), and which satisfies the following properties.

\[
\int_1^\infty \phi(\lambda)d\lambda = 0, \quad \int_1^\infty \lambda^k\phi(\lambda)d\lambda = 0, \quad \text{for} \quad k = 1, 2, ...
\]

With this definition, we can extend our symbol by

\[
F_{1,i}(y, \eta, k) = \sigma_{1,i}(y, \eta, k) \quad (k < -t\psi'(y))
\]

\[
= \int_1^\infty \sigma_{1,i}(y, \eta, (1 - 2\lambda)(k + t\psi'(y)) - t\psi'(y))\phi(\lambda)d\lambda \quad (k > -t\psi'(y))
\]

\[
F_{2,i}(y, \eta, k) = \sigma_{2,i}(y, \eta, k) \quad (k < t\psi'(y))
\]

\[
= \int_1^\infty \sigma_{2,i}(y, \eta, (1 - 2\lambda)(k + t\psi'(y)) - t\psi'(y))\phi(\lambda)d\lambda \quad (k < -t\psi'(y))
\]

Then

\[
\lim_{k \to -t\psi'(y)} F_{1,i}(y, \eta, k) = \int_1^\infty \sigma_{1,i}(y, \eta, k)\phi(\lambda)d\lambda
\]

Also

\[
\lim_{k \to -t\psi'(y)} d_k F_{1,i}(y, \eta, k) = \int_1^\infty \lim_{k \to -t\psi'(y)} (1 - 2\lambda)^i d_k \sigma_{1,i}(y, \eta, (1 - 2\lambda)(k + t\psi'(y)) - t\psi'(y))\phi(\lambda)d\lambda
\]

\[
= d_k^i \sigma_{1,i}(y, \eta, k) \int_1^\infty (1 - 2\lambda)^i \phi(\lambda)d\lambda
\]

\[
= d_k^i \sigma_{1,i}(y, \eta, k)
\]
The same argument says

\[ \lim_{k \to -t\psi'(y)} d_k^1 F_{2,1}(y, \eta, k) = d_k^1 \sigma_{2,1}(y, \eta, k) \]

Now we want to find the size of symbols. First we will estimate \( \sigma_{1,1}(y, \eta, k) \) and \( \sigma_{2,1}(y, \eta, k) \). Then using those estimates we can do the same for the extended symbols. We expect \( \sigma_{1,1}(y, \eta, k), \sigma_{2,1}(y, \eta, k) \) give same estimates.

Claim

\[ |(\frac{\partial}{\partial y})^N (\frac{\partial}{\partial \eta})^M d_k^1 \sigma_{i,1}(y, \eta, k)| \leq \frac{C_{N,M}}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k - i\eta|)^{M+m+1}} \times (1 + \frac{t\psi''(y)}{\sqrt{t\psi''(y)} + |t\psi'(y) + k - i\eta|})^N \quad i = 1, 2 \]

We will work on \( \sigma_{1,1}(y, \eta, k) \) from now on. First, from the above expression and \( \psi''(y - \xi s) \sim e^{i(y-\xi s)} \), we have

\[ h(y, y - s) \sim e^{i\psi} e^{i s} - 1 + s \quad s \]

\[ \frac{1}{4} e^{i\psi} \leq h(y, y - s) \leq \frac{1}{2} e^{i\psi} \quad \text{for } 0 \leq s \leq 2 \]

\[ e^{i\psi} \frac{1}{2s} \leq h(y, y - s) \leq e^{i\psi} \frac{1}{s} \quad \text{for } s \geq 2 \]

Then

\[ |\sigma_{1,1}(y, \eta, k)| \leq \int_0^2 e^{-t\psi''(y)s^2 + (t\psi'(y) + k)s} ds \]

\[ + \int_2^\infty e^{-t\psi''(y)s^2 + (t\psi'(y) + k)s} ds \quad (k < -t\psi'(y)) \]

\[ \leq \frac{2}{\sqrt{t\psi''(y)} + |t\psi'(y) + k|} \]

On the other hand, after putting absolute value in the integral we get

\[ |\sigma_{2,1}(y, \eta, k)| \leq \frac{1}{\sqrt{t\psi''(y)} + |t\psi'(y) + k|} \]

From now on for convenience let's call

\[ Q(t, y, s, k, \eta) = e^{i(\psi(y) - \psi(s)) + k(y - s) + i(s - y)\eta} \]

Then \( Q(t, y, y, k, \eta) = 1, Q(t, y, \infty, k, \eta) = 0. \) Now if we integrate by parts, we have

\[ \sigma_{1,1}(y, \eta, k) = \frac{1}{i\eta} Q(t, y, s, k, \eta)|^s_\infty - \frac{1}{i\eta} \int (t\psi'(s) + k)Q(t, y, s, k, \eta) ds \]

\[ 20 \]
The first term gives us $\frac{1}{\eta}$ and if we replace

$$t\psi'(s) + k = (t\psi'(y) + k) + h(y, s)(s - y)$$

The 2nd term equals $I_1 + I_2$, where

$$|I_1| = \frac{1}{|i\eta|} \int (t\psi'(y) + k)Q(t, y, s, k, \eta)ds \leq \frac{C}{|i\eta|}$$

$$|I_2| = \frac{1}{|i\eta|} \int_0^\infty th(y, y - s)se^{-th(y,y-s)s^2+(t\psi'(y)+k)s}ds$$

$$\leq \frac{1}{|i\eta|} \int_0^\infty t\psi''(y)s^{-t\psi''(y)s^2+(t\psi'(y)+k)s}ds + \frac{1}{|i\eta|} \int_0^\infty e^{-t\psi''(y)s+(t\psi'(y)+k)s}ds$$

$$\leq \frac{C}{|i\eta|}$$

This way we get $|\sigma_{1,z}(y, \eta, k)| \leq \frac{C}{|i\eta|}$ and if we combine two estimates, we get

$$|\sigma_{1,z}(y, \eta, k)| \leq C(|t\psi'(y) + k - i\eta|)^{-1} \text{ for all } y < 0.$$ 

We want the same estimate for $\sigma_{2,z}(y, \eta, k)$. Since we have half of what we need we will try integration of parts.

$$\sigma_{2,z}(y, \eta, k) = \int_{-\infty}^y g(s)Q(t, y, s, k, \eta)ds = I_1 + I_2 + I_3$$

$$I_1 = \frac{1}{i\eta} g(s)Q|_{-\infty}^y = \frac{1}{i\eta} g(y)$$

Then $|I_1| \leq \frac{1}{|i\eta|}$ since $|g|_{\infty} \leq 1$

$$I_2 = -\frac{1}{i\eta} \int g'(s)Q(t, y, s, k, \eta)ds$$

Then $|I_2| \leq \frac{C}{|i\eta|}$ since $g'(s)$ vanishes except (0,1).

$$I_3 = \frac{1}{i\eta} \int g(s)(t\psi'(s) + k)Q(t, y, s, k, \eta)ds$$

We can show $|I_3| \leq \frac{C}{|i\eta|}$ by the same reason as in the $|\sigma_{1,z}(y, k, \eta)| \leq \frac{C}{|i\eta|}$. So after getting terms together we get

$$|\sigma_{2,z}(y, \eta, k)| \leq \frac{C}{|i\eta|}$$

Now we want to differentiate $\sigma_{1,z}$ with respect to $\eta N$ times and get an estimate. First,

$$(\partial/\partial \eta)^N \sigma_{1,z}(y, \eta, k) = \int_{-\infty}^y (s - y)^N Q(t, y, s, k, \eta)ds$$
If we put the absolute sign inside the integral and change \( y \cdot s = s' \), we have two terms \( I_1 + I_2 \). We will estimate them separately.

\[
I_1 = \int_0^1 s^N e^{-(\psi''(y)s^2 + (\psi'(y)+k)s)} \, ds
\]

Now

\[
I_1 \leq \int_0^1 s^N e^{-\psi''(y)s^2} \, ds
\]

Let \( u = t\psi''(y)s^2 \). Then using \( s = \left( \frac{u}{t\psi''(y)} \right)^{1/2} \), \( ds = \frac{u^{-1/2}}{\sqrt{t\psi''(y)}} \, du \) we get

\[
I_1 \leq (\sqrt{t\psi''(y)})^{-1-N}
\]

On the other hand

\[
I_1 \leq \int_0^1 s^N e^{(t\psi'(y)+k)s} \, ds
\]

In this case by changing \( u = (t\psi'(y) + k)s \), we get

\[
I_1 \leq (t\psi'(y) + k)^{-1-N}
\]

So if we combine the above two estimates, we get

\[
I_1 \leq \frac{1}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k|)^{N+1}}
\]

Now

\[
I_2 = \int_1^\infty s^N e^{-\psi''(y)s^2 + (t\psi'(y)+k)s} \, ds
\]

which implies

\[
|\partial/\partial \eta|^N \sigma_1,\xi(y, \eta, k)| \leq \frac{C}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k|)^{N+1}}
\]

By the same argument as in \( I_1 \) case we get the estimate

\[
I_2 \leq \frac{1}{(\sqrt{t\psi''(y)} + |t\psi'(y) + K|)^{N+1}}
\]

Now we can check that the same argument works for \( \partial/\partial \eta \sigma_2,\xi \) and get

\[
|\partial/\partial \eta|^N \sigma_2,\xi(y, \eta, k)| \leq \frac{C}{(\sqrt{t\psi'(y)} + |t\psi'(y) + k|)^{N+1}}
\]

Claim

\[
|\partial/\partial \eta|^N \sigma_1,\xi(y, \eta, k)| \leq \frac{C}{(|\eta|)^{N+1}}
\]
We first want to check when \( N=1 \).

\[
(\partial / \partial \eta) \sigma_{1,1}(y, \eta, k) = \int_{-\infty}^{\gamma} (s - y)e^{i(\psi(y) - \psi(s)) + k(y-s) + i(s-y)\eta} \, ds
\]

Let \( e^{i\eta} = dg, \quad (s - y)e^{i(\psi(y) - \psi(s)) + k(y-s) - i\nu} = f \). Then after integration by parts we get terms of type \( I_1 + I_2 + I_3 \) where

\[
I_1 = \frac{1}{i\eta} \int e^{i(\psi(y) - \psi(s)) + k(y-s) + i(s-y)\eta} \, ds
\]

\[
I_2 = \frac{1}{i\eta} \int (s - y)(t\psi'(y) + k)e^{i(\psi(y) - \psi(s)) + k(y-s) + i(s-y)\eta} \, ds
\]

\[
I_3 = \frac{1}{i\eta} \int (s - y)(t\psi'(y) - \psi'(s))e^{i(\psi(y) - \psi(s)) + k(y-s) + i(s-y)\eta} \, ds
\]

The same argument to show \( |\sigma_{1,1}(y, \eta, k)| \leq \frac{1}{|i\eta|} \) tells us

\[
|I_1| \leq \frac{1}{|i\eta|}
\]

For other terms we perform integration by parts one more time and get terms of the following type.

\[
A_1 = \frac{1}{(i\eta)^2} \int_{-\infty}^{\gamma} (t\psi'(y) + k)e^{i(\psi(y) - \psi(s)) + k(y-s) + i(s-y)\eta} \, ds
\]

\[
A_2 = \frac{1}{(i\eta)^2} \int_{-\infty}^{\gamma} (s - y)(t\psi''(y))e^{i(\psi(y) - \psi(s)) + k(y-s) + i(s-y)\eta} \, ds
\]

\[
A_3 = \frac{1}{(i\eta)^2} \int_{-\infty}^{\gamma} (s - y)(t\psi'(y) + k)^2e^{i(\psi(y) - \psi(s)) + k(y-s) + i(s-y)\eta} \, ds
\]

\[
A_4 = \frac{1}{(i\eta)^2} \int_{-\infty}^{\gamma} (s - y)(t\psi'(y) - t\psi'(s))^2e^{i(\psi(y) - \psi(s)) + k(y-s) + i(s-y)\eta} \, ds
\]

\[
A_5 = \frac{1}{(i\eta)^2} \int_{-\infty}^{\gamma} (s - y)(t\psi''(y) - t\psi''(s))e^{i(\psi(y) - \psi(s)) + k(y-s) + i(s-y)\eta} \, ds
\]

\( A_1 \) is bounded by an integral of the form

\[
\frac{1}{|i\eta|^2} \int_0^\infty ae^{-as} \, ds \leq \frac{C}{|i\eta|^2} \quad a = t\psi'(y) + k
\]

\( A_2 \) is bounded by an integral of the form

\[
\frac{1}{|i\eta|^2} \int_0^\infty ase^{-as^2} \, ds \leq \frac{C}{|i\eta|^2} \quad a = t\psi''(y)
\]

\( A_3 \) is bounded by an integral of the form

\[
\frac{1}{|i\eta|^2} \int_0^\infty a^2se^{as} \, ds \leq \frac{C}{|i\eta|^2} \quad a = t\psi'(y) + k
\]

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$A_4$ is bounded by an integral of the form
\[
\frac{1}{|i\eta|^2} \int_0^\infty a^2 s^3 e^{-as^2} ds \leq \frac{C}{|i\eta|^2} \quad a = t\psi''(y)
\]

$A_5$ is bounded by an integral of the form
\[
\frac{1}{|i\eta|^2} \int_0^\infty ase^{-as^2} ds \leq \frac{C}{|i\eta|^2} \quad a = t\psi''(y)
\]

With this help we can get the general formula for $(\partial/\partial \eta)^N \sigma_{1,i}(y, \eta, k)$ i.e. after integration by parts $N$ or $N+1$ times we get the following.

\[
A_1 = \frac{1}{(i\eta)^N} \int e^{(\psi(y) - \psi(s) + k(y-s) + i(s-y)\eta} ds
\]

\[
A_2 = \frac{C}{(i\eta)^{N+1}} \sum_{m=0}^{N+1} \int (s - y)^m (t\psi'(y) + k)^{m+1} e^{(\psi(y) - \psi(s) + k(y-s) + i(s-y)\eta} ds
\]

\[
A_3 = \frac{C}{(i\eta)^{N+1}} \sum_{m=0}^{N+1} \int (s - y)^m (t\psi'(y) - \psi'(s))^{m+1} e^{(\psi(y) - \psi(s) + k(y-s) + i(s-y)\eta} ds
\]

Then $|A_1| \leq \frac{C}{|i\eta|^{N+1}}$ and terms appearing in $A_2$ are bounded by
\[
\frac{C}{|i\eta|^{N+1}} \int_0^\infty s^m (t\psi'(y) + k)^{m+1} e^{(t\psi'(y) + k)s} ds \leq \frac{C}{|i\eta|^{N+1}}
\]

after change of variables i.e. putting $u = (t\psi'(y) + k)s$ gives the desired result. On the other hand, terms appearing in $A_3$ are bounded by
\[
\frac{C}{|i\eta|^{N+1}} \int_0^1 (t\psi''(y))^{m+1} s^{m+1} e^{-im\psi'(s)} ds
\]

\[
+ \frac{C}{|i\eta|^{N+1}} \int_1^\infty (t\psi''(y))^{m+1} s^m e^{-im\psi'(s)} ds
\]

\[
\leq \frac{C}{|i\eta|^{N+1}}
\]

after change of variables i.e. putting $u = t\psi''(y)s^2$ gives the desired result.

From now on we want to find an estimate for $(\partial/\partial y)^N \sigma_{1,i}(y, \eta, k)$ with the help from the previous discussion. First, Let's look at $(\partial/\partial y)\sigma_{1,i}(y, \eta, k)$.

\[
(\partial/\partial y)\sigma_{1,i}(y, \eta, k) = 1 + \int_{-\infty}^y (t\psi'(y) + k - i\eta)e^{(\psi(y) - \psi(s) + k(y-s) + i(s-y)\eta} ds
\]

\[
= 1 + I_1 + I_2
\]

\[
I_1 = \int_{-\infty}^y (t\psi'(y) + k - i\eta)e^{(\psi(y) - \psi(s) + k(y-s) + i(s-y)\eta} ds
\]

\[
I_2 = \int_y^\infty \cdots
\]

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\[ I_2 = \int_{-\infty}^{\nu} (t\psi'(s) - t\psi'(\nu)) e^{i(\psi(y) - \psi(s)) + k(y-s) + i(y-v)\eta} ds \]

Since

\[ I_1 = \int_{-\infty}^{\nu} ( - \partial/\partial s) e^{i(\psi(y) - \psi(s)) + k(y-s) + i(y-v)\eta} ds \]
\[ = - e^{i(\psi(y) - \psi(s)) + k(y-s) + i(y-v)\eta} \bigg|_{s = \nu} = -1 \]

\( I_1 \) cancels 1 and we have only \( I_2 \). Now if we use Taylor series expansion, then \( I_2 \) equals

\[ - \int_{-\infty}^{\nu} t\psi''(y)(s - y)e^{i(\psi(y) - \psi(s)) + k(y-s) + i(y-v)\eta} ds \]
\[ + \int_{-\infty}^{\nu} \theta(y, s)(s - y)^2 e^{i(\psi(y) - \psi(s)) + k(y-s) + i(y-v)\eta} ds \]

The first term is familiar. It is the same term appearing in \( (\partial/\partial \eta)\sigma_1,\eta(y, \eta, k) \) multiplied by \( t\psi''(y) \). So in this case we can tell it is bounded by

\[ \frac{t\psi''(y)}{|\eta|} \]

The second term, after putting absolute sign in the integral, bounded by

\[ \int_0^1 t\psi''(y)s e^{-t\psi''(y)s^2} + (t\psi'(y) + k)s ds + \int_{\nu}^{\infty} t\psi''(y)s e^{-t\psi''(y)s} + (t\psi'(y) + k)s ds \]

which is also bounded by

\[ \frac{t\psi''(y)}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k|)^2} \]

So if we combine two, we get

\[ |(\partial/\partial \eta)\sigma_1,\eta(y, \eta, k)| \leq \frac{Ct\psi''(y)}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k - i\eta|)^2} \]

If we try second order differentiation, we have

\[ (\partial/\partial \eta)^2 \sigma_1,\eta(y, \eta, k) = \int (t\psi'(y) - t\psi'(s))^2 e^{i(\psi(y) - \psi(s)) + k(y-s) + i(y-v)\eta} ds \]
\[ + \int (t\psi''(y) - t\psi''(s)) e^{i(\psi(y) - \psi(s)) + k(y-s) + i(y-v)\eta} ds \]

In general Nth derivative has the following formula.

\[ (\partial/\partial \eta)^N \sigma_1,\eta(y, \eta, k) = I_1 + I_2 + I_3 + I_4 + ... \]
\[ I_1 = \int (t\psi'(y) - t\psi'(s))^N e^{i(\psi(y) - \psi(s)) + k(y-s) + i(y-v)\eta} ds \]
\[ I_2 = \int (t\psi^{(N)}(y) - t\psi^{(N)}(s))e^{i(\psi(y) - \psi(s)) + k(y-s) + i(s-y)\eta} ds \]

\[ I_3 = \sum_{j=1}^{N} \int (t\psi'(y) - t\psi'(s))^j(t\psi^{(N-j)}(y) - t\psi^{(N-j)}(s))e^{i(\psi(y) - \psi(s)) + k(y-s) + i(s-y)\eta} ds \]

\[ I_m = \sum_j \int (t\psi^{(m)}(y) - t\psi^{(m)}(s))^j(t\psi^{(N-mj)}(y) - t\psi^{(N-mj)}(s))e^{i(\psi(y) - \psi(s)) + k(y-s) + i(s-y)\eta} ds \]

\[ I_1, \text{ after Taylor series expansion, has two components i.e.} \]

\[ I_1 = \int (t\psi''(y))^N (s - y)^N e^{i(\psi(y) - \psi(s)) + k(y-s) + i(s-y)\eta} ds \]

\[ + \int (t\eta(y, s))^N (s - y)^2 e^{i(\psi(y) - \psi(s)) + k(y-s) + i(s-y)\eta} ds \]

The first term is exactly \((t\psi''(y))^N(\partial/\partial \eta)^N \sigma_{1,i}(y, \eta, k)\). So the same argument says it has bound

\[ \frac{(t\psi''(y))^N}{|i\eta|^{N+1}} \]

On the other hand, the second term, after putting absolute sign in the integral, is bounded by

\[ \int_0^1 (t\psi''(y))^N s^N e^{-t\psi''(y)s^2 + t\psi'(y)s + k} ds \]

\[ + \int_1^\infty (t\psi''(y))^N s^N e^{-t\psi''(y)s + t\psi'(y)s + k} ds \]

\[ \leq \frac{C(t\psi''(y))^N}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k|)^{N+1}} \text{ by rescaling} \]

So after getting terms together we have

\[ |I_1| \leq C_1 \frac{(t\psi''(y))^N}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k - i\eta|)^{N+1}} \]

Now for \(I_2\), using the fact that \(\psi^{(l)}(y) \leq \psi''(y)\) we can show that

\[ |I_2| \leq C_2 \frac{t\psi''(y)}{(\sqrt{t\psi''(y)} + \sqrt{t\psi'(y) + k - i\eta})^2} \]

Now for \(I_3\), using Taylor series expansion, we get two terms which are

\[ I_3 = \int (t\psi''(y))^j t\psi^{(N-j+1)}(y)(y - s)^{j+1} e^{i(\psi(y) - \psi(s)) + k(y-s) + i(s-y)\eta} ds \]

\[ + \int (t\eta(y, s))^j t\eta(y, s)(s - y)^{j+1} e^{i(\psi(y) - \psi(s)) + k(y-s) + i(s-y)\eta} ds, \text{ where} \]
\[ g(y, s) = \int_0^1 (1 - \xi)\psi^{(N-j+1)}(y + \xi(s - y))d\xi \]

The first term is bounded by

\[ C\frac{(t\psi''(y))^{j+1}}{|i\eta|^{j+2}}. \]

On the other hand, the second term is bounded by

\[ \int_0^1 (t\psi''(y))^{j+1}s^{j+1}e^{-t\psi''(y)s^2+((\psi'(y)+k)s)}ds \]

\[ + \int_1^\infty (t\psi''(y))^{j+1}s^{j+1}e^{-t\psi''(y)s^2+((\psi'(y)+k)s)}ds \]

\[ \leq \frac{C(t\psi''(y))^{j+1}}{(t\psi''(y) + |t\psi'(y) + k|)^{j+2}}. \]

This way we can check each of \( I_m \) separately and \( (\partial/\partial y)^N\sigma_{1,l}(y, \eta, k) \) are bounded by the following.

\[ |(\partial/\partial y)^N\sigma_{1,l}(y, \eta, k)| \leq \sum_{j=1}^N C_j \frac{(t\psi''(y))^j}{(t\psi''(y) + |t\psi'(y) + k - i\eta|)^{j+1}} \]

On the other hand, if we treat \( k \) as a continuous variable and differentiate then we get \((s-y)\) factor each time we differentiate. This is the same phenomenon as we differentiate with respect to \( \eta \). so in this case we have

\[ |d_k^{m}\sigma_{1,l}(y, \eta, k)| \leq \frac{C}{(t\psi''(y) + |t\psi'(y) + k - i\eta|)^{m+1}} \]

Now if we combine all the information we have, we can tell

\[ |(\partial/\partial y)^N(\partial/\partial \eta)^M d_k^m \sigma_{1,l}(y, \eta, k)| \leq \frac{C}{(t\psi''(y) + |t\psi'(y) + k - i\eta|)^{M+m}}(1 + \frac{(t\psi''(y))}{(t\psi''(y) + |t\psi'(y) + k)} \]

(10)

Now we want to get the same estimate for \( (\partial/\partial \eta)^N\sigma_{2,l}(y, \eta, k) \).

First, let \( N=1 \).

\[ (\partial/\partial \eta)\sigma_{2,l}(y, \eta, k) = \int_y^\infty g(s)(s - y)e^{((\psi'(y)-\psi(s)) + k(s^2 - (s-y)\eta)ds = I_1 + I_2 + I_3} \]

\[ I_1 = \int \frac{1}{i\eta} \int g(s)e^{((\psi'(y)-\psi(s)) + k(s^2 - (s-y)\eta) ds} \]

\[ I_2 = \int \frac{1}{i\eta} \int g'(s)(s - y)e^{((\psi'(y)-\psi(s)) + k(s^2 - (s-y)\eta) ds} \]

\[ I_3 = \int \frac{1}{i\eta} \int g(s)(s - y)(t\psi'(y) + k)e^{((\psi'(y)-\psi(s)) + k(s^2 - (s-y)\eta) ds} \]

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We know already that $|I_1| \leq \frac{1}{|i\eta|}$. For $I_2$ we perform integration by parts one more time and get terms of following.

$$I_2 = A_1 + A_2 + A_3$$

$$A_1 = \frac{1}{(i\eta)^2} \int g''(s)(s - y) e^{(\psi(y) - \psi(s)) + h(y-s) + i(s-y)n} ds$$

$$A_2 = \frac{1}{(i\eta)^2} \int g'(s) e^{t(\psi(y) - \psi(s)) + h(y-s) + i(s-y)n} ds$$

$$A_3 = \frac{1}{(i\eta)^2} \int g'(s)(s - y)(t\psi'(y) + k) e^{t(\psi(y) - \psi(s)) + h(y-s) + i(s-y)n} ds$$

If we notice that derivatives of $g'(s)$ vanishes except the unit interval we can easily show that

$$|A_1| \leq \frac{C}{|i\eta|^2}$$

From $I_3$, after integration by parts, we get same terms as in $I_2$ except the following

$$\frac{1}{(i\eta)^2} \int g(s)(s - y)(t\psi'(y) + k)^2 e^{t(\psi(y) - \psi(s)) + h(y-s) + i(s-y)n} ds$$

which is the same type as in $(\partial/\partial \eta)\sigma_{1,i}(y, \eta, k)$. So this is also bounded by

$$\frac{C}{|i\eta|^2}$$

In general terms appearing in the $(\partial/\partial \eta)^N \sigma_{2,i}(y, \eta, k)$ has either same type as in $(\partial/\partial \eta)^N \sigma_{1,i}(y, \eta, k)$ or of the following type

$$I_1 = \frac{1}{(i\eta)^{N+1}} \int g'(s)(s - y)^m(t\psi'(y) + K)^m e^{t(\psi(y) - \psi(s)) + h(y-s) + i(s-y)n} ds$$

or

$$I_2 = \frac{1}{(i\eta)^{N+1}} \int g''(s)(s - y) e^{t(\psi(y) - \psi(s)) + h(y-s) + i(s-y)n} ds$$

Using the fact that $g''(s)$ vanishes except the unit interval we can easily show

$$|I_i| \leq \frac{C}{|i\eta|^{N+1}} \quad i = 1, 2$$

After getting terms together we can show that

$$|(\partial/\partial \eta)^N \sigma_{2,i}(y, \eta, k)| \leq \frac{C_N}{|i\eta|^{N+1}}$$

And after combining with the other half, we have

$$|(\partial/\partial \eta)^N \sigma_{2,i}(y, \eta, k)| \leq \frac{C_N}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k - i\eta|)^{N+1}}$$

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Now same argument as in differentiating with respect to \( \eta \) gives

\[
|d_k^m \sigma_{2,i}(y, \eta, k)| \leq \frac{C_m}{(\sqrt{t \psi''(y)} + |t \psi'(y) + k - i\eta|)^{m+1}}
\]

Now we want to differentiate \( \sigma_{2,i}(y, \eta, k) \) with respect to \( y \) variable \( N \) times and try to get similar estimates. First, let \( N=1 \).

\[
(\partial/\partial y)\sigma_{2,i}(y, \eta, k) = -g(y) + \int_y^\infty g(s)(t \psi'(y) + k - i\eta)e^{i(\psi(y)-\psi(s))+(s-y)\eta} ds
\]

\[
= -g(y) + I_1 + I_2
\]

\[
I_2 = \int_y^\infty g(s)(-\partial/\partial s)e^{i(\psi(y)-\psi(s))+(s-y)\eta} ds
\]

\[
I_3 = \int g(s)(t \psi'(y) - t \psi'(s))e^{i(\psi(y)-\psi(s))+(s-y)\eta} ds
\]

After integration by parts

\[
I_2 = g(y) + \int g'(s)e^{i(\psi(y)-\psi(s))+(s-y)\eta} ds
\]

which cancels \(-g(y)\) and the second term is the same type as \( \sigma_{2,i}(y, \eta, k) \). On the other hand, \( I_3 \) is the same term appearing in \((\partial/\partial y)\sigma_{1,i}(y, \eta, k)\). So after getting terms together we get the bound

\[
|(\partial/\partial y)\sigma_{1,i}(y, \eta, k)| \leq \frac{1}{\sqrt{t \psi''(y)} + |t \psi'(y) + k - i\eta|} + C \frac{t \psi''(y)}{(\sqrt{t \psi''(y)} + |t \psi'(y) + k - i\eta|)^2}
\]

In general \((\partial/\partial y)^N\sigma_{2,i}(y, \eta, k)\) has same type of terms as in \((\partial/\partial y)^N\sigma_{1,i}(y, \eta, k)\) plus terms in \((\partial/\partial y)^{N-1}\sigma_{2,i}(y, \eta, k)\). So we have the bound

\[
|(\partial/\partial y)^N\sigma_{2,i}| \leq \sum_{j=1}^{N} \frac{(t \psi''(y))^{j+1}}{(\sqrt{t \psi''(y)} + |t \psi'(y) + k - i\eta|)^{j+1}}
\]

After combining estimates with respect to \( y, k \) and \( \eta \) variables we finally have

\[
|(\partial/\partial y)^N(\partial/\partial \eta)^M d_k^m \sigma_{2,i}(y, \eta, t)| \leq \frac{C_{N,M,k}}{(\sqrt{t \psi''(y)} + |t \psi'(y) + k - i\eta|)^{M+m+1}}(1 + \frac{(t \psi''(y))}{(\sqrt{t \psi''(y)} + |t \psi'(y) + k - i\eta|)^{N+1}})
\]

Now using \((10')\), we want same estimates for extended symbols. First, when \( k > -t \psi'(y) \)

\[
|(\partial/\partial \eta)^N F_{1,i}(y, \eta, k)| = \int_1^\infty (\partial/\partial \eta)^N \sigma_{1,i}(y, \eta, (1-2\lambda)(k + t \psi'(y)) - t \psi'(y)) d\lambda |
\]

\[
\leq \int_1^\infty \frac{C_N \phi(\lambda) d\lambda}{(\sqrt{t \psi''(y)} + |t \psi'(y) + (1-2\lambda)(k + t \psi'(y)) - t \psi'(y) - i\eta|)^{N+1}}
\]

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\begin{align*}
&\leq \frac{C_N}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k - i\eta|)^{N+1}} \int_1^\infty \phi(\lambda) d\lambda \quad \text{(since } \lambda \geq 1) \\
&\leq \frac{C_N}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k - i\eta|)^{N+1}} \\
&\left| (\partial / \partial y)^M F_{1,1}(y, \eta, k) \right| \leq \sum_i \int_1^\infty (1 - 2\lambda)^i \left( (\partial / \partial y)^M - (t\psi''(y))^i \right) \sigma_{1,1} \\
&\quad \sigma_{1,1}(y, \eta, (1 - 2\lambda)(k + t\psi'(y)) - t\psi'(y)) \phi(\lambda) d\lambda \\
&\leq \frac{C_M(t\psi''(y))^M}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k - i\eta|)^{M+1}} \sum_i \int_1^\infty (1 - 2\lambda)^i \phi(\lambda) d\lambda \\
&\leq C_M \frac{(t\psi''(y))^M}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k - i\eta|)^{M+1}} \\
\end{align*}

Also the same argument gives

\begin{align*}
\left| d^n F_{1,1}(y, \eta, k) \right| &= \left| \int_1^\infty (1 - 2\lambda)^n d^n \sigma_{1,1}(y, \eta, (1 - 2\lambda)(k + t\psi'(y)) - t\psi'(y)) \phi(\lambda) d\lambda \\
&\leq \frac{C_m}{(\sqrt{t\psi''(y)} + |t\psi'(y) + k - i\eta|)^{m+1}} \\
\end{align*}

The same estimate is true for \( F_{2,1}(y, \eta, k) \), when \( k < -t\psi'(y) \). (From now on for the sake of simplicity we will replace \( F_{1,1}(y, \eta, k) \) by \( \sigma_{1,1}(y, \eta, k) \) and \( F_{2,1}(y, \eta, k) \) by \( \sigma_{2,1}(y, \eta, k) \). With these estimates, we can write \( B_1 \) as follows.

\begin{align*}
B_{1,i}f(y, w) &= \sum_k \int \sigma_{1,1}(y, \eta, k) \hat{f}(\eta, .)(w) e^{iwm} d\eta \\
B_{1,2}f(y, w) &= \sum_k \int \sigma_{2,1}(y, \eta, k) \hat{f}(\eta, .)(w) e^{iwm} d\eta \\
\end{align*}

With these definitions we want to show the following.

\[ \|B_{1,i}f\|_{L^p(R \times S, dx)} \leq C\|f\|_{L^q(R \times S, dx)} \quad i = 1, 2 \]

Now using duality at (11) we get the spherical restriction theorem we need;

\[ \|\pi_{M,N}g\|_{L^p(S, C^m)} \leq C(N^{\frac{q}{2}}(N - M + 1)^{\frac{q}{2}})^{1/2} 1/q \|g\|_{L^q(S, C^m)} \]

(11')

for \( q' = p = \frac{6n - 4}{3n + 2}, \quad q = \frac{6n - 4}{3n - 6} \)

With this theorem we introduce partition of unity \( \{\phi_\rho\} \) satisfying (12) and define

\[ \sigma_{1,1}^\rho(y, \eta, k) = \phi_\rho\left(\frac{1}{\sqrt{a + |t\psi'(y) + k - i\eta|}}\right) \sigma_{1,1}(y, \eta, k) \]

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Then $\sigma_{l,(y, \eta, k)}$ satisfies (13) and followingly we have
\[
|| (\partial/\partial \eta)^j \sigma_{l,(y, \eta, k)} || \leq C_j (2^{\beta} \sqrt{a})^{-1-j}
\]
\[
|| (\partial/\partial \eta)^j (\sigma_{l,(y, \eta, k)}^{\beta} - \sigma_{l,(y, \eta, k+1)}^{\beta}) || \leq C_j (2^{\beta} \sqrt{a})^{-2-j}
\]
uniformly for $y < 0$. Now define
\[
F_{l}^{\beta} f(y, w) = \sum_k \frac{1}{2\pi} \int \sigma_{l,(y, \eta, k)}^{\beta} f(\eta, .)(w) e^{i\eta m} d\eta
\]
Then $B_{l} = \sum_{\beta=0}^{N} F_{l}^{\beta}$.
We want to estimate $F_{l}^{\beta}$ separately.
Let $M = [-t \psi'(y) - 2^{\beta} \sqrt{a}]$, $M' = [M + 2 \times 2^{\beta} \sqrt{a}] + 1$. Denote
\[
T_{l}^{\beta}(y, \eta)g(w) = \sum_k F_{l}^{\beta}(y, \eta, k) \pi_k g(w)
\]
Here $F_{l}^{\beta} = 0$ unless $M \leq k \leq M'$. A summation by parts gives
\[
T_{l}^{\beta}(y, \eta) = \sum_{M}^{M'} (F_{l}^{\beta}(y, \eta, k) - F_{l}^{\beta}(y, \eta, k + 1)) \pi_{M,k} \quad \text{for} \quad M \leq k \leq M'
\]
Now (11') implies
\[
\| \pi_{M,k} g \|_{L^2(S;C^m)} \leq C (t^{\frac{a^2}{2}} (2^{\beta} \sqrt{a})^{n/2})^{1/2 - 1/4} \| g \|_{L^\infty(S;C^m)} \quad \text{for} \quad \beta \leq N - 1
\]
Therefore by (14')
\[
|| (\partial/\partial \eta)^j T_{l}^{\beta}(y, \eta) \pi_{M,k} g ||_{L^2(S;C^m)} \leq C_j (2^{\beta} \sqrt{a})^{-1-j} (t^{\frac{a^2}{2}} (2^{\beta} \sqrt{a})^{n/2})^{1/2 - 1/4} \| g \|_{L^\infty(S;C^m)}
\]
uniformly for $y < 0$.
Define for $z=y-y'$
\[
K_{l}^{\beta}(y, z) = \frac{1}{2\pi} \int (\partial/\partial \eta)^j T_{l}^{\beta}(y, \eta) \frac{1}{(iz)^j} e^{iz} d\eta
\]
Then since the length of the interval in $\eta$ where $T_{l}^{\beta}$ is non-zero is less than $2 \times 2^{\beta} \sqrt{a},$
\[
\| K_{l}^{\beta}(y, z) g \|_{L^2(S;C^m)} \leq C_j (2^{\beta} \sqrt{a})^{-j} (t^{\frac{a^2}{2}} (2^{\beta} \sqrt{a})^{n/2})^{1/2 - 1/4} \| g \|_{L^\infty(S;C^m)}
\]
Now if we take the value $j=1, 10$ we get
\[
\| K_{l}^{\beta}(y, z) g \|_{L^2(S;C^m)} \leq C (1 + |2^{\beta} \sqrt{a}|)^{-10} (t^{\frac{a^2}{2}} (2^{\beta} \sqrt{a})^{n/2})^{1/2 - 1/4} \| g \|_{L^\infty(S;C^m)}
\]
Note that
\[
F_{l}^{\beta} f(y, w) = \int K_{l}^{\beta}(y, y - y') f(y', .)(w) dy'
\]
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We can apply Young's inequality as before and get for $\beta \leq N - 1$

$$\|F_1^\beta f\|_{L^3(R-xS, dx)} \leq C \| (1 + |2^\beta \sqrt{a} z|)^{-\frac{10}{3n}} \|L^r(R, ds) ((t^{\frac{2n-6}{3n-2}}(2^\beta \sqrt{a})^{n/2})^{1/r-1/3}) \|f\|_{L^p(R-xS, dx)}$$

for $\frac{1}{r} + \frac{1}{p} = \frac{1}{2} + 1$

Note that

$$\| (1 + |2^\beta \sqrt{a} z|)^{-\frac{10}{3n-2}} \|_{L^r(ds)} \leq C (2^\beta \sqrt{a})^{-1/r}$$

Then for $\frac{1}{r} = \frac{1}{3} - \frac{1}{r} = \frac{2}{3n-2}$, and $\frac{1}{r} = \frac{3n-4}{3n-2}$ we finally have

$$\|F_1^\beta f\|_{L^3(R-xS, dx)} \leq C (2^\beta \sqrt{a})^{-1/r} \left( \left(2^\beta \sqrt{a} \right)^{n/2} \left( \phi''(y) \right)^{n/2} \right)^{1/r} \|f\|_{L^3(R-xS, dx)}$$

Since $2^\beta$ has negative power, if we sum the series in $\beta$ we get exactly same bound as $L^2 \to L^r$ estimate, i.e.

$$\| \sum_{\beta} F_1^\beta f\|_{L^3(R-xS, e^{\eta x} dw dy)} \leq C \|f\|_{L^3(R-xS, e^{(n-1/2)x} dw dy)}$$

Now the case $\beta = N$, we need a different estimate for $F_1^N$. We had

$$F_1^N f(y, w) = \sum_k \int \sigma_i^N(y, \eta, k) \tilde{f}(\eta, \tau)(w)e^{iy\eta}d\eta$$

Since $\sigma_i^N$ had support where

$$|t\psi'(y) + k - i\eta| \geq 2^N \sqrt{a},$$

we had

$$|t\psi'(y) + k - i\eta| > c(1 + |\eta| + |k|) \text{ uniformly for } y < 0.$$  

Hence

$$|d^j_i \partial_{\eta} \sigma_i^N(y, \eta, k)| \leq C_{j,m}(1 + |\eta| + |k|)^{-1-j-1} \quad j = 0, 1, ..$$  

(15)

Here $d^j_i$ is a difference operator of order $l$ in the $k$ variable. The above estimate means that $\sigma_i^N$ is a standard symbol in the $(y, \eta, k)$ variables of order $-1$. So $F_1^N$ is a standard pseudodifferential operator of order $-1$, Taylor p296 and we can write $F_1^N$ as

$$F_1^N f(y, w) = \int_{S \times R} K_1((w, y), (w', y')) f(w', y') dw' dy'$$

Then the corresponding kernel $K_1$ has bounds

$$|K_1((w', y')(w, y))| \leq C(|w - w'| + |y - y'|)^{-n+1}$$
Hence $F^N_i$ is a bounded operator from $L^p \rightarrow L^q$ for $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$ (Stein,[9],p128) i.e.

$$||F^N_i f||_{L^q(R^{-x}S,dx)} \leq C||f||_{L^p(R^{-x}S,dx)} \quad (*)$$

On the otherhand, compactness of $(w, w')$ and $n \geq 3$ gives us

$$\int |K_i((w', y'), (w, y))| dw' dy' \leq C \quad \text{and} \quad \int |K_i((w', y'), (w, y))| dy wy \leq C$$

Now we can apply Young's inequality to get $L^p \rightarrow L^p$ boundedness i.e.

$$||F^N_i f||_{L^q(R^{-x}S,dx)} \leq C||f||_{L^p(R^{-x}S,dx)} \quad (**).$$

Now if we interporate the above two estimates (*) and (**), we get

$$||F^N_i f||_{L^q(R^{-x}S,dx)} \leq C||f||_{L^p(R^{-x}S,dx)}$$

for all $p \leq q \leq \tilde{p}$. And in particular this holds for $q=2$. So if we combine this with $\beta \leq N - 1$ case we have

$$||B_{1,i} f||_{L^2(R^{-x}S,a(n+\frac{1}{2})y dy) dx} \leq C||f||_{L^p(R^{-x}S,dx)}$$

Exactly same process shows us same inequality holds for $B_{2,i}$. Now define

$$P_1 f = \sum_{k \leq l} \pi_k f$$

$$P_2 f = \sum_{k > l} \pi_k f$$

, then

$$f = P_1 f + P_2 f, \text{ and } P_1 B_{1,i} f = P_1 f, \text{ and } P_2 B_{2,i} f = P_2 f.$$ Now

$$||f||_{L^p(R^{-x}S,a(n+\frac{1}{2})y dy dw)} = ||P_1 f + P_2 f||_{L^p(R^{-x}S,a(n+\frac{1}{2})y dy dw)}$$

$$\leq ||P_1 f||_{L^p(R^{-x}S,a(n+\frac{1}{2})y dy dw)} + ||P_2 f||_{L^p(R^{-x}S,a(n+\frac{1}{2})y dy dw)}$$

$$= ||P_1 B_{1,i} f||_{L^p(R^{-x}S,a(n+\frac{1}{2})y dy dw)} + ||P_2 B_{2,i} f||_{L^p(R^{-x}S,a(n+\frac{1}{2})y dy dw)}$$

$$\leq ||B_{1,i} f||_{L^p(R^{-x}S,a(n+\frac{1}{2})y dy dw)} + ||B_{2,i} f||_{L^p(R^{-x}S,a(n+\frac{1}{2})y dy dw)}$$

$$\leq 2C||A_i f||_{L^p(R^{-x}S,dx)}$$

Now for the future use, we would like to note the following corollary also holds.

$$||e^{iy} f||_{L^p(R^{-x}S,dx)} \leq C||f||_{L^p(R^{-x}S,dx)} \quad f \in C^\infty_0(R^{-x}S)$$

We can apply exactly same argument as the $L^p \rightarrow L^2$ estimate. The idea is to prove slightly stronger estimate i.e.$(L^p(R^{-x}S) \rightarrow L^p(L^2(S),dy)$ estimate.) In this context we want to show the following

$$||B_{i} g||_{L^p(L^2(S),dy)} \leq C||g||_{L^p(R^{-x}S,dx)} \quad i = 1,2$$
First we have
\[ \|K_1^\theta(y,z)g\|_{L^2(S)} \leq C(1 + |2^\theta \sqrt{az}|)^{-10}(t^{-23/2}(2^\theta \sqrt{a})^{n/2})^{1/2} \|g\|_{L^p(S)} \]
and
\[ F_1^\theta f(y,w) = \int K_1^\theta(y,y-y')f(y',.)dy' \]

**Lemma**

Let \( H(y,y') \) be a bounded operator from \( L^p(S) \) to \( L^p(S) \) of operator norm \( \leq h(y-y') \) for each \( y, y' \in R \). Suppose that \( h \in L^1(R) \). Then

\[ Tf(y,w) = \int H(y,y')f(y')(w)dy' \]

satisfies

\[ \|Tf\|_{L^p(L^2(S),dy)} \leq \|h\|_{L^1(R)}\|f\|_{L^p(R-xS)} \]

**Proof**

\[ \|Tf\|_{L^p(L^2(S),dy)} = \int_{-\infty}^\infty \left( \int_{-\infty}^\infty H(y,y')f(y',.)dy' \right)^2dw \]
\[ \leq \int_{-\infty}^\infty \left( \int_{-\infty}^\infty |H(y,y')f(y',.)|^2dw \right)^{1/2}dy' \]
\[ \leq \int_{-\infty}^\infty \left( \int_{-\infty}^\infty h(y-y')\|f(y',.)\|_{L^p(S)}dy' \right)^pdy \]
\[ \leq \|h(y-y')\|_{L^1(R)}\|f\|_{L^p(R-xS)} \]

Note that
\[ \|(1 + |2^\theta \sqrt{az}|)^{-10}\|_{L^1(da)} \leq C(2^\theta \sqrt{a})^{-1} \]

and this gives
\[ \sum_{\beta}^{N-1} F_1^\theta f\|_{L^p(L^2(S),dy)} \leq C\|f\|_{L^p(R-xS_a^{a-i/2})}dydw \]

On the other hand, for \( \beta = N \), we need only
\[ \|F_1^N f\|_{L^p(R-xS,da)} \leq C\|f\|_{L^p(R-xS,da)} \quad (***) \]

After combining these two, we get
\[ \|B_{i+1}f\|_{L^p(L^2(S),dy)} \leq C\|f\|_{L^p(R-xS_a^{a-i/2})}dydw \quad i = 1, 2 \quad (18'') \]

No we will prove our inequality. First, from the definition of \( P_i \)'s i=1,2, it is easy to check
\[ \|P_if\|_{L^p(L^2(S),dy)} \leq \|f\|_{L^p(L^2(S),dy)} \quad \text{and} \]
\[ \|f\|_{L^p(R-xS,da)} \leq C\|f\|_{L^p(L^2(S),dy)} \]

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Then as before

\[ \| f \|_{L^p(R \times S, dx)} \leq \| P_1 f \|_{L^p(R \times S, dx)} + \| P_2 f \|_{L^p(R \times S, dx)} = \| P_1 B_1 A_1 f \|_{L^p(R \times S, dx)} + \| P_2 B_2 A_1 f \|_{L^p(R \times S, dx)} \leq C \| P_1 B_1 A_1 f \|_{L^p(L^2(S), dy)} + C \| P_2 B_2 A_1 f \|_{L^p(L^2(S), dy)} \leq C \| B_1 A_1 f \|_{L^p(L^2(S), dy)} + C \| B_2 A_1 f \|_{L^p(L^2(S), dy)} \leq 2C \| A_1 f \|_{L^p(R \times S, dx)} \]

From now on we will use this estimate to prove Corollary 2.

Corollary 2

It suffices to prove that \( u \equiv 0 \) on a neighborhood of \(-\infty\). Let \( \eta \) be a cut off function such that \( \eta \equiv 1 \) for \( y \in (-\infty, -2\rho), \eta \equiv 0 \) for \( y > -\rho \) for sufficiently large \( \rho \).

Then for \( f = \eta \rho \), (18) is equivalent to the following.

\[ \| e^{\psi} \eta u \|_{L^p(S_\rho, dx)} \leq C \| e^{\psi} D(\eta u) \|_{L^p(U, dx)} \]

where \( S_\rho = (-\infty, -\rho) \times S^{n-1} \).

Since \( D(\eta u) = (D\eta)u + \eta(Du) \), \( D\eta \equiv 0 \) for \( x \notin S_\rho \),

\[ \| e^{\psi} D(\eta u) \|_{L^p(U, dx)} \leq C \| D\eta \|_{\infty} \| e^{\psi} u \|_{L^p(S_\rho, dx)} \]

Furthermore, since \( (D+V)u = 0 \), on \( U \),

\[ \| e^{\psi} Du \|_{L^p(U, dx)} \leq \| e^{\psi} Vu \|_{L^p(S_\rho, dx)} + \| e^{\psi} Du \|_{L^p(U \setminus S_\rho, dx)} \]

After getting terms together, we have

\[ \| e^{\psi} \eta u \|_{L^p(S_\rho, dx)} \leq C''(\| D\eta \|_{\infty} + \| V \|_{L^p(S_\rho)} \| e^{\psi} u \|_{L^p(S_\rho, dx)} + C'' \| e^{\psi} Du \|_{L^p(U \setminus S_\rho, dx)} \]

For sufficiently large \( \rho \), we can make sure

\[ C''(\| D\eta \|_{\infty} + \| V \|_{L^p(S_\rho)}) < 1. \]

Then we can make cancellation of the first term on the right hand side against the left hand side. Now after cancellation, we have

\[ \| e^{\psi} u \|_{L^p(S_\rho, dx)} \leq C \| e^{\psi} Du \|_{L^p(U \setminus S_\rho)} \]

Then since \( \eta \equiv 1 \) for \( y \in (-\infty, -2\rho) = S_{\rho/2} \),

\[ \| e^{\psi} u \|_{L^p(S_{\rho/2}, dx)} \leq C \| e^{\psi(-\rho)} Du \|_{L^p(U \setminus S_\rho)} \]

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Now the fact that \( \psi \) is a decreasing function gives us
\[
||e^{t(\psi(-2\rho) - \psi(-\rho))}u||_{L^2(s_{\rho/2}, ds)} \leq C ||Du||_{L^2(U, dx)} < \infty.
\]
So as \( t \to \infty \), we are forced to \( u \equiv 0 \) on \( S_{\rho/2} \).

The Laplace operator

Proof of Theorem 3
We can decompose the right hand side of (3) i.e.
\[
(e^{t\psi(y)}e^{2\psi(y)}\Delta e^{-t\psi(y)})f(y, w) = \sum_k (\partial / \partial y - t\psi'(y) - k)(\partial / \partial y - t\psi'(y) + k + n - 2)\xi_k f(y, .)(w)
\]
Let's denote this as \( (A_t f)(y, w) \).
We are familiar with the first component of \( A_t f \), i.e. the Dirac operator has the same expression. On the other hand the second component has a nice property, i.e. whose symbol \( i\eta - t\psi'(y) + k + n - 2 \) never vanishes. For this reason, we will first work on the main part of \( A_t \) which is
\[
\sum_k (\partial / \partial y - ay + b)(\partial / \partial y - t\psi'(y) + k + n - 2)\xi_k \quad \text{with}
\]
\[
a = t\psi''(y), \quad b = t\psi''(y)y - t\psi'(y) - k
\]
Let's denote this operator as \( B_t \).
Now if we let \( \sigma_t(y, \eta, k) \) be the symbol of the left inverse of \( \partial / \partial y - ay + b \), then
\[
\sigma_t(y, \eta, k) \quad \text{with} \quad \frac{\sigma_t(y, \eta, k)}{i\eta - t\psi'(y) + k + n - 2}
\]
is a left parametrix. We want to inspect the property of \( \sigma_t(y, \eta, k) \). Since
\[
\sigma_t(y, \eta, k) \sim \frac{1}{\sqrt{a} + |t\psi'(y) + k - i\eta|}, \quad a = t\psi''(y) \sim te^y
\]
If we choose some large \( N_0 \), depending on \( t \) such that \( te^{-2N_0} = \frac{1}{100} \), then for \( y > -N_0 \), \( \sigma_t(y, \eta, k) \leq \frac{1}{\sqrt{a}} < 0(1) \).
On the other hand, for \( y \leq -N_0 \), \( \sqrt{a} \) is vanishing fast as \( y \) goes to \( -\infty \). But we can choose \( t \) such that \( |t - k| > 1/4 \). Also in this range \( t(\psi'(y) + 1) \sim te^y << 1/10 \).

Then \( |t\psi'(y) + k - i\eta| \geq | -t + k - i\eta| - |t(\psi'(y) + 1)| \) and \( \sigma_t(y, \eta, k) \leq \frac{1}{|t\psi'(y) + k - i\eta|} \leq c. \)
In either case,
\[
\sigma_t(y, \eta, k) \sim 0(1) \quad \text{and} \quad \frac{\sigma_t(y, \eta, k)}{i\eta - t\psi'(y) + k + n - 2} \sim t^{-1}.
\]
For \( f \in C_0^\infty(R \times S) \), denote
\[
F_t f(y, w) = \sum_k \int \frac{\sigma_t(y, \eta, k)}{i\eta - t\psi(y) + k + n - 2} e^{i\eta \cdot x_k} f(\eta, .)(w) d\eta.
\]

First we want to show the following inequality.
\[
\|F_t f\|_{L^r(I \times S, dx)} \leq C e^{-t \nu/2} \|f\|_{L^p(I \times S, ds)} \quad \text{for } f \in C_0^\infty(I \times S).
\]  \( (19) \)

Because of the behavior of \( \sigma_t(y, \eta, k) \), we want to proceed separately depending on the size of \( y \).

As before we introduce partition of unity \( \{\phi_\beta\}_{\beta=0}^N \) of the positive real axis satisfying (12). First, define
\[
\sigma_t^\beta(y, \eta, k) = \phi_\beta\left(\frac{1}{\sqrt{\alpha}} |t\psi(y) + k - i\eta|\right) \sigma_t(y, \eta, k) \quad y > -N_0.
\]

\[
\sigma_t^\beta(y, \eta, k) = \phi_\beta(|t\psi(y) + k - i\eta|) \sigma_t(y, \eta, k) \quad \text{for } y < -N_0.
\]

From now on we will work on the first case (i.e. \( y > -N_0 \)). Then from (10) it is easy to show
\[
|\langle \partial/\partial \eta \rangle^j \frac{\sigma_t^\beta(y, \eta, k)}{i\eta - t\psi(y) + k + n - 2}| \leq (2^\beta \sqrt{\alpha})^{1-j+1} \beta \leq N - 1 \quad (10')
\]

In the case \( \beta = N \), since \( \sigma_t^N \) is supported where \(|t\psi(y) + k - i\eta| \geq t/400\), we have \(|t\psi(y) + k - i\eta| > c(t + |k| + |\eta|)\) uniformly for \( y \in I_j \). So we get
\[
|\langle \partial/\partial \eta \rangle^j \frac{\sigma_t^N(y, \eta, k)}{i\eta - t\psi(y) + k + n - 2}| \leq C_j \mu(1 + |k| + |\eta|)^{-2-j+1} \quad (10'')
\]

Observe that since \( \sigma_t^\beta(y, \eta, k) \) for \( k \leq N - 1 \) is supported where \(|t\psi(y) + k - i\eta| \leq 2^\beta \sqrt{\alpha}\), there are at most \( 22^\beta \sqrt{\alpha} \) non-zero terms in the sum over \( k \), and the value of \( k \) in each case is comparable to \( t \).

Now we need Sogge's theorem for spherical harmonics.

**Theorem**

Let \( 1 \leq s \leq 2(n - 1)/n, r = (n - 1)s'/(n + 1) \). Then
\[
\|\xi_t f\|_{L^r(S^n)} \leq C k^{1/s'} \|f\|_{L^s(S^n)}
\]
Using the duality, this is equivalent to

\[ \| \xi_k f \|_{L^{r'}(S^n)} \leq C k^{1/r'} \| f \|_{L^r(S^n)} \]

From the conditions of \( r, s \) we have the relation \( 1/s - 1/r > 2/n \). Also, \( 1/r' - 1/s' > 2/n \). Now if we interpolate above two inequalities, we get

\[ \| \xi_k f \|_{L^r(S^n)} \leq C k^{1/r'} \| f \|_{L^r(S^n)} \quad \text{for} \]

\[ t = t/r + (1 - t)/s', \quad 1/p = t/s + (1 - t)/r' \quad \text{and} \]

\[ 1/r' = 1 - 1/r > \frac{n^2 + 2n - 4}{2n(n-1)} \quad \text{and} \quad 1/q - 1/p > 2/n . \]

Now from the above estimate (19) and (10') we get

\[ \| \sum \frac{\sigma_i^\beta(y, \eta, k)}{t \psi'(y) + k - i \eta + n - 2} \xi_k f \|_{L^r(S^n)} \leq (2^\beta \sqrt{\alpha})^{-i} t^{-1} t^{1/s'} \| f \|_{L^p(S^n)}. \]

Let

\[ F_1^\beta f(y, w) = \sum \frac{1}{2\pi} \int \frac{\sigma_i^\beta(y, \eta, k)}{t \psi'(y) + k - i \eta + n - 2} \xi_k f(\eta, \cdot)(w) e^{i\eta} d\eta \quad \beta = 0, 1, \ldots. \]

Then \( F_1^\beta f(y, w) = \sum \frac{1}{2\pi} \int K_1^\beta (y, y') f(y')(w) dy' \), where

\[ K_1^\beta (z) = \sum \frac{1}{2\pi} \int \frac{\sigma_i^\beta(y, \eta, k)}{t \psi'(y) + k - i \eta + n - 2} e^{it\eta} d\eta \xi_k \]

\[ = \sum \frac{1}{2\pi} \int (\partial/\partial \eta)^j \frac{\sigma_i^\beta(y, \eta, k)}{t \psi'(y) + k - i \eta + n - 2 (iz)^j} d\eta \xi_k \quad \text{for} \quad k \leq N - 1. \]

The integration in \( \eta \) is over an interval of length \( 2 \times 2^\beta \sqrt{\alpha} \). Therefore, \( K_1^\beta (z) \) is a bounded operator from \( L^p(S^n) \rightarrow L^q(S^n) \) whose operator norm is bounded by

\[ C j 2^\beta \sqrt{\alpha} |z|^{-j} (2^\beta \sqrt{\alpha}) t^{-1} t^{1/s'} \]

using the values \( j = 0, 10 \) we find that the norm is bounded by

\[ C (2^\beta \sqrt{\alpha}) t^{-1+1/s'} (1 + |2^\beta \sqrt{\alpha} z|)^{-10} \]

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Next, let $1/l + 1/p = 1/q + 1$, and apply Young's theorem and get
\[
\|F_i^\theta f\|_{L^q(I \times S, ds)} \leq (2^\theta \sqrt{\alpha})^{-1+1/q'} \|(1 + |2^\theta \sqrt{\alpha} z|)^{-10}\|_{L^1(R, dz)} \|f\|_{L^p(I \times S, ds)} \\
\leq (2^\theta \sqrt{\alpha})^{-1+1/q'} (2^\theta \sqrt{\alpha})^{-1/\delta} \|f\|_{L^p(I \times S, ds)}
\]
Since $2^\theta$ has power $1 - 1/l = 1/p - 1/q = 2/n + \delta$, and $-1 + 1/s' < -n/2n - 2$, we get
\[
\|\sum_{\theta} F_i^\theta f\|_{L^1(I \times S, ds)} \leq \sum_{\theta} (2^\theta \sqrt{\alpha})^{\frac{n}{2} + \delta} \|f\|_{L^1(I \times S, ds)} \\
\leq (1) \quad C t^{\frac{n}{2} + \delta} \|f\|_{L^1(I \times S, ds)} \\
\leq C t^{\delta} \|f\|_{L^1(I \times S, ds)}
\]
(1) holds since $\sum_{\theta=0}^{N-1} 2^\theta \sim \sqrt{\alpha}$. Now we want to estimate $F_i^N$.
It is obvious from (10") that the operators $(\partial/\partial y)^2 F_i^N$ and $\Delta_s F_i^N$ (with symbols
\[
-\eta^2 \frac{\sigma_i^N(y, \eta, k)}{i\eta - t\psi'(y) + k + n - 2} \quad \text{and} \quad -k(k + n - 2) \frac{\sigma_i^N(y, \eta, k)}{i\eta - t\psi'(y) + k + n - 2},
\]
are standard zero order multipliers.
They are therefore bounded from $L^p$ to $L^p$ for $1 < p < \infty$. Sovolev theorem [4] implies
\[
\|h\|_{L^q(I \times S)} \leq C \|((\partial/\partial y)^2 h\|_{L^p(I \times S)} + \|\Delta_s h\|_{L^p(I \times S)} \quad \text{for} \quad 1/p - 1/q \leq 2/n \\
\text{for all} \quad h \in C_0^\infty((-\infty, -N_0) \times S).
\]
So we find that
\[
\|F_i^N f\|_{L^q((-\infty, -N_0) \times S)} \leq C \|f\|_{L^p((-\infty, -N_0) \times S)} \quad (19)
\]
Since we have an estimate for the main term we will work on the remainder term and try to get similar estimate. Since
\[
F_i B_i f(y, w) = f(y, w) \\
= \sum \int \int \frac{\sigma_i^\theta(y, \eta, k)}{i\eta - t\psi'(y) + k + n - 2}(\partial/\partial y' - a(y)y' + b(y)) \\
\times (\partial/\partial y' - t\psi'(y) + k + n - 2)f(y', \eta)(w) e^{(y-y')\eta} dy' \eta \xi_k
\]
the difference $A_i - B_i$ appears in $R_i f(y, w) = F_iA_i f(y, w) - F_i B_i f(y, w)$ as the integrand. If we compute
\[
(\partial/\partial y' - t\psi'(y') - k)(\partial/\partial y' - t\psi'(y') + k + n - 2) \\
-(\partial/\partial y' - a(y)y' + b(y))(\partial/\partial y' - t\psi'(y) + k + n - 2) \\
= ((\partial/\partial y' - t\psi'(y') - k) - (\partial/\partial y' - a(y)y' + b(y)))(\partial/\partial y' - t\psi'(y') + k + n - 2) \\
+(\partial/\partial y' - a(y)y' + b(y))((\partial/\partial y' - t\psi'(y') + k + n - 2) - (\partial/\partial y' - t\psi'(y) + k + n - 2))
\]
\[
\begin{align*}
= t(y - y')^2 g(y, y')(\partial/\partial y' - t\psi'(y') + k + n - 2) + (\partial/\partial y' - a(y)y' + b(y))(t\psi'(y) - t\psi'(y')) \\
= t(y - y')^2 g(y, y')(\partial/\partial y' - t\psi'(y) + k + n - 2) + t(y - y')^2 g(y, y')(t\psi'(y) - t\psi'(y')) \\
+ t(y - y') h(y, y') (\partial/\partial y' - a(y)y' + b(y)) \quad \text{for} \\
& \begin{align*}
g(y, y') &= \int_0^1 (1-s) \psi''((1-s)y + sy') ds, & \text{and} \\
h(y, y') &= \int_0^1 (1-s) \psi''((1-s)y + sy') ds.
\end{align*}
\end{align*}
\]

Then
\[
R_4 f(y, w) = R_{1,4} f(y, w) + R_{2,4} f(y, w) + R_{3,4} f(y, w)
\]

\[
R_{1,4} f(y, w)
\]
\[
= \sum_k \int \int \frac{t(y - y')^2 \sigma_4^2(y, \eta, k)}{i\eta - t\psi'(y) + k + n - 2} g(y, y')(\partial/\partial y' - t\psi'(y) + k + n - 2) f(y',.,(w)e^{i(y - y')n} dy'd\eta_d\eta_k
\]
\[
= \sum_k \int \int t(y - y')^2 \sigma_4(y, \eta, k) g(y, y') f(y',.,(w)e^{i(y - y')n} dy'd\eta_d\eta_k
\]

\[
R_{2,4} f(y, w)
\]
\[
= \sum_k \int \int \frac{\sigma_4(y, \eta, k)}{i\eta - t\psi'(y) + k + n - 2} (\partial/\partial y' - a(y)y' + b(y))(t\psi'(y) - t\psi'(y')) f(y',.,(w)e^{i(y - y')n} dy'd\eta_d\eta_k
\]
\[
= \sum_k \int \int \frac{t(y - y') h(y, y')}{i\eta - t\psi'(y) + k + n - 2} f(y',.,(w)e^{i(y - y')n} dy'd\eta_d\eta_k
\]

\[
R_{3,4} f(y, y')
\]
\[
= \sum_k \int \int \frac{\sigma_4(y, \eta, k)}{i\eta - t\psi'(y) + k + n - 2} t(y - y')^2 g(y, y') h(y, y') f(y',.,(w)e^{i(y - y')n} dy'd\eta_d\eta_k
\]

We want to show
\[
\|R_4 f\|_{L^p(\mathbb{T} \times \mathbb{R}^d, \mu)} \leq C\|A_4 f\|_{L^p(\mathbb{T} \times \mathbb{R}^d, \mu)}.
\]

For this we first define \(F_{1,4}^\beta, F_{2,4}^\beta, F_{3,4}^\beta\) for \(\beta \leq N - 1\) as follows.

\[
F_{1,4}^\beta f(y, w) = \sum_k \int \int \frac{\sigma_4^\beta(y, \eta, k)}{i\eta - t\psi'(y) + k + n - 2} t g(y, y') (\partial/\partial \eta)^2 \sigma_4^\beta(y, \eta, k) f(y',.,(w)e^{i(y - y')n} dy'd\eta
\]

\[
F_{2,4}^\beta f(y, w) = \sum_k \int \int \frac{\sigma_4^\beta(y, \eta, k)}{i\eta - t\psi'(y) + k + n - 2} \frac{t(y - y') h(y, y')}{i\eta - t\psi'(y) + k + n - 2} f(y',.,(w)e^{i(y - y')n} dy'd\eta_d\eta_k
\]

\[
F_{3,4}^\beta f(y, w) = \sum_k \int \int \frac{\phi_4^\beta(y, \eta, k) \sigma_2^\beta(y, \eta, k)}{(i\eta - t\psi'(y) + k + n - 2)^2} (y - y')^2 g(y, y') h(y, y') f(y',.,(w)e^{i(y - y')n} dy'd\eta_d\eta_k
\]
With these definitions, we want to show

\[ ||\sum_{\beta} F_{i,1,2}(y,w)\|_{L^p(\{x,s,de\})} \leq C\|f\|_{L^p(\{x,s,de\})} \quad \text{for } i = 1,2,3... \text{ separately.} \]

First, we can rewrite \( F_{i,1,2}(y,w) \) as

\[ F_{i,1,2}(y,w) = \int K_{i,1,2}(y,y')f(y',\cdot)(w)dy', \quad \beta \leq N - 1 \]

where

\[
K_{i,1,2}(y,y') = t g(y,y') \sum_k \int \frac{\sigma_i(y,\eta, k)}{i\eta - t\psi'(y) + k + n - 2} (\partial/\partial \eta)^2 \sigma_i(t,\eta, k) e^{i(y-y')\eta} d\eta \xi_k
\]

\[
= t g(y,y') \sum_k \int (\partial/\partial \eta)^2 \sigma_{i,1,2}(y,\eta, k) \frac{1}{i(y-y')} e^{i(y-y')\eta} d\eta \xi_k \quad \text{where}
\]

\[
\sigma_{i,1,2}(y,\eta, k) = \frac{\sigma_i(y,\eta, k)}{i\eta - t\psi'(y) + k + n - 2} (\partial/\partial \eta)^2 \sigma_i(y,\eta, k)
\]

Then using (10') we can easily check that

\[
t ||g||_{\infty} |(\partial/\partial \eta)^2 \sigma_{i,1,2}(y,\eta, k)| \leq C_j t^{-1}(2^\alpha \sqrt{a})^{-3-j}
\]

(22)

Observe that since \( \sigma_{i,1,2}(y,\eta, k) \) for \( \beta \leq N - 1 \) is supported where \(|i\eta - t\psi'(y) - k| \leq 2^\alpha \sqrt{a}\), there are at most \( 22^\alpha \sqrt{a} \) non-zero terms in the sum over \( k \) and the value of \( k \) in each case is comparable to \( t \). Hence by Sogge's theorem and (21)

\[
||tg \sum_k (\partial/\partial \eta)^2 \sigma_{i,1,2}(y,\eta, k)\|_{L^p(S^n)} \leq C_j (2^\alpha \sqrt{a})^{-1-j} t^{-1+j+1/\nu'} ||f||_{L^p(S^n)}
\]

for \( \beta = 0,1,...N - 1 \). In the expression of \( K_{i,1,2}(y,z) \) when \( z = y - y' \), we can see the integration in \( \eta \) is over an interval of length \( \leq 22^\alpha \sqrt{a} \). Therefore, \( K_{i,1,2}(y,z) \) is a bounded operator from \( L^p(S^n) \) to \( L^q(S^n) \) whose operator norm is bounded by

\[
C_j |2^\alpha \sqrt{a} z|^{-j} t^{-1+1/\nu'}
\]

Using the values \( j = 0 \) and 10 we find that the norm is bounded by

\[
C t^{-1+1/\nu'} (1 + |2^\alpha \sqrt{a} z|)^{-10}.
\]

Next, let \( i/1 + 1/p = 1/q + 1 \). Then after applying Young's inequality,
On the other hand when $\beta = N$, it behaves like an ordinary fractional integral operator of order 2 and we can show

$$\|F_{1,i}^\beta f\|_{L^p(I \times S^d, \omega)} \leq C t^{-1+1/p'} \| (1 + |2^\beta \sqrt{a} z|)^{-10} \| f \|_{L^p(I \times S^d, \omega)}$$

$$= C t^{-1+1/p'} (2^\beta \sqrt{a})^{-1/10} \| f \|_{L^p(I \times S^d, \omega)}$$

$$\leq C' t^{-1/2} \| f \|_{L^p(I \times S^d, \omega)}$$

(21')

If we sum the series in $\beta$ then we get

$$\| \sum_{\beta} F_{1,i}^\beta f \|_{L^p(I \times S^d, \omega)} \leq C' \| f \|_{L^p(I \times S^d, \omega)}$$

Secondly, we want to show $\| F_{2,i} f \|_{L^p(I \times S^d, \omega)} \leq C \| f \|_{L^p(I \times S^d, \omega)}$.

As before, we can rewrite $F_{2,i} f$ as

$$F_{2,i} f(y, w) = \int K_{2,i}^\beta(y, y') f(y', .)(w) d\eta$$

where,

$$K_{2,i}^\beta(y, y') = \int \sigma_i^\beta(y, \eta, k) (y - y') e^{i(y - y') \eta \xi_k} d\eta d\xi_k$$

$$= \int (\partial/\partial \eta) \sigma_i^\beta(y, \eta, k) e^{i(y - y') \eta \xi_k} d\eta d\xi_k \text{ for } \beta \leq N - 1$$

Let's denote

$$\alpha_i^\beta(y, \eta, k) = (\partial/\partial \eta) \sigma_i^\beta(y, \eta, k) / (i \eta - t \psi'(y) + k + n - 2)^2.$$ 

Then using (10'), it's easy to check that

$$t |h|_{\infty} (\partial/\partial \eta)^j \alpha_i^\beta(y, \eta, k) \leq C_j t^{-1}(2^\beta \sqrt{a})^{-2-j}$$

(22')

The above and Sogge's theorem implies

$$\| \sum_k (\partial/\partial \eta)^j \alpha_i^\beta(y, \eta, k) \|_{L^p(S^n)} \leq C_j t^{-1+1/p'} (2^\beta \sqrt{a})^{-1-j} \| f \|_{L^p(S^n)}.$$ 

So $K_{2,i}^\beta(y, z)$ is a bounded operator from $L^p(S^n)$ to $L^p(S^n)$, whose operator norm is bounded by

$$t^{-1+1/p'}(2^\beta \sqrt{a})^{-j} \text{ for } z = y - y'$$

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and after applying Young's inequality, we get

$$
\|F_{2,1}^{\theta}f\|_{L^q(I\times S,d\sigma)} \leq C t^{-1+1/q'} \|f\|_{L^p(I\times S,d\sigma)} \left(1 + \left|2\sqrt{a}z\right|\right)^{-10} \|f\|_{L^p(I\times S,d\sigma)}
$$

and this implies

$$
\|\sum_{\beta=0}^{N-1} F_{2,1}^{\theta}f\|_{L^q(I\times S,d\sigma)} \leq C' t^{-1/2} \|f\|_{L^p(I\times S,d\sigma)}
$$

(23)

And as before $F_{2,1}^N$ behaves like an ordinary fractional integral operator and we get the same estimate.

Finally, we want to show

$$
\|F_{3,1}^{\theta}f\|_{L^q(I\times S,d\sigma)} \leq C \|f\|_{L^p(I\times S,d\sigma)}.
$$

For this we can rewrite $F_{3,1}^{\theta}f$ as

$$
F_{3,1}^{\theta}f(y,w) = \int K_{3,1}^{\theta}(y,y')f(y',\cdot)(w)dy',
$$

where

$$
K_{3,1}^{\theta}(y,y') = t^2 g(y,y')h(y,y') \sum_k \int \tilde{\sigma}_{3,4}(y,\eta,k)e^{i(y-y')\eta}d\eta
$$

and

$$
\tilde{\sigma}_{3,4} = \frac{\phi(y,\eta,k)\sigma(y,\eta,k)(\partial/\partial \eta)^3\sigma(y,\eta,k)}{(i\eta - t\psi(y) + k + n - 2)^2}
$$

Then using (10'), it is easy to check

$$
t^2 |g|_{\infty} |h|_{\infty} |(\partial/\partial \eta)^3 \tilde{\sigma}_{3,4}| \leq C t^{-2} (2\sqrt{a})^{-1-j}
$$

(22'')

From this and Sogge's theorem, it follows

$$
\|t^2 g h \sum_k (\partial/\partial \eta)^j \tilde{\sigma}_{3,4}(y,\eta,k)\xi_k f\|_{L^q(S^n)} \leq C t^{-2+1/p'} (2\sqrt{a})^{-j} \|f\|_{L^p(S^n)}
$$

and as before when $z=y-y'$ $K_{3,1}^{\theta}(y,z)$ is a bounded operator from $L^p(S^n)$ to $L^q(S^n)$ with norm bounded by

$$
C_j t^{-2+1/p'} (2\sqrt{a}) |2\sqrt{a}z|^{-j}.
$$

Using the values $j=0$ and $10$ we find that the norm is bounded by

$$
C't^{-2+1/p'} (2\sqrt{a})(1+|2\sqrt{a}z|)^{-10}.
$$
Then for $1/q + 1 = 1/p + 1/p'$, Young's theorem gives the following.

$$
\| \sum_{\beta=0}^{N-1} F_{3,1}^\beta f \|_{L^q(I \times S, d\alpha)} \leq C t^{-2+1/p'} (2^\alpha \sqrt{a})^{1-1/p} \| f \|_{L^p(I \times S, d\alpha)}
$$

$$
= C t^{-2+1/p'} t^{2+\epsilon} \| f \|_{L^p(I \times S, d\alpha)}
$$

$$
= C' \| f \|_{L^p(I \times S, d\alpha)}
$$

And as before $F_{3,4}^N$ behaves like an ordinary fractional integral operator of order 2 and we get the same estimate as $F_{1,4}^N$, i.e.

$$
\| F_{3,1}^N f \|_{L^q(I \times S, d\alpha)} \leq C \| f \|_{L^p(I \times S)}
$$

Now if we denote $F_1^1 f(y, w) = \sum_{i=1}^3 F_{i,1}^1 f(y, w)$, then from the estimate

$$
\| F_1^1 f \|_{L^q(I \times S, d\alpha)} \leq C \| f \|_{L^p(I \times S, d\alpha)} \tag{24}
$$

we get the second remainder term, i.e.

$$
R_2^1 f(y, w) = F_1^1 A_1 f(y, w) - R_1^1 f(y, w), \quad R_1^1 = R_1
$$

Let

$$
V_i(y, y', \eta, k) = t(y - y')^2 g(y, y') \sigma_i(y, \eta, k) + \frac{t(y - y') h(y, y')}{i \eta - t \psi'(y) + k + n - 2}
$$

$$
+ \frac{t^2(y - y')^3 g(y, y') h(y, y') \sigma_i(y, \eta, k)}{i \eta - t \psi'(y) + k + n - 2}
$$

Then from the following expressions

$$
F_1^1 f(y, w) = \sum_k \int \int \frac{\sigma_i(y, \eta, k)}{i \eta - t \psi'(y) + k + n - 2} V(y, y', \eta, k) f(y', .)(w) e^{i(y - y')^\eta} dy' d\eta \xi_k, \text{ and}
$$

$$
R_1 f(y, w) = \sum_k \int \int V(y, y', \eta, k) f(y', .)(w) e^{i(y - y')^\eta} dy' d\eta \xi_k
$$

we get

$$
R_2^1 f(y, w) = \sum_k \int \int \frac{\sigma_i(y, \eta, k)}{i \eta - t \psi'(y) + k + n - 2} V(y, y', \eta, k) [A_i - B_i] f(y', .)(w) e^{i(y - y')^\eta} dy' d\eta \xi_k
$$

$$
= \sum_k \int \int V(y, y', \eta, k)^2 f(y', .)(w) e^{i(y - y')^\eta} dy' d\eta \xi_k
$$

Define

$$
F_1^2 f(y, w) = \sum_k \int \int \frac{\sigma_i(y, \eta, k)}{i \eta - t \psi'(y) + k + n - 2} V(y, y', \eta, k)^2 f(y', .)(w) e^{i(y - y')^\eta} dy' d\eta \xi_k
$$

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Then using the same techniques as before we can show that

\[ ||F_i^2 f||_{L^p(I \times S, e^{\eta \rho} dy dw)} \leq C ||f||_{L^p(I \times S, e^{\eta \rho} dy dw)} \]  \hspace{2cm} (25)

In this way we have a next remainder term, which we call \( R^2_i f(y, w) \) coming from

\[ R^2_i f(y, w) = F^2_i A_1^i f(y, w) - R^2_i f(y, w) \]

Now \( R^2_i f \) has the following form.

\[ R^2_i f(y, w) = \sum_k \int \int V(y, y', \eta, k)^3 f(y', \cdot)(w)e^{i(y-y')^* \eta} dy' \eta \xi_k \]

**Claim**

1. \( ||R^2_i f||_{L^p(I \times S, e^{\eta \rho} dy dw)} \leq c ||f||_{L^p(I \times S, e^{\eta \rho} dy dw)} \quad c < 1/2 \)

2. \( ||f||_{L^q(I \times S, dw)} \leq Ce^{-\eta/2} ||f||_{L^p(I \times S, dw)} \) for \( f \in C_0^\infty(I \times S) \) for \( 1/p - 1/q \leq 2/n \)

**Proof of Claim**

We want to show this is true for each term separately. First, let

\[ R^2_{i,2}(y, w) = \sum_k \int \int \phi(y, \eta, k)^3(y - y')^3 g^3(y, y') \sigma_i^3(y, \eta, k)f(y', \cdot)(w)e^{i(y-y')^* \eta} dy' \eta \xi_k \]

Now let's denote \( \sigma^3_{i,2}(y, \eta, k) = \phi(y, \eta, k)(y - y')^3 \sigma_i^3(y, \eta, k) \) then

\[ R^2_{i,2}(y, w) = \sum_k \int \int t^3 g^3(y, y') \phi(y, \eta, k)(\partial/\partial \eta)^3 \sigma_i^3(y, \eta, k)f(y', \cdot)(w)e^{i(y-y')^* \eta} dy' \eta \xi_k \]

and from the fact that \( \sigma_i^3 \in S^{-3} \), and (10') we can check that

\[ |t^3|g^3_{\infty}(\partial/\partial \eta)^3 \sigma^3_{i,2} | \leq C_j t^3 |g|^3_{\infty}(2^\alpha \sqrt{a})^{-3-j} \leq C_j (2^\alpha \sqrt{a})^{-3-j} \]  \hspace{2cm} (26)

which implies

\[ ||t^3|g^3_{\infty}(\partial/\partial \eta)^3 \sigma^3_{i,2}(y, \eta, k) \xi_k f||_{L^p(S)} \leq C_j (2^\alpha \sqrt{a})^{-3-j} t^a ||f||_{L^p(S)} \]  \hspace{2cm} for some \( \alpha < 1 \)

On the other hand, we can write \( R^2_{i,2}(y, w) = \int K^0_{i,2}(y, y')f(y')dy' \), where

\[ K^0_{i,2}(y, y') = t^3 g^3(y, y') \int \sigma^3_{i,2}(y, \eta, k)e^{i(y-y')^* \eta} d\eta \xi_k \]

Then the above estimate implies \( K^0_{i,2}(y, y') \) is a bounded operator from \( L^q(S) \to L^p(S) \), whose operator norm bounded by

\[ t^a (2^\alpha \sqrt{a})^{-1} |2^\alpha \sqrt{a}|^{-j} \]

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Then by the usual procedure (including Young's theorem) we can show that

\[ ||R^{\delta,3}_{1,t}f||_{L^q(I \times S, dx)} \leq t^2 (2^\delta \sqrt{a})^{-1} ||(1 + |2^\delta \sqrt{a}z|)^{-10}||_{L^1(R,dz)} ||f||_{L^q(I \times S, dx)} \]
\[ \leq t^2 (2^\delta \sqrt{a})^{-2} ||f||_{L^q(I \times S, dx)} \]
\[ \leq C_j 2^{-2\delta t - 1 + \alpha} e^{-\gamma t} ||f||_{L^q(I \times S, dx)} \]

Since \( y > -N_0 \), this means that

\[ ||\sum_{\beta} R^{N,3}_{1,t} f||_{L^q(I \times S, dx)} \leq c ||f||_{L^q(I \times S, dx)} \text{ for } c < 1 \]  \hspace{1cm} (27) \]

Now for \( \beta = N \), we need different estimate. By definition,

\[ R^{N,3}_{1,t} f(y, w) = \sum_{k} \int \int t^2 g(y, y')^2 \phi(y, \eta, k)(\partial/\partial \eta)^i \sigma^N(y, \eta, k)^3 f(y', \cdot)(w)e^{i(y-y')n} dy' d\eta \xi_k \]

Then

\[ |t^2 (\partial/\partial \eta)^i (\partial/\partial y)^j \phi(y, \eta, k)(\partial/\partial \eta)^6 \sigma^N(y, \eta, k)^3| \leq Ct^{-2} (1 + |k| + |\eta|)^{-2 - j - l} \]  \hspace{1cm} (28) \]

It follows from (10') that \( (\partial/\partial y)^2 R^{N,3}_{1,t} \) and \( \Delta S R^{N,3}_{1,t} \) are standard zero order multipliers and therefore bounded from \( L^p \) to \( L^p \). \( 1 < p < \infty \). Now Sobolev's theorem, we have

\[ ||h||_{L^q(I \times S, dx)} \leq C(||(\partial/\partial y)^2 h||_{L^q(I \times S, dx)} + ||\Delta S h||_{L^q(I \times S, dx)}) \]

Combining above arguments, we find that

\[ ||R^{N,3}_{1,t} f||_{L^q(I \times S, dx)} \leq C t^{-2} e^{-t} ||f||_{L^q(I \times S, dx)} \]

Secondly, let's define \( R^{\delta,3}_{2,t} \) as

\[ R^{\delta,3}_{2,t} f(y, w) = \sum_{k} \int \int \phi(y, \eta, k) \frac{t^3 (y-y')^3 h^2(y, y')}{(i\eta - t\psi(y) + k + n - 2)^3} f(y', \cdot)(w)e^{i(y-y')n} dy' dw \xi_k \]

If we denote

\[ \sigma^{\delta,3}_{2,t}(y, \eta, k) = \frac{\phi(y-y')^3}{(i\eta - t\psi(y) + k + n - 2)^3}, \]  \hspace{1cm} (29) \]

then

\[ |t^3 (\partial/\partial \eta)^j \sigma^{\delta,3}_{2,t}(y, \eta, k)| \leq C_j (2^\delta \sqrt{a})^{-3 - j} \]

which implies

\[ ||\sum_{k} t^3 (\partial/\partial \eta)^j \sigma^{\delta,3}_{2,t}(y, \eta, k) \xi_k f||_{L^q(S)} \leq C_j t^2 (2^\delta \sqrt{a})^{-3 - j} ||f||_{L^p(S)} \alpha << 1 \]

On the other hand, we can write \( R^{\delta,3}_{2,t} f(y, w) = \int K^{\delta,3}_{2,t}(y, y') f(y', .) dy' \), where

\[ K^{\delta,3}_{2,t}(y, y') = h(y, y')^2 \int \sigma^{\delta,3}_{2,t}(y, \eta, k)e^{i(y-y')n} d\eta \xi_k \]

Then the above estimate implies that when \( z = y - y' \) \( K^{\delta,3}_{2,t}(y, z) \) is a bounded operator from \( L^q(S) \) to \( L^p(S) \) whose operator norm bounded by \( t^2 e^{\alpha (2^\delta \sqrt{a})^{-1} |2^\delta \sqrt{a}z|^{-j}} \). Now by the same
method as $R_{1,i}^3$ we can show that

$$
||R_{1,i}^3 f||_{L^t(x \times S_{de})} = C t^{-1+\gamma} ||f||_{L^t(x \times S_{de})}
$$

And for $\beta = N$, we get the same estimate as $R_{1,i}^N$. All other terms appearing in $R_i^3 f$ are combinations of $R_{1,i}^3$ and $R_{2,i}^3$ and they give the same estimates. So we get

$$
||R_i^3 f||_{L^t(x \times S_{de})} \leq c' ||f||_{L^t(x \times S_{de})} \quad c' < 1/2
$$

Now we want to show the second half of the claim. Since we have

$$
||\sum_\beta F_i^\beta f||_{L^t(x \times S)} \leq C t^\delta ||f||_{L^t(x \times S)} \quad \text{for } 1/p - 1/q \geq 2/n
$$

take $\delta$ small enough and use interpolation theorem coming from

$$
||F_i^\beta f||_{L^2(x \times S_{de})} \leq t^{-1}(2\sqrt{a})^{-1} ||f||_{L^2(x \times S_{de})},
$$

which is

$$
||\sum_\beta F_i^\beta f||_{L^2(x \times S_{de})} \leq \sum_\beta t^{-1}(2\sqrt{a})^{-1} ||f||_{L^2(x \times S_{de})}
\leq C t^{-3/2}(\psi''(y))^{-1/2} ||f||_{L^2(x \times S_{de})}
\leq C' e^{-u/2} ||f||_{L^2(x \times S_{de})}
$$

And similarly use the same interpolation for terms like

$$
||\sum_\beta F_i^\beta f||_{L^t(x \times S)} \leq C ||f||_{L^t(x \times S)} \quad i = 2, 3, \quad 1/p - 1/q \geq 2/n
$$

we gain some power of $t$ after interpolation. Now from the relation

$$
f = F_i A_i f - R_i^1 f, \quad \text{and } R_i^1 f = F_i^{-1} A_i f - R_i^{-1} f, \quad i = 2, 3
$$

we get

$$
||f||_{L^t(x \times S)} \leq \sum_{i=1}^{N-1} F_i^0 A_i f||_{L^t(x \times S)} + ||R_i^N A_i f||_{L^t(x \times S)}
+ \sum_{i=1}^{N-1} F_i^0 A_i f||_{L^t(x \times S)} + ||R_i^N A_i f||_{L^t(x \times S)}
+ \sum_{i=1}^{N-1} F_i^0 A_i f||_{L^t(x \times S)} + ||R_i^N A_i f||_{L^t(x \times S)} + ||R_i^2 f||_{L^t(x \times S)}
\leq (1) C e^{-u/2} ||A_i f||_{L^t(x \times S)} + t^{-1+\alpha} ||f||_{L^t(x \times S)}
$$
Inequality (1) follows from (21), (21') and above interpolation argument. Now since the second term of the last inequality has negative power of $t$ we can cancel that term again the left hand side and get the desired estimate (19) for $1/p - 1/q = 2/n - \epsilon$.

From now on we will work on the second case (i.e. $y < -N_0$). First, introduce $(\phi_\beta)_0^N$, satisfying (12) and $2^N < t/10 < 2^{N+1}$. We had two symbol estimate i.e.

$$|((\partial/\partial \eta))^j \sigma^\beta_i(y, \eta, k)_{i\eta - t\psi'(y) + k + n - 2}| \leq (2^\beta)^{-1-j-t^{-1}} \beta \leq N - 1$$

In the case $\beta = N$, since $\sigma^N_i(y, \eta, k)$ is supported where $|t\psi'(y) + k - i\eta| \geq t/400$, we have $|t\psi'(y) + k - i\eta| > c(t + |k| + |\eta|)$ uniformly for $y \in I_j$. So we get

$$|((\partial/\partial \eta))^j \delta^N_i(y, \eta, k)_{i\eta - t\psi'(y) + k + n - 2}| \leq C_{j,i}(1 + k + |\eta|)^{-2-j-t^{-1}}$$

Observe that since $\sigma^\beta_i(y, \eta, k)$ for $k \leq N - 1$ is supported where $|t\psi'(y) + k - i\eta| \leq 2$ there are at most $22^\beta$ non-zero terms in the sum over $k$, and the value of $k$ in each case comparable to $t$.

Now from the above estimates (20), (10') and Sogge's theorem, we get

$$\|\sum_k ((\partial/\partial \eta))^j \sigma^\beta_i(y, \eta, k)_{i\eta - t\psi'(y) + k + n - 2}\|L^s(S^n) \leq (2^\beta)^{-j-t^{-1}}\|f\|L^s(S^n).$$

Let

$$F^\beta_i(y, w) = \sum_k \frac{1}{2\pi} \int \frac{\sigma^\beta_i(y, \eta, k)}{t\psi'(y) + k - i\eta + n - 2} f_\xi(\eta, \cdot)(\omega) e^{im\eta} d\eta \quad \beta = 0, 1, \ldots$$

Then $F^\beta_i(y, w) = \sum_k \frac{1}{2\pi} \int K^\beta_i(y, y') f(y')(\omega) dy'$, where

$$K^\beta_i(y, z) = \sum_k \frac{1}{2\pi} \int \frac{\sigma^\beta_i(y, \eta, k)}{t\psi'(y) + k - i\eta + n - 2} e^{im\eta} d\eta \xi_k$$

$$= \sum_k \frac{1}{2\pi} \int (\partial/\partial \eta)^j \frac{\sigma^\beta_i(y, \eta, k)}{t\psi'(y) + k - i\eta + n - 2} (iz)^j d\eta \xi_k \quad \text{for } k \leq N - 1.$$

the integration in $\eta$ is over an interval of length $2 \times 2^\beta$. Therefore, $K^\beta_i(y, z)$ is a bounded operator from $L^p(S^n) \rightarrow L^q(S^n)$ whose operator norm is bounded by

$$C_{j} |2^\beta z|^{-j}(2^\beta)^{-t^{-1}}\|f\|$$

using the values $j = 0, 10$ we find that the norm is bounded by

$$C(2^\beta) t^{-1+1/\nu}(1 + |2^\beta z|)^{-10}$$

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Next, let \( 1/l + 1/p = 1/q + 1 \), and apply Young's theorem and get

\[
\|F_{\beta}f\|_{L^r(I\times S, dx)} \leq (2^\beta)^{1-1/l + 1/r'} \|(1 + |2^\beta z|)^{-10}\|L^r(R, dx)\|f\|_{L^p(I\times S, dx)} \\
\leq (2^\beta)^{1-1/l + 1/r'} (2^\beta)^{-1/l} \|f\|_{L^p(I\times S, dx)}
\]

Since \( 2^\beta \) has power \( 1 - 1/l = 1/p - 1/q = 2/n + \delta \), and \(-1 + 1/s' < \frac{-n}{2(n-2)} \), we get

\[
\|\sum_{\beta} F_{\beta} f\|_{L^r(I\times S, dx)} \leq \sum_{\beta} (2^\beta)^{2/n + \delta - n/(2n-2)} \|f\|_{L^r(I\times S, dx)} \leq (1) C t^{(\frac{2}{n} + \delta) - \frac{2n}{2n-2}} \|f\|_{L^p(I\times S, dx)} \leq C't^\delta \|f\|_{L^p(I\times S, dx)}
\]

(1) holds since \( \sum_{\beta=0}^{N} 2^\beta \approx t \). Now we want to estimate \( F_i^N \).

It is obvious from \((10)^\circ\) that the operators \((\partial/\partial y)^3 F_i^N\) and \(\Delta_s F_i^N\) (with symbols

\[
-\frac{\eta^2}{i\eta - t\psi'(y) + k + n - 2} \quad \text{and} \quad -k(k+n-2) \frac{\eta^2}{i\eta - t\psi'(y) + k + n - 2}, \quad \text{respectively}
\]

are standard zero order multipliers.

They are therefore bounded from \(L^p\) to itself for \(1 < p < \infty\). Sobolev theorem \([4]\) implies

\[
\|h\|_{L^p(I\times S)} \leq C \|\partial/\partial y^3 h\|_{L^p(I\times S)} + \|\Delta_s h\|_{L^p(I\times S)} \quad \text{for} \quad 1/p - 1/q \leq 2/n
\]

for all \(h \in C^\infty_0((-\infty,-N_0) \times S)\). So we find that

\[
\|F_{\beta} f\|_{L^p((-\infty,-N_0) \times S)} \leq C \|f\|_{L^p((-\infty,-N_0) \times S)}
\]

(21)

From now on we will work on the remainder term and try to get similar estimate as \(y > -N_0\) case. Since

\[
F_i B_i f(y,w) = f(y,w),
\]

if we denote

\[
R_i f(y,w) = F_i A_i f(y,w) - F_i B_i f(y,w)
\]

as the remainder term, then as before

\[
R_{\alpha} f(y,w) = R_{\alpha_1} f(y,w) + R_{\alpha_2} f(y,w) + R_{\alpha_3} f(y,w), \quad \text{and}
\]

\[
R_{\alpha_1} f(y,w) = \sum_k \int \int t(y-y')^3 g(y,y') \sigma_i(y,\eta, k) f(y',\cdot)(w) e^{(y-y')^2 \eta dy'd\eta d\xi_k}
\]

\[
R_{\alpha_2} f(y,w) = \sum_k \int \int \frac{t(y-y') h(y,y')}{i\eta - t\psi'(y) + k + n - 2} f(y',\cdot)(w) e^{(y-y')^2 \eta dy'd\eta d\xi_k}
\]

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\[ R_{3,t}f(y,w) = \sum_k \int \int \frac{t^2 \sigma^2(y, \eta, \kappa)(y-y')^3}{i\eta - t\psi'(y) + k + n - 2} g(y, y') h(y, y') f(y, ..)(w) e^{i(y-y')^n} dy' d\eta' \xi_k \]

We want to show

\[ ||R_{t}f||_{L^q(I \times S, dx)} \leq C ||A_{t}f||_{L^p(I \times S, dx)} \]

For this we define \( F_{1,t}^\beta, F_{2,t}^\beta, F_{3,t}^\beta \) for \( \beta \leq N - 1 \) as before.

\[ F_{1,t}^\beta f(y,w) = \sum_k \int \int \frac{t g(y, y') \phi^\beta(y, \eta, \kappa)}{i\eta - t\psi'(y) + k + n - 2} (y-y')^2 \sigma^2(y, \eta, \kappa) f(y, ..)(w) e^{i(y-y')^n} dy' d\eta' \xi_k \]

\[ F_{2,t}^\beta f(y,w) = \sum_k \int \int \frac{\sigma^2(y, \eta, \kappa)}{i\eta - t\psi'(y) + k + n - 2} \frac{t(y-y') h(y, y')}{i\eta - t\psi'(y) + k + n - 2} f(y, ..)(w) e^{i(y-y')^n} dy' d\eta' \xi_k \]

\[ F_{3,t}^\beta f(y,w) = \sum_k \int \int \frac{\phi^\beta(y, \eta, \kappa) \sigma^2(y, \eta, \kappa)}{(i\eta - t\psi'(y) + k + n - 2)^2} (y-y')^3 g(y, y') h(y, y') f(y, ..)(w) e^{i(y-y')^n} dy' d\eta' \xi_k \]

And then we want to show

\[ ||\sum_{\beta} F_{i,t}^\beta f||_{L^q(I \times S, dx)} \leq C ||f||_{L^p(I \times S, dx)} \text{ for } i = 1, 2, 3... \text{ separately.} \]

First, we can rewrite \( F_{1,t}^\beta f \) as

\[ F_{1,t}^\beta f(y,w) = \int K_{1,t}^\beta(y, y') f(y, ..)(w) dy', \ \beta \leq N - 1 \]

where

\[ K_{1,t}^\beta(y, y') = t g(y, y') \sum_k \int \frac{\phi^\beta(y, \eta, \kappa)}{i\eta - t\psi'(y) + k + n - 2} (\partial/\partial \eta)^2 \sigma^2(t, \eta, \kappa) e^{i(y-y')^n} d\eta' \xi_k \]

\[ = t g(y, y') \sum_k \int (\partial/\partial \eta)^j \tilde{\sigma}_{1,t}^\beta(y, \eta, \kappa) \frac{1}{(i(y-y'))^j} e^{i(y-y')^n} d\eta' \xi_k, \ \text{where} \]

\[ \tilde{\sigma}_{1,t}^\beta(y, \eta, \kappa) = (\partial/\partial \eta)^2 \phi^\beta(y, \eta, \kappa) \sigma^2(y, \eta, \kappa) \]

Then using (10') we can easily check that

\[ t ||g||_{\infty}(\partial/\partial \eta)^j \tilde{\sigma}_{1,t}^\beta(y, \eta, \kappa) | \leq C_j e^{t \psi(2^\beta) - 4-j} \]

(22')

Observe that since \( \tilde{\sigma}_{1,t}^\beta(y, \eta, \kappa) \) for \( \beta \leq N - 1 \) is supported where \( |i\eta - t\psi'(y) - k| \leq 2^\beta \), there are at most \( 2 \times 2^\beta \) non-zero terms in the sum over \( k \) and the value of \( k \) in each case is comparable to \( t \).

Hence by Sogge's theorem and (22')

\[ ||t \sum_k (\partial/\partial \eta)^j \tilde{\sigma}_{1,t}^\beta(y, \eta, \kappa) \xi_k f||_{L^q(S^n)} \leq C_j (2^\beta)^{-3-j} t^{1/\nu} ||f||_{L^p(S^n)} \text{ for } \beta = 0, 1, ..., N - 1. \]

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In the expression of $K_{1,t}^\beta(y, z)$ when $z = y - y'$, we can see that the integration in $\eta$ is over an interval of length $\leq 22^\beta$. Therefore, $K_{1,t}^\beta(y, z)$ is a bounded operator from $L^p(S^n)$ to $L^q(S^n)$ whose operator norm is bounded by

$$C_j e^{c^\psi(2^\beta)^{-2} |2^\beta z|^{-j}} t^{1/p'}.$$

Using the values $j=0$ and $10$ we find that the norm is bounded by

$$C t^{1/p'} e^{c^\psi(2^\beta)^{-2}(1+|2^\beta z|)^{-10}}.$$

Next, let $1/p + 1/p = 1/q + 1$. Then after applying Young's inequality,

$$||F_{1,t}^\beta f||_{L^q(I \times S, dx)} \leq C t^{1/p'} (2^\beta)^{-2} e^{c^\psi(1+|2^\beta z|)^{-10}} ||f||_{L^p(I \times S, dx)}$$

$$= C t^{1/p'} (2^\beta)^{-2 - 1/p'} e^{c^\psi} ||f||_{L^p(I \times S, dx)}$$

$$\leq (1) C ||f||_{L^p(I \times S, dx)}$$

(1) holds since $e^{c^\psi}$ vanishes fast as $y$ goes to $-\infty$. On the other hand when $\beta = N$, it behaves like ordinary fractional integral operator of order 2 and we can show

$$||F_{1,t}^N f||_{L^q(I \times S, dx)} \leq C' ||f||_{L^p(I \times S, dx)}^{1/p - 1/q} \leq 2/n$$

(21')

If we sum the series in $\beta$ then we get

$$||\sum_{\beta} F_{1,t}^\beta f||_{L^q(I \times S, dx)} \leq C' ||f||_{L^p(I \times S, dx)}$$

Secondly, we want to show $||F_{2,t} f||_{L^q(I \times S, dx)} \leq C ||f||_{L^p(I \times S, dx)}$.

As before, we can rewrite $F_{2,t}^\beta f$ as

$$F_{2,t}^\beta f(y, w) = \int K_{2,t}^\beta(y, y') f(y', .)(w) d\eta \quad \text{where,}$$

$$K_{2,t}^\beta(y, y') = t h(y, y') \int \frac{\sigma_{\eta}(y, \eta, k) (y - y')}{i \eta - t \psi'(y) + k + n - 2} e^{i(u - v')^n} d\eta \xi_k$$

$$= t h(y, y') \int \frac{(\partial/\partial \eta) \sigma_{\eta}(y, \eta, k)}{(i \eta - t \psi'(y) + k + n - 2)^2} e^{i(u - v')^n} d\eta \xi_k \quad \text{for } \beta \leq N - 1$$

Let's denote

$$\tilde{\sigma}_{2,t}^\beta(y, \eta, k) = \frac{(\partial/\partial \eta) \sigma_{\eta}(y, \eta, k)}{(i \eta - t \psi'(y) + k + n - 2)^2}$$

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Then using (10'), it's easy to check that
\[ t|[(\partial / \partial \eta)j\tilde{a}_{2,t}(y, \eta, k)| \leq C_j t^{-1}(2^j)^{2-j} \]

The above and Sogge's theorem implies
\[ ||t \sum_k (\partial / \partial \eta)j\tilde{a}_{2,t}(y, \eta, k)\xi_k||_{\mathcal{L}^q(S^n)} \leq C_j t^{-1+1/p'}(2^j)^{-1-j}||f||_{\mathcal{L}^p(S^n)}. \]

So \( K_{2,t}(y, z) \) is a bounded operator from \( \mathcal{L}^p(S^n) \) to \( \mathcal{L}^q(S^n) \), whose operator norm is bounded by \( t^{-1+1/p'}e^{e^{u}|2^j z|} \), when \( z = y - y' \), and after applying Young's inequality, we get
\[ ||F_{2,t}^\beta f||_{\mathcal{L}^q(I \times S, dx)} \leq Ct^{-1+1/p'}e^{e^{u}|||(1 + |2^j z|)|^{-10}||\mathcal{L}^p(R, dx)||f||_{\mathcal{L}^p(I \times S, dx)}, \]
and this implies
\[ \sum_{\beta=0}^{N-1} F_{2,t}^\beta f||_{\mathcal{L}^q(I \times S, dx)} \leq C^||f||_{\mathcal{L}^p(I \times S, dx)} \quad (23') \]

And as before \( F_{2,t}^N \) behaves like a ordinary fractional operator and we get
\[ ||F_{2,t}^N f||_{\mathcal{L}^q(I \times S, dx)} \leq C||f||_{\mathcal{L}^p(I \times S, dx)} \]

The above two inequalities imply
\[ ||F_{2,t} f||_{\mathcal{L}^q(I \times S, dx)} \leq C||f||_{\mathcal{L}^p(I \times S, dx)} \]

Finally, using the same method, we can show
\[ ||F_{3,t} f||_{\mathcal{L}^q(I \times S, dx)} \leq Ct^{-1/2}||f||_{\mathcal{L}^p(I \times S, dx)}. \]

Now if we denote \( F_{i}^2 f(y, w) = \sum_{i=1}^{3} F_{i} f(y, w) \), then from the estimate
\[ ||F_{i}^2 f||_{\mathcal{L}^q(I \times S, dx)} \leq Ct^{-1/2}||f||_{\mathcal{L}^p(I \times S, dx)} \quad (24') \]
we get the second remainder term as before i.e.
\[ R_{i}^2 f(y, w) = F_{i}^2 A_t f(y, w) - R_{i} f(y, w), \quad R_{i} = R_t \]
\[ R_{i}^2 f(y, w) = \sum_k \int \int V(y, y', \eta, k)^2 f(y, .)(w)e^{i(y'-\eta)\eta} dy'd\eta \xi_k \]

Define
\[ F_{i}^3 f(y, w) = \sum_k \int \int \frac{\sigma_i(y, \eta, k)}{i \eta - t its(y) + k + n - 2} V(y, y', \eta, k)^2 f(y', .)(w)e^{i(y'-\eta)\eta} dy'd\eta \xi_k \]

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Then using the same techniques as before we can show that

\[ ||F^3_t f||_{L^q(I \times S, e^{\nu y} dy dw)} \leq C ||f||_{L^p(I \times S, e^{\nu y} dy dw)} \]  \hspace{0.5cm} (25')

In this way we have the next remainder term, which we call \( R^3_t f(y, w) \) coming from

\[ R^3_t f(y, w) = F^3_t A_t f(y, w) - R^2_t f(y, w) \]

\( R^3_t f \) has the following form.

\[ R^3_t f(y, w) = \sum_k \int \int V(y, y', \eta, k) f(y', \eta)(w) e^{i(y-y')^n} dy' d\eta \xi_k \]

**Claim**

1. \( ||R^3_t f||_{L^q(I \times S, e^{\nu y} dy dw)} \leq c ||f||_{L^q(I \times S, e^{\nu y} dy dw)} \quad c << 1 \)
2. \( ||f||_{L^q(I \times S, dx)} \leq C e^{-\nu y/2} ||f||_{L^p(I \times S, dx)} \quad \text{for} \ 1/p - 1/q \leq 2/n \)

**Proof of Claim** We want to show this is true for each term separately. First, let

\[ R^3_{1, i} f(y, w) = \sum_k \int \int (y-y')^3 \phi_\rho(y, \eta, k) g^3(y, y') \sigma^3_t(y, \eta, k) f(y', \eta)(w) e^{i(y-y')^n} dy' d\eta \xi_k \]

Now if we denote \( \sigma^3_{1,i}(y, \eta, k) = \phi_\rho(y, \eta, k)(y-y')^3 \sigma^3_t(y, \eta, k) \) then

\[ R^3_{1, i} f(y, w) = \sum_k t^3 \int \int g^3(y, y') \phi_\rho(y, \eta, k) (\partial/\partial \eta)^3 \sigma^3_t(y, \eta, k) f(y', \eta)(w) e^{i(y-y')^n} dy' d\eta \xi_k \]

and from the fact that \( \sigma^3_t \in S^{-3} \), and (10') we can check that

\[ |(\partial/\partial \eta)^3 \sigma^3_{1,i}(y, \eta, k)| \leq C_j (2^\theta)^{-9-j} \]  \hspace{0.5cm} (26')

which implies

\[ ||\sum_k (\partial/\partial \eta)^i \sigma^3_{1,i}(y, \eta, k) \xi_k f||_{L^q(S)} \leq C_j (2^\theta)^{-6-j} ||f||_{L^q(S)} \quad \text{for some} \ \alpha < 1 \]

On the other hand, we can write \( R^3_{1, i} f(y, w) = \int K^3_{1, i}(y, y') f(y') dy' \). where

\[ K^3_{1, i}(y, y') = t^3 g^3(y, y') \int \sigma^3_{1,i}(y, \eta, k) e^{i(y-y')^n} d\eta \xi_k \]

Then the above estimate implies \( K^3_{1, i}(y, z) \) is a bounded operator from \( L^q(S) \to L^q(S) \), whose operator norm bounded by

\[ t^{\alpha+3} e^{\nu (2^\theta)^{-7} |2^\theta z|^{-j}}. \]

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Then by the usual procedure (including Young's theorem) we can show that
\[ ||R_{1,i}^{3,3} f||_{L^q(I \times S, dx)} \leq t^{\alpha + 3} e^{\epsilon_2(2^\beta)^{-1}} ||(1 + |2^\beta z|)^{-10}||_{L^1(R, dx)} ||f||_{L^q(I \times S, dx)} \]
\[ \leq t^{\alpha + 3} e^{\epsilon_2(2^\beta)^{-2}} ||f||_{L^q(I \times S, dx)} \]
\[ \leq C^2 \rho^2 t^{3 + \alpha} e^{\epsilon_2||f||_{L^q(I \times S, dx)}} \]
Since \( y < -N_0 \), this means as \( y \) goes to \(-\infty\),
\[ ||\sum_{\beta} R_{1,i}^{3,3} f||_{L^q(I \times S, dx)} \leq c||f||_{L^q(I \times S, dx)} \quad \because c << 1 \quad (27') \]
and for \( \beta = N \), we need different estimate. From the definition,
\[ R_{1,i}^{N,3} f(y, w) = \sum_k \int \int t^3 g(y, y')^3 \phi_\beta(y, \eta, k)(\partial/\partial \eta)^6 \sigma_i^N(y, \eta, k)^3 f(y', .)(w)e^{i(y-y')\eta} dy' d\eta \xi_k \]
Then,
\[ |t^3(\partial/\partial \eta)^i(\partial/\partial y)^j \phi_\beta(y, \eta, k)(\partial/\partial \eta)^6 \sigma_i^N(y, \eta, k)^3| \leq t^{-2}(1 + |k| + |\eta|)^{-2-j-i} \quad (28') \]
It follows from \( (28') \) that \( (\partial/\partial y)^2 R_{1,i}^{N,3} \) and \( \Delta_s R_{1,i}^{N,3} \) are standard zero order multipliers and therefore bounded from \( L^p \) to \( L^p \) for \( 1 < p < \infty \). Now by Sobolev’s theorem we get
\[ ||h||_{L^q(I, x)} \leq C(||(\partial/\partial y)^2 h||_{L^q(I, x)} + ||\Delta_s h||_{L^q(I, x)}) \]
Combining these two argument we find that
\[ ||R_{1,i}^{N,3} f||_{L^q(I \times S, dx)} \leq C t^{-2} e^{-\epsilon_2} ||f||_{L^q(I \times S)} \]
Now define
\[ R_{2,i}^{3,3} f(y, w) = t^3 \int \int h(y, y')^3 \sum_k \frac{\phi_\beta(y, \eta, k)(y - y')^3}{(i\eta - t\psi(y) + k + n - 2)^3} f(y', .)(w)e^{i(y-y')\eta} dy' d\eta \xi_k \]
If we denote
\[ \sigma_0^{2,3}(y, \eta, k) = \frac{(y - y')^3 \phi_\beta(y, \eta, k)}{(i\eta - t\psi(y) + k + n - 2)^3} \], then
\[ |t^3(\partial/\partial \eta)^i \sigma_0^{2,3}| \leq C_j (2^\beta)^{-j-3} \quad \text{which implies} \quad (29') \]
\[ ||t^3 \sum_k (\partial/\partial \eta)^j \sigma_0^{2,3}(y, \eta, k) \xi_k f||_{L^q(S)} \leq C_j (2^\beta)^{-2-j+3} ||f||_{L^p(S)} \quad \text{for some } \alpha < 1 \]
On the other hand, we can write
\[ R_{2,i}^{3,3} f(y, w) = \int K_{2,i}^{3,3}(y, y') e^{i(y-y')\eta} dy', \text{ where} \]
\[ K_{2,i}^{3,3}(y, y') = i^3 t^3 h^3(y, y') \int \sigma_0^{2,3}(y, \eta, k) e^{i(y-y')\eta} d\eta \xi_k \]
Then the above estimate implies when \( z = y - y' \), \( K^{\beta,3}_{2,i}(y, z) \) is a bounded operator from \( L^q(S) \) to \( L^q(S) \), whose operator norm is bounded by

\[
Ce^{\psi t^a(2^\beta)^{-1}|2^\beta z|^{-i}}.
\]

Then we can easily show that

\[
\| \sum_{\beta} R^{\beta,3}_{2,i} f \|_{L^q(I \times S, dx)} \leq c' \| f \|_{L^q(I \times S, dx)} \quad \text{for } c' << 1 \quad (30')
\]

And for \( \beta = N \), we get the same estimate as \( R^{N,3}_{1,i} \) since \( R^{N,3}_{2,i} \) is an ordinary fractional integral operator of order 2. Now all other terms appearing in \( R^{\beta,3}_{1,i} f \) are combinations of \( R^{\beta,3}_{1,i} \) and \( R^{\beta,3}_{2,i} \) and they give same estimates and we get

\[
\| R^{\beta,3}_{1,i} f \|_{L^q(I \times S, dx)} \leq c'' \| f \|_{L^q(I \times S, dx)} \quad \text{for } c'' < 1/2 \quad (31')
\]

Now the second half of the claim comes from the same interpolation argument as \( y > -N_0 \) case and we have (19).

Since we have the estimate

\[
\| f \|_{L^q(I \times S, e^{-\gamma|z|^2} dydw)} \leq C \| A_I f \|_{L^p(I \times S, dx)} , \quad I = I_j
\]

we want to extend this result to the whole \( R^+ \times S \).

First we want to remind one result we had before.

\[
\| e^{\psi t} f \|_{L^2(R^+ \times S, e^{\gamma|z|^2} dydw)} \leq C' \| e^{\psi t} e^{\nu} D f \|_{L^p(R^+ \times S, dz)} \quad (18')
\]

Now by replacing \( f = Dg \), the above inequality \( (18') \) implies

\[
\| e^{\psi t} D f \|_{L^2(R^+ \times S, e^{\gamma|z|^2} dydw)} \leq C \| e^{\psi t} e^{\nu} \Delta f \|_{L^p(R^+ \times S, e^{\gamma|z|^2} dydw)}
\]

and

\[
\| e^{\psi t} e^{\nu} D f \|_{L^2(R^+ \times S, e^{\gamma|z|^2} dydw)} \leq C \| e^{\psi t} e^{2\nu} \Delta f \|_{L^p(R^+ \times S, e^{\gamma|z|^2} dydw)} \quad (18''\text{"})
\]

Since

\[
|Df|_{L^2}^2 = |\sum_j \alpha_j \partial f / \partial x_j|_{L^2}^2 = |\sum_j \partial f / \partial x_j|_{L^2}^2 = |\nabla f|_{L^2}^2,
\]

we get,

\[
\| e^{\psi t} \nabla f \|_{L^2(R^+ \times S, e^{\gamma|z|^2} dydw)} \leq C \| e^{\psi t} \Delta f \|_{L^p(R^+ \times S, e^{\gamma|z|^2} dydw)} \quad (2')
\]

On the other hand, we had \( (18'') \) i.e.

\[
\| e^{\psi t} f \|_{L^p(R^+ \times S, e^{\gamma|z|^2} dydw)} \leq C \| e^{\psi t} D f \|_{L^p(R^+ \times S, e^{\gamma|z|^2} dydw)} \quad f \in C_0^\infty(R^+ \times S) \quad (18''\text{"})
\]

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By replacing \( f = Dg \), (18") becomes

\[
||e^{t\psi}e^{y}Df||_{L^p(R^{n}\times S\sigma(n+\nu)d\nu dy dw)} \leq C||e^{t\psi}e^{y}\Delta f||_{L^p(R^{n}\times S\sigma(n+\nu)d\nu dy dw)}
\]  \( (18\prime) \)

If we combine above two inequalities, we have

\[
||e^{t\psi}f||_{L^p(R^{n}\times S\sigma(n+2\nu)d\nu dy dw)} \leq C||e^{t\psi}e^{y}\Delta f||_{L^p(R^{n}\times S\sigma(n+\nu)d\nu dy dw)}
\] \( (32) \)

Now with these estimates, we can continue.

We can rewrite \( f = \sum_j \phi_j f \), for \( \{\phi_j\} \) partition of unities, having support in \((-\frac{1}{2}, -\frac{1}{2})\) and for each \( x \in R^- \), there are only finitely many \( \phi_j \)'s such that \( \phi_j(x) \neq 0 \). Then,

\[
||f||_{L^q(R^{n}\times S\sigma(n+2\nu)d\nu dy dw)} = \left|\sum_j \phi_j f\right|_{L^q(R^{n}\times S\sigma(n+2\nu)d\nu dy dw)}
\]

\[
\leq \left( \sum_j ||\phi_j f||_{L^q(I_j \times S\sigma(n+2\nu)d\nu)}^q \right)^{1/q}
\]

\[
\leq \left( \sum_j ||A_t(\phi_j f)||_{L^p(I_j \times S\sigma(n+\nu)d\nu dy dw)}^q \right)^{1/q}
\] \( (33) \)

Since

\[
A_t(\phi_j f) = (\phi_j'' f) + \sum_k (1 \phi_j''(\partial/\partial y - t\psi'(y) - k)\xi_k f \\
+ \sum_k \phi_j''(\partial/\partial y - t\psi'(y) + k + n - 2)\xi_k f + \phi_j A_t f
\]

We want to estimate each of them separately.

First,

\[
\sum_j ||\phi_j'' f||_{L^p(I_j \times S\sigma(n+2\nu)d\nu dy dw)} \leq C||f||_{L^p(R^{n}\times S\sigma(n+2\nu)d\nu dy dw)}
\]

\[
\leq C'||A_t f||_{L^p(R^{n}\times S\sigma(n+\nu)d\nu dy dw)} \text{ by (32)}
\]

Also

\[
\sum_j ||\phi_j A_t f||_{L^p(I_j \times S\sigma(n+2\nu)d\nu dy dw)} \leq C||A_t f||_{L^p(R^{n}\times S\sigma(n+\nu)d\nu dy dw)}
\]

Claim  We can get similar estimates for the intermediate terms, i.e.

\[
\sum_j ||\phi_j'' f||_{L^p(I_j \times S\sigma(n+2\nu)d\nu dy dw)} \leq C||A_t f||_{L^p(R^{n}\times S\sigma(n+\nu)d\nu dy dw)} \text{ (*)}
\]

\[
\sum_j ||\phi_j A_t f||_{L^p(I_j \times S\sigma(n+2\nu)d\nu dy dw)} \leq C||A_t f||_{L^p(R^{n}\times S\sigma(n+\nu)d\nu dy dw)} \text{ (**)}
\]

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Then after getting terms together, and from (33) we get
\[
\|f\|_{L^q(R - x S, e^{n+2n+2\alpha \epsilon/p})dydw)} \leq \left( \sum_j \|A_j(\phi_j f)\|_{L^p(I_j \times S, e^{(n+2\alpha \epsilon/p)}dydw)} \right)^{1/q}
\]
\[
\leq C\|A f\|_{L^p(R - x S, dx)}
\]
\[
= C\|e^{t^2\psi} e^{2y} \Delta f\|_{L^p(R - x S, dx)}
\]

The first inequality is from (33), the second is from (*), (**), and the third from the fact that
\[
L^q \supset L^p \text{ for } q > p
\]

Now if we choose small \( \epsilon \), then \( e^{2\psi} \leq e^{2\epsilon q+2\alpha \epsilon/p} \). So the above inequality is equivalent to
\[
\|e^{t^2\psi} f\|_{L^q(R - x S, dx)} \leq C\|e^{t^2\psi} \Delta f\|_{L^p(R - x S, dx)}
\]
(3)

Now we will prove the claim. Since
\[
\sum_j \|\phi_j \sum_k (\partial/\partial y - t^2\psi(y) - k)\xi_k f\|_{L^p(I_j \times S, e^{(n+2\alpha \epsilon)}dydw)}
\]
\[
\leq C\|\sum_k (\partial/\partial y - t^2\psi(y) - k)\xi_k f\|_{L^p(R - x S, e^{(n+2\alpha \epsilon)}dydw)}
\]
\[
= C\|e^{t^2\psi} e^{2y} De^{-t^2\psi} f\|_{L^p(R - x S, e^{n+\alpha \epsilon})dydw)}
\]
\[
\leq C'\|e^{t^2\psi} e^{2y} \Delta e^{-t^2\psi} f\|_{L^p(R - x S, e^{n+\alpha \epsilon})dydw)}
\]

The last inequality follows from (18'\alpha)

On the other hand, (**) is equivalent to
\[
\sum_j \|\phi_j f\|_{L^p(R - x S, e^{n+2\alpha \epsilon})dydw)} \leq C\|f\|_{L^p(R - x S, e^{n+2\alpha \epsilon})dydw)}
\]
\[
\leq C\|\sum_k (\partial/\partial y - t^2\psi(y) - k)\xi_k f\|_{L^p(R - x S, e^{n+\alpha \epsilon})dydw)}
\]

But the second inequality is just (18').

**Strong Unique Continuation Theorem**

**Proof of Corollary 3**

We have two Carleman type estimates.

\[
\|e^{t^2\psi} u\|_{L^q(R - x S, e^{n\alpha \psi})dydw)} \leq C\|e^{t^2\psi} \Delta u\|_{L^p(R - x S, e^{n\alpha \psi})dydw)}.
\]
(3)

\[
\|e^{t^2\psi} \nabla u\|_{L^2(R - x S, e^{n\alpha \psi})dydw)} \leq C\|e^{t^2\psi} \Delta u\|_{L^p(R - x S, e^{n\alpha \psi})dydw)}.
\]
(2')

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for $p = \frac{3n-4}{3n+2}$ and $\frac{1}{p} - \frac{1}{q} = \frac{2}{n} - \epsilon$, for small $\epsilon > 0$.

It suffices to show $u \equiv 0$ in a neighborhood of $y = -\infty$. Now let's introduce a cut off function $\eta \in C_0^\infty(R)$ such that $\eta \equiv 1$ for $y \in (-\infty, -2\rho)$, $\eta \equiv 0$ for $y > -\rho$.

Let $f = \eta u$. Then by limiting process, (3') and (2') gives

$$
\|e^{t\psi}\nabla f\|_{L^2(S_p, e^{\eta y}dydw)} + \|e^{t\psi} f\|_{L^q(S_p, e^{\eta y}dydw)} \leq C\|e^{t\psi} \Delta f\|_{L^p(R^n, e^{\eta y}dydw)}
$$

for $S_p = \{x \in R^n; x = e^yw, y < -4\rho\}$. Then the right side is bounded by

$$
C\|e^{t\psi} \Delta f\|_{L^p(R^n \setminus S_p)} + C\|e^{t\psi} \nabla f\|_{L^p(S_p)} + C\|e^{t\psi} \Delta f\|_{L^p(S_p)}.
$$

$$
\leq C\|e^{t\psi} \Delta f\|_{L^p(R^n \setminus S_p)} + C\|v\|_{L^1(S_p)}\|e^{t\psi(y)} \nabla f\|_{L^2(S_p)} + C\|w\|_{L^{\frac{n}{n+1}}(S_p)}\|e^{t\psi(y)} f\|_{L^q(S_p)}.
$$

Now if we choose $\rho$ large enough, we can assume

$$
\|v\|_{L^1(S_p)},\|w\|_{L^{\frac{n}{n+1}}(S_p)} < \frac{1}{2C}.
$$

This means we can cancel the second and third term, i.e.

$$
C\|v\|_{L^1(S_p)}\|e^{t\psi(y)} \nabla f\|_{L^2(S_p)}, \text{ and } C\|w\|_{L^{\frac{n}{n+1}}(S_p)}\|e^{t\psi(y)} f\|_{L^q(S_p)}
$$

can be canceled against

$$
\|e^{t\psi(y)} \nabla f\|_{L^2(S_p, e^{\eta y}dydw)}, \text{ and } \|e^{t\psi(y)} f\|_{L^q(S_p, e^{\eta y}dydw)}
$$

Then after cancellation,

$$
\|e^{t\psi} \nabla f\|_{L^2(S_p, dx)} + \|e^{t\psi} f\|_{L^q(S_p, dx)} \leq C\|e^{t\psi} \Delta f\|_{L^p(R^n \setminus S_p, dx)}.
$$

Since $\psi(y)$ is a decreasing function, for $y \notin \Pi_1(supp(f) \setminus S_p)$ we have $\psi(y) < \psi(\rho')$ for $\rho' \in \{y \in R; y < -4\rho\}$. 

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This implies
\[ \|e^{t(\psi(y)-\psi(\rho'))} f\|_{L^q(S_\rho, e^{\psi(y)}dydw)} \leq C\|\Delta f\|_{L^p(R^n, e^{\psi(y)}dydw)} \leq C'. \]

Since \( \psi(y) - \psi(\rho') \geq 0 \), as \( t \to \infty \) we are forced to \( f \equiv 0 \) on \( S_\rho \).

References