ANALYSIS AND DESIGN OF GAIN SCHEDULED CONTROL SYSTEMS

BY

JEFF S. SHAMMA

This report is based on the unaltered thesis of Jeff S. Shamma submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy at the Massachusetts Institute of Technology in May 1988. This research was conducted at the M.I.T. Laboratory for Information and Decision Systems with support provided by NASA Ames and Langley Research Centers under grant NASA/NAG 2-297.

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ANALYSIS AND DESIGN OF_GAIN SCHEDULED CONTROL SYSTEMS

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ABSTRACT

Gain scheduling has proven to be a successful design methodology in many engineering applications. The idea is to construct a global feedback control system for a time-varying and/or nonlinear plant from a collection of local linear time-invariant designs. However in the absence of a sound analysis, these designs come with no guarantees on the robustness, performance, or even nominal stability of the overall gain scheduled design.

We present such an analysis for three types of gain scheduling situations (1) a linear parameter-varying plant scheduling on its exogenous parameters, (2) a nonlinear plant scheduling on a prescribed reference trajectory, and (3) a nonlinear plant scheduling on the current plant output. Conditions are given which guarantee that the stability, robustness, and performance properties of the fixed operating point designs carry over to the global gain scheduled design. These conditions confirm and formalize popular notions regarding gain scheduled designs, such as the scheduling variable should "vary slowly" and "capture the plant's nonlinearities."

Finally, an alternate design framework is proposed which removes the "slowly varying" restriction on gain scheduled systems. This framework formally addresses certain fundamental feedback issues which previously have been ignored in standard gain scheduled designs.

Thesis Supervisor: Dr. Michael Athans
Title: Professor of Systems Science and Engineering

- 2 -
to my parents and grandparents
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Chapter 1
Introduction

1.1 Motivation

Feedback control is a method to force some physical process, or plant, to conform to a desired behavior. Through feedback, one can obtain the desired behavior with only partial and imprecise knowledge of the plant. However, the complexity of most plants forces one to construct oversimplified and approximate models for the purpose of analysis and design of a feedback control system. One class of models for which both analysis and design are well understood is the class of linear time-invariant plants. That is, if the plant is linear time-invariant, then one can design an excellent feedback control system in a relatively straightforward manner. Unfortunately, however, rarely is a linear time-invariant description of a plant's dynamics adequate - hence the need to seek alternate methods for design of controllers for systems with widely varying nonlinear and/or time-varying-parameter dependent dynamics.

One popular engineering method which has enjoyed much success is called Gain Scheduling (e.g. [53]). The idea is to select several operating points which cover the range of
the plant's dynamics. Then, at each of these points, the designer makes a linear time-invariant approximation to the plant and designs a linear compensator for each linearized plant. In between operating points, the parameters (gains) of the compensators are then interpolated, or scheduled, thus resulting in a global compensator.

Despite the lack of a sound theoretical analysis, gain scheduling is a design methodology which is known to work in many engineering applications (e.g. jet engines, submarines, and aircraft). However, in the absence of such an analysis, these designs come with no guarantees. More precisely, one typically cannot assess *a priori* the guaranteed stability, robustness (i.e. quality in the presence of uncertainties), and performance properties of gain scheduled designs. Rather, any such properties are inferred from extensive computer simulations. Furthermore, given an unsuccessful gain scheduled design, there are no guidelines on how to alter the design parameters and/or performance specifications to finally achieve a successful design.

In other words, a complete and systematic design methodology has yet to emerge. In its place, a collection of intuitive ideas have developed into heuristic guidelines for gain scheduled designs. For example, two primary such guidelines are "the scheduling variable should vary slowly," and "the scheduling variable should capture the plants nonlinearities." Thus, a sound analysis of various gain scheduling schemes would prove very useful in better understanding these designs.

Hopefully, such an analysis would (1) formalize the popular notions regarding the design of gain scheduled control systems, (2) enable one to give guarantees on the behavior and performance of gain scheduled designs, and (3) give guidelines on how to alter design parameters and/or performance specifications to achieve a successful design. In this sense, this analysis would be used towards the ultimate goal to develop a complete and systematic gain scheduling design framework.
Chapter 1: Introduction

1.2 Contribution of Thesis

The contributions of this thesis may be divided into two areas: (1) analysis and (2) design of gain scheduled control systems.

Firstly, this thesis presents the first formal and rigorous analysis of gain scheduling. Three forms of gain scheduling are identified and analyzed. They are (1) a parameter-dependent linear plant scheduling on its time-varying parameters, (2) a nonlinear plant scheduling on some prescribed reference trajectory, and (3) a nonlinear plant scheduling on the current plant output.

In each case, the fundamental issue is the question of guaranteed properties of the overall gain scheduled system. More precisely, given that the local operating point designs are based on linear time-invariant approximations to the actual plant, one is able to design controllers such that these local designs have guaranteed excellent feedback properties, such as command following, disturbance rejection, robustness to unmodeled dynamics, etc. However, the actual plant is nonlinear and/or time-varying. Thus, it is reasonable to ask under what conditions do the desired feedback properties of the local designs "carry over" to the overall gain scheduled design.

In fact, the overall design may exhibit none of these properties - even nominal stability. In this thesis, conditions are given which guarantee that the overall gain scheduled system exhibits the desired feedback properties of the fixed operating point designs. In doing so, the heuristic guidelines of "scheduling on a slow variable" and "capturing the nonlinearities" are transformed into precise and quantitative statements regarding the behavior of gain scheduled designs. Thus, one now has new insights into gain scheduling which go beyond the original statements of the heuristic guidelines. For example, one outcome of the analysis is that a gain scheduled design for the F-8 research aircraft [54] is shown to have nominal parameter-varying stability.
Chapter 1: Introduction

Secondly, this thesis brings to mind an important issue which has been ignored in the current gain scheduling framework. That is, the restriction of scheduling on a slow variable is primarily because gain scheduling does not explicitly address certain fundamental concepts behind feedback control. To elaborate, feedback control design is in essence the process of inverting the undesirable dynamics of the plant and replacing them with desirable dynamics. However, any such inversion must be done in a closed-loop stable manner. That is, there are parts of the plant's dynamics which the designer is "stuck with", i.e. they cannot be inverted without resulting in an unstable closed loop system. For example, in the case of linear time-invariant plants, this inversion amounts to an unstable pole/zero cancellation.

In this thesis, an alternate gain scheduling design framework for linear parameter-varying plants is suggested which better addresses these fundamental feedback issues. The design process consists of two stages:

1. Finding desirable dynamics - Since feedback control consists of replacing undesirable dynamics with desirable dynamics, the natural question is what are desirable dynamics? Two desirable characteristic are

   (a) the dynamics should maintain stability in the presence of fast parameter-variations, and

   (b) the dynamics should have some inherent robustness properties to unmodeled dynamics.

   Two candidate target loops, the time-varying integrator and time-varying Kalman filter, are presented which exhibit both of these properties - hence are suitable candidates for desirable dynamics.

2. Inverting undesirable dynamics - It is then shown how one can asymptotically invert and replace the undesirable plant dynamics using concepts of loop operator recovery [28].

   Two simple yet revealing simulation examples exhibit both of these stages.

   Finally, this design framework coupled with the analysis of nonlinear gain scheduled systems is used to present a new design procedure which guarantees global nominal stability
Chapter 1: Introduction

for a class of nonlinear plants. The design is demonstrated by the example of a single link driven by a flexible shaft.

1.3 Previous Work and Related Literature

Despite the wide-spread use of gain scheduling, any formal documentation is quite limited. This is probably due to both the aforementioned lack of any theoretical analysis as well as the very applied nature of implementation of gain scheduled controllers. In any case, gain scheduled designs for aircraft can be found in [53, 54].

Since there has been no formal analysis of gain scheduled systems, the analysis presented here primarily relies on general stability theory for feedback systems. Some standard references are [18, 48, 57]. The methods used in this thesis range from those of state-space analysis of differential equations (e.g. [8, 17, 35, 36, 45, 58, 60]) to those of input/output analysis of feedback systems (e.g. [2, 13, 20, 19, 39, 45, 49, 61]). One area of stability theory used in this thesis which is not found in the standard feedback control literature is that of Volterra Integrodifferential Equations (e.g. [11, 16]). Such equations are similar to ordinary differential equations with the exception that they contain an integral operator in the right-hand-side. It turns out that such equations provide a sufficiently general framework to describe many gain scheduled situations.

Given the lack of a theoretical analysis of gain scheduling, there is, of course, no previous work on gain scheduling as a systematic design methodology, per se. Recall that gain scheduling consists of constructing a global feedback control system from many local control designs based on local approximations, or linearizations, of the plant dynamics. Some related work is that of exact feedback linearization (e.g. [33] and references contained therein) pseudolinearization [46], and extended linearization [6], all for nonlinear plants.

The concept of extended linearization seems to be most closely related to gain scheduling.
Chapter 1: Introduction

The idea is to perform many linear state-feedback designs based on linearized dynamics and then integrate the feedback gains in order to achieve some desired performance criterion. The particular criterion achieved in [6] is that the closed-loop eigenvalues resulting from a linearization of the dynamics are independent of the particular equilibrium point.

The idea behind exact feedback linearization is to find a state-feedback law and a transformation of variables such that the transformed plant dynamics are linear. The less restrictive concept of pseudolinearization requires that this same process of state-feedback and transformation results in a nonlinear system that has the following property. The linear system which results from linearization about an equilibrium point is independent of the particular equilibrium point. That is although the new plant dynamics are not linear as in exact feedback linearization, a linearization of the dynamics leads to the same linear system regardless of the equilibrium point.

In addition to the requisite state-feedback, each of these methods is lacking in the sense that they only address stabilization of nonlinear systems. That is, they give no guarantees on important feedback issues such as robust performance, robust stability, disturbance rejection, etc.

The design approach taken in this thesis is can be viewed as a special case of Loop Operator Recovery (LOR) [28]. The work of [28] generalized to nonlinear systems the LQG/LTR design methodology [20, 55] for linear time-invariant systems. The idea here is the fundamental concept of stable approximate inversion of undesirable plant dynamics and replacement with desirable dynamics, where the desirable dynamics have guaranteed stability, robustness, and performance properties. In this sense, the LOR methodology is simply an execution of fundamental feedback control concepts as applied to nonlinear systems.
1.4 Organization of Thesis

This thesis is organized into six chapters. In Chapter 2, the mathematical notations and definitions used throughout the thesis are established. Furthermore, some requisite background material on stability of dynamical systems is presented.

Chapter 3 begins the first of two main chapters on guaranteed properties of gain scheduled control systems. In Chapter 3, the systems under consideration are linear plants with time-varying parameter-dependent dynamics. Conditions are given which guarantee that the parameter-varying systems maintain the desired feedback properties of the frozen parameter designs.

In Chapter 4, the analysis continues with nonlinear gain scheduled control systems. The two systems under consideration are scheduling on a prescribed reference trajectory and scheduling on the plant output. Again, conditions are given which guarantee that the overall nonlinear gain scheduled system maintains the feedback properties of the local operating point designs.

Throughout Chapters 3 and 4, the heuristic guidelines behind gain scheduled designs are justified and made quantitative. Almost all of the analysis is presented in the framework of Volterra Integrodifferential Equations, which can suitably describe general gain scheduled systems in the presence of infinite-dimensional modelling errors.

Chapter 5 discusses the design of gain scheduled control systems. A new design framework is presented for linear parameter-varying plants which remedies some severe limitations of standard gain scheduling and better addresses the fundamental philosophy behind feedback control. This new procedure is demonstrated via numerical simulations. Using the analysis of nonlinear gain scheduled systems, this framework is then used to present a new design procedure which guarantees global nominal stability for a certain class of nonlinear
Chapter 1: Introduction

systems. The procedure is demonstrated via a nonlinear simulation.

Finally, concluding remarks are given in Chapter 6.
Chapter 2
Mathematical Preliminaries

2.1 Introduction

This chapter presents the notation and definitions to be used throughout the thesis and presents some background stability theory. In each section, general references are given which cover all of the material presented except for certain particular results, in which case the specific references are supplied. Since most of the material in this chapter is standard, the explanatory discussion will be brief.

2.2 Notation and Definitions

First, some notation and mathematical definitions specialized for the purpose of this thesis are presented. For further details on the material in this section, see references [18, 23, 47].
### Table 2.2-1 List of Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}$</td>
<td>set of integers</td>
</tr>
<tr>
<td>$\mathbb{Z}^+$</td>
<td>set of non-negative integers</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>set of real numbers</td>
</tr>
<tr>
<td>$\mathbb{R}^+$</td>
<td>set of non-negative real numbers</td>
</tr>
<tr>
<td>$\mathbb{R}^n$</td>
<td>set of ordered $n$-tuples of real numbers</td>
</tr>
<tr>
<td>$\mathbb{R}^{nxm}$</td>
<td>set of $n$ by $m$ matrices with elements in $\mathbb{R}$</td>
</tr>
<tr>
<td>$\mathbb{C}, \mathbb{C}^n, \mathbb{C}^{nxm}$</td>
<td>complex analogs of $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{nxm}$</td>
</tr>
<tr>
<td>$\text{Re}[z]$</td>
<td>real part of complex number $z$</td>
</tr>
<tr>
<td>RHP / LHP</td>
<td>Right / Left Half Plane</td>
</tr>
<tr>
<td>$x^T / A^T$</td>
<td>vector / matrix transpose</td>
</tr>
<tr>
<td>$x^H / A^H$</td>
<td>vector / matrix complex conjugate transpose</td>
</tr>
<tr>
<td>$A_{ij}$</td>
<td>$ij^{th}$ element of matrix $A$</td>
</tr>
<tr>
<td>$| x | / | A |$</td>
<td>Euclidean norm on $\mathbb{R}^n$ or $\mathbb{C}^n$, i.e. $(x^H x)^{1/2}$, and its induced matrix norm on $\mathbb{R}^n$ or $\mathbb{C}^n$</td>
</tr>
<tr>
<td>$\mathcal{B}(x_o, \varepsilon)$</td>
<td>the set ${ x \in \mathbb{R}^n \mid | x - x_o | &lt; \varepsilon }$</td>
</tr>
<tr>
<td>$\lambda_i(A)$</td>
<td>$i^{th}$ eigenvalue of matrix $A$</td>
</tr>
<tr>
<td>$\sigma_i(A)$</td>
<td>$i^{th}$ singular value of matrix $A$, i.e. $\lambda_i(A^H A)^{1/2}$</td>
</tr>
<tr>
<td>$\sigma_{\text{max/min}}(A)$</td>
<td>maximum / minimum singular value of matrix $A$</td>
</tr>
</tbody>
</table>
Table 2.2-1 List of Notation, cont.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{det}(A) )</td>
<td>determinant of matrix ( A )</td>
</tr>
<tr>
<td>( Df )</td>
<td>derivative of function ( f : \mathbb{R}^n \to \mathbb{R}^m )</td>
</tr>
<tr>
<td>( D_i f )</td>
<td>derivative with respect to ( i^{th} ) variable of ( f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_i} \times \ldots \times \mathbb{R}^{n_k} \to \mathbb{R}^m )</td>
</tr>
<tr>
<td>( D^* f )</td>
<td>Dini derivative of ( f : \mathbb{R} \to \mathbb{R} )</td>
</tr>
<tr>
<td>( \dot{x} )</td>
<td>time derivative of ( x : \mathbb{R} \to \mathbb{R} )</td>
</tr>
<tr>
<td>( L_p, L_{pe} )</td>
<td>Lebesgue / extended Lebesgue function spaces, ( f : \mathbb{R}^+ \to \mathbb{R}^n )</td>
</tr>
<tr>
<td>( l_p )</td>
<td>appropriately summable sequence spaces, ( f : \mathbb{Z}^+ \to \mathbb{R}^n )</td>
</tr>
<tr>
<td>( B, B_e )</td>
<td>set of bounded measurable functions, ( f : \mathbb{R}^+ \to \mathbb{R}^n ), and its extension.</td>
</tr>
<tr>
<td>( | \cdot |<em>{L_p}, | \cdot |</em>{l_p}, | \cdot |_B )</td>
<td>norms in ( L_p, l_p, B )</td>
</tr>
<tr>
<td>( P_T )</td>
<td>truncation operator</td>
</tr>
<tr>
<td>( \omega_{T, \sigma} )</td>
<td>truncation and exponential weighting operator</td>
</tr>
<tr>
<td>( \hat{f} )</td>
<td>Laplace transform of ( f )</td>
</tr>
<tr>
<td>( \mathcal{A}(\sigma), \mathcal{A}_{n \times m}(\sigma) )</td>
<td>algebra of impulse responses</td>
</tr>
<tr>
<td>( \hat{\mathcal{A}}(\sigma), \hat{\mathcal{A}}_{n \times m}(\sigma) )</td>
<td>algebra of transfer functions</td>
</tr>
<tr>
<td>( f \ast g )</td>
<td>convolution of ( f ) and ( g )</td>
</tr>
</tbody>
</table>
Table 2.2-1 List of Notation, cont.

\[ \| \cdot \|_{\mathcal{A}(\sigma)} \] norm in \( \mathcal{A}(\sigma) \)

\[ e^{A t} \] matrix exponential

\[ \Phi(t, \tau) \] state transition matrix

\( s(t ; x_0, t_0), s(t ; x_0) \) solution of an ordinary differential equation

\( s(t ; P, \phi, t_0) \) solution of a Volterra integrodifferential equation

\( \tilde{V}_{(eq. \#)} \) a Lyapunov function or functional \( V \) evaluated along the trajectory of the system (equation \#)

\( \lim \sup \) limit superior

\( \text{ess sup} \) essential supremum

RHS / LHS Right Hand Side / Left Hand Side

\( a \equiv b \) \( a \) is defined as \( b \)

\[ \blacksquare \] end of proof

---

**Definition 2.2-1 Lipschitz Continuity**

The function \( f : \mathbb{R}^n \to \mathbb{R}^m \) is said to be *locally Lipschitz continuous* at \( x_0 \) with constant \( L \) if there exist constants \( L \) and \( \epsilon > 0 \) such that

\[ |f(x) - f(x_0)| \leq L |x - x_0|, \quad \forall x \in B(x_0, \epsilon) \]  

(2.2-1)

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Chapter 2: Mathematical Preliminaries

The function \( f : \mathbb{R}^n \to \mathbb{R}^m \) is said to be *globally Lipschitz continuous* with constant \( L \) if

\[
| f(x) - f(x') | \leq L |x - x'|, \quad \forall \ x, x' \in \mathbb{R}^n
\]  

(2.2-2)

**Definition 2.2-2 Dini Derivative**

Let \( f : \mathbb{R} \to \mathbb{R} \). The *Dini derivative* of \( f \) at \( x_o \), denoted \( D^+f(x_o) \), is

\[
D^+f(x_o) \equiv \limsup_{h \to 0^+} \frac{f(x_o + h) - f(x_o)}{h}
\]  

(2.2-3)

**Remark** A sufficient condition for the existence of the Dini derivative is that the function \( f \) be locally Lipschitz continuous at \( x_o \).

**Definition 2.2-3 \( L_p / L_p \) Spaces**

Let \( f : \mathbb{R}^+ \to \mathbb{R}^n \) be measurable. The function \( f \) is said to be in \( L_p, p \in [1, \infty) \), if

\[
\| f \|_{L_p} \equiv \left( \int_0^\infty |f(t)|^p \, dt \right)^{1/p} < \infty
\]  

(2.2-4)

The function \( f \) is said to be in \( L_\infty \) if

\[
\| f \|_{L_\infty} \equiv \operatorname{ess \, sup}_{t \geq 0} |f(t)| < \infty
\]  

(2.2-5)

Similarly, let \( f : \mathbb{Z}^+ \to \mathbb{R}^n \). The function \( f \) is said to be in \( l_p, p \in [1, \infty) \), if
\[ \| f \|_{L_p} = \left( \sum_{n=0}^{\infty} |f(n)|^p \right)^{1/p} < \infty \quad (2.2-6) \]

The function \( f \) is said to be in \( L_\infty \) if
\[ \| f \|_{L_\infty} = \sup_{n \in \mathbb{Z}^+} |f(n)| < \infty \quad (2.2-7) \]

**Definition 2.2-4 Truncation Operator**

Let \( f : \mathbb{R}^+ \to \mathbb{R}^n \). The truncation operator \( P_T \) maps \( f \) into \( P_T f : \mathbb{R}^+ \to \mathbb{R}^n \) where
\[ (P_T f)(t) = \begin{cases} f(t), & t \leq T \\ 0, & t > T \end{cases} \quad (2.2-8) \]

**Definition 2.2-5 Extended \( L_p \) Spaces**

Let \( f : \mathbb{R}^+ \to \mathbb{R}^n \). The function \( f \) is said to be in the extended space \( L_{pe}, p \in [1, \infty] \), if
\[ P_T f \in L_p, \quad \forall \ T \geq 0 \quad (2.2-9) \]

**Definition 2.2-6 Space of Bounded Functions \( B \) and its Extension \( B_e \)**

Let \( f : \mathbb{R}^+ \to \mathbb{R}^n \) be measurable. The function \( f \) is said to be in the space of bounded functions \( B \) if
\[ \| f \|_{B} \equiv \sup_{t \in \mathbb{R}^{+}} |f(t)| < \infty \] (2.2-10)

The function \( f \) is said to be in the extended space \( B_e \) if
\[ P_{\tau} f \in B, \quad \forall \tau \geq 0 \] (2.2-11)

**Definition 2.2-7** Truncation and Exponential Weighting Operator

Let \( f : \mathbb{R}^{+} \rightarrow \mathbb{R}^{n} \). The truncation and exponential weighting operator \( \mathbf{W}_{T, \sigma} \) maps \( f \) into \( \mathbf{W}_{T, \sigma} f : \mathbb{R}^{+} \rightarrow \mathbb{R}^{n} \) where
\[ (\mathbf{W}_{T, \sigma} f)(t) = \begin{cases} e^{-\sigma(t-T)} f(t), & t \leq T \\ 0, & t > T \end{cases} \] (2.2-12)

**Definition 2.2-8** [12, 18] Algebra of Impulse Responses and Transfer Functions

\( \mathcal{A}(\sigma) \) denotes the set whose elements are of the form
\[ f(t) = \begin{cases} f_a(t) + \sum_{i=0}^{\infty} f_i \delta(t - t_i), & t \geq 0 \\ 0, & t < 0 \end{cases} \] (2.2-13)

where \( f_a : \mathbb{R}^{+} \rightarrow \mathbb{R}, t_i \geq 0, f_i \in \mathbb{R} \), and
\[ \| f \|_{A(\sigma)} \equiv \int_{0}^{\infty} |f_a(t) e^{-\sigma t}| dt + \sum_{i=0}^{\infty} |f_i e^{-\sigma t_i}| < \infty \] (2.2-14)

\( \mathcal{A}^{n \times m}(\sigma) \) denotes the set of \( n \) by \( m \) matrices whose elements are in \( \mathcal{A}(\sigma) \). Let \( \Delta \in \)
\( A^{n \times m}(\sigma) \) and define \( \Delta' \in \mathbb{R}^{n \times m} \) as \( \Delta'_{ij} = \| \Delta_{ij} \| \mathcal{A}(\sigma) \). Then \( \| \Delta \| _{A(\sigma)} \) is defined as

\[
\| \Delta \| _{A(\sigma)} \equiv \| \Delta' \| \quad (2.2-15)
\]

Let \( f \in \mathcal{A}(\sigma) \) be as in (2.2-13) and let \( g \in \mathcal{A}(\sigma) \) be given by

\[
g(t) = \begin{cases} 
  g_a(t) + \sum_{k = 0}^{\infty} g_k \delta(t-t_k'), & t \geq 0 \\
  0, & t < 0
\end{cases} \quad (2.2-16)
\]

Then the convolution \( f \ast g \) is defined as

\[
(f \ast g)(t) = (f_a \ast g_a)(t) + \sum_{t_k' \leq t} f_a(t - t_k')g_k + \sum_{t_i \leq t} f_g(t - t_i) + \sum_{t_i + t_k' \leq t} f_g \delta(t - (t_i + t_k'))
\quad (2.2-17)
\]

Convolution of matrices is defined similarly.

Let \( \Delta \in \mathcal{A}^{n \times m}(\sigma) \) and \( u \in L_p, p \in [1, \infty] \). Then one has that

\[
\| \Delta \ast u \| _{L_p} \leq \| \Delta \| _{\mathcal{A}(\sigma)} \| u \| _{L_p} \quad (2.2-18)
\]

Finally, \( \mathcal{A}(\sigma) \) and \( \mathcal{A}^{n \times m}(\sigma) \) are defined as the set of Laplace transforms of elements of \( \mathcal{A}(\sigma) \) and \( \mathcal{A}^{n \times m}(\sigma) \), respectively.

### 2.3 Background Stability Theory

This section gives some background theory on internal stability and input/output stability. See references [18, 48, 57] for further details.
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2.3.1 Internal Stability

The first part of this section addresses internal or Lyapunov stability for systems of the form

\[ \dot{x}(t) = f(x(t), t), \quad x(t_0) = x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R}^+ \]  \hspace{1cm} (2.3-1)

under the following assumptions

**Assumption 2.3-1**

(1) \( f \) is such that (2.3-1) has a unique solution over \([t_0, \infty)\) which depends \hspace{1cm} (2.3-2)

continuously on the initial conditions \( x_0, t_0 \).

(2) \( f(x, t) = 0 \) \hspace{1cm} \( \forall \ t \geq 0. \) \hspace{1cm} (2.3-3)

**Notation** The notation \( s(t; x_0, t_0) \) is used to denote the solution to (2.3-1). This notation will be used only when it becomes necessary to emphasize and explicitly state the dependence on the initial conditions. Otherwise, simply \( x(t) \) is used.

**Definition 2.3-1 Local Exponential Stability**

The nonlinear system (2.3-1) under Assumption 2.3-1 is said to be **locally exponentially stable** if there exist constants \( m, \lambda, \) and \( \epsilon > 0 \) such that

\[ |s(t; x_0, t_0)| \leq me^{-\lambda(t-t_0)} |x_0|, \quad t \geq t_0, \ \forall \ x_0 \in B(x_0; \epsilon) \]  \hspace{1cm} (2.3-4)

The system is said to be **locally exponentially stable uniformly** if the constants \( m, \lambda, \) and \( \epsilon \) are independent of \( t_0 \). That is...
\[ |s(t; x_0, t_0)| \leq me^{-\lambda(t-t_0)} |x_0|, \quad t \geq t_0, \forall x_0 \in B(x_0; \varepsilon), \forall t_0 \in \mathcal{R}^+ \] (2.3-5)

**Definition 2.3-2 Global Exponential Stability**

The nonlinear system (2.3-1) under Assumption 2.3-1 is said to be *globally exponentially stable* if there exist constants \( m \) and \( \lambda > 0 \) such that

\[ |s(t; x_0, t_0)| \leq m e^{-\lambda(t-t_0)} |x_0|, \quad t \geq t_0, \forall x_0 \in \mathcal{R}^n \] (2.3-6)

The system is said to be *globally exponentially stable uniformly* if the constants \( m \) and \( \lambda \) are independent of \( t_0 \). That is

\[ |s(t; x_0, t_0)| \leq m e^{-\lambda(t-t_0)} |x_0|, \quad t \geq t_0, \forall x_0 \in \mathcal{R}^n, \forall t_0 \in \mathcal{R}^+ \] (2.3-7)

**Remark** In case the dynamics of (2.3-1) are linear, then local exponential stability are global exponential stability are equivalent. In case the dynamics of (2.3-1) are autonomous (i.e. not an explicit function of time), then stability and uniform stability are equivalent.

**Definition 2.3-3 Derivative along a Trajectory**

Let \( V : \mathcal{R}^n \times \mathcal{R}^+ \to \mathcal{R}^+ \) be continuously differentiable. Define \( \tilde{V}_{(2.3-1)}(t) : \mathcal{R}^+ \to \mathcal{R}^+ \) as

\[ \tilde{V}_{(2.3-1)}(t) \equiv V(s(t; x_0, t_0), t) \] (2.3-8)

Then the *time derivative of V along a trajectory* of (2.3-1) is defined as

\[ \frac{d}{dt} \tilde{V}_{(2.3-1)}(t) = D_1 V(s(t; x_0, t_0), t) f(s(t; x_0, t_0), t) + D_2 V(s(t; x_0, t_0), t) \] (2.3-9)
Notation The notation $\tilde{V}_{(eq. \#)}(t)$ will be used generically to denote a Lyapunov function evaluated along a trajectory of the specified equation number.

Definition 2.3.4 Dini Derivative along a Trajectory

Let $V : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+$. Define $\tilde{V}_{(2.3.1)}(t) : \mathbb{R}^+ \to \mathbb{R}^+$ as in (2.3-8). The Dini derivative of $V$ along a trajectory of (2.3-1) is defined as

$$
D^+ \tilde{V}_{(2.3.1)}(t) = \limsup_{h \to 0^+} \frac{V(s(t + h \mid x_0, t_0), t + h) - V(s(t \mid x_0, t_0), t)}{h} \tag{2.3-10}
$$

Theorem 2.3-1[18] Bellman-Gronwall Inequality

Let

(i) $f, g : \mathbb{R}^+ \to \mathbb{R} \tag{2.3-11}$

(ii) $k : \mathbb{R}^+ \to \mathbb{R}^+ \tag{2.3-12}$

(iii) $vk : \mathbb{R}^+ \to \mathbb{R} \tag{2.3-13}$

all be locally integrable. Under these conditions, if $u \in L_{loc}$ satisfies

$$
u(t) \leq f(t) + g(t) \int_0^t k(\tau)u(\tau) \, d\tau, \quad \forall t \in \mathbb{R}^+ \tag{2.3-14}$$

then

$$
u(t) \leq f(t) + g(t) \int_0^t k(\tau)f(\tau)e^{-\int_{\tau}^t k(\xi)g(\xi) \, d\xi} \, d\tau, \quad \forall t \in \mathbb{R}^+ \tag{2.3-15}$$
In the special case where \( c \geq 0 \) and \( k : \mathbb{R}^+ \to \mathbb{R}^+ \) is locally integrable, then

\[
u(t) \leq c + \int_0^t k(\tau) \nu(\tau) \, d\tau, \quad \forall \, t \in \mathbb{R}^+
\]

implies

\[
u(t) \leq ce^t \quad \forall \, t \in \mathbb{R}^+
\]

**Theorem 2.3-2**[45, 60] **Relationship between Exponential Stability and Lyapunov Functions**

Consider the nonlinear system (2.3-1) under Assumption 2.3-1. Assume that \( x \mapsto f(x, t) \) is globally Lipschitz continuous uniformly in time. That is, there exists a constant \( L \) such that

\[
|f(x, t) - f(x', t)| \leq L |x - x'|, \quad \forall \, x, x' \in \mathbb{R}^n, \forall \, t \in \mathbb{R}^+
\]

Under these conditions, the following statements are equivalent:

(A) The nonlinear system (2.3-1) is globally exponentially stable uniformly, i.e. there exist constants \( m \) and \( \lambda > 0 \) such that

\[
|s(t; x_0, t_0)| \leq me^{-\lambda(t - t_0)} |x_0|, \quad t \geq t_0, \forall \, x_0 \in \mathbb{R}^n, \forall \, t_0 \in \mathbb{R}^+
\]

(B) There exists a continuous function \( V : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+ \) and constants \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0 \) such that

\[
\alpha_1 |x| \leq V(x, t) \leq \alpha_2 |x|, \quad \forall \, x \in \mathbb{R}^n, \forall \, t \in \mathbb{R}^+
\]

\[
D^+ \hat{V}_{(2.3-1)}(t) \leq -\alpha_3 |s(t; x_0, t_0)|, \quad t \geq t_0
\]

\[
|V(x, t) - V(x', t)| \leq \alpha_4 |x - x'|, \quad \forall \, x \in \mathbb{R}^n, \forall \, t \in \mathbb{R}^+
\]

where \( \hat{V}_{(2.3-1)} \) is defined as in (2.3-8).
Proof of $A$ implies $B$

Choose any $\gamma \in (0, 1)$, and let $V: \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+$ be defined as

$$V(x, t) = \sup_{\delta \in \mathbb{R}^+} e^{\nu \delta} |s(t + \delta; x, t)|$$  \hspace{1cm} (2.3-23)$$

Clearly

$$|x| \leq V(x, t) \leq m|x|$$  \hspace{1cm} (2.3-24)$$

which proves the bounds on $V$ in (2.3-20).

It is now shown that $V$ satisfies the Lipschitz condition (2.2-22). First note that exponential stability implies

$$e^{\gamma \delta} |s(t + \delta; x, t)| \leq e^{\gamma \delta} me^{-\lambda \delta} |x| \leq me^{-\lambda(1 - \gamma \delta)} |x|$$  \hspace{1cm} (2.3-25)$$

Since $\gamma \in (0, 1)$, (2.3-25) implies that the supremum in (2.3-23) equivalently can be taken over a finite interval $\delta \in [0, T]$ where

$$T = \frac{\ln m}{\lambda(1 - \gamma)}$$  \hspace{1cm} (2.3-26)$$

Thus

$$V(x, t) - V(x', t) = \sup_{\delta \in [0, T]} e^{\gamma \delta} |s(t + \delta; x, t) - s(t + \delta; x', t)|$$

$$\leq \sup_{\delta \in [0, T]} e^{\gamma \delta} |s(t + \delta; x, t) - s(t + \delta; x', t)|$$  \hspace{1cm} (2.3-27)$$

$$\leq \sup_{\delta \in [0, T]} e^{\gamma \delta} |s(t + \delta; x, t) - s(t + \delta; x', t)|$$  \hspace{1cm} (2.3-28)$$

By definition of $s$,

$$s(t + \delta; x, t) - s(t + \delta; x', t) = x - x' + \int_t^{t + \delta} \hat{f}(s(\tau; x, t), \tau) - f(s(\tau; x', t), \tau) d\tau$$  \hspace{1cm} (2.3-29)$$

Bounding (2.3-29) and using the Lipschitz condition (2.3-18) on $f$, it follows that
\[ |s(t + \delta ; x, t) - s(t + \delta ; x', t)| \leq |x - x'| + \int_{t}^{t + \delta} L |s(\tau ; x, t) - s(\tau ; x', t)| d\tau \] 

(2.3-30)

After applying the Bellman-Gronwall inequality to (2.3-30) and substituting into (2.3-28),

\[ V(x, t) - V(x', t) \leq |x - x'| e^{(T + \lambda)} \] 

(2.3-31)

Repeating (2.3-27) - (2.3-31) with \( x \) and \( x' \) reversed proves the desired global Lipschitz condition with respect to \( x \) on \( V \) in (2.3-22). Continuity of \( t \mapsto V(x, t) \) can be shown using similar reasoning.

Finally, to prove the negative definiteness condition (2.3-21), recall the definition

\[ \mathcal{D} \tilde{V}_{(2.3-1)}(t) = \lim_{h \to 0^+} \sup_{h} \frac{V(s(t + h ; x_0, t_0) + h) - V(s(t ; x_0, t_0), t)}{h} \] 

(2.3-32)

Evaluating (2.3-32) term-by-term gives

\[ V(s(t + h ; x_0, t_0), t + h) = \sup_{\delta \in [0,T]} e^{\gamma \delta} |s(t + h + \delta ; s(t + h ; x_0, t_0), t + h)| \] 

(2.3-33)

and

\[ V(s(t ; x_0, t_0), t) = \sup_{\delta \in [0,T]} e^{\gamma \delta} |s(t + \delta ; s(t ; x_0, t_0), t)| \] 

(2.3-34)

Taking the supremum over a subset of \([0, T]\) in (2.3-34) and exploiting the properties of \( s \), one has that

\[ V(s(t ; x_0, t_0), t) \geq \sup_{\delta \in [h,T]} e^{\gamma \delta} |s(t + \delta ; s(t ; x_0, t_0), t)| \]

(2.3-35)

\[ = \sup_{\eta \in [0,T]} e^{\gamma h} e^{\gamma \eta} |s(t + h + \eta ; s(t ; x_0, t_0), t)| \]

\[ = e^{\gamma h} \sup_{\eta \in [0,T]} e^{\gamma \eta} |s(t + h + \eta ; s(t + h ; x_0, t_0), t + h)| \]
\[ e^{\gamma h} V(s(t + h; x_o, t_o), t + h) \]

Thus,

\[ V(s(t + h; x_o, t_o), t + h) \leq e^{-\gamma h} V(s(t; x_o, t_o), t) \tag{2.3-36} \]

Substituting (2.3-36) into (2.3-32)

\[ D^+ \tilde{V}_{(2.3-1)}(t) = \lim_{h \to 0^+} \frac{e^{-\gamma h} - 1}{h} - \gamma \alpha |s(t; x_o, t_o)| \tag{2.3-37} \]

which completes the proof of A implies B.

**Proof of B implies A**

Condition (2.3-21) assures that along any trajectory of (2.3-1), the function \( \tilde{V} \) is monotonically decreasing. Using the bounds (2.3-20), this implies

\[ \tilde{V}_{(2.3-1)}(t) \leq \tilde{V}_{(2.3-1)}(t_0) + \int_{t_0}^{t} D^+ \tilde{V}_{(2.3-1)}(\tau) d\tau \leq \tilde{V}_{(2.3-1)}(t_0) - \int_{t_0}^{t} \frac{\alpha_3}{\alpha_2} \tilde{V}_{(2.3-1)}(\tau) d\tau \tag{2.3-38} \]

Using the Bellman-Gronwall inequality,

\[ \tilde{V}_{(2.3-1)}(t) \leq \tilde{V}_{(2.3-1)}(t_0) e^{\frac{\alpha_3}{\alpha_2} (t - t_0)}, \quad t \geq t_0 \tag{2.3-39} \]

Substituting the bounds on \( V \) in (2.3-39), uniform global exponential stability follows with

\[ m = \frac{\alpha_2}{\alpha_1} \text{ and } \lambda = \frac{\alpha_3}{\alpha_2}. \tag{2.3-40} \]
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Remark 1 Theorem 2.3-2 and its proof should be studied carefully since it outlines methods which will be used extensively in the forthcoming analyses.

Remark 2 This equivalence between exponential stability and Lyapunov stability is also shown in reference [8] with the exception that the bounds (2.3-20)-(2.3-21) on $V$ are quadratic, i.e. proportional to $|x|^2$.

2.3.2 Input-Output Stability

The remainder of this section is devoted to background material on input/output (I/O) stability specialized for the purposes of this thesis.

Definition 2.3-5 Input/Output Operators and Unbiased Operators

Let $L_{p_1}$ and $L_{p_2}$ be extended $L_p$ spaces. An input/output operator, $H$, is any mapping such that

$$H : L_{p_1} \rightarrow L_{p_2}$$

(2.3-41)

An I/O operator is said to be unbiased if

$$H(0) = 0$$

(2.3-42)

Definition 2.3-6 Finite-Gain Stability / Operator Norm

The unbiased I/O operator $H : L_{p_1} \rightarrow L_{p_2}$ is said to be finite-gain stable if there exists a
constant $K$ such that

$$\|P_T H u\|_{L_{p^2}} \leq K \|P_T u\|_{L_{p^1}}, \quad \forall T \geq 0, \forall u \in L_{p^1}$$  \hspace{1cm} (2.3-43)

The operator norm of $H$, denoted $\|H\|$, is the smallest such constant which satisfies (2.3-43).

Definition 2.3-7 Finite-Gain Incremental Stability / Incremental Operator Norm

The unbiased I/O operator $H : L_{p^1} \rightarrow L_{p^2}$ is said to be finite-gain incrementally stable if there exists a constant $L$ (for Lipschitz) such that

$$\|P_T (H u - H u')\|_{L_{p^2}} \leq L \|P_T (u - u')\|_{L_{p^1}}, \quad \forall T \geq 0, \forall u, u' \in L_{p^1}$$  \hspace{1cm} (2.3-44)

The incremental operator norm of $H$, denoted $\|H\|_{\delta}$, is the smallest such constant which satisfies (2.3-44).

Definition 2.3-8[8, 58] Small Signal Finite-Gain Stability

The unbiased I/O operator $H : L_{p^1} \rightarrow L_{p^2}$ is said to be small-signal finite-gain stable if there exist constants $K$ and $\varepsilon \geq 0$ such that $\forall T \geq 0, \forall u \in L_{p^1}$

$$\|P_T u\|_{L_{p^1}} \leq \varepsilon$$  \hspace{1cm} (2.3-45)

implies

$$\|P_T H u\|_{L_{p^2}} \leq K \|P_T u\|_{L_{p^1}}$$  \hspace{1cm} (2.3-46)
The next theorem addresses the stability of the feedback configuration in Fig. 2.3-1. The feedback equations are given by

\[ e_1 = u_1 + H_2 e_2 \]  \hspace{1cm} (2.3-47) \]
\[ e_2 = u_2 + H_1 e_1 \]  \hspace{1cm} (2.3-48) \]

where \( H_1, H_2 \) are unbiased operators which satisfy

\[ H_1, H_2 : L_{pe} \rightarrow L_{pe} \quad \text{for some } p \in [1, \infty) \] (2.3-49)

Essentially, it will be shown that under certain conditions on \( H_1 \) and \( H_2 \), the size of the errors \( e_{1,2} \) (or equivalently the outputs \( y_{1,2} \)) can be bounded by the size of the inputs \( u_{1,2} \).

---

**Figure 2.3-1** Standard Feedback Configuration

Before proceeding with the theorem, the following assumption is made in order to avoid the questions of existence, uniqueness, etc. of the solutions to (2.3-47)-(2.3-48).

**Assumption 2.3-2** Given any \( u_{1,2} \in L_{pe} \), there exist unique \( e_{1,2} \in L_{pe} \) which satisfy (2.3-47)-(2.3-48).
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**Theorem 2.3-3[62] Small-Gain Theorem**

Consider the feedback configuration of Fig. 2.3-1 with feedback equations (2.3-47)-(2.3-48) under Assumption 2.3-2, and let the operators $H_1$ and $H_2$ be finite-gain stable. Under these conditions, if there exists a constant $\gamma$ such that

$$\|H_1\| \|H_2\| \leq \gamma < 1 \quad (2.3-50)$$

then

$$\|P_T e_1\|_{L_p} \leq (1 - \gamma)^{-1} \left( \|P_T u_1\|_{L_p} + \|H_2\| \|P_T u_2\|_{L_p} \right), \forall T \geq 0 \quad (2.3-51)$$

$$\|P_T e_2\|_{L_p} \leq (1 - \gamma)^{-1} \left( \|P_T u_2\|_{L_p} + \|H_1\| \|P_T u_1\|_{L_p} \right), \forall T \geq 0 \quad (2.3-52)$$

**Proof** Applying the finite-gain stability of $H_1$ and $H_2$ to (2.3-47)-(2.3-48), it follows that

$$\|P_T e_1\|_{L_p} \leq \|P_T u_1\|_{L_p} + \|H_2\| \|P_T e_2\|_{L_p} \quad \forall T \geq 0 \quad (2.3-53)$$

$$\|P_T e_2\|_{L_p} \leq \|P_T u_2\|_{L_p} + \|H_1\| \|P_T e_1\|_{L_p} \quad \forall T \geq 0 \quad (2.3-54)$$

Combining (2.3-53)-(2.3-54),

$$\|P_T e_1\|_{L_p} \leq \|P_T u_1\|_{L_p} + \|H_2\| \left( \|P_T u_2\|_{L_p} + \|H_1\| \|P_T e_1\|_{L_p} \right) \quad (2.3-55)$$

$$\leq \|P_T u_1\|_{L_p} + \|H_2\| \|P_T u_2\|_{L_p} + \gamma \|P_T e_1\|_{L_p} \quad (2.3-56)$$

Similarly

$$\|P_T e_2\|_{L_p} \leq \|P_T u_2\|_{L_p} + \|H_1\| \|P_T u_1\|_{L_p} + \gamma \|P_T e_2\|_{L_p} \quad (2.3-57)$$

Equations (2.3-51)-(2.3-52) immediately follow since $\gamma < 1$. \[\blacksquare\]
Remark  Note that (2.3-51)-(2.3-52) can be used to bound the outputs $y_{1,2}$ since
\begin{align*}
y_1 &= e_2 - u_2 \quad (2.3-58) \\
y_2 &= e_1 - u_1 \quad (2.3-59)
\end{align*}

For the purpose of completeness, a generalization of the small-gain theorem which considers a larger class of feedback configurations that in Fig. 2.3-1 is now presented. This generalization is done at the cost of slightly stronger assumptions on the operator $H_1$. The approach taken here is motivated by the general sector-stability criteria of [48].

![Figure 2.3-2 Generalized Feedback Configuration](image)

The system under consideration is shown in Fig. 2.3-2. The feedback equations are
\begin{align*}
e_1 &= H_2 e_2 \quad (2.3-60) \\
e_2 &= H_1[u] e_1 \quad (2.3-61)
\end{align*}

where $H_1$ and $H_2$ are defined as follows. Let $\mathcal{L}_{p_0}$, $\mathcal{L}_{p_1}$, and $\mathcal{L}_{p_2}$ be extended $\mathcal{L}_p$ spaces. Then
(1) $H_2 : L_{p_2e} \rightarrow L_{p_1e}$ is an unbiased I/O operator. \hfill (2.3-62)

(2) $H_1[\cdot] : L_{p_1e} \rightarrow L_{p_2e}$ is a family of (not necessarily unbiased) I/O operators \hfill (2.3-63)

indexed by the variable $u \in L_{p_0e}$

(3) $H_1[0] : L_{p_1e} \rightarrow L_{p_2e}$ is an unbiased I/O operator. \hfill (2.3-64)

The fundamental difference between this representation and the standard feedback configuration is the way in which the inputs enter into the equations. Unlike the standard case, the inputs are not restricted to enter additively into the feedback loop. Rather, the input $u$ enters by selecting the forward loop operator $H_1$ out of a possible family of operators.

In Fig. 2.3-3, it is shown how the generalized configuration can be applied to the standard case. In terms of the generalized configuration, the system input is $u = (u_1, u_2)$, the system 'errors' are $(y_2, e_2)$, and the forward loop operator $\tilde{H}_1[u]$ is defined as

$$\tilde{H}_1[(u_1, u_2)] y_2 = u_2 + H_1(u_1 + y_2)$$ \hfill (2.3-65)

![Figure 2.3-3: Alternate Representation of Standard Configuration](image)

Before proceeding with the theorem, the following assumptions are made.
**Assumption 2.3-3** Given any \( u \in L_{p_0} \), there exist unique \( (e_1, e_2) \in L_{p_1} \times L_{p_2} \) which satisfy (2.3-60)-(2.3-61).

**Assumption 2.3-4** There exists a constant \( K \) such that the following is true. For any \( u, e_1, e_2 \) which satisfy

\[
e_2 = H_1[u] e_1
\]

there exist \( z_1, z_2 \in L_{p_1} \times L_{p_2} \) such that

\[
z_2 = H_1[0] z_1
\]

\[
\| P_T(e_1 - z_1) \|_{L_{p_1}} \leq K \| P_T u \|_{L_{p_0}}, \quad \forall \; T \geq 0
\]

\[
\| P_T(e_2 - z_2) \|_{L_{p_2}} \leq K \| P_T u \|_{L_{p_0}}, \quad \forall \; T \geq 0
\]

**Remark** Essentially, Assumption 2.3-4 restricts in a quantitative manner the degree to which the operators \( H_1[u] \) can deviate from \( H_1[0] \). In the original terminology of reference [48], \( H_1[u] \) is finite-gain stable about the set \( H_1[0] \). From (2.3-65) it is easy to see that the standard configuration satisfies this assumption.

**Theorem 2.3-4 Generalized Small-Gain Theorem**

Consider the feedback configuration of Fig. 2.3-3 with feedback equations (2.3-60)-(2.3-61)
Chapter 2: Mathematical Preliminaries

under Assumptions 2.3-3 and 2.3-4, and let the operator \( H_1[0] \) be finite-gain incrementally stable and \( H_2 \) finite-gain stable. Under these conditions, if there exists a constant \( \gamma \) such that

\[
\| H_1[0] H_2 \| \leq \gamma < 1
\] (2.3-70)

then \( \forall T \in \mathbb{R}^+ \)

\[
\| P_T e_1 \|_{L_{p_1}} \leq (1 - \gamma)^{-1} \left( 1 + \| H_1[0] \|_{\delta} \right) K \| P_T u \|_{L_{p_0}} \| H_2 \|
\] (2.3-71)

\[
\| P_T e_2 \|_{L_{p_2}} \leq (1 - \gamma)^{-1} \left( 1 + \| H_1[0] \|_{\delta} \right) K \| P_T u \|_{L_{p_3}}
\] (2.3-72)

Proof Let \( e_1 \) and \( e_2 \) be solutions to (2.3-60)-(2.3-61). Since \( e_2 = H_1[u] e_1 \), let \( z_1 \) and \( z_2 \) be approximations to \( e_1 \) and \( e_2 \) as in Assumption 2.3-4. Then,

\[
e_2 = H_1[u] e_1 = H_1[0] e_1 + \{ H_1[u] e_1 - z_2 \} + \{ H_1[0] z_1 - H_1[0] e_1 \}
\] (2.3-73)

Thus

\[
\| P_T e_2 \|_{L_{p_2}} \leq \| P_T H_1[0] e_1 \|_{L_{p_2}} + K \| P_T u \|_{L_{p_0}} + \| H_1[0] \|_{\delta} K \| P_T u \|_{L_{p_0}}
\] (2.3-74)

\[
= \| P_T H_1[0] H_2 e_2 \|_{L_{p_2}} + (1 + \| H_1[0] \|_{\delta}) K \| P_T u \|_{L_{p_0}}
\]

\[
\leq \gamma \| P_T e_2 \|_{L_{p_2}} + (1 + \| H_1[0] \|_{\delta}) K \| P_T u \|_{L_{p_0}}
\]

which implies the bound (2.3-72) since \( \gamma < 1 \). Equation (2.3-71) follows immediately from finite-gain stability of \( H_2 \).

Remark 1 Note the stronger assumption on the operators is that \( H[0] \) is required to be finite-gain incrementally stable, rather than just finite-gain stable.
Remark 2 As is typical of small-gain theorems, the main idea is in the stable (and possibly causal) invertibility of the operator $I - H_1(0)H_2$. The small-gain condition is simply a sufficient condition to achieve this.

Remark 3 Note that with only a slight modification, Theorem 2.3-4 can be applied to the case where the input $u$ and errors $e_{1,2}$ belong to any extended normed vector space. In fact, it may be possible to extend Theorem 2.3-4 to cover the more general cases where the signals are in any extended normed spaces and $H_1$ and $H_2$ are relations instead of operators.
Chapter 3
Analysis of Linear Parameter-Varying Gain Scheduled Control Systems

3.1 Introduction

This chapter addresses the issue of guaranteed properties for parameter-varying gain scheduled control systems. Consider a plant of the form

\[ \dot{x}(t) = A(\theta(t)) \ x(t) + B(\theta(t)) \ u(t) \quad (3.1-1) \]
\[ y(t) = C(\theta(t)) \ x(t) \quad (3.1-2) \]

These equations represent a linear plant whose dynamics depend on a vector of time-varying exogenous parameters, \( \theta \), which take their values in some prescribed set \( \theta(t) \in \Theta \). One example of a physical system whose (linearized) dynamics take the form of (3.1-1)-(3.1-2) is an aircraft, where the time-varying parameter is typically dynamic pressure [e.g. 54].

Gain scheduled controllers for such plants typically are designed as follows. First, the designer selects a set of parameter values, \( \{\theta_i\} \), which represent the range of the plant's dynamics, and designs a linear time-invariant compensator for each. Then, in between operating points, the compensators are interpolated such that for all frozen values of the
parameters, the closed loop system has excellent feedback properties, such as nominal stability, robustness to unmodeled dynamics, and robust performance (Fig. 3.1-1).

\[ \theta \rightarrow K(s, \theta) \rightarrow P(s, \theta) \rightarrow L(s) \rightarrow y \]

**Figure 3.1-1 A Linear Plant Scheduling on Exogenous Parameters**

Since the parameters are actually time-varying, none of these properties need carry over to the overall time-varying closed loop system. Even in the simplest case of nominal stability (i.e. no unmodeled dynamics), *parameter time-variations can be destabilizing*.

In this chapter, conditions are given which guarantee that the closed loop system will retain the feedback properties of the frozen-time designs. These conditions formalize various heuristic ideas which have guided successful gain scheduled designs. For example, one primary such guideline is "the scheduling variables should vary slowly with respect to the system dynamics." Note that this idea is simply a reminder that the original designs were based on *linear time-invariant approximations* to the actual plant. In this sense, these approximations must be sufficiently faithful to the true plant if one expects the global design to exhibit the desired feedback properties. In fact, it is this idea which proves most fundamental in the forthcoming analysis.
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Organization of Chapter

Section 3.2 addresses the issue of nominal stability. In Section 3.2.1, the mechanism of time-varying instability arising from frozen-time stability is discussed. Section 3.2.2 discusses the general problem of stability of slowly-varying linear systems. In Section 3.3, the issues of robust stability and robust performance are addressed. The formal problem statement is given in Section 3.3.1. Section 3.3.2 presents background material on Volterra integrodifferential equations. In Section 3.3.3, conditions are given which guarantee time-varying robustness/performance given frozen-time robustness/performance. The conditions are presented from both a state-space and input-output viewpoint. Finally, concluding remarks are given in Section 3.4.

3.2 Nominal Stability

3.2.1 The Mechanism of Time-Varying Instability

Even in the simplest case of nominal stability, it is well known that time-variations can destabilize a frozen-time stable system. For example, consider the linear system [1, 59].

\[
\frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -a \sin 2t & -a \cos 2t \\ -a \cos 2t & a \sin 2t \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
\]  

(3.2-1)

where \( a \) is a constant parameter. The eigenvalues of the dynamics matrix are given by

\[
\lambda_{1,2} = \frac{a - 2 \pm \sqrt{a^2 - 4}}{2}
\]

(3.2-2)

which are constant and lie in the left-half complex plane for all \( a < 2 \). However, the state
transition matrix is given by

\[
\Phi(t, 0) = \begin{bmatrix}
    e^{(a-1)t} \sin t & e^{-t} \sin t \\
    -e^{(a-1)t} \sin t & e^{-t} \cos t
\end{bmatrix}
\]

(3.2-3)

Thus, it is seen that for the values of \( a, 1 < a < 2 \), the time-varying dynamics of (3.2-1) are unstable, even though the frozen-time dynamics are stable. Conversely, reference [51] contains an example where the time-varying dynamics are stable, while the frozen-time dynamics are unstable.

The following example, taken from [56], better illustrates how time-varying instability can arise from frozen-time stability. Consider the variable structure system given by

\[
\ddot{x}(t) + \omega(t)^2 x(t) = 0, \quad \omega(t) = \begin{cases} 
    \omega_1, & nT \leq t < (n+1)T, \ n \text{ even} \\
    \omega_2, & nT \leq t < (n+1)T, \ n \text{ odd}
\end{cases}
\]

(3.2-4)

for some interval \( T \). The system equations switch between two oscillatory, hence marginally stable, systems. As shown in Fig. 3.2-1, a careful selection of initial conditions and switching interval can result in an unstable state trajectory.

This instability phenomena can be explained as follows. The two individual systems each experience an amplification/attenuation cycle of the state vector in their trajectories. The switching is timed so that the time-varying system's state trajectory always experiences the individual amplification phases, resulting in an unstable trajectory.
This reasoning can also be used to explain the instability found in (3.2-1). That is, although the frozen-time systems are stable, each frozen-time trajectory experiences a certain amount of amplification of the initial conditions before the state decays to zero. As in the previous example, the time-variations are introduced in such a way that the time-varying state trajectory always experiences these amplification phases.

3.2.2 Stability of Slowly Varying Systems

Suppose that one has carried out the gain scheduled design procedure outlined in the introduction for some linear parameter-varying plant. Then along any particular parameter vector trajectory, the closed loop unforced dynamics are of the form

\[ \dot{x}(t) = A(t) x(t), \quad x(0) = x_0 \in \mathbb{R}^n, \quad t \geq 0. \]  

(3.2-5)

where A now represents the closed loop dynamics matrix.
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In this section, conditions are given which guarantee time-varying stability of (3.2-5) given frozen-time stability. In relation to the gain scheduled design, these conditions can be used to place restrictions on the nature of allowable parameter variations in order to guarantee nominal stability. These conditions summarize and extend previous results found in [17, 18, 35, 57]. Furthermore, the approach taken here extends almost immediately to the analysis of time-varying robustness and performance.

First, assumptions on (3.2-1) and a preliminary lemma are presented.

Assumption 3.2-1 The dynamics matrix $A : \mathbb{R}^+ \to \mathbb{R}^{n \times n}$ is bounded and globally Lipschitz continuous with constant $L_A$, i.e.

$$|A(t) - A(\tau)| \leq L_A |t - \tau| \quad \forall \ t, \tau \in \mathbb{R}^+.$$  \hspace{1cm} (3.2-6)

Lemma 3.2-1 [7] Consider the linear system

$$\dot{x}(t) = A_o x(t) + \delta A(t) x(t), \quad x(0) = x_o \in \mathbb{R}^n, \ t \geq 0.$$  \hspace{1cm} (3.2-7)

Suppose that for some $m, \lambda$, and $k \geq 0$

$$|e^{A_o t}| \leq me^{-\lambda t},$$  \hspace{1cm} (3.2-8)

$$|\delta A(t)| \leq k, \quad \forall \ t \geq 0.$$  \hspace{1cm} (3.2-9)

Under these conditions,

$$|x(t)| \leq me^{-\lambda t} \cdot k \cdot x_o \quad \forall \ t \geq 0, \ x_o \in \mathbb{R}^n.$$  \hspace{1cm} (3.2-10)

Proof The solution to (3.2-6) is given by
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\[
x(t) = e^{A_0(t)} x_0 + \int_0^t e^{A(t-\tau)} \delta A(\tau) \| x(\tau) \| d\tau
\]  

(3.2-11)

Using (3.2-8) and (3.2-9),

\[
\| x(t) \| \leq m e^{-\lambda t} \| x_0 \| + \int_0^t m e^{-\lambda(t-\tau)} k \| x(\tau) \| d\tau
\]  

(3.2-12)

Multiplying by \( e^{-\lambda t} \) and applying the Bellman-Gronwall inequality,

\[
\| x(t) \| \leq m e^{-(\lambda - mk)t} \| x_0 \|.
\]  

(3.2-13)

The usefulness of Lemma 3.2-1 is that it allows one to guarantee exponential stability of a perturbed time-varying system (3.2-7) given that the unperturbed time-invariant system \( (\delta A(t) \equiv 0) \) is exponentially stable.

The main result of this section is now presented.

**Theorem 3.2-1** Consider the linear system of (3.2-5) under Assumption 3.2-1. Assume that at each instant in time (1) \( A(t) \) is a stable matrix and (2) there exist constants \( m \) and \( \lambda \geq 0 \) such that

\[
\| e^{A(t)\tau} \| \leq m e^{-\lambda \tau}, \quad \forall \tau, \tau \geq 0.
\]  

(3.2-14)

Under these conditions, given any \( \eta \in [0, \lambda] \),

\[
L_A \leq \frac{(\lambda - \eta)^2}{4 m \ln m}
\]  

(3.2-15)

implies
\[ |x(t)| \leq m e^{-\gamma t} |x_0|, \quad \forall \ t \geq 0, \ x_o \in \mathbb{R}^n \] (3.2-16)

**Proof** Consider approximating \( A(t) \) in (3.2-5) by the piecewise constant matrix

\[ A_{pc}(t) = A(nT), \quad nT \leq t < (n+1)T, \ n = 0,1,2, \ldots \] (3.2-17)

where \( T \) is to be chosen. Rewriting (3.2-5),

\[ \dot{x}(t) = A_{pc}(t) x(t) + \{A(t) - A_{pc}(t)\} x(t). \] (3.2-18)

Now choose

\[ T = \frac{2 \ln m}{\lambda - \eta} \] (3.2-19)

Then for all time \( t \geq 0 \),

\[ |A(t) - A_{pc}(t)| \leq L_A T \leq \frac{\lambda - \eta}{2m} \] (3.2-20)

where \( L_A \) is chosen according to (3.2-15). It follows from Lemma 3.2-1 that on \( nT \leq t < (n+1)T \),

\[ |x(t)| \leq m e^{-\frac{\lambda + \eta}{2} (t - nT)} |x(nT)| \] (3.2-21)

\[ \leq m e^{-\frac{\lambda + \eta}{2} (t - nT)} \left\{ m e^{-\frac{\lambda + \eta}{2}} \right\}^n |x_0| \]

\[ = m e^{-\frac{\eta}{2} (t - nT)} \left\{ m e^{-\frac{\lambda - \eta}{2}} \right\}^n |x_0| \]

\[ \leq m e^{-\eta t} |x_0| \]

which completes the proof. \[ \blacksquare \]

**Remark** The particular value in the condition (3.2-15) results from some flexibility in
choosing the interval $T$ in (3.2-17). This flexibility is exploited in (3.2-19) where the interval $T$ is chosen to maximize the RHS of (3.2-16).

Theorem 3.2-1 states that a time-varying system retains its frozen-time exponential stability (to a lessened degree) provided that the time-variations are sufficiently slow. It is stressed that the only restriction Theorem 3.2-1 imposes on the dynamics is on the rate of the variations. That is, the variations themselves may be large - provided that the dynamics change slowly. In terms of the original problem of a parameter-varying linear system, this means that the range of parameter variations may be large provided that the time derivative (e.g. with $p$ parameters)

$$\left| \frac{dA(\theta(t))}{dt} \right| = \left| \dot{\theta}_1(t) \frac{\partial A(\theta(t))}{\partial \theta_1} + \ldots + \dot{\theta}_p(t) \frac{\partial A(\theta(t))}{\partial \theta_p} \right|$$

(3.2-22)

is small.

Conversely, Theorem 3.2-1 is not immediately applicable in the case of rapidly varying dynamics over a small range; in which case, Lemma 3.2-1 is more appropriate. In the case of rapidly varying dynamics over a large range, it may be possible to combine Theorem 3.2-1 with the method of averaging [e.g. 29, 36] to decompose the dynamics into a "slowly-varying averaged part" and a "perturbation part."

The reasoning behind Theorem 3.2-1 can be outlined as follows. First, the time-varying dynamics are approximated by piecewise constant dynamics given by (3.2-17). On each piecewise constant interval, the linear system is in the form of a nominally stable time-invariant part and a time-varying perturbation. Using Lemma 3.2-1, the state will decay exponentially provided that the perturbation is sufficiently small - or equivalently, the constant approximation is accurate. When the current constant approximation is no longer accurate, a new approximation is made.

Now suppose that the approximation (3.2-17) is sufficiently accurate so that the state
decays exponentially on each interval. Although the state decays on each interval, it may experience a certain amount of amplification before its eventual decay. This amplification imposes a constraint on the approximation. Namely, the approximation must be valid long enough to allow sufficient decay to protect against the possible amplification of the next interval (Fig. 3.2-2). Thus the approximation is required to be accurate for a certain length of time which depends on the nature of the frozen-time exponential stability. It is this restriction which translates into the "slowly varying" restriction (3.2-15).

![Figure 3.2-2 Interval-by-Interval Exponential Decay](image)

A key parameter of the frozen-time exponential stability is the "overshoot parameter" of (3.2-14). For example, in the special case where \( m = 1 \), one has that the time-variations may be arbitrarily fast. In terms of the previous discussion, the piecewise-constant approximations must be accurate for only infinitesimal intervals since there is never an amplification of the state vector.

The case where \( m = 1 \) also is in agreement with the discussion in Section 3.2.1. Recall it was stated that instability may arise when time-variations are introduced in such a way that the time-varying trajectory always experiences an amplification phase of a frozen-time trajectory.
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When $m = 1$, none of the frozen-time systems experience any amplification; hence, instability cannot occur.

Given the above explanations behind Theorem 3.2-1, it becomes evident how possible conservatism can arise. The sufficient condition given by (3.2-15) prevents any "cascade of amplification phases" by uniformly enforcing that the amplification of all the frozen-time systems be sufficiently small in relation to the system's speed of response. However for time-varying instability to occur, it is required that a certain amount of "directional coincidence" of the amplification phases be present. In other words, it is possible that the frozen-time systems may exhibit a great deal of overshoot but not in such a way to cause this cascade of amplification phases. This is demonstrated by the damped Mathieu equation [62]

$$\frac{d^2 y}{dt^2} + 0.2 \frac{dy}{dt} + (0.01 + a)y + 3.2 \cos 2ty = 0$$ (3.2-23)

which becomes unstable for $a \in [3.2, 3.8]$ and then for $a \geq 5$.

It is now shown how the constants $m$ and $\lambda$ can be calculated.

**Lyapunov Method** Assume that the frozen-time linear system

$$\dot{x}(t) = A(\tau)x(t), \quad x(0) = x_0 \in \mathbb{R}^n, \quad t \geq 0$$ (3.2-24)

is exponentially stable. This implies that there exist symmetric positive definite matrices $K(\tau)$ and $Q(\tau)$ such that

$$K(\tau)A(\tau) + A^T(\tau)K(\tau) = -Q(\tau)$$ (3.2-25)

Under these conditions,

$$|x(t)| \leq me^{-\lambda t} |x_0|$$ (3.2-26)

where
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\[
m = \left( \frac{\sigma_{\text{max}}(K(\tau))}{\sigma_{\text{min}}(K(\tau))} \right)^{1/2}
\]

(3.2-27)

\[
\lambda = \frac{1}{2} \frac{\sigma_{\text{min}}(Q(\tau))}{\sigma_{\text{max}}(K(\tau))}
\]

(3.2-28)

This can be shown using reasoning similar to Theorem 2.3-2 with \( V(x) = x^T K(\tau) x \). The only difference is that in this case the bounds on \( V \) are quadratic.

Matrix Exponential Method Assume that the frozen-time system of (3.2-24) is exponentially stable. Assume further that \( A(\tau) \) has \( n \) linearly independent eigenvectors. Then the solution to (3.2-24) can be written as

\[
x(t) = e^{A(\tau)t} x_o = V(\tau) e^{\Lambda(\tau)t} V^{-1}(\tau) x_o
\]

(3.2-29)

where \( V(\tau) \) and \( \Lambda(\tau) \) are the eigenvector and eigenvalue matrices, respectively. Bounding \( x(t) \) in (3.2-30),

\[
| x(t) | \leq m e^{-\lambda t} | x_o |
\]

(3.2-30)

where

\[
m = \frac{\sigma_{\text{max}}(V(\tau))}{\sigma_{\text{min}}(V(\tau))}
\]

(3.2-31)

\[
\lambda = \max_i \text{Re}[\lambda_i(A(\tau))]
\]

(3.2-32)

When using either of these two methods to satisfy Theorem 3.2-1, it is required that the worst case values for \( m \) and \( \lambda \) over all time are taken. In the case of a parameter-varying
linear system, these constants should be evaluated for the worst case parameter values. This is reasonable since the only assumptions made on the parameters are on their admissible values and the magnitude of their time-derivatives; informally, a worst case parameter trajectory is as likely as a constant parameter trajectory.

Before presenting some examples, the approach (specialized to linear systems) found in references \cite{18, 35, 57} is presented for completeness and comparison purposes.

**Theorem 3.2-2** Consider the linear system of (3.2-5) under Assumption 3.2-1. Assume that there exist symmetric positive definite matrices $\mathbf{K} : \mathbb{R}^+ \rightarrow \mathbb{C}^{n \times n}$ and $\mathbf{Q} : \mathbb{R}^+ \rightarrow \mathbb{C}^{n \times n}$ such that

(i) $\mathbf{K}$ is continuously differentiable for all $t > 0$.

(ii) There exist constants $\alpha_1$, $\alpha_2$, and $\alpha_3 > 0$ such that for all $t \geq 0$

$$\alpha_1 \leq \sigma_{\text{min}}(\mathbf{K}(t)) \leq \sigma_{\text{max}}(\mathbf{K}(t)) \leq \alpha_2$$

$$\lambda_{\text{min}}(\mathbf{Q}(t) - \dot{\mathbf{K}}(t)) \geq \alpha_3$$

(iii) $\mathbf{K}(t)\mathbf{A}(t) + \mathbf{A}^T(t)\mathbf{K}(t) = -\mathbf{Q}(t)$ $\forall t \geq 0.$

Under these conditions, the linear system of (3.2-5) is exponentially stable.

**Proof** Conditions (3.2-33) and (3.2-35) state that $V(\mathbf{x}) = \mathbf{x}^T \mathbf{K}(\tau) \mathbf{x}$ is a time-invariant Lyapunov function for each $\tau$-frozen time system. It is shown that

$$V(\mathbf{x}, t) = \mathbf{x}^T K(t) \mathbf{x}$$

is a time-varying Lyapunov function for the linear system (3.2-5) under the slowness condition (3.2-34). Using (3.2-35), the time derivative of $V$ along solutions of (3.2-5) is given by

$$\frac{d\tilde{V}_{(3.2-5)}(t)}{dt} = \mathbf{x}^T(t) \{ \mathbf{K}(t)\mathbf{A}(t) + \mathbf{A}^T(t)\mathbf{K}(t) + \dot{\mathbf{K}}(t) \} \mathbf{x}(t)$$
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\[ = x^T(t) \left\{ -Q(t) + \dot{K}(t) \right\} x(t). \]

From (3.2-34), it follows that

\[ \frac{d\tilde{V}(t)}{dt} \leq -\alpha_3 |x(t)|^2 \]  \hspace{1cm} (3.2-38)

along solutions of (3.2-5); hence \( V \) is a quadratic Lyapunov function for (3.2-5). Using reasoning similar to Theorem 2.3-2, it can be shown that

\[ |x(t)| \leq \left( \frac{\alpha_2}{\alpha_1} \right)^{1/2} - \frac{1}{2} (\alpha_2/\alpha_1)^t e \]  \hspace{1cm} (3.2-39)

Example 3.2-1 Recall the linear time-varying system

\[ \frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -a \sin 2t & -a \cos 2t \\ -a \cos 2t & a \sin 2t \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \]  \hspace{1cm} (3.2-40)

It was shown that for constant values of \( a \), \( 1 < a < 2 \), the time-varying dynamics of (3.2-40) are unstable, even though the frozen-time dynamics are stable. For constant values of \( a < 1 \), one has that the time-varying dynamics are exponentially stable.

The stability of (3.2-40) is now analyzed using the theorems of Section 3.2-3. Suppose one tried the "Lyapunov Method" to calculate the overshoot parameter \( m \). Using \( V(x) = x^T x \) as a Lyapunov function candidate, one has that along trajectories of (3.2-40)

\[ \frac{d\tilde{V}(t)}{dt} = x^T(t) \left\{ A(t) + A^T(t) \right\} x(t) \]  \hspace{1cm} (3.2-41)

However, inspection of (3.2-41) reveals that

\[ \lambda_{1,2}(A(t) + A^T(t)) = -2, 2a - 2 \]  \hspace{1cm} (3.2-42)

Thus for values of \( a < 1 \), \( A(t) + A^T(t) \) is negative definite. In terms of the overshoot
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parameter $m$, this means $m = 1$, and stability is guaranteed using condition (3.2-15). In terms of Theorem 3.2-2, this means that $V(x) = x^T x$ is a time-invariant Lyapunov function; hence, stability is guaranteed using condition (3.2-32). In fact, these theorems allow one to make much stronger statements regarding the stability of (3.2-40). They state that even if the parameter $a$ is time-varying, stability is guaranteed provided that $a < 1$ for all time.

Example 3.2-2 (F-8 Research Aircraft) In this example, the gain scheduled control system suggested for the F-8 in [54] is examined. The linearized longitudinal state dynamics are given by

$$\begin{bmatrix} \frac{d}{dt} q \\ \alpha \\ \delta_e \end{bmatrix} = \begin{bmatrix} M_q & M_\alpha & M_\delta \\ 1 & Z_\alpha & Z_\delta \\ 0 & 0 & -a \end{bmatrix} \begin{bmatrix} q \\ \alpha \\ \delta_e \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \delta_c$$

(3.2-43)

with measurements

$$\begin{bmatrix} q \\ n_e \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ dM_q & dM_\alpha - Z_\alpha V & dM_\delta - Z_\delta V \end{bmatrix} \begin{bmatrix} q \\ \alpha \\ \delta_e \end{bmatrix}$$

(3.2-44)

Table 3.2-1 provides the definitions for the equation coefficients. These coefficients are functions of a time-varying parameter, the elevator surface effectiveness $M_{\delta_0}$, which (for the purposes of this example) ranges from $M_{\delta_0} \in [-25, -5]$. The elevator effectiveness is related to the dynamic pressure by

$$\bar{q} = -23 M_{\delta_0}$$

(3.2-45)

The parameter-varying compensator is
\[
\frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -2.38 & 0 & 0 \\ 0 & -2.38 & 0 \\ -324G_{c^*} & G_{c^*} & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 2.38 & 0 \\ 0 & 2.38 \\ 648G_{c^*} & 0 \end{bmatrix} \begin{bmatrix} q \\ n_z \end{bmatrix}
\]

(3.2-46)

\[
\delta_c = \begin{bmatrix} -324G_{c^*} & G_{c^*} & 2.3 \\ \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 648G_{c^*} & 0 \end{bmatrix} \begin{bmatrix} q \\ n_z \end{bmatrix}
\]

(3.2-47)

where the gain \( G_{c^*} \) varies as

\[
G_{c^*} = -\frac{0.0155}{M_{80}}
\]

(3.2-48)

---

**Table 2.2-1 F-8 Coefficients**

<table>
<thead>
<tr>
<th>( q )</th>
<th>pitch rate (rad/sec)</th>
<th>( M_q, M_{\alpha}, M_{\delta}, )</th>
<th>( Z_{\alpha}V, Z_{\delta}V, V )</th>
<th>dynamic coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>total angle of attack (rad)</td>
<td>( M_q = -0.23 + 0.0028M_{80} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( d_e )</td>
<td>elevator position (rad)</td>
<td>( M_{\alpha} = 0.61M_{80} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( d_c )</td>
<td>elevator command (rad)</td>
<td>( V = 200(-M_{80})^{1/2} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( n_z )</td>
<td>normal acceleration (ft/sec²)</td>
<td>( Z_{\alpha}V = 53M_{80} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( a )</td>
<td>actuator bandwidth</td>
<td>( Z_{\delta}V = 7.7M_{\delta} )</td>
<td>(1 + 0.016M_{80} + 0.0002M_{80}²)M_{80}</td>
<td></td>
</tr>
<tr>
<td>( d )</td>
<td>accelerometer displacement from CG (15 ft)</td>
<td>( M_{\delta} = )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( q )</td>
<td>dynamic pressure (psf)</td>
<td>( q = -23M_{80} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( M_{80} )</td>
<td>surface effectiveness</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Combining (3.2-43)-(3.2-48), one obtains the closed loop dynamics which are denoted as
\[
\dot{x}(t) = A_c(M_{\delta_0}(t)) x(t)
\]  
(3.2-49)

Thus, it is seen that the closed loop dynamics are a function of the time-varying elevator effectiveness, \(M_{\delta_0}\), or equivalently of the dynamic pressure, \(\overline{q}\). Using simulations of severe aircraft maneuvers [3], one obtains the following estimate of the maximum time-derivative of the dynamics pressure:

\[
| \dot{\overline{q}} | \leq 9.2 \text{ lb/ft}^2/\text{sec}
\]  
(3.2-50)

Using (3.2-45), it follows that

\[
| M_{\delta_0} | \leq 0.4 \text{ units/sec}
\]  
(3.2-51)

**Stability via Theorem 3.2-1** In order to apply Theorem 3.2-1, one first must obtain the exponential stability constants \(m\) and \(\lambda\) such that

\[
| x(t) | \leq me^{-\lambda t} | x_0 |
\]  
(3.2-52)

Using the "matrix exponential method", one has that

\[
m = 3.15 \times 10^4 \text{ and } \lambda = 2.08
\]  
(3.2-53)

Substituting (3.2-52) into the stability condition (3.2-15) states that exponential stability is guaranteed if

\[
| M_{\delta_0} | \leq 3.14 \times 10^{-8} \text{ units/sec}
\]  
(3.2-54)

which is far from the actual rate of variations in (3.2-51). Now consider the change of variables

\[
w = T x
\]  
(3.2-55)

\[
\dot{w} = T A_c(M_{\delta_0}(t)) T^{-1} w
\]  
(3.2-56)

\[
T = \text{diag}\left(\frac{\pi}{180}, \frac{\pi}{180}, \frac{\pi}{180}, \frac{\pi}{180}, \frac{1}{32.2}, \frac{\pi}{180}\right)
\]  
(3.2-57)
which simply changes angle units to degrees and acceleration units to \( g \)'s. Applying the "matrix exponential method" to (3.2-56), one obtains

\[
| \dot{M}_{\delta_0} | \leq 2.78 \times 10^{-4} \text{ units/sec} \tag{3.2-58}
\]

Although (3.2-58) is still far from the actual time-variations of (3.2-51), the dramatic change from (3.2-54) to (3.2-58) points out the extreme sensitivity of Theorem 3.2-1 to the scaling of the state variables. Thus when using Theorem 3.2-1, one should try to use an "optimal" scaling of the state variables. Such an optimization was attempted using a pattern search [32] on the scaling \( T \). However, the bounds obtained on \( \dot{M}_{\delta_0} \) were still two orders-of-magnitude too slow.

**Stability via Theorem 3.2-2** Recall that Theorem 3.2-2 states that a frozen-parameter Lyapunov function can be used as a time-varying parameter Lyapunov function provided that the time-variations are sufficiently slow. Let \( \mathbf{R}(M_{\delta_0}) \) and \( \Lambda(M_{\delta_0}) \) denote the eigenvectors and eigenvalues of \( A_{\delta_0}(M_{\delta_0}) \), respectively. Now define the time-varying Lyapunov function candidate

\[
V(x, t) = x^H K(t) x \tag{3.2-59}
\]

where

\[
K(M_{\delta_0}(t)) \equiv \mathbf{R}^H(M_{\delta_0}(t)) \mathbf{R}(M_{\delta_0}(t)) \tag{3.2-60}
\]

Then

\[
Q(M_{\delta_0}(t)) = \mathbf{R}^H(M_{\delta_0}(t)) (\Lambda(M_{\delta_0}(t)) + \Lambda^H(M_{\delta_0}(t))) \mathbf{R}(M_{\delta_0}(t)) \tag{3.2-61}
\]

Substituting (3.2-60) and (3.2-61) into the negative definiteness condition (3.2-34), one obtains that

\[
| \dot{M}_{\delta_0} | \leq 0.4 \text{ units/sec} \tag{3.2-62}
\]

implies

\[
\lambda_{\min}(Q(M_{\delta_0}(t)) - \dot{K}(M_{\delta_0}(t))) \geq 2.7 \times 10^{-8} \tag{3.2-63}
\]

(as compared to a frozen-parameter value \( \lambda_{\min}(Q(-25)) = 2.9 \times 10^{-8} \)) which in turn
guarantees exponential stability.

It is noted that upon examining the definition of the frozen-parameter Lyapunov function (3.2-60)-(3.2-61), one sees that the slowness condition (3.2-62) states that the time-variations of the eigenvectors of the closed loop dynamics should be sufficiently slow. In this sense, the instability of (3.2-1) in the presence of constant eigenvalues comes as no surprise.

Although Theorem 3.2-1 ultimately failed in showing stability of the F-8, it reveals an important point which traditionally has been ignored in gain-scheduled designs. Namely, the selection of state variables in the realization of the compensator is crucial. More precisely, suppose that one has designed a linear time-invariant compensator $K(s, \theta)$ for each value of the parameter. The notation $K(s, \theta)$ is used to stress that such designs are typically done with an emphasis on the input/output frequency domain aspects of the compensator. Now let two realizations of the compensator be given by

$$\dot{x}_k = A(\theta) x_k + B(\theta) z$$

(3.2-64)

$$u = C(\theta) x_k$$

(3.2-65)

and

$$\dot{w}_k = T(\theta) A(\theta) T^{-1}(\theta) w_k + T(\theta) B(\theta) z$$

(3.2-64)

$$u = C(\theta) T^{-1}(\theta) w_k$$

(3.2-65)

Although each realization results in the same frozen-parameter compensator $K(s, \theta)$, they result in two different parameter-varying compensators. To see this, note that the term $T(\theta) T^{-1}(\theta) w_k$ is missing in the RHS of (3.2-64) which would then make the two realizations time-varying equivalent. Given the possible sensitivity of time-varying stability to the scaling of state variables, it may be that an alternate realization of the frozen-parameter compensators could mean the difference between parameter-varying stability or instability. Hence, the selection of the realization of the frozen-parameter compensators is crucial.
3.3 Robust Stability and Robust Performance

3.3.1 Problem Statement

In the previous section, it was shown that the nominal stability of frozen-parameter gain scheduled designs can be lost in the presence of parameter time-variations. However, if the parameter time-variations are sufficiently slow then nominal stability is maintained.

In this section, the issues of guaranteed robustness and performance in the presence of parameter time-variations are addressed. As in the case of nominal stability, given a frozen-parameter design with desirable robustness and performance properties, these properties may be lost since the parameters are actually time-varying.

To set up the problem, consider the block diagram of Fig. 3.3-1. This represents a standard unity feedback configuration where $P(\theta)$ is a finite-dimensional linear parameter-varying model of the plant, and $K(\theta)$ is a finite-dimensional linear parameter-varying gain scheduled compensator. It is assumed that uncertainties enter at three different points in the feedback loop:

(1) $\Delta_u \equiv$ unmodelled actuator dynamics

(2) $\Delta_y \equiv$ unmodelled sensor dynamics

(3) $\Delta_p \equiv$ an artificial uncertainty which represents a performance specification
These uncertainties are modelled as unknown but bounded stable time-invariant linear systems (possibly infinite-dimensional) which are independent of the exogenous parameters (see [21] for a detailed discussion on how various performance specifications may be represented as uncertainties in the feedback loop). By performing appropriate block diagram manipulations, Fig. 3.3-1 may be transformed into Fig. 3.3-2. In this figure, \( H(\theta) \) represents a stable parameter-varying linear system which depends on only the plant model \( P(\theta) \) and compensator \( H(\theta) \); \( \Delta \) represents a block-diagonal linear system which depends on only the uncertainties \( \Delta_u, \Delta_y, \) and \( \Delta_p \). More generally, any linear system with linear uncertainties can be transformed to the form of Fig. 3.3-3 where \( H(\theta) \) represents a stable finite-dimensional parameter-varying linear system and \( \Delta \) represents a block diagonal linear system which depends on only the uncertainties.

Given the block diagram of Fig. 3.3-3, one can use various analysis tools (e.g. small-gain theorem [13, 20] or \( \mu \)-value [19, 49]) to guarantee robust stability and robust performance for any frozen parameter values. However, this analysis is insufficient since the parameters are actually time-varying. Since the uncertainties are possibly infinite-dimensional (e.g. time-delays, flexible structures, etc.), one cannot use the state-space tools of Section 3.2 - hence the need for new stability tests.
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Figure 3.3-2 Unity Feedback Configuration after Loop Transformations

Figure 3.3-3 General Block Diagram for Robustness / Performance Analysis

Let $H(\theta)$ have the following state-space realization

$$
\dot{x}(t) = A(\theta(t)) \ x(t) + B(\theta(t)) \ e(t)
$$

(3.3-1)

$$
y(t) = C(\theta(t)) \ x(t)
$$

(3.3-2)

Furthermore, let the I/O relationship of $\Delta$ be given by
\[ y'(t) = \int_0^t \Delta(t - \tau) y(\tau) \, d\tau \]  

(3.3-3)

Then, the feedback equations are

\[ \dot{x}(t) = A(\theta(t)) \, x(t) + \int_0^t B(\theta(t)) \, \Delta(t - \tau) \, C(\theta(\tau)) \, x(\tau) \, d\tau \]  

(3.3-4)

This equation represents a type of linear Volterra integrodifferential equation (VIDE). As in the case of nominal stability, it will be shown that the stability of (3.3-4), hence the robustness and performance properties, are maintained in the presence of parameter time-variations provided that the variations are sufficiently slow.

### 3.3.2 Volterra Integrodifferential Equations

Before time-varying robustness and performance are discussed, some facts are presented regarding equations of the form in (3.3-4). Evaluating (3.3-4) along any parameter vector trajectory, one has that

\[ \dot{x}(t) = A(t) \, x(t) + \int_0^t B(t) \, \Delta(t - \tau) \, C(t) \, x(\tau) \, d\tau \]  

(3.3-5)

where A, B, and C have been appropriately redefined. This is the general form of time-varying VIDE's and will be the object of all of the forthcoming analysis. Note that any conditions imposed on (3.3-5) can be translated immediately into conditions on the parameter-varying (3.3-4).

It was stated that equation (3.3-5) falls under the class of linear VIDE's. In fact, under assumptions to be stated on \( \Delta \), (3.3-5) actually represents a combination of VIDE's and linear delay-differential equations. Thus, both types of equations are treated under the same
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framework. VIDE's and their stability have been studied in, for example, [11, 25, 26, 27, 42, 43, 44], and delay-differential equations in [16, 22, 30].

In this section, assumptions on (3.3-5) are given, a definition of exponential stability is introduced, and a sufficient condition for exponential stability in the case of time-invariant A, B, and C matrices is presented. Finally, a perturbational result analogous to Lemma 3.2-1 is presented.

Consider the VIDE

\[ \dot{x}(t) = A(t)x(t) + \int_0^t B(t) \Delta(t - \tau) C(\tau)x(\tau) d\tau, \quad t > t_0 \]  

(3.3-6)

with initial condition

\[
\begin{align*}
    x(t) &= \phi(t), & 0 &\leq t \leq t_0, \phi \in B_e \\
    x(t_0^+) &= \phi(t_0)
\end{align*}
\]

(3.3-7)

Note that an initial condition for (3.3-6) consists of both an initial time, \( t_0 \), and an initial function \( \phi \). Typically, the only case of interest is \( t_0 = 0 \). However, the concept of an initial function is quite useful in analyzing the stability of (3.3-6).

The following assumptions are made on (3.3-6):

Assumption 3.3-1 The matrices \( A : \mathbb{R}^+ \rightarrow \mathbb{R}^{nxn} \), \( B : \mathbb{R}^+ \rightarrow \mathbb{R}^{nxm} \), and \( C : \mathbb{R}^+ \rightarrow \mathbb{R}^{pxn} \) are bounded and globally Lipschitz continuous with constants \( L_A, L_B, \) and \( L_C \), respectively.

Assumption 3.3-2 For some \( \sigma \geq 0 \), \( \Delta \in \mathcal{A}^{mxp}(-\sigma) \).
Assumption 3.3-2 states the the uncertainties $\Delta$ are finite-gain stable as operators from $L_{\infty}$ to $L_{\infty}$. In the case of a rational $\hat{\Delta}$, this is equivalent to requiring that $\hat{\Delta}$ is bounded and analytic in the complex RHP, $\text{Re}[s] > -\sigma$, e.g. $\hat{\Delta} \in H^\infty$ in case $\sigma = 0$. In the case of infinite-dimensional $\Delta$'s, Assumption 3.3-2 is slightly stronger (see [9] for an example where $\hat{\Delta} \in H^\infty$ and does not satisfy Assumption 3.3-2).

VIDE's containing an integral operator as in Assumption 3.3-2 have been studied in [14, 15, 41], and references contained in [16]. Reference [14] establishes existence and uniqueness of solutions in the case of time-invariant $A$, $B$, and $C$ matrices. Existence in the case of time-varying $A$, $B$, and $C$ matrices is similarly shown using standard contraction mapping techniques, and is omitted here. It is noted, however, that these proofs rely heavily on Assumption 3.3-2. However, this assumption is not-at-all necessary for actual existence and uniqueness, e.g. the case where $\hat{\Delta}$ is rational.

In the case of time-invariant $A$, $B$, and $C$ matrices, solutions to (3.3-6) can be explicitly characterized as follows:

**Theorem 3.3-1** [14, 15] Consider the VIDE

$$\begin{align*}
\dot{x}(t) &= A \ x(t) + \int_{0}^{t} B \ \Delta(t-\tau) \ C \ x(\tau) \ d\tau + f(t), & t > t_o \\
\end{align*}$$

$$\begin{cases}
x(t) = \phi(t), & 0 \leq t \leq t_o, \ \phi \in B_e \\
x(t_o^+) = \phi(t_o)
\end{cases}$$

(3.3-8) (3.3-9)

under Assumptions 3.3-1 and 3.3-2. Here, $f \in L_{\infty}$ is an exogenous input. In this case where
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A, B, and C are constant matrices, the unique solution to (3.3-8) is given by

\[ x(t + t_o) = R(t) x(t_o) + \int_0^t R(t - \tau) \left[ f(\tau + t_o) + F(\tau + t_o) \right] d\tau, \quad t > 0 \]  

(3.3-10)

where

\[ F(t + t_o) = \int_0^{t_o} B \Delta(t + t_o - \tau) C \phi(t) d\tau, \quad t > 0 \]  

(3.3-11)

and \( R \) is the unique matrix satisfying

\[ R(t) = I + \int_0^t \left\{ A R(\tau) + \int_0^\tau B \Delta(\tau - \beta) C R(\beta) d\beta \right\} d\tau, \quad t > 0, R(0^+) = I, \]  

(3.3-12)

The matrix \( R \) is called the 'resolvent matrix,' and is analogous to the standard matrix exponential. Note that (3.3-12) implies that \( R \) satisfies almost everywhere

\[ \dot{R}(t) = A R(t) + \int_0^t B \Delta(t - \tau) C R(\tau) d\tau, \quad t > 0 \]  

(3.3-13)

A definition of exponential stability for (3.3-6) is now introduced. Note that the integral operator in the RHS of (3.3-6) depends on all of the previous values of \( x \) and not just the current value \( x(t) \). For example, it is possible that although \( x(t_o) = 0 \), the solution \( x(t) \neq 0 \) for all time. This memory in the dynamics prevents one from using the standard definitions of exponential stability.

**Definition 3.3-1** Let \( \omega_{T,o} \) be defined as in Definition 2.2-7. The VIDE (3.3-6) is said to be *exponentially stable uniformly* if there exist constants \( m, \lambda, \) and \( \beta > 0 \) where \( \beta \geq \lambda \)
such that

\[ |x(t)| \leq m e^{-\lambda(t-t_0)} \| \mathbf{W}_{t_0,\beta} \phi \|_B, \quad t \geq t_0, \forall t_0 \in \mathbb{R}^+ , \forall \phi \in B \ \|_e \]  \hspace{1cm} (3.3-14)

It is stressed that the constants \( m, \lambda, \) and \( \beta \) are independent of \( t_0 \) and \( \phi \).

This definition implies that not only does the state decay exponentially but also with a magnitude which is proportional to an \textit{exponentially weighted supremum} of the initial function \( \phi \) (Fig. 3.3-4). This weighting essentially introduces the notion of a forgetting factor in the dynamics of (3.3-6). That is, the value of the initial function at \( t \ll t_0 \) does little to effect the state dynamics. The convention \( \beta \geq \lambda \) follows from the reasoning that solutions to (3.3-6) cannot decay faster than they are forgotten. This convention is in agreement with the case of no integral operators in the RHS (i.e. ordinary differential equations).

![Figure 3.3-4 Visualization of \( \| \mathbf{W}_{T,\beta} \phi \|_B \)](image)

The following theorem gives a sufficient condition for exponential stability in the case where \( A, B, \) and \( C \) are constant as in (3.3-8).
Theorem 3.3-2 Consider the VIDE

\[
\dot{x}(t) = A \ x(t) + \int_0^t B \ \Delta(t - \tau) \ C \ x(\tau) \ d\tau, \quad t > t_o
\]  \hfill (3.3-15)

\[
\begin{align*}
\begin{cases}
    x(t) = \phi(t), & 0 \leq t \leq t_o, \ \phi \in B_e \\
    x(t_o^+) = \phi(t_o)
\end{cases}
\end{align*}
\]  \hfill (3.3-16)

A sufficient condition for uniform exponential stability is that there exist a constant \( \beta > 0 \) such that

\[
s \mapsto (sI - A - B \ \hat{\Delta}(s) \ C)^{-1} \in \mathcal{A}^{nxn}(-2\beta)
\]  \hfill (3.3-17)

\[
\hat{\Delta} \in \mathcal{A}^{mxp}(-2\beta)
\]  \hfill (3.3-18)

Proof It is first shown that the resolvent matrix \( R \) is bounded by a decaying exponential. Taking the Laplace transform of (3.3-13) shows that

\[
\hat{R}(s) = (sI - A - B \ \hat{\Delta}(s) \ C)^{-1}
\]  \hfill (3.3-19)

It follows by hypothesis (3.3-17) that \( R \in \mathcal{A}^{nxn}(-2\beta) \). Since \( R \) contains no impulses, \( R \in L_1 \), and hence \( \hat{R} \in L_1 \) from (3.3-13). These two imply that \( R \in L_\infty \). Now, write \( R \) as

\[
R(t) = R'(t) \ e^{-\beta t}
\]  \hfill (3.3-20)

Clearly, \( R' \in \mathcal{A}^{nxn}(-\beta) \). Using the same arguments as above along with

\[
\hat{R}'(s) = (A + \beta I) \ R'(s) + \int_0^t B \ \Delta(t - \tau) \ C \ e^{\beta(t - \tau)} \ R'(\tau) \ d\tau, \quad t > t_o, \ R'(0^+) = I
\]  \hfill (3.3-21)

it follows that \( R \) and \( \hat{R} \in L_1 \), hence \( R' \in L_\infty \). Thus from (3.3-20), it follows that there exists
a constant $k_1$, for example $\| R^* \|_{\mathcal{L}_\infty}$, such that
\[
\| R(t) \| \leq k_1 e^{-\beta t}, \quad \forall \ t \geq 0 \tag{3.3-22}
\]

Now, recall that the solution to (3.3-15) is given by
\[
x(t + t_o) = R(t) x(t_o) + \int_0^t R(t - \tau) F(t + t_o) d\tau, \quad t > 0 \tag{3.3-23}
\]

where
\[
F(t + t_o) = \int_0^{t_o} \Delta(t + t_o - \tau) C e^{-\beta(t + t_o - \tau)} \phi(\tau) d\tau, \quad t > 0 \tag{3.3-24}
\]

It is now shown that $F$ is also bounded by a decaying exponential. Rewriting (3.3-24),
\[
F(t + t_o) = \int_0^{t_o} \Delta(t + t_o - \tau) e^{\beta(t + t_o - \tau)} C e^{-\beta(t + t_o - \tau)} \phi(\tau) d\tau \tag{3.3-25}
\]
\[
= e^{-\beta t} \int_0^{t_o} \Delta(t + t_o - \tau) e^{\beta(t + t_o - \tau)} C e^{-\beta(t + t_o - \tau)} \phi(\tau) d\tau \tag{3.3-26}
\]

Since $\Delta \in \mathcal{A}^{\text{max}}(-2\beta)$, it follows from (3.3-26) that there exists a constant $k_2 > 0$, for example $k_2 = \| B \| \| \Delta \|_{\mathcal{A}(\beta)} \| C \|$, such that
\[
\| F(t + t_o) \| \leq k_2 e^{-\beta t} \| L_{t_o, \beta} \phi \|_B \tag{3.3-27}
\]

Substituting (3.3-22) and (3.3-27) into (3.3-23),
\[
\| x(t + t_o) \| \leq k_1 e^{-\beta t} \| x(t_o) \| + \int_0^t k_1 e^{-\beta(t - \tau)} k_2 e^{-\beta \tau} \| L_{t_o, \beta} \phi \|_B d\tau \tag{3.3-28}
\]
\[
\leq k_1 e^{-\beta t} \| \Phi_{t_0, \beta} \phi \|_B \{ 1 + k_2 t \}
\]
\[
\leq k_1 \left\{ 1 + \frac{2k_2}{e} \right\} e^{-(\beta/2)t} \| \Phi_{t_0, \beta} \phi \|_B \quad (3.3-29)
\]

Since (3.3-29) is true for arbitrary \( \phi \) and \( t_0 \), it follows that (3.3-15) is uniformly exponentially stable.

Regarding the hypotheses of Theorem 3.3-2, it seems that condition (3.3-17) is also necessary. This is because exponential stability implies that \( \mathbf{R} \) itself decays exponentially. Thus, \( \mathbf{R} \in \mathcal{A}^{n \times n}(-2\beta) \) for some \( \beta > 0 \). Condition (3.3-17) immediately follows since it deals with the Laplace transform of \( \mathbf{R} \). In the case of no integral operator, i.e. \( \Delta = 0 \), condition (3.3-17) reduces to the standard statement that the eigenvalues of \( \mathbf{A} \) lie in the open complex LHP. Otherwise, it is worth noting that '\( \mathbf{A} \) is a stable matrix' is not assumed in proving the exponential stability of (3.3-15).

As for condition (3.3-18), note that it is slightly stronger than the standing Assumption 3.3-2. Specifically, Assumption 3.3-2 guarantees only that \( \Delta \in \mathcal{A}^{m \times p}(0) \), and it was stated that this is used in the existence and uniqueness proofs in [14]. As is the case for the existence and uniqueness proofs, condition (3.3-18) is not necessary for exponential stability. Once again, one can easily find a counterexample in the case where \( \Delta \) is rational.

Theorem 3.3-2 is novel in that it takes a state-space approach, rather than input-output, to the robustness of time-invariant linear systems. This approach was chosen since it corresponds to the original motivation of parameter-varying gain scheduled systems, as in (3.3-4). Nevertheless, the standard results on robustness can be obtained from the previous theorem. Rewriting \( \hat{\mathbf{R}}(s) \) in (3.3-19),
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\[ \hat{R}(s) = (I - (sI - A)^{-1} B \hat{\Delta}(s) C)^{-1} (sI - A)^{-1} \]  

(3.3-30)

Now suppose that \( A \) is a stable matrix; thus \( s \mapsto (sI - A)^{-1} \in \hat{\mathcal{A}}^{nxn}(-2\beta) \) for some \( \beta > 0 \). Assume further that \( \hat{\Delta} \in \mathcal{A}^m \). Thus,

\[ s \mapsto (I - (sI - A)^{-1} B \hat{\Delta}(s) C) \in \mathcal{A}^{nxn}(-2\beta) \]  

(3.3-31)

Under these conditions, \( \hat{R} \in \hat{\mathcal{A}}^{nxn}(-2\beta) \) if and only if [18, 31]

\[ \inf_{\Re(s) \geq -2\beta} \det(I - (sI - A)^{-1} B \hat{\Delta}(s) C) = \det(I - C(sI - A)^{-1} B \hat{\Delta}(s)) > 0 \]  

(3.3-32)

However, a sufficient condition for (3.3-32) is that

\[ |C((-2\beta + j\omega)I - A)^{-1} B \hat{\Delta}(-2\beta + j\omega)| \leq \gamma < 1, \quad \forall \omega \in \mathbb{R} \]  

(3.3-33)

As \( \beta \to 0 \), condition (3.3-30) approaches the standard small-gain robustness condition for time-invariant linear systems. However, unlike previous results, Theorem 3.3-2 gives some quantitative indication of the degree of robust stability.

Recall that the original motivation for studying VIDE's is that these equations that arise in the analysis of robustness and performance. Therefore, it is important that one is able to verify the conditions (3.3-17)-(3.3-18) for exponential stability.

First, consider condition (3.3-18). This assumption is stronger than requiring \( \Delta \) to be a stable linear time-invariant system. More precisely, (3.3-18) implies that the impulse response of \( \Delta \) decays exponentially with a rate of at least \(-2\beta\). This may be given the following physical interpretation. Let the mapping \( \Delta' : L_{pe} \to L_{pe} \) be defined as

\[ (\Delta' u)(t) = \int_{0}^{t} \Delta(t - \tau)e^{2\beta(t - \tau)} u(\tau) d\tau \]  

(3.3-34)

Then (3.3-18) is equivalent to the mapping \( \Delta' \) being finite-gain stable for any \( p \in [1, \infty] \). In
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terms of the original , rewriting (3.3-34) gives

\[
(\Delta' u)(t) = e^{2\beta t} \int_0^t \Delta(t - \tau)e^{-2\beta\tau}u(\tau)\,d\tau
\]  
(3.3-35)

Thus, it is seen that the operator \( \Delta' \) may be decomposed as follows:

1. Multiply the input \( u \) be a decaying exponential:

\[
u'(t) = e^{-2\beta t}u(t)
\]  
(3.3-36)

2. Pass the modified input \( u' \) through \( \Delta \)

3. Multiply \( (\Delta u')(t) \) by a growing exponential:

\[
(\Delta' u)(t) = e^{2\beta t}(\Delta u')(t)
\]  
(3.3-37)

Then condition (3.3-18) is equivalent to the sequence of operations described in (3.3-35)-(3.3-37) mapping \( u \mapsto \Delta' u \) being finite-gain stable for any \( p \in [1, \infty] \). In case \( \hat{\Delta} \) is rational, this amounts to \( \hat{\Delta} \) not having any poles in the complex RHP, \( \text{Re}[s] \geq -2\beta \). Otherwise, \( \hat{\Delta} \in \mathcal{A}_{\text{msp}}(-2\beta) \) must either be assumed or somehow verified experimentally. This is discussed further in Section 3.3.3.

Now consider the condition (3.3-17). Recall that this stated the Laplace transform of the resolvent matrix \( \overline{R} \) should satisfy

\[
\overline{R} = s \mapsto (sI - A - B \hat{\Delta}(s)C)^{-1} \in \mathcal{A}^{n\times n}(-2\beta)
\]  
(3.3-38)

In order to verify (3.3-38), first let the matrix \( A \) have no eigenvalues in the complex RHP, \( \text{Re}[s] \geq -2\beta \), for some \( \beta > 0 \). This is in agreement with the gain scheduled case since the linear system \( H \) in (3.3-1)-(3.3-2) is designed to be stable for every frozen value of the parameter. Second, let \( \Delta \) satisfy the condition (3.3-18), namely \( \Delta \in \mathcal{A}_{\text{msp}}(-2\beta) \). Then using the same reasoning as in (3.3-30)-(3.3-32), \( \overline{R} \) satisfies (3.3-38) if and only if

\[
\inf_{\text{Re}[s] \geq -2\beta} \det(I - (sI - A)^{-1}B\hat{\Delta}(s)C) = \det(I - C(sI - A)^{-1}B\hat{\Delta}(s)) > 0
\]  
(3.3-39)
which, under these conditions, is equivalent to

$$\inf_{\omega \in \mathbb{R}} \det \left( \mathbf{I} - \mathbf{C} [(-2\beta + j\omega)\mathbf{I} - \mathbf{A}]^{-1} \mathbf{B} \hat{\Delta}(-2\beta + j\omega) \right) > 0 \quad (3.3-40)$$

Now, condition (3.3-17) is in the form where it can be verified using standard tools from linear time-invariant robustness analysis. For example, suppose that $\Delta$ has been normalized [21] such that

$$|\hat{\Delta}(-2\beta + j\omega)| \leq 1, \quad \forall \omega \in \mathbb{R} \quad (3.3-41)$$

Then one can use either of the following two methods to verify (3.3-40):

**Small-Gain Condition**[13, 20] Let $\Delta$ satisfy (3.3-41). Then a sufficient condition for (3.3-40) is that

$$|\mathbf{C} [(-2\beta + j\omega)\mathbf{I} - \mathbf{A}]^{-1} \mathbf{B}| \leq \gamma < 1, \quad \forall \omega \in \mathbb{R} \quad (3.3-42)$$

**$\mu$-value**[19] Recall that in the combined analysis of robustness and performance, the uncertainty $\Delta$ was block-diagonal (e.g. Fig. 3.3-2). In this case, one can use the $\mu$-value nonconservatively to determine (3.3-40). More precisely, in case $\Delta$ can be any I/O stable linear time-invariant system which satisfies (3.3-41), a necessary and sufficient condition for (3.3-40) is that

$$\mu(\mathbf{C} [(-2\beta + j\omega)\mathbf{I} - \mathbf{A}]^{-1} \mathbf{B}) \leq \gamma < 1, \quad \forall \omega \in \mathbb{R} \quad (3.3-43)$$

To summarize, satisfying the conditions for exponential stability (3.3-17)-(3.3-18) would
require the following knowledge about $\Delta$. First, knowledge of the degree of I/O stability

$$\Delta \in \tilde{\mathcal{A}}^{m\times p}(-2\beta)$$  \hspace{1cm} (3.3-44)

and second, a bound on $\Delta$ off the $j\omega$-axis, e.g. for some $l(\omega)$,

$$|\hat{\Delta}(-2\beta + j\omega)| \leq l(\omega), \hspace{1cm} \forall \omega \in \mathbb{R}$$  \hspace{1cm} (3.3-45)

It is stressed that neither of these two may be inferred from the more common $j\omega$-axis bound on $\Delta$

$$|\hat{\Delta}(j\omega)| \leq l(\omega), \hspace{1cm} \forall \omega \in \mathbb{R}$$  \hspace{1cm} (3.3-46)

(consider multiplying $\Delta$ any all-pass filter with a stable pole of magnitude $\leq 2\beta$). This inadequacy of (3.3-46) is discussed further in Section 3.3.3.

Before closing this section, a perturbational theorem which is analogous to Lemma 3.2-1 is presented. Informally, this theorem states that an exponentially stable time-invariant VIDE maintains its exponential stability in the presence of sufficiently small (possibly time-varying and/or nonlinear) perturbations.

**Theorem 3.3-3** Consider the VIDE

$$\dot{x}(t) = A x(t) + \int_{0}^{t} B \Delta(t - \tau) C x(\tau) d\tau + (g x)(t), \hspace{1cm} t > t_0$$  \hspace{1cm} (3.3-47)

$$\begin{cases} x(t) = \phi(t), & 0 \leq t \leq t_0, \phi \in \mathcal{B}_{e} \\ x(t_0^+) = \phi(t_0) \end{cases}$$  \hspace{1cm} (3.3-48)

Here, $g$ represents an integral operator on $x$. Let

$$s \mapsto (sI - A - B \hat{\Delta}(s) C)^{-1} \in \tilde{\mathcal{A}}^{n\times n}(-2\beta)$$  \hspace{1cm} (3.3-49)

$$\hat{\Delta} \in \tilde{\mathcal{A}}^{m\times p}(-2\beta)$$  \hspace{1cm} (3.3-50)
hold true. Assume further that there exist constants \( k > 0 \) and \( \alpha \geq \beta \), where \( \beta \) is from (3.3-49)-(3.3-50), such that

\[
| (g)(t) | \leq k \| \phi_{t,\alpha} x \|_{B}, \quad t \geq 0, \forall x \in B_e \tag{3.3-51}
\]

Let \( k_1 \) be as in (3.3-22). Under these conditions,

\[
k < \frac{\beta}{k_1} \tag{3.3-52}
\]

implies that (3.3-47) is uniformly exponentially stable.

**Proof** Define \( z(t) \equiv x(t + t_0) \). As in (3.3-10)

\[
z(t) = R(t) z(0) + \int_{0}^{t} R(t - \tau) \left[ F(t + t_0) + (g)(\tau) \right] d\tau, \quad t > 0 \tag{3.3-53}
\]

where

\[
F(t + t_0) = \int_{0}^{t_0} B \Delta(t_0 - \tau) C \phi(\tau) d\tau, \quad t > 0 \tag{3.3-54}
\]

As before, there exist \( k_1 \) and \( k_2 > 0 \) such that

\[
| R(t) | \leq k_1 e^{\beta t} \tag{3.3-55}
\]

\[
| F(t + t_0) | \leq k_2 e^{-\beta t} \| \phi \|_{B} \tag{3.3-56}
\]

Substituting (3.3-55), (3.3-56), and (3.3-51) into (3.3-53),

\[
| z(t) | \leq \int_{0}^{t} k_1 e^{\beta(t - \tau)} \left[ k_2 e^{-\beta \tau} \| \phi \|_{B} + k \| \phi \|_{B} \right] d\tau + k_1 e^{\beta t} | z(0) | \tag{3.3-57}
\]

Since \( \alpha \geq \beta \),
\[ |z(t)| \leq \int_0^t k_1 e^{-\beta(t-\tau)} \left( k_2 e^{-\beta\tau} \| \mathbf{w}_{\tau, t_0, \beta} \phi \|_{\mathbf{B}} + k \| \mathbf{w}_{\tau+t_0, t_0, \beta} x \|_{\mathbf{B}} \right) d\tau + k_3 e^{-\beta t} |z(0)| \quad (3.3-58) \]

Using Definition 2.2-7 of the truncation and exponential weighting operator,

\[ \| \mathbf{w}_{\tau+t_0, \beta} x \|_{\mathbf{B}} = \sup_{\xi \in [0, \tau+t_0]} e^{-\beta(t_0+\xi)} x(\xi) \quad (3.3-59) \]

\[ \leq e^{-\beta t} \left\{ \sup_{\xi \in [0, t]} e^{-\beta(t-t_0)} \phi(\xi) + \sup_{\xi \in [t, t+t_0]} e^{-\beta(t-t_0)} x(\xi) \right\} \]

Thus

\[ e^{\beta t} |z(t)| \leq k_1 \left( 1 + (k+k_2) t \right) \| \mathbf{w}_{t_0, \beta} \phi \|_{\mathbf{B}} + \int_0^t k_1 k \sup_{\xi \in [0, t]} e^{\beta \xi} z(\xi) d\tau \quad (3.3-60) \]

Since the RHS of (3.3-60) is a nondecreasing function of time,

\[ \sup_{\xi \in [0, t]} e^{\beta \xi} z(\xi) \leq k_1 \left( 1 + (k+k_2) t \right) \| \mathbf{w}_{t_0, \beta} \phi \|_{\mathbf{B}} + \int_0^t k_1 k \sup_{\xi \in [0, t]} e^{\beta \xi} z(\xi) d\tau \quad (3.3-61) \]

Rewriting (3.3-61),

\[ f(t) \leq k_1 + k_2 t + k_3 \int_0^t f(\tau) d\tau \quad (3.3-62) \]

where, \( k_1, k_2, k_3 \), and \( f \) are defined in the obvious manner. Applying the Bellman-Gronwall inequality to (3.3-62),

\[ f(t) \leq \left\{ k_1 + \frac{k_2}{k_3} \right\} e^{k_3 t} \frac{k_2}{k_3} \quad (3.3-63) \]

Thus,
\[ |z(t)| \leq \left\{ \left[ k_1 + 1 + \frac{k_2}{k} \right] e^{-\beta k_2 t} \right\} \left[ 1 + \frac{k_2}{k} \right] e^{-\beta k} \| \mathbf{w} \|_{t_0, t} \phi \|_B \]  

Uniform exponential stability then follows from (3.3-52).

Note that (3.3-64) implies that for some \( m_k \) (which depends on \( k, k_1, \) and \( k_2 \)),

\[ |x(t + t_o)| \leq m_k e^{-\beta k_2 t_o/2} \| \mathbf{w} \|_{t_0, t} \phi \|_B \]  

However, \( m_k \) should be defined carefully so that as \( k \to 0 \) (i.e. as \( g \to 0 \)), one has that

\[ m_k \to k_1 \left[ 1 + \frac{2k_2}{e} \right] \]  

which is the case in (3.3-29) where \( g = 0 \).

### 3.3.3 Robustness and Performance of Slowly-Varying Linear Systems

In this section, the results of Section 3.2 are extended to VIDE's. Namely, it is shown if the VIDE (3.3-7) is exponentially stable for all frozen values of time, then the time-varying VIDE is exponentially stable for sufficiently slow time-variations. In terms of the original motivation of guaranteed properties for gain-scheduled control systems, this means that robustness to unmodelled dynamics and robust performance in the sense of Figs. 3.3-2 and 3.3-3 is maintained provided that the parameter variations are sufficiently slow.

Consider the VIDE

\[ \dot{x}(t) = A(t) x(t) + \int_0^t B(t) \Delta(t - \tau) C(\tau) x(\tau) d\tau, \quad t > t_o \]  

(3.3-67)
with initial condition

\[
\begin{aligned}
\begin{cases}
    x(t) &= \phi(t), \\
    x(t_o^+) &= \phi(t_o)
\end{cases}
\end{aligned}
\quad 0 \leq t \leq t_o, \phi \in B_e
\] (3.3-68)

Before proceeding with the theorem, some assumptions and definitions are given.

**Assumption 3.3-3** There exists a constant \( \beta > 0 \) such that

\[
s \mapsto (sI - A(\tau) - B(\tau) \hat{\Delta}(s) C(\tau))^{-1} \in \mathcal{A}^{\text{max}} (-2\beta), \quad \forall \tau \in \mathcal{R}^+
\] (3.3-69)

\[
\hat{\Delta} \in \mathcal{A}^{\text{max}} (-2\beta)
\] (3.3-70)

This assumption guarantees exponential stability for all frozen values of time.

**Definition 3.3-2** From Assumption 3.3-1, let \( k_B \) and \( k_C \) satisfy

\[
|B(t)| \leq k_B, \quad \forall t \in \mathcal{R}^+
\] (3.3-71)

\[
|C(t)| \leq k_C, \quad \forall t \in \mathcal{R}^+
\] (3.3-72)

Then define

\[
K \equiv L_A + L_B \parallel_{\mathcal{A}(\cdot \beta)}^\Delta k_C + k_B \parallel_{\mathcal{A}(\cdot \beta)}^\Delta L_C
\] (3.3-73)

Essentially, \( K \) is measure of the rate of time-variations in (3.3-67), i.e. as \( K \to 0 \), (3.3-67) approaches a time-invariant VIDE.

**Definition 3.3-3** Let \( R_{e}^t \) denote the resolvent matrix of (3.3-21) associated with the frozen matrices \( A(\tau), B(\tau) \) and \( C(\tau) \). Then define
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\[ K_1 \equiv \sup_{\tau \in \mathcal{R}} \| R^\tau \|_{L_\infty} \]  \hspace{1cm} (3.3-74)

\[ K_2 \equiv k_B \| \Delta \|_{\mathcal{A}(\beta)} k_C \]  \hspace{1cm} (3.3-75)

These constants represent worst case values of their \( \tau \)-frozen analogs in (3.3-22) and (3.3-27).

The question of slowly time-varying stability of linear VIDE's is now addressed. The proof closely follows that of Theorem 3.2-1.

**Theorem 3.3-4** Consider the VIDE (3.3-67) under Assumptions 3.3-1 and 3.3-3 and Definition 3.3-2 and 3.3-3. Under these conditions, given any \( \eta \in (0, \beta) \), (3.3-67) is uniformly exponentially stable with a decay rate of \( \eta/2 \) for sufficiently small \( K \), or equivalently, sufficiently slow time-variations in \( A, B, \) and \( C \).

**Proof** Let \( t_n \) denote \( t_0 + nT \), where \( T \) is some constant interval to be chosen. As in the proof of Theorem 3.2-1, the VIDE (3.3-67) first is analyzed on the intervals \( t_n \leq t \leq t_{n+1} \). Approximating \( A, B, \) and \( C \) be piecewise constant matrices, one has that

\[ \dot{x}(t) = A(t_n) x(t) + \int_{t_n}^{t} B(t_n) \Delta(t - \tau) C(t_n) x(\tau) d\tau + \int_{0}^{t_n} B(t) \Delta(t - \tau) C(\tau) x(\tau) d\tau + (g_n x)(t) \]  \hspace{1cm} (3.3-76)
where

\[
(g_n x)(t) = \left[ A(t) - A(t_n^+) \right] x(t) + \int_{t_n^+}^{t} \left[ B(t) - B(t_n^+) \right] \Delta(t - \tau) C(t_n^+) x(\tau) \, d\tau + \int_{t_n^+}^{t} B(t) \Delta(t - \tau) \left[ C(\tau) - C(t_n^+) \right] x(\tau) \, d\tau
\]  \tag{3.3-77}

Then

\[
| (g_n x)(t) | \leq KT \| W_{t_n,\beta} x \|_{B}, \quad t_n \leq t \leq t_{n+1}
\]  \tag{3.3-78}

Thus using Theorem 3.3-3, it is seen that if

\[
KT \leq \frac{\beta - \eta}{2K_1}
\]  \tag{3.3-79}

then

\[
| x(t) | \leq \left\{ \left[ K_1 + 1 + \frac{K_2}{KT} \right] e^{-(\beta + \eta)(t - t_n)/2} - \left[ 1 + \frac{K_2}{KT} \right] e^{-\beta(t - t_n)} \right\} \| W_{t_n,\beta} x \|_{B}
\]  \tag{3.3-80}

or as in (3.3-65)

\[
| x(t) | \leq m_{KT} e^{-\frac{1}{2} \frac{\beta + \eta}{2} (t - t_n)} \| W_{t_n,\beta} x \|_{B}
\]  \tag{3.3-81}

In order to guarantee (3.3-79), choose

\[
T = \frac{4 \ln m_{KT}}{\beta - \eta}
\]  \tag{3.3-82}

\[
K \leq \frac{(\beta - \eta)^2}{8 K_1 \ln m_{KT}}
\]  \tag{3.3-83}

Now, (3.3-81) implies that
\[
\| \mathbf{w}_{t_{n+1}, \beta} x \|_{\mathcal{L}_\infty} \leq m_K e^{\frac{1}{2} \frac{\beta + \eta}{2} T} \| \mathbf{w}_{t_n, \beta} x \|_B
\]  

(3.3-84)

Substituting (3.3-82) and (3.3-84) into (3.3-81),

\[
| x(t) | \leq m_K e^{-\frac{1}{2} \left( \frac{\eta + \beta - \eta}{2} \right) (t - t_0 - nT)} \left( m_K e^{-\frac{1}{2} \left( \frac{\beta + \eta}{2} \right) T} \right)^n \| \mathbf{w}_{t_0, \beta} \phi \|_B
\]

\[
\leq m_K e^{-\frac{1}{2} \left( \frac{\eta + \beta - \eta}{2} \right) (t - t_0)} \left( m_K e^{-\frac{1}{2} \left( \frac{\beta + \eta}{2} \right) T} \right)^n \| \mathbf{w}_{t_0, \beta} \phi \|_B
\]

\[
\leq m_K e^{-\frac{n}{2} (t - t_0)} \| \mathbf{w}_{t_0, \beta} \phi \|_B
\]  

(3.3-85)

which completes the proof.

\[\square\]

**Remark** It is noted that Theorem 3.3-4 can be extended in a straightforward manner to address *time-varying* unmodeled dynamics of the form \( \Delta(t, \tau) = G(t) \Delta'(t - \tau) H(\tau) \).

The idea behind Theorem 3.3-4 is essentially the same as in the case of time-varying nominal stability. That is, the time-varying VIDE (3.3-67) is approximated by piecewise constant VIDE's (3.3-76) which are exponentially stable. Thus on each interval, the time-varying VIDE is decomposed into a constant part and a time-varying perturbation. Using Theorem 3.3-3, the solution will decay provided that the constant approximation is sufficiently accurate. Furthermore, this approximation must remain valid long enough to guard against any overshoot in the next interval (Fig. 3.2-3), which is guaranteed for sufficiently slow
approximations. Once again, a key parameter is the measure of overshoot $m_{kT}$. In case $m_{kT} = 1$, then stability is guaranteed for arbitrarily fast time-variations.

As in the case of nominal stability, the conditions one needs to check in order to verify time-varying robustness/performance are (1) the frozen-time systems are stable and (2) the time-variations are sufficiently slow. In the case of nominal stability, both of these conditions may easily be checked by examining the time-varying dynamics matrix $A$. However, the presence of the integral operator $\Delta$ makes matters more complicated.

First of all, one must verify the frozen-time stability conditions of Assumption 3.3-3

$$s \mapsto (sI - A(t) - B(t) \hat{\Delta}(s) C(t))^{-1} \in \mathcal{A}^{n \times n}_{\text{max}} (-2\beta), \quad \forall \tau \in \mathbb{R}^+ \quad (3.3-86)$$

$$\hat{\Delta} \in \mathcal{A}^{m \times p}_{\text{max}} (-2\beta) \quad (3.3-87)$$

As discussed in Section 3.3.2, this requires knowledge about (1) the degree of I/O stability of $\Delta$ for (3.3-87) and (2) some bound on $\hat{\Delta}$ off of the $j\omega$-axis for (3.3-86). Furthermore, neither of these can be inferred from the more common assumption of a $j\omega$-axis bound on $\hat{\Delta}$.

In the absence of a priori knowledge of (3.3-86)-(3.3-87), these conditions somehow must be verified experimentally. In Section 3.3.2, the I/O exponential stability of $\Delta$ was given a physical interpretation in terms of exponentially modified inputs and outputs to $\Delta$ (see the discussion for (3.3-34)-(3.3-37)). This interpretation stated that the I/O exponential stability of $\Delta$ is equivalent to the finite-gain stability of the operator

$$(\Delta'u)(t) = e^{\frac{2\beta t}{3}} \int_0^t \Delta(t - \tau)e^{-2\beta \tau} u(\tau) d\tau \quad (3.3-88)$$

Informally, this interpretation may be used to verify (3.3-86)-(3.3-87) as follows:

**Frozen-time Stability Verification**

(1) Inject sufficiently rich signals $u$ which decay exponentially at a rate of $-2\beta$ into $\Delta$. 

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(2) Multiply the output from (1) by a growing exponential with rate $2\beta$.

(3) Obtain the operator norm of $\Delta'$ (if it exists). In case $\| \Delta' \|$ does not exist, then $\Delta$ does not have I/O exponential stability of degree of $2\beta$. In terms of a rational $\hat{\Delta}$, this means that $\hat{\Delta}$ has a stable pole with a real part $> -2\beta$. It is stressed that even if $\| \Delta' \|$ does not exist, all physically realized signals remain bounded since $\Delta$ is stable and the multiplication of (2) is only computational.

(4) Using methods from system identification (e.g. [40]), obtain a bound on the frequency response of $\Delta'$. This corresponds to the desired bound on the frequency response of $\Delta \text{ off}$ of the $j\omega$-axis. One can use the methods discussed in Section 3.3.2, namely the small-gain condition or $\mu$-value, to verify (3.3-86).

In addition to verifying the frozen-time exponential stability, one must guarantee that the time-variations are sufficiently slow. From (3.3-83), this means that

$$
K \leq \frac{(\beta - \eta)^2}{8 K_1 \ln m_{KT}}
$$

(3.3-89)

where

$$
K \equiv L_A + L_B \| \Delta \|_{\mathcal{A}(\beta)} k_C + k_B \| \Delta \|_{\mathcal{A}(\beta)} L_C
$$

(3.3-90)

and $m_{KT}$ depends on

$$
K_1 \equiv \sup_{\tau \in \mathcal{R}^+} \| R\tau \|_{L_{\infty}}
$$

(3.3-91)

$$
K_2 \equiv k_B \| \Delta \|_{\mathcal{A}(\beta)} k_C
$$

(3.3-92)

Thus, in addition to the bounds and Lipschitz constants of $A$, $B$, and $C$, one needs
(1) the operator norm $\| \Delta \|_{\mathcal{L}(\mathcal{H})}$

(2) the time-domain bound $\| t \mapsto e^{\beta t} R(t) \|_{L^1}$.

The operator norm in (1) can be found using the frozen-time stability verification described above (with $\beta$ replacing $2\beta$). Similarly, since this procedure guarantees exponential stability, it may be used to show that the time-domain bound (2) is finite. Unfortunately, however, it does not seem possible to use this procedure to obtain even an upper bound on (2). This imposes a major restriction on the immediate applicability of Theorem 3.3-4.

In light of these difficulties in verifying the exact conditions of Theorem 3.3-4, it seems that one is limited to making qualitative statements regarding guaranteed robustness and performance properties for parameter-varying gain scheduled systems. Nevertheless, Theorem 3.3-4 still provides useful insight into the design of these systems. Namely, it is shown that if one wants increased time-varying robustness/performance guarantees, then the frozen-time robustness tests should be met with increased margin (e.g. Fig. 3.3-5, where the frozen-time stability test is the small-gain condition). Qualitatively, a larger margin implies increased exponential stability of the frozen-time VIDE's which implies greater margin against time-variations.

![Figure 3.3-5 Increased Stability Margin with Small-Gain Condition](image-url)
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All of the aforementioned difficulties regarding verification of exponential stability can be avoided by using the more classical I/O small-gain approach. However, it will be shown that even the small-gain theorem suffers from verification problems. This time, the problem lies in the parameter-varying system $H(\theta)$ rather than in the uncertainties $\Delta$.

To see this, recall the block diagram of Fig. 3.3-3. The forward-loop operator, $H(\theta)$, denotes a finite-dimensional parameter-varying linear system which is frozen-time stable; the feedback-loop operator, $\Delta$, denotes a possibly infinite-dimensional stable linear time-invariant system which represents various robustness and performance requirements (as in Fig. 3.3-2). Let $H$ have the following state-space realization

$$\dot{x}(t) = A(\theta(t)) x(t) + B(\theta(t)) e(t) \quad (3.3-93)$$

$$y(t) = C(\theta(t)) x(t) \quad (3.3-94)$$

where (for ease of discussion) the dynamics depend on a single parameter, $\theta$. Let $\theta$ satisfy the following conditions

$$\theta_{\text{min}} \leq \theta(t) \leq \theta_{\text{max}} \quad (3.3-95)$$

$$|\dot{\theta}(t)| \leq \alpha, \quad \forall \ t \in \mathbb{R}^+ \quad (3.3-96)$$

$\theta_{\text{min}}$, $\theta_{\text{max}}$, and $\alpha$ are given constants. Finally, suppose that the uncertainty $\Delta$ has been normalized [21] so that

$$|\hat{\Delta}(j\omega)| \leq 1, \quad \forall \ \omega \in \mathbb{R} \quad (3.3-97)$$

Under these conditions, one can use the small-gain theorem to guarantee I/O stability for any frozen value of $\theta$. More precisely, stability of the above feedback system is guaranteed provided that

$$|C(\theta)(j\omega I - A(\theta))^{-1}B(\theta)| \leq \gamma < 1 \quad \forall \ \omega \in \mathbb{R} \quad (3.3-98)$$

which is easily verified as in Fig. 3.3-5.

The problem is considerably more complicated for time-varying $\theta$. Let $H(\theta)$ be the I/O operator described by
\[ y'(t) = \int_0^t C(\theta(\tau)) \Phi_\theta(t, \tau) B(\theta(\tau)) y(\tau) d\tau \quad (3.3-99) \]

where \( \Phi_\theta(t, \tau) \) denotes the transition-matrix associated with a certain parameter trajectory. Using the small-gain theorem, stability is guaranteed in the case of a time-varying \( \theta \) if

\[ \| H(\theta) \|_{L_p} \leq \gamma < 1 \quad \forall \text{admissible } t \mapsto \theta(t) \quad (3.3-100) \]

In other words, in order to prove stability using the small-gain theorem, one must guarantee that the I/O norms for all time-varying operators \( H(\theta) \) with admissible parameter trajectories satisfy (3.3-100).

To summarize, although the small-gain approach to time-varying robustness/performance does not require hard-to-obtain information about the uncertainties, it does require calculation of the I/O norm for families of linear parameter-varying systems. As discussed in Section 3.2, even the problem of determining nominal stability for families of parameter-varying systems is highly nontrivial.

In light of these difficulties, any application of the small-gain theorem is likely to be limited to the following. Suppose that the parameter-varying linear system is known to be uniformly exponentially stable for all admissible parameter trajectories (e.g. using the methods of Section 3.2). Then, there exists constants \( m \) and \( \lambda \) such that

\[ | \Phi_\theta(t, \tau) | \leq m e^{-\lambda(t-\tau)} , \quad \forall \ t, \tau \geq 0 \quad (3.3-101) \]

for all admissible parameter trajectories. Assuming that the matrices \( B \) and \( C \) are bounded as functions of \( \theta \), it follows from (3.3-100) that

\[ \| H(\theta) \|_{L_p} \leq \frac{m | C | | B |}{\lambda} \quad \forall \text{admissible } t \mapsto \theta(t) \quad (3.3-102) \]

Thus, one can use (3.3-102) in (3.3-100) to give a conservative condition to guarantee time-varying robustness/performance. This approach suffers in that it completely ignores that the
feedback system of Fig. 3.3-3 was designed to be stable for all frozen values of \( \theta \). In this sense, it offers no new insights into controller design for parameter-varying plants.

**Example 3.3-1** This example demonstrates the use of the small-gain theorem in the simple case of a scalar system with a single parameter, and points out a limitation of the state-space approach of Theorem 3.3-4. Consider the scalar VIDE

\[
\dot{x}(t) = -a(\theta(t)) x(t) + \int_{0}^{t} b(\Delta(t - \tau)) c x(t) \, d\tau
\]  \hspace{1cm} (3.3-103)

where \( a, b, \) and \( c \) form a state-space realization for a stable \( H(\theta) \) as in Fig. 3.3-6, and the parameter, \( \theta \), satisfies

\[
\theta_{\text{min}} \leq \theta(t) \leq \theta_{\text{max}}
\]  \hspace{1cm} (3.3-104)

![Diagram for Small-Gain Approach to VIDE Stability](image)

**Figure 3.3-6** Diagram for Small-Gain Approach to VIDE Stability
Furthermore, suppose that the uncertainty, Δ, has been normalized so that

\[ |\hat{\Delta}(j\omega)| \leq 1 \leq 1 \]  

(3.3-105)

Under these conditions, the small-gain theorem guarantees stability for all frozen-values of \( \theta \) provided that

\[ \left| \frac{c b}{j\omega - a(\theta)} \right| \leq \left| \frac{c b}{a(\theta)} \right| \leq \gamma < 1, \quad \forall \omega \in \mathcal{R} \]  

(3.3-106)

As in (3.3-100), the small-gain theorem guarantees stability for all time-varying trajectories provided that

\[ \| H(\theta) \|_{L_p} \leq \gamma < 1 \quad \forall \text{admissible } t \mapsto \theta(t) \]  

(3.3-107)

Using the definition of \( H(\theta) \)

\[ \gamma(t) = \int_0^t c e^{-\tau} b u(\tau) d\tau \]  

(3.3-108)

it follows that

\[ \| H(\theta) \|_{L_p} \leq \max_{\theta_{\min} \leq \theta \leq \theta_{\max}} \left| \frac{c b}{a(\theta)} \right| \leq \gamma < 1 \]  

(3.3-109)

Thus, the frozen-parameter stability condition (3.3-106) is the same as the time-varying stability criterion. In other words, the small-gain theorem guarantees that in the special case of (3.3-103), the parameter time-variations may be arbitrarily fast provided that the system is frozen-parameter stable for all parameter values (3.3-104). In terms of the previous discussion on the applicability of the small-gain theorem, the calculation of the I/O norm for a family of parameter-varying linear systems, \( H(\theta) \), is trivial in the scalar case of one state-variable.

It is worth mentioning here that the state-space approach of Theorem 3.3-4 in general will not allow arbitrary time-variations even in the scalar case of (3.3-103). This can be explained as follows. From condition (3.3-83), Theorem 3.3-4 allows arbitrary time-variations only if
the overshoot parameter $m_{kT} = 1$. In general, this is not the case for scalar VIDE's as demonstrated in the case where

$$H(s) = \frac{1}{s + 1}$$

and

$$\hat{\Delta}(s) = 10 \frac{(s + 1)^2}{(s^2 + 14s + 14)}$$

which results in a resolvent matrix

$$\hat{R}(s) = \frac{1}{s + 1} + \frac{10}{(s + 2)^2} \iff R(t) = e^{-t} + 10t \ e^{-2t}$$

(3.3-112)

3.4 Concluding Remarks

This chapter has addressed the nominal stability, robust stability, and robust performance of parameter-varying linear systems in the context of gain scheduling. The results may be summarized as follows. Essentially, it was shown that given a gain scheduled system which has excellent feedback properties for all frozen values of the parameter, these properties are maintained provided that the parameter time-variations are sufficiently slow.

In each case, sufficient conditions on the parameter time-variations were given which guarantee stability of the overall gain scheduled system. Thus, the heuristic guideline of "scheduling on a slow variable" has been transformed into quantitative statements.

In spite of the possible conservatism and difficulty of verification of these conditions, the value of the results is that they have identified the key parameters which affect the quality of the overall gain scheduled system. For example in the case of nominal stability, a key parameter is
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the overshoot $m$ of the frozen-parameter designs. In identifying this parameter, new insights were obtained on how to perform the frozen-parameter designs, such as selection of the compensator realization. In the robust stability / robust performance case, it was shown that not only must one guarantee the I/O stability of the frozen-parameter designs, but also one should have some idea of the degree of internal exponential stability. In order to guarantee such exponential stability, one should either (1) evaluate frequency-domain inequalities off of the $j\omega$-axis or (2) satisfy the standard $j\omega$-axis inequalities with an increased margin of sufficiency.

Thus, although the stability conditions might not be explicitly verified, the insights they provide go much beyond the "slow varying" guideline. Furthermore, the sufficiency of the conditions is simply a reminder that the designs were based on time-invariant approximations to the actual time-varying plant. If these approximations are inaccurate, then one should not demand guarantees on the overall gain scheduled system.
Chapter 4
Analysis of Nonlinear Gain Scheduled Control Systems

4.1 Introduction

This chapter continues the discussion of guaranteed properties for gain scheduled control systems, with the focus of attention now on nonlinear plants. A typical gain scheduled design procedure for nonlinear plants is as follows. First, the designer selects several operating points which cover the range of the plants dynamics. In contrast to the linear parameter-varying case, these operating points are usually indexed by some combination of state variables or reference state trajectories. Then, at each of these operating points, the designer constructs a linear time-invariant approximation to the plant and designs a linear compensator for each linearized plant. In between operating points, the parameters (gains) of the compensators are then interpolated, or scheduled, thus resulting in a global compensator.

Since the local designs are based on linear time-invariant approximations to the plant, the designer can guarantee that each at local operating point, the feedback system has excellent feedback properties such as robust stability, robust performance, and of course nominal
stability. Since the actual system is nonlinear, the overall gain scheduled system need not have any of these properties (even nominal stability). In other words, one typically cannot assess a priori the guaranteed stability, robustness, and performance properties of gain scheduled designs. Rather, any such properties are inferred from extensive computer simulations.

In addition to simulations, gain scheduled designs are guided by heuristic rules-of-thumb. The two most fundamental guidelines are: (1) the scheduling variable should vary slowly and (2) the scheduling variable should capture the plant's nonlinearities. As in the linear parameter-varying case, these guidelines are simply reminders that the local operating point designs were based on linear time-invariant approximations to the actual plant. Thus, these approximations must be sufficiently accurate if one expects the local feedback properties to carry over to the overall gain scheduled system.

In this chapter, two types of nonlinear gain scheduled systems are analyzed: (1) a nonlinear plant scheduling on a reference trajectory and (2) a nonlinear plant scheduling on the plant output. In each case, conditions are given which guarantee that the overall gain scheduled system will retain the feedback properties of the local designs. These conditions formalize the rules-of-thumb which have guided successful gain scheduled designs. Again, the most fundamental idea behind the analysis is that the original designs are based on linear time-invariant approximations of a nonlinear plant.

Organization of Chapter

Section 4.2 addresses the first of two nonlinear gain scheduled situations, scheduling on a reference trajectory. The formal problem statement is given in Section 4.2.1. In Section 4.2.2, conditions are given which guarantee the stability, robustness, and performance of the overall gain scheduled design. Section 4.3 presents the second situation, scheduling on the plant output. Section 4.3.1 explains the output-scheduled design process. Section 4.3.2 addresses the issue of nominal stability. In Section 4.3.3, conditions are given which guarantee that the
robust stability and robust performance properties of the local designs carry over to the overall design. Finally, concluding remarks are given in Section 4.4. Throughout this chapter, the tools developed in Chapter 3 form the backbone of the analysis.

4.2 Scheduling on a Reference Trajectory

4.2.1 Problem Statement

Consider the block diagram of Fig. 4.2-1. This figure shows a standard unity feedback configuration in which the command trajectory, $r^*$, is generated by passing a reference control signal, $u^*$, through a model of the plant, $P_m$. This reference control signal may be the outcome of a nonlinear optimal control problem, or some other off-line design process. The control input, $u$, to the actual plant, $P$, then consists of the reference control, $u^*$, and a small perturbational control, $\delta u$. In the ideal situation of no modeling errors, disturbances, or other uncertainties, the perturbational control $\delta u = 0$, and perfect command tracking is achieved, i.e. $y = r^*$.

![Figure 4.2-1 Scheduling on a Prescribed Reference Trajectory](image)
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Such perfect knowledge is rare, hence the need for feedback and compensator design. Now consider the block diagram of Fig. 4.2-2. This diagram represents the feedback system of Fig. 4.2-1 in the presence of three modeling errors: (1) \( \Delta_y \), unmodeled sensor dynamics, (2) \( \Delta_u \), unmodeled actuator dynamics, and (3) \( \Delta_p \), an artificial uncertainty which corresponds to a performance specification [21].

![Block Diagram](image)

**Figure 4.2-2** Scheduling in the Presence of Robustness / Performance Uncertainties

A gain scheduled approach to control design for Fig. 4.2-2 would be as follows. Let the plant model, \( P_m \), be given by

\[
\begin{align*}
\dot{x}(t) &= f(x(t)) + B \ u(t), \quad x(0) = x_0 \in \mathbb{R}^n \quad (4.2-1) \\
y(t) &= C \ x(t) \quad (4.2-2)
\end{align*}
\]

Equations (4.2-1)-(4.2-2) are quite general since many systems may be put into the above form by selecting state variables as outputs and augmenting dynamics at the plant input. Applying the reference command input, \( u^* \),

\[
\begin{align*}
\dot{x}^*(t) &= f(x^*(t)) + B \ u^*(t), \quad x^*(0) = x_0^* \in \mathbb{R}^n \quad (4.2-3) \\
r^*(t) &= y^*(t) = C \ x^*(t) \quad (4.2-4)
\end{align*}
\]

Now, define
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\[ \delta x(t) = x(t) - x^*(t) \quad (4.2-5) \]
\[ \delta y(t) = y(t) - y^*(t) \quad (4.2-6) \]
\[ \delta u(t) = u(t) - u^*(t) \quad (4.2-7) \]

Then, subtracting (4.2-3) from (4.2-1) and linearizing about \( x^*(t) \),

\[ \delta \dot{x}(t) = Df(x^*(t)) \delta x(t) + B \delta u(t) + \delta f(t, \delta x(t)), \quad \delta x(0) = x_o - x_o^* \in \mathbb{R}^n \quad (4.2-8) \]
\[ \delta y(t) = C \delta x(t) \quad (4.2-9) \]

where

\[ \delta f(t, \delta x(t)) = f(x(t)) - \{ f(x^*(t)) + Df(x^*(t)) \delta x(t) \} \quad (4.2-10) \]

These equations may be decomposed into (1) a linear time-varying plant and (2) a nonlinear residual from the linearization. Let \( \delta P \) denote the nonlinear time-varying perturbational plant (4.2-8)-(4.2-10). Furthermore, let \( \delta P_r \) denote the linear frozen-time plant

\[ \delta \dot{x}(t) = Df(x^*(\tau)) \delta x(t) + B \delta u(t) \quad \delta x(0) = x_o - x_o^* \in \mathbb{R}^n \quad (4.2-11) \]
\[ \delta y(t) = C \delta x(t) \quad (4.2-12) \]

Then a gain scheduled approach would be to design a compensator for (4.2-11)-(4.2-12) so that for all frozen-values of time, the feedback system of Fig. 4.2-3 achieves robust stability and robust performance.

Figure 4.3-3 Block Diagram for Frozen-Time Designs
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In this situation, the design process may be given two different viewpoints. First, the time-varying plant (4.2-8)-(4.2-9) may be viewed as parameter-varying where the parameter is time. Thus, the time-invariant design plant (4.2-11)-(4.2-12) may be viewed as a frozen-parameter plant with parameter value \( \tau \). This would be the approach in case the reference command trajectory \( r^* \) is known \textit{a priori} for all time. Now suppose that the command trajectory \( r^* \) is not known beforehand. Rather, it may be taken from a known \textit{family} of possible command trajectories. This family in turn implies a family of possible values of the reference state-trajectory. That is for some set \( \mathbf{X} \), \( x^*(t) \in \mathbf{X} \). In this case, the time-varying plant (4.2-8)-(4.2-9) may be viewed as parameter-varying with the \( n \) parameters being the reference state-trajectory. Thus, the time-invariant design plant (4.2-11)-(4.2-12) may be viewed as a frozen-parameter plant with frozen parameter values \( x^*(\tau) \). In either case, one can guarantee the desired condition that for all \textit{frozen-values} of time, the feedback system of Fig. 4.2-3 achieves robust stability and robust performance.

Since the original plant model, \( \delta P \), is nonlinear and time-varying, none of the desired feedback properties - including nominal stability - of the frozen-time designs may be present in the overall gain-scheduled system. In Section 4.2.2, conditions are given which \textit{guarantee} the robust stability and robust performance of the global gain-scheduled design.

4.2.2 Stability, Robustness, and Performance Analysis

Suppose that one has carried out the gain scheduled design procedure outlined in Section 4.2.1. Then at each instant of time, one has designed a finite-dimensional compensator which stabilizes the feedback configuration of Fig. 4.2-3. Let the resulting \textit{time-varying} compensator have the following state-space realization

\[
\dot{x}_k(t) = A_k(t) x_k(t) + B_k(t) e(t)
\]  

(4.2-13)
\[ \delta u(t) = C_k(t) \, x_k(t) \]  

(4.2-14)

Using (4.2-8)-(4.2-10) along with (4.2-13)-(4.2-14), the feedback equations of Fig. 4.2-2 are given by

\[
\begin{align*}
\begin{bmatrix}
\dot{x}_k(t) \\
\end{bmatrix} &=
\begin{bmatrix}
\mathbf{D}_f(x^*(t)) & \mathbf{B} & C_k(t) \\
- \mathbf{B}_k(t) & \mathbf{A}_k(t) & 0 \\
\end{bmatrix}
\begin{bmatrix}
\delta x(t) \\
\delta f(t, \delta x(t)) \\
\end{bmatrix} +
\begin{bmatrix}
\mathbf{B} \, (\Delta_u u^*) \,(t) \\
- \mathbf{B}_k(t) \, (\Delta_{py} \, r^* + \Delta_y r^*) \,(t) \\
\end{bmatrix} \\
\int_0^t \begin{bmatrix}
\mathbf{B} & 0 \\
0 & - \mathbf{B}_k(t) \\
\end{bmatrix}
\begin{bmatrix}
\Delta_p(t - \tau) \\
0 \\
\Delta_{py}(t - \tau) + \Delta_p(t - \tau) + \Delta_y(t - \tau) \\
C & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
\mathbf{C}_k(\tau) \\
\delta x(\tau) \\
x_k(\tau) \\
\end{bmatrix}
\, d\tau
\end{align*}
\]

(4.2-15)

where

\[ \Delta_p = \Delta_p \ast (I - \Delta_p)^{-1} \]  

(4.2-16)

\[ \Delta_{py} = \Delta_p \ast \Delta_y \]  

(4.2-17)

Rewriting (4.2-15),

\[ \dot{z}(t) = A(t) \, z(t) + \int_0^t B(t \, \Delta(t - \tau) \, C(\tau) \, z(\tau) \, d\tau + \delta F(t, z(t)) + d(t) \]  

(4.2-18)

where \( A, B, C, \delta F, \Delta, \) and \( d \) are defined in the obvious manner. Note that the feedback equations may be decomposed into (1) a linear time-varying VIDE, (2) a nonlinear residual of the linearization, and (3) an exogenous disturbance. In the context of scheduling on a reference trajectory, stability of (4.2-18) means that the overall gain scheduled feedback system maintains the desired feedback properties of the frozen-time designs.

The stability of (4.2-18) will be shown as follows. Recall that the compensator (4.2-13) - (4.2-14) was designed so that the VIDE

\[ \dot{z}(t) = A(t) \, z(t) + \int_0^t B(t \, \Delta(t - \tau) \, C(\tau) \, z(\tau) \, d\tau \]  

(4.2-19)
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is stable for all \textit{frozen} \( A, B, \) and \( C. \) Using results from Chapter 3, it is shown that (4.2-19) is exponentially stable for sufficiently slow time-variations. Given this time-varying exponential stability, a \textit{Lyapunov functional} for (4.2-19) is constructed. This generalizes the concept of "Converse Theorems of Lyapunov" for ordinary differential equations (e.g. [8, 45]). This Lyapunov functional is then used to give guaranteed stability margins for (4.2-18).

\textbf{Step 1 Slowly Time-Varying Stability of (4.2-19)}

Since (4.2-19) is precisely the class of equations addressed in Section 3.3.3, one can use Theorem 3.3-4 to guarantee stability for sufficiently slow time-variations as follows:

\textbf{Assumption 4.2-1} The matrices \( A, B, \) and \( C \) are bounded and globally Lipschitz continuous.

\textbf{Assumption 4.2-2} There exists a constant \( \beta > 0 \) such that

\[ s \mapsto (sI - A(\tau) - B(\tau) \hat{\Delta}(\tau) C(\tau))^{-1} \in \mathcal{A}^{nm}(-2\beta), \quad \forall \tau \in \mathbb{R}^+ \]  
\tag{4.2-20}

\[ \hat{\Delta} \in \mathcal{A}(-2\beta) \]  
\tag{4.2-21}

The following theorem is a direct consequence of Theorem 3.3-4.

\textbf{Theorem 4.2-1} Consider the linear time-varying VIDE (4.2-19) under Assumptions 4.2-1-
4.2-2. Under these conditions, (4.2-19) is uniformly exponentially stable for sufficiently slow time-variations in $A$, $B$, and $C$.

In terms of the reference state-trajectory, $x^*$, this slowness condition on the dynamics of (4.2-19) states that $x^*$ itself should vary slowly. In light of the discussion in Section 3.4, this comes as no surprise since the designs were based on frozen values of $x^*(t)$.

**Step 2 Construction of a Lyapunov Functional**

Assume now that one has satisfied Theorem 4.2-1 to guarantee the time-varying stability of (4.2-19). Let

$$s(t ; \phi, t_0)$$

(4.2-22)

denote the solution to (4.2-19) with initial conditions $(\phi, t_0)$. From the definition of uniform exponential stability, there exist constants $m$, $\lambda$, and $\beta$ where $\beta \geq \lambda$ such that for any initial condition $(\phi, t_0)$

$$|s(t ; \phi, t_0)| \leq m \lambda^{(t - t_0)} \| \omega_{t_0, \beta} \phi \|_B$$

(4.2-23)

**Theorem 4.2-2** Consider the linear time-varying VIDE (4.2-19). Suppose that (4.2-19) is exponentially stable and satisfies (4.2-23). Under these conditions, given any $\gamma \in (0, 1)$ there exists a function $V : B_e \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which satisfies

$$\| \omega_{t, \beta} x \|_B \leq V(x, t) \leq m \| \omega_{t, \beta} x \|_B$$

(4.2-24)
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\[ l \| V(x, t) - V(x', t) \| \leq m \| W_{t, \beta}(x - x') \|_B \]  \hspace{1cm} (4.2-25)

Furthermore, let \( \widetilde{V}_{(4.2-19)} \) denote \( V \) evaluated along trajectories of (4.2-19), i.e.

\[ \widetilde{V}_{(4.2-19)}(t) \equiv V(s(t; \phi, t_o), t), \quad t \geq t_o \]  \hspace{1cm} (4.2-26)

Then \( V \) satisfies

\[ D^+ \widetilde{V}_{(4.2-19)}(t) \leq -\gamma \| W_{t, \beta} s(\cdot; \phi, t_o) \|_B, \quad t \geq t_o \]  \hspace{1cm} (4.2-27)

**Proof** It is first shown that the exponential stability condition (4.2-23) also implies that

\[ \| W_{t, \beta} s(\cdot; \phi, t_o) \|_B \leq me^{-\lambda(t - t_o)} \| W_{t_o, \beta} \phi \|_B, \quad t \geq t_o \]  \hspace{1cm} (4.2-28)

Using the definition of the LHS of (4.2-28),

\[ \sup_{\xi \in [0, t]} e^{-\beta(t - \xi)} s(\xi; \phi, t_o) \leq \max \left\{ \sup_{\xi \in [0, t_o]} e^{-\beta(t - \xi)} s(\xi; \phi, t_o), \sup_{\xi \in (t_o, t]} e^{-\beta(t - \xi)} s(\xi; \phi, t_o) \right\} \]  \hspace{1cm} (4.2-29)

\[ \leq \max \left\{ e^{-\beta(t - t_o)} \sup_{\xi \in [0, t_o]} e^{-\beta(t_o - \xi)} s(\xi; \phi, t_o), \sup_{\xi \in (t_o, t]} e^{-\beta(t - \xi)} me^{-\lambda(t - \xi)} \| W_{t_o, \beta} \phi \|_B \right\} \]

\[ \leq \max \left\{ e^{-\beta(t - t_o)} \| W_{t_o, \beta} \phi \|_B, me^{-\lambda(t - t_o)} \| W_{t_o, \beta} \phi \|_B \right\} = me^{-\lambda(t - t_o)} \| W_{t_o, \beta} \phi \|_B \]

The remainder of the proof resembles that of Theorem 2.3-2 for nonlinear ordinary differential equations. Let \( \gamma \in (0, 1) \). Then define

\[ V(x, t) \equiv \sup_{t \geq t} \left\{ e^{\gamma(t - \tau)} \| W_{t, \beta} s(\cdot; P_{t, x}, t) \|_B \right\} \]  \hspace{1cm} (4.2-30)

The bounds (4.2-24) on \( V \) immediately follow from the exponential stability condition (4.2-28). To see that \( V \) satisfies the Lipschitz condition (4.2-25), note that
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\[ V(x, t) - V(x', t) = \sup_{\tau \geq t} \left\{ e^{\gamma \alpha(t - \tau)} \| W_{\tau, \beta} s(\cdot; P_t x, t) \| B \right\} \]  
(4.2-31)

\[ - \sup_{\tau \geq t} \left\{ e^{\gamma \alpha(t - \tau)} \| W_{\tau, \beta} s(\cdot; P_t x', t) \| B \right\} \]

\[ \leq \sup_{\tau \geq t} \left\{ e^{\gamma \alpha(t - \tau)} \| W_{\tau, \beta} (s(\cdot; P_t x, t) - s(\cdot; P_t x', t)) \| B \right\} \]  
(4.2-32)

Repeating (4.2-31)-(4.2-32) with \( x \) and \( x' \) reversed, it follows that

\[ |V(x, t) - V(x', t)| \leq \sup_{\tau \geq t} \left\{ e^{\gamma \alpha(t - \tau)} \| W_{\tau, \beta} (s(\cdot; P_t x, t) - s(\cdot; P_t x', t)) \| B \right\} \]  
(4.2-33)

By linearity of (4.2-19),

\[ s(\cdot; P_t x, t) - s(\cdot; P_t x', t) = s(\cdot; P_t (x - x'), t) \]  
(4.2-34)

Thus

\[ |V(x, t) - V(x', t)| \leq \sup_{\tau \geq t} \left\{ e^{\gamma \alpha(t - \tau)} \| W_{\tau, \beta} (s(\cdot; P_t (x - x'), t) \| B \right\} \]  
(4.2-35)

Condition (4.2-25) then follows from exponential stability.

Finally, to prove the negative definiteness condition (4.2-27), recall the definition

\[ D^+ V_{(4.2-19)}(t) = \lim_{h \to 0^+} \sup_{h} \frac{V(s(t + h; \phi, t_0), t + h) - V(s(t; \phi, t_0), t)}{h} \]  
(4.2-36)

Evaluating (4.2-36) term by term gives

\[ V(s(t + h; \phi, t_0), t + h) = \sup_{\tau \geq t + h} \left\{ e^{\gamma \alpha(t - \tau + h)} \| W_{\tau, \beta} s(\cdot; s(t + h; \phi, t_0), t + h) \| B \right\} \]  
(4.2-37)

and
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\[ V(s(t; \phi, t_0), t) = \sup_{\tau \geq t} \left\{ e^{\gamma_{\lambda}(\tau - t)} \| W_{\tau, \beta} S(\cdot; s(t; \phi, t_0), t) \|_B \right\} \quad (4.2-38) \]

Exploiting the properties of \( V \) and \( s \),

\[ V(s(t; \phi, t_0), t) \geq \sup_{\tau \geq t + h} \left\{ e^{\gamma_{\lambda}(\tau - t)} \| W_{\tau, \beta} S(\cdot; s(t; \phi, t_0), t) \|_B \right\} \quad (4.2-39) \]

\[ \geq e^{\gamma_{\lambda} h} \sup_{\tau \geq t + h} \left\{ e^{\gamma_{\lambda}(\tau - (t + h))} \| W_{\tau, \beta} S(\cdot; s(t + h; \phi, t_0), t + h) \|_B \right\} \]

\[ \geq e^{\gamma_{\lambda} h} V(s(t + h; \phi, t_0), t + h) \quad (4.2-40) \]

Thus,

\[ V(s(t + h; \phi, t_0), t + h) \leq e^{-\gamma_{\lambda} h} V(s(t; \phi, t_0), t) \quad (4.2-41) \]

Condition (4.2-27) follows from substituting (4.2-41) into (4.2-36) and letting \( h \to 0 \) (cf. (2.3-37)).

As mentioned earlier, Theorem 4.2-2 represents a type of "converse theorem of Lyapunov" [8, 45]. It is noted that the existence of a function which satisfies (4.2-24)-(4.2-27) can be used to prove uniform exponential stability of (4.2-19). Thus, Theorem 4.2-2 is also a statement of the equivalence of uniform exponential stability and existence of Lyapunov functionals (as in Theorem 2.3-2 for ordinary differential equations). Finally, it is noted that Theorem 4.2-2 does not require that the exponential stability of (4.2-19) is due to slow time-variations.
Step 3 Proof of Stability of the Overall Gain Scheduled System

Recall that the feedback configuration of Fig. 4.2-2 leads to dynamics of the form

$$
\dot{z}(t) = A(t)z(t) + \int_{0}^{t} B(t - \tau) C(\tau) z(\tau) d\tau + \delta F(t, z(t)) + d(t)
$$

(4.2-42)

In light of Steps 1 and 2, these equations may be viewed as perturbations (\(\delta F\) and \(d\)) on an exponentially stable time-varying VIDE. Using the Lyapunov functional of Theorem 4.2-2, conditions will be placed on \(\delta F\) and \(d\) to guarantee the boundedness of solutions to (4.2-42).

First, the following assumption is made on \(\delta F\).

Assumption 4.2-3 There exists a constant \(k_{\delta F} \geq 0\) such that

$$
|\delta F(t, z)| \leq k_{\delta F} |z|^2, \quad \forall t \in \mathbb{R}^+, \forall z \in \mathbb{R}^n
$$

(4.2-43)

This quadratic bound reflects that \(\delta F\) is a residual from a linearization.

The stability of (4.2-42) is now addressed. Let \(s'(t; \phi, t_0)\) denote the solution to (4.2-42) with initial condition \((\phi, t_0)\).

Theorem 4.2-3 Consider the nonlinear VIDE (4.2-42). Let the linear time-varying VIDE (4.2-19) be exponentially stable. Let \(V\) be defined as in Theorem 4.2-2. Then given any \(\gamma' \in (0, 1)\),

$$
\|W_{t_0, \beta} \phi\|_B \leq \frac{\gamma'}{m^2 k_{\delta F}}
$$

(4.2-44)
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and

\[ \| d \|_{L_{\infty}} \leq \frac{(\gamma \lambda)^2}{m^2 k_{8F}} (1 - \gamma') \gamma' \]  
(4.2-45)

together imply

\[ \| s'(t ; \phi, t_0) \| \leq \frac{\gamma \lambda}{mk_{8F}} \gamma', \quad t \geq t_0 \]  
(4.2-46)

**Proof** Let \( \vec{V}_{(4.2-42)} \) denote \( V \) evaluated along trajectories of (4.2-42):

\[ \vec{V}_{(4.2-42)}(t) \equiv V(s(t ; \phi, t_0), t), \quad t \geq t_0 \]  
(4.2-47)

It is first shown that for \( t \geq t_0 \),

\[ D^+ \vec{V}_{(4.2-42)}(t) \leq -\gamma \lambda \sum_{(4.2-42)}(t) + mk_{8F} \sum_{(4.2-42)}^2(t) + m \| d \|_{L_{\infty}} \]  
(4.2-48)

By definition of \( D^+ \vec{V}_{(4.2-42)}(t) \)

\[ D^+ \vec{V}_{(4.2-42)}(t) = \limsup_{h \to 0^+} \frac{V(s(t + h ; \phi, t_0), t + h) - V(s(t ; \phi, t_0), t)}{h} \]  
(4.2-49)

\[ = \limsup_{h \to 0} \frac{V(s(t + h ; s(t ; \phi, t_0), t), t + h) - V(s(t ; \phi, t_0), t)}{h} + \]

\[ \limsup_{h \to 0^+} \frac{V(s'(t + h ; s'(t ; \phi, t_0), t), t + h) - V(s(t + h ; s'(t ; \phi, t_0), t), t + h)}{h} \]

\[ \leq -\gamma \lambda \| W_{t,0} s'(\cdot ; \phi, t_0) \|_{\mathcal{B}} + \]

\[ \limsup_{h \to 0^+} \frac{m \| \mathcal{W}_{t+h,0} (s'(\cdot ; s'(t ; \phi, t_0), t) - s(\cdot ; s(t ; \phi, t_0), t)) \|_{\mathcal{B}}}{h} \]

Applying standard bounding techniques in conjunction with Assumption 4.2-1 and (4.2-21), it
can be shown that
\[ |s'(t + h; s'(t; \phi, t_o), t) - s(t + h; s'(t; \phi, t_o), t)| \leq h(k_{SF}^2 s(t; s'(t; \phi, t_o), t)^2 + \|d\|_{L_{\infty}}) + \text{higher order terms in } h \] (4.2-50)
Substituting (4.2-50) into (4.2-49) yields the desired bound
\[ \mathbf{D}^T \hat{V}_{(4.2-42)}(t) \leq -\gamma \lambda \hat{V}_{(4.2-42)}(t) + mk_{SF}^2 \hat{V}_{(4.2-42)}^2(t) + m\|d\|_{L_{\infty}} \] (4.2-51)
Now given \( \gamma' \in (0, 1) \), let \( d \) be bounded as in (4.2-45), i.e.
\[ \|d\|_{L_{\infty}} \leq \frac{(\gamma \lambda)^2}{m^2 k_{SF}^2} (1 - \gamma') \gamma' \] (4.2-52)
Furthermore, suppose that at time \( t \)
\[ \hat{V}_{(4.2-42)}(t) = \frac{\gamma \lambda}{mk_{SF}^2} \gamma' \] (4.2-53)
Substituting (4.2-52) and (4.2-53) into (4.2-51), it follows that
\[ \mathbf{D}^T \hat{V}_{(4.2-42)}(t) \leq 0 \] (4.2-54)
Thus by definition of the Dini derivative, if \( d \) is bounded as in (4.2-52) and
\[ \hat{V}_{(4.2-42)}(t_o) \leq \frac{\gamma \lambda}{mk_{SF}^2} \gamma' \] (4.2-55)
satisfies (4.2-53), then (4.2-53) holds for all time. These conditions immediately translate into (4.2-44)-(4.2-46) which completes the proof.

Theorem 4.2-3 can be interpreted as a type of small-signal finite-gain stability result [8, 58]. It states that provided the disturbance, \( d \), is sufficiently small, then the mapping \( d \mapsto s'(\cdot; \phi, t_o) \) is finite-gain stable with gain \( \frac{m}{\gamma \lambda (1 - \gamma')} \). However, recall the definition of \( d \)
Thus, the condition "d sufficiently small" essentially states the intuitive condition that the reference trajectories \(u^*\) and \(r^*\) should not excite the unmodeled actuator or sensor dynamics.

To summarize, it has been shown that a gain scheduled approach applied to the feedback system of Fig. 4.2-2 has guaranteed robustness and performance properties under the following conditions. First of all, it is required that the reference trajectory \(x^*\) is sufficiently slow. This comes as no surprise since the gain scheduled designs are based on linear time-invariant approximations to the plant. The restriction of slow variations simply states that such a frozen-time approximation should be accurate. Since the system is actually nonlinear, the internal stability is only local. As the nonlinearities approach zero (i.e. \(k_{\phi} \rightarrow 0\)), one has that the internal stability approaches global internal stability. Again, the restriction that nonlinearities impose are reminders that the design plants are linear time-invariant. The nonlinearities place another restriction on the feedback system, this time on the reference trajectories \(u^*\) and \(r^*\). Namely, from (4.2-56) it is required that these reference trajectories do not excite the unmodeled dynamics. For example, if the reference control trajectory, \(u^*\), has significant frequency content in the region of unmodeled actuator dynamics, then one cannot make demands on the resulting stability and performance of the closed loop gain-scheduled system. In fact, since the reference control trajectory is fedforward to the plant, it is unlikely that any control strategy can remedy this situation.
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4.3 Scheduling on the Plant Output

4.3.1 Problem Statement

Consider the plant model given by

\[
\frac{d}{dt} \begin{bmatrix} y \\ z \end{bmatrix} = f(y, z) + B u, \quad y(t) \in \mathbb{R}^m, \quad z(t) \in \mathbb{R}^{n-m}, \quad u(t) \in \mathbb{R}^m
\]  

(4.3-1)

where the plant output, \( y \), is explicitly a state variable. The following assumption is made on (4.3-1).

**Assumption 4.3-1** \( f : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m \) satisfies

\[
f(0, 0) = 0
\]  

(4.3-2)

**Assumption 4.3-2** There exist unique continuously differentiable functions

\[
u_{eq} : \mathbb{R}^m \to \mathbb{R}^m
\]  

(4.3-3)

\[
z_{eq} : \mathbb{R}^m \to \mathbb{R}^{n-m}
\]  

(4.3-4)

which satisfy

\[
0 = f(y, z_{eq}(y)) + B u_{eq}(y)
\]  

(4.3-5)

Assumption 4.3-2 essentially states that one has a family of equilibrium conditions...
parameterized by the output, \( y \). In terms of gain scheduling, each of these equilibrium conditions is a possible "operating condition."

A gain scheduled approach to controlling (4.3-1) would be as follows. The plant linearized about a possible operating point, \( y_o \), is given by

\[
\frac{d}{dt} \begin{bmatrix} y - y_o \\ z - z_{eq}(y_o) \end{bmatrix} = D f(y_o, z_{eq}(y_o)) \begin{bmatrix} y - y_o \\ z - z_{eq}(y_o) \end{bmatrix} + B (u - u_{eq}(y_o)) \tag{4.3-6}
\]

Thus at each operating point, one would design a compensator based on a local linear time-invariant approximation (4.3-6). This procedure would result in a family of linear time-invariant compensators \( \{A_k(y_o), B_k(y_o), C_k(y_o)\} \) parameterized by the operating condition \( y_o \). These compensators are then used as in Fig. 4.3-1. In this figure, the current operating condition is instantaneously updated as the current plant output. Thus, the compensator dynamics would evolve as

\[
\dot{x}_k(t) = A_k(y(t)) x_k(t) + B_k(y(t)) e(t) \tag{4.3-7}
\]

\[
\delta u(t) = C_k(y(t)) x_k(t) \tag{4.3-8}
\]

![Figure 4.3-1 Scheduling on the Plant Output](image-url)

This procedure leads to feedback equations of the form
\[
\begin{bmatrix}
y \\
z - z_{eq}(y) \\
x_k
\end{bmatrix}
= \begin{bmatrix}
0 & \mathcal{D}_z f(y, z_{eq}(y)) & B_y C_k(y) \\
0 & \mathcal{D}_z f(y, z_{eq}(y)) & B_z C_k(y) \\
-B_k(y) & 0 & A_k(y)
\end{bmatrix} \begin{bmatrix}
y \\
z - z_{eq}(y) \\
x_k
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
B_k(y)
\end{bmatrix} r + 
\]

\[
\begin{bmatrix}
\delta f_y(y, z) \\
\delta f_z(y, z) \\
0
\end{bmatrix}
= \begin{bmatrix}
0 \\
-d \frac{dz_{eq}(y)}{dt} \\
0
\end{bmatrix} 
\]

where

\[
\delta f(y, z) = f(y, z) - \left( f(y, z_{eq}(y)) + \mathcal{D}_z f(y, z_{eq}(y)) (z - z_{eq}(y)) \right) \quad (4.3-10)
\]

and the subscripts \(y\) and \(z\) denote decomposition of the matrix functions into their \(y\) and \(z\) components, respectively. Explicitly evaluating the time derivative of \(z_{eq}(y)\), (4.3-9) takes the form

\[
\begin{bmatrix}
y \\
z - z_{eq}(y) \\
x_k
\end{bmatrix}
= \begin{bmatrix}
\delta f_y(y, z) \\
\delta f_z(y, z) - \mathcal{D}_z z_{eq}(y) \delta f_y(y, z) \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
B_k(y)
\end{bmatrix} r(i) + 
\]

\[
\begin{bmatrix}
0 & \mathcal{D}_f f(y, z_{eq}(y)) & B_y C_k(y) \\
0 & \mathcal{D}_f f(y, z_{eq}(y)) - \mathcal{D}_z z_{eq}(y) \mathcal{D}_f f(y, z_{eq}(y)) & B_z C_k(y) - \mathcal{D}_z z_{eq}(y) B_y C_k(y) \\
-B_k(y) & 0 & A_k(y)
\end{bmatrix} \begin{bmatrix}
y \\
z - z_{eq}(y) \\
x_k
\end{bmatrix}
\]

which represents the nominal feedback equations for Fig. 4.3-1. It is noted that the "dynamics matrix" \(A(y)\) of (4.3-11) differs from the closed loop dynamics matrix which would occur from applying the compensator dynamics \([A_k(y_o), B_k(y_o), C_k(y_o)]\) to linearized plant dynamics (4.3-6). This difference (discussed further in Chapter 5) leads to the question of
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what one should use as "design matrices" for the plant.

The gain scheduled designs have the property that for each linearized frozen operating condition, the feedback system has excellent stability, robustness, and performance properties. However, the actual gain scheduled system will have a time-varying (endogenous) scheduling variable evolving under nonlinear dynamics. In the following sections, conditions are given for the nominal stability, robust stability, robust performance, and disturbance rejection properties of the fixed operating point designs to carry over to the actual gain scheduled system of Fig. 4.3-1.

4.3.2 Nominal Stability

The nominal gain scheduled feedback equations (4.3-11) of Fig. 4.3-1 may be put in the form

\[
\begin{bmatrix}
\dot{y}(t) \\
\dot{v}(t)
\end{bmatrix} = A(y(t)) \begin{bmatrix} y(t) \\ v(t) \end{bmatrix} + \delta F(y(t), v(t)) + B(y(t)) r(t), \quad y(t) \in \mathbb{R}^m, \; v(t) \in \mathbb{R}^p
\]

(4.3-12)

where \( A, B, \delta F \) are defined in the obvious manner, and \( v = [ (z - z_{eq}(y))^T x_k^T ]^T \).

Furthermore, define

\[
x = \begin{bmatrix} y \\ v \end{bmatrix}
\]

(4.3-13)

The following assumptions are made on (4.3-12)

Assumption 4.3-3 The matrix \( A \) is bounded with constant \( k_A \) and Lipschitz continuous with constant \( L_A \).
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Assumption 4.3-4  The constant eigenvalues of \( A(y) \) are uniformly bounded away from the closed complex RHP for all constant \( y \).

Assumption 4.3-5  The linearization residual satisfies

\[
|\delta F(y, v)| \leq k_{\delta F} |v|^2, \quad \forall \ y \in \mathbb{R}^m, \forall \ v \in \mathbb{R}^p
\]  (4.3-14)

It is important to note that from definition (4.3-10), the residual \( \delta F \) results only from the linearization of the function \( z \mapsto f(y, z) \) of the original dynamics (4.3-1). This implies that if the dynamics of (4.3-1) are linear in \( z \), then \( k_{\delta F} \equiv 0 \). In this sense, the notion of "the scheduling variable capturing the plant's nonlinearities" is precisely quantified.

Before examining the stability of (4.3-12), consider the unforced quasi-linear equations

\[
\begin{bmatrix}
\dot{y} \\
\dot{v}
\end{bmatrix} = A(y) \begin{bmatrix}
y \\
v
\end{bmatrix}, \quad x(0) = x_0
\]  (4.3-15)

From Assumption 4.3-4, it follows that solutions of (4.3-15) will decay provided that the time-variations in (4.3-15) are sufficiently slow. This is quantified in the following theorem, which is a direct consequence of Theorem 3.2-1.

Theorem 4.3-1  Consider (4.3-15) under Assumptions 4.3-3 - 4.3-4. Under these conditions, there exist constants \( m, \lambda, \) and \( \varepsilon > 0 \) such that
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\[ |\dot{y}(t)| \leq \varepsilon, \quad \forall \ t \in [0, T] \Rightarrow |x(t)| \leq me^{-\lambda t} |x_0|, \quad \forall \ t \in [0, T] \tag{4.3-16} \]

This is used in showing exponential stability of (4.3-15) as follows.

**Theorem 4.3-2** Consider (4.3-15) under Assumptions 4.3-3-4.3-4. Let \( L_y \) be such that

\[ |\dot{y}| \leq L_y |x| \tag{4.3-17} \]

Under these conditions, given any \( \gamma \in (0, 1) \)

\[ |x_0| \leq \rho \equiv \frac{\gamma e}{k_A \ln L_y} m \tag{4.3-18} \]

implies

\[ |x(t)| \leq me^{-\lambda t} |x_0|, \quad \forall \ t \in \mathbb{R}^+ \tag{4.3-19} \]

**Proof** Using the Bellman-Gronwall inequality, one can show that condition (4.3-18) implies that

\[ |\dot{y}(t)| \leq \varepsilon, \quad t \in \left[0, \frac{1}{\lambda} \ln m + \frac{1}{k_A} \frac{1}{\ln \frac{1}{\gamma}} \right] \tag{4.3-20} \]

During this interval, Theorem 4.3-1 assures that the state decays exponentially according to (4.3-16). However, the restriction (4.3-18) on the initial condition guarantees that at the end of the interval

\[ \left| x \left( \frac{1}{\lambda} \ln m + \frac{1}{k_A} \frac{1}{\ln \frac{1}{\gamma}} \right) \right| \leq |x_0| \tag{4.3-21} \]
Repeating this process, it follows that $|\dot{y}(t)| \leq \epsilon$ for all time which in turn implies (4.3-19).

Theorems 4.2-1 - 4.2-2 essentially state the following. Since equation (4.3-15) is the result of gain scheduling on the output, $y$, one has that $A(y)$ is a stable matrix. Using the results from Chapter 3, Theorem 4.1-1 states that if the scheduling variable, $y$, varies slowly, then local exponential stability is guaranteed. Theorem 4.2-2 then quantifies how slowly $y$ should vary relative to the system dynamics. Namely, the slower the time-variations of the output $y$ (i.e. as $L_y \rightarrow 0$), then from (4.3-18) the larger the neighborhood for local exponential stability. Furthermore, the size of this neighborhood increases as the size of allowable time-variations increases (i.e. as $\epsilon \rightarrow \infty$). This dependence is important since the parameter $\epsilon$ is a function of the frozen operating point closed loop designs. Furthermore, it is stressed that the local restriction (4.3-18) is not due to the linearization of the dynamics in the gain scheduled designs. Rather, it is because the gain scheduled designs were based on frozen values of the scheduling variable, $y$, which is actually time-varying.

The stability of the nominal gain scheduled dynamics (4.3-12) is now addressed. First note that (4.3-12) may be decomposed into (1) locally exponentially stable dynamics $A(y)$, (2) a nonlinear residual $\delta F$, and (3) an exogenous input $r$. This is precisely the same sort of decomposition which existed of the Volterra equation (4.2-18) in Section 4.2. Thus, stability of (4.3-12) may be shown in the same manner; namely use the local exponential stability of (4.3-15) to construct a Lyapunov function and use this Lyapunov function to prove local exponential stability in the presence of $\delta F$ and small-signal finite-gain stability in the presence of $r$. Unlike Section 4.2, the equations of interest here are ordinary differential equations.
Step 1 Construction of a Lyapunov Function

Let \( s(t; x_0) \) denote the solution to (4.3-15) under initial conditions \( x(0) = x_0 \).

Assumption 4.3-6 Let \( m \) and \( \rho \) be as in (4.3-18)-(4.3-19). There exists a constant \( L_{Ax} \) such that

\[
|A(y) x - A(y') x'| \leq L_{Ax} |x - x'|, \quad \forall |x|, |x'| \leq m \rho \tag{4.3-22}
\]

Theorem 4.3-3 Consider the locally exponentially stable system (4.3-15) under Assumptions 4.3-3 - 4.3-6. Under these conditions given any \( \gamma \in (0, 1) \), there exists a Lyapunov function \( V : \mathbb{R}^n \rightarrow \mathbb{R}^+ \) such that

\[
|x| \leq V(x) \leq m |x|, \quad \forall |x| \leq \rho \tag{4.3-23}
\]

\[
|V(x) - V(x')| \leq L_V |x - x'|, \quad \forall |x|, |x'| \leq \rho \tag{4.3-24}
\]

where

\[
L_V = e^{(\gamma \lambda + L_{Ax}) T} \tag{4.3-25}
\]

\[
T = \frac{\ln m}{(1 - \gamma \lambda)} \tag{4.3-26}
\]

Furthermore, let \( \tilde{V}_{(4.3-15)} \) denote \( V \) evaluated along \( s(t; x_0) \). Then

\[
D^+ \tilde{V}_{(4.3-15)}(t) \leq -\gamma \lambda \tilde{V}_{(4.3-15)}(t), \quad \forall |x_o| \leq \rho \tag{4.3-27}
\]

Proof Define \( V \) as

\[
V(x) \equiv \sup_{t \geq 0} e^{\gamma \lambda t} |s(t; x)| \tag{4.3-28}
\]

The remainder of the proof essentially follows that of Theorem 2.3-2 with the exception that
the dynamics of (4.3-15) are only locally exponentially stable - hence the local restrictions on $V$ in (4.3-23)-(4.3-24) and (4.3-27).

\[ \frac{\gamma \lambda}{L \sqrt{k \delta_F}} \gamma' < \rho \]  
(4.3-29)

Step 2 Nominal Stability of the Gain Scheduled System

Let $s'(t; x_o)$ denote the solution to (4.3-12) under initial conditions $x(0) = x_o$.

Theorem 4.3-4 Consider the nominal gain scheduled feedback equations (4.3-12) under Assumptions 4.3-3 - 4.3-6. Let $V$ be defined as in Theorem 4.3-3. Let $\gamma' > 0$ be such that

Under these conditions,

\[ |x_o| \leq \frac{\gamma \lambda}{mL \sqrt{k \delta_F}} \gamma' \]  
(4.3-30)

and

\[ \| B(y) r \|_{L_\infty} \leq \frac{(\gamma \lambda)^2}{L^2 \sqrt{k \delta_F}} (1 - \gamma') \gamma' \]  
(4.3-31)

together imply

\[ |s'(t; x_o)| \leq \frac{\gamma \lambda}{L \sqrt{k \delta_F}} \gamma', \quad t \geq 0 \]  
(4.3-32)

Proof Let $\tilde{V}_{(4.3-12)}$ denote $V$ evaluated along $s'(t; x_o)$. Then using methods similar to those of Theorem 4.2-3, it can be shown that for $\tilde{V}_{(4.3-12)}(t) < \rho$
\[ \mathbf{D}^+ \tilde{V}_{(4.3-12)}(t) \leq -\gamma \tilde{V}_{(4.3-12)}(t) + L_V k_{\text{SF}} \tilde{V}_{(4.3-12)}^2(t) + L_V \parallel \mathbf{B}(y) \parallel_r \parallel_{L_\infty} \]  

(4.3-33)

Now let \( r \) be bounded as in (4.3-31). Furthermore, suppose that at time \( t \)

\[ \tilde{V}_{(4.3-12)}(t) = \frac{\gamma \lambda}{L_V k_{\text{SF}}} \gamma' \]  

(4.3-34)

From (4.3-29), the bound (4.3-33) holds. Furthermore, substituting (4.3-31) and (4.3-34) into (4.3-33), one has that

\[ \mathbf{D}^+ \tilde{V}_{(4.3-12)}(t) \leq 0 \]  

(4.3-35)

It then follows that

\[ \tilde{V}_{(4.3-12)}(t_0) \leq \frac{\gamma \lambda}{L_V k_{\text{SF}}} \gamma' \]  

(4.3-36)

implies (4.3-32) holds for all time, which completes the proof.

As in Theorem 4.2-3, Theorem 4.3-4 can be interpreted as a type of small-signal finite-gain stability result. In case \( r \equiv 0 \), then using the Lyapunov function of Theorem 4.3-3, it is easy to show that the nominal gain scheduled dynamics (4.3-12) are locally exponentially stable. This is not done here since Theorem 4.3-4 captures the essence of the underlying stability of the nominal gain scheduled dynamics.

To summarize, it has been shown that the gain scheduled system of Fig. 4.3-1 has guaranteed nominal stability under the following conditions. First of all, the rate of variations in the scheduling variable, \( y \), should be slow (Theorem 4.3-1). This comes as no surprise since the designs were based on frozen values of \( y \). The slowness condition then translates into local exponential stability (Theorem 4.3-2) of the quasi-linear dynamics of (4.3-15). As the
allowable rate of variations in $y$ increases, then the local stability of (4.3-15) approaches global stability. Thus one should perform the frozen designs to allow for time-variations as large as possible. In moving from the quasi-linear dynamics of (4.3-15) to the full gain scheduled dynamics (4.3-12), another restriction is placed on the locality of the stability. Namely that of the linearization. That is, as the magnitude of the linearization residual approaches zero (i.e. as $k_{δF} \to 0$) then the local stability of the full gain scheduled dynamics approaches that of the quasi-linear dynamics (4.3-15).

However, recall that the residual $δF$ results only from the linearization of the function $z \mapsto f(y, z)$ of the original dynamics (4.3-1). Thus, Theorems 4.3-1 - 4.3-4 state that one can guarantee global stability (i.e. global exponential stability and finite-gain stability without a small-signal restriction) if (1) the scheduling variable $y$ contains all of the nonlinearities and (2) the gain scheduled designs are such that the allowable time-variations are arbitrary. Note that these two conditions are namely the limiting cases of the two primary gain scheduling guidelines of "scheduling on a slow variable" and "capturing the nonlinearities". However, the guidelines have now been transformed into quantitative statements about the resulting gain scheduled dynamics.

4.3.3 Robust Stability and Robust Performance

Consider the block diagram of Fig. 4.3-2. This represents the nonlinear gain scheduled system of Fig. 4.1 in the presence of plant input unmodeled dynamics and artificial unmodeled dynamics which represent performance specifications. In this section, it is shown how one can guarantee that the robust stability of the frozen operating condition designs carries over to the full gain scheduled system of Fig. 4.3-2.

Following the gain scheduling design procedure outlined in Section 4.3.1, the feedback equations of Fig. 4.3-2 are given by
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\[
\begin{bmatrix}
    \frac{dy}{dt} \\
    \frac{dz}{dt} - z_{eq}(y) \\
    \frac{dx_k}{dt}
\end{bmatrix}
= \begin{bmatrix}
    0 & D_2 f_y(y, z_{eq}(y)) & B_y \\
    0 & D_2 f_z(y, z_{eq}(y)) - Dz_{eq}(y) D_2 f_y(y, z_{eq}(y)) & B_z - Dz_{eq}(y) B_y \\
    -B_k(y) & 0 & A_k(y)
\end{bmatrix}
\begin{bmatrix}
    y \\
    z - z_{eq}(y) \\
    x_k
\end{bmatrix}
\]

\[
\begin{bmatrix}
    \delta f_y(y, z) \\
    \delta f_z(y, z) - Dz_{eq}(y) \delta f_y(y, z) \\
    0
\end{bmatrix}
+ \begin{bmatrix}
    B_y \\
    B_z - Dz_{eq}(y) B_y \\
    0
\end{bmatrix}
\int_0^t \Delta_\mu(t - \tau) u_{eq}(y(\tau)) d\tau +
\]

\[
\begin{bmatrix}
    y(\tau) \\
    z(\tau) - z_{eq}(y(\tau)) \\
    x_k(\tau)
\end{bmatrix}
\]

which may be put in the form

\[
\frac{d}{dt}\begin{bmatrix}
    y \\
    v
\end{bmatrix} = A(y)\begin{bmatrix}
    y \\
    v
\end{bmatrix} + \delta F(y, v) + (gx)(i) + \int_0^t B(y(i)) \Delta(t - \tau) C(y(\tau)) x(\tau) d\tau + d(i) \quad (4.3-38)
\]

where \( A, \delta F, x \) and \( v \) are defined as in (4.3-12) - (4.3-13) and \( B, C, \Delta, \) and \( d \) are defined in the obvious manner.

In the remainder of this section, conditions are given which guarantee the stability of (4.3-38). In terms of gain scheduling, this means that the robustness and performance properties of the frozen operating point designs carry over to the full gain scheduled system of Fig. 4.3-2. Both the theorems and proofs closely follow those for the nominal stability analysis of Section 4.3.2.
The following assumptions are made on (4.3-38).

**Assumption 4.3-7** The matrices $A$, $B$, $C$ are bounded and globally Lipschitz continuous.

**Assumption 4.3-8** There exists a constant $\beta > 0$ such that

$$s \mapsto (sI - A(y) - B(y)A(s)C(y))^{-1} \in \mathcal{A}(-2\beta), \quad \forall y \in \mathcal{R}^m \quad (4.3-39)$$

$$\hat{\Delta} \in \mathcal{A}(-2\beta) \quad (4.3-40)$$

**Assumption 4.3-9** There exists a constant $k_\varepsilon$ such that

$$| (g\varepsilon)(t) | \leq k_\varepsilon \| \nu \|_{\delta, \beta} \| x \|_{B^*} \quad t \geq 0, \forall x \in B_e \quad (4.3-41)$$
Before examining the stability of (4.3-38), consider the unforced quasi-linear equations

\[
\frac{d}{dt} \begin{bmatrix} y \\ v \end{bmatrix} = A(y) \begin{bmatrix} y \\ v \end{bmatrix} + \int_{0}^{t} B(y(\tau)) \Delta(t - \tau) C(y(\tau)) \, d\tau, \quad t > t_o
\]

(4.3-42)

under initial conditions

\[
\begin{cases}
  x(t) = \phi(t), & 0 \leq t \leq t_o, \phi \in B_e \\
  x(t_o^+) = \phi(t_o)
\end{cases}
\]

(4.3-43)

Assumption 4.3-8 reflects that (4.3-42) is a product of a gain scheduled design. That is, (4.3-42) is exponentially stable for all frozen values of the scheduling variable \( y \). Thus, Assumptions 4.3-7 - 4.3-8 together imply that solutions of (4.3-42) will decay provided that the time variations in \( y \) are sufficiently slow. This is quantified in the next theorem which is analogous to Theorem 4.3-1. The proof follows from an immediate application of Theorem 3.3-4.

**Theorem 4.3-5** Consider (4.3-42) under Assumptions 4.3-7 - 4.3-8. Under these conditions, there exist constants \( m, \lambda, \) and \( \epsilon > 0 \) such that

\[
| \dot{y}(t) | \leq \epsilon, \quad \forall \, t \in [t_o, t_o + T] \Rightarrow | x(t) | \leq me^{-\lambda(t - t_o)} \| \Omega \|_{0, \beta} \| \phi \|_{B_e}, \quad \forall \, t \in [t_o, t_o + T]
\]

(4.3-44)

As in the case of nominal stability, Theorem 4.3-5 will be used to prove local exponential stability for (4.3-42). First, a lemma regarding the boundedness of solutions of (4.3-42) is given.
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Lemma 4.3-1 Consider (4.3-42) under Assumptions 4.3-7 - 4.3-8. Under these conditions, there exists a continuous monotonically increasing function $f: \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(0) = 1$ and

$$\| \mathbf{w}_{t_o, \beta} \mathbf{x} \|_B \leq f(t - t_o) \| \mathbf{w}_{t_o, \beta} \Phi \|_B$$  \hspace{1cm} (4.3-45)

Proof The solution to (4.3-42) is given by

$$x(t) = x_o + \int_{t_o}^{t} A(y(\tau)) + \int_{t_o}^{\tau} B(y(\xi)) \Delta(\tau - \xi) C(y(\xi)) x(\xi) d\xi d\tau + \int_{t_o}^{t} F(\tau) d\tau$$ \hspace{1cm} (4.3-46)

where

$$F(\tau) = \int_{\tau}^{t_o} B(y(\xi)) \Delta(\tau - \xi) C(y(\xi)) \Phi(\xi) d\xi$$ \hspace{1cm} (4.3-47)

Switching the order of integration in (4.3-46) yields

$$x(t) = x_o + \int_{t_o}^{t} \left( A(y(\tau)) + \int_{\tau}^{t_o} B(y(\xi)) \Delta(\xi - \tau) C(y(\tau)) d\xi \right) x(\tau) d\tau + \int_{t_o}^{t} F(\tau) d\tau$$ \hspace{1cm} (4.3-48)

Now in the proof of Theorem 3.3-2, it was shown that $F$ can be bounded by a decaying exponential

$$|F(\tau)| \leq \alpha e^{-\beta (\tau - t_o)} \| \mathbf{w}_{t_o, \beta} \Phi \|_B$$ \hspace{1cm} (4.3-49)

Furthermore, Assumptions 4.3-7 - 4.3-8 guarantee that for some $K \geq 0$,

$$\left| A(y(\tau)) + \int_{\tau}^{t} B(y(\xi)) \Delta(\xi - \tau) C(y(\tau)) d\xi \right| \leq K, \quad \forall \, t \geq \tau \geq t_o$$ \hspace{1cm} (4.3-50)

Bounding (4.3-48), substituting (4.3-49)-(4.3-50), and applying the Bellman-Gronwall
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inequality, it follows that

\[ |x(t)| \leq f(t-t_0) \| \mathbf{W}_{t_0, \beta} \phi \|_{\mathcal{B}}, \quad t \geq t_0 \]  \hspace{1cm} (4.3-51)

where \( f \) is a continuous monotonically increasing function \( f: \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( f(0) = 1 \).

Finally, (4.3-45) follows using the same arguments found in (4.2-29).

The stability of (4.3-42) is now addressed. Note the resemblance to Theorem 4.3-2.

**Theorem 4.3-6** Consider (4.3-42) under Assumptions 4.3-7 - 4.3-8. Let \( L_y \) be such that

\[ |\dot{y}(t)| \leq L_y \| \mathbf{W}_{t, \beta} x \|_{\mathcal{B}}, \quad t \geq t_0 \]  \hspace{1cm} (4.3-52)

Under these conditions, given any \( \gamma \in (0, 1) \)

\[ \| \mathbf{W}_{t_0, \beta} \phi \|_{\mathcal{B}} \leq \rho \equiv \frac{\gamma e}{L_y f \left( \frac{\ln m}{\lambda} \right)} \]  \hspace{1cm} (4.3-53)

implies

\[ |x(t)| \leq me^{-\lambda(t-t_0)} \| \mathbf{W}_{t_0, \beta} \phi \|_{\mathcal{B}} \]  \hspace{1cm} (4.3-54)

**Proof** Let \( \delta > 0 \) be such that

\[ \frac{f \left( \frac{\ln m}{\lambda} + \delta \right)}{f \left( \frac{\ln m}{\lambda} \right)} = \frac{1}{\gamma} \]  \hspace{1cm} (4.3-55)
Then combining (4.3-53) with (4.3-45), it follows that

\[ |\dot{y}(t)| \leq \varepsilon, \quad t \in [t_0, t_0 + \frac{\ln m}{\lambda} + \delta] \quad (4.3-56) \]

During this interval, Theorem 4.3-5 assures that the state decays exponentially according to (4.3-44). However, the restriction (4.3-53) on the initial condition guarantees that at the end of the interval (say at time T)

\[ \| W_{T, \beta} x \|_B \leq \| W_{t_0, \beta} \phi \|_B \quad (4.3-57) \]

Repeating this process, it follows that \(|\dot{y}(t)| \leq \varepsilon\) for all time which in turn implies (4.3-54).

In words, Theorem 4.3-6 states that if the quasi-linear dynamics (4.3-42) are stable for all frozen-\( y \) (which is the case for gain scheduling) then (4.3-42) is locally exponentially stable. It is stressed that the local nature of the stability is not due to the linearization. Rather, it is because of the time-\( \text{variations} \) in the scheduling variable, \( y \). Thus, the local exponential stability approaches global stability as the relative rate of time-\( \text{variations} \) decreases (\( L_y \rightarrow 0 \)) or the size of allowable time-\( \text{variations} \) increases (\( \varepsilon \rightarrow 0 \)).

The robust stability and robust performance of Fig. 4.3-2 is now addressed. First, note that the closed loop equations (4.3-38) may be decomposed into (1) locally stable quasi-linear dynamics, (2) a nonlinear residual \( \delta F \), (3) a nonlinear functional perturbation (\( g x \)), and (4) an exogenous input \( d \). Similarly to the previous sections, the method of proof is to construct a Lyapunov function based on the stability of (4.3-42) and to use this Lyapunov function to show guaranteed stability margins.
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Step 1 Construction of Lyapunov Functional

Let $s(t; \phi, t_0)$ denote the solution to (4.3-42) under initial conditions (4.3-43).

Assumption 4.3-10 Let $m$ and $\rho$ be as in (4.3-53)-(4.3-54). There exists a constant $L$ such that

$$
\| \mathbf{W}_{t,\beta} \mathbf{x} \|_B, \| \mathbf{W}_{t,\beta} \mathbf{x}' \|_B \leq m\rho, \quad \mathbf{x}, \mathbf{x}' \in B_e
$$

implies

$$
\left\| \left\{ \mathbf{A}(y(t)) \mathbf{x}(t) + \int_0^t \mathbf{B}(y(t)) \Delta(t - \tau) \mathbf{C}(y(\tau)) \mathbf{x}(\tau) d\tau \right\} \right. - \\
\left. \left\{ \mathbf{A}(y'(t)) \mathbf{x}'(t) + \int_0^t \mathbf{B}(y'(t)) \Delta(t - \tau) \mathbf{C}(y'(\tau)) \mathbf{x}'(\tau) d\tau \right\} \right\|_B \leq L \| \mathbf{W}_{t,\beta} (\mathbf{x} - \mathbf{x}') \|_B
$$

Theorem 4.3-7 Consider the quasi-linear dynamics (4.3-42) under Assumption 4.3-10. Suppose that (4.3-42) is locally exponentially stable and satisfies (4.3-53)-(4.3-54). Under these conditions, given any $\gamma \in (0, 1)$, there exists a functional $V : B_e \times \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies

$$
\| \mathbf{W}_{t,\beta} \mathbf{x} \|_B \leq V(\mathbf{x}, t) \leq m \| \mathbf{W}_{t,\beta} \mathbf{x} \|_B, \quad \| \mathbf{W}_{t,\beta} \mathbf{x}' \|_B \leq \rho
$$

$$
| V(\mathbf{x}, t) - V(\mathbf{x}', t) | \leq L_V \| \mathbf{W}_{t,\beta} (\mathbf{x} - \mathbf{x}') \|_B, \quad \| \mathbf{W}_{t,\beta} \mathbf{x} \|_B, \| \mathbf{W}_{t,\beta} \mathbf{x} \|_B \leq \rho
$$
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where

\[ L_V = e^{(\gamma \lambda + L)T} \]  \hspace{1cm} (4.3-62)

\[ T = \frac{\ln m}{(1 - \gamma)\lambda} \]  \hspace{1cm} (4.3-63)

Furthermore, let \( \tilde{V}_{(4.3-42)} \) denote \( V \) evaluated along trajectories of (4.2-19), i.e.

\[ \tilde{V}_{(4.3-42)}(t) \equiv V(s(t; \phi, t_0), t), \quad t \geq t_0 \]  \hspace{1cm} (4.3-64)

Then \( V \) satisfies

\[ \mathbf{D}^T \tilde{V}_{(4.3-42)}(t) \leq -\gamma \lambda \| \mathbf{W}_{t, \beta} s(\cdot; \phi, t_0) \|_B, \quad t \geq t_0, \| \mathbf{W}_{t, \beta} \phi \|_B \]  \hspace{1cm} (4.3-65)

Proof Define

\[ V(x, t) = \sup_{\tau \geq t} \left\{ e^{\gamma \lambda (\tau - t)} \| \mathbf{W}_{\tau, \beta} s(\cdot; \mathbf{P}_x, t) \|_B \right\} \]  \hspace{1cm} (4.3-66)

The bounds (4.3-60) and (4.3-65) follow immediately from local exponential stability (4.3-54) and methods used in Theorem 4.2-2. The Lipschitz condition (4.3-61) follows from a combination of methods used in Theorem 4.2-2 and Theorem 2.3-2. Namely, exponential stability implies that the supremum in (4.3-66) may be taken over the finite interval \( \tau \in [t, t + T] \) where \( T \) is defined in (4.3-63). Using the Bellman-Gronwall inequality with Assumption 4.3-10 to bound the RHS of

\[ | V(x, t) - V(x', t) | \leq \sup_{\tau \geq t} \left\{ e^{\gamma \lambda (\tau - t)} \| \mathbf{W}_{\tau, \beta} (s(\cdot; \mathbf{P}_x, t) - s(\cdot; \mathbf{P}_{x'}, t)) \|_B \right\} \]  \hspace{1cm} (4.3-67)

yields the desired result.

\[ \square \]

Step 2 Proof of Robust Stability and Robust Performance

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Let \( s'(t; \phi, t_0) \) denote the solution to (4.3-38) with initial condition \((\phi, t_0)\).

**Theorem 4.3-8** Consider the gain scheduled dynamics (4.3-38) of Fig. 4.3-2 under Assumption 4.3-7 - 4.3-10. Let \( V \) be defined as in Theorem 4.3-7. Furthermore, let \( \eta \) and \( \gamma' \) satisfy

\[
\eta \equiv \gamma \lambda - L_V k_g > 0 \tag{4.3-68}
\]

\[
\frac{n}{L_V k_{DF}} \gamma' < \rho, \quad \gamma' \in (0, 1) \tag{4.3-69}
\]

(Note that (4.3-68) imposes a requirement on the dynamics, namely that \( \eta > 0 \), hence it may not be satisfied in general.) Under these conditions

\[
\| \mathbf{w}_{t_0, \beta} \|_B \leq \frac{n}{mL_V k_{DF}} \gamma' \tag{4.3-70}
\]

and

\[
\| d \|_{L_\infty} \leq \frac{n^2}{L_V k_{DF}} (1 - \gamma') \gamma' \tag{4.3-71}
\]

together imply

\[
| s'(t; \phi, t_0) | \leq \frac{n}{L_V k_{DF}} \gamma', \quad t \geq t_0 \tag{4.3-72}
\]

**Proof** Let \( \tilde{V}_{(4.3-38)} \) denote \( V \) evaluated along \( s'(t; \phi, t_0) \). Following the proof of Theorem 4.3-3, it can be shown that

\[
\mathbf{D} \tilde{V}_{(4.3-38)}(t) \leq -\gamma \lambda \tilde{V}_{(4.3-38)}(t) + L_V k_g \tilde{V}(t) + L_V k_{DF} \tilde{V}_{(4.3-38)}^2(t) + L_V \| d \|_{L_\infty} \tag{4.3-73}
\]
\[ \leq - \eta \tilde{V}_{(4.3-38)}(t) + L_{V} k_{6F} \tilde{V}_{(4.3-38)}^{2}(t) + L_{V} \| d \|_{L_{\infty}} \]  

(4.3-74)

The remainder of the proof follows from the same arguments of the proof of Theorem 4.2-3 and Theorem 4.3-4.

Thus, it has been shown that the gain scheduled system of Fig. 4.3-2 has guaranteed robust stability and robust performance properties under the following conditions. First of all are the familiar (and now quantitative) heuristic guidelines

(1) Slow variations in the scheduling variable (Theorem 4.3-6).

(2) Small nonlinearities not captured by the scheduling variable (i.e. small \( k_{6F} \)).

In addition to these restrictions, one requires that

\[ \eta \equiv \gamma \lambda - L_{V} k_{g} > 0 \]  

(4.3-75)

In words, this means that the degree of exponential stability must be large enough to overcome the nonlinear functional perturbation, \((gx)\), in (4.3-38). In terms of the original definition of \((gx)\),

\[ (gx)(t) = \begin{bmatrix} B_{y} \\ B_{z} - Dz_{eq}(y) B_{y} \end{bmatrix} \int_{0}^{t} \Delta_{u}(t - \tau) u_{eq}(y(\tau)) d\tau \]  

(4.3-76)

this perturbation is a product of the equilibrium control exciting the unmodeled actuator dynamics. The restriction due to the presence of (4.3-76) is typical of such forms of "precompensation". For example a gain scheduled design, such as Fig. 4.3-1, relies on the accuracy of the linearization through \( u_{eq} \). Thus, it is implicitly assumed that unmodeled dynamics occur at the compensator output - which is the case in the frozen-\( y \) designs (4.3-42).
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In terms of Fig. 4.3-3, the loop properties broken at (a) the compensator output and (b) the plant input have different properties. That is, it is possible that non-destabilizing unmodeled dynamics at breaking point (a) destabilize the feedback system when placed at breaking point (b). Note that by augmenting integrators at the plant input, one eliminates the equilibrium control feedback of Fig. 4.3-1 \( (u_{eq} = 0) \). This is discussed further in Chapter 5.

![Diagram of Loop Properties](image)

**Figure 4.3-3 Loop Breaking Points with Different Properties**

Finally, another restriction imposed by Theorem 4.3-8 is the familiar small-signal condition

\[
\|d\|_{L_{\infty}} \leq \frac{n^2}{L_Y^2 k_{\delta F}} (1 - \gamma') \gamma' 
\]

(4.3-77)

Recall that \( d \) is defined as

\[
d(t) = \int_0^t \begin{bmatrix} 0 \\ 0 \\ B_k(y(t)) \end{bmatrix} (I - \Delta_p)^{-1} (t - \tau) r(\tau) \, d\tau 
\]

(4.3-78)

where \( \Delta_p \) is an artificial uncertainty which represents a performance specification. In terms of the linear control designs, stability in the presence of \( \Delta_p \) is equivalent to
\[
|\hat{\Delta}_p(j\omega)\hat{S}(j\omega)| < 1, \quad \forall \omega
\]  
(4.3-79)

where \( S \) is the standard unity feedback sensitivity function (e.g. [55]). In general, \( \hat{\Delta}_p(j\omega) \) can be large in the regions where good command following is expected. Thus, the small-signal condition (4.3-77) states that the reference command should not have significant frequency content in the regions where \( (I - \hat{\Delta}_p(j\omega))^{-1} \) is large - or equivalently poor command following is expected. Note that since \( \Delta_p = 0 \) is feasible, (4.3-77) also translates into a small-signal condition on \( r \) itself.

Before closing this section, it is noted that this analysis did not address the possible presence of unmodeled dynamics at the plant output. This is because including such unmodeled dynamics would result in feedback equations which become overly algebraically cumbersome. However, it is easy to show (using the Lyapunov function established in Section 4.3.2) that the gain scheduled feedback system of Fig. 4.3-1 does maintain stability in the presence of possible plant output disturbances (Fig. 4.3-4), i.e. the slightest disturbance will not destabilize the system. This stability margin is important since such a disturbance alters both the equilibrium control and the value of the scheduling variable used in the compensator. As would be expected, the margins of stability are proportional to the degree frozen-\( y \) stability and inversely proportional to the time-variations in \( y \) and the size of the non-output nonlinearities (cf. (4.3-31)). For the sake of completeness, the following theorem formalizes the nature of the output-disturbance stability margin. No proof is given since it does not add any insights into the gain scheduled designs beyond those already obtained in this section and Section 4.3.2.

**Theorem 4.3-9** Consider the feedback system of Fig. 4.3-4. Suppose that a gain scheduled design procedure outlined in Section 4.3.2 has been carried out. Under these conditions, the feedback system of Fig. 4.3-4 is small-signal finite gain stable from \((r, d)\) to the states of the
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plant and compensator.

Figure 4.3-4 Scheduling in the Presence of a Plant Output Disturbance

4.4 Concluding Remarks

This chapter has presented a formal analysis of two types of nonlinear gain scheduled control systems, (1) scheduling on a reference trajectory and (2) scheduling on the plant output. In both cases, conditions were given which guarantee that certain stability, robustness, and performance properties of the frozen operating condition designs carry over to the overall gain scheduled design.

The main results may be summarized as follows. In the case of scheduling on a reference trajectory, given that the feedback system (4.2-18) is stable for all frozen values of time, the robust stability and robust performance is maintained provided that (1) the reference trajectory varies slowly and (2) the reference trajectory does not excite unmodelled dynamics. In the case of scheduling on the plant output (4.3-38), conditions were given which essentially verify and formalize the two common gain scheduling guidelines of "scheduling on a slow variable" and "capturing the plant's nonlinearities." That is, in the limiting cases where the rate of the output
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time-variations approaches zero and the non-output nonlinearities approach zero, then the feedback properties of the gain scheduled designs approach those of the frozen-time designs.

Although the systems analyzed were all of particular unity-feedback configurations, the tools developed are applicable to general feedback systems whose dynamics may be put into the quasi-linear Volterra integrodifferential equations (4.2-18) or (4.3-38). In fact, it is this need for a quasi-linear form of the closed-loop dynamics which restricted the analysis to scheduling on a state-variable which is explicitly a plant output (as in (4.3-1)).

In this setting of stability theory for VIDE's, the two main methods of analysis which were generalized from their ordinary differential equation counterparts were Lyapunov Stability / Exponential Stability equivalence [8] and small-signal finite-gain stability [8, 58].

The main limitations of these results are as follows. Firstly, verification of the theorems dealing with infinite-dimensional unmodeled dynamics requires hard-to-obtain information on the uncertainties, as discussed in Chapter 3. Secondly, even if the sufficient conditions are verified, they are apt to be conservative, which is typical of Lyapunov analyses of nonlinear systems. However, the conservatism of the stability conditions may be viewed as a reminder of the origins of the designs. Namely, the original gain scheduled designs were based on linear time-invariant approximations to the nonlinear plant.

In spite of these limitations, they do add valuable insight beyond the original heuristic guidelines. For example in Section 4.2, the danger of reference trajectories exciting unmodeled dynamics was revealed in a quantitative manner. Furthermore, the theorems are useful in that they help to identify various parameters which in turn improve the feedback properties of the gain scheduled design. For example in the case of scheduling on the plant output, one can now say in a quantitative manner what variables one should try to use as scheduling variables. Thus, although the sufficient conditions may not be explicitly verified, one can still use them to gain new insights for design purposes.

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Chapter 5
Design of Gain Scheduled Control Systems

5.1 Introduction

This chapter presents an alternate framework for gain scheduled control design for both linear parameter-scheduled systems and nonlinear output-scheduled systems. Essentially, this framework states that gain scheduling should conform explicitly to the fundamental philosophy behind feedback control: (1) stable inversion of undesirable dynamics of the plant and (2) replacement with desirable dynamics, which will be called a "target loop". Such a procedure of control design will be called "loop shaping".

Standard gain scheduling captures this philosophy in only a limited manner. To see this, consider a unity feedback configuration of a parameter-varying gain scheduled system (Fig. 5.1-1). The standard gain scheduled design approach is to design a parameter-varying compensator $K(\theta)$ such that for each frozen value of the parameters, the resulting feedback system has excellent properties. More fundamentally, the process of inversion and replacement is performed for each frozen parameter value. Thus, the resulting $P(\theta)K(\theta)$ approximately
equals some desired target loop dynamics, say $L_d(\theta)$, for each fixed $\theta$. However, as parameter time-variations are introduced, then

1. The target loop dynamics, $L_d(\theta)$, as a time-varying system no longer may be desirable.
2. The inversion of $P(\theta)$ as a time-varying system is no longer accomplished by $K(\theta)$.
3. The undesirable dynamics of $P(\theta)$ may no longer be invertible in a stable manner.

![Figure 5.1-1 Scheduling on an Exogenous Parameter in a Unity Feedback Configuration](image)

These statements simply reiterate that nominal stability, robust stability, and robust performance may be lost in the presence of parameter time-variations. However from Chapter 3, one can say that if the time-variations are sufficiently slow, then the desired feedback properties are maintained. Said differently, if the parameter time-variations are sufficiently slow then

1. The target loop dynamics, $L_d(\theta)$, are still satisfactory.
2. The inversion of $P(\theta)$ is approximately accomplished by $K(\theta)$.
3. The undesirable dynamics of $P(\theta)$ retain their stable invertibility.

These three statements imply that in case the parameter time-variations are sufficiently slow, standard gain scheduling implicitly conforms to loop shaping. To remedy this shortcoming of slow variations, it is necessary to recognize the following:
(1) \( L_\theta(\theta) \) should be chosen to have guaranteed stability and robustness properties as a time-varying system as well as for fixed values of \( \theta \).

(2) \( P(\theta) \) is a time-varying system, hence any stable inversion of its dynamics must be done in a time-varying manner.

In this chapter, a gain scheduling framework is presented which addresses these points. First, some specific candidates for time-varying target loops are presented. These candidates are such that they have guaranteed stability and robustness properties as time-varying systems without recourse to a slow time-variation analysis; hence, they allow arbitrarily fast time-variations. Second, stable inversion of undesirable dynamics of the plant is discussed. Through a example, it is shown that this inversion must be viewed in a time-varying framework since time-variations can ruin stable invertibility of undesirable dynamics. This philosophy of time-varying loop shaping can be viewed as a special case of the Loop Operator Recovery method of [28] for nonlinear systems which generalized the LQG/L.TR design methodology [20, 55] for linear time-invariant systems.

Finally, this design framework is coupled with the analysis of nonlinear gain scheduled systems to present a new design procedure which guarantees global nominal stability for a class of nonlinear plants.

Organization of Chapter

Section 5.2 discusses gain scheduled control design for linear parameter-varying plants. In Section 5.2.1, two candidates for target loop dynamics are presented: (1) the time-varying integrator and (2) the time-varying Kalman filter. It is shown that both of these systems allow arbitrarily fast time-variations and have robustness properties which can be analyzed without recourse to a slow time-variation analysis. Section 5.2.2 discusses inverting undesirable plant dynamics in a time-varying setting. Depending on the selection of the target loop, it is shown that one can recover the desired dynamics if the appropriate portions of the plant undesirable
dynamics can be inverted in a stable manner. Section 5.2.3 contains simulations which demonstrate both stages of loop shaping. In Section 5.3, the nonlinear analysis of Chapter 4 is coupled with the design framework of Chapter 5.2 to present a gain scheduling design procedure for nonlinear plants. Section 5.2.1 shows how a class of nonlinear plants may be put in the form of quasi-linear parameter-varying plants, where the parameter is in fact the plant output. Section 5.3.2 then discusses the parallel issues of selecting target dynamics and replacing undesirable dynamics. These methods are demonstrated by numerical example in Section 5.3.3. Finally, concluding remarks are given in Section 5.4.

5.2 Scheduling on an Exogenous Parameter

This section discusses the design of gain scheduled control systems for linear parameter-varying plants in a unity feedback configuration (Fig. 5.1-1) via loop shaping. Let the plant to be controlled \( P(\theta) \) have the following state-space realization

\[
\dot{x}(t) = A(\theta(t)) x(t) + B(\theta(t)) u(t), \quad x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m
\]  

\[y(t) = C(\theta(t)) x(t), \quad y(t) \in \mathbb{R}^m\]  

where the dynamics matrix \( A \) is a continuous function of a scalar (for ease of discussion) parameter which satisfies

\[
\theta_{\text{min}} \leq \theta(t) \leq \theta_{\text{max}}, \quad |\dot{\theta}(t)| \leq \alpha
\]  

The meaning of "plant" used here is not only a model of the physical process but also any augmented dynamics added for design purposes (e.g. integrator dynamics).

As applied to Fig. 5.1-1, loop shaping consists of finding time-varying target loop dynamics, \( L_d(\theta) \), and performing a stable inversion in order to recover these dynamics. Of course the closed loop system must maintain the desired feedback properties for any admissible
parameter trajectory (5.2-3).

5.2.1 Selection of Target Loop Dynamics

Consider the block diagram of Fig. 5.2-1. This shows a unity feedback configuration where the forward loop \( L_d(\theta) \) represents desirable target loop dynamics. As mentioned in the introduction, standard gain scheduled designs are such that \( L_d(\theta) \) has excellent feedback properties for admissible frozen values of the parameter. However, in Chapter 3 it was shown that a major restriction of such gain scheduled designs is that of slow parameter time-variations. More precisely, since the designs were based on frozen values of the parameter, one can give guarantees on the resulting time-varying stability, robustness, and performance only if the parameter time-variations are slow. In terms of loop shaping, such a loss of the desired feedback properties means that the target loop \( L_d(\theta) \) is no longer desirable as a time-varying system.

![Figure 5.2-1 Target Loop Dynamics](image)

This section discusses the issue of selection of target loop dynamics which are desirable for arbitrary admissible parameter trajectories (5.1-3) as well as for constant parameter trajectories as in the case of standard gain scheduling. As a prerequisite, the target loop dynamics should
have the property that stability and robustness of the resulting time-varying system can be analyzed without recourse to a slow time-variation analysis. This can be accomplished by (1) selecting dynamics which are simple in structure, hence easy to analyze, or (2) selecting dynamics whose structure has inherent stability and robustness properties. Furthermore, the target loop dynamics should have enough flexibility to satisfy frozen-parameter specifications which vary with the parameter (e.g., the desired closed loop bandwidth varies with $\theta$).

In view of these restrictions, two possible target loop candidates are proposed which individually satisfy the stated prerequisites. They are the time-varying integrator and the time varying Kalman filter.

**Definition 5.2-1 The Time-Varying Integrator**

The *Time Varying Integrator* is defined as the dynamical system $e \rightarrow y$ given by

\[
\dot{z}(t) = \Gamma(\theta(t)) e(t), \quad z(t) \in \mathbb{R}^m
\]

\[
y(t) = z(t)
\]

(5.2-4)

(5.2-5)

where $\Gamma$ is a diagonal matrix with continuous elements $\gamma_i$ which satisfy

\[
\gamma_i(\theta) > 1, \ \forall \ \theta, \ \forall \ i
\]

(5.2-6)

In words, the time-varying integrator is simply a diagonal block of integrators with a time-varying gain in each channel (Fig. 5.2-2). Choosing the time-varying integrator as target loop dynamics $L_d(\theta)$ in Fig. 5.2-1 immediately leads to the following advantages:

2. Flexibility in satisfying frozen-parameter performance specifications which vary with the parameter. For example, in each channel the designer can let the bandwidth vary with $\theta$. 

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(3) Because of the simple structure, calculation of input/output properties of the closed loop system for robustness calculations is simplified considerably.

\[
\Gamma(\theta)
\]

\[
\begin{bmatrix}
\begin{array}{c}
\hat{d}_1(\theta) \\
\hat{d}_2(\theta) \\
\vdots \\
\hat{d}_m(\theta)
\end{array}
\end{bmatrix}
\]

\[
\begin{array}{c}
\mathbf{r} \\
\mathbf{e}
\end{array}
\rightarrow
\int
\rightarrow
\mathbf{y}
\]

**Figure 5.2-2** The Time-Varying Integrator

To illustrate this last point, consider the block diagram of Fig. 5.2-3. The block \( \Delta \) represents unmodeled dynamics which can be any I/O operator

\[
\Delta : \mathcal{L}_{\infty} \rightarrow \mathcal{L}_{\infty}
\]  

(5.2-7)

which satisfies the finite-gain stability criterion

\[
\| \Delta \| \leq \eta < 1
\]  

(5.2-8)

Note that \( \Delta \) may be any infinite-dimensional time-varying nonlinear system, as long as it satisfies (5.2-7) - (5.2-8). The following theorem states Fig. 5.2-3 with \( \mathbf{L}_d(\theta) \) a time-varying integrator maintains stability for all \( \Delta \) satisfying (5.2-7)-(5.2-8).
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**Figure 5.2-3** Robustness Analysis of the Target Loop

**Theorem 5.2-1** Consider the feedback system of Fig. 5.2-3, where \( L_d(\theta) \) denotes the time-varying integrator of Definition 5.2-1. Furthermore, let \( \Delta \) be any I/O operator which satisfies (5.2-7)-(5.2-8). Under these conditions, the feedback system of Fig. 5.2-3 is I/O stable for arbitrary admissible parameter trajectories.

**Proof** Using block diagram manipulations, Fig. 5.2-3 may be transformed to the feedback system of Fig. 5.2-4.

**Figure 5.2-4** Alternate Diagram for Target Loop Robustness Analysis

From the small gain theorem, it follows that Fig. 5.2-4 is I/O stable if

\[
\| L_d(\theta) (I + L_d(\theta))^{-1} \| \leq 1
\]  

(5.2-9)
where the above norm is the operator norm of

\[ L_d(\theta) (I + L_d(\theta))^{-1} : L_\infty \to L_\infty \]  
\hspace{1cm} (5.2-10)

In order to show (5.2-9), consider an element of the diagonal operator in (5.2-10)

\[ \dot{w}_i(t) = -\gamma_i(\theta(t))w_i(t) + \gamma_i(\theta(t))v_i(t) \]  
\hspace{1cm} (5.2-11)

A standard result from linear systems theory is that the norm of the mapping \(v_i \mapsto w_i\) as an operator on \(L_\infty\) is given by (dropping the \(i\) subscript)

\[ \lim_{t \to \infty} \int_0^t e^{-\gamma(\theta(\xi))} d\xi \]  
\hspace{1cm} (5.2-12)

From the closed-loop stability of the time-varying integrator, the limit in (5.2-12) is guaranteed to exist. Furthermore, (5.2-12) is precisely the steady-state solution of the differential equation

\[ \dot{w}_i(t) = -\gamma_i(\theta(t))(w_i(t) - 1) \]  
\hspace{1cm} (5.2-13)

In order for the limit of (5.2-12) to coincide with the equilibrium of (5.2-13), it follows that

\[ \lim_{t \to \infty} \int_0^t e^{-\gamma(\theta(\xi))} d\xi \]  
\hspace{1cm} (5.2-14)

Repeating this process for each diagonal element of \(L_d(\theta) (I + L_d(\theta))^{-1}\) proves condition (5.2-9) which, via the small gain theorem, completes the proof.

To summarize, it has been shown that the time-varying integrator has guaranteed nominal stability for arbitrary time-variations, is flexible enough to meet design specifications which vary with \(\theta\), and has inherent robustness properties without recourse to a slow-variation analysis. In this sense, the time-varying integrator satisfies the necessary requirements for target loop dynamics in a loop shaping framework.
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Another candidate for a time-varying target loop which satisfies the requisite target loop properties is the time-varying Kalman filter. First, some definitions for time-varying linear systems are given.

Definition 5.2-2[34] Observability and Controllability

Consider the time-varying linear system

\[
\dot{x}(t) = A(t) x(t) + B(t) u(t) \quad (5.2-15)
\]

\[
y(t) = C(t) x(t) \quad (5.2-16)
\]

Let \( \Phi(t, \tau) \) denote the state-transition matrix corresponding to (5.2-15). The matrix pair [A, C] is said to be **uniformly completely observable** if there exist constants \( \kappa_1, \kappa_2, \) and \( h > 0 \) such that the **observability grammian**

\[
W(t_0, t_1) \equiv \int_{t_0}^{t_1} \Phi^T(\tau, t_1) C^T(\tau) C(\tau) \Phi(\tau, t_1) d\tau \quad (5.2-17)
\]

satisfies

\[
\kappa_1 < \lambda_{\min}(W(t_0, t_0 + h)) < \lambda_{\max}(W(t_0, t_0 + h)) < \kappa_2, \quad \forall t_0 \in \mathbb{R}^+ \quad (5.2-18)
\]

The matrix pair [A, B] is said to be **uniformly completely controllable** if there exist constants \( \kappa_1, \kappa_2, \) and \( h > 0 \) such that the **controllability grammian**

\[
C(t_0, t_1) \equiv \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) B^T(\tau) \Phi^T(t_1, \tau) d\tau \quad (5.2-19)
\]

satisfies

\[
\kappa_1 < \lambda_{\min}(C(t_0, t_0 + h)) < \lambda_{\max}(C(t_0, t_0 + h)) < \kappa_2, \quad \forall t_0 \in \mathbb{R}^+ \quad (5.2-20)
\]
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Assumption 5.2-1 The parameter-varying plant, \( P(\theta) \), given by (5.2-1)-(5.2-3) is uniformly completely controllable and uniformly completely observable for all admissible parameter trajectories.

The time-varying Kalman Filter Loop is now defined.

Definition 5.2-3 The Time-Varying Kalman Filter Loop

Consider the parameter varying plant, \( P(\theta) \), given by (5.2-1)-(5.2-3) under Assumption 5.2-1. The Time Varying Kalman Filter Loop (Fig. 5.2-5) is defined as the dynamical system \( e \mapsto y \) given by

\[
\dot{z}(t) = A(\theta(t)) z(t) + H_{\theta}(t) e(t), \quad z(t) \in \mathbb{R}^n, \quad e(t) \in \mathbb{R}^m
\]  
(5.2-21)

\[
y(t) = C(\theta(t)) z(t), \quad y(t) \in \mathbb{R}^m
\]  
(5.2-22)

\[
H_{\theta}(t) = \frac{1}{\mu(\theta(t))} \Sigma(t) C^T(\theta(t))
\]  
(5.2-23)

\[
\dot{\Sigma}(t) = A(\theta(t)) \Sigma(t) + \Sigma(t) A^T(\theta(t)) + N(\theta(t))N^T(\theta(t)) - \frac{1}{\mu(\theta(t))} \Sigma(t) C^T(\theta(t))C(\theta(t)) \Sigma(t)
\]  
(5.2-24)

\[
\Sigma(0) = \Sigma_0 = \Sigma_0^T \in \mathbb{R}^{n \times n}, \quad \Sigma_0 > 0
\]  
(5.2-25)

where the design parameters, \( N \) and \( \mu \), are continuous functions.
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\[ \mu : [\theta_{\text{min}}, \theta_{\text{max}}] \to (0, \infty), \quad N : [\theta_{\text{min}}, \theta_{\text{max}}] \to \mathbb{R}^{n \times m} \]  

(5.2-26)

with the pair \([A, N]\) is uniformly completely controllable for all admissible parameter trajectories.

![Diagram of the Time-Varying Kalman Filter Loop](image)

**Figure 5.2-5** The Time-Varying Kalman Filter Loop

**Remark** The notation \(H_{\theta(t)}\) is used to express the implicit dependence of \(H\) on the parameter \(\theta\). That is, \(H\) depends on the parameter trajectory rather than the current parameter value.

The time-invariant Kalman filter previously has been recognized as a target loop with excellent feedback properties for linear time-invariant designs [4, 55]. From (5.2-24), it is seen that the time-varying Kalman filter is simply a collection of time-invariant Kalman filter target loops, parameterized by \(N(\theta)\) and \(\mu(\theta)\), which evolve according to the matrix Riccati differential equation (5.2-24).

Using the time-varying Kalman filter as a target loop in Fig. 5.2-1 leads to the following advantages. First of all as a frozen-parameter target loop, the Kalman filter has sufficient...
flexibility to meet frozen-parameter specifications which vary with the parameter. Furthermore, the time-varying Kalman filter has the following guaranteed stability and robustness properties:

**Theorem 5.2-2 Nominal Stability for Arbitrarily Fast Time-Variations**

Consider the feedback system of Fig. 5.2-1 where \( L_d(\theta) \) denotes the time-varying Kalman filter of Definition 5.2-3. Furthermore, let Assumption 5.2-1 hold. Under these conditions, the closed-loop system of Fig. 5.2-1 is exponentially stable for arbitrary admissible parameter variations.

**Proof** The closed-loop dynamics under investigation are given by

\[
\dot{x}(t) = (A(\theta(t)) - H_\theta(t) C(\theta(t))) x(t), \quad x(0) = x_0
\]

(5.2-27)

It is shown that

\[
V(x, t) = x^T \Sigma^{-1}(t) x
\]

(5.2-28)

is a quadratic Lyapunov function for (5.2-27) which then implies exponential stability [8].

From Assumption 5.2-1 and by definition of the time-varying Kalman filter, the matrix pairs \([A, C]\) and \([A, N]\) are uniformly completely observable and controllable, respectively. Let \( h \) be the corresponding positive constant in (5.2-18) and (5.2-20). Under these conditions, it can be shown that [10]

\[
\left( C^{-1}(t, t-h) + W(t, t-h) \right)^{-1} \leq \Sigma(t) \leq \left( W^{-1}(t, t-h) + C(t, t-h) \right), \quad t > h
\]

(5.2-29)

for all \( \Sigma_0 \). Now define \( V \) as in (5.2-28). From (5.2-29) it follows that there exist constants \( \alpha_1 \) and \( \alpha_2 > 0 \) such that

\[
\alpha_1 |x|^2 \leq V(x, t) \leq \alpha_2 |x|^2, \quad \forall x \in \mathbb{R}^n, \quad t > h
\]

(5.2-30)

Taking the time-derivative of \( V \) along trajectories of (5.2-27) and using (5.2-24), it can be
shown that for \( t > h \)

\[
\frac{d}{dt} \tilde{V}_{(5.2-27)}(t) = -x^T(t) \left( \frac{1}{\mu(\theta(t))} C^T(\theta(t)) C(\theta(t)) + \Sigma^{-1}(t) N(\theta(t)) N^T(\theta(t)) \Sigma^{-1}(t) \right) x(t)
\]  

(5.2-31)

Substituting (5.2-31) into

\[
\tilde{V}(t) = V(h) + \int_h^t \frac{d}{d\tau} \tilde{V}_{(5.2-27)}(\tau) d\tau
\]  

(5.2-32)

along with uniform complete observability (5.2-18), it follows that

\[
\tilde{V}(t) \leq V(h) - \kappa_1 |x(t)|^2 , \quad t > h
\]  

(5.2-33)

Exponential stability then follows from (5.2-30) and the Bellman-Gronwall inequality.

\[
\]

**Theorem 5.2-3 Guaranteed Robustness Properties**

Consider the feedback system of Fig. 5.2-3 where \( L_\Delta(\theta) \) denotes the time-varying Kalman filter of Definition 5.2-3. Let Assumption 5.2-1 hold. Furthermore, let \( \Delta \) be any I/O operator which satisfies

\[
\Delta : L_2 \rightarrow L_2 \quad (5.2-34)
\]

\[
\| \Delta \| \leq \eta < 1/2 \quad (5.2-35)
\]

Under these conditions, the feedback system of Fig. 5.2-3 is I/O stable for arbitrary admissible parameter trajectories.

**Proof** As before, Fig. 5.2-3 may be transformed to the feedback system of 5.2-4. From the
small-gain Theorem, it follows that Fig. 5.2-3 is I/O stable if
\[ \|L_d(\theta)(I + L_d(\theta))^{-1}\| \leq 2 \]  
(5.2-36)

where the above norm is the operator norm of
\[ L_d(\theta)(I + L_d(\theta))^{-1} : L_2 \rightarrow L_2 \]  
(5.2-37)

However, one of the guaranteed properties of the time-varying Kalman filter [28] is
\[ \| (I + L_d(\theta))^{-1} \| \leq 1 \]  
(5.2-38)

Condition (5.2-36) then follows using
\[ L_d(\theta)(I + L_d(\theta))^{-1} + (I + L_d(\theta))^{-1} = I \]  
(5.2-39)

To summarize, two possible candidates for target loops in a loop shaping framework for gain scheduled designs have been presented, the time-varying integrator and the time-varying Kalman filter. First, both candidates have sufficient flexibility to meet various specifications which vary with the parameter. Furthermore, unlike standard gain scheduling, it was shown that these target-loops have guaranteed stability and robustness properties for arbitrarily fast time-varying parameter trajectories as well as frozen parameter trajectories.

### 5.2.2 Plant Inversion and Recovery of Target Loop Dynamics

The second stage of loop shaping involves inverting the undesirable dynamics of the plant and replacing them with desirable dynamics, i.e. the target loop. Following [20, 28, 55] this process of inverting undesirable plant dynamics will be call "target loop recovery".

One obvious restriction of target loop recovery is that it must lead to a stable closed-loop system. That is, there are portions of the plant's dynamics which can never be inverted while leading to a stable closed loop system. Rather, the designer must cope with any such dynamics
in the best manner possible. For example from classical control, it is well known that any unstable pole/zero cancellations between the plant and compensator lead to an unstable closed-loop system. Thus, unstable poles and zeros of the plant cannot be removed; hence, they impose a fundamental limitation on the achievable performance of the closed loop system [24].

In this section, it is shown how one can perform target loop recovery of the time-varying integrator and time-varying Kalman filter. In light of the previous discussion, such recovery will only be possible under certain invertibility restrictions on the plant. It should be mentioned here that the results presented in this section are specialized cases of the more general methods of Loop Operator Recovery found in [28]. Furthermore, since the purpose of this chapter is to emphasize and demonstrate the fundamental control philosophy of loop shaping (i.e. inversion and replacement) as applied to gain scheduling, the results of [28] are not presented in their fullest generality.

Recovery of Time-Varying Integrator Dynamics via Formal Loop Shaping

Formal Loop Shaping as presented in [28] is a method to completely invert and replace all plant dynamics. Figure 5.2-6 shows the structure of the compensator in a Formal Loop Shaping feedback configuration specialized for time-varying integrator target-loop dynamics. Note the presence of a new element, \( G_\theta(t) \), in the compensator. The following theorem states how to select \( G_\theta(t) \) in this specialized presentation of Formal Loop Shaping to achieve target loop recovery. Again, the notation \( G_\theta(t) \) reflects that \( G \) depends on the parameter trajectory and not necessarily the current parameter value.
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Theorem 5.2-4[28] Consider the block diagram of Fig. 5.2-6. Here $P(\theta)$ denotes the plant as defined in (5.2-1)-(5.2-2) and $L_\theta(\theta)$ denotes the time-varying integrator as in Definition 5.2-1. Suppose that along any admissible parameter trajectory the matrix $G_\theta(t)$ satisfies

$$
\lim_{\rho \to 0} \sqrt{\rho} \ G_\theta(t) = V_\theta(t) \ C(\theta(t))
$$

(5.2-40)

for some invertible $V_\theta(t)$. Under these conditions, for any $p \in [1, \infty]$

$$
\lim_{\rho \to 0} \| P_T ( P(\theta) K(\theta) v - L_\theta(\theta) v ) \|_{L_p} = 0, \ \forall v \in L_{pe}, T \geq 0
$$

(5.2-41)

In words, Theorem 5.2-4 essentially states that the forward loop in Fig. 5.2-6 asymptotically approaches (non-uniformly) the time-varying integrator target loop. However, this does not imply that the closed loop system of Fig. 5.2-6 is stable. For example in the classical control case, such a limiting behavior may be due to an approximate unstable pole/zero cancellation. However, Theorem 5.2-5 leads to conditions for stable recovery of the target-loop.

Theorem 5.2-5[2] Consider the feedback system of Fig. 5.2-7, where $P$ and $Q$ are I/O operators on $L_p$. If

(i) $P$ is finite-gain incrementally stable, and

(ii) $Q$ is finite-gain stable

then the closed loop system is finite-gain stable.
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**Proof** Writing the feedback equations of Fig. 5.2-7,

\[ u = Q(r - P(w + u) + Pu) \]  
(5.2-42)

Then for all \( T \in \mathbb{R}^+ \),

\[ \| P_Tu \|_{L_p} \leq \| Q \| \{ \| P_Tr \| + \| P \| \delta \| P_Tw \|_{L_p} \} \]  
(5.2-43)

and

\[ \| P_Ty \|_{L_p} \leq \| P \| \delta \{ \| P_Tu \| + \| P_Tw \|_{L_p} \} \]  
(5.2-44)

\[ \square \]

![Diagram](image)

**Figure 5.2-7** Diagram for Stability Analysis of Formal Loop Shaping

Applying Theorem 5.2-5 to recovery of the target integrator loop, one has that the recovery procedure of Theorem 5.2-4 leads to a stable feedback system under the following conditions:

(i) The plant \( P(\theta) \) is stable for all admissible parameter trajectories. Note that linearity of \( P(\theta) \) allows one to relax the original requirement of incremental stability.

(ii) \( Q \) stable translates into the requirement that the time-varying system

\[ \dot{s}(i) = (A(\theta(i)) - B(\theta(i)) G_\theta(i)) s(i) \]  
(5.2-45)

is exponentially stable for all admissible parameter trajectories (which must be satisfied in...
addition to (5.2-40)).

These conditions are discussed further at the end of this section.

Recovery of Time-Varying Kalman Filter Dynamics via the Model Based Compensator

The Model Based Compensator (MBC) shown in Fig. 5.2-8 is simply a state estimation / state feedback compensator (e.g. [38]) adapted for the purposes of command following. The design parameters in this compensator are the two matrices $H_{\theta}(t)$ and $G_{\theta}(t)$. In the context of recovery of time-varying Kalman filter loop dynamics, $H_{\theta}(t)$ denotes the Kalman filter gain matrix (5.2-23), and $G_{\theta}(t)$ is a new element which is used to achieve recovery. The following theorem states how to use the MBC to achieve target loop recovery.

![Diagram](image)

**Figure 5.2-8** Recovery of Time-Varying Kalman Filter via Model Based Compensator

**Theorem 5.2-6[28]** Consider the block diagram of Fig. 5.2-8. Here $P(\theta)$ denotes the plant as defined in (5.2-1)-(5.2-2). Let $L_{\Delta}(\theta)$ denote the time-varying integrator as in
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Definition 5.2-3 with filter gain $H_\theta(t)$. Suppose that along any admissible parameter trajectory the matrix $G_\theta(t)$ satisfies

$$\lim_{\rho \to 0} \sqrt{\rho} \ G_\theta(t) = V_\theta(t) \ C(\theta(t))$$

(5.2-46)

for some invertible $V_\theta(t)$. Under these conditions, for any $p \in [1, \infty]$

$$\lim_{\rho \to 0} \| P_T (P(\theta)K(\theta) v - L_d(\theta) v) \|_{L_p} = 0, \ \forall \ v \in L_{pe}, T \geq 0$$

(5.2-47)

Thus, it is seen that the forward loop in Fig. 5.2-8 asymptotically approaches (non-uniformly) the time-varying Kalman filter target loop. As in the case of recovery of the time-varying integrator, this does not imply that the closed loop system of Fig. 5.2-8 is stable. The following theorem gives conditions for stable recovery.

Theorem 5.2-7 Consider the feedback system of Fig. 5.2-8 under the recovery procedure of Theorem 5.2-6. Under these conditions, Fig. 5.2-8 is stable if and only if

$$\dot{s}(t) = (A(\theta(t)) - B(\theta(t)) \ G_\theta(t)) s(t)$$

(5.2-48)

is exponentially stable for all admissible parameter trajectories.

Proof Let $x_k$ denote the states of the compensator in Fig. 5.2-8. The (unforced) feedback equations then take the form

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ x_k(t) \end{bmatrix} = \begin{bmatrix} A(\theta(t)) & -B(\theta(t))G_\theta(t) \\ -H_\theta(t)C(\theta(t)) & A(\theta(t)) - B(\theta(t))G_\theta(t) - H_\theta(t)C(\theta(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ x_k(t) \end{bmatrix}$$

(5.2-49)

which can be transformed to
\[
\frac{d}{dt} \begin{bmatrix} x(t) \\ x(t) - x_k(t) \end{bmatrix} = \begin{bmatrix} A(\theta(t)) - B(\theta(t))G_\theta(t) & B(\theta(t))G_\theta(t) \\ 0 & A(\theta(t)) - H_\theta(t)C(\theta(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ x(t) - x_k(t) \end{bmatrix}
\tag{5.2-50}
\]

In the form (5.2-50), it is easy to see that the stability of Fig. 5.2-8 is completely determined by the stability of

\[
A(\theta(t)) - H_\theta(t)C(\theta(t)) \tag{5.2-51}
\]

\[
A(\theta(t)) - B(\theta(t))G_\theta(t) \tag{5.2-52}
\]

In fact, this is the acclaimed separation principle of state estimation / state feedback control structures (e.g. [28, 37, 48]), and is a reminder of the origins of the MBC.

Now the dynamics matrix (5.2-51) is precisely the closed loop dynamics of the target Kalman filter loop, which via Theorem 5.2-2, is known to be stable. Thus, stability of Fig. 5.2-8 rests with (5.2-52).

\[
\text{Discussion of Target Loop Recovery}
\]

This section has presented the process of target loop recovery for (1) the time-varying integrator via a Formal Loop Shaping structure (Fig. 5.2-6) and (2) the time-varying Kalman filter via a MBC structure (Fig. 5.2-8). In both cases, target loop recovery was guaranteed provided that one can find a matrix \( G_\theta \) such that for all admissible parameter trajectories

\[
\lim_{\rho \to 0} \sqrt[\rho]{G_\theta(t)} = V_\theta(t)C(\theta(t)) \tag{5.2-53}
\]

Although (5.2-53) guarantees that the forward loop \( P(\theta)K(\theta) \) asymptotically approaches the target loop \( L_d(\theta) \), in general it does not guarantee that the closed loop system will be stable during the recovery process. Thus, additional contraints were imposed in order to guarantee stability.

In the case of the time-varying integrator, stable recovery is guaranteed provided that
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(1) \( P(\theta) \) is stable for all admissible parameter trajectories.

(2) The time-varying system

\[
\dot{s}(t) = (A(\theta(t)) - B(\theta(t)) \, G_\theta(t)) \, s(t)
\]

is exponentially stable for all admissible parameter trajectories.

In the case of the time-varying Kalman filter, condition (5.2-54) alone is sufficient to guarantee stable recovery.

Both conditions of a stable plant and (5.2-54) can be given important interpretations in terms of inversion of unstable plant dynamics. Suppose that one has found a matrix \( G_\theta \) which satisfies (5.2-53) for all admissible parameter trajectories. This means that as \( \rho \) diminishes, the forward loop \( P(\theta)K(\theta) \) approaches the target loop \( L_d(\theta) \) (Fig. 5.2-9).

**Figure 5.2-9 Asymptotic Target Loop Recovery**

In the case of the time-varying Kalman filter, this means that portions of the plant dynamics are being inverted in favor for those of the target Kalman filter. However, an inspection of Fig. 5.2-5 reveals that the dynamics matrix of the Kalman filter target loop is \textit{identical} to the dynamics matrix of the plant, namely \( A(\theta(t)) \). Informally, the inverted portion of the plant dynamics does not contain the possibly unstable dynamics from \( A(\theta(t)) \). In the case of a
constant parameter trajectory, this means that the compensator only inverts the zeros of the plant. However from classical control theory, unstable zeros of the plant cannot be inverted in a closed-loop stable manner. Assuming that an analogous restriction exists for time-varying systems, the stability of (5.2-54) can be viewed as a restriction on the "zeros" of the plant. This is in fact the case for linear time-invariant systems, as formalized in the following theorem.

**Theorem 5.2-8[37]** Consider the time-invariant linear system

\[
\dot{x}(t) = A \, x(t) + B \, u(t), \quad x \in \mathbb{R}^n, \, u \in \mathbb{R}^m
\]  

(5.2-55)

\[
y(t) = C \, x(t), \quad y \in \mathbb{R}^m
\]  

(5.2-56)

with transfer function matrix

\[
P(s) = C \, (sI - A)^{-1} \, B
\]  

(5.2-57)

where it is assumed that the matrix pairs \([A, B]\) and \([A, C]\) are controllable and observable, respectively. Let \(K\) be the solution to the algebraic Riccati equation

\[
0 = -ZA - ATZ - CT^T C + \frac{1}{\rho} ZBB^T Z, \quad \rho > 0
\]  

(5.2-58)

and let \(G\) be given by

\[
G = \frac{1}{\rho} B^T Z
\]  

(5.2-59)

Then

\[
\dot{x}(t) = (A - BG)x(t)
\]  

(5.2-60)

is a stable system. Furthermore,

\[
limit_{\rho \to 0} \sqrt{\rho} \, G = VC, \quad V^T V = I
\]  

(5.2-61)
if and only if all of the zeros of $P(s)$ have strictly negative real parts.

Theorem 5.2-8 states that for linear time-invariant systems, one can find a matrix $G$ which simultaneously satisfies the stability condition (5.2-60) and the asymptotic condition (5.2-61) if the plant zeros are minimum phase. In this sense, the analogous time-varying restrictions (5.2-53) and (5.2-54) may be viewed as a minimum-phasedness, or more appropriately, a stable invertibility condition on the plant. In fact, since a matrix $G_{\theta}$ which simultaneously satisfies (5.2-53) and (5.2-54) implies a stable recovery process, the existence of such a matrix has been proposed as a definition of minimum-phasedness [28]. If one were to accept this definition, then testing the stable invertibility of a time-varying system is just as difficult as testing the stability of a time-varying system since stability of (5.2-54) is a requirement.

Recall that an additional restriction was imposed on the plant dynamics in the case of recovery of the time-varying integrator. Namely, it was required that the plant $P(\theta)$ is stable for all admissible parameter trajectories. In light of the previous discussion, this restriction can be explained as follows. While the target Kalman filter loop has the same dynamics matrix $A(\theta)$ as the plant, the time-varying integrators do not. Hence, loop recovery via the Formal Loop Shaping structure of Fig. 5.2-6 attempts to invert all of the plant dynamics. Since only stable dynamic cancellations are allowed, the plant dynamics should be stable in order to guarantee a stable closed-loop system.

5.2.3 Simulation Examples

In this section, two numerical simulations are presented which demonstrate the inadequacy of standard gain scheduling and show how such inadequacies are resolved via a time-varying loop shaping approach.
Example 5.2-1 The Time-Varying Oscillator

The purpose of this example is to demonstrate the importance of selecting target loop dynamics which have desirable properties in the presence of parameter time-variations.

The parameter-varying plant, \( P(\theta) \), to be controlled has two poorly damped eigenvalues which oscillate along the imaginary axis. Let \( P(\theta) \) have the following state-space representation

\[
\frac{d}{dt} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -0.5\theta \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{5.2-62}
\]

\[
y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \tag{5.2-63}
\]

where

\[-1 \leq \theta(t) \leq 1, \quad |\dot{\theta}(t)| \leq 2 \tag{5.2-64}\]

The eigenvalues as a function of \( \theta \) are

\[
\lambda_{1,2} = -0.1 \pm j \sqrt{0.99 + 0.5\theta} \tag{5.2-65}
\]

which implies that \( P(\theta) \) is stable for all frozen parameter values.

A standard gain scheduled design for \( P(\theta) \) is now presented. First, to improve command following at low frequencies, an integrator is augmented at the plant input. This augmented plant, \( P_{aug}(\theta) \), will be the object of the compensator design of \( K(\theta) \). In the actual implementation, the augmented integrator will actually be part of the overall compensator \( K_{aug}(\theta) \) (Fig. 5.2-10). The augmented plant \( P_{aug}(\theta) \) has the following state-space realization (suppressing the explicit dependence on time)
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\[
\frac{d}{dt} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -0.5\theta & -0.2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} v
\]

(5.2-66)

\[
y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ u \end{bmatrix}
\]

(5.2-67)

which will be denoted as

\[
\frac{d}{dt} x_{\text{aug}}(t) = A(\theta(t)) x_{\text{aug}}(t) + B_{\text{aug}} v(t)
\]

(5.2-68)

\[
y(t) = C_{\text{aug}} x_{\text{aug}}(t)
\]

(5.2-69)

\[
\begin{array}{c}
\text{Figure 5.2-10 Augmenting Integrator Dynamics}
\end{array}
\]

Let \( x_k \) denote the state of the compensator \( K(\theta) \) (Fig. 5.2-10). The frozen parameter compensators for the augmented plant are LQG/LTR designs given by the MBC
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\[
\frac{d}{dt} x_k(t) = \left( A_{aug}(\theta(t)) - B_{aug} G(\theta(t)) - H(\theta(t)) C_{aug} \right) x_k(t) - H(\theta(t)) (r(t) - y(t))
\]  \(\text{(5.2-70)}\)

\[
v(t) = - G(\theta(t)) x_k(t)
\]  \(\text{(5.2-71)}\)

where the matrices \(H\) and \(G\) are given by

\[
H(\theta) = \frac{1}{\mu(\theta)} \Sigma(\theta) C_{aug}^T
\]  \(\text{(5.2-72)}\)

\[
G(\theta) = \frac{1}{\rho} B_{aug}^T Z(\theta)
\]  \(\text{(5.2-73)}\)

with \(\Sigma\) and \(Z\) the solutions to the algebraic Riccati equations

\[
0 = \Sigma A_{aug}^T(\theta) + A_{aug}^T(\theta) \Sigma + N(\theta) N^T(\theta) - \frac{1}{\mu(\theta)} \Sigma(\theta) C_{aug}^T C_{aug} \Sigma(\theta)
\]  \(\text{(5.2-74)}\)

\[
0 = - Z(\theta) A_{aug}(\theta) - A_{aug}^T(\theta) Z(\theta) - C_{aug}^T C + \frac{1}{\rho} Z(\theta) B_{aug} B_{aug}^T Z(\theta)
\]  \(\text{(5.2-75)}\)

and the design parameters chosen as

\[
N(\theta) = \begin{bmatrix} 1 \\ 0 \\ 2.0 + 1.5\theta \end{bmatrix}, \quad \mu(\theta) = 10^{-2}, \quad \rho = 10^{-8}
\]  \(\text{(5.2-76)}\)

For frozen parameter values, this selection of design parameters leads to a forward loop frequency response \(P_{aug}(\theta) K(\theta)(j\omega)\) which is independent of \(\theta\) (Fig. 5.2-11). This is achieved by selecting \(N(\theta)\) such that the resulting compensator zeros cancel the lightly-damped plant poles. From the frequency response of Fig. 5.2-11, it is clear that the frozen-parameter designs have good feedback properties.
Figure 5.2-11 Frequency Response of Frozen-Parameter Designs
Since the frozen parameter designs are the product of an LQG/LTR procedure, one has that the mapping \( e \mapsto y \) given by
\[
\dot{z}(t) = A_{\text{aug}}(\theta(t)) z(t) + H(\theta(t)) e(t)
\]
was explicitly chosen as a target loop (as in Fig. 5.2-1) for each frozen value of \( \theta \). However, since LQG/LTR results in a MBC structure, one has that (5.2-78) is in fact a target loop for \textit{time-varying} parameter trajectories (Theorem 5.2-6). Furthermore, since the gains \( G(\theta) \) were chosen using the algebraic Riccati equation (5.2-75), from Theorem 5.2-8 it follows that the the asymptotic property (5.2-46) is achieved (i.e. \( \rho^{1/2} G(\theta) \to VC \)) and recovery takes place.

The following phenomena has taken place. A collection of good frozen-parameter designs has implicitly selected a parameter-varying target loop which may or may not have good feedback properties. In order to evaluate the properties of the implicitly chosen target-loop, consider applying a unit step to the closed loop system with the implicitly chosen target loop (5.2-77)-(5.2-78) in the forward loop (as in Fig. 5.2-1). Choosing a parameter trajectory as
\[
\theta(t) = \cos 2t
\]
(5.2-79)
yields a closed-loop step response shown in Fig. 5.2-12 which is unstable. Since target-loop recovery is occurring, it is not surprising that the actual closed-loop step response with the compensator as in (5.2-70)-(5.2-71) is also unstable (Fig. 5.2-13). It should be noted that the selection of (5.2-79) as a parameter-trajectory was not coincidental. In fact, this trajectory makes the frozen-time stable plant (5.2-62) unstable.

This instability demonstrates the need to explicitly chose target loops which have good feedback properties in the presence of admissible parameter-variations. It is now shown how to use time-varying loop shaping to achieve this goal.
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Figure 5.2-12 Closed-Loop Step Response of Implicitely Selected Target Loop

Figure 5.2-13 Closed-Loop Step Response of Recovered Implicit Target Loop
(1) Selection of Target Loop Dynamics

The target loop dynamics will be those of the time-varying Kalman filter loop of Definition 5.2-3, where N and μ are chosen as in (5.2-76). Note that in case of a constant parameter trajectory, this leads to the same target loop as the standard gain scheduled designs. Thus, selecting the time-varying Kalman filter as a target-loop presents no sacrifice in terms of the constant parameter designs. The difference is that as parameter time-variations are introduced, the time-varying Kalman filter loop maintains nominal stability and a guaranteed degree of robustness. This is evident from Fig. 5.2-14 which shows the closed-loop step response of the time-varying Kalman filter loop using the same destabilizing parameter trajectory (5.2-79).

(2) Recovery of Target Loop Dynamics

Following Theorem 5.2-6, a MBC structure is used to recover the Kalman filter target loop. Recall that recovery is achieved if the matrix \( G_\theta \) in Fig. 5.2-6 satisfies the asymptotic property

\[
\lim_{\rho \to 0} \sqrt{\rho} \ G_\theta(t) = V_\theta(t) \ C_{aug} \tag{5.2-80}
\]

Choosing \( G(\theta) \) as in the Riccati equation (5.2-73) and (5.2-75) and defining

\[
G_\theta(t) = G(\theta(t)) \tag{5.2-81}
\]

leads to target loop recovery via Theorem 5.2-8. This is demonstrated in Fig. 5.2-15 which shows the resulting closed-loop step response using the MBC of Theorem 5.2-8.

(3) Stability of the Recovery Process

Using Theorem 5.2-7, one can guarantee a closed-loop stable recovery process via the stability of the time-varying system

\[
\dot{s}(t) = ( A_{aug}(\theta(t)) - B_{aug} \ G(\theta(t)) ) \ s(t) \tag{5.2-82}
\]


**Figure 5.2-14** Closed-Loop Step Response of Time-Varying Kalman Filter Target Loop

**Figure 5.2-15** Closed-Loop Step Response of Recovered Time-Varying Kalman Filter Loop
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As discussed in Chapter 3, checking the stability of a time-varying system is a highly nontrivial matter. It turns out that in this particular example, the stability of (5.2-82) is easy to prove. First, note that for fixed values of $\theta$, the augmented plant (5.2-66)-(5.2-67) has no finite zeros. Using optimal root-locus arguments [5] and the definition of $G(\theta)$, it can be shown that as $\rho$ approaches 0 in (5.2-73) and (5.2-75)

$$\text{Re} \lambda_i[A_{aug}(\theta) - B_{aug} G(\theta)] \to -\infty$$

(5.2-83)

Thus, the rate of exponential decay of the frozen parameter systems of (5.2-82) becomes large. Using results from Section 3.2, it then follows that (5.2-82) is exponentially stable.

Example 5.2-2 Plant with Time-Varying Oscillatory Zeros

The purpose of this example is to demonstrate the importance of recognizing stable invertibility for time-varying systems.

The parameter varying plant to be controlled has two poorly damped zeros which oscillate along the imaginary axis. Let the plant $P(\theta)$ have the following state-space representation, denoted $\{A(\theta), B, C\}$,

$$\frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & (2 - \theta(t))^2 & 1 + 0.5\theta + (2 - \theta(t))^2 \\ 1 & 0 & 0.2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

(5.2-84)

$$y(t) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

(5.2-85)

where

$$-1 \leq \theta(t) \leq 1, \quad |\dot{\theta}(t)| \leq 2$$

(5.2-86)
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The transfer function for frozen values of \( \theta \) is

\[
\frac{s^2 + 0.2s + (1 + 0.5\theta)}{s(s - (2 - \theta))(s + (2 - \theta))}
\]  

(5.2-87)

whose zeros are given by

\[-0.1 \pm j \sqrt{0.99 + 0.5\theta}\]  

(5.2-88)

Note that for frozen values of \( \theta \), the plant zeros are minimum phase; hence the plant has a stable inverse.

The time-varying loop shaping design for \( P(\theta) \) is now presented. Essentially, an LQG/LTR design is done for each fixed value of \( \theta \) (Fig. 5.2-16). As in Example 5.2-1, the dynamics then evolve according to the time-varying Kalman filter.

(1) Selection of Target Loop Dynamics

The target loop is chosen to be the time-varying Kalman filter loop of Definition 5.2-3, where the design parameters \( N \) and \( \mu \) for selected values of \( \theta \) are given in Table 5.2-1. In between these values, \( N \) is obtained by linear interpolation. Figure 5.2-17 shows a closed-loop step response of the time-varying Kalman filter along the parameter trajectory

\[ \theta(t) = \cos 2t \]  

(5.2-89)

<table>
<thead>
<tr>
<th>Table 5.2-1 Design Parameters of Kalman Filter Target Loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta )</td>
</tr>
<tr>
<td>( N )</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>( \mu )</td>
</tr>
</tbody>
</table>
Figure 5.2-16 Typical Frequency Response of Frozen Parameter Designs ($\theta = 0$)
(2) Recovery of Target Loop Dynamics

Following Theorem 5.2-6, a MBC structure is used to recover the Kalman filter target loop. In order to satisfy the asymptotic property, the matrix $G_\theta$ is chosen to be an explicit function of $\theta$ (i.e. $G_\theta(t) = G(\theta(t))$) given by

$$G(\theta) = \frac{1}{\rho} B^T Z(\theta)$$  \hspace{1cm} (5.2-91)

$$0 = -Z(\theta) A(\theta) - A^T(\theta) Z(\theta) - C^T C + \frac{1}{\rho} Z(\theta) B B^T Z(\theta), \quad \rho = 10^{-8}$$  \hspace{1cm} (5.2-92)

Combining Theorem 5.2-6 and Theorem 5.2-8, it follows that such a choice of $G_\theta(t)$ achieves recovery, as demonstrated in Fig. 5.2-18.

(3) Stability of Recovery Process

Figure 5.2-18 demonstrates that the input/output behavior of the target Kalman filter loop has been recovered. Now from (5.2-85), the plant output, $y$, is the sum of the states $x_2$ and $x_3$. Upon examining the time-responses of these states (Fig. 5.2-19), it becomes evident that the recovery process is unstable. Since the time-varying Kalman filter target loop is guaranteed to be nominally stable (Theorem 5.2-2), this instability is not due to a poor choice of target loop dynamics.

It turns out that along the parameter-trajectory (5.2-89), the time-varying plant $P(\theta)$ cannot be inverted in a stable manner. This happens even though the plant is minimum phase for all frozen parameter values. Intuitively, this can be explained as follows. Recall that for frozen parameter values, the plant has two lightly-damped zeros. The values of these zeros (5.2-88) are precisely the values of the frozen-parameter eigenvalues in Example 5.2-1. Now upon inverting $P(\theta)$, these frozen-parameter zeros become poorly damped frozen parameter poles. Since the parameter trajectory (5.2-89) causes the dynamics of Example 5.2-1 (5.2-61) to become unstable, it follows intuitively that this trajectory also causes the plant to have an unstable inverse.

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Figure 5.2-17 Closed-Loop Step Response Time-Varying Kalman Filter Target Loop

Figure 5.2-18 Closed-Loop Step Response of Recovered Time-Varying Kalman Filter Loop
Figure 5.2-19 Hidden Instability of Target Loop Recovery

The unstable invertibility of $P(\theta)$ can be seen more clearly as follows. Let the plant dynamics (5.2-84)-(5.2-85) be denoted by

$$\dot{x}(t) = A(\theta(t)) x(t) + B \ u(t)$$  \hspace{1cm} (5.2-93)

$$y(t) = C \ x(t)$$  \hspace{1cm} (5.2-94)

Then using standard techniques of state-space inversion [50], one can construct the following inverse system $\tilde{P}^{-1}(\theta)$

$$\dot{w}(t) = (I - B(CB)^{-1}C) \ A(\theta(t)) \ w(t) + B(CB)^{-1} y(t)$$  \hspace{1cm} (5.2-95)

$$u(t) = -(CB)^{-1}CA(\theta(t)) \ v(t) + (CB)^{-1} \dot{y}(t)$$  \hspace{1cm} (5.2-96)
Note that $\tilde{P}^{-1}(\theta)$ is not exactly the inverse of the plant $P(\theta)$ since the input is the derivative of $y$. However, an approximate $P^{-1}(\theta)$ can be constructed using $\tilde{P}^{-1}(\theta)$ as shown in Fig. 5.2-21.

As simulation of the approximate plant inversion in Fig. 5.2-21 was performed using
\[ y^*(t) = \sin(0.5t), \quad \tau = .01 \]  \hspace{1cm} (5.2-97)
with the parameter trajectory as in (5.2-89). Figure 5.2-22 shows that an approxiamte inversion is indeed taking place. Upon examination of the output of the approximate inverse system (Fig. 5.2-23), it becomes clear that along the selected parameter trajectory (5.2-89), the plant $P(\theta)$ has an unstable inverse. This unstable invertibility is in constrast to the constant-parameter stable invertibility of the plant.

In terms of Fig. 5.2-20, the following phenomena has occured. An unbounded input, $u$, to the (unstable!) plant produces a bounded plant output, $y$. Note that this is precisely the phenomena which occurs in linear time-invariant non-minimum phase systems.
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Figure 5.2-21 Error of Approximate Inversion

Figure 5.2-22 Instability of $P^{-1}(\theta)$ for Time-Varying Parameter Trajectory

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5.3 Scheduling on the Plant Output

5.3.1 Problem Formulation

In this section, the analysis of nonlinear output scheduling in Section 4.3 is combined with the loop shaping ideas of Section 5.2 to give a new design framework for nonlinear gain scheduling.

The plant dynamics under consideration are

\[
\frac{d}{dt} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = f(y(t)) + A(y(t)) \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} + B u(t) \tag{5.3-1}
\]

where

\[
y(t) \in \mathbb{R}^m, \ z(t) \in \mathbb{R}^{n-m}, \ u(t) \in \mathbb{R}^m,
\]

Note that in (5.3-1), all plant nonlinearities are contained in the output, \( y \). Thus, (5.3-1) is not as general as the dynamics considered in Section 4.3.

The following assumptions are made on (5.3-1).

**Assumption 5.3-1** \( f : \mathbb{R}^m \to \mathbb{R}^n \) satisfies

\[
f(0) = 0. \tag{5.3-3}
\]

**Assumption 5.3-2** There exists a unique continuously differentiable function

\[
z_{eq} : \mathbb{R}^m \to \mathbb{R}^{n-m} \tag{5.3-4}
\]
which satisfies

$$0 = f(y) + A(y) \begin{bmatrix} y \\ z_{eq}(y) \end{bmatrix}$$

(5.3-5)

Assumption 5.3-2 states that one has a family of equilibrium conditions parameterized by the output, \( y \). In constrast to Section 4.3, these equilibrium conditions are achieved only through the state. That is, there is no equilibrium control value.

This condition of no equilibrium control can always be achieved by augmenting integrators to the plant dynamics as in Example 5.2-1. For example, suppose that to achieve equilibrium, one requires some non-zero function \( u_{eq} : \mathbb{R}^m \rightarrow \mathbb{R}^{n-m} \) such that

$$0 = f(y) + A(y) \begin{bmatrix} y \\ z_{eq}(y) \end{bmatrix} + B \ u_{eq}(y)$$

(5.3-6)

Then by augmenting integrators at the plant input,

$$u = \int v$$

(5.3-7)

one obtains plant dynamics of the form

$$\frac{d}{dt} \begin{bmatrix} y(t) \\ z(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} f(y(t)) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} A(y(t)) & B & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} v(t)$$

(5.3-8)

which clearly satisfy Assumption 5.3-2 (after obvious reselection of variables).

The advantages of no equilibrium control feedback are clear. As pointed out in Section 4.3, the presence of such feedback violates the implicit assumption of gain scheduling that unmodeled actuator dynamics occur at the compensator output rather than the plant input (Fig. 4.3-3). Similarly in terms of loop shaping, such feedback also destroys recovering the desired
properties of the target loop. This can be seen in Fig. 5.3-1. If one were to use a loop shaping framework for control design, then the resulting forward loop would have good feedback properties, such as robustness to unmodelled dynamics, at breaking point \((a)\). However, actual unmodeled dynamics actually occur at the plant output \((b)\). If there were no equilibrium control feedback, then breaking the loop at \((a)\) or \((b)\) are equivalent. Finally, as an informal testimony against feeding back the equilibrium control, it fails the intuitive test of "what if the plant were linear?" That is, such equilibrium control feedback is not done in linear time-invariant designs and hence it is not a natural extension of any linear time-invariant concepts.

![Diagram](image)

**Figure 5.3-1** Loss of Target Loop Properties due to Precompensation

Proceeding with the problem formulation, using Assumptions 5.3-1 and 5.3-2 the system dynamics (5.3-1) may be put in the form

\[
\frac{d}{dt} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = A(y(t)) \begin{bmatrix} y(t) \\ z(t) - z_{eq}(y(t)) \end{bmatrix} + B u(t) \tag{5.3-9}
\]

Let the functions \(f\), \(A\), and \(B\) of (5.3-1) be partitioned as
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\[
f(y) = \begin{bmatrix} f_y(y) \\ f_z(y) \end{bmatrix}, \quad A(y) = \begin{bmatrix} A_{yy}(y) & A_{yz}(y) \\ A_{zy}(y) & A_{zz}(y) \end{bmatrix}, \quad B = \begin{bmatrix} B_y \\ B_z \end{bmatrix}
\]

(5.3-10)

Then the feedback equations (5.3-9) may be put in the form (suppressing explicit dependence on time)

\[
\frac{d}{dt} \begin{bmatrix} y \\ z - z_{eq}(y) \end{bmatrix} = \begin{bmatrix} A_{yy}(y) & A_{yz}(y) \\ A_{zy}(y) & A_{zz}(y) \end{bmatrix} \begin{bmatrix} y \\ z - z_{eq}(y) \end{bmatrix} + \begin{bmatrix} B_y \\ B_z - D z_{eq}(y) B_y \end{bmatrix} u
\]

(5.3-11)

\[
y = \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} y \\ z - z_{eq}(y) \end{bmatrix}
\]

(5.3-12)

which takes the form

\[
\dot{x}(t) = A_{des}(y(t)) x(t) + B_{des}(y(t)) u(t)
\]

(5.3-13)

\[
y(t) = C_{des} x(t)
\]

(5.3-14)

where \( x, A_{des}, B_{des}, \) and \( C_{des} \) are defined accordingly.

The original dynamics of (5.3-1) are now in the form of a "quasi-linear parameter-varying system," where the parameter is actually part of the state-variable. In Section 5.2, it was shown how to design controllers for of parameter-varying linear systems such that one maintains the desired feedback properties in the presence of arbitrarily fast parameter-variations. Since (5.3-12) takes the form of a parameter-varying linear system, one can also design a controller for (5.3-1) which guarantees global nominal stability and robust stability.
5.3.2 Linear Parameter-Varying Analogies

In this section, it is shown informally how the parameter-varying concepts of target loop selection and target loop recovery carry over to plants of the quasi-linear form of (5.3-13). Essentially, the ideas and concepts follow exactly from their parameter-varying counterparts. For this reason, the discussion here concentrates on the differences encountered in addressing the quasi-linear form (5.3-13).

Selection of Target Loop Dynamics

Since the "parameter" in the quasi-linear form (5.3-13) is the plant output, the difference in the target loop dynamics is that it now depends on a state-variable (Fig. 5.3-2) rather than an exogenous parameter (Fig. 5.2-1).

![Figure 5.3-2 Output-Varying Target Loop](image)

Thus, simply replacing $\theta$ by $y$ in Section 5.2 naturally leads to the following:

1. A definition of an "output-varying" integrator with the same stability and robustness properties (Fig. 5.3-3).

2. A definition of a "output-varying" Kalman filter loop with the same stability and
robustness properties (Fig. 5.3-4).

**Figure 5.3-3** Output-Varying Loop Integrator with Nondestabilizing Unmodeled Dynamics

**Figure 5.3-4** Output-Varying Kalman Filter Loop with Nondestabilizing Unmodeled Dynamics

**Recovery of Target Loop Dynamics**

Most of the differences between the quasi-linear case (5.3-13) and the true parameter-varying case of Section 5.2 are in recovery of the target-loop dynamics.
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(1) Recovery of Output-Varying Integrator via Formal Loop Shaping: Consider replacing \( \theta \) with \( y \) in the formal loop shaping structure of Fig. 5.2-6. Furthermore, let \( G_y(t) \) denote the analogous gain matrix. Then, it can be shown [28] that the output-varying integrator is asymptotically recovered provided that

\[
\lim_{\rho \to 0} \sqrt{\rho} \ G_y(t) = V_y(t) \ C_{des}, \quad V_y(t) \text{ invertible} \tag{5.3-15}
\]

For example, this asymptotic property can be achieved (Theorem 5.2-8) by using an output-index Riccati structure

\[
G(y) = \frac{1}{\rho} B_{des}^T Z(y) \tag{5.3-16}
\]

\[
0 = -Z(y) A_{des}(y) - A_{des}^T(y) Z(y) - C_{des}^T C_{des} + \frac{1}{\rho} Z(y) B_{des}(y) B_{des}^T(y) Z(y) \tag{5.3-17}
\]

(2) Recovery of Output-Varying Kalman Filter Loop via MBC Structure: As before, consider replacing \( \theta \) with \( y \) in the MBC structure of Fig. 5.2-8. Furthermore, let \( G_y(t) \) denote the analogous gain matrix. It turns out that in this case, the asymptotic property (5.3-15) no longer guarantees target loop recovery [28].

This lack of recovery can be explained as follows. Given the MBC structure, Theorem 5.2-6 gives conditions such that along any given parameter trajectory, the recovery process of Fig. 5.2-9 occurs. However, this requires that the parameter-varying forward loop, \( P(\theta)K(\theta) \), and the parameter-varying target loop, \( L_d(\theta) \), evolve along the same parameter trajectory. In the quasi-linear setup, the resulting "parameter trajectory", or more appropriately "output trajectory," is a function of the forward loop (e.g. Fig. 5.3-1); i.e. the parameter is not exogenous. Thus at each step of the recovery process, the forward loop is evolving along a different output trajectory and recovery need not occur.

This can be seen more explicitly as follows. Let \( P(y) \) denote the plant (5.3-12) evolving along any given output trajectory. Thus, one has that

\[
y = P(y) u, \quad \forall u \tag{5.3-18}
\]
Similarly, let $K(y)$ and $L_d(y)$ denote the MBC compensator and the Kalman filter target loop dynamics evolving along a given output trajectory. Then in order to guarantee recovery, one must show that for any signal $e$

$$y = P(y)K(y)e \rightarrow y^* \equiv L_d(y^*)e$$  \hspace{1cm} (5.3-19)

Replacing $\theta$ with $y$ in Theorem 5.2-6 only guarantees that

$$y = P(y)K(y)e \rightarrow L_d(y)e$$ \hspace{1cm} (5.3-20)

That is, the forward loop approaches the target loop evaluated along the resulting output trajectory of the forward loop.

Although target loop recovery does not occur in the strict sense, (5.3-20) does imply that as the asymptotic condition (5.3-15) progresses, the forward loop always approximates some output-varying Kalman filter loop. Thus, the guaranteed robustness properties are still present after the attempted recovery.

(3) **Stability of the Recovery Process**: Recall that in the parameter-varying case, two conditions arose in showing stability of the recovery process. These conditions translate into the following.

First of all, to guarantee stable recovery of the output-varying integrator, one of two requirements is that the plant be incrementally stable. In terms of (5.3-13), this means that the unforced dynamics

$$\dot{x}(t) = A_{des}(y(t))x(t)$$  \hspace{1cm} (5.3-21)

should be exponentially stable. Note that the linear parameter-varying analysis of Section 3.2 could be used to assess stability of (5.3-21) (cf. Theorem 4.3-2). However, such an analysis is apt to be overly restrictive. This is because viewing (5.3-21) as a parameter-varying system ignores the increased structure of an admissible parameter-trajectory actually being a state-trajectory.
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The second condition to guarantee a stable recovery of the output-varying integrator dynamics is as follows. Let $G_y$ satisfy the asymptotic condition (5.3-15). Then the nonlinear system

$$\dot{x}(t) = (A_{des}(y(t)) - B_{des}(y(t)) G_y(t)) x(t)$$

with $y$ as in (5.3-14) must be exponentially stable. This is also the condition needed for a stable recovery attempt of the output-varying Kalman filter loop. As in the case of testing incremental stability of the plant, any parameter-varying analysis of (5.3-22) is apt to be conservative. For example, recall that the stability of (5.3-22) can be given the interpretation of stable invertibility of the plant. However, it is conceivable that a parameter-varying plant $P(\theta)$ has an unstable inverse whereas the corresponding output-varying plant $P(y)$ always has a stable inverse. This is because the parameter-trajectory which caused the unstable invertibility of $P(\theta)$ can never be generated as an output of $P(y)$.

5.3.3 A Simulation Example

The purpose of this section is to demonstrate some of the ideas behind a quasi-linear approach to a nonlinear output-varying design. It is stressed that this example is demonstrative and is not meant to promote a complete design methodology. The system under consideration is a single link coupled to a rotational inertia by a flexible shaft (Fig. 5.3-5). The idea is to control the link through torques on the rotational inertia with only a measurement of the link angle. This physical system is a simplified case of a proposed model for robotic manipulators with flexible joints [52].
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Figure 5.3-5 Nonlinear Design Example

The equations of motion are given by (suppressing explicit dependence on time)

\[
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{T}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\theta_1 \\
\theta_2 \\
T
\end{bmatrix} + \begin{bmatrix}
\frac{mgL}{J_1} \sin y \\
-k \frac{k}{J_1} \\
-k \frac{k}{J_2}
\end{bmatrix} \begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{T}
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} u
\]

(5.3-23)

\[
y = \begin{bmatrix}
1 & 0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\theta_1 \\
\theta_2 \\
\dot{\theta}_1 \\
\dot{\theta}_2 \\
T
\end{bmatrix}
\]

(5.3-24)

where an integrator has been augmented at the plant input in order to avoid feedback of the
equilibrium control. The parameters of (5.3-23) are given in Table 5.3-1.

Table 5.3-1 Physical Parameters of Nonlinear Example

- $\theta_1, \theta_2 =$ angles measured from vertical (rad)
- $T =$ torque input (N-m)
- $k =$ rotational spring constant (N-m/rad) = 100
- $J_1, J_2 =$ rotational inertias (kg-m$^2$) = 1
- $m =$ link mass (kg) = 1

Note that these equations take the desired form

$$\frac{d}{dt} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = f(y(t)) + A(y(t)) \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} + B \ u(t)$$  \hspace{1cm} (5.3-25)

Furthermore, they satisfy Assumptions 5.3-1 and 5.3-2 with

$$z_{eq}(y) = \begin{bmatrix} y - \frac{mgL \sin y}{k} \\ 0 \\ 0 \\ -mgL \sin y \end{bmatrix}$$  \hspace{1cm} (5.3-26)

Following the procedure of (5.3-9) - (5.3-12), the system dynamics (5.3-23) may be put in the form
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\begin{equation}
\frac{d}{dt} \begin{bmatrix}
\theta_1 \\
\theta_2 - \theta_{2,eq}(y) \\
\dot{\theta}_1 \\
\dot{\theta}_2 \\
T - T_{eq}(y)
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & \frac{mgL \cos y}{k} & 1 & 0 \\
0 & \frac{k}{J_1} & 0 & 0 & 0 \\
0 & -\frac{k}{J_2} & 0 & 0 & 0 \\
0 & 0 & mgL \cos y & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\theta_2 - \theta_{2,eq}(y) \\
\dot{\theta}_1 \\
\dot{\theta}_2 \\
T - T_{eq}(y)
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\text{1}
\end{bmatrix} u (5.3-27)
\end{equation}

\begin{equation}
y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\theta_2 - \theta_{2,eq}(y) \\
\dot{\theta}_1 \\
\dot{\theta}_2 \\
T - T_{eq}(y)
\end{bmatrix} (5.3-28)
\end{equation}

which may be expressed as the quasi-linear

\begin{equation}
\dot{x}(t) = A_{des}(y(t)) x(t) + B_{des}(y(t)) u(t) (5.3-29)
\end{equation}

\begin{equation}
y(t) = C_{des} x(t) (5.3-30)
\end{equation}

The original plant dynamics are now in the form of applying the output-varying analogies of Section 5.3.2.

(1) Selection of Target Loop Dynamics

The target loop is chosen to be an output-varying Kalman filter, which is the dynamical system $e \mapsto y$ defined as

\begin{equation}
\dot{w}(t) = A_{des}(y(t)) w(t) + H_y(t) e(t) (5.3-31)
\end{equation}

\begin{equation}
y(t) = C_{des} w(t) (5.3-32)
\end{equation}
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\[ H_y(t) = \frac{1}{\mu(y(t))} \Sigma(t) C_{des}^T \]  

(5.3-33)

\[ \dot{\Sigma}(t) = A_{des}(y(t))\Sigma(t) + \Sigma(t)A_{des}^T(y(t)) + N(y(t))N^T(y(t)) - \frac{1}{\mu(y(t))} \Sigma(t)C_{des}^T C_{des} \Sigma(t) \]  

(5.3-34)

Table 5.3-2 shows values of the design parameters N and \( \mu \). In between these values, a linear interpolation is performed.

<table>
<thead>
<tr>
<th>( \theta_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>( N )</td>
</tr>
<tr>
<td>+ 0.01426</td>
</tr>
<tr>
<td>+ 0.044311</td>
</tr>
<tr>
<td>+ 0.388490</td>
</tr>
<tr>
<td>- 0.062159</td>
</tr>
<tr>
<td>+ 0.918510</td>
</tr>
<tr>
<td>( \mu )</td>
</tr>
<tr>
<td>( 10^{-8} )</td>
</tr>
</tbody>
</table>

(2) Attempted Recovery of Target Loop

As discussed in Section 5.3.2, the linear parameter-varying methods of recovering a target Kalman filter loop do not carry over to the output-varying case. However, it was shown that as the recovery process progresses, the forward loop approaches some Kalman filter loop and hence has good feedback properties. Thus, the linear parameter-varying recovery methods can still be used in a meaningful manner.

With this in mind, the MBC structure of Fig. 5.2-8 can be adapted to an output-varying
framework. Let \( x_k \) denote the states of the MBC. Then the output-varying MBC is given by

\[
\frac{d}{dt} x_k(t) = \left( \begin{array}{c}
A_{des}(y(t)) - B_{des}(y(t)) G(y(t)) - H_y(t) C_{des} \end{array} \right) x_k(t) - H_y(t) (r(t) - y(t)) \tag{5.3-35}
\]

\[
u(t) = - G(y(t)) x_k(t) \tag{5.3-36}
\]

with \( H_y \) as in (5.3-33). In order to guarantee the asymptotic property (5.3-15), \( G \) is chosen according to the output-indexed Riccati equations (5.3-16)-(5.3-17) with \( \rho = 10^{-14} \). This selection of \( G \) guarantees that the forward loop approximates some Kalman filter loop; hence it approaches the target loop's guaranteed feedback properties. Note that this procedure of a compensator design leads to an LQG/LTR compensator for each frozen output value. Figure 5.3-5 shows a typical frequency response of such a frozen-output design.

(3) Stability of Recovery Attempt

As mentioned in Section 5.2, the recovery process is guaranteed to be stable if the nonlinear system

\[
\dot{x}(t) = (A_{des}(y(t)) - B_{des}(y(t)) G(y(t))) x(t) \tag{5.3-37}
\]

It turns out that in this example, parameter-varying analysis of (5.3-37) is not conservative. This is because the transfer function of (5.3-29)-(5.3-30) treated as a frozen-y system is minimum phase. Thus using optimal root-locus arguments [5], it can be shown that as \( \rho \) approaches 0 in (5.3-16), the frozen-y eigenvalues of (5.3-37) satisfy

\[
\text{Re} \lambda_i \left[ A_{des}(y(t)) - B_{des}(y(t)) G(y(t)) \right] \to -\infty \tag{5.3-38}
\]

which then guarantees stability using the results of Section 3.2.

Finally, Fig. 5.3-7 shows the closed-loop response to an initial condition

\[
\theta_1 = 3, \; \theta_2, \; \dot{\theta}_1, \; \dot{\theta}_2, \; T = 0 \tag{5.3-37}
\]

and Fig. 5.3-8 shows the closed-loop response to the sinusoidal input
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Figure 5.3-6 Typical Frequency Response of Frozen-Output Designs ($\theta_1 = 0$)
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Figure 5.3-7 Closed-Loop Response to Initial Condition

Figure 5.3-8 Closed-Loop Response to Sinusoidal Command
\[ r(t) = 3 \sin t \]  
(5.3-40)

which commands the link angle throughout the range of the frozen-\(y\) designs.

### 5.4 Concluding Remarks

This chapter has presented an alternate framework for design of linear parameter-varying and nonlinear output-varying gain scheduled control systems. Essentially, it is stated that standard gain scheduling only implicitly complies to the fundamental concept of loop shaping, i.e. inversion of undesirable dynamics and replacement with desirable dynamics. For this reason, standard gain scheduling is limited to slow variations in the scheduling variable. Thus, if one is to remove this restriction, then gain scheduling should explicitly address the fundamental concepts behind loop shaping.

The formal emphasis of this chapter is on the linear parameter-varying case. It is shown that if one wishes to allow fast variations in the scheduling parameter, then a frozen-parameter design approach is inadequate in itself. In terms of loop shaping, if the parameter variations are fast then the designer must address the following.

First, one should find desirable dynamics, or target loops, which have inherent stability and robustness properties without recourse to a slow-variation analysis. Furthermore, such dynamics should have enough flexibility to satisfy frozen-parameter specifications in case the parameter-trajectory were in fact constant. Two candidates for such target loops are presented, the time-varying integrator and the time-varying Kalman filter loop. It is shown that each satisfies the aforementioned requirements of desirable dynamics. Second, if the parameter-variations are fast, then one must recognize that the plant is in fact time-varying, hence any inversion of its dynamics in order to approximate a target loop must be done in a time-varying context. These notions are simply the time-varying generalizations of unstable pole/zero
cancellations in linear time-invariant systems. Two numerical examples are presented which clearly demonstrate the need to address gain scheduling in this loop shaping context.

This chapter also addresses gain scheduled control design for a nonlinear plant scheduling on the plant output. Essentially, it is shown that a class of nonlinear plants, namely those whose nonlinearities depend entirely on the plant output, can be transformed to a quasi-linear parameter-varying plant where the parameter is the plant output. In this form, the loop shaping ideas for linear-parameter varying systems carry over to nonlinear output-varying systems in a straightforward manner. This leads to a nonlinear feedback system which has guaranteed global stability and robustness properties. In case the nonlinearities do not depend entirely on the output, then one can form a "best linear approximation" to the dynamics at the cost of losing the global stability properties.
Chapter 6
Conclusion

The main contribution of this thesis is that it has led to a formal and insightful understanding of gain scheduled control systems. Such an understanding is necessary if gain scheduling is ever to evolve into a formal and systematic design methodology.

In the analysis portion of this thesis, the fundamental question was that of guaranteed properties. That is, under what conditions does the global feedback control system maintain the excellent feedback properties of the local operating point designs. Towards this end, various conditions were given which lead to such guaranteed properties. It turns out that these conditions are simply rigorous statements of heuristic ideas which have guided successful gain scheduled designs. They are "schedule on a slow variable" and "schedule on a variable which captures the plant nonlinearities."

In retrospect, these guidelines are simply a reminder of the gain scheduling design process. That is, a global gain scheduled design is based on linear time-invariant approximations of the plant dynamics. If these approximation are inaccurate, then one cannot demand guarantees on the overall performance of the gain scheduled designs.
Unlike the heuristic guidelines, the conditions given in this thesis lead to deeper and more detailed insights into gain scheduling. These insights are maintained despite the possible conservatism and difficulty of verification of the various stability conditions. For example in the case of scheduling on an exogenous parameter, these conditions led to a precise quantification of the notion of a slow variable, the effects of selection of state-variables, and the need to satisfy frozen parameter robustness / performance tests with an increased margin. The examination of a nonlinear plant scheduling on a prescribed trajectory led the issue of reference trajectories exciting unmodeled dynamics. In the case of a scheduling on the plant output, these conditions led to a quantification of a slow output variable, a quantification of capturing the plant nonlinearities, the need to use an alternate "design model" in the design process, and the adverse affect of feeding back the equilibrium control as a form of precompensation.

It turns out that the restriction of scheduling on a slow variable can be given an alternate interpretation in the framework of fundamental feedback control. Namely, given that a gain scheduled design is based on time-invariant plants, the fundamental process of stable plant inversion and replacement of desirable dynamics can be disrupted in case the time-variations are fast.

Thus, the restriction of slow time-variations is because standard gain scheduling only implicitly addresses these concepts of inversion and replacement. If this restriction is to be removed, then the gain scheduled design process must recognize the presence of time-variations. That is, a frozen-time approach is inadequate.

In the design portion of this thesis, an alternate design philosophy which addresses the issue of fast time-variations was presented. Two simple examples revealed how fast time-variations can destroy both the selection of desirable dynamics and the stable invertibility of the plant, hence the need to address these issues explicitly. Towards this end, the concept of selection of desirable dynamics and stable inversion of undesirable dynamics while explicitly recognizing the time-varying nature of the problem was demonstrated.
Chapter 6: Conclusion

The attention is now directed towards future research. The results for parameter-varying linear systems formed the backbone of the entire thesis; e.g. both scheduling on a prescribed reference trajectory and scheduling on the plant output were put in the form of parameter-varying systems plus nonlinear perturbations. Thus, it seems that a better understanding of such systems would prove most valuable.

Further analysis of the case where parameter-variations are slow might lead to less conservative and more easily verifiable stability conditions. However, it is unlikely that any new insights into gain scheduling are to be obtained through such an analysis. In other words, scheduling on a slow parameter can be considered well understood and conceptually complete.

The more interesting case is where the parameter-variations are fast and a frozen-parameter design approach is inadequate. In this case, the importance of frozen-parameter designs is greatly reduced to a degenerate condition of the time-varying design, namely that the actual parameter-trajectory is constant. Thus, the designer loses the convenience of dealing with only time-invariant systems and must explicitly deal with feedback issues in their time-varying form.

Towards this end of understanding parameter-varying systems, research is needed in better understanding stability and the related issue of stable invertibility without recourse to a slow-variation analysis. More precisely, one needs to come up with tighter conditions for the stability of parameter-varying systems. Furthermore, it is unlikely that the classical stability analysis methods will lead to anything beyond conservative bounds, hence the need for taking nonstandard approaches. For example if the exogenous parameter is viewed as an independent input, it may be that the differential geometric tools of nonlinear reachability can be applied to give alternate stability conditions. Another approach would be to use nonlinear optimal control theory to calculate the input/output norm of a parameter-varying system, hence allowing the use of classical small-gain conditions to analyze robustness and performance.

In addition to stability and stable invertibility, the concepts of "unstable dynamics cancellations" for parameter-varying systems needs to be addressed. For example, exactly what
are the appropriate extensions of unstable pole/zero cancellations. Furthermore in the spirit of [24], what are the feedback limitations which such unstable dynamics impose. This would lead towards a more complete theoretical understanding of the nature of parameter-varying systems.

From a design perspective, research is needed in determining precisely what constitutes general desirable dynamics in a time-varying setting. That is given that a system is time-varying, then one does not have the "lead/lag" insights of classical control to determine what constitutes a desirable feedback loop. Thus, research is needed towards developing a general ability to generate classes of quality time-varying feedback systems for the purpose of target loops. For example, in the design chapter of this thesis, two candidates were proposed for their inherent stability and robustness properties. It may be that the outcomes of optimal disturbance rejection designs in the sense of [63] may lead towards this goal. In fact, a stated objective of [63] was to be able to generate classical lead/lag concepts as the outcome of an appropriately posed optimization.

Finally to round off the goal of understanding rapidly parameter-varying systems, it would be valuable to apply some of the concepts of Chapter 5 via extensive case studies to systems where a frozen-time design approach is inadequate in order to get some sense of the applicability of these ideas.

As mentioned earlier, all of these concepts have immediate implications on scheduling on a reference trajectory and scheduling on the plant output, since both cases may be analyzed from a parameter-varying viewpoint. Although the case of scheduling on a reference trajectory is almost an immediate extension of parameter-varying concepts, it turns out that scheduling on the plant output can be handled by a parameter-varying viewpoint, but doing so may lead to some conservatism (e.g. as stated in Chapter 5, an output-varying system may be stable while the analogous parameter-varying system is unstable). Thus, the above outline of research as applied to nonlinear systems should isolate which aspects are unique to nonlinear systems and not to their linear parameter-varying counterparts.
References


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