MODELLING AND ACTIVE MODIFICATION OF
WAVE SCATTERING IN STRUCTURAL NETWORKS

by

DAVID W. MILLER

S.B., Massachusetts Institute of Technology
(1982)
S.M., Massachusetts Institute of Technology
(1985)

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Signature of Author ______________________________________

Department of Aeronautics and Astronautics
May 20, 1988

Certified by _____________________________________________

Prof. Andreas H. von Flotow
Thesis Supervisor

Certified by _____________________________________________

Prof. Edward F. Crawley
Committee Chairman

Certified by _____________________________________________

Prof. Steven R. Hall
Thesis Committee

Certified by _____________________________________________

Prof. Wallace E. Vander Velde
Thesis Committee

Accepted by _____________________________________________

✓ Prof. Harold Y. Wachman
Chairman, Department Graduate Committee

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ABSTRACT

This work extends a frequency domain formalism for the modelling of wave propagation in structures composed of arbitrary assemblages of one dimensional waveguides (members) interconnected at boundaries (junctions). A method is illustrated for evaluating member and junction power flow characteristics. These power flow models, in conjunction with member transmission and junction scattering models, are used in the derivation of control algorithms which employ collocated measurements to actively alter the reflection and transmission properties of a junction.

The member and junction models are local models of structural behavior. The member models are used to describe the wave transmission characteristics of either uniform members or periodic members for wavelengths considerably greater than the length of periodicity. In either case, the member is demarcated by two junctions. The junction models are composed of boundary and attached member dynamics. The accuracy of the junction description is only sensitive to errors in the modelling of the boundary and attached members. Relations governing member and junction power flow are formulated. These are used to identify the mechanisms which propagate energy in members and the energy dissipation characteristics of junctions.

A theory is developed for the active control of elastic wave propagation in structures using discrete point devices. Attention is focused on the active modification of the scattering behavior of discrete locations in a structural network. First, the feedback necessary to achieve specified closed-loop reflection and transmission coefficients can be determined. Second, optimal control is formulated to maximize junction power dissipation.

A network wave description is formulated by assembling component dynamical models. Net power flow reveals the paths through which energy travels from disturbance sources to damping mechanisms. Total power flow identifies those components which contain the most vibrational energy. Both quantities can be used to guide the placement of wave controllers.

Experiments were conducted which illustrate the differences and similarities between a sample wave control design and rate feedback.

The application of these control techniques is limited to objectives which can be posed in terms of local behavior. Such objectives range from maximizing junction power dissipation to shunting departing power along preselected members. The control can be derived based upon models which may be accurate descriptions of local behavior in frequency regimes where global models are unable to accurately describe global behavior. This indicates that this type of local wave control might compliment the performance of global control by providing stabilizing power dissipation throughout a bandwidth exceeding that of the global control.

Thesis Supervisor: Dr. Andreas H. von Flotow
Assistant Professor of Aeronautics and Astronautics
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>List of Figures</td>
<td>6</td>
</tr>
<tr>
<td>List of Tables</td>
<td>8</td>
</tr>
<tr>
<td>1. Introduction</td>
<td>9</td>
</tr>
<tr>
<td>1.1 Modelling Options for Complex Structures</td>
<td>10</td>
</tr>
<tr>
<td>1.2 Control Options for Complex Structures</td>
<td>13</td>
</tr>
<tr>
<td>1.3 A Wave Approach</td>
<td>15</td>
</tr>
<tr>
<td>2. Component Dynamics and Power Flow</td>
<td>19</td>
</tr>
<tr>
<td>2.1 Local Wave Dynamics</td>
<td>19</td>
</tr>
<tr>
<td>2.1.1 Member Dynamics Formulation</td>
<td>19</td>
</tr>
<tr>
<td>2.1.2 Junction Dynamics Formulation</td>
<td>22</td>
</tr>
<tr>
<td>2.2 Component Power Flow</td>
<td>27</td>
</tr>
<tr>
<td>2.2.1 Member Power Flow</td>
<td>28</td>
</tr>
<tr>
<td>2.2.2 Junction Power Flow</td>
<td>34</td>
</tr>
<tr>
<td>3. Junction Control</td>
<td>40</td>
</tr>
<tr>
<td>3.1 Actuator and Measurement Selection</td>
<td>43</td>
</tr>
<tr>
<td>3.2 Specify Closed-Loop Scattering Characteristics</td>
<td>43</td>
</tr>
<tr>
<td>3.2.1 Full Specification</td>
<td>45</td>
</tr>
<tr>
<td>3.2.2 Partial Specification</td>
<td>47</td>
</tr>
<tr>
<td>3.3 Optimal Junction Control</td>
<td>50</td>
</tr>
<tr>
<td>3.3.1 Noncausal Solution</td>
<td>53</td>
</tr>
<tr>
<td>3.3.2 Causal, Fixed Form Parameter Optimization</td>
<td>57</td>
</tr>
<tr>
<td>3.3.3 Causal Solution Using Wiener-Hopf Techniques</td>
<td>59</td>
</tr>
<tr>
<td>3.3.4 Causal Solution Using Noncollocated Feedback</td>
<td>61</td>
</tr>
<tr>
<td>3.4 Examples of Junction Control Formulation</td>
<td>62</td>
</tr>
<tr>
<td>3.5 Summary</td>
<td>88</td>
</tr>
<tr>
<td>4. Network Dynamics and Power Flow</td>
<td>90</td>
</tr>
<tr>
<td>4.1 Network Dynamics</td>
<td>90</td>
</tr>
<tr>
<td>4.2 Network Power Flow</td>
<td>92</td>
</tr>
<tr>
<td>4.2.1 Net Power Flow</td>
<td>92</td>
</tr>
<tr>
<td>4.2.2 Total Power Flow</td>
<td>93</td>
</tr>
<tr>
<td>4.3 Controller Placement</td>
<td>103</td>
</tr>
<tr>
<td>4.4 Network Control Issues</td>
<td>111</td>
</tr>
</tbody>
</table>
LIST OF FIGURES

1.1 Three frequency regimes for modelling and control 9
1.2 Model of proposed dual keel space station 11
1.3 Generic wave junction composed of members and a body 16
2.1 Complex wave numbers for bending stress rate damping 20
2.2 Free-end of Bernoulli-Euler beam 25
2.3 Compression rod with dashpot at right end and two external inputs 37
3.1 Illustration of moments and forces required to confine strain energy, associated with unit deflection, within a bounded region 41
3.2 Illustration of discrete force distribution required to localize beam strain energy associated with unit deflection 41
3.3 Feedforward of incoming wave models 44
3.4 Feedback of cross-sectional physical measurements 44
3.5 Use of physical measurement feedback to mimic the feedforward of incoming wave mode amplitudes 45
3.6 Feedforward of incoming wave mode information 65
3.7 Junction of three compression rods. Actuators include relative displacement between members 1 and 2 and junction force 69
3.8 Magnitude and phase of beam transfer function from unit forcing. Actuator dynamics are not included 76
3.9 Net power flowing out of the left junction. Actuator dynamics are not included 76
3.10 Magnitude and phase of beam transfer function from unit forcing. Actuator dynamics are included 76
3.11 Net power flowing out of the left junction. Actuator dynamics are included 76
3.12 Magnitude and phase for the optimal compensator derived using the Wiener-Hopf technique 80
3.13 Comparison of open and closed-loop beam transfer function using Wiener-Hopf compensator 81
3.14 Boundary condition at Bernoulli-Euler beam midsection 82
4.1 Model of proposed dual keel space station 98
4.2 Magnitude of net power flow in the members as a result of a disturbance at junction 1 99
4.3 Direction of net power flow through the model for ten different frequency ranges 100
4.4 Magnitude of total power flow in the members as a result of a disturbance at junction 1. 102
4.5 Magnitude of net power flow in the members of the controlled structure as a result of a disturbance at junction 1 104
4.6 Direction of net power flow through the controlled structure for seven different frequency ranges

4.7 Magnitude of total power flow in the members of the controlled structure as a result of a disturbance at junction 1

5.1 Schematic of beam, actuator, shaker and sensors with feedback and data acquisition equipment

5.2 Top and side views of moment actuator and rotary acceleration sensor

5.3 Matrix code for evaluating model transfer function

5.4 Open-loop vs. closed-loop beam responses for indicated rate feedback gains

5.5 Open-loop and closed-loop beam responses for anticausal wave control

5.6 Open-loop and closed-loop beam responses for optimal/causal wave control

5.7 Comparison of closed-loop beam responses using rate feedback at the maximum stable gain and optimal/causal wave control

5.8 Schematics of analog computer programs for both feedback compensators

5.9 Measured transfer functions of circuits shown in Fig. 5.8 used to approximate rate feedback and wave control

5.10 Characteristics of half integrator circuit

5.11 Comparisons of open-loop, wave control and rate feedback beam responses

5.12 Fitted damping ratio vs. modal frequency curves for various gain levels for rate feedback and wave control

5.13 Free and controlled decays for the mode at 2.62 Hz and the mode at 27.7 Hz for rate feedback at a gain of 2 Nm sec² and optimal/causal wave control

A.1 Rod termination with attached resonator

A.2 Junction of two rods and two beams

A.3 Rod and beam junction

A.4 Rod midsection with attached resonator

B.1 One degree of freedom, second order, oscillatory system

B.2 Region of closed-loop pole locations for all combinations of diagonal state penalty matrix

B.3 Two degree of freedom spillover model
LIST OF TABLES

4.1 Specifications used for space station model analysis 98
4.2 Relative rankings of critical total power members 105
5.1 Properties of pinned/free brass beam used in control tests 114
5.2 Control hardware specifications 116
5.3 Closed-loop modal damping ratios using both compensators 135
5.4 Comparison of beat period with propagation delay 138
1. INTRODUCTION

In recent years, the dynamic analysis and control of complex structural networks has received increasing attention. From submarines to airframes to large space structures, more attention is being focused on modelling and controlling these structures well into the frequency range where elastic deformations resonate. These modelling and control issues are highly coupled because active control makes unprecedented demands on modelling fidelity. Therefore, predicting the response of structures in frequency regimes where many of the structure's natural modes are excited is an ongoing topic of research.

Motivations for such study have historically included interest in the acoustic behavior of buildings and vehicles\(^1\). More modern concerns include the modelling and control of complex space structures, the unintentional interaction between active controllers and elastic deformation of structures, and the intentional control of structural dynamics and acoustics. The concern becomes how to accurately predict and control the behavior of these complex systems in three frequency regimes (Figure 1.1): that encompassing the fundamental elastic modes; the midrange where global modes become dense in the frequency spectrum; and the acoustic regime where local deformations are dominant and modal descriptions are inappropriate.

![Diagram of frequency regimes](image)

**Figure 1.1 Three frequency regimes for modelling and control**

The modelling and control of elastic deformation in the fundamental regime has become important in the areas of flutter suppression, stabilization of aeroelastic systems, gust alleviation, rigid spacecraft motion control, and global level performance enhancement for some precision space structures. While these applications have required models of only the lowest elastic modes, analyses\(^2\)\(^3\) for some future space missions show that hundreds of modes of an elastic spacecraft can contribute significantly to performance degradation.
At the high end of the frequency spectrum, the "acoustic" regime characterizes that frequency region where so many local and global modes interact that attempting to accurately describe the behavior in terms of modes becomes impossible. Instead, elements of acoustic analysis are employed. Structural motion is described in terms of waves transiting from component to component and reflecting from structural discontinuities. In such approaches, the concept of vibrational power transmission is often employed\(^4\).

Large flexible spacecraft are a class of structures for which accurate modelling, for both response prediction and active control, is important. These structures may be of such large geometric dimension, as dictated by mission objectives, and low mass, as dictated by launch constraints, that they exhibit low stiffness and high modal density. Structures such as the proposed Space Station (Figure 1.2) may have fundamental modes below 0.1 Hz with the response becoming modally rich, or more appropriately called acoustic, at frequencies as low as 10 Hz. Therefore, methods may be needed for characterizing structural behavior in all frequency regimes.

Unlike the station, some of these proposed systems have performance requirements, such as shape control and dynamic isolation, which far exceed that which is attainable through passive means. This combination of modelling difficulty and stringent performance specifications, with the added limitation of being unable to accurately test these large systems in a one-gravity environment, presents the design engineer with a particularly difficult problem. Therefore, the pressing issue becomes that of modelling these complex dynamic systems, across these three regimes, with sufficient accuracy as to enable prediction of on-orbit performance and to allow confidence in the stability and performance of any structural controller that may be required.

1.1 Modelling Options for Complex Structures

The first step in addressing this modelling and control problem is to analyze the strengths and limitations of the available modelling techniques. Modal descriptions of dynamic behavior are employed in the majority of today's structural analysis schemes. As would be expected, this has lead to their wide use for design of proposed structural control systems. Since modal parameters, particularly eigen-shapes, are extremely sensitive to parameter perturbations of the structure\(^5\) and this sensitivity increases with modal density, what might be considered an acceptable level of uncertainty in the modelling of fundamental behavior can result in a global description which becomes inaccurate as the structure's modal frequency spectrum becomes quite dense, as in the midrange. Since modal density increases with mode number, this sensitivity has prompted one analyst to suggest\(^6\) that it is possible to make the modal model too complex (of too large a dimension). This is
Figure 1.2 Model of proposed dual keel space station
primarily due to the fact that many of these modes are considerably beyond the range where they may be confidently modelled. Thus, one would eventually face the problem of controlling structural dynamics which are well beyond the frequency range in which modal analysis is applicable. Since not only control performance but stability depends on the accuracy of these models, additional models may be needed to supplement modal descriptions.

Statistical Energy Analysis (SEA)\(^7\) and Asymptotic Modal Analysis (AMA)\(^8\) attempt to resolve this modeling problem by sacrificing detailed knowledge of the response in favor of root mean square (RMS) amplitudes of points or of entire components, averaged over a broad frequency range to include many of the structure's natural modes of vibration. The response is then viewed in a thermodynamic manner by estimating energy balance relationships between structural components. The assumptions upon which these approaches are based become valid only as the number of participating modes becomes very large. This requirement for high modal density, when compared to modal analysis' requirement for sparse spectrum dynamics, illustrates the point that they are only applicable in two very different regimes.

Traditional acoustic approaches\(^1\), based entirely upon spatially local models without attempting to synthesize global dynamics, are limited by a complete lack of global information. Approximations introduced in such approaches, such as use of a simple impedance to model a complicated portion of the structure, may lack the fidelity demanded by new applications.

Methods such as travelling waves\(^9\) and component modes using transfer mobilities\(^10\) have been proposed to bridge the midrange between FEM and SEA. These techniques provide a two tier modelling approach. On the local level, the component models can be capable of accurately resolving the details of local structural response well into the high frequency, acoustic regime. When the component models are assembled on the global level, global structural response can be analyzed. This resembles SEA by analyzing behavior on a local and component scale while their global models retain the modelling sensitivity of the FEM. In many of these emerging local modelling techniques, interest is being focused on power flow from component to component\(^10\)\(^11\)\(^12\).

As with any modelling approach, wave dynamics are based on spatially local models. The reflection and transmission coefficients are relatively insensitive to modelling errors, depending only to first order upon local parameter perturbations and insensitive to remote perturbations. The model only needs to describe the frequency properties of the local material behavior. This includes the boundary dynamics and the transmission characteristics of the attached members.
1.2 Control Options for Complex Structures

The second issue is that of active structural control based upon these various modelling options. Active control of structural dynamics\textsuperscript{11,13,14} and of structural acoustics\textsuperscript{12,15} is a possibility receiving much recent attention. Typically, modal models originate from discretization techniques such as finite elements. These models become cumbersome as a high degree of accuracy is required in order to resolve densely spaced or higher frequency modes. The resulting control not only becomes hardware intensive, but also computationally prohibitive. As models become more complex and higher frequency modes are retained, the increased processing burden forces a reduction in controller bandwidth. These are diametrically opposed tendencies.

A proposed solution involves using parallel and hierarchic control\textsuperscript{16,17} to simultaneously process different portions of the task while matching the control rates to the dynamic rate of the modes controlled at each level\textsuperscript{16}. One such scheme is named HAC/LAC\textsuperscript{17} for high authority/low authority control. The high authority level implements higher order controllers on the slower modes. The low authority control is used to provide a margin of stability in the event that spillover, due to the truncated structural description in the high authority model, might otherwise destabilize the structure. Spillover\textsuperscript{18} refers to the deviation between actual and predicted performance in the event that control effort is expended on (control spillover), and measurements include (measurement spillover), unmodelled dynamics. This deviation can be stabilizing or destabilizing. Such LAC stabilization typically has a form similar to direct velocity feedback between dual (collocated and of like type) sensors and actuators. This has been shown\textsuperscript{19,20} to be unconditionally stabilizing if the matrix of feedback gains is positive definite. However, this result ignores sensor and actuator dynamics. Although the feedback gain matrix may, in principle, be full, experience has shown\textsuperscript{21} that a restriction to local velocity feedback (a diagonal gain matrix) results in negligible degradation in performance.

The use of modal models to derive control has invariably led to the statement of control objectives in terms of modes. Typically, a state-space, linear, time invariant model is created which contains anywhere from fifty to one hundred hopefully accurate modes. This is sometimes called the evaluation model. Methods\textsuperscript{22} have been developed to identify the dominant modes through which the output variables are disturbed, controlled and measured. Those modes, identified as influencing the output variables the most, are retained in what is called the design model. The control design procedure is iterative using the design model to formulate the feedback and the evaluation model to check the stability and evaluate performance. Control and measurement spillover\textsuperscript{23} from unmodelled modes
creates the need for the evaluation step. This spillover problem originates from the fact that discrete point hardware is being used to control a finite set of the structural modes.

An alternative statement of objective might involve controlling properties at a discrete location using discrete point hardware. Instead of identifying modes which interact with the output variables, it might be beneficial to identify load paths down which energy propagates to disturb the output variables. Actively altering the transmission properties of a load path is a local objective that could be achieved based upon a local description.

Insights into controller design are limited by the model used. Models which contain characteristics of some reduced set of structural standing modes are only one type of structural description. Waves\textsuperscript{24}, boundary descriptions\textsuperscript{25}, P.D.E. controllers\textsuperscript{26 27}, component modes\textsuperscript{16}, and impedances are others and should be investigated. The choice of model should depend on control objective, attainable modelling accuracy and the extent to which the design can be implemented.

Many structural control problems can be classified in terms of implementation aspects. First, unless some distributed system can be efficiently manufactured\textsuperscript{28}, only a small number of devices are available for implementing control. Prevention of destabilizing spillover often involves accurate modal sensing or actuating which can only be achieved with large numbers of instruments. Second, larger stability margins are needed to allow increased performance. Typically, passive damping or collocated, dual rate feedback provides this stability margin and spillover may reduce this margin as performance is increased. Third, the computational burden must be reduced from a control rate and reliability viewpoint. Simpler models, perhaps based upon local or component dynamics, are needed, particularly for the LAC level. Perhaps the incorporation of passive damping mechanisms may provide a major proportion of the damping performance required of the LAC level. Fourth, it is necessary to have models which are accurate in the frequency region in which modal analysis is inappropriate. Finally, it may be beneficial to introduce control options which are not available using modal models or when using available control formulation techniques such as Linear Quadratic Regulator (LQR).

A wave approach is analyzed in this work because it satisfies many of these implementation constraints. In addition, characterization of wave mode power flow in a structure is a useful and physically pleasing quantity for purposes of active control design\textsuperscript{12 13 29}. The technique can pinpoint important disturbance transmission paths and allow analysis of the energetic interaction of disturbance sources and control actuation. Control can also be based directly upon concepts of power flow\textsuperscript{11 15}, the control objective being the minimization of component power flow.
1.3 A Wave Approach

The work reported here approaches the modelling problem from a wave control point of view where the reflection/transmission properties of performance critical locations are actively altered in order to meet mission requirements regarding dynamic isolation, energy shunting, and power dissipation. An appropriate application of this wave viewpoint might be for structures which undergo discrete point disturbances from either docking, thruster firing, impacts, machinery vibrations, or crew movements. Another is for structures which have performance objectives which vary as a function of location. Either situation might necessitate the use of dynamic isolation, disturbance rejection, or energy shunting.

A wave approach views a disturbance as generating vibrational energy which propagates away from the disturbance location. This energy may propagate along structural members described by partial differential equations (P.D.E.s) and encounter junctions with other members. These boundaries are often governed by ordinary differential equations (O.D.E.s). The control objective may be to remove the energy from the disturbance location as quickly as possible (spatially or temporally), prevent it from disturbing some performance critical location or shunt it to some region of the structure where it may be more effectively dissipated. In this way, the vibrational energy resulting from the disturbance may be regarded as the new disturbance to other portions of the structure. From this perspective, the control problem can not only be viewed as feedback of motion measurements but as the feedforward of incoming wave disturbances. While this viewpoint is illustrative, the application of wave models is not restricted to systems based on their disturbance nature but on their ability to support waves. The control approach presented herein is applicable to situations where the control objective can be posed in terms of local behavior.

This thesis extends an analytical formalism for the analysis of dynamics, power flow and control which is applicable to a specific class of structures (Figure 1.3): networks of slender structural members (waveguides) interconnected at boundaries (wave junctions). The analysis of these members and junctions, often referred to as components in this work, is based upon travelling wave concepts. This work draws on a prior publication\textsuperscript{9} in which a formalism for the frequency and time domain analysis of local and global dynamics was presented.

The main contribution of this work is not the development of another procedure for the calculation of transcendental transfer functions\textsuperscript{30}, rather it is the exploitation of wave propagation analyses for illumination of structural dynamics, power flow and control.
Figure 1.3 Generic wave junction composed of members and a body

methodologies. The type of junction wave model envisioned is an input/output model of a location on a structure relating waves outgoing from that location along attached members to those incoming and to those generated by external inputs applied at that location. These attached members, whose far ends are demarcated by other junctions, may be of arbitrarily short length without altering the accuracy of the junction description as long as the member models are true representations of the member transmission properties. Otherwise, these models may only be valid for specific frequency ranges. The nonhomogeneous external inputs can be made a function of the junction or member response to alter the homogeneous characteristics. Using this description, the reflection/transmission characteristics of the location, or wave junction, can be actively altered in order to vary the path of power departure or reduce the total emanating power.

An analysis of power flow in terms of propagating waves is provided. This is done to create an alternative perspective on vibrational energy. Energy in a structure, besides constantly changing form between kinetic and potential, actually flows from one location to another. This existence of power flow in structures can be illustrated with a simple example. Consider a free-free, uniform, undamped beam vibrating in a single natural mode. The kinetic energy has a spatial distribution, at a particular frequency, that is a function of the velocity mode shape and the distribution of potential energy is a function of the strain mode shape. Accordingly, a free end can possess kinetic energy density but can never possess potential energy density. Therefore, during one cycle of the beam's motion, the total energy density at the free end is time varying. Since the beam as a whole
conserves energy, the energy must propagate away from the free end and then return after one period of the motion. Wave propagation theory is a physically appealing way of describing this phenomenon. Not only are the relations simple, they add a spatial quality to the behavior which can be intuitively pleasing.

Energy is a quadratic function of the classic structural response variables; deflection, strain, and stress. These depend linearly, for linear elastic structures, upon the excitation. Since associated power is a bilinear function of the response variables, the total power is not the sum of the power flows due to each excitation source separately. A convenient approach to calculating power flow is to solve linearly for the response, and to subsequently extract the power flow information.

Power flow analyses are offered at every level. Structural members can enable power flow, both through the presence of travelling waves, and also through the interaction of evanescent modes; which exhibit an exponential rather than oscillatory spatial amplitude dependence. A junction generates, dissipates, or transmits power between members. This thesis demonstrates an analytical technique, using local models, for determining whether a junction is dissipative or nondissipative.

Actively controlled junctions are used to minimize total emanating power or power departing in specific directions. Prior work has shown that, in special cases, compensators designed for active absorption of travelling waves can be very similar to direct velocity feedback. But in general they can be quite different. Since a wave model is a local description, a guarantee of system stability can not be based upon knowledge of global structural behavior. Instead, stability is judged based on power generation/dissipation properties of the active junction at various frequencies. If the open-loop junction is stable and the control is dissipative for all frequencies and incoming wave mode mixes, the active junction can never behave as a power source and contribute to the onset of instability. This stability condition is similar to that of positive definite, dual rate feedback in that it is positive real. Since junction control is used to alter discrete point properties, only a few actuators and sensors may be required to carry out most wave control objectives.

Global control performance is dependent upon the importance of the load path containing the junction. In this work, a global, frequency domain model of the dynamics of the structural network is assembled from models of local component behavior; dispersion relations of wave propagation along members, and scattering behavior of junctions. This global model leads to transcendental transfer functions from external excitations to response variables; both deflections and internal stresses. Network power flow information can then be extracted directly from these variables and be used to evaluate
performance or place actuators. The formalism is based upon matrix manipulation allowing easy numerical implementation.

The following chapters discuss in detail the concepts raised here. Chapter 2 presents the derivation of the component wave descriptions and power flow. Chapter 3 presents two junction control techniques. Chapter 4 addresses network issues related to power flow and controller performance. Chapter 5 presents experimental results using a sample junction controller. Chapter 6 presents conclusions and recommendations for furthering this research field. Appendix A gives the details of a numerical example presented in Chapter 4. Appendix B discusses various issues associated with the development of member controllers.
2. COMPONENT DYNAMICS AND POWER FLOW

2.1 Local Wave Dynamics

Modelling wave propagation through structures of arbitrary complexity can become impractical. However it is invariably possible to find many components in any structure for which a wave propagation viewpoint is feasible. This chapter considers two such components; a slender, one-dimensional, spatially uniform elastic member and a junction of an arbitrary number of these members (Fig. 1.3). The members are viewed as waveguides along which a set of discrete, decoupled travelling wave modes may propagate. These travelling wave modes are generally coupled to one another at junctions. Junction dynamics are described by frequency dependent scattering coefficients which govern wave reflection and transmission. Since the remainder of this work builds upon the travelling wave description of member and junction dynamics, a brief summary is provided. Most of the dynamics formulation in Sections 2.1.1 and 2.1.2 was originally presented in References 9 is included here for completeness.

2.1.1 Member Dynamics Formulation The behavior of a structural member, at every frequency \( \omega \), can be decomposed into the superposition of a finite number of independent wave modes. The independent wave modes are found by Fourier transforming the governing equation of motion (P.D.E.) and forming a state-space description governing spatial behavior (Eq. 2.1).

\[
\frac{dy}{dx} = A(\omega) y = A(\omega) \begin{bmatrix} u_m \\ f_m \end{bmatrix}
\]

where the vectors \( u \) and \( f \) correspond to the deflections and internal stresses at an arbitrary cross-section of the member, respectively, and the subscript, \( m \), identifies these coordinates as being associated with a member description. These quantities will be referred to as cross-sectional variables.

Fourier transforming the spatial variable \( x \) in this equation results in a spatial eigen problem whose eigenvalues, \( k_j \), are the allowable spatial frequencies, or wave numbers, supported by the medium as independent wave modes (Eq. 2.2). The imaginary constant, \( i \), is the square root of negative one. Since wave numbers are continuous functions of temporal frequency, wave motion of a specific wavelength can only exist at a certain frequency.

\[
\{ ik L - A(\omega) \} y = 0
\]

19
The characteristic equation of Eq. 2.1, used to solve for the wave numbers, is called the dispersion relation in wave terminology.

The combination of the temporal and spatial Fourier transforms, used to create Eqs. 2.1 and 2.2 from the original P.D.E., calls for the $j^{th}$ wave mode at frequency $\omega$ to have the form

$$\tilde{w}_j(x,t) = w_j e^{ik_j x} e^{i\omega t} = w_j e^{ik_j x + i\omega t}$$  \hspace{1cm} 2.3

Real $k_j$'s yield unattenuated propagating waves, imaginary $k_j$'s yield evanescent modes, and complex $k_j$'s yield propagating waves which may amplify or attenuate. Figure 2.1 shows the complex wave number plane. Since these members are assumed to be conservative or dissipative, energy cannot be generated. Therefore, because of conservation of energy, propagating waves in the types of members discussed in this work may not amplify. This requires that the complex wavenumbers lie in the second and fourth quadrants. This can be seen by substituting test points from each of these quadrants into Eq. 2.3.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{complex_wave_numbers.png}
\caption{Complex wave numbers for bending stress rate damping}
\end{figure}
The superposition of a solution like Eq. 2.3 for each of the wave numbers found in Eq. 2.2 describes the motion of the structure at every frequency. A further superposition, by inverse Fourier transform, yields the temporal response.

\[ u_m(x,t) = \int_{-\infty}^{\infty} u_m(x,\omega) \, d\omega = \int_{-\infty}^{\infty} \sum_{j=1}^{n} w_j e^{ikx} e^{-i\omega t} \, d\omega \]  \hspace{1cm} 2.4

Therefore, arbitrary motion of the member is described by the continuous superposition (integral) of all temporal frequency components which, at each frequency, are composed of a finite set of decoupled wave modes or deflection patterns. This is similar to a modal description where the mode shape, at a specific frequency, is the superposition of this finite set of wave modes whose relative proportions are determined by the member and boundary interactions. In addition, it is this interaction that creates preferential frequencies which correspond to the standing modes of a structure. Standing modes correspond to frequencies at which the steady-state waves, after circumnavigating the structure, return to constructively interfere with themselves. This results in a modal description where physical motion equals an infinite sum of modal motions at discrete frequencies, as opposed to an infinite integral over all frequency as with a wave description.

A transmission matrix\(^9\, ^{14}\) can be used to describe the transmission of a wave set travelling in each direction along a member. As shown in Eq. 2.5, this matrix relates the wave mode amplitudes, \(w(x_2)\) at a position \(x_2\), to those at a position \(x_1\) (Eq. 2.7 in Reference 9).

\[ w(x_2,\omega) = \xi(x_2,x_1,\omega) \, w(x_1,\omega) \]  \hspace{1cm} 2.5

The matrix \(\xi\) is a spatial state transition matrix where wave mode amplitudes are the states. It is diagonal since the wave modes propagate independently. The entries, for a uniform member, are transcendental functions of the path length, \(x_2 - x_1\), and the frequency, \(\omega\). For a passive, one-dimensional member consisting of a reciprocal material, two identical \(\xi\) matrices exist. Each corresponds to wave mode sets propagating in opposite directions.

A similar matrix equation can be used to describe wave behavior in a periodic member, the important difference being that the path length \(x_2 - x_1\), for which Eq. 2.5 is defined, is fixed at multiples of the periodic bay length. This topic is presented in depth in References 9 and 33 and, though applicable to the following discussion, is not dealt with here.

At an arbitrary member cross-section, a frequency-dependent, invertible matrix, \(Y_m\), can be used to transform from member deflections, \(u_m\), and internal stresses, \(f_m\), to the leftward and rightward wave mode amplitudes, \(w_1\) and \(w_r\) (Eq. 2.6). This
transformation, $Y_m$, contains the frequency dependent eigenvectors, of the matrix A, associated with each of the eigenvalues, $k_j$. Thus, the wave mode amplitude vector is grouped according to the direction of propagation and exponential decay for the travelling and evanescent modes, respectively (Eq. 2.3 in Reference 9).

$$y_m(\omega) = \begin{bmatrix} u_m \\ f_m \end{bmatrix} = \begin{bmatrix} Y_{ul} & Y_{ur} \\ Y_{fl} & Y_{fr} \end{bmatrix} \begin{bmatrix} w_i \\ w_r \end{bmatrix} = Y_m(\omega) w_m(\omega)$$  

This transformation is a characteristic of the member. The number of cross-sectional variables equals the number of independent wave modes; making $Y_m$ square. Associated with each wave mode is its eigen-shape, a frequency-dependent mix of member deflections and internal stresses (each row of $Y_m(\omega)^{-1}$).

2.1.2 Junction Dynamics Formulation A junction (Fig. 1.3) is defined as a body to which are attached one or more members. This body can correspond to a discontinuity in the structure, which results in the scattering of the incoming waves, or to an artificially chosen location internal to a uniform member. The junction can also correspond to a point where an external input may be introduced, in which case waves are generated at that point. Any junction description can include internal dynamics.

The boundary condition at the junction is used to derive the wave scattering behavior. This condition may be a function of frequency, $\omega$, and describes how boundary motions interact with member stresses and external inputs. Such a relation has the general form

$$B(\omega) y(\omega) = \begin{bmatrix} B_u(\omega) & B_f(\omega) \end{bmatrix} \begin{bmatrix} u(\omega) \\ f(\omega) \end{bmatrix} = Q(\omega)$$  

In this boundary description, the vector $u$ contains all of the member deflections ($u_m$) and $f$ contains all of the internal member stresses ($f_m$) that exist at the boundary interface. The square sub-matrices $B_u$ and $B_f$ contain the homogeneous dynamics of the boundary while $Q$ is a vector of external inputs (forces, moments, and/or relative and absolute member deflections) acting on the boundary. This boundary relation can be transformed into a relation governing the local wave behavior.

A full rank transformation can be made from the junction’s physical variables, $u$ and $f$, to wave mode coordinates $w$, as follows:

$$y(\omega) = \begin{bmatrix} u \\ f \end{bmatrix} = \begin{bmatrix} Y_{ui} & Y_{uo} \\ Y_{fi} & Y_{fo} \end{bmatrix} \begin{bmatrix} w_i \\ w_o \end{bmatrix} = Y(\omega) w(\omega)$$  

22
where the junction transformation matrix, $Y$, has been partitioned into square sub-matrices, and the vector of wave mode amplitudes has been partitioned according to each wave mode's association with the junction: incoming and outgoing ($w_i$ and $w_o$). Non-propagating waves, such as evanescent modes, are partitioned according to their causal relationship to the junction: the near field being created at the junction and included in $w_o$ and the far field created elsewhere and included in $w_i$. As seen in Eq. 2.8b, this transformation is a characteristic of the members attached to the junction and contains all of the corresponding member transformation matrices. Equation 2.8b simply reveals the member matrices, corresponding to the $p$ members, contained within the junction matrices of Eq. 2.8a. A simple pointing matrix, $T$, is used to regroup the member wave modes in terms of incoming to and outgoing from the junction. Each wave mode associated with this transformation propagates along one of the attached members; independent of the remainder of the member response.

Substituting Eq. 2.8a into the boundary relation (Eq. 2.7) gives

\[
\begin{bmatrix} B_u & B_f \end{bmatrix} \begin{bmatrix} Y_{ui} & Y_{uo} \\ Y_{fi} & Y_{fo} \end{bmatrix} \begin{bmatrix} w_i \\ w_o \end{bmatrix} = Q, \tag{2.9}
\]

which is an expression of the boundary condition in wave mode coordinates. Equation 2.9 can be rearranged to give the input/output junction relation describing how outgoing waves result from both the scattering of incoming waves and the generation by external inputs;

\[
w_o(\omega) = S(\omega) w_i(\omega) + \psi(\omega) Q(\omega) \tag{2.10a}
\]

where

\[
S = \left( B_u Y_{wo} + B_f Y_{fo} \right)^{-1} \left[ B_u Y_{ui} + B_f Y_{fi} \right] \tag{2.10b}
\]

\[
\psi = \left( B_u Y_{wo} + B_f Y_{fo} \right)^{-1} \tag{2.10c}
\]
In this junction description, the matrices $S$ and $\psi$ represent homogeneous and nonhomogeneous wave behavior and are called the scattering and generation matrices, respectively. Both may be complex and frequency dependent. This description contains only local junction and member dynamics and does not contain information about other portions of the structure.

These junction descriptions can be used at any network discontinuity. This includes reflection points, disturbance points, material property changes, actuator locations, terminations, member intersections, etc. In addition, junctions may be defined at any structural cross-section where response information is desired. In other words, they must be used to describe any structural discontinuity which is not a part of a periodic member, any external discrete point input, and may be used at any desired structural cross-section where discrete point control might be applied. Equation 2.10a provides an input/output relation which can be combined with member transmission equations similar to Eq. 2.5 to monitor waves as they travel throughout a structure. In this work, a structure as a whole is often referred to as a structural network; an arbitrarily complex assemblage of one-dimensional members and junctions. Network topics are dealt with in detail in Chapter 4.

It may be useful, for some applications, to transform from wave modes which propagate independently in a member to those which propagate independently through a junction. Although, in general, the expression in Eq. 2.10a is fully coupled, similarity transformations can be made for both the incoming and outgoing wave mode vectors as follows

$$w_o = T_o \overline{w}_o \quad 2.11a$$

$$w_i = T_i \overline{w}_i \quad 2.11b$$

Substituting these transformations into Eq. 2.10a gives

$$\overline{w}_o = T_o^1 S T_i \overline{w}_i + T_o^1 \psi Q \quad 2.12$$

If the transformation matrices $T_o$ and $T_i$ are chosen to be equal to the left and right eigenvector matrices of $S$ and appropriately scaled, then the resulting scattering matrix is diagonal with entries corresponding to the eigenvalues of $S$. If the scattering matrix is hermitian then

$$T_o = T_i = T \quad 2.13$$

An eigenvalue of the scattering matrix, which corresponds to a wave mode mix which traverses the junction independent of the other wave mixes, reveals that wave's change in
magnitude and phase. These eigenvalues do not, in general, indicate the dissipative nature of that junction.

**Example 2.1** This example illustrates the derivation of the component relations for a Bernoulli-Euler beam model. A free end and externally applied inputs are shown in Fig. 2.2. The governing P.D.E. is

\[
EI \frac{d^4v}{dx^4} + \rho A \frac{d^2v}{dt^2} = 0
\]

In state-space form (Eq. 2.1), with primes denoting spatial derivatives, this becomes

\[
\begin{bmatrix}
\dot{v} \\
\dot{v}' \\
\ddot{v}'' \\
\dddot{v}'''
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{EI} \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
v \\
v' \\
\ddot{v}' \\
\dddot{v}''
\end{bmatrix} = Ay
\]

The dispersion relation (Eq. 2.2) is

\[
k^4 - \frac{\rho A}{EI} \omega^2 = 0
\]

![Figure 2.2 Free-end of Bernoulli-Euler beam](image)

The roots of this equation reveal that the undamped Bernoulli-Euler beam model supports one evanescent and one propagating wave mode in each direction. The four wave solutions characterize the motion at each frequency as

\[
v = w_{rp} e^{-ikx + i\alpha x} + w_{re} e^{ikx + i\alpha x} + w_{lp} e^{ikx + i\alpha x} + w_{le} e^{-ikx + i\alpha x}
\]

where the parameter

\[
k = \sqrt[4]{\frac{\rho A}{EI}} \sqrt{\omega}
\]

is a positive, real quantity and the subscripts and constants are defined as

rp: rightward propagating
re: right end evanescent
lp: leftward propagating
le: left end evanescent
ρ : mass density
A : cross-sectional area
E : modulus of elasticity
I : cross-sectional moment of inertia

The evanescent modes, \(w_{re}\) and \(w_{le}\), are analogous to the hyperbolic terms which appear in the model's exact mode shapes.

The member transformation matrix in Eq. 2.6 is found by evaluating each cross-sectional variable in the vector \(y\) (Eq. b), using Eq. d, with the local origin \((x=0)\) located at the cross-section. This gives

\[
\begin{bmatrix}
\nu \\
\nu' \\
-EI \nu''' \\
EI \nu''
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 & 1 \\
\text{ik} & \text{k} & -\text{ik} & -\text{k} \\
\text{iEIk}^3 & -\text{EI}k^3 & -\text{iEIk}^3 & \text{EI}k^3 \\
\text{-EI}k^2 & \text{EI}k^2 & \text{-EI}k^2 & \text{EI}k^2
\end{bmatrix}
\begin{bmatrix}
w_{lp} \\
w_{re} \\
w_{rp} \\
w_{le}
\end{bmatrix}
\]

Notice, in conjunction with Eq. e, the frequency dependence of this transformation.

Assuming a uniform beam of length 1, the member transmission matrix (Eq. 2.5) from one end of the beam to the other is

\[
\xi = \begin{bmatrix}
e^{-\text{ik}l} & 0 \\
0 & e^{-\text{k}l}
\end{bmatrix}, \quad \text{where} \quad w = \begin{bmatrix}w_{lp} \\
w_{re}\end{bmatrix} \quad \text{or} \quad w = \begin{bmatrix}w_{rp} \\
w_{le}\end{bmatrix}
\]

Notice that the propagating wave undergoes a phase change while the evanescent mode attenuates with distance.

The next step in describing the wave dynamics is to derive the boundary condition for the left end of the beam (Eq. 2.7). The boundary condition for the free end is

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\text{-EI} \nu''' \\
\text{EI} \nu''
\end{bmatrix}
\begin{bmatrix}
\nu \\
\nu' \\
\nu'''
\end{bmatrix} =
\begin{bmatrix}
F \\
M
\end{bmatrix}
\]

where \(F\) and \(M\) are external force and moment, respectively.

For the left end of the beam in Fig. 2.2, the incoming and outgoing wave mode amplitudes are defined as
\[
\begin{bmatrix}
w_{i1} \\
w_{o1}
\end{bmatrix} =
\begin{bmatrix}
w_{ip} \\
w_{re} \\
w_{rp} \\
w_{le}
\end{bmatrix}
\]

The transformation matrix for the member (Eq. f) is the same as that for the junction (Eqs. 2.8a and 2.8b) since only one member is attached to this junction. Substituting Eqs. f and i into Eq. h and solving for the scattering and wave generation matrices, as defined by Eqs. 2.10b and 2.10c, respectively, gives

\[
S_L = \begin{bmatrix}
-i & 1 + i \\
1 - i & i
\end{bmatrix}
\]

\[
\psi_L = \frac{1 + i}{2EIk^3} \begin{bmatrix}
1 & -k \\
1 & -ik
\end{bmatrix}
\]

Since \( S_L \) is fully populated, it can be diagonalized using the eigenvector transformation from wave modes which propagate along the member independently to those which traverse the junction independently. As yet, no information has been revealed about the dissipation characteristics of these junctions. The following section addresses this issue.

2.2 Component Power Flow

Vibrational energy propagates through structures. This movement of energy results in power flow. The concept of power flow, besides being intuitively pleasing, can be used for damping analysis and control design for both members and junctions. For cases where the incoming and outgoing wave modes propagate energy independently, a junction described by a scattering matrix of the form

\[
\begin{bmatrix}
w_{o1} \\
w_{o2}
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{bmatrix}
\begin{bmatrix}
w_{i1} \\
w_{i2}
\end{bmatrix}
\]

is dissipative if and only if

\[
|w_{o1}|^2 + |w_{o2}|^2 < |w_{i1}|^2 + |w_{i2}|^2
\]

and all of the wave mode amplitudes are scaled consistently, such as equal magnitudes indicating equal power flow\(^{34}\). However, this is not always the case, particularly when evanescent modes exist. Therefore, a method is required which identifies how waves
propagate energy. Sections 2.2.1 and 2.2.2 detail such methods for members and junctions, respectively. Chapter 4 discusses power flow in a structural network.

### 2.2.1 Member Power Flow

Power at a member cross-section is equal to the product of the deflection velocities and collocated stresses. These velocities and stresses must also be of like type: velocity and force; rotational rate and moment. The local, instantaneous power flow in a member is given by

\[
\text{Power} = \frac{\partial u_m(x,t)}{\partial t}^T f_m(x,t)
\]

(2.16)

where \(u_m\) and \(f_m\) are member deflections and internal forces. If all of the processes are ergodic, then time averaging is equivalent to ensemble averaging. The processes are assumed to be stationary since the disturbances are stationary and the structural properties, which transport the disturbance to the cross-section in the form of waves, are time invariant. Therefore, the expected value of power flow at an arbitrary cross-section, \(x\), is found by integrating Eq. 2.16 over all time. This quantity can be expressed in terms of the spectral components of the response variables through the use of the Power Theorem, a variation of Parseval's Theorem.

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \frac{\partial u_m(x,t)}{\partial t}^T f_m(x,t) \, dt = \int_{-\infty}^{\infty} i\omega \ u_m(x,\omega)^H f_m(x,\omega) \, \frac{d\omega}{2\pi}
\]

(2.17)

This expression is complex because the analytic parts have been appended. Power is simply the real part of this expression.

Though power is a nonlinear quantity enabling instantaneous interaction between response variables of different frequencies, Eq. 2.17 illustrates that it is the interaction between identical frequency components that results in steady-state power flow. Thus, one can analyze the steady-state power associated with each separate frequency by using the integrand of the right hand side of Eq. 2.17.

To analyze real power in terms of wave coordinates, elements of the member transformation matrix \(Y_m\) in Eq. 2.6 can be substituted into the real part of the integrand of the right hand term in Eq. 2.17 to get

\[
P_{AVG} = \text{Re}(i\omega \ u_m^H f_m) = \text{Re}(i\omega \ w_m^H \begin{bmatrix} Y^{H}_{fl} & Y^{H}_{fr} \end{bmatrix} \begin{bmatrix} Y_{fl} & Y_{fr} \end{bmatrix} w_m)
\]

(2.18)
\[ P_{AVG} = \frac{i}{2} \omega \mathbf{w}_m^H \begin{bmatrix} Y_{ul}^{H} Y_{fl} & Y_{ul}^{H} Y_{fr} \\ Y_{ur}^{H} Y_{fl} & Y_{ur}^{H} Y_{fr} \end{bmatrix} \mathbf{w}_m \]

where \( \mathbf{w}_m \) is the vector of wave mode amplitudes, and the superscript \( H \) denotes the hermitian or complex conjugate transpose. We term this expression \( P_{AVG} \) since it equals the time average power flow when only one frequency component is present. Taking the real part of the term in parentheses yields

\[ P_{AVG} = \frac{i}{2} \omega \mathbf{w}_m^H \begin{bmatrix} Y_{ul}^{H} Y_{fl} & Y_{ul}^{H} Y_{fr} \\ Y_{ur}^{H} Y_{fl} & Y_{ur}^{H} Y_{fr} \end{bmatrix} - \mathbf{w}_m \]

\[ \begin{bmatrix} Y_{fl}^{H} Y_{ul} & Y_{fr}^{H} Y_{ur} \\ Y_{fl}^{H} Y_{ul} & Y_{fr}^{H} Y_{ur} \end{bmatrix} \]

since only the symmetric portion of the real part and the anti-symmetric portion of the imaginary part contribute to the real power.

This expression has the form

\[ P_{AVG}(\omega) = \frac{1}{2} \omega \mathbf{w}_m(\omega)^H \mathbf{P}_m(\omega) \mathbf{w}_m(\omega) \]

where the matrix \( \mathbf{P} \) is given by

\[ \mathbf{P}_m = i\omega \begin{bmatrix} Y_{ul}^{H} Y_{fl} & Y_{ul}^{H} Y_{fr} \\ Y_{ur}^{H} Y_{fl} & Y_{ur}^{H} Y_{fr} \end{bmatrix} - \begin{bmatrix} Y_{fl}^{H} Y_{ul} & Y_{fr}^{H} Y_{ur} \\ Y_{fl}^{H} Y_{ul} & Y_{fr}^{H} Y_{ur} \end{bmatrix} \]

This matrix is a function of the member dynamics, specifically of the transformation (Eq. 2.6) from physical coordinates to wave mode coordinates. Since \( \mathbf{P}_m \) is hermitian, \( P_{AVG} \) is real for any mix of wave modes.

If the \( \mathbf{P} \) matrix is not diagonal, a transformation can be used to diagonalize it by transforming from wave modes which propagate independently to those which propagate power independently.

**Example 2.2.** This example illustrates the derivation of the power matrix for a rod in compression described by the classical wave equation.

\[ EA \frac{\partial^2 u}{\partial x^2} - \rho A \frac{\partial^2 u}{\partial t^2} = 0 \]
where \( E \) is the modulus of elasticity, \( \rho \) is the mass per unit volume, and \( A \) is the cross-sectional area of the rod. The axial displacement, \( u \), of the rod is composed of a leftward and rightward travelling wave mode.

For the undamped case, the spatial state-space description and dispersion relations are

\[
\begin{bmatrix}
    u' \\
    E A u''
\end{bmatrix} =
\begin{bmatrix}
    0 & \frac{1}{E A} \\
    -\rho A \omega^2 & 0
\end{bmatrix}
\begin{bmatrix}
    u \\
    E A u'
\end{bmatrix}
\]

\[ k^2 - \frac{\rho}{E} \omega^2 = 0 \]

The motion at a given frequency decomposed in terms of wave modes is

\[ u(x, \omega) = w_l e^{i k x + i \alpha x} + w_r e^{-i k x + i \alpha x} \quad \text{with} \quad k = \sqrt{\frac{\rho}{E}} \omega \]

The transformation, Eq. 2.6, is

\[
\begin{bmatrix}
    u \\
    E A u'
\end{bmatrix} =
\begin{bmatrix}
    1 & 1 \\
    i k E A & -i k E A
\end{bmatrix}
\begin{bmatrix}
    w_l \\
    w_r
\end{bmatrix}
\]

Applying Eq. 2.21 gives the member power matrix as

\[ P_m = 2 \alpha k E A \begin{bmatrix}
    -1 & 0 \\
    0 & 1
\end{bmatrix} \]

Notice that the leftward propagating wave has been defined as carrying negative power. This sign designation is arbitrary and may be altered as needed. Since \( P_m \) is diagonal, the wave modes that propagate the disturbance independently also propagate power independently.

For a damped case, the response at frequency \( \omega \) has the form

\[ u(x, \omega) = w_l e^{i (k_R + i k_I) x + i \alpha x} + w_r e^{-i (k_R + i k_I) x + i \alpha x} \]

Now the wave number is complex. Due to conservation of energy, the real and imaginary parts, \( k_R \) and \( k_I \), must be of opposite sign. This corresponds to the second and fourth quadrants in Fig. 2.1.

The transformation in Eq. 2.6 is

\[
\begin{bmatrix}
    u \\
    E A u'
\end{bmatrix} =
\begin{bmatrix}
    1 & 1 \\
    (-k_I + i k_R) E A & (-k_I + i k_R) E A
\end{bmatrix}
\begin{bmatrix}
    w_l \\
    w_r
\end{bmatrix}
\]

30
Applying Eq. 2.21 gives

\[
P_m = 2\omega EA \begin{bmatrix} -k_R & ik_i \\ -ik_i & k_R \end{bmatrix} \]

Notice that if the two wave mode amplitudes at the cross-section are in phase, the waves propagate power independently. However, if they are not in phase, their interaction propagates power. The direction of propagation depends on which amplitude leads the other.

This member power matrix can be diagonalized using the transformation

\[
\begin{bmatrix} w_I \\ w_r \end{bmatrix} = \frac{1}{\sqrt{2(k_R^2+k_I^2)+2k_R\sqrt{k_R^2+k_I^2}}} \begin{bmatrix} \sqrt{k_R^2+k_I^2} + k_R & ik_I \\ ik_I & \sqrt{k_R^2+k_I^2} + k_R \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}
\]

Notice, for zero damping, that this transformation matrix is the identity matrix since the independent wave modes propagate power independently in the undamped case. To get the diagonal matrix

\[
\bar{P} = 2\omega EA \sqrt{k_R^2 + k_I^2} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}
\]

**Example 2.3.** This example illustrates the derivation of the power matrix for the Bernoulli-Euler beam model. Substituting the member transformation in Eq. f of Example 2.1 into Eq. 2.21 gives

\[
P_m = 4\omega k^3 EI \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 1 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} \quad \text{where} \quad w_m = \begin{bmatrix} w_{lp} \\ w_{re} \\ w_{rp} \\ w_{le} \end{bmatrix}
\]

Notice that $P_m$ is hermitian resulting in real power being associated with any combination of complex wave mode amplitudes. The parameter $k$ is defined in Eq. e of Example 2.1. The leftward and rightward propagating wave modes propagate power independently but the evanescent wave modes, while not propagating power independently, may propagate power through their interaction. Again, positive power flow is arbitrarily defined to be towards the right, in the direction of increasing $x$ in Fig. 2.2.
Since the $P_m$ matrix is not diagonal, a transformation can be used to transform from independent disturbance wave modes to independent power wave modes.

$$
\begin{bmatrix}
w_{p1} \\
w_{p2} \\
w_{p3} \\
w_{p4}
\end{bmatrix} = 
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -0.707 & 0 & 0.707i \\
0 & 0 & 1 & 0 \\
0 & -0.707i & 0 & 0.707
\end{bmatrix}
\begin{bmatrix}
w_{lp} \\
w_{re} \\
w_{rp} \\
w_{te}
\end{bmatrix}
$$

The importance of this transformation, which diagonalizes the power matrix, is discussed in the section on Total Power Flow.

For the damped case, the motion at any frequency can be decomposed as

$$
v = w_{rp}e^{-k_ix} + w_{re}e^{-(k_i + ik_R)x + i\omega t} + w_{lp}e^{-(k_i - ik_R)x + i\omega t} + w_{te}e^{-(k_i + ik_R)x + i\omega t}
$$

This form for the wave modes can be found using bending strain rate damping which is a linear function of the time derivative of the bending strain (curvature rate). This has the form

$$
EI \frac{d^4v}{dx^4} - c \frac{d^3v}{dx^3 \frac{dt}{dt}} + \rho A \frac{d^2v}{dt^2} = 0
$$

The state-space equation is

$$
\begin{bmatrix}
v' \\
v'' \\
-EIv'''
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{EI} \\
0 & 0 & 0 & \frac{-i\omega}{EI}
\end{bmatrix}
\begin{bmatrix}
v \\
v' \\
-EIv'''
\end{bmatrix}
$$

The dispersion relation is

$$
k^4 + \frac{i\omega}{EI}k^2 - \frac{\omega^2\rho A}{EI} = 0 \quad \text{or} \quad k^2 = -\frac{i\omega}{2EI} + \frac{\omega}{2} \sqrt{\frac{4\rho A EI}{E^2I^2} - c^2}
$$

the square of the wave numbers are shown by the solid circles in Fig. 2.1 while the four complex wave numbers are shown by the solid squares for a representative value of light damping.

The quantity $k_R + ik_I$ corresponds to the first branch indicated by the right most square in the figure. Therefore, $k_R$ is a positive real quantity and $k_I$ is negative real throughout this example. The trajectories indicate the locus of wave numbers for positive
values of c. The arrows indicate the changes in the locations of the four wave numbers as the damping level is increased. Notice that for large damping, the four roots coalesce and then depart along forty-five degree trajectories. As stated in Section 2.1.1, wave numbers are prohibited from existing in the first and third quadrants due to conservation of energy. Complex wave numbers in these quadrants indicate propagating waves which amplify.

The member transformation matrix (Eq. 2.6) is evaluated for each cross-sectional coordinate using Eq. c.

\[
Y_m = \begin{bmatrix}
1 & 1 & 1 & 1 \\
-k_i + ik_r & k_r - ik_i & -(-k_i + ik_r) & -(k_r - ik_i) \\
\text{-EI}(-k_i + ik_r)^3 & \text{-EI}(k_r - ik_i)^3 & \text{EI}(-k_i + ik_r)^3 & \text{EI}(k_r - ik_i)^3 \\
\text{EI}(-k_i + ik_r)^2 & \text{EI}(k_r - ik_i)^2 & \text{EI}(-k_i + ik_r)^2 & \text{EI}(k_r - ik_i)^2
\end{bmatrix}
\]

Then, using Eq. 2.21, the member power matrix is

\[
P_m = 4\omega (k_r^2 - k_i^2) EI \begin{bmatrix}
-k_r & 0 & ik_i & 0 \\
0 & -k_i & 0 & ik_r \\
-ik_i & 0 & k_r & 0 \\
0 & -ik_r & 0 & k_i
\end{bmatrix}
\]

where \( w_m = \begin{bmatrix}
w_{lp} \\
w_{re} \\
w_{rp} \\
w_{le}
\end{bmatrix}\)

Notice that when the magnitude of \( k_r \) equals the magnitude of \( k_i \), there is no power flow in the member for any wave mode set. If \( k_i \) has the same frequency dependence as \( k_r \), this can occur at all frequencies. For this example, there is no power flow as long as the damping strength (c) equals or exceeds \( 2 (\rho AEI)^{1/2} \). This occurs because the wave numbers lie on the forty-five degree lines in the figure making \( k_r = k_i \). Notice that this is a very large level of damping.

The transformation which diagonalizes this power matrix is
\[
\begin{align*}
\begin{bmatrix}
-w_{lp} \\
w_{re} \\
w_{rp} \\
w_{le}
\end{bmatrix}
= 
\begin{bmatrix}
\frac{-k_I}{d_1} & 0 & \frac{-k_I}{d_3} & 0 \\
0 & \frac{k_R}{d_2} & 0 & \frac{k_R}{d_4} \\
-i \frac{r-k_R}{d_1} & 0 & i \frac{r+k_R}{d_3} & 0 \\
0 & i \frac{r-k_I}{d_2} & 0 & -i \frac{r+k_I}{d_4}
\end{bmatrix}
\begin{bmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4
\end{bmatrix}
\end{align*}
\]

where
\[
\begin{align*}
d_1 &= \sqrt{2r^2 - 2k_R r} \\
d_2 &= \sqrt{2r^2 - 2k_I r} \\
d_3 &= \sqrt{2r^2 + 2k_R r} \\
d_4 &= \sqrt{2r^2 + 2k_I r} \\
r &= \sqrt{k_R^2 + k_I^2}
\end{align*}
\]

This gives the decoupled power matrix as
\[
P = (k_R^2 - k_I^2) r
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

2.2.2 Junction Power Flow In certain applications, such as discrete point control, it is useful to determine how much power is departing a junction as a result of both incoming power and external inputs. When the junction is non-reciprocal due to the action of active control, the question becomes non-trivial and the formal analysis developed here becomes necessary. A junction power matrix, \( P_j \), can be assembled, using an expression equivalent to Eq. 2.21 for each member attached to the junction giving
\[
P_{\text{AVG}} = \frac{1}{2} \begin{bmatrix}
P_{m_1}^H & \cdots & w_{m_p}^H \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots
\end{bmatrix}
\begin{bmatrix}
w_{m_1} \\
\cdots \\
w_{m_p}
\end{bmatrix}
\]

2.22a
Care must be taken to ensure that positive power flow is associated with outgoing power on each of the attached members. Junction power flow is then the sum of incoming and outgoing power yielding negative flow for dissipative junctions: more power arrives than departs. Equation 2.22a can be rearranged to give

\[
P_{AVG} = \frac{1}{2} \left[ w_i^H \phantom{^H} w_o^H \right] P_j \begin{bmatrix} w_i \\ w_o \end{bmatrix}
\]

2.22b

The junction dynamics in Eq. 2.10a can be substituted to get

\[
P_{AVG} = \frac{1}{2} \left[ w_i^H \phantom{^H} w_i^H S^H + Q^H \psi^H \right] P_j \begin{bmatrix} w_i \\ w_i + \psi Q \end{bmatrix}
\]

\[
= \frac{1}{2} \left\{ w_i^H \left[ P_{11} + S^H P_{12} \right] w_i^H + P_{12} S + S^H P_{22} S \right\} w_i
\]

\[
+ \left[ w_i^H \left[ P_{12} + S^H P_{22} \right] \psi Q + Q^H \psi^H \left[ P_{12} + P_{22} S \right] w_i \right]
\]

\[
+ Q^H \psi^H P_{22} \psi Q \}
\]

Notice that the entries of \( P_j \), whose partitions are shown in the expanded expression, must be arranged to correspond with the ordering of wave modes in \( w_i \) and \( w_o \).

In the expanded expression, all three terms are hermitian yielding real power quantities. The first term corresponds to homogeneous junction power. This is used to identify the junction damping characteristics by comparing outgoing to incoming power flow. The second term determines how external inputs in \( Q \) interact with incoming waves. This can cause coupling with other inputs that generate these incoming waves. The third term represents power generation due to the local input, \( Q \). This is always positive if outgoing power is defined in the positive sense.

If the external input \( Q \) is zero, then only the first term in Eq. 2.23 is nonzero giving

\[
P_{AVG} = \frac{1}{2} w_i^H \begin{bmatrix} I & S^H \end{bmatrix} P_j \begin{bmatrix} I \\ S \end{bmatrix} w_i = \frac{1}{2} w_i^H P_{OL} w_i
\]

2.24

and the junction scattering matrix can be investigated for dissipation using the eigenvalues of the open-loop power matrix (\( P_{OL} \)).

This is done in three steps. First, the frequency dependence of the eigenvalues of \( P_{OL} \) determine those frequency ranges in which the junction might generate (positive eigenvalues), conserve (zero eigenvalues), and dissipate (negative eigenvalues) power. Often, the eigenvalues will not all have the same form (positive, zero or negative) at a given
frequency and it will be necessary to determine which motions correspond to each eigenvalue in order to better determine the actual behavior of the junction. Therefore, the second step involves finding the eigenvectors of $P_{OL}$ which correspond to each eigenvalue. This identifies the wave mode mixes whose associated power is amplified, conserved and dissipated by the junction. The third step involves determining the mix of incoming waves, or the statistics thereof, to identify the actual behavior of the junction. This third step typically requires accurate knowledge of the entire structure and disturbances, which may not be available. Therefore, in the control formulation presented in Chapter 3, either these statistics are employed or it is deemed sufficient to create as many negative eigenvalues in the closed-loop junction power flow matrix as possible.

For a conservative junction, Eq. 2.24 always yields zero average power flow, independent of the incoming waves, since the amount of power arriving at the junction always equals the amount departing. If negative power is defined as flowing into a junction then, for a dissipative junction acting as a power sink, $P_{AVG}$ is negative indicating that more power arrives than departs. The opposite is true if the junction is acting as a homogeneous power source. In Chapter 3, the junction power matrix, $P_{j}$, will be used to formulate junction controllers which minimize junction power flow.

**Example 2.4.** This example derives the various junction power matrices for the right end and midpoint of the compression rod in Fig. 2.3 (Junctions B and C). At the right end, with the damper of strength $c$ attached, the scattering matrix is simply a scalar reflection coefficient;

$$S = \frac{kEA - \omega c}{kEA + \omega c} = \tau$$  \hspace{1cm} a

which is 1 if $c = 0$, 0 if $c = A(pE)^{1/2}$ (using the parameter defined in Eq. d of Example 2.2), and -1 if $c$ is large and effectively clamping the end.

The member power matrix in Eq. f of Example 2.2 is equal to the junction power matrix in this example, since only one member is attached. Substituting it and Eq. a in Eq. 2.24 gives

$$P_{AVG} = \sum w^H_i \omega kEA (1 - \tau^2) w_i$$  \hspace{1cm} b

Since Eq. f of Example 2.2 defines positive power as flowing into the junction and the magnitude of $\tau$ is restricted to be less than or equal to unity, Eq. b is never negative for any incoming wave mode amplitude or phase. Thus, the junction never behaves as a source. The junction exhibits maximum dissipation when $\tau = 0$. This is the well-known matched termination: no reflection ($S = 0$).
Four wave modes and two external excitations, $\Delta_C$ and $F_B$, can exist at the midpoint junction in Fig. 2.3. The boundary condition is

\[
\begin{bmatrix}
  -1 & 0 & 1 & 0 \\
  0 & -1 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
  u_L \\
  \frac{\partial u_L}{\partial x} \\
  u_R \\
  \frac{\partial u_R}{\partial x}
\end{bmatrix}
= \begin{bmatrix}
  D_C \\
  F_B 
\end{bmatrix}
\]

where the subscripts, R and L, refer to cross-sectional variables immediately to the right and left of junction B, respectively. $\Delta_C$ is a commanded difference between the displacements of the left and right hand sides of the junction and $F_B$ is an external force. The junction transformation matrix is

\[
\begin{bmatrix}
  u_L \\
  \frac{\partial u_L}{\partial x} \\
  u_R \\
  \frac{\partial u_R}{\partial x}
\end{bmatrix}
= \begin{bmatrix}
  1 & 0 & 1 & 0 \\
  -ikEA & 0 & ikEA & 0 \\
  0 & 1 & 0 & 1 \\
  0 & ikEA & 0 & -ikEA
\end{bmatrix}
\begin{bmatrix}
  w_{3B} \\
  w_{4B} \\
  w_{1B} \\
  w_{2B}
\end{bmatrix}
\]

The junction dynamical relation is

\[
\begin{bmatrix}
  w_{1B} \\
  w_{2B}
\end{bmatrix}
= \begin{bmatrix}
  0 & 1 \\
  1 & 0
\end{bmatrix}
\begin{bmatrix}
  w_{3B} \\
  w_{4B}
\end{bmatrix}
- \frac{1}{2ikEA}
\begin{bmatrix}
  ikEA & 1 \\
  -ikEA & 1
\end{bmatrix}
\begin{bmatrix}
  \Delta_C \\
  F_B
\end{bmatrix}
\]
Defining power flowing into the junction as negative gives,

\[
P_{\text{AVG}} = \alpha k E A \begin{bmatrix} w_{3B}^H & w_{4B}^H & w_{1B}^H & w_{2B}^H \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_{3B} \\ w_{4B} \\ w_{1B} \\ w_{2B} \end{bmatrix}
\]

Evaluating Eq. 2.23 with Eqs. e and f yields

\[
P_{\text{AVG}} = \begin{bmatrix} w_{3B}^H & w_{4B}^H \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_{3B} \\ w_{4B} \end{bmatrix}
\]

\[
+ \frac{i \omega}{2} \begin{bmatrix} w_{3B}^H & w_{4B}^H \end{bmatrix} \begin{bmatrix} -ik E A & 1 \\ ik E A & 1 \end{bmatrix} \begin{bmatrix} D_C \\ F_B \end{bmatrix} - \begin{bmatrix} D_C^H & F_B^H \end{bmatrix} \begin{bmatrix} ik E A & -ik E A \\ 1 & 1 \end{bmatrix} \begin{bmatrix} w_{3B} \\ w_{4B} \end{bmatrix}
\]

\[
+ \frac{\omega}{2k E A} \begin{bmatrix} \Delta_C^H & F_B^H \end{bmatrix} \begin{bmatrix} (k E A)^2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta_C \\ F_B \end{bmatrix}
\]

The first term shows that the junction is conservative. The second term governs the coupling between the incoming waves and the external inputs \( \Delta_C \) and \( F_B \). The third term always yields power flow out of the junction as a result of the local inputs. The net effect of the external inputs is determined by both the second and third terms. These inputs can either impart or extract power depending on the sign of the sum of these two terms.

**Example 2.5** This example shows how the member power matrix can be found numerically for a slightly more complex problem: a Timoshenko beam. From Reference 35, the governing equations of motion are

\[
GA_S \left( \frac{\partial \phi}{\partial x} + \frac{\partial^2 \nu}{\partial x^2} \right) - \rho A \frac{\partial^2 \nu}{\partial t^2} = 0
\]

\[
GA_S \left( \phi + \frac{\partial \nu}{\partial x} \right) - EI \frac{\partial^2 \phi}{\partial x^2} + \rho l \frac{\partial^2 \phi}{\partial t^2} = 0
\]

Transforming the temporal variable gives

\[
GA_S \left( \frac{\partial \phi}{\partial x} + \frac{\partial^2 \nu}{\partial x^2} \right) + \rho A \omega^2 \nu = 0
\]
\[ GA_s \left( \phi + \frac{\partial v}{\partial x} \right) - Ei \frac{\partial^2 \phi}{\partial x^2} - \mu i \omega^2 \phi = 0 \]

Expressing these coupled equations in spatial state-space form,

\[
\begin{bmatrix}
- \frac{\partial v}{\partial x} \\
\frac{\partial \phi}{\partial x} \\
GA_s \left( \frac{\partial \phi}{\partial x} + \frac{\partial^2 v}{\partial x^2} \right) \\
EI \frac{\partial^2 \phi}{\partial x^2}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & - \frac{1}{GA_s} & 0 \\
0 & 0 & 0 & \frac{1}{EI} \\
\rho A \omega^2 & 0 & 0 & 0 \\
0 & - \rho J \omega^2 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
-\nu \\
\phi \\
GA_s \left( \frac{\partial v}{\partial x} + \phi \right) \\
EI \frac{\partial \phi}{\partial x}
\end{bmatrix}
\]

Transforming the spatial variable using the wave number (k) as the transform variable, and looking at the various frequency limits gives

\[ k = \mp i \sqrt{\frac{\rho A}{EI}} \sqrt{\omega} , \mp \sqrt{\frac{\rho A}{EI}} \sqrt{\omega} \text{ at low frequencies} \]

\[ k = \mp i \sqrt{\frac{\rho A}{GA_s}} \omega , \mp i \sqrt{\frac{\rho J}{EI}} \omega \text{ at high frequencies} \]

\[ k = 0 , 0 , \mp i \sqrt{\frac{GA_s}{EI} + \frac{\rho A}{\rho J}} \text{ at } \omega = \sqrt{\frac{GA_s}{\rho J}} \]

Notice that at low frequencies, the Timoshenko beam behaves like a B-E beam with two evanescent modes and two propagating modes. At high frequencies, the beam behaves more like a rod in that the beam becomes non-dispersive and all the modes are propagating modes. However, now two modes propagate in each direction. One wave is a bending wave while the other is a shear wave. Of course, the transition is gradual and the solutions in Eq. d are only accurate in the limit. The transition between evanescent behavior and propagating behavior occurs at a specific cutoff frequency shown in Eq. d.
3. JUNCTION CONTROL

The majority of available structural control hardware measures response and exerts actuator inputs at a discrete point location. Therefore, they directly influence discrete point properties in the structure. Since, in this chapter, the control of wave junctions is being derived to alter the discrete point scattering properties of a junction, it can have wide applicability in the field of active structural control. The generic representation of a junction, given in Fig. 1.3, applies to any number of structural locations. For example, such a location may be a single member end condition (pinned, clamped, sliding, or free), an intersection of several members (e.g., the intersection of several truss beams), or an arbitrary location on a uniform member. A junction description may also include internal dynamics, as long as a consistent input/output relation can be derived.

The basic objective behind junction control is to actively alter the wave scattering properties. Control objectives can range from the complete absorption of incoming energy to the lossless shunting of this energy along pre-selected paths. A wide variety of control objectives lie within these two extremes. A matched termination, which exhibits no reflection or transmission, is an example of the former. A clamped condition, blocking energy flow, is an example of the latter. An active junction which prevents energy from departing along a particular member, while absorbing some fraction of the total incoming energy, is an example of the application of both control objectives. In other words, junction control can be used to extract energy from the structure or shunt energy to non-critical locations.

A simple example of how these types of controllers might be used is illustrative. For the Bernoulli-Euler beam model shown in Fig. 2.2, it can be seen in Fig. 3.1 that if a unit displacement or rotation is imposed at some cross-section, then the resulting strain energy can be confined within a finite region through the proper application of both force and moment on either side of the cross-section. Particular types of junction control are simply the dynamic extension of this static example. As might be expected, these levels of force and moment correspond to the entries of a stiffness finite element because they are defined as those internal member stresses required at the element nodes to ensure that the member is stress free outside of the region bounded by the element. This is a static example of generating internal stresses which prohibit energy, created by motions within the region, from extending out of that region. Using junction controllers at the boundaries, both lossless clamping and termination matching can prevent energy from escaping some defined region whose extent is demarcated by these active junctions. For the same B-E beam example, Figure 3.2 illustrates that when using only a subset of the applicable
Figure 3.1  Illustration of moments and forces required to confine beam strain energy, associated with unit deflection, within a bounded region.

Figure 3.2  Illustration of discrete force distribution required to localize beam strain energy associated with unit deflection.
external inputs, force in this case, only a portion of the deflection strain energy can be confined within the region. Each applied force is only capable of confining a fraction of the strain energy. This example illustrates an important point. If half of the cross-sectional coordinates at a junction are able to be commanded, through the external inputs, then an active junction can be created such that no energy can penetrate that cross-section.

A generalization can be made at this point. The number of unique actuators needed to enable arbitrary definition of the closed-loop scattering matrix is equal to half of the total number of cross-sectional variables in the attached members. In other words, if half of the cross-sectional variables are commanded through actuators and the other half, or an equal number of wave mode amplitudes, are measured, the feedback matrix has the same dimensions as the scattering matrix allowing inversion of the wave generation matrix and therefore complete and arbitrary specification (selection) of the closed-loop scattering matrix entries. This required actuator set may include control forces, control stresses, commanded deflections, or any mix of these. These actuators may directly affect half of the cross-sectional coordinates chosen in the boundary relation or may directly affect quantities which are related to these coordinates through some similarity transformation. In addition, those cross-sectional coordinates which are not directly influenced by the actuators must be measured for use in the feedback. In this way, control is maintained over all of the junction's cross-sectional variables because they are either measured or actuated. If this is not the case, the closed-loop scattering behavior can not be arbitrarily specified.

Two types of control law formulations are presented in this chapter. The first is similar to "pole placement" in multivariable control where, instead of specifying the locations of the closed-loop poles and elements of the closed-loop mode shapes, the closed-loop scattering characteristics of a junction are specified. Once some or all of these closed-loop reflection and transmission coefficients are chosen, the appropriate control is derived. In the second formulation, optimal control is defined, and the resulting compensation derived. This is done by finding that compensator which minimizes a frequency domain cost functional composed of quadratic penalties on plant transfer function properties, such as power flow, and control effort.

Throughout most of this chapter, it is assumed that the measurements are collocated with the junction. This implies that the type of controllers formulated involve the feedback of collocated junction measurements, through frequency dependent compensators, to the actuators influencing that junction. Often in the following discussion, the control is first posed in terms of the feedforward of incoming wave mode amplitudes. This is then transformed to equivalent feedback of the junction's cross-sectional variables. The feedback matrices F and G correspond to wave mode amplitude feedforward and cross-
sectional measurement feedback, respectively. The first half of this chapter contains theory and the second half contains extensive examples which illustrate the junction control design procedure for various systems.

3.1 Actuator and Measurement Selection

A junction, as seen in the previous sections, corresponds to a cross-section of the structure. Associated with the junction is a finite set, at each frequency, of incoming, outgoing, near field, and far field evanescent waves. The number of supported waves is identical to the number of cross-sectional variables which exist at the junction.

These cross-sectional variables can either be measured, commanded through the use of actuators or both. While wave mode amplitudes can be measured in certain cases, with varying degrees of difficulty (References 12, 13 and 15), they can also be expressed in terms of the amplitudes of the cross-sectional variables. Due to ease of measurement, equivalent feedback of cross-sectional variables will be used to alter the junction's wave scattering characteristics. Similarly, instead of actuating wave mode amplitudes, actuators that influence cross-sectional variables will be used. This point is not as important as the measurement issue because physical actuators which command cross-sectional quantities appear in the junction relation (Eq. 2.10a) while the measured cross-sectional variables do not. Ultimately, while control objectives will be posed in terms of waves, the control will be expressed in terms of the feedback of physical measurements to physical actuators. The number of available actuators and measurements equals the total number of cross-sectional variables at the junction. The actuated variables are useless as measurements since their quantities are known or directly commanded (controlled) by the actuators. Other variables, such as free end stress when no force actuation is present, are always zero and thus useless as measurements. Therefore, the partial differential equation descriptions of the members which attach to the junction determine the number of cross-sectional variables and, accordingly, the number and type of unique control actions which are available.

3.2 Specify Closed-Loop Scattering Characteristics

The waves incoming to a junction can be thought of as a disturbance to that junction. Conceptually, the disturbance, whether defined as all incoming waves or only those incoming on a particular member, is measured and fed to the actuators at the junction in order to reduce or eliminate the departing waves. This corresponds to controlling all departing waves or only those departing along a particular member. Figure 3.3 illustrates this control architecture with F as the feedforward compensator and the junction transfer...
function matrices shown. This structure assumes that measurements of the incoming wave mode amplitudes at the junction interface are available.

Since this is typically not the case, it is desirable to use cross-sectional variables as measurements. In this chapter, the vector \( u \) contains measurable cross-sectional quantities while \( f \) contains those cross-sectional quantities that are commanded by the actuators. If the control is to have the form of feedback of cross-sectional quantities, the amplitudes of the cross-sectional variables available as measurements \((u)\) can be expressed in terms of both incoming and outgoing wave mode amplitudes by using the transformation in Eq. 2.8. The use of these as measurements provides feedforward of incoming wave mode amplitudes and feedback of outgoing wave mode amplitudes as shown in Figure 3.4. In this figure, \( G \) is the control compensator. If \( G \) is causal, only exhibits left half complex plane dynamics (stable for positive time), then the actuator commands \((Q)\) are physically realizable because the quantities in \((u)\) are measurable.
Rearrangement of the block diagram in Fig. 3.4 yields a structure identical to that in Fig. 3.3 (see Figure 3.5). This illustrates that cross-sectional variables can be used as feedback measurements to mimic the feedforward of incoming wave mode amplitudes. Notice that the block diagram in Fig. 3.5 has the same structure as that in Fig. 3.3 with a feedback stage in the feedforward path. Thus, Figure 3.5 represents the disturbance rejection problem where incoming waves are actively rejected using the feedback of physical, cross-sectional measurements to physical actuators.

![Block Diagram](image)

**Figure 3.5** Use of physical measurement feedback to mimic the feedforward of incoming wave mode amplitudes

3.2.1 Full Specification As stated at the beginning of this chapter, in order to be able to completely and independently select all of the scattering coefficients in the closed-loop scattering matrix, all of the unique actuator types in \( Q \) must be available. The number of unique actuators equals the number of incoming (or outgoing) wave modes, making \( \psi \) square. Full specification can be performed if an equal number of measurements are used, making the feedforward matrix the same dimension as the scattering matrix. In addition, those cross-sectional variables not directly commanded by \( Q \) must be measured. Returning to Fig. 3.3, the transfer function matrix for the open-loop system (\( F = 0 \)) is seen to be equivalent to Eq. 2.10a.

\[
W_o = SW_i + \psi Q
\]

Letting the control have the form

\[
Q = F(\omega)W_i
\]

then
\[ w_o = \{ S + \psi F \} w_i = S_{CL} w_i \quad 3.3 \]

If the entries in the closed-loop scattering matrix are selected, then the feedforward compensator matrix that achieves this behavior is

\[ F = \psi^1 (S_{CL} - S) \quad 3.4 \]

If the closed-loop scattering matrix is specified to be the zero matrix resulting in no reflection or transmission of incoming waves then a matched termination results (Eq. 3.5).

\[ F = -\psi^1 S \quad 3.5 \]

This development assumes that the incoming wave mode amplitude vector is directly measurable, as indicated in Fig. 3.3 and Eq. 3.2.

On the other hand, using feedback of cross-sectional measurements, as indicated in Fig. 3.5, gives

\[ Q = G(\omega) u \quad 3.6 \]

while substituting for the u vector using the transformation in Eq. 2.8 and substituting the result into Eq. 3.1, gives

\[ w_o = (I - \psi G Y_{uo})^{-1} (S + \psi G Y_{ui}) w_i = S_{CL} w_i \quad 3.7 \]

Solving for G gives

\[ G = \psi^1 (S_{CL} - S)(Y_{ui} + Y_{uo} S_{CL}) \quad 3.8 \]

with the matched termination being created by

\[ G = -\psi^1 S Y_{ui} \quad 3.9 \]

Therefore, the incoming wave mode amplitude and cross-sectional variable compensators are related, using Eqs. 3.4 and 3.8, by

\[ G = F (Y_{ui} + Y_{uo} S_{CL}) \quad 3.10 \]

\[ = F (Y_{ui} + Y_{uo} S) + FY_{uo} \psi F \]

This expresses the equivalence between the systems in Figs. 3.3 and 3.4 as shown in Fig. 3.5.

Not all specifications of the closed-loop scattering matrix yield causal compensators in G, but certain control objectives can guarantee causal solutions. For example, suppose the free end of a beam is being controlled. If the control actuators are made to mimic the internal moments and forces that would exist at this location given that the beam did not terminate but continued for an indefinite length, the control would be causal. This causality
is guaranteed because the required set of actuators are available to generate the internal beam stresses that would otherwise exist at this location.

3.2.2 Partial Specification Typically, not all actuator degrees of freedom are available to completely specify the nature of the closed-loop scattering matrix. Under such circumstances, the available degrees of freedom may be used to specify specific entries. However, there is no guarantee that the resulting closed-loop scattering matrix will be dissipative. While certain entries are chosen, the resulting control can create other entries which may cause the closed-loop system to generate power and behave as a source. Therefore, the closed-loop scattering matrix should be inserted into the corresponding junction power flow equation (Eq. 2.24) to identify the closed-loop power flow behavior. If $P_{CL}$ has any positive eigenvalues, for incoming power defined in a negative sense, a redesign of the control may be advisable.

By restricting the choice of closed-loop scattering matrices, the available feedback terms are also being restricted. Otherwise, full specification would be possible. The following equations show the solutions for various partial specification options. The superscript D denotes quantities that are specified. Given that $S_{CL}$ in Eq. 3.3 is partitioned as

$$
\begin{bmatrix}
S_{CL_{11}} & S_{CL_{12}} \\
S_{CL_{21}} & S_{CL_{22}}
\end{bmatrix} = \begin{bmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{bmatrix} + \begin{bmatrix}
\psi_{11} & \psi_{12} \\
\psi_{21} & \psi_{22}
\end{bmatrix} \begin{bmatrix}
F_{11} & F_{12} \\
F_{21} & F_{22}
\end{bmatrix}
$$

there are five different situations. An alternative to any of these techniques would be the use of a weighted pseudo inverse.

In the first, a row of submatrices in $S_{CL}$ are specified in addition to a row of feedback submatrices. This situation could arise if it is desired to eliminate a set of outgoing waves, zeroing rows in $S_{CL}$, with a subset of allowable actuators, zeroing rows in $F$. This situation can be described in matrix form as

$$
\begin{bmatrix}
S_{CL_{11}}^D & S_{CL_{12}}^D \\
S_{CL_{21}} & S_{CL_{22}}
\end{bmatrix} = \begin{bmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{bmatrix} + \begin{bmatrix}
\psi_{11} & \psi_{12} \\
\psi_{21} & \psi_{22}
\end{bmatrix} \begin{bmatrix}
F_{11} & F_{12} \\
F_{21} & F_{22}^D
\end{bmatrix}
$$

where the specified submatrices need not be square. The solution has the form
\[
\begin{bmatrix}
F_{11} & F_{12} \\
S_{CL11} & S_{CL22}
\end{bmatrix} = R
\begin{bmatrix}
S_{CL11}^D & S_{CL12}^D \\
F_{21}^D & F_{22}^D
\end{bmatrix} + T S
\]

where
\[
R = \begin{bmatrix}
\psi_{11}^1 & -\psi_{11}^1 \psi_{12} \\
\psi_{21}^1 & \psi_{21}^1 \psi_{12} - \psi_{21}^1 \psi_{11}^1 \psi_{12}
\end{bmatrix}
\text{ and } T = \begin{bmatrix}
-\psi_{11}^1 & 0 \\
-\psi_{21}^1 \psi_{11}^1 & I
\end{bmatrix}
\]

The unknown feedforward and scattering terms are found using Eq. 3.13.

The second situation occurs when some actuators and wave amplitude measurements are not available or constrained. This situation can be represented as
\[
\begin{bmatrix}
S_{CL11}^D & S_{CL12}^D \\
S_{CL21} & S_{CL22}
\end{bmatrix} = S + \psi
\begin{bmatrix}
F_{11} & F_{12}^D \\
F_{21}^D & F_{22}^D
\end{bmatrix}
\]

The solution is
\[
\begin{bmatrix}
F_{11} \\
S_{CL21}
\end{bmatrix} = R
\begin{bmatrix}
S_{CL11}^D \\
F_{21}^D
\end{bmatrix} + T
\begin{bmatrix}
S_{11} \\
S_{21}
\end{bmatrix}
\]
\[
\begin{bmatrix}
S_{CL12} \\
S_{CL22}
\end{bmatrix} = \psi
\begin{bmatrix}
F_{12}^D \\
F_{22}^D
\end{bmatrix} + I
\begin{bmatrix}
S_{12} \\
S_{22}
\end{bmatrix}
\]

The third and most unlikely situation arises when certain measurements are not available to certain actuators. This situation has the form
\[
\begin{bmatrix}
S_{CL11}^D & S_{CL12}^D \\
S_{CL21}^D & S_{CL22}^D
\end{bmatrix} = S + \psi
\begin{bmatrix}
F_{11} & F_{12} \\
F_{21}^D & F_{22}^D
\end{bmatrix}
\]

The solution is
\[
\begin{bmatrix}
F_{11} \\
F_{21}
\end{bmatrix} = \psi^1 \begin{bmatrix}
S_{CL_{11}}^D - S_{11} \\
S_{CL_{21}}^D - S_{21}
\end{bmatrix}
\]

\[
\begin{bmatrix}
F_{12} \\
S_{CL_{22}}
\end{bmatrix} = R \begin{bmatrix}
S_{CL_{12}}^D \\
F_{22}^D
\end{bmatrix} + T \begin{bmatrix}
S_{12} \\
S_{22}
\end{bmatrix}
\]

The fourth situation corresponds to full specification of the closed-loop scattering matrix (Eq. 3.4) and the fifth corresponds to no control.

Three possible objectives exist when partially specifying the closed-loop scattering matrix. It must be emphasized that this type of design procedure is ad-hoc with guidance only coming from the physical insight of the control engineer.

1.) First, the objective may be to simply dissipate vibrational power that flows into the junction. This can be achieved by minimizing the amplitude of one or several outgoing propagating waves. This can be done by zeroing one or several of the entries in the row of \(S_{CL}\) corresponding to that outgoing wave. For example, it may be better to zero the entry which relates that outgoing wave to an incoming propagating wave, rather than to an incoming evanescent wave because the propagating waves dominate power flow, particularly at higher frequencies where the characteristic lengths of the evanescent modes become insignificant. Another example is if a known disturbance source lies on one of several attached members, it may be advantageous to zero the column entries in \(S_{CL}\) corresponding to the generation of outgoing waves by the incoming, propagating wave on that particular member. Again, the junction power relation should be used to identify the dissipative nature.

2.) A second control objective is pure energy shunting. Since this objective involves removing energy spatially without altering the dissipative properties of the open-loop junction, it is at the other extreme from the first objective of extracting energy temporally. With this in mind, it is desirable to maintain the junction power matrix of the open-loop junction \(P_{OL} = P_{CL}\) and merely change the directions of transmission through that junction. In the closed-loop junction power description, Eq. 2.24 with \(S_{CL}\) replacing \(S\), there are two types of terms. The first type corresponds to the open-loop power characteristics and the second type describes how the applied control alters these characteristics. If the objective is to maintain the open-loop characteristics, then the sum of
the terms of the second type should equal the null matrix. This constraint, when substituting $S_{CL}$ from Eq. 3.3 into Eq. 2.24, gives the condition

$$F^H ( \psi^H P_{oi} + \psi^H P_{oo} S ) + ( P^H_{oi} \psi + S^H P_{oo} \psi ) F + F^H \psi^H P_{oo} \psi F = 0 \quad 3.19$$

or

$$F^H A^H + A F + F^H B F = 0$$

where $F$ is the wave mode amplitude feedforward matrix and the submatrices of the open-loop junction power matrix ($P_i$) are used. Therefore the feedforward, while zeroing the appropriate entries in $S_{CL}$, must also satisfy this constraint.

3.) The first two cases represent the extremes of specifying the closed-loop characteristics. The third type of control objective would involve a combination of the first two. For example, the objective might be to minimize power flowing out of the junction while requiring that no power depart along a particular attached member. This objective includes both energy shunting and power minimization. Once the shunting requirement is met, the remaining control degrees of freedom may be used to minimize power by specifying other entries in the closed-loop scattering matrix, or by using an optimal power minimization scheme as explained in the following section.

3.3 Optimal Junction Control

Typically, optimal control is based upon minimizing some index describing a characteristic of the system being controlled\textsuperscript{36}. For most of this discussion of optimal junction control, power flow will be the quantity minimized. For the junction control problem, incoming power is defined as negatively flowing. If more power arrives at the active junction than departs, the net power flow is negative. Thus, minimization of junction power flow reduces the amount of outgoing power resulting from incoming power. In Section 2.2.2, power flow was shown to be a quadratic quantity in terms of the amplitudes of the wave modes incoming to the junction. Since power flow is expressed in terms of its frequency components, a frequency domain formulation will be used.

In Section 2.2, power flow was shown to be a bilinear function of the cross-sectional variables. After transforming to wave coordinates, power flow becomes a quadratic function of the wave mode amplitudes. This quadratic expression describes average power flow at each frequency. The minimization of power flow at each frequency can be balanced by a quadratic penalty on the steady-state control effort employed. Summing over all frequencies yields the total steady-state power flow.

The control optimization problem becomes the minimization of the expected steady-state power flow. Placing the quadratic power term in Eq. 2.20 back into the frequency
integral of Eq. 2.17, adding a quadratic control effort penalty and taking the expected value of the resulting integral relation, gives the cost functional as

$$J = \frac{1}{2} E \left( \int_{-\infty}^{\infty} \left( w^H P w + Q^H R Q \right) d\omega \right)$$  \hspace{1cm} 3.20a

$$= \frac{1}{2} \int_{-\infty}^{\infty} \text{trace} \left( E \left( P w w^H + R Q Q^H \right) \right) d\omega$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \text{trace} \left( P(\omega) \Phi_{ww}(\omega) + R(\omega) \Phi_{QQ}(\omega) \right) d\omega$$

where

$$\Phi_{ww} = E \left( w w^H \right), \quad \Phi_{QQ} = E \left( Q Q^H \right)$$  \hspace{1cm} 3.20b

The optimal control matrix that minimizes this cost, subject to various constraints, will be composed of frequency dependent compensators. As in Reference 37, the following discussion outlines several procedures for minimizing this specific type of frequency-shaped cost functional.

The cost, $J$, in Eq. 3.20 is a scalar quantity corresponding to the expected value of the integral of two quadratic terms. The first term is the inner product of the wave mode amplitudes weighted by the junction power flow matrix. The second term is the inner product of the transformed control input amplitudes weighted by a hermitian control effort penalty matrix.

The next step involves defining the appropriate elements of the feedforward structure shown in Fig. 3.5. The block diagram element defined as $K$ in Eq. 3.21 transforms the incoming wave modes into the cross-sectional motions that would exist at the junction if no control were applied.

$$K = Y_{ui} + Y_{uo} S$$  \hspace{1cm} 3.21

The matrix $Y_{uo}$ in the feedback loop of the feedforward path in Fig. 3.5 corresponds to the additional junction motions generated by the control inputs.

This feedback loop and gain matrix, $G$, can be condensed into a single transfer function matrix given by

$$H = G \left( I - Y_{uo} \psi G \right)^{-1}$$  \hspace{1cm} 3.22
The matrix $H$ corresponds to the transfer function matrix of the elements enclosed by the dashed boundary in Fig. 3.5. The transfer function relation from incoming to outgoing wave mode amplitudes, in terms of these defined quantities, is then given by

$$w_o = (S + \psi HK) w_i = S_{CL} w_i$$  \hspace{1cm} (3.23)

With these definitions in place, the cost functional given by Eq. 3.20 can be rewritten in terms of the incoming and outgoing wave mode power spectral densities. Partitioning the junction wave mode amplitude vector as

$$w = \begin{bmatrix} w_i \\ w_o \end{bmatrix}$$  \hspace{1cm} (3.24)

gives a cost which can be expressed in terms of the auto- and cross-power spectral density functions for the incoming and outgoing waves.

$$J = \frac{1}{2} \int_{-\infty}^{\infty} \text{trace} \left( P \begin{bmatrix} \Phi_{w_i w_i} & \Phi_{w_i w_o} \\ \Phi_{w_o w_i} & \Phi_{w_o w_o} \end{bmatrix} + RHK \Phi_{w_i w_i} K^H H^H \right) d\omega$$  \hspace{1cm} (3.25)

These power spectral density (PSD) matrices can be expressed in terms of the PSD matrix of the incoming waves by substituting the transfer function relation given in Eq. 3.23. This gives the cost as

$$J = \frac{1}{2} \int_{-\infty}^{\infty} \text{trace} \left( P \begin{bmatrix} \Phi & \Phi(S + \psi HK)^H \\ (S + \psi HK)\Phi & (S + \psi HK)(S + \psi HK)^H \end{bmatrix} + RHK \Phi K^H H^H \right) d\omega$$  \hspace{1cm} (3.26)

where Eq. 3.27 has been used to simplify this expression.

$$\Phi = \Phi_{w_i w_i}$$  \hspace{1cm} (3.27)

The optimization step involves minimizing the trace of Eq. 3.26. This is done, using the calculus of variations, by perturbing the feedforward matrix, $H$, by a frequency dependent perturbation matrix, $\eta$, scaled by a small parameter, $\varepsilon$ (Eq. 3.28).

$$H \Rightarrow H(\omega) + \varepsilon \eta(\omega)$$  \hspace{1cm} (3.28)

The procedure involves showing that the optimal feedforward compensator matrix, $H$, when subjected to small "allowable" perturbations, yields a stationary cost. "Allowable" perturbations depend on the constraints imposed upon the optimization problem. These constraints will be defined as needed in the following work and differ between Sections 3.3.1 and 3.3.3.

52
The condition that the optimal compensator matrix must satisfy is found by minimizing the cost with respect to the small parameter \( \epsilon \). The relation governing the optimal compensator matrix, \( H \), is then found by allowing \( \epsilon \) to approach zero. Equation 3.21 shows the first variation of the cost, \( J \), as

\[
\delta J = \frac{1}{2} \int_{-\infty}^{\infty} \text{trace} \left\{ P \begin{bmatrix} 0 & \Phi K^H \eta^H \psi^H \\ \psi \eta K \Phi & \psi \eta K \Phi (S + \psi HK)^H + (S + \psi HK) \Phi K^H \eta^H \psi^H \end{bmatrix} \right\} d\omega + R \eta K \Phi K^H \eta^H + RHK \Phi K^H \eta^H \right\} d\omega 
\]

The expression in Eq. 3.29 can be simplified by evaluating the trace of the product of the power matrix and the matrix quantity shown. Using the partitions of the power matrix, given in Eq. 3.30,

\[
P = \begin{bmatrix} P_{ii} & P_{io} \\ P_{oi} & P_{oo} \end{bmatrix}
\]

yields an equivalent expression for the first variation of the cost as

\[
\delta J = \frac{1}{2} \int_{-\infty}^{\infty} \text{trace} \left\{ P_{io} \psi \eta K \Phi + P_{oi} \Phi K^H \eta^H \psi^H \right. \\
+ \left. P_{oo} \psi \eta K \Phi (S + \psi HK)^H + P_{oo} (S + \psi HK) \Phi K^H \eta^H \psi^H \right. \\
+ \left. R \eta K \Phi K^H \eta^H + RHK \Phi K^H \eta^H \right\} d\omega 
\]

Using the fact that the trace of the hermitian of a matrix equals the hermitian of the trace of that matrix, Eq. 3.31 can be simplified to have the form

\[
\delta J = \int_{-\infty}^{\infty} \Re \left\{ \text{trace} \left( \eta^H (\psi^H (P_{oi} + P_{oo} S) + (\psi^H P_{oo} \psi + R) HK^H ) \Phi K^H \right) \right\} d\omega 
\]

where the real part of the trace is retained.

At this point, the optimization problem can proceed in several directions based upon the perturbations allowed. Sections 3.3.1 through 3.3.4 discuss several of these directions.

3.3.1 Noncausal Solution For the feedforward gain matrix \( H \) to be optimal, it must make the cost stationary for allowable perturbations given by \( \eta^H \). At present, the
optimal noncausal solution is being sought and no constraint is being placed upon \( H \). Therefore, the optimal gain matrix \( H \) must make Eq. 3.32 satisfy
\[
\frac{\partial J}{\partial \psi} = 0
\]
for any arbitrary perturbation given by \( \eta^H \). This indicates that \( \eta \), and therefore \( H \), may contain either right or left half complex plane dynamics, or both. The resulting feedforward compensator may contain both left and right half plane dynamics requiring the control to anticipate future information. Such solutions can not be implemented in a real time control environment. Equation 3.33 is satisfied if the feedforward gain matrix is given by
\[
HK = -(\psi^H P_{oo} \psi + R)^{-1} \psi^H [P_{oi} + P_{oo} S] = F
\]
This causes Eq. 3.24 to equal the trace of a zero matrix for arbitrary \( \eta^{-H} \).

Notice that the product \( HK \) equals the feedforward matrix \( F \) shown in Fig. 3.3. The gain matrix \( G \) which feeds cross-sectional junction motions to control inputs, and represents the compensator that is actually implemented in practice, can be found by solving Eq. 3.34 for \( H \), by inverting \( K \), and substituting \( H \) into Eq. 3.35. Equation 3.35 can be obtained from Eq. 3.22.
\[
G = (I + HY_{uo} \psi)^{-1} H
\]

Equation 3.34 gives the compensator that relates the amplitudes of the control actuation as a linear function of the incoming wave mode amplitudes at the junction interface. Solving for \( H \) and substituting \( H \) into Eq. 3.35 gives the equivalent compensator that relates the amplitudes of the control inputs to the junction deflections. Both frequency dependent compensators are functions of the junction and attached member dynamics.

The second variation of Eq. 3.26 with respect to the parameter \( \epsilon \), after allowing \( \epsilon \) to approach zero, is given by
\[
\psi^H P_{oo} \psi + R
\]
\( P_{oo} \) is hermitian and positive semidefinite, since outgoing waves propagate energy away from this junction. If \( R \) is chosen to make Eq. 3.36 positive definite, then the control in Eq. 3.34 satisfies not only the first order necessary conditions, but also satisfies the second order necessary conditions.

The following discussion is a form of stability analysis. However, stability is a global characteristic and cannot be judged directly based on a local model and subsequent control. Instead, the feedback in Eq. 3.34 guarantees that the no energy can be introduced to the structure through the closed-loop, homogeneous dynamics of the junction if the
open-loop junction is conservative or dissipative. The control is guaranteed to not contribute to the onset of instability because it contributes at least a negative semidefinite term to the closed-loop junction power flow matrix \(P_{CL}\). In other words, the feedback is positive real. This can be seen by substituting Eqs. 3.34 and 3.23 into the junction power matrix.

\[
P_{CL} = \begin{bmatrix} I & S_{CL}^H \\ \end{bmatrix} \begin{bmatrix} P_{ii} & P_{io} \\ P_{oi} & P_{oo} \\ \end{bmatrix} \begin{bmatrix} I \\ S_{CL} \end{bmatrix}
\]

\[= P_{OL} + (S^H P_{oo} + P_{oi}^H) (\Gamma^H P_{oo} \Gamma - 2 \Gamma) (P_{oi} + P_{oo} S)
\]

where

\[
\Gamma = \psi(\psi^H P_{oo} \psi + R)^{-1} \psi^H
\]

is positive definite and hermitian. Since \(P_{OL}\) is assumed to be zero (conservative), negative semidefinite, or negative definite, all that is required is that

\[
\Gamma^H P_{oo} \Gamma - 2 \Gamma
\]

be negative semidefinite. Substituting Eq. 3.38 into 3.39 yields

\[
\Gamma^H P_{oo} \Gamma - 2 \Gamma = \psi(\psi^H P_{oo} \psi + R)^{-1} \psi^H P_{oo} \psi(\psi^H P_{oo} \psi + R)^{-1} \psi^H
\]

\[ - 2\psi(\psi^H P_{oo} \psi + R)^{-1} \psi^H
\]

\[= -\psi(\psi^H P_{oo} \psi + R)^{-1} (\psi^H P_{oo} \psi + 2R)(\psi^H P_{oo} \psi + R)^{-1} \psi^H
\]

which is always negative definite since the center term in the quadratic is always positive definite as seen in Eq. 3.36.

Thus, if each open-loop junction is conservative or behaves as a power sink and each controller is derived in this manner, the closed-loop structure is guaranteed to be stable. However, this does not consider junction modelling error. In addition, if the open-loop junction generates power, this control is not guaranteed to yield a negative semidefinite closed-loop junction power matrix. Though an energy generating junction is unlikely to exist in a passive structure, it may arise if the optimal control is being applied to a junction which has received some sort of specified control, to shunt energy, which is not guaranteed to be dissipative. Only the contribution of the control to the closed-loop junction power matrix is guaranteed to be at least negative semidefinite.

Unlike the linear quadratic regulator, an energy generating closed-loop junction controller does not result in infinite cost because this problem formulation assumes that the
incoming wave modes are uncorrelated with the outgoing wave modes. In an actual structure, they are always correlated because the outgoing waves, after circumnavigating the structure, return as incoming waves. Therefore, the control will attempt to maximize junction power dissipation but will not guarantee stability if the open-loop junction is unstable. This might be remedied by correlating the outgoing and incoming waves. This, however, requires knowledge of the entire structure.

On the other hand, dissipation is not guaranteed for arbitrary selection of the P matrix in Eq. 3.20. If a hermitian wave mode penalty matrix is chosen, other than the power matrix, with the general form

\[ P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \]  

then the condition in Eq. 3.37 becomes

\[ P_{CL} = P_{OL} + (S^H P_{22} + P_{21}^H) \Gamma^H P_{oo} \Gamma (P_{21} + P_{22} S) - (S^H P_{22} + P_{21}^H) \Gamma^H (P_{oi} + P_{oo} S) \]

\[ - (S^H P_{oo} + P_{oi}^H) \Gamma (P_{21} + P_{22} S) \]  

\[ = P_{OL} + A^H \Gamma^H P_{oo} \Gamma A - A^H \Gamma B - B^H \Gamma A \]  

where

\[ \Gamma = \psi (\psi^H P_{22} \psi + R)^{-1} \psi^H \]  

For the contribution of the control to the junction power to be negative definite, the penalty submatrices P_{21} and P_{22} cannot be arbitrary but must be appropriately chosen. In conclusion, any penalty matrix which makes the expression in Eq. 3.42 at least negative semidefinite guarantees that the closed-loop junction is dissipative at all frequencies for any set of incoming wave modes.

Several characteristics are readily visible from the solution in Eq. 3.34. First, if the term \( \psi^H P_{oo} \psi \) is of full rank (invertible) then it is possible to formulate control with R as the zero matrix. Such situations exist when no mechanisms exist to draw power towards the active junction. In other words, the controller must wait for the power to arrive in order to dissipate it. In addition, the amount of effort expended is on the order of the incoming power. Since the arriving power is assumed finite, the control effort is finite.

Another such situation sometimes occurs when not all of the unique types of actuators are used. This situation might be expressed as
\[ Q = T \tilde{Q} \]

where \( T \) is a matrix containing ones and zeros which indicate which types of actuators in \( Q \) are actually available. The matrix \( T \) has more rows than columns. If all types are being used, then the matrix \( T \) is the identity matrix. Using Eq. 3.44, the control is

\[ \tilde{Q} = -(T^H \psi^H P_{\omega \omega} \psi T + R)^{-1} T^H \psi^H (P_{\omega i} + P_{\omega \omega} S) \omega_i = F(\omega) \omega_i \]

As before, cheap control (\( R = 0 \)) is possible when the term \( T^H \psi^H P_{\omega \omega} \psi T \) is of full rank. This is possible for certain \( T \) matrices when \( P_{\omega \omega} \) is not of full rank.

A second characteristic of the expression in Eq. 3.34 is that if four conditions are satisfied; \( R \) is the zero matrix, all unique actuators are used (\( T \) is the identity matrix), no evanescent behavior exists (\( P_{\omega i} \) is the null matrix), and \( \psi^H P_{\omega \omega} \psi \) is of full rank; then the optimal control law yields the matched termination shown in Eq. 3.5. However, the matched termination is only a subset of the complete range of optimal solutions. If evanescent behavior exists, the controller can create a near field, outgoing evanescent wave with the proper phase so as to draw more power towards the junction. By enlarging the amplitude of the near field, more power can be attracted. Thus, cheap control is not an option since a zero control effort penalty matrix would result in the controller driving the near field mode to infinite amplitude in order to attract maximum power.

Thus far, an optimal procedure have been presented for deriving control laws based upon control objectives posed in terms of waves. The primary problem with these control laws is that there has been little guarantee that the compensator will be causal. Of course, any realizable compensator must be causal. Therefore, the next three sections discuss two different techniques for finding causal solutions.

### 3.3.2 Causal, Fixed Form Parameter Optimization

In a fixed form parameter optimization problem, the form of the control compensator matrix is selected, such as order and the frequency dependence of magnitude and phase, and the various gains are left arbitrary. The optimal set of gains corresponds to that set which minimizes some performance index.

In this section, the performance index is the trace of the integrand given in Eq. 3.26. The compensator to be optimized is \( G \) in Fig. 3.5 relating junction deflections to control inputs. This compensator, and therefore the fixed form, is constrained to be causal. Notice in Figure 3.5 that the element in the feedback loop, \( Y_{\omega \omega} \psi \), must always be causal. Therefore, if the feedback matrix \( H \) is constrained to be causal, then the relation in Eq. 3.35 guarantees that the implemented compensator, \( G \), is also causal. Thus, the fixed
form, parameter optimization problem has gone from optimizing the gain for a fixed, causal form for G to optimizing the gain for a fixed, causal form for H.

The solution procedure starts with selecting the fixed form for H. First of all, this form must be causal. Beyond this constraint, the control engineer is free to select the form. Several factors may guide in the selection of this form. First, available sensor hardware may dictate which feedback terms may be used. Second, a form may be chosen which has the same frequency dependence as some characteristic property of the junction; such as the characteristic impedance. Third, the noncausal solution given by Eqs. 3.34 and 3.35 can be used to select the appropriate form by retaining the frequency dependence of the magnitude and adjusting the phase such that the new form is causal. Fourth, the form can be selected such that it positive real. This last factor is advantageous since it guarantees that the closed-loop junction will not generate power.

Once the form is chosen, the set of gains which minimize the chosen performance index are found. In this work and in the examples in Section 3.4, the trace of Eq. 3.26 is used. This is given by

$$J = \frac{1}{2} \int_{-\infty}^{\infty} \text{trace} \left\{ P_{ii} \Phi + P_{io} (S + \psi HK) \Phi + P_{oi} (S + \psi HK)^H \right. $$

$$+ P_{oo} (S + \psi HK) (S + \psi HK)^H + RHK \Phi K^H H^H \bigg \} \, d\omega$$

3.46

The trace of the integrand of Equation 3.46 can be rearranged to yield

$$\text{trace} \left[ \{ P_{ii} + P_{io} (S + \psi HK) + (S + \psi HK)^H P_{oi} \right. $$

$$+ (S + \psi HK)^H P_{oo} (S + \psi HK) + K^H H^H RHK \} \Phi \bigg \}$$

3.47

Thus, the optimization includes knowledge of, or assumptions about, the incoming wave mode statistics contained in $\Phi$.

This is an optimization procedure which minimizes the expectation of steady-state power flow plus control effort based upon knowledge of, or assumptions as to, the statistics of the incoming wave modes. In addition, with the choice of a positive real form, the optimal compensator is guaranteed not to destabilize the structure, if ideal actuator and sensor dynamics are assumed. However, the solution is not the globally optimal solution because the optimization has been constrained by the choice of the form. A Wiener-Hopf approach can be used to find the optimal free form, causal compensator.
3.3.3 Causal Solution Using Weiner-Hopf Techniques The optimal, causal, free form solution to this frequency domain control formulation can be found by using Weiner-Hopf techniques. Much of the following discussion on Weiner-Hopf techniques was extracted from Reference 38. The cost defined in Eq. 3.20 equals the integral over all frequency of the trace of the expected power flow plus control effort. In the following procedure, analytic continuation is employed to enable the various frequency dependent relationships to be valid throughout the complex Laplace plane. In this new domain, the integral in Eq. 3.20 is evaluated along the infinite extent of the imaginary axis.

The fundamental difference between the free form causal solution and the free form noncausal solution presented in Section 3.3.1 is the definition of "allowable" perturbations. In the noncausal solution, the perturbing matrix, \( \eta \), was permitted to be arbitrary. In the problem at hand, the optimal feedforward compensator matrix, \( H \), is constrained to be causal and stable. Therefore, it must be analytic in the right half of the complex Laplace plane: it must be right half plane analytic (RHPA). Therefore, the matrix \( \eta \) which perturbs \( H \) from its optimal form must itself be causal. In other words, in the search for the optimal, causal compensator, only causal perturbations are permitted. Since \( \eta \) is RHPA, \( \eta^H \) is LHPA.

Thus the optimal compensator, \( H \), is that compensator which, for arbitrary LHPA perturbations in \( \eta^H \), causes the expression in Eq. 3.48 to possess no RHPA terms with left half plane dynamics.

\[
\left( \psi^H ( P_{oi} + P_{oo} S ) + ( \psi^H P_{oo} \psi + R ) H K \right) \Phi K^H \tag{3.48}
\]

If such terms do exist in this expression, then the integral in Eq. 3.32, when the contour is closed about the left half plane, will be nonzero for some RHPA \( \eta \), and the stationary cost constraint (Eq. 3.33) will not be satisfied. Therefore, the expression in Eq. 3.48 must be equal to some LHPA function.

\[
\psi^H ( P_{oi} + P_{oo} S ) \Phi K^H + ( \psi^H P_{oo} \psi + R ) H_{RHPA} K \Phi K^H = A_{LHPA} \tag{3.49}
\]

Notice that the term \( \Phi K^H \) post-multiplies both terms and that \( H_{RHPA} \), which replaces \( H \) in the second term, is pre and post multiplied by quadratic functions.

The Wiener-Hopf technique proceeds as follows. First, it is observed for simplicity, that Eq. 3.49 has the form

\[
H_N + H_D H_{RHPA} H_C = A_{LHPA} \tag{3.50}
\]

Second, the terms on either side of \( H_{RHPA} \) can be spectrally factored into their RHPA and LHPA parts. Since both \( H_D \) and \( H_C \) are quadratic, these factors can be hermitians of each
other yielding the singularities contained in the RHPA and LHPA parts as mirror images. Performing this factorization yields

$$H_N + H_{DLHPA} H_{DRHPA} H_{RHPA} H_{CRHPA} H_{CLHPA} = A_{LHPA} \quad 3.51$$

In the third step, Eq. 3.51 is pre- and post-multiplied by the inverses of $H_{DLHPA}$ and $H_{CLHPA}$, respectively. This operation yields Eq. 3.52 whose third term is strictly LHPA, whose second term is strictly RHPA and whose first term can be a mix of both right and left half plane dynamics.

$$H_{DLHPA}^{-1} H_N H_{CLHPA}^{-1} + H_{DRHPA} H_{RHPA} H_{CRHPA} = H_{DLHPA}^{-1} A_{LHPA} H_{CLHPA}^{-1} \quad 3.52$$

The fourth step involves factoring the first term, which may have both right and left half plane singularities, into the sum of a LHPA and a RHPA term (Eq. 3.53).

$$H_{DLHPA}^{-1} H_N H_{CLHPA}^{-1} = NTP\left(H_{DLHPA}^{-1} H_N H_{CLHPA}^{-1}\right) + PTP\left(H_{DLHPA}^{-1} H_N H_{CLHPA}^{-1}\right) \quad 3.53$$

The LHPA and RHPA terms are transforms of negative and positive time functions, respectively. Accordingly, these factors are preceded by NTP and PTP for negative time part and positive time part, respectively.

It can be seen in Eq. 3.52, when the first term is factored as shown in Eq. 3.53, that two decoupled relations exist: one governing the RHPA functions and one governing the LHPA functions. $H_{RHPA}$ only appears in the RHPA relation. Solving this relation for $H_{RHPA}$ gives

$$H_{RHPA} = -H_{DRHPA}^{-1} \left(PTP\left(H_{DLHPA}^{-1} H_N H_{CLHPA}^{-1}\right)\right) H_{CRHPA}^{-1} \quad 3.54$$

As desired, $H_{RHPA}$ only contains RHPA functions.

It was observed in Section 3.3.2 that the derivation of the implemented compensator, $G$ (Eq. 3.27), yields a causal compensator if $H$ is causal. Therefore, the final step involves evaluating $G$ using $H_{RHPA}$ in place of $H$ in Eq. 3.35.

Several examples in Section 3.4 illustrate the use of this technique. Several issues were kept in mind when performing these examples. First, notice that while the noncausal solution was independent of the incoming wave mode statistics, this causal solution often is not.

Second, It was stated that the term in Eq. 3.48 must be LHPA to ensure that a contour of integration encircling the left half plane does not enclose any singularities. In
addition, the frequency dependence of this term must decay faster than 1/ω in order that the integral along the enclosing contour of infinite radius is zero. Often this may involve proper frequency shaping of Φ. From a physical perspective, one would expect the amplitudes of the incoming waves to have a decaying dependence on frequency to support the fact that the total junction power flow is finite.

Third, it was seen in previous examples that the junction matrices often contain irrational transfer functions.

Fourth, the junction model, and therefore the control formulation, contains no information about the rest of the structure. In an attempt to minimize power flow in a certain frequency range, the Wiener-Hopf solution may result in amplification of power in another. As far as this solution is concerned, this is not a problem. However, the finite extent of any structure makes the return of the emanating power an eventuality and an instability can occur. Therefore, an iterative design approach may be required which first solves the Wiener-Hopf problem, then checks junction power flow and repeats the cycle if the first solution proves to be inadequate. The Wiener-Hopf technique can be extended to include a constraint that the form of the optimal, causal compensator must result in a closed-loop junction power flow matrix which has no positive eigenvalues at any frequency. Another approach may be to correlate the incoming with the outgoing waves. Either additional restriction would ensure the closed-loop stability of any structure that is stable in the open-loop. However, the latter requires knowledge of the entire structure while the former does not.

This outlines a Wiener-Hopf approach to the junction wave control problem. Given that future information is not made available to the controller, as it was in the noncausal solution, a reduction in performance might be expected. This solution technique is illustrated in closed form in the examples of Section 3.4.

3.3.4 Causal Solution Using Noncollocated Feedback A noncausal solution can be viewed in several ways. It's transform is analytic in the left half of the complex, Laplace plane. It also exhibits an anticipatory nature by requiring the convolution of future junction motions into the present control commands. If the collocated feedback restriction is maintained, then the anticipatory portion of the control must be eliminated as was done using Weiner-Hopf techniques (Section 3.3.3). But, as can be observed in the P.D.E. descriptions of the members attached to the junction, the vibrational energy that will cause junction motion at some time in the future is, at present, distributed along the attached members and is approaching the junction. Therefore, all that is required in order to identify the future junction motions is to observe "upstream" of the junction. In this context, the
term "upstream" has a specific meaning. It refers to measuring the wave motion at locations on the junction's members which are not coincident with the junction. If the observed wave motion is propagating energy towards the junction, then the observation is made "upstream". If the observed wave motion is propagating energy away from the junction, then the observation has been made "downstream". Therefore, the objective of this section is to demonstrate a method for implementing the noncausal solutions found in Section 3.3.1 by realizing the anticipatory portion of the control through noncollocated feedback.

3.4 Examples of Junction Control Formulation

This section presents examples which illustrate the derivation of various junction controllers for four different junctions. Examples 3.1 and 3.2 apply the concepts of this chapter to a nondispersive, undamped rod in compression. The first deals with the application of control to a free end and the second applies it to a cross-section at an arbitrary location within the rod. Examples 3.3 and 3.4 apply junction control to a dispersive, undamped, Bernoulli-Euler beam model. As with the rod, control is applied to a free end in Example 3.3 and internal to the beam in Example 3.4. The examples include references to the equations given in the previous sections of this chapter which govern each of the example steps.

Example 3.1 This first example involves the derivation of optimal, causal wave junction control for the free end of a uniform, undamped, compression rod undergoing axial displacement $u(t)$ (see the left end of Fig. 2.3). This example derives optimal junction control of this free end using no control effort penalty in part 1 (cheap control), several examples of frequency weighted control effort penalties in part 2, and a noncollocated feedback technique for implementing the matched termination in part 3.

1.) The rod properties from Eqs. 2.2, 2.6, 2.7, 2.10a, and 2.21 are

\[ S = 1 ; \quad \psi = \frac{i}{kEA} ; \quad \begin{bmatrix} B_u & B_f \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} ; \quad k = \sqrt{\frac{\rho}{E}} \omega \]

\[ P_m = \alpha kEA \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} ; \quad \begin{bmatrix} Y_{ui} & Y_{uo} \\ Y_{fi} & Y_{fo} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ ikEA & -ikEA \end{bmatrix} \]

For cheap control ($R = 0$) with $F = 1/s^4$, Eq. 3.50 gives

\[ -\frac{2}{s^3} + \frac{4}{A\sqrt{\rho E}} H_{RHPA} \frac{1}{s^4} = A_{LHPA} \quad ; \quad H_{RHPA} = \frac{1}{2} A\sqrt{\rho E} s \quad ; \quad G = A\sqrt{\rho E} s \]
This is the matched termination solution. Notice that neither term on the left hand side of the first equation contains transforms of negative time functions. Thus, the unconstrained noncausal solution is also the causal solution. Since $\Phi$ is a real scalar, the solution is independent of $\Phi$ as long as the necessary frequency behavior is achieved.

2.) Frequency weighting of the control effort penalty can be used to derive controllers which exert different levels of authority in different frequency intervals. The following present three such cases. For the first case, let the control penalty be

$$ R = r^2 \left( -s^2 + \beta^2 \right) $$

where $s^2 = -\omega^2$. This has constant control penalty at low frequencies, and the penalty increases with frequency above the corner frequency at $\beta$. The ESD, $\Phi$, is the same as in case (1). Eq. 3.49 gives

$$ \frac{-2}{s^3} + \frac{1 + r^2 A \sqrt{\rho E}}{A \sqrt{\rho E}} (-s^2 + \beta^2) \frac{H_{RHPA}}{s^4} = \frac{4}{4} A_{LHPA} $$

Defining

$$ \alpha^2 = \frac{1}{A \sqrt{\rho E}} ; \quad \sigma = \sqrt{\beta^2 + \frac{\alpha^2}{r^2}} $$

the intermediate solution steps using Eqs. 3.52, 3.53, 3.54 and 3.35 are

$$ -\frac{1}{rs(-s + \sigma)} + (s + \sigma) r H_{RHPA} \frac{2}{s^2} = \frac{s^2 A_{LHPA}}{2r(-s + \sigma)} $$

$$ -\frac{1}{rs} \sigma - \frac{1}{rs(-s + \sigma)} + (s + \sigma) r H_{RHPA} \frac{2}{s^2} = \frac{s^2 A_{LHPA}}{2r(-s + \sigma)} $$

$$ H_{RHPA} = \frac{s}{2r^2 \sigma(s + \sigma)} $$

$$ G = \frac{s}{2r^2 \sigma(s + \sigma) - \alpha^2} $$

At low frequencies, $G$ is rate feedback with a gain that approaches the matched termination solution as $r$ approaches zero. At high frequencies, where the control effort tapers off as dictated by the control effort pe; alty, the control has the form of displacement feedback at a gain which diminishes with increasing $r$.

For the second case, let the control penalty be
\[ R = \frac{-r^2 s^2}{-s^2 + \beta^2} \]

Now the control effort penalty increases with frequency at low frequencies and approaches a constant at high frequencies. Then Eq. 3.49 gives

\[-\frac{2}{s^3} + (\alpha^2 + r^2) \frac{s^2 + \alpha^2 \beta^2}{s^2 + \beta^2} H_{RHPA} \frac{4}{s^4} = A_{LHPA} \]

and

\[ G = \frac{s (s + \beta)}{\alpha(2 \sqrt{\alpha^2 + r^2} - \alpha) s + \alpha^2 \beta} \]

As \( r \) approaches zero, the compensator approaches that of the matched termination. The control yields rate feedback at both low and high frequencies and behaves as a matched termination at low frequencies for any value of \( r \).

\[ G \equiv A \sqrt{\rho E} \quad \text{at low frequencies} \]

\[ G \equiv \frac{s}{\alpha (2 \sqrt{\alpha^2 + r^2} - \alpha)} \quad \text{at high frequencies} \]

For the third case, let the control penalty be a simple positive constant

\[ R = r^2 \]

The resulting control compensator is

\[ G = \frac{s}{\alpha^2 + 2r^2} \]

This yields rate feedback throughout the frequency spectrum with gain less than that of the matched termination. Again, the compensator matches the termination as the control effort penalty approaches zero. These three cases show that, while frequency weighting can readily be applied to the junction control problem, care must be taken in the selection of the weighting. Notice that in case 1, conservative forces are implemented at high frequencies in order to yield optimal, causal control as dictated by the weighting. The weighting in case 2 allows a broader range of energy absorption.

3.) This final portion of the example illustrates the use of noncollocated measurements in the optimal control of the free end of an undamped, uniform compression
rod. As shown in Fig. 3.6, the measurements will be assumed to be taken at a point a distance \( z \) from the rod end. The junction relation for the rod end is

\[
\tilde{w}_p = \tilde{w}_p + \frac{1}{ikEA} F
\]

and the transmission relation is

\[
\zeta = e^{ikz}
\]

**Figure 3.6 Feedforward of incoming wave mode information**

A new junction can be defined which encompasses the length of rod between the measurement location and the end. Denoting the wave mode amplitudes at the measurement location by a tilde, the new junction relation is

\[
\tilde{w}_p = e^{2ikz} \tilde{w}_p + \frac{e^{ikz}}{ikEA} F
\]

Applying Eq. 3.16 with no control effort penalty, the control force is

\[
F = -i\omega A\sqrt{\rho E} e^{-ikz} \tilde{w}_p
\]

Using the inverse of the member transformation matrix (Eq. e of Example 2.2) defines the rightward propagating wave mode amplitude in terms of the cross-sectional variables.

\[
\tilde{w}_p = \frac{1}{2} \tilde{u} - \frac{1}{2ikEA} \tilde{f}
\]

Substituting Eq. t into Eq. s, using the dispersion relation, gives the control command as

\[
F = \frac{1}{2} e^{-i\omega t} (-i\omega A\sqrt{\rho E} \tilde{u} + \tilde{f})
\]

Notice that, when substituting for \( k \) with the dispersion relation, the noncollocation distance, \( z \), divided by the wave speed, \((E/\rho)^{1/2}\), gives a time delay. Therefore, the control consists of measuring the cross-sectional variables and issuing the command after a time delay.
\[ \Delta t = \frac{z}{\sqrt{E/\rho}} \]

The measurement of the stress is used to determine the propagation direction of the wave causing the velocity term. If this wave were leftward propagating and the stress measurement were excluded, then the control would only absorb half of the incoming wave. However, if any of the velocity measurement were composed of a rightward propagating wave, then the actuator would increase the energy content of the rod every \(2\Delta t\) because it would be issuing control inputs to create outgoing waves which will chase but never catch the rightward propagating wave. Therefore, when using these noncollocated measurements, it is important to identify the propagation direction in order to only identify incoming waves.

**Example 3.2** This example presents the derivation of various junction controllers for the intersection of several compression rods. In the first part, conservative shunting is demonstrated for the uniform intersection of two identical compression rods. The second part formulates optimal control for the same junction. The third part illustrates a more complex shunting problem at the intersection of three identical rods.

1.) The objective of this first part is to take the compression rod problem from Example 2.4 and alter the transmission coefficients at the midpoint (junction B) with the available control while maintaining a conservative junction. Specifically, the wave departing to the right will be eliminated. The general two by two feedforward matrix between incoming waves and junction force and relative displacement is

\[
F = \begin{bmatrix}
F_{11} & F_{12} \\
F_{21} & F_{22}
\end{bmatrix}
\]

Substituting into Eq. e from Example 2.4 gives the closed-loop scattering matrix

\[
S_{CL} = \begin{bmatrix}
\frac{-ikEAF_{11} + F_{21}}{2ikE} & 1 - \frac{ikEAF_{12} + F_{22}}{2ikE} \\
1 - \frac{ikEAF_{11} + F_{21}}{2ikE} & \frac{-ikEAF_{12} + F_{22}}{2ikE}
\end{bmatrix}
\]

To eliminate the rightward outgoing wave, it is necessary to set the bottom row to zero. This gives the relations

\[
F_{21} = 2ikEA + ikEA F_{11}
\]

\[
F_{22} = ikEA F_{12}
\]
which results in an intermediate junction relation expressed in terms of the remaining unchosen feedforward terms

\[ w_o = \begin{bmatrix} -1 \\ 0 \end{bmatrix} w_i - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \end{bmatrix} w_i \]

Notice that the choice of the remaining feedback terms cannot alter the primary objective of cancelling the rightward, outgoing wave. Solving the constraint in Eq. 3.19 gives three relations

\[
F_{11}^H + F_{11} + F_{11}^H F_{11} = 0 \quad e
\]

\[-F_{12}^H - F_{12} + F_{12}^H F_{12} = 0\]

\[F_{11}^H = \frac{F_{12}^H}{1 - F_{12}^H}\]

The feedforward terms \( F_{11} \) and \( F_{12} \) must satisfy these constraints in order that the closed-loop junction be conservative.

Since Eq. e allows an entire range of feedback compensators to be chosen, one objective might be to transform the feedback and constraints into equivalent feedback of junction motion measurements. This derivation has been performed with the assumption that wave mode amplitudes can be measured. Since this is not often the case, the addition of a causality constraint for equivalent feedback of cross-sectional measurements may reduce the range of control possibilities. Measurement or actuator limitations may also reduce the available options. For example, \( F_{11} = F_{12} = 0 \) satisfies the constraints while also causing \( F_{22} = 0 \). In other words, the one feedback compensator \( F_{21} \) can achieve the primary objective while maintaining a conservative junction. It becomes obvious, by studying the scattering matrix in Eq. d, that the intermediate junction is conservative. When only propagating waves exist, this observation is trivial. This technique becomes more useful when simple observation of the scattering matrix does not reveal its power transmission nature. The constraint conditions can be solved numerically at each frequency, using Riccati solver algorithms, for more complex problems.

2.) This second part involves deriving optimal wave control at junction B in Fig. 2.3 using the two control inputs included in the junction dynamical relation (Eq. e of Example 2.4). Rewriting the junction power matrix from Eq. f of Example 2.4 gives
\[ P_{\text{AVG}} = \omega kEA \begin{bmatrix} w_{3B}^H & w_{4B}^H & w_{1B}^H & w_{2B}^H \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_{3B}^f \\ w_{4B}^f \\ w_{1B}^f \\ w_{2B}^f \end{bmatrix} \]

Since the submatrix \( P_{oi} \) of the power matrix is null, the wave generation matrix \( \psi \) is square and non-singular, and the control effort penalty \( R \) is chosen to be the null matrix, Eq. 3.34 gives the matched termination solution given by Eq. 3.5. Using the power relation above and the junction dynamical relation from Example 2.4, the control in terms of incoming wave mode amplitudes is

\[
\begin{bmatrix} \Delta_C \\ F_B \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ ikEA & ikEA \end{bmatrix} \begin{bmatrix} w_{3B}^f \\ w_{4B}^f \end{bmatrix}
\]

This feedback can be transformed into equivalent feedback of cross-sectional variables at the junction by inverting the transformation in Eq. d of Example 2.4, post multiplying this inverted transformation by a further transformation, and selecting the top two rows describing the incoming wave modes. The "further" transformation is introduced to change from cross-sectional quantities in the inverted transformation to a vector of quantities that describe these variables that are directly actuated and those that need to be measured. Since the control actuators command the differences between the displacements \( (\Delta_C) \) and stresses \( (F_B) \) at the left and right sides of junction B, respectively, these are included in the new cross-sectional vector

\[
\begin{bmatrix} u_L \\ EA \frac{\partial u_L}{\partial x} \\ u_R \\ EA \frac{\partial u_R}{\partial x} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_R - u_L \\ EA \left( \frac{\partial u_R}{\partial x} - \frac{\partial u_L}{\partial x} \right) \\ u_R + u_L \\ EA \left( \frac{\partial u_R}{\partial x} + \frac{\partial u_L}{\partial x} \right) \end{bmatrix}
\]

Knowing that

\[
\begin{bmatrix} u_R - u_L \\ EA \left( \frac{\partial u_R}{\partial x} - \frac{\partial u_L}{\partial x} \right) \end{bmatrix} = \begin{bmatrix} \Delta_C \\ F_B \end{bmatrix}
\]

68
the control becomes

\[
\begin{bmatrix}
\Delta C \\
F_B
\end{bmatrix} = \begin{bmatrix}
0 & \frac{2}{i \omega A \sqrt{\rho E}} \\
2 i \omega A \sqrt{\rho E} & 0
\end{bmatrix} \begin{bmatrix}
\frac{u_R + u_i}{2} \\
\frac{EA}{2} \left( \frac{\partial u_R}{\partial x} + \frac{\partial u_L}{\partial x} \right)
\end{bmatrix} j
\]

Notice that the optimal feedback derived using the noncausal technique is causal and
between dual sensors and actuators.

3.) This final part of Example 3.2 looks at the shunting options at a junction of
three identical compression rods (Fig. 3.7). The boundary condition is

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
u_1 \\
EA u'_1 \\
u_2 \\
EA u'_2 \\
u_3 \\
EA u'_3
\end{bmatrix} = \begin{bmatrix} 0 \\ \Delta \\ F \end{bmatrix}
\]

and the junction relation is

\[
\begin{bmatrix}
w_{lp_1} \\
w_{rp_2} \\
w_{rp_3}
\end{bmatrix} = \frac{1}{3}
\begin{bmatrix}
-1 & 2 & 2 \\
2 & -1 & 2 \\
2 & 2 & -1
\end{bmatrix}
\begin{bmatrix}
w_{rp_1} \\
w_{lp_2} \\
w_{lp_3}
\end{bmatrix} + \frac{1}{3iEA} \begin{bmatrix} ikEA & 1 \\ -2ikEA & 1 \\ ikEA & 1 \end{bmatrix} \begin{bmatrix} \Delta \\ F \end{bmatrix}
\]

Figure 3.7 Junction of three compression rods. Actuators include relative
displacement between members 1 and 2 and junction force
The junction power matrix and wave mode vector are

\[
P = \omega k E A \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} ; \quad w = \begin{bmatrix}
w_{rp_1} \\
w_{lp_2} \\
w_{lp_3} \\
w_{rp_2} \\
w_{rp_3}
\end{bmatrix}
\]

For the case where only force actuation is used and wave modes can be measured or estimated;

\[
\begin{bmatrix}
A \\
F
\end{bmatrix} = 3i k E A \begin{bmatrix}
0 & 0 & 0 \\
F_{21} & F_{22} & F_{23}
\end{bmatrix} w_i
\]

the constraint is

\[
F_{23} = F_{22} = F_{21} ; \quad F_{21}^H = \frac{-F_{21}}{1 + 3F_{21}}
\]

This gives a closed-loop scattering matrix equal to

\[
S_{CL} = \begin{bmatrix}
-\frac{1}{3} + F_{21} & \frac{2}{3} + F_{21} & \frac{2}{3} + F_{21} \\
\frac{2}{3} + F_{21} & -\frac{1}{3} + F_{21} & \frac{2}{3} + F_{21} \\
\frac{2}{3} + F_{21} & \frac{2}{3} + F_{21} & -\frac{1}{3} + F_{21}
\end{bmatrix}
\]

The only value of $F_{21}$ that zeroes entries in the closed-loop scattering matrix while satisfying the constraint is $F_{21} = -2/3$ ($F_{21} = F_{22} = F_{23} = -2/3$) yielding

\[
\begin{bmatrix}
w_{lp_1} \\
w_{rp_2} \\
w_{rp_3}
\end{bmatrix} = \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix} \begin{bmatrix}
w_{rp_1} \\
w_{lp_2} \\
w_{lp_3}
\end{bmatrix}
\]
as the closed-loop junction relation. Notice that, for a finite feedback gain, each of the three compression rods is dynamically isolated from the others and only reflection of waves, onto the same member over which they arrived, occurs.

The derivation of various junction controllers for the nondispersive rod has yielded rather simple and, in most cases, causal solutions using the noncausal procedure. The following two examples illustrate the increase in complexity involved in formulating solutions for dispersive media such as a beam undergoing transverse deflection.

Example 3.3 This example uses the same Bernoulli-Euler beam model as Example 2.1. Parts one and two perform partial and total specification of $S_{CL}$, respectively. The third part illustrates the junction control formulation that gives classical rate feedback as the optimal compensator. The fourth, fifth and sixth parts derive optimal noncausal and causal junction control.

1.) In the first part, the closed-loop (1,1) scattering matrix entry will be set to zero assuming that only one physical deflection measurement is available. This results in no outgoing propagating wave being created by the incoming propagating wave. Since the characteristic attenuation length of the evanescent mode is inversely proportional to the square root of frequency (Eqs. d and e of Example 2.1), this is the dominant entry of the scattering matrix, particularly at higher frequencies.

Feedback of a physical deflection can be achieved by using Eq. 3.6. The gain for which the closed loop (1,1) scattering matrix entry is zero can be extracted in closed form or numerically at various values of frequency. If a transverse displacement is chosen in conjunction with moment actuation, giving feedback of the form

$$M(\omega) = G(\omega) u(\omega) = g(\omega) v(\omega)$$

then the closed loop scattering matrix is

$$S_{CL} = \frac{1}{1 + \tau(1+i)} \begin{bmatrix} -i - \tau(1-i) & 1 + i - 2\tau \\ 1 - i - 2i\tau & i + \tau(1-i) \end{bmatrix}$$

where

$$\tau = \frac{(1+i)g}{2EIk^2}$$

Setting the (1,1) entry to zero and using the definition from Eq. e of Example 2.1 gives

$$\tau = \frac{-i}{1-i} \text{ or } g(\omega) = -i\sqrt{\rho AEI} \omega$$

71
This corresponds to feedback of transverse velocity to torque through a gain equal to \((\rho AE)\frac{1}{2}\). Notice that the moment actuator and transverse velocity measurement are not dual. The resulting closed loop scattering matrix is

\[
S_{CL} = \begin{bmatrix}
0 & i \\
-i & 0
\end{bmatrix}
\]

The power flowing out of the closed loop junction, at each frequency, can be found by substituting Eq. a from Example 2.3 and Eq. e into Eq. 3.37 to get

\[
P_{AVG} = \frac{1}{2} 2\omega k^3 EI w_i^\prime \begin{bmatrix}
-1 & 1 \\
1 & 1
\end{bmatrix} w_i = \frac{1}{2} w_i^\prime P_{CL} w_i
\]

This quadratic yields a real value for average power. The eigenvalues of the matrix are 1.414 and -1.414. This means that \(P_{CL}\) is an indefinite matrix and may amplify certain incoming wave mode mixes. Therefore, junction dissipation depends on the mix of incoming wave modes, and, unfortunately, on the dynamics of the remainder of the beam.

2.) The objective of this second part is to use both end force and moment actuation to match the end, thus preventing any reflection of energy at this junction. Using Eq. 3.5, the matched feedforward matrix using wave mode amplitudes is

\[
F = -EI k^2 \begin{bmatrix}
-i k & k \\
1 & -1
\end{bmatrix}
\]

This feedforward can be transformed to the equivalent feedback of cross-sectional measurements at the junction. To this end, Eqs. 3.3 and 3.10 give

\[
G = -EI \begin{bmatrix}
(1 - i)k^3 & ik^2 \\
-ik^2 & -(1 + i)k
\end{bmatrix} = -EI \begin{bmatrix}
c_0^2(1 - i)^{\frac{3}{2}} & c_0^2 j\omega \\
c_0^2 j\omega & c_0(1 + i)^{\frac{1}{2}}
\end{bmatrix}
\]

where \(c_0\) is the constant in Eq. e of Example 2.1. This equation could have been found directly using Eq. 3.9. Notice that all of the terms in the right hand matrix are analytic in the right half side of the complex, Laplace plane. However, the diagonal entries can only be approximated through the use of rational transfer functions. Fortunately, the behavior of these terms can be accurately approximated over wide frequency ranges given adequate resources. A circuit which approximates this transfer function is described in Chapter 5 and in Reference 24. The solution is causal. This might be expected because the control is mimicking the internal forces and moments that would exist in the junction if the attached
beam member did not terminate but continued uniformly through the junction to infinity. This type of junction, mimicked by this control, exhibits no reflection and is typically referred to as a "matched termination."

Since the closed-loop scattering matrix is the zero matrix, Eq. 3.37 and Eq. a of Example 2.3 give the closed-loop junction power flow matrix as

\[
P_{CL} = 2\omega k^3 EI \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}
\]

This matrix is negative semidefinite and therefore the junction never behaves as a source. The zero eigenvalue corresponds to that incoming wave mode set that results in no displacement or rotation of the junction.

3.) This third part uses the presented optimal wave control formulation to derive a form of control which has found wide applicability in the fields of control due to its robustness and performance characteristics: colocated, dual rate feedback. Assuming that only a moment actuator is available at the left end, rate feedback of tip rotation through a gain \( c \) will be chosen and, with \( P_j \) being used in the cost functional, the form of the control effort penalty matrix will be found.

The control, in terms of cross-sectional variables, has the form

\[
M = \begin{bmatrix} 0 & i\omega \end{bmatrix} \begin{bmatrix} v \\ v' \end{bmatrix}
\]

The closed-loop scattering matrix is

\[
S_{CL} = \frac{1}{1 + 2 \left( 1 + i \right) \tau} \begin{bmatrix} -i + 2 \left( 1 + i \right) \tau & 1 + i \\ 1 - i & i + 2 \left( 1 + i \right) \tau \end{bmatrix}
\]

where the frequency dependent parameter \( \tau \) is

\[
\tau = \frac{\frac{c}{2} \omega^2}{\frac{1}{2EI^4 \rho A^4}}
\]

and the closed-loop junction power matrix is

\[
P_{CL} = \frac{16}{1 + 4 \tau + 8 \tau^2} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}
\]
Notice that dual rate feedback always yields a negative semidefinite closed-loop junction power matrix. However, since \( \tau \) is a function of frequency, the power dissipation characteristics are also frequency dependent.

An entire range of \( P_{21} \) and \( P_{22} \) matrices yield this control solution. Requiring that the \( P_{21}(2,2) \) and \( P_{22}(1,1) \) entries to be nonzero and equal to the power submatrices \( P_{01} \) and \( P_{00} \) (Eq. a of Example 2.3), and providing a control effort penalty equal to

\[
Control \text{ effort penalty} = \frac{R}{2 (EI)^2 k^4}
\]

gives

\[
R = \frac{1 - 2 (1 - i) \tau}{4 \tau}
\]

It can be seen in Eq. o that collocated rate feedback is equivalent to less penalty on control effort at high frequencies. The corner frequency of the control effort penalty increases in frequency as \( \tau \), and \( c \), are increased. In other words, the higher the rate feedback gain used, the less low frequency waves are controlled. This shows that specific levels of dual rate feedback are only optimal, in the sense of wave mode power flow, for certain frequency ranges and suggests a technique for selecting \( c \). Similar results for dual rate feedback of deflection at the end of a B-E beam are presented in Reference 24.

4.) Optimal noncausal control can be derived for the same beam end. If only moment actuation is used, cheap control can be derived (\( R = 0 \)). The wave mode amplitude feedforward gain matrix found using Eq. (3.34) is

\[
F = EI k^2 (1 + i) \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}
\]

The equivalent feedback matrix in terms of cross-sectional measurements (Eq. 3.35) is

\[
G = \frac{1}{2} \begin{bmatrix} (EI)^{3/4} (\rho A)^{1/4} & \frac{1}{\omega^2} \\ 0 & (-1 + i) \omega^2 \end{bmatrix}
\]

This only calls for rotation feedback and does so through a frequency dependent compensator which is similar to a half differentiator but with a 90 degree phase shift. The half differentiator exhibits half of the magnitude slope (1/2) and phase shift (45 degrees) that a full differentiator exhibits. This results in a closed loop scattering matrix of

\[
S_{CL} = \begin{bmatrix} 0 & 1 \\ -i & 1 + i \end{bmatrix}
\]

The resulting net power is
\[ P_{AVG} = \omega^2 EI w_i^H \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} w_i = w_i^H P_{CL} w_i \]

This closed loop power matrix has eigenvalues equal to 0 and -2. Therefore, the matrix is negative semidefinite and energy is never generated at the junction.

Figure 3.8 compares the closed-loop responses with the open-loop response using the feedbacks given in Eqs. d and q. The open loop system is provided with about 0.5% damping through the addition of a linear dashpot at the right junction. Note that the feedback in both cases result in 135 degrees phase lag and a logarithmic magnitude rolloff slope of -3/2 above 1 rad/sec. This matches the receptance relating force to displacement at the right end of a semi-infinite beam. In other words, at high frequencies (above 1 rad/sec in Fig. 3.8), where evanescent modes become insignificant, the beam behaves as if it were semi-infinite. Fig. 3.9 illustrates that the corresponding power flow out of the left junction is negative for all frequencies. Noting Fig. 3.9, the performance is slightly better at low frequencies for the optimal feedback, since evanescent waves are being exploited to increase junction power dissipation.

Figure 3.10 compares the closed loop transfer functions to the open loop transfer function with the dynamics of a torque wheel actuator included (Table 3.1). The actuator dynamics were ignored during control design, but need not be. Note that the controller derived by setting \( S_{CL}(1,1) = 0 \) results in an instability at 72 rad/sec. This instability was verified using a finite element model and is seen in Fig. 3.11 where the power flow out of the junction becomes a positive quantity near 67 rad/sec. An interesting feature can be seen by comparing Fig. 3.8 with 3.10 and Fig. 3.9 with 3.11. Note that when the compensator is derived based upon the correct model, resonant behavior appears to vanish. When not based upon the exact model, resonant behavior still exists, since wave cancelling is not exact, and instability occurs for the compensator in Eq. d.

The primary drawback to this solution is that the compensator is anticausal. While a half differentiator can be approximated with relative accuracy over a broad frequency range (see Chapter 5), a ninety degree phase shifter cannot, given the collocated feedback restriction. Therefore, a causal solution is required.

5.) This fifth part illustrates the use of a fixed form parameter optimization technique. The form of the anticausal compensator in Eq. q will be used with the exception that the phase is now 45 degrees. Since Eq. 3.35 does not change the causal nature of \( H \), the fixed form with variable gain is

\[ H = \alpha EI k (1 + i) [0 \ 1] \]
Figure 3.8 Magnitude (a) and phase (b) of beam transfer function from unit forcing at the right end to colocated transverse displacement for a) $S_{CL}(1,1)=0$ control, b) optimal noncausal control, and c) open loop. Actuator dynamics are not included.

Figure 3.9 Net power flowing out of the left junction for unit amplitude forcing at the right end for a) $S_{CL}(1,1)=0$ control and b) optimal noncausal control. Actuator dynamics are not included.

Figure 3.10 Magnitude (a) and phase (b) of beam transfer function from unit forcing at the right end to colocated transverse displacement for a) $S_{CL}(1,1)=0$ control, b) optimal noncausal control, and c) open loop. Actuator dynamics are included.

Figure 3.11 Net power flowing out of the left junction for unit amplitude forcing at the right end for a) $S_{CL}(1,1)=0$ control and b) optimal noncausal control. Actuator dynamics are included.
Notice that not only is this form positive real, it also causes the closed-loop junction power matrix in Eq. s to have the same frequency dependence as the open-loop power matrix. The eigenvalues are independent of frequency.

Using this form and the junction matrices in Example 2.1 in Eq. 3.46, for \( \Phi \) equal to the identity matrix, gives the trace as

\[
\text{trace} = -8 \alpha + 16 \alpha^2 \quad \frac{\partial \text{trace}}{\partial \alpha} = -8 + 32 \alpha = 0 \quad \alpha = \frac{1}{4}
\]

Substituting this minimizing value for \( \alpha \) into Eq. t and substituting this compensator into Eq. 3.35 gives

\[
G = \frac{EI}{2} \begin{bmatrix} 1 + i \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{3}{(EI)^4} (\rho A)^{\frac{1}{4}} (1 + i) \sqrt{\omega} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}
\]

The closed-loop scattering matrix for this system is

\[
S_{CL} = \frac{1}{2} \begin{bmatrix} 1 - i & 1 + i \\ 1 - i & 1 + i \end{bmatrix}
\]

and a closed-loop junction power matrix of

\[
P_{CL} = \omega^3 EI \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}
\]

Notice that the causal solution does not zero the \((1,1)\) entry of the closed-loop scattering matrix. In addition, while the closed-loop junction power matrix is still negative semidefinite, the eigenvalues are now 0 and -1 meaning that, for identical incoming wave mode sets (the eigenvectors are unchanged), the causal control absorbs half as much power as the noncausal control in part four.

6.) This fifth part illustrates the Wiener-Hopf solution to the optimal junction control of the free end of the B-E beam. The junction properties can be found in Examples 2.1 and 2.3. Extreme care must be exercised in the solution procedure to ensure that the LHPA and RHPA functions are properly handled.

First, the wave number relation will be expressed in terms of the complex Laplace variable, \( s \), as

\[
k = 4\sqrt{\rho A/\omega} = c_0 \sqrt{s} \sqrt{4-\sqrt{s}}
\]

where

\[
s = \sigma + i \omega
\]
The right side of Eq. y is the analytical continuation in the complex plane of the value of the left side on the imaginary axis. Notice that the substitution of $s = i\omega$ for positive and negative values of $\omega$ yield the principle square root of $\omega$. All roots in Eq. y are principle roots. The branch cuts corresponding to the fourth roots in Eq. y cut the entire real axis in the complex plane. The portion given by $s^{1/4}$ is RHPA and $(-s)^{1/4}$ is LHPA.

Second, the control will be constrained to only include the feedback of junction rotation to moment. Since only the rotation coordinate is desired as a function of the wave mode amplitudes, the $K$ matrix in Eq. 3.21 is given by

$$K = 2c_o 4\sqrt{s} 4\sqrt{-s} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Third, an appropriate form for $\Phi$ must be chosen in order to provide the required frequency behavior of the function in Eq. 3.49. It will also be assumed that no far field evanescent wave exists. Testing various forms,

$$\Phi = \frac{1}{(1 - s/o_n)^2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{(1 + s/o_n)^2}$$

exhibits satisfactory behavior.

Now, each of the terms in Eq. 3.50 can be evaluated, remembering that $A_{LHPA}$ and $H_{RHPA}$ are as yet unknown. The other three terms are given by

$$H_N = \frac{-4c_o^2 \sqrt{s} \sqrt{-s} s}{(1 - s^2/o_n^2)^2}$$

$$H_D = \frac{4\sqrt{s} 4\sqrt{-s}}{c_e E I}$$

$$H_C = \frac{8c_o \sqrt{s} \sqrt{-s}}{(1 - s^2/o_n^2)^2}$$

Equations ad and ae can be spectrally factored as

$$H_{DLHPA} H_{DRHPA} = \frac{4\sqrt{s}}{\sqrt{c_e E I}} \frac{4\sqrt{-s}}{\sqrt{c_o E I}}$$

$$H_{HRPA} H_{CLHPA} = \frac{2\sqrt{2} c_o \sqrt{s}}{(1 + s/o_n)^2} \frac{2\sqrt{2} c_o \sqrt{-s}}{(1 - s/o_n)^2}$$

Then
\[ H_{DLHPA}^{1} H_{N} H_{CLHPA}^{1} = \frac{\sqrt{2}}{(1 + s/\omega_n)^2} \left( \sqrt{2} \sqrt{\sqrt{s} - s} \right) s \]

Notice that the first term in this expression is RHPA and the second term contains both RHPA and LHPA parts. The positive time part of the expression in Eq. ah is given by

\[ PTP \left( H_{DLHPA}^{1} H_{N} H_{CLHPA}^{1} \right) = \frac{\sqrt{2}}{(1 + s/\omega_n)^2} \left( \sqrt{2} \sqrt{\sqrt{s} - s} \right) s - \frac{1}{4} \left( \omega_n + 5 s \right) \sqrt{\omega_n} \]

This can be verified by evaluating the remainder of Eq. ah minus Eq. ai. This remainder is LHPA. Evaluating Eq. 3.46 gives \( H_{RHPA} \) as

\[ H_{RHPA} = -\frac{1}{2} c_o EI \left( \sqrt{2} \sqrt{s} - \frac{1}{4} \left( \omega_n + 5 s \right) \sqrt{\omega_n} \right) \]

Figure 3.12 compares the magnitude and phase of this compensator for a corner frequency, \( \omega_n \), equal to 100 radians per second (solid lines) to those of the noncausal compensator from Eq. q (dotted lines). Notice that the phase of the compensator in Eq. aj equals that of the noncausal solution near the corner frequency. The magnitudes are identical at about 13 and 700 radians per second, with the Wiener-Hopf solution having less magnitude between these frequencies. In addition, since the magnitude of \( \Phi \) is constant below \( \omega_n \) and power increases as \( \omega^{5/2} \) for identical amplitudes, this solution technique may attempt to minimize power at frequencies near the corner frequency, \( \omega_n \).

This behavior might suggest that the solution is a tradeoff between an attempt to mimic the high performance noncausal solution near \( \omega_n \) and providing broadband power dissipation. If only the power at the frequency \( \omega_n \) were penalized, the Wiener-Hopf compensator would exhibit magnitude and phase identical to the noncausal solution evaluated at this frequency. Instead, the phase is close but the magnitude is less because the causal solution is distributing effort over a larger range.

Using wave models, a transfer function was derived between force and transverse displacement at the far end of the beam with that end modelled as a free boundary. The beam properties were \( EI = 31.1 \text{ N m}^2 \), \( \rho A = 2.85 \text{ Kg/m} \) and length = 7.32 m. Figure 3.13 compares the open-loop magnitude characteristics to those when the loop is closed using the compensator given in Eq. aj. Notice that good performance is achieved between 50 and 1000 rad/sec. However, the light damping associated with a mode near 30 rad/sec, with the support of phase information, that modes below this frequency are unstable. This occurs because the Wiener-Hopf solution was not constrained to guarantee a negative
Figure 3.12 Magnitude (a) and phase (b) for the optimal compensator derived using the Wiener-Hopf technique
Figure 3.13 Comparison of open (dotted) and closed-loop (solid) beam transfer function using Wiener-Hopf compensator
semidefinite closed-loop junction power flow matrix for all frequencies. Ensuring that this instability does not occur is a topic for future research.

**Example 3.4** This example derives junction control for the union of the two identical B-E beam segments shown in Fig. 3.14. The first part illustrates a conservative energy shunting solution. The second part uses active control to zero a chosen outgoing wave. The third part applies optimal control around the shunting junction from the second part. The fourth, fifth, and sixth parts yield noncausal, causal, and noncollocated solutions, respectively.

![Boundary condition at Bernoulli-Euler beam midsection](image)

**Figure 3.14** Boundary condition at Bernoulli-Euler beam midsection

1.) This first part analyzes the shunting options in the middle of a uniform Bernoulli-Euler beam when only a control moment is available. The boundary condition is

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
\end{bmatrix}
\begin{bmatrix}
v_l \\
v_l' \\
-Elv_l'' \\
v_r \\
v_r' \\
-Elv_r'' \\
Elv_r'' \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
M \\
\end{bmatrix}
\]

The junction relation is
\[
\begin{bmatrix}
w_{lp_1} \\
w_{re_1} \\
w_{rp_2} \\
w_{le_2}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix} \begin{bmatrix}
w_{rp_1} \\
w_{le_1} \\
w_{lp_2} \\
w_{re_2}
\end{bmatrix} + \frac{1}{4EI_k^2} \begin{bmatrix}
-1 \\
1 \\
-1 \\
1
\end{bmatrix} M b
\]

where the member power matrix and wave mode vector are given by

\[
P = 2\alpha k^2 EI
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -i & 0 & 0 & 0 & 0
\end{bmatrix}
; \quad w = \begin{bmatrix}
w_{rp_1} \\
w_{le_1} \\
w_{lp_2} \\
w_{re_2} \\
w_{lp_1} \\
w_{re_1} \\
w_{rp_2} \\
w_{le_2}
\end{bmatrix}
\]

Power incoming to the junction is defined to be negative.

With feedforward of the form

\[
M = 4iEI_k^2 \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \end{bmatrix} w_i
\]

the constraint in Eq. 3.19 gives

\[
F_{11}^H = \frac{-i F_{11}}{-i + 2 F_{11}} \quad ; \quad F_{12} = -i F_{11} \quad ; \quad F_{13} = -F_{11} \quad ; \quad F_{14} = i F_{11}
\]

Thus, the closed-loop scattering matrix, guaranteed to be conservative, is

\[
\begin{bmatrix}
w_{lp_1} \\
w_{re_1} \\
w_{rp_2} \\
w_{le_2}
\end{bmatrix} = \begin{bmatrix}
-iF_{11} & -F_{11} & 1 + iF_{11} & F_{11} \\
iF_{11} & F_{11} & -iF_{11} & -F_{11} \\
1 + iF_{11} & F_{11} & -iF_{11} & -F_{11} \\
-iF_{11} & 1 - F_{11} & iF_{11} & F_{11}
\end{bmatrix} \begin{bmatrix}
w_{rp_1} \\
w_{le_1} \\
w_{lp_2} \\
w_{re_2}
\end{bmatrix}
\]
While a whole range of solutions exist for $F_{11}$, the only solutions which zero entries in $S_{CL}$ while satisfying the constraints are

$$F_{11} = 0, i$$

The nontrivial solution ($F_{11} = i, F_{12} = 1, F_{13} = -i, F_{14} = -1$) gives a closed-loop scattering matrix

$$S_{CL} = \begin{bmatrix}
1 & -i & 0 & i \\
-1 & i & 1 & 1 - i \\
0 & i & 1 & -i \\
1 & 1 - i & -1 & i
\end{bmatrix}$$

Notice that incoming, propagating waves cannot penetrate the junction and depart as propagating waves. Given the stipulations that the closed-loop junction be conservative and that only one control input be used, Eq. h is the only nontrivial solution if the objective is to zero some entries.

2.) This part uses the same model as the previous part but adds an extra control input: transverse force. The new junction relation is

$$\begin{bmatrix}
w_{lp_1} \\
w_{re_1} \\
w_{rp_2} \\
w_{le_2}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix} \begin{bmatrix}
w_{rp_1} \\
w_{le_1} \\
w_{lp_2} \\
w_{re_2}
\end{bmatrix} + \frac{1}{4iEIk^3} \begin{bmatrix}
1 & -ik \\
-i & ik \\
1 & ik \\
-i & -ik
\end{bmatrix} \begin{bmatrix}
F \\
M
\end{bmatrix}$$

With feedback of the form

$$\begin{bmatrix}
F \\
M
\end{bmatrix} = 4iEIk^3 \begin{bmatrix}
F_{11} & F_{12} & F_{13} & F_{14} \\
F_{21} & F_{22} & F_{23} & F_{24}
\end{bmatrix} \begin{bmatrix}
w_i \\
k
\end{bmatrix}$$

the outgoing propagating wave in member 2 (see Figure 3.14) can be eliminated by letting

$$F_{21} = i (1 + F_{11}) ; \quad F_{22} = i F_{12} ; \quad F_{23} = i F_{13} ; \quad F_{24} = i F_{14}$$

This gives an intermediate junction relation
\[
\begin{bmatrix}
  w_{rP_1} \\
  w_{re_1} \\
  w_{rP_2} \\
  w_{le_2}
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 1 & 0 \\
  -1 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0 \\
  1 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  w_{rP_1} \\
  w_{le_1} \\
  w_{rP_2} \\
  w_{re_2}
\end{bmatrix} +
\begin{bmatrix}
  2 \\
  -1 - i \\
  0 \\
  1 - i
\end{bmatrix}
\begin{bmatrix}
  F_{11} & F_{12} & F_{13} & F_{14}
\end{bmatrix}
\]

Notice that the remaining control degrees of freedom cannot alter the third row in the scattering matrix. The remaining feedback terms will be derived in part 3 using optimal techniques.

3.) Continuing part 2, the wave mode amplitude feedback using the noncausal solution with no control effort penalty is

\[ F = -\frac{1}{4} \begin{bmatrix} 2 & 1 + i & 2 & 1 - i \end{bmatrix} \]

giving a closed-loop scattering matrix of

\[
S_{CL} =
\begin{bmatrix}
  0 & -\frac{1 + i}{2} & 0 & -\frac{1 - i}{2} \\
  -\frac{1 - i}{2} & i & \frac{1 + i}{2} & 3 \\
  0 & 0 & 0 & 0 \\
 \frac{1 + i}{2} & \frac{1}{2} & -\frac{1 - i}{2} & i
\end{bmatrix}
\]

The closed-loop junction power matrix is

\[
P_{CL} = 2\omega k^3 E I
\begin{bmatrix}
  -1 & -\frac{1 - i}{2} & 0 & -\frac{1 + i}{2} \\
  -\frac{1 + i}{2} & \frac{1}{2} & -\frac{1 - i}{2} & i \\
  0 & -\frac{1 + i}{2} & -1 & -\frac{1 - i}{2} \\
  -\frac{1 - i}{2} & \frac{i}{2} & \frac{1 + i}{2} & \frac{1}{2}
\end{bmatrix}
\]

This junction power matrix has eigenvalues of 0, -2, -1.62, and 0.62, as compared to 0, 0, -1.73, and 1.73 for the intermediate junction power matrix. Therefore, neither the intermediate or closed-loop junction power matrices are at least negative semidefinite. This is not a surprise for the intermediate case where no attention was paid to junction power flow. However, in the optimal case where the control is guaranteed to increase the junction

85
dissipation by adding a negative semidefinite junction power matrix to the open-loop power matrix, the control degrees of freedom were insufficient to make the closed-loop junction power matrix negative semidefinite. This occurred because the intermediate scattering matrix, used in the control, was indefinite (see \( \lambda \) eigenvalues).

Remember that closed-loop junction dissipation is guaranteed only if the open-loop junction, the intermediate junction in this example, is conservative or dissipative. In order to achieve a negative definite closed-loop junction power matrix, it would be necessary to either change the type of control input or supplement the control with the addition of another type of actuator. In this example, two additional unique actuators would have to be added in order to completely specify the closed-loop scattering matrix but the addition of one would probably be sufficient to guarantee closed-loop dissipation.

4.) In this part, optimal control is formulated for the external moment at the B-E beam midsection shown in Fig. 3.14. Using the \( S, y, P_{oi}, \) and \( P_{oo} \) matrices, the optimal noncausal control is

\[
M = -2 \, EI \, k^2 \, \begin{bmatrix} 1 & -i & -1 & i \end{bmatrix} \, w_i
\]

Using the transformation matrix between wave mode amplitudes and cross-sectional variables, shown in Eq. q, and the boundary relation,

\[
Y = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
-i k & -k & 0 & 0 & ik & k & 0 & 0 \\
-i E i k^3 & E i k^3 & 0 & 0 & i E i k^3 & -E i k^3 & 0 & 0 \\
-E i k^2 & E i k^2 & 0 & 0 & -E i k^2 & E i k^2 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & ik & k & 0 & 0 & -ik & -k \\
0 & 0 & i E i k^3 & -E i k^3 & 0 & 0 & -i E i k^3 & E i k^3 \\
0 & 0 & -E i k^2 & E i k^2 & 0 & 0 & -E i k^2 & E i k^2
\end{bmatrix}
\]

the equivalent feedback of cross-sectional variables is

\[
M = -2 \, i \, (1 + i) \, (EI) \, \left( \rho A \right) \, \frac{1}{4} \, \frac{1}{\omega^2} \, \nu_i
\]

The closed-loop scattering and junction power flow matrices are
\[ S_{CL} = \frac{1}{2} \begin{bmatrix} 1 & -i & 1 & i \\ -1 & i & 1 & 2i \\ 1 & i & 1 & -i \\ 1 & 2i & -1 & i \end{bmatrix} ; \quad P_{CL} = \omega k^3 EI \begin{bmatrix} -1 & i & 1 & -i \\ -i & -1 & i & 1 \\ 1 & -i & -1 & i \\ i & 1 & -i & -1 \end{bmatrix} \]

with \( P_{CL} \) being negative semidefinite as required.

This feedback could be easily approximated, if the initial ninety degree phase shift \((i)\) were not present. If this were the case, the feedback would be a constant times the half derivative of rotation of the junction. However, the phase shift contains dynamics which are not analytic in either plane.

5.) An implementable, causal optimal solution can be found using the fixed form parameter optimization technique. The chosen form equals the compensator given in Eq. \( r \) multiplied by \((-1)^{1/2} = i\). The feedback solution is given by

\[ M = -2 (EI)^4 (\rho A)^4 (1 + i) \frac{1}{\omega^2} v_1 \]

The closed-loop scattering and junction power flow matrices are

\[ S_{CL} = \frac{1}{4} \begin{bmatrix} 1-i & -1-i & 3+i & 1+i \\ -1+i & 1+i & 1-i & 3-i \\ 3+i & 1+i & 1-i & -1-i \\ 1-i & 3-i & -1+i & 1+i \end{bmatrix} ; \quad P_{CL} = \frac{1}{2} \omega k^3 EI \begin{bmatrix} -1 & i & 1 & -i \\ -i & -1 & i & 1 \\ 1 & -i & -1 & i \\ i & 1 & -i & -1 \end{bmatrix} \]

Again, the closed-loop junction power flow matrix is negative semidefinite. The feedback compensator in Eq. \( t \) now has the proper magnitude and phase properties indicative of a function that is analytic in the right half plane (RHPA). Notice, however, that by requiring a causal solution with a form similar to the noncausal solution, half of the power dissipation performance has been sacrificed.

6.) Since future junction rotation information is required in the noncausal solution, an alternative to Wiener-Hopf would be to sense upstream in both directions. In the nondispersive compression rod case, sensing at a point upstream provided a time delay, linear phase lead. However, in a dispersive medium such as the B-E beam, the phase lead is neither a constant or linear phase lead.

In this case, it is better to allow spatially distributed sensing upstream of the junction. This would require that the control moment be a convolution of the noncollocated state information with a feedback kernel. This feedback has the form
\[ M(0, \omega) = \int_{-\infty}^{\infty} G(z, \omega) v'(z, \omega) \, dz \]

Transforming this convolution and assuming that the feedback at every noncollocated measurement is proportional to the half derivative of rotation, the transform of \( G(z, \omega) \) simply has to be equal to \((-1)^{1/2} = i\). Inverse transforming the constant \( i \) gives

\[ \int_{-\infty}^{\infty} i e^{-ikz} \, dk = \frac{\pi}{z} \]

Therefore, the feedback in the form of Eq. 4 is

\[ M(0, \omega) = -\frac{2}{\pi} \frac{3}{(EI)^4} (\rho A)^4 \int_{-\infty}^{\infty} \frac{1}{z} (1 + i \sqrt{\omega}) v'(z, \omega) \, dz \]

Notice that the noncollocated information phase shifts the spatial deflection pattern by ninety degrees. But, this is equivalent to a temporal phase shift of the same amount. However, it can only be implemented in the spatial domain because all information is available.

### 3.5 Summary

This chapter has presented several methods for deriving discrete point and local structural controllers. The control objectives have been formulated in terms of waves incoming to and outgoing from these locations. Instead of trying to control a few global structural modes which affect a physical property, the wave controllers model the entire spectrum of wave modes. Since the entire frequency behavior is modelled, this allows the derivation of algorithms which have adequate characteristics to enable proper controller rolloff without generating destabilizing spillover effects. This broad bandwidth type of control often results in benign functions of frequency which are more easily implemented than individual mode controllers.

Since junction control is implemented on a spatially local scale, control objectives can be stated in terms of altering how energy is transported from one location to another through the active junction. This enables the creation of structural "diodes" which dynamically isolate one region from another. Alternately, matched terminations can be created which, depending on the rest of the structure, can extract all energy in a finite time interval.

The wave formulation provides the use of all cross-sectional, measurable quantities to perform specified and optimal control. The specified control does not guarantee power
conservation or dissipation for all incoming wave mode sets. An iterative design approach is needed to derive the control, check junction power flow, and then redesign if necessary. Neither specified or noncausal optimal control guarantee causality. Therefore, for the optimal approach, several causal solutions were presented. A fixed form parameter optimization procedure was used where the chosen compensator form was restricted to be causal. The manner in which the Weiner-Hopf approach was constrained guaranteed a causal solution but did not guarantee that the junction behaved as a power sink at all frequencies. Noncollocated feedback, continuously distributed and therefore difficult to implement, enables the globally optimal solutions to be implemented.

All of these junction controllers do not contribute to instability if they do not behave as energy sources. The optimal, noncausal control is guaranteed to create a power sink given that the open-loop junction was at least conservative.

Now that local performance of these controllers has been determined, a method is needed to examine open and closed-loop global performance. One important issue that requires a global perspective is actuator placement. If the controller performs well locally, absorbing all impinging energy, but is placed in a remote structural location where only a small fraction of the total energy in the structure is able to penetrate, the controller will not significantly alter the global response. For the noncausal and fixed form parameter control formulations, global stability can be guaranteed without the use of global models if the every constituent member and junction does not generate power. This is done on the local level by guaranteeing local power dissipation. Global issues are addressed in detail in Chapter 4.
4. NETWORK DYNAMICS AND POWER FLOW

A structural network description is needed in order to determine the manner in which the local junction and member controllers manifest themselves on the global scale. In addition, it can provide insight into characteristics such as damping, structural modes, dynamic load paths. The important issue of placement determines whether a local controller, while possessing adequate local performance, contributes significantly to the global closed-loop behavior. To this end, this chapter introduces three concepts: network dynamics, net power flow, and total power flow. The network dynamical formulation derives transfer functions from disturbances to arbitrarily selected wave modes or cross-sectional variables. This is done by monitoring how the steady-state wave modes travel throughout the structural network to eventually close upon themselves.

As in Statistical Energy Analysis\(^7\), where component dynamical interactions are used to characterize vibrational power flow through the structure, network wave dynamics can be used to identify net and total power flow in junctions and members. Net power flow is the difference in flow in one direction as compared to another. This helps to identify power sources and sinks because, in most cases, steady-state power flows from source to sink. This quantity is most beneficial in analyzing the passive or active dissipation characteristics of a structure. Total power flow is the sum of all power flow in a component, independent of direction. This quantity is best used to identify, as a function frequency, those network paths which support the most vibrational energy. This can be used for passive design analysis or for selecting control locations. For example, if the control objective is to absorb as much vibrational energy as possible in a particular frequency range, it would be most useful to place the controller on the structural component which exhibits the most energy, or power flow, in that same frequency range.

The following sections discuss these three network concepts in detail along with illustrative examples. The final section discusses some other useful ways that this network information might be employed.

4.1 Network Dynamics

A global network description can be assembled from a collection of member descriptions, Eq. 2.5, and junction descriptions, Eq. 2.10a. This frequency domain model can then yield transfer functions between arbitrarily selected disturbance and response locations. The result is identical to that achieved through techniques based upon the exact solution of the governing partial differential equations and boundary conditions\(^30\). The technique used in this section is unique in that it builds upon component descriptions in
terms of wave behavior. This network dynamics formulation was previously presented in Reference 9.

To assemble a global model, one first chooses the cross-sections at which the response information is desired; response cross-sections. Then, a member model must be prepared in the form of Eq. 2.5 for each member segment that either connects response cross-sections to adjacent ones or to adjacent junctions. A junction model in the form of Eq. 2.10a must be derived for each junction. These can then be assembled into a large matrix equation of the form

\[ \hat{W} = S \hat{W} + \Psi Q \]  

where

\[ \hat{W} = \hat{W}(w_1,..,w_n) \] is a large vector of all wave modes that can exist at all response cross-sections.

\[ S = S(\xi_1,..,\xi_n,\xi_1,..,\xi_m) \] is a matrix whose nonzero entries describe how wave modes in \( \hat{W} \) result from others in \( \hat{W} \) through junction scattering and member transmission.

\[ \Psi = \Psi(\xi_1,..,\xi_n,\psi_1,..,\psi_p) \] is a matrix relating how external disturbances generate waves in \( \hat{W} \).

\[ Q = \Omega(Q_1,..,Q_p) \] is the vector of all disturbance inputs.

Typically, the diagonal entries in \( S \) must be zero since the process of scattering and propagation described by \( S \) is causal. Just as with the FEM, care must be taken when assembling the global matrices, \( S \) and \( \Psi \). An automated approach with local and global wave mode numbering systems is possible. This method is illustrated in Example 4.1.

Equation 4.1 is now a global network model with the dependent variables being the complex amplitudes of the wave modes at specified locations in the network. This equation can be solved to yield the steady-state wave mode transfer functions.

\[ \hat{W}(\omega) = (I - S(\omega))^{-1} \Psi(\omega) \quad \Omega(\omega) = H(\omega) \quad \Omega(\omega) \]  

where \( I \) is the identity matrix of compatible dimension. Each element of \( H(n,m) \) corresponds to a steady-state wave mode \( w_n \) resulting from a particular disturbance \( Q_m \).

The phase closure principle states that the roots of the determinant of \( (I - S(\omega)) \) correspond to the frequencies of those steady-state wave trains which, after circumnavigating the network, return to constructively interfere with themselves. This corresponds to a total spatial phase change of \( 2n\pi \). The frequencies yielding zero entries in \( (I - S)^{-1}\Psi \) correspond to those wave trains which destructively interfere with a total phase change of \( (2n+1)\pi \). The mathematically exact poles and mode shapes of an arbitrarily complex structural network can, in principle, be found using this technique.
Equation 4.2 gives the response in terms of wave mode amplitudes at all of the chosen response cross-sections in the network. The response in terms of physical variables at any point in the network can be reconstructed using the transformation in Eq. 2.6 and, if necessary, Eq. 2.5.

4.2 Network Power Flow

The tools developed thus far can be used to obtain the global steady-state power behavior of a structural network. Using the wave mode transfer function matrices of Eq. 4.2, the steady-state amplitudes can be substituted into an Eq. 2.19 and 2.22b for each member and junction, respectively.

Two types of power information, based on the disturbance inputs in $Q$, can be found. First, the magnitude and sign of net power flow in each member identifies the manner in which power is being transported in and extracted from the structure. The magnitude of a member's net power flow indicates the degree to which power flowing in one direction exceeds that flowing in the other. The sign reveals the direction of this net flow. In such an analysis, zero net power flow can either indicate no member motion or member motion composed of equal amounts of power flowing in opposite directions. For this reason, a second quantity of interest.

Total member power flow reveals the total amount of power being carried in a member, independent of direction. This is directly related to the energy of member motion. Total power information can be used to aid in the placement of actuators, identifying major dynamic load carrying members, predicting maximum over stresses, or estimating member fatigue life. In other words, it reveals those critical paths through which most of the power in the structure flows.

Power information can also be obtained at the junctions. This can be used to find levels of imparted or extracted power in addition to characterizing interactions between external excitations. Though this is an easy task to perform using the tools developed, the subsequent sections only focus on member power flow.

4.2.1 Net Power Flow There are basically three steps in finding the net power flow in the network members of interest. First, choose a cross-sectional location on each member of interest and assemble a vector containing all of the wave mode amplitudes that exist at all of the chosen cross-sections. Second, solve for the steady-state amplitudes of these wave modes using Eq. 4.2. Separate this equation into equations governing the wave mode vector for each individual cross-section.
\[
\begin{bmatrix}
w_1 \\
w_2 \\
\vdots \\
w_n \\
\end{bmatrix}
= \begin{bmatrix}
H_1 \\
H_2 \\
\vdots \\
H_n \\
\end{bmatrix} Q
\]

Third, evaluate member power flow with Eq. 2.19 to get

\[\text{Net Power} = \frac{1}{2} w_m^H P_m w_m = \frac{1}{2} Q^H H_n^H P_m H_n Q\]

\[= \frac{1}{2} Q^H P_{\text{net}} Q\]

Solving a relation of this type for each member gives a representation of how energy is being imparted, distributed and dissipated in the structure. Notice that the net power information associated with each member is contained in a square matrix of the same order as the number of disturbances. This allows the relative contributions from each input, separately and in combination, to be determined.

4.2.2 Total Power Flow  Total power flow is quite a different quantity from net power flow. To characterize total power flow, the absolute values of the power carried by the various independent power waves must be summed. This can be done for an individual member by transforming from wave modes which propagate disturbances independently to those which propagate power independently (independent power waves).

As demonstrated in Chapter 2, a similarity transformation of the form

\[w_m = T_P w_p\]

enables diagonalization of \(P_m\) in Eq. 2.20. Each wave group in \(w_p\) propagates power independently. Substituting this relation into Eq. 2.20 gives a member net power relation of the form

\[P_{AVG} = \frac{1}{2} w_p^H T_P^H P_m T_P w_p = \frac{1}{2} w_p^H P_p w_p\]

This new power matrix \(P_p\) is diagonal with real entries, the eigenvalues of the hermitian matrix \(P_m\). If these eigenvalues are replaced by their absolute values, the transfer function relations (Eq. 4.2) can be substituted to determine the steady-state total power flow.

93
Total Power = \frac{1}{2} w_p^H \text{abs}(P_p) w_p \\
= \frac{1}{2} Q^H H_n^T P_T \text{abs}(P_p) T_p^H H_n Q \\
= \frac{1}{2} Q^H P_{tot} Q

This is identical to the third step used to determine net power flow in networks. Examples of this transformation (Eq. 4.5) are given in Examples 2.2 and 2.3.

The results of this analysis enable the identification of those critical members which exhibit the greatest energy of vibration resulting from the total disturbance to the network. Net and total power flow analyses can also be used at junctions, excitations, and dissipation points to analyze scattered, input, and dissipated power. The following two examples illustrate the member power flow analysis for two structural networks.

Example 4.1 This example presents a closed form solution to network power flow, building upon Examples 2.2 and 2.4. As seen in Figure 2.3, the network consists of a uniform rod in compression with two input locations; one located at the left end, F_A, and the other at the midpoint, F_B and ΔC. The right end of the rod is attached to a linear viscous damper of strength (c). The objective of this example is to determine the net and total power flow at the 1/4 and 3/4 length cross-sections; denoted by (a) and (b), respectively. Since the rod has been given no distributed damping, the solutions are invariant to position changes about (a) and (b) as long as they remain in their respective rod segments. The cross-section positions were only chosen to simplify the derivation. Distributed damping could be introduced.

The component dynamical relations are

Member transmission matrices:

\[ \xi_{Aa} = \xi_{aB} = \xi_{bB} = \xi_{bC} = \xi = e^{-ikl/4} \quad \text{a} \]

Member power matrix:

\[ P = 2\alpha kEA \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{b} \]

Junction A:

\[ w_{1A} = w_{2A} + \frac{1}{ikEA} F_A \quad \text{c} \]

Junction B:
\[
\begin{bmatrix}
    w_{1B} \\
    w_{2B}
\end{bmatrix} =
\begin{bmatrix}
    0 & 1 \\
    1 & 0
\end{bmatrix}
\begin{bmatrix}
    w_{3B} \\
    w_{4B}
\end{bmatrix} - \frac{1}{2ikE\alpha} \begin{bmatrix}
    ikE \alpha & 1 \\
    -ikE \alpha & 1
\end{bmatrix}
\begin{bmatrix}
    \Delta_c \\
    F_B
\end{bmatrix}
\]

Junction C:
\[
w_{1C} = \frac{kE\alpha - \omega_c}{kE\alpha + \omega_c} w_{2c} = \tau w_{2c}
\]

The external excitation, \( \Delta_c \), is zero in this example. Positive power flow is defined towards the right. Assembling these dynamical relations we get a network model in the form of Eq. 4.1
\[
\begin{bmatrix}
    w_{1a} \\
    w_{2a} \\
    w_{1b} \\
    w_{2b}
\end{bmatrix} = \xi^2 \begin{bmatrix}
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & \tau \\
    1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    w_{1a} \\
    w_{2a} \\
    w_{1b} \\
    w_{2b}
\end{bmatrix} + \frac{\xi}{2ikE\alpha} \begin{bmatrix}
    2 & 0 \\
    0 & -1 \\
    0 & 0 \\
    0 & -1
\end{bmatrix}
\begin{bmatrix}
    F_A \\
    F_B
\end{bmatrix}
\]

This can be solved to find the transfer function matrices which relate external inputs to waves at the two response cross-sections.
\[
\begin{bmatrix}
    w_{1a} \\
    w_{2a}
\end{bmatrix} = \frac{\xi}{1 - \xi \tau \frac{\xi^2}{2ikE\alpha}} \begin{bmatrix}
    2 & -\xi (1 + \xi^4 \tau) \\
    2\xi^6 \tau & -(1 + \xi^4 \tau)
\end{bmatrix}
\begin{bmatrix}
    F_A \\
    F_B
\end{bmatrix}
\]
\[
\begin{bmatrix}
    w_{1b} \\
    w_{2b}
\end{bmatrix} = \frac{\xi}{1 - \xi \tau \frac{\xi^2}{2ikE\alpha}} \begin{bmatrix}
    2\xi^4 \tau & -\xi^2 (1 + \xi^4 \tau) \\
    2\xi^2 & -(1 + \xi^4 \tau)
\end{bmatrix}
\begin{bmatrix}
    F_A \\
    F_B
\end{bmatrix}
\]

Notice that if \( \tau = 1 \) (\( c = 0 \)), the wave motion is unbounded when \( k = n\pi/l \), as would be expected. Also notice that a zero exists between \( F_B \) and the two wave modes when \( k = (2n-1)\pi/l \). These zeros are independent of the damping.

The next step is to substitute Eqs. g and h into Eq. 4.4 to get the net power flow in each of the members.
\[
P_{\text{net}_a} = c_0 \begin{bmatrix}
    2 (1 - \tau^2) & \tau^2 e^{ikl/2} + 2i\tau \sin(3kl/2) - e^{ikl/2} \\
    \tau^2 e^{ikl/2} - 2i\tau \sin(3kl/2) - e^{ikl/2} & 0
\end{bmatrix}
\]
\[
P_{\text{net}_b} = c_0 (1 - \tau^2) \begin{bmatrix}
    2 & -2 \cos(kl/2) \\
    -2 \cos(kl/2) & 1 + \cos(kl)
\end{bmatrix}
\]

95
where
\[ c_0 = \frac{1}{A \sqrt{\rho E}} \frac{1}{1 - 2\tau \cos(2kl) + \tau^2} \]

Several interesting and expected features can be seen in this solution, remembering that the magnitude of \( \tau \) is less than or equal to one (\( \tau = 1 \) for a free end condition and \( \tau = -1 \) for a clamped end condition). First, if \( F_A = 0 \) then there is zero net flow in member (a) and the net flow in (b) is towards the damper at all frequencies. Second, if \( F_B = 0 \) then the flows in the two rod sections are equal and towards the damper at all frequencies. Third, if there is no damper \( c = 0 \) (\( \tau = 1 \)) then there is no net flow in member (b), but net flow can exist in member (a) due to the interaction between the two disturbances \( F_A \) and \( F_B \);

\[ P_{net_a} = \frac{i}{2A \sqrt{\rho E}} \frac{\sin(kl/2) + \sin(3kl/2)}{1 - \cos(2kl)} (F_A^H F_B - F_A F_B^H) \]

The magnitude of net power flow depends on the magnitudes of both \( F_A \) and \( F_B \) and their relative phase (+/- 90 degrees maximizes the flow). The direction of flow, from \( F_A \) to \( F_B \) or vice versa, depends on which disturbance leads the other and this depends on the intervening wave trains. Three wave-to-rod length scale ratios, \( kl/2, 2kl, \) and \( 3kl/2 \), determine which disturbance leads. This last result is important because it illustrates that the net power in a member is not simply the sum of the net flows due to each disturbance source, separately. Just like evanescent waves, external disturbances can propagate power through their interaction and should not be analyzed separately.

The next step is to derive the total power flow in each member. This is done using Eq. 4.7 to get

\[ P_{tot_a} = c_0 \begin{bmatrix} 2 \left( 1 + \tau^2 \right) & -\tau^2 e^{ikl/2} - 2\tau \cos(3kl/2) - e^{ikl/2} \\ -\tau^2 e^{-ikl/2} - 2\tau \cos(3kl/2) - e^{ikl/2} & 1 + 2\tau \cos(kl) + \tau^2 \end{bmatrix} \]

\[ P_{tot_b} = c_0 \left( 1 + \tau^2 \right) \begin{bmatrix} 2 & -2 \cos(kl/2) \\ -2 \cos(kl/2) & 1 + \cos(kl) \end{bmatrix} \]

Several interesting features can be seen in this solution. First, if \( F_A = 0 \) and the magnitude of \( \tau \) is less than unity then the total power in member (a) far exceeds that in member (b) at most frequencies making member (a) the critical member. Second, if \( F_B = 0 \), both members are equally critical because the same power flows through each member enroute to the damper.
The ratio of net-to-total power flow in each member yields information as to that fraction of total power which manifests itself as net flow. By definition, the ratio magnitude must always be less than or equal to unity. For example, if $F_B = 0$ then the power ratios, $\eta$, are identical and independent of frequency.

$$\eta_a = \eta_b = \frac{1 - \tau^2}{1 + \tau^2}$$

If $F_A = 0$ then

$$\eta_a = 0$$

$$\eta_b = \frac{1 - \tau^2}{1 + \tau^2}$$

These ratios define the degree of flow directionality. In other words, the closer $\eta$ is to zero, the more balanced the flow in both directions: the flows are equal. The closer the magnitude of $\eta$ is to unity, the more the power flow exhibits a directional preference. Notice that if there is no damping ($\tau = 1, -1$) then there is no net flow, $\eta_a = \eta_b = 0$, and power flow is balanced in both directions. If the right end of the rod is impedance matched ($\tau = 0$ for a matched termination), there is no reflection at that end and the three $\eta$'s in Eqs. o and q equal unity indicating that net power flow equals total power flow in the respective members for the given disturbance configuration. Notice that the sign of $\eta$ indicates the direction of net flow. When more than one disturbance is present, the relative magnitudes and phases between disturbances must be included in the evaluation of Eqs. 4.4 and 4.7.

Example 4.2. The second example involves the numerical determination of net and total power flow in a more complex structure. The example structure, shown in Figures 1.2 and 4.1, resembles the proposed dual keel space station. The horizontal members (a,b,d,f,g, and i) are modeled as undamped torsion rods and the vertical members (c,e,h, and j) as undamped Bernoulli-Euler beams. Material damping could easily be introduced through the use of complex moduli. The positions marked by the hatching in Fig. 4.1 each contain a one degree-of-freedom torsional resonant body with relative motion dampers, modeling discrete flexibility in the crew modules and solar arrays shown in Fig. 1.2. At junctions 1 and 6, a torsional disturbance can be applied between the body and rod, modeling solar panel motor drives. At junction 9 a disturbance can be introduced which is inertially referenced and acts on the body, modeling crew motion. For simplicity, all power quantities were determined at the midsections of the members. The parameter values used are listed in Table 4.1. Appendix A contains the details of the problem formulation.
Figure 4.1 Model of proposed dual keel space station. Numerals and letters denote junctions and members, respectively. Shaded junctions contain internal dynamics. Horizontal members are modeled as torsion rods and vertical members as Bernoulli-Euler Beams.

Table 4.1. Specifications Used for Space Station Model Analysis

\[
\begin{align*}
GJ & = 3.45\times10^6 \text{ lb ft}^2 \\
\rho J & = 43.95 \text{ lb sec}^2 \\
EI & = 9.86\times10^7 \text{ lb ft}^2 \\
\rho A & = 0.781 \text{ lb sec}^2 \\
l_{12} & = l_{23} = l_{34} = l_{45} = l_{56} = l_{78} = l_{27} = 150 \text{ ft} \\
l_{29} & = l_{59} = 75 \text{ ft} \\
I_p & = 2.08\times10^6 \text{ lb ft sec}^2 \\
K_p & = 8.22\times10^5 \text{ lb ft} \\
C_p & = 1.05\times10^5 \text{ lb ft sec} \\
I_m & = 1.60\times10^7 \text{ lb ft sec}^2 \\
K_m & = 6.32\times10^8 \text{ lb ft} \\
C_m & = 1.41\times10^7 \text{ lb ft sec}
\end{align*}
\]

The first numerical case involved only the disturbance at junction 9. As expected from symmetry, nonzero net power flow only occurred in members a, b, f, and g and flowed away from junction 9 toward the dampers at junctions 1 and 6, at all frequencies. Net power flow in the other members was zero. These results served to check the computer program.

The second case involved only a disturbance at junction 1. Figures 4.2a and 4.2b show the magnitude of the net flow in the members and Fig. 4.3 shows the direction of net
Figure 4.2 Magnitude of net power flow in the members as a result of a disturbance at junction 1. Some members have identical flow due to symmetry.
flow for various frequency ranges. Consistent with its definition, the net flow into a junction equals the sum of power dissipated at and departing from that junction. Comparing the frequencies at which the flow changes direction (Fig. 4.3) with the structural behavior shown in Fig. 4.2 indicates that these flow reversals correspond to resonances and zeros of the structural response.

Interesting and unexpected behaviors can be seen in this case. First, as would be expected, net power flows away from the disturbance at junction 1 and towards the damper at junction 6, at all frequencies. Nonintuitive are the direction reversals in the other
members which sometimes cause the damper at junction 9 to absorb power from both
directions (parts 4, 7, and 10 in Fig. 4.3).

Most significant is the occurrence of power circulation. While one might expect the
net flow in member (a) to be the largest, since subsequent damping and splitting at
junctions should reduce individual member flow, net power flow is actually largest in other
members for certain frequency ranges. As seen in part 2 of Fig. 4.3, net flow is a
maximum in member (b) (dotted curve in Fig. 4.2a), and for a portion of the frequency
range in part 3, net flow is a maximum in member f. In part 8 of Fig. 4.3, the net flow in
members (a) and (g) are almost identical meaning the flows in members (b) and (f),
through the damper at junction 9, are negligible (see Figs. 4.2a and 4.2b). The effects in
parts 2 and 3 are due to storage of energy in the structure in a circulating fashion so as to
result in net flow.

Several researchers have mathematically demonstrated the existence of
power circulation, or power vortices, in acoustic media and have presented necessary and
sufficient conditions for their existence. Perhaps the closed contour associated with the
dual keel design is necessary for the existence of this phenomenon.

Figures 4.4a through 4.4c show the magnitudes of total power flow in each of the
members, (a) through (g), with those for members (h) through (j) omitted due to
symmetry. Table 4.2 lists the members with the most total power flow in order of
magnitude, for various frequency ranges. Notice that the frequencies corresponding to a
reordering of critical paths in Table 4.2 seldom match those corresponding to direction
changes in net flow in Fig. 4.3. This emphasizes the significant difference in information
provided by net and total power flow.

From Table 4.2, member (a) is the critical path for the widest range of frequencies.
For a decade, between 0.4 Hz and 4 Hz, it is by far the critical member. This is
characterized by a small power ratio, $\eta$, in this range (compare Fig. 4.2a with 4.4a). In
frequency ranges where member (a) does not contain the most power, it is often a close
second. Members (b), (f), and (d) are the critical paths during other portions of the
frequency range calculated. Members (c) and (e) are never critical in the range analyzed,
from 0.01 Hz to 10 Hz. Notice, in Figs. 4.2a, 4.4a and 4.4b, that while the net flow
through members (a) and (c) are always similar, within an order of magnitude, the total
flow is very disproportionate.

A second analysis was performed with junctions 3 and 4 controlled as matched
terminations. In other words, the scattering matrices of these two junctions equaled the
null matrix. A white noise disturbance, with identical RMS amplitude level as that used in
the first analysis, was applied at junction 1. Figure 4.5, 4.6 and 4.7 show the magnitude

101
Figure 4.4 Magnitude of total power flow in the members as a result of a disturbance at junction 1. Some members have identical flow due to symmetry.
of net power flow, the direction of net power flow, and the magnitude of total power flow, respectively.

The most notable and expected result was that member d, between the two matched junctions, exhibits both zero net and zero total power flow. It is dynamically isolated from the rest of the station. In addition, the energy from a disturbance on member d will not reflect at the active junctions. Several other interesting results also occur.

As seen in Fig. 4.6, the net flows in members a, c, e and g are towards junctions 2, 3, 4 and 6, respectively. Power circulation, while prohibited in the members c, d and e, still occurs in the other end of the keel. Fig. 4.5 shows that the net flow in member a is almost always equal to that in member c. This implies that a high percentage of the power imparted to the structure at junction 1 is drained from the structure at the matched junction (3). For this reason, little of the imparted power participates in circulation and neither member b or j exceed member a in net power flow.

Figure 4.5, when compared with Figs. 4.2a and b, shows that the matching control restricts the damper at junction 9 to participating in energy dissipation only at higher frequencies. In the controlled case, a significant difference in net power flow only occurs above 2 rad/sec (Fig. 4.5c). In the uncontrolled case, a significant difference occurred as low as 0.4 rad/sec (Figs. 4.2a and b). Notice that these frequencies correspond, in Figs. 4.3 and 4.6, to the first occurrence of net flow in members b and f being towards junction 9. Notice that the isolated resonance of the module at junction 9 is at about 6 rad/sec.

In general, the control highly damped some resonances while it eliminated others. Most significant is that high performance is achieved in member d using a controller on each of the two load carrying paths which join member d to the rest of the structure. This control is derived independent of a network model and performance is independent of changes made to other portions of the structure.

4.3 Controller Placement

Network transfer functions and power flow are not only useful from the perspective of disturbance response, but have significance with regards to actuator placement. As discussed in Section 4.2, the identification of major total or net power flow paths enables the identification of structural regions where control might be most effectively implemented. While wave control can provide high local performance, absorbing all impinging energy, if these controllers are not placed on critical paths, their global performance may prove to be mediocre.

This performance discrepancy arises due to the use of local models. Since the remainder of the structure is assumed to be unknown when generating the local junction
Figure 4.5 Magnitude of net power flow in the members of the **controlled structure** as a result of a disturbance at junction 1. Some members have identical flow due to symmetry.
Figure 4.6  Direction of net power flow through the controlled structure for seven different frequency ranges. Frequency units are radians/second

Table 4.2.  Relative Rankings of Critical Total Power Members

<table>
<thead>
<tr>
<th>Frequency Range (rad/sec)</th>
<th>Member</th>
<th>Max.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01-0.14</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>0.14-0.18</td>
<td>b</td>
<td>a</td>
</tr>
<tr>
<td>0.18-0.19</td>
<td>b</td>
<td>f</td>
</tr>
<tr>
<td>0.19-0.21</td>
<td>b</td>
<td>f</td>
</tr>
<tr>
<td>0.21-0.23</td>
<td>b</td>
<td>f</td>
</tr>
<tr>
<td>0.23-0.25</td>
<td>f</td>
<td>a</td>
</tr>
<tr>
<td>0.25-0.29</td>
<td>f</td>
<td>a</td>
</tr>
<tr>
<td>0.29-0.31</td>
<td>f</td>
<td>d</td>
</tr>
<tr>
<td>0.31-0.34</td>
<td>b</td>
<td>a</td>
</tr>
<tr>
<td>0.34-1.10</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>1.10-1.76</td>
<td>a</td>
<td>d</td>
</tr>
<tr>
<td>1.76-1.93</td>
<td>a</td>
<td>d</td>
</tr>
<tr>
<td>1.93-7.91</td>
<td>a</td>
<td>d</td>
</tr>
<tr>
<td>7.91-8.29</td>
<td>d</td>
<td>a</td>
</tr>
<tr>
<td>8.29-10.00</td>
<td>a</td>
<td>d</td>
</tr>
</tbody>
</table>

105
Figure 4.7 Magnitude of total power flow in the members of the controlled structure as a result of a disturbance at junction 1. Some members have identical flow due to symmetry.
control, no information is available about the incoming wave mode set. It is assumed that each of the incoming wave modes is just as likely to arrive at the junction as any of the other incoming wave modes. However, as mentioned in Section 3.3, it may be possible to obtain some a priori information as to the structure of the incoming wave mode vector. Another form of information can also be used. It can be determined whether a particular cross-section does not support specific motions. This knowledge can then be used to determine whether a chosen actuator has control effectiveness at this location. This approach is best described starting with an example.

Suppose that an axial force actuator is available to perform junction wave control at an arbitrary position on a nondispersive, undamped compression rod. The relation describing the wave scattering at an arbitrary position along this member is insensitive to location because as far as the model is concerned, the member is infinite in length to the left and right of the junction. The associated power flow matrix for the members are also insensitive since they only govern the local power flow properties. This leads to a control compensator and closed-loop junction power flow matrix which are also insensitive. The only sensitivity is that of performance. The examination of the closed-loop power matrix eigenvalues will reveal that certain incoming wave mode mixes are dissipated more than others. Therefore, from a performance perspective, it would be preferable to place the controller at a position where the energy dissipation would be maximized: where the incoming wave mode vector corresponding to the largest negative eigenvalue, guaranteed to be real, of the closed-loop junction power flow matrix exists. Determining this position makes no sense until the compression member is viewed as a finite length medium which preferentially supports specific wave mode mixes.

For this example, it is assumed that the compression rod is of finite length with a clamped end condition at one end and a free condition at the other. Suppose that the wave controller is moved towards the free end. While performance as judged on the local level is unchanged, the controller is being moved to a location where the energy in the medium manifests itself more in the form of axial velocity rather than strain. In the limit, as the controller is moved to the free end, strain cannot exist and the junction displacement, through which the actuator force performs work, is maximized. In the other limit, where the control force is exerted at the clamped end, junction displacement is prohibited and the controller can have no performance because it can perform no work. Obviously, global performance is very sensitive to placement.

This change in global performance is caused by a change in the wave mode vector that is incoming to the junction as the junction is displaced from one end to the other. At the free end, the incoming wave mode mix is aligned with the direction in the closed-loop
power flow matrix which corresponds to maximum dissipation. At the clamped end, it is aligned with a direction corresponding to a zero eigenvalue since no dissipation can be achieved. Obviously, the closed-loop junction power flow matrix is negative semidefinite since the incoming wave mode set that results in zero junction displacement causes no control effectiveness. However, if both axial force and stress actuators are available, the closed-loop junction power flow matrix will be negative definite because no incoming wave mode can create a physical junction motion through which neither actuator can perform work.

For the general junction control problem, proper actuator placement can be performed using several tools. First, network power flow should be used to identify those junctions and members which carry the most power throughout the entire frequency range or within a specific range. Alternatively, the controllers can be placed on the dominant or all dynamic load carrying members which transmit power between the various disturbance sources and the region that is performance critical. In the former case, the objective is to extract as much energy from the structure in a given time. In the latter, the objective is to minimize the amount of disturbance energy that reaches the critical location, perhaps to the point of total dynamic isolation.

Second, once the general regions of placement have been determined, there is typically freedom as to the exact placement. The next consideration is the type of control hardware available. If all unique actuator types, as described in Section 4.1, are available, performance becomes rather insensitive to placement within the region because the closed-loop junction power flow matrix will be negative definite. However, if all unique actuator types are not available, the issues raised in the previous example become important. Then, several techniques can be used to resolved this problem.

One way to solve the placement of a less than complete actuator set is to describe the region as a finite domain junction and investigate how the effectiveness of the actuators change as their position is varied within the junction. This can be done by first describing the local, discrete point junction relation which is insensitive to position. Then, the new distributed but finite domain junction can be described by using this local junction, member transmission, and other junction descriptions of all the components which reside within the domain. This new relation describes the amplitudes of the departing wave modes as a function of the incoming wave modes and, in particular, the actuators. The question then becomes, how effective are the actuators at generating, and therefore controlling these waves, as the position of the actuator set is varied within the junction. This is found by evaluating the new junction wave generation matrix. Typically, actuator effectiveness, and therefore performance, will be a function of frequency.
Another way to resolve this issue is to find the wave mode vector (eigenvector) corresponding to the largest negative eigenvalue of $P_{CL}$. Using the member transformation matrix in Eq. 2.6, the corresponding cross-sectional variable set can be found. At this point, the transfer function relations in Section 4.1 can be used to determine that location in the desired region that exhibits this type of motion.

Finally, engineering insight may be used directly by understanding that performance is directly related by the amount of work that the controller can perform. This is at the location which exhibits the maximum motion that is dual, of like type, with the actuators available. Typically, this difference in performance only becomes significant as the actuator position approaches a discontinuity in the member. Example 4.3 illustrates the method for determining controller placement once a region of placement has been selected.

**Example 4.3** This example looks at the placement of a wave controller on the undamped compression rod used in Example 2.4. In that example, control force and relative displacement actuators were used to absorb power at junction B while a dashpot was attached to junction C. In this example, junction C will be a free end with no damping and actuator effectiveness will be investigated as junction B is moved closer to junction C. The effectiveness will be determined in two ways. First, a new junction relation, which includes the free end, will be formulated. Then, the $P_{CL}$ matrix will be used.

By defining the new junction to encompass the rod length from junction B to C, the only outgoing wave is $w_{1B}$. Since actuator effectiveness is being investigated, only the new wave generation matrix will be investigated. As can be seen in Eq. a, the outgoing wave $w_{1B}$ has two components: that which is generated in the leftward direction by the actuators and that which is the reflection off the free end of the rightward generated wave. Describing this second part in terms of the transmission matrix and actuator commands yields

$$w_{1B} = -\frac{1}{2ikEA} \begin{bmatrix} ikEA & 1 \end{bmatrix} \begin{bmatrix} A_C \\ F_B \end{bmatrix} + w_{4B} \quad \text{a}$$

$$w_{1B} = -\frac{1}{2ikEA} \left( \begin{bmatrix} ikEA & 1 \end{bmatrix} + \xi^2 \begin{bmatrix} -ikEA & 1 \end{bmatrix} \right) \begin{bmatrix} A_C \\ F_B \end{bmatrix} \quad \text{where} \quad \xi = e^{-ikl} \quad \text{b}$$

where $l$ is the variable length between junctions B and C. Substituting for the transmission matrix gives
\[ w_{1B} = -\frac{1}{2ikEA} \left[ ikE \left( 1 - e^{2ikl} \right) 1 + e^{-2ikl} \right] \frac{\Delta_C}{F_B} \]

For junction B being well internal to the member, l being large, the effectiveness of each actuator varies periodically with frequency and no general determination of effectiveness can be made except that at those frequencies where one is effective, the other is not. This is the advantage of having all types of actuators available. If junction B coincides with the free end then Eq. c becomes

\[ w_{1B} = -\frac{1}{2ikEA} \left[ \begin{array}{c} 0 \\ 2 \end{array} \right] \frac{\Delta_C}{F_B} \]

Therefore, the relative displacement actuator is ineffective while the force actuator is effective at all frequencies at the free end. A direct analogy can be made between this result and the controllability of modes in the middle of the rod and at the free end where every mode undergoes displacement.

A second procedure is to judge placement based upon the closed-loop junction power flow matrix. If only the commanded relative displacement actuator (\(\Delta_C\)) is used, Eq. 3.26 gives the optimal, noncausal control and closed-loop scattering matrices as

\[ \Delta_C = \begin{bmatrix} -1 & 1 \\ w_{3B} & w_{4B} \end{bmatrix} \quad ; \quad S_{CL} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \]

The closed-loop power matrix (\(P_{CL}\)), found using Eq. 3.37, is

\[ P_{CL} = \frac{1}{2} \alpha kEA \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \]

The eigenvalues of the matrix in Eq. f are 0 and -2. The wave mode eigenvector, and associated cross-sectional vector, that correspond to the zero eigenvalue are

\[ \begin{bmatrix} w_{lp} \\ w_{rp} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad ; \quad \begin{bmatrix} u \\ EAu' \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} \]

The same vectors for the -2 eigenvalue are

\[ \begin{bmatrix} w_{lp} \\ w_{rp} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad ; \quad \begin{bmatrix} u \\ EAu' \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} ikEA \end{bmatrix} \]
Notice that the maximum power dissipation occurs when the junction motion is aligned with the eigenvector associated with the -2 eigenvalue. From Eq. h, this is when the motion at the junction is only composed of strain. It can be seen, in Eq. g, that no dissipation occurs when only displacement occurs at the junction. The conclusion, consistent with Eq. d, is that the $\Delta_C$ actuator is ineffective at the free end.

A similar analysis using only the force actuator ($F_B$) shows that this actuator exhibits maximum effectiveness when located at the free end. The optimal control, for no control effort penalty, and closed-loop scattering matrices are

$$F_B = ikEA \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} w_{3B} \\ w_{4B} \end{bmatrix}$$

$$S_{CL} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

The closed-loop power matrix is

$$P_{CL} = \frac{1}{2} \alpha kEA \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

The eigenvalues of this matrix are the same as those for the matrix in Eq. f. However, the associated eigenvectors have switched. Thus, the force actuator's effectiveness is opposite to that of the relative displacement actuator. This fact suggests that these two actuators should be used together since one is effective where the other is not.

In conclusion, these two examples of determining actuator placement within a predetermined region simply confirm that which is intuitively obvious: for maximum work and therefore effectiveness, actuators should be placed at positions where they act through the largest dual deformation.

4.4 Network Control Issues

A network view of the control objective helps determine the specific directions in which junctions should block power flow and members should direct power. For example, if vibration sensitive equipment is located on the dual keel's bottom torsional member, and only a subset of the total unique actuators are used at junctions 2 and 5, then the concepts in Sections 3.2 and 3.3 should be employed. First, as many of the transmission coefficients from incoming waves to outgoing waves along members $h$ and $j$ should be set to zero. The most critical transmission coefficients are those associated with the incoming waves on members which contain the dominant disturbance sources, such as crew motion disturbance along members $b$ and $f$. The remaining actuator degrees of freedom should be used to minimize junction power flow.
If the disturbance location is not well defined, controller switching can be used. Sensors can be used to identify the location of a new disturbance to the network. With knowledge of the performance critical locations and the various wave control solutions for each junction, the junctions near the disturbance can be reconfigured to best deal with the disturbance by both absorbing and shunting the energy.
5. EXPERIMENTS

An experimental phase of research was conducted for three reasons. First, the experimental structure exhibits, by definition, true dynamics. The models thereof are simply "best" approximations to these dynamics. Often in structural control, the design and evaluation models are finite order and do not exhibit the true behavior of the structure. Second, implementation aspects introduce issues which are often overlooked in the analytical stage. Certain actuators and sensors are more feasible than others. Hardware lags and dynamics often generate instabilities which may not be foreseen. Third, experimental data reveals the "realistic" status of the tested technology by demonstrating the maximum achieved performance and identifying those mechanisms which limit that performance.

A series of experiments were conducted to compare two types of compensators used to control the pinned end of a beam modelled as a Bernoulli-Euler beam. The first type of feedback compensator provided conventional rate feedback of pinned end rotation to collocated moment. Tests using this compensator will be subsequently referred to as rate feedback tests. The second type of compensator was derived using the optimal/causal wave junction control technique outlined in Chapter 4. This will subsequently be referred to as wave control. The actual compensator that was used will be derived in Section 5.2. These two types of feedback compensators were chosen for comparison because both involve collocated control using a dual actuator/sensor pair with the feedback being of a positive definite nature. Since dual rate feedback is a conventional way in which to actively enhance the damping of structural systems, it seems to be the appropriate test case against which this unconventional wave technique should be compared.

This chapter is divided into six sections. Section 5.1 describes the experimental apparatus which includes the controlled structure and controlling hardware. The specifications are provided for all components including the data acquisition equipment. Section 5.2 presents the derivation of the wave control compensator. The model is based upon the available hardware and structure. The assumptions made in the formulation of the model are listed explicitly. Models are also formulated for the other members and junctions in the structure. This is done to enable the calculation of beam model transfer functions. Section 5.3 presents numerically calculated transfer functions based upon the models formulated in Section 5.2. These calculations are performed for the open-loop structure and closed-loop structure using both rate feedback and wave control. Section 5.4 outlines the experimental procedure used to obtain test data of the beam's transfer function from shaker to transverse displacement at the uncontrolled end. This enables the test data to be
compared directly to the transfer function calculations found in Section 5.3. In addition, measured transfer functions of the two control compensators are given. Section 5.5 presents the test results using both compensators. Section 5.6 summarizes the theoretical and experimental comparison of these two control approaches.

5.1 Experimental Setup

The controlled structure used in the tests consisted of a twenty-four foot brass beam, composed of four six foot sections bolted end to end, suspended horizontally from six sets of cables. The beam was situated such that the lowest bending moment of inertia was about the vertical axis and the lowest frequency modes of vibration consisted of motion transverse to the beam's length and in the horizontal plane. The suspension was configured to suppress vertical as well as torsional motion. The properties of the beam are listed in Table 5.1. Since the ratio of beam thickness to length was quite small, a Bernoulli Euler model was used.

Table 5.1 Properties of pinned/free brass beam used in control tests.

<table>
<thead>
<tr>
<th>Beam dimensions</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Length:</td>
<td>7.32 m.</td>
</tr>
<tr>
<td>Width:</td>
<td>10.2 cm.</td>
</tr>
<tr>
<td>Thickness:</td>
<td>0.3175 cm.</td>
</tr>
<tr>
<td>Thickness/Length</td>
<td>0.0004</td>
</tr>
</tbody>
</table>

**Beam Properties**

- $EI$: 31.1 N m$^2$
- $\rho A$: 2.85 kg/m

**Open-loop modal frequencies and damping ratios of modes tested**

<table>
<thead>
<tr>
<th>Elastic Mode Number</th>
<th>Frequency (Hz)</th>
<th>Damping Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2.62</td>
<td>0.0039</td>
</tr>
<tr>
<td>6</td>
<td>3.72</td>
<td>0.0048</td>
</tr>
<tr>
<td>8</td>
<td>6.41</td>
<td>0.0045</td>
</tr>
<tr>
<td>9</td>
<td>8.17</td>
<td>0.0037</td>
</tr>
<tr>
<td>10</td>
<td>9.98</td>
<td>0.0028</td>
</tr>
<tr>
<td>11</td>
<td>12.02</td>
<td>0.0026</td>
</tr>
<tr>
<td>12</td>
<td>14.29</td>
<td>0.0023</td>
</tr>
<tr>
<td>13</td>
<td>16.72</td>
<td>0.0036</td>
</tr>
<tr>
<td>14</td>
<td>19.31</td>
<td>0.0020</td>
</tr>
<tr>
<td>15</td>
<td>22.03</td>
<td>0.0025</td>
</tr>
<tr>
<td>16</td>
<td>24.69</td>
<td>0.0023</td>
</tr>
<tr>
<td>17</td>
<td>27.70</td>
<td>0.0020</td>
</tr>
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</table>
The boundary conditions at the two ends of the beam are shown in Figure 5.1. The end with the shaker was free in the sense that it was able to rotate and translate. The other end, where the control was applied, was pinned in the sense that this end was allowed to rotate but not translate. The assumptions made as to the importance of retaining shaker and control actuator dynamics in the models are listed in Section 5.2.

Collocated with the shaker was a linear accelerometer which was used as the output measurement of the system during the control tests. The shaker was of an inertial reaction, pivoting proof-mass type as described in Reference 21. The specifications of the data acquisition accelerometer and shaker are listed in Table 5.2.

Located at the pinned end were the control actuator and linear accelerometer used as the feedback measurement (Table 5.2 and Fig. 5.2). The actuator consisted of a torque motor whose outer casing, permanent magnet, was clamped to the laboratory frame. Clamped to both ends of the rotating armature shaft was a solid U-shaped arm which was also clamped to the beam end. The accelerometer, mounted on the solid arm near to the position where the arm clamped to the beam, provided a measurement of the inertial rotary
<table>
<thead>
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<th>Shaker</th>
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<tr>
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<td>Pittman</td>
</tr>
<tr>
<td>Model number</td>
<td>7214</td>
</tr>
<tr>
<td>Type</td>
<td>DC Servo</td>
</tr>
<tr>
<td>Torque constant</td>
<td>0.0357 N m/amp</td>
</tr>
<tr>
<td>Power Amplifier</td>
<td>EG&amp;G</td>
</tr>
<tr>
<td>Manufacturer</td>
<td>PA-223</td>
</tr>
<tr>
<td>Model</td>
<td>Current Source</td>
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<td>Type</td>
<td>-1.87 Amps/volt</td>
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<table>
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<tr>
<th>Data Acquisition Accelerometer</th>
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<td>Endevco</td>
</tr>
<tr>
<td>Model number</td>
<td>2262-25</td>
</tr>
<tr>
<td>Type</td>
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</tr>
<tr>
<td>Excitation voltage</td>
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</tr>
<tr>
<td>Gain</td>
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<tr>
<td>Signal amplifier gain</td>
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<tr>
<td>Corner frequency</td>
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<tr>
<td>Damping</td>
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<td>Model number</td>
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</tr>
<tr>
<td>Type</td>
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<tr>
<td>Excitation voltage</td>
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<tr>
<td>Gain</td>
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</tr>
<tr>
<td>Signal amplifier gain</td>
<td>1000 volts/volt</td>
</tr>
<tr>
<td>Corner frequency</td>
<td>1200 Hz</td>
</tr>
<tr>
<td>Damping</td>
<td>light</td>
</tr>
<tr>
<td>Distance from motor pivot</td>
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<table>
<thead>
<tr>
<th>Moment actuator</th>
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<tbody>
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<td>Motor manufacturer</td>
<td>PMI</td>
</tr>
<tr>
<td>Model number</td>
<td>U-9</td>
</tr>
<tr>
<td>Type</td>
<td>Torque</td>
</tr>
<tr>
<td>Torque constant</td>
<td>0.0212 N m/amp</td>
</tr>
<tr>
<td>Armature plus arm rotary inertia</td>
<td>0.000146 Kg m^2</td>
</tr>
<tr>
<td>Motor diameter</td>
<td>0.1048 m</td>
</tr>
<tr>
<td>Motor thickness</td>
<td>0.0345 m</td>
</tr>
<tr>
<td>Power amplifier</td>
<td>EG&amp;G</td>
</tr>
<tr>
<td>Manufacturer</td>
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<tr>
<td>Model</td>
<td>Current Source</td>
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<tr>
<td>Type</td>
<td>-2.08 Amps/volt</td>
</tr>
<tr>
<td>Gain</td>
<td></td>
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</tbody>
</table>
acceleration of the actuator armature shaft. Since the actuator provided an inertially referenced moment, this sensor/actuator pair were considered dual\textsuperscript{20}.

As will be discussed in the following sections, this pair was not dual at frequencies where the compliance of the laboratory surface, to which the actuator was clamped, became significant. In the case where the outer actuator casing is allowed to translate due to surface compliance, the accelerometer also measures linear acceleration. In the case where it is allowed to rotate due to surface compliance, the inertially referenced acceleration
measurement was fed to relative moment between two rotating components. In both cases, the sensor and actuator are no longer a dual pair. It was determined that this compliance was the cause of an instability in both the rate feedback and wave control systems at low gain. This determination was based upon the observation that a surface resonance coincided with the frequency of the instability. In a moderately successful attempt to inhibit this instability, damping was introduced to the interface between this surface and the laboratory floor using a viscoelastic (foam) layer. This prevented instability at low feedback gains. The ensuing higher gain instability defined the maximum level of feedback gain that was employed in the tests described in Section 5.5. A possible solution to this instability problem is to employ a truly dual sensor, such as a tachometer, to measure true armature rotational rate. However, this sensor was not available at the time of the tests. As will be indicated in Section 5.5, reasonable performance was achieved using the existing hardware.

Two types of support equipment were also used (Fig. 5.1). First, equipment that resided in the feedback loop consisted of a signal amplifier for the accelerometer, an analog computer on which the feedback compensators were programmed and a power amplifier which drove the moment actuator. The power amplifier, operating as a current source, commanded moment because the output current tracked input voltage from the analog computer (Pace TR-48). Therefore, the electrodynamics of the actuator were ignored.

The second type of equipment consisted of components which resided outside of the feedback loop and were used for data acquisition. This included the signal amplifier for the accelerometer collocated with the shaker, a Nicolet 2090 Digital Oscilloscope to record time histories of the beam response and a Signology SP-20 Signal Processing Peripheral (spectrum analyzer) to record response data and process frequency domain information. The components are shown in Figure 5.1 and the specifications are listed in Table 5.2.

With this hardware configuration in place, a wave model was formulated for the controlled junction and other beam components. This model was employed in the derivation of the wave control compensator.

5.2 Problem Solution

5.2.1 Model Assumptions Six assumptions were made in the formulation of the wave models. These assumptions are as follows:

1.) The brass beam, which comprises the one member attached to the pinned junction, was modelled as a Bernoulli Euler beam.
2.) Excluding the end conditions, the beam was assumed to be uniform for its entire length. In Reference 21 it was shown that the bolt connections between the six foot beam sections did not result in the uniform model differing significantly from the actual system. The pendulous stiffness of the suspension cables only has a noticeable effect in the first five elastic modes. Since these two factors were not significant in altering the accuracy of a modal model, it was assumed that they would also be insignificant in altering the accuracy of a wave model. In either case, these components do not alter the accuracy of the junction model used to formulate the control since the closest suspension point was located two feet away. They only alter the accuracy of the transmission models used to simulate the transfer function properties of the structure as a whole.

3.) The rotary inertia of the armature shaft was assumed to be negligible making the boundary condition model at the controlled end a pinned condition with zero concentrated inertia. This assumption was supported by the fact that the motor armature is a laser etched wafer specifically designed to minimize rotary inertia in order to maximize response characteristics. Since an inertial property is being excluded, this assumption becomes invalid at high frequencies. However, as long as the sensor/actuator pair remain dual, this modelling error only results in a deviation between predicted and actual performance and will not induce an instability. This fact is true for both collocated rate feedback and the implemented wave control which, as will be seen in Section 5.2.2, also has a positive real nature.

4.) The fact that the beam is not directly attached to the armature shaft, but terminates at a rigid arm some distance from the shaft, was not modelled. As in Assumption 3, this modelling error can become significant at higher frequencies where wavelengths are on the order of the length of the rigid arm. Again, this will only alter performance and not stability given dual hardware.

5.) Often in the excitation of the beam using the shaker, it has been observed that the centripetal force, caused by the rotation of the pivoting arm, acts axially on the beam to parametrically excite double frequency motion. This nonlinear effect was not accounted for since is does not effect the stability of the closed-loop system. This might not be the case if this shaker were used as a control actuator.

6.) The surface to which the outside motor casing of the actuator was attached was assumed rigid and not compliant. This assumption was erroneous at higher frequencies and resulted in a gain limiting instability.
5.2.2 Derivation of Wave Control Compensator  With these assumptions in mind, a junction model was formulated for control purposes and the other components of the beam were modelled for to allow calculation of the model transfer function. Using the pinned condition assumption, the boundary condition at the controlled end is

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
v \\
v' \\
-Elv'''
\end{bmatrix} = \begin{bmatrix}
v_C \\
0
\end{bmatrix}
\]

where the primes denote spatial derivatives. Notice that there are two unique types of actuators that can be used at this junction. However, commanded displacement was not available and only moment was used.

Since only one member was attached to this junction, the member transformation given in Eq. f of Example 2.1 was also the junction transformation matrix. Substituting this transformation into Eq. 5.1 gives the junction relation as

\[
\begin{bmatrix}
w_{rp} \\
w_{le}
\end{bmatrix} = \begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
w_{lp} \\
w_{re}
\end{bmatrix} + \frac{1}{2El\kappa^2}
\begin{bmatrix}
-1 \\
1
\end{bmatrix} M
\]

Notice that the propagating and evanescent waves that are decoupled in the member are also decoupled in the junction. This is related to the fact that the mode shapes of a pinned/pinned B-E beam do not contain hyperbolic terms.

Solving for the noncausal control using Eqs. 3.34 and 3.35, using no control effort penalty (R = 0) and the junction power matrix given in Eq. a of Example 2.3 (also the member power matrix), gives

\[
M = 2EI\kappa^2
\begin{bmatrix}
-1 & i \\
i & -1 + i
\end{bmatrix}
\]

\[
\begin{bmatrix}
w_{lp} \\
w_{re}
\end{bmatrix} = \begin{bmatrix}
\frac{3}{(EI)^4} (\rho A)^4 (-1 + i) \omega^2 \\
\frac{1}{2EI}\kappa^3
\end{bmatrix}
\]

This gives the closed-loop junction scattering and power relations as

\[
S_{CL} = \begin{bmatrix}
0 & -i \\
-1 & -1 + i
\end{bmatrix} ; \quad P_{CL} = \frac{1}{2EI}\kappa^3
\begin{bmatrix}
-1 & i \\
-i & -1
\end{bmatrix}
\]

Notice, however, that the control compensator feeding back junction rotation in Eq. 5.3 is anticausal.

Using the fixed form parameter optimization technique outlined in Section 3.3.2, with the form of the compensator equalling the anticausal solution shifted by ninety degrees of phase, gives the optimal/causal, fixed form compensator as
\[
M = (EI)^4 (\rho A)^4 (1 + i) \omega^2 \begin{bmatrix} 1 & 0 \\ -EI \nu'' \\
\end{bmatrix}
\]

This solution calls for the feedback of the half derivative of pinned end rotation to collocated moment. In the experiment, however, pinned end rotational acceleration was available. Therefore, the wave control compensator implemented in the experiment was

\[
M = - (EI)^4 (\rho A)^4 (1 + i) \omega^2 \cdot \frac{3}{2} a'
\]

where \( a' \) is the Fourier transform of the rotational acceleration.

For the rate feedback experiments, the feedback compensator was given by

\[
M = c_0 i \omega^1 a'
\]

Notice that the gain of the rate feedback is left as an unknown. The reason for this will be clarified in Section 5.3.

### 5.2.3 Wave Models of Other System Components

The other system components consist of the beam member and the junction which includes the shaker. From Assumptions 1 and 2, the member transmission matrix is simply that of a Bernoulli-Euler beam member of length \( l = 7.32 \text{ m} \) (24 ft) from Table 5.1 given by

\[
\xi = \begin{bmatrix}
    e^{-ikl} & 0 \\
    0 & e^{-ikl}
\end{bmatrix}
\text{ where } k = \sqrt{\frac{\rho A}{EI}} \sqrt{\omega}
\]

The beam parameters given in variable form in Eq. 5.8 are listed in Table 5.1.

The boundary condition for the other junction includes the shaker dynamics. The equations of motion which describe the linearized coupling between a pivoting proof-mass actuator and a Bernoulli-Euler beam are presented in Reference 21. The coupling is quite complex and is not described here since the shaker dynamics do not affect the control aspects that need to be observed. The shaker dynamics simply frequency shape the disturbance input. However, the Matrix\( _{x} \) transfer function code listed in Fig. 5.3 shows how these dynamics are included in the analysis.

### 5.3 Transfer Function Calculations

The intent of this section is to predict and compare the performance of the closed-loop systems when using the compensators given in Eqs. 5.6 and 5.7. The network dynamical formulation described in Chapter 4 was used to find the transfer functions from the shaker to collocated transverse displacement for the system in open-loop, closed-loop
// [Omega, H, J] = Xfer(npts, wli, wfl)

inc=0.4342945*(log(wfl)-log(wli))/(npts-1);
wl=wli;
l=7.3152
e=2.718281828;
llee=eye(2,2);

for i=1:npts,
    k=0.5503117*(w**0.5);
    all=[1 1;jay*k k];
    bel=[1 1;-1*jay*k -k];
    gal=31.11*(k**2)*[jary*k 1];
    del=31.11*(k**2)*[-1*jay*k 1];
    alr=[1 1;-1*jay*k -k];
    ber=[1 1;jay*k k];
    gar=31.11*(k**2)*[-1*jay*k k];
    der=31.11*(k**2)*[jay*k -k];
    c=inv(ber-alr*inv(gar)*del);
    d=inv(der-gar*inv(alr)*ber);

    EE=[e**(-1)*jary*k] 0 0 e**(-1)*k*] ;

    t=0.00205*(w**2)-jary*0.326*w-0.0129;
    u=0.00142*(w**2)*jay*0.326*w-0.0129;
    c0=(-0.0003662*(w**4)/t)+0.62942*(w**2);
    c1=(-0.000012193*(w**2)/t)-0.0235*(w**2);
    c2=0.019137*(w**2)/t;
    d0=(u*0.019137*(w**2)/t)-0.04266*(w**2);
    d1=(u*0.0063714*(w**2)/t)+0.00585*(w**2);
    d2=(-u/t)+1;

    Bur=[-0.5*((d0/d2)+(c0/c2)); -0.5*((d1/d2)+(c1/c2))];
    Bur=[Bur;((c0/c2)-(d0/d2)) ((c1/c2)-(d1/d2))];
    Bfr=[1(2*d2); 1(2*c2); 1/d2 -1/c2];
    Bul=[1 0 0 0];
    Bfl=[0 0 0 1];

    Sr=inv(Bur*ber+Bfr*der)*(Bur*alr+Bfr*gar);
    Psr=inv(Bur*ber+Bfr*der)*[1 0];
    Ssr=inv(Bul*bel+Bfl*der)*(Bul*alr+Bfl*gar);
    Pss=inv(Bul*bel+Bfl*der);

    G=inv((c-d*inv(Bfr)*bur)*(inv(ll-Sr*EE*sl*ee)*Psrd*inv(Bfr)*[1 0]);
    H(i,1)=G(1,1); T=

    T=[0 0 1 0 0 0 0 0 0 0 0 0 1];
    Y=inv(T)*[all bel;gal del];
    at=y([1;2],[1;2]);
    bt=y([1;2],[3;4]);

    F=[0 0 1 0];

    sl=inv((ll-pls*fr*bt)*(sl+pls*fr*at));
    G=inv((c-d*inv(Bfr)*bur)*(inv(ll-Sr*EE*sl*ee)*Psrd*inv(Bfr)*[1 0]);
    J(i,1)=G(1,1);
    Omega(i,1)=w;
    w=10**(inc+0.4342945*log(w));

    retf

Figure 5.3 Matrix code for evaluating model transfer function
using rate feedback and closed-loop using wave control. The transfer function code, written as a user-defined-function for the computer aided control design program known as Matrixx is shown in Figure 5.3.

The transfer functions were derived for the open-loop structure, several gain levels using rate feedback (Eq. 5.7), the anticausal wave control compensator (Eq. 5.3) and the causal wave control compensator (Eq. 5.6). The wave models selected for this structure are mathematically exact solutions of the governing ordinary and partial differential equations of motion. This makes them infinite order allowing the transfer function to be calculated up to any frequency without sacrificing mathematical accuracy.

Figure 5.4 compares the closed-loop transfer function magnitude for several values of indicated rate feedback gain to that of the open-loop beam. Light damping was introduced to the open-loop beam through the shaker dynamics. Notice that at low rate feedback gain, the higher modes are more damped than the lower modes. As the gain is increased, the damping in the lower modes is increased while that in the higher modes is decreased. This was also found to be true for a different beam configuration in Reference 24. This indicates that at a given gain, damping is maximized in only a particular region of the frequency spectrum and that region varies with gain. Therefore, a particular level of rate feedback can only be considered optimal, in the sense that it maximizes damping performance, for a particular frequency regime. This result was obtained for a similar problem in Part 3 of Example 3.3.

This phenomenon can be explained on the basis of work performed by the control on the structure. For a given frequency, the work equals the generalized control force times the dual motion through which it is exerted. Since the control guarantees this force to be in phase with collocated generalized rate, the control is nonconservative yielding net work over a cycle of the motion. At low gain, the motion is large while the control action is small. The opposite is true at high gain. The gain dependence of these two actions gives a damping behavior as follows: damping, which is initially increased at low gain, decreases at high gain because the control effectively clamps the end of the structure. This implies that there is a maximum level of damping that can be achieved at a particular frequency and a corresponding optimal gain which achieves this damping. Figure 5.4 indicates that this optimal gain is a function of frequencies. In fact, it is higher at lower frequencies. In other words, the impedance of the control does not match the impedance of the structure. The prediction that clamping is occurring at high gains is supported by the fact that the closed-loop resonances occur at higher frequencies than their open-loop counterparts (Clamped/free frequencies being higher than pinned/free frequencies).
Figure 5.4  Open-loop (dotted) vs. closed-loop (solid) beam responses for indicated rate feedback gains.
Figure 5.5 compares the closed-loop beam response using the anticausal compensator given by Eq. 5.3 with that in the open-loop. Notice that if this compensator could be implemented, damping would not only be increased, but no semblance of resonant behavior would exist. Figure 5.6 compares closed-loop response using optimal/causal wave control, consisting of a three halves integrator with a gain of 24.2 N m sec\(^2\) (Eq. 5.6 and Table 5.1), to open-loop response. Notice that, unlike rate feedback, the active damping enhancement is more broadband.

Remembering that the optimal rate feedback gain is lower at higher frequencies implies that it might be desirable to have a frequency dependent rate feedback gain which diminishes with frequency. The gain of the portion of the wave control compensator that is in phase with velocity does just that. However, since the wave control solution has been constrained to be causal, the compensator satisfies both this constraint and the need for frequency dependent rate feedback by introducing some displacement feedback. Unfortunately, this active stiffening does not alter the power dissipation characteristics; the objective of the control formulation. Remember that this displacement feedback is introduced at no cost since no control effort penalty matrix was used.

Figures 5.4b and 5.6 also correspond to the predicted response at the maximum gain levels that were used in the actual tests. Figure 5.7 compares the closed-loop magnitudes using the two different compensators at their maximum stable gains. The dotted curve and solid curve correspond to wave control and rate feedback, respectively. Notice that the wave control provides more damping at low frequencies than this maximum gain of rate feedback. The opposite is true at higher frequencies. Since rate feedback at this gain effectively clamps the motion at frequencies higher than those shown in Fig. 5.7, one can expect that wave control provides more damping at the highest frequencies. Notice that rate feedback provides more damping than wave control in a particular range.

5.4 Experimental Procedure

Two types of tests were performed using the hardware described in Section 5.1: quasi-white noise excitation to obtain frequency domain transfer functions and controlled decays of beam motion to better reveal damping properties.

In the first set of tests, the shaker was excited using a random signal composed of approximately constant power spectral density in the frequency range from 0.5 Hz to 50.0 Hz. and zero outside of this range. Fifteen ensembles of response data measured by the accelerometer collocated with the shaker, consisting of 4096 data points each, were transformed by the Signology spectrum analyzer using an FFT technique. These transforms were then averaged. Flattop filtering was used to obtain reliable resonant
Figure 5.5  Open-loop (solid) and closed-loop (dotted) beam responses for anticausal wave control

Figure 5.6  Open-loop (dotted) and closed loop (solid) beam responses for optimal/causal wave control
Figure 5.7  Comparison of closed-loop beam responses using rate feedback (solid) at the maximum stable gain and optimal/causal wave control (dotted)
amplitudes and therefore damping information. A test as just described was performed for the beam in open-loop, for the beam in closed-loop using the maximum rate feedback gain before the onset of an instability at 150 Hz, and the maximum wave control feedback gain before the onset of an instability at 150 Hz. This instability is described in the latter part of this section. The maximum wave control gain was slightly higher than the optimal given in Eq. 5.6 using the beam member values in Table 5.1.

The second set of tests consisted of calculating damping ratios for twelve modes at various feedback gains using both compensators. The procedure involved setting the feedback gain, exciting the controlled beam harmonically at a frequency corresponding to one of the selected modes and abruptly turning off the excitation to allow controlled decay of the motion. The accelerometer data was stored on the Nicolet Digital Oscilloscope to allow post processing of the damping ratio information.

Approximations of the rate feedback and wave control compensators given by Eqs. 5.6 and 5.7 were programmed on the analog computer. The circuit diagrams are given in Figure 5.8 and the phase and magnitude components of the measured transfer functions are shown in Figure 5.9. Notice that in each circuit, a stabilized double integrator was employed to reject the ever present DC bias in the accelerometer signal (collocated with the actuator). This circuit element had second order dynamics at 0.1 Hz with a damping ratio of 0.7071. The "first integral" of this circuit was then considered the rotational rate measurement of the pinned junction.

For the wave control tests, this rate estimate was then fed through a circuit which approximates a half integrator well beyond the frequency range over which data was taken. The circuit of this half integrator with its corresponding transfer function information, is shown in Figure 5.10. This design was borrowed from Reference 24.

For the rate feedback tests, the rate estimate was initially fed straight to the power amplifier which drives the actuator. This, however, caused a 1200 Hz. instability at rather low gain. This was identified to be caused by the second order dynamics of the lightly damped accelerometers. The large magnitude of the accelerometer transfer function around 1200 Hz provided little gain margin. To remedy this, a first order low pass filter with a corner frequency of 100 Hz. was placed in the rate feedback compensator to achieve more gain margin (Fig. 5.8b). This accounts for the additional phase lag which becomes significant near 20 Hz. in the phase plot for the rate feedback compensator in Fig. 5.9c. The addition of this filter significantly reduces the higher frequency damping performance that would be predicted from Figure 5.7a. The noise in the plots of Fig. 5.9 at high frequency is due to the fact that the low gain of these compensators at these frequencies has caused the output signal-to-noise ratio to be quite small.
Figure 5.8 Schematics of analog computer programs for both feedback compensators
Figure 5.9 Measured transfer functions of circuits shown in Fig. 5.8 used to approximate rate feedback and wave control
Figure 5.10 Characteristics of half integrator circuit
Requiring the low pass filter in the rate feedback compensator raises an interesting issue. The instability that it suppresses can be caused by two mechanisms. First, the rate feedback compensator rolls off slower with frequency than does the wave compensator. This increases the probability of this type of hardware lag instability when using the rate feedback compensator. In addition, beam motion near this frequency, which comprises an element of this unstable loop, might have been undergoing damping reduction as this low gain was increased to the onset of instability. This latter point, while not verified as a contributor to this particular instability, suggests that the effective high frequency damping reductions associated with increased gain can quickly contribute to an instability such as that associated with sensor/actuator lags.

As stated earlier, the maximum gains used in both sets of tests were just below those which caused an instability at 150 Hz. Though not verified, this instability was thought to be due to the compliance of the surface to which the actuator's motor casing was clamped. The use of a tachometer, stiffer surface or more highly damped surface might be used to remedy this situation.

5.5 Test Results

5.5.1 Random Excitation Tests. The first series of tests involved random excitation of the beam. Figure 5.11 shows the magnitudes of the transfer functions from shaker input to collocated transverse displacement. Displacement information was retrieved by dividing acceleration data by the square of the frequency. Figures 5.11a and 5.11b compare the open-loop transfer function (dotted curve) to the closed-loop (solid curve) using the maximum stable gain level of rate feedback. Figures 5.11c and 5.11d show the same information when using the optimal wave control gain; which was equal to the optimal gain given in Eq. 5.6.

Notice that horizontally adjacent plots in each figure have the same vertical scale and correspond to two portions of the frequency range between 0.9 Hz and 30 Hz for the same test. Fifteen beam modes are visible in this range requiring the splitting of the range in order to reveal the details of this modally rich behavior. Information above 30 Hz. is not included because the accelerometer was unable to sense more consecutive modes without the close proximity of modal nodes corrupting the signal.

One to thirty hertz is a relatively narrow range in which to attempt to identify the subtle and broadband differences between the two types of compensators. The data, however, shows indications of these differences. Notice in Figure 5.11 that the control stiffens the modes of the beam as expected. In addition, both resulted in about an order of
Figure 5.11 Comparisons of open-loop, wave control and rate feedback beam responses
magnitude increase in damping. Figures 5.11e and 5.11f, overplotting the two closed-loop transfer function magnitudes (dotted for wave and solid for rate), shows that the additional displacement feedback present in the wave control performs additional stiffening. This figure also reveals that the wave control causes the higher modes to exhibit less damping. This is expected from the more negatively sloped frequency dependence of the rate feedback portion. Figure 5.11, when compared with Figures 5.4b, 5.6, 5.7b and 5.7c, show fair correlation between theory and experiment.

5.5.2 Controlled Decay Tests The second series of tests consisted of controlled decays of individual beam modes. These modes are twelve of those shown in Fig. 5.11. Various gain levels were used for each of the two compensators. Table 5.3 lists the damping ratios for each mode at the gain levels used. The gains were varied in approximately a linear fashion. The data at each gain level was then fitted using curves exhibiting the best correlation with the data. This is plotted in Figure 5.12 with each curve illustrating the fitted modal (frequency) dependence of damping at a given gain. Each curve corresponds to a different gain.

Notice that the best fit for the rate feedback data was achieved using a linear curve. Damping in the lowest mode increases almost linearly with gain. The change in slope of the fitted lines indicates that the damping increase as a function of gain is not as great in the higher modes. In addition, the high modes do not exhibit a strictly linear increase in damping with gain. The fitted curves actually imply that damping performance in the higher frequency modes becomes insensitive to gain variation. It might be further observed that damping actually decreases at high gain. However, this observation is questionable in light of the scatter in the raw damping data (Table 5.3).

The wave control data is best approximated using a logarithmic curve. No damping performance reversal is indicated by the fitted curves. The curves corresponding to the highest two gains indicate that an insensitivity of damping performance to gain increase is occurring because the optimal gain is being approached.

Figure 5.13 displays the free and controlled decays for two modes using both compensators at the maximum stable gain. Notice that this data supports the observation that the wave control provides more damping in the low mode and less in the high mode. Notice that a beat phenomenon occurs which is more significant in closed-loop. Table 5.4 compares the period of this beat with the time required for the outgoing power at that frequency to return as incoming power. This time is simply twice the length of the beam divided by the energy propagation or group velocity \((d\omega/dk)\) at that frequency. For a B-E beam, the group velocity is twice the phase velocity. The fact that there is good correlation
Table 5.3  Closed-loop modal damping ratios using both compensators.

**Rate Feedback:**

<table>
<thead>
<tr>
<th>Freq (Hz)</th>
<th>0.344</th>
<th>0.688</th>
<th>1.032</th>
<th>1.376</th>
<th>1.720</th>
<th>2.064</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.62</td>
<td>0.0120</td>
<td>0.0136</td>
<td>0.0177</td>
<td>0.0238</td>
<td>0.0254</td>
<td>0.0301</td>
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<tr>
<td>3.71</td>
<td>0.0105</td>
<td>0.0157</td>
<td>0.0203</td>
<td>0.0257</td>
<td>0.0334</td>
<td>0.0366</td>
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<tr>
<td>6.41</td>
<td>0.0099</td>
<td>0.0129</td>
<td>0.0228</td>
<td>0.0243</td>
<td>0.0298</td>
<td>0.0384</td>
</tr>
<tr>
<td>8.17</td>
<td>0.0082</td>
<td>0.0171</td>
<td>0.0171</td>
<td>0.0243</td>
<td>0.0251</td>
<td>0.0299</td>
</tr>
<tr>
<td>9.98</td>
<td>0.0091</td>
<td>0.0159</td>
<td>0.0263</td>
<td>0.0247</td>
<td>0.0277</td>
<td>0.0336</td>
</tr>
<tr>
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<td>0.0185</td>
<td>0.0206</td>
<td>0.0174</td>
<td>0.0202</td>
<td>0.0257</td>
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<td>0.0134</td>
<td>0.0168</td>
<td>0.0181</td>
<td>0.0234</td>
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<td>0.0109</td>
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<tr>
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<td>0.0137</td>
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<td>0.0151</td>
<td>0.0164</td>
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</table>

Fitted curve: $\zeta = B_0 + B_1 \omega$

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<th>$B_0$</th>
<th>0.0106</th>
<th>0.0155</th>
<th>0.0211</th>
<th>0.0265</th>
<th>0.0310</th>
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<td>-0.00014</td>
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<td>-0.00058</td>
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Correlation:

-0.65 -0.58 -0.67 -0.97 -0.88 -0.88

**Wave Control:**

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<th>Freq (Hz)</th>
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<th>9.81</th>
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<th>24.54</th>
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<td>0.0123</td>
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<td>0.0339</td>
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<td>0.0085</td>
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Fitted curve: $\ln(\zeta) = B_0 + B_1 \ln(\omega)$

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<th>$B_0$</th>
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<td>-0.414</td>
<td>-0.485</td>
<td>-0.437</td>
<td>-0.510</td>
<td>-0.560</td>
<td>-0.498</td>
<td>-0.520</td>
</tr>
<tr>
<td>Correlation</td>
<td>-0.86</td>
<td>-0.95</td>
<td>-0.83</td>
<td>-0.94</td>
<td>-0.88</td>
<td>-0.94</td>
<td>-0.81</td>
</tr>
</tbody>
</table>

Optimal gain from Eq. 5.6 = 24.204 N m sec$^2$
a) Using rate feedback at the various gains listed in the legend

b) Using wave control at the various gains listed in the legend

**Figure 5.12** Fitted damping ratio vs. modal frequency curves for various gain levels for rate feedback and wave control
Figure 5.13  Free and controlled decays for the mode at 2.62 Hz and the mode at 27.7 Hz for rate feedback at a gain of 2 Nmsec\(^2\) and optimal/causal wave control.
Table 5.4 Comparison of beat period with propagation delay.

<table>
<thead>
<tr>
<th>Modal Frequency</th>
<th>Energy Speed</th>
<th>Time Delay</th>
<th>Number of Cycles</th>
<th>Actual Cycles per Beat</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.62 Hz</td>
<td>14.74 m/s</td>
<td>0.99 s</td>
<td>2.6 cyc</td>
<td>3</td>
</tr>
<tr>
<td>27.7 Hz</td>
<td>48.00 m/s</td>
<td>0.30 s</td>
<td>8.5 cyc</td>
<td>9.5</td>
</tr>
</tbody>
</table>

Propagation distance = 14.63 m

between the number of actual temporal cycles and the number that the model predicts would occur, during the time required for outgoing power to return, might imply that the beating is a result of the time delays in the beam.

A possible explanation for the beating might be as follows. In steady-state, the amount of power extracted by the controlled junction equals the amount introduced by the shaker, leaving the amplitude of the beam response at any cross-section a constant. The power flowing towards the actuator exceeds that flowing away. Once the shaker is turned off, it no longer introduces power into the beam and a discontinuity occurs in the amount of power flowing towards the actuator, and the amplitude of the motion at the shaker drops. The motion amplitude is then steady until that discontinuity arrives back at the shaker position after travelling twice the length of the beam. Therefore, the motion drops after integral multiples of the time required for the energy associated with waves at that modal frequency to traverse twice the length of the beam. The higher the damping, and corresponding power flow reduction, the larger the beat amplitude. This is a plausible explanation for the observed beating behavior in terms of a wave train at a single frequency. The same motion could be equivalently described by some superposition of the responses of all standing modes when excited at that frequency. Since the damping is not proportional damping, the standing modes have complex mode shapes.

5.6 Summary

With the hardware limitations placed upon the formulation of the wave controller, it and rate feedback exhibit nearly identical behavior, especially over the small frequency range measured. In this example, the differences between the two techniques are subtle. From a gross perspective, both the demonstrated rate feedback and wave control exhibit good performance. Both have a positive definite nature and involve a dual sensor/actuator pair.

From the perspective of damping performance, the model transfer function showed rate feedback to be optimal only over a certain frequency range. In addition, it has the undesirable property of decreasing damping in higher modes, where hardware lags might
be significant, when attempting to increase damping performance in lower modes. This feature might have coupled with other effects to induce the hardware and surface compliance instabilities.

The wave control compensator used in these tests did not exhibit the need for the low pass filter that reduced rate feedback performance at higher frequencies. The numerical examples revealed that wave control provides significant damping over a wide frequency range, limited to the extent that the models are applicable, because the rate feedback portion is frequency shaped. Unfortunately, achieving this frequency dependence carries with it the considerable exertion of control effort in the form of displacement feedback. The wave control solution was derived without penalty on control effort but in many applications this may be a wasteful and therefore expensive expenditure of effort. This additional stiffening does, however, move the damped resonant frequencies higher in the frequency spectrum. This may be considered desirable because it reduces the resonant amplitude of motion associated with a given energy of vibration.

Again, these differences are of higher order. To truly exhibit first order differences between these two types of control approaches, a larger percentage of the unique junction actuator types should be employed. For this experiment, that would correspond to both control moment and commanded transverse displacement.
6. CONCLUSIONS

This work has extended a formalism for the frequency domain modelling of spatially local wave behavior in a specific class of structures: networks of one-dimensional members intersecting at junctions. A power flow analysis was developed which enables the identification of certain aspects of junction energy dissipation and member transport mechanisms prior to the construction of a global model: such a global model, when assembled, is sensitive to modelling errors, particularly for high-frequency behavior.

The concept of power flow, besides being used to judge component (member and junction) dissipation, was used in the formulation of junction controllers. Optimal junction control was formulated based upon the minimization of power departing a junction as a function of the arriving power. Closed-loop junction power flow was used to analyze the dissipation of active junctions designed to alter specific scattering characteristics for the purpose of power shunting.

The component wave models were assembled to provide network models to be used in modelling global structural behavior. These network wave models were shown to be equivalent to modal models enabling the identification of mode shapes, modal frequencies, and damping. In addition to providing these classical quantities, these network models can be used for analyzing global power flow. This enables the understanding of the mechanisms through which energy is imparted to, circulates about, and drains from these structures. These network models also aid in properly placing local controllers and identifying the dominant dynamic load paths which connect mission critical regions to the rest of the structure, particularly the disturbance locations.

Finally, an experimental demonstration was used to identify some of the advantages and limitations of junction control. The closed-loop damping performance using a wave control compensator was compared to that using various gains of collocated rate feedback. Since both involve collocated feedback of a positive real nature, this comparison was deemed useful.

The following discussion comprises the main conclusions of each of the preceding chapters. This is followed by a discussion of future work that would be instrumental in advancing this research.

6.1 Specific Results

This section presents the specific results of each chapter and explains how they are prerequisite to the concepts developed in the subsequent chapters. The main contribution of Chapter 2, on component dynamics and power flow, is the formulation of the power
flow properties of the two selected components. Any network model is founded upon descriptions of local behavior. The accuracy of such a global model is sensitive to errors made in modelling each of its components. The approach taken in this work was to use the subset of global information that can be obtained on the local scale. The accuracy of the resulting component model is completely insensitive to errors in modelling other regions of the structure. A useful form of local information was found to be the manner in which these components dissipate and transport energy.

The resulting power flow relations describe how the uncoupled wave modes, which superimpose at each frequency, participate in the transfer of energy from one region of a structure to another. While several other modelling approaches have developed the concept of power flow, it is felt by the author that the wave approach provides an alternative perspective that is both insightful and physically intuitive. Junction power flow, in particular, characterizes the various ways in which a junction can interact with its unmodelled neighbor junctions. This is beneficial in that some local information is obtained, such as the available energy generative, conservative, and dissipative properties, without a priori knowledge of the global model.

Power flow, especially a formalism for representing power flow in a generic component, is a useful and physical quantity for minimization in a control application, as shown in Chapter 3. Not only is it a physically obvious quadratic index, but the knowledge of the various ways in which junction dynamics can interact with incoming power enables the application of control which guarantees that, in the closed-loop, every type of interaction is dissipative to some extent. The application of the controllers derived in Ch. 3 is restricted to performance objectives which can be posed in terms of local behavior. Some objectives which have this feature are energy absorption, dynamic isolation, energy shunting and disturbance rejection.

In Chapter 3, discrete point control algorithms were derived based upon two concepts. The first, which employed feedforward degrees of freedom to alter pre-selected entries in the closed-loop scattering matrix, was shown to allow isolation of chosen members from various wave modes impinging on the junction. These types of solutions range from the conservative shunting of outgoing power along pre-selected paths to the complete absorption of specific wave modes. Unfortunately, the manner in which the control was derived did not involve monitoring the closed-loop power characteristics of the junction. No guarantee that the closed-loop junction would not generate power could be offered with this approach. Rather, investigation of the closed-loop junction power flow matrix must be performed subsequent to the control design. This can result in an iterative design procedure if the initial control design causes the closed-loop junction to generate
power. In addition, arbitrary selection of feedforward measurements can result in actuator/sensor pairs that are not dual: collocated and of like type.

On the other hand, optimal junction control based on the unconstrained minimization of junction power flow does provide a guarantee of power conservation, or absorption, given that the open-loop junction does not generate power. In addition, the examples support the claim that the optimal control often yields feedback between actuators and sensors that are dual. This result, when combined with the fact that the feedback is often collocated and always local, implies a degree of robustness in the design due to its positive real nature.

It was shown that when penalizing other wave mode properties, care must be taken in selecting the penalty matrix because the resulting control, while minimizing the cost functional, may contribute to destabilizing the network by creating a junction that generates power for certain frequency ranges. The unfortunate property of the unconstrained, noncausal solution is that often the optimal compensator is not implementable.

For this reason, three solution techniques are discussed which are constrained to yield causal compensators, the first two of which are dependent upon assumptions or knowledge of the incoming wave mode statistics. The first technique involved a fixed form optimization. The dynamics of the compensators are chosen by the control designer; based upon knowledge of the structural characteristics, the robustness of the feedback form (a form with a positive real nature), control objectives and plain intuition; and the gains are left variable. In this work, these gains are optimized based upon the minimization of the trace of expected power flow plus control effort. Since the form is selected before the optimization, this procedure yields suboptimal solutions.

The second technique involves a Wiener-Hopf approach. It was assumed that all of the wave processes are stationary. This assumption is valid if the disturbance statistics are stationary, the structural properties are time invariant, the control compensator is linear and time invariant and the compensator has been in operation for an assumed infinite duration. The only constraint on the compensator form is a causality constraint. Otherwise, the procedure is free to select a compensator matrix which minimizes the expected value of the sum of power flow plus control effort. It was shown in an example that this somewhat free selection of form can result in an optimal solution which allows the generation of power in one frequency range in order to maximize dissipation in another. Therefore, an iterative design procedure must be employed in order to verify junction dissipation. This problem could be solved by realizing that the incoming and outgoing waves are correlated. Quantizing this, however, would require a model of the entire structure. An alternative
would be to append a constraint requiring that the Wiener-Hopf compensator result in a closed-loop junction power matrix that is at least negative semidefinite at all frequencies.

The third technique involves using noncollocated measurements to allow the implementation of noncausal solutions which would otherwise not be implementable. These noncollocated terms can provide the information anticipated by the anticausal portion of the control. This approach can compromise the assumption made in Chapter 1 that the solutions provide strictly local control.

Several observations were made as to the types of achievable solutions. When all junction actuators are available, high, broadband performance with regards to energy dissipation and isolation is possible because a matched termination, or better as in the case where evanescent modes are present, can be achieved. As junction control is derived using fewer and fewer actuators, either the achievable performance diminishes until the resulting control has an appearance similar to collocated rate feedback or the high performance becomes restricted to a narrower range of frequencies.

Wave junction control can take advantage of combining the robustness of collocated feedback of a positive real nature with the flexibility enabled when allowing dynamic compensation. Certain characteristics of the local performance and dissipation of the junction control can be determined without knowledge of global structural behavior. If the closed-loop power matrix is at least negative semidefinite and the design depends only upon local dynamics, errors in the modelling of distant portions of the structure can not change a junction, which behaves as a power sink, into a power source. Global performance, however, is dependent upon the interactions of the junction with the rest of the structure. By maximizing junction dissipation at all frequencies, high performance vibration suppression can be achieved; perhaps even to the extent of eliminating resonant behavior in the structure. It was shown that designing compensators without regard to actuator dynamics causes a degradation in performance. The actuator dynamics may, however, be included in the boundary conditions to prevent this.

Several disadvantages must also be faced in using this scheme. Its applicability is limited to local objectives. The compensators are typically complex functions of frequency. These can be difficult to implement, and may become more difficult when actuator dynamics are modelled. Many extensions to this theory of active control of wave propagation are possible. One might, for example, attempt to adapt such controllers by observation of their performance. One might attempt dynamic estimation of incoming wave modes, using partial measurements and theory yet to be developed.

In Chapter 4, local dynamical models were assembled into network descriptions. Given the disturbance inputs, the actual wave mode amplitudes at the junctions and in the
members can be found. With this information, the actual values for net and total power flow in the components can be derived. These network models are as sensitive to modelling errors as equivalent modal techniques.

Net power flow subtracts out the balanced flow in components, left and right in a member and in and out of a junction, to extract information concerning the actual steady-state generation, redistribution, and dissipation of energy in a structure. Net power obeys a conservation property which requires that the net flow out of a junction equals the sum of that which arrives and that which is generated, or perhaps dissipated, at that junction. These net power quantities reveal the paths, which may vary with frequency, through which vibrational energy is transmitted from sources to sinks. It was observed, in an example, that these directions do vary with frequency and that the transitions are demarcated by structural resonances and zeros. It was also observed that net power flow can be created by the circulation of power around a closed path. This approach to power flow needs to be compared to mathematically approximate derivations of power flow as in SEA. The wave approach developed here may prove to be more restrictive in its applicability due to the class of structures analyzed.

Total component power flow retains the balanced portion of the flow to find the total magnitude of flow traversing a member cross-section or junction. While the transfer function relations provide similar information directly, the quantification of total power flow requires the monitoring of both deflection and stress coordinates. The technique presented in Chapter 4 extracts this information without the added burden of transforming from the wave states back to physical, cross-sectional coordinates. This information can be particularly useful in the placement of some junction controllers. In addition, power flow information can be used to determine component power coupling and power imbalance.

While an explicit example of component power coupling was not given, power imbalance was shown to be that fraction of the total component power flow that is unbalanced. The larger this ratio is, the more the component power exhibits a directional preference. In a member, this corresponds to a dominant flow direction which is usually towards sinks in the network. For a junction, this might be used as some frequency dependent damping parameter expressing the loss factor associated with the junction.

A set of experiments was conducted. The feedback consisted of feeding rotational acceleration of a pinned end of a beam through two types of compensators. The output of these compensators was used to command moment about the pinned end. The first compensator was a simple integrator to provide collocated, dual rate feedback. The second compensator was found using the fixed form parameter optimization scheme outlined in
Chapter 3. The output of the compensator was approximately the half integral of rotational velocity.

Since both tests employed a dual sensor/actuator pair and both feedbacks had a positive real nature, it was felt that these tests would provide a consistent comparison of a wave control approach to a classical approach for dissipating structural energy: collocated rate feedback. It was found in the model and the experiment that a given gain of rate feedback was only optimal for a limited frequency range; above which the gain was too high and clamped the beam and below which the gain was insufficient to maximize damping. The optimal gain for the wave compensator, however, seemed to maximize damping in modes spanning a broader frequency range.

It was felt, but not substantiated, that high gain rate feedback, in conjunction with unmodelled control hardware dynamics and the corresponding low damping in the higher frequency modes, can easily result in an instability for certain structural configurations. This was thought to be the cause of an instability near the frequency of the sensor (accelerometer) dynamics. The portion of the wave control compensator output in phase with rotational velocity had a frequency dependent gain which was more appropriately matched to the impedance of the beam. In order to achieve this, the wave compensator also included a portion of the output in phase with rotation which stiffened the modes of the beam.

With only one actuator and sensor being used and with the compensator form selected, the gross performance of the wave control was similar to that of the rate feedback over the frequency range tested. However, employing an additional actuator and sensor at the pinned end, in conjunction with a free form control solution, could greatly improve the wave control performance.

6.2 General Results

This work has addressed the issue of modelling and control of complex structures from a wave point of view. The wave approach was motivated by the need for broadband control of elastic deformations, especially in frequency regimes where modal models are inappropriate. In order to achieve this, local control of wave scattering was investigated. The restriction to local control using local models restricts the application of these techniques to local objectives.

From a modelling accuracy perspective, local wave models are insensitive to the modelling of distant regions of the structure. Therefore, it may be argued that they can be more confidently used to capture local behavior in frequency regimes where modal approaches are unable to accurately capture global behavior. Just as modal techniques are
only capable of modelling homogeneous dynamics if the disturbance nature is unknown, local wave techniques can only capture homogeneous component behavior if the incoming wave modes which disturb that component are unknown. These incoming wave modes can be predicted by modelling the structure and structural disturbances as a whole or their statistics can be used to generate a probabilistic description of the incoming waves.

The wave approach to structural control does not achieve every control objective. Objectives must be formulated in terms of local behavior. The absorption of local energy is an example of such an objective. Preventing energy from penetrating a cross-section is another. Aligning one location with another is not. Therefore, it might be most appropriate to combine wave control with some type of global modal control. The wave control, while performing its primary objective, would provide the stability margin in the modes truncated from the global model. This margin is often a prerequisite to the application of modal control based upon truncated descriptions.

From an implementation perspective, determining where and when the nonlinearities in a structure become important is an emerging concern, especially to control engineers. One desirable solution would be to design as linear a structure as possible and then design linear control. This solution is impractical in the light of launch, assembly and deployment constraints. However, it is foreseeable that a portion of an arbitrarily complex and highly nonlinear structure could be constructed to behave in a very linear manner. If, like wave control, it were possible to restrict the implementation and model to be associated only with this linear segment, the nonlinearities in the structure could be ignored. These nonlinearities would only contribute to the composition of the incoming wave modes. An example would be an active truss segment which is designed to behave in a nearly linear fashion while being an integral part of an otherwise nonlinear truss.

It should be noted that local controllers designed individually can destabilize the structure as a whole if they are not constrained to yield closed-loop power matrices without positive eigenvalues. This destabilization can result either from energy shunting objectives or Wiener-Hopf solutions without power constraints. Restricting each local controller to not allow positive junction power flow at any frequency is a conservative approach. An active junction can be permitted to generate power as long as the excess power is dissipated elsewhere. However, the assumption that only local models are used in the formulation of the control makes this conservative approach a necessity in assuring the stability of the structure in the absence of a global model.
6.3 Future Work

Various avenues are available for pursuing further research in this area. Member wave control needs to be formulated for continuous and periodic members. Constrained by available hardware, this would most likely take the form of highly distributed, discrete point approximations to spatially continuous control. Some topics in this area are discussed in Appendix B. Typically in the design of a control algorithm, the structure is discretized and discrete point control is derived. An alternative would be to mathematically derive the exact solution for continuous control and discretize the control in some optimum fashion. Since exact solutions for complex systems are difficult to find, it might be more appropriate to accumulate a set of functional dependencies, with their discrete approximations, which relate specific feedback forms to specific control objectives. Many of these may be functions of the material properties. In addition, it would be useful to formulate a technique for selecting actuator and sensor types which best compliment the control objective. An example in Appendix B shows that a specific sensor/actuator set can minimize the complexity of a feedback architecture without compromising performance.

Another direction for research in the area of member control involves appending an additional constraint to modal control. Typically, some finite set of modes are retained in a design model. Cost functionals are formulated for each mode and the actual control is found. A major problem can exist with this procedure. First, assume that a finite set of physical sensors and actuators are called upon to control a set of modes that are closely spaced in the frequency domain. Second, assume that the control objectives differ significantly from mode to mode. The feedback algorithm will be hard pressed to achieve these different objectives using physical hardware. This difficulty increases as the model spacing becomes smaller or the number of control devices is reduced.

The formulation of modal objectives needs to be constrained as a function of the frequency separation and available hardware. In addition, the added constraint that the control authority roll off sufficiently fast as to not destabilize a neighboring unmodelled mode introduces an additional burden to the iterative design procedure. The appended constraint should restrict the diversity of objectives such that they can be achieved with the available hardware. Furthermore, the resulting control should not be required to taper off authority within such a limited frequency range. Rather, is should be constrained to be stably interacting with neighboring, unmodelled modes.

Wave control approaches can be formulated which create preferential directions for power flow in members. It would be instructive to find that form of the LQR cost functional that yields closed-loop members exhibiting this behavior.
Extensions can be made to the component modelling work. A procedure for experimental identification of junction scattering characteristics would be useful. This analysis of point junctions and one-dimensional members could be extended to two-dimensional waveguides and to one-dimensional junctions.

Several extensions can be made to the junction control problem. First, numerical techniques for solving the Wiener-Hopf problem can be used. Second, the nonstationary solution to the optimal junction control problem can be solved for situations where the incoming wave mode statistics are not stationary and real-time reconfiguration of the junction control is performed. Third, it would be useful to relate optimal wave junction control to LQR modal based solutions. Since both techniques can be used to derive feedback for identical systems, it must be possible to either find the same solution using either technique or explain why one approach yields solutions that the other does not. Fourth, outgoing wave mode measurements might be used to adapt the feedforward loop by comparing downstream measurements with predicted output. Fifth, time domain analysis should be developed. Sixth, the correlation between incoming and outgoing waves can be used in the Wiener-Hopf formulation. Seventh, a condition can be appended which constrains the Wiener-Hopf solution into yielding negative closed-loop power flow at all frequencies. Eighth, a local modelling error sensitivity and control robustness analysis would be useful.

While the local wave models are rather insensitive to modelling errors, the network models exhibit the same limiting sensitivities as modal models. Perhaps spatial and frequency averaging techniques could be employed, as in SEA and AMA. Energy monitoring schemes could be developed which identify disturbances in real-time and reconfigure the various junction and member controllers to either contain this energy in the proximity of the disturbance or shunt it to noncritical locations to be dissipated.

As with any proposed structural control scheme, experimental demonstrations are needed to simulate these methods on truly infinite order systems which cannot be exactly modelled. Experiments not only develop the sensor, computation and actuator technology, they demonstrate the level to which the technology is implementable and identify the mechanisms, that might have been overlooked, which limit the achievable performance.
REFERENCES


APPENDIX A

This appendix presents the details of the problem presented in Example 4.2. It demonstrates how the frequency domain power flow analysis, on both the local and global levels, is performed numerically. The first step is to generate the boundary conditions for each of the junctions in Figure 4.1. Equations A.1 through A.9 list these conditions with only force and moment disturbances made available at the junctions. Figure 4.1 should be used to identify the letter designations corresponding to the members attached to each junction. The primes denote spatial partial derivatives.

Junction 1 (see Figure A.1):

\[
\begin{bmatrix}
k_1 + i\omega c_1 & \frac{1}{\omega^2 l_1} (k_1 + i\omega c_1 - \omega^2 l_1) \\
\end{bmatrix}
\begin{bmatrix}
\theta_a \\
GJ\theta'_a \\
\end{bmatrix} = \tau_1
\]

A.1

Junction 2 (see Figure A.2):

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure_a2}
\caption{Junction of two rods and two beams}
\end{figure}
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
y_c \\
y_j \\
y_a \\
y_b \\
0 \\
0 \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
M \\
F \\
\end{bmatrix}
\]

where

\[
y_c = \begin{bmatrix}
v_c \\
v'_c \\
-EIv''_c \\
\end{bmatrix};
\quad
y_j = \begin{bmatrix}
v_j \\
v'_j \\
-EIv''_j \\
\end{bmatrix};
\quad
y_a = \begin{bmatrix}
\theta_a \\
GJ\theta'_a \\
\end{bmatrix};
\quad
y_b = \begin{bmatrix}
\theta_b \\
GJ\theta'_b \\
\end{bmatrix}
\]

In the following equations, the ordering of the variables within the cross-sectional vectors is the same.

Junction 3 (see Figure A.3):

![Diagram of a rod and beam junction](image)

**Figure A.3 Rod and beam junction**
\[
\begin{bmatrix}
0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\nu_c \\
\nu'_c \\
-Elv'''_c \\
Elv''_c \\
\theta_d \\
GJ\theta'_d
\end{bmatrix} =
\begin{bmatrix}
0 \\
M_3 \\
F_3
\end{bmatrix}
\]

Junction 4 (mirror image of Figure A.3):

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\nu_e \\
\nu'_e \\
-Elv'''_e \\
Elv''_e \\
\theta_d \\
GJ\theta'_d
\end{bmatrix} =
\begin{bmatrix}
0 \\
M_4 \\
F_4
\end{bmatrix}
\]

Junction 5 (see Figure A.2):

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
y_e \\
y_h \\
y_f \\
y_g \\
\theta_g \\
M_5 \\
F_5
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
M_5 \\
F_5
\end{bmatrix}
\]

where the ordering of the cross-sectional variables in \( y \) is the same as in Eq. A.2

Junction 6 (mirror image of Figure A.1):

\[
\begin{bmatrix}
k_6 + i\omega c_6 & -\frac{1}{\omega^2 I_6} (k_6 + i\omega c_6 - \omega^2 I_6)
\end{bmatrix}
\begin{bmatrix}
\theta_g \\
GJ\theta'_g
\end{bmatrix} = \tau_6
\]

Junction 7 (mirror image of Figure A.3):

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
y_j \\
y_i
\end{bmatrix} =
\begin{bmatrix}
0 \\
M_7 \\
F_7
\end{bmatrix}
\]
Junction 8 (mirror image of Figure A.3):

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
y_h \\
y_i
\end{bmatrix}
=
\begin{bmatrix}
0 \\
M_8 \\
F_8
\end{bmatrix}
\]  \hspace{1cm} A.8

Junction 9 (see Figure A.4):

\[
\begin{bmatrix}
1 & 0 & -1 & 0 \\
-\omega^2 I_9 & \frac{k_g + i\omega c_g - \omega^2 I_9}{k_g + i\omega c_g} & 0 & \frac{-k_g + i\omega c_g - \omega^2 I_9}{k_g + i\omega c_g}
\end{bmatrix}
\begin{bmatrix}
y_b \\
y_f
\end{bmatrix}
=
\begin{bmatrix}
0 \\
\tau_g
\end{bmatrix}
\]  \hspace{1cm} A.9

Figure A.4 Rod midsection with attached resonator

The next step is to generate the rod and beam member relations. The rod cross-
section to wave transformation matrix is

\[
\begin{bmatrix}
\theta \\
GJ\theta',
\end{bmatrix}
= \begin{bmatrix}
1 & 1 \\
i k_T GJ & -i k_T GJ
\end{bmatrix}
\begin{bmatrix}
\theta_{lp} \\
\theta_{rp}
\end{bmatrix}
= Y_T w_T
\]  \hspace{1cm} A.10

and the rod transmission matrix for half a rod length is the scalar

\[
\xi_T = e^{-i \frac{k_T l}{2}}
\]  \hspace{1cm} A.11

where

\[
k_T = \sqrt{\frac{\rho}{G}} \omega
\]  \hspace{1cm} A.12
The corresponding beam matrices are

\[
\begin{bmatrix}
v \\
v' \\
-Elv'' \\
-E lv''
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 & 1 \\
-ik_B & k_B & -ik_B & -k_B \\
iElk_B^3 & -Elk_B^3 & -iElk_B^3 & EElk_B^3 \\
\end{bmatrix} \begin{bmatrix}
v_{lp} \\
v_{re} \\
v_{rp} \\
v_{le}
\end{bmatrix} = Y_B \bar{w}_B \tag{A.13}
\]

\[
\bar{\xi}_B = \begin{bmatrix}
\frac{k_B l}{e^{i \omega t}} & 0 \\
0 & \frac{k_B l}{e^{i \omega t}}
\end{bmatrix} \tag{A.14}
\]

where

\[
k_B = 4\sqrt{\frac{\rho A}{EI}} \sqrt{\omega} \tag{A.15}
\]

Notice that the transmission matrices contain transmission lengths equal to half of the member length (each member has been chosen to be the same length). Since the response cross-sections were chosen to be at the midpoints of the members, these are the only transmission matrices that are required.

The next step is to generate the junction transformation matrices. The following equations illustrate how this is done, with the numerical subscripts identifying the junction number. Notice the pointing matrices which rearrange the wave mode vector into incoming and outgoing.

\[
Y_1 = Y_T \tag{A.16}
\]

\[
Y_2 = \begin{bmatrix}
Y_B & 0 & 0 & 0 \\
0 & Y_B & 0 & 0 \\
0 & 0 & Y_T & 0 \\
0 & 0 & 0 & Y_T^{-1}
\end{bmatrix} \begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix} \tag{A.17}
\]

where
\[ T_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} ; \quad T_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \]

\[ T_{21} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} ; \quad T_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ Y_3 = \begin{bmatrix} Y_B & 0 \\ 0 & Y_T \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ Y_4 = \begin{bmatrix} Y_B & 0 \\ 0 & Y_T \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \]

\[ Y_5 = Y_2 \]

\[ Y_6 = Y_T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

\[ Y_7 = \begin{bmatrix} Y_B & 0 \\ 0 & Y_T \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]
\[ Y_8 = \begin{bmatrix} Y_B & 0 \\ 0 & Y_T \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \]

\[ Y_9 = \begin{bmatrix} Y_T & 0 \\ 0 & Y_T \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

The next step is to set up the transfer function relation. This is first done by choosing each junction in turn and relating the outgoing waves to the junction's incoming waves and external inputs. The difference between this and the normal junction relation (Eq. 2.10a) is that the incoming and outgoing wave amplitudes are those at the junction's adjacent response cross-sections. Though all of the rod and beam transmission matrices are identical, the letter designations for the junction's attached members are included for clarity. Equations A.26 through A.34 illustrate this for each of the junctions, consecutively.

\[ w_{oa} = \xi_{Ta} (S_1 \xi_{Ta} w_{ia} + \psi_1 Q_1) \quad \text{A.26} \]

\[ \begin{bmatrix} w_{oc} \\ w_{oj} \\ w_{oa} \\ w_{ob} \end{bmatrix} = \begin{bmatrix} \xi_{Bc} & 0 & 0 & 0 \\ 0 & \xi_{Bj} & 0 & 0 \\ 0 & 0 & \xi_{Ta} & 0 \\ 0 & 0 & 0 & \xi_{Tb} \end{bmatrix} \begin{bmatrix} \xi_{Bc} & 0 & 0 & 0 \\ 0 & \xi_{Bj} & 0 & 0 \\ 0 & 0 & \xi_{Ta} & 0 \\ 0 & 0 & 0 & \xi_{Tb} \end{bmatrix} + \psi_2 Q_2 \quad \text{A.27} \]

\[ \begin{bmatrix} w_{oc} \\ w_{od} \end{bmatrix} = \begin{bmatrix} \xi_{Bc} & 0 \\ 0 & \xi_{Td} \end{bmatrix} \begin{bmatrix} \xi_{Bc} & 0 \\ 0 & \xi_{Td} \end{bmatrix} \begin{bmatrix} w_{ic} \\ w_{id} \end{bmatrix} + \psi_3 Q_3 \quad \text{A.28} \]

\[ \begin{bmatrix} w_{oe} \\ w_{od} \end{bmatrix} = \begin{bmatrix} \xi_{Be} & 0 \\ 0 & \xi_{Td} \end{bmatrix} \begin{bmatrix} \xi_{Be} & 0 \\ 0 & \xi_{Td} \end{bmatrix} \begin{bmatrix} w_{ie} \\ w_{id} \end{bmatrix} + \psi_4 Q_4 \quad \text{A.29} \]
\[
\begin{bmatrix}
w_{oe} \\
w_{oh} \\
w_{of} \\
w_{og}
\end{bmatrix} = \begin{bmatrix}
\xi_{Be} & 0 & 0 & 0 \\
0 & \xi_{Bh} & 0 & 0 \\
0 & 0 & \xi_{Tf} & 0 \\
0 & 0 & 0 & \xi_{Tg}
\end{bmatrix} \begin{bmatrix}
\xi_{Be} & 0 & 0 & 0 \\
0 & \xi_{Bh} & 0 & 0 \\
0 & 0 & \xi_{Tf} & 0 \\
0 & 0 & 0 & \xi_{Tg}
\end{bmatrix} \begin{bmatrix}
w_{ie} \\
w_{ih} \\
w_{if} \\
w_{ig}
\end{bmatrix} + \psi_5 Q_5 \quad \text{(A.30)}
\]

\[
w_{og} = \xi_{Tg} (S_6 \xi_{Tg} \ w_{ig} + \psi_6 Q_6) \quad \text{(A.31)}
\]

\[
\begin{bmatrix}
w_{oj} \\
w_{oi}
\end{bmatrix} = \begin{bmatrix}
\xi_{Bj} & 0 \\
0 & \xi_{Tj}
\end{bmatrix} \begin{bmatrix}
\xi_{Bj} & 0 \\
0 & \xi_{Tj}
\end{bmatrix} \begin{bmatrix}
w_{ij} \\
w_{ii}
\end{bmatrix} + \psi_7 Q_7 \quad \text{(A.32)}
\]

\[
\begin{bmatrix}
w_{oh} \\
w_{oi}
\end{bmatrix} = \begin{bmatrix}
\xi_{Bh} & 0 \\
0 & \xi_{Tj}
\end{bmatrix} \begin{bmatrix}
\xi_{Bh} & 0 \\
0 & \xi_{Tj}
\end{bmatrix} \begin{bmatrix}
w_{ih} \\
w_{ii}
\end{bmatrix} + \psi_8 Q_8 \quad \text{(A.33)}
\]

\[
\begin{bmatrix}
w_{ob} \\
w_{of}
\end{bmatrix} = \begin{bmatrix}
\xi_{Tb} & 0 \\
0 & \xi_{Tf}
\end{bmatrix} \begin{bmatrix}
\xi_{Tb} & 0 \\
0 & \xi_{Tf}
\end{bmatrix} \begin{bmatrix}
w_{ib} \\
w_{if}
\end{bmatrix} + \psi_9 Q_9 \quad \text{(A.34)}
\]

Notice that the ordering of the outgoing wave mode vectors does not equal the ordering of the incoming wave mode vectors. But, every wave mode vector that is incoming to a particular junction is outgoing from another junction. Therefore, a reordering matrix can be generated relating the incoming wave vectors to the outgoing equivalents.

Placing the last nine equations in a single matrix form, rearranging the incoming vector to be equivalent to the outgoing vector, using the reordering matrix, the result has the form of Eq. 4.1. This equation was solved numerically, along with its constituent equations given in this appendix, at every desired frequency. Notice the torsion and beam wave numbers (Eqs. A.12 and A.15) are different for a given temporal frequency. The wave mode amplitudes, at a given frequency, are assembled for a given cross-section and substituted into the appropriate net and total member power flow matrices, evaluated at the same frequency. The results are presented in Example 4.2.

This numerical problem finds the mathematically exact solution to the governing partial differential equations plus boundary conditions. Unlike FEM, the further subdivision of the components, with the addition of the corresponding transmission or
scattering relations, adds no new information or accuracy. It does, however, allow the extraction of response information at more response cross-sections.

A.1 Computer Code

\[ ([P_{\text{net}}, P_{\text{tot}}, h, \Omega}] = \text{NPF}(w_i, w_f, npts) \]

\[ \text{inc} = 0.4342945 \times (\log(w_f) - \log(w_i)) / (npts - 1); \]

\[ e = 2.718281828; \]
\[ G = 0.00345; p = 0.0439; p_A = 780; E_l = 0.0986; l_1 = 0.15; l_2 = 0.075; \]
\[ l_p = 2.08; K_p = 0.82; C_p = 0.105; \]
\[ l_m = 16; K_m = 631.65; C_m = 14.07; \]

\[ w = w_i; \]
\[ i = j; \]

\[ a = \text{ones}(28, 28); a(1, 6) = 1; a(2, 8) = 1; a(3, 9) = 1; a(4, 21) = 1; \]
\[ a(5, 22) = 1; a(6, 1) = 1; a(7, 27) = 1; a(8, 2) = 1; a(9) = 1; a(10, 13) = 1; \]
\[ a(11, 14) = 1; a(12, 15) = 1; a(13, 10) = 1; a(14, 11) = 1; a(15, 12) = 1; \]
\[ a(16, 24) = 1; a(17, 25) = 1; a(18, 28) = 1; a(19, 20) = 1; a(20, 19) = 1; a(21, 4) = 1; \]
\[ a(22, 5) = 1; a(23, 26) = 1; a(24, 16) = 1; a(25, 17) = 1; a(26, 23) = 1; \]
\[ a(27, 7) = 1; a(28, 18) = 1; \]

\[ u = [1 0 0 0 0 -0.7071 0 i^0.7071; 0 1 0 1 0 0 -i^0.7071 0 0.7071]; \]

\[ u = u^* u; \]

\[ B_2 = [1 0 0 0 -1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0]; \]
\[ B_2 = [B_2, 0 1 0 0 0 0 0 -1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0]; \]
\[ B_2 = [B_2, 0 0 0 0 0 1 0 1 0 -1; 0 0 1 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0]; \]
\[ B_3 = [0 1 0 0 -1 0; 0 0 1 0 -1; 0 0 1 0 0 0]; \]
\[ B_4 = [0 1 0 0 -1 0; 0 0 1 0 1 0 0 1 0 0 0]; \]
\[ B_5 = B_2; \]
\[ B_7 = [0 1 0 0 -1 0; 0 0 0 0 -1 0 -1; 0 0 1 0 0 0]; \]
\[ B_8 = [0 1 0 0 -1 0; 0 0 0 1 1 0 1; 0 0 1 0 0 0]; \]

\[ T_2 = \text{eye(2,2)} \times \text{ones}(2,10); \text{eye}(4,4) \times \text{ones}(4,2); \]
\[ T_2 = [T_2, 0 \times \text{ones}(2,2); \text{eye}(2,2) \times \text{ones}(2,8); 0 \times \text{ones}(1,10) 1 0]; \]
\[ T_2 = [T_2, 0 \times \text{ones}(1,4) 1 0; \text{ones}(1,7); 0 \times \text{ones}(1,5) 1 0 \times \text{ones}(1,8); 0 \times \text{ones}(1,11) 1]; \]
\[ T_3 = [0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1]; \]
\[ T_4 = [0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0]; \]
\[ T_6 = [0 1 1 0]; \]
\[ T_7 = [1 0 0 0 0 0; 1 0 0 0 0; 0 0 0 1 0; 0 0 0 0 1; 0 0 1 0 0 0; 0 0 0 0 1]; \]
\[ T_8 = [1 0 0 0 0; 1 0 0 0 0; 0 0 0 1 0; 0 0 0 0 1; 0 0 0 0 1; 0 0 1 0 0 0]; \]
\[ T_9 = [0 0 1 0; 0 1 0 0; 0 1 0 0; 0 0 0 1]; \]

for \( j = 1 : npts, \ldots \)

\[ k_b = ((p_A / E_l)^{0.25}) \times (w^{0.5}); \ldots \]
\[ k_t = ((p_J / G_J)^{0.5}) \times w; \ldots \]

160
B1=[Kp+i*w*Cp,(Kp+i*w*Cp-lp*w**2)/(lp*w**2)];... FREQUENCY
B6=[Kp+i*w*Cp,-(Kp+i*w*Cp-lp*w**2)/(lp*w**2)];... DEPENDENT
s=(Km+i*w*Cm-lm*w**2)/(Km+i*w*Cm);... BOUNDARY
B9=[1 0 -1 0;-lm*w**2 t 0 -l];... CONDITIONS

Yt=[1 1;ikt*GJ -ikt*GJ];... BEAM AND ROD
Yb=[1 1 1;ikb kb -kb;i*El*kb**3 -El*kb**3 -i*El*kb**3 El*kb**3];... TRANSFORMATION
Yb=[Yb,-El*kb**2 El*kb**2 -El*kb**2 El*kb**2];... MATRICIES

t=B1*Yt;...
[n,m]=size(t);...
P1=inv(t([1+m/2:m]));...
S1=P1'*t([1:m/2]);...
t=[Yb,0*ones(4,8);0*ones(4,4),Yb,0*ones(4,4);0*ones(2,8),Yt,0*ones(2,2)];...
t=[t;0*ones(2,10),Yt];...
t=B2*t*T2;...
[n,m]=size(t);...
P2=inv(t([1+m/2:m]));...
S2=P2'*t([1:m/2]);...
n=[Yb,0*ones(4,2);0*ones(2,4),Yt];...
t=B3*t*t3;...
[n,m]=size(t);...
P3=inv(t([1+m/2:m]));...
S3=P3'*t([1:m/2]);...
t=B4*t*t4;...
[n,m]=size(t);...
P4=inv(t([1+m/2:m]));...
S4=P4'*t([1:m/2]);...

Poo=[2*w*El*kb**3 0 0 0 0 0 0 0 0 w*GJ*kt];... DERIVATION OF CONTROLLED
Poi=[0 0 0 0 -2*i*w*El*kb**3 0 0 0 0];... JUNCTIONS
t=[0 0 1 0 0 1];...
m=P3*t;...
S3=S3-m*inv(m'*Poo*m)*m*(Poi+Poo*S3);...
m=P4*t;...
S4=S4-m*inv(m'*Poo*m)*m*(Poi+Poo*S4);...

P5=P2;...
S5=S2;...
t=B6*Yt*T6;...
[n,m]=size(t);...
P6=inv(t([1+m/2:m]));...
S6=P6'*t([1:m/2]);...
t=B7*t*t7;...
[n,m]=size(t);...
P7=inv(t([1+m/2:m]));...
S7=P7'*t([1:m/2]);...
t=B8*t*T8;...
[n,m]=size(t);...
P8=inv(t([1+m/2:m]));...
S8=P8'*t([1:m/2]);...
t=B9*[Yt,0*ones(2,2);0*ones(2,2),Yt]*T9;...
[n,m]=size(t);...
P9=inv(t([1+m/2:m]));...
clear Yb Yu;...

Zt = e**(-i*kt*l/2);...  
Ztbf = e**(-i*kt*l/2);...  
Zb = [e**(i*kb*l/2) 0;0 e**(-kb*l/2)];...  

CAUSAL DESCRIPTION OF WAVE MODES  
DEPARTING A JUNCTION THROUGH THE

ROD AND BEAM MEMBER  
TRANSMISSION MATRICES

clear B1 B6 B9 t tt n m;...

w1 = Zt*S1*Zt;...

CAUSAL DESCRIPTION OF WAVE MODES  
DEPARTING A JUNCTION THROUGH THE

x1 = Zt*P1;...

NEIGHBORING RESPONSE CROSS-  
SECTIONS AS A FUNCTION OF THOSE

clear S1 P1;...

APPROACHING THE SAME JUNCTION  
THROUGH THESE CROSS-SECTIONS

tt = [Zb,0*ones(2,4);0*ones(2,2),Zb,0*ones(2,2);0 0 0 0 Zt 0;0 0 0 0 Ztbf];...

w25 = tt*S2*tt;...

w1516 = tt*S7*tt;...

w1718 = tt*S8*tt;...

w1920 = tt*S9*tt;...

clear S9 P9 tt;...

y = [w1,0*ones(1,27);0*ones(6,1),w25,0*ones(6,21)];...

NEIGHBORING RESPONSE CROSS-  
SECTIONS AS A FUNCTION OF THOSE

y = [l;0*ones(3,7),w67,0*ones(3,18);0*ones(3,10),w89,0*ones(3,15)];...  

APPROACHING THE SAME JUNCTION  
THROUGH THESE CROSS-SECTIONS

y = [y;0*ones(6,13),w1013,0*ones(6,9);0*ones(1,19),w14,0*ones(1,8)];...

SCATTERING MATRIX

y = [y;0*ones(3,20),w1516,0*ones(3,5);0*ones(3,23),w1718,0*ones(3,2)];...

y = [y;0*ones(2,26),x1920];...

GENERATION MATRIX

y = [y;0*ones(3,7),x67,0*ones(3,18);0*ones(3,10),x89,0*ones(3,15)];...

ASSEMBLY OF THE

y = [y;0*ones(6,13),x1013,0*ones(6,9);0*ones(1,19),x14,0*ones(1,8)];...

NETWORK MATRIX

y = [y;0*ones(3,20),x1516,0*ones(3,5);0*ones(3,23),x1718,0*ones(3,2)];...

ASSEMBLY OF THE

x = [x;0*ones(2,26),x1920];...

162
y=y*a;...

\[ h = \text{inv}(\text{eye}(28,28) - y) \times \ldots \]

\[ h = \ast\{1000;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0\};\ldots \]

\[ hh(:, :) = h; \ldots \]

clear y x t t w1 x 1 w25 x 25 w67 x67 w89 x89 w1013 x1013 w14 x14;\ldots

clear w1516 x1516 w1718 x1718 w1920 x1920 Zb Zl Zblf;\ldots

\[ Pt = w^* k^* G J^{-1} [1:0:0:1]; \ldots \]

\[ Pb = 2^* w^* (k b^* 3)^* E I^{-1} [-1 0 0 0; 0 0 0 i; 0 0 1 0; 0 -i 0 0]; \ldots \]

\[ p(1,1) = 0.5^* [h(6,1); h(1,1)]^* P t^* [h(6,1); h(1,1)]; \ldots \]

\[ p(2,1) = 0.5^* [h(27,1); h(7,1)]^* P t^* [h(27,1); h(7,1)]; \ldots \]

\[ t = 0.5^* [h(8,1); h(9,1); h(2,1); h(3,1)]^* P b; \ldots \]

\[ p(3,1) = t^* [h(8,1); h(9,1); h(2,1); h(3,1)]; \ldots \]

\[ p(4,1) = 0.5^* [h(13,1); h(10,1)]^* P t^* [h(13,1); h(10,1)]; \ldots \]

\[ t = 0.5^* [h(11,1); h(12,1); h(14,1); h(15,1)]^* P b; \ldots \]

\[ p(5,1) = t^* [h(11,1); h(12,1); h(14,1); h(15,1)]; \ldots \]

\[ p(6,1) = 0.5^* [h(18,1); h(28,1)]^* P t^* [h(18,1); h(28,1)]; \ldots \]

\[ p(7,1) = 0.5^* [h(20,1); h(19,1)]^* P t^* [h(20,1); h(19,1)]; \ldots \]

\[ t = 0.5^* [h(16,1); h(17,1); h(24,1); h(25,1)]^* P b; \ldots \]

\[ p(8,1) = t^* [h(16,1); h(17,1); h(24,1); h(25,1)]; \ldots \]

\[ p(9,1) = [h(26,1); h(23,1)]^* P t^* [h(26,1); h(23,1)]; \ldots \]

\[ t = 0.5^* [h(4,1); h(5,1); h(21,1); h(22,1)]^* P b; \ldots \]

\[ p(10,1) = t^* [h(4,1); h(5,1); h(21,1); h(22,1)]; \ldots \]

\[ \text{Pnet}(; :) = p; \ldots \]

clear Pb Pt p t;\ldots

\[ p(1,1) = 0.5^* [h(6,1); h(1,1)]^* [h(6,1); h(1,1)]; \ldots \]

\[ p(2,1) = 0.5^* [h(27,1); h(7,1)]^* [h(27,1); h(7,1)]; \ldots \]

\[ t = 0.5^* [h(8,1); h(9,1); h(2,1); h(3,1)]^* u; \ldots \]

\[ p(3,1) = t^* [h(8,1); h(9,1); h(2,1); h(3,1)]; \ldots \]

\[ p(4,1) = 0.5^* [h(13,1); h(10,1)]^* [h(13,1); h(10,1)]; \ldots \]

\[ t = 0.5^* [h(11,1); h(12,1); h(14,1); h(15,1)]^* u; \ldots \]

\[ p(5,1) = t^* [h(11,1); h(12,1); h(14,1); h(15,1)]; \ldots \]

\[ p(6,1) = 0.5^* [h(18,1); h(28,1)]^* [h(18,1); h(28,1)]; \ldots \]

\[ p(7,1) = 0.5^* [h(20,1); h(19,1)]^* [h(20,1); h(19,1)]; \ldots \]

\[ t = 0.5^* [h(16,1); h(17,1); h(24,1); h(25,1)]^* u; \ldots \]

\[ p(8,1) = t^* [h(16,1); h(17,1); h(24,1); h(25,1)]; \ldots \]

\[ p(9,1) = [h(26,1); h(23,1)]^* [h(26,1); h(23,1)]; \ldots \]

\[ t = 0.5^* [h(4,1); h(5,1); h(21,1); h(22,1)]^* u; \ldots \]

\[ p(10,1) = t^* [h(4,1); h(5,1); h(21,1); h(22,1)]; \ldots \]

\[ \text{Ptot}(; :) = p; \ldots \]

clear p t h;\ldots

\[ \Omega(1,1) = w; \ldots \]

\[ w = 10^* \{\text{inc} + 0.4342945 \times \text{log}(w)\}; \]

\[ \text{refl} \]

163
APPENDIX B: MEMBER CONTROL

This chapter discusses several issues associated with the control of one-dimensional members (waveguides) described by partial differential equations (PDE's). Member control, whether in the context of modal control, continuum control or wave control, has received extensive attention\textsuperscript{16} 26 27 28. Therefore this chapter does not present a straightforward control design procedure for P.D.E. systems, it is included in this document in order to present various concepts which have not received sufficient attention in previous work. The contribution of this chapter is aimed at motivating future research and addressing some of the underlying principles of present control design techniques. While Chapter 3 presents a formalism for deriving junction control, this appendix takes a more open ended approach to the discussion of member control in terms of structural continuum, modes and waves.

There are three basic concepts that are motivated in this chapter. These concepts are presented to help in interpreting control solutions to more complex problems. These four concepts are: the use of typical section problems to guide in the design of control for complex systems, the proper selection of control hardware, the derivation of fundamental functional feedback relationships, and the generation of directional closed-loop power flow behavior. These are described in more detail in the following paragraphs.

1.) During the design of control for complex systems, the control engineer looks for elements of the feedback solution which "make sense." The expression "make sense" implies that the engineer has some basic understanding of the physics involved and that some elements of the solution exhibit features which are obviously appropriate and could have been selected through engineering insight prior to the control optimization. This insight often arises from the actual analysis of the underlying physics. To this end, two typical section problems are presented. Typical section, borrowed from the field of aeroelasticity, refers to a simple system which exhibits some physical behavior similar to its more complex counterpart. The first problem presents control solutions applicable to the control of standing and wave modes in structures. The second illustrates the inherent problem associated with truncating dynamics to facilitate the derivation of control. This is a typical section problem for spillover.

2.) Control, whether distributed continuously or discretely, can be implemented on structural members to alter their power transmission behavior. Structural systems contain response variables which are typically not employed in discrete models. These include strain, curvature, shear deformation, etc. and their corresponding rates, in addition
to the commonly used displacement and rotation. These quantities, which might be more easily measured, are available for exploitation. The same is true in actuator selection. In addition to force and moment there exist stress, curvature, etc. actuator possibilities. Sensor and actuator selection should be based both on performance and simplicity of the device and feedback structure.

3.) The optimal feedback for an arbitrarily complex structure often is itself, complex. The ensuing difficulty in interpreting exactly how the feedback structure achieves the control objective makes it desirable to have a repertoire of functional feedback forms which, when combined in a particular fashion, account for this complex form. In other words, it might be possible to decompose a complex feedback kernel into a set of simpler functional forms. These functional forms might have scale lengths over which the control deems information to be important. These scale lengths might be a function of passive (positive real) damping or could vary as a function of control objective. Not only scale length, but functional shape is important.

Naturally, if the structural model is first discretized and then the feedback is derived, these functional forms can become obscured. However, if the control is first derived and then the spatially continuous feedback functions are optimally discretized, several benefits arise. This idea of functional dependence ties in with proper selection of control hardware (paragraph 1). The purpose of a noncollocated feedback kernel might be to approximate other state variables or actuator types. If this fact is understood and recognized in a design, a simple change of sensor or actuator might greatly simplify the complexity of the feedback structure.

4.) The fourth concept deals with the creation of directional closed-loop power flow behavior. In structural continua described by partial differential equations (PDEs), there are multiple dimensions through which vibrational energy may be extracted. Not only temporal but spatial dimensions may be used. While all forms of structural damping remove energy from the entire structure over time, directional power flow behavior can be created to shunt energy to noncritical portions of the structure, thus removing the energy spatially. In this context, "directional" refers to a structure's dynamic characteristics which effect the propagation of energy in different ways depending on the direction of propagation. While this behavior is not common in passive structures, this attribute can be obtained through active means.

The rest of this chapter discusses and illustrates these concepts in more detail. Various optimal and suboptimal techniques are used for deriving continuously distributed
control for structural members. The methods of separation of variables and functional analysis are used.

B.1 Typical Section Problems

Typical section is a term borrowed from the field of aeroelasticity. A typical section problem refers to the simplest problem which provides a basic understanding of the fundamental issues associated with a more complex problem. For member control, the issue is the physical understanding of the Linear Quadratic Regulator solution.

Finite element or assumed mode models are useful in modelling the homogeneous dynamics of structures. The application of control alters these homogeneous dynamics. If this control is applied in such a way as not to couple the modes in the model to those not modelled, the model retains its accuracy and applicability. However if this coupling does occur, the original model no longer retains the fidelity it originally possessed. This is the inherent limitation behind the use of finite element models and extensive work has been performed to accommodate this problem. This limitation is not confined to finite element models or modal models for that matter. This problem is inherent in any reduced order model and can emerge in the use of wave models if these models do not represent the actual behavior throughout the entire frequency spectrum.

This raises two issues. First, it would be beneficial to understand the control possibilities that are available if it were possible to not couple one mode, be it standing or propagating, to another. This is the topic of the typical section control problem in Section B.1.1. Second, given that the control engineer is unable to prevent coupling, it is important to understand the fundamental ways in which instability can occur if this coupling occurs between the control and modes not retained in the model. These two very related problems are important because they help to visualize the physics at work in more complex situations. In addition, their applicability is far reaching in that they describe physics which occur in many complex structural control problems.

B.1.1 Typical Section Control Problem. The response of P.D.E. members can be decoupled in terms of modes or waves. In either case, it can be shown that this decoupling results in a set of uncoupled, second order, harmonic oscillators. Therefore, the LQR solution for optimal control of a harmonic oscillator is the typical section control problem.

The general Linear Quadratic Regulator equations are as follows. The system dynamics are contained in a state-space description

$$\frac{dz(t)}{dt} = A z(t) + B u(t)$$

B.1
The cost functional has the general form

\[ J = \frac{1}{2} \int_{-\infty}^{\infty} \left( z^T Q z + u^T R u \right) dt \quad \text{B.2} \]

The steady-state Ricatti equation and control function are

\[ PA + A^T P + Q - PB \beta^{-1} B^T P = 0 \quad \text{B.3} \]

\[ u = -\beta^{-1} B^T P z = -F z \]

where P is the solution to the steady-state Ricatti equation.

The one degree of freedom, second order, harmonic oscillator is shown in Fig. B.1. The state space equation is

\[
\begin{bmatrix}
\dot{x} \\
\dot{\dot{x}}
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
-\frac{k}{m} & -\frac{c}{m}
\end{bmatrix} 
\begin{bmatrix}
x \\
\dot{x}
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
\frac{1}{m}
\end{bmatrix} f 
\quad \text{B.4}
\]

![Figure B.1 One degree of freedom, second order, oscillatory system](image)

Since energy minimization and vibration suppression is the main objective, the matrices in the cost functional are chosen as

\[ Q = \begin{bmatrix} n & 0 \\ 0 & \alpha \end{bmatrix} \quad ; \quad R = \beta \quad \text{B.5} \]

The solution to the Ricatti equation is

\[ P_{11} = -ck\beta \left( 1 - \sqrt{1 + \frac{1}{k^2 \beta}} \sqrt{1 - \frac{2\ km}{c^2 \left( 1 - \sqrt{1 + \frac{1}{k^2 \beta}} \right) + \frac{1}{\beta} \frac{\alpha}{c^2 \beta}} } \right) \quad \text{B.6a} \]

\[ P_{12} = -km\beta \left( 1 - \sqrt{1 + \frac{1}{k^2 \beta}} \right) \quad \text{B.6b} \]

\[ P_{22} = -cm\beta \left( 1 - \sqrt{1 - \frac{2\ km}{c^2 \left( 1 - \sqrt{1 + \frac{1}{k^2 \beta}} \right) + \frac{1}{\beta} \frac{\alpha}{c^2 \beta}} } \right) \quad \text{B.6c} \]

167
The displacement and velocity feedback gains are

\[ F_x = -k \left( 1 - \sqrt{1 + \frac{1}{k^2} \frac{\nu}{\beta}} \right) \]  \hspace{1cm} B.7a

\[ F_x = -c \left( 1 - \sqrt{1 - 2 \frac{km}{c^2} \left( 1 - \sqrt{1 + \frac{1}{k^2} \frac{\nu}{\beta}} \right) \frac{1}{c^2} \frac{\alpha}{\beta}} \right) \]  \hspace{1cm} B.7b

respectively. This control gives the closed-loop poles as

\[ s = -\frac{c}{2m} \sqrt{1 - 2 \frac{km}{c^2} \left( 1 - \sqrt{1 + \frac{1}{k^2} \frac{\nu}{\beta}} \right) \frac{1}{c^2} \frac{\alpha}{\beta}} + \frac{1}{c^2} \frac{\alpha}{\beta} \]  \hspace{1cm} B.8

\[ + i \frac{c}{2m} \sqrt{-1 + 2 \frac{km}{c^2} \left( 1 + \sqrt{1 + \frac{1}{k^2} \frac{\nu}{\beta}} \right) \frac{1}{c^2} \frac{\alpha}{\beta}} - \frac{1}{c^2} \frac{\alpha}{\beta} \]

The region of possible pole locations (Eq. B.8) in the s-plane is shown in Fig. B.2 by the shading. Four interesting cases exist. First, if \( \nu = 0 \) then there is no penalty on displacement and, as seen in Eq. 4.6, only velocity feedback occurs. This result is intuitive because the controller is not concerned with position and simply attempts to slow the motion. The expected locus of pole locations, that of a damped system, is shown in the figure. Second, if \( \alpha = 0 \), then only displacement penalty exists. Notice that for large control effort penalties (\( \beta \)), the little control effort that is expended goes towards damping the system, moving the poles directly away from the vertical axis. As more control effort is permitted, the locus of pole locations becomes asymptotic to the forty-five degree lines extending from the origin. Poles on these lines have damping ratios of \( \zeta = 0.7071 \). This result is intuitive since the system desires to converge on zero displacement as quickly as possible (minimum response time).

These first two important cases represent the limiting cases for the state penalty matrix, and their corresponding closed-loop pole loci bound the shaded region. The third case is that of energy penalty where \( \nu = k \) and \( \alpha = m \). As compared to pure velocity feedback, some stiffening occurs. But eventually, the system does become critically damped. The fourth case is that relationship between displacement and velocity penalties that causes the imaginary part of the closed-loop poles to remain constant and equal to the imaginary part of the open-loop poles. The two state penalty terms are related by

\[ \frac{\nu}{\beta} = \frac{k}{m} \frac{\alpha}{\beta} + \frac{1}{4m^2} \frac{\alpha^2}{\beta^2} \]  \hspace{1cm} B.9
This penalty is the boundary between whether the closed-loop poles become critically damped, intersect the real axis, or remain complex for all levels of control effort penalty. Note that for critical damping, one pole moves back towards the imaginary axis. If the displacement penalty is greater than that given in Eq. B.9, both poles continue to move leftward.

For systems whose response can be expressed as a sum of the responses of uncoupled second order systems, this typical section control solution illustrates the variety of control possibilities, for diagonal state penalty, for each uncoupled system. If it were
possible to control each of these uncoupled subsystems independently, deriving control
would simply entail choosing the ratios $\alpha/\beta$ and $\nu/\beta$, deriving the feedback and
implementing the control. If the sensing and actuating of physical variables is truly
continuously distributed and the exact eigenvector transformations can be found between
the physical measurements and modal coordinates, then each of the subsystems can be
controlled independently. Typically, these types of actuators and sensors do not exist and
the control engineer is restricted to sensing and actuating a finite number of physical
motions. The transformation between physical and modal variables becomes inexact and
spillover occurs.

Spillover is a major problem in structural control. This problem can be dealt with in
two ways. First, if the stability margin is known, the maximum stable gain can be
determined. Second, control solutions can be limited to only those which guarantee that the
spillover will not create an instability. This latter alternative is often too restrictive by
limiting the feedback structure to dual feedback to provide positive real damping. It then
becomes difficult to justify the complexity of implementing active control instead of passive
damping. Therefore, it might be better to understand the physical properties of the
structure and control which govern the transition from stabilizing spillover, which causes
performance differing from that predicted, to destabilizing spillover which can cause an
instability. If the cost functional, and therefore control objective, varies radically between
neighboring subsystems (modes) in the frequency domain, then the subsystem feedback
gains, and therefore physical variable feedback gains, must be rapidly varying functions of
frequency if the interval is narrow. Often, this is not achievable and the system becomes
unstable. What is needed are smooth transitions between control objectives for adjacent
modes which are compatible with the available hardware.

B.1.2 Typical Section Spillover Problem This section derives the typical section
solution for spillover where the higher mode of a two mode system has been truncated
from the control design model. Control is derived based upon a single mode control
model, using the results of the typical section control problem in Section B.1.1, and
evaluated against the two mode evaluation model. A stability condition is derived for a
specific feedback configuration which is a function of the control objective, modal
frequency separation and damping in the unmodelled mode.

Using the two mode system shown in Figure B.3, the governing equations of
motion are

$$m \ddot{q}_1 = Q_1 + c_2 ( \dot{q}_2 - \dot{q}_1 ) + k_2 ( q_2 - q_1 ) - c_1 \dot{q}_1 - k_1 q_1$$  B.10
\[ m \ddot{q}_2 = Q_2 - c_2 (\dot{q}_2 - \dot{q}_1) - k_2 (q_2 - q_1) - c_1 \dot{q}_2 - k_1 q_2 \]

In state-space form these become

\[
\begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_1 + c_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}
\]

Figure B.3 Two degree of freedom spillover model

Using the decoupling transformation

\[
\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}
\]

gives the decoupled system as

\[
\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \frac{1}{m} \begin{bmatrix} c_1 & 0 \\ 0 & c_1 + 2c_2 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \frac{1}{m} \begin{bmatrix} k_1 & 0 \\ 0 & k_1 + 2k_2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}
\]

Since spillover is being investigated, only one actuator should be available. Therefore, \( Q_2 \) in Eq. B.17 will be assumed to be equal to zero and this equation will be the evaluation model.

Control will be derived based upon a model of the lower frequency mode. This mode has the generalized coordinate \( \eta_1 \). Therefore, the design model in state-space form is
\[
\begin{bmatrix}
\dot{\eta}_1 \\
\dot{\eta}_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-\omega_1^2 & -2\zeta_1 \omega_1
\end{bmatrix} \begin{bmatrix}
\eta_1 \\
\eta_2
\end{bmatrix} + \begin{bmatrix}
0 \\
\frac{1}{m\sqrt{2}}
\end{bmatrix} Q_1 \quad \text{where} \quad \omega_1^2 = \frac{k_1}{m}, \quad 2\zeta_1 \omega_1 = \frac{c_1}{m} \quad \text{B.14}
\]

Solving for the feedback using the technique illustrated in Section B.1.1 gives

\[F_{\eta_1} = -m\sqrt{2} \omega_1^2 \left(1 - \sqrt{1 + \frac{\nu}{2\beta m^2 \omega_1^4}}\right)\quad \text{B.15}\]

\[F_{\eta_2} = -2\sqrt{2} \zeta_1 \omega_1 m \left(1 - \sqrt{1 + \frac{2P_{12} + \alpha}{8\beta \zeta_1^2 \omega_1^4 m^2}}\right)\]

where \(\nu\) and \(\alpha\) are the diagonal modal displacement and modal velocity penalties, respectively, and

\[P_{12} = -2\beta m^2 \omega_1^2 \left(1 - \sqrt{1 + \frac{\nu}{2\beta m^2 \omega_1^4}}\right)\]

The spillover analysis is performed with no displacement penalty (\(\nu=0\)). Thus, only rate feedback is used. Since the lower frequency mode corresponds to the motions of the two masses being of equal magnitude and in phase, the assumption that \(\eta_1 = 1.414 \eta_2\) will be made. Notice that the sensor/actuator pair are noncollocated. This is required for an instability to occur when using rate feedback. The damper \(c_1\) equals zero in the following analysis.

Substituting the closed-loop design model into the evaluation model and solving for the condition that two poles of the system lie on the imaginary axis in the complex plane, gives

\[
\sqrt{\frac{\alpha}{\beta}} = 0 \quad , \quad 2\sqrt{2} \zeta_2 \omega_2 \frac{\omega_2^2 - \omega_1^2 + \frac{4}{\zeta_1} \omega_2^2 \omega_1^2}{\omega_2^2 \left(1 - \frac{4}{\zeta_2^2}\right)} - \omega_1^2 \quad \text{B.16}
\]

The first solution corresponds to the fact that the low mode is undamped in the open loop.

Equation B.20 contains several interesting properties. First, if the lower mode is a rigid body mode, \(\omega_1 = 0\), then Eq. B.20 becomes

\[
\sqrt{\frac{\alpha}{\beta}} = 0 \quad , \quad 2\sqrt{2} \omega_2 \frac{\zeta_2}{1 - \frac{4}{\zeta_2^2}} \quad \text{B.17}
\]

and for light damping in the second mode, this becomes

172
This situation characterizes the control penalty ratio that brings a system to the verge of instability when destabilizing spillover occurs between a rigid body controller and the first elastic mode. As indicated in Eq. B.22, more performance may be achieved in the control of the rigid body mode if there is either a large frequency separation, large \( \omega_2 \) as is typically the case in the control of rigid single or multi-body systems, or large damping in the unmodelled mode, large \( \zeta_2 \) as might be required in large flexible space structures. The exact amount of required damping is given by Eq. B.21. When \( \zeta_2 \) exceeds 0.5, which is quite large, no feedback level can destabilize the unmodelled elastic mode.

These typical section problems yield control solutions and spillover conditions based upon modal and wave descriptions.

### B.2 Actuator/Sensor Selection and Functional Form

This section presents the functional derivation of optimal control, using the LQR formulation, for the one-dimensional wave equation. This solution is used to illustrate the difference in feedback structure that can be achieved through the use of different actuators. In addition, Example B.1 a functional form which may be common to many energy minimization solutions.

**Example B.1** For an undamped, uniform rod undergoing axial compression, the governing equation of motion is

\[
E \frac{\partial^2 u}{\partial x^2} - \rho \frac{\partial^2 u}{\partial t^2} = -q - \frac{\partial \tau}{\partial x}
\]

where both distributed body force and stress have been included as available actuation because they represent the complete set of unique nonhomogeneous terms. Defining the states to be the energy variables

\[
v_1 = \frac{\partial u}{\partial t} \quad ; \quad v_2 = \frac{\partial u}{\partial x}
\]

the state-space representation is

\[
\begin{bmatrix}
\frac{\partial}{\partial x} [v_1] \\
[\frac{\partial}{\partial x} [v_2]
\end{bmatrix}
= \begin{bmatrix}
0 & E \frac{\partial}{\partial x} \\
\frac{\partial}{\partial x} & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
+ \begin{bmatrix}
\frac{1}{\rho} & \frac{1}{\rho} \frac{\partial}{\partial x} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
q \\
\tau
\end{bmatrix}
\]

with a cost functional penalizing energy density and control effort.
\[ J = \frac{1}{2} \int_0^\infty \int_{-\infty}^\infty \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \rho & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + [q \tau] \begin{bmatrix} \beta & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} q \\ \tau \end{bmatrix} \, dx \, dt \] B.22

To find the optimal control, we wish to solve the steady-state Ricatti Equation

\[ 0 = PA + A^*P + Q - PB(R^{-1}B^*P \quad \text{where} \quad * \text{ denotes adjoint} \] B.23

with \( P \) being the convolution

\[ P \eta(.) (x) = \int_{-\infty}^{\infty} K(x - w) \eta(w) \, dw \] B.24

The cost matrix \( P \) is therefore a convolution of the collocated and noncollocated states with a kernel matrix of the form

\[ K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \] B.25

Since \( P \) must be symmetric, the following must hold

\[ K(x - w) = K^T(w - z) \] B.26

\[ K_{12}(z) = K_{21}(-z) \]

\[ K_{11}, K_{22} \text{ are even functions} \]

In addition, \( P \) is positive definite. It can be shown that

\[ A^* = \begin{bmatrix} 0 & -\frac{\partial}{\partial x} \\ \frac{E}{\rho} \frac{\partial}{\partial x} & 0 \end{bmatrix} \] B.27

\[ B^* = \begin{bmatrix} \frac{1}{\rho} & 0 \\ -\frac{1}{\rho} \frac{\partial}{\partial x} & 0 \end{bmatrix} \] B.28

By using the integration by parts technique, the various portions of Eq. 2.16 can be assembled.

174
0 = \int_{-\infty}^{\infty} \left\{ \begin{bmatrix} K'_{12} & \frac{E}{\rho} K'_{11} \\ K'_{22} & \frac{E}{\rho} K'_{21} \end{bmatrix} \begin{bmatrix} K'_{21} & K'_{22} \\ \frac{E}{\rho} K'_{11} & \frac{E}{\rho} K'_{12} \end{bmatrix} \begin{bmatrix} \rho \delta(x - w) & 0 \\ 0 & E \delta(x - w) \end{bmatrix} \right\} du \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} dw

The integrand yields four integral equations which can be solved using Laplace Transforms

\[ \bar{K}_{11} = \sqrt{\frac{E}{\rho} s^2 \left( 1 - \sqrt{1 - \frac{1}{Es^2} \left( \frac{1}{\beta} - \frac{s^2}{\alpha} \right)} \right) + \frac{1}{\rho} \left( \frac{1}{\beta} - \frac{s^2}{\alpha} \right)^2} \]  

\[ \bar{K}_{12} = -\bar{K}_{21} = \frac{E}{\rho} s^2 \left( 1 - \sqrt{1 - \frac{1}{Es^2} \left( \frac{1}{\beta} - \frac{s^2}{\alpha} \right)} \right) \frac{1}{\rho^2 \left( \frac{1}{\beta} - \frac{s^2}{\alpha} \right)} \]  

\[ \bar{K}_{22} = \frac{E}{\rho} \sqrt{1 - \frac{1}{Es^2} \left( \frac{1}{\beta} - \frac{s^2}{\alpha} \right)} \bar{K}_{11} \]  

Note that if s = ik then the conditions in Eq. 2.18 are satisfied because their transforms have identical properties. Note also that these relations have a positive definite quality as required of P.

If no stress actuation is desired (\( \alpha = \infty \)) then the relations become

\[ \bar{K}_{11} = \sqrt{\frac{E}{\rho} s^2 \left( 1 - \sqrt{1 - \frac{1}{E\beta s^2}} \right) + \frac{1}{\rho \beta}} \]
\[
K_{12} = K_{21} = \frac{E_s}{\rho} \left( 1 - \sqrt{1 - \frac{1}{E_s^2 \beta}} \right) \frac{1}{\rho^2 \beta}
\]

\[
K_{22} = \frac{E}{\rho} \sqrt{1 - \frac{1}{E_s^2 \beta}} K_{11}
\]

The frequency dependence of the transforms is mixed and not easily inverse transformed. However, if body force actuation is not desired \((\beta = \infty)\) then

\[
K_{11} = \sqrt{\frac{E_s^2}{\rho^4 \alpha} \left( 1 - \sqrt{1 + \frac{1}{E \alpha}} \right) - \frac{1}{\rho \alpha}}
\]

\[
K_{12} = K_{21} = \frac{E}{\rho} \left( 1 - \sqrt{1 - \frac{1}{E \alpha}} \right) \frac{s}{\rho^2 \alpha}
\]

\[
K_{22} = \frac{E}{\rho} \sqrt{1 + \frac{1}{E \alpha}} K_{11}
\]

The transformed form of the feedback is

\[
u = R^{-1} B^* P = \begin{bmatrix}
\frac{K_{11}}{\rho \beta} & \frac{K_{12}}{\rho \beta} \\
\frac{s K_{11}}{\rho \alpha} & \frac{s K_{12}}{\rho \alpha}
\end{bmatrix}
\]

This feedback form, in conjunction with Equations 2.22, gives a feedback system whose inverse transform requires a complex form of noncollocated feedback. But, in conjunction with Equations 2.23, the inverse transforms yield simpler convolution kernels. The use of stress actuation results in simpler feedback kernels. The inability to control rigid body motion is the only associated loss of performance. The strain-to-stress feedback kernel is a spatial delta function giving colocated feedback and the velocity-to-stress feedback kernel has a \(1/(x-w)\) dependence.

A numerical example was performed, using only stress actuation, in order to verify this solution. The Linear Quadratic Regulator technique was used to derive the full state
feedback matrix for a finite element model of the rod. Numerous elements were used in the model to enable the identification of the noncollocated feedback gains for actuators internal to the rod. This minimizes the influence of the boundaries on the feedback terms. The strain feedback terms are collocated as in the continuum solution. Fig. B.4 compares the velocity feedback kernel with a row of the feedback matrix corresponding to the center stress actuator. Notice the good correlation between the continuum and discrete gains.

![Graph showing comparison of exact and discrete models](image)

**Figure B.4** Overplot of exact gain value (1/x) obtained from continuum solution and numerical approximation from finite element model

The closed-loop system can be transformed to find the closed-loop wave behavior. This will reveal the manner in which a typical, physical objective such as energy minimization manifests itself in terms of wave motion. The closed-loop rod is governed by the equation

\[
E \frac{\partial^2 u}{\partial x^2} - \rho \frac{\partial^2 u}{\partial t^2} = -\frac{\partial}{\partial x} \left( \int_{-\infty}^{\infty} \frac{\mu}{w} \frac{\partial u(w)}{\partial t} \, dw + v \frac{\partial u}{\partial x} \right)
\]

\[\text{B.34}\]

Fourier transforming the temporal and spatial variables gives

\[-k^2 E \bar{u} + \omega^2 \rho \bar{u} = -i \omega \mu u_0 e^{i\alpha x} \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \frac{e^{i k w}}{x - w} \, dw + v k^2 \bar{u}\]

\[\text{B.35}\]

Assuming \(k\) to be real and using the fact that the principle value of the first term on the right hand side of Eq. B.31 is given by
\[ P \int_{-\infty}^{\infty} \frac{e^{ikw}}{x - w} \, dw = i\pi e^{ikx} \quad k > 0 \]

\[ = -i\pi e^{ikx} \quad k < 0 \]

The dispersion relation gives

\[ s = i\omega = -\frac{k}{2} \sqrt{\frac{2E}{\rho} \left( -1 + \sqrt{1 + \frac{1}{E\alpha}} \right) + \frac{1}{\rho\alpha}} + \frac{ik}{2} \sqrt{\frac{2E}{\rho} \left( 1 + \sqrt{1 + \frac{1}{E\alpha}} \right) - \frac{1}{\rho\alpha}} \]

This form is identical to the spring-mass typical section solution problem, for \( c = 0, \nu = k \) and \( \alpha = m \), where stress replaces force and strain replaces displacement.

This result might be expected. Since the response of a continuum rod can be decoupled in terms of the wave modes at each frequency, if the spatial variables are transformed and the relation is placed in temporal state space form then the response at each frequency behaves as a harmonic oscillator. The same is true if a finite structure is decoupled in terms of its second order modes. Therefore, the typical section control solution presented in Section B.1.1 for a simple harmonic oscillator is equally applicable to the formulation of control for standing and wave modes.

Returning to the actuator issue, if only distributed force actuation is used (\( \alpha \) becomes large) then the transform of the feedback is

\[ \bar{u} = \begin{bmatrix} -\rho \sqrt{\frac{2E}{\rho} \frac{k^2}{\beta} \left( -1 + \sqrt{1 + \frac{1}{E\beta k^2}} \right)} + \frac{1}{\rho\beta} & -ikE \left( 1 - \sqrt{1 + \frac{1}{E\beta k^2}} \right) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} \]

This has the same form as for stress actuation except now there is a wave number dependence, and equivalent frequency dependence, which accounts for the sense in which the frequency behavior of force actuation differs from that of stress actuation. Therefore, these are not unique actuators in this case and can be used interchangeably. Subsequently, the choice of the most appropriate actuator might be decided based upon the simplicity of the feedback structure. This would favor stress actuation.

If the control penalties for the force and stress actuated systems are related by

\[ \beta k^2 = \alpha \]

then the costs, using either actuator separately, are identical.

This example illustrates that proper selection of the actuator type can greatly simplify the control. In addition, it would be enlightening to determine whether much, if
any, performance is compromised when using the stress actuators instead of the force actuators. This loss of performance is probably only associated with the rigid body energy. If not, then perhaps much of the noncollocated control effort goes towards using the force actuators to simulate stress actuation. Notice that when stress actuators are used in conjunction with velocity and strain states in the energy minimization problem, the form of the control simplifies tremendously. Perhaps this is a result of the fact that the Ricatti Equation has been given the energy variables as measurements.

B.3 Directional Control

Direction behavior implies that, in closed-loop, the member has preferential directions for energy flow. Such a member can be thought of as inhibiting the flow of power in one direction while draining power in the other. In either light, the objective is to remove energy from one location by transporting it to another.

If the sole purpose of active control is for energy dissipation, it has to compete with numerous passive damping techniques from the perspectives of performance, weight, and robustness. To date, there has been no substantial verification that active control wins on the latter two points. Even the first point is in dispute. One of the true advantages of an active approach is that it enables closed-loop behaviors that are simply not achievable through existing passive techniques. One such behavior is the creation of directional energy propagation characteristics. These include energy barriers, energy shunting, and preferred directions for energy drainage. In this section, this concept is applied to the active control of structural members.

B.3.1 Specifying the Dispersion Relation Control solutions can be found by specifying the nature of the dispersion relation. This is similar to a pole placement technique where the form of an O.D.E.'s characteristic equation is specified. Instead of specifying the closed-loop response time and damped frequency (the real and imaginary parts of the poles) the spatial frequency and characteristic attenuation lengths (the real and imaginary parts of the wave number) are specified. The following example briefly illustrates several possibilities using this technique.

Example B.2 Various interesting solutions can be found by specifying the nature of the dispersion relation for an undamped, uniform rod in compression. The feedback is collocated with the distributed force actuator in each of the following cases. Distributed rate feedback yields waves which attenuate in both directions (Eq. B.40).
\[ E \frac{\partial^2 u}{\partial x^2} - \rho \frac{\partial^2 u}{\partial t^2} = -q = \beta \frac{\partial u}{\partial x} \] \quad k = \pm \sqrt{\omega \frac{\rho \omega - i\beta}{E}} \tag{B.40}

The proper ratio of displacement feedback in conjunction with rate can be used to maintain the real part of the open-loop wave number. Notice that the characteristic attenuation length is independent of frequency in the relation

\[ E \frac{\partial^2 u}{\partial x^2} - \rho \frac{\partial^2 u}{\partial t^2} = -q = \beta \frac{\partial u}{\partial x} + \frac{\beta^2}{4\rho} u \quad k = \sqrt{\frac{\rho}{E}} \omega - i \frac{\beta}{2\sqrt{\rho E}} \tag{B.41} \]

The introduction of strain rate maintains the nondispersive behavior but energy propagates at different speeds in different directions (wave speed \( c_0 = \partial \omega / \partial k \) in Eq. B.42).

\[ E \frac{\partial^2 u}{\partial x^2} - \rho \frac{\partial^2 u}{\partial t^2} = -q = -\mu \frac{\partial^2 u}{\partial x \partial t} \quad k = -\omega \frac{\mu}{2E} \left[ 1 \pm \sqrt{1 + \frac{4\rho E}{\mu^2}} \right] \tag{B.42} \]

Strain feedback causes stiffening of the structure in addition to causing the wave amplitude to vary as it propagates. This variation in in the form of attenuation in one direction and amplification in the other. Strain feedback is the spatial analog to rate feedback.

\[ E \frac{\partial^2 u}{\partial x^2} - \rho \frac{\partial^2 u}{\partial t^2} = -q = -\eta \frac{\partial u}{\partial x} \quad k = \frac{i\eta}{2E} \left[ 1 \pm \sqrt{1 - \frac{4\rho E}{\eta^2 \omega^2}} \right] \tag{B.43} \]

With the addition of the correct amount of displacement feedback, attenuation/amplification occurs without the additional structural stiffening associated with the strain term, notice that the real part of the closed-loop wave number equals the open-loop wave number.

\[ E \frac{\partial^2 u}{\partial x^2} - \rho \frac{\partial^2 u}{\partial t^2} = -q = -\eta \frac{\partial u}{\partial x} - \frac{\eta^2}{4E} u \quad k = \frac{i\eta}{2E} \pm \sqrt{\frac{\rho}{E}} \omega \tag{B.44} \]

These are some simple examples of the behavior that can theoretically be achieved using the proper response coordinates and actuation to alter the dispersion properties of the one-dimensional wave equation. Of particular interest is that in some cases performance is a function of propagation direction.

A generalization may be made concerning the directionality of the control. There are three stages in a generic output feedback structure. The first stage involves the measurement of particular motion quantities. These quantities can be of two types: the first being directional (displacement, rotation, etc.) and the second being nondirectional (strain, curvature, etc.).
The second stage is the convolution of collocated and noncollocated measurements through a feedback kernel to generate the actuator command. In Example B.1 the strain-to-stress feedback kernel was a Dirac delta function of the actuator position: collocated feedback. This kernel can be symmetric (even) or nonsymmetric (odd) about the actuator position.

The third stage is the actuation of certain structural variables. As with the measurements, these are of two types: directional (force, moment, etc.) and nondirectional (bending moment, shear force, etc.). If a directional measurement (e.g., displacement, velocity, rotation, etc.) is fed in a collocated (even) fashion to a nondirectional actuator (e.g., stress, curvature) then the resulting closed-loop propagation characteristics are directional. The reverse causes the same behavior (e.g., strain to force). Otherwise, if a directional measurement is fed to a directional actuator (e.g., velocity to force) or the opposite (strain to stress), isotropic behavior is maintained.

Nonsymmetric feedback kernels change the directionality of the measurement. Notice, in the stress actuator solution, that strain is fed in a collocated fashion to stress. This results in a dual sensor/actuator pair which does not generate directional behavior. Rate feedback to stress is a the feedback of a directional measurement to a nondirectional actuator. Still, this does not result in directional closed-loop behavior because the noncollocated measurements are fed through a nonsymmetric feedback kernel.

This is a useful concept when selecting sensors and actuators for a particular application. Specifically, some feedback kernels simply change one measurement into another measurement which could have been sensed directly. Not only distributed, but discrete point control exhibits these same properties.

B.3.2 Optimal and Suboptimal Directional Control The following two examples illustrate ways in which control objectives can be formulated in order to find feedback structures which exert different levels of control depending on the direction of wave propagation.

Example B.3 This example derives continuously distributed control for a uniform, assumed infinite length rod in compression using some elements of optimal regulator theory. While the total solution may not be globally optimal, the results accentuate some of the features of viewing the problem in terms of waves. The governing equation is given by

\[ E \frac{\partial^2 u}{\partial x^2} - \rho \frac{\partial^2 u}{\partial t^2} = -q \]  

B.45

Using separation of variables, the input and response can be decomposed as

181
\[ q(x,t) = Q(x) T(t) + \varphi(t) X(x) \]  
\[ u(x,t) = X(x) T(t) \]

This form has been chosen to give the input, \( q \), a state dependent controller form. Substituting and rearranging yields two coupled equations, one of which governs temporal behavior and one which governs spatial behavior.

\[ E \frac{\partial^2 X}{\partial x^2} - \eta^2 X = -Q(x) \]

\[ \rho \frac{\partial^2 T}{\partial t^2} - \eta^2 T = \varphi(t) \]

These can be written in state-space form

\[ \frac{\partial}{\partial t} \begin{bmatrix} T \\ \frac{\partial T}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \eta^2 \rho & 0 \end{bmatrix} \begin{bmatrix} T \\ \frac{\partial T}{\partial t} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\partial \varphi}{\partial t} \end{bmatrix}, \quad \varphi \propto \begin{bmatrix} T \\ \frac{\partial T}{\partial t} \end{bmatrix} \]

\[ \frac{\partial}{\partial x} \begin{bmatrix} X \\ \frac{\partial X}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{\eta^2}{E} & 0 \end{bmatrix} \begin{bmatrix} X \\ \frac{\partial X}{\partial x} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{\partial Q}{\partial x} \end{bmatrix}, \quad Q \propto \begin{bmatrix} X \\ \frac{\partial X}{\partial x} \end{bmatrix} \]

Thus, the feedback has the form

\[ q(x,t) = -(F_{x1} + F_{T1}) X T - F_{x2} \frac{\partial X}{\partial x} T - F_{T2} \frac{\partial T}{\partial t} \]

\[ = -(F_{x1} + F_{T1}) u - F_{x2} \frac{\partial u}{\partial x} - F_{T2} \frac{\partial u}{\partial t} \]

Notice the existence of strain in the feedback quantities. In addition, the displacement feedback has two possible contributions, one from the temporal problem and one from the spatial problem. It should be pointed out that the sensing and actuation are restricted to be continuous in time and space and the feedback to be memoryless, fixed gain, and collocated. This makes the solution suboptimal.

A cost functional can be written for each second order relation which, when combined, penalize portions of the structural energy.

\[ J_T = \frac{1}{2} \int_{0}^{\infty} \int_{0}^{\infty} \left[ T \frac{\partial T}{\partial t} \begin{bmatrix} -\kappa & 0 \\ 0 & \rho \end{bmatrix} \begin{bmatrix} T \\ \frac{\partial T}{\partial t} \end{bmatrix} + \frac{1}{\beta} \varphi^2 \right] dt \, dx \]
\[ J_X = \frac{1}{2} \int \int_{\mu}^\infty T^2 \left[ X \left( \frac{\partial X}{\partial x} \right) \kappa \begin{bmatrix} 0 & 1 \\ 0 & E \end{bmatrix} \begin{bmatrix} X \\ \frac{\partial X}{\partial x} \end{bmatrix} + \frac{1}{\alpha} Q^2 \right] dx \, dt \]  

B.54

\[ J_X + J_T = \text{Energy} \]  

B.55

Notice the \( \kappa \) terms cancel to give the total energy in the semi-infinite length starting at \( \mu \) and extending in the positive \( x \) direction. Both the temporal and spatial problems have the same form as the typical section problem in Section B.1. Using the typical section solution for the temporal problem, the resulting displacement and velocity feedback are given by

\[ F_{T1} = -\left( 1 - \sqrt{1 - \kappa \beta} \right) \]  

B.56

\[ F_{T2} = \left( -2 + 2 \sqrt{1 - \kappa \beta} + \rho \beta \right) \]  

B.57

The solutions to the spatial problem are identical with \(-\kappa \) and \( E \) replacing \( \kappa \) and \( \rho \), respectively. Three interesting solutions exist (\( \kappa /\rho \) large, \( \kappa /\rho \) small and \(-\kappa \beta = 0.25 \rho \beta \) (4 + \( \rho \beta \)). When combined with the three solutions of the spatial problem, a total of nine interesting solutions exist. The solution corresponding to simple energy minimization calls for \( \kappa = 0 \). The resulting feedback terms are

\[ F_{T1} = 0 \]  

B.58

\[ F_{T2} = \sqrt{\rho \beta} \]  

B.59

\[ F_{X1} = 0 \]  

B.60

\[ F_{X2} = \sqrt{E \alpha} \]  

B.61

The dispersion relation of the resulting closed loop partial differential equation, using

\[ u = w e^{ikx + i\omega t} \]  

B.62

is

\[ -k^2 E + \omega^2 \rho + i \kappa \eta - i \omega \beta = 0 \]  

B.63

This results in a real and imaginary equation

\[ \text{Re:} \quad \omega^2 \rho = k^2 E \]  

B.64

\[ \text{Im:} \quad \omega \beta = k \eta \]  

B.65
which, when solved simultaneously, yield two wave modes supported by the closed-loop rod as

\[ k = \sqrt{\frac{\rho}{E}} \omega, \quad -\sqrt{\frac{\rho}{E}} \omega + \frac{i \beta}{\sqrt{\rho E}} \]  B.66

Note that the first wave mode travels in the negative x direction and has the same propagation characteristics as waves in the uncontrolled rod (i.e., no control is expended on this mode). The wave mode travelling in the other direction is attenuated. The closed-loop rod exhibits directional behavior. One root corresponds to waves propagating unattenuated in the negative direction, out of the penalized portion of the structure, with propagation characteristics identical to the open-loop rod. The other root causes attenuation in the direction towards the penalized portion. Thus, since energy in only a portion of the structure was penalized, the control causes the energy to drain out of that portion by creating directional behavior.

**Example B.4** An example can be formulated where the control is not restricted to collocated measurement feedback yet directional closed-loop behavior results. Such a result occurs when the same procedure as that in Example B.1 is used given that the energy is penalized only to one side of the stress actuator. The only difference between this and the former example arises in the state penalty matrix which is now given by

\[ (Q u(.)) (x) = \int_{-\infty}^{\infty} \begin{bmatrix} \gamma H(x - w) & 0 \\ 0 & \varepsilon H(x - w) \end{bmatrix} \begin{bmatrix} u_1(w) \\ u_2(w) \end{bmatrix} dw \]  B.67

where \( H(x - w) \) is the Heaviside function which is unity for nonnegative values of its argument. Therefore, Eq. B.36 represents a penalty on the squares of the states to the left of the actuation position (x). Notice that the locations on the rod that are penalized are relative to the particular actuator location being analyzed.

The feedback to stress actuation, transformed in the spatial domain, is given by

\[ \tilde{\tau} = \begin{bmatrix} \sqrt{2\rho E} \left( 1 - \sqrt{1 + \frac{\varepsilon}{\alpha E^2 s}} \right) - \frac{\gamma}{\alpha s} - E \left( 1 - \sqrt{1 + \frac{\varepsilon}{\alpha E^2 s}} \right) \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} \]  B.68

The inverse transforms of both feedback gains give kernels which are odd (nonsymmetric) functions of noncollocated distance between measurement and actuator. This indicates that, while the velocity feedback part maintains isotropic behavior, the feedback of strain through a nonsymmetric kernel causes directional behavior. As shown previously, this
causes the wave propagation and attenuation characteristics to become a function of propagation direction.