GREEN'S FUNCTIONS FOR PLANE-LAYERED ELASTOSTATIC AND
VISCOELASTIC REGIONS WITH APPLICATION TO 3-D CRACK ANALYSIS

by

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ABSTRACT

A new generalized image method is derived and used to obtain
Green's functions for bonded elastic halfspaces and for a region
consisting of an elastic plate perfectly bonded to either two
elastic halfspaces or (through the use of the correspondence
principle) to an elastic halfspace and a (Maxwell) viscoelastic
halfspace.

The newly derived Green's functions are implemented in a
program based on an "enhanced" boundary element method (BEM) that
features arbitrary displacement discontinuity specification
capability along crack surfaces. Using the BEM, stress intensity
factors for sample 3-D cracks in infinite space and for elliptic
part-through cracks in a plate are obtained. Also using the BEM, a
preliminary model of a fault region at Parkfield is analyzed.

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to my family
past, present and future
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Chapter 1: Introduction and Overview

The study of cracking processes in plane-layered regions is relevant to the understanding of several physical phenomena and engineering problems such as the growth of hydrofractures in oil-bearing "layers" in the earth, cracking in plane-layered composites as well as the growth of part-through cracks in pressure vessels of large radii of curvature. However, the direct motivation for the research effort that led to this thesis was the perceived need of a layered earth model for studying 3-D aspects of deformations near fault zones. It was also desirable for the model to (if possible) account for the viscoelastic coupling of the lithosphere with the asthenosphere (first two layers in the earth). A brief description for the required layered-earth model would be that of an elastic plate (representing the lithosphere) having a traction free upper surface, and a perfect bond with a viscoelastic halfspace (representing the asthenosphere) at the lower surface. Slip surfaces (over which possible constitutive relations would be enforced) would be allowed to occur in the lithosphere, and an accurate evaluation of surface displacements would be a major concern of the analysis. Owing to the complexity of the 3-D model and the geometry of the fault regions which are to be modelled, classical numerical tools such as the Finite Element Method (FEM) or the regular Boundary Element Method (BEM with substructuring) were deemed to be unfeasible. The FEM and regular BEM would require excessively high levels of discretization due to the presence of infinite boundaries and the expected high gradients of elastic fields that are present near some regions of fault zones, as well as due to the elongated and complex geometry of faults at plate boundaries.

A possibly feasible numerical technique for analyzing slip surfaces in the layered-earth model described above would be the
BEM coupled with the Green's functions which automatically satisfy boundary conditions at the free surface and at the lithosphere-asthenosphere interface. However, the required Green's functions are not available in the literature and hence, the first step in building a feasible 3-D model for faulting processes is to derive the appropriate Green's functions. The derivation of Green's functions for complex geometries is usually a major task, and hence the major effort of this thesis is the presentation and derivation of the appropriate Green's functions. Once the Green's functions are available, their implementation is demonstrated within a BEM scheme, and the verification of the usefulness and accuracy of the ensuing numerical tool is demonstrated using sample (relatively simple) problems.

The scope of this thesis covers the development of the appropriate Green's functions for analyzing slip surfaces in the layered-earth model described above and its implementation in a BEM formulation. The BEM formulation to be presented only uses the elastic Green's functions (i.e. the plate and halfspace are elastic), but a kinematic (i.e. only slip and/or opening boundary conditions on the fault or crack surface can be specified) implementation of the viscoelastic Green's functions has also been performed with the restriction that slip and/or opening is to be applied at time "zero" and then maintained indefinitely. The scope of this thesis however, is not limited to the layered earth model described above. The thesis discusses a new method for deriving (antiplane, 2-D and 3-D) Green's functions for bonded elastic halfspaces and for a layered-medium consisting of an elastic plate perfectly bonded to two elastic halfspaces (which can be specialized to a plate free of tractions at the upper and lower surfaces); In addition, use of the correspondence principle allows the associated viscoelastic Green's functions to be obtained and is demonstrated for a specific specialization.
The structure of this thesis is as follows: Chapter 2 covers the presentation of all the theoretical aspects pertaining to the derivation of the required Green's functions. The presentation of chapter 2 starts with a literature review of previous theoretical efforts of deriving Green's functions, presents a new algorithm for deriving Green's functions in bonded halfspaces, uses the algorithm to obtain the Green's functions for a layered-region, and discusses the method of obtaining the associated viscoelastic Green's functions for a specific viscoelastic model. Chapter 3 covers background, formulation and implementation aspects of the Boundary Element Method as adapted to the specific crack analysis in the layered-region considered. Chapter 4 covers some application problems using the newly derived Green's functions and the BEM formulated; preliminary (simple) modelling of geophysically motivated problems are also presented in chapter 4. Finally, chapter 5 summarizes the major findings and contributions of the thesis, and suggests topics which are deemed to require further research. Throughout the main body of the thesis, an effort has been made to keep the discussions to main themes and to leave algebraically intensive derivations, peripheral theoretical and numerical studies as well as miscellaneous notes to appendices.
Chapter 2: Theory and derivation of elastostatic and viscoelastic Green's functions in plane-layered regions

2.1 Introduction / literature review

The importance of point sources (Green's functions are point sources) for some given (linear) governing differential equations and boundary conditions lies in two main reasons. First, any localized process when viewed from a sufficient distance can be modelled as some suitably chosen point sources. Second, Green's functions can be used to reframe the governing differential equations and boundary conditions in an integral equation form. The integral equation form can, for example, be used as the basis for numerically analyzing a large class of problems using the boundary element method.

This chapter presents an algorithm for the derivation of point sources of elastostatics and viscoelastostatics in multi-layered media assuming the point source in infinite space is known. The method is similar to the image method that is familiar when deriving Green's functions in plane layered media where there is only one unknown scalar field in the governing equations such as in heat conduction, potential flow and electrostatics problems. The algorithm is then used to derive the Green's functions for any point source in a region consisting of an elastic layer perfectly bonded to two elastic halfspaces.

Background

There are many known Green's functions for halfspace problems in elastostatics. Most of the known Green's functions are specialized for a single halfspace having a stress free surface (a special case of bonded elastic halfspaces when one of the regions
has zero rigidity). A brief survey of some of these known solutions with occasional comments on the method of derivation will be done in this section, then a discussion of some of the analytic methods used to derive Green's functions in multilayered halfspaces, and a mentioning of a few known results for layered systems with some comments will follow.

The most used point source solutions are the point force, the dislocation and the nuclei of strain (or double couple) solutions. The point force solution for 2-D plane problems in a halfspace with a free surface (Mellan 1932), 3-D problem in a halfspace with a free surface (Mindlin 1936) and 3-D problem in bonded elastic halfspaces (Rongved 1955) are known. Rongved obtained the Green's function through the use of the Papkovich-Neuber potentials and arguments from harmonic analysis. The resulting solution is in the form of the sum of a point force solution in infinite space and some point sources at the image point with respect to the interface plane.

The screw dislocation (e.g. Rybicki 1971) and edge dislocation (e.g. Freund and Barnett (1976a,b) presented it in a convenient form, whereas Dmowska and Kostrov (1973) presented the solution earlier) for a halfspace with a free surface and a general dislocation line intersecting a free surface (Yoffe 1961) are also known. The screw dislocation problem is obtained by the method of images (since there is only one field variable), while Freund and Barnett solved the edge dislocation problem through the use of complex analysis and the Mushkelishvili potentials.

There are six nuclei of strain sources. The solution to the first (double couple in a plane parallel to the free surface) was given by Steketee (1958), the remaining five sources were given by Maruyama (1964). Maruyama used image nuclei of strain sources to
cancel the tangential component of the surface traction on the free
surface. He then used the Boussinesq solution (in Galerkin vector
representation) and the remaining normal tractions on the free
surface in a Hankel/Fourier transformed space to obtain the rest of
the fields after which he transformed the solution back to real
space. This procedure is highly specific to half space problems
with a free surface and cannot be generalized to multiple layered
systems. In addition, note that the nuclei of strain in a halfspace
could alternatively be obtained from combining appropriate
derivatives of Mindlin's point force solution using the reciprocity
theorem (see chapter 3) as a guide. Next, the known techniques to
systematically treat sources in multiple layered media will be
considered.

A systematic formulation for the derivation of the fields due
to 3-D sources in a layered halfspace was presented by Ben-Menahem
and Singh (1968) and later refined by Singh (1970) and
independently by Sato (1971). The formulation makes use of the
analogue of Hansen's eigenvector expansion for electromagnetic
problems (1935) applied to elastostatic and dynamic problems,
combined with the Haskell-Thompson transfer matrix technique and
the Pekeris (1955) 'source condition' at the level of the
discontinuity. The formulation leads to a solution for the field
variables of the form:

\[
\sigma_{ij}, u_i \sim \left( \sum_{s=0}^{\infty} \alpha_s k_s \exp(\beta_s k z) \right) \frac{J_p(kr)}{kr} e^{iC_s f(p)} \theta
\]

(2.1)

where:

\[ \alpha_s, \beta_s, \gamma_q, \delta_q \]

are constants dependent on the indices \( s \)

and \( q \)
\( J_p \) is Bessel's function of the first kind and of order \( p \) 
\( r \) is the radial (cylindrical) coordinate 
\( z \) is the \( z \) cylindrical coordinate 
\( \gamma_q \) are constants that depend on the thickness of the layers and on the location of the source 
\( m, n \) are integers that depend on the number of layers being considered 
\( k \) is an integration variable

In the above integrals, \( \alpha_i \)'s and \( \gamma_i \)'s may also depend on both the elastic properties of the half space and the layer.

Sato and Matsu'ura (1973) and Jovanovich et al. (1974 a,b) made use of the above mentioned formulation to calculate surface deformations by numerically integrating the ensuing expressions. Such evaluations require special numerical techniques and significant amounts of computational effort. Furthermore, no field values were computed inside the layer or halfspace. To accomplish such evaluations requires significantly more effort (both in further algebraic manipulations and in computations and special numerical treatments). This observation is especially true for field points close to the source point. The importance of having the field variables being available everywhere occurs when a boundary element/integral equation formulation for processes occurring in a region consisting of such multi-layered media is required.

A formulation in the same spirit as the Ben-Menahem and Singh formulation for multi-layered 2-D problems using the Airy stress function was presented by Singh and Garg (1985). Although the original formulation is applicable to 3-D, 2-D and antiplane
problems, the specialized 2-D formulation is less complex.

Simpler but more specialized point source solutions in multi-layered media are available. For example, Rybicki (1971) presented the solution to a screw dislocation (can be specialized from his expressions) in a region consisting of an elastic layer with a free surface and perfectly bonded to an elastic halfspace. Rybicki used the method of images to derive his solution. Stagni and Lizzio (1986) used the method of images (in terms of Muskhelishvili potentials) to obtain the solution to a point force in a plate with either traction free or rigidly clamped surfaces. Rundle and Jackson (1977) presented an approximate solution for a 3-D double couple source parallel to the free surface in a region consisting of an elastic layer perfectly bonded to an elastic halfspace. The approximate solution was obtained by using Steketee's 1958 solution combined with the use of an image method (using Rybicki's technique) on an "antiplane" part of the point source being considered.

Rundle and Jackson compared their solution for the surface deformations to that obtained by the direct integration of the improper integrals of Jovannovich et al. and found errors up to 14% while varying the thickness of the plate and the location of the source. However, the errors are less than 5% for the parameteric values they need in their specific application. Note that they take the rigidity contrast between the halfspace and elastic layer to be 10 to 1, a choice that would favor their approximation. The most severe case corresponds to a plate with upper and lower surfaces to be stress free.

Rundle and Jackson also obtained the (approximate) response of the point source they considered with a viscoelastic instead of an elastic halfspace through the use of the correspondence principle.
The direct application of the correspondence principle to their elastic solution is possible because the material parameters are kept separate from the geometric coordinates (in the form of a sum of material parameters multiplied by a function of the coordinates). This procedure is not directly applicable to the solution presented in the form of improper integrals (2.1) because the material parameters and the geometric coordinates are intermixed in the denominator of the integrand, but is applicable using the method described in this chapter.

Finally, the existence of an image method for perfectly bonded elastic halfspaces in terms of the Papkovich-Neuber potentials (Aderogba 1977) has to be mentioned. Aderogba presents the algorithm for obtaining the four image potentials which involves multiple integrations with respect to the coordinate perpendicular to the interface plane and differentiation with respect to all three coordinates. The algorithm presented in this thesis involves 3 potentials only, and only differentiation of the potentials with respect to the coordinate perpendicular to the interface plane (as well as multiplication by scalars) is required. This distinction is especially important while contemplating the repeated use of the image algorithm to obtain the fields due to point sources in regions consisting of an elastic layer perfectly bonded to two elastic halfspaces. Nevertheless, the architecture of obtaining Green's functions for plate regions using the Papkovich-Neuber potentials has been discussed by Rice 1985. The relationships between the Papkovich-Neuber potentials and the potentials used in the present algorithm will be mentioned in the next section.

2.2. Presentation of the image method

Preliminary Considerations
The image method considered here is dependent upon expressing the displacements in terms of potentials. The specific potentials used are the analogue to Hansen's potentials for elastostatics and dynamics. Unlike Ben-Menahem and Singh (1968) the potentials are not expanded in terms of eigenfunctions, instead the algorithm operates directly on the potentials. Note however, that the derivation of the algorithm makes use of the eigenfunction expansion technique (see Appendix C).

Specifically, the displacement field is expressed in terms of the Hansen potentials $\varphi_1$, $\varphi_2$ and $\varphi_3$ in the following way:

$$u(h,\delta,\varphi_R,\varphi_L) = N(h,\varphi_1) + F(\delta,h,\varphi_2) + M(h,\varphi_3)$$

$$\varphi_R = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \quad \varphi_L = \begin{bmatrix} \varphi_3 \end{bmatrix}$$

$$N(h,\varphi_1) = \nabla \varphi_1(x,y,z-h)$$

$$F(\delta,h,\varphi_2) = + 2 \hat{e}_z \frac{\partial}{\partial z} \varphi_2(x,y,z-h)$$

$$- \nabla \varphi_2(x,y,z-h)$$

$$- 2 \cdot \delta \cdot (z-h) \frac{\partial}{\partial z} \varphi_2(x,y,z-h)$$

$$M(h,\varphi_3) = \nabla \times \left[ \hat{e}_z \varphi_3(x,y,z-h) \right]$$

where:

$$\nabla$$ is the gradient operator

$$\nabla \times$$ is the curl operator

$$\nabla^2 \varphi_1 = \nabla^2 \varphi_2 = \nabla^2 \varphi_3 = 0$$

$$\delta = \frac{\lambda + \mu}{\lambda + 3\mu}$$
\( \lambda \) is the Lame constant

\( \mu \) is the shear modulus

\( h \) is a scalar for shifting the z-coordinate

Note that the potentials \( \varphi_1, \varphi_2 \) and \( \varphi_3 \) have to be harmonic in order for \( N, F \) and \( M \) to satisfy equilibrium. The subscripts "R" and "L" on the Hansen potentials \( \varphi_R \) and \( \varphi_L \) respectively are used in order to be consistent with the previous work of Ben-Menahem and Singh (1968). The Cartesian components for the displacements, strains and stresses are given in Appendix A.

The Hansen potentials can be related to the more familiar Papkovitch-Neuber potentials (\( \psi \) and \( \Omega \)) which are defined as:

\[
2\mu \cdot u_j = -\frac{1}{\delta} \cdot \psi_j + \chi_k \cdot \psi_{k,j} + \Omega, j
\]  
(2.2.1)

The Hansen potentials are related to the Papkovitch-Neuber potentials as follows:

\[
\varphi_1 = \Omega
\]
\[
\frac{\partial^2}{\partial z^2} \varphi_2 = -\frac{1}{4\mu \delta} \cdot \nabla \cdot \psi
\]  
(2.2.2)
\[
\frac{\partial^2}{\partial z^2} \varphi_3 = -\frac{1+\delta}{2\mu \delta} \cdot \epsilon_z \cdot \nabla \psi
\]

In order for the Hansen potentials to be useful for deriving new Green's functions, the method for obtaining these potentials given an elastic field satisfying equilibrium have to be described. Note the following:

\[
\nabla \cdot N = \nabla \cdot M = 0
\]
\[
\nabla \cdot F = 2 \cdot (1-\delta) \cdot \frac{\partial^2}{\partial z^2} \varphi_2
\]
\[ \nabla \times \mathbf{N} = 0 \]  

\[ \nabla \times \mathbf{F} = 2 \cdot (1+\delta) \frac{\partial^2 \varphi_2}{\partial y \partial z} \mathbf{e}_x - 2 \cdot (1+\delta) \frac{\partial^2 \varphi_2}{\partial x \partial z} \mathbf{e}_y \]

\[ \nabla \times \mathbf{M} = \frac{\partial^2 \varphi_3}{\partial x \partial z} \mathbf{e}_x + \frac{\partial^2 \varphi_3}{\partial y \partial z} \mathbf{e}_y + \frac{\partial^2 \varphi_3}{\partial z^2} \mathbf{e}_z \]

Therefore, if given a displacement field \( \mathbf{u} \), it is possible to calculate the following:

\[ \nabla \cdot \mathbf{u} = 2 \cdot (1-\delta) \frac{\partial^2 \varphi_2}{\partial z^2} \]  

(2.4)

and

\[ (\nabla \times \mathbf{u}) \cdot \mathbf{e}_z = \frac{\partial^2 \varphi_3}{\partial z^2} \]  

(2.5)

From the above relations it is found that:

\[ \varphi_2 = \int \int dz \int dz \frac{\nabla \cdot \mathbf{u}}{2 \cdot (1-\delta) \mathbf{2} \cdot (1-\delta)} + z \cdot F_2(x,y) + G_2(x,y) \]

(2.6)

\[ \varphi_3 = \int dz \int dz [(\nabla \times \mathbf{u}) \cdot \mathbf{e}_z + z \cdot F_3(x,y) + G_3(x,y)] \]

The \( F_1 \)'s and \( G_1 \)'s are chosen such that \( \varphi_2 \) and \( \varphi_3 \) are harmonic in the required region. Note, for the image method one should choose all the singularities of the potentials to occur in the region where the source occurs. This is made clearer in appendix E when examples of the use of the algorithm are considered. Finally,
once $\varphi_2$ and $\varphi_3$ are determined, whatever remains is ascribed to $\varphi_1$. If the given displacement field does satisfy equilibrium, the field should be expressible in terms of these three potentials (see Ben-Menahem and Singh 1968, and Morse and Feshbach 1953).

The Hansen potentials for a point force, a line force perpendicular to the z-direction, and an edge dislocation perpendicular to the z-direction are given in Appendix B. It will be shown later (as is already known) that there is no need to obtain the potentials for a purely antiplane deformation field, since the image field involves a field of a similar nature as the source.

The Algorithm

Now the image algorithm and the notation associated with it will be described. Consider two elastic halfspaces perfectly bonded along an interface plane at $z=0$ (see figure 2.1). The material properties of region 1 are described by $\mu_1$ and $\delta_1$, and of region 2 by $\mu_2$ and $\delta_2$. Next, define the following:

$$
\bar{\varphi}(x,y,z) = \varphi(x,y,-z)
$$

$$
\gamma = \frac{\mu_2}{\mu_1}
$$

$$
a = \frac{(\delta_1 + 1)}{(\delta_1 + \gamma)}
$$

$$
b = \frac{(\delta_1 + 1)}{(\gamma \cdot \delta_2 + 1)}
$$

Note that if $\varphi$ is harmonic then $\bar{\varphi}$ is also harmonic and hence can be used as a Hansen potential for $\mathcal{N}$, $\mathcal{F}$ and $\mathcal{M}$.

The algorithm states that given the representation for a point source in infinite space of elastic constants similar to those of
Figure 2.1
region 1 at the location x-y=0 and z=h described by the displacement field:

\[ u^0 = u^0(h, \delta_1, \varphi_R^0, \varphi_L^0) \quad (2.8) \]

then the displacement fields in regions 1 and 2 for a similar point source in region 1 at x-y=0 and z=h are given by:

\[ u^1 = u^0 + u(-h, \delta_1^0, \varphi_R^{-1}, \varphi_L^{-1}) \]
\[ u^2 = u(h, \delta_2, \varphi_R^2, \varphi_L^2) \quad (2.9) \]

where:

\[ \varphi_R^1 = R_R(-h, a, b, \delta_1^0) \cdot \varphi_R^0 \]
\[ \varphi_L^1 = R_L(\gamma) \cdot \varphi_L^0 \]
\[ \varphi_R^2 = T_R(h, a, b, \delta_2, \delta_1) \cdot \varphi_R^0 \]
\[ \varphi_L^2 = T_L(\gamma) \cdot \varphi_L^0 \]

\[
R_R(-h, a, b, \delta_1) = \begin{bmatrix}
-2\delta_1(1-a)h\frac{\partial}{\partial z} & +(1-b) - 4\delta_1^2(1-a)h^2 \frac{\partial^2}{\partial z^2} \\
+(1-a) & +2\delta_1(1-a)h\frac{\partial}{\partial z}
\end{bmatrix}
\]

\[
T_R(h, a, b, \delta_2, \delta_1) = \begin{bmatrix}
+\delta_1 & -2(\delta_2-b\delta_1a)h\frac{\partial}{\partial z} \\
0 & +b
\end{bmatrix}
\]
$$R_L(\gamma) = \begin{bmatrix} 1 - \gamma \\ 1 + \gamma \end{bmatrix} \quad T_L(\gamma) = \begin{bmatrix} \frac{+2}{1+\gamma} \end{bmatrix}$$

(2.10)

Note that:

$$\varphi_R^{-1} = R_R(-h, a, b, \delta_1) \varphi_R^0 = R_R(-h, a, b, \delta_1) \varphi_R^{-2}$$

and:

$$R_R(-h, a, b, \delta_1) = R_R(+h, a, b, \delta_1)$$

(2.11)

Note that the $\varphi_L^{1,2}$ are simple multiplicatives of $\varphi_L^0$. The case when $\varphi_R^0 = 0$ corresponds to the purely anti-plane problem, and thus, the algorithm reduces to the scalar image method for that case.

In the above algorithm, the displacement fields $u^0$, $u(-h, \delta_1, \varphi_R^{-1}, \varphi_L^{1})$, and $u(+h, \delta_1, \varphi_R^{-2}, \varphi_L^{2})$ will be called the "original" source, the "reflected" image and the "transmitted" image respectively. Thus, the displacement field due to a point source near an interface is expressed as the sum of the "original" source and the "reflected" image in the halfspace containing the point source and as a "transmitted" image in the halfspace not containing the point source. The above terminology of "image" sources will be helpful in describing the formulation for the derivation of the displacement field due to a point source in the layered region (shown in figure 2.2).

The derivation of the above algorithm is given in Appendix C. The approach used in deriving the image algorithm in Appendix C follows the classical approach of obtaining an eigenfunction representation of the displacement fields in bonded elastic halfspaces. The new feature that the derivation employs is in using the form of the eigenfunction representation of the solution.
displacement fields (for an arbitrary point source) to determine how image potentials can be related to the infinite space potentials. An analytic check of the algorithm (making sure the displacement and traction interface conditions are satisfied) using the Cartesian components is shown in Appendix D. Finally some sample known solutions are rederived in Appendix E; namely screw dislocation in a half space with a free surface, the Boussinesq and Cerruti point force normal and tangential (respectively) to a free surface, Flamant’s line force normal to a free surface, a line force tangential to a free surface, and finally Mindlin’s solution of a point force interior to a halfspace.

In anticipation of applying the above algorithm to the derivation of point sources for a region consisting of an elastic layer perfectly bonded to two elastic halfspaces, consider the effect of shifting the interface plane from \( z=0 \) to \( z=H \) on the form of the terms in the algorithm.

Assume the interface is at \( z=H \). Define a new coordinate \( z' = z-H \). If \( z = h \) is the location of the source point \( (h > H) \) then \( z' = h-H \) is the location of the source point in terms of the new coordinate, and \( z' = H-h \ (z = 2H-h) \) is the location of the image of the source point with respect to the interface. In terms of \( z' \), the algorithm is applicable as shown above with \( h_{\text{effective}} = h-H \). Now, reexpress \( z' \) in terms of \( z \). Therefore, for the case when \( z=H \) is the interface plane one gets:

\[
\begin{align*}
given: \quad & u^0 = u^0(h, \delta_1, \varphi_R^0, \varphi_L^0) \\
then: \quad & u^1 = u^0 + u(2H-h, \delta_1, \varphi_R, \varphi_L^0) \\
& \quad + u^0(h, \delta_2, \varphi_R^2, \varphi_L^2)
\end{align*}
\]
where:

\[ \varphi_R = \mathcal{R}_R(H-h,a,b,\delta_1) \cdot \varphi_R^0 \]

\[ \varphi_L = \mathcal{T}_L(\gamma) \cdot \varphi_L^0 \]

\[ \varphi_R = \mathcal{R}_R(-H+h,a,b,\delta_2,\delta_1) \cdot \varphi_R^0 \]

\[ \varphi_L = \mathcal{T}_L(\gamma) \cdot \varphi_L^0 \]

(2.12)

2.3. Point sources in a region consisting of an elastic layer perfectly bonded to two elastic halfspaces

In this section, the method of derivation of the displacement field due to a point source in a region consisting of an elastic plate \((0<z<H)\) perfectly bonded to two elastic halfspaces (see figure 2.2) is considered. The location of the point source is allowed to be either in the plate or in one of the halfspaces. The case of the source being in the elastic plate and the case of the source being in one of the halfspaces have to be treated separately. The elastic parameters used to characterize each region are chosen to be \(\mu_1\) and \(\delta_1\), where \(\delta\) is defined as \(\delta = (\lambda+\mu)/(\lambda+3\mu)\).

In order to simplify the presentation, define the following terms (this notation is suggested from private notes by Rice 1985 of an outline of using the image method combined with the Papkovich-Neuber potentials to solve the same problem, although a detailed description of the implementation is not performed):

\[ \mu^+ = \mu_3 \quad \mu^- = \mu_2 \quad \gamma^\pm = \mu^\pm/\mu_1 \]
\[ \delta^+ = \delta_3 \quad \delta^- = \delta_2 \]

\[ a^\pm = (\delta_1^+1)/(\delta_1^+\gamma^+) \quad b^\pm = (\delta_1^+1)/(\gamma^+\delta_1^+1) \]

\[ R_R^+(h) = R_R(h, \delta_1^+, a^+, b^+) \quad R_L^+ = R_L(\gamma^+) \]

\[ T_R^+(h) = T_R(h, a^+, b^+, \delta^-, \delta_1) \quad T_L^+ = T_L(\gamma^+) \]

\[ \gamma^* = \mu_1/\mu_2 \]

\[ a^* = (\delta_2^+1)/(\delta_2^+\gamma^*) \quad b^* = (\delta_2^+1)/(\gamma^+\delta_1^+1) \]

\[ R_R^*(h) = R_R(h, \delta_2^+, a^*, b^*) \quad R_L^* = R_L(\gamma^*) \]

\[ T_R^*(h) = T_R(h, a^*, b^*, \delta_1, \delta_2) \quad T_L^* = T_L(\gamma^*) \]

(2.13)

Case I: source is in region 1 (the elastic plate), and \( h<H \)

Figure 2.3 denotes the location of the original and reflected image source singularities contributing to the displacement field in region 1 (left series of points) and the location of the transmitted image source singularities contributing to the displacement field in region 2 (right series of points). Referring to figure 2.3, the way these sources are generated will next be described. Consider an original source (denoted by "0" in the left series of points) in region 1; in order for that source to satisfy boundary conditions on the lower interface, an image has to be reflected (denoted by "-0" left series of points) and another has to be transmitted("-0" right series of points). Both the original source and the first reflected source have to satisfy the boundary
Figure 2.3
conditions on the upper interface which requires 2 reflected sources (left series of points, denoted by "+0" and "+-1" for the original source and the source denoted by "-0" respectively) and 2 transmitted image sources contributing to region 3 (location of the singularities of sources contributing to region 3 are not shown in figure 2.3). Consequently, the sources that are denoted by "+-1" and "+0" (left series of points) require 2 reflected image sources (denoted by "-1" and "+-1" in the left series of points) and 2 transmitted image sources (denoted by "-1" and "+-1" in the right series of points) in order to satisfy the boundary conditions at the lower interface. This procedure continues and it is noticed that each time new image sources are generated in order to "fix" boundary conditions on one of the interfaces, further image sources are required to "fix" the boundary conditions on the other interface. Thus the reflected image sources denoted by "-m", "+m", "-m" and "+-m" (left series of points) and the transmitted image sources denoted by "-m" and "+-m" (right series of points) are generated. Hence, we can deduce the following:

Given a point source in region 1 (h<H) in the form:

\[ u^0 = u(h, \delta_1, \varphi_R, \varphi_L) \]

Then:

\[ u^1 = u^0 + u(-h, \delta_1, \varphi_R, \varphi_L(0), \varphi_L(0)) + \sum_{m=1}^{\infty} u(-2mh-h, \delta_1, \varphi_R, \varphi_L(0), \varphi_L(0)) \]

\[ + \sum_{m=1}^{\infty} u(2mh+h, \delta_1, \varphi_R, \varphi_L(0), \varphi_L(0)) + u(2h-h, \delta_1, \varphi_R, \varphi_L(0), \varphi_L(0)) \]
\[ + \sum_{m=1}^{\infty} u(2(m+1)H-h, \delta_1, C_{\varphi_R}^1(m), C_{\varphi_L}^1(m)) \]
\[ + \sum_{m=1}^{\infty} u(-2mH+h, \delta_1, C_{\varphi_R}^1(m), C_{\varphi_L}^1(m)) \]  \hspace{1cm} (2.14)

\[ u^2 = \sum_{m=1}^{\infty} u(-2(m-1)H-h, \delta_2, C_{\varphi_R}^2(m), C_{\varphi_L}^2(m)) + u(h, \delta_2, C_{\varphi_R}^2(0), C_{\varphi_L}^2(0)) \]
\[ + \sum_{m=1}^{\infty} u(-2mH+h, \delta_2, C_{\varphi_R}^2(m), C_{\varphi_L}^2(m)) \]  \hspace{1cm} (2.15)

where:

\[ C_{\varphi_R}^1(0) = \frac{R^+}{R^-}(-h) \cdot \varphi_0 \]
\[ C_{\varphi_L}^1(0) = \frac{R^+}{R^-} \cdot \varphi_0 \]
\[ C_{\varphi_R}^1(m) = \frac{R^+}{R^-}((2m-1)H-h) \cdot C_{\varphi_R}^1(m-1) \]
\[ C_{\varphi_L}^1(m) = \frac{R^+}{R^-} \cdot C_{\varphi_L}^1(m) \]
\[ C_{\varphi_R}^1(m) = \frac{R^+}{R^-}(-2mh-h) \cdot C_{\varphi_R}^1(m) \]
\[ C_{\varphi_L}^1(m) = \frac{R^+}{R^-} \cdot C_{\varphi_L}^1(m) \]

\[ C_{\varphi_R}^2(0) = \frac{R^-}{R^-}(H-h) \cdot \varphi_0 \]
\[ C_{\varphi_L}^2(0) = \frac{R^-}{R^-} \cdot \varphi_0 \]
\[ C_{\varphi_R}^2(m) = \frac{R^+}{R^-}(-2mH+h) \cdot C_{\varphi_R}^2(m-1) \]
\[ C_{\varphi_L}^2(m) = \frac{R^+}{R^-} \cdot C_{\varphi_L}^2(m) \]
\[ C_{\varphi_R}^2(m) = \frac{R^+}{R^-}((2m+1)H-h) \cdot C_{\varphi_R}^2(m) \]
\[ C_{\varphi_L}^2(m) = \frac{R^+}{R^-} \cdot C_{\varphi_L}^2(m) \]

\[ C_{\varphi_R}^2(0) = \frac{T^-}{T^-}(-H+h) \cdot \varphi_0 \]
\[ C_{\varphi_L}^2(0) = \frac{T^-}{T^-} \cdot \varphi_0 \]
\[ C_{\varphi_R}^2(m) = \frac{T^-}{T^-}(-(2m+1)H+h) \cdot C_{\varphi_R}^2(m) \]
\[ C_{\varphi_L}^2(m) = \frac{T^-}{T^-} \cdot C_{\varphi_L}^2(m) \]
\[ C_{\varphi_R}^2(m) = \frac{T^-}{T^-}(-2mH+h) \cdot C_{\varphi_R}^2(m-1) \]
\[ C_{\varphi_L}^2(m) = \frac{T^-}{T^-} \cdot C_{\varphi_L}^2(m-1) \]  \hspace{1cm} (2.16)

It may be useful to keep in mind that the material points inside region 1 "see" only the repeatedly reflected sources (plus the original source), whereas the material points inside region 2 "see" only the transmitted sources. From the above recursive relations for the "image" potentials it is desirable to obtain
their direct relation to the infinite space potentials $\varphi^0_R$ and $\varphi^0_L$. This can be done by induction and the final results are:

\[
\begin{align*}
\varphi^0_R(0) &= R^+_R(h) \cdot \varphi^0_R \\
\varphi^1_R(m) &= \left[ \prod_{k=1}^{m} \left[ R^+_R(2kH+h) \cdot R^-_R((2k-1)H+h) \right]_k \right] \cdot R^+_R(h) \cdot \varphi^0_R \\
\varphi^1_L(0) &= R^+_L \cdot \varphi^0_L \\
\varphi^1_L(m) &= \left[ \prod_{k=1}^{m} \left[ R^+_L \cdot R^-_L \right]_k \right] \cdot R^+_L \cdot \varphi^0_L \\
\varphi^1_R(m) &= \left[ \prod_{k=1}^{m} \left[ R^-_R(\cdot(2k-1)H-h) \cdot R^+_R(\cdot(2k-2)H-h) \right]_k \right] \cdot \varphi^0_R \\
\varphi^1_L(m) &= \left[ \prod_{k=1}^{m} \left[ R^-_L \cdot R^+_L \right]_k \right] \cdot \varphi^0_L \\
\varphi^1_R(0) &= R^-_R(\cdotH+h) \cdot \varphi^0_R \\
\varphi^1_R(m) &= \left[ \prod_{k=1}^{m} \left[ R^-_R(\cdot(2k+1)H+h) \cdot R^+_R(\cdot2kH+h) \right]_k \right] \cdot R^+_R(\cdotH+h) \cdot \varphi^0_R \\
\varphi^1_L(0) &= R^-_L \cdot \varphi^0_L \\
\varphi^1_L(m) &= \left[ \prod_{k=1}^{m} \left[ R^-_L \cdot R^+_L \right]_k \right] \cdot R^-_L \cdot \varphi^0_L \\
\varphi^1_R(m) &= \left[ \prod_{k=1}^{m} \left[ R^+_R(2kH-h) \cdot R^-_R((2k-1)H-h) \right]_k \right] \cdot \varphi^0_R \\
\varphi^1_L(m) &= \left[ \prod_{k=1}^{m} \left[ R^+_L \cdot R^-_L \right]_k \right] \cdot \varphi^0_L
\end{align*}
\]

(2.17)

Once the $\varphi^1$ potentials are known in terms of the $\varphi^0$
potentials, the $\phi^2$ potentials are obtained by one further matrix operation as given previously (2.16).

Note that whereas the $R_L$ and $T_L$ matrices consist of one scalar per matrix, the $R_R$ and $T_R$ matrices are 2x2 matrix operators involving $\frac{\delta}{\delta z}$ and $\frac{\delta^2}{\delta z^2}$ operators as well as constants.

One subtle point when deriving any given point source for this case (i.e. when the point source is in the elastic layer) is that the reflected image potentials denoted by "-m" and "+-m", left series of points in figure 2.3, have to have all their singularities in the region $z>0$. In addition, the reflected image potentials denoted by "+m" and "+-m", left series of points in figure 2.3, as well as the transmitted image potentials denoted by "-m" and "+-m", right series of points in figure 2.3, have to have all their singularities in the region $z<H$. These conditions are imposed on the choice of the potentials defining the original source, in order not to introduce any further singularities in any given region through the use of the algorithm. This condition can be satisfied due to the flexibility in choosing the potentials.

Case II: source is in region 2 (the lower parts, etc) and $h>H$

Figure 2.4 denotes the location of the (one) transmitted and (all other) reflected image source singularities contributing to the displacement field in region 1 (left series of points) and the location of the original, (one) reflected and (all other) transmitted image source singularities contributing to the displacement field in region 2 (right series of points). Referring to figure 2.4, the way these sources are generated will next be described. Consider an original source (denoted by "0" right series of points) in region 2; in order for that source to satisfy
Figure 2.4
boundary conditions on the lower interface (between region 1 and 2), an image has to be reflected (denoted by "-0" right series of points) and another has to be transmitted ("0*" left series of points). The source denoted by "0*" has to satisfy boundary conditions on the upper interface which requires a reflected source ("+0" left series of points)) and a transmitted image source contributing to region 3 (location of the source of singularities contributing to region 3 are not shown in figure 2.4).

Consequently, the source denoted by "+0" (left series of points) require a reflected image source (denoted by "-+1" left series of points) and a transmitted image source (denoted by "-+1" right series of points) in order to satisfy the boundary conditions at the lower interface. This procedure continues and it is noticed that each time new image sources are generated in order to "fix" the boundary conditions on one of the interfaces, further image sources are required to "fix" the boundary conditions on the other interface. Thus the reflected image sources denoted by "+m" and "-+m" (left series of points) and the transmitted image sources denoted by "-+m" (right series of points) are generated. Hence one can deduce the following:

Given a point source in region 2 \((h>H)\) in the form:

\[
u^0 = u(h, \delta, \varphi_R, \varphi_L)
\]

Then:

\[
u^1 = u(h, \delta, \varphi_R, \varphi_L) + u(-h, \delta, \varphi_R(0), \varphi_L(0)) + \sum_{m=-1}^{\infty} u(-2mH-h, \delta, \varphi_R(m), \varphi_L(m))
\]
\[ u^2 = u^0 + u(2H-h, \delta_2, \varphi^2_R(0), \varphi^2_L(0)) + \sum_{m=1}^\infty u(-2(m-1)H-h, \delta_2, \varphi^2_R(m), \varphi^2_L(m)) \]  
\[ \text{(2.19)} \]

where:

\[ \varphi^0_R = T^*_R(h-H) \cdot \varphi^0_R \]

\[ \varphi^{1+}_R(0) = R^+_R(-h) \cdot T^*_R(h-H) \cdot \varphi^0_R \]

\[ \varphi^{1+}_R(m) = R^+_R((2m-1)H+h) \cdot \varphi^{1+}_{R(m-1)} \]

\[ \varphi^{1+}_R(m) = R^+_R(-2mH-h) \cdot \varphi^{1+}_{R(m-1)} \]

\[ \varphi^{2-}_R(0) = R^*_R(H-h) \cdot \varphi^0_R \]

\[ \varphi^{2-}_R(m) = T^*_R(-(2m-1)H-h) \cdot \varphi^{2-}_{R(m-1)} \]

\[ \varphi^{2-}_L(0) = R^*_L \cdot \varphi^0_L \]

\[ \varphi^{2-}_L(m) = T^*_L \cdot \varphi^{2-}_{L(m-1)} \]

\[ \varphi^0_L = T^*_L \cdot \varphi^0_L \]

\[ \varphi^{1+}_L(0) = R^+_L \cdot \varphi^0_L \]

\[ \varphi^{1+}_L(m) = R^+_L \cdot \varphi^{1+}_{L(m-1)} \]

\[ \varphi^{1+}_L(0) = R^+_L \cdot \varphi^0_L \]

\[ \varphi^{1+}_L(m) = R^+_L \cdot \varphi^{1+}_{L(m-1)} \]

From the above recursive relations for the "image" potentials their direct relation to the infinite space potentials \( \varphi^0_R \) and \( \varphi^0_L \) can be obtained. This can be done by induction and the final results are:

\[ \varphi^{1+}_R(0) = R^+_R(h) \cdot T^*_R(-h+H) \cdot \varphi^0_R \]

\[ \varphi^{1+}_R(m) = \left[ \prod_{k=1}^m \left[ R^+_R(2kH+h) \cdot R^-_R((2k-1)H+H) \right] \right] \cdot R^+_R(h) \cdot T^*_R(-h+H) \cdot \varphi^0_R \]

\[ \varphi^{1+}_L(0) = R^+_L \cdot T^*_L \cdot \varphi^0_L \]

\[ \varphi^{1+}_L(m) = \left[ \prod_{k=1}^m \left[ R^+_L \cdot R^-_L \right] \right] \cdot R^+_L \cdot T^*_L \cdot \varphi^0_L \]
\begin{align*}
\mathbf{z}_{\mathbf{R}}^{-1}+(m) &= \left[ \prod_{k=1}^{m} \left[ \mathbf{R}^{-} \cdot (2k-1)\mathbf{h} \cdot \mathbf{R}^{+} \cdot (-2k-2)\mathbf{h} \right] \right] \mathbf{z}_{\mathbf{R}}^{(h-H)} \cdot \mathbf{z}_{\mathbf{R}}^{0} \\
\mathbf{z}_{\mathbf{L}}^{-1}+(m) &= \left[ \prod_{k=1}^{m} \left[ \mathbf{R}^{-} \cdot \mathbf{R}^{+} \right] \right] \mathbf{z}_{\mathbf{L}}^{*} \cdot \mathbf{z}_{\mathbf{L}}^{0}.
\end{align*}

(2.21)

Again, once the $\varphi^1$ potentials are known in terms of the $\varphi^0$ potentials, the $\varphi^2$ potentials are obtained by one further matrix operation as given previously (2.20).

When deriving any given point source for this case (i.e. when the point source is in the lower halfspace) the potentials have to have all their singularities in the region $z>H$, in order for the reflected images not to introduce any further singularities within any given region.

Note that whereas the $\mathbf{R}_{\mathbf{L}}$ and $\mathbf{T}_{\mathbf{L}}$ matrices consist of one scalar per matrix, the $\mathbf{R}_{\mathbf{R}}$ and $\mathbf{T}_{\mathbf{R}}$ matrices are 2x2 matrix operators involving $\frac{\partial}{\partial z}$ and $\frac{\partial^2}{\partial z^2}$ operators as well as constants. The product of $\mathbf{R}_{\mathbf{L}}$, $\mathbf{T}_{\mathbf{L}}$ and $\mathbf{R}_{\mathbf{R}}$, $\mathbf{T}_{\mathbf{R}}$ matrices are relatively simple to obtain because the operators are linear and the "component" operators of the $\mathbf{R}_{\mathbf{R}}$ and $\mathbf{T}_{\mathbf{R}}$ matrices involve simple operations (differentiation with respect to the "z" coordinate as well as multiplication by constants). By contrast, the use of the similar "repeated imaging" approach to derive solutions for multi-layered regions using the Papkovich-Neuber (P-N) potentials is much more difficult. This is because the analogous "image algorithm" using the P-N potentials involves differentiation with respect to "x", "y" and "z" coordinates, multiple integration with respect to "z", as well as multiplication by constants and the "x" and "y" coordinates. Using the P-N potentials, repeated imaging leads to rather severe
complexities when contemplating an application to a specific source.

2.4 Convergence studies

The solution for a given point source as obtained in the above formulation for a plate region, is in the form of an infinite series of "image" sources. In practice, such a series has to be truncated after a certain number of terms. In this section, the effect of the type of source, location of the source and material properties on the number of terms required for an accurate solution is studied. In addition, a method which accelerates the convergence is proposed and is shown to significantly improve the performance of this method. For simplicity, antiplane sources which only involve a scalar displacement field and which are located inside the plate are studied, and guidelines are inferred for the more general 3-D case. The formulas that are needed to calculate the displacement fields within the plate (region 1 in figure 2.2) for antiplane sources with the original source being in the plate are (2.14) and those formulas involving the $\psi_L$ potentials in (2.17). Recall that for antiplane sources, the vector image method reduces to the (well known) scalar image method.

A general statement that can be made concerning a truncated series of the type considered in this chapter, is that the resulting approximate stress and displacement fields are asymptotic to the exact fields in the limit as we get closer to the location of the source and/or as the material properties tend to become homogeneous. The reason is that the fields associated with the truncated terms (with large $m$ in (2.14)) have algebraic singularities (with respect to position) which are farther from the region of definition than the nontruncated terms. These fields are also multiplied by parameters dependent on the material properties
which tend to zero as the region of definition (plate and halfspaces) tends to become materially homogeneous. For a given source, material properties and a given truncation level, the region close to the source where the approximation is accurate depends on the severity of the source singularity; for example, a source whose displacement singularity is \(1/R^2\) is expected to be accurate over a larger region in the vicinity of the source than a source whose singularity is \(\arctan(z/x)\). For point sources whose displacement fields tend to zero far from the source (such as the nuclei of strain sources) an "acceptable" solution is required to be accurate up to a distance of a few "plate thicknesses" (defined as "H") away. In the subsequent analysis this region of "required accurate solution" is taken to be 3-5 H; at farther distances the displacement fields are expected to be negligible.

For the antiplane problem, the values of \(u_y(x,z)=0\) and \(u_x-y=0\). When summing the series of image sources that give the value of \(u_y(x,z)\) the image sources are added in groups of four which correspond to \(+m\), \(-m\), \(+m\) and \(-m\) in the left series of points in figure 2.3; the exception is the first group which consists of the original (denoted by "0" left series of points) and the first two reflected sources ("-0" and "+0" left series of points). In figures 2.5-2.8, the displacement field \(u_y(x,z)\) obtained by summing the first 1,2,3 and 4 groups of image sources (displacement fields) are denoted by \((m=1,2,3\text{ and }4)\).

Accelerating Convergence

The algebra involved in the derivation of 3-D displacement fields for the layered region is considerably more complicated than for the antiplane (scalar imaging) case. Thus, the maximum amount of information should be extracted from the fields associated with the first few groups of image sources. This requires the
implementation of a "suitable" extrapolation technique. Therefore, before discussing the results of the convergence studies, an "acceleration technique" which was implemented to improve the convergence of the fields derived using this "repeated imaging" method will be presented.

A specific extrapolation technique for series that was determined "suitable" and that was thus used is Richardson's extrapolation method (e.g. see Bender and Orszag (1978)). The main idea in Richardson's extrapolation is next described. Consider a function defined as follows:

$$F(x) = \sum_{k=1}^{\infty} f(x,k)$$  \hspace{1cm} (2.22)

Then define a truncated series $F(x,n)$ as:

$$F(x,n) = \sum_{k=1}^{n} f(x,k)$$  \hspace{1cm} (2.23)

Here $F(x)$ is the "exact" function that is to be determined and $F(x,n)$ are considered to be "approximations". In order to construct an extrapolation scheme, one postulates that:

$$F(x,n) = F(x) + \text{error}(x,n)$$  \hspace{1cm} (2.24)

where:

$$\text{error}(x,n) = \sum_{k=1}^{\infty} C_k(x) \cdot (1/n)^k$$  \hspace{1cm} (2.25)

That is we postulate that as more terms are added in the series defining $F(x)$, the error decreases as a polynomial in inverse powers of the number of terms added (i.e. "1/n"). A (nonrigorous but "intuitive") way to justify this assumption is to
expand the truncated terms as a Taylor series in powers of "1/n" (this would be possible for some class of error(x,n) which tend to zero as n->∞).

Next the error(x,n) function is truncated say from the "p'th" term on. This means that there are (for a fixed "x") "p" unknown constants C_k(x) in the error(x,n) function (viewed as a function of "n"). Knowing "p+1" consecutive terms of F(x,n) (say 1≤n≤p+1), one can then solve for the values of C_k(x) as well as F(x). In the calculations presented in figures 2.5-2.8, the first four groups of image sources have been used to obtain an extrapolation. Using F(x,k) where (1≤k≤4), one can get the following extrapolation formula:

\[
F(x) = R_1^3 = \frac{32}{4} \cdot F(x,4) - \frac{27}{2} \cdot F(x,3) \\
+ 4 \cdot F(x,2) - \frac{1}{6} \cdot F(x,1)
\]  
(2.26)

and using F(x,100) and F(x,101), one can obtain the following extrapolation formula:

\[
F(x) = R_{100}^1 = 101 \cdot F(x,101) - 100 \cdot F(x,100)
\]  
(2.27)

The subscript "a" and superscript "b" in \( R_a^b \) denote the starting term and the number of terms after the first term used to obtain the extrapolation respectively.

Results of the convergence studies

In the convergence studies conducted using the antiplane sources, two types of displacement singularities were considered. The first is that of a screw dislocation ("arctan" singularity) whose closed form solution is available (Tse and Rice 1986), and
the second is that displacement field whose original source is given by \( u_y(x,z) = 1/R^2 - 2x^2/R^4 (R^2-x^2+z^2) \) (which from hereon and in figures 2.5-2.8 will be referred to as a "1/R^2" source); the convergence of the "1/R^2" source is expected to characterize the convergence of the "nuclei of strain" (from hereon referred to as NOS) in the general 3-D problem because of the similar singularities.

In what follows the effect of the material properties, depth of the source (denoted by "h"), horizontal distance from the source and type of the source on the convergence of the displacement field \( u_y(x,z=0) \) (surface displacement) for a point source placed at the location \( x=0 \) and \( z=h \) in a plate thickness "H" will be discussed.

Effect of Material Property Contrasts

For antiplane problems the relevant material property contrasts are \( \gamma^\pm \) (this can be seen from the \( R_L \) matrix operator in (2.12)) which are taken to be equal and referred to as \( \gamma \). Figure 2.5 shows how the surface displacements (at \( x=0 \) and source depth \( h=0.5\cdot H \)) for the "1/R^2" source vary with \( \gamma \). The "best estimate" for the "exact" displacement field for this source is obtained by summing up the contribution to the displacement field of the first 100 group of images (denoted by \( u(k=100) \)). Note that \( R_1^3 \) gives an excellent estimate (much less than 1% error in most cases) for all values of \( \gamma \). Finally, the slowest convergence rate for the image formulation is encountered when \( \gamma=0 \). Therefore, for the rest of the convergence studies, a value of \( \gamma=0 \) will be used in order to be conservative with respect to the estimated accuracy of the method.

Note that the errors encountered at \( \gamma=0.1 \) (material contrast
Figure 2.5
is 10 to 1) has half as much error for the same level of truncation as for $\gamma=0$. This relates to the comment mentioned in relation to Rundle and Jackson’s work in the background section.

Effect of Depth of the Source

The depth of the source (as measured by "h") has a significant effect on the accuracy of the solution as is seen in figure 2.6, in which the surface displacement multiplied by $h^2$ (for the figure's resolution purposes) is plotted at (x=0) for the $1/R^2$ source. As we get farther from the source (deeper source h -> H), more images are needed to obtain an accurate approximation. However, the $R_1^3$ extrapolation is very accurate for all values of "h/H".

Effect of horizontal distance from the source for 2 types of sources

The surface displacement for 2 types of sources ("1/R^2" source and "arctan" screw dislocation source) are each plotted for 2 different source depths (h/H=0.1 and h/H=0.9) in figures 2.7a,b and 2.8a,b.

For both type of sources, the $R_1^3$ extrapolation is accurate throughout the range 0≤x/H≤5 when the source is "shallow" (i.e. h/H=0.1). However, when the source is "deep" (i.e. h/H=0.9) the $R_1^3$ extrapolation is reasonably accurate for the "1/R^2" source for the same range 0≤x/H≤5, whereas the "arctan" source is only acceptable up to x/H=3. Note that both the "exact" and the $R_1^3$ extrapolation tend to zero as x/H->∞ for the "1/R^2" source. In contrast, the "exact" displacement field goes to a finite value while the $R_1^3$
Figure 2.6
Figure 2.7(a)
Figure 2.7(b)
Figure 2.8(a)
ARCTAN SOURCE
h/H=0.9 \gamma=0.0

EXACT
U(R)
U(n=1,2,3,4)

SURFACE DISPLACEMENT

Figure 2.8(b)
extrapolation tend to zero (very slowly) as \( x/H \to \infty \) for the "arctan" source.

The above results suggest that for the derivation of NOS displacement fields in a plate, the first four groups of images combined in a \( R_l^3 \) extrapolation should yield an accurate approximation to the exact fields within a vicinity \( x/H \leq 5 \) from the source. This procedure has been implemented when plotting the solutions for the NOS in an elastic plate shown in figures 2.9-2.20, to be presented next.

2.5 Nuclei of Strain - Numerical examples

In this section, NOS displacement fields will be related to the stresses due to a point force. After that, the relevant relations to be used from part 2.3 for the derivation of the displacement fields due to NOS in a plate (a special case of the layered media where the top and bottom surfaces have zero rigidity) will be described. Finally, some numerical solutions for the displacements at the surface of a plate and a halfspace due to a distribution of embedded NOS will be presented and discussed.

Note however, that the details required for the actual implementation of the multiple imaging scheme to specific sources are given in the appendices. Appendix F presents a formalism for the multiple imaging scheme that is easier to implement and manipulate, appendix G and H give the explicit components of the product of matrix operators ("R" part) and ("L" part) respectively required for obtaining up till the fourth group of images and up till any group of images respectively. Appendix I discusses some special aspects of implementation related to nuclei of strain in a plate (see also below), finally appendix J lists all the functions required in the evaluation of the displacement and stresses of NOS
in a plate.

As mentioned in section 2.1, any distribution of slippage on an internal surface in a plate can be modelled as a distribution of NOS. The displacement at a point \( x' \) in the \( i^{th} \) direction due to a nucleus of strain \((k,l)\) at a location \( x \) is equal to (by reciprocity, e.g. see chapter 3) the \((k,l)\) stress component at point \( x \) due to a point force at point \( x' \) acting in the \( i^{th} \) direction. In general we have (e.g. see Mura (1982)):

\[
    u_k(x') = \int \left[ g^k(x,x') \cdot \Pi(x) \right] \cdot \Delta u(x) \cdot dS
    \tag{2.28}
\]

where \( u_k(x') \) is the \( k^{th} \) displacement component at point \( x' \) due to a distribution of displacement discontinuity \( \Delta u(x) \) on a given slip surface. Also, \( g^k(x,x') \) is the stress tensor at point \( x \) due to a point force acting in the \( k^{th} \) direction at point \( x' \), and \( \Pi(x) \) is the normal vector defined on the slip surface.

Taking \( \Delta u(x) \) to be \( \overline{\Delta u} \cdot \delta(x-x_0) \cdot A \) (where \( \delta(x) \) is the Dirac delta and "A" is an effective area), one obtains that the displacement field at a point \( x \) due to a NOS (defined by \( \overline{\Delta u} \), \( A \) and \( \Pi \)) at a point \( x_0 \) is given by:

\[
    u_k(x) = - (g^k(x_0,x) \cdot \Pi(x_0)) \cdot \overline{\Delta u}(x_0) \cdot A \tag{2.29}
\]

Therefore, if the stress fields of point forces are known in a certain region, the displacement fields of NOS are also known.
The stress field of point sources in a plate are obtained from the displacement field of point forces. The displacement field of point forces are obtained using the potentials for the point force given in appendix B, combined with formulas 2.15 and 2.17 in this chapter. More details and special considerations that arise in deriving NOS displacement as well as stress fields in a plate are discussed in appendix I.

Equations 2.14 and 2.17 are used to calculate some components of the surface displacement fields \((z=0, -3 \leq x/H \leq 3, -3 \leq y/H \leq 3)\) for several NOS located at \(x_0 = (0, 0, h=0.5H)\) in an elastic plate (top and bottom surfaces are traction free). The choice of the component of the displacement fields and type of sources are given in Table 2.1 (also \(\mu=\lambda\) is taken). These displacement fields are shown in figures (b) of 2.9-2.20. For comparisons, the exact displacement fields for halfspaces (Mindlin's solution 1936; top surface is traction free) are shown in figures (a) of 2.9-2.20.

In general, the figures show that the level of the displacements are higher in the plate than in the halfspace, while the qualitative features are the same. This is expected because of the higher compliance of the plate and the similarity of the sources. One exception is the "z component" of the displacement (figures 2.20a and b) due to a "closing strain" in the "z" direction on a surface parallel to the "z" plane. This type of source tries to attract the material from above and below its level into the source. The existence of a halfspace acts as an anchor from below which causes the z-displacement on the surface in the case of the halfspace to be higher than for the case of the plate.
<table>
<thead>
<tr>
<th>Figure</th>
<th>$k$ displacement</th>
<th>$n$</th>
<th>$\mu \cdot \Delta n \cdot A$</th>
<th>$\mu \cdot A \cdot (\Delta u_x, \Delta u_y, \Delta u_z)$</th>
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<tr>
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<td>z-displacement</td>
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<td>(10,0,0)</td>
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<tr>
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<td>(0,0,2)</td>
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</tbody>
</table>
Figure 2.9(a)
Figure 2.10(a)
Figure 2.11(b)
Figure 2.12(b)
Figure 2.13(a)
Figure 2.16(a)
Figure 2.17(a)
Figure 2.17(b)
2.6 Viscoelastic Green's Functions

The viscoelastic Green's functions that will be considered in this thesis are the displacement and stress fields due to a NOS located in an elastic plate (0 ≤ z ≤ H) with a free surface at z = 0 and a perfectly bonded lower (see figure 2.2) viscoelastic halfspace at z ≥ H.

The viscoelastic model used for the lower halfspace is that of a Maxwell fluid, and will be summarized below. Once the model has been described the steps for deriving the response due to a suddenly introduced and maintained NOS acting in the plate region will be discussed within the context of the correspondence principle.

Define the deviatoric part of the stress and strain components as:

\[
\begin{align*}
    s_{ij} &= \sigma_{ij} - \frac{1}{3}\cdot\sigma_{kk}\cdot\delta_{ij} \\
    e_{ij} &= \varepsilon_{ij} - \frac{1}{3}\cdot\varepsilon_{kk}\cdot\delta_{ij}
\end{align*}
\]  

(2.30) \hspace{1cm} (2.31)

Then the elastic stress-strain relations can be written as:

\[
\begin{align*}
    s_{ij} &= 2\mu\cdot e_{ij} \\
    \sigma_{kk} &= (2\mu + 3\lambda)\cdot\varepsilon_{kk} = 3K\cdot\varepsilon_{kk}
\end{align*}
\]

(2.32) \hspace{1cm} (2.33)

A Maxwell viscoelastic model is taken in the form:

\[
\begin{align*}
    \frac{1}{2\mu}\cdot s_{ij} + \frac{1}{\eta_s}\cdot s_{ij} &= \dot{e}_{ij} \\
    \frac{1}{3K}\cdot \sigma_{kk} + \frac{1}{\eta_d}\cdot \sigma_{kk} &= \dot{\varepsilon}_{kk}
\end{align*}
\]

(2.34) \hspace{1cm} (2.35)

where a dot over a variable indicates rate with respect to time and
\( \eta_s \) and \( \eta_d \) are the shearing and volumetric viscosities respectively (\( \eta_d \) will be specialized to \( \infty \) later).

The Laplace transform of a function \( f(t) \) is defined as:

\[
L[f(t)] = \int_0^\infty e^{-st}f(t)dt = \tilde{f}(s)
\]  

(2.36)

Laplace transforming equations (2.34) and (2.35) and using well-known properties of the Laplace transform one obtains:

\[
\bar{s}_{ij} = 2\bar{\mu}(s) \cdot e_{ij}
\]  

(2.37)

\[
\bar{\mu}(s) = \frac{s}{2(\frac{s}{2\mu} + \frac{1}{\eta_s})}
\]  

(2.38)

\[
\bar{\sigma}_{kk} = 3\bar{K}(s) \cdot \bar{\epsilon}_{kk}
\]  

(2.39)

\[
\bar{K}(s) = \frac{s}{3(\frac{s}{3K} + \frac{1}{\eta_d})}
\]  

(2.40)

Defining the following parameters:

\[
\omega_s = \frac{2\mu}{\eta_s} ; \quad \omega_d = \frac{3K}{\eta_d}
\]  

(2.41)

we obtain:

\[
\bar{\mu}(s) = \frac{\mu s}{(s+\omega_s)}
\]  

(2.42)

\[
\bar{K}(s) = \frac{K s}{(s+\omega_d)}
\]  

(2.43)

If we are considering a quasistatic problem where the inertial terms are negligible, the only time dependence occurs in the
boundary conditions ("load application") and constitutive relations. The constitutive relations have been shown in (2.37) and (2.39) to be similar in form in the Laplace transformed space to the elastic stress-strain relations in real space (2.32) and (2.33), but with the shear and bulk moduli replaced with moduli dependent on the Laplace variable "s". The Laplace transformed equations of equilibrium and compatibility have exactly the same form as in real space. If the Laplace transformed boundary conditions are everywhere proportional (the proportionality constant can be a function of "s") to the boundary conditions in real space, then we get a perfect correspondence between the elastostatic and the viscoelastostatic problem. The solution in Laplace space to the viscoelastic problem is obtained by replacing the "load" term with the Laplace transformed "load", and the shear and bulk moduli by the corresponding Laplace variable dependent moduli discussed above (for our model they are given by (2.42) and (2.43) in the elastic solution. Finally, the solution in real space to the viscoelastic problem is obtained by Laplace inverting the solution given in Laplace space. This is the essence of the correspondence principle (e.g. see Christensen 1982).

Within the context of the image method and for the case when the NOS lies in the plate, field variables are only required in the plate, and where only the lower halfspace is viscoelastic, the only instances where we have moduli in Laplace transformed space that are dependent on the Laplace variable "s" are in the "reflection" operators, and specifically in combinations of terms denoted by:

\[
\begin{align*}
\bar{A}^-(s) &= 1 - \bar{a}^-(s) = 1 - \frac{\delta+1}{\delta+\gamma(s)} \quad (2.44) \\
\bar{B}^-(s) &= 1 - \bar{b}^-(s) = 1 - \frac{\delta+1}{\gamma(s) \cdot \bar{\delta}_2(s) + 1} \quad (2.45) \\
\frac{1-\gamma(s)}{1+\gamma(s)} & \quad (2.46)
\end{align*}
\]
where \( \gamma \) and \( \bar{\gamma} \) refer to \( \gamma^- \) and \( \bar{\gamma}^- \) in the above and what follows. After some algebra we can obtain:

\[
\bar{A}^- = A^- - \frac{a \gamma s}{\delta + \gamma} \frac{1}{(s + \delta) \cdot \omega s} \tag{2.47}
\]
\[
\bar{B}^- = B^- - \frac{Q_2(s)}{P_3(s)} \tag{2.48}
\]

where:
\[
Q_2(s) = + b^- \gamma \cdot (\delta s \omega_s + \alpha + \beta - \delta s \alpha - 7 \delta s \beta) \cdot s^2
\]
\[
+ b^- \gamma \cdot (\delta \omega_s^2 + \delta \omega_s \omega_d + \alpha \omega_s
\]
\[
+ \beta \omega_d - 2 \delta \omega_s - 7 \delta \omega_d - 7 \delta s \omega_s) \cdot s
\]
\[
+ b^- \gamma \delta s \omega_s \cdot (\omega_s \omega_d - \alpha \omega_s - 7 \beta \omega_d)
\]
\[
P_3(s) = + (\gamma \delta s + 1) \cdot s^3
\]
\[
+ (\gamma \delta s \omega_s + \gamma \delta s \omega_d - \gamma \alpha - \gamma \beta + 2 \omega_s + \omega_d - \alpha - 7 \beta) \cdot s^2
\]
\[
+ (\gamma \delta s \omega_s \omega_d - \alpha \gamma s - \beta \gamma d + 2 \omega_s \omega_d + \omega_s^2
\]
\[
- 2 \omega_s - 7 \beta \omega_d - 7 \beta \omega_s) \cdot s
\]
\[
+ \omega_s \cdot (\omega_s \omega_d - \alpha \omega_s - 7 \beta \omega_d)
\]

\[
\alpha = \frac{3K_2 \omega_d}{3K_2 + 7 \mu_2}
\]
\[
\beta = \frac{\mu_2 \omega_s}{3K_2 + 7 \mu_2}
\]
\[
3K_2 = 2 \mu_2 + 3 \lambda_2
\]
\[
\omega_s = \frac{2 \mu_2}{\eta_s}
\]
\[
\omega_d = \frac{3K_2}{\eta_d}
\]

\[
\frac{1 - \gamma}{1 + \gamma} = \frac{1 - \gamma}{1 + \gamma} \left[ 1 + \frac{2 \gamma s \omega_s - \frac{1}{1 + \gamma}}{s + \delta} \right] \tag{2.49}
\]

Now we specialize our viscoelastic model to the case where the volumetric deformations are purely elastic and only the deviatoric deformations are viscoelastic. The simplifications that we get are:

\[
\omega_d = 0 \quad \omega_s = \omega \tag{2.50}
\]
\[ Q_2(s)/s = + b^* \gamma \cdot (\delta \omega + \beta - 7 \delta \omega) \cdot s \]
\[ + b^* \gamma \cdot (\delta \omega^2 - 7 \delta \omega) \]  
(2.51)

\[ P_3(s)/s = + (\gamma \delta \omega + 1) \cdot s^2 \]
\[ + (\gamma \delta \omega - \gamma \beta + 2 \omega - 7 \beta) \cdot s \]
\[ + (\omega^2 - 7 \beta \omega) \]  
(2.52)

The zero of \( Q_2(s)/s \) is:

\[ s_Q = - \frac{9K_2^2 + 3K_2 \mu_2}{9K_2^2 + 6K_2 \mu_2 + 7 \mu_2} \cdot \omega \]  
(2.53)

The root \( s_Q \) is always negative as can be seen in (2.53). The zeroes of \( P_3(s)/s \) are:

\[ s_{p1,2} = \frac{-(3\gamma K_2 + 6K_2 + 7 \mu_2) \pm \sqrt{9 \gamma^2 K_2^2 + 49 \mu_2^2 + 30 \gamma \mu_2 K_2}}{2 \cdot (3 \gamma K_2 + \gamma \mu_2 + 3K_2 + 7 \mu_2)} \cdot \omega \]  
(2.54)

The roots \( s_{p1,2} \) are always distinct and are negative real numbers, since the determinant is always positive and less (in absolute value) than the first group in brackets in the numerator in (2.54).

If we define:

\[ \Omega = \frac{\omega}{1 + \gamma} \]  
(2.55)

\[ \Omega_A = \frac{\delta \omega}{\delta + \gamma} \]  
(2.56)

\[ \Omega_{B1} = - s_{p1} \]  
(2.57)

\[ \Omega_{B2} = - s_{p2} \]  
(2.58)
\[ c = \frac{1}{a^\gamma} \frac{\gamma \omega}{\delta + \gamma} \quad (2.59) \]

\[ d = \frac{1}{b^\gamma} \frac{\gamma \omega \cdot (9K_2^2 + 6K_2\mu_2 + 7\mu_2^2)}{(3K_2^2 + 7\mu_2) \cdot (3\gamma K_2 + \gamma \mu_2 + 3K_2 + 7\mu_2)} \frac{\Omega_{B1} + s}{\Omega_{B1} - \Omega_{B2}} \quad (2.60) \]

\[ e = \frac{1}{b^\gamma} \frac{\gamma \omega \cdot (9K_2^2 + 6K_2\mu_2 + 7\mu_2^2)}{(3K_2^2 + 7\mu_2) \cdot (3\gamma K_2 + \gamma \mu_2 + 3K_2 + 7\mu_2)} \frac{\Omega_{B2} + s}{\Omega_{B2} - \Omega_{B2}} \quad (2.61) \]

\[ p = \frac{1 - \gamma}{1 + \gamma} \quad (2.62) \]

\[ q = \frac{2\gamma \omega}{(1 + \gamma)^2} \quad (2.63) \]

Then we obtain the following simplified expressions:

\[ \overline{A^-}_A = 1 - c \cdot \frac{1}{s + \Omega_A} \quad (2.64) \]

\[ \overline{B^-}_B = 1 - d \cdot \frac{1}{s + \Omega_{B1}} + e \cdot \frac{1}{s + \Omega_{B2}} \quad (2.65) \]

\[ \frac{1 + \gamma}{1 - \gamma} = p + q \cdot \frac{1}{s + \Omega} \quad (2.66) \]

The above expressions indicate that there are in general 4 characteristic time scales for the response of a point source located in a plate perfectly bonded to one Maxwellian (viscoelastic in deviatoric deformation and elastic in volumetric deformation) halfspace. The Laplace transform of a suddenly applied and then maintained unit excitation (the Heaviside function) is \((1/s)\). Therefore, within the context of the multiple imaging scheme the Laplace inverse of the expressions:

\[ \frac{1}{s} \cdot (\overline{A^-}_A)^n \cdot (\overline{B^-}_B)^m \quad (2.67) \]
are required for suddenly applied and maintained point sources, where "n", "m" and "k" are integers. The above Laplace inversions can be obtained by transforming all rational expressions into partial fractions form. The Laplace inversion of the terms required up till the fourth image set for (2.67) are given in appendix K and up till any required term for (2.68) in appendix L. Once the Laplace Inverse transform of (2.67) and (2.68) are obtained, their values (as a function of time) are used in the matrix operators given in appendices G and H instead of \((A^-)^n \cdot (B^-)^n\) and \((1-\gamma^-)^k/(1+\gamma^-)^k\) and the required viscoelastic solution is thus obtained following the same way as for the elastic solution using the "multiple imaging" scheme.

In order to acquire some insight into the behavior of the viscoelastic solutions derived, consider the Laplace inverse of the terms:

\[
L^{-1}\left[ \frac{1}{s} \frac{\bar{A}^-}{A^-} \right] = (1 - \frac{c}{\bar{\Omega}_A}) + \frac{c}{\bar{\Omega}_A} \cdot e^{-\bar{\Omega}_A t}
\]

\[
L^{-1}\left[ \frac{1}{s} \frac{\bar{B}^-}{B^-} \right] = (1 - \frac{d}{\bar{\Omega}_{B1}} + \frac{e}{\bar{\Omega}_{B2}}) + \frac{d}{\bar{\Omega}_{B1}} \cdot e^{-\bar{\Omega}_{B1} t} - \frac{e}{\bar{\Omega}_{B2}} \cdot e^{-\bar{\Omega}_{B2} t}
\]

\[
L^{-1}\left[ \frac{1}{s} \frac{1-\gamma^-}{1+\gamma^-} \right] = (p + \frac{q}{\bar{\Omega}}) - \frac{q}{\bar{\Omega}} \cdot e^{-\bar{\Omega} t}
\]

After some algebra, the terms that are not multiplying the exponential functions in (2.69), (2.70) and (2.71) can be
simplified to the following:

\[ 1 - \frac{c}{\Omega_A} = -\frac{1}{\delta} \cdot \frac{1}{A^-} \] (2.72)
\[ 1 - \frac{d}{\Omega_{B1}} + \frac{3}{\Omega_{B2}} = -\delta \cdot \frac{1}{B^-} \] (2.73)
\[ p + \frac{q}{\Omega} = \gamma \] (2.74)

Therefore, the short time response of (2.69), (2.70) and (2.71) (i.e. \( t \to 0^+ \)) corresponds to the halfspace being purely elastic and the long time response (from (2.72), (2.73) and (2.74)) corresponds to a stress free boundary condition on the lower interface; this can be seen by noting that if \( \gamma = 0 \) then \( A^- = \frac{1}{\delta} \), \( B^- = \delta \) and \( \frac{1-\gamma}{1+\gamma} \cdot 1 \).

The Laplace inverse of (2.67) and (2.68) with "n+m" and "k" greater than 1 respectively do not introduce new time scales in the exponential terms, and the long time response still reduces to the case when the lower boundary is traction free. However, one obtains polynomials in "t" (whose degrees increase with "n+m" and "k") multiplying the exponential terms occurring in (2.69), (2.70) and (2.71).

Noting the following statements: i) The effective time scale of \( te^{-\lambda t} \) is longer than the effective time scale of \( t^n e^{-\lambda t} \) when \( m > n \), ii) the components of the product of the reflection matrices for "farther off" images contain polynomials in "t" with higher degrees multiplying the exponential terms, and iii) the relative contribution of "farther off" images increases as we get farther from the source location in the plate for fixed values of moduli, we can deduce that the effective relaxation rate decreases as we go farther away from the source region. This concept could be
associated with the stress diffusion process that is commonly referred to in geophysical applications related to earthquake phenomena.

As a numerical example of the above remarks, figures 2.21 and 2.22 show the surface displacement components \( u_x \) and \( u_z \) respectively for embedded nuclei of strain (or moment sources) located at \( x-y=0 \) and \( z=0.5H \) and having the normal vector \( \mathbf{n} = (0,1,0) \) and \( (1,0,0) \) respectively and Displacement Discontinuity (DD) components \( \Delta u = (1/\mu A) \cdot (1,0,0) \) and \( \Delta u = (1/\mu A) \cdot (0,0,1) \) respectively, where \( \mu \) is the shear modulus and "A" is an "effective area". The surface displacements are plotted for \( y-z=0 \) and \( 0 \leq x/H \leq 4.0 \); the surface displacements are antisymmetric with respect to reflection of the \( x \)-axis. The times \( t \) at which the curves are plotted are \( \omega t = 0.0, 0.5, 1.0, 2.0, 3.0, 4.0, 5.0, 10.0, 15.0 \) and \( 30.0 \) for figure 2.21 and \( \omega t = 0.0, 1.0, 3.0, 5.0, 7.0, 10.0, 12.5 \) and \( 30.0 \) for figure 2.22; for both figures, the level of displacement increases with time (at least) in the range of \( 0 \leq x/H \leq 1.0 \).

The "diffusion" aspect of the displacement fields can be recognized with the help of the following observations. In figure 2.21, consider two separate locations, one relatively "close" to the source (e.g. at \( x/H=0.5 \)) and one relatively "far" from the source (e.g. at \( x/H=4.0 \)). While the displacement seems to increase most at "early" times after which this increase "slows down" for the closer location, the displacement increases relatively "faster" at "later" times for the "farther" location. The above remarks can be interpreted as being due to a "disturbance" which reaches and "saturates" the displacement field closer to the source and "diffuses" to reach farther off points from the source at "later" times. The time increments at which curves are plotted in figure 2.21 are not equal; specifically, the last time step \( \omega t = 30.0 \) is
Mxy Moment Source

Figure 2.21
Figure 2.22
chosen so that no significant change in the displacement magnitudes (for all locations $0 \leq x/H \leq 4.0$) is noticed beyond that time step. The diffusion aspect in figure 2.22 is more readily apparent. A "disturbance" in the displacement magnitudes "seems" to be "penetrating" (or diffusing) farther from the source with increase in time. Again (for clarity of the figures), the time increments at which curves are plotted in figure 2.22 are not equal and $\omega t = 30.0$ is chosen so that no significant change in the displacement magnitudes (for all location $0 \leq x/H \leq 4.0$) is noticed beyond that time step.

In chapter 4.4, results from this section are again used to compare the time-dependent surface uplift due to a finite dislocated "patch" in an elastic plate underlain by a viscoelastic substrate, and will serve to clarify further details of the viscoelastic fields derived.
Chapter 3: Boundary element formulation for 3-D crack analysis using NOS as fundamental solutions

In this chapter, a boundary element formulation for 3-D cracks in plane layered (as well as infinite space) regions will be presented. The formulation relies on the use of Nuclei of Strain (NOS) Green's functions derived in chapter 2. As a numerical technique, the method could be classified as an indirect boundary element method, a surface integral scheme, a body force method or a superposition/collocation method, all of which are basically equivalent.

The chapter starts with a brief survey of specialized numerical techniques that have been used to analyze 3-D cracks in plane layered media. Next, a review of the representation theorem which serves as the theoretical basis of the method used will be presented, leading into the specific form of the formulation used, and the "new features" that have been implemented. The major difficulty in constructing boundary element schemes (or solving singular integral equations) is the method of treating the singular integrals; therefore, a section in this chapter will discuss how this aspect was handled, and will specify the merits and disadvantages of the approach used. Note that throughout this chapter the summation convention for repeated indices will be implicitly used (although an explicit summation symbol will sometimes be employed).

3.1 Introduction

Methods for the fully 3-D accurate stress analysis of irregularly shaped cracks in plane layered media has received considerable research effort recently, but has been limited to a single interface with the crack being planar (both intersecting and
nonintersecting the interface). Murakami and Nemat-Nasser (1982,1983) dealt with surface flaws using the "body force" method for cracks that are perpendicular to the surface, while Lee and Keer (1986) solved the problem of a 3-D crack terminating at a general interface (also using the body force method) for cracks that are perpendicular to the interface surface. Finally, Murakami (1985) analyzed the generally oriented (and generally loaded) crack in a (traction free at the surface) halfspace. In addition to the above mentioned numerical solutions for irregularly shaped, arbitrarily loaded cracks in (bonded or traction free) halfspaces, Arin and Erdogan (1971) presented the numerical solution for a penny-shaped crack in an elastic layer bonded to two halfspaces (the plane of the crack being parallel to the interface planes).

3.2 Representation Theorem

In this section a rederivation of the representation theorem for elastostatic media will be presented. The rederivation is done to introduce notation (that will be used in this thesis from hereon), to have easy referral to some relations that will be discussed and to present some comments that will be relevant in later sections.

Consider a (for now finite) three dimensional region \(V\) having a boundary \(S\) and inside which a surface \(S_c\) is defined (see figure 3.1). Over this region, two elastic fields (1) and (2) having the same linear stress-strain relations are specified:

\[
\begin{align*}
\mathbf{u}^{(1)}_i &= \sigma^{(1)}_{ij} = C_{ijkl} \mathbf{e}^{(1)}_{kl} \\
\mathbf{u}^{(2)}_i &= \sigma^{(2)}_{ij} = C_{ijkl} \mathbf{e}^{(2)}_{kl}
\end{align*}
\]
where \( u^{(1)}_{ij}, \epsilon^{(1)}_{ij}, \sigma^{(1)}_{ij}, u^{(2)}_{i}, \epsilon^{(2)}_{ij} \) and \( \sigma^{(2)}_{ij} \) are the displacement, strain and stress components of elastic fields (1) and (2) respectively, and \( C_{ijkl} \) is the linear "stiffness" elasticity tensor relating the stresses to the strains.

Now specialize the elastic fields (1) and (2) as follows. Let the first elastic field (1), correspond to a unit point force applied at point \( x \) acting in a direction "p". Denote by \( G^p_I(x,x') \), \( G^p_{ij}(x,x') \) the components of the displacement and stress fields respectively for elastic field (1) evaluated at point location "x". Let the second elastic field (2) correspond to any elastic field that allows some distribution of slippage and/or opening to occur over the surface \( S^c \) such that no net load is applied on \( S^c \) (Somigliana condition, e.g. see Maruyama 1964) and such that there are zero body forces being exerted in "v" (also drop the "(2)" superscript for elastic fields (2)). The Somigliana condition can be written as:

\[
\sigma_{ij}^+ \cdot \hat{n}_i^+ + \sigma_{ij}^- \cdot \hat{n}_i^- = 0 \tag{3.3}
\]

where the "\( \pm \)" superscripts on \( \sigma_{ij} \) (and all subsequent field variables) aim at differentiating between the values of \( \sigma_{ij} \) (and all subsequent field variables) when approaching the surface \( S^c \) from two different sides (see figure 3.2), and \( \hat{n}_i^\pm \) are the normal vectors to the surface \( S^c \) when approached from the "positive" and the "negative" sides shown in figure 3.2.

If we exclude an infinitesimal sphere \( V(x) \) surrounding the location of the point force, then we can form the following integral:
This is the most complete text of the thesis available. The following page(s) were not included in the copy of the thesis deposited in the Institute Archives by the author.

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\[ \int_{V-V'_{\varepsilon}} C_{ijkl} \varepsilon^{(1)}_{ij} \varepsilon^{(2)}_{kl} \, dV \]  

(3.4)

Using relation (3.1) and (3.2) alternatingly in the expression (3.4) we obtain the reciprocal relation:

\[ \int_{V-V'_{\varepsilon}} \sigma^{(1)}_{ij} \varepsilon^{(2)}_{ij} \, dV = \int_{V-V'_{\varepsilon}} \sigma^{(2)}_{ij} \varepsilon^{(1)}_{ij} \, dV \]  

(3.5)

Using the definition for the strain ("infinitesimal strain"), and the symmetry of the stress tensor (and the specialized notation discussed above for fields (1) and (2)) we can obtain from (3.5):

\[ \int_{V-V'_{\varepsilon}} \left[ (G^p_{ij} \cdot u_1)_{,i} + G^p_{ij, i} \cdot u_1 \right] \, dV = \int_{V-V'_{\varepsilon}} \left[ \left( \sigma_{ij} \cdot G^p_{ij} \right)_{,i} + \sigma_{ij, i} \cdot G^p_{ij} \right] \, dV \]  

(3.6)

The conditions imposed on the elastic fields (1) and (2) imply that \( \sigma_{ij, i} = G^p_{ij, i} = 0 \) inside "V-V'_{\varepsilon}". Therefore, using the divergence theorem, the symmetry of the stress tensor and equation (3.6) we can obtain:

\[ \int_{S} G^p_{ij} \cdot n_j \cdot u_1 \cdot dS + \int_{S_{\varepsilon}} G^p_{ij} \cdot n_j \cdot u_1 \cdot dS + \int_{S_{-}} G^p_{ij} \cdot n_j \cdot u_1 \cdot dS \]

\[ + \int_{S^+_{c}} G^p_{ij} \cdot n_j \cdot u_1^+ \cdot dS = \int_{S} \sigma_{ij} \cdot n_j \cdot G^p_{ij} \cdot dS + \int_{S_{\varepsilon}} \sigma_{ij} \cdot n_j \cdot G^p_{ij} \cdot dS \]
\[ + \int_{S_c^-} \sigma_{j1}^- \cdot n_j^- \cdot G_{j1}^p^- \cdot dS + \int_{S_c^+} \sigma_{j1}^+ \cdot n_j^+ \cdot G_{j1}^p^+ \cdot dS \] (3.7)

As the radius of the sphere of region \( V_\epsilon \) tend to zero we obtain the following results:

\[ \int_{S_\epsilon} G_{j1}^p(x,x') \cdot n_j(x') \cdot u_1(x') \cdot dS = u_1(x) \cdot \int_{S_\epsilon} G_{j1}^p(x,x') \cdot n_j(x') \cdot dS \]

\[ = u_1(x) \cdot \delta_{1p} - u_p(x) \] (3.8)

\[ \int_{S_\epsilon} \sigma_{j1}(x') \cdot n_j(x') \cdot G_{j1}^p(x,x') \cdot dS = \sigma_{j1}(x) \cdot \int_{S_\epsilon} n_j(x') \cdot G_{j1}^p(x,x') \cdot dS \]

\[ \rightarrow \sigma_{j1}(x) \cdot (0) \rightarrow 0 \] (3.9)

To obtain the results in (3.8) and (3.9), the explicit expressions for the Green's functions must be written and the integrals must be evaluated (as a function of \( \epsilon \)), after which the limiting operation \( \epsilon \rightarrow 0 \) is performed.

In addition, from the choice of elastic fields (1) and (2) we have:

\[ G_{j1}^p^+ \cdot n_j^+ + G_{j1}^p^- \cdot n_j^- = 0 \quad \text{on } S_c \] (3.10)

\[ u_1 = \begin{cases} u_1^* + \Delta_{1j}^+/2 & \text{on } S_c^+ \\ u_1^* - \Delta_{1j}^+/2 & \text{on } S_c^- \end{cases} \] (3.11)

\[ G_{j1}^p^+ = G_{j1}^p^- \quad \text{on } S_c \] (3.12)

In (3.11), \( \Delta_{1j}^+ \) is the displacement discontinuity vector.
across the surface $S_c$, and $u_\perp^*$ is the average displacement vector (of the two faces of $S_c$) along $S_c$. Defining $T_\perp$ to be a traction component (i.e. $T_\perp = \sigma_\perp \cdot n_\perp$), and using relations (3.3) and (3.8-12) in (3.7) we obtain the "representation theorem":

$$u_p(x) = \int_S G_{j\perp}^p(x,x') \cdot n_j(x') \cdot u_\perp(x') \cdot ds$$

$$+ \int_S G_{\perp}^p(x,x') \cdot T_\perp(x') \cdot ds - \int_{S_c} G_{j\perp}^p(x,x') \cdot n_j^+(x') \cdot \Delta_\perp(x') \cdot ds$$

(3.13)

In order to handle "infinite" boundaries, the integrands that are being integrated over the "infinite" surfaces have to go to zero "reasonably fast". For our purposes, if all "elastic disturbances" are occurring due to slippage on a finite surface $S_c$, or at finite locations on the boundaries $S$ then equation (3.13) also handles the "infinite boundaries" case.

Two remarks will now be mentioned. First, if the region of interest is a composite region made of two or more linear elastic regions with common boundaries, then equation (3.13) would still be obtained assuming that relations (3.1) and (3.2) hold for both the "point force" solution (elastic field (1)) as well as the elastic field of interest (elastic field (2)); the surface $S$ is then interpreted to be the outermost boundaries of the composite region. For example, for the case of an elastic plate perfectly bonded to two elastic halfspaces, the outermost boundaries are all at "infinity" (and all integrals over "−" vanish) if we use the Green's function for the composite region of an elastic plate perfectly bonded to two halfspaces. A rephrasing of the above remark is that when a Green's function for a composite region is
employed, then all "internal" boundaries are "transparent" and should not be included in the representation relation (3.13).

The second remark is that since relation (3.13) is a representation for a linear elastic field, we can superpose whatever additional fields (that do not necessarily "go to zero reasonably fast") we require to define a given problem. For example, when studying mode I cracks in infinite space, the elastic fields for the unknown opening distribution on the crack surface can be superposed with known far-field tensile elastic fields, in order to load the crack with other than local pressure.

Therefore, for the case when the Green's function for the composite plate region is available and additional far-field elastic fields (due to any elastic disturbances not produced by the displacement discontinuity distribution on the crack surface) are superposed, the specialized representation relation reduces to:

\[
\mathbf{u}_p(x) = \mathbf{u}_p^{\text{far-field}}(x) - \int_{S_c} \mathbf{G}^{p}_{ij}(x,x') \cdot \mathbf{n}_j^+(x') \cdot \Delta_i(x') \cdot dS
\]  

\[
(3.14)
\]

From relation (3.14), a representation relation for the stresses can be obtained (by combining the gradients of terms in equation (3.14)). The representation relation for the stresses can be written as:

\[
\sigma_{pq}(x) = \sigma_{pq}^{\text{far-field}}(x) - \int_{S_c} \mathbf{G}^{pq}_{ij}(x,x') \cdot \mathbf{n}_j^+(x') \cdot \Delta_i(x') \cdot dS
\]  

\[
(3.15)
\]
where:

\[
G_{ij}^{pq}(x,x') = 2\mu \left[ \frac{\partial}{\partial x_p} G_{ij}^p(x,x') + \frac{\partial}{\partial x_q} G_{ij}^q(x,x') \right] \\
+ \lambda \cdot \delta_{pq} \sum_{k=1}^{3} \frac{\partial}{\partial x_k} G_{ij}^k(x,x')
\]

(3.16)

Note that the differentiation operation (3.16) can be performed before the integration in (3.15) only when the integrand is continuous (i.e., x not on \( S_c \), see section 3.5 for more details and remarks). The functions "\( G_{ij}^p(x,x') \)" and "\( G_{ij}^{pq}(x,x') \)" could be interpreted as the displacement and stress fields at point "\( x \)" due to Nuclei Of Strain (NOS) located at "\( x' \)" (refer to section 2.5).

Finally, the representation relations (3.14) and (3.15) apply to cases when the point "\( x \)" is totally within a region "\( V \)" (this includes locations on any "internal boundaries") or when "\( x \)" is tending to (but not being on) the surface "\( S_c \). The distinction between a point "tending to" and "being on" is important, because the integrals involved might (and do) exist for the first case but not for the latter case (this can be demonstrated both analytically as well as numerically).

3.3 Boundary Element Formulation

In this section the use of relations (3.14) and (3.15) in a "boundary element formulation" for the analysis of 3-D cracks in a 3-D layered region will be presented. Consider a layered region (shown in figure 3.3) having a surface "\( S_c \)" across which displacement can be discontinuous. Assume that the following boundary conditions (with respect to some fixed or local coordinate system) are imposed:
\[ Q^c(x) \cdot T(x) + Q^b(x) \cdot \Delta(x) = R(x) \quad \text{on } S_c \quad (3.17) \]

where \( Q^c(x) \) and \( Q^b(x) \) are given vector functions and \( R(x) \) is a given scalar function of position on the surface "\( S_c \)". and \( T(x) \) and \( \Delta(x) \) are the tractions and the displacement discontinuity (slip and/or opening) on the surface "\( S_c \)" respectively. Note that for a general problem where "displacement discontinuities" can occur in any of three directions (i.e. slip in two directions as well as opening), there would be three sets of boundary conditions imposed in the form of (3.17) at every point on "\( S_c \)". Using the above general form (3.17) for boundary condition specification, a traction, slip/opening, "friction" as well as "spring" boundary conditions can be specified on "\( S_c \)". Casting equation (3.17) in a vectorial form allows one to refer the boundary conditions to any specific local or global coordinate system in a systematic and clear manner (e.g. see appendix N).

The next step in this formulation is to decide on the manner of discretizing the "unknown" slip/opening quantities. That is on the assumptions on how the slip and opening distributions vary over the surface "\( S_c \)". This step is rather important. The assumptions concerning slip/opening distributions are especially important to the accuracy of the solution for the "crack front" part of "\( S_c \)". For example, for "sharp" embedded (not intersecting any boundaries) cracks, the asymptotic slip/opening in a direction normal to the crack front (lying in the crack plane) is known to vary as \( r^{1/2} \) (where "\( r \)" is measured from the crack front); the numerical solution exhibits markedly improved accuracy when this theoretical result is accommodated in the slip opening distribution assumptions.

There usually is a compromise to be made between the generality of assumptions made concerning the displacement
discontinuity (DD) fields and the ease of implementation. This is because an arbitrary DD field involves the evaluation of integrals of singular Green's functions multiplied by arbitrary "shape functions" describing the DD field distribution. Such singular integrals are significantly easier to obtain when the "shape functions" are not arbitrary. This is because specialized analytic/numerical treatments could then be performed more easily.

In the literature, there are two ways of assuming the variation of slip/opening over the surface (or surfaces) $S_c$. The first is to assume one continuous function (or sum of functions) over the whole surface (or each of the surfaces) $S_c$; this approach will be referred to as a global interpolation approach. The second approach is to divide the surface (or surfaces) $S_c$ into "elements" (which in this formulation are chosen to be straight-edged planar triangles, see figure 3.4) and to assume a different function (or sum of functions) over each "element"; this approach will be referred to as a local interpolation approach.

In 2-D problems, Erdogan and Gupta (1972), Krenk (1975-6) and Narendran and Cleary (1984) used the global interpolation approach, while Crouch (1976), Crawford and Curran (1982), Gerasoulis (1982) and Fares and Li (1986) used the local interpolation approach. In 3-D problems, Murakami and Nemat-Nasser (1982, 1983), Murakami (1985), and Lee and Keer (1986) used a mixture of a global and local interpolation approach; the variation of slip/opening was assumed to be a product of a globally defined function (involving a variable measuring the closest distance to the crack front) multiplied by piecewise constant triangular "patches" or "elements".

As alluded to above, the choice of slip/opening distribution influences the choice of method used to integrate the singular
integrals involved. For example, in 2-D problems with the global interpolation approach, specific choices of assumed functional distribution for the slip/opening leads to an efficient integration scheme (e.g. see Erdogan and Gupta (1972)). Note that the local interpolation approach is more general than the global approach if the "shape functions" can be arbitrarily chosen; this is because a "global shape function" could then be accommodated using the local interpolation approach to correspond to the global interpolation approach's assumed slip/opening distribution.

The present formulation employs the local interpolation approach, with the slip/opening distribution allowed to be arbitrarily chosen. The surface $S_c$ is first divided into non-overlapping triangles that completely cover the surface $S_c$ except for a small mismatch between the smooth continuous boundary of $S_c$ and the straight edges of the triangles used (see figure 3.4). Local coordinate axis are also associated with each triangle. "Shape functions" (any number) are then associated with each triangle (and with respect to the local coordinate system) over which slip/opening is allowed to occur. These "shape functions" could be defined as follows:

$$
P(x) = \begin{cases} 
0 & \text{if the shape function is not associated with the triangle under consideration} \\
 f(x^L) & \text{if the shape function is associated with the triangle under consideration} 
\end{cases}
$$

(3.18)

where $P(x)$ is the "shape function" defined over all $S_c$ and $f(x^L)$ is a functional distribution referred to a local coordinate system (with coordinates denoted by $x^L$).

The next step in the formulation is to decide upon the number
Figure 3.4
of degrees of freedom used and how they are spatially allocated. In contrast to common practice in the literature, the geometry descretization (the triangles defining the crack surface), the displacement discontinuity (DD) "shape functions" and the unknown parameters that determine the DD magnitudes (i.e. the "degrees of freedom") are not "rigidly" linked in the present formulation; specifically there may be more than one "shape function" (not ascribed to the same triangle) associated with (or "controlled by") the same degree of freedom. This feature allows more flexibility in specifying symmetry conditions and kinematic constraints. For example, in order to impose radial symmetry for a penny-shaped crack under tensile loading (see figure 3.5) all DD "shape functions" for triangles that lie in concentric regions surrounding the center of that penny-shaped crack would be controlled by the same set of parameters (note however that the "shape functions" would be referred to the local coordinate system for each triangle). Formally, the slip variation along any point on the surface "S_c" is specified by:

$$\Delta_j(x) = \sum_{\ell=1}^{m} \left[ \begin{array}{c} n(x) \\ k(x) \end{array} \right] \cdot b^\ell \right]$$

(3.19)

where $b^\ell$ "denotes" a "degree of freedom" (DOF), $m$ denotes the total number of DOF for a given problem, and $n$ denotes the total number of "shape functions" associated with a given DOF. Note that "$n$" can vary for each separate degree of freedom, and that any given DOF can have either a known or unknown magnitude.

Once the DD distribution has been specified, the stresses (and hence tractions) at any point on the surface "S_c" can be obtained. Using the notation above, the tractions on the surface "S_c" can be written as (equation 3.15):
\[ T_j(x) = \sum_{\ell=1}^{m} \left[ \left( \sum_{k=1}^{n^{(\ell)}} \left[ \int_{S_c}^{\infty} p^{k\ell}_{q}(x',G^{ij}_{pq}(x,x')\cdot n_p(x')\cdot dS(x') \right] \cdot n_1(x) \right) \cdot b^{\ell} \right] + \sum_{\ell=m+1}^{s+m+1} \left[ \int_{S_c}^{\infty} \sigma_{ij}^{\text{far-field}}(x) \cdot n_1(x) \right) \cdot b^{\ell} \right] \] (3.20)

where the summation notation has been used for the indices "p", "q", and "j", and "s" is the total number of "far-field" stresses defined for the given problems. The equations (3.19) and (3.20) can be rewritten as:

\[ \Delta(x) = \sum_{\ell=1}^{m} M_{2}^{b}(x) \cdot b^{\ell} \] (3.21)

\[ \mathcal{I}(x) = \sum_{\ell=1}^{m} M_{2}^{t}(x) \cdot b^{\ell} \] (3.22)

and:

\[ \left[ M_{2}^{b}(x) \right]_{j} = \sum_{k=1}^{n^{(\ell)}} p^{k\ell}_{j}(x) \] (3.23)

\[ \left[ M_{2}^{t}(x) \right]_{j} = \sum_{k=1}^{n^{(\ell)}} \left[ \int_{S_c}^{\infty} p^{k\ell}_{q}(x',G^{ij}_{pq}(x,x')\cdot n_p(x')\cdot dS(x') \right] \cdot n_1(x) \] if \( 1 \leq \ell \leq m \)

and

\[ \left[ \sigma_{ij}^{\text{far-field}}(x) \right]^{\ell} \cdot n_1(x) \] if \( m < \ell \leq m + s + 1 \) (3.24)
Note that \( \mathbf{M}_x^b(x) \) and \( \mathbf{M}_x^t(x) \) represent true (3-D) vector fields defined over "\( S_c \)".

Now that the DD and tractions on "\( S_c \)" have been related to the "degrees of freedom" (DOF) for a given problem, we still have to set up an algorithm whereby those DOF "\( b^f \)" with unknown magnitudes can be determined. Usually, the DOF associated with the DD variation along "\( S_c \)" are unknown, whereas the DOF associated with the "far-field" stresses are known. The algorithm chosen for this formulation is to simply satisfy boundary conditions along the surface "\( S_c \)" at as many points as as there are unknown DOF; this is usually referred to as a collocation procedure. Collocating at the required number of points, we can write sets of equations denoted by:

\[
\mathbf{M} \cdot \mathbf{b} = \mathbf{R} \quad (3.25)
\]

where:
\[
\mathbf{b}_i = b_i^f \quad (i.e. \ a \ list \ of \ all \ the \ DOF) \quad (3.26)
\]
\[
\mathbf{R}_i = R(x_i) \quad (refer \ to \ equation \ (3.17)) \quad (3.27)
\]
\[
\mathbf{M}_{ij} = \mathbf{C}^t(x_i) \cdot \mathbf{M}_j^t(x_i) + \mathbf{C}^b(x_i) \cdot \mathbf{M}_j^b(x_i) \quad (3.28)
\]

Now the system of equations (3.25) could be repartitioned so that all the known DOF are placed in \( \mathbf{b}^{\text{known}} \) and all the unknown DOF are placed in \( \mathbf{b}^{\text{unknown}} \), and the columns of \( \mathbf{M} \) rearranged into \( \mathbf{M}^{\text{part 1}} \) and \( \mathbf{M}^{\text{part 2}} \) containing the coefficients (from (3.28)) multiplying the \( \mathbf{b}^{\text{known}} \) and \( \mathbf{b}^{\text{unknown}} \) respectively. Doing that we obtain:

\[
\mathbf{M}^{\text{part 1}} \cdot \mathbf{b}^{\text{unknown}} = \mathbf{R} - \mathbf{M}^{\text{part 2}} \cdot \mathbf{b}^{\text{known}} \quad (3.29)
\]
Relation (3.29) represents a system of simultaneous equations which can be inverted to obtain the unknown DOF. Once the magnitude of all unknown DOF are determined, all elastic fields can be obtained everywhere on "$S_c$" and in "$\Gamma$".

3.4 Integration Schemes

The BEM formulation presented in the previous section requires the evaluation of integrals of the form given in equation (3.24). The "shape functions" $p_{pq}^{k1}(x')$ and the normal vector $\mathbf{n}(x')$ are assumed to be smooth functions over the surface "$S_c$". However, the Green's function $G_{pq}^{ij}(x, x')$ is singular. In fact, when the point $x$ lies on "$S_c$", the integrals as written in (3.24) should be interpreted as:

$$
\lim_{\epsilon \to 0} \left[ \int_{S_c} p_{pq}^{k1}(x') \cdot G_{pq}^{ij}(x + \epsilon \cdot \hat{n}(x), x') \cdot \mathbf{n}(x') \cdot ds(x') \right] \quad (3.30)
$$

In the above integral, the order of doing the operations "taking the limit as $\epsilon \to 0$" and "integrating over the surface $S_c$" cannot be interchanged. For a general "shape function", interchanging the above-mentioned operations causes the integral not to exist (the value of the integral becomes unbounded). It is also worth noting that the above integral (3.30) cannot be broken down into the evaluation of a certain function of "$x$" added to a "principal value" integral (i.e. an integral over "$S_c$" excluding an infinitesimal neighborhood around "$x$"), although this procedure can be done for the integrals which give the displacement fields. The previous remark will be elucidated when considering a specific example to be treated in this section. It is worth noting at this point that the problems associated with evaluating the severely singular integrals that determine the tractions over the crack
surface can be somewhat alleviated so that one would require to evaluate Principal Value (PV) integrals instead. Weaver (1977) and Budiansky and Rice (1979) use integration by parts in order to obtain expressions for the tractions involving PV integrals only. Bui (1977) achieves the same objective by using potentials as unknown functions to be determined over the surface of the crack. Bui evaluates the PV integrals numerically by employing a symmetric numerical integration scheme centered around the singular point. The integration techniques to be discussed next are not similar to those employed by Bui (1977) since the singular integrals involved are not of the PV type.

In the literature, the most common way of implementing 2-D and 3-D BEM formulations (not necessarily crack problems) is by "fixing" the choice of the "shape functions" (e.g. constant, linear and parabolic variation), and then performing the integrations analytically over an element. The analytic integrations turn out to be as what would formally be obtained by first setting $\varepsilon$ (in (3.30)) to zero, doing the symbolic integrations and evaluating the functions obtained at the required limits. The above "analytic" procedure, though not rigorous, effectively removes that part of the integrals that becomes unbounded.

General description for the integration scheme

The integration of (3.30) is actually done across $S_c$ one planar, straight-edged triangle at a time and therefore the discussion will center about integrating the integrand of (3.30) over a planar, straight-edged triangle. The Green's functions for the layered region is divided into the sum of the infinite space Green's function (which contains all the singular terms when $x$ lies on the triangle) and the "inhomogeneous" part (which contains nonsingular but possibly highly varying terms; for an exception to
this statement see the last paragraph of the last part of this section). The integration procedure is divided into two types. The first type of procedure is used when integrating the "inhomogeneous" part of the Green's function (multiplied by the other terms in (3.30)) or when "x" does not lie on the triangle; in both these cases, the integrands are non-singular but possibly highly varying; the integrand becomes more and more "highly varying" as the singularity (source or image sources) approach being on the triangle. The second type of integration procedure is used when integrating the "infinite" space singular Green's function multiplied by the "other terms" in (3.30) when the point x lies on the triangle to be integrated over. Each type of procedure will next be described separately.

Integration for non-singular but highly varying integrands

In order to discuss the integration procedure in an easier manner, assume we have a triangle with vertices "A", "B" and "C" in the plane z=0 (x,y,z represent a locally chosen coordinate system) over which we have to integrate:

\[
\iint_{\text{triangle}} P(x,y) \cdot G(x,y;x_o,y_o,z_o) \cdot dx dy
\]

(3.31)

where \( P(x,y) \) is a "shape function" and \( G(x,y;x_o,y_o,z_o) \) is a "typical" component of a Green's function having the "source" of singularity located outside the triangle at a point \( (x_o,y_o,z_o) \).

The "type 1" integration procedure is taken from the literature (Lee and Keer (1986), Murakami and Nemat-Nasser (1983)) and is described in the following steps:

1. Subdivide triangle "ABC" into 4 equal area subtriangles (by joining the midpoints of the sides) if the following condition
is violated:

\[
\text{distance from } x_o \text{ to center of gravity of triangle "ABC" } > \alpha \sqrt{\text{Area of triangle "ABC"}} \tag{3.32}
\]

where "\(\alpha\)" is a chosen real number which affects the subdivision process.

2. If triangle subdivision has been performed, repeat step 1 for each of the subtriangles produced. Repeat step 2 until condition (3.32) is satisfied for all subtriangles.

3. Use Gaussian Quadrature to integrate the integrand of (3.31) over all subtriangles produced and sum up the values. The sum is the value of the required integral in (3.31).

There are many reasons why the above integration procedure was determined to be suitable to the type of integrand being considered. The main reasons are that the method increases the integration effort as the source gets closer to the triangle (from a low level effort for far-away sources), and that the method concentrates the integration effort at those regions having the highest variation in the integrand where they are most needed (i.e. closest to the source). For example, figures 3.6-3.14 show the subdivision of a triangle with \(\alpha = 1\), 2 and 3 and for 3 different locations of the source point. Vertices "A", "B" and "C" are located at \((x,y) = (-0.5,0), (0.5,0)\) and \((0,\sqrt{3}/2)\) respectively and source points \((x_o,y_o,z_o) = (0,0.01\cdot\sqrt{\text{Area}},0), (0.5 + 0.01\cdot\sqrt{\text{Area}},0)\) and \((0,\sqrt{3}/6,0.01\cdot\sqrt{\text{Area}})\) for the sets of figures 3.6-3.8, 3.9-3.11 and 3.12-3.14 respectively. The total number of triangles "created" for \(\alpha = 1\) shown in figures 3.6, 3.9 and 3.12 are 37, 13 and 67 respectively, for \(\alpha = 2\) shown in figures 3.7, 3.10 and 3.13 are 100, 34 and 241 respectively and for \(\alpha = 3\) shown in figures 3.8, 3.11 and 3.14 are 217, 82 and 469 respectively. Another nice feature of the integration procedure (and specifically the
Figure 3.7
Figure 3.11
Integration Points vs Alpha

Number of Points vs $\alpha$

- $x = y = 0.6$
- $x = y = 1.0$

Figure 3.15
subdivision process) is that it is scale invariant; that is it all coordinates (i.e. triangle vertices as well as the point $X_0$) are scaled with the same (real) number, the integration effort (and subdivision process) stays the same.

However, there are some problems with the subdivision condition (3.32). First, the condition is "discrete"; that is there is a very sudden transition between when a triangle will be subdivided and when it will not. This feature is reflected in the accuracy of the integration which sometimes suddenly jumps (e.g. as depicted by the logarithm of the percentage error, see appendix S)) from one level to another for small changes in source point location or in the subdivision parameter "$\alpha". For example, fig 3.15 shows the number of integration points required for a triangle "ABC" with vertices at (0,0), (1,0) and (0,1) respectively and two source locations (these are at $x-y=0.6$ and $x-y=1.0$) as a function of the subdivision parameter "$\alpha". Note that the "discreteness" sensitivity decreases as "$\alpha" increases.

The second problem with the subdivision process is that the subdivision process is sensitive to triangle shape and might not adequately subdivide a triangle that is overly "elongated". This point can best be demonstrated by showing three concentric circular contours around several triangles' center of gravity (figures 3.16-3.19); a source point inside the circles labelled "1", "2" and "3" corresponds to condition (3.32) being violated for $\alpha = 1$, 2 and 3 respectively. Notice that as the triangle becomes more and more elongated, a source point could approach the given triangle from certain directions without the given triangle being subdivided even as the subdivision parameter "$\alpha" increases. Such an occurrence imply that "overly elongated" triangles would have "regions of innaccuracy" for a fixed value of the subdivision parameter "$\alpha". 
When using this "type 1" integration procedure there are two parameters that have to be selected. These are the Gaussian integration order and subdivision parameter "\( \alpha \)". The actual values of the integration order and "\( \alpha \)" would have to be "calibrated" to the specific source singularity and the required accuracy. Appendix "R" describes a model for estimating integration effort required given a subdivision parameter "\( \alpha \)" (with empirical confirmation of the model); the main result is that integration effort (or number of triangles "created") is proportional to "\( \alpha^2 \)" and a (non-isotropic) logarithmic function of "closeness of source point". Appendix "S" performs empirical accuracy studies of the "type 1" integration procedure for several severities of source singularity; from those accuracy studies, an integration order of 4 and a subdivision parameter of 3-4 was found to be "adequate" (usually less than 0.001% errors) for all type of the integrands considered.

Integration for "singular" integrands

In this section the integration of the integrand in (3.15) will be discussed when the point \( \mathbf{x} \) lies on "\( S_c \)". Recall that (3.15) was obtained by formally combining gradients of terms with respect to \( \mathbf{x} \) on both sides of (3.14) and interchanging the required sequence of "differentiation" and "integration" of the operations on the left-hand side of (3.14). This interchange is allowed only when the integrand is continuous, which is the case when \( \mathbf{x} \) is not on "\( S_c \)". Since \( u_p(\mathbf{x}) \) and \( \sigma_{pq}(\mathbf{x}) \) are assumed to be continuous on "\( V-S_c \)", then the limit of the integral in (3.15) as \( \mathbf{x} \) approaches being on "\( S_c \)" exists. As mentioned previously, taking the limit of the integrand for \( \mathbf{x} \) being on the surface "\( S_c \)" and then integrating leads to divergent results (the integral is unbounded).

The discussion to follow will aim at achieving two main
objectives. The first aim is to understand the limit-integration process, and the second aim is to describe a numerical procedure to integrate expressions of the form:

\[
\lim_{z \to 0} \iint_{\text{planar surface } S_c} P(x,y) \cdot G(x,y;x_o,y_o,z_o) \cdot dx \, dy
\]  \hspace{1cm} (3.33)

where the point \((x_o,y_o,z=0)\) lies on \(S_c\) and \(G(x,y;x_o,y_o,z_o)\) is a representative Green's function that occurs in (3.15). Here as in (3.31) the coordinate system (represented by \(x,y,z\)) is a locally chosen one.

In order to achieve the objectives of this section, an example which simplifies the discussion without losing the essential features of the required operations will be treated. Let \(S_c\) include the point \((x,y) = (0,0)\) and \((x_o,y_o) = (0,0)\); Now assume that we are required to obtain the traction \(T_z(x=0,y=0,z=0)\) due to a uniform opening discontinuity (of magnitude one) over \(S_c\). The integral that would have to be performed (aside from a multiplication by a constant) is:

\[
I = \lim_{z \to 0} \iint \left[ \frac{1}{r^3} - 3 \cdot \frac{z^2}{r^5} \cdot \frac{1}{r} \right] \cdot dx \, dy \hspace{1cm} (3.34)
\]

where: \(r^2 = x^2 + y^2 + z^2\) \hspace{1cm} (3.35)

(3.34) will first be performed over a specialized surface \(S_c\), and a discussion of integrating over a general (planar) \(S_c\) will be handled afterwards.

Choose \(S_c\) to be a circle in the \(z=0\) plane with center at the
origin and radius \( r \). Let \( R \), \( \theta \) and \( z \) be the cylindrical coordinates and transform (3.34) to this coordinate system (see figure 3.20); we obtain:

\[
I_1 = \lim_{z \to 0} \int_0^a \int_0^{2\pi} \frac{dR \cdot d\theta}{R^2 + (z + z)^2} \left[ \frac{1}{(R^2 + z^2)^{3/2}} - 3 \cdot \frac{z}{(R^2 + z^2)^{5/2}} \right]
\]

\[(3.36)\]

Integrating (3.36) we obtain:

\[
I_1 = \lim_{z \to 0} \left[ -2\pi \cdot \frac{(a^2 + z^2)^{1/2}}{2} + 2 \cdot \frac{z}{(a^2 + z^2)^{3/2}} + 2 \cdot \frac{1}{z} - 2 \cdot \frac{1}{z} \right]
\]

\[(3.37)\]

where the underlined terms are due to the first term in the integrand in (3.36) (i.e. due to the \( 1/r^3 \) term). Therefore, (3.36) gives the final value of:

\[
I_1 = -\frac{2\pi \cdot a}{a}
\]

\[(3.38)\]

There are several interesting remarks that will be made and explained at this point. First notice that the value of \( I_1 \) is negative (and decreasing in absolute value as \( a \) increases), while the first term in the integrand in (3.36) is always positive. This means that the contribution of the term \( 3 \cdot \frac{z^2}{r^5} \) in the integrand in (3.36) has (in absolute value) an overall larger contribution than the term \( 1/r^3 \) which only "catches up" in the limit as \( a \to \infty \); this is the case even though as \( z \to 0 \), the contribution of \( 3 \cdot \frac{z^2}{r^5} \) must come from an infinitesimally small neighborhood of the origin.
Figure 3.20
This situation also prevents us from being able to write (3.36) in
the form of a principal value integral of the limit of the
integrand in (3.36) as \( z \to 0 \) added to some function of \( x(\vec{Q}) \). That
is of the form:

\[
\lim_{z \to 0} \int_{\frac{a}{z}}^{2\pi} d\theta \int_{0}^{R} \frac{1}{R^3} + f(Q)
\]

A form like (3.39) cannot be obtained because the integral in
(3.39) is unbounded (the second term in the integrand of (3.36) is
an essential "feature"). The integration of (3.34) can be done
numerically (however at "great expense", see appendix S) whereas
neglecting the second term in (3.34) makes the integrations
"undefined" (both numerically and analytically). Now that some
understanding of the nature of the singular integral involved in
(3.34) has been gained, the method by which the integral (3.34)
(over a more general region) is actually handled in the
implementation of the numerical scheme used will next be explained.

For a simply connected, convex and planar \( S_c \) (surrounding
the origin) subdivide the boundary \( S_c \) into two parts (as shown in
figure (3.21)) such that each of the two parts have as their
endpoints the coordinate points on the boundary of \( S_c \) having
minimum \( y_{\text{min}} \) and maximum \( y_{\text{max}} \) values of \( y \); \( c^+ \) intersects
the positive x-axis and \( c^- \) intersects the negative x-axis. The
surface integral (3.34) will now be converted to a line integral
(this procedure, although specialized is in the spirit of the work
done by Maruyama (1964)). Using Green's theorem (for converting
area integrals to line integrals in the plane), we can obtain:
Figure 3.21
\[ I = \lim_{z \to 0} \int_{C^+} f(x, y, z) \cdot dy + \lim_{z \to 0} \int_{C^-} f(x, y, z) \cdot dy \quad (3.40) \]

where:

\[ f(x, y, z) = \int dx \cdot \left[ \frac{1}{r^3} - 3z \cdot \frac{1}{r^5} \right] \quad (3.41) \]

\[ \Rightarrow f(x, y, z) = \frac{x}{(y^2 + z^2) \cdot r} - \frac{z \cdot x \cdot (2x^2 + 3y^2 + 3z^2)}{(y^2 + z^2)^2 \cdot r^3} \quad (3.42) \]

In order to evaluate "I" in (3.40) add and subtract the function \( g(x, y, z) \) defined as:

\[ g(x, y, z) = \frac{1}{y^2 + z^2} - 2z^2 \cdot \frac{1}{(y^2 + z^2)^2} - 3z^2 \cdot \frac{y^2}{x^2 \cdot (y^2 + z^2)^2} \quad (3.43) \]

from the function \( f(x, y, z) \) and regroup terms in the following way:

\[ f(x, y, z) = \frac{x \pm r}{(y^2 + z^2) \cdot r} \cdot \frac{2z^2 \cdot (x^2 \pm r^3)}{(y^2 + z^2)^3 \cdot r^3} - \frac{3z^2 \cdot (x \cdot (y^2 + z^2) \pm r \cdot y^2)}{x^2 \cdot (y^2 + z^2)^2 \cdot r^3} \]

\[ \pm \frac{3z^2 y^2}{x^2 \cdot (y^2 + z^2)^2} \pm \frac{2z^2}{(y^2 + z^2)^2} \pm \frac{1}{y^2 + z^2} \quad (3.44) \]

where the lower sign is taken on "C^+" and the upper sign is taken on "C^-". Noting that when the intercept of "C^+" with the x-axis is not through the origin, then near \( y=0 \) we have that

\[ x \pm r = O(y^2) \] for "C^+". This means that the first term in (3.44) is
regular (or has a removable singularity) as \( z \to 0 \), hence it is allowable to interchange the "limit" and "integration" operations for the first term in (3.44) and hence the first term does not present any difficulties of being integrated (either numerically or analytically). Using asymptotic analysis (and after some algebra) the limit of the integrals as \( z \to 0 \) of the second, third and fourth terms in (3.44) over "\( C^\pm \)" are of the order \( O(z) \), \( O(z^2) \) and \( O(z) \) respectively and hence "go to zero". What remains is to integrate the last two terms in (3.44) over "\( C^\pm \)", and these "happen to be" independent of "\( x \)"; hence we have definite integrals of the form:

\[
\lim_{z \to 0} \int_{y_{\min}}^{y_{\max}} \left[ \frac{2z^2}{(y^2 + z^2)^2} - \frac{1}{y^2 + z^2} \right] \cdot dy = \lim_{z \to 0} \left[ \frac{y_{\max}}{y_{\max}^2 + z^2} - \frac{y_{\min}}{y_{\min}^2 + z^2} \right] = \frac{1}{y_{\max}} - \frac{1}{y_{\min}} \tag{3.45}
\]

Therefore, the integral "I" in (3.40) reduces to:

\[
I = \int_{C^+} \frac{x-R}{y^2 R} \cdot dy + \int_{C^-} \frac{x+R}{y^2 R} \cdot dy - 2 \cdot \left( \frac{1}{y_{\max}} - \frac{1}{y_{\min}} \right) \tag{3.46}
\]

where: \( R^2 = x^2 + y^2 \) \( \tag{3.47} \)

The result (3.46) can be found to reduce to the correct value (given by (3.38)) when "\( S_c \)" is a circle with radius "\( a \)" and center at the origin. Note that there is a "removable" singularity at \( y = 0 \) in the integrands in (3.46), specifically the integrands in (3.46) over "\( C^\pm \)" have to be interpreted as:
\[
\frac{x+R}{y^2R} = \begin{cases} 
\frac{x+R}{y^2R} & \text{when } y \neq 0 \\
-\frac{1}{2x^2} & \text{when } y = 0
\end{cases} \tag{3.48}
\]

The conversion of the surface integrals of the form (3.33) for the "infinite" space Green's functions multiplied by constant, linear and parabolic "shape functions" to the "line integral" form are given in appendix Q. Once the integration of (3.15) for constant, linear and parabolic "shape functions" can be accurately obtained, the integration of (3.15) for an arbitrary but smooth "shape function" can also be obtained using the "type 2" procedure to be described next.

Consider an integral of the form given in (3.33) with \( S_c \) being a planar triangle with straight edges. To integrate (3.33) perform the following steps:

1. Expand \( P(x,y) \) in a Taylor series up till quadratic terms around the point \((x_0, y_0)\), and rewrite \( P(x,y) \) as:

\[
P(x,y) = C_0 + C_1 \cdot (x-x_0) + C_2 \cdot (y-y_0) + C_3 \cdot (x-x_0)^2 + C_4 \cdot (x-x_0) \cdot (y-y_0) + C_5 \cdot (y-y_0)^2 + \bar{P}(x,y) \tag{3.49}
\]

where:

\[
C_0 = P(x_0, y_0)
\]

\[
C_1 = \left. \frac{\partial}{\partial x} P(x,y) \right|_{x_0,y_0}
\]

\[
C_2 = \left. \frac{\partial}{\partial y} P(x,y) \right|_{x_0,y_0}
\]

\[
C_3 = \frac{1}{2} \left. \frac{\partial^2}{\partial x \partial y} P(x,y) \right|_{x_0,y_0}
\]

\[
C_4 = \left. \frac{\partial^2}{\partial x \partial y} P(x,y) \right|_{x_0,y_0}
\]
\[ C_5 = \frac{1}{2} \left. \frac{\partial^2}{\partial y^2} P(x, y) \right|_{x_0, y_0} \]  

(3.50)

2. The integral in (3.33) with \( P(x, y) \) replaced by \( \tilde{P}(x, y) \) can be shown to be continuous (with a removable singularity, but with possibly high gradients) at \( z = 0 \), and hence can be (somewhat) efficiently handled using the integration procedure "type 1" with a small subdivision parameter "\( \alpha \)" (good accuracy of less than 0.01% overall error was achieved with \( \alpha = 1 - 2 \)) and a small value for "\( z \)" (\( z \) was taken to be \( 0.01 \cdot \sqrt{\text{Area of triangle}} \)).

3. The constant, linear and parabolic "components" of \( P(x, y) \) in (3.49) can be done using the conversion to line integration procedure (the line integrations are done numerically; the accuracy of the numerical line integrations was checked by comparing with some analytically obtained results for specific rectangular regions). Adding the results obtained in step 2 and 3, the required value of the integral in (3.33) is obtained.

Note that when the "shape function" is restricted to be at most a quadratic expression in "\( x \)" and "\( y \)" , the integration procedure "type 2" can "skip" step 2 altogether.

A final remark concerning the integration of integrals of the form (3.15) when the region under consideration is an elastic plate bonded to two elastic halfspaces and the point \( x \) is on \( S_c \) is required. This remark concerns the special (but important) case of interface cracks (that is cracks lying along the boundary between the elastic plate and halfspace). For this case, the first "image" of the "infinite" space Green's function also lies on \( S_c \) and therefore contributes to the "singular terms" when \( x \) approaches
being on \( S_\Omega \). This case has not been explicitly studied in this thesis (although the integrations can still be performed inefficiently using "type 1" integration procedure with a high subdivision parameter "\( \alpha \". However, the technique presented in this section for the "infinite" space source can be extended to the "first image" source (by first transforming the surface integrals of the "first image" source multiplied by constant, linear and parabolic variations of the "shape function" into line integrals).
Chapter 4: Application problems for the boundary element method

4.1 Introduction

Chapter 3 presented a boundary element formulation for studying 3-D cracks in infinite and layered regions; in this chapter, applications of that formulation will be considered. Part of the presentation will be in the form of individual numerical studies which aim at improving the understanding of the numerical method being used; these studies have the additional objective of determining whether the numerical formulation is sound and the Green's functions are accurate enough to be used in a general numerical technique. The last two sections of this chapter are concerned with demonstrating the usefulness of this method in studying earthquake source mechanics problems; these geophysically motivated problems are to be considered as preliminary and aim at highlighting the features that would be obtained by using a 3-D elastic or viscoelastic model.

4.2 Infinite space problems

The formulation presented in chapter 3 is most easily applied to 3-D cracks in infinite space; Application problems for these types of geometries will be considered in this section. Studying specific 3-D crack geometries in infinite space allows us to compare numerically obtained stress intensity factors with analytically derived ones. Furthermore, by studying infinite space problems first and then layered space problems, errors that might be due to the newly derived Green's functions could then be more easily isolated. Fortunately, as will be shown in this chapter, the boundary element formulation and the newly derived Green's functions seem to perform well for both infinite space and layered
space regions. In what follows, several individual numerical studies will be presented and discussed, which (hopefully) improve the understanding of the details of the method.

In this boundary element method, both the geometry and the way in which the unknown functions are represented have to be "discretized". The geometry discretization requires the use of straight-edged planar triangles which are required to cover (as much as possible) the surface of the crack without overlap; a mismatch between the (possibly non-straight) smooth boundary of the crack and the straight edges of the border triangles (e.g. see figure 3.4) are expected to introduce some errors in the numerical results; in order to get an idea of the amount of error introduced due to geometry discretization, a case study of (unit radius) circular cracks with an increasing number of (equal length) straight edges per quadrant will next be performed, and the results discussed in the light of a first order theory for "slightly non-circular cracks" (Gao and Rice 1987).

Consider a series of (unit radius) circular cracks with 6, 9, 12, 18 and 24 (equal length) straight edges per quadrant shown in figures 4.1-4.5 respectively. Radial symmetry is used such that crack opening variations have to be specified over "basic sectors" only; the basic sector for figure 4.1(a) is shown in figure 4.1(b). In figure 4.1(b), the local axis systems corresponding to triangles 1-2, 3-4, 5-6 and 7-10 are systems 1, 2, 3 and 4 respectively shown on figure 4.1(b). The border triangles 1-6 have $\bar{\gamma}_L$ and $y_L \bar{\gamma}_L$ opening (multiplied by factors to be determined) variations, triangles 7-9 have $\gamma_L$ and $y_L^2$ opening (multiplied by factors to be determined) variations, and triangle 10 has $\gamma_L$ and $y_L^2$ opening (multiplied by factors to be determined) variations; this implies a total of 8 "degrees of freedom". The opening functional variations
Figure 4.1(b)
Figure 4.4
for the circular cracks shown in figures 4.2-4.5 are performed in a manner analogous to the circular crack of figure 4.1.

The stress intensity factor (SIF) for circular cracks is constant over the crack front whereas the exact SIF for the discretized geometry is expected to vary periodically over each edge along the crack front (since the geometric discretization of the circular crack is not exactly radially symmetric). Using the "first order" theory for slightly circular cracks we can estimate the range of SIF that is obtained by exactly solving slightly circular cracks having the discretized geometries shown in figures 4.1-4.5 under a tensile load; having that estimated range of SIF, a comparison with the SIF for the (exactly) circular crack being discretized under the same load will give an estimate (in the form of an interval) of the theoretically expected errors in the SIF values due to geometric discretization of the circular crack. Of course, the actual numerical errors will be due to both geometric and functional discretizations; however, by allowing enough "flexibility" in the functional variations used for the opening (as detailed above), the numerical SIF value obtained is expected to reflect the average error due to the geometry discretization only. The next discussion will describe how an a-priori estimate for the SIFs of the discretized geometries of the circular cracks can be obtained.

From Gao and Rice's 1987 paper, the following expression for the stress intensity factor (K) of a slightly circular crack (reference radius \( a^0 \)) with a harmonic variation from "circularity" of wavelength \( \lambda \) and amplitude "A" and a far-field stress "\( \sigma \)" can be written as:

\[
K = K^0 + \Delta K
\]

(4.1)

where:
\[ \Delta K = \left( \frac{dK_0}{da} - \frac{\pi}{\lambda} K_0 \right) \cdot A \cdot e^{2\pi is/\lambda} \]  \hspace{1cm} (4.2)

\[ K_0 = \frac{2}{\pi} \cdot \sigma \cdot \left| a^0 \right| \]  \hspace{1cm} (4.3)

and "s" is the arc length along the crack front. Denoting by "n" the number of straight line segments per quadrant, and by \( \Delta \theta \) the angle subtended by each line segment and assuming that "n" is "large", the following estimates can be made:

\[ \lambda = \frac{\pi \cdot a^0}{2n} \]  \hspace{1cm} (4.4)

\[ A = a^0 \cdot (1 - \cos \frac{\Delta \theta}{2}) \approx a^0 \cdot \frac{\pi^2}{8n^2} \]  \hspace{1cm} (4.5)

where:

\[ \Delta \theta = \frac{\pi}{2n} \]  \hspace{1cm} (4.6)

Using (4.4) and (4.5) and assuming an average value for \( |\exp(2\pi is/\lambda)| \) of 0.5, then the following estimate can be obtained:

\[ \frac{\Delta K}{K_0} \approx \frac{\pi^2}{32n} \]  \hspace{1cm} (4.7)

For "n" equal 6, 9, 12, 18 and 24, the estimate (4.7) gives 0.00514, 0.00343, 0.00257, 0.00171 and 0.00129 respectively, while the numerical results gives 0.00612, 0.00476, 0.00247, 0.00184 and 0.00121 respectively; it is noticed that the estimated error is reasonably good especially for the larger values of "n" and in spite of the "rough" manner in which the estimates were obtained. At any rate, the "lesson" to be learned from this "case study" is that the convergence of the SIF values with the number of straight line segments per arc length for uniform curvature is slow (a mere 1/n convergence for circular cracks, see (4.7)). The best way to alleviate this shortcoming is to use non-straight line segments to
discretize the crack boundary (e.g. employing "triangles" with curved sides). However, an alternative method has been employed in this thesis; the alternative method is based on the "guess" that even though the exact SIF values along a "straight line-segments" approximation for a smooth boundary would vary from the exact SIF values for the smooth boundary, the SIF values at the midpoints of the line-segments should be very close to the SIF values at the corresponding points along the smooth boundary. The preceding statement can be exploited in a numerical scheme by allowing the SIF values to vary along a line-segment (by introducing the appropriate degrees of freedom) and reporting the SIF values at only the midpoint of that segment; the "appropriate" degrees of freedom have to be introduced even though the actual geometry might not require such degrees of freedom (as in the circular crack case). This remark will be further elaborated at the end of this section when discussing elliptic cracks in infinite space under tensile loadings. However, the next item to be discussed will deal with "errors" that occur due to functional discretization.

The way in which the unknown displacement discontinuity (DD) vector is assumed to vary along the surface of the crack (or cracks) is also a type of discretization. The assumptions on the DD distributions have to be made (in this formulation) over each triangle separately. If two (or more) adjacent triangles have a common local axis and have the same distributions multiplied by the same degrees of freedom, then the assumed DD distribution will be continuous over the region composed of the union of the two (or more) triangles; otherwise, incompatibility in DD variation over the crack surface (or surfaces) can occur. However, it is known in boundary element methods that it is possible to obtain accurate results for a continuous function assuming (a range of) piecewise continuous variations for that function. An example of the above statement showing the "opening" profile for a circular crack (in
infinite space) under tensile loading for different types of (discontinuous) "opening" variation over the crack surface will next be presented.

Refer again to figures 4.1(a,b) and to the same local axis that were assigned previously while discussing geometric discretization effects (see figure 4.1(b) for a graphical representation of the local axis). One choice for the assignment of opening variations was also used above (in connection with geometric discretizations); the triangles 1-6 had $\sqrt{y_L}$ and $y_L \cdot \sqrt{y_L}$ opening (multiplied by factors to be determined) variations, triangles 7-9 had 1, $y_L$ and $y_L^2$ opening (multiplied by factors to be determined) variations, and triangle 10 had 1, $y_L$ and $y_L^2$ opening (multiplied by factors to be determined) variations; call this opening functional variation "parabolic" in view of the assumed parabolic variation (in $y_L$) of the "non-border" triangles. Similarly, call "constant" and "linear" opening functional variations those opening functional variations with triangles 7-9 and 10 having constant and linear variations (in $y_L$) respectively, while keeping the same opening functional variation for triangle 1-6 as for the parabolic case. The constant, linear and parabolic opening functional variations have 4, 6 and 8 "degrees of freedom" respectively, and the ensuing slip variations are plotted in figure 4.6 as a function of one minus radial distance (the circle being of unit radius). The percent errors in the SIF values obtained using the constant, linear and parabolic opening functional variations are 12.18%, 6.69% and 6.12% respectively. Note, that the improvement in either the SIF values or in the opening profiles are considerable from the constant to the linear opening functional variation but much less important from the linear to the parabolic opening functional variation (in spite of having the same increase in degrees of freedom). In addition, as the degrees of freedom
Opening Variation; penny crack

Figure 4.6
increase, the discontinuities along the border of domains over which different opening variations are specified decrease. The above observations suggest that reasonably accurate results (assuming geometric discretization error sources have been accounted for) can be obtained if the assumed DD functional variation over the crack surface can reasonably approximate the (expected qualitative) exact DD functional variation. Furthermore, improvements in the accuracy of results versus effort also seem to lead to diminishing returns beyond a threshold level of effort (i.e. as the effort increases, progressively smaller gains in accuracy is obtained beyond a "critical" level of adequate funtional variation representation). Having discussed aspects of both geometric and functional variation discretizations, the next discussion will center along the interaction between the geometric and functional variation discretizations.

The way in which geometry is discretized and the assumptions concerning the displacement discontinuity functional variation should not (always) be made independently. This is especially true when discontinuities in the DD variations are allowed to occur between triangles; this is due to the fact that (some) of the internal edges of triangles become lines of potential DD functional discontinuity. Since the actual DD distribution is (usually) known to be continuous over a crack's surface, it is desirable to choose (if possible) the "lines of potential DD discontinuity" in those directions in which the assumed DD variation can "best fit" the actual DD variation while introducing the least amount of incompatibility. Again a tensile loaded circular crack (in infinite space) under two different internal geometry discretizations will be used as a case study for errors arising due to "inter-triangle" DD variation discontinuity.

Consider again figure 4.1(a,b) with the "parabolic" opening
functional variation discussed previously. For comparison, the border triangles are now discretized in a "different" way as shown in figure 4.7(a) with figure 4.7(b) representing a "basic sector". Note that figure 4.7(a) has the same number of line segments per quadrant as figure 4.1(a). The internal triangles 6-8 and 9 have parabolic opening variation (in $y_L$) whereas the triangles 1-5 have $\sqrt{y_L}$ and $y_L \cdot \sqrt{y_L}$ opening variation such that triangle 1-2, 3 and 4-5 have local axis 1, 2 and 3 respectively. The number of "degrees of freedom" for this discretization is the same as that for 4.1(a) (i.e. 8 "degrees of freedom"). The SIF obtained for the "parabolic" case of figure 4.1 has 6.12% error whereas the SIF obtained for an analogous "parabolic" functional discretization case for figure 4.7 has 9.78% error. This 4.66% error increase for the discretization of figure 4.7(a) is wholly due to the orientation of the line segments common to triangles 2 and 3, and the orientation of the line segment common to triangles 4 and 5 (for the "basic sector"); along these line segments there is incompatibility in opening for a uniform variation in SIF along the boundary (using the prescribed functional variation assumptions stated above). Note that the discretization used for figure 4.1(a,b) is compatible (between the border triangles) when the SIF is uniform along the boundary. The above example shows that the internal discretization (especially near the crack front) has to be carefully dealt with in order to avoid significant numerical errors.

A final example problem of an elliptic 3-D crack in infinite space under tensile loading will be discussed. This final example has several aims. First, the geometry will exhibit a case where SIF values vary over the crack front and will be a preliminary check on the adequacy of the discretization for the part-through elliptic cracks in a plate that will be performed in the next section. Also, aspects of the previous case studies will be further discussed.
Figure 4.7(b)
Consider an elliptic crack (aspect ratio 2:1) with a
discretization as shown in figure 4.8(a), where figure 4.8(b)
represents a "basic quadrant" such that quarter space symmetry can
be employed. The exact SIF values along the crack front can be
found in the Tada et al Handbook (1973), and is plotted as the
(non-labeled) solid line in figures 4.9-4.11.

Note that the internal discretization in figure 4.8(a) is such
that all lines of "potential DD discontinuity" are perpendicular to
the crack front; this allows compatibility to "more readily" occur
along those line segments. Two cases for the opening functional
variation is specified. The first case of opening functional
variation is specified over the quadrant 4.8(b) such that triangles
1-2, 3-4, 5-6, 7-8, 9-10, 11-12, 13-14, 15-16 and 17-18 have \( \sqrt{y_L} \),
\( y_L \cdot \sqrt{y_L} \) and \( x_L \cdot \sqrt{y_L} \) opening (multiplied by factors to be determined)
variation (\( y_L \) being the direction perpendicular to the crack front)
and triangles 19-20, 22-23 and 21 with 24 each have quadratic
opening variation in two directions (i.e. 1, \( x_L \), \( y_L \), \( x_L^2 \), \( x_L \cdot y_L \) and
\( y_L^2 \) degrees of freedom). The total number of degrees of freedom for
case one (functional discretization) is therefore 45. The second
case of opening functional discretization is such that triangles
1-2, 3-4, 5-6, 7-8, 9-10, 11-12, 13-14, 15-16 and 17-18 have a
\( \sqrt{y_L} \), \( y_L \cdot \sqrt{y_L} \), \( x_L \cdot \sqrt{y_L} \) and \( x_L^2 \cdot \sqrt{y_L} \) opening (multiplied by factors to be
determined) variation (\( y_L \) being the direction perpendicular to the
crack front) and the same quadratic opening variations over the
internal triangles as in case one. The total number of degrees of
freedom for case two is therefore 54.

The SIF results for the first case of functional
discretization is shown in figure 4.9, and the SIF results for the
second case of functional discretization is shown in figure 4.10
and 4.11. The SIF results in figure 4.9 are reasonably well (around
Figure 4.9
Elliptic Crack; Infinite Space

Stress Intensity Factor

Theta / (π/2)

Figure 4.10
Elliptic Crack; Infinite Space

Stress Intensity Factor

\[ \frac{\theta}{(\pi/2)} \]

Figure 4.11
3% errors), whereas the SIF results in figure 4.11 are significantly better (mostly less than 1% error). Figure 4.11 plots the numerical SIF values of figure 4.10 at only the midpoint of the line segments; Note that by introducing additional degrees of freedom for the opening variation along the crack front (case 2 versus case 1 functional discretization), the SIF values at the midpoint of the line segments used to discretize the crack front improves significantly, (whereas the SIF values at other locations might in some cases drastically deteriorate); this relates to the discussion concerning the improvement of SIF values along the crack front while using line segments to discretize a non-straight boundary.

The above numerical studies do not exhaustively exhibit the features of this specific implementation of the boundary element method, however they are intended to familiarize the reader with certain aspects of this specific implementation. Some of the studies that have been omitted have already been studied in related boundary element schemes (such as convergence with further discretization and collocation point location sensitivity). The general boundary element method literature is too vast to be adequately reviewed in this thesis, and is only an incidental part of this thesis (e.g. see Banerjee and Rutterfield's 1981 textbook for a general overview and an extensive cataloguing of references of further specialized topics, or Brebia's 1984 textbook for more recent publications and studies).

4.3 Part-through cracks in a plate

In this section some part-through cracks in an elastic plate (traction free upper and lower surfaces) will be numerically analyzed using the Boundary Element Method (BEM) and the new Green's functions. Two types of part-through cracks will be
analyzed. The first type of cracks are purely antiplane cracks under antiplane shear deformation which are closely approximated in 3-D by "very long" 3-D rectangular cracks in a plate under in-plane shear deformation (see figure 4.12). The second type of part-through cracks are semi-elliptic cracks (see figure 4.16) under tensile loading.

The reason for first analyzing antiplane cracks is to gain some confidence in the accuracy of the performance of the newly derived Green's function for simple problems whose analytic solutions are known (e.g., Tada et al. Handbook). The geometry of the rectangles used is chosen so that the length to penetration "a" (see figure 4.12) is 30:1 (as depicted in figure 4.13) which is expected to produce a (very nearly) antiplane deformation close to the middle of the rectangle. Three depth penetration to plate thickness values are chosen, and these are 0.1, 0.5 and 0.9. The geometric discretization for these antiplane cracks is shown in figure 4.13. Using two degrees of freedom with slip variations \( \sqrt{y_L} \) and \( y_L \cdot \sqrt{y_L} \) (\( y_L \) is the distance from the crack front), the errors in the numerically obtained SIF results are 1.80%, 0.05% and 12.97% for "a/H" ratios of 0.1, 0.5 and 0.9 respectively. The errors in the SIF values corresponding to the first two values of "a/H" are deemed acceptable (especially since only 2 degrees of freedom are used). In order to determine whether "acceptable" SIF values can be obtained for the "a/H" ratio of 0.9, further discretizations are performed. Figures 4.14 and 4.15 show two further geometric discretizations of the antiplane crack when "a/H" ratio equals 0.9. The "upper row" of triangles containing the crack front has 2 degrees of freedom associated with it (corresponding to \( \sqrt{y_L} \) and \( y_L \cdot \sqrt{y_L} \) slip variation), whereas the "lower row (or rows)" in figures 4.14 and 4.15 have parabolically varying slip. A total of 5 and 8 degrees of freedom for the slip variation discretizations of
figures 4.14 and 4.15 respectively is obtained. The errors in the numerically obtained SIF results are 9.94% and 5.53% for the 5 and 8 degrees of freedom discretizations when "a/H" equals 0.9. Since the errors seem to be decreasing significantly with the increase in degrees of freedom, this preliminary use of the new Green's functions coupled with the BEM is considered to be successful.

The next type of part-through cracks are semi-elliptic cracks under tensile loading. There are no analytic results for the SIF values of these crack problems. However, a numerical analysis using the Finite Element Method (FEM) has been performed by Raju and Newman (1979) for an extensive combination of "a/c" and "a/H" ratios (see figure 4.16). The FEM model used by Raju and Newman employs up to 6900 degrees of freedom and is considered to yield the most accurate SIF values in the literature (for semi-elliptic part-through cracks in a plate). In what follows SIF results from numerical simulations using the present BEM formulation will be presented for three different ratios of "a/c" and two different ratios of "a/H". The geometric discretization of the semi-elliptic cracks with "aspect" ratios "a/c" equal to 0.4, 1.0 and 2.0 are shown in figures 4.17, 4.18(a) and 4.19. The two penetration to thickness ratios "a/H" considered in this study are equal to 0.2 and 0.8. The opening variation discretization is similar to that discussed for the case of the infinite space elliptic crack (per quadrant) with one additional degree of freedom associated with triangles 1-2 and 25-26 corresponding to an opening variation of $x_L^2 \cdot |y_L|$ ($y_L$ is the direction perpendicular to the crack front); this implies that a total of 46 degrees of freedom has been used. Even though the number of degrees of freedom used in this BEM is very much lower than the number used for the FEM, the computational effort (using the current "prototype" BEM computer program) per degree of freedom is much higher for the BEM than for the FEM, making the direct comparison of "degrees of freedom" for "comparable
accuracy" not very significant.

The results for the SIF values for the various combinations of ratios "a/c" and "a/H" (the ratios are included as inserts in the figures) are shown in figures 4.20 to 4.25 (a) and (b), and are compared with Raju and Newman's results. Figures (a) contain SIF values at the endpoints as well as midpoints of each line segment along the crack front, whereas figures (b) show the SIF values at only the midpoints of each line segment. The SIF values are normalized (following Raju and Newman) by the factor \( \sigma \sqrt{\pi a/Q} \) where \( \sigma \) is the applied far-field tensile stress, "a" is the crack penetration depth and Q is defined as:

\[
Q = \left( \int \int 1 - k^2 \cdot \sin^2 \theta \cdot d\theta \right)^2 \tag{4.8}
\]

where:

\[
k^2 = \begin{cases} 
1 - a^2/c^2 & \text{if } a/c \leq 1 \\
1 - c^2/a^2 & \text{if } a/c > 1
\end{cases} \tag{4.9}
\]

The "normalized" SIF results are plotted versus \( \theta \) (theta) divided by \( \pi/2 \), where "\( \theta \)" can be geometrically interpreted as shown in figure 4.16 (the contour of the semi-elliptic cracks can be duplicated on an x-y plot by plotting the (x,y) points having the values \( (c \cdot \cos \theta, a \cdot \sin \theta) \) with "\( \theta \)" varying between 0 and \( \pi \).

The present numerical SIF results show good agreement with Raju and Newman's results except at points lying on the first one or two line segments forming the crack front that are closest to the crack-free surface intersection (i.e. approximately for "\( \theta/(\pi/2) \)" greater than 0.2).

In order to determine the reason for the high apparent errors in the SIF values for "small \( \theta \)" values, the semi-circular crack
SIF Elliptic Part-Through Crack

\[ K_I \text{ normalized} \]

\[ a/c = 0.4 \]

\[ a/H = 0.8 \]

- \( k_I^* \text{ present} \)
- \( k_I^* \text{ Raju-New} \)

\[ \theta/(\pi/2) \]

**Figure 4.20(a)**
SIF Elliptic Part-Through Crack

\[ K_l \text{ normalized} \]

\[ a/c = 0.4 \]
\[ a/H = 0.8 \]

\[ \theta/(\pi/2) \]

*Figure 4.20(b)*
SIF for Elliptic Part-Through Crack

\[ K_I \text{ normalized} \]

\[ \frac{a}{c} = 1.0 \]
\[ \frac{a}{H} = 0.8 \]

- \( k_{I*} \text{ present} \)
- \( k_I^* \text{ Raj-New} \)

\( \theta/(\pi/2) \)

Figure 4.21(a)
SIF for Elliptic Part-Through Crack

\[ K_I \text{ normalized} \]

\( a/c = 1.0 \)

\( a/H = 0.8 \)

\( \theta/(\pi/2) \)

Figure 4.21(b)
SIF Elliptic Part-Through Crack

\[
\frac{K_I \text{ normalized}}{a/c = 2.0, \ a/H = 0.8}
\]

\[
\frac{\theta}{\pi/2}
\]

Figure 4.22(a)
SIF Elliptic Part-Through Crack

![Graph showing KI normalized vs. \(\theta/(\pi/2)\) for different values of \(a/c\) and \(a/H\).]

- $a/c = 2.0$
- $a/H = 0.8$

Legend:
- $k1^*$ present
- $k1^*$ Raju-New

Figure 4.22(b)
SIF Elliptic Part-Through Crack

theta/(pl/2)

Figure 4.23(a)
SIF Elliptic Part-Through Crack

\[ a/c = 0.4 \]
\[ a/H = 0.2 \]

- \( k_1 \) * Raju-New
- \( k_1 \) * present

\( \theta/(\pi/2) \)

Figure 4.23(b)
SIF Elliptic Part-Through Crack

\[ \frac{K_1}{K_{ref}} \]

\[ \theta/(\pi/2) \]

Figure 4.24(a)
SIF Elliptic Part-Through Crack

$K_I$ normalized

$\theta/(\pi/2)$

Figure 4.24(b)
SIF Elliptic Part-Through Crack

\[ K_I \text{ normalized} \]

\[ \frac{a}{c} = 1.0 \]
\[ \frac{a}{H} = 0.2 \]

\[ \theta/(\pi/2) \]

Figure 4.24(c)
SIF Elliptic Part-Through Crack

\[ a/c = 1.0 \]
\[ a/H = 0.2 \]

\( K_I \) normalized

\( \theta/(\pi/2) \)

Figure 4.24(d)
SIF Elliptic Part-Through Crack

\[ K_I \text{ normalized} \]

\[ \frac{a}{c} = 1.0 \]
\[ \frac{a}{H} = 0.2 \]

- \( kI \text{* New-Raj} \)
- \( kI \text{* 13/quad} \)
- \( kI \text{* 9/quad} \)

Figure 4.24(e)
SIF Elliptic Part-Through Crack

Figure 4.25(a)

theta/(pi/2)
SIF Elliptic Part-Through Crack

\[ K_I \text{ normalized} \]

\[ a/c = 2.0 \]

\[ a/H = 0.2 \]

\[ \theta/(\pi/2) \]

Figure 4.25(b)
with \( a/c = 1.0 \) and \( a/H = 0.2 \) is further discretized as shown in figure 4.18(b) with 67 degrees of freedom; Referring to figure 4.18(b), the border triangles have linearly varying SIF parametrization and internal triangles 9-10, 30-31, 33-34 and 32 with 35 have parabolically varying opening variation, and triangle 11 has a linear opening variation specified over it (symmetry specification is implicitly assumed). The results for this "higher discretization" case is shown in figures 4.24(c) and (d) (where 4.24(c) plots SIF values at endpoints as well as midpoints of line segments forming the crack front, whereas 4.24(d) plots SIF values at only the midpoints of the same line segments). The results shown in figures 4.24(c) and (d), and the comparison of SIF results with the previous discretization shown in figure 4.24(e) indicate that the problem at small\( \theta \) was due to "lack of resolution" (i.e. too few degrees of freedom to account for the opening variation).

4.4 Viscoelastic surface displacements due to a dislocated embedded patch

A time-dependent model consisting of slippage (or dislocation) along a surface in an elastic plate overlying a viscoelastic foundation is usually used in geophysical applications to study (or to take into account) postseismic surface deformations (i.e. surface deformations that occur "after" an earthquake). Two-dimensional "thrust" slippage and antiplane "strike slip" (both assume infinitely long faults) viscoelastic models have been repeatedly considered by many researchers (e.g. Nur and Mavko 1974, Smith 1974, Savage and Prescott 1978, Thatcher and Rundle 1979, Lehner et al. 1981, Li and Rice 1987 and Reilinger 1986).

In this section, the surface uplift due to uniform "thrust" slippage that occurs and is maintained along an inclined finite rectangular patch (two different lengths are considered) in a plate
will be presented. The geometry of the side and top views of those
two patches are shown in figure 4.26. The aim of this study is to
present surface deformations due to a 3-D viscoelastic model of a
layered earth, and to try and discern (if any) deformation feature
differences between a 2-D and a 3-D model. The short patch
represents a 3-D model and the long patch represents an approximate
2-D model. The calculation of the time-dependent surface uplift is
obtained by the integration of the viscoelastic Green's functions
discussed in section 2.6 over the surface of the patches using the
integration techniques discussed in section 3.4.

The geometric parameters of the model are as shown in figure
4.26 (the short and long patches having lengths of 1.2H and 5H
respectively). Unit slippage (i.e. 1H) is prescribed to occur
uniformly over the surface of each patch; the surface uplift plots
to be presented in this section (being linearly proportional to the
amount of slippage along the patch) can be alternately viewed as
being normalized by the slippage value. The elastic properties are
such that the plate and the halfspace have identical shear modulus
"G" and Poisson' ratio "ν"; the shear modulus "G" is taken to be
"1" (arbitrary units) and Poisson's ratio "ν" is taken to be 0.25.
The units of the shear modulus does not affect the surface
deformation values but serves to specify how much "force" has to be
applied at the opposing slipping surfaces in order to produce the
slippage (in geophysical applications, the shear modulus is said to
affect the "moment" and "stress drop" on the patches). Finally, the
viscoelastic model for the halfspace is the same as discussed in
section 2.6; The volumetric deformation is purely elastic and the
deviatoric deformation is a Maxwell model with a viscosity
parameter η = 20 (arbitrary units); the ratio of "η/(2G)"
determines the characteristic time "τ" (time units depend on the
units of both "η" and "G"). Referring to section 2.6, the above
values of "η/G" and "ν" imply ω = 2G/η = 0.1, Ω = 0.5ω, Ωₐ = 0.33ω,
Figure 4.26
\[ \Omega_{B1} = 0.9\omega \text{ and } \Omega_{B2} = 0.24\omega; \] the preceding values of \( \Omega \), \( \Omega_A \), \( \Omega_{B1} \) and \( \Omega_{B2} \) show that the four distinct exponential time-scales lie within an interval of "1/\omega" to "4.2/\omega" for a model having the elastic properties of the plate and halfspace equal, and with Poisson's ratio equal to 0.25.

The surface uplift values along the x-axis (z=0, y=0 in figure 4.26) for \(-4.2 \leq x/H \leq 4.2\) and up till sufficiently long times (so that "steady state" has occurred in the range \(-0.7 \leq x/H \leq 0.7\)) are presented in figures 4.27 to 4.35 for both the short (figures 4.27 to 4.31) and long patch (figure 4.32) as well as for comparisons between the two patches (figures 4.33 to 4.35). A discussion of these results will be next presented.

First consider the uplift values due to the short patch. For convenience the deformation pattern will be divided into three sections. These sections numbered 1 to 3 lie roughly between \(-4.2 \leq x/H \leq -0.6\), \(-0.6 \leq x/H \leq 0.4\) and \(0.4 \leq x/H \leq 4.2\) respectively. The material lying at \(x/H \leq 0.3\) is being "pushed" onto a narrow "gap" between the free surface and the closest edge of the rectangular patch to that free surface. This "squeezing" effect causes a sharp peak in the uplift values to occur in section "2". However, the material lying at \(x/H \leq 0.3\) is being "anchored" to a halfspace which is steadily relaxing, and hence with time, the uplift values in section "2" move in the direction of "imposed slippage" on the upper surface of the patch. Finally, material is being driven below (or being pushed under) the patch (starting from \(x \leq 0.3\) to \(x \geq -0.3\)) and this material is being resisted with a "floor" which is steadily relaxing; hence the uplift values in section "3" move in the direction of "imposed slippage" on the lower surface of the patch. The evolution of uplift with time is quite complex in detail, as can be seen in figures 4.28 to 4.31. Some features that can be discerned in those figures is an exponential decay of
Surface Uplift vs Position (Short Patch)

Figure 4.27
Surface Uplift vs Time

Figure 4.28
Surface Uplift vs Time

Figure 4.29
Surface Uplift vs Time

Figure 4.30
Surface Uplift vs Time

Figure 4.31
Surface Uplift vs Position (Long Patch)

Figure 4.32
Surface Uplift \((t/\tau = 0.0)\)

Figure 4.33
Surface Uplift ($t/\tau = 5.0$)

$x / H$

Figure 4.34
Surface Uplift (t/tr = 30.0)

Figure 4.35
surface deformation towards a "steady state" in section "1" (figures 4.29 and 4.30), as well as some diffusional behavior which can be deduced from the "more elongated" and "longer lived activity" time behavior of locations that are farther off from the patch (figures 4.28 and 4.31); however, from the convergence studies of chapter 2, the numerical accuracy of the surface deformations of "faraway" locations are not very certain, and exact judgement on the results cannot be made. The next discussion will compare the "short" (3-D) versus the "long" (2-D) surface uplifts.

The surface uplift for the "long" patch is shown in figure 4.32, and is not qualitatively much different from the surface uplift due to the "short" patch. A general observation is that the surface uplift values for the "long" patch has higher peaks and varies more with time than the "short" patch; this observation is probably due to the higher compliance inherent in a longer patch (by having less "side supports"). Figures 4.33 to 4.35 show comparisons at short, intermediate and long time values between the uplift values of the "short" and "long" patch. The higher "inherent compliance" of the longer patch causes the "anchoring" in section "1" (of the plots) to be less effective and the "driving under" in section "3" (of the plots) to be more effective than for the shorter patch.

4.5 Preliminary model of an interplate fault region at Parkfield

The fault section of the San Andreas fault near Parkfield has been the focus of extensive research. This extensive research is mainly due to the relatively short recurrence interval and repeatable character of moderate earthquakes occurring near Parkfield. According to Bakun and McEvilly (1979, 1984), at least five earthquakes of similar magnitude (M=5.5-6) and epicenter have occurred at 21±8 year intervals (1881, 1901, 1922, 1934, 1966).
Consequently, the Parkfield area has become heavily instrumented for geodetic and seismological measurements in anticipation of the next Parkfield earthquake. Geodetic measurements (e.g. King et al. 1987), seismicity distributions data (e.g. Buhr and Lindh 1982) and surface mappings (e.g. Brown 1970) have been extensively reported in the literature, and have been used to reveal the through depth fault geometry near Parkfield (Harris and Segall 1987).

The use of mechanical models to study fault regions usually involve the determination of elastic fields due to slip distributions over a predetermained fault surface and can either be kinematic or non-kinematic in nature. Kinematic models use surface deformations (horizontal and/or vertical change in displacements) near the fault trace as a direct constraint to obtain the slip distributions, irrespective of the ensuing stress distribution on the fault surface. Non-kinematic models use frictional boundary conditions on the fault surface in order to obtain the slip distributions. In non-kinematic models, surface deformations are indirectly used in order to constrain the parameters that specify the boundary conditions. For example, using a frictional law which specifies that all slipping regions of the fault have a fixed frictional strength, surface deformation data would be used to delineate the slipping versus non-slipping (or locked) regions of the fault surface. Kinematic models are usually simpler to use, and hence deformation studies of faults occurring in complex layered media can be more easily implemented for kinematic models than for non-kinematic models. Non-kinematic models employ more knowledge concerning the fundamental behavior of solid materials, and (for models that include a wide enough characterization of frictional behavior) may be used to extrapolate deformation behavior that is yet to occur. Hence, non-kinematic models unlike kinematic models have the potential of predicting the occurrence of an earthquake. Although such prediction capabilities have not yet been realized
there does exist detailed forecast models that awaits verification (e.g. Stuart et al. 1985).

Several researchers have studied the fault region near Parkfield using non-kinematic models. Quasi 3-D non-kinematic preliminary models using the line-spring technique have been used by Li and Fares (1986) and by Tse et al (1985). However, the line-spring technique is not considered accurate enough when sudden changes in geometry occurs, or when determining elastic fields at locations that are too close to the fault. A detailed 3-D non-kinematic forecast model of the Parkfield region has been developed by Stuart et al (1985). Stuart et al's model uses slip patches in a halfspace with a free surface and phenomenological frictional (slip weakening) boundary conditions. A shortcoming of the frictional boundary conditions used by Stuart et al is that the frictional relations are not based on experimental data related to the frictional properties of known solid materials (such as rock) that occur in fault regions. Another shortcoming is that a halfspace model does not take into account the significant change in material properties between the lithospheric plate and the underlying asthenosphere (the first two layers in the earth). A model using slippage distribution on a fault surface lying in a plate over a viscoelastic halfspace would more appropriately model the lithosphere-asthenosphere coupling as noted by several researchers based on evidence of postseismic relaxation studies (e.g. Li and Rice 1987). Assuming the rate of load application on the fault region is applied at a much slower rate than that at which the asthenosphere relaxes, the asthenosphere could be assumed to be fully relaxed at steady state loading conditions. However, the assumption of a fully relaxed asthenosphere (as compared with a partially relaxed asthenosphere) cannot be conclusively demonstrated, especially in view of the relatively short recurrence time of Parkfield earthquakes as discussed in Tse et al (1985).
Nevertheless, a model using slippage distribution on a fault surface lying in a plate (with both surfaces traction free) is in general a more appropriate model than a model using slippage on a fault in a halfspace. This is because the recurrence time interval of Parkfield earthquakes (21 years) is longer than the 10 to 16 years estimated characteristic relaxation time of the asthenosphere (Li and Rice 1987), which suggests that significant relaxation of the asthenosphere would have occurred at times around 70-90% of the complete earthquake cycle.

In this thesis, a preliminary non-kinematic 3-D model of the fault region near Parkfield is implemented and is shown in figure 4.36. The model assumes slippage to occur over a surface lying in a plate (with both plate surfaces traction free), and with simple frictional boundary conditions specifying that all slipping regions have the same constant level of frictional resistance. As noted earlier, such a frictional boundary condition requires the determination of which region of the fault is slipping and which is locked. The determination of the geometry of the slipping region of the fault requires extensive studies and has not been performed in this thesis and hence the fault model is termed preliminary. The geometry of the slipping region in the fault region near Parkfield has been determined by referring to previous kinematic studies by Harris and Segall 1987, whereas the length of the slipping (or "creeping") zone north of Parkfield has been determined from surface deformation data along the central section of the San Andreas fault (see figures 4.37 (a,b)). In determining the geometry of the slipping region of the fault, the thickness of the plate "H" is taken to be 20-22 km., which implies that the length of the slipping zone is around 8H (i.e. 160-180 km.). Finally, the geometry details at the end x=8.75H of the slipping fault region is assumed to have a negligible effect on the fault region near Parkfield, and hence have been assumed to be the simplest possible
Figure 4.36
(as shown in figure 4.37(a)). The discretization of the fault region specified above will next be discussed.

The geometric discretization of the Parkfield region is shown in figures 4.37 (a,b) using a total of 74 triangles. A total of 98 degrees of freedom (DOF) associated with slippage in the "x" direction is specified over the slipping part of the fault region. The allocation of DOF over the fault region is such that 30 DOF are allocated for triangles 1 to 28 (figure 4.37 (a,b)), 10 DOF are allocated for triangles 29 to 31, 10 DOF are allocated for triangles 33 to 40, 16 DOF are allocated for triangles 41 to 54, 10 DOF are allocated for triangles 55 to 60, 8 DOF are allocated for triangles 61 to 64 and 14 DOF are allocated for triangles 65 to 74. The degrees of freedom allocated to triangles 65 to 74 all specify a variation of slippage in the "x" direction only (i.e. uniform with respect to the "z" direction), whereas the slippage over other triangles are allowed to vary slightly more in the "x" than in the "z" direction. Note that the association of slip distributions over triangles that are close to an edge (i.e. adjacent to a locked region of the fault) are such that a \( \sqrt{r} \) variation is always included (where "r" is the distance from the edge), and such that the slippage is zero at all edges. Finally, the results to be presented next are such that displacement measures are normalized by "\( \sigma H/G \)" and stress intensity factors (KIII or KII) are normalized by "\( \sigma \sqrt{H} \)" (where "\( \sigma \)" is the far-field shear, "G" is the shear modulus of the plate and "H" is the thickness of the plate).

The variation of the surface slip along the fault (i.e. the displacement mismatch along the surface \( z=0 \) at opposing sides of the fault) is shown in figure 4.38 (a,b). As can be noticed in figure 4.38 (a), the surface slip distribution is very similar to the thickness averaged slip of a mode II plane stress crack whose length is around 8.3H, except along the interval from 0≤\( x/H \)≤1.75
This is the most complete text of the thesis available. The following page(s) were not included in the copy of the thesis deposited in the Institute Archives by the author:

pg. 210
Surface Slip vs x-axis

Figure 4.38(b)
Lower Surface Slip vs $x$-axis

Figure 4.39
(as can be inferred from the maximum slip as well as from the stress intensity factor KII at x=8.75H). The detail of surface slip from 0≤x/H≤1.75 (figure 4.38 (b)) consists of a relatively constant surface slip level between 0.2≤x/H≤1.6 whose magnitude can be approximately obtained from an antiplane analysis of a doubly cracked plate as in the appendix of Tse et al (1985) with a far-field stress of 1.5 to 2 times the actual far-field stress. Rather narrow transition intervals at each end (0≤x/H≤0.2 and 1.6≤x/H≤1.75) are observed in figure 4.38 (b). The variation of the lower surface slip (i.e. the displacement mismatch along the surface z=H at opposing sides of the fault) is shown in figure 4.39. The transition interval between the through thickness uniformly cracked slip region and the semi-locked region (1.6≤x/H≤2.0) shows an oscillation which is probably due to an insufficient level of discretization at the lower section of the plate but is not expected to significantly affect the upper surface slip results which are of primary concern. Note that the lower surface slip distribution does not contain a region for which an antiplane analysis could adequately model the slip variation (i.e. the slip distribution between -1.25≤x/H≤1.75 does not contain an interval over which the slip is relatively flat).

The stress intensity factor (SIF) values along the upper edge (z/H=0.05, 0≤x/H≤1.75) shown in figure 4.40 and along the lower edge (z/H=0.5, -1.25≤x/H≤1.75) shown in figure 4.42 are consistent with the observed slip distributions along the upper and lower surfaces. An antiplane analysis of a doubly cracked specimen would yield a normalized KIII of 0.56 and 1.4 for the upper and lower edges respectively. The SIF values also show especially high values near the edge of the semi-locked and through thickness slipping region of the fault (i.e. at x/H=1.75). Finally, the mode II stress intensity factors along the edge between the semi-locked and through thickness slipping region of the fault (i.e. x/H=1.75,
Figure 4.40
KII vs z-axis

Figure 4.41
$K_{III}$ vs $x$-axis

Figure 4.42
0.05 ≤ z/H ≤ 0.5) shows a normalized KII level of around 4.6 which for a nominal crack length of 8.3H implies a far-field stress level amplification of around 1.3.

Finally, the (normalized) surface uplift values near the fault trace at y/H = 0.01 and -0.5 ≤ x/H ≤ 2.25 are shown in figure 4.43. Note that surface uplift values can only be obtained using a 3-D model since uplift occurs due to pinching effects introduced by gradients of slippage along the fault. The maximum change in uplift across the fault trace is at most around 10% the change in surface slip values across the same locations on the fault trace. As expected, the surface uplift is highest at the location where the gradients in surface slippage is highest (i.e. at x/H = 1.75) and reduces to a low level along the interval where surface slip is relatively flat (i.e. at 0.2 ≤ x/H ≤ 1.6). Finally, the surface uplift in the interval -0.5 ≤ x/H ≤ 1.0 is observed to increase in the negative "x" direction with a small hump at x/H = 0.0. This observation could be explained by pointing out to the relative increase in gradients of slippage (and hence pinching) that occurs when the semi-locked surface of the fault (-1.25 ≤ x/H ≤ 1.75) is locked at its upper edge at x/H = 0.0 (and hence the hump) and then locked at its lower edge at x/H = -1.25 (and hence the gradual increase of surface uplift in the negative "x" direction which is expected to decrease again after x/H = -1.25 is reached).

A final remark is that the Parkfield fault model that has been presented is preliminary and serves only to demonstrate 3-D deformation features that cannot be accurately obtained using simpler models. Extensive parametric studies have to be performed before comparisons with geodetic surface deformation data can be performed.
Surface uplift

Figure 4.43
Chapter 5: Conclusion and further recommendations

In this chapter the major contributions of this thesis will be summarized and recommendations for topics that require further research and implementation will be suggested.

The major contributions of this thesis are:

1. A systematic method for deriving point source elastic fields in bonded elastic halfspaces using an "extended image method" for 3-D, 2-D as well as antiplane deformation.
2. Extending the multiple imaging scheme used for deriving Green's functions for scalar fields (antiplane) problems in plane-layered regions to apply to the derivation of 2-D and 3-D Green's functions in a plane-layered elastostatic region.
3. The actual derivation of specific Green's functions (nucleii of strain) for an elastic plate perfectly bonded to two elastic halfspaces or for an elastic plate perfectly bonded to an elastic halfspace and a viscoelastic halfspace.
4. Discussions elucidating the nature of the viscoelastic Green's functions that can be obtained.
5. A "new" Boundary Element Method (BEM) formulation for 3-D crack problems and clarifications of certain details of its implementation (such as the integration of "singular" Green's functions multiplied by arbitrary "shape functions", effect of geometric discretization on numerical results, etc.)

There are several "areas" in which the above mentioned contributions could be "built upon". The first area of recommended research concerns the further development of the numerical "tool" which is required for geophysical applications. First, the programming of the Green's functions could be made more efficient.
(as suggested in section 4.3) and the programming of the BEM improved in a manner which optimizes its use on a "supercomputer"; although the preceding statement refers to a "detail" in technical implementation (which has not been dealt with in this thesis), it is nevertheless an important suggestion. Second, the viscoelastic Green's functions could be implemented in a BEM formulation. Beyond the above recommendations concerning the development of the required numerical "tool" for the fault model, other modelling aspects (such as fault constitutive relations and details of fault geometry and "asperity" locations and shapes) are outside the context of this thesis.

The second area of recommended research concerns further studies related to the derivation of Green's functions. The method of deriving the "image algorithm" suggests that analogous "image algorithms" might be obtained for spherical interface elastostatic problems, for elastodynamic plane-layered problems, and perhaps even for poroelastic plane-layered problems.

Finally, the last area of recommended research concerns the refinement of the BEM formulation. As suggested in section 4.2, the development of triangular "elements" with curvilinear sides might be useful. Another interesting topic would be the possible handling of "inclusions" and/or "cutouts" in the layered region through "substructuring"; The "inclusions" and/or "cutouts" would have to be explicitly discretized in addition to any cracks that are present in the layered region.
References


Yoffe E.H. (1961), "A Dislocation at a Free Surface.", Phil. Mag., 6, 1147-1155.
Appendix A: Cartesian components of the displacements, gradients of the displacements, strains and stresses in terms of Hansen's potentials

Displacements:

\[ u(0, \delta, \varphi_R, \varphi_L) = e_x \left[ \frac{\partial}{\partial x} \varphi_1 - \frac{\partial}{\partial x} \varphi_2 - 2 \delta z \frac{\partial^2}{\partial x \partial z} \varphi_2 + \frac{\partial}{\partial y} \varphi_3 \right] + e_y \left[ \frac{\partial}{\partial y} \varphi_1 - \frac{\partial}{\partial y} \varphi_2 - 2 \delta z \frac{\partial^2}{\partial y \partial z} \varphi_2 - \frac{\partial}{\partial x} \varphi_3 \right] + e_z \left[ \frac{\partial}{\partial z} \varphi_1 + \frac{\partial}{\partial z} \varphi_2 - 2 \delta z \frac{\partial^2}{\partial z^2} \varphi_2 \right] \]

(A.1)

Gradients of the displacements:

\[ \frac{\partial}{\partial x} u_x = \frac{\partial^2}{\partial x^2} \varphi_1 - \frac{\partial^2}{\partial x^2} \varphi_2 - 2 \delta z \frac{\partial^3}{\partial x \partial z} \varphi_2 + \frac{\partial^2}{\partial x \partial y} \varphi_3 \]

\[ \frac{\partial}{\partial y} u_x = \frac{\partial^2}{\partial x \partial y} \varphi_1 - \frac{\partial^2}{\partial x \partial y} \varphi_2 - 2 \delta z \frac{\partial^3}{\partial x \partial y \partial z} \varphi_2 + \frac{\partial^2}{\partial y^2} \varphi_3 \]

\[ \frac{\partial}{\partial z} u_x = \frac{\partial^2}{\partial x \partial z} \varphi_1 - (1+2\delta) \frac{\partial^2}{\partial z^2} \varphi_2 - 2 \delta z \frac{\partial^3}{\partial z \partial x} \varphi_2 + \frac{\partial^2}{\partial y \partial z} \varphi_3 \]

\[ \frac{\partial}{\partial x} u_y = \frac{\partial^2}{\partial x \partial y} \varphi_1 - \frac{\partial^2}{\partial x \partial y} \varphi_2 - 2 \delta z \frac{\partial^3}{\partial x \partial y \partial z} \varphi_2 - \frac{\partial^2}{\partial x^2} \varphi_3 \]

\[ \frac{\partial}{\partial y} u_y = \frac{\partial^2}{\partial y^2} \varphi_1 - \frac{\partial^2}{\partial y^2} \varphi_2 - 2 \delta z \frac{\partial^3}{\partial y^2 \partial z} \varphi_2 - \frac{\partial^2}{\partial y \partial y} \varphi_3 \]

\[ \frac{\partial}{\partial z} u_y = \frac{\partial^2}{\partial y \partial z} \varphi_1 - (1+2\delta) \frac{\partial^2}{\partial z \partial y} \varphi_2 - 2 \delta z \frac{\partial^3}{\partial z \partial y} \varphi_2 - \frac{\partial^2}{\partial x \partial z} \varphi_3 \]

\[ \frac{\partial}{\partial x} u_z = \frac{\partial^2}{\partial x \partial z} \varphi_1 + \frac{\partial^2}{\partial x \partial z} \varphi_2 - 2 \delta z \frac{\partial^3}{\partial x \partial z} \varphi_2 \]

\[ \frac{\partial}{\partial y} u_z = \frac{\partial^2}{\partial y \partial z} \varphi_1 + \frac{\partial^2}{\partial y \partial z} \varphi_2 - 2 \delta z \frac{\partial^3}{\partial y \partial z} \varphi_2 \]
\[ \frac{\partial u_z}{\partial z} = \frac{\partial^2 \varphi}{\partial z^2} + (1-2\delta) \cdot \frac{\partial^2 \varphi}{\partial x^2} - 2 \cdot \delta \cdot z \cdot \frac{\partial^3 \varphi}{\partial z^3} \]

\[ \text{(A.2)} \]

**Strains:**

\[ \varepsilon_{xx} = \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} - 2 \cdot \delta \cdot z \cdot \frac{\partial^3 \varphi}{\partial x \partial y \partial z} + \frac{\partial^2 \varphi}{\partial x \partial y^3} \]

\[ \varepsilon_{xy} = \frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^2 \varphi}{\partial y \partial z} - 2 \cdot \delta \cdot z \cdot \frac{\partial^3 \varphi}{\partial x \partial y \partial z} + \frac{1}{2} \left( \frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \varphi}{\partial x^2} \right) \]

\[ \varepsilon_{xz} = \frac{\partial^2 \varphi}{\partial x \partial z} - \delta \cdot \frac{\partial^2 \varphi}{\partial x \partial z} - 2 \cdot \delta \cdot z \cdot \frac{\partial^3 \varphi}{\partial z \partial x} + \frac{1}{2} \cdot \frac{\partial^2 \varphi}{\partial y \partial z \partial x} \]

\[ \varepsilon_{yy} = \frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \varphi}{\partial z^2} - 2 \cdot \delta \cdot z \cdot \frac{\partial^3 \varphi}{\partial y \partial z} - \frac{\partial^2 \varphi}{\partial x \partial y^3} \]

\[ \varepsilon_{yz} = \frac{\partial^2 \varphi}{\partial y \partial z} - \delta \cdot \frac{\partial^2 \varphi}{\partial y \partial z} - 2 \cdot \delta \cdot z \cdot \frac{\partial^3 \varphi}{\partial z \partial y} - \frac{1}{2} \cdot \frac{\partial^2 \varphi}{\partial x \partial z} \]

\[ \varepsilon_{zz} = \frac{\partial^2 \varphi}{\partial z^2} + (1-2\delta) \cdot \frac{\partial^2 \varphi}{\partial z^2} - 2 \cdot \delta \cdot z \cdot \frac{\partial^3 \varphi}{\partial z^3} \]

\[ \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 2 \cdot (1-\delta) \cdot \frac{\partial^2 \varphi}{\partial z^2} \]

\[ \text{(A.3)} \]

**Stresses:**

\[ \sigma_{xx} = 2\mu \cdot \frac{\partial^2 \varphi}{\partial x^2} - 2\mu \cdot \frac{\partial^2 \varphi}{\partial y^2} - 4\mu \delta \cdot z \cdot \frac{\partial^3 \varphi}{\partial x \partial y \partial z} + 2\mu \cdot (3\delta - 1) \cdot \frac{\partial^2 \varphi}{\partial z^2} \]

\[ + 2\mu \cdot \frac{\partial^2 \varphi}{\partial x \partial y^3} \]

\[ \sigma_{xy} = 2\mu \cdot \frac{\partial^2 \varphi}{\partial x \partial y} - 2\mu \cdot \frac{\partial^2 \varphi}{\partial x \partial y^3} - 4\mu \delta \cdot z \cdot \frac{\partial^3 \varphi}{\partial x \partial y \partial z} + \mu \cdot \frac{\partial^2 \varphi}{\partial y^2} \]

\[ + \mu \cdot \left( \frac{\partial^2 \varphi}{\partial x \partial y^3} - \frac{\partial^2 \varphi}{\partial x^2} \right) \]

\[ \sigma_{xz} = 2\mu \cdot \frac{\partial^2 \varphi}{\partial x \partial z} - 2\mu \delta \cdot \frac{\partial^2 \varphi}{\partial x \partial z} - 4\mu \delta \cdot z \cdot \frac{\partial^3 \varphi}{\partial z^2 \partial x} + \mu \cdot \frac{\partial^2 \varphi}{\partial y \partial z} \]

\[ + \mu \cdot \frac{\partial^2 \varphi}{\partial x \partial z} \]
\begin{align*}
\sigma_{yy} &= 2\mu \cdot \frac{\partial^2}{\partial y^2} \phi_1 - 2\mu \cdot \frac{\partial^2}{\partial y^2} \phi_2 - 4\mu \delta \cdot z \cdot \frac{\partial^3}{\partial y^2 \partial z} \phi_2 + 2\mu \cdot (3\delta - 1) \cdot \frac{\partial^2}{\partial z^2} \phi_2 \\
&\quad - 2\mu \cdot \frac{\partial}{\partial x \partial y} \phi_3 \\
\sigma_{yz} &= 2\mu \cdot \frac{\partial^2}{\partial y \partial z} \phi_1 - 2\mu \delta \cdot \frac{\partial^2}{\partial y \partial z} \phi_2 - 4\mu \delta \cdot z \cdot \frac{\partial^3}{\partial z^2 \partial y} \phi_2 - \mu \cdot \frac{\partial^2}{\partial x \partial z} \phi_3 \\
\sigma_{zz} &= 2\mu \cdot \frac{\partial^2}{\partial z^2} \phi_1 + 2\mu \delta \cdot \frac{\partial^2}{\partial z^2} \phi_2 - 4\mu \delta \cdot z \cdot \frac{\partial^3}{\partial z^3} \phi_2
\end{align*}

(A.4)

Note: $2\lambda \cdot (1 - \delta) = 2\mu \cdot (3\delta - 1)$
Appendix B: Sample potentials for some point sources

Point Force:

The displacement field due to a point force at the origin can be written as:

\[
\mathbf{u}_i = \frac{1}{4\pi \mu (1+\delta)} \left[ p_i \frac{1}{r} + \delta \cdot p_k \cdot x_k \cdot x_i \cdot \frac{1}{r^3} \right] \tag{B.1}
\]

where:

\[
\delta = \frac{\lambda + \mu}{\lambda + 3\mu} = \frac{1}{\kappa} = \frac{1}{3-4\nu}
\]

\[
r^2 = x^2 + y^2 + z^2
\]

\[p_i\] is the magnitude of the point force in the \(i^{th}\)-direction.

For the point force it can then be checked that the Hansen potentials are:

\[
\varphi_1 = \frac{\beta}{2} \left[ p_1 \cdot \frac{x}{r^2} + p_2 \cdot \frac{y}{r^2} \pm p_3 \cdot \ln(r^2) \right]
\]

\[
\varphi_2 = \frac{\beta}{2} \left[ -p_1 \cdot \frac{x}{r^2} - p_2 \cdot \frac{y}{r^2} \pm p_3 \cdot \ln(r^2) \right] \tag{B.2}
\]

\[
\varphi_3 = \beta \left[ p_1 \cdot (1+\delta) \cdot \frac{y}{r^2} - p_2 \cdot (1+\delta) \cdot \frac{x}{r^2} \right]
\]

where:

\[
\beta = 1/[4\pi \mu (1+\delta)]
\]

Note that if the upper (lower) "sign" is chosen in one expression, the upper (lower) "signs" must be chosen throughout for all the potentials. Also note that taking \(r+z\) (\(r-z\)) in the expressions makes the potentials (but not necessarily the
displacements) singular when \( x=y=0 \) and \( z<0 \) (\( z>0 \)).

**Line forces at \( x=0 \) parallel to the \( z=0 \) plane**

The displacement field due to a line force can be written (for plain strain) as:

\[
\begin{align*}
    u_1 &= \frac{\alpha}{4\pi\mu\delta} \left[ -p_1 \cdot \ln \xi + \delta \cdot p_k \cdot x_k \cdot x_1 \cdot \frac{1}{\xi^2} \right] \quad \text{for } i, k = 1, 3 \\
    u_2 &= 0 \\
    \xi^2 &= x^2 + z^2
\end{align*}
\]

where:
\[
\alpha = \frac{\lambda + \mu}{\lambda + 2\mu} \quad \delta = \frac{\lambda + \mu}{\lambda + 3\mu}
\]

\( p_1 \) and \( p_3 \) are the magnitude of the line forces

For the line force it can be checked that the Hansen potentials are:

\[
\begin{align*}
    \varphi_1 &= \frac{\alpha}{8\pi\mu\delta} \left[ -p_1 \cdot \left[ z \cdot \arctan\left( \frac{z}{x} \right) \right] - x \cdot \ln \xi + (1+\delta) \cdot x \right] \\
    \varphi_2 &= \frac{\alpha}{8\pi\mu\delta} \left[ -p_1 \cdot \left[ z \cdot \arctan\left( \frac{z}{x} \right) \right] - x \cdot \ln \xi + (1+\delta) \cdot x \right] \\
    \varphi_3 &= 0
\end{align*}
\]

\[ (B.4) \]
Dislocations parallel to the z=0 plane:

The displacement due to a dislocation along the y-axis (plane strain) can be written as:

\[ u_1 = \left[ \frac{d_1}{2\pi} \cdot \arctan \left( \frac{z}{x} \right) - \epsilon_{ik} \frac{d_k}{2\pi} \cdot 1 \ln \xi \right] \]

\[ - \frac{a}{4\pi \mu \delta} \left[ -(2\mu \epsilon_{ik} d_k) \cdot 1 \ln \xi + (2\mu \epsilon_{nk} d_k) \cdot \delta \cdot x_n \cdot x_i \cdot \frac{1}{\xi^2} \right] \]

for \( i, k, n = 1, 3 \)

and:

\[ u_2 = 0 \]  \hspace{1cm} (B.5)

where:

\[ \epsilon_{ik} = \begin{cases} 
+1 & \text{for } i = 1, k = 3 \\
-1 & \text{for } i = 3, k = 1 \\
0 & \text{otherwise} 
\end{cases} \]

\( d_1 \) and \( d_3 \) are the slip magnitude of the dislocations

We note that the terms in the second brackets expressing the displacements are of the form of line force expressions with equivalent magnitudes of \( 2\mu \epsilon_{ik} d_k \) and thus their Hansen potentials are already known. The Hansen potentials for the terms in the first bracket can be shown to be:

\[ \varphi_1^{\text{first bracket}} = \frac{1}{2\pi} \left[ d_1 \left[ z \cdot \ln \xi - z + x \cdot \arctan \left( \frac{z}{x} \right) \right] \right. \]

\[ + d_3 \left[ z \cdot \arctan \left( \frac{z}{x} \right) - x \cdot \ln \xi + x \right] \]

\[ \varphi_2^{\text{first bracket}} = 0 \]

\[ \varphi_3^{\text{first bracket}} = 0 \]  \hspace{1cm} (B.6)
Appendix C: Derivation of the image method algorithm

The eigenfunction expansion method for elasticity problems in layered media was first formulated by Ben-Menahem and Singh 1968. We have used this method to derive the algorithm discussed in this paper. The notation (as far as possible) is the same as in the 1968 reference paper, although some new temporary terms have been defined in order to simplify the algebra for this specific implementation.

Any elastic displacement field satisfying the equilibrium equations:

\[ \nabla^2 \mathbf{u} + (1 + \lambda/\mu) \cdot \nabla \cdot \mathbf{u} = 0 \]  \hspace{1cm} (C.1)

can be written as the sum of \( N, F \) and \( M \):

\[ N = \nabla^2 \varphi_1 \]
\[ F = 2 \cdot \mathbf{e}_z \cdot \frac{\partial}{\partial z} \varphi_2 - \nabla \varphi_2 - 2 \cdot \mathbf{e}_z \cdot \nabla \cdot \frac{\partial}{\partial z} \varphi_2 \]  \hspace{1cm} (C.2)
\[ M = \nabla \times [\mathbf{e}_z \cdot \varphi_3] \]

where: \( \delta \equiv (\lambda + \mu)/(\lambda + 3\mu) \)

\[ \nabla^2 \varphi_1 = \nabla^2 \varphi_2 = \nabla^2 \varphi_3 = 0 \]

Using the method of the separation of variables in cylindrical coordinates on the potentials \( \varphi_1, \varphi_2 \) and \( \varphi_3 \) in the form:

\[ \varphi = R(r) \cdot F(\theta) \cdot Z(z) \]  \hspace{1cm} (C.3)

We get:

\[ \varphi = \exp(\pm kr) \cdot J_m(\pm k\theta) \cdot \exp(\pm \pm \theta) \]

where:

\( J_m \) is Bessel's function of the first kind \( m \)th order
Reexpressing \( p_1, p_2 \) and \( p_3 \) in the above form and carrying out the \( v \) and \( v \cdot x \) and \( \frac{\partial}{\partial z} \) operations, we get:

\[
\psi = \sum_{m=1}^{\infty} \int_{0}^{\infty} \left[ A_m^+ \cdot N_m^+ + A_m^- \cdot N_m^- + B_m^+ \cdot F_m^+ + B_m^- \cdot F_m^- + C_m^+ \cdot M_m^+ + C_m^- \cdot M_m^- \right] \cdot dk
\]

(C.4)

where: \( A_m^+, B_m^+ \) and \( C_m^+ \) are constants dependent on 'm' and on "h" only.

\[
N_m^+ = \exp(\pm k z) \cdot \left[ \pm p_m + B_m \right]
\]

\[
P_m^+ = \exp(\pm k z) \cdot \left[ (\pm z-2\pi k z) \cdot p_m - (1 \pm 2\pi k z) \cdot B_m \right]
\]

\[
M_m^+ = \exp(\pm k z) \cdot C_m
\]

and:

\[
p_m = e_z \cdot J_m(\text{k}r) \cdot \exp(\text{i}m\theta)
\]

\[
B_m = (e_r \cdot \frac{\partial}{\partial k r} + e_\theta \cdot \frac{1}{k r} \cdot \frac{\partial}{\partial \theta}) J_m(\text{k}r) \cdot \exp(\text{i}m\theta)
\]

\[
C_m = (e_r \cdot \frac{1}{k r} \cdot \frac{\partial}{\partial \theta} - e_\theta \cdot \frac{\partial}{\partial k r}) J_m(\text{k}r) \cdot \exp(\text{i}m\theta)
\]

(C.5)

In the above expressions for \( p_m, B_m \) and \( C_m \) there is the implicit understanding that we can consider either the real or imaginary components of the expressions seperately.

From the above expressions for the displacements, we can find the expressions for the tractions at a plane \( z=\text{constant} \), and we rewrite the above as:
\[ u = \sum_{m=1}^{\infty} \int_{0}^{\infty} (u^R_m + u^L_m) \cdot dk \]  
(C.6)

\[ T^Z = \sum_{m=1}^{\infty} \int_{0}^{\infty} (T^R_m + T^L_m) \cdot dk \]  
(C.7)

where:

\[ u^R_m = x_m \cdot p_m + y_m \cdot b_m \]
\[ T^R_m = 2k \cdot x_m \cdot p_m + 2k \cdot y_m \cdot b_m \]
\[ u^L_m = z_m \cdot c_m \]
\[ T^L_m = k \cdot z_m \cdot c_m \]  
(C.8)

and:

\[ x_m = A_m^+ \cdot \exp(kz) - A_m^- \cdot \exp(-kz) \]
\[ + B_m^+ \cdot (1-2\delta kz) \cdot \exp(kz) + B_m^- \cdot (-1-2\delta kz) \cdot \exp(-kz) \]

\[ y_m = A_m^+ \cdot \exp(kz) + A_m^- \cdot \exp(-kz) \]
\[ + B_m^+ \cdot (-1-2\delta kz) \cdot \exp(kz) + B_m^- \cdot (-1+2\delta kz) \cdot \exp(-kz) \]

\[ z_m = c_m^+ \cdot \exp(kz) + c_m^- \cdot \exp(-kz) \]

\[ X_m = \mu \cdot A_m^+ \cdot \exp(kz) + \mu \cdot A_m^- \cdot \exp(-kz) \]
\[ + \mu \delta \cdot (1-2\delta kz) \cdot \exp(kz) + \mu \delta \cdot (1+2\delta kz) \cdot \exp(-kz) \]

\[ Y_m = \mu \cdot A_m^+ \cdot \exp(kz) - \mu \cdot A_m^- \cdot \exp(-kz) \]
\[ + \mu \delta \cdot (-1-2\delta kz) \cdot \exp(kz) + \mu \delta \cdot (1-2\delta kz) \cdot \exp(-kz) \]

\[ Z_m = c_m^+ \cdot \exp(kz) - c_m^- \cdot \exp(-kz) \]  
(C.9)
We notice that the $u^R_m$ components are uncoupled from the $u^L_m$ in the sense that the $A^\pm_m$, $B^\pm_m$ coefficients do not affect the $u^L_m$ components and the $C^\pm_m$ coefficients do not affect the $u^R_m$ components. We therefore treat the $u^R_m$ and the $u^L_m$ components separately when analyzing a specific problem in terms of the Hansen potentials.

Now we consider the specific geometry shown in figure C.1. The region consists of two elastic materials separated by a planar interface. The material elasticity parameters used to characterize the regions are taken to be $\mu_1$, $\delta_1$, and $\mu_2$, $\delta_2$. A point source exists at the position $z=-h$. We are required to find the displacement fields for region 1 ($z<0$) and for region 2 ($z>0$) under the influence of the point source, such that the displacements and the tractions are continuous across the interface plane ($z=0$).

In what follows, we are manipulating $u^R_m$ and $u^L_m$ in equation C.6 for a fixed 'm', but the 'm' subscript will be dropped. First, we express the displacement and traction (on a z-plane) for $(z+h)>0$ (which includes $z=0$) of a point source of arbitrary nature (using the eigenfunction expansion method and expressing the m'th component in matrix form) in the following way:

$$
\begin{bmatrix}
u(P) \\
u(B) \\
T(P) \\
T(B)
\end{bmatrix}
= \begin{bmatrix}
-1 & -1 - 26_1k \cdot (z+h) \\
+1 & -1 + 26_1k \cdot (z+h) \\
+2k\mu_1 & 2k\mu_1\delta_1 \cdot [1+2k \cdot (z+h)] \\
-2k\mu_1 & +2k\mu_1\delta_1 \cdot [1-2k \cdot (z+h)]
\end{bmatrix}
\cdot \begin{bmatrix}
A^-_0 \\
B^-_0 \end{bmatrix} \cdot \exp(-k|z+h|)
$$
Figure C.1
\[
\begin{bmatrix}
\frac{u(C)}{T(C)}
\end{bmatrix}_0 = \begin{bmatrix}
+1 \\
-k\mu_1
\end{bmatrix} \cdot \frac{C_0}{\exp(-k|z+h|)}
\]

(C.10)

and we define:

\[
\begin{bmatrix}
\frac{S_1}{S_2} \\
\frac{S_3}{S_4}
\end{bmatrix} = -1 \cdot \begin{bmatrix}
\frac{u(P)}{u(B)} \\
\frac{T(P)}{T(B)}
\end{bmatrix}_0 
\begin{bmatrix}
\frac{S_5}{S_6}
\end{bmatrix} = -1 \cdot \begin{bmatrix}
\frac{u(C)}{T(C)}
\end{bmatrix}_0 
\]

(C.11)

Now the elastic fields in region 1 are expressible as:

\[
\begin{bmatrix}
\frac{u(P)}{u(B)} \\
\frac{T(P)}{T(B)}
\end{bmatrix}_1 = \begin{bmatrix}
+1 \\
+2k\mu_1
\end{bmatrix} \cdot \begin{bmatrix}
+1 - 2\delta_1 k \cdot z \\
+2k\mu_1 \delta_1 \cdot (1-2kz)
\end{bmatrix} \cdot \begin{bmatrix}
A_1^+ \\
B_1^+
\end{bmatrix} \cdot \exp(kz) 
\]

\[
+ \begin{bmatrix}
\text{terms due to the point source as given above}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{u(C)}{T(C)}
\end{bmatrix}_1 = \begin{bmatrix}
+1 \\
+k\mu_1
\end{bmatrix} \cdot \frac{C_1^+}{\exp(kz)} + \begin{bmatrix}
\text{terms due to the point source as given above}
\end{bmatrix}
\]

(C.12)

And the elastic fields in region 2 are expressible as:
\[
\begin{bmatrix}
    u(P) \\
    u(B) \\
    T(P) \\
    T(B)
\end{bmatrix}
\text{2} 
= 
\begin{bmatrix}
    -1 & -1 - 2\delta_1 kz \\
    +1 & -1 + 2\delta_1 kz \\
    +2k\mu_1 & +2k\mu_1\delta_1 \cdot (1+2kz) \\
    -2k\mu_1 & +2k\mu_2\delta_2 \cdot (1-2kz)
\end{bmatrix}
\cdot 
\begin{bmatrix}
    A^-_2 \\
    B^-_2
\end{bmatrix}
\cdot \exp(-kz)
\]

\[
\begin{bmatrix}
    u(C) \\
    T(C)
\end{bmatrix}
\text{2} 
= 
\begin{bmatrix}
    +1 \\
    -k\mu_2
\end{bmatrix}
\cdot 
\begin{bmatrix}
    C^-_2
\end{bmatrix}
\cdot \exp(-kz)
\]

\begin{equation}
\text{(C.13)}
\end{equation}

Applying the condition that \( u_m \) and \( T_m \) are to be continuous (for each \( m \)) along the interface plane \( z=0 \), we get:

\[
\begin{bmatrix}
    +1 \\
    +1 \\
    +1 \\
    +1
\end{bmatrix}
\cdot 
\begin{bmatrix}
    +1 \\
    -1 \\
    +2k\mu_1 \\
    +2k\mu_1\delta_1 \\
    -2k\mu_2 \\
    -2k\mu_2\delta_2 \\
    +2k\mu_1 \\
    -2k\mu_1\delta_1 \\
    +2k\mu_2 \\
    -2k\mu_2\delta_2
\end{bmatrix}
\cdot 
\begin{bmatrix}
    A^+_1 \\
    B^+_1 \\
    A^-_2 \\
    B^-_2
\end{bmatrix}
\quad = 
\begin{bmatrix}
    S_1 \\
    S_2 \\
    S_3 \\
    S_4
\end{bmatrix}
\]

\[
\begin{bmatrix}
    +1 \\
    -k\mu_1 \\
    -k\mu_2
\end{bmatrix}
\cdot 
\begin{bmatrix}
    C^+_1 \\
    C^-_1 \\
    C^-_2
\end{bmatrix}
\quad = 
\begin{bmatrix}
    S_5 \\
    S_6
\end{bmatrix}
\]

\begin{equation}
\text{(C.14)}
\end{equation}

Now we solve for \( A^+_1, B^+_1, C^+_1 \) and \( A^-_2, B^-_2, C^-_2 \) by inverting the 4x4 and 2x2 system of equations. We obtain:
\[
\begin{bmatrix}
A_1^+ \\
B_1^+ \\
A_2^- \\
B_2^-
\end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix}
+4/2 - \gamma - \delta_1 & +4/2 - \gamma - \delta_1 & +7/2 + \delta_1 & +1/2 + \delta_1 \\
+4/2 - \gamma - \delta_1 & -4/2 + \gamma - \delta_1 & +7/2 + \delta_1 & +1/2 + \delta_1 \\
+\gamma \delta_2 & +\gamma & +\gamma \delta_1 & +\gamma \\
+\gamma & +\gamma \delta_2 & +\gamma & +\gamma \\
\end{bmatrix} \begin{bmatrix}
S_1 \\
S_2 \\
S_3 \\
S_4
\end{bmatrix}
\]

\[
\begin{bmatrix}
C_1^+ \\
C_2^-
\end{bmatrix} = \frac{1}{\gamma + 1} \begin{bmatrix}
\gamma + 1 & \gamma + 1 \\
-1 & -1
\end{bmatrix} \begin{bmatrix}
S_5 \\
S_6
\end{bmatrix}
\]

(C.15)

where:
\[
\gamma \equiv \mu_2 / \mu_1
\]
\[
\Delta \equiv 2 \cdot (\gamma + \delta_1) \cdot (\gamma \delta_2 + 1)
\]

Expressing the S's in terms of \( A_0^- \), \( B_0^- \) and \( C_0^- \), and simplifying the expressions we get:

\[
\begin{bmatrix}
A_1^+ \\
B_1^+ \\
A_2^- \\
B_2^-
\end{bmatrix} = \begin{bmatrix}
0 & 1 - b \\
1 - a & 2 \gamma_1 \cdot (1 - a) \cdot k \phi \\
a & 2 \gamma_1 a \cdot k \phi \\
0 & b
\end{bmatrix} \begin{bmatrix}
A_0^- \\
B_0^-
\end{bmatrix} \cdot \exp(-k \phi)
\]

\[
\begin{bmatrix}
C_1^- \\
C_2^-
\end{bmatrix} = \begin{bmatrix}
\gamma + 1 \\
\gamma + 1 \\
\gamma + 1 \\
\gamma + 1
\end{bmatrix} \begin{bmatrix}
C_0^- 
\end{bmatrix} \cdot \exp(-k \phi)
\]

(C.16)

where:
\[
a \equiv (\delta_1 + 1) / (\gamma + \delta_1)
\]

(b) \( (\delta_1 + 1) / (\gamma \cdot \delta_2 + 1) \)

(C.17)

Therefore we find that the displacement field (for a given m)
in region 1 and region 2 can be written as:

\[
\left[ \frac{u(P)}{u(B)} \right]_1 = \begin{bmatrix} +1 \\ +1 \end{bmatrix} \begin{bmatrix} +1-2\delta_1 kz \\ -1-2\delta_1 kz \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1-a \end{bmatrix} \cdot \begin{bmatrix} 1-b \\ 2\delta_1 \cdot (1-a) \cdot kh \end{bmatrix} \cdot \frac{A^-}{B^-} \cdot \exp[k(z-h)] \\
\begin{bmatrix} A_0^- \\ B_0^- \end{bmatrix} \cdot \exp[k(z-h)]
\]

+ \left[ \text{source terms} \right]

\[
\left[ \frac{u(C)}{u(B)} \right]_1 = \begin{bmatrix} +1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1-\gamma}{\gamma+\gamma} \end{bmatrix} \cdot \begin{bmatrix} C^- \end{bmatrix} \cdot \exp[k(z-h)] + \left[ \text{source terms} \right]
\]

(C.18)

\[
\left[ \frac{u(P)}{u(B)} \right]_2 = \begin{bmatrix} -1 \\ +1 \end{bmatrix} \begin{bmatrix} -1-2\delta_1 kz \\ -1+2\delta_1 kz \end{bmatrix} \cdot \begin{bmatrix} +a \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2\delta_1 \cdot kh \\ +b \end{bmatrix} \cdot \frac{A^-}{B^-} \cdot \exp[-k(z+h)]
\]

\[
\left[ \frac{u(C)}{u(B)} \right]_2 = \begin{bmatrix} +1 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{\gamma+\gamma} \end{bmatrix} \cdot \begin{bmatrix} C^- \end{bmatrix} \cdot \exp[-k(z+h)]
\]

(C.19)

The \( y(C) \) terms are in a form from which we can deduce the algorithm, however, the \( y(P) \) and \( y(B) \) terms have to be further manipulated. We now try to express the \( y(P) \) and \( y(B) \) terms in the following manner:

\[
\left[ \frac{u(P)}{u(B)} \right]_1 = \begin{bmatrix} +1 \\ +1 \end{bmatrix} \begin{bmatrix} +1-2\delta_1 k \cdot (z-h) \\ -1-2\delta_1 k \cdot (z-h) \end{bmatrix} \cdot \begin{bmatrix} A^+ \\ B^+ \end{bmatrix} \cdot \exp[k(z-h)] \\
\begin{bmatrix} A_a^+ \\ B_a^+ \end{bmatrix}
\]

+ \frac{\partial}{\partial z} \begin{bmatrix} +1 \\ +1 \end{bmatrix} \begin{bmatrix} +1-2\delta_1 k \cdot (z-h) \\ -1-2\delta_1 k \cdot (z-h) \end{bmatrix} \cdot \begin{bmatrix} A_b^+ \\ B_b^+ \end{bmatrix} \cdot \exp[k(z-h)]
\[ + \frac{\partial^2}{\partial z^2} \begin{bmatrix} +1 \\ +1 \end{bmatrix} \cdot \begin{bmatrix} A_c^+ \\ \end{bmatrix} \cdot \exp[k(z-h)] + \text{source terms} \]

\[ \begin{bmatrix} u(P) \\ u(B) \end{bmatrix}_2 = \begin{bmatrix} -1 \\ +1 \end{bmatrix} \begin{bmatrix} -1-\delta_1 k \cdot (z+h) \\ +1-1+2\delta_1 k \cdot (z+h) \end{bmatrix} \cdot \begin{bmatrix} A_a^- \\ B_a^- \end{bmatrix} \cdot \exp[-k(z+h)] \]

\[ + \frac{\partial}{\partial z} \begin{bmatrix} -1 \\ +1 \end{bmatrix} \begin{bmatrix} -1-\delta_1 k \cdot (z+h) \\ +1-1+2\delta_1 k \cdot (z+h) \end{bmatrix} \cdot \begin{bmatrix} A_b^- \\ B_b^- \end{bmatrix} \cdot \exp[-k(z+h)] \]

\[ + \frac{\partial^2}{\partial z^2} \begin{bmatrix} -1 \\ +1 \end{bmatrix} \cdot \begin{bmatrix} A_c^- \end{bmatrix} \cdot \exp[-k(z+h)] \]

(C.21)

Noting that:

\[ \frac{\delta^n}{\delta z^n} \exp[k(z-h)] = k^n \cdot \exp[k(z-h)] \]

and

\[ \frac{\delta^n}{\delta z^n} \exp[-k(z+h)] = (-k)^n \cdot \exp[-k(z+h)] \]

(C.22)

We obtain:

\[ A_a^+ = (1-b) \cdot B_0^- \quad B_a^+ = (1-a) \cdot A_0^- \]

\[ A_b^+ = -2\delta_1 \cdot (1-a) \cdot h \cdot A_0^- + 4\delta_1^2 \cdot (1-a) \cdot h \cdot B_0^- \]  

(C.23)

\[ B_b^+ = 2\delta_1 \cdot (1-a) \cdot h \cdot B_0^- \quad A_c^+ = -4\delta_1^2 \cdot (1-a) \cdot h^2 \cdot B_0^- \]
and:

\[
\begin{align*}
A_a^{-} &= a \cdot A_0^{-} \\
B_a^{-} &= b \cdot B_0^{-} \\
A_b^{-} &= 2 \cdot (\delta_1 b - \delta_1 a) \cdot h \cdot B_0^{-} \\
B_b^{-} &= A_c^{-} = 0
\end{align*}
\]  

(C.24)

Since the above relations are true for each component of a potential, then they must be true for the whole potential and we get:

if:

\[
y^0 = N(-h, r_1^0) + F(\delta_1, -h, r_2^0) + M(-h, r_3^0)
\]

Then:

\[
y^1 = y^0 + N(h, (1-b) \cdot r_2^0) + \frac{\partial}{\partial z} N(h, -2\delta_1 \cdot (1-a) \cdot h \cdot r_1^0)
\]

\[
+ \frac{\partial^2}{\partial z^2} N(h, -4\delta_1^2 \cdot (1-a) \cdot h^2 \cdot r_2^0)
\]

\[
+ F(\delta_1, h, (1-a) r_1^0)
\]

\[
+ \frac{\partial}{\partial z} F(\delta_1, h, 2\delta_1 \cdot (1-a) \cdot h \cdot r_2^0)
\]

\[
+ M(h, \frac{1-\gamma}{1+\gamma} r_3^0)
\]

(C.25)

\[
y^2 = N(-h, a r_1^0) + \frac{\partial}{\partial z} N(-h, 2 \cdot (\delta_1 b - \delta_1 a) \cdot h \cdot r_2^0)
\]

\[
+ F(\delta_1, -h, b r_2^0)
\]

\[
+ M(-h, \frac{2}{1+\gamma} r_3^0)
\]

(C.26)
Noting that:

\[ \frac{\delta^n}{\delta z^n} N(h, \text{cst} \cdot \varphi) = N(h, \text{cst} \cdot \frac{\delta^n}{\delta z^n} \varphi) \]

and

\[ \frac{\delta}{\delta z} F(\delta, h, \text{cst} \cdot \varphi) = N(h, -2\delta, \text{cst} \cdot \frac{\delta}{\delta z} \varphi) + F(\delta, h, \text{cst} \cdot \frac{\delta}{\delta z} \varphi) \]  \hspace{1cm} (C.27)

where: \text{cst} is a constant

One obtains the algorithm given in chapter 2.3 of this thesis (two minor differences are: i) The statement of the algorithm in the paper considers region 1 to be at \( z>0 \) and hence \( z=+h \) instead of \( z=-h \) to be the location of the source point and ii) A formalism in terms of matrix operators is implemented in chapter 2.3).
Appendix D: Analytic check of the image method algorithm

Referring to figure D.1 and using the notation defined in the main text, the image algorithm states that if:

\[ u^0 = \mathcal{H}(h, \varphi_1^0) + \mathcal{E}(\delta_1^0, h, \varphi_2^0) + \mathcal{M}(h, \varphi_3^0) \]

Then:

\[ u^1 = u^0 + u^1(-h, \delta_1^1, \varphi_R^1, \varphi_L^1) \]
\[ u^2 = u^2(h, \delta_2^2, \varphi_R^2, \varphi_L^2) \]

where:

\[ \varphi_R^1 = \varphi_R^0(h, a, b, \delta_1^1) \cdot \varphi_R^0 \]
\[ \varphi_L^1 = \varphi_L^0(\gamma) \cdot \varphi_L^0 \]
\[ \varphi_R^2 = \varphi_R^0(h, a, b, \delta_2^2, \delta_1^1) \cdot \varphi_R^0 \]
\[ \varphi_L^2 = \varphi_L^0(\gamma) \cdot \varphi_L^0 \]

(D.1)

We have already expressed the Cartesian components of the displacement and stress fields of any given Hansen potentials (Appendix A). In this section, we will check whether the Cartesian components of the displacement and stress fields (in terms of Hansen's potentials) that are generated by the image algorithm satisfy the conditions of displacement and traction continuity along the interface plane.

In order to simplify the checking of the algorithm, we will consider the following 3 cases separately:

i) \( \varphi_1 = \varphi_0; \quad \varphi_2 = \varphi_3 = 0 \)

ii) \( \varphi_2 = \varphi_0; \quad \varphi_1 = \varphi_3 = 0 \)

iii) \( \varphi_3 = \varphi_0; \quad \varphi_1 = \varphi_2 = 0 \)
Figure D.1
We will also note the following:

\[
\frac{\partial}{\partial z} \varphi = \left[ \frac{\partial}{\partial (-z)} \varphi \right] \quad \Rightarrow \quad \frac{\partial}{\partial z} \varphi \bigg|_{z=0} = -\frac{\partial}{\partial z} \varphi \bigg|_{z=0} \tag{D.2}
\]

\[
\frac{\partial}{\partial x} \varphi \bigg|_{z=0} = \frac{\partial}{\partial x} \varphi \bigg|_{z=0} \quad \text{and} \quad \frac{\partial}{\partial y} \varphi \bigg|_{z=0} = \frac{\partial}{\partial y} \varphi \bigg|_{z=0}
\]

Also define:

\[
\frac{\partial^n}{\partial (x,y,z)^n} \varphi \bigg|_{z=0} = \frac{\partial^n}{\partial (x,y,z)^n} \varphi \bigg|_{z=0} \tag{D.3}
\]

\[
\frac{\partial^n}{\partial (x,y,z)^n} \varphi \bigg|_{z=0} = \frac{\partial^n}{\partial (x,y,z)^n} \varphi \bigg|_{z=0}
\]

Case I: \( \varphi_1 = \varphi_0 \); \( \varphi_2 = \varphi_3 = 0 \)

\[
u^1 \bigg|_{z=0} = \hat{e}_x \left[ \frac{\partial}{\partial x} \varphi + 2 \delta_1 \cdot (1-a) \cdot h \cdot \frac{\partial^2}{\partial x \partial z} \varphi + (1-a) \cdot \left( \frac{\partial}{\partial y} \varphi - 2 \delta_1 \cdot h \cdot \frac{\partial^2}{\partial y \partial z} \varphi \right) \right]
+ \hat{e}_y \left[ \frac{\partial}{\partial y} \varphi + 2 \delta_1 \cdot (1-a) \cdot h \cdot \frac{\partial^2}{\partial y \partial z} \varphi + (1-a) \cdot \left( \frac{\partial}{\partial z} \varphi + 2 \delta_1 \cdot h \cdot \frac{\partial^2}{\partial z \partial z} \varphi \right) \right]
+ \hat{e}_z \left[ \frac{\partial}{\partial z} \varphi + 2 \delta_1 \cdot (1-a) \cdot h \cdot \frac{\partial^2}{\partial z \partial z} \varphi + (1-a) \cdot \left( \frac{\partial}{\partial z} \varphi + 2 \delta_1 \cdot h \cdot \frac{\partial^2}{\partial z \partial z} \varphi \right) \right] \tag{D.4}
\]

\[
u^2 \bigg|_{z=0} = \hat{e}_x \left[ a \cdot \frac{\partial}{\partial x} \varphi \right] + \hat{e}_y \left[ a \cdot \frac{\partial}{\partial y} \varphi \right] + \hat{e}_z \left[ a \cdot \frac{\partial}{\partial z} \varphi \right] \tag{D.5}
\]

We notice that the displacement field is continuous across \( z=0 \) (the interface plane). Next we consider the \( T^z \) traction continuity
(equilibrium).

\[
T_{z=0}^{z_1} = \hat{e}_x \cdot \left[ 2\mu_1 \frac{\partial^2}{\partial x \partial z} \varphi + 2\delta_1 (1-a) \cdot 2\mu_1 h \cdot \frac{\partial^3}{\partial x \partial \varphi} 
+ (1-a) \cdot (-2\mu_1 \delta_1 \frac{\partial^2}{\partial x \partial z} \varphi - 4\mu_1 \delta_1 h \cdot \frac{\partial^3}{\partial x \partial \varphi}) \right] 
+ \hat{e}_y \cdot \left[ 2\mu_1 \frac{\partial^2}{\partial y \partial z} \varphi + 2\delta_1 (1-a) \cdot 2\mu_1 h \cdot \frac{\partial^3}{\partial y \partial \varphi} 
+ (1-a) \cdot (-2\mu_1 \delta_1 \frac{\partial^2}{\partial y \partial z} \varphi - 4\mu_1 \delta_1 h \cdot \frac{\partial^3}{\partial y \partial \varphi}) \right] 
+ \hat{e}_z \cdot \left[ 2\mu_1 \frac{\partial^2}{\partial z \varphi} + 2\delta_1 (1-a) \cdot 2\mu_1 h \cdot \frac{\partial^3}{\partial z \varphi} 
+ (1-a) \cdot (2\mu_1 \delta_1 \frac{\partial^2}{\partial z \varphi} - 4\mu_1 \delta_1 h \cdot \frac{\partial^3}{\partial z \varphi}) \right] 
\]

(D.6)

\[
T_{z=0}^{z_2} = \hat{e}_x \cdot \left[ 2\mu_2 a \cdot \frac{\partial^2}{\partial x \partial z} \varphi \right] 
+ \hat{e}_y \cdot \left[ 2\mu_2 a \cdot \frac{\partial^2}{\partial y \partial z} \varphi \right] 
+ \hat{e}_z \cdot \left[ 2\mu_2 a \cdot \frac{\partial^2}{\partial z \varphi} \right] 
\]

(D.7)

Noting that:

\[2\mu_2 \cdot [1 + (1-a) \cdot \delta_1] = 2\mu_2 a\]

since

\[
2\mu_1 \left[ \frac{\gamma + \delta_1 + \delta_1 \gamma - \delta_1}{\gamma + \delta_1} \right] = 2\mu_2 \cdot \frac{\delta_1 + 1}{\gamma + \delta_1}
\]

(D.8)

We find that the tractions across the interface plane \((z=0)\)
Case II: \( \omega_2 = \omega_0 : \omega_1 = \omega_3 = 0 \)

\[
\begin{array}{c}
\begin{aligned}
u^1 \bigg|_{z=0} &= \hat{e}_x \left[ - \frac{\partial}{\partial x} \varphi + 2 \delta_1 \cdot \frac{\partial}{\partial x} \varphi + (1-b) \cdot \frac{\partial}{\partial x} \varphi - 4 \delta_1 \cdot (1-a) \cdot h^2 \frac{\partial}{\partial x} \frac{\partial}{\partial z} \varphi \\
&\quad - 2 \delta_1 \cdot (1-a) \cdot h \cdot \left( - \frac{\partial}{\partial x} \varphi - 2 \delta_1 \cdot h \cdot \frac{\partial}{\partial x} \frac{\partial}{\partial z} \varphi \right) \right] \\
&+ \hat{e}_y \left[ - \frac{\partial}{\partial y} \varphi + 2 \delta_1 \cdot \frac{\partial}{\partial y} \varphi + (1-b) \cdot \frac{\partial}{\partial y} \varphi - 4 \delta_1 \cdot (1-a) \cdot h^2 \frac{\partial}{\partial y} \frac{\partial}{\partial z} \varphi \\
&\quad - 2 \delta_1 \cdot (1-a) \cdot h \cdot \left( - \frac{\partial}{\partial y} \varphi - 2 \delta_1 \cdot h \cdot \frac{\partial}{\partial y} \frac{\partial}{\partial z} \varphi \right) \right] \\
&+ \hat{e}_z \left[ \frac{\partial}{\partial z} \varphi + 2 \delta_1 \cdot \frac{\partial}{\partial z} \varphi + (1-b) \cdot \frac{\partial}{\partial z} \varphi - 4 \delta_1 \cdot (1-a) \cdot h^2 \frac{\partial}{\partial z} \frac{\partial}{\partial z} \varphi \\
&\quad - 2 \delta_1 \cdot (1-a) \cdot h \cdot \left( \frac{\partial}{\partial z} \varphi - 2 \delta_1 \cdot h \cdot \frac{\partial}{\partial z} \frac{\partial}{\partial z} \varphi \right) \right]
\end{aligned}
\end{array}
\]

(D.9)

\[
\begin{array}{c}
\begin{aligned}
u^2 \bigg|_{z=0} &= \hat{e}_x \left[ -2 \cdot h \cdot (\delta_2 \cdot b - \delta_1 \cdot a) \cdot \frac{\partial}{\partial x} \frac{\partial}{\partial z} \varphi + b \cdot \left( \frac{\partial}{\partial x} \varphi + 2 \delta_2 \cdot h \cdot \frac{\partial}{\partial x} \frac{\partial}{\partial z} \varphi \right) \right] \\
&+ \hat{e}_y \left[ -2 \cdot h \cdot (\delta_2 \cdot b - \delta_1 \cdot a) \cdot \frac{\partial}{\partial y} \frac{\partial}{\partial z} \varphi + b \cdot \left( \frac{\partial}{\partial y} \varphi + 2 \delta_2 \cdot h \cdot \frac{\partial}{\partial y} \frac{\partial}{\partial z} \varphi \right) \right] \\
&+ \hat{e}_z \left[ -2 \cdot h \cdot (\delta_2 \cdot b - \delta_1 \cdot a) \cdot \frac{\partial}{\partial z} \frac{\partial}{\partial z} \varphi + b \cdot \left( \frac{\partial}{\partial z} \varphi + 2 \delta_2 \cdot h \cdot \frac{\partial}{\partial z} \frac{\partial}{\partial z} \varphi \right) \right]
\end{aligned}
\end{array}
\]

(D.10)

We notice that the displacement field is continuous across \( z=0 \)
(the interface plane). Next we consider the \( \Gamma^z \) traction continuity
(equilibrium).

\[
\Gamma^z \bigg|_{z=0} = \hat{e}_x \left[ -2 \mu_1 \delta_1 \cdot \frac{\partial}{\partial x} \varphi + 4 \mu_1 \delta_1 \cdot h \cdot \frac{\partial}{\partial x} \frac{\partial}{\partial z} \varphi + 2 \mu_1 \cdot (1-b) \cdot \frac{\partial}{\partial x} \frac{\partial}{\partial z} \varphi \right]
\]
\[
- 4\delta_1^2 \cdot (1-a) \cdot 2\mu_1 \cdot h^2 \cdot \frac{\partial^4 -}{\partial x \partial z^3 \varphi} \\
- 2\delta_1 \cdot (1-a) \cdot h \cdot (-2\mu_1 \delta_1 \cdot \frac{\partial^3 -}{\partial x \partial z^2 \varphi} - 4\mu_1 \delta_1 \cdot h \cdot \frac{\partial^4 -}{\partial x \partial z^3 \varphi}) \\
+ \hat{e}_y \left[ -2\mu_1 \delta_1 \cdot \frac{\partial^2}{\partial y \partial z \varphi} + 4\mu_1 \delta_1 \cdot h \cdot \frac{\partial^3}{\partial y \partial z^2 \varphi} + 2\mu_1 \cdot (1-b) \cdot \frac{\partial^2 -}{\partial y \partial z \varphi} \\
- 4\delta_1^2 \cdot (1-a) \cdot 2\mu_1 \cdot h^2 \cdot \frac{\partial^4 -}{\partial y \partial z^3 \varphi} \\
- 2\delta_1 \cdot (1-a) \cdot h \cdot (-2\mu_1 \delta_1 \cdot \frac{\partial^3 -}{\partial y \partial z^2 \varphi} - 4\mu_1 \delta_1 \cdot h \cdot \frac{\partial^4 -}{\partial y \partial z^3 \varphi}) \\
+ \hat{e}_z \left[ 2\mu_1 \delta_1 \cdot \frac{\partial^2}{\partial z^2 \varphi} + 4\mu_1 \delta_1 \cdot h \cdot \frac{\partial^3}{\partial z^3 \varphi} + 2\mu_1 \cdot (1-b) \cdot \frac{\partial^2 -}{\partial z^2 \varphi} \\
- 4\delta_1^2 \cdot (1-a) \cdot 2\mu_1 \cdot h^2 \cdot \frac{\partial^4 -}{\partial z^4 \varphi} \\
- 2\delta_1 \cdot (1-a) \cdot h \cdot (2\mu_1 \delta_1 \cdot \frac{\partial^3 -}{\partial z^3 \varphi} - 4\mu_1 \delta_1 \cdot h \cdot \frac{\partial^4 -}{\partial z^4 \varphi}) \right] \\
\]
\]

\[
T_{z=0}^{z^2} = \hat{e}_x \left[ -2h \cdot (\delta_2 \cdot \delta_1 a) \cdot 2\mu_2 \cdot \frac{\partial^3}{\partial x \partial z^2 \varphi} \\
+ b \cdot (-2\mu_2 \delta_2 \cdot \frac{\partial^2}{\partial x \partial z \varphi} + 4\mu_2 \delta_2 \cdot h \cdot \frac{\partial^3}{\partial x \partial z^2 \varphi}) \right] \\
+ \hat{e}_y \left[ -2h \cdot (\delta_2 \cdot \delta_1 a) \cdot 2\mu_2 \cdot \frac{\partial^3}{\partial y \partial z^2 \varphi} \\
+ b \cdot (-2\mu_2 \delta_2 \cdot \frac{\partial^2}{\partial y \partial z \varphi} + 4\mu_2 \delta_2 \cdot h \cdot \frac{\partial^3}{\partial y \partial z^2 \varphi}) \right] \\
+ \hat{e}_z \left[ -2h \cdot (\delta_2 \cdot \delta_1 a) \cdot 2\mu_2 \cdot \frac{\partial^3}{\partial z^3 \varphi} \\
+ b \cdot (2\mu_2 \delta_2 \cdot \frac{\partial^2}{\partial z^2 \varphi} + 4\mu_2 \delta_2 \cdot h \cdot \frac{\partial^3}{\partial z^3 \varphi}) \right] \\
\]
\]

(D.11)

Noting that:
\[ 2\mu_1 \cdot (\delta_1 + (1-b)) = 2\mu_2 \cdot b \cdot \delta_2 \]

since

\[ 2\mu_1 \cdot \left[ \frac{\gamma \delta_2 \delta_1 + \gamma \delta_2}{1 + \gamma \delta_2} \right] = 2\mu_2 \cdot \frac{\delta_1 \delta_2 + \delta_2}{1 + \gamma \delta_2} \]

and

\[ 4\mu_1 [\delta_1 + \delta_1^2 (1-a)] = 4\mu_2 \delta_1 \cdot a \]

(D.13)

We find that the tractions across the interface plane \((z=0)\) are continuous.

Case III: \(\varphi_3 = \varphi_0; \varphi_1 = \varphi_2 = 0\)

\[ u^1 \bigg|_{z=0} = e_x \cdot \left[ \frac{d}{dy} \varphi + \frac{1-\gamma}{1+\gamma} \frac{\partial}{\partial y} \varphi \right] + e_y \cdot \left[ -\frac{\partial}{\partial x} \varphi - \frac{1-\gamma}{1+\gamma} \frac{\partial}{\partial x} \varphi \right] + e_z \cdot \left[ 0 \right] \]

(D.14)

\[ u^2 \bigg|_{z=0} = e_x \cdot \left[ \frac{2}{1+\gamma} \frac{\partial}{\partial y} \varphi \right] + e_y \cdot \left[ -\frac{2}{1+\gamma} \frac{\partial}{\partial x} \varphi \right] + e_z \cdot \left[ 0 \right] \]

(D.15)

We notice that the displacement field is continuous across \(z=0\) (the interface plane). Next we consider the \(I^r\) traction continuity (equilibrium).

\[ I^r \bigg|_{z=0} = e_x \cdot \left[ \mu_1 \frac{\partial^2}{\partial y \partial z} \varphi + \frac{1-\gamma}{1+\gamma} \mu_1 \frac{\partial^2}{\partial y \partial z} \varphi \right] \]
\[
+ \hat{e}_y \left[ -\mu_1 \frac{\partial^2 \varphi}{\partial x \partial z} - \frac{1-\gamma}{1+\gamma} \mu_1 \frac{\partial^2 \varphi}{\partial x^2} \right] + \hat{e}_z \left[ 0 \right]
\]
(D.16)

\[
\Gamma_{z=0}^z = \hat{e}_x \left[ \mu_2 \frac{2}{1+\gamma} \frac{\partial^2 \varphi}{\partial y \partial z} \right] + \hat{e}_y \left[ -\mu_2 \frac{2}{1+\gamma} \frac{\partial^2 \varphi}{\partial x \partial z} \right] + \hat{e}_z \left[ 0 \right]
\]
(D.17)

Noting that:

\[
\mu_1 \left[ 1 - \frac{1-\gamma}{1+\gamma} \right] = \mu_2 \frac{2}{1+\gamma}
\]
(D.18)

We find that the tractions across the interface plane (z=0) are continuous.

Therefore, all three cases check correctly, and the algorithm is correct.
Appendix E: Derivation of some sample Green's functions through the use of the image algorithm

In this section, we consider displacement fields in Cartesian components for some sample point source problems on or in a halfspace with a free surface. The solutions that will be rederived are readily available (and established) in the literature and hence serve as an empirical check (see appendix D for an analytic check) of the algorithm. In addition, these specific examples help clarify details of the application of the algorithm.

By considering a halfspace problem with a free surface, we obtain the following simplifications:

$$\gamma = 0 \quad 1-a = -1/\delta_1 \quad 1-b = -\delta_1$$

call:

$$\delta = \delta_1$$

then:

$$R_R(h, a, b, \delta_1) = \begin{bmatrix} -2h \frac{\partial}{\partial z} & -\delta + 4\delta h^2 \frac{\partial^2}{\partial z^2} \\ -\frac{1}{\delta} & +2h \frac{\partial}{\partial z} \end{bmatrix} \quad R_L(\gamma) = \begin{bmatrix} 1 \end{bmatrix}$$

(E.1)

The following example problems will be considered:

I. Screw dislocation (Antiplane problem).

II. Line force (x=0, z=0) acting on a free surface.
   i) The line force is in the x direction.
   ii) The line force is in the z direction (Flamant’s solution).

III. Point force acting on a free surface.
    i) The point force is in the x direction (Cerruti’s solution).
ii) The point force is in the z direction (Boussinesq solution).

IV. Point force acting interior to the halfspace with a free surface (Mindlin's solution).
   i) The point force is in the x direction.
   ii) The point force is in the z direction.

I. Screw dislocation (antiplane problem)

For the antiplane problem, all that is required is to obtain the image potential with respect to the interface plane since the $\mathbf{R}_L$ matrix is the identity operator. Also, we notice that getting the image of a given $\mathbf{M}$ type (see main text) displacement field is equal to the $\mathbf{M}$ displacement field of the image potential describing that field (i.e. $\mathbf{M}(h,\varphi) = \mathbf{M}(-h,\varphi)$), and hence we can directly operate on a given displacement field when using the algorithm for a purely antiplane problem. This corresponds to the scalar field image method for the antiplane case.

As an example we consider the field due to a screw dislocation in the plane perpendicular to the x-z plane at location $z-h$ and $x=0$. The field due to the dislocation in infinite space is:

$$u_y = \arctan((z-h)/x) \quad (E.2)$$

The image field will be $\overline{u}_y$ which implies that the combined fields give:

$$u_y = \arctan((z-h)/x) - \arctan((z+h)/x) \quad (E.3)$$

Of course this is but a simple application of the scalar image method.
II. Line force \((x=0, z=0)\) acting on a free surface.

For this problem the potentials are given in appendix B. We note that \(\varphi_3 = 0\) (i.e. there is no antiplane mode in a plane strain problem as is trivially known).

We note that for the case when the point source is at the interface (i.e. \(h=0\)), we get a significant simplification in the \(R_R\) matrix operator in the following way:

\[
R_R(0, a, b, \delta_1) = \begin{bmatrix}
0 & -\delta \\
\cdots & \cdots \\
1/\delta & 0
\end{bmatrix}
\]  

(E.4)

This means that for the displacement fields all we need to calculate are the following:

\[
\begin{bmatrix}
\frac{\partial}{\partial x'} & \frac{\partial}{\partial z'}
\end{bmatrix}
\begin{bmatrix}
\varphi_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{\partial}{\partial x'} & \frac{\partial}{\partial z'} & \frac{\partial^2}{\partial x \partial z'} & \frac{\partial^2}{\partial z'^2}
\end{bmatrix}
\begin{bmatrix}
\varphi_1
\end{bmatrix}
\]

Now we perform the above differentiations for the image potentials due to a line force. These potentials are linear combinations of the following functions:

\[
\overline{\varphi}_A = (z \cdot \arctan(z/x) - x \cdot \ln \xi + (1+\delta) \cdot x)/2 - \varphi_A
\]

\[
\overline{\varphi}_C = (z \cdot \ln \xi - z + x \cdot \arctan(z/x))/2 - \varphi_C
\]

(E.5)

where: \(\xi^2 = x^2 + z^2\)

We get:

\[
\frac{\partial}{\partial x} \overline{\varphi}_A = -\ln \xi/2 + \delta/2 \quad \quad \frac{\partial}{\partial x} \overline{\varphi}_C = [\arctan(z/x)]/2
\]
\[
\frac{\delta}{\delta z^2} \varphi_A = \frac{1}{2} \arctan(z/x) \\
\frac{\delta^2}{\delta x \delta z} \varphi_A = \frac{1}{2} (z/x^2) \\
\frac{\delta^2}{\delta z^2} \varphi_A = \frac{1}{2} (x/z^2) \\
\frac{\delta}{\delta z} \varphi_C = \frac{1}{2} \ln \xi \\
\frac{\delta^2}{\delta x \delta z} \varphi_C = \frac{1}{2} \left( \frac{x}{\xi^2} \right) \\
\frac{\delta^2}{\delta z^2} \varphi_C = \frac{1}{2} \left( \frac{z}{\xi^2} \right)
\]

(E.6)

1) Case when the force is acting in the x direction:

We get:

\[
u = \nu^0 + \left[ \frac{\alpha}{(4\pi\mu\delta)} \right] \cdot \left[ e_x \cdot \left[ \frac{-\delta \cdot \ln \xi/2 - \ln \xi/(2\delta) - z^2/\xi^2 + \text{constants}}{2} \right] + e_z \cdot \left[ \delta \cdot \arctan(z/x)/2 - \arctan(z/x)/(2\delta) + xz/\xi^2 \right] \right]
\]

(E.7)

Noting the following identities:

\[
z^2/\xi^2 = 1 - x^2/\xi^2 \\
\delta/2-1/(2\delta) = -\mu \cdot (1+\delta)/(\lambda+\mu) \\
\delta/2+1/(2\delta)+1 = (1+\delta)^2/(2\delta) \\
\alpha = 2 \cdot \delta/(\delta+1)
\]

(E.8)

We find that the above solution coincides with that given in Love 1927 (article 151), except for a rigid body motion.

1) Case when the force is acting in the z direction (Flamant's solution):

We get:
\[ u = u^0 - \left[ \frac{\alpha}{4\pi \mu \delta} \right] \left[ \begin{array}{c}
\hat{e}_x \cdot \left[ \delta \cdot \text{arctan}(z/x) / 2 - \text{arctan}(z/x) / (2\delta) - \frac{xz}{\xi^2} \right] \\
\hat{e}_z \cdot \left[ \delta \cdot \ln \xi / 2 + \ln \xi / (2\delta) - \frac{z^2}{\xi^2} \right] \end{array} \right] \]

(E.9)

Noting the same identities mentioned above (E.8), we find that the above solution coincides with that given in Love 1927 (article 151), except for a rigid body motion.

III. Point force acting on a free surface.

For this problem the potentials for the point source in infinite space are given in appendix B. However, we have a choice of where to locate the singularities of the potentials. Since we do not want the image potentials to introduce any new sources inside the halfspace, we choose the infinite space potentials to have all their singularities in that halfspace.

Again, if we are only interested in the case when the point force acts on the free surface (h=0), we get equation (E.4). This means that all we need to calculate are the following:

\[ \begin{bmatrix}
\frac{\partial}{\partial x'} \frac{\partial}{\partial y'} \frac{\partial}{\partial z'}
\end{bmatrix} \begin{bmatrix}
\varphi_2
\end{bmatrix} \quad \begin{bmatrix}
\frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \frac{\partial^2}{\partial x' \partial z'} \frac{\partial^2}{\partial y' \partial z'} \frac{\partial^2}{\partial z^2}
\end{bmatrix} \begin{bmatrix}
\varphi_1
\end{bmatrix} \begin{bmatrix}
-1
\end{bmatrix} \]

Now we perform the above differentiations for the image potentials due to a point force. These potentials are linear combinations of the following functions:
\[ \phi_A = x/[2 \cdot (r+z)] \]  
\[ \phi_C = -[\ln(r+z)]/2 \]  

(E.10)

In addition, we have an antiplane potential for this case which is a linear multiple of the following function:

\[ \phi_B = y/[2 \cdot (r+z)] \]  

(E.11)

and we will also have to calculate:

\[ \left[ \frac{\partial}{\partial x'}, \frac{\partial}{\partial y} \right] \cdot \phi_B \]

We get:

\[ \frac{\partial}{\partial x} \phi_A = \left[ 1/(r+z) - x^2/[r \cdot (r+z)^2] \right]/2 \]

\[ \frac{\partial}{\partial y} \phi_A = \left[ -(xy)/[r \cdot (r+z)^2] \right]/2 \]

\[ \frac{\partial}{\partial z} \phi_A = \left[ -x/[r \cdot (r+z)] \right]/2 \]

\[ \frac{\partial^2}{\partial x \partial z} \phi_A = \left[ -1/[r \cdot (r+z)] + x^2/[r^3 \cdot (r+z)] + x^2/[r^2 \cdot (r+z)^2] \right]/2 \]

\[ \frac{\partial^2}{\partial y \partial z} \phi_A = \left[ (xy)/[r^3 \cdot (r+z)] + (xy)/[r^2 \cdot (r+z)^2] \right]/2 \]

\[ \frac{\partial^2}{\partial z^2} \phi_A = \left[ x/r^3 \right]/2 \]

(E.12)

\[ \frac{\partial}{\partial x} \phi_C = \left[ -x/[r \cdot (r+z)] \right]/2 \]

\[ \frac{\partial^2}{\partial x \partial z} \phi_C = \left[ x/r^3 \right]/2 \]
This is the most complete text of the thesis available. The following page(s) were not included in the copy of the thesis deposited in the Institute Archives by the author:

pg 261
\[ + \hat{e}_z \left[ -\frac{\delta \cdot x}{2r \cdot (r+z)} \right. \\
+ \frac{x}{2\delta r \cdot (r+z)} \\
\left. + \frac{(xz)}{r^3} \right] \]  
\]

Noting the following identities:

\[
\frac{x^2 z}{[r^3 \cdot (r+z)]} + \frac{x^2 z}{[r^2 \cdot (r+z)^2]} = \frac{x^2}{r^3} - \frac{x^2}{[r \cdot (r+z)^2]} \\
z/[r \cdot (r+z)] = \frac{1}{r} - \frac{1}{(r+z)} \\
y^2/[r \cdot (r+z)^2] = \frac{2}{(r+z)} - \frac{1}{r} - \frac{x^2}{[r \cdot (r+z)^2]} \\
(xyz)/[r^3 \cdot (r+z)] + (xyz)/[r^2 \cdot (r+z)^2] = \frac{(xy)}{r^3} - \frac{(xy)}/[r \cdot (r+z)^2] \\
-\delta/2 + 1/(2\delta) = \mu \cdot (1+\delta)/(\lambda+\mu) \]

We find that the above result coincides with the published results (e.g. Love 1927, page 243).

ii) Case when the force is acting in the z direction:

We get:

\[ u = u^0 - \left[ \frac{1}{4\pi \mu (1+\delta)} \right] \hat{e}_x \left[ -\frac{\delta \cdot x}{2r \cdot (r+z)} \right. \\
+ \frac{x}{2\delta r \cdot (r+z)} - \frac{(xz)}{r^3} \right] + \hat{e}_y \left[ -\frac{\delta \cdot y}{2r \cdot (r+z)} \right. \\
\left. + \frac{y}{2\delta r \cdot (r+z)} - \frac{(yz)}{r^3} \right] \]
\[ + e_z \left[ -\delta/(2\pi) - 1/(2\delta r) - z^2/r^3 \right] \]

(E.17)

Noting the identities in (E.16) concerning material parameters, we find that the above result coincides with the published results (e.g. Love 1927, page 191).

IV. Point force acting interior to a halfspace with a free surface.

The infinite space source potentials for this case are the same as for case III. However, since h=0 in general, the matrix operator involved in the algorithm (E.1) requires us to further calculate in addition to the terms shown in case III, the following functions:

\[ \frac{\partial}{\partial z} \left[ \frac{\partial}{\partial x'} \frac{\partial}{\partial y'} \frac{\partial}{\partial z} \right] \begin{bmatrix} -\psi_0 \\ \varphi_1 \end{bmatrix} \quad \frac{\partial^2}{\partial z^2} \left[ \frac{\partial}{\partial x'} \frac{\partial}{\partial y'} \frac{\partial}{\partial z} \right] \begin{bmatrix} -\psi_0 \\ \varphi_2 \end{bmatrix} \]

\[ \frac{\partial}{\partial z} \left[ \frac{\partial}{\partial x'} \frac{\partial}{\partial y'} \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} \right] \begin{bmatrix} -\psi_0 \\ \varphi_2 \end{bmatrix} \]

There are some repetition in the suggested functions to be calculated since the partial differentiation operations are commutative when the function is sufficiently smooth. Now we perform the additional required differentiations:

\[ \frac{\partial^3}{\partial z^2 \partial x} \psi_A = \left[ \frac{1}{r^3} - \frac{3}{x^2}r^5 \right]/2 \]

\[ \frac{\partial^3}{\partial z^2 \partial y} \psi_A = \left[ \frac{-3}{y}r^5 \right]/2 \]

(E.18)
\[
\frac{\partial^3}{\partial z^3 \varphi_A} = \left[ -3 \cdot (xz)/r^5 \right]/2
\]

\[
\frac{\partial^3}{\partial z^2 \partial x} \varphi_C = \left[ -3 \cdot (xz)/r^5 \right]/2
\]

\[
\frac{\partial^3}{\partial z^2 \partial y} \varphi_C = \left[ -3 \cdot (yz)/r^5 \right]/2
\]

\[
\frac{\partial^3}{\partial z^3 \varphi_C} = \left[ 1/r^3 - 3 \cdot z^2/r^5 \right]/2
\]

(E.19)

1) Case when the force is acting in the x direction:

Defining:

\[
x'_2 = x^2 + y^2 + (z+h)^2
\]

We get after simplification and the use of identities similar to those given in (E.16), but with z replaced by (z+h) and "r" replaced by "r₂" wherever they occur:

\[
\hat{u} = \hat{u}^0 + \left[ \frac{1}{4\pi \mu (1+\delta)} \right] \left[ \hat{e}_x \cdot \left[ \frac{\delta/2 + 1/(2\delta) + 1 - 2 \cdot (1+\delta) + (1+\delta)}{r_2+z+h} \right]
+ \left[ \frac{-\delta/2 - 1/(2\delta) - 1 + (1+\delta)}{r_2} \cdot x^2/(r_2 \cdot (z+h)^2) \right]
+ \left[ \frac{1}{1 + (1+\delta)} \right]/r_2
+ \left[ \frac{x^2/r_2^3 + 2 \delta h z \cdot (1/r_2^3 - 3 \cdot x^2/r_2^5)}{r_2} \right]
+ \hat{e}_y \cdot \left[ \frac{\delta/2 - 1/(2\delta) - 1 + (1+\delta)}{r_2} \cdot (xy)/(r_2 \cdot (z+h)^2) \right]
+ \left[ \frac{(xy)/r_2^3 + 2 \delta h z \cdot (-3 \cdot xy/r_2^5)}{r_2} \right] \right]
\]
This is the most complete text of the thesis available. The following page(s) were not included in the copy of the thesis deposited in the Institute Archives by the author:

pg 265
\[
\frac{\delta}{2+1/(2\delta)} = \delta \cdot \left[ 8 \cdot (1 - \nu)^2 - (3 - 4\nu) \right] \tag{E.23}
\]

We find that the above result (E.22) coincides with the solution first obtained by Mindlin (1936) and shown in Mura (1982).
Appendix F: Modified formalism for the multiple imaging scheme

In this appendix a different formalism for the repeated imaging method than the main text will be defined. This formalism, although being farther from the spirit in which the method was derived, is easier to manipulate, and will be used from hereon (e.g. in appendices G, H, I, K and L).

Define the following operator matrices:

\[
\begin{align*}
R_{Ln}^\pm (q,1) & = \begin{cases}
I & \text{if } n = 0 \\
R_{Ln}^\pm (q) & \text{if } n = 1 \\
\sum_{k=1}^{n/2} \left[ R_{Ln}^\pm ((2k-1) \cdot 1+q) \cdot R_{Ln}^\pm ((2k-2) \cdot 1+q) \right]_k & \text{if } n \text{ is even} \\
\sum_{k=1}^{(n-1)/2} \left[ R_{Ln}^\pm (2k+1+q) \cdot R_{Ln}^\pm ((2k+1) \cdot 1+q) \right]_k \cdot R_{Ln}^\pm (q) & \text{if } n \text{ is odd}
\end{cases}
\end{align*}
\]  

(F.1)

Then equation (2.14) from the main text can be rewritten as:
\[ u_1 = - u(\ +h, \delta_1, \varphi_R^0, \varphi_L^0) \]

\[ + \sum_{m=1}^{\infty} u(-2(m-1)H-h, \delta_1, R^+_{R(2m-1)} (+h, +H) \cdot \varphi_R^0, R^+_{L(2m-1)} \cdot \varphi_L^0) \]

\[ + \sum_{m=1}^{\infty} u(2(m-1)H+h, \delta_1, R^+_{R(2m-2)} (-h, -H) \cdot \varphi_R^0, R^+_{L(2m-2)} \cdot \varphi_L^0) \]

\[ + \sum_{m=1}^{\infty} u(2H-h, \delta_1, R^-_{R(2m-1)} (-H+h, -H) \cdot \varphi_R^0, R^-_{L(2m-1)} \cdot \varphi_L^0) \]

\[ + \sum_{m=1}^{\infty} u(-2(m-1)H+h, \delta_1, R^-_{R(2m-2)} (+H-h, +H) \cdot \varphi_R^0, R^-_{L(2m-2)} \cdot \varphi_L^0) \]

(F.3)

Now for each component in each of the infinite sums we have a displacement field of the form:

\[ u = u(\ p, \delta_1, \varphi_R, \varphi_L) \]  

(F.4)

where:

\[ \varphi_R = \begin{bmatrix} \varphi_1(x-x', y-y', z-p; q) \\ \varphi_2(x-x', y-y', z-p; q) \end{bmatrix} \]  

(F.5)

\[ \varphi_L = \begin{bmatrix} \varphi_3(x-x', y-y', z-p; q) \end{bmatrix} \]  

(F.6)

\[ p = p(h) \quad q = q(h) \]  

(F.7)

The stresses derived for each of those potentials can be written as follows:

\[ \sigma_{xx} = 2\mu \cdot \frac{\partial^2}{\partial x^2} \varphi_1 - 2\mu \cdot \frac{\partial^2}{\partial x^2} \varphi_2 - 4\mu \delta \cdot (z-p) \cdot \frac{\partial^3}{\partial x^2 \partial z} \varphi_2 \]

\[ + 2\mu \cdot (3\delta - 1) \cdot \frac{\partial^2}{\partial z^2} \varphi_2 + 2\mu \cdot \frac{\partial^2}{\partial x \partial y} \varphi_3 \]
\[
\begin{align*}
\sigma_{xy} &= 2\mu \frac{\partial^2}{\partial x \partial y} \psi_1 - 2\mu \frac{\partial^2}{\partial x \partial y} \psi_2 - 4\mu \delta \cdot (z-p) \frac{\partial^3}{\partial x \partial y \partial z} \varphi_2 \\
&\quad + \mu \left( \frac{\partial^2}{\partial y^2} \varphi_3 - \frac{\partial^2}{\partial x^2} \varphi_3 \right) \\
\sigma_{xz} &= 2\mu \frac{\partial^2}{\partial x \partial z} \psi_1 - 2\mu \delta \cdot \frac{\partial^2}{\partial x \partial z} \psi_2 - 4\mu \delta \cdot (z-p) \frac{\partial^3}{\partial z^2} \varphi_2 \\
&\quad + \mu \frac{\partial^2}{\partial y \partial z} \varphi_3 \\
\sigma_{yy} &= 2\mu \frac{\partial^2}{\partial y^2} \psi_1 - 2\mu \frac{\partial^2}{\partial y^2} \psi_2 - 4\mu \delta \cdot (z-p) \frac{\partial^3}{\partial y^2 \partial z} \varphi_2 \\
&\quad + 2\mu \cdot (3\delta - 1) \frac{\partial^2}{\partial z^2} \varphi_2 - 2\mu \frac{\partial^2}{\partial x \partial y} \varphi_3 \\
\sigma_{yz} &= 2\mu \frac{\partial^2}{\partial y \partial z} \psi_1 - 2\mu \delta \cdot \frac{\partial^2}{\partial y \partial z} \psi_2 - 4\mu \delta \cdot (z-p) \frac{\partial^3}{\partial z^2} \varphi_2 - \mu \frac{\partial^2}{\partial x \partial z} \varphi_3 \\
\sigma_{zz} &= 2\mu \frac{\partial^2}{\partial z^2} \psi_1 + 2\mu \delta \cdot \frac{\partial^2}{\partial z^2} \psi_2 - 4\mu \delta \cdot (z-p) \frac{\partial^3}{\partial z^3} \varphi_2 \\
\end{align*}
\] (F.8)

When deriving the nuclei of strain (NOS) in a plate, we will need the derivative of the above stresses with respect to "x'", "y'" and "h". These are given by:

\[
\begin{align*}
\frac{\partial}{\partial x'} \begin{bmatrix} \sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yy}, \sigma_{yz}, \sigma_{zz} \end{bmatrix} &= \frac{\partial}{\partial x} \begin{bmatrix} \sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yy}, \sigma_{yz}, \sigma_{zz} \end{bmatrix} \\
\frac{\partial}{\partial y'} \begin{bmatrix} \sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yy}, \sigma_{yz}, \sigma_{zz} \end{bmatrix} &= \frac{\partial}{\partial y} \begin{bmatrix} \sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yy}, \sigma_{yz}, \sigma_{zz} \end{bmatrix} \\
\frac{\partial}{\partial h} \begin{bmatrix} \sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yy}, \sigma_{yz}, \sigma_{zz} \end{bmatrix} &= - \frac{dp}{dh} \frac{\partial}{\partial z} \begin{bmatrix} \sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yy}, \sigma_{yz}, \sigma_{zz} \end{bmatrix} + \frac{dq}{dq} \frac{\partial}{\partial z} \begin{bmatrix} \sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yy}, \sigma_{yz}, \sigma_{zz} \end{bmatrix} \\
\end{align*}
\] (F.9, F.10, F.11)
\[ \frac{\partial}{\partial z} \left[ (z-p) \cdot \varphi(x-x', y-y', z-p; q) \right] = \varphi(x-x', y-y', z-p; q) + (z-p) \cdot \frac{\partial}{\partial z} \varphi(x-x', y-y', z-p; q) \] (F.12)


Appendix G: Components of products of matrix operators ("R" part)

In appendix F, the product operators \( R^\pm_{Rn} (q, l) \) were defined. In this appendix, the explicit components of those product operators for obtaining \( R^+_{Rn} (q, l) \) and \( \frac{\partial}{\partial q} R^+_{Rn} (q, l) \) up till the fourth group of images (refer to main text section 2.3) will be given in this appendix. The \( R^-_{Rn} (q, l) \) and \( \frac{\partial}{\partial q} R^-_{Rn} (q, l) \) matrix operators can be obtained from the \( R^+_{Rn} (q, l) \) and \( \frac{\partial}{\partial q} R^+_{Rn} (q, l) \) by interchanging the "+" and "-" signs wherever they occur. In what follows, the operator \( \frac{\partial^n}{\partial z^n} \) will be referred to as "\( \partial^n \)", \( (A^+)^n \) and \( (B^+)^n \) will be referred to as \( A^+n \) and \( B^+n \) respectively.

First the components of the matrix operators \( R^+_{Rn} (q, l) \):

\[
R^+_{R1}(q, l) = \begin{bmatrix}
\frac{c_1 \partial}{c_4} & \frac{c_2 + c_3 \partial^2}{c_5 \partial} \\
\end{bmatrix}
\]

\( c_1 = 2A^+q \)

\( c_2 = B^+ \)

\( c_3 = -4A^+q^2 \)

\( c_4 = A^+ \)

\( c_5 = -2A^+q \)

\[
R^+_{R2}(q, l) = \begin{bmatrix}
\frac{d_1 + d_2 \partial^2}{d_5 \partial} & \frac{d_3 \partial + d_4 \partial^3}{d_6 + d_7 \partial^2} \\
\end{bmatrix}
\]

\( d_1 = A^+B^- \)

\( d_2 = -4A^+A^-q \cdot (1+q) \)
\[ d_3 = 2 \delta A^- B^+ \cdot (1+q) - 2 \delta A^+ B^- q \]
\[ d_4 = 8 \delta^3 A^- A^+ \cdot (1+q) \cdot q \cdot 1 \]
\[ d_5 = -2 \delta A^- A^+ \cdot 1 \]
\[ d_6 = A^- B^+ \]
\[ d_7 = 4 \delta^2 A^- A^+ q \cdot 1 \]

\[ \mathbb{R}^+_{R3}(q,1) = \begin{bmatrix} \frac{e_1 \theta + e_2 \theta^3}{e_6 + e_7 \theta^2} & \frac{e_3 + e_4 \theta^2 + e_5 \theta^4}{e_8 \theta + e_9 \theta^3} \end{bmatrix} \]
\[ e_1 = 2 \delta A^+ B^- \cdot (21+q) - 2 \delta A^- A^+ \cdot 1 \]
\[ e_2 = 8 \delta^3 A^+ A^- \cdot (21+q) \cdot q \cdot 1^2 \]
\[ e_3 = A^- B^+ \]
\[ e_4 = -4 \delta^2 A^- A^+ B^- \cdot (21^2) - 4 \delta^2 A^+ B^- \cdot (21+q) \cdot q \]
\[ e_5 = -16 \delta^4 A^- A^+ B^- q \cdot 1^2 \cdot (21+q) \]
\[ e_6 = A^+ B^- \]
\[ e_7 = 4 \delta^2 A^- A^+ \cdot 1 \cdot 1^2 \]
\[ e_8 = -2 \delta A^- A^+ B^- \cdot 1 - 2 \delta A^+ B^- q \]
\[ e_9 = -8 \delta^3 A^- A^+ q \cdot 1^2 \]

\[ \mathbb{R}^+_{R4}(q,1) = \begin{bmatrix} \frac{f_1 + f_2 \theta^2 + f_3 \theta^4}{f_7 \theta + f_8 \theta^3} & \frac{f_4 \theta + f_5 \theta^3 + f_6 \theta^5}{f_9 + f_{10} \theta^2 + f_{11} \theta^4} \end{bmatrix} \]
\[ f_1 = A^+ 2 B^- \cdot 2 \]
\[ f_2 = -4 \delta^2 A^- A^+ B^- \cdot (21^2 + q) \cdot 1 - 4 \delta^2 A^- A^+ B^- \cdot 1 \cdot (31+q) \]
\[ f_3 = -16 \delta^4 A^- A^+ B^- \cdot 2 \cdot 1^3 \cdot (31+q) \]
\[ f_4 = 2 \delta A^- B^+ \cdot (31+q) - 2 \delta A^+ B^- \cdot 2 q - 2 \delta A^- B^- A^+ B^+ \cdot 1 \]
\[ f_5 = 8 \delta^3 A^- A^+ B^- \cdot (31+q) \cdot (1^2 + q) + 8 \delta^3 A^- A^+ B^- q \cdot (21^2 + q) \]
\[ f_6 = 328 A^{-2} A^2 + 2 \cdot q_1^3 \cdot (31 + q) \]
\[ f_7 = -28 A^{-2} B^{-2} \cdot 1 - 28 A^{-2} B^{-2} \cdot 1 \]
\[ f_8 = -8 A^{-2} A^2 + 2 \cdot 1^3 \]
\[ f_9 = A^{-2} B + 2 \]
\[ f_{10} = 48^2 A^{-2} A^2 + 2 \cdot B + 2 \cdot (1^2 + q_1) + 8 A^{-2} B^{-2} \cdot 1 \cdot (31^2 + q_1) \]
\[ f_{11} = 16 A^{-2} A^2 + 2 \cdot q_1^3 \]

\[ E_{15}^+ (q_1) = \frac{g_1 + g_2 + g_3 + g_4 + g_5 + g_6 + g_7}{g_8 + g_9 + g_{10} + g_{11} + g_{12} + g_{13}} \]
\[ E_{12} = 2 A^{-3} B^{-2} \cdot (41 + q) - 2 A^{-2} B^2 \cdot A^{-2} - 2 A^{-2} B^{-2} \cdot 1 \]
\[ E_{13} = 8 A^{-2} A^{-2} \cdot (21^2) \cdot (41 + q) + 8 A^{-2} A^{-2} \cdot (21^2) \cdot (31^2 + q_1) \]
\[ E_{14} = 2 A^{-2} B + 3 \]
\[ E_{15} = -4 A^{-2} A^{-2} \cdot (31^2) - 4 A^{-2} B^2 \cdot A^{-2} \cdot (41^2) - 4 A^{-2} B^2 \cdot (41 + q) \]
\[ E_{16} = -16 A^{-2} A^{-2} \cdot (21^2 + q_1)^2 - 16 A^{-2} A^{-2} \cdot (21^2) \cdot q \cdot (41 + q) \]
\[ E_{17} = -6 A^{-2} A^{-2} \cdot (41 + q) \]
\[ g_8 = A^{-2} B^{-2} \]
\[ g_9 = 4 A^{-2} A^{-2} \cdot (21^2) + 4 A^{-2} B^2 \cdot A^{-2} \cdot 1^2 \]
\[ g_{10} = 16 A^{-2} A^{-2} \cdot 1^4 \]
\[ g_{11} = -2 A^{-2} B^2 \cdot (41 + q) - 2 A^{-2} B^{-2} \cdot 1^4 \]
\[ g_{12} = -8 A^{-2} B^2 \cdot (1^2 + q_1^2) - 8 A^{-2} B^2 \cdot (21^2) \cdot q \]
\[ g_{13} = -32 A^{-2} A^{-2} \cdot q \cdot 1^4 \]
\[
\mathbb{R}^+_6(q, 1) = \left[ \begin{array}{c}
1 \quad 1_2 \delta^2 + 1_3 \delta^4 + 1_4 \delta^6 \\
1_9 \delta + 1_10 \delta^3 + 1_11 \delta^5 \\
1_5 \delta + 1_6 \delta^3 + 1_7 \delta^5 + 1_8 \delta^7 \\
1_12 + 1_13 \delta^2 + 1_14 \delta^4 + 1_15 \delta^6 
\end{array} \right]
\]

\[i_1 = \text{A}^3 \text{B}^3 - 3\]

\[i_2 = -4\delta^2 \text{A}^3 \text{B}^3 \text{A}^2 \text{B}^2 \cdot (31^2 + q_1) - 4\delta^2 \text{A}^2 \text{B}^2 \text{A}^2 \text{B} \cdot (41^2 + q_1) - 4\delta^2 \text{A}^2 \text{B} \text{A}^2 \text{B} \cdot (51^2 + q_1)\]

\[i_3 = -16\delta^4 \text{A}^3 \text{A}^2 \text{B}^2 \cdot (91^4 + 2q_1^3) - 16\delta^4 \text{A}^2 \text{B}^2 \cdot (101^4 + 2q_1^3)\]

\[i_4 = -64\delta^6 \text{A}^3 \text{A}^2 \text{B}^2 \cdot (51^6 + q_1^5)\]

\[i_5 = 26\delta^2 \text{A}^3 \text{B}^3 \cdot (51^2 + q_1) - 26\delta^2 \text{A}^2 \text{B}^2 \text{A}^2 \text{B} - 1 - 26\delta^2 \text{A}^2 \text{B}^2 \text{A}^2 \text{B} - 2 \cdot 1 - 26\delta^2 \text{A}^2 \text{B}^2 \cdot (51^2 + q_1) - 26\delta^2 \text{A}^2 \text{B}^2 \cdot (51^2 + q_1) - 26\delta^2 \text{A}^2 \text{B}^2 \cdot (51^2 + q_1)\]

\[i_6 = 8\delta^2 \text{A}^3 \text{B}^3 \cdot (51 + q_1) \cdot (q_1 + 21^2) + 8\delta^2 \text{A}^2 \text{B}^2 \text{A}^2 \text{B} - (41 + q_1) \cdot (q_1 + 1^3)\]

\[i_7 = 32\delta^4 \text{A}^3 \text{A}^2 \text{B}^2 \cdot (51 + q_1) \cdot (2q_1^3 + 1^4) + 32\delta^4 \text{A}^3 \text{A}^2 \text{B}^2 \cdot (91^4 + 2q_1^3)\]

\[i_8 = 128\delta^6 \text{A}^3 \text{A}^2 \text{B}^2 \cdot 15 \cdot (51 + q_1)\]

\[i_9 = -26\delta^2 \text{A}^3 \text{B}^3 \cdot (1 - 26\delta^2 \text{A}^2 \text{B}^2 \text{A}^2 \text{B} - 1 - 26\delta^2 \text{A}^2 \text{B}^2 \cdot (51^2 + q_1) - 26\delta^2 \text{A}^2 \text{B}^2 \cdot (51^2 + q_1)\]

\[i_10 = -8\delta^2 \text{A}^3 \text{B}^3 \cdot (21^3) - 8\delta^2 \text{A}^2 \text{B}^2 \text{A}^2 \text{B} - (21^3)\]

\[i_11 = -32\delta^4 \text{A}^3 \text{A}^2 \text{B}^2 \cdot 15\]

\[i_12 = \text{A}^3 \text{B}^3\]

\[i_13 = 4\delta^2 \text{A}^3 \text{B}^3 \cdot (q_1 + 21^2) + 4\delta^2 \text{A}^2 \text{B}^2 \text{A}^2 \text{B} - (q_1 + 1^2) + 4\delta^2 \text{A}^2 \text{B}^2 \cdot q_1\]

\[i_14 = 16\delta^4 \text{A}^3 \text{A}^2 \text{B}^2 \cdot (2q_1^3 + 1^4) + 16\delta^4 \text{A}^3 \text{A}^2 \text{B}^2 \cdot (2q_1^3)\]

\[i_15 = 64\delta^6 \text{A}^3 \text{A}^2 \text{B}^2 \cdot q_1^5\]

\[
\mathbb{R}^+_7(q, 1) = \left[ \begin{array}{c}
1_1 \delta + 1_2 \delta^3 + 1_3 \delta^5 + 1_4 \delta^7 \\
1_10 + 1_11 \delta^2 + 1_12 \delta^4 + 1_13 \delta^6 \\
1_15 + 1_16 \delta^2 + 1_17 \delta^4 + 1_18 \delta^6 + 1_19 \delta^8 \\
1_14 \delta + 1_15 \delta^3 + 1_16 \delta^5 + 1_17 \delta^7 
\end{array} \right]
\]
\[ J_1 = 2 \delta A^{+4} B^{-3} \cdot (61 + q) - 2 \delta A^{+3} B^+ A^- B^{-2} \cdot 1 \]
\[ - 2 \delta A^{+2} B^+ A^- B^{-1} \cdot 1 - 2 \delta A^{+3} A^- 3 \cdot 1 \]
\[ J_2 = 8 \delta A^{+4} B^- - 2 \cdot (31^2) \cdot (61 + q) + 8 \delta A^{+3} B^A^- B^{-2} \cdot (21^2) \cdot (51 + q) \]
\[ + 8 \delta A^{+2} B^+ A^- 3 \cdot 1^2 \cdot (41 + q) \]
\[ J_3 = 32 \delta A^{+4} A^- B^{-3} \cdot (31^4) \cdot (61 + q) + 32 \delta A^{+3} B^A^- A^- 3 \cdot 1^4 \cdot (111 + 2q) \]
\[ J_4 = 128 \delta A^{+4} A^+ 3 \cdot 1^6 \cdot (61 + q) \]
\[ J_5 = A^{+3} B^+ 4 \]
\[ J_6 = 4 \delta A^{+2} B^+ A^- 3 \cdot (41^2) - 4 \delta A^{+2} B^+ A^- 2 \cdot (51^2) \]
\[ - 4 \delta A^+ 2 B^A^- - 2 \cdot (61^2) - 4 \delta A^{+4} B^- 3 \cdot q \cdot (61 + q) \]
\[ J_7 = -16 \delta A^{+4} B^+ A^- 3 \cdot (q^2 1^2 + 6q1^3 + 111^4) \]
\[ - 16 \delta A^{+3} B^A^- A^- 2 \cdot (2q^2 1^2 + 12q1^3 + 121^4) - 16 \delta A^{+4} A^- B^{-2} \cdot (31^2) \cdot q \cdot (61 + q) \]
\[ J_8 = 6 \delta A^{+4} A^+ A^- 3 \cdot (2q^2 1^4 + 12q1^5 + 61^6) - 6 \delta A^{+4} A^- B^{-2} \cdot (31^4) \cdot q \cdot (61 + q) \]
\[ J_9 = -25 \delta A^{+4} A^- A^- 3 \cdot 1^6 \cdot q \cdot (61 + q) \]
\[ J_{10} = A^{+4} B^+ 3 \]
\[ J_{11} = 4 \delta A^{+4} A^+ B^- 2 \cdot (31^2) + 4 \delta A^{+3} B^A^- A^- B^{-2} \cdot (21^2) + 4 \delta A^{+2} B^+ A^- 3 \cdot 1^2 \]
\[ J_{12} = 16 \delta A^{+4} A^- B^{-2} \cdot (31^4) + 16 \delta A^{+3} B^A^- A^- 3 \cdot (21^4) \]
\[ J_{13} = 6 \delta A^{+3} B^A^- A^- 3 \cdot 1^6 \]
\[ J_{14} = -2 \delta A^+ B^+ A^- 3 \cdot 1 - 2 \delta A^{+2} B^+ A^- B^{-1} \cdot 1 - 2 \delta A^{+3} B^A^- B^{-2} \cdot 1 - 2 \delta A^{+4} B^- 3 \cdot q \]
\[ J_{15} = -8 \delta A^{+2} B^+ A^- 3 \cdot (q + 21^2) - 8 \delta A^{+3} B^A^- A^- B^{-2} \cdot (21) \cdot (q^2 + 1^2) \]
\[ - 8 \delta A^{+4} A^- B^{-2} \cdot (3q1^2) \]
\[ J_{16} = -32 \delta A^{+3} B^A^- A^- 3 \cdot 1 \cdot (2q1^3 + 1^4) - 32 \delta A^{+4} A^- B^{-2} \cdot (3q1^4) \]
\[ J_{17} = -128 \delta A^{+4} A^- A^- 3 \cdot q1^6 \]
Next the components of the matrix operators $\frac{\partial}{\partial q_{R^n}}(q,1)$:

$$\frac{\partial}{\partial q_{R^1}}(q,1) = \begin{bmatrix} \frac{c_1 \partial}{c_4} & \frac{c_2 + c_3 \partial^2}{c_5 \partial} \end{bmatrix}$$

$c_1 = 2\delta A^+$
$c_2 = 0$
$c_3 = -4\delta^2 \cdot A^+ \cdot 2q$
$c_4 = 0$
$c_5 = -2\delta \delta^+$

$$\frac{\partial}{\partial q_{R^2}}(q,1) = \begin{bmatrix} \frac{d_1 + d_2 \partial^2}{d_5 \partial} & \frac{d_3 \partial + d_4 \partial^3}{d_6 + d_7 \partial} \end{bmatrix}$$

$d_1 = 0$
$d_2 = -4\delta^2 A^- A^+ \cdot 1$
$d_3 = 2\delta A^- B^+ - 2\delta A^+ B^-$
$d_4 = 8\delta^3 A^- A^+ \cdot (2q + 1) \cdot 1$
$d_5 = 0$
$d_6 = 0$
$d_7 = 4\delta^2 A^- A^+ \cdot 1$

$$\frac{\partial}{\partial q_{R^3}}(q,1) = \begin{bmatrix} \frac{e_1 \partial + e_2 \partial^2}{e_6 + e_7 \partial^2} & \frac{e_3 + e_4 \partial^2 + e_5 \partial^4}{e_8 \partial + e_9 \partial^3} \end{bmatrix}$$

$e_1 = 2\delta A^2 B^-$
$e_2 = 8\delta^3 A^- A^2 \cdot 1^2$
$e_3 = 0$
$e_4 = -4\delta^2 A^2 B^- \cdot (21 + 2q)$
$e_5 = -16\delta^4 A^- A^2 \cdot 1^2 \cdot (21 + 2q)$
$e_6 = 0$
\[ e_7 = 0 \]
\[ e_8 = -2\delta A^2 B^{-} \]
\[ e_9 = -8\delta^3 A^- A^2 + 2 \]

\[
\frac{\partial R^+}{\partial q^{R4}}(q,1) = \left[ \begin{array}{c}
\frac{f_1 + f_2\vartheta^2 + f_3\vartheta^4}{f_7\vartheta + f_8\vartheta^3} \\
\frac{f_4\vartheta + f_5\vartheta^3 + f_6\vartheta^5}{f_9 + f_{10}\vartheta^2 + f_{11}\vartheta^4}
\end{array} \right]
\]
\[ f_1 = 0 \]
\[ f_2 = -4\delta^2 A^- A^2 B^- + 4\delta^2 A^- A^2 B^+ + 1 \]
\[ f_3 = -16\delta^4 A^2 - 2 + 3 \]
\[ f_4 = 2\delta A^{-2} B^+ + 2 - 2\delta A^2 B^{-2} \]
\[ f_5 = 8\delta^3 A^{-2} B^+ + (41^2 + 2q) + 8\delta^3 A^- B^+ + (21^2 + 2q) \]
\[ f_6 = 32\delta^5 A^{-2} B^+ + 1^3 (31 + 2q) \]
\[ f_7 = 0 \]
\[ f_8 = 0 \]
\[ f_9 = 0 \]
\[ f_{10} = 4\delta^2 A^{-2} A^B^+ + 1 + 4\delta^2 A^- B^- A^{+2} + 1 \]
\[ f_{11} = 16\delta^4 A^{-2} A^2 + 2 + 3 \]

\[
\frac{\partial R^+}{\partial q^{R5}}(q,1) = \left[ \begin{array}{c}
g_1 \vartheta + g_2\vartheta^2 + g_3\vartheta^4 \\
g_8 + g_9\vartheta^2 + g_{10}\vartheta^4
\end{array} \right] \left[ \begin{array}{c}
g_4 + g_5\vartheta^2 + g_6\vartheta^4 + g_7\vartheta^6 \\
g_{11} + g_{12}\vartheta^3 + g_{13}\vartheta^5
\end{array} \right]
\]
\[ g_1 = 2\delta A^{-3} B^{-2} \]
\[ g_2 = 8\delta^3 A^- A^B^- + (21^2) + 8\delta^3 A^2 B^+ A^{-2} + 2 \]
\[ g_3 = 32\delta^5 A^{-2} A^B^+ + 1^4 \]
\[ g_4 = 0 \]
\[ g_5 = -4\delta^2 A^+ B^- (41 + 2q) \]
\[ g_6 = -16\delta^4 A^2 B^+ A^{-2} (41^2 + 2q) - 16\delta^4 A^3 B^- (21^2) (41 + 2q) \]
\[ g_7 = -64\delta^6 A^3 - 1^4 (41 + 2q) \]
\[ g_8 = 0 \]
\[ g_9 = 0 \]
\[ g_{10} = 0 \]
\[ g_{11} = -2 \delta A^3 B^{-2} \]
\[ g_{12} = -8 \delta A^2 B + A^{-2} B^{-1} - 8 \delta A^3 B^{-2} \] (21^2)
\[ g_{13} = -32 \delta A^3 B^{-2} B^{-1} \]

\[ \frac{\partial R^+}{\partial q} (q, 1) = \begin{pmatrix}
1 + \frac{2}{2} \delta^2 + \frac{3}{4} \delta^4 + \frac{1}{4} \delta^6 & \frac{1}{2} \delta + \frac{1}{6} \delta^3 + \frac{1}{7} \delta^5 + \frac{1}{8} \delta^7 \\
\frac{1}{2} \delta + \frac{1}{11} \delta^3 + \frac{1}{12} \delta^5 & \frac{1}{13} \delta^2 + \frac{1}{14} \delta^4 + \frac{1}{15} \delta^6
\end{pmatrix} \]

\[ i_1 = 0 \]
\[ i_2 = -4 \delta A^3 B^{-2} \] (21^2)
\[ i_3 = -16 \delta A^2 B + A^{-2} B^{-1} \] (21^2)
\[ i_4 = -64 \delta A^3 B^{-3} \]
\[ i_5 = 2 \delta A^{-3} B^3 - 2 \delta A^3 B^{-3} \]
\[ i_6 = 8 \delta A^3 B + A^{-2} B^{-1} \] (71^2 + 2q1) + \[ 8 \delta A^3 B + A^{-2} B^{-1} \] (51^2 + 2q1)
\[ + 8 \delta A^3 B^{-2} \] (31^2 + 2q1)
\[ i_7 = 32 \delta A^5 + A^2 B^{-1} \] (111^4 + 4q1^3) + \[ 32 \delta A^5 + A^{-2} B^{-1} \] (91^4 + 4q1^3)
\[ i_8 = 128 \delta A^7 + A^{-3} \] (51 + 2q)
\[ i_9 = 0 \]
\[ i_{10} = 0 \]
\[ i_{11} = 0 \]
\[ i_{12} = 0 \]
\[ i_{13} = 4 \delta A^2 B + A^{-3} B^{-1} + 4 \delta A^2 B + A^{-2} B^{-1} + 4 \delta A^3 B^{-2} \] (21^2)
\[ i_{14} = 16 \delta A^2 B + A^{-3} B^{-1} \] (21^2) + \[ 16 \delta A^3 B^{-2} \] (21^2)
\[ i_{15} = 64 \delta A + A^{-3} \] (15)
\[ \frac{\partial R^+_n(q,\lambda)}{\partial q} = \begin{bmatrix} j_1 \partial^2 + j_2 \partial^3 + j_3 \partial^5 + j_4 \partial^7 \\ j_{10} + j_{11} \partial^2 + j_{12} \partial^4 + j_{13} \partial^6 \end{bmatrix} \begin{bmatrix} j_5 + j_6 \partial^2 + j_7 \partial^4 + j_8 \partial^6 + j_9 \partial^8 \\ j_{14} \partial + j_{15} \partial^3 + j_{16} \partial^5 + j_{17} \partial^7 \end{bmatrix} \]

\[
\begin{align*}
j_1 &= 2\delta A^4 B^{-3} \\
j_2 &= 8\delta A^4 A^{-2} B^{-2} \cdot (31^2) + 8\delta A^3 B A^{-3} \cdot (21^2) + 8\delta A^2 B^2 A^{-3} \cdot (21^2) \\
j_3 &= 32\delta A^5 A^{-2} B^{-3} \cdot (31^2) + 32\delta A^3 B A^{-3} \cdot (21^2) \\
j_4 &= 128\delta A^7 A^{-3} \cdot 1^2 \\
j_5 &= 0 \\
j_6 &= -4\delta A^4 B^{-3} \cdot (61^2+2q) \\
j_7 &= -16\delta A^4 A^2 B A^{-3} \cdot (31^2) - 16\delta A^3 B A^{-2} B^{-1} \cdot (4q1^2+121^3) \\
&\quad - 16\delta A^4 A^{-4} A^{-2} \cdot (31^2) \cdot (61^2+2q) \\
j_8 &= -64\delta A^6 A^3 B A^{-3} \cdot (4q1^4+121^5) + 64\delta A^4 A^{-2} B^{-1} \cdot (31^2) \cdot (61^2+2q) \\
j_9 &= -256\delta A^8 A^{-3} \cdot 1^2 \cdot (61^2+2q) \\
j_{10} &= 0 \\
j_{11} &= 0 \\
j_{12} &= 0 \\
j_{13} &= 0 \\
j_{14} &= -2\delta A^4 B^{-3} \\
j_{15} &= -8\delta A^3 A^{-2} B A^{-3} \cdot 1^2 + 8\delta A^3 B^2 A^{-2} \cdot (21^2) + 8\delta A^4 A B^{-2} \cdot (31^2) \\
j_{16} &= -32\delta A^5 A^3 B A^{-3} \cdot (21^2) - 32\delta A^5 A^3 B A^{-3} \cdot (31^2) \\
j_{17} &= -128\delta A^7 A^4 A^{-3} \cdot 1^6 
\end{align*}
\]

The explicit components of the matrix operators \( R^+_n(q,\lambda) \) have been numerically checked by assigning numerical values to \( \delta \), 
"A\(^\pm\)", "B\(^\pm\)", "q", "1", and "\(\partial\)", performing the matrix products specified in equations (F.1) and comparing to the numerical values.
using the explicit components given above. The explicit components of the matrix operators $\frac{\partial}{\partial q} R^+_n (q,1)$ is numerically checked by comparing the differencing approximation of a derivative on $R^+_n (q,1)$ to the explicit components given above.
Appendix H: Components of products of matrix operators ("L" part)

In appendix F, the product operators $R^+_{Ln}$ were defined. In this appendix, the explicit components of those product operators for obtaining $R^+_{Ln}$ up till the any group of images (refer to main text section 2.3) will be given in this appendix.

First define the following parameters:

$$y^\pm = \frac{1 - y_\pm}{1 + y_\pm} \quad (H.1)$$

Then we have:

$$R^+_{Ln} = \begin{cases} 
1 & \text{if } n = 0 \\
y^+ & \text{if } n = 1 \\
n/2 & \text{if } n \text{ is even} \\
(y^+) (y^-) & \text{if } n \text{ is odd} \\
(n+1)/2 & \text{if } n \text{ is even} \\
(n-1)/2 & \text{if } n \text{ is odd}
\end{cases} \quad (H.2)$$

$$R^-_{Ln} = \begin{cases} 
1 & \text{if } n = 0 \\
y^- & \text{if } n = 1 \\
n/2 & \text{if } n \text{ is even} \\
(y^-) (y^+) & \text{if } n \text{ is odd} \\
(n+1)/2 & \text{if } n \text{ is even} \\
(n-1)/2 & \text{if } n \text{ is odd}
\end{cases} \quad (H.3)$$
Appendix I: Notes on the derivation of nucleii of strain in a plate

The derivation of displacement and stress fields of nucleii of strain (NOS) in a plate are obtained indirectly from the stresses and (linear) combinations of the gradient of stresses with respect to "load application location" of point forces in a plate through the reciprocity relation (see section 2.5, as well as chapter 3.1). For convenience, the Hansen potentials for point forces (given in appendix B), are rewritten in matrix notation below:

\[ \varphi_R^0 = + p_1 \begin{bmatrix} +1 \\ -1 \end{bmatrix} \frac{\beta}{2} \varphi_A + p_2 \begin{bmatrix} +1 \\ -1 \end{bmatrix} \frac{\beta}{2} \varphi_C + p_3 \begin{bmatrix} +1 \\ +1 \end{bmatrix} \frac{\beta}{2} \varphi_B \]

(I.1)

\[ \varphi_L^0 = + p_1 \begin{bmatrix} +1 \\ -1 \end{bmatrix} \beta \cdot (1+\delta) \cdot \varphi_C + p_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix} \beta \cdot (1+\delta) \cdot \varphi_A + p_3 \begin{bmatrix} 0 \\ \end{bmatrix} \]

(I.2)

or

\[ \varphi_R^0 = + p_1 \begin{bmatrix} +1 \\ -1 \end{bmatrix} \frac{\beta}{2} \varphi_A + p_2 \begin{bmatrix} +1 \\ -1 \end{bmatrix} \frac{\beta}{2} \varphi_C + p_3 \begin{bmatrix} -1 \\ -1 \end{bmatrix} \frac{\beta}{2} \varphi_B \]

(I.3)

\[ \varphi_L^0 = + p_1 \begin{bmatrix} +1 \\ -1 \end{bmatrix} \beta \cdot (1+\delta) \cdot \varphi_C + p_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix} \beta \cdot (1+\delta) \cdot \varphi_A + p_3 \begin{bmatrix} 0 \\ \end{bmatrix} \]

(I.4)

where:

\[ \beta = \frac{1}{4\pi \mu (1+\delta)} \]

\[ \varphi_A = \frac{\kappa}{r+z} \]

\[ \varphi_A = \frac{\kappa}{r-z} \]
\[ \varphi_C = \frac{y}{r+z} \quad \varphi_C = \frac{y}{r-z} \]
\[ \varphi_B = + \ln(r+z) \quad \varphi_B = + \ln(r-z) \]

(I.5)

Note that both the \( \varphi \)'s and \( \varphi \)'s are potentials that will give the same displacement field due to a point force, with the singularity located at a single point only (the location of the point force). However, the functions of which they are linearly combined (i.e. \( \varphi_A, \varphi_A, \varphi_B, \varphi_B, \varphi_C \) and \( \varphi_C \)), when not combined in the precise way that produces the Hansen potentials for a point force, introduce singularities in the displacement fields at either \( z<0 \) for the \( \varphi \) potentials or at \( z>0 \) for the \( \varphi \) potentials. Since the image algorithm is expected to recombine the constituent potentials of the point force and their \( \frac{\partial^n}{\partial z^n} \) derivatives, care has to be taken so as not to introduce any singular behavior in the displacement field other than at the location of the point force. This is effectively done by using the \( \varphi \) potentials at the "+m" and "-m" image sources (see figure 2.3) and the \( \varphi \) potentials at the "-m" and "+m" image sources.

Finally, note the following relations that exist between the functions \( \varphi_A \) and \( \varphi_C \), and the \( \varphi \) and \( \varphi \) functions.

\[ \varphi_A(x,y,z) = \varphi_C(y,x,z) \quad \text{(I.6)} \]

\[ \frac{\partial^{m+n+p}}{\partial x^m \partial y^n \partial z^p} \varphi_C(x,y,z) = \frac{\partial^{m+n+p}}{\partial \xi^m \partial \eta^n \partial z^p} \varphi_C(\eta,\xi,z) \bigg|_{\eta=y \xi-x} \quad \text{(I.7)} \]

The above relations are useful when programming the elastic NOS fields on a computer, since only the \( \varphi_A \) and \( \varphi_B \) functions and
their required partial derivatives need to be explicitly
programmed. The above relations imply that the functions \( \varphi_A, \varphi_B, \varphi_C \)
and \( \varphi_C \) and their partial derivatives can be evaluated in terms of
\( \varphi_A \) and \( \varphi_B \) and their partial derivatives.
Appendix J: Functions required for the evaluation of
displacement and stress fields of nuclei of strain in a plate

In appendix I, it was shown that the functions $\varphi_A$ and $\varphi_B$ and
(a set of) their partial derivatives were required for obtaining
the displacement and stress fields of nuclei of strain (NOS) in a
plate. In this appendix, the above functions and their partial
derivatives that are required to obtain up till the fourth set of
images for the "R part" and up till any set of images for the "L
part" (see section 2.3 for the meaning of the "R" and "L" part) are
given.

Partial derivatives of the function $\varphi - \varphi_A = \frac{x}{r+z}$:

$$
\varphi = \frac{x}{r+z}
\frac{\partial}{\partial z} \varphi = -\frac{x}{r \cdot (r+z)}
\frac{\partial^2}{\partial z^2} \varphi = \frac{x}{r^3}
\frac{\partial^3}{\partial z^3} \varphi = -\frac{3xz}{r^5}
\frac{\partial^4}{\partial z^4} \varphi = -3x \left( \frac{1}{r^5} - \frac{5z^2}{r^7} \right)
\frac{\partial^5}{\partial z^5} \varphi = -15x \left( -3 \cdot \frac{z}{r^7} + 7 \cdot \frac{z^3}{r^9} \right)
\frac{\partial^6}{\partial z^6} \varphi = -15x \left( -3 \cdot \frac{1}{r^7} + 42 \cdot \frac{z^2}{r^9} - 63 \cdot \frac{z^4}{r^{11}} \right)
\frac{\partial^7}{\partial z^7} \varphi = -315x \left( 5 \cdot \frac{z}{r^9} - 30 \cdot \frac{z^3}{r^{11}} + 33 \cdot \frac{z^5}{r^{13}} \right)
\frac{\partial^8}{\partial z^8} \varphi = -315x \left( 5 \cdot \frac{1}{r^9} - 135 \cdot \frac{z^2}{r^{11}} + 495 \cdot \frac{z^4}{r^{13}} - 429 \cdot \frac{z^6}{r^{15}} \right)
\frac{\partial^9}{\partial z^9} \varphi = -2835x \left( -35 \cdot \frac{z}{r^{11}} + 385 \cdot \frac{z^3}{r^{13}} - 1001 \cdot \frac{z^5}{r^{15}} + 715 \cdot \frac{z^7}{r^{17}} \right)
$$
\[
\frac{\partial^{10}}{\partial z^{10}} = -14175x \cdot \left(-7 \cdot \frac{1}{r^{11}} + 308 \cdot \frac{z^2}{r^{13}} - 2002 \cdot \frac{z^4}{r^{15}} + 4004 \cdot \frac{z^6}{r^{17}} - 2431 \cdot \frac{z^8}{r^{19}} \right)
\]

\[
\frac{\partial^{11}}{\partial z^{11}} = -155925x \cdot \left( 63 \cdot \frac{z}{r^{13}} - 1092 \cdot \frac{z^3}{r^{15}} + 4914 \cdot \frac{z^5}{r^{17}} - 7956 \cdot \frac{z^7}{r^{19}} + 4199 \cdot \frac{z^9}{r^{21}} \right)
\]

\[
\frac{\partial^{12}}{\partial z^{12}} = -467775x \cdot \left( 21 \cdot \frac{1}{r^{13}} - 1365 \cdot \frac{z^2}{r^{15}} + 13650 \cdot \frac{z^4}{r^{17}} - 46410 \cdot \frac{z^6}{r^{19}} + 62985 \cdot \frac{z^8}{r^{21}} - 29393 \cdot \frac{z^{10}}{r^{23}} \right)
\]

\[
\frac{\partial}{\partial x} \varphi = \frac{1}{r+z} - \frac{x^2}{r \cdot (r+z)^2}
\]

\[
\frac{\partial^2}{\partial x \partial z} \varphi = \frac{-1}{r \cdot (r+z)} + \frac{x^2}{r^3 \cdot (r+z)^2} + \frac{x^2}{r^2 \cdot (r+z)^2}
\]

\[
\frac{\partial^3}{\partial x \partial z^2} \varphi = \frac{1}{r^3} - \frac{3x^2}{r^5}
\]

\[
\frac{\partial^4}{\partial x \partial z^3} \varphi = \frac{-3z}{r^5} + 15x^2 \cdot \frac{z}{r^7}
\]

\[
\frac{\partial^5}{\partial x \partial z^4} \varphi = -3 \cdot \left( \frac{1}{r^5} - 5 \cdot \frac{z^2}{r^7} \right) - 3x^2 \cdot \left( -5 \cdot \frac{1}{r^7} + 35 \cdot \frac{z^2}{r^9} \right)
\]

\[
\frac{\partial^6}{\partial x \partial z^5} \varphi = -15 \cdot \left( -3 \cdot \frac{z}{r^7} + 7 \cdot \frac{z^3}{r^9} \right) - 15x^2 \cdot \left( 21 \cdot \frac{z}{r^9} - 63 \cdot \frac{z^3}{r^{11}} \right)
\]

\[
\frac{\partial^7}{\partial x \partial z^6} \varphi = -15 \cdot \left( -3 \cdot \frac{1}{r^7} + 42 \cdot \frac{z^2}{r^9} - 63 \cdot \frac{z^4}{r^{11}} \right)
\]

\[
- 15x^2 \cdot \left( 21 \cdot \frac{1}{r^9} - 378 \cdot \frac{z^2}{r^{11}} + 693 \cdot \frac{z^4}{r^{13}} \right)
\]

\[
\frac{\partial^8}{\partial x \partial z^7} \varphi = -315 \cdot \left( 5 \cdot \frac{z}{r^9} - 30 \cdot \frac{z^3}{r^{11}} + 33 \cdot \frac{z^5}{r^{13}} \right)
\]

\[
- 315x^2 \cdot \left( -45 \cdot \frac{z}{r^{11}} + 330 \cdot \frac{z^3}{r^{13}} - 429 \cdot \frac{z^5}{r^{15}} \right)
\]
\[ \frac{\partial^9 \varphi}{\partial x \partial z} = -315 \cdot \left( \frac{5}{r^9} \right) - 135 \cdot \frac{z^2}{r^{11}} + 495 \cdot \frac{z^4}{r^{13}} - 429 \cdot \frac{z^6}{r^{15}} \]
\[ \quad - 315x^2 \cdot \left( -45 \cdot \frac{1}{r^{11}} + 1485 \cdot \frac{z^2}{r^{13}} - 6435 \cdot \frac{z^4}{r^{15}} + 6435 \cdot \frac{z^6}{r^{17}} \right) \]
\[ \frac{\partial^{10} \varphi}{\partial x \partial z} = -2835 \cdot \left( -35 \cdot \frac{z}{r^{11}} + 385 \cdot \frac{z^3}{r^{13}} - 1001 \cdot \frac{z^5}{r^{15}} + 715 \cdot \frac{z^7}{r^{17}} \right) \]
\[ \quad - 2835x^2 \cdot \left( 385 \cdot \frac{z}{r^{13}} - 5005 \cdot \frac{z^3}{r^{15}} + 15015 \cdot \frac{z^5}{r^{17}} - 12155 \cdot \frac{z^7}{r^{19}} \right) \]
\[ \frac{\partial^{11} \varphi}{\partial x \partial z} = -14175 \cdot \left( -7 \cdot \frac{1}{r^{11}} + 308 \cdot \frac{z^2}{r^{13}} - 2002 \cdot \frac{z^4}{r^{15}} \right. \]
\[ \quad + 4004 \cdot \frac{z^6}{r^{17}} - 2431 \cdot \frac{z^8}{r^{19}} \]
\[ \quad - 14175x^2 \cdot \left( 77 \cdot \frac{1}{r^{13}} - 4004 \cdot \frac{z^2}{r^{15}} \right. \]
\[ \quad + 30030 \cdot \frac{z^4}{r^{17}} - 68068 \cdot \frac{z^6}{r^{19}} + 46189 \cdot \frac{z^8}{r^{21}} \right) \]
\[ \frac{\partial^{12} \varphi}{\partial x \partial z} = -155925 \cdot \left( 63 \cdot \frac{z}{r^{13}} - 1092 \cdot \frac{z^3}{r^{15}} \right. \]
\[ \quad + 4914 \cdot \frac{z^5}{r^{17}} - 7956 \cdot \frac{z^7}{r^{19}} + 4199 \cdot \frac{z^9}{r^{21}} \right) \]
\[ \quad - 155925x^2 \cdot \left( -819 \cdot \frac{z}{r^{15}} + 16380 \cdot \frac{z^3}{r^{17}} \right. \]
\[ \quad - 83538 \cdot \frac{z^5}{r^{19}} + 151164 \cdot \frac{z^7}{r^{21}} - 88179 \cdot \frac{z^9}{r^{23}} \right) \]
\[ \frac{\partial^2 \varphi}{\partial x^2} = \frac{-3x}{r \cdot (r+z)^2} + \frac{x^3}{r^3 \cdot (r+z)^2} + \frac{2x^3}{r^2 \cdot (r+z)^3} \]
\[ \frac{\partial^3 \varphi}{\partial x^2 \partial z} = \frac{3x}{r^3 \cdot (r+z)} + \frac{3x}{r^2 \cdot (r+z)^2} - \frac{3x^3}{r^4 \cdot (r+z)^2} - \frac{3x^3}{r^5 \cdot (r+z)} - \frac{2x^3}{r^3 \cdot (r+z)^3} \]
\[ \frac{\partial^4 \varphi}{\partial x^2 \partial z^2} = \frac{-9x}{r^5} + \frac{15x^3}{r^7} \]
\[ \frac{\partial^5 \varphi}{\partial x^2 \partial z^3} = \frac{45x}{r^7} - \frac{105x^3}{r^9} \]
\[
\frac{\partial^6 \varphi}{\partial x^2 \partial z^4} = -9x \cdot (\frac{5}{r^7} + 35 \cdot \frac{2}{r^9}) - 3x^3 \cdot (\frac{35}{r^9} - 315 \cdot \frac{2}{r^{11}})
\]
\[
\frac{\partial^7 \varphi}{\partial x^2 \partial z^5} = -45x \cdot (21 \cdot \frac{z}{r} - 63 \cdot \frac{z^3}{r^{11}}) - 15x^3 \cdot (-189 \cdot \frac{z}{r^{11}} + 693 \cdot \frac{z^3}{r^{13}})
\]
\[
\frac{\partial^8 \varphi}{\partial x^2 \partial z^6} = -45x \cdot (21 \cdot \frac{1}{r} - 378 \cdot \frac{z^2}{r^{11}} + 693 \cdot \frac{z^4}{r^{13}})
\]
\[
- 15x^3 \cdot (-189 \cdot \frac{1}{r^{11}} + 4158 \cdot \frac{z^2}{r^{13}} - 9009 \cdot \frac{z^4}{r^{15}})
\]
\[
\frac{\partial^9 \varphi}{\partial x^2 \partial z^7} = -945x \cdot (-45 \cdot \frac{z}{r^{11}} + 330 \cdot \frac{z^3}{r^{13}} - 429 \cdot \frac{z^5}{r^{15}})
\]
\[
- 315x^3 \cdot (495 \cdot \frac{z}{r^{13}} - 4290 \cdot \frac{z^3}{r^{15}} + 6435 \cdot \frac{z^5}{r^{17}})
\]
\[
\frac{\partial^{10} \varphi}{\partial x^2 \partial z^8} = -945x \cdot (-45 \cdot \frac{1}{r^{11}} + 1485 \cdot \frac{z^2}{r^{13}} - 6435 \cdot \frac{z^4}{r^{15}} + 6435 \cdot \frac{z^6}{r^{17}})
\]
\[
- 315x^3 \cdot (495 \cdot \frac{1}{r^{13}} - 19305 \cdot \frac{z^2}{r^{15}} + 96525 \cdot \frac{z^4}{r^{17}} - 109395 \cdot \frac{z^6}{r^{19}})
\]
\[
\frac{\partial^{11} \varphi}{\partial x^2 \partial z^9} = -8505x \cdot (385 \cdot \frac{z}{r^{13}} - 5005 \cdot \frac{z^3}{r^{15}} + 15015 \cdot \frac{z^5}{r^{17}} - 12155 \cdot \frac{z^7}{r^{19}})
\]
\[
- 2835x^3 \cdot (-5005 \cdot \frac{z}{r^{15}} + 75075 \cdot \frac{z^3}{r^{17}} - 255255 \cdot \frac{z^5}{r^{19}} + 230945 \cdot \frac{z^7}{r^{21}})
\]
\[
\frac{\partial^{12} \varphi}{\partial x^2 \partial z^{10}} = -42525x \cdot (77 \cdot \frac{1}{r^{13}} - 4004 \cdot \frac{z^2}{r^{15}} + 30030 \cdot \frac{z^4}{r^{17}} - 68068 \cdot \frac{z^6}{r^{19}} + 46189 \cdot \frac{z^8}{r^{21}})
\]
\[
- 14175x^3 \cdot (-1001 \cdot \frac{1}{r^{15}} + 60060 \cdot \frac{z^2}{r^{17}} - 510510 \cdot \frac{z^4}{r^{19}}
\]
\[
+ 1293292 \cdot \frac{z^6}{r^{21}} - 969969 \cdot \frac{z^8}{r^{23}})
\]
\[
\frac{\partial^3 \varphi}{\partial x^3 \varphi} = \frac{-3}{r \cdot (r+z)^2} + \frac{6x^2}{r^3 \cdot (r+z)^2} + \frac{12x^2}{r^2 \cdot (r+z)^3} - \frac{6x^4}{r^4 \cdot (r+z)^3}
\]
\[
- \frac{3x^4}{r^5 \cdot (r+z)^2} - \frac{6x^4}{r^3 \cdot (r+z)^4}
\]
\[
\frac{\partial^4 \varphi}{\partial x^3 \partial z} = \frac{3}{r^3 \cdot (r+z)} - \frac{18x^2}{r^5 \cdot (r+z)} - \frac{18x^2}{r^4 \cdot (r+z)^2} + \frac{3}{r^2 \cdot (r+z)^2} - \frac{12x^2}{r^3 \cdot (r+z)^3}
\]
\[+ \frac{15x^4}{r^6 \cdot (r+z)^2} + \frac{12x^4}{r^5 \cdot (r+z)^3} + \frac{15x^4}{r^7 \cdot (r+z)} + \frac{6x^4}{r^4 \cdot (r+z)^4} \]
\[
\frac{\partial^5 \varphi}{\partial x^3 \partial z^2} = -\frac{9}{5} + 90x^2 \cdot \frac{1}{r} - 105x^4 \cdot \frac{1}{r^9} \]
\[
\frac{\partial^6 \varphi}{\partial x^3 \partial z^3} = 45 \cdot \frac{z}{r^7} - 630x^2 \cdot \frac{z}{r^9} + 945x^4 \cdot \frac{z}{r^{11}} \]
\[
\frac{\partial^7 \varphi}{\partial x^3 \partial z^4} = -9 \cdot (-5 \cdot \frac{1}{r^7} + 35 \cdot \frac{z^2}{r^9}) - 18x^2 \cdot (35 \cdot \frac{1}{r^9} - 315 \cdot \frac{z^2}{r^{11}})
\]
\[\quad - 3x^4 \cdot (-315 \cdot \frac{1}{r^{11}} + 3465 \cdot \frac{z^2}{r^{13}}) \]
\[
\frac{\partial^8 \varphi}{\partial x^3 \partial z^5} = -45 \cdot (21 \cdot \frac{z}{r^9} - 63 \cdot \frac{z^3}{r^{11}}) - 90x^2 \cdot (-189 \cdot \frac{z}{r^{11}} + 693 \cdot \frac{z^3}{r^{13}})
\]
\[\quad - 15x^4 \cdot (2079 \cdot \frac{z}{r^{13}} - 9009 \cdot \frac{z^3}{r^{15}}) \]
\[
\frac{\partial^9 \varphi}{\partial x^3 \partial z^6} = -45 \cdot (21 \cdot \frac{1}{r^9} - 378 \cdot \frac{z^2}{r^{11}} + 693 \cdot \frac{z^4}{r^{13}})
\]
\[\quad - 90x^2 \cdot (-189 \cdot \frac{1}{r^{11}} + 4158 \cdot \frac{z^2}{r^{13}} - 9009 \cdot \frac{z^4}{r^{15}})
\]
\[\quad - 15x^4 \cdot (2079 \cdot \frac{1}{r^{13}} - 54054 \cdot \frac{z^2}{r^{15}} + 135135 \cdot \frac{z^4}{r^{17}}) \]
\[
\frac{\partial^{10} \varphi}{\partial x^3 \partial z^7} = -945 \cdot (-45 \cdot \frac{z}{r^{11}} + 330 \cdot \frac{z^3}{r^{13}} - 429 \cdot \frac{z^5}{r^{15}})
\]
\[\quad - 1890x^2 \cdot (495 \cdot \frac{z}{r^{13}} - 4290 \cdot \frac{z^3}{r^{15}} + 6435 \cdot \frac{z^5}{r^{17}})
\]
\[\quad - 315x^4 \cdot (-6435 \cdot \frac{z}{r^{15}} + 64350 \cdot \frac{z^3}{r^{17}} - 109395 \cdot \frac{z^5}{r^{19}}) \]
\[
\frac{\partial^{11}}{\partial x^3 \partial z^8} \varphi = -945 \cdot \left( -45 \cdot \frac{1}{r^{11}} + 1485 \cdot \frac{2}{r^{13}} - 6435 \cdot \frac{4}{r^{15}} + 6435 \cdot \frac{6}{r^{17}} \right )
\]

\[-1890 \frac{x^2}{r^{11}} \cdot \left( 495 \cdot \frac{1}{r^{13}} - 19305 \cdot \frac{2}{r^{15}} + 96525 \cdot \frac{4}{r^{17}} - 109395 \cdot \frac{6}{r^{19}} \right )
\]

\[-315 \frac{x^4}{r^{13}} \cdot \left( -6435 \cdot \frac{1}{r^{15}} + 289575 \cdot \frac{2}{r^{17}} - 1640925 \cdot \frac{4}{r^{19}} + 2078505 \cdot \frac{6}{r^{21}} \right )
\]

\[
\frac{\partial^{12}}{\partial x^3 \partial z^9} \varphi = -8505 \cdot \left( 385 \cdot \frac{z^3}{r^{13}} - 5005 \cdot \frac{z^5}{r^{15}} + 15015 \cdot \frac{z^7}{r^{17}} - 12155 \cdot \frac{z^9}{r^{19}} \right )
\]

\[-17010 \frac{x^2 z^3}{r^{13}} \cdot \left( -5005 \cdot \frac{z^5}{r^{15}} + 75075 \cdot \frac{z^7}{r^{17}} - 255255 \cdot \frac{z^9}{r^{19}} + 230945 \cdot \frac{z^{11}}{r^{21}} \right )
\]

\[-2835 \frac{x^4 z^3}{r^{15}} \cdot \left( 75075 \cdot \frac{z^7}{r^{19}} - 1276275 \cdot \frac{z^9}{r^{21}} + 4849845 \cdot \frac{z^{11}}{r^{23}} - 4849845 \cdot \frac{z^{13}}{r^{25}} \right )
\]

\[
\frac{\partial}{\partial y} \varphi = -\frac{xy}{r \cdot (r+z)^2}
\]

\[
\frac{\partial^2}{\partial y \partial z} \varphi = -\frac{xy}{r^3 (r+z)} + \frac{xy}{r^2 (r+z)^2}
\]

\[
\frac{\partial^3}{\partial y^3} \varphi = -3xy \cdot \frac{1}{r^5}
\]

\[
\frac{\partial^4}{\partial y^4} \varphi = 15xy \cdot \frac{z}{r^7}
\]

\[
\frac{\partial^5}{\partial y^5} \varphi = -3xy \left( -5 \cdot \frac{1}{r^7} + 35 \cdot \frac{z^2}{r^9} \right )
\]

\[
\frac{\partial^6}{\partial y^6} \varphi = -15xy \left( 21 \cdot \frac{z}{r^9} - 63 \cdot \frac{z^3}{r^{11}} \right )
\]

\[
\frac{\partial^7}{\partial y^7} \varphi = -15xy \left( 21 \cdot \frac{1}{r^9} - 378 \cdot \frac{z^2}{r^{11}} + 693 \cdot \frac{z^4}{r^{13}} \right )
\]

\[
\frac{\partial^8}{\partial y^8} \varphi = -315xy \left( -45 \cdot \frac{z}{r^{11}} + 330 \cdot \frac{z^3}{r^{13}} - 429 \cdot \frac{z^5}{r^{15}} \right )
\]

\[
\frac{\partial^9}{\partial y^9} \varphi = -315xy \left( -45 \cdot \frac{1}{r^{11}} + 1485 \cdot \frac{z^2}{r^{13}} - 6435 \cdot \frac{z^4}{r^{15}} + 6435 \cdot \frac{z^6}{r^{17}} \right )
\]

\[
\frac{\partial^{10}}{\partial y^{10}} \varphi = -2835xy \left( 385 \cdot \frac{z}{r^{13}} - 5005 \cdot \frac{z^3}{r^{15}} + 15015 \cdot \frac{z^5}{r^{17}} - 12155 \cdot \frac{z^7}{r^{19}} \right )
\]
\[
\frac{\partial^{12}\varphi}{\partial y \partial z} = -155925xy \cdot (-819 \cdot \frac{z}{r^{15}} + 16380 \cdot \frac{z^3}{r^{17}} - 83538 \cdot \frac{z^5}{r^{19}} + 151164 \cdot \frac{z^7}{r^{21}} - 88179 \cdot \frac{z^9}{r^{23}})
\]
\[
\frac{\partial^2 \varphi}{\partial y^2} = \frac{-x}{r \cdot (r+z)^2} + \frac{2xy^2}{r^2 \cdot (r+z)^3}
\]
\[
\frac{\partial^3 \varphi}{\partial y^3 \partial z} = \frac{x}{r \cdot (r+z)} + \frac{x}{r^2 \cdot (r+z)^2} - \frac{3xy^2}{r^3 \cdot (r+z)^2} - \frac{3xy^2}{r^4 \cdot (r+z)^2} - \frac{2xy^2}{r^5 \cdot (r+z)^3}
\]
\[
\frac{\partial^4 \varphi}{\partial y^4 \partial z^2} = -3x \cdot \frac{1}{r^5} + 15xy^2 \cdot \frac{1}{r^7}
\]
\[
\frac{\partial^5 \varphi}{\partial y^5 \partial z^3} = 15x \cdot \frac{z}{r} - 105xy^2 \cdot \frac{z}{r^9}
\]
\[
\frac{\partial^6 \varphi}{\partial y^6 \partial z^5} = -3x \cdot (-5 \cdot \frac{1}{r^7} + 35 \cdot \frac{z}{r^9}) - 3xy^2 \cdot (35 \cdot \frac{1}{r^9} - 315 \cdot \frac{z}{r^{11}})
\]
\[
\frac{\partial^7 \varphi}{\partial y^7 \partial z^6} = -15x \cdot (21 \cdot \frac{z}{r^9} - 63 \cdot \frac{z^3}{r^{11}}) - 15xy^2 \cdot (-189 \cdot \frac{z}{r^{11}} + 693 \cdot \frac{z^3}{r^{13}})
\]
\[
\frac{\partial^8 \varphi}{\partial y^8 \partial z^8} = -15x \cdot (21 \cdot \frac{1}{r^9} - 378 \cdot \frac{z^2}{r^{11}} + 693 \cdot \frac{z^4}{r^{13}})
\]
\[
- 15xy^2 \cdot (-189 \cdot \frac{1}{r^{11}} + 4158 \cdot \frac{z^2}{r^{13}} - 9009 \cdot \frac{z^4}{r^{15}})
\]
\[
\frac{\partial^9 \varphi}{\partial y^9 \partial z^9} = -315x \cdot (-45 \cdot \frac{z}{r^{11}} + 330 \cdot \frac{z^3}{r^{13}} - 429 \cdot \frac{z^5}{r^{15}})
\]
\[
- 315xy^2 \cdot (495 \cdot \frac{z}{r^{13}} - 4290 \cdot \frac{z^3}{r^{15}} + 6435 \cdot \frac{z^5}{r^{17}})
\]
\[
\frac{\partial^{10} \varphi}{\partial y^{10} \partial z^{10}} = -315x \cdot (-45 \cdot \frac{1}{r^{11}} + 1485 \cdot \frac{z^2}{r^{13}} - 6435 \cdot \frac{z^4}{r^{15}} + 6435 \cdot \frac{z^6}{r^{17}})
\]
\[
- 315xy^2 \cdot (495 \cdot \frac{1}{r^{13}} - 19305 \cdot \frac{z^2}{r^{15}} + 96525 \cdot \frac{z^4}{r^{17}} - 109395 \cdot \frac{z^6}{r^{19}})
\]
\[ \frac{\partial^{11}}{\partial y^2 \partial z^9} \phi = -2835x \cdot \left( \frac{385}{r^{13}} \cdot \frac{z}{r^{15}} + 5005 \cdot \frac{z^3}{r^{15}} + 15015 \cdot \frac{z^5}{r^{17}} - 12155 \cdot \frac{z^7}{r^{19}} \right) \\
- 2835xy \cdot \left( -5005 \cdot \frac{z}{r^{15}} + 75075 \cdot \frac{z^3}{r^{17}} - 255255 \cdot \frac{z^5}{r^{19}} + 230945 \cdot \frac{z^7}{r^{21}} \right) \\
\frac{\partial^{12}}{\partial y^2 \partial z^{10}} \phi = -14175x \cdot \left( \frac{77}{r^{13}} \cdot \frac{1}{r^{15}} - 4004 \cdot \frac{z^2}{r^{15}} + 30030 \cdot \frac{z^4}{r^{17}} - 68068 \cdot \frac{z^6}{r^{19}} + 46189 \cdot \frac{z^8}{r^{21}} \right) \\
- 14175xy \cdot \left( -1001 \cdot \frac{1}{r^{15}} + 60060 \cdot \frac{z^2}{r^{17}} - 510510 \cdot \frac{z^4}{r^{19}} + 1293292 \cdot \frac{z^6}{r^{21}} - 969969 \cdot \frac{z^8}{r^{23}} \right) \\
\frac{\partial^3}{\partial y^3 \partial z} \phi = \frac{3xy}{r^3 \cdot (r+z)^2} + \frac{6xy}{r^2 \cdot (r+z)^3} - \frac{6xy^3}{r^4 \cdot (r+z)^2} - \frac{3xy^3}{r^3 \cdot (r+z)^3} - \frac{6xy^3}{r^3 \cdot (r+z)^4} \\
\frac{\partial^4}{\partial y^4 \partial z} \phi = -\frac{9xy}{r^5 \cdot (r+z)} - \frac{9xy}{r^4 \cdot (r+z)^2} - \frac{6xy^3}{r^3 \cdot (r+z)^3} + \frac{15xy^3}{r^6 \cdot (r+z)^2} \\
+ \frac{12xy^3}{r^5 \cdot (r+z)^3} + \frac{15xy^3}{r^7 \cdot (r+z)} + \frac{6xy^3}{r^4 \cdot (r+z)^4} \\
\frac{\partial^5}{\partial y^5 \partial z^2} \phi = 45xy \cdot \frac{1}{r^7} - 105xy^3 \cdot \frac{1}{r^9} \\
\frac{\partial^6}{\partial y^6 \partial z^3} \phi = -315xy \cdot \frac{z}{r^9} + 945xy^3 \cdot \frac{z}{r^{11}} \\
\frac{\partial^7}{\partial y^7 \partial z^4} \phi = -9xy \cdot \left( \frac{35}{r^9} - \frac{315}{r^{11}} \cdot \frac{z^2}{r^9} \right) - 3xy^3 \cdot \left( -315 \cdot \frac{1}{r^{11}} + 3465 \cdot \frac{z^2}{r^{13}} \right) \\
\frac{\partial^8}{\partial y^8 \partial z^5} \phi = -45xy \cdot \left( -189 \cdot \frac{z}{r^{11}} + 693 \cdot \frac{z^3}{r^{13}} \right) \\
- 15xy^5 \cdot \left( 2079 \cdot \frac{z}{r^{13}} - 9009 \cdot \frac{z^3}{r^{15}} \right) \\
\frac{\partial^9}{\partial y^9 \partial z^6} \phi = -45xy \cdot \left( -189 \cdot \frac{1}{r^{11}} + 4158 \cdot \frac{z^2}{r^{13}} - 9009 \cdot \frac{z^4}{r^{15}} \right) \\
- 15xy^5 \cdot \left( 2079 \cdot \frac{1}{r^{13}} - 54054 \cdot \frac{z^2}{r^{15}} + 135135 \cdot \frac{z^4}{r^{17}} \right) \]
\[
\begin{align*}
\frac{\partial^{10}}{\partial y^3 \partial z} \varphi &= -945 xy \cdot \left( 495 \cdot \frac{z}{r^{13}} - 4290 \cdot \frac{z^3}{r^{15}} + 6435 \cdot \frac{z^5}{r^{17}} \right) \\
&\quad - 315 xy^3 \cdot \left( -6435 \cdot \frac{z}{r^{15}} + 64350 \cdot \frac{z^3}{r^{17}} - 109395 \cdot \frac{z^5}{r^{19}} \right) \\
\frac{\partial^{11}}{\partial y^3 \partial z} \varphi &= -945 xy \cdot \left( 495 \cdot \frac{1}{r^{13}} - 19305 \cdot \frac{z^2}{r^{15}} + 96525 \cdot \frac{z^4}{r^{17}} - 109395 \cdot \frac{z^6}{r^{19}} \right) \\
&\quad - 315 xy^3 \cdot \left( -6435 \cdot \frac{1}{r^{15}} + 289575 \cdot \frac{z^2}{r^{17}} - 1640925 \cdot \frac{z^4}{r^{19}} + 2078505 \cdot \frac{z^6}{r^{21}} \right) \\
\frac{\partial^{12}}{\partial y^3 \partial z} \varphi &= -8505 xy \cdot \left( -5005 \cdot \frac{z}{r^{15}} + 75075 \cdot \frac{z^3}{r^{17}} \\
&\quad - 255255 \cdot \frac{z^5}{r^{19}} + 230945 \cdot \frac{z^7}{r^{21}} \right) \\
&\quad - 2835 xy^3 \cdot \left( 75075 \cdot \frac{z}{r^{17}} - 1276275 \cdot \frac{z^3}{r^{19}} + 4849845 \cdot \frac{z^5}{r^{21}} - 4849845 \cdot \frac{z^7}{r^{23}} \right)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2}{\partial x \partial y} \varphi &= -\frac{y}{r \cdot (r+z)^2} + \frac{x^2 y}{r^3 \cdot (r+z)^2} + \frac{2x^2 y}{r^2 \cdot (r+z)^3} \\
\frac{\partial^3}{\partial x \partial y \partial z} \varphi &= -\frac{y}{r^3 \cdot (r+z)} + \frac{y}{r^2 \cdot (r+z)^2} - \frac{3x^2 y}{r^5 \cdot (r+z)} - \frac{3x^2 y}{r^4 \cdot (r+z)^2} - \frac{2x^2 y}{r^3 \cdot (r+z)^3} \\
\frac{\partial^4}{\partial x \partial y \partial z^2} \varphi &= -3y \cdot \frac{1}{r^5} + 15x^2 y \cdot \frac{1}{r^7} \\
\frac{\partial^5}{\partial x \partial y \partial z^3} \varphi &= +15y \cdot \frac{z}{r^7} - 105x^2 y \cdot \frac{z}{r^9} \\
\frac{\partial^6}{\partial x \partial y \partial z^4} \varphi &= -3y \cdot \left( -5 \cdot \frac{1}{r^7} + 35 \cdot \frac{z^2}{r^9} \right) - 3x^2 y \cdot \left( 35 \cdot \frac{1}{r^9} - 315 \cdot \frac{z^2}{r^{11}} \right) \\
\frac{\partial^7}{\partial x \partial y \partial z^5} \varphi &= -15y \cdot \left( 21 \cdot \frac{z}{r^{11}} - 63 \cdot \frac{z^3}{r^{13}} \right) - 15x^2 y \cdot \left( -189 \cdot \frac{z}{r^{11}} + 693 \cdot \frac{z^3}{r^{13}} \right) \\
\frac{\partial^8}{\partial x \partial y \partial z^6} \varphi &= -15y \cdot \left( 21 \cdot \frac{1}{r^{13}} - 378 \cdot \frac{z^2}{r^{15}} + 693 \cdot \frac{z^4}{r^{17}} \right) \\
&\quad - 15x^2 y \cdot \left( -189 \cdot \frac{1}{r^{13}} + 4158 \cdot \frac{z^2}{r^{15}} - 9009 \cdot \frac{z^4}{r^{17}} \right)
\end{align*}
\]
\frac{\partial^8 \phi}{\partial x^2 \partial y^2 \partial z^5} = -45xy \left( -189 \frac{z}{r^{11}} + 693 \frac{z^3}{r^{13}} \right) - 15x^3y \left( 2079 \frac{z}{r^{13}} - 9009 \frac{z^3}{r^{15}} \right)

\frac{\partial^9 \phi}{\partial x^2 \partial y^2 \partial z^6} = -45xy \left( -189 \frac{1}{r^{11}} + 4158 \frac{z^2}{r^{13}} - 9009 \frac{z^4}{r^{15}} \right) - 15x^3y \left( 2079 \frac{1}{r^{13}} - 54054 \frac{z^2}{r^{15}} + 135135 \frac{z^4}{r^{17}} \right)

\frac{\partial^{10} \phi}{\partial x^2 \partial y^2 \partial z^7} = -945xy \left( 495 \frac{z}{r^{13}} - 4290 \frac{z^3}{r^{15}} + 6435 \frac{z^5}{r^{17}} \right) - 315x^3y \left( -6435 \frac{z}{r^{15}} + 64350 \frac{z^3}{r^{17}} - 109395 \frac{z^5}{r^{19}} \right)

\frac{\partial^{11} \phi}{\partial x^2 \partial y^2 \partial z^8} = -945xy \left( 495 \frac{1}{r^{13}} - 19305 \frac{z^2}{r^{15}} + 96525 \frac{z^4}{r^{17}} - 109395 \frac{z^6}{r^{19}} \right) - 315x^3y \left( -6435 \frac{1}{r^{15}} + 289575 \frac{z^2}{r^{17}} - 1640925 \frac{z^4}{r^{19}} + 207850 \frac{z^6}{r^{21}} \right)

\frac{\partial^{12} \phi}{\partial x^2 \partial y^2 \partial z^9} = -8505xy \left( -5005 \frac{z}{r^{15}} + 75075 \frac{z^3}{r^{17}} - 255255 \frac{z^5}{r^{19}} + 230945 \frac{z^7}{r^{21}} \right) - 2835x^3y \left( 75075 \frac{z}{r^{17}} - 1276275 \frac{z^3}{r^{19}} + 4849845 \frac{z^5}{r^{21}} - 4849845 \frac{z^7}{r^{23}} \right)

\frac{\partial^3 \phi}{\partial x \partial y^2 \partial z} = \frac{2}{(r+z)^3} - \frac{z^2}{r \cdot (r+z)^2} - \frac{2z^2}{r^2 \cdot (r+z)} - \frac{6x^2 y^2}{r^4 \cdot (r+z)^3} - \frac{3x^2 y^2}{r^5 \cdot (r+z)^2} - \frac{6x^2 y^2}{r^3 \cdot (r+z)^4}
\[
\frac{\partial^4 \varphi}{\partial x \partial y^2 \partial z} = \frac{1}{r^3 \cdot (r+z)} + \frac{1}{r^2 \cdot (r+z)^2} - \frac{3 \cdot (x^2 + y^2)}{r^5 \cdot (r+z)} - \frac{3 \cdot (x^2 + y^2)}{r^6 \cdot (r+z)^2} \\
- \frac{2 \cdot (x^2 + y^2)}{r^3 \cdot (r+z)^3} + \frac{15 \cdot x^2 y^2}{r^6 \cdot (r+z)^2} + \frac{12 \cdot x^2 y^2}{r^5 \cdot (r+z)^3} + \frac{15 \cdot x^2 y^2}{r^7 \cdot (r+z)} + \frac{6 \cdot x^2 y^2}{r^4 \cdot (r+z)^4}
\]

\[
\frac{\partial^5 \varphi}{\partial x \partial y^2 \partial z} = -3 \cdot \frac{1}{r^5} + 15 \cdot (x^2 + y^2) \cdot \frac{1}{r^7} - 105 \cdot x^2 y^2 \cdot \frac{1}{r^9}
\]

\[
\frac{\partial^6 \varphi}{\partial x \partial y^2 \partial z} = 15 \cdot \frac{z}{r^7} - 105 \cdot (x^2 + y^2) \cdot \frac{z}{r^9} + 945 \cdot x^2 y^2 \cdot \frac{z}{r^{11}}
\]

\[
\frac{\partial^7 \varphi}{\partial x \partial y^2 \partial z} = -3 \cdot \left( -5 \cdot \frac{1}{r^7} + 35 \cdot \frac{z^2}{r^9} \right) - 3 \cdot (x^2 + y^2) \cdot \left( 35 \cdot \frac{1}{r^9} - 315 \cdot \frac{z^2}{r^{11}} \right)
\]

\[
- 3 \cdot x^2 y^2 \cdot \left( -315 \cdot \frac{1}{r^{11}} + 3465 \cdot \frac{z^2}{r^{13}} \right)
\]

\[
\frac{\partial^8 \varphi}{\partial x \partial y^2 \partial z} = -15 \cdot \left( 21 \cdot \frac{z}{r^9} - 63 \cdot \frac{z^3}{r^{11}} \right)
\]

\[
- 15 \cdot (x^2 + y^2) \cdot \left( -189 \cdot \frac{z}{r^{11}} + 693 \cdot \frac{z^3}{r^{13}} \right)
\]

\[
- 15 \cdot x^2 y^2 \cdot \left( 2079 \cdot \frac{z}{r^{13}} - 9009 \cdot \frac{z^3}{r^{15}} \right)
\]

\[
\frac{\partial^9 \varphi}{\partial x \partial y^2 \partial z} = -15 \cdot \left( 21 \cdot \frac{1}{r^9} - 378 \cdot \frac{z^2}{r^{11}} + 693 \cdot \frac{z^4}{r^{13}} \right)
\]

\[
- 15 \cdot (x^2 + y^2) \cdot \left( -189 \cdot \frac{1}{r^{11}} + 4158 \cdot \frac{z^2}{r^{13}} - 9009 \cdot \frac{z^4}{r^{15}} \right)
\]

\[
- 15 \cdot x^2 y^2 \cdot \left( 2079 \cdot \frac{1}{r^{13}} - 54054 \cdot \frac{z^2}{r^{15}} + 135135 \cdot \frac{z^4}{r^{17}} \right)
\]

\[
\frac{\partial^{10} \varphi}{\partial x \partial y^2 \partial z} = -315 \cdot \left( -45 \cdot \frac{z}{r^{11}} + 330 \cdot \frac{z^3}{r^{13}} - 429 \cdot \frac{z^5}{r^{15}} \right)
\]

\[
- 315 \cdot (x^2 + y^2) \cdot \left( 495 \cdot \frac{z}{r^{13}} - 4290 \cdot \frac{z^3}{r^{15}} + 6435 \cdot \frac{z^5}{r^{17}} \right)
\]

\[
- 315 \cdot x^2 y^2 \cdot \left( -6435 \cdot \frac{z}{r^{15}} + 64350 \cdot \frac{z^3}{r^{17}} - 109395 \cdot \frac{z^5}{r^{19}} \right)
\]
\[
\frac{\partial^2 \varphi}{\partial x \partial y \partial z} = -315 \cdot \left( -45 \cdot \frac{1}{r^{11}} + 1485 \cdot \frac{z^2}{r^{13}} - 6435 \cdot \frac{z^4}{r^{15}} + 6435 \cdot \frac{z^6}{r^{17}} \right) \\
- 315 \cdot (x^2 + y^2) \cdot \left( 495 \cdot \frac{1}{r^{13}} - 19305 \cdot \frac{z^2}{r^{15}} + 96525 \cdot \frac{z^4}{r^{17}} - 109395 \cdot \frac{z^6}{r^{19}} \right) \\
- 315 \cdot x^2 y^2 \cdot \left( -6435 \cdot \frac{1}{r^{15}} + 289575 \cdot \frac{z^2}{r^{17}} - 1640925 \cdot \frac{z^4}{r^{19}} + 2078505 \cdot \frac{z^6}{r^{21}} \right)
\]

\[
\frac{\partial^3 \varphi}{\partial x \partial y \partial z} = -2835 \cdot \left( 385 \cdot \frac{z}{r^{13}} - 5005 \cdot \frac{z^3}{r^{15}} + 15015 \cdot \frac{z^5}{r^{17}} - 12155 \cdot \frac{z^7}{r^{19}} \right) \\
- 2835 \cdot (x^2 + y^2) \cdot \left( -5005 \cdot \frac{z}{r^{15}} + 75075 \cdot \frac{z^3}{r^{17}} - 255255 \cdot \frac{z^5}{r^{19}} + 230945 \cdot \frac{z^7}{r^{21}} \right) \\
- 2835 \cdot x^2 y^2 \cdot \left( 75075 \cdot \frac{z}{r^{17}} - 1276275 \cdot \frac{z^3}{r^{19}} + 4849845 \cdot \frac{z^5}{r^{21}} - 4849845 \cdot \frac{z^7}{r^{23}} \right)
\]

Partial derivatives of the function \( \varphi = \varphi_B = \ln(r+z) \):

\[
\varphi = \ln(r+z) \\
\frac{\partial \varphi}{\partial z} = \frac{1}{r} \\
\frac{\partial^2 \varphi}{\partial z^2} = -\frac{z}{r^3} \\
\frac{\partial^3 \varphi}{\partial z^3} = -\frac{1}{r^3} + 3 \cdot \frac{z^2}{r^5} \\
\frac{\partial^4 \varphi}{\partial z^4} = 3 \cdot \left( 3 \cdot \frac{z^5}{r^5} - 5 \cdot \frac{z^7}{r} \right) \\
\frac{\partial^5 \varphi}{\partial z^5} = 3 \cdot \left( 3 \cdot \frac{1}{r^5} - 30 \cdot \frac{z^2}{r^7} + 35 \cdot \frac{z^4}{r^9} \right) \\
\frac{\partial^6 \varphi}{\partial z^6} = 15 \cdot \left( -15 \cdot \frac{z}{r^7} + 70 \cdot \frac{z^3}{r^9} - 63 \cdot \frac{z^5}{r^{11}} \right) \\
\frac{\partial^7 \varphi}{\partial z^7} = 45 \cdot \left( -5 \cdot \frac{1}{r^7} + 105 \cdot \frac{z^2}{r^9} - 315 \cdot \frac{z^4}{r^{11}} + 231 \cdot \frac{z^6}{r^{13}} \right)
\]
\[ \frac{\partial^8 \varphi}{\partial z^8} = 315 \cdot \left( 35 \cdot \frac{z}{r^9} - 315 \cdot \frac{z^3}{r^{11}} + 693 \cdot \frac{z^5}{r^{13}} - 429 \cdot \frac{z^7}{r^{15}} \right) \]

\[ \frac{\partial^9 \varphi}{\partial z^9} = 315 \cdot \left( 35 \cdot \frac{1}{r^9} - 1260 \cdot \frac{z^2}{r^{11}} + 6930 \cdot \frac{z^4}{r^{13}} - 12012 \cdot \frac{z^6}{r^{15}} + 6435 \cdot \frac{z^8}{r^{17}} \right) \]

\[ \frac{\partial^{10} \varphi}{\partial z^{10}} = 2835 \cdot \left( -315 \cdot \frac{z}{r^{11}} + 4620 \cdot \frac{z^3}{r^{13}} - 18018 \cdot \frac{z^5}{r^{15}} \right. \\
\left. + 25740 \cdot \frac{z^7}{r^{17}} - 12155 \cdot \frac{z^9}{r^{19}} \right) \]

\[ \frac{\partial^{11} \varphi}{\partial z^{11}} = 14175 \cdot \left( -63 \cdot \frac{1}{r^{11}} + 3465 \cdot \frac{z^2}{r^{13}} - 30030 \cdot \frac{z^4}{r^{15}} \right. \\
\left. + 90090 \cdot \frac{z^6}{r^{17}} - 109395 \cdot \frac{z^8}{r^{19}} + 46189 \cdot \frac{z^{10}}{r^{21}} \right) \]

\[ \frac{\partial^{12} \varphi}{\partial z^{12}} = 155925 \cdot \left( 693 \cdot \frac{z}{r^{13}} - 15015 \cdot \frac{z^3}{r^{15}} + 90090 \cdot \frac{z^5}{r^{17}} \right. \\
\left. - 218790 \cdot \frac{z^7}{r^{19}} + 230945 \cdot \frac{z^9}{r^{21}} - 88179 \cdot \frac{z^{11}}{r^{23}} \right) \]

\[ \frac{\partial \varphi}{\partial x} = \frac{x}{r \cdot (r+z)} \]

\[ \frac{\partial^2 \varphi}{\partial x \partial z} = -\frac{x}{r^3} \]

\[ \frac{\partial^3 \varphi}{\partial x \partial z^2} = 3x \cdot \frac{z}{r^5} \]

\[ \frac{\partial^4 \varphi}{\partial x \partial z^3} = 3x \cdot \frac{1}{r^5} - 15x \cdot \frac{z^2}{r^7} \]

\[ \frac{\partial^5 \varphi}{\partial x \partial z^4} = 3x \cdot \left( -15 \cdot \frac{z}{r^7} + 35 \cdot \frac{z^3}{r^9} \right) \]

\[ \frac{\partial^6 \varphi}{\partial x \partial z^5} = 3x \cdot \left( -15 \cdot \frac{1}{r^7} + 210 \cdot \frac{z^2}{r^9} - 315 \cdot \frac{z^4}{r^{11}} \right) \]

\[ \frac{\partial^7 \varphi}{\partial x \partial z^6} = 15x \cdot \left( 105 \cdot \frac{z}{r^9} - 630 \cdot \frac{z^3}{r^{11}} + 693 \cdot \frac{z^5}{r^{13}} \right) \]

\[ \frac{\partial^8 \varphi}{\partial x \partial z^7} = 45x \cdot \left( 35 \cdot \frac{1}{r^9} - 945 \cdot \frac{z^2}{r^{11}} + 3465 \cdot \frac{z^4}{r^{13}} - 3003 \cdot \frac{z^6}{r^{15}} \right) \]

\[ \frac{\partial^9 \varphi}{\partial x \partial z^8} = 315x \cdot \left( -315 \cdot \frac{z}{r^{11}} + 3465 \cdot \frac{z^3}{r^{13}} - 9009 \cdot \frac{z^5}{r^{15}} + 6435 \cdot \frac{z^7}{r^{17}} \right) \]
\[
\frac{\partial^{10}}{\partial x \partial z^{10}} = 315x \cdot (-315 \cdot \frac{1}{r^{11}} + 13860 \cdot \frac{z^2}{r^{13}} - 90090 \cdot \frac{z^4}{r^{15}} \\
+ 180180 \cdot \frac{z^6}{r^{17}} - 109395 \cdot \frac{z^8}{r^{19}} )
\]
\[
\frac{\partial^{11}}{\partial x \partial z^{10}} = 2835x \cdot ( 3465 \cdot \frac{z}{r^{13}} - 60060 \cdot \frac{z^3}{r^{15}} + 270270 \cdot \frac{z^5}{r^{17}} \\
- 437580 \cdot \frac{z^7}{r^{19}} + 230945 \cdot \frac{z^9}{r^{21}} )
\]
\[
\frac{\partial^{12}}{\partial x \partial z^{11}} = 14175x \cdot ( 693 \cdot \frac{1}{r^{13}} - 45045 \cdot \frac{z^2}{r^{15}} + 450450 \cdot \frac{z^4}{r^{17}} \\
- 1531530 \cdot \frac{z^6}{r^{19}} + 2078505 \cdot \frac{z^8}{r^{21}} - 969969 \cdot \frac{z^{10}}{r^{23}} )
\]
\[
\frac{\partial^2}{\partial x^2 \partial z^2 \varphi} = \frac{1}{r \cdot (r+z)} - \frac{x^2}{r^3 \cdot (r+z)} - \frac{x^2}{r^2 \cdot (r+z)^2}
\]
\[
\frac{\partial^3}{\partial x^2 \partial z^2 \varphi} = -\frac{1}{r^3} + 3 \cdot \frac{x^2}{r^5} \cdot \frac{1}{5}
\]
\[
\frac{\partial^4}{\partial x^2 \partial z^2 \varphi} = 3 \cdot \frac{z^5}{r^5} - 15 \cdot \frac{x^2}{r^7} \cdot \frac{z}{r}
\]
\[
\frac{\partial^5}{\partial x^2 \partial z^2 \varphi} = 3 \cdot \frac{1}{r^5} - 15 \cdot \frac{z^2}{r^7} + x^2 \cdot (-15 \cdot \frac{1}{r^7} + 105 \cdot \frac{z^2}{r^9})
\]
\[
\frac{\partial^6}{\partial x^2 \partial z^2 \varphi} = 3 \cdot (-15 \cdot \frac{z}{r^7} + 35 \cdot \frac{z^3}{r^9}) + 3 \cdot x^2 \cdot (105 \cdot \frac{z}{r^9} - 315 \cdot \frac{z^3}{r^{11}})
\]
\[
\frac{\partial^7}{\partial x^2 \partial z^2 \varphi} = 3 \cdot (-15 \cdot \frac{1}{r^7} + 210 \cdot \frac{z^2}{r^9} - 315 \cdot \frac{z^4}{r^{11}})
\]
\[
+ 3 \cdot x^2 \cdot (105 \cdot \frac{1}{r^9} - 1890 \cdot \frac{z^2}{r^{11}} + 3465 \cdot \frac{z^4}{r^{13}})
\]
\[
\frac{\partial^8}{\partial x^2 \partial z^2 \varphi} = 15 \cdot (105 \cdot \frac{z}{r^9} - 630 \cdot \frac{z^3}{r^{11}} + 693 \cdot \frac{z^5}{r^{13}})
\]
\[
+ 15 \cdot x^2 \cdot (-945 \cdot \frac{z}{r^{11}} + 6930 \cdot \frac{z^3}{r^{13}} - 9009 \cdot \frac{z^5}{r^{15}})
\]
\[ \frac{\partial^9 \phi}{\partial x^2 \partial z^7} = 45 \cdot \left( 35 \cdot \frac{1}{r^9} - 945 \cdot \frac{z^2}{r^{11}} + 3465 \cdot \frac{z^4}{r^{13}} - 3003 \cdot \frac{z^6}{r^{15}} \right) \\
+ 45 \cdot x^2 \cdot \left( -315 \cdot \frac{1}{r^{11}} + 10395 \cdot \frac{z^2}{r^{13}} - 45045 \cdot \frac{z^4}{r^{15}} + 45045 \cdot \frac{z^6}{r^{17}} \right) \\
+ 315 \cdot x^2 \cdot \left( 3465 \cdot \frac{z}{r^{13}} - 45045 \cdot \frac{z^3}{r^{15}} + 135135 \cdot \frac{z^5}{r^{17}} - 109395 \cdot \frac{z^7}{r^{19}} \right) \\
- 315 \cdot \frac{1}{r^{11}} + 13860 \cdot \frac{z^2}{r^{13}} - 90090 \cdot \frac{z^4}{r^{15}} \\
+ 180180 \cdot \frac{z^6}{r^{17}} - 109395 \cdot \frac{z^8}{r^{19}} \\
+ 315 \cdot x^2 \cdot \left( 3465 \cdot \frac{1}{r^{13}} - 180180 \cdot \frac{z^2}{r^{15}} + 1351350 \cdot \frac{z^4}{r^{17}} \\
- 3063060 \cdot \frac{z^6}{r^{19}} + 2078505 \cdot \frac{z^8}{r^{21}} \right) \\
\frac{\partial^{12} \phi}{\partial x^2 \partial z^{10}} = 2835 \cdot \left( 3465 \cdot \frac{z}{r^{13}} - 60060 \cdot \frac{z^3}{r^{15}} + 270270 \cdot \frac{z^5}{r^{17}} \\
- 437580 \cdot \frac{z^7}{r^{19}} + 230945 \cdot \frac{z^9}{r^{21}} \right) \\
+ 2835 \cdot x^2 \cdot \left( -45045 \cdot \frac{z}{r^{15}} + 900900 \cdot \frac{z^3}{r^{17}} - 4594590 \cdot \frac{z^5}{r^{19}} \\
+ 8314020 \cdot \frac{z^7}{r^{21}} - 4849845 \cdot \frac{z^9}{r^{23}} \right) \\
\frac{\partial^3 \phi}{\partial x^3} = -3x \cdot \frac{1}{r^3 \cdot (r+z)} + 3x \cdot \frac{1}{r^2 \cdot (r+z)^2} + 3x^3 \cdot \frac{1}{r^4 \cdot (r+z)^2} + 3x^3 \cdot \frac{1}{r^5 \cdot (r+z)^2} + 2x^3 \cdot \frac{1}{r^3 \cdot (r+z)^3} \\
\frac{\partial^4 \phi}{\partial x^3 \partial z} = 9x \cdot \frac{1}{r^5} - 15x^3 \cdot \frac{1}{r^7} \\
\frac{\partial^5 \phi}{\partial x^3 \partial z^2} = -45x \cdot \frac{z}{r^7} + 105x^3 \cdot \frac{z}{r^9} \\
\frac{\partial^6 \phi}{\partial x^3 \partial z^3} = -3x \cdot (-15 \cdot \frac{1}{r^7} + 105 \cdot \frac{z^2}{r^9}) + x^3 \cdot (105 \cdot \frac{1}{r^9} - 945 \cdot \frac{z^2}{r^{11}})
\[ \frac{\partial^7}{\partial x^3 \partial z^4} \varphi = 9x \cdot \left( 105 \cdot \frac{z}{r} - 315 \cdot \frac{z^3}{r^2} \right) + 3x \cdot \left( -945 \cdot \frac{z}{r} + 3465 \cdot \frac{z^3}{r^2} \right) \]

\[ \frac{\partial^8}{\partial x^3 \partial z^5} \varphi = 9x \cdot \left( 105 \cdot \frac{1}{r} - 1890 \cdot \frac{z^2}{r^3} + 3465 \cdot \frac{z^2}{r^3} \right) \]

\[ + 3x \cdot \left( -945 \cdot \frac{1}{r} + 20790 \cdot \frac{z^2}{r^3} - 45045 \cdot \frac{z^2}{r^3} \right) \]

\[ \frac{\partial^9}{\partial x^3 \partial z^6} \varphi = 45x \cdot \left( -945 \cdot \frac{z}{r^2} + 6930 \cdot \frac{z^3}{r^4} - 9009 \cdot \frac{z^5}{r^4} \right) \]

\[ + 15x \cdot \left( 10395 \cdot \frac{z}{r^3} - 90090 \cdot \frac{z^3}{r^5} + 135135 \cdot \frac{z^5}{r^5} \right) \]

\[ \frac{\partial^{10}}{\partial x^3 \partial z^7} \varphi = 135x \cdot \left( -315 \cdot \frac{1}{r^3} + 10395 \cdot \frac{z^2}{r^4} - 45045 \cdot \frac{z^4}{r^6} + 45045 \cdot \frac{z^6}{r^6} \right) \]

\[ + 45x \cdot \left( 3465 \cdot \frac{1}{r^3} - 135135 \cdot \frac{z^2}{r^4} + 675675 \cdot \frac{z^4}{r^6} - 765765 \cdot \frac{z^6}{r^6} \right) \]

\[ \frac{\partial^{11}}{\partial x^3 \partial z^8} \varphi = 945x \cdot \left( 3465 \cdot \frac{z}{r^3} - 45045 \cdot \frac{z^3}{r^4} + 135135 \cdot \frac{z^5}{r^6} - 109395 \cdot \frac{z^7}{r^6} \right) \]

\[ + 315x \cdot \left( -45045 \cdot \frac{z^2}{r^4} + 675675 \cdot \frac{z^4}{r^6} - 2297295 \cdot \frac{z^6}{r^6} - 2078505 \cdot \frac{z^6}{r^6} \right) \]

\[ \frac{\partial^{12}}{\partial x^3 \partial z^9} \varphi = 945x \cdot \left( 3465 \cdot \frac{1}{r^3} - 180180 \cdot \frac{z^2}{r^4} + 351350 \cdot \frac{z^4}{r^6} - 3063060 \cdot \frac{z^6}{r^6} + 2078505 \cdot \frac{z^8}{r^6} \right) \]

\[ + 315x \cdot \left( -45045 \cdot \frac{1}{r^4} + 2702700 \cdot \frac{z^2}{r^6} - 2297295 \cdot \frac{z^4}{r^6} - 2297295 \cdot \frac{z^4}{r^6} \right) \]

\[ + 58198140 \cdot \frac{z^6}{r^6} - 43648605 \cdot \frac{z^8}{r^6} \]

\[ \frac{\partial}{\partial y} \varphi = \frac{y}{r \cdot (r+z)} \]

\[ \frac{\partial^2}{\partial y \partial z} \varphi = -\frac{y^3}{r} \]

\[ \frac{\partial^3}{\partial y \partial z^2} \varphi = 3y \cdot \frac{z}{r^5} \]

\[ \frac{\partial^4}{\partial y \partial z^3} \varphi = 3y \cdot \frac{1}{r^5} - 15y \cdot \frac{z^2}{r^7} \]
\[
\begin{align*}
\frac{\partial^5 \phi}{\partial y \partial z^4} & = 3y \cdot (-15 \cdot \frac{z}{r^7} + 35 \cdot \frac{z^3}{r^9}) \\
\frac{\partial^6 \phi}{\partial y \partial z^5} & = 3y \cdot (-15 \cdot \frac{1}{r^7} + 210 \cdot \frac{z^2}{r^9} - 315 \cdot \frac{z^4}{r^{11}}) \\
\frac{\partial^7 \phi}{\partial y \partial z^6} & = 15y \cdot (105 \cdot \frac{z}{r^9} - 630 \cdot \frac{z^3}{r^{11}} + 693 \cdot \frac{z^5}{r^{13}}) \\
\frac{\partial^8 \phi}{\partial y \partial z^7} & = 45y \cdot (35 \cdot \frac{1}{r^9} - 945 \cdot \frac{z^2}{r^{11}} + 3465 \cdot \frac{z^4}{r^{13}} - 3003 \cdot \frac{z^6}{r^{15}}) \\
\frac{\partial^9 \phi}{\partial y \partial z^8} & = 315y \cdot (-315 \cdot \frac{z}{r^{11}} + 3465 \cdot \frac{z^3}{r^{13}} - 9009 \cdot \frac{z^5}{r^{15}} + 6435 \cdot \frac{z^7}{r^{17}}) \\
\frac{\partial^{10} \phi}{\partial y \partial z^9} & = 315y \cdot (-315 \cdot \frac{1}{r^{11}} + 13860 \cdot \frac{z^2}{r^{13}} - 90090 \cdot \frac{z^4}{r^{15}} \\
& \quad + 180180 \cdot \frac{z^6}{r^{17}} - 109395 \cdot \frac{z^8}{r^{19}}) \\
\frac{\partial^{11} \phi}{\partial y \partial z^{10}} & = 2835y \cdot (3465 \cdot \frac{z}{r^{13}} - 60060 \cdot \frac{z^3}{r^{15}} + 270270 \cdot \frac{z^5}{r^{17}} \\
& \quad - 437580 \cdot \frac{z^7}{r^{19}} + 230945 \cdot \frac{z^9}{r^{21}}) \\
\frac{\partial^{12} \phi}{\partial y \partial z^{11}} & = 14175y \cdot (693 \cdot \frac{1}{r^{13}} - 45045 \cdot \frac{z^2}{r^{15}} + 450450 \cdot \frac{z^4}{r^{17}} - 1531530 \cdot \frac{z^6}{r^{19}} \\
& \quad + 2078505 \cdot \frac{z^8}{r^{21}} - 969969 \cdot \frac{z^{10}}{r^{23}}) \\
\frac{\partial^2 \phi}{\partial y^2 \partial z^0} & = -\frac{1}{r \cdot (r+z)} - \frac{y^2}{r^3 \cdot (r+z)} - \frac{y^2}{r^2 \cdot (r+z)^2} \\
\frac{\partial^3 \phi}{\partial y^2 \partial z} & = \frac{1}{r^3} + 3y^2 \cdot \frac{1}{r^5} \\
\frac{\partial^4 \phi}{\partial y^2 \partial z^2} & = 3 \cdot \frac{z}{r^5} - 15y^2 \cdot \frac{z}{r^7} \\
\frac{\partial^5 \phi}{\partial y^2 \partial z^3} & = 3 \cdot \frac{1}{r^5} - 15 \cdot \frac{z^2}{r^7} + y^2 \cdot (-15 \cdot \frac{1}{r^7} + 105 \cdot \frac{z^2}{r^9}) \\
\frac{\partial^6 \phi}{\partial y^2 \partial z^4} & = 3 \cdot (-15 \cdot \frac{z}{r^7} + 35 \cdot \frac{z^3}{r^9}) + 3y^2 \cdot (105 \cdot \frac{z}{r^9} - 315 \cdot \frac{z^3}{r^{11}})
\end{align*}
\]
\[ \frac{\partial^7}{\partial y^2 \partial z^5} \varphi = 3 \cdot (-15 \cdot \frac{1}{r^7} + 210 \cdot \frac{z^2}{r^9} - 315 \cdot \frac{z^4}{r^{11}} ) \\
+ 3 \cdot y^2 \cdot (105 \cdot \frac{1}{r^9} - 1890 \cdot \frac{z^2}{r^{11}} + 3465 \cdot \frac{z^4}{r^{13}} ) \\
\]
\[ + 15 \cdot y^2 \cdot (-945 \cdot \frac{z}{r^{11}} + 6930 \cdot \frac{z^3}{r^{13}} - 9009 \cdot \frac{z^5}{r^{15}} ) \]
\[ \frac{\partial^8}{\partial y^2 \partial z^6} \varphi = 15 \cdot (105 \cdot \frac{z}{r^9} - 620 \cdot \frac{z^3}{r^{11}} + 693 \cdot \frac{z^5}{r^{13}} ) \\
+ 15 \cdot y^2 \cdot (-945 \cdot \frac{z}{r^{11}} + 6930 \cdot \frac{z^3}{r^{13}} - 9009 \cdot \frac{z^5}{r^{15}} ) \]
\[ + 45 \cdot y^2 \cdot (-315 \cdot \frac{1}{r^{11}} + 10395 \cdot \frac{z^2}{r^{13}} - 45045 \cdot \frac{z^4}{r^{15}} + 45045 \cdot \frac{z^6}{r^{17}} ) \]
\[ \frac{\partial^9}{\partial y^2 \partial z^7} \varphi = 45 \cdot (35 \cdot \frac{1}{r^9} - 945 \cdot \frac{z^2}{r^{11}} + 3465 \cdot \frac{z^4}{r^{13}} - 3003 \cdot \frac{z^6}{r^{15}} ) \\
+ 45 \cdot y^2 \cdot (-315 \cdot \frac{1}{r^{11}} + 10395 \cdot \frac{z^2}{r^{13}} - 45045 \cdot \frac{z^4}{r^{15}} + 45045 \cdot \frac{z^6}{r^{17}} ) \]
\[ + 315 \cdot y^2 \cdot (3465 \cdot \frac{z}{r^{13}} - 45045 \cdot \frac{z^3}{r^{15}} + 135135 \cdot \frac{z^5}{r^{17}} - 109395 \cdot \frac{z^7}{r^{19}} ) \]
\[ \frac{\partial^{10}}{\partial y^2 \partial z^8} \varphi = 315 \cdot (-315 \cdot \frac{z}{r^{11}} + 3465 \cdot \frac{z^3}{r^{13}} - 9009 \cdot \frac{z^5}{r^{15}} + 6435 \cdot \frac{z^7}{r^{17}} ) \\
+ 315 \cdot y^2 \cdot (3465 \cdot \frac{z}{r^{13}} - 45045 \cdot \frac{z^3}{r^{15}} + 135135 \cdot \frac{z^5}{r^{17}} - 109395 \cdot \frac{z^7}{r^{19}} ) \]
\[ + 315 \cdot y^2 \cdot (3465 \cdot \frac{1}{r^{13}} - 180180 \cdot \frac{z^2}{r^{15}} + 1351350 \cdot \frac{z^4}{r^{17}} - 3063060 \cdot \frac{z^6}{r^{19}} + 2078505 \cdot \frac{z^8}{r^{21}} ) \]
\[ \frac{\partial^{12}}{\partial y^2 \partial z^{10}} \varphi = 2835 \cdot (3465 \cdot \frac{z}{r^{13}} - 60060 \cdot \frac{z^3}{r^{15}} + 270270 \cdot \frac{z^5}{r^{17}} - 437580 \cdot \frac{z^7}{r^{19}} + 230945 \cdot \frac{z^9}{r^{21}} ) \\
+ 2835 \cdot y^2 \cdot (-45045 \cdot \frac{z}{r^{15}} + 900900 \cdot \frac{z^3}{r^{17}} - 4594590 \cdot \frac{z^5}{r^{19}} \\
+ 8314020 \cdot \frac{z^7}{r^{21}} - 4849845 \cdot \frac{z^9}{r^{23}} ) \]
\[ \frac{\partial^3 \varphi}{\partial y^3 \partial z} = \frac{-3y}{r^3 \cdot (r+z)} - \frac{3y}{r^2 \cdot (r+z)^2} + \frac{3y}{r^4 \cdot (r+z)^2} + \frac{3y}{r^5 \cdot (r+z)} + \frac{2y}{r^3 \cdot (r+z)^3} \]

\[ \frac{\partial^4 \varphi}{\partial y^4 \partial z} = 9y \cdot \frac{1}{r^5} - 15y^3 \cdot \frac{1}{r^7} \]

\[ \frac{\partial^5 \varphi}{\partial y^5 \partial z^2} = -45y \cdot \frac{z}{r^7} + 105y^3 \cdot \frac{z}{9r^9} \]

\[ \frac{\partial^6 \varphi}{\partial y^6 \partial z^3} = 3y \cdot (-15 \cdot \frac{1}{r^7} + 105 \cdot \frac{z^2}{9r^9}) + y^3 \cdot (105 \cdot \frac{1}{9r^9} - 945 \cdot \frac{z^2}{11r^11}) \]

\[ \frac{\partial^7 \varphi}{\partial y^7 \partial z^4} = 9y \cdot (105 \cdot \frac{z}{9r^9} - 315 \cdot \frac{z^3}{11r^11}) + 3y^3 \cdot (-945 \cdot \frac{z}{11r^{11}} + 3465 \cdot \frac{z}{13r^{13}}) \]

\[ \frac{\partial^8 \varphi}{\partial y^8 \partial z^5} = 9y \cdot (105 \cdot \frac{1}{9r^9} - 1890 \cdot \frac{z^2}{11r^{11}} + 3465 \cdot \frac{z}{13r^{13}}) + 3y^3 \cdot (-945 \cdot \frac{1}{11r^{11}} + 20790 \cdot \frac{z^2}{13r^{13}} - 45045 \cdot \frac{z^4}{15r^{15}}) \]

\[ \frac{\partial^9 \varphi}{\partial y^9 \partial z^6} = 45y \cdot (-945 \cdot \frac{z}{11r^{11}} + 6930 \cdot \frac{z^3}{13r^{13}} - 9009 \cdot \frac{z^5}{15r^{15}}) + 15y^3 \cdot (10395 \cdot \frac{z}{r^{13}} - 90090 \cdot \frac{z^3}{r^{15}} + 135135 \cdot \frac{z^5}{r^{17}}) \]

\[ \frac{\partial^{10} \varphi}{\partial y^{10} \partial z^7} = 135y \cdot (-315 \cdot \frac{1}{11r^{11}} + 10395 \cdot \frac{z^2}{r^{13}} - 45045 \cdot \frac{z}{15r^{15}} + 45045 \cdot \frac{z^6}{11r^{17}}) \]

\[ + 45y^3 \cdot (3465 \cdot \frac{1}{13r^{13}} - 135135 \cdot \frac{z^2}{15r^{15}} + 675675 \cdot \frac{z^4}{17r^{17}} - 765765 \cdot \frac{z^6}{19r^{19}}) \]

\[ \frac{\partial^{11} \varphi}{\partial y^{11} \partial z^8} = 945y \cdot (3465 \cdot \frac{z}{13r^{13}} - 45045 \cdot \frac{z^3}{15r^{15}} + 135135 \cdot \frac{z^5}{17r^{17}} - 109395 \cdot \frac{z^7}{19r^{19}}) \]

\[ + 315y^3 \cdot (-45045 \cdot \frac{z}{15r^{15}} + 675675 \cdot \frac{z^3}{17r^{17}} - 2297295 \cdot \frac{z^5}{19r^{19}} + 2078505 \cdot \frac{z^7}{21r^{21}}) \]
\[
\frac{\partial^{12}}{\partial y^{3} \partial z^{9}} y^6 = 945y \cdot \left( 3465 \cdot \frac{1}{r^{13}} - 180180 \cdot \frac{z^2}{r^{15}} + 1351350 \cdot \frac{z^4}{r^{17}} - 3063060 \cdot \frac{z^6}{r^{19}} + 2078505 \cdot \frac{z^8}{r^{21}} \right) \\
+ 315y^3 \cdot \left( -45045 \cdot \frac{1}{r^{15}} + 2702700 \cdot \frac{z^2}{r^{17}} - 22972950 \cdot \frac{z^4}{r^{19}} + 58198140 \cdot \frac{z^6}{r^{21}} - 43648605 \cdot \frac{z^8}{r^{23}} \right)
\]

\[
\frac{\partial^2}{\partial x \partial y^2} y^6 = -\frac{xy}{r^3 \cdot (r+z)} - \frac{xy}{r^2 \cdot (r+z)^2}
\]

\[
\frac{\partial^3}{\partial x \partial y \partial z} y^6 = 3xy \cdot \frac{1}{r^5}
\]

\[
\frac{\partial^4}{\partial x \partial y^2 \partial z} y^6 = -15xy \cdot \frac{z}{r^7}
\]

\[
\frac{\partial^5}{\partial x \partial y^3 \partial z} y^6 = xy \cdot \left( -15 \cdot \frac{1}{r^7} + 105 \cdot \frac{z^2}{r^9} \right)
\]

\[
\frac{\partial^6}{\partial x \partial y^4 \partial z} y^6 = 3xy \cdot \left( 105 \cdot \frac{z}{r^9} - 315 \cdot \frac{z^3}{r^{11}} \right)
\]

\[
\frac{\partial^7}{\partial x \partial y^5 \partial z} y^6 = -15xy \cdot \left( -945 \cdot \frac{z}{r^{11}} + 6930 \cdot \frac{z^3}{r^{13}} - 9009 \cdot \frac{z^5}{r^{15}} \right)
\]

\[
\frac{\partial^8}{\partial x \partial y^6 \partial z} y^6 = 45xy \cdot \left( -315 \cdot \frac{1}{r^{11}} + 10395 \cdot \frac{z^2}{r^{13}} - 45045 \cdot \frac{z^4}{r^{15}} + 45045 \cdot \frac{z^6}{r^{17}} \right)
\]

\[
\frac{\partial^9}{\partial x \partial y^7 \partial z} y^6 = 315xy \cdot \left( 3465 \cdot \frac{z}{r^{13}} - 45045 \cdot \frac{z^3}{r^{15}} + 1351350 \cdot \frac{z^5}{r^{17}} - 3063060 \cdot \frac{z^6}{r^{19}} + 2078505 \cdot \frac{z^8}{r^{21}} \right)
\]

\[
\frac{\partial^{11}}{\partial x \partial y^9 \partial z} y^6 = 315xy \cdot \left( 3465 \cdot \frac{1}{r^{13}} - 180180 \cdot \frac{z^2}{r^{15}} + 1351350 \cdot \frac{z^4}{r^{17}} - 3063060 \cdot \frac{z^6}{r^{19}} + 2078505 \cdot \frac{z^8}{r^{21}} \right)
\]
\[
\frac{\partial^{12}}{\partial x \partial y \partial z} - 2835 \cdot xy \cdot ( -45045 \cdot \frac{z}{r^{15}} + 900900 \cdot \frac{z^3}{r^{17}} - 4594590 \cdot \frac{z^5}{r^{19}} \\
\quad + 8314020 \cdot \frac{z^7}{r^{21}} - 4849845 \cdot \frac{z^9}{r^{23}} )
\]

\[
\frac{\partial^3}{\partial x \partial y \partial z^2} \frac{x}{r^3 \cdot (r+z)} - \frac{x}{r^2 \cdot (r+z)^2} + \frac{3xy^2}{r^5 \cdot (r+z)} + \frac{3xy^2}{r^4 \cdot (r+z)^2} + \frac{2xy^2}{r^3 \cdot (r+z)^3}
\]

\[
\frac{\partial^4}{\partial x \partial y \partial z^2} \frac{x}{r} - 15xy \cdot \frac{1}{r^5} - 15xy \cdot \frac{1}{r^7}
\]

\[
\frac{\partial^5}{\partial x \partial y \partial z^2} \frac{x}{r} - 15xy \cdot \frac{z}{r^7} + 105xy \cdot \frac{z}{r^9}
\]

\[
\frac{\partial^6}{\partial x \partial y \partial z^2} \frac{x}{r} - 15xy \cdot \frac{z^2}{r^7} + 105 \cdot \frac{z^2}{r^9} + xy^2 \cdot (105 \cdot \frac{1}{r^9} - 945 \cdot \frac{z^2}{r^11})
\]

\[
\frac{\partial^7}{\partial x \partial y \partial z^3} \frac{x}{r} - 15xy \cdot \frac{z^3}{r^9} + 3xy^2 \cdot (945 \cdot \frac{z}{r^11} + 3465 \cdot \frac{z^3}{r^13})
\]

\[
\frac{\partial^8}{\partial x \partial y \partial z^3} \frac{x}{r} - 15xy \cdot \frac{z^4}{r^9} + 1900 \cdot \frac{z^4}{r^11} + 3465 \cdot \frac{z^4}{r^13}
\]

\[
\frac{\partial^9}{\partial x \partial y \partial z^3} \frac{x}{r} - 15xy \cdot \frac{z^5}{r^9} + 6930 \cdot \frac{z^5}{r^11} + 9009 \cdot \frac{z^5}{r^{13}}
\]

\[
\frac{\partial^{10}}{\partial x \partial y \partial z^3} \frac{x}{r} - 15xy \cdot \frac{z^6}{r^9} + 10395 \cdot \frac{z^6}{r^{11}} + 135135 \cdot \frac{z^6}{r^{15}}
\]

\[
\frac{\partial^{11}}{\partial x \partial y \partial z^4} \frac{x}{r} - 45xy \cdot \frac{z^7}{r^9} + 10395 \cdot \frac{z^7}{r^{11}} + 45045 \cdot \frac{z^7}{r^{15}} + 45045 \cdot \frac{z^7}{r^{17}}
\]

\[
\frac{\partial^{12}}{\partial x \partial y \partial z^4} \frac{x}{r} - 45xy \cdot \frac{z^8}{r^9} + 135135 \cdot \frac{z^8}{r^{11}} + 675675 \cdot \frac{z^8}{r^{17}} + 765765 \cdot \frac{z^8}{r^{19}}
\]

\[
\frac{\partial^{13}}{\partial x \partial y \partial z^4} \frac{x}{r} - 315xy \cdot \frac{z^9}{r^9} + 45045 \cdot \frac{z^9}{r^{11}} + 135135 \cdot \frac{z^9}{r^{15}} + 109395 \cdot \frac{z^9}{r^{17}}
\]

\[
\frac{\partial^{14}}{\partial x \partial y \partial z^4} \frac{x}{r} - 315xy \cdot \frac{z^{10}}{r^9} + 675675 \cdot \frac{z^{10}}{r^{11}} + 2297295 \cdot \frac{z^{10}}{r^{17}} + 2078505 \cdot \frac{z^{10}}{r^{21}}
\]
\[
\frac{\partial^{12}}{\partial x \partial y^2 \partial z} \phi = 315x \cdot \left( 3465 \cdot \frac{1}{r^{13}} - 180180 \cdot \frac{z^2}{r^{15}} + 1351350 \cdot \frac{z^4}{r^{17}} \right) \\
- 3063060 \cdot \frac{z^6}{r^{19}} + 2078505 \cdot \frac{z^8}{r^{21}} \\
+ 315xy \cdot \left( -45045 \cdot \frac{1}{r^{15}} + 2702700 \cdot \frac{z^2}{r^{17}} - 22972950 \cdot \frac{z^4}{r^{19}} \right) \\
+ 58198140 \cdot \frac{z^6}{r^{21}} - 43648605 \cdot \frac{z^8}{r^{23}} \\
\frac{\partial^3}{\partial x \partial y \partial z} \phi = \frac{-y}{r^3 \cdot (r + z)} - \frac{y \cdot 3x^2 y}{r^5 \cdot (r + z)^2} + \frac{3x^2 y}{r^4 \cdot (r + z)^2} + \frac{2x^2 y}{r^3 \cdot (r + z)^3} \\
\frac{\partial^4}{\partial x^2 \partial y \partial z} \phi = 3y \cdot \frac{1}{r^5} - 15x^2 y \cdot \frac{1}{r^7} \\
\frac{\partial^5}{\partial x^2 \partial y^2 \partial z} \phi = -15y \cdot \frac{z}{r^7} + 105x^2 y \cdot \frac{z}{r^9} \\
\frac{\partial^6}{\partial x^2 \partial y^2 \partial z} \phi = y \cdot \left( -15 \cdot \frac{1}{r^7} + 105 \cdot \frac{z^2}{r^9} \right) + x^2 y \cdot \left( 105 \cdot \frac{1}{r^9} - 945 \cdot \frac{z^2}{r^{11}} \right) \\
\frac{\partial^7}{\partial x^2 \partial y^2 \partial z} \phi = 3y \cdot \left( 105 \cdot \frac{z}{r^9} - 315 \cdot \frac{z^3}{r^{11}} \right) + 3x^2 y \cdot \left( -945 \cdot \frac{z}{r^{11}} + 3465 \cdot \frac{z^3}{r^{13}} \right) \\
\frac{\partial^8}{\partial x^2 \partial y^2 \partial z} \phi = 3y \cdot \left( 105 \cdot \frac{1}{r^9} - 1890 \cdot \frac{z^2}{r^{11}} + 3465 \cdot \frac{z^4}{r^{13}} \right) \\
+ 3x^2 y \cdot \left( -945 \cdot \frac{1}{r^{11}} + 20790 \cdot \frac{z^2}{r^{13}} - 45045 \cdot \frac{z^4}{r^{15}} \right) \\
\frac{\partial^9}{\partial x^2 \partial y^2 \partial z} \phi = 15y \cdot \left( -945 \cdot \frac{z}{r^{11}} + 6930 \cdot \frac{z^3}{r^{13}} - 9009 \cdot \frac{z^5}{r^{15}} \right) \\
+ 15x^2 y \cdot \left( 10395 \cdot \frac{z}{r^{13}} - 90090 \cdot \frac{z^3}{r^{15}} + 135135 \cdot \frac{z^5}{r^{17}} \right) \\
\frac{\partial^{10}}{\partial x^2 \partial y^2 \partial z} \phi = 45y \cdot \left( -315 \cdot \frac{1}{r^{11}} + 10395 \cdot \frac{z^2}{r^{13}} - 45045 \cdot \frac{z^4}{r^{15}} + 45045 \cdot \frac{z^6}{r^{17}} \right) \\
+ 45x^2 y \cdot \left( 3465 \cdot \frac{1}{r^{13}} - 135135 \cdot \frac{z^2}{r^{15}} + 675675 \cdot \frac{z^4}{r^{17}} - 765765 \cdot \frac{z^6}{r^{19}} \right)
\]
\[ \frac{\partial^{11}}{\partial x^2 \partial y \partial z} \Phi = 315y \cdot \left( \frac{3465}{r^{13}} \cdot \frac{z}{r^{13}} - \frac{45045}{r^{15}} \cdot \frac{z^3}{r^{15}} + \frac{135135}{r^{17}} \cdot \frac{z^5}{r^{17}} - \frac{109395}{r^{19}} \cdot \frac{z^7}{r^{19}} \right) \\
+ 315x^2 y \cdot \left( -\frac{45045}{r^{15}} \cdot \frac{z}{r^{15}} + 675675 \cdot \frac{z^3}{r^{17}} - \frac{2297295}{r^{19}} \cdot \frac{z^5}{r^{19}} + 2078505 \cdot \frac{z^7}{r^{21}} \right) \\
\frac{\partial^{12}}{\partial x^2 \partial y \partial z} \Phi = 315y \cdot \left( \frac{3465}{r^{13}} \cdot \frac{1}{r^{13}} - \frac{180180}{r^{15}} \cdot \frac{z^2}{r^{15}} + \frac{1351350}{r^{17}} \cdot \frac{z^4}{r^{17}} \\
- \frac{3063060}{r^{19}} \cdot \frac{z^6}{r^{19}} + 2078505 \cdot \frac{z^8}{r^{21}} \right) \\
+ 315x^2 y \cdot \left( -\frac{45045}{r^{15}} \cdot \frac{1}{r^{15}} + 2702700 \cdot \frac{z^2}{r^{17}} - \frac{22972950}{r^{19}} \cdot \frac{z^4}{r^{19}} \\
+ \frac{58198140}{r^{21}} \cdot \frac{z^6}{r^{21}} - 43648605 \cdot \frac{z^8}{r^{23}} \right) \]

The above formulas for the partial derivatives are checked in a computer program by means of comparison to finite difference approximation of partial derivatives operators.
Appendix K: Laplace inverting components of products of matrix operators ("R" part)

In section 2.6 (equation (2.67)) it was noted that the Laplace inverse of terms of the form:

$$\frac{1}{s} \cdot (\overline{A}^-)^n \cdot (\overline{B}^-)^m$$  \hspace{1cm} (K.1)

are required, where:

$$\overline{A}^- = 1 - c \cdot \frac{1}{s + \Omega_A}$$  \hspace{1cm} (K.2)

$$\overline{B}^- = 1 - d \cdot \frac{1}{s + \Omega_{B1}} + e \cdot \frac{1}{s + \Omega_{B2}}$$  \hspace{1cm} (K.3)

The Laplace inverse of (K.1) can be obtained once (K.1) has been reduced to partial fractions form, that is to terms of the form:

$$\sum_{k=1}^{P} a_k \cdot \frac{1}{(s + \Omega_k)^k}$$  \hspace{1cm} (K.4)

since:

$$L^{-1} \left[ \frac{1}{(s + \Omega_k)^k} \right] = t^{k-1} \cdot e^{-\Omega_k t} \cdot \frac{1}{(n-1)!}$$  \hspace{1cm} (K.5)

In this appendix, the terms of the form (K.1) will be reduced to partial fractions form for the terms required up till obtaining the fourth group of images (see section 2.3) for the "R" matrix operators. The expressions are given below:
\[
\frac{1}{s} \left[ \frac{A}{A} \right]^2 = \left(1 - \frac{c}{\Omega_A}\right) \cdot \frac{1}{s} + \frac{c}{\Omega_A} \cdot \frac{1}{s + \Omega_A}
\]

\[
\frac{1}{s} \left[ \frac{A}{A} \right]^3 = \frac{1}{s} \left[ \frac{A}{A} \right] - \frac{1}{s} \left[ \frac{A}{A} \right] \cdot c \cdot \frac{1}{s + \Omega_A}
\]

\[
\frac{1}{s} \left[ \frac{A}{A} \right]^4 = \frac{1}{s} \left[ \frac{A}{A} \right] - 3 \frac{1}{s} \left[ \frac{A}{A} \right] \cdot c \cdot \frac{1}{s + \Omega_A}
\]

\[
\frac{1}{s} \left[ \frac{A}{A} \right] \cdot c \cdot \frac{1}{s + \Omega_A} = \left(1 - \frac{c}{\Omega_A}\right) \cdot \frac{1}{s - \frac{1}{s + \Omega_A}} + \left[ \frac{c}{\Omega_A} \right]^2 \cdot \Omega_A \cdot \frac{1}{(s + \Omega_A)^2}
\]

\[
\frac{1}{s} \left[ \frac{A}{A} \right] \cdot c^2 \cdot \frac{1}{(s + \Omega_A)^2} = \left(1 - \frac{c}{\Omega_A}\right) \cdot \left[ \frac{c}{\Omega_A} \right]^2 \cdot \left(\frac{1}{s - \frac{1}{s + \Omega_A}} - \frac{\Omega_A}{(s + \Omega_A)^2}\right)
\]

\[
\frac{1}{s} \left[ \frac{A}{A} \right] \cdot c^3 \cdot \frac{1}{(s + \Omega_A)^3} = \left(1 - \frac{c}{\Omega_A}\right) \cdot \left[ \frac{c}{\Omega_A} \right]^3 \cdot \left(\frac{1}{s - \frac{1}{s + \Omega_A}} - \frac{\Omega_A}{(s + \Omega_A)^2} - \frac{\Omega_A}{(s + \Omega_A)^3}\right)
\]

\[
\frac{1}{s} \left[ \frac{A}{A} \right] \cdot \left[ \frac{B}{B} \right] = \frac{1}{s} \left[ \frac{A}{A} \right] - \frac{1}{s} \left[ \frac{A}{A} \right] \cdot d \cdot \frac{1}{s + \Omega_{B1}} + \frac{1}{s} \left[ \frac{A}{A} \right] \cdot e \cdot \frac{1}{s + \Omega_{B2}}
\]

\[
\frac{1}{s} \left[ \frac{A}{A} \right]^2 = \frac{1}{s} \left[ \frac{A}{A} \right] \cdot \left[ \frac{B}{B} \right] - \frac{1}{s} \left[ \frac{A}{A} \right] \cdot \left[ \frac{B}{B} \right] - \frac{1}{s} \left[ \frac{A}{A} \right] \cdot c \cdot \frac{1}{s + \Omega_A}
\]
\[
\begin{align*}
&\frac{1}{s} \left[ \begin{array}{c}
A^- \\
B^-
\end{array} \right] \cdot d \cdot \frac{1}{s + \Omega_A} + \frac{1}{s} \left[ \begin{array}{c}
A^- \\
B^-
\end{array} \right] \cdot c \cdot d \cdot \frac{1}{s + \Omega_A} - \frac{1}{s} \left[ \begin{array}{c}
A^- \\
B^-
\end{array} \right] \cdot c \cdot e \cdot \frac{1}{s + \Omega_B} - \frac{1}{s} \left[ \begin{array}{c}
A^- \\
B^-
\end{array} \right] \cdot c \cdot e \cdot \frac{1}{s + \Omega_B} \\
&+ 2 \cdot \frac{1}{s} \left[ \begin{array}{c}
A^- \\
B^-
\end{array} \right] \cdot c \cdot d \cdot \frac{1}{s + \Omega_A} - 2 \cdot \frac{1}{s} \left[ \begin{array}{c}
A^- \\
B^-
\end{array} \right] \cdot c \cdot e \cdot \frac{1}{s + \Omega_B} \\
&+ \frac{1}{s} \left[ \begin{array}{c}
A^- \\
B^-
\end{array} \right] \cdot c^2 \cdot \frac{1}{(s + \Omega_A)^2} - \frac{1}{s} \left[ \begin{array}{c}
A^- \\
B^-
\end{array} \right] \cdot c^2 \cdot d \cdot \frac{1}{(s + \Omega_A)^2} \cdot \frac{1}{s + \Omega_B} \\
&+ \frac{1}{s} \left[ \begin{array}{c}
A^- \\
B^-
\end{array} \right] \cdot c^2 \cdot e \cdot \frac{1}{(s + \Omega_A)^2} \cdot \frac{1}{s + \Omega_B}
\end{align*}
\]

where:

\[
\begin{align*}
&\frac{1}{s} \left[ \begin{array}{c}
A^- \\
B^-
\end{array} \right] \cdot d \cdot \frac{1}{s + \Omega_B} = (1 - \frac{c}{\Omega_A}) \cdot \frac{d}{\Omega_B} \cdot \left( \frac{1}{s} - \frac{1}{s + \Omega_B} \right) \\
&+ \frac{c}{\Omega_A} \cdot \frac{d}{\Omega_A - \Omega_B} \cdot \left( \frac{1}{s + \Omega_B} - \frac{1}{s + \Omega_A} \right)
\end{align*}
\]

\[
\begin{align*}
&\frac{1}{s} \left[ \begin{array}{c}
A^- \\
B^-
\end{array} \right] \cdot e \cdot \frac{1}{s + \Omega_B} = (1 - \frac{c}{\Omega_A}) \cdot \frac{e}{\Omega_B} \cdot \left( \frac{1}{s} - \frac{1}{s + \Omega_B} \right) \\
&+ \frac{c}{\Omega_A} \cdot \frac{e}{\Omega_A - \Omega_B} \cdot \left( \frac{1}{s + \Omega_B} - \frac{1}{s + \Omega_A} \right)
\end{align*}
\]

\[
\begin{align*}
&\frac{1}{s} \left[ \begin{array}{c}
A^- \\
B^-
\end{array} \right] \cdot c \cdot d \cdot \frac{1}{s + \Omega_B} = (1 - \frac{c}{\Omega_A}) \cdot \frac{c}{\Omega_B} \cdot \left( \frac{1}{s} - \frac{1}{s + \Omega_B} \right) \\
&- (1 - \frac{c}{\Omega_A}) \cdot \frac{d}{\Omega_A - \Omega_B} \cdot \left( \frac{1}{s + \Omega_B} - \frac{1}{s + \Omega_A} \right) \\
&+ \left[ \frac{c}{\Omega_A} \right]^2 \cdot \frac{d}{(\Omega_A - \Omega_B)^2} \cdot \left( - \frac{1}{(s + \Omega_A)^2} \cdot \frac{1}{s + \Omega_B} \right) + \frac{1}{s + \Omega_B}
\end{align*}
\]

\[
\begin{align*}
&\frac{1}{s} \left[ \begin{array}{c}
A^- \\
B^-
\end{array} \right] \cdot c \cdot e \cdot \frac{1}{s + \Omega_B} = (1 - \frac{c}{\Omega_A}) \cdot \frac{c}{\Omega_B} \cdot \left( \frac{1}{s} - \frac{1}{s + \Omega_B} \right) \\
&- (1 - \frac{c}{\Omega_A}) \cdot \frac{e}{\Omega_B} \cdot \left( \frac{1}{s + \Omega_B} - \frac{1}{s + \Omega_A} \right) \\
&+ \left[ \frac{c}{\Omega_A} \right]^2 \cdot \frac{e}{(\Omega_A - \Omega_B)^2} \cdot \left( - \frac{1}{(s + \Omega_A)^2} \cdot \frac{1}{s + \Omega_B} \right) + \frac{1}{s + \Omega_B}
\end{align*}
\]
\[
\frac{1}{s} \left[ \frac{A}{A} \right] \cdot c^2 \cdot \frac{1}{d} \cdot \frac{1}{s+\Omega_A} \cdot \frac{1}{s+\Omega_{B1}} = (1-\frac{c}{\Omega_A}) \cdot \left[ \frac{c}{\Omega_A} \right] \cdot \frac{d}{\Omega_{B1}} \cdot \left( \frac{1}{s} - \frac{1}{s+\Omega_{B1}} \right) \\
- (1-\frac{c}{\Omega_A}) \cdot \left[ \frac{c}{\Omega_A} \right] \cdot \frac{d}{\Omega_A \cdot \Omega_{B1}} \cdot \left( \frac{1}{s+\Omega_{B1}} - \frac{1}{s+\Omega_A} \right) \\
+ (1-\frac{c}{\Omega_A}) \cdot \left[ \frac{c}{\Omega_A} \right] \cdot \frac{d}{\Omega_A \cdot \Omega_{B1}} \cdot \frac{1}{\Omega_A \cdot \Omega_{B1}} \\
\left( \frac{\Omega_A \cdot \Omega_{B1}}{(s+\Omega_A)^2} \right) + \frac{1}{s+\Omega_A} - \frac{1}{s+\Omega_{B1}} \\
- \left[ \frac{c}{\Omega_A} \right]^3 \cdot \left[ \frac{\Omega_A}{\Omega_A \cdot \Omega_{B1}} \right] \cdot \frac{d}{\Omega_A \cdot \Omega_{B1}} \\
\left( \frac{\Omega_A \cdot \Omega_{B1}}{(s+\Omega_A)^3} \right) + \frac{1}{s+\Omega_A} - \frac{1}{s+\Omega_{B1}} \\
\frac{1}{s} \left[ \frac{A}{A} \right] \cdot c^2 \cdot e \cdot \frac{1}{(s+\Omega_A)^2} \cdot \frac{1}{s+\Omega_{B2}} = (1-\frac{c}{\Omega_A}) \cdot \left[ \frac{c}{\Omega_A} \right] \cdot \frac{e}{\Omega_{B2}} \cdot \left( \frac{1}{s} - \frac{1}{s+\Omega_{B2}} \right) \\
- (1-\frac{c}{\Omega_A}) \cdot \left[ \frac{c}{\Omega_A} \right] \cdot \frac{e}{\Omega_A \cdot \Omega_{B2}} \cdot \left( \frac{1}{s+\Omega_{B2}} - \frac{1}{s+\Omega_A} \right) \\
+ (1-\frac{c}{\Omega_A}) \cdot \left[ \frac{c}{\Omega_A} \right] \cdot \frac{e}{\Omega_A \cdot \Omega_{B2}} \cdot \frac{1}{\Omega_A \cdot \Omega_{B2}} \\
\left( \frac{\Omega_A \cdot \Omega_{B2}}{(s+\Omega_A)^2} \right) + \frac{1}{s+\Omega_A} - \frac{1}{s+\Omega_{B2}} \\
- \left[ \frac{c}{\Omega_A} \right]^3 \cdot \left[ \frac{\Omega_A}{\Omega_A \cdot \Omega_{B2}} \right] \cdot \frac{e}{\Omega_A \cdot \Omega_{B2}} \\
\left( \frac{\Omega_A \cdot \Omega_{B2}}{(s+\Omega_A)^3} \right) + \frac{1}{s+\Omega_A} - \frac{1}{s+\Omega_{B2}} \\
\frac{1}{s} \left[ \frac{B}{B} \right] = (1-\frac{d}{\Omega_{B1}}) \cdot \frac{e}{\Omega_{B2}} \cdot \frac{1}{(s+\Omega_{B1})^2} \cdot \frac{1}{s+\Omega_{B1}} \cdot \frac{1}{\Omega_{B2}} \cdot \frac{1}{s+\Omega_{B2}} \\
\frac{1}{s} \left[ \frac{B}{B} \right]^2 = \frac{1}{s} \left[ \frac{B}{B} \right] - \frac{1}{s} \left[ \frac{B}{B} \right] \cdot \frac{1}{s+\Omega_{B1}} \cdot \frac{1}{s} \left[ \frac{B}{B} \right] \cdot \frac{1}{e \cdot \frac{1}{s+\Omega_{B2}}} \\
\frac{1}{s} \left[ \frac{B}{B} \right]^3 = \frac{1}{s} \left[ \frac{B}{B} \right] - 2 \cdot \frac{1}{s} \left[ \frac{B}{B} \right] \cdot \frac{1}{s+\Omega_{B1}} \cdot \frac{1}{s} \left[ \frac{B}{B} \right] \cdot \frac{1}{e \cdot \frac{1}{s+\Omega_{B2}}} 
\]
\[\begin{align*}
&- 2 \cdot \frac{1}{s} \left[ \frac{\dot{B}}{B} \right] \cdot d \cdot \mathbf{e} \cdot \frac{1}{s+\Omega_{B1}} \cdot \frac{1}{s+\Omega_{B2}} + \frac{1}{s} \left[ \frac{\dot{B}}{B} \right] \cdot d^2 \cdot \frac{1}{(s+\Omega_{B1})^2} \\
&+ \frac{1}{s} \left[ \frac{\dot{B}}{B} \right] \cdot d \cdot \mathbf{e} \cdot \frac{1}{(s+\Omega_{B2})^2} \\
&\frac{1}{s} \left[ \frac{\dot{B}}{B} \right]^4 - \frac{1}{s} \left[ \frac{\dot{B}}{B} \right] - 3 \cdot \frac{1}{s} \left[ \frac{\dot{B}}{B} \right] \cdot d \cdot \frac{1}{s+\Omega_{B1}} + 3 \cdot \frac{1}{s} \left[ \frac{\dot{B}}{B} \right] \cdot d^2 \cdot \frac{1}{(s+\Omega_{B1})^2} \\
&- 6 \cdot \frac{1}{s} \left[ \frac{\dot{B}}{B} \right] \cdot d \cdot \mathbf{e} \cdot \frac{1}{s+\Omega_{B1}} \cdot \frac{1}{s+\Omega_{B2}} + 3 \cdot \frac{1}{s} \left[ \frac{\dot{B}}{B} \right] \cdot d^2 \cdot \frac{1}{(s+\Omega_{B1})^2} \\
&+ 3 \cdot \frac{1}{s} \left[ \frac{\dot{B}}{B} \right] \cdot d \cdot \mathbf{e} \cdot \frac{1}{(s+\Omega_{B2})^2} + 3 \cdot \frac{1}{s} \left[ \frac{\dot{B}}{B} \right] \cdot d^2 \cdot \frac{1}{s+\Omega_{B1}} \cdot \frac{1}{(s+\Omega_{B2})^2} \\
&- \frac{1}{s} \left[ \frac{\dot{B}}{B} \right] \cdot d^3 \cdot \frac{1}{(s+\Omega_{B1})^3} + \frac{1}{s} \left[ \frac{\dot{B}}{B} \right] \cdot \mathbf{e}^3 \cdot \frac{1}{(s+\Omega_{B2})^3}
\end{align*}\]

where:

\[\begin{align*}
&\frac{1}{s} \left[ \frac{\dot{B}}{B} \right] \cdot d \cdot \frac{1}{s+\Omega_{B1}} = (1 - \frac{d}{\Omega_{B1}} - \frac{e}{\Omega_{B2}}) \cdot \frac{d}{\Omega_{B1}} \cdot (\frac{1}{s} - \frac{1}{s+\Omega_{B1}}) \\
&+ \left[ \frac{d}{\Omega_{B1}} \right]^2 \cdot \frac{1}{(s+\Omega_{B1})^2} - \frac{\mathbf{e}}{\Omega_{B2} \cdot \Omega_{B1} \cdot \Omega_{B2}} \cdot (\frac{1}{s+\Omega_{B1}} - \frac{1}{s+\Omega_{B1}}) \\
&\frac{1}{s} \left[ \frac{\dot{B}}{B} \right] \cdot \mathbf{e} \cdot \frac{1}{s+\Omega_{B2}} = (1 - \frac{d}{\Omega_{B1}} - \frac{e}{\Omega_{B2}}) \cdot \frac{e}{\Omega_{B2}} \cdot (\frac{1}{s} - \frac{1}{s+\Omega_{B2}}) \\
&+ \frac{d}{\Omega_{B1} \cdot \Omega_{B1} \cdot \Omega_{B2}} \cdot (\frac{1}{s+\Omega_{B2}} - \frac{1}{s+\Omega_{B1}}) - \left[ \frac{e}{\Omega_{B2}} \right]^2 \cdot \frac{1}{(s+\Omega_{B2})^2} \\
&\frac{1}{s} \left[ \frac{\dot{B}}{B} \right] \cdot d \cdot \mathbf{e} \cdot \frac{1}{s+\Omega_{B1}} \cdot \frac{1}{s+\Omega_{B2}} = (1 - \frac{d}{\Omega_{B1}} - \frac{e}{\Omega_{B2}}) \cdot \frac{d}{\Omega_{B1}} \cdot \frac{e}{\Omega_{B2}} \cdot \frac{1}{s+\Omega_{B1}} \cdot \frac{1}{s+\Omega_{B2}} \\
&\left( \frac{\Omega_{B2} \cdot \Omega_{B1} \cdot \Omega_{B2} \cdot \Omega_{B1} \cdot \Omega_{B2}}{s+\Omega_{B1}} - \frac{1}{s+\Omega_{B1}} \cdot \frac{1}{s+\Omega_{B2}} + (\Omega_{B1} \cdot \Omega_{B2} \cdot \Omega_{B1} \cdot \Omega_{B2}) \cdot \frac{1}{s} \right) \\
&+ \frac{d}{\Omega_{B1} \cdot \Omega_{B1} \cdot \Omega_{B2}} \cdot \frac{d}{\Omega_{B1} \cdot \Omega_{B2}} \cdot \frac{e}{\Omega_{B1} \cdot \Omega_{B2}} \cdot \frac{1}{(s+\Omega_{B1})^2} - \frac{1}{s+\Omega_{B1} \cdot \Omega_{B2} + 1} \cdot \frac{1}{s+\Omega_{B2}}
\end{align*}\]
\[-\frac{e}{\bar{\Omega}_B} \cdot \frac{d}{\bar{\Omega}_B} \cdot \frac{e}{\bar{\Omega}_B} \cdot \frac{1}{\bar{\Omega}_B} \cdot \frac{1}{(s+\Omega_B)^2} - \frac{1}{s+\Omega_B} + \frac{1}{s+\Omega_B} \]

\[\frac{1}{s} \left[ \bar{B}^- \right] \cdot \frac{1}{(s+\Omega_B)^2} = \left( 1 - \frac{d}{\bar{\Omega}_B} \frac{e}{\bar{\Omega}_B} \right) \cdot \left[ \frac{d}{\bar{\Omega}_B} \left( \frac{d}{\bar{\Omega}_B} \right)^2 \right].\]

\[-\frac{e}{\bar{\Omega}_B} \cdot \left( \frac{1}{s+\Omega_B} \right)^2 - \frac{1}{s+\Omega_B} + \frac{1}{s} \right) + \left[ \frac{d}{\bar{\Omega}_B} \left( \frac{d}{\bar{\Omega}_B} \right)^2 \right] \cdot \frac{1}{(s+\Omega_B)^3} \]

\[-\frac{e}{\bar{\Omega}_B} \left( \frac{d}{\bar{\Omega}_B} \left( \frac{d}{\bar{\Omega}_B} \right)^2 \right)^2 \cdot \left( -\frac{1}{(s+\Omega_B)^2} - \frac{1}{s+\Omega_B} + \frac{1}{s} \right) - \left[ \frac{e}{\bar{\Omega}_B} \right]^3 \cdot \frac{1}{(s+\Omega_B)^3} + \frac{d}{\bar{\Omega}_B} \left( \frac{e}{\bar{\Omega}_B} \right)^2 \cdot \left( \frac{1}{\bar{\Omega}_B} \cdot \frac{1}{(s+\Omega_B)^2} - \frac{1}{s+\Omega_B} + \frac{1}{s+\Omega_B} \right)\]

\[\left( \left( -\frac{1}{(s+\Omega_B)^2} - \frac{1}{s+\Omega_B} + \frac{1}{s} \right) - \frac{e}{\bar{\Omega}_B} \left( \frac{d}{\bar{\Omega}_B} \left( \frac{d}{\bar{\Omega}_B} \right)^2 \right)^2 \cdot \left( \frac{1}{(s+\Omega_B)^3} \right) - \left( \frac{1}{(s+\Omega_B)^2} \right)^2 - \left( \frac{1}{s+\Omega_B} \right)^2 + \frac{1}{s} \right)\]

\[\left( \frac{1}{\bar{\Omega}_B} \cdot \frac{1}{(s+\Omega_B)^2} - \frac{1}{\bar{\Omega}_B} \cdot \frac{1}{(s+\Omega_B)^2} \right) + \left( \frac{1}{s+\Omega_B} \right)^2 + \left( \frac{1}{s} \right)^2\]
\[
\frac{1}{s} \left[ B^\alpha \right] \cdot e \cdot e^2 \cdot \frac{1}{s + \Omega_{B_1}} \cdot \frac{1}{(s + \Omega_{B_2})^2} = \\
(1 - \frac{d}{\Omega_{B_1} - \Omega_{B_2}}) \cdot e^2 \cdot \frac{1}{\Omega_{B_2}} \cdot \frac{1}{\Omega_{B_1}} \cdot \left[ \frac{1}{\Omega_{B_1} - \Omega_{B_2}} \right]^2 \cdot \\
(-\Omega_{B_1} \cdot (\Omega_{B_1} - \Omega_{B_2}) \cdot \Omega_{B_2} \cdot \frac{1}{(s + \Omega_{B_2})^2} + \Omega_{B_1} \cdot (2\Omega_{B_2} - \Omega_{B_1}) \cdot \frac{1}{s + \Omega_{B_2}} \\
- \Omega_{B_2}^2 \cdot \frac{1}{s + \Omega_{B_1}} + (\Omega_{B_1} - \Omega_{B_2})^2 \cdot \frac{1}{s} ) \\
+ \frac{e}{\Omega_{B_2}} \cdot \left[ \frac{e}{\Omega_{B_1} - \Omega_{B_2}} \right]^2 \cdot \frac{d}{\Omega_{B_1} - \Omega_{B_2}} \cdot \\
(- (\Omega_{B_1} - \Omega_{B_2})^2 \cdot \frac{1}{(s + \Omega_{B_2})^3} + (\Omega_{B_1} - \Omega_{B_2}) \cdot \frac{1}{(s + \Omega_{B_2})^2} \cdot \frac{1}{s + \Omega_{B_2}} + \frac{1}{s + \Omega_{B_1}} ) \\
+ \frac{d}{\Omega_{B_1}} \cdot \left[ \frac{e}{\Omega_{B_1} - \Omega_{B_2}} \right]^2 \cdot \frac{d}{\Omega_{B_1} - \Omega_{B_2}} \cdot \\
( (\Omega_{B_1} - \Omega_{B_2}) \cdot \frac{1}{(s + \Omega_{B_1})^2} + (\Omega_{B_1} - \Omega_{B_2}) \cdot \frac{1}{(s + \Omega_{B_2})^2} + 2 \cdot \frac{1}{s + \Omega_{B_1}} - 2 \cdot \frac{1}{s + \Omega_{B_2}} ) \\
\frac{1}{s} \left[ B^\alpha \right] \cdot e^3 \cdot \frac{1}{s + \Omega_{B_1}}^3 = (1 - \frac{d}{\Omega_{B_1} - \Omega_{B_2}}) \cdot \left[ \frac{d}{\Omega_{B_1}} \right]^3 \cdot \\
(-\Omega_{B_1}^2 \cdot \frac{1}{(s + \Omega_{B_1})^3} - \Omega_{B_1} \cdot \frac{1}{(s + \Omega_{B_1})^2} \cdot \frac{1}{s + \Omega_{B_1}} + \frac{1}{s} ) \\
+ \left[ \frac{d}{\Omega_{B_1}} \right]^4 \cdot \text{e}^3 \cdot \frac{1}{(s + \Omega_{B_1})^4} - \frac{e}{\Omega_{B_2}} \cdot \left[ \frac{d}{\Omega_{B_1} - \Omega_{B_2}} \right]^3 \cdot \\
(- (\Omega_{B_1} - \Omega_{B_2})^2 \cdot \frac{1}{(s + \Omega_{B_1})^3} - (\Omega_{B_1} - \Omega_{B_2}) \cdot \frac{1}{(s + \Omega_{B_2})^2} \cdot \frac{1}{s + \Omega_{B_1}} + \frac{1}{s + \Omega_{B_2}} ) \\
\frac{1}{s} \left[ B^\alpha \right] \cdot e^3 \cdot \frac{1}{s + \Omega_{B_2}}^3 = (1 - \frac{d}{\Omega_{B_1} - \Omega_{B_2}}) \cdot \left[ \frac{e}{\Omega_{B_2}} \right]^3 \cdot \\
(-\Omega_{B_2}^2 \cdot \frac{1}{(s + \Omega_{B_2})^3} - \Omega_{B_2} \cdot \frac{1}{(s + \Omega_{B_2})^2} \cdot \frac{1}{s + \Omega_{B_2}} + \frac{1}{s} ) \\
- \left[ \frac{e}{\Omega_{B_2}} \right]^4 \cdot \text{e}^3 \cdot \frac{1}{(s + \Omega_{B_2})^4} - \frac{d}{\Omega_{B_1}} \cdot \left[ \frac{e}{\Omega_{B_1} - \Omega_{B_2}} \right]^3 \cdot \\
(- (\Omega_{B_1} - \Omega_{B_2})^2 \cdot \frac{1}{(s + \Omega_{B_2})^3} + (\Omega_{B_1} - \Omega_{B_2}) \cdot \frac{1}{(s + \Omega_{B_2})^2} \cdot \frac{1}{s + \Omega_{B_2}} + \frac{1}{s + \Omega_{B_1}} ) \]
\[
\frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix}^2 = \frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix} - \frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix} \cdot d \cdot \frac{1}{s+\Omega_{B1}} + \frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix} \cdot e \cdot \frac{1}{s+\Omega_{B2}} \\
- \frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix} \cdot c \cdot \frac{1}{s+\Omega_{A}} + \frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix} \cdot c \cdot d \cdot \frac{1}{s+\Omega_{A}} \cdot \frac{1}{s+\Omega_{B1}} \\
- \frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix} \cdot c \cdot e \cdot \frac{1}{s+\Omega_{A}} \cdot \frac{1}{s+\Omega_{B2}} \\
\]

\[
\frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix}^3 = \frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix} + 2 \cdot \frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix} \cdot d \cdot \frac{1}{s+\Omega_{B1}} + 2 \cdot \frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix} \cdot e \cdot \frac{1}{s+\Omega_{B2}} \\
- 2 \cdot \frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix} \cdot d \cdot e \cdot \frac{1}{s+\Omega_{B1}} \cdot \frac{1}{s+\Omega_{B2}} + \frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix} \cdot d^2 \cdot \frac{1}{(s+\Omega_{B1})^2} \\
+ \frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix} \cdot e^2 \cdot \frac{1}{(s+\Omega_{B2})^2} - \frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix} \cdot c \cdot \frac{1}{s+\Omega_{A}} \cdot \frac{1}{s+\Omega_{B1}} + 2 \cdot \frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix} \cdot c \cdot d \cdot \frac{1}{s+\Omega_{A}} \cdot \frac{1}{s+\Omega_{B1}} \\
- 2 \cdot \frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix} \cdot c \cdot e \cdot \frac{1}{s+\Omega_{A}} \cdot \frac{1}{s+\Omega_{B2}} + \frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix} \cdot c^2 \cdot \frac{1}{s+\Omega_{A}} \cdot \frac{1}{s+\Omega_{B2}} \\
\]

where:

\[
\frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix} \cdot c \cdot \frac{1}{s+\Omega_{A}} = \left(1 - \frac{d}{\Omega_{B1}} \cdot \frac{e}{\Omega_{B2}} - \frac{1}{s+\Omega_{A}} + \frac{1}{s} \right) \\
+ \frac{d}{\Omega_{B1}} \cdot \frac{e}{\Omega_{B2}} \cdot \left( \frac{1}{s+\Omega_{A}} - \frac{1}{s+\Omega_{B1}} \right) - \frac{e}{\Omega_{B2}} \cdot \frac{c}{\Omega_{A}} \cdot \left( \frac{1}{s+\Omega_{B2}} - \frac{1}{s+\Omega_{A}} \right) \\
\]

\[
\frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix} \cdot c \cdot \frac{1}{s+\Omega_{A}} \cdot \frac{1}{s+\Omega_{B1}} = \left(1 - \frac{d}{\Omega_{B1}} \cdot \frac{e}{\Omega_{B2}} \right) \cdot \frac{c}{\Omega_{A}} \cdot \frac{d}{\Omega_{B1}} \cdot \frac{1}{s+\Omega_{B1}} \\
\left( \frac{1}{\Omega_{B1}} - \frac{1}{s+\Omega_{B1}} \right) - \frac{\Omega_{A}}{s+\Omega_{B1}} + \left( \frac{\Omega_{A}}{\Omega_{B1}} \right) \cdot \frac{1}{s} \\
+ \frac{d}{\Omega_{B1}} \cdot \frac{c}{\Omega_{B1}} \cdot \frac{d}{\Omega_{A}} \cdot \left( \Omega_{A} \cdot \Omega_{B1} \right) \cdot \frac{1}{(s+\Omega_{B1})^2} - \frac{1}{s+\Omega_{B1}} + \frac{1}{s+\Omega_{A}} \\
- \frac{e}{\Omega_{B2}} \cdot \frac{c}{\Omega_{A}} \cdot \frac{d}{\Omega_{B1}} \cdot \frac{1}{s+\Omega_{B2}} \cdot \left( \Omega_{B1} \cdot \Omega_{B2} \right) \cdot \frac{1}{s+\Omega_{A}} - \left( \Omega_{A} \cdot \Omega_{B2} \right) \cdot \frac{1}{s+\Omega_{B1}} + \left( \Omega_{A} \cdot \Omega_{B1} \right) \cdot \frac{1}{s+\Omega_{B2}} \\
\]

\[
\frac{1}{s} \begin{bmatrix} B^- \\ A^+ \end{bmatrix} \cdot c \cdot e \cdot \frac{1}{s+\Omega_{A}} \cdot \frac{1}{s+\Omega_{B2}} = \left(1 - \frac{d}{\Omega_{B1}} \cdot \frac{e}{\Omega_{B2}} \right) \cdot \frac{c}{\Omega_{A}} \cdot \frac{e}{\Omega_{B1}} \cdot \frac{1}{s+\Omega_{B2}} \\
- \frac{e}{\Omega_{B1}} \cdot \frac{c}{\Omega_{B1}} \cdot \frac{e}{\Omega_{A}} \cdot \frac{1}{s+\Omega_{B2}} \\
\]
\[
\begin{align*}
&\left( \Omega_B \cdot s + \Omega_A \cdot \frac{1}{s+\Omega_B} - \Omega_A \cdot \frac{1}{s+\Omega_B} + (\Omega_A \cdot \Omega_B \cdot \frac{1}{s}) \right) \\
&- e \cdot \frac{c}{\Omega_B} \cdot \frac{e}{\Omega_A \cdot \Omega_B} \cdot \left( \frac{(\Omega_A \cdot \Omega_B)}{(s+\Omega_B)^2} \right) - \frac{1}{s+\Omega_B} + \frac{1}{s+\Omega_A} \\
&+ \frac{d}{\Omega_B} \cdot \frac{c}{\Omega_A \cdot \Omega_B} \cdot \frac{e}{\Omega_B \cdot \Omega_A} \cdot \frac{1}{\Omega_A \cdot \Omega_B} \\
&\left( \frac{(\Omega_B \cdot \Omega_B)}{(s+\Omega_B)^2} + \frac{(\Omega_A \cdot \Omega_B)}{s+\Omega_B} + \frac{(\Omega_A \cdot \Omega_B)}{s+\Omega_B} \right) \\
&\frac{1}{s} \cdot \left[ \frac{B}{B} \right] \cdot c \cdot d \cdot e \cdot \frac{1}{s+\Omega_A} \cdot \frac{1}{s+\Omega_B} \cdot \frac{1}{s+\Omega_B} \\
&\left( 1 - \frac{d}{\Omega_B} \cdot \frac{e}{\Omega_A \cdot \Omega_B} \cdot \frac{c}{\Omega_B \cdot \Omega_A} \cdot \frac{1}{\Omega_B \cdot \Omega_A} \cdot \frac{1}{\Omega_B \cdot \Omega_A} \right) \\
&\left( - (\Omega_A \cdot \Omega_B) \cdot \Omega_B \cdot \Omega_B \cdot \frac{1}{s+\Omega_A} + (\Omega_A \cdot \Omega_B) \cdot \Omega_A \cdot \Omega_B \cdot \frac{1}{s+\Omega_B} \\
&- (\Omega_A \cdot \Omega_B) \cdot \Omega_A \cdot \Omega_B \cdot \frac{1}{s+\Omega_B} + \frac{(\Omega_A \cdot \Omega_B)}{(s+\Omega_B)^2} \right) \\
&+ \frac{d}{\Omega_B} \cdot \frac{c}{\Omega_B \cdot \Omega_B} \cdot \frac{d}{\Omega_A \cdot \Omega_B} \cdot \frac{e}{\Omega_B \cdot \Omega_A} \cdot \frac{1}{\Omega_B \cdot \Omega_A} \cdot \frac{1}{\Omega_B \cdot \Omega_A} \\
&\left( - (\Omega_A \cdot \Omega_B) \cdot (\Omega_B \cdot \Omega_B) \cdot (\Omega_A \cdot \Omega_B) \cdot \frac{1}{(s+\Omega_B)^2} \right) \\
&+ (\Omega_A \cdot \Omega_B) \cdot (2 \cdot \Omega_B \cdot \Omega_A \cdot \Omega_B) \cdot \frac{1}{s+\Omega_B} - (\Omega_B \cdot \Omega_B)^2 \cdot \frac{1}{s+\Omega_A} \\
&+ (\Omega_A \cdot \Omega_B)^2 \cdot \frac{1}{s+\Omega_B} \right) \\
&\frac{1}{s} \cdot \left[ \frac{B}{B} \right] \cdot c \cdot d \cdot e \cdot \frac{1}{s+\Omega_A} \cdot \frac{1}{s+\Omega_B} \cdot \frac{1}{s+\Omega_B} \\
&\left( 1 - \frac{d}{\Omega_B} \cdot \frac{e}{\Omega_A \cdot \Omega_B} \cdot \frac{c}{\Omega_B \cdot \Omega_A} \cdot \frac{1}{\Omega_B \cdot \Omega_A} \cdot \frac{1}{\Omega_B \cdot \Omega_A} \right) \\
&\left( - (\Omega_A \cdot \Omega_B) \cdot (2 \cdot \Omega_B \cdot \Omega_A \cdot \Omega_B) \cdot \frac{1}{s+\Omega_B} - (\Omega_B \cdot \Omega_B)^2 \cdot \frac{1}{s+\Omega_A} \\
&+ (\Omega_A \cdot \Omega_B)^2 \cdot \frac{1}{s+\Omega_B} \right) \right]
\end{align*}
\]
\[
\begin{align*}
- \frac{\Omega_{B1}}{\Omega - \Omega_{B1}} \cdot \frac{1}{\Omega_A} + (\Omega_{A} - \Omega_{B1})^2 \cdot \frac{1}{s} \\
- \frac{d}{\Omega_{B1}} \cdot \frac{d^2}{\Omega_A - \Omega_{B1}} \cdot \frac{c}{\Omega_A - \Omega_{B1}} \cdot (-(\Omega_{A} - \Omega_{B1})^2 \cdot \frac{1}{(s+\Omega_{B1})^3} \\
+ (\Omega_{A} - \Omega_{B1}) \cdot \frac{1}{(s+\Omega_{B1})^2} \cdot \frac{1}{s+\Omega_{B1}} + \frac{1}{s+\Omega_{A}} \\
- \frac{e}{\Omega_{B2}} \cdot \frac{c}{\Omega_{B1} - \Omega_{B2}} \cdot \frac{d^2}{\Omega_A - \Omega_{B1}} \cdot \frac{1}{\Omega_A - \Omega_{B2}} \cdot \frac{1}{\Omega_{B1} - \Omega_{B2}} \\
(\Omega_{A} - \Omega_{B1}) \cdot (\Omega_{B1} - \Omega_{B2}) \cdot (\Omega_{A} - \Omega_{B2}) \cdot \frac{1}{(s+\Omega_{B1})^2} \\
+ (\Omega_{A} - \Omega_{B2}) \cdot (2 \cdot \Omega_{B1} - \Omega_{A} - \Omega_{B2}) \cdot \frac{1}{s+\Omega_{B1}} - (\Omega_{B1} - \Omega_{B2})^2 \cdot \frac{1}{s+\Omega_{A}} \\
+ (\Omega_{A} - \Omega_{B1})^2 \cdot \frac{1}{s+\Omega_{B2}} \\
\frac{1}{s} \left[ \begin{array}{c} \Omega_{B1} \\
\Omega_{B2} \\
\end{array} \right]^2 \cdot \frac{e^2}{\Omega_{B1} - \Omega_{B2}} \cdot \frac{1}{\Omega_A} \cdot \frac{1}{(s+\Omega_{B2})^2} = (1 - \frac{d}{\Omega_{B1}} \cdot \frac{e}{\Omega_{B2}}) \cdot \frac{c}{\Omega_A} \left[ \frac{\Omega_{B1}}{\Omega_{B2}} \right]^2 \left[ \frac{\Omega_A - \Omega_{B2}}{\Omega_A - \Omega_{B1}} \right]^2. \\
(\Omega_{A} - \Omega_{B2}) \cdot \Omega_{A} \cdot \frac{1}{(s+\Omega_{B2})^2} + (\Omega_{A} - \Omega_{B2}) \cdot \frac{1}{s+\Omega_{B2}} \\
- \Omega_{B2} \cdot \frac{1}{s+\Omega_{A}} + (\Omega_{A} - \Omega_{B2})^2 \cdot \frac{1}{s} \\
+ \frac{e}{\Omega_{B2}} \cdot (\Omega_{A} - \Omega_{B2})^2 \cdot \frac{c}{\Omega_A - \Omega_{B1}} \cdot (-(\Omega_{A} - \Omega_{B2})^2 \cdot \frac{1}{(s+\Omega_{B2})^3} \\
+ (\Omega_{A} - \Omega_{B2}) \cdot \frac{1}{(s+\Omega_{B2})^2} \cdot \frac{1}{s+\Omega_{B2}} + \frac{1}{s+\Omega_{A}} \\
+ \frac{d}{\Omega_{B1}} \cdot \frac{c}{\Omega_{B1} - \Omega_{B2}} \cdot \frac{e^2}{\Omega_A - \Omega_{B1}} \cdot \frac{1}{\Omega_{B1} - \Omega_{B2}} \\
(\Omega_{A} - \Omega_{B2}) \cdot (\Omega_{B1} - \Omega_{B2}) \cdot (\Omega_{A} - \Omega_{B1}) \cdot \frac{1}{(s+\Omega_{B2})^2} \\
+ (\Omega_{A} - \Omega_{B1}) \cdot (2 \cdot \Omega_{B2} - \Omega_{A} - \Omega_{B1}) \cdot \frac{1}{s+\Omega_{B2}} - (\Omega_{B1} - \Omega_{B2})^2 \cdot \frac{1}{s+\Omega_{A}} \\
+ (\Omega_{A} - \Omega_{B2})^2 \cdot \frac{1}{s+\Omega_{B1}} \\
\end{align*}
\]
\[
\frac{1}{s} \begin{bmatrix} \frac{B}{B} \end{bmatrix}^2 \begin{bmatrix} A \end{bmatrix}^2 = \frac{1}{s} \begin{bmatrix} \frac{B}{B} \end{bmatrix} \cdot \frac{1}{s} \begin{bmatrix} \frac{B}{B} \end{bmatrix} \cdot d \cdot \frac{1}{s + \Omega_{B1}} + \frac{1}{s} \begin{bmatrix} \frac{B}{B} \end{bmatrix} \cdot e \cdot \frac{1}{s + \Omega_{B2}}
\]

\[
- 2 \cdot \frac{1}{s} \begin{bmatrix} \frac{B}{B} \end{bmatrix} \cdot c \cdot \frac{1}{s + \Omega_A} + 2 \cdot \frac{1}{s} \begin{bmatrix} \frac{B}{B} \end{bmatrix} \cdot c \cdot d \cdot \frac{1}{s + \Omega_A} \cdot \frac{1}{s + \Omega_{B1}}
\]

\[
- 2 \cdot \frac{1}{s} \begin{bmatrix} \frac{B}{B} \end{bmatrix} \cdot c \cdot e \cdot \frac{1}{s + \Omega_A} \cdot \frac{1}{s + \Omega_{B2}} + \frac{1}{s} \begin{bmatrix} \frac{B}{B} \end{bmatrix} \cdot c^2 \cdot \frac{1}{(s + \Omega_A)^2}
\]

\[
- \frac{1}{s} \begin{bmatrix} \frac{B}{B} \end{bmatrix} \cdot c^2 \cdot d \cdot \frac{1}{(s + \Omega_A)^2} \cdot \frac{1}{s + \Omega_{B1}} + \frac{1}{s} \begin{bmatrix} \frac{B}{B} \end{bmatrix} \cdot c^2 \cdot e \cdot \frac{1}{(s + \Omega_A)^2} \cdot \frac{1}{s + \Omega_{B2}}
\]

where:

\[
\frac{1}{s} \begin{bmatrix} \frac{B}{B} \end{bmatrix} \cdot c^2 \cdot \frac{1}{(s + \Omega_A)^2} = (1 - d \cdot e) \cdot \left( \frac{c}{\Omega_A} \right)^2 \cdot \left( -\Omega_A \cdot \frac{1}{(s + \Omega_A)^2} - \frac{1}{s + \Omega_A} + \frac{1}{s} \right)
\]

\[
+ \frac{d}{\Omega_{B1}} \cdot \left[ \frac{c}{\Omega_A - \Omega_{B1}} \right]^2 \cdot \left( -\Omega_A \cdot \frac{1}{(s + \Omega_A)^2} - \frac{1}{s + \Omega_A} + \frac{1}{s + \Omega_{B1}} \right)
\]

\[
- \frac{e}{\Omega_{B2}} \cdot \left[ \frac{c}{\Omega_A - \Omega_{B2}} \right]^2 \cdot \left( -\Omega_A \cdot \frac{1}{(s + \Omega_A)^2} - \frac{1}{s + \Omega_A} + \frac{1}{s + \Omega_{B2}} \right)
\]

\[
\frac{1}{s} \begin{bmatrix} \frac{B}{B} \end{bmatrix} \cdot c^2 \cdot d \cdot \frac{1}{(s + \Omega_A)^2} \cdot \frac{1}{s + \Omega_{B1}} = (1 - d \cdot e) \cdot \left( \frac{c}{\Omega_A} \right)^2 \cdot \frac{d}{\Omega_{B1}} \cdot \left[ \frac{1}{\Omega_A - \Omega_{B1}} \right]^2 \cdot \left( \Omega_A \cdot \Omega_{B1} \cdot \frac{1}{(s + \Omega_A)^2} + \Omega_{B1} \cdot (2 \cdot \Omega_A - \Omega_{B1}) \cdot \frac{1}{s + \Omega_A} \right)
\]

\[
- \Omega_A \cdot \frac{1}{s + \Omega_{B1}} + (\Omega_A \cdot \Omega_{B1})^2 \cdot \frac{1}{s}
\]

\[
+ \frac{d}{\Omega_{B1}} \cdot \left[ \frac{c}{\Omega_A - \Omega_{B1}} \right]^2 \cdot \frac{d}{\Omega_A - \Omega_{B1}}
\]

\[
(\Omega_A - \Omega_{B1}) \cdot \frac{1}{(s + \Omega_A)^2} + (\Omega_A - \Omega_{B1}) \cdot \frac{1}{(s + \Omega_{B1})^2} + 2 \cdot \frac{1}{s + \Omega_A} - 2 \cdot \frac{1}{s + \Omega_{B1}}
\]

\[
- \frac{e}{\Omega_{B2}} \cdot \frac{c}{\Omega_A - \Omega_{B2}} \cdot \frac{d}{\Omega_A - \Omega_{B1}} \cdot \frac{1}{\Omega_A - \Omega_{B2}}
\]

\[
(\Omega_A - \Omega_{B1}) \cdot (\Omega_A - \Omega_{B2}) \cdot (\Omega_{B1} - \Omega_{B2}) \cdot \frac{1}{(s + \Omega_A)^2}
\]

\[
+ (\Omega_{B1} - \Omega_{B2}) \cdot (2 \cdot \Omega_A - \Omega_{B1} - \Omega_{B2}) \cdot \frac{1}{s + \Omega_A}
\]
\[
\frac{1}{s} \left[ \frac{B_s}{B} \right] \cdot c^2 \cdot e \cdot \frac{1}{(s+\Omega_A)^2} \cdot \frac{1}{s+\Omega_B} = \left(1 + \frac{d}{\Omega_{B1}} + \frac{e}{\Omega_{B2}} \right) \cdot \frac{c}{\Omega_A} \cdot \frac{e}{\Omega_{B2}} \cdot \left( \frac{1}{s+\Omega_B} \right)^2.
\]

\[
\left( \frac{\Omega_A - \Omega_{B2}}{\Omega_A - \Omega_{B2}} \right) \cdot \frac{1}{(s+\Omega_A)^2} + \frac{\Omega_{B2}}{s+\Omega_A} \cdot \left( 2 \cdot \Omega_A - \Omega_{B2} \right) \cdot \frac{1}{s+\Omega_A} \cdot \frac{\Omega_A}{\Omega_A - \Omega_{B2}} \cdot \frac{\Omega_A - \Omega_{B2}}{\Omega_A - \Omega_{B2}} ^2.
\]

The above partial fraction expressions are checked numerically by setting values for "c", "d", "e", "\Omega_A", "\Omega_{B1}", "\Omega_{B2}" and "s", calculating the (nonexpanded) expressions in (K.1) and comparing to the result obtained by using the (expanded) partial fractions expressions.

Note, the cases when \( \Omega_{B1} - \Omega_{B2} \) and/or \( \Omega_A - \Omega_{B1} \) and/or \( \Omega_A - \Omega_{B2} \) are treated by "shifting" "\Omega_A", "\Omega_{B1}" or "\Omega_{B2}" by a value of "\( \epsilon \cdot \Omega_A \)" or "\( \epsilon \cdot \Omega_{B1} \)" respectively. For example, if \( \Omega_{B1} - \Omega_{B2} \) \( \Omega_{B1} + \epsilon \cdot \Omega_{B1} \) is set to "\( \Omega_{B1} + \epsilon \cdot \Omega_{B1} \)" and "\( \Omega_{B2} \)" is left unchanged; this procedure introduces relative errors of the order of "\( (\epsilon \cdot \Omega_{B1} \cdot t)^2 \)" in the final expressions, and hence the expressions would be
"accurate" as long as $\Omega_{B1} \cdot t \ll 1/\epsilon$. This procedure was done in order to avoid further algebraic complications.
Appendix L: Laplace inverting components of products of matrix operators ("L" part)

In section 2.6 (equation 2.68) it was noted that the Laplace inverse of terms of the form:

\[
\frac{1}{s} \left[ \frac{1 - \gamma}{1 + \gamma} \right]^m
\]

(L.1)

are required, where:

\[
\left[ \frac{1 - \gamma}{1 + \gamma} \right] = p + q \cdot \frac{1}{s + \Omega}
\]

(L.2)

The Laplace inverse transform of (L.1) is described next. First, use the binomial theorem to rewrite (L.1) as:

\[
\frac{1}{s} \left[ \frac{1 - \gamma}{1 + \gamma} \right]^m = \sum_{k=0}^{m} {m \choose k} \cdot p^{m-k} \cdot q^{k} \cdot \frac{1}{s} \cdot \frac{1}{(s + \Omega)^k}
\]

(L.3)

where:

\[
{m \choose k} = \frac{m!}{k!(m-k)!}
\]

Next using a result from Abramowitz and Stegun (1970):

\[
L^{-1} \left[ \frac{\Omega^k}{s} \cdot \frac{1}{(s + \Omega)^k} \right] = H(t) \cdot \left[ 1 - e^{-\Omega t} \cdot \sum_{\ell=0}^{k-1} \frac{(\Omega t)^\ell}{\ell!} \right]
\]

(L.4)

where \( H(t) \) is the Heaviside function, that is:

\[
H(t) = \begin{cases} 
0 & \text{if } t < 0 \\
1 & \text{if } t > 0 
\end{cases}
\]

(L.5)
Therefore, combining (L.3) and (L.4) we obtain:

\[
L^{-1}\left[ \frac{1}{s} \left[ \frac{1}{1 + \gamma} \right]^m \right] = \sum_{k=0}^{\infty} \binom{m}{k} p^{m-k} \cdot \left( \frac{q}{\Omega} \right)^k \cdot \left[ 1 - e^{-\Omega t} \sum_{l=0}^{k-1} \left( \Omega t \right)^l \cdot \frac{1}{l!} \right] \cdot H(t)
\]

Note that Rundle and Jackson (1977) obtain a different expression for the Laplace inverse of (L.1) due to some algebraic errors.
Appendix M: Displacement and Stresses of
Nuclei of Strain in "Infinite" Space

The displacement fields of Nuclei of Strain (NOS) in
"infinite" space are given by:

\[ u_{ij}^{k}(x, \xi) = \mu \cdot \beta \cdot \frac{1}{r^3} \left[ (\delta + 1) \cdot \delta_{ik} \cdot (x_j - \xi_j) \\
+ (\delta - 1) \cdot \delta_{jk} \cdot (x_i - \xi_i) \\
- (\delta - 1) \cdot \delta_{ij} \cdot (x_k - \xi_k) \\
- 6\delta \cdot (x_k - \xi_k) \cdot (x_i - \xi_i) \cdot (x_j - \xi_j) \cdot \frac{1}{r^2} \right] \]

(M.1)

where: \( r = |x - \xi| \)
\( \beta = 1/\{4\pi \cdot \mu \cdot (1+\delta)\} \)
\( \delta = (\lambda + \mu)/(\lambda + 3\mu) \)
\( \mu \) is the shear modulus
\( \lambda \) is Lamé's constant
\( \xi \) is the location of the nuclei of strain
\( x \) is the location at which displacements are to be evaluated

(ij) subscripts on "u" denote the type of the nuclei
      of strain

(\( i \)) superscript on "u" denote the displacement
      components

\( \delta \) is Kronecker's delta

The stress fields of Nuclei of Strain in "infinite" space are given by:

\[ \sigma_{ij}^{qr}(x, \xi) = \mu^2 \cdot \beta \cdot (\delta - 1) \cdot \frac{1}{r^3} \left[ 2 \cdot \delta_{iq} \delta_{jr} + 2 \cdot \delta_{jq} \delta_{ir} - 2 \cdot \delta_{ij} \delta_{qr} \right] \]
\begin{equation}
+ \mu^2 \cdot \beta \cdot (\delta - 1) \cdot \frac{1}{r^5} \left[ - 3 \cdot \delta_{1q} \cdot (x_j - \xi_j) \cdot (x_r - \xi_r) \\
- 3 \cdot \delta_{1r} \cdot (x_j - \xi_j) \cdot (x_q - \xi_q) \\
- 3 \cdot \delta_{jq} \cdot (x_1 - \xi_1) \cdot (x_r - \xi_r) \\
- 3 \cdot \delta_{jr} \cdot (x_1 - \xi_1) \cdot (x_q - \xi_q) \\
+ 6 \cdot \delta_{1j} \cdot (x_q - \xi_q) \cdot (x_r - \xi_r) \right] \\
+ \mu^2 \cdot \beta \cdot \epsilon \cdot \frac{1}{r^5} \left[ - 12 \cdot \delta_{qr} \cdot (x_1 - \xi_1) \cdot (x_j - \xi_j) \\
- 6 \cdot \delta_{1r} \cdot (x_j - \xi_j) \cdot (x_q - \xi_q) \\
- 6 \cdot \delta_{1q} \cdot (x_j - \xi_j) \cdot (x_r - \xi_r) \\
- 6 \cdot \delta_{jr} \cdot (x_1 - \xi_1) \cdot (x_q - \xi_q) \\
- 6 \cdot \delta_{jq} \cdot (x_1 - \xi_1) \cdot (x_r - \xi_r) \right] \\
+ \mu^2 \cdot \beta \cdot \epsilon \cdot \frac{1}{r^7} \left[ 60 \cdot (x_q - \xi_q) \cdot (x_r - \xi_r) \cdot (x_1 - \xi_1) \cdot (x_j - \xi_j) \right] \\
+ \lambda \cdot \mu \cdot \beta \cdot (\delta - 1) \cdot \frac{1}{r^3} \left[ 2 \cdot \delta_{1q} \delta_{qr} \right] \\
+ \lambda \cdot \mu \cdot \beta \cdot (\delta - 1) \cdot \frac{1}{r^5} \left[ - 6 \cdot \delta_{qr} \cdot (x_1 - \xi_1) \cdot (x_j - \xi_j) \right] \tag{H.2}
\end{equation}

where: (ij) subscripts on "\( \sigma \)" denote the type of nuclei of strain

(qr) superscripts on "\( \sigma \)" denote the stress components
Appendix N: Effect of Rotating Axis on Tensor Components

Any object in the main body of this thesis that was referred to as being a "true" vector or tensor (as compared with list of numbers or a matrix) have components in a given orthogonal system related to components (at the same points) in another orthogonal system by specific linear relations (e.g. see any textbook on vector and tensor analysis).

Let \( \hat{e}_i^{(1)} \) and \( \hat{e}_i^{(2)} \) denote the basis (physical) vectors for the orthogonal systems (1) and (2) respectively. Define the following scalar products:

\[
\alpha_{ij}^{(1,2)} = \hat{e}_i^{(1)} \cdot \hat{e}_j^{(2)} \tag{N.1}
\]

Then a tensor component \( C_{ijkl...p}^{(2)} \) in system (2) is related to tensor components \( C_{qrs...v}^{(1)} \) in system (1) by:

\[
C_{ijkl...p}^{(2)} = \alpha_{qi}^{(1,2)} \alpha_{rj}^{(1,2)} \alpha_{sk}^{(1,2)} ... \alpha_{vp}^{(1,2)} C_{qrs...v}^{(1)} \tag{N.2}
\]

where the summation notation has been used in (N.2).
Appendix 0: Displacement Field
Corresponding to a Constant Stress Field

In some of the application problems in chapter 4, a constant far-field stress was "applied" (refer to equations (3.15)); in order to obtain the displacement field for those problems, the displacement field due to a constant far-field stress is required (refer to equation (3.14)). This appendix gives the displacement field due to a constant stress state.

Assume that in a given coordinate system, a constant stress field \( \sigma_{ij} \) is specified, then:

\[
\begin{align*}
u_i &= \varepsilon_{ik} x_k + e_{ikl} \omega_l x_1 + u_i^0 \\
\varepsilon_{ij} &= \frac{1}{2\mu} \left[ \sigma_{ij} - \delta_{ij} \frac{\lambda}{3\lambda + 2\mu} \sigma_{kk} \right]
\end{align*}
\]

where \(\mu\) and \(\lambda\) are the shear modulus and Lame's constant respectively, and \((\omega_x, \omega_y, \omega_z)\) and \((u_x^0, u_y^0, u_z^0)\) are arbitrary constants which can be interpreted as rotation components and rigid body translation components respectively.
Appendix P: Stress Intensity Factors
As a function of slip near a crack front

The plane-strain and antiplane asymptotic crack-opening variations were used to obtain the Stress Intensity Factors (SIF) near a 3-D planar crack front. Given a point "P" located at a crack front, define a local coordinate system with coordinates \((x,y,z)\) and origin at point "P" such that the \(x\)-axis is tangent to the crack front, the \(z\)-axis is perpendicular to the crack plane and the \(y\)-axis is determined by the right-hand rule (see figure P.1). Let \((\Delta_x, \Delta_y, \Delta_z)\) be the slip components on the crack surface with respect to the local coordinate system defined above. The SIF values are then given by:

\[
K_I = \lim_{y \to 0} \frac{\Delta_z(x=0,y)}{\sqrt{|y|}} \cdot \frac{\sqrt{2\pi} \cdot \mu \cdot \delta}{1 + \delta} \quad (P.1)
\]

\[
K_{II} = \lim_{y \to 0} \frac{\Delta_y(x=0,y)}{\sqrt{|y|}} \cdot \frac{\sqrt{2\pi} \cdot \mu \cdot \delta}{1 + \delta} \quad (P.2)
\]

\[
K_{III} = \lim_{y \to 0} \frac{\Delta_x(x=0,y)}{\sqrt{|y|}} \cdot \frac{\sqrt{\pi} \cdot \mu}{2 \cdot \sqrt{2}} \quad (P.3)
\]

where \(\mu\) and \(\lambda\) are the shear modulus and Lamé's constant respectively and \(\delta = (\lambda + \mu)/(\lambda + 3\mu)\).
crack-front

$S_c$

point $P$

Figure P.1
Appendix D: Integrating singular surface integrals by transforming them to non-singular line integrals plus "other" terms

In this appendix, the transformation of some "singular" surface integrals into non-singular line integrals plus "other" terms will be presented. The surface \( S_c \) over which the surface integrations are to be performed are taken to be planar which without loss of generality is taken to be on the plane \( z=0 \). Linear combinations of the surface integrals that are given in this appendix give the stresses at points on \( S_c \) due to "infinite" space nuclei of strain (NOS) Green's functions (given in appendix M) multiplied by at most quadratic "shape functions". In what follows, the limiting operation \( (z->0) \) and the surface integration operation are (inappropriately) interchanged on the left hand side of the equations, and are to be interpreted "appropriately" (see section 3.4). The required surface integrals and their transformations are as follows:

\[
\iiint_{S_c} \frac{1}{r^3} \cdot dx \cdot dy = \int_{C^+} \frac{x-R}{y^2 R} \cdot dy + \int_{C^-} \frac{x+R}{y^2 R} \cdot dy - 2 \cdot (\frac{1}{y_{\text{max}}} - \frac{1}{y_{\text{min}}}) \quad (Q.1)
\]

\[
\iiint_{S_c} \frac{x}{r^3} \cdot dx \cdot dy = \int_{C^+} \frac{-1}{R} \cdot dy - \int_{C^-} \frac{1}{R} \cdot dy \quad (Q.2)
\]

\[
\iiint_{S_c} \frac{-y}{r^3} \cdot dx \cdot dy = \int_{C^+} \frac{x-R}{y R} \cdot dy + \int_{C^-} \frac{x+R}{y R} \cdot dy + 2 \cdot \ln |y_{\text{max}}/y_{\text{min}}| \quad (Q.3)
\]
\[ \iint_{S_c} \frac{x^2}{r^3} \, dx \, dy = \int_{C^+} \left[ \ln(R+x) - \frac{x}{R} \right] \, dy + \int_{C^-} \left[ \ln((R+x)/y^2) - \frac{x}{R} \right] \, dy \\
- 2 \left[ \langle y_{\max} \cdot \ln|y_{\max}| - y_{\max} \rangle - \langle y_{\min} \cdot \ln|y_{\min}| - y_{\min} \rangle \right] \]

(Q.4)

\[ \iint_{S_c} \frac{xy}{r^3} \, dx \, dy = \int_{C^+} \frac{-y}{R} \, dy - \int_{C^-} \frac{y}{R} \, dy \] (Q.5)

\[ \iint_{S_c} \frac{y^2}{r^3} \, dx \, dy = \int_{C^+} \frac{x-R}{R} \, dy + \int_{C^-} \frac{x+R}{R} \, dy + 2 \cdot (y_{\max} - y_{\min}) \] (Q.6)

\[ \iint_{S_c} \frac{x^2}{r^5} \, dx \, dy = \int_{C^+} \left[ \frac{x-R}{3y^2 R} - \frac{x}{3R^3} \right] \, dy + \int_{C^-} \left[ \frac{x+R}{3y^2 R} - \frac{x}{3R^3} \right] \, dy \\
- \frac{2}{3} \left( \frac{1}{y_{\max}} - \frac{1}{y_{\min}} \right) \] (Q.7)

\[ \iint_{S_c} \frac{xy}{r^5} \, dx \, dy = \int_{C^+} \frac{-y}{3R^3} \, dy + \int_{C^-} \frac{-y}{3R^3} \, dy \] (Q.8)

\[ \iint_{S_c} \frac{y^2}{r^5} \, dx \, dy = \int_{C^+} \left[ 2 \cdot \frac{x-R}{3y^2 R} + \frac{x}{3R^3} \right] \, dy + \int_{C^-} \left[ 2 \cdot \frac{x+R}{3y^2 R} + \frac{x}{3R^3} \right] \, dy \\
- \frac{4}{3} \left( \frac{1}{y_{\max}} - \frac{1}{y_{\min}} \right) \] (Q.9)
\[
\iiint_{S} \frac{x^3}{r^5} \, dx \, dy = \int_{C^+} \left[ \frac{-1}{R} + \frac{y^2}{3R^3} \right] \, dy + \int_{C^-} \left[ \frac{-1}{R} + \frac{y^2}{3R^3} \right] \, dy \tag{Q.10}
\]

\[
\iiint_{S} \frac{x^2y}{r^5} \, dx \, dy = \int_{C^+} \left[ \frac{x-R}{3yR} - \frac{xy}{3R^3} \right] \, dy + \int_{C^-} \left[ \frac{x+R}{3yR} - \frac{xy}{3R^3} \right] \, dy
\]

\[+ \frac{2}{3} \ln(|y_{\text{max}}/y_{\text{min}}|) \tag{Q.11}\]

\[
\iiint_{S} \frac{x^4}{r^5} \, dx \, dy = \int_{C^+} \left[ \ln(R+x) - \frac{4x}{3R} + \frac{xy^2}{3R^3} \right] \, dy
\]

\[+ \int_{C^-} \left[ \ln[(R+x)/y^2] - \frac{4x}{3R} + \frac{xy^2}{3R^3} \right] \, dy
\]

\[- 2 \cdot \left( y_{\text{max}} \cdot \ln|y_{\text{max}}| - y_{\text{max}} \right) - (y_{\text{min}} \cdot \ln|y_{\text{min}}| - y_{\text{min}}) \] \tag{Q.12}

\[
\iiint_{S} \frac{x^3y}{r^5} \, dx \, dy = \int_{C^+} \left[ -\frac{y}{R} + \frac{y^3}{3R^3} \right] \, dy + \int_{C^-} \left[ -\frac{y}{R} + \frac{y^3}{3R^3} \right] \, dy \tag{Q.13}
\]

\[
\iiint_{S} \frac{x^2y^2}{r^5} \, dx \, dy = \int_{C^+} \left[ \frac{x-R}{3R^3} - \frac{xy}{3R^3} \right] \, dy + \int_{C^-} \left[ \frac{x+R}{3R^3} - \frac{xy}{3R^3} \right] \, dy
\]

\[+ \frac{2}{3} \cdot (y_{\text{max}} - y_{\text{min}}) \tag{Q.14}\]

\[
\iiint_{S} \frac{xy^2}{r^5} \, dx \, dy = \int_{C^+} \frac{-y^2}{3R^3} \, dy + \int_{C^-} \frac{-y^2}{3R^3} \, dy \tag{Q.15}
\]
\[
\begin{align*}
\iint_{S_c} \frac{y^3}{r^5} dx dy &= \int_{C^+} \left[ 2 \cdot \frac{x-R}{3yR} + \frac{xy}{3y^3} \right] dy + \int_{C^-} \left[ 2 \cdot \frac{x+R}{3yR} + \frac{xy}{3y^3} \right] dy \\
&+ \frac{4}{3} \ln \left( \frac{y_{\text{max}}}{y_{\text{min}}} \right) \\
&= (Q.16) \\
\iint_{S_c} \frac{xy^3}{r^5} dx dy &= \int_{C^+} -\frac{y^3}{3r^3} dy + \int_{C^-} -\frac{y^3}{3r^3} dy \\
&= (Q.17) \\
\iint_{S_c} \frac{y^4}{r^5} dx dy &= \int_{C^+} \left[ 2 \cdot \frac{x-R}{3r} + \frac{xy^2}{3r^3} \right] dy + \int_{C^-} \left[ 2 \cdot \frac{x+R}{3r} + \frac{xy^2}{3r^3} \right] dy \\
&+ \frac{4}{3} \left( y_{\text{max}} - y_{\text{min}} \right) \\
&= (Q.18)
\end{align*}
\]

where: 
\[r^2 = x^2 + y^2 + z^2\]
\[r^2 = x^2 + y^2\]

\(C^\pm\) are defined in section 3.4 (e.g. see figure 3.21).

Note that all integrands having a "\(y^n\)" (where "n" is an integer) in the denominator have a "removable" singularity at \(y=0\) and must be suitably defined for numerical and analytical manipulations (see section 3.4 for more details).
Appendix R: Number of triangles subdivided versus subdivision parameter and geometry

In this appendix, the number of triangles produced by the triangle subdivision algorithm specified in section 3.4 ("type 1" integration process) will be analytically and empirically studied.

Consider a triangle with vertices "A", "B" and "C", an arbitrary point "P" not lying on the triangle, a constant real number "a" and a subdivision condition specified by:

If: distance from point "P" to the center of gravity of
the triangle ≤ a·Area of triangle
Then: subdivide the triangle into 4 equal-area triangles by
joining the midpoints together (R.1)

It is required to find the total number of subtriangles
obtained if condition (R.1) is first applied on triangle "ABC" and
then on all ensuing subtriangles until condition (R.1) is satisfied
for all subtriangles in the final subdivision configuration of
triangle "ABC" (for subdivision "patterns" see figures 3.6-3.14).

An approximate relation for the total number of triangles
obtained will first be derived, after which empirical studies will
be performed and compared to the approximate analytic estimate.

Define the following scalar functions:

ρ(x) = number of triangles/ unit area
-> 1/ρ(x) = area / unit triangle (R.2)

The definitions (R.2) are ambiguous since the "number of
triangles" is not a continuous function of position and area is not
defined "at a point". However, we interpret "number of triangles per unit area" at a point as an average value over a circular region (with some "length scale") centered around that point.

The subdivision process stops for a given triangle when the distance from point "P" to the center of gravity of that triangle is between "1" and "2" times \( \alpha \cdot \sqrt{\text{Area of triangle}} \) (since the subdivision process reduces the area of a triangle by a factor of "4"). Calling "r" the distance from point "P" to a point \( \mathbf{x} \) we have:

\[
r = \frac{1 \to 2 \cdot \alpha \cdot \sqrt{\text{Area of triangle at } \mathbf{x} \text{ / unit triangle}}}{\text{i.e. } r = [1 \to 2] \cdot \alpha \cdot \sqrt{1/\rho}}
\]

(R.3)

Note that "r" is determined only within an interval for any given point \( \mathbf{x} \); when a large number of "r" values is considered, we expect to obtain an average value of "r", that is:

\[
r_{\text{average}} \approx 1.5 \cdot \alpha \cdot \sqrt{1/\rho}
\]

(R.4)

Now using (R.3) we can obtain:

\[
\rho \approx 2.25 \cdot \left( \frac{\alpha}{r} \right)^2
\]

\[
\Rightarrow \text{total number of triangles} = \left( \int \frac{1}{r^2} \cdot dA \right) \cdot 2.25\alpha^2
\]

(R.5)

Note that the total number of triangles vary as the square of "\( \alpha \)" and as some logarithmic function (since \( \frac{1}{r^2} \) has units of "1/area") of the distance "P" to the triangle "ABC".
Now some empirical results will be presented and compared to the above theoretical "model". Consider vertices of a triangle located at \((x, y, z) = (-0.5, 0, 0), (0.5, 0, 0), (0, \sqrt{3}/2, 0)\). The dependence of the "total number of triangles" on \(\alpha\) and on the position \(P\) will now be plotted, and a curve fitting of the empirical results will be performed. Figures R.1 and R.2 are plots which show the empirical dependence of the number of triangles on \(\alpha\) with a "least-square" (straight line) fit of the empirical data when point \(P\) is located at \((x, y, z) = (0, \sqrt{3}/6, 0.01 \cdot \sqrt{\text{area of triangle}})\) and \((0, -0.01 \cdot \sqrt{\text{area of triangle}}, 0)\) respectively. For both cases, the goodness of fit ratio \(R\) turns out to be equal to "1.00" indicating that a straight line fits the data very well. Furthermore, the magnitude of \(\left[ 2.25 \cdot \frac{1}{\sqrt{2r}} \cdot \text{dA} \right]\) for triangle figures R.1 and R.2 are 56.4 and 25.9 respectively which compare well with the "least-squares" slopes of 53.6 and 24.9 respectively. Figures R.3 and R.4 are plots which show the empirical dependence of the number of triangles on the logarithm of some measure of the distance of point \(P\) from the triangle which are taken to be \(z\) and \(-y\) (for figures R.3 and R.4) respectively. The coordinates \((x, y) = (0, \sqrt{3}/6)\) for figure R.3 and \((x, z) = (0, 0)\) for figure R.4 and \(\alpha = 3\) for both figures. "Least-square" (straight) lines are plotted for figures R.3 and R.4 which have goodness of fit ratios \(R\) equal to "1.00" indicating that the total number of triangles "created" does seem to vary with the logarithm of some measure of the distance of point \(P\) to the triangle to be subdivided.
Triangles vs Alpha

\[ y = 4.8982 + 53.5685x \quad R = 1.00 \]

Number of Triangles

alpha^2

Figure R.1
Triangles vs Alpha

\[ y = 0.9738 + 24.8583x \quad R = 1.00 \]

Number of Triangles

\[ \text{empirical data} \]

alpha^2

Figure R.2
Triangles vs $z$

$y = -77.46 + 105.415x$  $R = 1.00$

$-\ln(z)$

Figure R.3
Triangles vs $y$

$y = -40.9217 + 51.0626x \quad R = 1.00$

Figure R.4
Appendix S: Empirical accuracy of triangle subdivision method

In this appendix a (nonexhaustive) empirical study of the accuracy of the "type 1" integration process discussed in section 3.4 will be performed. A more extensive study was performed (by considering different triangle shapes, different "directions of approach" and different singularity types) but was not documented due to the excessive amounts of data generated and the difficulty of processing the data into relevant plots. However, the results presented in this appendix are representative.

The studies to be presented will be performed with respect to integrations over a fixed triangle with vertices \((x,y,z) = (0,0,0), (1,0,0)\) and \((0,1,0)\). Also, three types of integrands will be considered having different "severities" of the singularities;

These are \(\frac{\partial}{\partial z} \left[ \frac{z}{r^3} \right], \frac{\partial}{\partial z} \left[ \frac{z \cdot (x-x_0)}{r^3} \right]\) and \(\frac{\partial}{\partial z} \left[ \frac{z \cdot (x-x_0)^2}{r^3} \right]\), where \(x_0\) is the "x" coordinate of a "fixed" point and "r" is the distance from that "fixed" point to a point on the triangle; the integrands will be referred to as "sources" "1/r^3", "(x-x_0)/r^3" and "(x-x_0)^2/r^3" respectively and the "fixed" point will be referred to as the source point. Finally, two "classes" of operations will be discussed; the first class corresponds to "type 1" integration process (see section 3.4) with a source point not lying on the triangle. The second class of "operations" corresponds to numerically obtaining the limit of a "type 1" integration process for a source point approaching the surface of integration (i.e. being on the triangle).

Regular "type 1" integration process

For each kind of source, two plots are shown; the first set of
plots (figures S.1-3) show the variation of percentage of error with the subdivision parameter "α" (with z=0 and (x,y) given on the figures), and the second set of plots (figures S.4-6) show the variation of the percentage error with location of the source point along the diagonal x=y (with z=0 and "α" given on the figures). For all cases considered in this appendix, the "exact" value of the integrals are obtained using line integrations (refer to main text) with very high line-subdivision parameters and Gaussian integration orders.

The logarithm of the error seems to vary as (figures S.1-3) a decreasing step-like function of "α"; The step-like behavior of the error could be associated with step-like behavior of the triangle subdivision process which was explained in section 3.4. The general trend of the results indicate that the logarithm of the percentage error vary linearly with the subdivision parameter "α". Also, in general, for the same level of accuracy, a lower "α" value is required for a "weaker singularity". The variation of the logarithm of the error is an "erratic" function of the position of the source (this could also be related to the subdivision process); When α=0 (i.e. no subdivision and simple Gaussian quadrature), the accuracy improves as the source gets farther away from the triangle (this can be shown analytically). Also as "α" gets larger, the "erratic" behavior of percentage error versus position decreases and the general level of accuracy improves.

Source points approaching integration surface from above

For each kind of source, two plots are shown; the first set of plots (figures S.7-9) show the variation of the percentage error with the subdivision parameter "α" (x-y=1/3 and "z" given on the figures), and the second set of plots (figures S.10-12) show the variation of the percentage error with location of the source point.
along x-y=1/3 (z varying and "α" given on the figures). The "exact" value of the integrals are obtained using line integration and correspond to the limit z->0.

For a fixed value of "z", the percentage error is seen to (erratically) decrease with increase in "α" until a certain value of "α" is reached after which the error becomes somewhat constant (this is especially apparent for the weakest source "(x-x_o)^2/r^3"); In addition, the percentage error decreases with decrease in "z" (the 
"(x-x_o)/r^3" source does show the trends mentioned above but at still higher values of "α" than are shown in figure S.3). The theoretical results of section 3.4 suggest that the error level (at the "constant error" region, i.e. high "α" values) are proportional to "z" (i.e. 0(z)). Figures S.9 and S.12 for the "(x-x_o)^2/r^3" source does corroborate the previous statement. However, figures S.10 and S.11 suggest that the error for the "1/r^3" and "(x-x_o)/r^3" sources vary as O(z^2) (perhaps due to cancellation effects of the O(z) terms); this observation is obtained by comparing ratios of errors versus ratios of "z" for each of figures S.10 and S.11.

Note that the significance of reaching a constant level of error for high enough values of "α" is that the numerical integration errors become much less important than the errors due to the (necessarily) finite values of "z" used (recall that the integrands of the "[1,(x-x_o),(x-x_o)^2]/r^3" sources have integrands "[1,(x-x_o),(x-x_o)^2]/r^3 - 3z^2*[1,(x-x_o),(x-x_o)^2]/r^5"; setting z=0 in the above integrands give progressively divergent (growing) results for higher and higher values of "α"). Finally, a plot of integration points used (for "type 1" integration process) versus "α" for two location of "z" (shown in figures S.13) indicate the excessive computational effort required for those integrals where
the limit of a source point approaching the integration surface is needed.
Percent Error vs Alpha \((1/r^3)\)

![Graph of Percent Error vs Alpha](graph)

- \(x = y = 0.6\)
- \(x = y = 1.0\)

Figure S.1
Percent Error vs Alpha \([(x-x0)/r^3]\)

Figure S.2
Percent Error vs Alpha \[\left(\frac{(x-x_0)^2}{r^3}\right)\]

\[\begin{align*}
\text{Percent Error} & \\
\text{alpha} & \\
\end{align*}\]

Figure S.3
Percent Error vs Distance ($1/r^3$)

$x = y$

Figure S.4
Percent Error vs Distance \([((x-x_0)/r^3)]\)

![Graph showing percent error vs distance for different values of alpha.]

- Alpha = 0
- Alpha = 1
- Alpha = 2
- Alpha = 3

\(x = y\)

**Figure S.5**
Percent Error vs Distance \([((x-x_0)^2/r^3]\)

\[x = y\]

Figure 3.6
Percent Error vs Alpha ($1/r^3$)

![Graph showing percent error vs alpha with markers for z = 0.01 and z = 0.001.]

alpha

Figure S.7
Percent Error vs Alpha \(\frac{(x-x_0)}{r^3}\)

- \(z = 0.01\)
- \(z = 0.001\)

**Figure S.8**
Percent Error vs Alpha $[(x-x_0)^2/r^3]$
Percent Error vs \( z \rightarrow 0 \) \((1/r^3)\)

Fig. 5.10
Percent Error vs $z \rightarrow 0 \left[ \frac{(x-x_0)}{r^3} \right]$

* $\alpha = 1$
* $\alpha = 2$
* $\alpha = 3$
* $\alpha = 4$

**Figure S.11**
Percent Error vs $z \to 0 \ [(x-x_0)^2/r^3]$
Integration Points vs Alpha

Figure S.13