INTERACTION BETWEEN A TWO-DIMENSIONAL WAKE AND THE FREE SURFACE AT LOW FROUDE NUMBERS

by

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Abstract

The purpose of this work is to study the physical mechanisms involved in the generation of large-scale vortices in the wake of steadily moving, two-dimensional, floating objects.

The problem of vortex formation is considered for two types of objects, a thin hydrofoil and a cylinder. A linear stability analysis of the time-averaged flow is performed. For this purpose the dispersion relation of the problem is studied. The dispersion relation is derived from the inviscid Orr-Sommerfeld equation, subject to the dynamic and the kinematic boundary conditions at the free surface.

It is found that the wake is unstable to small perturbations. However, for low Froude numbers the instability is convective, i.e. all disturbances are carried away as they grow, leaving the wake undisturbed. The instability becomes absolute, leading to a self-sustained oscillation, only at high Froude numbers. It is also found that the wake will respond to the excitation provided by a monochromatic wave with a spatially growing mode, provided that the wave is within two certain ranges of frequencies; one at low frequencies and one at high frequencies. The high-frequency waves disturb mostly the free surface, so it is concluded that viscous two dimensional wakes disturb the free surface primarily with patterns that have short wavelengths.

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Chapter 1

Introduction

The interaction between the viscous wake of a floating object and the ocean surface is a problem of considerable importance for a variety of engineering applications, such as ship stability and resistance. Recently, the interest in this problem has been strengthened, in connection with the SAR (Synthetic Aperture Radar) imaging problem [Hammond 85] where the patterns resulting from the interaction between the viscous wake and the ocean surface are a possible means for ship detection.

The general three-dimensional problem of the interaction between the ship wake and the ocean surface is an extremely difficult problem. In this work the interaction between the wake of a two-dimensional floating object and the ocean surface is considered. The results and the conclusions for this case will give us an insight for the three-dimensional case.

The floating objects examined in this work are a thin NACA003 hydrofoil, representative of streamlined bodies, and a cylinder, representative of bluff bodies. It is found that the wakes behind these two types of objects have similar stability properties.

The main questions of interest are:

a) The possible appearance of "spontaneous" flow patterns in the wake as a result of the wake instability.

b) The response of the wake to the excitation provided by waves in the ocean.

The first question can be answered by studying the physical character of the instability, i.e. whether the instability is absolute or convective. The instability is called absolute, if any randomly excited disturbance grows in time at any location in space. The instability is called convective, if the disturbances are convected away from their point of
excitation, leading eventually to decaying motions at every specific point in space. The method for the distinction between absolute and convective instability was first developed for the study of space-time evolution of plasma instabilities [Bers 83], and only recently applied to fluid problems [Triantafyllou 86].

In the present thesis, it is found that, for low Froude numbers, the instability is always of the convective type; an absolute instability exists in the wake of floating objects only for high Froude numbers which are usually of limited practical interest. From this we can conclude that the presence of the free surface has a stabilizing effect on the wake, since it transforms into convective the absolute instability that is known to exist in wakes in infinite fluid at high Reynolds numbers.

For the second question, since the wake is convectively unstable, it will respond to the excitation provided by a monochromatic wave with a spatially growing mode, provided that the wave is within a certain range of frequencies. It is found that there are two distinct such ranges of frequencies; one at low frequencies and one at high frequencies, separated by a range of frequencies where the response is stable. It is also found that the high-frequency waves, owing to their high Froude numbers (based on their wavelengths), disturb mostly the free surface. From this we can conclude that viscous two dimensional wakes disturb the free surface primarily with patterns that have short wavelengths.
Chapter 2

Stability Analysis in the Wake of a Floating Two-Dimensional Object

The formulation of the stability problem in the wake behind a floating two-dimensional object is described in this chapter.

2.1 Absolute-Convective Instability

Within linear theory, the distinction between absolute and convective instability can be made by studying the dispersion relation of the medium [Bers 83]. The response of the medium, $G(x,t)$, to an impulsive excitation $\delta(x)\delta(t)$ can be written as an inverse Fourier-Laplace integral of the form:

$$G(x,t) = \int_{L} d\omega \frac{e^{-i\omega t}}{2\pi} \int_{F} dk \frac{e^{ikx}}{2\pi} \frac{1}{D(k,\omega)}$$

(2.1)

where $D(k,\omega)$ is the dispersion relation of the medium, $\omega=\omega_r+i\omega_i$ is the complex frequency, $k=k_r+ik_i$ is the complex wave number, $\omega_i$ is the temporal growth rate and $k_i$ is the spatial growth rate. Finally, $L$ and $F$ are appropriate integration contours in the $\omega$-plane and the $k$-plane respectively. In general, evaluation of the double integral in (2.1) is difficult. For the purpose of distinguishing between absolute and convective instability, however, only the behavior of (2.1) as $t \to \infty$ is required. It can be shown [Bers 83] that the asymptotic behavior of $G(x,t)$ is determined by the "pinch-point" type of double root, $(\omega_0, k_0)$, of the dispersion relation with the highest imaginary part of $\omega_0$. Such a double root of the dispersion relation satisfies the equations:

$$D(\omega_0,k_0) = 0 \quad \quad \frac{\partial D(\omega_0,k_0)}{\partial k} = 0$$

(2.2)
plus the "pinching" condition, which can be mathematically expressed as a requirement that
the mapping in the k-plane of the $\omega_\tau=$constant line passing through $\omega_0$ intersects the k-real
axis an odd number of times [Bers 83]. We will call such a point a "critical point" of the
dispersion relation. When the $\omega_0$ of the critical point has a positive imaginary part, the
response of the medium is given by:

$$G(x,t) = e^{ik_0 x} e^{-i\omega_0 t} \frac{e^{-i\omega_0 t}}{t^{1/2}}$$  (2.3)

and consequently the instability is absolute. When the $\omega_0$ of the critical point has a
negative imaginary part, the instability is convective. In the case of an absolute instability,
the real part of $\omega_0$ gives the preferred frequency of the instability, and the real part of $k_0$ the
preferred wave number.

The double roots of the dispersion relation satisfy the equations (2.2). Usually the
dispersion relation is a complicated expression and it is not an easy task to find its double
roots by directly solving the equations (2.2). So another method is used for the calculation
of the critical point of the dispersion relation [Triantafyllou 86]. The dispersion relation
can be solved to yield the frequency as a function of the wave number, in the form $\omega=\omega(k)$.
From the equations (2.2) we have that:

$$\omega_0 = \omega(k_0) \quad \quad \frac{d\omega(k_0)}{dk} = 0$$  (2.4)

For an analytic function $\omega(k)$ the second equation of (2.4) implies that in the neighborhood
of $k_0$, $\omega$ behaves like a quadratic function of $k$:

$$\omega(k) - \omega(k_0) - \frac{1}{2} \frac{d^2 \omega(k_0)}{dk^2} (k-k_0)^2$$  (2.5)

So an orthogonal grid in the k-plane is mapped on the $\omega$-plane in the typical fashion of a
quadratic function. In Figure 2-1 the mapping \( \omega = k^2 \) on the \( \omega \)-plane is shown, for which \( k_0 = \omega_0 = 0 \). In the general case, \( k_\omega = \text{constant} \) lines will group around the critical point in a similar manner. Then the double roots of the dispersion relation can be identified from this similarity. After that, it remains to verify that these double roots are indeed critical points of the dispersion relation. This can readily be done by observing the topological form of the mappings in the \( \omega \)-plane. In Figure 4-9, it can be seen that the \( \omega_\perp = \text{constant} \) line passing through \( \omega_0 \) intercepts the mapping of the \( k \)-real axis into the \( \omega \)-plane only once. Consequently, the inverse mapping through the dispersion relation of this \( \omega_\perp = \text{constant} \) line into the \( k \)-plane will intercept the \( k \)-real axis only once, satisfying the pinching requirement [Triantafyllou 87]. For example see Figure 4-9, where the double root is covered by the map of the \( k \)-real axis only once.

2.2 Dispersion Relation of the Wake of a Floating Two-Dimensional Object

We consider the interaction between the wake of a floating two-dimensional object and the ocean surface. It is assumed that the waves generated by the floating object in its wake are negligible. This can be justified if the Froude number \( Fr_\perp = U_\infty \sqrt{g l} \) is low, where \( U_\infty \) is the free-flow velocity and \( l \) is the length of the object. Then the average position of the ocean surface is flat and the average velocity profiles are half the velocity profiles in the wake of the "double body".

We choose a system of coordinates fixed to the body: the \( x \)-axis parallel and the \( y \)-axis normal to the oncoming flow. The wake of a floating object is a slowly diverging flow. For the stability analysis of the wake, it is assumed that locally the properties of the wake are adequately represented by the stability properties of a parallel flow having the same average velocity profile. So the analysis of the wake is decomposed into several equivalent parallel flow problems. For each parallel flow problem the dispersion relation consists of the Rayleigh equation [Drazin 81]:
\[
(U - \frac{\omega}{k}) \left( \frac{d^2 \Phi}{dy^2} - k^2 \Phi \right) - \frac{d^2 U}{dy^2} \Phi = 0
\]  
(2.6)

where \( \omega \) is the complex frequency of the unsteady flow, \( k \) is the complex wave number of the unsteady flow, \( U(y) \) is the mean flow velocity profile and \( \Phi(y) \) is the stream function of the unsteady flow. The pressure field \( P(y) \) due to the unsteady flow can be expressed in terms of the stream function as follows:

\[
P = \frac{dU}{dy} \Phi - (U - \frac{\omega}{k}) \frac{d\Phi}{dy}
\]  
(2.7)

In (2.6) and (2.7) all parameters are non-dimensional. Co-ordinates \( x \) and \( y \) are non-dimensionalized with respect to the half-width \( b \) of the mean flow profile, velocities with respect to the free-flow velocity \( U_\infty \) and time with respect to the quantity \( b/U_\infty \). Finally, the pressure is non-dimensionalized with respect to the quantity \( \rho U_\infty^2 \), where \( \rho \) is the fluid density, and the stream function with respect to the quantity \( bU_\infty \).

In Figure 2-2 is shown the general geometry of our problem and the mean flow velocity profile at some point \( x^* \) behind the body. The boundary conditions are the kinematic and the dynamic boundary condition at the free surface and the condition for a decaying solution as \( y \to -\infty \). The linearized, non-dimensional, kinematic and dynamic boundary conditions at the free surface are:

\[
v_1 = \frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} \quad \text{at} \quad y=0
\]  
(2.8)

\[
\eta = P_1 \frac{U_\infty^2}{gb} \quad \text{at} \quad y=0
\]  
(2.9)

where \( v_1(x,y,t) = v(y)e^{i(kx - \omega t)} \) is the normal velocity of the unsteady flow, \( \eta(x,t) = Ae^{i(kx - \omega t)} \)
is the free-surface elevation due to the unsteady flow and \( P_1(x,y,t) = P(y)e^{i(\kappa x - \omega t)} \) is the pressure due to the unsteady flow. Let (2.8), (2.9) together with (2.7) become:

\[
k \Phi = (\omega - kU)A \quad \text{at} \quad y = 0 \quad (2.10)
\]

\[
\frac{A}{Fr^2} = \frac{dU}{dy} \Phi - \left(U - \frac{\omega}{k}\right) \frac{d\Phi}{dy} \quad \text{at} \quad y = 0
\]

where \( Fr = U_\infty / \sqrt{gb} \). This is the general form of the boundary conditions at the free surface. It is considered that \( dU/dy = 0 \) at the free surface, so that the physical requirement of zero mean shear at the ocean surface is satisfied. Then the boundary conditions become:

\[
k \Phi = (\omega - kU)A \quad \text{at} \quad y = 0 \quad (2.12)
\]

\[
\frac{kA}{Fr^2} = (\omega - kU) \frac{d\Phi}{dy} \quad \text{at} \quad y = 0
\]

The condition as \( y \to -\infty \) is:

\[
\Phi \to 0 \quad \text{as} \quad y \to -\infty \quad (2.14)
\]

We note that in equation (2.13) the Froude number is based on the half width \( b \) of the wake. So for slender bodies \( (b << l) \), the assumption that \( Fr_l \) is low does not necessarily mean that \( Fr \) is also low. For \( Fr=0 \) equations (2.12) and (2.13) give \( A = 0 \) and \( \Phi(0) = 0 \). This situation is equivalent to the unbounded flow around the double body of the submerged part of the floating object with the condition that \( \Phi(y) \) is an antisymmetric function of \( y \). This is called the antisymmetric mode of the instability in the wake of an unbounded flow. For \( Fr=\infty \) equation (2.13) gives \( d\Phi/dy = 0 \). This situation corresponds to the symmetric mode of the instability in the wake of an unbounded flow. As it is shown in [Triantafyllou 86], the antisymmetric mode is convectively unstable, while the symmetric mode is absolutely
unstable. Given that the former corresponds to Fr=0 and the latter to Fr=∞, a transition from convective to absolute instability is expected at some finite value of the Froude number.
Figure 2-1: Plot of $\omega=k^2$ in the $\omega$-plane. The solid lines are the mapping of the lines with $k_1=$constant, while the interrupted lines are the mapping of the lines with $k_r=$constant.
Figure 2-2: The floating object, and the characteristic form of the average velocity profiles in its wake.
Chapter 3

Wake Model

In this chapter we consider the instability of a wake with a piecewise-linear velocity profile. Such an approximation for the velocity profile contains the basic features determining the stability properties of the wake; at the same time, the calculations are simple enough to allow a parametric investigation of the problem.

3.1 The Piecewise-Linear Velocity Profile

The piecewise-linear velocity profile considered is shown in Figure 3-1. It represents a fitting for an actual velocity profile in the wake of a floating object. The fitting is performed such that the piecewise-linear velocity profile has the same maximum velocity deficit and the same maximum shear as the actual velocity profile. In Figures 3-2 and 3-3, the piecewise-linear approximations to the velocity profiles in the near-wakes of a thin hydrofoil (at x*/l=0.003) and a cylinder (at x*/d=1) are shown.

In Figure 3-1, $U_\infty$ is the free-flow velocity, $U_0$ is the velocity at $y=0$ and $b_o$ is the half-width of the wake. Note that:

$$b_o = \frac{H_o + h_o}{2} \quad (3.1)$$

and the Froude number is defined to be:

$$Fr = \frac{U_\infty}{\sqrt{g b_o}} \quad (3.2)$$

From Figure 3-2, the curve-fitting for the hydrofoil wake gives: $H_0=1.75b$, $h_0=0.25b$, 

\[ U_o = 0.0012U_\infty, \text{ where } b \text{ is the half-width of the actual velocity profile. Note that } b_o = b \text{ since the definition is the same for both, i.e. they are equal to the } y\text{-coordinate where } (U_\infty - U)/(U_\infty - U_o) = 0.5. \text{ From Figure 3-3, the curve-fitting for the cylinder wake gives: } H_o = 0.64d, h_o = 0.18d, U_o = 0.018U_\infty, \text{ where } d \text{ is the diameter of the cylinder.} \]

### 3.2 Dispersion relation

We introduce the non-dimensional parameters \( H, h, u_o, \omega, k \) and \( \Phi \), defined as follows:

\[
H = \frac{H_o}{b_o}, \quad h = \frac{h_o}{b_o}, \quad u_o = \frac{U_o}{U_\infty}, \quad \omega = \frac{\omega\cdot b_o}{U_\infty}, \quad k = k\cdot b_o, \quad \Phi = \frac{\Phi\cdot b_o}{U_\infty}
\]

(3.3)

where \( \omega^* \) is the dimensional angular frequency, \( k^* \) is the dimensional wave number and \( \Phi^* \) is the dimensional stream function.

When \( U(y) \) is piecewise-linear, Rayleigh's equation (2.6) is reduced to:

\[
\Phi'' - k^2 \Phi = 0
\]

(3.4)

where the primes denote derivatives with respect to \( y \). Equation (3.4) has a general solution of the form:

\[
\Phi(y) = C\cdot \cosh(ky) + S\cdot \sinh(ky)
\]

(3.5)

In addition, we have to satisfy continuity conditions at the nodes 2 and 3, where the slope of the average velocity is discontinuous (Figure 3-1). From Section 23 of [Drazin 81], these conditions are:

\[
\Delta \left( (U - \frac{\omega}{k})\Phi' - U'\Phi \right) = 0
\]

(3.6)

\[
\Delta (\Phi) = 0
\]

(3.7)
The first one is the continuity of pressure and the second one is the continuity of the normal velocity.

The boundary conditions, as mentioned in Chapter 2, are the free-surface boundary conditions at \( y=0 \) and the condition as \( y \to -\infty \). The non-dimensionalized free-surface boundary conditions are:

\[
k \Phi = (\omega - kU)A \quad (3.8)
\]

\[
\frac{kA}{Fr^2} = (\omega - kU)\Phi' \quad (3.9)
\]

satisfied at \( y=0 \). The other boundary condition is:

\[
\Phi \to 0 \quad \text{as} \quad y \to -\infty \quad (3.10)
\]

Let \( \Phi_1, \Phi_2, \Phi_3 \) be the values of the stream function at the nodes 1, 2, 3 respectively. Then we write equation (3.5) in the form:

a) Between points 1 and 2:

\[
\Phi(y) = \frac{\Phi_1 \sinh(k(y+h)) - \Phi_2 \sinh(ky)}{\sinh(kh)} \quad (3.11)
\]

b) Between points 2 and 3:

\[
\Phi(y) = \frac{\Phi_2 \sinh(k(y+H)) - \Phi_3 \sinh(k(y+h))}{\sinh(k(H-h))} \quad (3.12)
\]

c) Beyond point 3:

\[
\Phi(y) = \Phi_3 e^{k(y+H)} \quad (3.13)
\]
Thus the continuity condition (3.7) is satisfied at nodes 2 and 3, and it remains to apply the dynamic continuity condition (3.6) and the boundary conditions at the free surface.

The free-surface boundary conditions become:

\[ u_0 k A + k \Phi_1 = \omega A \]  
(3.14)

\[ k A + Fr^2 u_0 k^2 \frac{cosh(kh)}{sinh(kh)} \Phi_1 - Fr^2 u_0 k^2 \frac{1}{sinh(kh)} \Phi_2 = \omega (Fr^2 k \frac{cosh(kh)}{sinh(kh)} \Phi_1 - Fr^2 k \frac{1}{sinh(kh)} \Phi_2) \]  
(3.15)

The continuity condition at node 2 (y=-h) becomes:

\[ \frac{u_0 k}{sinh(kh)} \Phi_1 + (\Delta - u_0 k \frac{cosh(kh)}{sinh(kh)} - u_0 k \frac{cosh(k(H-h))}{sinh(k(H-h))}) \Phi_2 + \frac{u_0 k}{sinh(k(H-h))} \Phi_3 = \omega (\frac{1}{sinh(kh)} \Phi_1 - (\frac{cosh(kh)}{sinh(kh)} + \frac{cosh(k(H-h))}{sinh(k(H-h))}) \Phi_2 + \frac{1}{sinh(k(H-h))} \Phi_3) \]  
(3.16)

where \( \Delta = (u_0 - 1)/(H-h) \) is the non-dimensional derivative of the velocity profile between points 2 and 3.

Finally, the continuity condition at node 3 (y=-H) becomes:

\[ \frac{k}{sinh(k(H-h))} \Phi_2 - (\Delta + k \frac{cosh(k(H-h))}{sinh(k(H-h))} + k) \Phi_3 = \]  
(3.17)

\[ \omega (\frac{1}{sinh(k(H-h))} \Phi_2 - (\frac{cosh(k(H-h))}{sinh(k(H-h))} + 1) \Phi_3) \]

The equations (3.14), (3.15), (3.16) and (3.17) define a generalized eigenvalue problem of the form:
\[ \bar{A}(k)\Phi = \omega \bar{B}(k)\Phi \]  

(3.18)

where \( \omega \) is the complex frequency, the complex matrices \( \bar{A}, \bar{B} \) depend on the complex wave number \( k \) and \( \Phi \) is the eigenvector:

\[ \Phi^T = [ A \; \Phi_1 \; \Phi_2 \; \Phi_3 ] \]  

(3.19)

Since the matrices \( \bar{A}, \bar{B} \) are of order 4, little computational effort is required to solve the eigenvalue problem.

3.3 Results for the Wake Model

The temporal instability curves for the near-wake of the hydrofoil and the cylinder are shown in Figures 3-4 and 3-5 respectively. For finite Froude numbers, we obtain always two distinct unstable branches, one for low wave numbers and one for high wave numbers. We will call the first Branch I and the second Branch II. For Fr=0, we obtain only Branch I. This is identical with the anti-symmetric mode of the unbounded flow, as we have already seen, and it is located near the origin, \( \omega_r=0 \). For small Froude numbers a second branch appears at high \( \omega_r \). As the Froude number increases, Branch I decreases in magnitude (has lower \( \omega_i \)), while Branch II increases in magnitude (has higher \( \omega_i \)) and moves closer to the origin. When Fr=\( \infty \), Branch II becomes identical with the symmetric mode of the unbounded flow.

Note that each branch has nonzero \( \omega_i \) for a small range of \( \omega_r \), and \( \omega_i=0 \) for all other \( \omega_r \). This can be shown clearly in Figure 3-6, where the eigenvalues of (3.18) have been plotted for a very small \( k_i \) (\( k_i=-0.05 \)) of the hydrofoil wake and at \( Fr=1.5 \). The fourth eigenvalue has not been plotted since it contains only negative \( \omega_r \)'s. The two branches correspond to the two unstable eigenvalues, while the third one is stable for all \( \omega_r \)'s.
Each of the two branches I and II has one critical point. This is shown in Figures 3-7 and 3-8 for Fr=0.5 and Fr=1.5 respectively. More specifically the critical point for Branch I is:

\[
Fr=0.5 \quad k_r=10.0 \quad k_i=-6.2 \quad \omega_r=0.347 \quad \omega_i=-0.00753 \tag{3.20}
\]

\[
Fr=1.5 \quad k_r=10.0 \quad k_i=-5.7 \quad \omega_r=0.347 \quad \omega_i=-0.00746 \tag{3.21}
\]

and for Branch II:

\[
Fr=0.5 \quad k_r=0.15 \quad k_i=-1.25 \quad \omega_r=2.137 \quad \omega_i=-1.134 \tag{3.22}
\]

\[
Fr=1.5 \quad k_r=0.70 \quad k_i=-1.30 \quad \omega_r=0.981 \quad \omega_i=-0.222 \tag{3.23}
\]

It will be shown in the next chapter that the quantitative prediction of the wake model for the critical point of Branch II is very good, while for the critical point of Branch I the prediction of the wake model is only qualitatively correct. So, for low Froude numbers the instability is convective, since both branches have critical points with negative \( \omega_i \). The consequences of that are:

a) All arbitrarily excited disturbances are convected away, leaving the wake, eventually, undisturbed. Therefore there is no "spontaneous" instability wave.

b) Since the wake is unstable, a persistent oscillatory excitation causes a spatially growing response.

The spatial instability curves (\( k_i \) versus \( k_r \) with \( \omega_i=0 \)) for the near-wake of the hydrofoil will also have two branches, one resulting from Branch I and one resulting from Branch II. See Figure 3-9 where this curve has been calculated for Fr=1.5. As for the temporal instability cases, there exist two distinct regions of spatial amplification, one at low wave numbers and one at high wave numbers.

Next, we investigate the evolution of the instability of the wake downstream. To this purpose, we consider the velocity profiles in the wake of the hydrofoil at the positions
x*/l=0.1 and x*/l=2, downstream from the trailing edge. The velocity profiles and their fitting is shown in Figures 3-10 and 3-11. At x*/l=0.1 the curve-fitting gives: \( H_0 = 1.75b_{0.1}, \) \( h_0 = 0.25b_{0.1}, \) \( U_0 = 0.23U_\infty, \) where \( b_{0.1} \) is the half-width of the wake at x*/l=0.1. At x*/l=2 the curve-fitting gives: \( H_0 = 1.8b_2, \) \( h_0 = 0.2b_2, \) \( U_0 = 0.956U_\infty, \) where \( b_2 \) is the half-width of the wake at x*/l=2. From [Mattingly 72] we have that \( b_{0.1}/b_{0.003} = 1.013 \) and \( b_2/b_{0.003} = 2.144. \) The stability results at x*/l=0.1 and x*/l=2 will be presented in non-dimensionalized form, based on \( b_{0.003}. \)

The temporal instability curves of the profiles at x*/l=0.1 and x*/l=2, and for various Froude numbers, can be seen in Figures 3-12 and 3-13. In comparison with Figure 3-4, we note that Branch II decays very rapidly in magnitude and moves to higher frequencies and wave numbers as we move downstream. The spatial instability curves follow the same trend. Their typical form is shown in Figure 3-14 where we have used \( k_r \) instead of \( \omega_r. \) Since the instability in the wake of the hydrofoil is convective for low Froude numbers, the wake will respond to an external excitation with a spatially growing mode, only if the wave number of the excitation is in the range A to B, or C to D. These wave number ranges depend on the Froude number and the distance x*/l downstream from the trailing edge, as we have already seen. For constant x*/l and increasing Froude number, the range AB decreases, while the range CD increases and moves to lower wave numbers. The magnitude of Branch I decreases, whereas the magnitude of Branch II increases. For constant Froude number and increasing x*/l, both ranges AB and CD decrease, and range CD moves to higher wave numbers. The magnitude of Branch I decreases slowly, while the amplitude of Branch II decreases very fast.

Finally, we examine the effect of the thickness of the shear layer on the wake instability. Thus we keep the velocity deficit constant and vary the shear layer thickness, as shown in Figure 3-15. The non-dimensional thickness \( \delta \) of the shear layer is defined to be:

\[
\delta = H - h
\]  
\[ \text{(3.24)} \]
In Figure 3-16, the temporal instability curves are shown for Froude number 1.5 and for various shear-layer thicknesses. As the thickness of the shear layer decreases, both Branch I and Branch II increase in magnitude (they have higher $\omega_i$), and their frequency ranges also increase. When the thickness reaches a value approximately equal to 0.68, the frequency ranges of the two branches start to overlap with each other. From there on only one unstable branch exists. We repeat the same calculations for Froude number 0.5 and the results are shown in Figure 3-17. Here Branch II is much smaller in magnitude than Branch I (it is so small that it does not show in the figure), and the shear layer must become very thin in order for the magnitude of Branch II to become relatively large. Thus for constant Froude number and decreasing shear-layer thickness both Branches increase in magnitude. This shows that vortex sheets ($\delta=0$) are an inadequate model for the stability properties of the wake, since they suppress Branch II. In order to properly study a wake free-surface interaction, finite-thickness layers must be used.
Figure 3-1: The piece-wise linear velocity profile.
Figure 3-2: Velocity profile at $x^*l=0.003$ behind the hydrofoil.

Figure 3-3: Velocity profile at $x^*/d=1$ behind the cylinder.
Figure 3-4: Temporal instability curves for the hydrofoil ($x^*/l=0.003$).
Curves 1-10: Branch I ($Fr=0, 0.5, 0.75, 1, 1.5, 2, 2.5, 3, 5, 10$).
Curves 11-20: Branch II ($Fr=0.5, 0.75, 1, 1.5, 2, 2.5, 3, 5, 10, \infty$).

Figure 3-5: Temporal instability curves for the cylinder ($x^*/d=1$).
Curves 1-9: Branch I ($Fr=0, 0.5, 0.75, 1, 1.5, 2, 3, 5, 10$).
Curves 10-18: Branch II ($Fr=0.5, 0.75, 1, 1.5, 2, 3, 5, 10, \infty$).
Figure 3-6: The eigenvalues for the hydrofoil \((x^*/l=0.003)\) at Fr=1.5, and for \(k_1=-0.05\).
Figure 3-7: Instability curves for the hydrofoil \((x*/l=0.003)\) at \(Fr=0.5\).
Curves 1-7: Branch I \((k_i=0, -0.5, -1, -2, -4, -6, -6.2)\).
Curves 8-13: Branch II \((k_i=0, -0.2, -0.5, -1, -1.2, -1.25)\).

Figure 3-8: Instability curves for the hydrofoil \((x*/l=0.003)\) at \(Fr=1.5\).
Curves 1-7: Branch I \((k_i=0, -0.5, -1, -3, -5, -5.6, -5.7)\).
Curves 8-12: Branch II \((k_i=0, -0.5, -1, -1.2, -1.3)\).
Figure 3-9: Spatial instability curve for the hydrofoil ($x^*/l=0.003$) at $Fr=1.5$. 
Figure 3-10: Velocity profile at $x*/l=0.1$ behind the hydrofoil.

Figure 3-11: Velocity profile at $x*/l=2$ behind the hydrofoil.
Figure 3-12: Temporal instability curves for the hydrofoil ($x^*$/l=0.1).
Curves 1-9: Branch I ($Fr=0, 0.5, 0.75, 1, 1.5, 2, 3, 5, 10$).
Curves 10-17: Branch II ($Fr=0.75, 1, 1.5, 2, 3, 5, 10, \infty$).

Figure 3-13: Temporal instability curves for the hydrofoil ($x^*$/l=2).
Curves 1-3: Branch I ($Fr=0, 10, 100$).
Curves 4-5: Branch II ($Fr=100, 1000$).
Figure 3-14: The typical form of the spatial instability curves.
Figure 3-15: Piece-wise linear velocity profiles with the same velocity deficit and various shear-layer thicknesses.
Figure 3-16: Temporal instability curves for the velocity profiles of Figure 3-15 at Fr=1.5. Curves 1-8: δ=0.2, 0.6, 0.66, 0.68, 0.8, 1, 1.4, 1.8.

Figure 3-17: Temporal instability curves for the velocity profiles of Figure 3-15 at Fr=0.5. Curves 1-8: δ=0.1, 0.2, 0.3, 0.4, 0.6, 0.8, 1, 1.2.
Chapter 4

Thin-Hydrofoil Wake

The stability analysis results in the wake of a floating, symmetric, thin hydrofoil NACA 0003 are given in this chapter. The hydrofoil (Figure 4-1) is assumed to be half-submerged, so that the velocity profile is half of the profile developing in infinite fluid, for which detailed measurements are available [Mattingly 72].

4.1 The Wake Velocity Profiles

The velocity profiles in the wake of the hydrofoil in infinite fluid are fitted with the following expression (up to a distance $x^*/l=0.2$ behind the trailing edge of the hydrofoil):

$$\frac{U^*(y^*)-U_\infty}{U_c-U_\infty} = \frac{1}{cosh^2(\sigma y^*/b)}$$

(4.1)

where $U_c$ is the center-line velocity, $b$ is the half-width of the wake and $\sigma=0.88137$, such that $(U^*(y^*)-U_\infty)/(U_c-U_\infty)=0.5$ at $y^*=b$. So, the non-dimensional velocity profile in the near-wake of the floating hydrofoil is:

$$U(y) = 1+(U_c-1)\frac{1}{cosh^2(\sigma y)} \quad y \leq 0$$

(4.2)

where $U(y)=U^*(y^*)/U_\infty$ and $U_c=U_c/U_\infty$. For the near-wake, the velocity profiles at $x^*/l=0.003$ and $x^*/l=0.1$ are reported [Mattingly 72] to have:

$$\begin{align*}
x^*/l & = 0.003 & b/l & = 0.010417 & u_c & = 0.0012 \\
x^*/l & = 0.1 & b/l & = 0.010552 & u_c & = 0.23
\end{align*}$$

(4.3) (4.4)

The corresponding velocity profiles are shown in Figures 4-2 and 4-3.
In the far-wake of the unbounded flow around the hydrofoil, the velocity profiles can be approximated with the similarity solution of Schlichting [Schlichting 60]. These velocity profiles are:

\[
\frac{U_{\infty} - U^*(y^*)}{U_{\infty}} = \frac{\sqrt{10}}{3.24} \left( \frac{x^*}{C_D l} \right)^{-1/2} \left( 1 - \left( \frac{y^*}{b} \right)^{3/2} \right)^2 \quad \text{for } \left| \frac{y^*}{b} \right| \leq 2.2675
\]

\[
U_{\infty} - U^*(y^*) = 0 \quad \text{for } \left| \frac{y^*}{b} \right| \geq 2.2675
\]

(4.5)

where \( C_D \) is the drag coefficient of the hydrofoil. For NACA 0003, \( C_D = 0.004 \). The half-width of the wake satisfies the similarity expression:

\[
\frac{b}{l} = \frac{1}{4} \left( \frac{C_D x^*}{l} \right)^{1/2}
\]

(4.6)

Note that the expressions (4.5) and (4.6) are correct for \( x^*/C_D l \geq 50 \). So, the non-dimensional velocity profile in the far-wake of the floating hydrofoil is:

\[
U(y) = 1 - \frac{\sqrt{10}}{3.24} \left( \frac{x^*}{C_D l} \right)^{-1/2} \left( 1 - \left( \frac{0.441}{y^{3/2}} \right)^{3/2} \right)^2 \quad \text{for } 0 \geq y \geq -2.2675
\]

\[
U(y) = 1 \quad \text{for } y \leq -2.2675
\]

(4.7)

The velocity profile at \( x^*/l = 2 \) has \( b/l = 0.02245 \) and \( x^*/C_D l = 500 \). The corresponding velocity profile is shown in Figure 4-4.

4.2 Numerical Treatment of the Dispersion Relation

The dispersion relation (2.6) together with the boundary conditions (2.12), (2.13) and (2.14) can be brought in the form of a generalized eigenvalue problem. Thus, for a given \( k \) we can obtain \( \omega \) as an eigenvalue and \( \Phi \) as an eigenfunction.
A 3-point finite-difference scheme is used to approximate the derivatives of $\Phi(y)$. The finite-difference grid is shown in Figure 4-5. At each point the finite-difference approximation of (2.6) is:

\[
U(y_i)k\Phi_{i-1}-k(2U(y_i)+h^2\frac{d^2U(y_i)}{dy^2}+h^2k^2U(y_i))\Phi_i+U(y_i)k\Phi_{i+1} = \\
\omega(\Phi_{i-1}-(2+k^2h^2)\Phi_i+\Phi_{i+1}) \quad i = 1, \ldots N
\]

(4.8)

where $h=y_{i+1}-y_i$ for every $i$. For a sufficiently low $y$, the velocity profile becomes $U(y)=1$ and $a^2U/\text{dy}^2=0$. So, (2.6) subject to (2.14) has a solution of the form:

\[
\Phi(y) = C_1e^{ky}
\]

(4.9)

Thus we have the truncation condition:

\[
\Phi(y+h) = \Phi(y)e^{kh} \quad \text{for} \quad y < -b
\]

(4.10)

The finite-difference forms of the boundary conditions (2.12), (2.13) at the free surface are:

\[
kU(0)A+k\Phi_1 = \omega A
\]

(4.11)

\[
kA - \frac{Fr^2kU(0)}{2h} \Phi_0 + \frac{Fr^2kU(0)}{2h} \Phi_2 = \omega \left( -\frac{Fr^2}{2h} \Phi_0 + \frac{Fr^2}{2h} \Phi_2 \right)
\]

(4.12)

where $\Phi_0$ is the value of the stream function at an artificial point introduced at a distance $h$ above the free surface.

The expressions (4.8), (4.11) and (4.12) together with the truncation condition (4.10) are combined to give us a generalized eigenvalue problem:
$$\overline{A}(k)\overline{\Phi} = \omega \overline{B}(k)\overline{\Phi} \quad (4.13)$$

where $\overline{A}(k)$, $\overline{B}(k)$ are complex matrices and functions of $k$ and $\overline{\Phi}$ is the eigenvector:

$$\overline{\Phi}^T = [ A \quad \Phi_0 \quad \Phi_1 \quad \Phi_2 \quad \cdots \quad \Phi_N ] \quad (4.14)$$

The eigenvalues and eigenvectors of (4.13) are determined numerically using the Q-Z algorithm [NAG 86].

4.3 Results for the Thin-Hydrofoil Wake

We first consider the wake profile at distance $x^*/l=0.003$. For this velocity profile, the instability curves for various Froude numbers are shown in Figures 4-6 to 4-12. From these figures, it can be verified that the actual velocity profile has the same instability properties as the wake model used in the previous section. Again, for finite Froude numbers there are two unstable branches, I and II. For Fr=0 there is only Branch I and it is identical with the antisymmetric mode in infinite fluid. For Fr=∞ there is only Branch II and it is identical with the symmetric mode in infinite fluid. The spatial instability curve for Fr=1.5 is shown in Figure 4-13.

For every finite Froude number, each Branch has its own critical point. The instability is characterized by the Branch with the higher $\omega_i$. The dependence of the two critical points on the value of the Froude number is given in Table 4-1. For small Froude numbers, Branch I has a critical point with higher $\omega_i$ than the critical point of Branch II. This can be seen in Figure 4-7 for Fr=0.5, where the $k_i=$constant curves of Branch II have gone well below the critical point of Branch I without forming any critical point yet. So for small Froude numbers the instability is characterized by Branch I and it is of the convective type.
Table 4-I: The critical points for the hydrofoil (x*/l=0.003) at various Froude numbers.

<table>
<thead>
<tr>
<th>Fr</th>
<th>Branch</th>
<th>k_r</th>
<th>k_i</th>
<th>\omega_r</th>
<th>\omega_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Anti-sym.</td>
<td>1.1</td>
<td>-2.60</td>
<td>0.217</td>
<td>-0.355</td>
</tr>
<tr>
<td>0.5</td>
<td>Branch I</td>
<td>1.1</td>
<td>-2.60</td>
<td>0.216</td>
<td>-0.361</td>
</tr>
<tr>
<td>1.5</td>
<td>Branch I</td>
<td>0.7</td>
<td>-2.70</td>
<td>0.173</td>
<td>-0.452</td>
</tr>
<tr>
<td>1.5</td>
<td>Branch II</td>
<td>0.3</td>
<td>-1.70</td>
<td>0.984</td>
<td>-0.310</td>
</tr>
<tr>
<td>2.5</td>
<td>Branch II</td>
<td>0.8</td>
<td>-1.50</td>
<td>0.733</td>
<td>-0.108</td>
</tr>
<tr>
<td>3.5</td>
<td>Branch II</td>
<td>1.2</td>
<td>-1.40</td>
<td>0.630</td>
<td>-0.031</td>
</tr>
<tr>
<td>4.5</td>
<td>Branch II</td>
<td>1.2</td>
<td>-1.20</td>
<td>0.577</td>
<td>0.0007</td>
</tr>
<tr>
<td>5.5</td>
<td>Branch II</td>
<td>1.2</td>
<td>-1.00</td>
<td>0.549</td>
<td>0.0158</td>
</tr>
<tr>
<td>\infty</td>
<td>Symmetric</td>
<td>1.1</td>
<td>-0.75</td>
<td>0.492</td>
<td>0.0450</td>
</tr>
</tbody>
</table>

As Froude number increases, Branch II provides critical points with increasingly higher \omega_i. On the other hand the \omega_i obtained from the critical point of Branch I decreases as Froude number increases, and at Fr=1.3 the two branches give critical points with comparable \omega_i's. The instability is still convective at that point. As Froude number increases further, Branch II becomes the more unstable mode. For Fr=1.5 (Figure 4-8) the critical points of both branches can be seen, while for Fr=2.5, Fr=3.5 and Fr=4.5 (Figures 4-9, 4-10 and 4-11 respectively) the critical point of Branch I is well below the critical point of Branch II. For Froude number approximately 4.5 (corresponding to a Fr=0.46) the instability becomes absolute. Finally, as Froude number tends to infinity, the critical point tends to the critical point of the symmetric mode. The variation of the \omega_i of the critical point with respect to Froude number is shown in Figure 4-14.

From Fr=0 to Fr=4.5, where the instability is convective, a persistent excitation, such as the one provided by ocean waves, will cause a spatially growing response of the wake if the frequency of the excitation is within the amplification regions of the spatial instability
curves. For every Froude number there are two such regions, one at low wave numbers and one at high wave numbers. These regions, for Fr=1.5, are shown in Figure 4-13. For Froude numbers greater than 4.5, the instability is absolute and the wake develops a self-sustained oscillation. In this case, the real part of the critical point frequency, \( \omega_r \), is the preferred frequency of the oscillation.

The time-averaged Reynolds stresses are given by:

\[
\tau_{xy}(y) = \rho u_1 v_1
\]  

(4.15)

where \( u_1(x,y,t) \) and \( v_1(x,y,t) \) are the velocity components of the unsteady flow. This can be shown to give:

\[
\frac{\tau_{xy}(y)}{e^{-2kx}} = \frac{1}{2} \rho \text{Im}(k\Phi \Phi^*)
\]

(4.16)

The stream function, the vorticity and the Reynolds stresses distribution for the modes with the maximum spatial growth rates from the spatial instability curve for Fr=1.5 are shown in Figures 4-15 to 4-20. The vorticity distribution \( \Omega(y) \) is evaluated using the relation:

\[
\Omega(y) = \frac{d^2U}{dy^2} \Phi(y)
\]

(4.17)

The magnitude plots are calculated within an arbitrary multiplicative constant, since the values of the stream function consist an eigenvector. Hence only relative comparisons can be made between the results for Branch I and II.

From the plots of the stream function distribution, we can notice that the ratio \( \Phi(0)/\Phi_{\max}(y) \) is always higher for Branch II than for Branch I, for the same Froude number. This means that the high frequency waves disturb more the free surface than the
low frequency ones, since the $y$-component of the velocity is proportional to the stream function ($v(y) = -ik\Phi(y)$). Also from the plots of the vorticity distribution, we can notice that there is a jump of 180 degrees in the phase at the point where the vorticity is zero.
Figure 4-1: The hydrofoil NACA003.
Figure 4-2: Velocity profile at $x^*/l=0.003$ behind the hydrofoil.

Figure 4-3: Velocity profile at $x^*/l=0.1$ behind the hydrofoil.
Figure 4-4: Velocity profile at $x^*/l=2$ behind the hydrofoil.
Figure 4-5: The finite-difference grid.
Figure 4-6: Instability curves for the hydrofoil \((x^*/l=0.003)\) at \(Fr=0\).
Curves 1-7: \(k_i=0, -0.5, -1, -1.5, -2, -2.5, -2.6\).

Figure 4-7: Instability curves for the hydrofoil \((x^*/l=0.003)\) at \(Fr=0.5\).
Curves 1-7: Branch I \((k_i=0, -0.5, -1, -1.5, -2, -2.5, -2.6)\).
Curves 8-11: Branch II \((k_i=0, -0.5, -1, -1.5)\).
Figure 4-8: Instability curves for the hydrofoil (x*/l=0.003) at Fr=1.5.
Curves 1-7: Branch I (k_i=0, -0.5, -1, -1.5, -2, -2.5, -2.7).
Curves 8-12: Branch II (k_i=0, -0.5, -1, -1.5, -1.7).

Figure 4-9: Instability curves for the hydrofoil (x*/l=0.003) at Fr=2.5.
Curves 1-4: Branch I (k_i=0, -0.5, -1, -1.5).
Curves 5-9: Branch II (k_i=0, -0.5, -1, -1.2, -1.5).
Figure 4-10: Instability curves for the hydrofoil (x*/l=0.003) at Fr=3.5.
Curve 1: Branch I (k₁=0).
Curves 2-7: Branch II (k₁=0, -0.2, -0.5, -1, -1.2, -1.4).

Figure 4-11: Instability curves for the hydrofoil (x*/l=0.003) at Fr=4.5.
Curve 1: Branch I (k₁=0).
Curves 2-6: Branch II (k₁=0, -0.2, -0.5, -1, -1.2).
Figure 4-12: Instability curves for the hydrofoil \((x^*/l=0.003)\) at \(Fr=5.5\).
Curve 1: Branch I \((k_1=0)\).
Curves 2-6: Branch II \((k_1=0, -0.2, -0.5, -0.8, -1)\).
Figure 4-13: Spatial instability curve for the hydrofoil (x*l=0.003) at Fr=1.5.
Figure 4-14: The variation of $\omega_{0i}$ with respect to the Froude number for the hydrofoil ($x^*/l=0.003$).
Figure 4-15: The stream function distribution for the hydrofoil ($x^*/l=0.003$) at 
Fr=1.5 (Branch I, $k_r=0.275$, $k_i=-0.069$, $\omega_r=0.072$, $\omega_i=0$).
The first plot shows the magnitude and the second the phase in degrees.
Figure 4-16: The vorticity distribution for the hydrofoil ($x^*/l=0.003$) at $Fr=1.5$ (Branch I, $k_l=0.275$, $k_i=-0.069$, $\omega_l=0.072$, $\omega_i=0$).

The first plot shows the magnitude and the second the phase in degrees.
Figure 4.17: The Reynolds stresses distribution for the hydrofoil \((x^*/l=0.003)\) at \(Fr=1.5\) (Branch I, \(k_t=0.275, k_i=-0.069, \omega_t=0.072, \omega_i=0\)).
Figure 4.18: The stream function distribution for the hydrofoil \((x^*/l=0.003)\) at \(Fr=1.5\) (Branch II, \(k_r=1.8, k_i=-0.068, \omega_r=1.084, \omega_i=0\)).

The first plot shows the magnitude and the second the phase in degrees.
Figure 4.19: The vorticity distribution for the hydrofoil ($x^*/l=0.003$) at $Fr=1.5$ (Branch II, $k_r=1.8$, $k_i=-0.068$, $\omega_r=1.084$, $\omega_i=0$).

The first plot shows the magnitude and the second the phase in degrees.
Figure 4-20: The Reynolds stresses distribution for the hydrofoil ($x^*/l=0.003$) at Fr=1.5 (Branch II, $k_r=1.8, k_i=-0.068, \omega_r=1.084, \omega_i=0$).
Chapter 5

Cylinder Wake

In this chapter the stability analysis results in the wake of a floating cylinder are given.

5.1 The Wake Velocity Profiles

The cylinder is assumed to be half submerged (Figure 5-1). The velocity profiles of the unbounded flow around a cylinder at Reynolds number 140,000 reported in [Cantwell 76] have been used for the stability calculations. The velocity profiles (experimental data) were fitted by the expression [Triantafyllou 86]:

\[ \frac{U^*(y^*)}{U_\infty} = 1 - A(1 - \tanh(\alpha \left( \frac{y^*}{d} \right)^2 - \beta)) \]  \hspace{1cm} (5.1)

where \( A, \alpha, \beta \) are fitting parameters which depend on the distance \( x^* \) behind the cylinder and \( d \) is the diameter of the cylinder. So the non-dimensional velocity profile in the near-wake of the floating cylinder is:

\[ U(y) = 1 - A(1 - \tanh(\alpha y^2 - \beta)) \quad y \leq 0 \]  \hspace{1cm} (5.2)

For the velocity profile at \( x^*/d = 1 \) the fitting parameters are (Figure 5-2):

\[ x^*/d = 1 \quad A = 0.75 \quad \alpha = 4.0 \quad \beta = 0.32 \]  \hspace{1cm} (5.3)
5.2 Results for the Cylinder Wake

The wake profile at station \(x*/d=1\) is considered. All parameters are non-dimensionalized with respect to the diameter \(d\) of the cylinder, instead of the half-width \(b\) of the wake. The instability curves for various Froude numbers are shown in Figures 5-3 to 5-8. As in the hydrofoil wake, we notice that there are two unstable branches, I and II, for every finite Froude number. For Fr=0 there is only Branch I and it is identical with the antisymmetric mode in infinite fluid, while for Fr=\(\infty\) there is only Branch II and it is identical with the symmetric mode in infinite fluid.

For every finite Froude number, each Branch has its own critical point, and the instability is characterized by the Branch with the higher \(\omega_i\). The dependence of the two critical points on the value of the Froude number is given in Table 5-I.

<table>
<thead>
<tr>
<th>Fr</th>
<th>Branch</th>
<th>(k_r)</th>
<th>(k_i)</th>
<th>(\omega_r)</th>
<th>(\omega_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Anti-sym.</td>
<td>2.3</td>
<td>-4.10</td>
<td>0.907</td>
<td>-1.0910</td>
</tr>
<tr>
<td>0.5</td>
<td>Branch I</td>
<td>2.7</td>
<td>-5.00</td>
<td>0.952</td>
<td>-1.1650</td>
</tr>
<tr>
<td>1</td>
<td>Branch II</td>
<td>1.9</td>
<td>-3.10</td>
<td>2.322</td>
<td>-0.4800</td>
</tr>
<tr>
<td>1.5</td>
<td>Branch II</td>
<td>2.3</td>
<td>-2.80</td>
<td>1.863</td>
<td>-0.1740</td>
</tr>
<tr>
<td>2.5</td>
<td>Branch II</td>
<td>2.4</td>
<td>-2.20</td>
<td>1.550</td>
<td>-0.0045</td>
</tr>
<tr>
<td>3</td>
<td>Branch II</td>
<td>2.4</td>
<td>-2.20</td>
<td>1.488</td>
<td>0.02300</td>
</tr>
<tr>
<td>(\infty)</td>
<td>Symmetric</td>
<td>2.2</td>
<td>-1.83</td>
<td>1.338</td>
<td>0.08600</td>
</tr>
</tbody>
</table>

Also the variation of the \(\omega_i\) of the critical point with respect to Froude number is shown in Figure 5-9 and in principle agrees with the results from the thin hydrofoil: for small Froude numbers, Branch I is dominant, and the instability is convective; as the Froude number increases, Branch II becomes dominant, and, for a Froude number equal to 2.5, the instability becomes of the absolute type.
Hence, from Fr=0 to Fr=2.5, where the instability is convective, a persistent excitation of the wake will cause a spatially growing response, if the frequency of the excitation is within the amplification regions of the spatial instability curves. The spatial instability curve, for Fr=1, is shown in Figure 5-10. For Froude number greater than 2.5, the instability is absolute and the wake develops a self-sustained oscillation.

Typical results for the stream function, the vorticity and the Reynolds stresses distribution for the modes with the maximum spatial growth rates from the spatial instability curve for Fr=1, are shown in Figures 5-11 to 5-16.
Figure 5-1: The half-submerged cylinder.

Figure 5-2: Velocity profile at $x^*/d = 1$ behind the cylinder.
Figure 5-3: Instability curves for the cylinder ($x^*/d=1$) at $Fr=0$.
Curves 1-7: $k_i=0$, -0.5, -1, -2, -3, -4, -5.

Figure 5-4: Instability curves for the cylinder ($x^*/d=1$) at $Fr=0.5$.
Curves 1-6: Branch I ($k_i=0$, -1, -2, -3, -4, -5).
Curves 7-10: Branch II ($k_i=0$, -1, -2, -3).
Figure 5-5: Instability curves for the cylinder ($x^*/d=1$) at Fr=1.
Curves 1-3: Branch I ($k_i=0, -1, -2$).
Curves 4-8: Branch II ($k_i=0, -1, -2, -3, -3.1$).

Figure 5-6: Instability curves for the cylinder ($x^*/d=1$) at Fr=1.5.
Curves 1-3: Branch I ($k_i=0, -0.6, -1$).
Curves 4-8: Branch II ($k_i=0, -0.6, -1.4, -2.2, -2.8$).
Figure 5-7: Instability curves for the cylinder (x*/d=1) at Fr=2.5.
Curve 1: Branch I (k_i=0).
Curves 2-6: Branch II (k_i=0, -0.8, -1.6, -2, -2.4).

Figure 5-8: Instability curves for the cylinder (x*/d=1) at Fr=3.
Curve 1: Branch I (k_i=0).
Curves 2-5: Branch II (k_i=0, -1, -1.8, -2.2).
Figure 5-9: The variation of $\omega_{0i}$ with respect to the Froude number for the cylinder ($x^*/d=1$).
Figure 5-10: Spatial instability curve for the cylinder ($x^*/d=1$) at $Fr=1$. 
Figure 5-11: The stream function distribution for the cylinder (x*/d=1) at Fr=1 (Branch I, k_r=1.2, k_i=-0.278, \omega_r=0.502, \omega_i=0). The first plot shows the magnitude and the second the phase in degrees.
Figure 5-12: The vorticity distribution for the cylinder ($x*/d=1$) at $Fr=1$
(Branch I, $k_f=1.2$, $k_i=-0.278$, $\omega_f=0.502$, $\omega_i=0$).
The first plot shows the magnitude and the second the phase in degrees.
Figure 5-13: The Reynolds stresses distribution for the cylinder \((x^*/d=1)\) at \(Fr=1\) (Branch I, \(k_f=1.2, k_i=-0.278, \omega_f=0.502, \omega_i=0\)).
Figure 5.14: The stream function distribution for the cylinder \((x^*/d=1)\) at \(Fr=1\) (Branch II, \(k_x=4, k_y=-0.595, \omega_x=2.506, \omega_y=0\)).

The first plot shows the magnitude and the second the phase in degrees.
Figure 5-15: The vorticity distribution for the cylinder \((x^*/d=1)\) at \(Fr=1\) (Branch II, \(k_r=4, k_i=-0.595, \omega_r=2.506, \omega_i=0\)).

The first plot shows the magnitude and the second the phase in degrees.
Figure 5-16: The Reynolds stresses distribution for the cylinder ($x^*/d=1$) at $Fr=1$
(Branch II, $k_f=4$, $k_i=-0.595$, $\omega_f=2.506$, $\omega_i=0$).
Chapter 6

Time-Development of the unstable disturbances in the wake

The time-asymptotic shapes of the Green's function (pulse-shapes) for both the thin hydrofoil and the cylinder wakes have been calculated, and the results are presented in this chapter.

6.1 Calculation of the Pulse Shape

Additional insight about the time-development of the instability requires a study of the complete time-asymptotic shape of the Green's function given in (2.1). This shape is called the time-asymptotic pulse shape of the Green's function [Bers 83]. The time-asymptotic pulse shape shows the direction and propagation speed of unstable wave groups developing in the wake. In the case of an absolute instability (Figure 6-1), the time-asymptotic pulse shape always encompasses the origin of the excitation with its front and back ends moving in opposite directions. In the case of a convective instability (Figure 6-2) the pulse does not encompass the origin, and its front and back ends are moving in the same direction.

In order to find the time-asymptotic pulse shape, we consider the Lagrangian description of the Green's function (2.1), as seen from a moving observer. The time-asymptotic response, as seen from the moving frame, will be determined by the critical point of:

\[ D_V(k, \omega^m) = D(k, \omega^m + kV) = 0 \]  

(6.1)

where \( V \) is the velocity of the observer and \( \omega^m \) is the complex frequency, as seen from the
moving observer. Let the critical point of (6.2) be \( \omega_{m0}(V) = \omega_{m0}(V) + i\omega_{m01}(V) \). Then it can be shown, [Bers 83], that the time-asymptotic response satisfies:

\[
\log|G(V,t \to \infty)| \approx -\omega_{m01}(V)t
\] (6.2)

Hence a plot of \( \omega_{m01}(V) \) when scaled by \( t \) (in both \( \omega_{m01} \) and \( \nu \) coordinates) gives at the same time a plot of the logarithmic magnitude of the time-asymptotic Green's function versus the position \( x (x=Vt) \), i.e. the asymptotic pulse shape of the unstable evolution.

6.2 Pulse Shapes for the Thin Hydrofoil and the Cylinder Wakes

First the pulse shapes of the infinite fluid case are considered (Figures 6.3 and 6.4) for both the hydrofoil and the cylinder. For both wakes, the symmetric mode (i.e. the mode whose stream function \( \Phi(y) \) is an even function of \( y \)) is absolutely unstable, while the anti-symmetric mode (i.e. the mode whose stream function is an odd function of \( y \)) is convectively unstable. Almost the whole pulse shape is characterized by the symmetric mode and only its front end is characterized by the anti-symmetric mode.

Next, the pulse shapes for the floating hydrofoil and cylinder, and for various Froude numbers, are considered (Figures 6.5 and 6.6). For low Froude numbers, the instability is convective; so the pulse shape has moved away from the origin. The instability is characterized by Branch I. Note that the magnitude of the pulse shape decreases as Froude number increases. As Froude number increases even more, the instability starts to be characterized by Branch II and is still of the convective type. Now the magnitude of the pulse shape increases as the Froude number increases. At a certain Froude number the instability turns to absolute, and the pulse shape encompasses the origin. In the limit, as Froude number tends to infinity, the pulse shape corresponds to the symmetric mode of the infinite fluid. As a general remark, we see that, for all Froude numbers, the instability
disturbance propagates downstream of the object. Even when the instability is absolute, only a very small part of the disturbance propagates upstream and at a very low speed. From this we can conclude that the modelling of the wake as an independent shear flow, by neglecting the presence of the object producing the wake, is well justified, since there is basically no interaction between the development of the instability and the presence of the object.
Figure 6-1: The pulse shape in the case of absolute instability.

Figure 6-2: The pulse shape in the case of convective instability.
Figure 6-3: Pulse shapes for the hydrofoil ($x^*/l=0.003$).
Curve 1: Symmetric mode. Curve 2: Anti-symmetric mode.

Figure 6-4: Pulse shapes for the cylinder ($x^*/d=1$).
Curve 1: Symmetric mode. Curve 2: Anti-symmetric mode.
Figure 6-5: Pulse shapes for the hydrofoil (x*/l=0.003).
Curves 1-6: Fr=0, 0.5, 1.5, 2.5, 4.5, ∞.

Figure 6-6: Pulse shapes for the cylinder (x*/d=1).
Curves 1-4: Fr=0, 0.5, 1.5, ∞.
Chapter 7

Conclusions and Recommendation for Future work

The interaction between the ocean surface and the wake behind a floating, two-dimensional object was the topic in this thesis. In particular, it was investigated whether permanent flow patterns appear spontaneously, as a result of an absolute instability of the flow. It has been first shown that there are no such "spontaneous" flow patterns in the wake for low Froude numbers; so the presence of the ocean surface seems to have a stabilizing effect, since it is known that "spontaneous" patterns exist in the case of an infinite fluid (Karman street behind bluff bodies).

For low Froude numbers, the wake responds to the excitation provided by ocean waves with spatially growing modes, because of the convective instability of the flow. It has been found that for every Froude number there exist two distinct ranges of frequencies where the wake is unstable; one at low frequencies, and one at high frequencies. So the wake will respond with a spatially growing mode to a monochromatic ocean wave only if its frequency lies within these two frequency ranges. Two types of objects have been studied, one streamlined (hydrofoil) and one bluff (circular cylinder), and it has been found that their wakes have, qualitatively, similar stability properties.

We have also found that very thin shear layers are an inadequate model to study the stability properties of the wake, since they suppress Branch II.

The pulse shape calculation demonstrates that even for absolutely unstable flows the upstream propagation velocity is small, and hence the interaction between the object and the instability wave can be neglected. This justifies the approach taken in this thesis whereby the presence of the body is neglected in calculating the dynamic response of the wake.
In future work, the first improvement to be made is to consider the actual shape of the average free surface elevation, and hence actual velocity profiles should be measured behind objects moving near or on the free surface. This will enable us to study the stability properties at higher Froude numbers, and find out what is the effect of the free surface shape on the stability properties of the wake. The ultimate goal, of course, is to extend the study to three dimensional objects, and consider the stability properties of the wakes behind ships and offshore structures.
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