ON THE CONJUGATE LOCUS OF A RIEMANNIAN MANIFOLD

by

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ABSTRACT

On the Conjugate Locus of a Riemannian Manifold

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Submitted to the Department of Mathematics on August 22, 1966, in partial fulfillment of the requirement for the degree of Doctor of Philosophy.

The conjugate locus of a Riemannian manifold splits naturally into two subsets--the regular locus and the singular locus. The regular locus and those properties of the exponential map that depend on it, have been studied by J. H. C. Whitehead, S. B. Myers, L. J. Savage, F. W. Warner and others. The study of the singular locus is started in this work.

It is studied how the order of the conjugate points are distributed near a singular point p, for some types of intersection at p. In the case (the only one of which examples are known) (*) where the conjugate locus near p consists of two submanifolds intersecting in general position at p, the relations between the kernel of the differential of the exponential map and the tangent spaces to these submanifolds are described completely. This extends to the singular locus results of J. H. C. Whitehead and F. W. Warner for the regular locus. A characterization is given (in terms of the second differential of the exponential map) of the tangent space to the conjugate variety at a point p in the cases where p is regular and where p is as in (*) above. This is given on the assumption that M is a C^∞ manifold and relates to a result of H. Whitney for analytic varieties.

It is proved a sequence of results that eliminate the possibility of certain types of intersections at a conjugate point. All these results are not restricted to Riemannian manifolds, but hold for what F. W. Warner called a regular exponential map.

To prove the above results, some new techniques are developed in Differential Analysis. In particular, upper bounds are given for the order of the singularities of a C^∞ map ϕ of manifolds, in a given direction. This is given in terms of the dimension of certain subspaces of the null-space of the differential of ϕ.

Some problems and conjectures are stated in relation to the conjugate locus of a Riemannian manifold.

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INTRODUCTION

The conjugate locus of a Riemannian manifold (i.e. the set of singularities of its exponential map), as a differentiable variety, splits naturally into two subsets--the regular locus and the singular locus. The regular locus and those properties of the exponential map that depend on it have been studied by J. H. C. Whitehead in (5), F. W. Warner in (4) and others.

In this work we start the study of the singular locus. We study how the order of the conjugate points are distributed near a singular point p, for some admissible types of intersections at p. In the case (*) where the local conjugate variety at p consists of two submanifolds intersecting in general position we describe completely the relations between the kernel of the differential of the exponential map and the tangent spaces to these submanifolds. This extends to the singular locus results of J. H. C. Whitehead and F. W. Warner for the regular locus. We give a characterization (in terms of the second differential of the exponential map) of the tangent space to the conjugate variety at a point p in the cases where p is a regular conjugate point and when p is as in (*) above. This is given on the assumption that M is a C^{∞} manifold and relates to a result of H. Whitney in (7) for analytic varieties. We also prove a sequence of results that eliminate the possibility of certain types of intersections at a conjugate point.

These results are not restricted to Riemannian structures, but hold for what F. W. Warner called in (4) a regular exponential map. It was proved in (4) that the exponential map for a Finsler space is a regular exponential map.

In Section 2 we prove a sequence of lemmas that give upper bounds for the order of the singularities of a C^∞ map of manifolds, in a given direction. Our results on the conjugate locus are proved in Sections 3 and 4.

I thank F. W. Warner for reading a preliminary version of this work and for giving me some good suggestions. I thank D. Ebin for helping me to correct some mistakes.

§1. Preliminaries.

We are going to fix some notation and conventions that will be used throughout this work. Manifolds will be locally euclidean, second countable, Hausdorff spaces with a C^∞ differentiable structure. A submanifold N of a manifold M is a manifold N together with a 1:1 immersion of N into M. If m is a point of M the space of k-th order tangent vectors at m will be denoted by M_m^k. (c.f. (1) for definitions of higher order contact elements.). M_m, the space of first order tangent vectors will be considered as a manifold in the usual way. If $p \varepsilon M_m$, $(M_m)_p^k$ will denote the space of k-th order tangent vectors at p. If $f\colon M \to N$ is a differentiable map of manifolds, the k-th order differential of f will be denoted by $d_f^k d$; we suppress k if $k = 1$. The space of k-th order differentials at $m \varepsilon M$ will be denoted by ${}^kM_m$, and $\delta^k f\colon {}^kN_{f(m)} \to {}^kM_m$ will denote the dual map, corresponding to $d^k f$. If $f\colon M \to N$ is a differentiable map and $m \varepsilon M$ is a singularity of f, we denote by $N(m)$, the null-space of df at m.

$\mathrm{Ord}(m) = k$ will mean: order of m as a singularity of f equals k.

§2. On the singularities of differentiable maps.

In this section we prove a sequence of lemmas that give upper bounds for the orders of the singularities of a C^∞ map $\phi: M \to N$, near a singularity, $m \in M$ in a given direction $x \in M_m$. This is given in terms of the higher order differentials of ϕ at m. We give also a relation between the order of a singularity m of a C^∞ map $\phi: M \to M$ and the order of m as a zero of the Jacobian of this map.

We have a natural isomorphism $\dfrac{M_m^k}{M_m^{k-1}} \approx S^k(M_m)$ where $S^k(M_m)$ stands for the k-fold symetric tensor product $M_m \# \cdots \# M_m$. To see this we consider the diagram

$$\begin{tikzcd} \prod_{i=1}^{k} M_m \arrow[r, "\xi"] \arrow[dr, "\psi"'] & S^k(M_m) \arrow[d, "\overline{\psi}"] \\ & \dfrac{M_m^k}{M_m^{k-1}} \end{tikzcd}$$

where $\prod_{i=1}^{k} M_m$ denotes the k-fold cartesian product and ψ is defined by

$$\psi(x_1, \ldots, x_k) = \overline{x}_1 \cdots \overline{x}_k + M_m^{k-1}$$

(here $x_i \in M_m$ and $\overline{x}_i$ is any extension of x_i to a C^∞

vector field in some neighborhood of m). It is easily checked that ψ is a well-defined, k-linear, symmetric map and that $\overline{\psi}$, the induced map is an isomorphism.

Now let $\phi: M \to L$ be a C^∞ map. The k-order differential of ϕ at m,

$$d^k\phi: M_m^k \to L_{\phi(m)}^k$$

induces a linear map

$$\phi_m^k: \frac{M_m^k}{M_m^{k-1}} \to \frac{L_{\phi(m)}^k}{d^{k-1}\phi(M_m^{k-1})}$$

and using the above isomorphism we have

$$\phi_m^k: S^k(M_m) \to \frac{L_{\phi(m)}^k}{d^{k-1}\phi(M_m^{k-1})} .$$

We now define, associated with each direction $x \in M_m$ two subspaces $N_i(x)$, $i=1,2$, of the null-space $N(m)$ of $d\phi$ at m, by:

$$N_1(X) = [A \in N(m) | \phi_m^2(X \# A) = 0]$$

and

$$N_2(X) = [A \in N_1(X) | \phi_m^3(X \# X \# A) = 0] .$$

Let $k_i(x)$ be the dimension of $N_i(x)$, $i=1,2$. Thus

$$N_2(x) \subset N_1(x) \text{ and } k_2(x) \leq k_1(x) .$$

Let $\gamma: (a,b) \to M$, $a < 1 < b$ be any C^∞ curve in M such that $\gamma(1) = m$ and $\gamma_*(1) = X$, where $\gamma_*(1)$ stands for the tangent vector to γ at $\gamma(1)$. Let $\sigma = \phi \circ \gamma$ and thus σ is a C^∞ curve such that $\sigma(1) = \phi(m)$ and $\sigma_*(1) = d\phi(x)$. Let $u_1(t),\ldots,u_\ell(t)$ ($\ell = \dim L$) be C^∞ vector fields along σ which span $L_{\sigma(t)}$ for all t. Given $A \varepsilon N(m)$ we extend it to a C^∞ vector field $A(t)$ along γ and we let $Y(t) = d\phi(A(t))$. Thus:

$$Y(t) = \sum_{i=1}^{\ell} y_i(t)u_i(t)$$

where y_i are C^∞ functions of t and $Y(1) = 0$.

Define

$$\dot{Y}(1) = \sum_{i=1}^{\ell} y_i'(1)u_i(1)$$

and it is easy to see that since $Y(1) = 0$, $\dot{Y}(1)$ does not depend upon the choice of the particular basis $\{u_i(t);\ 1 \leq i \leq \ell\}$. Now we let X and A denote any extensions of $\gamma_*(t)$ and $A(t)$ to C^∞ vector fields in some neighborhood of M. Then $(XA)(m)$ is an element of M_m^2 and

$$X \# A = (XA)(m) + M_m \text{ as element of } \frac{M_m^2}{M_m}.$$ Moreover

$X \# A$ does not depend upon the choice of the extensions X and A and

$$\phi_m^2(X \# A) = d^2\phi((XA)(m)) + d\phi(M_m)$$

$$= \dot{Y}(1) + d\phi(M_m) \qquad \text{(I)}$$

In fact: $d^2\phi(XA)(m)f = X_m(A(f \circ \phi)) = \frac{d}{dt}\Big|_{t=1}(Y(t)f) = \dot{Y}(1)f$ for all C^∞ functions f at $\phi(m)$. Now we remark that if $A \in N_1(X)$ then we can find an extension $A(t)$ of A along γ such that $d^2\phi(XA) = 0$. In fact, let A_0 be any extension of A and let $d\phi(A_0(t)) = Z_0(t)$. Thus

$$d^2\phi(XA_0)(m) = \dot{Y}_0(1) \in d\phi(M_m) .$$

Thus we can find $z \in M_m$ such that $d\phi(Z) = \dot{Y}_0(1)$. Let $Z(t)$ be any extension of Z along γ. Take

$$A(t) = A_0(t) + (1-t)Z(t)$$

Now if $A \in N_1(X)$ and $A(t)$ is the above extension, then

$$d^3\phi((X^2A)(m)) = \ddot{Y}(1) = \sum_{i=1}^{\ell} y_i''(1)u_i(1) \qquad \text{(II)}$$

This follows because $Y(1) = \dot{Y}(1) = 0$ (and thus $y_i(1) = y_i'(1) = 0$, $1 \leq i \leq \ell$) and

$$d^3\phi(X^2A)(m)f = X_m^2(A(f \circ \phi)) = \frac{d^2}{dt^2}\Big|_{t=1}(Y(t)f)$$

for all C^∞ function f at $\phi(m)$. Moreover $\ddot{Y}(1)$ does not depend upon the choice of the basis $\{ u_i(t);\ 1 \leq i \leq \ell \}$.

Lemma 2.1. Let $\phi: M \to L$ be any C^∞ map and $A \varepsilon M_m - N_1(X)$, $m \varepsilon M$. Let $\gamma: (a,b) \to M$, $a < 1 < b$ be any smooth curve such that $\gamma(1) = m$, $\gamma_*(1) = X$ and $A(t)$ be any non-vanishing C^∞ vector field along γ such that $A(1) = A$. Let $Y(t) = d\phi(A(t))$ be the corresponding C^∞ vector field along $\sigma = \phi \circ \gamma$. Then there exists $\varepsilon > 0$ such that for $1 - \varepsilon < t < 1 + \varepsilon$

$Y(t) = f(t)e(t)$, where $e(t)$ is a non-vanishing C^∞ vector field along σ , $f(t)$ a C^∞ function and $f'(1) \neq 0$ if $f(1) = 0$.

Proof. The proof is exactly the same as in (4) for lemma 2.3. Let $u_i(t)$; $1 \leq i \leq \ell$ be smooth vector fields along σ spanning $L_{\sigma(t)}$ for all t . Thus

$$Y(t) = \sum_{i=1}^{\ell} y_i(t)u_i(t) , \qquad \text{where } y_i \text{ are } C^\infty$$

functions of t . If $A(1) \varepsilon N(m)$ we have $Y(1) = 0$ and since $X \not\varepsilon N_1(X)$ we have $\phi_m^2(X \# A) \neq 0$ and from (I) we see that $\dot{Y}(1) \neq 0$. Thus there exists $\varepsilon > 0$ such that $Y(t) \neq 0$ for $1 - \varepsilon < t < 1 + \varepsilon$ and not all $y_i'(1)$ are zero. It is easy to see that

$\sum_{i=1}^{\ell} y_i^2(t)$ is a non-negative C^∞ function whose zeros are all of second order. Thus by lemma 2.2 of (4) , this function has a C^∞ square root $f(t)$. Moreover $f(1)$ is an isolated zero of $f(t)$, of order one.

Define

$$e(t) = \frac{Y(t)}{f(t)} = \sum_{i=1}^{\ell} \frac{y_i(t)}{f(t)}\, u_i(t) \qquad \text{if} \quad t \neq 1$$

and (III)

$$e(t) = \sum_{i=1}^{\ell} \frac{y_i'(1)}{f'(1)}\, u_i(1)$$

Thus $e(t)$ is a non-vanishing vector field along σ and that e is C^∞ follows from the fact that if t_0 is a zero of Y, then on a neighborhood of t_0, $y_i(t) = (t - t_0)k_i(t)$ and $f(t) = (t - t_0)g(t)$, where $k_i(t)$ and $g(t)$ are C^∞ functions, $k_i(t_0) = y_i'(t_0)$, and $g(t_0) = f'(t_0) \neq 0$ ∎

<u>Lemma 2.2.</u> Let $\phi\colon M \to L$ be a C^∞ map, m be a singularity of order k for ϕ and $\gamma\colon (a,b) \to M$, $a < 1 < b$ be any smooth curve such that $\gamma(1) = m$ and $\gamma_*(1) = X$. Then there exists $\varepsilon > 0$ such that

$$\text{order } \gamma(t) \leq k_1(X) \leq k \quad \text{for all} \quad t \neq 1,$$

$1 - \varepsilon < t < 1 + \varepsilon$.

<u>Proof.</u> Let $C_1(X)$ be any complementary subspace for $N_1(X)$ in $N(p)$, i.e. $N(p) = N_1(X) \oplus C_1(X)$. Choose a basis $\{ A_i \,;\, 1 \leq i \leq d\}$ for M_m such that $\{ A_i \,;\, 1 \leq i \leq k\}$ be a basis for $N(p)$, $\{ A_i \,;\, 1 \leq i \leq k_1(X)\}$ basis for $N_1(X)$ and $\{ A_i \,;\, k_1(X) + 1 \leq i \leq k\}$ be a basis for $C_1(X)$.

Now let $\sigma = \phi \circ \gamma$ and extend $\{ A_i ; 1 \leq i \leq d \}$ to a basis $\{ A_i(t) ; 1 \leq i \leq d \}$ for $M_{\gamma(t)}$ along γ in such a way that the $A_i(t)$ be smooth vector fields. Let $Y_i(t) = d\phi(A_i(t))$, $k_1(X) + 1 \leq i \leq d$. Using lemma 2.1, we have:

$Y^i(t) = f_i(t)e_i(t)$, $k_1(X) + 1 \leq i \leq d$ where $e_i(t)$ are non-vanishing C^∞ vector fields along σ , $f_i(t)$ are C^∞ functions of t , $f_i(1) = 0$, $f_i'(1) \neq 0$ for $k_1(X) + 1 \leq 1 \leq k$ and $f_i(1) \neq 0$ for $k + 1 \leq i \leq d$. Thus we can find $\varepsilon > 0$ such that $f_i(t) \neq 0$ if $t \neq 1$, and $1 - \varepsilon < t < 1 + \varepsilon$, $k_1(X) + 1 \leq i \leq d$.

Now using (III) (c.f. proof of lemma 2.1) we see that
$\phi_m^2(X \# A_i(1)) = \frac{e_i(1)}{c_i} + d\phi(M_m)$ where $c_i \neq 0$
$k_1(X) + 1 \leq i \leq k$. Because $\phi_m^2|X \# C_1(X)$ is an isomorphism we see that $\{ e_i(1) ; k_1(X) + 1 \leq i \leq d \}$ is linearly independent. Thus we may assume that $\{ e_i(t) ; k_1(X) + 1 \leq i \leq d \}$ is linearly independent for $1 - \varepsilon < t < 1 + \varepsilon$. Now since $f_i(t) \neq 0$ for $k_1(X) + 1 \leq i \leq d$, $t \neq 1$, $1 - \varepsilon < t < 1 + \varepsilon$ we see that the order of $(\gamma(t)) \leq k_1(X)$ for $t \neq 1$, $1 - \varepsilon < t < 1 + \varepsilon$ ∎

Remark 2.1. Under the assumptions of lemma 2.2 if $\dim M = \dim L = d$, $N_1(X) = 0$ and $X \notin N(p)$, then we can find coordinate systems $x_1,\ldots,x_d$ and $y_1,\ldots,y_d$ on neighborhoods U and V of m and $\phi(m)$ respectively such that $\phi(U) \subset V$ and

$$d\phi\left(\frac{\partial}{\partial x_j}(\gamma(t))\right) = f_j(t) \frac{\partial}{\partial y_j}(\sigma(t))$$

for $1 \leq j \leq d$, where f_j are C^∞ functions of t and $f_j(t) \neq 0$ for all $t \in \gamma^{-1}(U)$ and $d - k + 1 \leq j \leq d$ and for $1 \leq j \leq k$, $t = 1$ is the only zero of f_j and $f_j'(1) > 0$.

Proof. Same as in (4) for lemma 2.5.

Lemma 2.3. Under the assumptions of lemma 2.1 if $A \in N_1(X) - N_2(X)$.

Then $Y(t) = f(t)e(t)$ for $1 - \varepsilon < t < 1 + \varepsilon$ where $e(t)$ is a non-vanishing C^∞ vector field along σ, $f(t)$ a C^∞ function and $t = 1$ is a second order zero of f i.e. $f(1) = f'(1) = 0$ and $f''(1) \neq 0$.

Proof. Let $A(t)$ be the extension given immediately before lemma 2.1. Thus

$$Y(1) = \sum_{i=1}^{\ell} y_i(1)u_i(1) = \dot{Y}(1) = \sum_{i=1}^{\ell} y_i'(1)u_i(1) = 0 .$$

Now since $X \notin N_2(X)$ using (II) we get $\ddot{Y}(1) \neq 0$. Thus $g(t) = \sum_{i=1}^{\ell} y_i^2(t)$ is a non-negative C^∞ function having a zero of order four at $t = 1$. Hence $g(t) = (t-1)^4 h(t)$

for $1 - \varepsilon < t < 1 + \varepsilon$, where h is a C^∞ function of t and $h(1) = g^{(4)}(1) \neq 0$. Thus $g(t) \neq 0$ if $t \neq 1$, $1 - \varepsilon < t < 1 + \varepsilon$ and $Y(t) \neq 0$ for the same values of t. We define a non-vanishing C^∞ vector field $e(t)$ along σ for $1 - \varepsilon < t < 1 + \varepsilon$ as follows:

$$e(t) = \frac{Y(t)}{f(t)} \quad \text{if} \quad t \neq 1$$

and (IV)

$$e(1) = \frac{Y(1)}{f''(1)}$$

where f is a C^∞ square root of g. To see that e is C^∞ we note that $y_i(t) = (t-1)^2 k_i(t)$ and $f(t) = (t-1)^2 h(t)$ where k_i and h are C^∞ functions, $k_i(1) = y_i''(1)$, and $h(1) = f''(1) \neq 0$ ∎

<u>Lemma 2.4.</u> Under the assumptions of lemma 2.2 there exists $\varepsilon > 0$ such that

order $\gamma(t) \leq k_2(X) \leq k_1(X) \leq k$ for all $t \neq 1$, $1 - \varepsilon < t < 1 + \varepsilon$.

Proof. Let $\{ A_i ; 1 \leq i \leq d \}$ be a basis for M_m such that

$\{ A_i ; 1 \leq i \leq k_2(X) \}$ is a basis for $N_2(X)$

$\{ A_i ; 1 \leq i \leq k_1(X) \}$ is a basis for $N_1(X)$

and $\{ A_i ; 1 \leq i \leq k \}$ be a basis for $N(p)$.

Now by lemmas 2.2 and 2.3 we have an extension $\{A_i(t) \; ; \; 1 \leq i \leq d\}$ of the above basis along γ such that if $Y_i(t) = d\phi(A_i(t))$ then:

$$Y_i(t) = f_i(t)e_i(t) \; , \; k_2(X) + 1 \leq i \leq d \qquad (*)$$

where f_i are C^∞ functions such that $f_i(1) \neq 0$ for $k + 1 \leq i \leq d$ and $t = 1$ is the only zero of f_i on the interval $1 - \varepsilon < t < 1 + \varepsilon$ for $k_2(X) + 1 \leq i \leq k$. Moreover $\{ e_i(1) \; ; \; k_1(X) + 1 \leq i \leq d\}$ is linearly independent. Now let $C_2(X)$ be the subspace spanned by $\{A_i \; ; \; k_2(X) + 1 \leq i \leq k_1(X)\}$. Thus

$$\phi^3_m | X \# X \# C_2(X) \qquad \text{is an isomorphism and}$$

$$\phi^3_m(X \# X \# A_i) = \frac{e_i(1)}{C_i} + d^2\phi(M^2_m) \qquad \text{for}$$

$k_2(X) + 1 \leq i \leq k_1(X)$ where $C_i \neq 0$ (this follows from (IV)).

Thus $\{ e_i(1) \; ; \; k_2(X) + 1 \leq i \leq d\}$ is linearly independent and thus we may assume it is linearly independent for $1 - \varepsilon < t < 1 + \varepsilon$. From this and (*) it follows that

$$\text{order } \gamma(t) \leq k_2(X) \leq k_1(X) \leq k \qquad \text{for}$$

$1 - \varepsilon < t < 1 + \varepsilon$, $t \neq 1$ ■

Now we consider C^∞ maps $\phi: M \to M$. Let m be a point of M . Then to any pair of coordinate systems $x_1,\ldots,x_d$ and $y_1,\ldots,y_d$ on neighborhoods U of m and V of $\phi(m)$

we associate a C^∞ function $J: U \to R$, the Jacobian of ϕ with respect to these coordinate systems. Let m be a singularity of order k and $X \in M_m$ be a direction such that $N_1(X) = 0$ and $X \notin N(m)$. Let γ be any smooth curve $\gamma: (a,b) \to M$, $a < 1 < b$ with $\gamma(1) = m$, $\gamma_*(1) = X$. We claim that $t = 1$ is a zero of order k of $J \circ \gamma$. To see this let $x_1,\ldots,x_d$ and $y_1,\ldots,y_d$ be the coordinate systems given by remark 2.1. Thus

$$\frac{d^i}{dt^i}(J \circ \gamma) = \frac{d^i}{dt^i}(f_1(t)\cdots fd(t)) = 0 \quad \text{if} \quad i < k$$

and (V)

$$\left.\frac{d^k}{dt^k}\right|_{t=1}(J \circ \gamma) = f_1'(1)\cdots f_k'(1)f_{k+1}(1)\cdots f_d(1) \neq 0$$

Now we show that this does not depend upon the choice of the coordinate systems. In fact, with respect to the above coordinate systems, we have:

$$d\phi\left(\frac{\partial}{\partial x_j}\right) = \sum_i a_{ij} \frac{\partial}{\partial y_i} \qquad \text{where} \quad A = (a_{ij}): U \to R^{d^2}$$

is a smooth map. Now if $(U',u_1,\ldots,u_d)$ and $(V',v_1,\ldots,v_d)$ is another pair of coordinate systems at m and $\phi(m)$, we have:

$$d\phi\left(\frac{\partial}{\partial u_j}\right) = \sum_i a_{ij}' \frac{\partial}{\partial v_i} \quad \text{where} \quad A' = (a_{ij}'): U' \to R^{d^2}$$

is a C^∞ map. Thus $\frac{\partial}{\partial x_j} = \sum_s C_{sj} \frac{\partial}{\partial u_s}$ where

$C = (C_{sj})\colon U \cap U' \to R^{d^2}$ is a smooth map. Similarly $\frac{\partial}{\partial y_j} = \sum_s d_{sj} \frac{\partial}{\partial v_s}$ and $D = (d_{sj})\colon V \cap V' \to R^{d^2}$ is a C^∞ map. Now $A = (D \circ \phi)A'C^{-1}$ (*)

and if $J = \det A$, $J' = \det A'$ and $f = \det(D \circ \phi)(\det C)^{-1}$ we have $J = f \cdot J'$ and

$$\frac{d^i}{dt^i}(J) = \sum_{r=0}^{i} \binom{i}{r} \frac{d^{i-r} f}{dt^{i-r}} \cdot \frac{d^i}{dt^i}(J') \blacksquare$$

Definition 2.1. We say that a C^∞ vector field X on a manifold M is transverse to a C^∞ map $\phi\colon M \to N$ if $N_1(X) = 0$ for all $m \in M$.

This roughly speaking, means that X is never tangent to the singular variety of ϕ. We now prove the existence of a useful coordinate system.

Lemma 2.5. If $m \in M$ is a singularity of order k of a C^∞ map $\phi\colon M \to L$, then there exists a coordinate system $(U, x_1, \ldots, x_d)$ at m such that $\frac{\partial}{\partial x_{d-k+i}}(q)$, $1 \leq i \leq k$ spans $N(q)$ for all $q \in U$, with ord $(q) = k$.

Proof. We first note that $d\phi\colon M_m \to L_{\phi(m)}$ and $\delta\phi\colon L_{\phi(m)} \to M_m$ have both the same rank $d - k$ (here $d = \dim M$, $\ell = \dim L$). Now let $(V, y_1, \ldots, y_\ell)$ be any coordinate system at $\phi(m)$. We note that since rank $(\delta\phi) =$ $d - k$, we may assume that $\delta\phi(dy_i) = d(y_i \circ \phi)$

are linearly independent at m, for $1 \leq i \leq d - k$.
Thus by (3) page I-18 we can find smooth functions x_{d-k+i}, $1 \leq i \leq k$ such that $(U; x_1, \ldots, x_d)$ is a coordinate system at m with $\phi(U) \subset V$ and $x_i = y_i \circ \phi$ for $1 \leq i \leq d - k$. We claim $\frac{\partial}{\partial x_{d-k+i}}(q) \in N(q)$ for $1 \leq i \leq d - k$ if $\operatorname{ord}(q) = k$.

In fact: $d\phi(\frac{\partial}{\partial x_{d-k+i}}(q))y_j = \frac{\partial x_j}{\partial x_{d-k+i}}(q) = 0$

for $1 \leq j \leq d - k$ and all $q \in U$. Now if $\operatorname{ord}(q) = k$, $q \in U$ we have

$$d(y_{d-k+j} \circ \phi) = \sum_{r=1}^{d-k} \lambda_r \, d(y_r \circ \phi) \quad \text{at } q$$

for some real numbers λ_r and thus

$$d\phi(\frac{\partial}{\partial x_{d-k+i}}(q))y_{d-k+j} = 0$$

for $1 \leq j \leq \ell - d + k$ ∎

<u>Corollary 2.1.</u> Under the assumptions of lemma 2.5 if $X \in N(m)$ then we can extend X to a smooth vector field X on some neighborhood U of m in such a way that $X_q \in N(q)$ for all $q \in U$ with $\operatorname{ord}(q) = k$.

Proof. Using the coordinate system given by lemma 2.5 if $X_m = \sum_{i=1}^{k} a_i \frac{\partial}{\partial x_i}(m)$ then take $X = \sum_{i=1}^{k} a_i \frac{\partial}{\partial x_i}$ ∎

§3. <u>On the conjugate locus of a Riemannian Manifold.</u>

In this section we prove our results on the conjugate locus. These results are not restricted to Riemannian structures but hold for what F. W. Warner called in (4) a regular exponential map. It was proved in (4) that the exponential map for a Riemannian manifold, and more generally for a Finsler space is a regular exponential map.

Let M be a d-dimensional manifold and m a fixed point in M. Let $e: M_m \to M$ be a C^∞ map and if p is a point in M_m we let $N(p)$ denote the null-space of de at p and r_p the tangent space at p to the ray through p.

<u>Definition 3.1.</u> A map $e: M_m \to M$ is called a regular exponential map if it satisfies the following:

(R1) e is C^∞ on M_m except possibly at the origin where it is at least C^1, and $de(r_*(t)) \neq 0$ for all t, where r is any ray and $r_*(t)$ its tangent vector at $r(t)$.

(R2) The radial vector field T is everywhere transverse to e, i.e. $N_1(T_p) = 0$ for all $p \in M_m$, $p \neq 0$.

(R3) For each non zero point p in M_m there exists a convex neighborhood U of p such that the number of singularities of e (counted with multiplicities) on $r \cap U$, for each ray r which intersects U, is constant and equals the order of p as a singularity of e.

The set of singularities of a regular exponential map $e: M_m \to M$ is called the <u>conjugate locus of e</u> and it is denoted by $C(m)$. The conjugate locus splits naturally into two subsets, the <u>regular locus</u>, denoted by $C^r(m)$ and the <u>singular locus</u>, denoted by $C^s(m)$. (See (4) for definitions of these loci.)
We need the following

<u>Definition 3.2.</u> The <u>intersection number</u> or <u>branching order</u> of a point p in M_m, is that positive integer $\#(p)$ such that there exists some convex neighborhood U of p having the following property: for all convex neighborhood V of p, $V \subset U$ and for each ray r, $r \cap V$ has at most $\#(p)$ distinct conjugate points and there exists some ray r such that $r \cap V$ has exactly $\#(p)$ conjugate points.

Thus if $\#(r \cap V)$ denotes the number of conjugate points on $r \cap V$, we have

$$\#(p) = \inf_{\substack{V \text{ convex} \\ p \in V \subset U}} \sup_{r} \#(r \cap V)$$

We remark that if p is a conjugate point of order k, then property (r.3) of the regular exponential map implies $\#(p) \leq k$.

One natural and important thing in the study of the conjugate locus is to know how the order of the conjugate points are distributed near a conjugate point p. In the case where p is a regular conjugate point, property (R.3) of a regular exponential map says that all points near p have the same order. It was proved by F. W. Warner ((4), Th. 3.2) that $C^r(m)$ is an open dense subset of $C(m)$, having a structure of $(d-1)$-dimensional submanifold of M_m such that the inclusion $i: C^r(m) \to Mm$ is a submanifold with the relative topology. Moreover $N(p) \subset C^r(m)p$ if p is a regular conjugate point of order $k \geq 2$. Those results, in the case where M is an analytic Riemannian manifold and $e: M_m \to M$ its exponential, were proved before by J. H. C. Whitehead in (5). (With the restriction that if $k > \frac{d}{2}$ then $N(p) \subset C^r(m)p)$.

In this section we use the techniques developed in §2 to prove some results in the case where p is a singular conjugate point. We solve completely the problem in the case where the local conjugate variety is a union of two $(d-1)$-dimensional submanifolds L^1 and L^2 intersecting in general position at a point p. This is the kind of singular point that is found in product of two Riemannian manifolds. We also prove a sequence of results that eliminate the possibility of certain types of intersections at a conjugate point p .

Given a point p in M_m we have as in §2, the linear map

$$e_p^2\colon (M_m)_p \# (M_m)_p \to \frac{M^2_{e(p)}}{de(M_m)_p}$$

and it restricts to $(M_m)_p \# N(p)$ as

$$e_p^2\colon (M_m)_p \# N(p) \to \frac{M_{e(p)}}{de(M_m)_p}$$

For each vector A in $N(p)$ we associate the linear map

$$\phi_A\colon (M_m)_p \to \frac{M_{e(p)}}{de(M_m)_p}$$

defined by $\phi_A(X) = e_p^2(X \# A)$.

If L_p is any linear subspace of $(M_m)_p$ such that $L_p \# N(p)$ is contained in the kernel of e_p^2 , then for each $A \in N(p)$ we have the linear map ϕ_A^* induced by ϕ_A

$$\phi_A^*\colon \frac{(M_m)_p}{L_p} \to \frac{M_{e(p)}}{de(M_m)_p} \quad .$$

Lemma 3.1. With the above notation, we have:

1) $\phi_A^* = 0 \Leftrightarrow A = 0$

2) If $X_i \varepsilon (M_m)_p$, $i=1,\ldots,n$ where $n = d - \dim L_p$ then $\{\overline{X}_1,\ldots,\overline{X}_n\}$ linearly independent $\Rightarrow \bigcap_{i=1}^{n} N_1(X_i) = 0$, where $\overline{X}_i = X_i + L_p$.

3) $\dim(L_p \cap N(p)) \geq k - n + 1$ if $k \geq n + 1$ where k = order of p as a conjugate point.

Proof. 1) If $A = 0$ then it is trivial to see that $\phi_A^* = 0$.

Assume that $\phi_A^* = 0$. Thus

$$0 = \phi_A^*(r_p) = e_p^2(r_p \# A)$$

and using property (R.2) of a regular exponential map we see that $A = 0$.

2) Follows from 1). In fact if A is a vector in $\bigcap_{i=1}^{n} N_1(X_i)$ and $\{\overline{X}_i;\ 1 \leq i \leq n\}$ is linearly independent then $\phi_A^* = 0$ and thus by 1) , $A = 0$.

3) Assume $\dim(L_p \cap N(p)) \leq k - n$ and let C be any complementary subspace for $L_p \cap N(p)$ in $N(p)$. Thus $\dim C \geq n$. Let $\{X_1,\ldots,X_n\}$ be a basis for C . Thus $\{\overline{X}_i;\ 1 \leq i \leq n\}$ is linearly independent in $\dfrac{(M_m)_p}{L_p}$ and by 2) we have $\bigcap_{i=1}^{n} N_1(X_i) = 0$. But since $k \geq n + 1$ we have $L_p \cap N(p) \neq 0$. Now if $A \varepsilon L_p \cap N(p)$ we get $\phi_A^*(\overline{X}_i) = 0$ for $1 \leq i \leq n$ and then by 1) , $A = 0$,

contradicting the fact that $L_p \cap N(p) \neq 0$ ∎

We assume from now on that M and e are C^∞ .

Theorem 3.1. Let p be a conjugate point of order k and $\#(p) = 2$ and suppose that the conjugate locus near p consists of two $(d-1)$-dimensional connected submanifolds L^1 and L^2 intersecting in general position at p . (Thus $L = L^1 \cap L^2$ is a $(d-2)$-dimensional submanifold.) Then there exists a convex neighborhood U of p such that:

a) $\mathrm{ord}\,(q) = \mathrm{ord}\,(p) = k$ for all $q \in L \cap U$

b) $\mathrm{ord}\,(q) = k_i$ (k_i constant) for all $q \in (L^i - L) \cap U$ and for each fixed $i = 1,2$. Moreover $k_1 + k_2 = k$.

c) If $k \geq 3$ then $\dim(L_p \cap N(p)) \geq k - 1$ and $L_p^i \supset N(p)$ for some i , $1 \leq i \leq 2$. Moreover if $\dim(L_p \cap N(p)) = k - 1$ then $k_i = 1$ for some i .

d) $L_p^i \cap N(p) \neq 0$ for $i = 1,2$, $k \geq 2$.

Proof. a) It follows from the proof of Th. 3.1 of (4) and the property (R.3) of the regular exponential map that there exists some convex neighborhood U of p such that a) holds.

b) and c). We first show $e_p^2(L_p \# N(p)) = 0$.

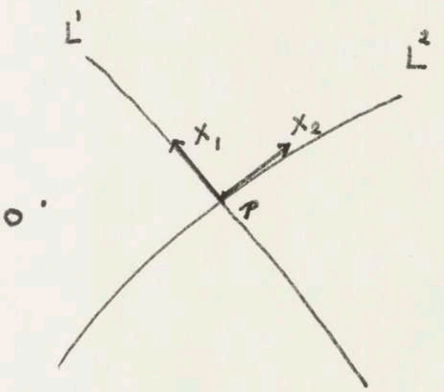
In fact, if $X_p \in L_p$ and $A_p \in N(p)$, let A denote the smooth extension of A_p given by Cor. 2.1. Now using (I), §2 and since $d^2e((XA)(p))f = X_p(A(f \circ e)) = 0$ we see that $e_p^2(X \# A) = 0$.

Now we note that since by assumption $\#(p) = 2$ we have $k \geq 2$ and for $k = 2$ the theorem is an immediate consequence of a) and the property (R.3) of e. Thus we may assume $k \geq 3$ and then by lemma 3.1 3), since $\dim L_p = d - 2$ we get

$$C = \dim(L_p \cap N(p)) \geq k - 1 .$$

We divide the proof into two cases

<u>1º case.</u> $c = k$ i.e., $N(p) \subset L_p$.



Choose $X_i \in L_p^i - L_p$ for $i = 1,2$ and thus $\{\overline{X}_1, \overline{X}_2\}$ is linearly independent in $\frac{(M_m)_p}{L_p}$. By lemma 3.1 part 2) we see that

$$N_1(X_1) \cap N_1(X_2) = 0 \qquad \text{(i)}$$

Now using lemma 2.2, we get

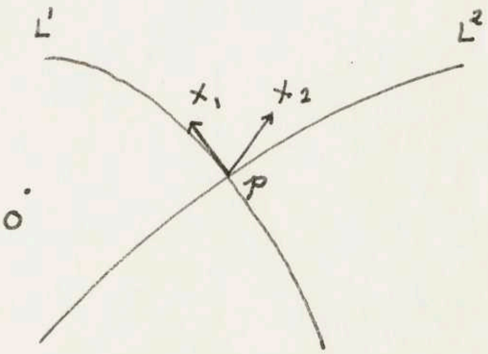
$$k_i = k_1(X_i) \geq \max_{q \in (L^i - L) \cap U} [\text{ord}(q)] \quad \text{for } i = 1,2 . \qquad \text{(ii)}$$

and by property (R.3) of e we see that

$$k_1 + k_2 \geq k \qquad \text{(iii)}$$

But (i) - (iii) implies $k_1 + k_2 = k$ and $\mathrm{ord}(q) = k_i$ for all $q \in (L^i - L) \cap U$ and for each $i = 1,2$.

<u>2º case.</u> $C = k - 1$. In this case it is possible to choose $X_i \in L^i_p - L_p$, X_i not in $N(p)$; for some i , say $i = 1$ and $X_2 \in N(p) - L_p$. Thus $\{\overline{X}_1, \overline{X}_2\}$ is linearly independent in $\frac{(M_m)_p}{L_p}$ and hence by lemma 2.2



part 2) we see that $N_1(X_1) \cap N_1(X_2) = 0$ (iv)

Now since $e^2_p | L_p \# N(p) = 0$, we see that $N_1(X_2) \supset L_p \cap N(p)$ and thus $k_2 = k_1(X_2) \geq k - 1$ (v)

But lemma 2.2 says

$$k_1 = k_1(X_1) \geq \max_{q \in (L'-L) \cap U}[\mathrm{ord}\,(q)] > 0 \qquad \text{(vi)}$$

Now from (iv) - (vi) we get that

$$k_1 + k_2 = k \qquad \text{and} \qquad k_1 = 1 .$$

Thus $\mathrm{ord}(q) = k_i$ for all $q \in (L^i - L) \cap U$ and each i , $1 \leq i \leq 2$. Now we claim that $L^i_p \supset N(p)$ for some i , $1 \leq i \leq 2$. For, otherwise we can choose $Y_i \supset N(p) - L^i_p$ and $X_i \in L^i_p - L_p$ for some $i = 1,2$ and thus by lemma 3.1 part 2) we get that

$$N_1(X_i) \cap N_1(Y_i) = 0 \text{ , } i \leq i \leq 2 \qquad \text{(vii)}$$

Then as in (v) above we see that

$$N_1(Y_i) \supset L_p \cap N(p) , \qquad 1 \leq i \leq 2$$

and thus $k_1(Y_i) \geq k - 1$, $1 \leq i \leq 2$ (viii)

and

$$k_i = k_1(X_i) \geq \max_{q \,\varepsilon\, (L^i - L) \cap U}[\mathrm{ord}\ (q)] > 0 \qquad \text{(ix)}$$

for $1 \leq i \leq 2$.

Now from (vii) and (ix) we see that $k_i = 1$ for $i = 1,2$, contradicting property (R.3) of e . Statement d) is just the fact that each submanifold L^i has co-dimension 1 ∎

<u>Corollary 3.1.</u> Under the assumptions of Theorem 3.2 if $k_i \geq 2$ for $i = 1,2$ then $d \geq k + 2$.

Proof. From part c) of the above theorem we see that since $k_i \geq 2$, then $N(p) \subset L_p$ and thus $d \geq k + 2$ since L has co-dimension 2 in M_m .

<u>Corollary 3.2.</u> Under the assumptions of Theorem 3.2 if $N(p) - L_p^i \neq 0$ for $i = 1,2$, then $k = 2$ and thus $k_1 = k_2 = 1$.

Proof. By part c) of the above theorem we see that since $N(p) - L_p^i \neq 0$ for $i = 1,2$, then $k = 2$ and thus by property (R.3) of e we have $k_1 = k_2 = 1$.

<u>Corollary 3.3.</u> Let $p \in M_m$ be a conjugate point of order k and $\#(p) = 3$ and suppose that the conjugate variety near p consists of three $(d-1)$-dimensional submanifolds L^i of M_m, $1 \leq i \leq 3$ intersecting at p in general position, i.e.

1) Each two submanifolds L^i and L^j are in general position for $i \neq j$, $1 \leq i$, $j \leq 3$. (Thus $L^{ij} = L^i \cap L^j$ is a $(d-2)$-dimensional submanifold of M_m, for $i \neq j$.)

2) L^{ij} and L^s are in general position for all $1 \leq i, j, s \leq 3, s \neq i,j$, $i \neq j$. (Thus $L = \bigcap_{i=1}^{3} L^i$ is a $(d-3)$-dimensional submanifold of M_m.)

Then:

a) For each ordered triple (i,j,k) of distinct integers, $1 \leq i, j, k \leq 3$ all points in $L^i - (L^{ij} \cup L^{ik})$ have the same order k_i.

b) For each pair (i,j) of distinct integers, $1 \leq i$, $j \leq 3$, all points in $L^{ij} - L$ have the same order $k_i + k_j$.

c) All points in L have the same order k.

Proof. a) Since $\dim L = d - 3$ we see that $L^i - L$ is connected for each $i = 1,2,3$. Now each point in $L^i - (L^{ij} \cup L^{ik})$ is a regular conjugate point and any two points p_1 and p_2 in $L^i - (L^{ij} \cup L^{ik})$ can be joined by an arc γ that will meet at most a (finite) number of singular conjugate points of the type described in Theorem 3.2. Thus we can cover γ by a finite number of convex neighborhoods given by property (R.3) of e. Using Theorem 3.2 we see that $\mathrm{ord}(p_1) = \mathrm{ord}(p_2) = k_i$.

b) follows from a) and property (R.3) of e.

c) follows from property (R.3) of e ∎

We now show by examples that if k is a singular conjugate point of order two then we can have the three cases:

a) $L_p \cap N(p) = 0$

b) $\dim (L_p \cap N(p)) = 1$

c) $\dim (L_p \cap N(p)) = 2$

To see this we note first that if M and N are Riemannian manifolds, $m \in M$, $n \in N$ and $e^1 \colon M_m \to M$, $e^2 \colon N_m \to N$ are the respective exponential maps then if C^1 and C^2 are the respective (first) conjugate locus of e^1 and e^2 then the conjugate locus of the exponential map

$$e: (M \times N)_{(m,n)} \longrightarrow M \times N$$

(here $M \times N$ is the product manifold with the product metric and we use the natural iso. $(M \times N)_{(m,n)} \simeq M_m \oplus N_n)$, is

$$(C^1 \times N_n) \cup (M_m \times C^2) \tag{*}$$

and the singular locus is given by

$$L = (C^1 \times N_n) \cap (M_m \times C^2) . \tag{**}$$

if both C^1 and C^2 have only regular conjugate points. This is well-known and can be found in (8) .

Now take $M = N = E^2$ the 2-dimensional Riemannian ellipsoid. The (1st) conjugate locus of E^2 is an ellipse E and the null spaces of the differential of e^1 are only tangent to this ellipse at the ends of its major and minor axes since the null space in the Riemannian case is always orthogonal to the rays in the tangent space.

1) To see a) take $p = q$ not at the end of one of the principal axes of E . Thus $N(p) \cap E_p = 0$ and $N(p,q) \approx N_p \oplus N_q$ and thus $N(p,q) \cap L_{(p,q)} = 0$ (L as in (**) above).

2) To see b) take p as in 1) above and q at the end of a principal axis.

3) To see c) take $p = q$ as in 2) above.

We call a differentiable variety V a "smooth cone at $p \varepsilon M_m$" if V is the image under a diffeomorphism $\phi: R^d \to M_m$, $\phi(0) = p$ of the algebraic variety C given by

$$C = [q \varepsilon R^d ; f(q) = 0]$$

where $f(x_1,\ldots,x_d) = \sum_{i=1}^{d-1} x_i^2 - x_d^2$. We have the following:

<u>Proposition 3.1.</u> Let p be a singular conjugate point, such that $\#(p) = 2$, and assume that the local conjugate variety at p is a smooth cone. Then all points in $V - p$ have the same order $\frac{k}{2}$ if k is the order of p .

Proof. Let $X \varepsilon R_o^d$, $d\phi(X) = r_*(1)$ where r is the ray through p in M_m (i.e. $r(t) = tp$) . We note that X is not tangent to any smooth arc in C since $r_*(1)$ is not tangent to any smooth arc in V (to see this use property (R.2) of e and lemma 2.2).

Let P be any 2-plane through 0 in R^d such that $X \varepsilon P_o$. Then $P \cap C$ consists of two C^∞ curves intersecting in general position at 0 . Let $\gamma^i: (a,b) \to M_m$, $\gamma^i(1) = p$ be their images under ϕ . Then $\{ \gamma^i_*(1) ; i = 1,2 \}$ is linearly independent so

$$r_*(1) = \lambda\gamma_*^1(1) + \mu\gamma_*^2(1)$$

where λ and μ are real numbers different from zero.

It follows from property (R.2) of e that

$$N_1(\gamma_*^1(1)) \cap N_1(\gamma_*^2(1)) = 0 .$$

Since $V - p$ has two connected components exachly consisting of regular points with the same order and since each curve γ^i has points in both components, using property (R.3) of e and lemma 2.2, we see that

$$N_1(\gamma_*^1(1)) \oplus N_1(\gamma_*^2(1)) = N(p)$$

and if $k_i = k_1(\gamma_*^i(1))$ then $k_1 = k_2 = \frac{k}{2}$ and thus all points in $V - p$ have the same order, $\frac{k}{2}$ ∎

To prove the next theorem we need

Lemma 3.2. With the assumptions of lemma 3.1 suppose that $\dim L_p = d - 2$ and let $X_i \in (M_m)_p$ such that $N_1(X_i) \neq 0$ for $i = 1,2,3$. If $\overline{X}_i = X_i + L_p$ are pairwise linearly independent in $\frac{(M_m)_p}{L_p}$, then the subspaces $N_1(X_i)$ are independent for $i = 1,2,3$.

Proof. It suffices to show that

$$N_1(X_3) \cap (N_1(X_1) + N_1(X_2)) = 0 . \tag{*}$$

To show this we note first, that since by assumption $\overline{X}_i$ are pairwise linear by independent, then

$$X_3 = \lambda_1 X_1 + \lambda_2 X_2 + X_o \qquad \text{for some reals}$$

$\lambda_1, \lambda_2 \neq 0$ and $X_o \in L_p$. (Recall $\dim L_p = d - 2$.)
Now let $A \in N_1(X_3) \cap (N_1(X_1) + N_1(X_2))$. Thus $A = A_1 + A_2$, $A_i \in N_1(X_i)$ $i = 1,2$, and hence

$$0 = e_p^2(X_3 \# A) = \lambda_1 e_p^2(X_1 \# A_2) + \lambda_2 e_p^2(X_2 \# A_1) .$$

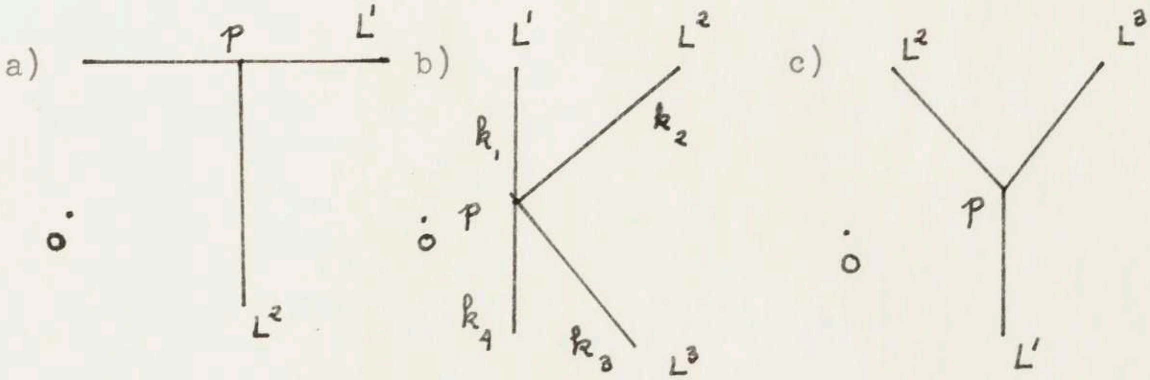
Thus to show (*) it suffices to show that $\{ e_p^2(X_1 \# A_2) , e_p^2(X_2 \# A_1) \}$ is linearly independent if $A_1, A_2 \neq 0$. To see this note that since $\dim L_p = d - 2$, using property (R.2) of e we have: $r_*(1) = \mu_1 X_1 + \mu_2 X_2$ for some reals $\mu_1, \mu_2 \neq 0$ (here $r(t) = tp$ is the ray through p in M_m). Now if C_i , $i = 1,2$ are any real numbers, we have:

$$e_p^2(r_*(1) \# (C_1 A_1 + C_2 A_2)) =$$

$$\mu_1 C_2 e_p^2(X_1 \# A_2) + \mu_2 C_1 e_p^2(X_2 \# A_1) = 0 .$$

This implies, by property (R.2) of e , that $C_1 A_1 + C_2 A_2 = 0$ and since by lemma 3.1 we know that $N_1(X_1) \cap N_1(X_2) = 0$, we get $C_1 = C_2 = 0$ ∎

<u>Theorem 3.2.</u> The local conjugate variety at a point p in M_m cannot look like the pictures



where: 1) each L^i is a $(d-1)$-dimensional submanifold of M_m (submanifolds with boundary at the intersection, except L^1 for pictures a) and b)).

2) L^i and L^j intersect in general position at p for $1 \leq i, j \leq 3$, $i \neq j$.

3) $L = \bigcap_{i=1}^{3} L^i$ is a submanifold and $\dim L \leq d - 2$.

Proof. 1º case. $\dim L = d - 2$.

We first prove that a) and c) cannot happen. In fact, in both cases using property (R.3) of e and lemma 2.2, we see that there exists $X_i \in L_p^i - L_p$ such that $N_1(X_i) = N(p)$ for some $i = 1,2,3$, say $i = 1$. Now since all points in L have order equal to the order of p, then as in the proof of theorem 3.1, we see that

$$e_p^2(L_p \# N(p)) = 0 .$$

Thus by lemma 3.1, if $X_2 \in L_p^2 - L_p$ then $N_1(X_1) \cap N_1(X_2) = 0$

which with $N_1(X_1) = N(p)$ implies $N_1(X_2) = 0$ and lemma 2.2 says $L^2 - L$ has no conjugate points, contradiction.

To prove that b) cannot exist we take $X_i \in L_p^i - L_p$, $i = 1,2,3$ and then by lemma 3.2 we see that the subspaces $N_1(X_i)$ are independent for $i = 1,2,3$. Now assume that the order of the pieces are as follows:

ord $(L^i - L) = k_i$ for $i = 2,3$.

L^1 has points of order k_1 and k_4.

Using property (R.3) of e and lemma 2.2 we see that

$$k_1(X_i) \geq k_i \quad \text{for} \quad i = 2,3 \;, \quad k_1(x_1) \geq k_1, k_4 \tag{i}$$

and

$$k_1 + k_2 = k \;, \qquad k = \text{ord}(p) \;. \tag{ii}$$

(see picture b)).

Since $N_1(X_i)$ are independent for $i = 1,2,3$ we have $k_1(X_1) + k_1(X_2) + k_1(X_3) \leq k$ and thus by (i) we see that

$$k_1 + k_2 + k_3 \leq k \;. \tag{iii}$$

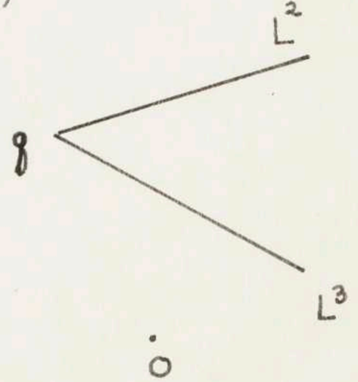
Now (ii) and (iii) implies $k_3 = 0$ which as before gives a contradiction.

<u>2 º case.</u> dim $L \leq d - 3$ (only for pictures b) and c) .

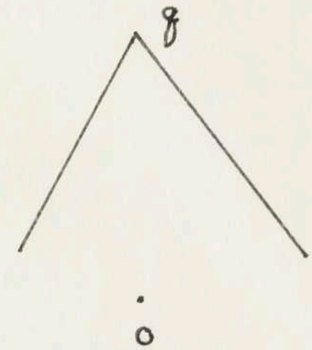
Since dim $L \leq d - 3$ and by assumption L^1 and L^2 intersect in general position at p , then $L^2 \cap L^3 - L \neq \emptyset$.

But if $q \in L^2 \cap L^3 - L$ the local conjugate variety at q looks like

(i)



(ii)



(i) contradicts property (R.3) of e and (ii) says q is a regular conjugate point and thus L^2 and L^3 are not in general position at q, contradicting the assumption that L^2 and L^3 are in general position at p ∎

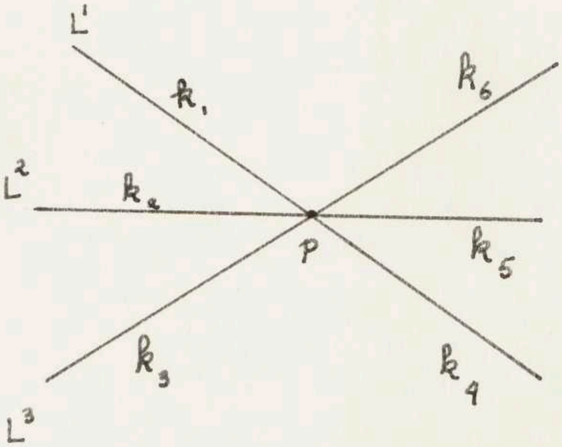
<u>Proposition 3.2.</u> Let $p \in M_m$ be a conjugate point of order k and $\#(p) = 3$. Suppose that the local conjugate variety at p consists of three $(d-1)$-dimensional submanifolds L^i, $i = 1,2,3$, of M_m and that

a) L^i and L^j intersect in general position at p for $1 \leq i, j \leq 3$, $i \neq j$.

b) $L = \bigcap_{i=1}^{3} L_p^i$ is a $(d-2)$-dimensional submanifold of M_m.

Then all points in $L^i - L$ have the same order k_i for $i = 1,2,3$ and $k_1 + k_2 + k_3 = k$.

Proof.



Choose $X_i \in L_p^i - L_p$ for $i = 1,2,3$. Since all points in L have the same order k , as in the proof of Theorem 3.1 we can show $e_p^2(L_p \# N(p)) = 0$. Now by lemma 3.2 the subspaces $N_1(X_i)$ are independent for $i = 1,2,3$. (i)

Let the orders of the pieces be as follows:

$L^i - L$ has two pieces of orders k_i and k_{i+3} for $i = 1,2,3$.

We have to show $k_i = k_{i+3}$ for $i = 1,2,3$.

Lemma 2.2 implies

$$k_1(X_i) \geq k_i \ , \ k_{i+3} \ , \ 1 \leq i \leq 3 \ . \qquad \text{(ii)}$$

On the other hand (i) , (ii) and property (R.3) of e say

$$k_1(X_1) + k_1(X_2) + k_1(X_3) = k \ . \qquad \text{(iii)}$$

But (ii) and (iii) implies

$$k_i = k_{i+3} \quad \text{for} \quad i = 1,2,3 \ \blacksquare$$

We remark that if $p \in M_m$ is a regular conjugate point and if L is the local conjugate submanifold at p then as in the proof of Theorem 3.1, we see that $e_p^2(L_p \# N(p)) = 0$. Then by lemma 3.1, we get $N(p) \subset L_p$ if $\operatorname{ord}(p) \geq 2$.

This is a result of F. W. Warner and J. H. C. Whitehead cited at the beginning of this section.

§4. Tangent Vectors to the Conjugate Variety

In this section we give a characterization of the tangent vectors to the conjugate variety at a point $p \varepsilon M_m$, in the cases where p is a regular conjugate point or when p is a singular conjugate point as in Theorem 3.1. We use here some notation and definitions of (7) .

Definition 4.1. A cone is a subset C of R^d with the following properties:

1) $\lambda X \varepsilon C$ for all $X \varepsilon C$ and $\lambda \varepsilon R$ (here R denotes the real numbers).

2) C is a closed set in R^d .

Given a point $p \varepsilon R^d$ we say that a subset C_p of R^d is a cone at p if

$$C_p = p + C = [p + X \ , \ X \varepsilon C]$$

where C is a cone in R^d .

Definition 4.2. Let $e: M_m \to M$ be a regular exponential map and p be a point in M_m . The conjugate cone at p , is the subset C_p of M_m given by

$$C_p = [X \varepsilon (M_m)_p \ ; \ e_p^2(X \# A) = 0 \ \text{ for some } \ A \varepsilon N_p \ , \ A \neq 0] \ .$$

It is easy to see from the linearity and continuity of the symmetric tensor product that C_p is actually a cone at p .

We note that $C_p = \bigcup_{\substack{A \varepsilon N_p \\ A \neq 0}} S_A$ where S_A is the linear subspace of $(M_m)_p$ given by

$$S_A = [X \varepsilon (M_m)_p \ ; \ e_p^2(X \# A) = 0] \ .$$

We denote by I_p the intersection

$$I_p = \bigcap_{A \varepsilon N_p} S_A$$

and by $i(p)$ the dimension of I_p . It follows from property (R.2) of e that

$$i(p) \leq \dim S_A \leq d - 1 \tag{4.1}$$

<u>Theorem 4.1.</u> Let $e\colon M_m \to M$ be a regular exponential map and $p \varepsilon M_m$. Then

1) $C^R(m)_p = C_p = I_p$ if p is a regular conjugate point.

2) $C_p = L_p^1 \cup L_p^2$, $I_p = L_p$ if p is a singular conjugate point as in Theorem 3.1 (here we use the notation of Theorem 3.1).

Proof. 1) Since $C^R(m)$ is a (d-1)-dimensional submanifold of M_m (c.f. (4) , Theorem 3.1), using inequality

(4.1) , it suffices to show that

$$C^R(m)_p \subset S_A \quad \text{for all} \quad A \in N_p \ .$$

To show this we note that given $A \in N_p$, if A is the extension of A given by Corollary 2.1, we have

$$e_p^2(X \# A) = 0 \quad \text{for all} \quad X \in (M_m)_p \ .$$

This follows exactly in the same way as in the proof of Theorem 3.1, and we recall only that since p is by assumption a regular conjugate point, then all conjugate points near p have order equal to the order of p .

2) We first note that lemma 2.2 implies $L_p^1 \cup L_p^2 \subset C_p$ and from the proof of Theorem 3.1 we have $e_p^2(L_p \# N_p) = 0$, i.e., $L_p \subset I_p$.

To show that $C_p \subset L_p^1 \cup L_p^2$ we show that if X is not in $L_p^1 \cup L_p^2$ then X is not in C_p . To see this take X not in L_p^i , and $X_i \in L_p^i - L_p$, for $i = 1,2$. Now from Theorem 3.1 we have

$$N_1(X_1) \oplus N_1(X_2) = N_p \qquad (*)$$

and by lemma 3.2

$$N_1(X) \cap (N_1(X_1) \oplus N_1(X_2)) = 0 \qquad (**)$$

But (*) and (**) implies $N_1(X) = 0$, i.e., X not in C_p .

To see that $I_p \subset L_p$ it suffices to note that Theorem 3.1 implies that if $Y \varepsilon L_p^i - L_p$, then $k_1(Y) = k_i < k$ and thus Y is not in I_p ∎

§5. Some Problems on the Conjugate Locus.

We need some definitions and notations.

We recall that a submanifold of M_m is a pair (L,ϕ) where L is a manifold, $\phi\colon L_i \to M_m$ is a 1:1 smooth map and $d\phi$ is 1:1 .

Definition 5.1. We say that two submanifold (L^i,ϕ^i) , $i = 1,2$ of M_m have a contact of order c (c being a positive integer) at a point $p \in M_m$ if the following is true:

1) $\phi^i(p_i) = p$, $p_i \in L^i$ for $i = 1,2$.

2) $d^r\phi^1((L^1)^r_p) = d^r\phi^2((L^2)^r_p)$ for $1 \leq r \leq c$.

If the submanifolds (L^i,ϕ^i) have contact of all orders at p , we say that they have an infinite contact there.

We say that a family $\{S_i\}$ of subsets of M_m is locally finite if each point $p \in M_m$ has a neighborhood U intersecting only a finite number of subsets S_i .

Definition 5.2. A subset S of M_m is said to be weakly stratified if it has the following properties:

1) S is a locally finite disjoint union of C^∞ submanifolds of M_m, $S = \bigcup_i L_i$. (The L_i are called the strata of S.)

2) The boundary $\partial L_i = \overline{L}_i - L_i$ (here $\overline{L}_i$ stands for the closure of L_i) of a stratum is a union of lower dimensional strata.

<u>Definition 5.3.</u> A subset S of M_m is said to be <u>strongly</u> <u>stratified</u> if it is a locally finite disjoint union of closed submanifolds.

Now we point out some problems on the conjugate locus of a Riemannian manifold.

<u>Main</u> <u>Conjecture</u>: the conjugate locus is, like the analytic varieties, a weakly stratified subset of M_m. Moreover, all the points in a same stratum have the same order.

A stronger conjecture is: The conjugate locus is a strongly stratified subset of M_m.

In relation with these conjectures a more specific problem is to know whether or not there exists a smooth cone as stated in Proposition 3.1. Of course a smooth cone is a weakly stratified subset, but not a strongly one.

We now state some more specific problems.

1) Is it possible near a conjugate point $p \in M_m$ for the conjugate locus to have two $(d-1)$-dimensional submanifolds having an infinite order of contact? Obviously, this is not possible for an analytic variety.

2) It would be very interesting to have examples of different types of intersections at a conjugate point.

In particular, is there an example of a conjugate point $p \in M_m$ near which the conjugate locus is a union of two $(d-1)$-dimensional submanifolds, not in general position at p ?

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BIOGRAPHY

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