DIFFERENTIAL GAMING MODELS
OF OLIGOPOLY

by

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A.B. University of Chicago
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Submitted to the Department of Economics
in Partial Fulfillment of the
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ABSTRACT

This thesis applies the theory of differential games to models of competition in an oligopoly. Conventional static Nash equilibrium requires each firm to act optimally, given the actions of its rivals. Because of the one-period nature of static models, no reaction is possible. The differential gaming approach requires each firm to choose an optimal equilibrium strategy, which specifies how it will act as a function of the state variables (e.g., the levels of capital of the firms, or the market share of each firm). Consideration of the reaction effect leads to equilibria which differ from the static Nash equilibrium.

The optimal control techniques used for solving differential games require the value functions to be continuous. This assumption eliminates supergame equilibria with implicit collusion, where a small deviation from equilibrium by one firm (i.e., cheating) is punished by reverting to competitive behavior. The differential games examined in this thesis describe the nature of competitive behavior in a multi-period context. It is shown that the steady state of the dynamic games is different from repeated Nash-Cournot or Nash-Bertrand behavior. Also, the consistent Conjectures equilibrium is shown to have no theoretical foundation.

Chapter 3 analyzes a linear-quadratic differential duopoly game, in which each firm can observe the current level of capital of the two firms, and may invest or divest, subject to convex adjustment costs. The Riccati equations are solved numerically for particular values of the parameters. The steady state of the perfect and the open-loop equilibrium are computed, and compared with static equilibria. In a perfect equilibrium, a rival will react to a quantity increase by accommodating. This makes aggressive
behavior more attractive, yielding an equilibrium output greater than the static Cournot-Nash level. When there is no reaction effect, as in the case of an open-loop equilibrium, or in a perfect equilibrium with very large adjustment costs, the steady state equals the Cournot equilibrium.

Chapter 4 analyzes a linear-quadratic model of price competition in the presence of goodwill effects, in which demand gradually shifts between firms in response to price differentials. Numerical solution of the Riccati equations shows that a rival will react to price decreases by fighting back. This makes aggressive behavior less attractive, yielding an equilibrium price above the static Bertrand-Nash level. When demand does not respond to price differentials, the steady-state price equals the monopoly level. The Bertrand-Nash equilibrium corresponds to the case where demand shifts instantaneously in response to price differentials and firms can precommit to a price.

Chapter 5 analyzes a probabilistic model of entry deterrence. It shows that, even when investment is reversible, capital is still a credible entry deterrent if investment is subject to convex adjustment costs. The optimal investment strategy for a monopolist faced with potential entry may be to deter entry completely, or to retard entry.

Thesis Supervisor: Dr. Robert S. Pindyck
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Finally, I would like to dedicate this thesis to the memory of Virginia Dubina, who introduced me to the world of academics, and who shaped my American persona.
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CHAPTER 1

INTRODUCTION

The title of the present thesis is "differential gaming models of oligopoly." To a reader unfamiliar with economic jargon, these words are mere gobbledygook. Indeed, economists are sometimes accused of clothing common sense into fancy language and passing it off as science. To dispel any such false accusations, I will try to explain the title of this thesis, as well as the reasons for its relevance, in terms which any layman will hopefully understand.

Oligopoly theory is concerned with the study of markets with few sellers. An example of such a market might be the automotive industry, with GM, Ford, and Chrysler as the major producers. Economists study this type of market in contrast to competitive markets and monopolistic markets. In competitive markets (e.g., grain), there are so many sellers that each seller's output has such an insignificant effect on price as to be negligible. Under competition, the price is determined by "the market" or "supply and demand." In a monopoly, there is only a single seller, who is free to produce as much (or as little) output as will maximize his profits.

The determination of output and prices in an oligopoly is much more complex, and less thoroughly understood than in
either monopoly or competition. This is due to the fact that with only a few sellers, each firm is sufficiently big to have a major impact on the profits of its rivals. As an example, if Ford is charging rock-bottom prices, and selling vast numbers of cars, this will clearly affect the demand for GM cars. In setting its own prices, the marketing department at GM presumably must spend some of its time guessing at the prices which its competition will be charging. This type of situation, where one firm can affect the profits of another, and both must guess about what actions the rival will take has been modeled through the use of game theory.

Game theory attempts to predict the outcome of situations in which each "player" can effect the "payoff" of his rival by his choice of an "action." It defines several "equilibrium" concepts, each of which corresponds to some "rational" way of predicting the behavior of the two players. The best-known such equilibrium concept is the "Nash equilibrium."

This Nash equilibrium concept was developed for the following abstract game. Two players will play a game exactly once. Both understand how a given choice of actions affects the payoffs of the two players. They simultaneously must choose their actions. The "rational" outcome, according to Nash, is for the two players to pick actions with the following property: each player is doing the best he can,
given the other player's actions.

Translating this abstract game back to the automotive example, each of the 3 automakers knows that a large production of automobiles will depress car prices. The production levels which they choose according to Nash are those one which allow each company's VP of marketing to go to his CEO and claim triumphantly: "Not only did I guess exactly how many cars the other two firms were going to produce, I even maximized our profits given this correct guess!" The fact that it is possible to calculate a set of outputs with this "correct in hindsight" property is usually one of the subjects of an intermediate microeconomics course.

Unfortunately, not all oligopolies appear to follow the predictions of the Nash gaming model. The OPEC oil cartel is an example of how a small group of sellers can band together and attempt to act as a monopolist. Recent history suggests that, although it is in the joint interest of all oil producers to keep price high by keeping production low, it can be difficult to prevent defection from the cartel output. This suggests that oligopoly behavior may fall into two separate categories: "collusive" behavior like that of OPEC during its prime, and "competitive" behavior, like the current OPEC, or like the automakers.

Much of the recent effort in game theory has been devoted to explaining how rational, Nash-like behavior can
result in collusive outcomes. The fundamental insight is that, contrary to the assumption of the "static" Nash model, oligopolists do not engage in a once-and-for-all game. Instead, they compete with each other repeatedly, day after day. Once this "dynamic" nature of many real-life games is recognized, it is possible to explain why a firm may want to depart from Nash behavior.

In a game which is played repeatedly, collusion may actually be self-interestedly rational behavior. Suppose OPEC has agreed on a set of production quotas which would keep the price of oil at a "more-than-competitive" level. It is a fact that each individual OPEC member has an incentive to exceed his quota. Also, OPEC clearly has no legal mechanism for enforcing its agreement.

Nevertheless, members of OPEC may find it in their own interest to forego the temptation of exceeding the quota. This is because, in a dynamic game, other producers would notice the transgression, and respond by also exceeding their quotas, leading to the collapse of the cartel. In other words, the short-term gain from exceeding the quota may be less than the negative effect of causing the collapse of the cartel.

The problem which remains to be solved is that, alas, repeated games typically have a multitude of perfect equilibria. Whereas traditional static game theory could predict only "competitive" behavior, and was unable to
predict "collusive" behavior, the new dynamic game theory predicts so many possible outcomes as to be indefinite.

To their credit, the dynamic gaming models encompass both competitive and collusive outcomes as rational equilibria. The collusive equilibria are "supported" by the threat that collusion may collapse, i.e. all players revert to competitive behavior. It has usually been assumed that "competitive" behavior in a dynamic game would correspond to the behavior which was rational in the single-period game, i.e. the good, old-fashioned Nash equilibrium. The major point of this thesis is that this is not the case.

More explicitly, static models fail to capture real-life oligopoly behavior because they fail to capture the reaction effect. The breakdown of implicit collusion is an extreme example of a reaction to a player's actions. But even in the absence of implicit collusion, the reaction effect will substantially affect the equilibrium.

The technique which this thesis uses to model this effect is called the theory of differential games. It is a particular kind of dynamic game, which is played continuously, rather than in discrete time increments. This theory was developed originally to find the best strategy which a missile should follow which is attempting to shoot down an enemy missile, which in turn is trying to evade. The fact that missiles obey the laws of physics, and that these laws provide for continuous, differentiable functions,
allows a sophisticated mathematical apparatus to be applied to this problem.

It will be argued in chapter 2 that differential game equilibria are a special case of general dynamic game equilibria, and that they represent the competitive rather than the collusive equilibrium. It will then be shown in chapters 3 and 4 that this "dynamic-competitive" equilibrium is different from the conventional Nash equilibrium. The characterization of this equilibrium is the main object of this thesis.

Returning to the automaker example, the triumphant VP of marketing earned the respect of his CEO because he appeared to have done the right thing, given what everyone else had done. But this situation is unlikely to persist over time. Suppose for a moment that, due to a scheduling error, Ford produces more cars than the Nash equilibrium level. The differential gaming model of chapter 3 shows that it is rational for GM and Chrysler to react by reducing their output slightly. This mitigating "reaction effect" leads Ford to be more aggressive than it would be in a static situation with no possibility of reaction.

The model of chapter 4 considers a situation in which GM, Ford, and Chrysler compete by setting prices, rather than output levels. In this case a similar result holds. Static Nash competition would predict that the three firms would compete quite aggressively, by lowering prices to
steal customers away from each other. The differential gaming model shows that if Ford cuts price, it is rational for GM and Chrysler to also decrease their price. This leads all the firms to be less aggressive in their price cutting, since they all realize that their price decreases will be immediately offset by the rivals' price decreases.

Chapter 5 examines a model of entry deterrence. This is a situation where a monopolist is currently the only seller in a market, but there are potential entrants into his industry. Of course, the incentives for entry by a new firm will depend on the profit which the entrant expects to make after entry. Since the post-entry market will involve two sellers, it will be an oligopoly, and is therefore amenable to analysis as a differential gaming model. The resulting model of entry deterrence exhibits some features different from conventional models.

The preceding pages have tried to explain both the content and the relevance of this thesis in the simplest possible terms. The remainder of this introduction will be more technical. Section 1.1 relates the analysis of this thesis to several well-known articles, including the Cournot and the Bertrand model. Section 1.2 summarizes the main issues and findings of this thesis, and section 1.3 gives a detailed plan of the organization of the remaining chapters.
1.1 Background

The unifying theme of the present thesis is the use of differential games as a tool for modeling oligopoly behavior. This approach is used in several contexts, each with its own body of literature. Rather than reviewing all the relevant literature for each area, this section introduces the differential gaming approach by comparing and contrasting it with the approach of a small number of well-known articles.

The static model relevant for comparison with chapter 3 is that of Cournot (1838). This comparison highlights the differences between static games and the differential gaming approach. It also previews the discussion of chapter 2, which explains the difference between the differential gaming approach and the "implicit collusion" equilibria of infinitely repeated games.

The next subsection examines the relation between the static Bertrand model and its dynamic version, presented in chapter 4. The dynamic model assumes that demand does not instantaneously shift in response to price differentials, to to the presence of a "goodwill effect."

Next, the differential gaming approach is compared to the "consistent conjectures" model of Bresnahan (1981), which attempts to capture the dynamic "reaction effect" in a static setting. It will be argued that the consistent conjectures approach does not provide a satisfactory
approximation to the dynamic problem.

The fourth set of models for comparison are those of Spence (1979) and Fudenberg and Tirole (1983a). The model of investment in a duopoly, presented in chapter 3, is closely related to these articles, and differs from them in the assumption that investment is reversible.

Finally, the analysis of entry deterrence in chapter 5 is closely related to the model of Kamien and Schwartz (1971). They assume that pre-entry price does not affect post-entry profits, but also assume that the likelihood of entry depends on the pre-entry price. It will be argued that these two assumptions are not consistent, but that their model can be extended in a way that meets this objection.

The remainder of this section will discuss the differences and commonalities between the present thesis and the above list of articles. The emphasis will be on explaining how the differential gaming approach modifies the gaming aspect of each of these well-known articles.

The Cournot Model

The reference to "dynamic gaming" in the title of this thesis is clearly in contrast to "static gaming" models, of which the best-known are those of Cournot (1838) and Bertrand (1883). The difference between these two models is that the Cournot model assumes that firms choose quantities, and the market determines the price, whereas the Bertrand
model assumes that firms select prices, and the market determines their respective demands.

The differential gaming models of this thesis can also be of the quantity-setting or the price-setting variety. Indeed chapters 3 and 5 are based on a quantity-setting model, whereas chapter 4 assumes price-setting firms. However, the differential gaming approach differs from the static models in assuming that there isn't just a single quantity (or price) to be chosen, but a sequence of quantities (or prices). I will use the Cournot quantity-setting model as the basis of the comparison, and then extend the comparison to the Bertrand model.

The Cournot model can be described as follows. There are two firms which produce a homogeneous good. Their level of output of this good is given by \( q_1 \) and \( q_2 \) respectively, and the market price of the good is determined by the (inverse) demand function \( P=P(Q) \), where \( Q=q_1+q_2 \). Each firm has a cost function \( C(q_i) \). Hence the profit maximization problem for firm \( i \) is to choose \( q_i \) as to maximize

\[
\pi_i = P(q_i+q_j)q_i - C(q_i)
\]

The formal solution of this problem is found from the following first-order condition

\[
0 = P(q_i+q_j) + \frac{dP}{dQ}(1+\frac{\partial q_j}{\partial q_i})q_i - \frac{dC}{dq_i}
\]

This equation can be solved implicitly for the profit-maximizing \( q_i^* \), which is given by \( q_i^* = f_i(q_j, \frac{\partial q_j}{\partial q_i}) \). Clearly, firm \( i \)'s optimal output cannot be determined without
knowledge of firm j's output. Similarly, firm j must "guess" firm i's output in order to determine its own optimal output. This interdependence is the salient feature of an oligopolistic market, and can be formalized into a gaming model.

However, the specification of the model is not quite complete. Will this game be played only once, or will it be played repeatedly, maybe even continuously? The Cournot assumption is that the game will be played only once. In addition, each firm chooses its output without knowledge of the other firm's output. This latter assumption implies that $\delta q_j/\delta q_i = 0$. In other words, there is no way for firm i's output to influence firm j's output, and vice versa. Of course, if the game were played repeatedly, firm i's current output might influence firm j's output in the next period, but this would have to be investigated in an explicitly multi-period game.

This discussion showed that, in a single period, simultaneous choice game (i.e. the Cournot assumption), firm i's optimal quantity depends on firm j's output in a way described by

$$q_i^* = f_i(q_j) \quad i=1,2 \quad i\neq j$$

What remains to be specified is how each firm will guess what the other's output will be. Assume that each firm knows the demand function, as well as its own and its rival's cost function. Then each firm has a full
understanding of the payoff structure of the game.

The equilibrium concept which has been commonly adopted for single-period, simultaneous choice games is that of Nash (1951). It defines equilibrium as follows:

**Definition:** \( \{q^N_i, q^N_j\} \) is a **Nash equilibrium** if

(i) \( q^N_i \) maximizes firm i's payoff, given \( q^N_j = q^N_j \), and

(ii) \( q^N_j \) maximizes firm j's payoff, given \( q^N_i = q^N_i \).

Formally, the Cournot-Nash equilibrium is found as the simultaneous solution of the two equations (1.1.3).

This equilibrium concept is very appealing: it says that each firm is doing as well as possible, given the actions of the other firm. If each firm indeed chooses to produce the Cournot-Nash quantity, neither firm will have any regrets. The management of each firm can tell its stockholders: we guessed correctly, and we made the most out of it. No other combination of outputs has this "no regrets" property.

The Cournot-Nash model thus yields a definite prediction of the levels of output and price in an oligopoly. Obviously, it is predicated upon the validity of the assumptions of the model. First, firms must fully understand the payoff structure of the game. Second, firms choose outputs rather than prices. Third, this choice is made simultaneously. Finally, firms act "rationally" in the manner specified by Nash.
The Cournot-Nash model is not the only oligopoly model which has been proposed. For example, the Bertrand model assumes instead that firms choose prices, yielding a different Bertrand-Nash equilibrium, which is discussed below. However, the Cournot model has been questioned most frequently due to its assumption that the game is played only once.

It is clear that most firms in an oligopoly must play the game repeatedly, day after day. One might think that the Cournot-Nash equilibrium, being "rational" for each individual play of the game, would also be rational for the repeated game. However, Friedman (1971) showed that in an infinitely repeated Cournot game, this is not the case. The reason for this is easily explained. It is well-known that if the two firms decided to jointly act as a monopolist, with each firm producing half of the monopoly output \( \frac{1}{2} q^M \), each could achieve higher profits than by setting the Cournot output. Of course, given that firm \( J \) produced \( \frac{1}{2} q^M \), firm \( I \) could have made even higher profits by setting \( q_i = f_i(\frac{1}{2} q^M) \), so the firm is not "rational" in the single-period Nash sense.

The reason why this behavior may nevertheless be rational is that in an infinitely repeated game implicit collusion may arise. Both firms realize that by producing the collusive output \( \frac{1}{2} q^M \) forever, their profits will be greater than by producing the Cournot output forever. Of
course, in any single period, there is an incentive to cheat, but "cheating" will likely result in "punishment," with the other firm reverting to the Cournot output. As long as the gain from single-period cheating are outweighed by the losses from punishment, it is rational to sustain the collusive output.

Much attention has recently been devoted to such infinitely repeated games. The difficulty with these models is pointed out by Kreps and Spence (1983): "In general, infinitely repeated games come with an embarassment of riches in terms of the number of possible noncooperative equilibria." In the above example of implicit collusion, the threat of reverting to single-period Cournot behavior might also be used to sustain an implicitly collusive output of \( \frac{M}{2q} \), as long as the gain from single-period cheating is outweighed by the loss from subsequent punishment.

The differential gaming approach taken by this thesis is one possible way of narrowing down the multitude of noncooperative equilibria of infinite horizon games. The assumptions are: (i) the game is played continuously over time, (ii) the time horizon is infinite, (iii) "reactions" to actions by the rival firm are subject to adjustment costs, and (iv) the equilibrium strategies are restricted to those which yield continuous value functions.

Chapter 2 will be devoted to a detailed explanation of the differential gaming assumptions, which restricts
strategies to those yielding continuous value functions. It will be argued that this restriction corresponds to "competitive" rather than "implicitly collusive" equilibria. Thus, the differential gaming approach does not eliminate the multiplicity of implicitly collusive equilibria. However, it does analyze the "failure to collude" or "breakdown of collusion" case. This is clearly an important case, since it is the threat which is used to support collusion.

The major finding of this thesis is that the "no collusion" equilibrium, described by the differential game equilibrium, is not the same as the single-period Nash equilibrium. The differential game model explicitly assumes that a firm can react to actions of its rival, albeit subject to adjustment costs, and subject to the "continuity restriction." At time $t+dt$, firm i's choice of quantity $q_i(t+dt)$ will generally depend upon firm j's choice at time $t$, $q_j(t)$. There is generally no reason to believe that, in the limit as $dt \to 0$, $\partial q_j / \partial q_i = 0$. On the other hand, the Cournot model, because of its single-period nature did not allow for $\partial q_j / \partial q_i = 0$.

There are two circumstances under which one would nevertheless expect the equilibrium to be equal to the prediction of the Cournot model. The first is the case where adjustment costs are infinitely large. In that case, although a "reaction" is technically possible, it will never
pay to react, which yields the same outcome as the Cournot model, where it is not possible to react.

The second case is given by the possibility that firms may be able to "precommit" themselves, for example by making binding contracts with a third party. If firm j can make credible that it will pick its output without any regard for the output chosen by firm i, then clearly no reaction is possible, just as in the Cournot model. This type of equilibrium strategy corresponds to "open loop" strategies in the parlance of optimal control theory.

The main finding of this thesis is that, with these two exceptions, the single-period Cournot model does not provide an approximation of multi-period oligopoly behavior, even when the possibility of implicit collusion is explicitly excluded. This is due to the fact that, independently of the possibility of implicit collusion, there will be a "reaction effect": a firm’s actions today will provoke a reaction from its rival tomorrow.

The Bertrand Model

The Bertrand model differs from the Cournot model by assuming that firms choose prices rather than quantities. In its simplest form, it assumes that both firms have identical, constant marginal cost c. It also assumes that the firm which charges the lower price gets the entire market demand. If both firms charge the same price, then
they split the total demand equally.

It is not difficult to see that these assumptions yield a different Nash equilibrium outcome than the Cournot model. Given that firm j is charging price $p_j$, firm i can capture the entire market demand by charging a price $p_i = p_j - \epsilon$, where $\epsilon$ is very small. It is well-known that the Nash equilibrium of this model has each firm charging price equal to marginal cost.

If this single-period game is replaced by an infinitely repeated game, there is the possibility of implicit collusion just like for the extension of the Cournot model. Again, a multiplicity of implicitly collusive equilibria will usually exist, each supported by the threat of "punishment" in the form of reverting to single-period Bertrand behavior. However, similarly to the dynamic extension of the Cournot model, the "competitive" (i.e. differential gaming) equilibrium of the dynamic Bertrand game is not the repeated charging of the Bertrand price.

The differential game model based on the Bertrand model assumes that demand does not instantaneously shift between firms in response to price differentials. This is a way of modeling the accumulated "goodwill," or of capturing the slow diffusion of information about prices. The Bertrand model would correspond to the extreme case where shifts in demand occur almost instantaneously.
The main result of chapter 4 is that, even when demand shifts occur instantaneously, the perfect Nash equilibrium strategies has each firm charging a price above marginal cost. The reason for this is that each firm explicitly recognizes that any attempt to undercut its rival would be almost instantaneously met with a lower price by the rival. Thus in the dynamic Bertrand model, the payoff from undercutting the rival's price is limited by the rival's reaction. This implies that firms will act less competitively than in the static Bertrand model, even when they are not implicitly colluding.

The Consistent Conjectures Model

The fact that firms are not generally engaged in one-period games has been recognized at least since Bowley (1924). In examining the stability of the Cournot equilibrium, he developed the concept of a "reaction function." The reaction function for the Cournot model is described mathematically by the equation (1.1.3). It describes the level of output which firm i would choose, if it knew that firm j was going to produce output q_j.

Bowley argued the stability of the Cournot equilibrium quantities q^C in a way illustrated by Figure 1.1.1, where firm i's reaction function is given by R_i. Suppose firm 1 were to produce output A. Then firm 2 would "respond" by choosing output B. But if firm 2 chooses B, then firm 1
would respond with output C, etc. Given some regularity assumption about the properties of the reaction functions, this argument is supposed to show that the outputs will converge to the Cournot level $q^C$.

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**Figure 1.1.1**

*Stability of Cournot Equilibrium*

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There is a fundamental problem with this argument. On the one hand, it assumes (with Cournot) that outputs are simultaneously determined in a single-period game. On the other hand, it assumes that each firm knows the other’s output before choosing its own, and that firms choose output sequentially.

In fact, the term "reaction function" is really a misnomer: it merely gives the optimal output if the other firm’s output were known. In a simultaneous-choice one-period game no chance for reaction is ever given. What
Bowley's argument persuasively shows is that \( q^C \) is indeed the Nash equilibrium. It is the only combination of outputs which lies on both "reaction functions," i.e. the only combination of outputs where each firm is maximizing profits, given the other's output.

The existence of the equation (1.1.3), alias the reaction function, has been the source of much confusion. After all, asks any particularly clever student in an introductory microeconomics course, how can you assume that \( \partial q_j / \partial q_i = 0 \) when equation (1.1.3) clearly shows a functional (and differentiable!) relation between the two variables?

The answer to this puzzle is quite simple. Equation (1.1.3) describes how firm i's output varies with its expectation about firm j's output. If it expects firm j to produce \( q_j^e \), it will be optimal to produce \( q_i = f_i(q_j^e) \). However, the term \( \partial q_j / \partial q_i \) in equation (1.1.2) clearly refers to how the actual output of firm j varies with firm i's output \( q_i \). With a more careful choice of notation, which distinguishes between expected and actual outputs, the puzzle would never arise.

Inevitably, one student remains skeptical. Isn't the entire point of the Nash equilibrium concept that for each firm, the expected output is equal to the actual output? Isn't that the precise definition of "no regrets?" Even this last objection can be met. Actual and expected output are equal only when both firms produce the Cournot-Nash output.
This does not imply that actual and expected output must be equal for non-equilibrium points.

The preceding argument should (hopefully) lay to rest the arguments surrounding the appropriate "conjectural variation" in the static, simultaneous choice quantity-setting model. The only formally correct choice is that of Cournot ($\partial q_j/\partial q_i = 0$). Recently, however, Bresnahan (1981) has revived the controversy. He asserts that the validity of the Bertrand versus the Cournot model can be decided on "theoretical grounds," by introducing the concept of "consistent conjectures."

The consistent conjectures approach can be explained by use of the "reaction function" puzzle. The particularly clever student discovers the supposed "inconsistency" of the Cournot assumption $\partial q_j/\partial q_i = 0$, and the fact that the reaction function suggests that $\partial q_j/\partial q_i \neq 0$. His solution to the puzzle is that Cournot must be wrong, and that firm $j$ does indeed react to firm $i$. He then uses the expression for the conjectural variation, $\partial q_j/\partial q_i$, obtained by differentiating the reaction function, and uses it the first-order condition (1.1.2). Solving (1.1.2) implicitly for $q_i$ will now lead to a new reaction function, which differs from the first one. Differentiating this new reaction function will yield yet another conjectural variation, and so on. Interestingly, infinite repetitions of this process yield a reaction function and a conjectural variation which are consistent.
Although this process has no theoretical foundation, one might think of it as a clever approximation of real-world dynamics. The Cournot assumption of a single-period game is clearly not realistic. In dynamic games firms will usually react to each other's actions. However, dynamic games are quite difficult to solve, so it would be extremely useful to have a simple, static model which accomplishes the same result, namely to capture the "reaction effect."

The usefulness of the consistent conjectures model must therefore be judged by whether it provides a useful approximation to a full-blown dynamic model. The analysis of such a dynamic model is precisely the objective of chapter 3. Unfortunately, the consistent conjectures equilibrium does not provide a better an approximation to the full dynamic equilibrium than the original Cournot model. The moral of the story is that dynamic effects cannot be modeled using static shortcuts.

**Models of Irreversible Investment**

The model of chapter 3 is most closely related to the analysis of Spence (1979) and Fudenberg and Tirole (1983a). Spence considers a model of investment of two firms. He assumes that the instantaneous profit of each firm is determined by the current level of capital (or capacity) of the two firms. Firms can increase their level of capital by investing, but there is an upper bound on the rate of
investment. Furthermore, investment is irreversible, and there is no depreciation. Although he doesn't explicitly formulate it as such, he does solve for the perfect equilibrium of a differential game.

The model of chapter 3 differs from that of Spence in two ways: investment is assumed to be reversible, and there is no bound on the rate of investment. Instead, investment is assumed to be subject to convex adjustment costs, which also implies that firms will invest at a finite rate, but the rate will be endogenous.

There are two possible interpretations of the model of chapter 3. One interpretation is simply as an extension of Spence's investment model to account for the possibility of reversible investment. The capital level can clearly decrease as a result of depreciation, but may also be reduced if a firm can sell its capital stock. This modification leads to substantially different results. In Spence's model, the firm with a headstart can invest beyond the Cournot level, and attain a steady-state equilibrium akin to Stackelberg leadership. In contrast, for the model of reversible investment, any headstart is only temporary: although the firm may overshoot the eventual steady-state level of capital, the two firms will have equal steady-state levels of capital if their costs are symmetric.

The model of chapter 3 can be also be thought of as a quantity-setting model by reinterpreting the variables. In
this context, profit is determined by the current level of output of the two firms, and the rate of change of output is subject to adjustment costs. This assumption attempts to capture the difficulty of changing operations planning and scheduling, as well as adjustments in the labor force. Under this interpretation, the Cournot model corresponds to the limiting case when adjustment costs are very small. The main result of the chapter will be that, even when friction is absent, the reaction effect alone suffices to yield an equilibrium different from the Cournot equilibrium.

The paper of Fudenberg and Tirole (1983a) also extended Spence's model, but in a different direction. They show that the investment game of Spence has not just one, but many perfect equilibria, some of which have an "early stopping" property. They point out that these equilibria are reminiscent of supergame equilibria: the two firms are at an equilibrium where both firms stop investing, and this equilibrium is supported by a credible threat to keep investing. It is important to note that this is a noncooperative equilibrium: each firm acts optimally given the other firm's credible threat. The only collusion comes in coordinating on a particular one of the infinite number of perfect equilibria.

The equilibria proposed by Fudenberg and Tirole have the property that the "value functions" of the firms are discontinuous at the early stopping equilibrium. A small
step away from the equilibrium state leads to discontinuously lower net profits. This is the "threat" which supports the "collusive" equilibrium. However, the threat is provided by strategies with continuous value functions.

This leads to an interpretation of equilibria with continuous value functions as the "default" equilibrium, when no coordination on a "more collusive" equilibrium can be reached. But equilibria with continuous value functions are precisely those which are admissible for differential games. The differential game approach is thus a way of characterizing "competitive" oligopoly equilibria.

The Entry Deterrence Model

Chapter 5 uses the investment model of chapter 3 as a building block of a model of entry deterrence. The analysis extends the entry deterrence model of Kamien and Schwartz (1971). The problem is that of a monopolist faced with uncertain entry. They assume that the probability of entry is lower, the higher the pre-entry price of the incumbent. However, the pre-entry price does not affect the post entry profits. They derive the optimal price path for the monopolist, and show that it is given by a single, constant limit price.

However, the assumptions of Kamien and Schwartz are not entirely consistent. On the one hand, they assume that the
pre-entry price does not affect the post-entry profits. But if this is the case, why would the pre-entry price give any indication of post-entry profits? The probability of entry presumably depends on the future profits expected by the entrant. If the pre-entry price does not affect post-entry profits, then it should also have no effect on the likelihood of entry.

Although the assumptions of Kamien and Schwartz may not be consistent, their method can be extended. Chapter 5 assumes that the probability of entry increases with the expected profit of the entrant, and the expected profit is affected by the incumbent's choice of pre-entry capacity. The reason for this dependence is given by the fact that, when investment is subject to adjustment costs, the speed and duration of the adjustment to the new post-entry equilibrium will depend on the level of capacity at the time of entry. This is where the model of chapter 3 is integrated into the analysis of entry deterrence.

An additional modification of the Kamien and Schwartz model is necessary when the monopolist has not yet reached its steady-state level of capacity. If the monopolist is subject to adjustment costs during his growth, his optimal pre-entry policy will not be stationary, i.e. there is no single optimal pre-entry level of capacity, but rather a growth path starting from when the monopolist first enters the market.
This section has indicated how the analysis of this thesis is related to some well-known papers. The next section summarizes the main issues and conclusions of this thesis.

1.2 Major Issues and Results

The previous section gave a lengthy description of the background which preceded the analysis of the present thesis. The aim of this section is to provide a concise summary of the major issues examined, as well as some of the major results.

The Reaction Effect

Static models of oligopoly, like the Cournot and the Bertrand model, are formally valid descriptions of single-period, simultaneous-choice games. However, most oligopolists face each other repeatedly, day after day. This suggests that dynamic games may provide a more realistic model of oligopoly.

In a static model, there is no possibility for one player to react to the actions of his rival, since both choose their actions without knowledge of the rival’s action. However, in a dynamic game, one would expect that today’s actions might provoke a reaction by the rival player in the next period. The formal modeling of this reaction effect and its consequences is one of the aims of this
thesis.

In the quantity-setting context of chapter 3, the differential gaming model predicts that a rival will react to a quantity increase by reducing his quantity somewhat, i.e. by "accomodating." This makes aggressive behavior more attractive, and yields equilibrium output which is greater than the static Cournot level.

The differential gaming approach also predicts that in the price-setting context of chapter 4, a rival will react to price decreases by also reducing price, i.e. by "fighting back." This makes aggressive behavior less attractive, and yields an equilibrium price above the static Bertrand level.

**Differential versus Dynamic Games**

Differential games are a special class of dynamic games. Differential games are those amenable to optimal control techniques. The use of these techniques requires that some additional structure be imposed. In particular, the equilibrium strategies of a differential game are required to yield value functions which are continuous in the values of the state variables.

It is argued in chapter 2 that the restriction of dynamic games to differential games eliminates "implicitly collusive" or "supergame" equilibria. These equilibria rely on discontinuities of the value function. An infinitesimal deviation from the collusive action leads to severe
punishment, with a discontinuously lower payoff. This is the "threat" necessary to support collusion.

Concentration on differential games thus strips away the issue of conclusion from dynamic games. What is left is a dynamic model of competition in a gaming situation. Because of the reaction effect, the dynamic model of oligopolistic competition usually differs from the corresponding static oligopoly model.

Open Loop versus Closed Loop Equilibria

Differential game theory distinguishes between "open loop" and "closed loop" strategies. An open loop strategy is one in which a player is allowed to vary his actions only as a function of time. An example might be a missile with a predetermined engine firing sequence. In contrast, a closed loop strategy is one which allows a player to choose his actions depending on the current value of the state variables. An example would be a heat-seeking missile, which can adjust its thrust on the basis of the position of its rival missile.

Open-loop strategies correspond to the notion of pre-commitment. Once the missile is launched, its path can no longer be modified. In economics, pre-commitment is usually difficult to make credible. If it is possible to observe the current state variables, it is difficult to persuade one's rival that this information will be ignored. However, if a
player is able to sign binding contract with a third party, specifying large penalties in case of a deviation from a predetermined sequence of actions, pre-commitment can be made credible.

The difference between open-loop and closed-loop equilibria is that, by their very nature, open-loop strategies do not allow for a reaction effect. One would therefore expect that open-loop differential gaming equilibria should yield similar results to the static models of competition.

Indeed, chapter 3 shows that the static Cournot equilibrium corresponds to the dynamic open-loop equilibrium of a quantity-setting model. Similarly, chapter 4 shows that the static Bertrand equilibrium corresponds to the limiting case of the dynamic open-loop equilibrium of a price-setting model, where the speed with which demand shifts as a result of price differentials becomes very large.

**Commitment and the Reversibility of Investment**

Recent articles on strategic investment have focused on investment in irreversible capital as a way of attaining a long-run competitive advantage by preempting one's rivals. However, capital does not usually have zero resale value. It is generally possible for firms to reduce their levels of capital, both through resale and through depreciation.

Modifying the assumption of irreversible investment to
one where capital can be reduced, but is subject to convex adjustment costs, yields the result that preemption is impossible in the long run. Nevertheless, a firm can act "strategically" in the short-run, by taking into account the fact that additional capital will have the effect of discouraging capacity additions by its rival.

Even though reversible capital prevents a firm from establishing long-run commitment to a high level of output, chapter 5 shows that it is nevertheless possible for a monopolist to deter entry.

1.3 Overview of the Thesis

This section provides an overview of the chapters of this thesis, and their subsections.

Chapter 2 discusses the theory of differential games and their relation to the more general class of dynamic games. Section 2.1 gives a mathematical exposition of the formulation of the problem, gives rigorous definitions of the terms, and states some of the major theorems concerning solutions of differential games. It also gives conditions under which equilibria exist and conditions under which they are unique. However, conditions under which a perfect Nash equilibrium is both unique and guaranteed to exist appear not to be unresolved. Section 2.2 discusses the relation between state-space games, dynamic games, and differential games. It argues that the restriction to differential games
is equivalent to the exclusion of implicitly collusive equilibria.

Chapter 3 presents a linear-quadratic differential gaming model with reversible investment. Section 3.1 clarifies the relation between the Cournot-Nash equilibrium, conjectural variations, reaction functions, and dynamic games. Section 3.2 presents the actual model and its assumptions. Section 3.3 uses the Pontryagin Maximum Principle to solve for the Riccati equations for this problem, and shows by brute force that the perfect Nash equilibrium is unique in the symmetric case. Section 3.4 provides some numerical examples of solutions. Section 3.5 examines some comparative static results, based on numerical examples which vary one parameter at a time. This section also shows that the consistent conjectures equilibrium does not provide an approximation of the dynamic model.

Chapter 4 presents a model of price competition when goodwill effects are present. This analysis of complementary to that of chapter 3, in that it examines a price-setting rather than a quantity-setting model. Section 4.1 presents a general model, and a linear-quadratic version of it. Section 4.2 derives the necessary conditions which must be satisfied by an open-loop and a perfect equilibrium of the general model. The resulting differential equations are investigated using a phase diagram. Section 4.3 derives the necessary conditions and the Riccati equations for the
linear-quadratic case. Section 4.4 gives some numerical examples of open-loop and perfect equilibria. Section 4.5 presents comparative-static results, based on numerical examples which vary one parameter at a time.

Chapter 5 is concerned with entry deterrence in the model of reversible strategic investment of chapter 3. Section 5.1 gives a brief introduction to the major issues in the theory of entry deterrence, such as the importance of credible threats and the value of commitments. Section 5.2 discusses some of the entry deterrence literature which emphasizes capital as a means of commitment. Section 5.3 modifies the entry deterrence model of Kamien and Schwartz, and section 5.4 combines this model with the reversible investment model of chapter 3. This section also presents the comparative static results.

Finally, chapter 6 concludes by pointing out some of the limitations of the thesis, as well as some directions in which it could be usefully extended.
CHAPTER 2

DIFFERENTIAL GAMES

This chapter gives a brief introduction to the theory of differential games, and sketches their relation to the broader class of dynamic games. It will be argued that equilibria of differential games with continuous value functions can be thought of as "competitive", while those with discontinuous value functions correspond to "implicitly collusive" or "supergame" equilibria.

The present chapter will use the term "dynamic game" to describe the repeated playing of "static games" over time. Dynamic games will be called "discrete" or "continuous," depending on whether they are formulated in discrete or continuous time, and will be "finite" or "infinite," depending upon the time horizon of the game.

Friedman (1971) defines the term "supergame" as the "playing of an infinite sequence of 'ordinary games' over time." However, the term "supergame" has been used in the recent literature to describe not just the game, but those of its equilibria associated with the notion of "threat" and "punishment." To avoid confusion between the game itself and this particular set of equilibria, the present chapter will use the term "infinite dynamic game" instead of "supergame."

Differential games are those continuous dynamic games to which optimal control techniques, such as Pontryagin's
maximum principle can be applied. The use of this technique requires that some additional structure be imposed: it is assumed that the payoff of the players at each moment depends on some "state variables," the motion of which is controlled by differential equations.\(^1\) A precise description of this approach is presented in section 2.1.

Differential games are a special case of what Fudenberg and Tirole (1983a) call a "state-space" game. In such a game, each firm's strategy is assumed to depend on the history of the game only through the current (payoff-relevant) state....Because both the payoffs and the strategies depend on the history only through the state we call this a "state-space" game.

Section 2.2 examines the relation between general dynamic games and state-space games. It will be argued that the state-space assumption is a useful restriction, and that many of the interesting features of general dynamic games also arise in state-space games. While at first glance it may appear that the state-space assumption allows history to play only a very limited role, the state variables can be interpreted as "sufficient statistics" about the past.

Section 2.2 also discusses the differences between general state-space games and the narrower class of differential games. Fudenberg and Tirole (1983a) consider a state-space investment game, and show the existence of a continuum of perfect equilibria with an "early-stopping" property. Given this non-uniqueness, an equilibrium requires
coordination over the set of equilibria, or the outcome may be no equilibrium at all. Using the early-stopping equilibria as an example, it will be argued that the differential game equilibrium is the "default" equilibrium, which "supports" the "collusive" equilibria.

The distinction between differential game equilibria and general state-space equilibria will be seen to depend on the continuity of the value function. Differential game equilibria have continuous value functions, whereas for the "collusive" equilibria, discontinuities in the value functions provide the "threats" and "punishments."

The motivation for this extended review of differential games and their relation to general dynamic games is given by the neglect which differential games have received. Section 2.2 argues that the state-space assumption imposes a useful restriction on the broader class of dynamic games, and that differential game equilibria are a focal equilibrium among the multiple state-space equilibria. Following this line of reasoning, one might expect differential games to be extensively used in economic models.

There are two possible reasons for the paucity of applications of differential games to economics. One is the fact that closed form solutions to differential games are generally very difficult to obtain. Indeed, the only case which is analytically tractable is the linear-quadratic
case. Nevertheless, general statements can sometimes be made about the steady state of differential games even when a closed-form solution cannot be obtained.

The second reason, I believe, is due to the division of labor among economists. Differential games involve the merging of game theory and optimal control, both of which are highly specialized subjects, with specialized journals. An example of this is given by the use of the term "perfectness" by game theorists, which corresponds to the idea of "feedback strategies," used by control theorists. Fudenberg and Tirole (1983b) note that

Perfectness is just a many-player version of the principle of optimality. Thus, it is not surprising that "perfectness" was independently formulated by optimal-control theorists in their study of nonzero-sum differential games.

As a historical note on this division of labor, I would like to mention two parallel developments. Isaacs (1965) is usually credited with the introduction of differential games, and his discussion of closed-loop Nash equilibria implicitly refers to perfect closed-loop Nash equilibria. In the same year, Selten (1965) developed the concept of perfect equilibrium. As a final paradox, Selten defines perfect equilibrium for general dynamic games. However, the bulk of his paper is devoted to an extensive study of a differential game, formulated in discrete time, with extensive use of dynamic programming techniques.
2.1 Introduction to Differential Games

The purpose of this section is to review some of the techniques and issues in the theory of differential games. These techniques will be applied in later chapters to some specific dynamic models of duopoly. Differential games are those state-space games which yield continuously differentiable value functions, so that optimal control techniques can be applied.

First, the general form of a two-person differential game is presented, and the Nash equilibrium is defined for dynamic games. The Nash equilibrium will generally depend on the information available to the players during the game, and this results in a distinction between open-loop and closed-loop Nash equilibria. It is then shown that, when players have closed-loop information, the Nash equilibrium concept is too broad, and must be refined to perfect Nash equilibrium. This refinement eliminates those multiple closed-loop Nash equilibria arising from "informational non-uniqueness." It is then shown that, if strategies are restricted to be analytic functions of the state variables, the perfect Nash equilibrium of a differential game with finite time horizon is unique. Finally, some existence results for linear-quadratic differential games are presented.
Formulation of the problem

The following paragraphs define the terms which are used to describe differential games: the state and control variables, the equation of motion, the objective function, the information structure, the set of strategies, and the Nash equilibrium concept.

A two-person differential game is one in which two players, player 1 and player 2, can affect the motion of a (vector-valued) state variable $x$, which has equations of motion given by the set of differential equations

$$\dot{x}(t) = f(x(t), u_1(t), u_2(t), t); \quad x(0) = x_0;$$

where $u_1(t), u_2(t)$ are the (vector-valued) control variables of players 1 and 2 respectively.

Player $i$ chooses his control variable $u_i(t)$ as to maximize his objective function

$$J_i(u_1(t), u_2(t)) = \int_0^T g_i(x(t), u_1(t), u_2(t), t) dt + F_i(x(T)).$$

The functions $f(\cdot)$, $g_i(\cdot)$, and $F_i(\cdot)$ are assumed to be continuously differentiable, and the control variables $u_1(t)$ and $u_2(t)$ are assumed to be piecewise continuous functions of time, belonging to the nonempty, compact control sets $U_1, U_2$ respectively, and $x(t)$ is confined to a nonempty, compact set of permissible states $X$.

The player’s choice of his control variables will depend on the information which is available to him at the time the choice must be made. This is formalized as follows:
Definition: The information structure of player $i$ is a set-valued function $\eta_i(t) = \{x(s), s \in S(t)\}$, where $S(t)$ is a subset of $[0, T]$, and $S(t)$ is nondecreasing in $t$.

(i) If $\eta_i(t) = \{x_0\}$, the information structure is of open loop form.

(ii) If $\eta_i(t) = \{x(t)\}$, the information structure is of pure feedback form.

(iii) If $\eta_i(t) = \{x(s), 0 \leq s \leq t\}$, the information structure is of closed loop form.

The open-loop information structure obviously applies when a player will not be able to observe the state variables after the start of the game. In contrast, if a player can observe (and recall) the values of the state variables after the start of the game, his information structure is of the closed-loop form.

Pure feedback information assumes that the player can observe the current state of the game, but cannot recall any of the previous values. This possibility is not empirically likely, but is introduced for a technical reason. The equation of motion indicates that the evolution of the state variables depends only on the current value of the state variables. Hence knowledge of past values of the state variables really adds no useful information. However, it is conceivable that a player may choose to ignore the current information, and base his strategy on past information. As will be shown below, if both players choose to follow such
an "informationally inferior" strategy, a plethora of "informationally non-unique" equilibria can arise. One possible way of assuring that strategies of the players are actually based on the current information is to pretend that they cannot recall past values.

**Definition:** A **strategy** for player $i$ is a (vector-valued) function $\sigma_i(\eta_i(t),t)$, which indicates how the player will act given his current information set, i.e. $u_i(t) = \sigma_i(\eta_i(t),t)$. The **strategy space** $\Sigma_i$ consists of the set of permissible strategies for player $i$.

As posed so far, this is not a well-defined problem. The objective function $J_i$ of player $i$ depends on his opponent's control function as well as his own. Thus, in general, the optimal control for player $i$ depends on the control chosen by player $j$. By analogy with static games, a solution is defined as follows:

**Definition:** A **Nash equilibrium** is a pair of strategies $(\sigma_1^*,\sigma_2^*)$ such that

- $J_1(\sigma_1^*(t),\sigma_2^*(t)) \geq J_1(\sigma_1(t),\sigma_2^*(t))$ for all $\sigma_1 \in \Sigma_1$,
- $J_2(\sigma_1^*(t),\sigma_2^*(t)) \geq J_2(\sigma_1^*(t),\sigma_2(t))$ for all $\sigma_2 \in \Sigma_2$.

Note that since a strategy is defined for a particular information structure, there will be different Nash equilibria for each type of information structure. This is the basis for the distinction between open-loop, closed-loop and perfect Nash equilibria.
Open-loop and Closed-loop Nash Equilibria

In the case where the information structure for both players is of the open-loop type, the Nash equilibrium concept is quite compelling. However, it will be shown that in the case where both players have closed-loop information structure, the Nash equilibrium concept is too broad, and needs to be further restricted.

Suppose that either player can observe only the initial state of the system, i.e. the information structure is of the open loop form. This means that both players must choose their entire strategy at the beginning of the game, without any possibility for revision, and the strategy space is given by functions which may depend only on the initial state of the system and on time. The solution to this problem can be obtained by using Pontryagin's maximum principle.

Theorem 2.1.1 If \( \sigma_1^*(x_0,t)=u_1^*(t), \sigma_2^*(x_0,t)=u_2^*(t) \) is an open-loop Nash equilibrium, and \( \{x^*(t), 0 \leq t \leq T\} \) is the corresponding trajectory, then there exist (vector-valued) costate function \( \lambda_i^*(t) \), such that for \( i=1,2; j \neq i \)

\[
\begin{align*}
\dot{x}^*(t) &= f(x^*(t), u_1^*(t), u_2^*(t), t); \quad x^*(0) = x_0 \\
u_i^*(t) &= \text{argmax } H_i(x^*(t), u_1^*(t), u_j^*(t), \lambda_i^*(t), t) \quad \text{for } u_i \in \Sigma_i \\
\lambda_i^*(t) &= \frac{\partial H_i}{\partial x} (x^*(t), u_1^*(t), u_2^*(t), \lambda_i^*(t), t) \\
\lambda_i^*(T) &= \frac{\partial F_i}{\partial x} (x^*(T)),
\end{align*}
\]

where \( H_i(x, u_1, u_2, \lambda_i, t) = g_i(x, u_1, u_2, t) + \lambda_i \cdot f(x, u_1, u_2, t) \).
Now suppose instead that either player can observe the current state \( x(t) \) of the system, and has memory about all previous states, \( \{x(s), 0 \leq s \leq t\} \). Then his information structure is of the closed-loop form, and the corresponding Nash equilibrium is given by:

**Theorem 2.1.2** If \( \sigma_i^*(x(t), x_0, t) = u_i^*(t) \) \( i=1,2 \) is a closed-loop Nash equilibrium, such that \( \sigma_i(x(t), x_0, t) \) is continuously differentiable, and \( \{x^*(t), 0 \leq t \leq T\} \) is the corresponding trajectory, then there exist (vector-valued) costate function \( \lambda_i^*(t) \), such that for \( i=1,2; j \neq i \)

\[
\dot{x}^* = f(x^*, u_1^*, u_2^*, t); \quad x^*(0) = x_0
\]

\[
u_i^* = \arg\max H_i(x^*, u_1^*, u_2^*, \lambda_i^*, t) \text{ for } u_i \in \Sigma_i
\]

\[
\lambda_i^* = - \frac{\partial H_i}{\partial x}(x^*, u_1^*, u_2^*, \lambda_i^*, t) - \frac{\partial H_i}{\partial u_i} \frac{\partial u_i}{\partial x}(x^*, u_1^*, u_2^*, t)
\]

\[
\lambda_i^*(T) = \frac{\partial F_i}{\partial x}(x^*(T)),
\]

where \( H_i(x, u_1, u_2, \lambda_i, t) = g_i(x, u_1, u_2, t) + \lambda_i \cdot f(x, u_1, u_2, t) \).

Comparing these necessary conditions with those for an open-loop Nash equilibrium reveals an additional term \( \frac{\partial u_j}{\partial x} \) in the costate equation, which captures the effect of a change in the state variable on the control variable of player \( i \)'s opponent.

Note that the equations of motion and the objective function are assumed to be continuously differentiable in the formulation of the problem. For the open-loop equilibrium, these assumptions are sufficient to allow application of Pontryagin's maximum principle. However, for
the closed-loop equilibrium, the strategies are assumed to be continuously differentiable functions of the state variables.\textsuperscript{6}

The reason for considering closed-loop Nash equilibria is that in many applications both players can instantaneously observe the values of the state variables. One would thus expect that, in a closed-loop equilibrium, the action of each player depends on the value of the state variables.

However, the definition of Nash equilibrium fails to guarantee that a player will actually use all the information available to him. This leads to a problem which Basar (1977) refers to as "informational non-uniqueness". He shows that there will generally be an uncountable number of closed-loop Nash equilibria.

The following example will show that this multiplicity of closed-loop equilibria arises because the definition of Nash equilibrium is too broad. As has also been noted by Fudenberg and Tirole (1983b), an open-loop Nash equilibrium will also be a closed-loop Nash equilibrium. To see this, suppose that player 2 chooses to ignore the current value of the state variable, and follows an open-loop strategy $\sigma_2(x_0,t)$. The optimal closed-loop response for player 1 will be some strategy $\sigma_1^*(x(t),x_0,t)$. But, given that $\sigma_2(x_0,t)$ is known, it is possible to determine the state $x(t)$ at any time from the initial state $x_0$. Hence player 1's optimal
strategy can equivalently be written in its open-loop representation as $\sigma^*_1(x(t), x_0, t) = \sigma^*_1(x_0, t)$. In other words, if player 2 chooses to ignore the value of the state variable, then player 1 might as well do the same.

There are undoubtedly situations in which a player may want to pre-commit himself to a certain strategy. Indeed, this can frequently lead to an advantage for a player. However, the issue arises whether pre-commitment is actually credible. Suppose that the two players have committed themselves to open-loop strategies $\sigma^*_1(x_0, t), \sigma^*_2(x_0, t)$ respectively, and that these strategies form an open-loop Nash equilibrium. Then as the state evolves over time, neither firm has any reason to deviate from its equilibrium strategy, because its strategies have to be optimal over any interval $[s, T], (s > 0)$ by the Principle of Optimality.

However, these strategies are optimal only along the equilibrium path. Suppose that a random shock were to perturb the system, causing $x(t)$ to lie off the original equilibrium path. Then a new open-loop Nash equilibrium with this new initial state would generally lead to strategies different from the original ones. Indeed, Basar (1976) has shown that, when the motion of the state variable is subject to noise, open-loop equilibria are not closed-loop equilibria, and any closed-loop Nash equilibrium must have strategies which are of pure feedback form.
It is in this sense that the definition of a closed-loop Nash equilibrium is deficient: some closed-loop strategies are optimal only along the equilibrium path, but would no longer be optimal in the presence of random shocks. This leads to a more stringent definition of Nash equilibrium, which is presented next.

Perfect Nash Equilibria

The preceding section showed that the definition of closed-loop Nash equilibrium is so broad as to encompass an infinite number of equilibria. In particular, if both players decide to ignore the information provided by the state variable, and follow an open-loop strategy, this results in a closed-loop Nash equilibrium. This open-loop strategy has the undesirable property that it is optimal only along the equilibrium path. This suggests a definition of equilibrium which eliminates strategies with this "razor's edge" property.

The necessary refinement was developed by Selten (1965) in the context of a discrete-time dynamic game. The additional requirement is precisely the one needed to eliminate strategies with the "razor's edge" property. The strategies of a perfect Nash equilibrium must be a Nash equilibrium independently of the initial state of the game. This property is formalized in the following definition:
Definition: The strategies \( \{ \sigma_i^* \in \Sigma_i \} \), \( i=1,2 \) are a perfect Nash equilibrium for the differential game with closed-loop information structure if there exist functions \( V_i : \mathbb{R}^n \times [0,T] \rightarrow \mathbb{R} \) which for all permissible values of \( x_0 \) satisfy

\[
V_i(x,t) \geq J_i(\sigma_1(x,x_0,t),\sigma_2(x,x_0,t)) \quad \text{for all } \sigma_1 \in \Sigma_1
\]

\[
V_2(x,t) \geq J_2(\sigma_1^*(x,x_0,t),\sigma_2(x,x_0,t)) \quad \text{for all } \sigma_2 \in \Sigma_2
\]

The strength of this definition lies in the requirement that the value functions \( V_i \) induced by the strategies must be optimal for any initial state of the game. Consequently, the perfect equilibrium strategies must also be optimal for any initial state. But that is to say that the perfect equilibrium strategies must be of pure feedback form:

\[
\sigma_i^*(x,x_0,t) = \sigma_i^*(x,t).
\]

This points out an alternative way of looking at the difference between a closed-loop equilibrium and a perfect equilibrium: a perfect equilibrium makes full use of all current information. Any perfect equilibrium must also be a closed-loop equilibrium, but only those closed-loop equilibria which are of pure feedback form are perfect equilibria. Thus the necessary conditions for a perfect Nash equilibrium will be identical to those for a closed-loop Nash equilibrium, with the additional requirement that the equilibrium strategies must be of pure feedback form. This requirement ensures that the "interaction term" of the costate equation be nonzero, i.e. the strategy of each
player is actually influenced by the current state of the game.

**An Equivalent Formulation of Perfection**

There is an equivalent way of looking at perfect Nash equilibria of a differential game which will provide a useful tool for verifying that a proposed equilibrium is actually a perfect equilibrium.

Suppose player 1 knows that player 2 plays according to the "reaction function" \( \tilde{\sigma}_2(x,t) \), which may or may not optimal for player 2. Then player 1 is faced with a conventional one-person optimal control problem. There are no gaming aspects to this problem, since player 1 now knows that the equations of motion are

\[
\dot{x}(t) = f(x(t), u_1(t), \tilde{\sigma}_2(x,t), t).
\]

Given this information, player 1 can find his optimal strategy \( \sigma_1^*(t) \), which will depend on the reaction function of player 2: \( \sigma_1^*(t) = \sigma_1^*(x, x_0, t; \tilde{\sigma}_2(x,t)) \). Although this strategy for player 1 could be written in open-loop form, \( \sigma_1^*(t) \) is required to be given in its feedback representation.

Now suppose that the one-person optimal control problem has a unique solution. Sufficient conditions for uniqueness of such a solution are well-known. Then one can define the following functional from the space of feedback strategies into itself: let \( L_1(\tilde{\sigma}_j) \) be the optimal feedback strategy of
player 1, given that player 2 is using the feedback strategy 
\( \tilde{\sigma}_j \), i.e. \( \sigma^*_1 = L_1(\tilde{\sigma}_j) \). Then a perfect equilibrium is a fixed
point of the following functional equations:

\[
\sigma_1 = L_1(L_2(\sigma_1)) \quad \text{and} \quad \sigma_2 = L_2(L_1(\sigma_2)).
\]

This is analogous to the static concept of Nash
equilibrium. In a static context, player 1 determines his
optimal action \( a_1 \) for any given action \( a_2 \) of his opponent,
i.e. he determines his reaction function \( a_1 = R_1(a_2) \). Again,
the Nash equilibrium is the fixed point defined by
\( a_1 = R_1(R_j(a_1)) \). The only difference is that in the dynamic
context the actions are chosen not from the set of real
numbers, but rather from a space of functions.

This approach gives a powerful tool for verifying that
a proposed solution is a perfect equilibrium. Suppose that
\( \{\sigma_i^*(x,t)\} \) is a candidate solution. One can simply solve
player 1's one-person optimal control problem, given
\( \sigma_j^*(x,t) \). Then \( \sigma_i^*(x,t) \) must be the resulting optimal control.
It is important to note that \( \sigma_i^*(x,t) \) must yield the optimum
over all closed-loop controls, not just the feedback
controls. However, it is then represented in feedback form.

This approach shows that the requirements for a perfect
equilibrium are quite formidable. The strategies of the each
player must be optimal, given the conjectured strategy of
his opponent, and the conjectures must be mutually
consistent. The existence of these equilibria is certainly
not a trivial question, and will be examined next.
Existence and Uniqueness of Equilibria

The primary reason for considering perfect equilibria rather than closed-loop equilibria is to eliminate the existence of multiple equilibria due to "informational non-uniqueness". It is reassuring to know that, at least for a particular class of differential games, perfectness implies uniqueness. The following theorem was proven by Papavassilopoulos and Cruz (1979):

Theorem 2.1.3: Suppose that:

(i) the time horizon of the game is finite,

(ii) the functions \( f, g_i, F_1 \) (i.e. the equation of motion and the objective functions) are analytic,

(iii) the admissible strategies are analytic functions of the state variables, then the perfect Nash equilibrium pair \( (\sigma_1^*(x,t), \sigma_2^*(x,t)) \) is unique, if it exists.

It is important to note the requirement that the strategies must be analytic functions of the state variables. Indeed, if discontinuous strategies are allowed, then Fudenberg and Tirole (1983a) have shown that multiple perfect equilibria may exist. Discontinuous strategies will be further discussed in section 2.2.

No uniqueness result appears to be known in the infinite horizon case. However, I believe that a uniqueness proof for the infinite horizon case with analytic
strategies might actually be obtained by making some assumptions about the boundedness of the functions \( f \) and \( g_i \).

So far I have been unable to obtain a formal proof, but I would like to outline my approach in the hope of providing intuition for why I believe the result to be true. The uniqueness proof in the finite horizon case relies on the following steps. If the strategy space is restricted to analytic functions, the definition of perfect equilibrium implies the two value functions must satisfy the Bellman equation of dynamic programming. This requirement consists of two coupled partial differential equations, with a terminal condition given by the transversality condition. Within the class of analytic functions, this initial value problem (where the "initial time" is \( T \)) has a unique solution by a theorem on partial differential equations. In order to extend this proof, one must show that the transversality conditions in the infinite horizon case provide an "endpoint condition", such that the "general solution" of the differential equations has a unique "particular solution."

The necessary conditions which a perfect equilibrium must satisfy give a place to look for a solution, but they do not assure the existence of a solution. Indeed, this area is still a subject of investigation. However, some results are known in the linear-quadratic case which will be discussed in detail.
Linear-quadratic Games

Linear-quadratic games are those for which the equation of motion is linear in the state and control variable, and the objective function is quadratic in the state and control variable. Instead of stating the most general form of a linear-quadratic game, the case in which the payoff is discounted, and the problem is autonomous is presented here.

A differential game of linear-quadratic form is one in which the objective of player $i$ is to choose $u_i$ as to maximize

$$J_i(u_i, u_j) = \int_0^T \left[ \frac{1}{2} x' C_i u_i + \frac{1}{2} x' D_i x + c_i' u_i + d_i' x + f_i \right] e^{-r t} dt + \frac{1}{2} x'(T) F_i x(T)$$

subject to $\dot{x} = Ax + B_1 u_1 + B_2 u_2$, $x(0) = x_0$.

There are $n$ state variables, and there are $r_i$ control variables for player $i$, so $x$ and $d_i$ are $nx1$ vectors, $u_i$ and $c_i$ are $r_i \times 1$ vectors, and the matrices have the following dimensions: $A$ and $D_i$ are $nxn$, $C_i$ is $r_i \times r_i$, and $B_i$ is $nx r_i$. For the problem to make sense, $C_i$ must be negative definite. In general, the elements of these matrices are allowed to be piecewise continuous functions of time, but will be assumed constant here.

The solution to the linear quadratic game is found by assuming that the costate variables are linear in the state, i.e. $\lambda_i = x' Q_i + q_i$ for some $Q_i, q_i$. One can then differentiate this expression with respect to time, and compare it with the costate equation. By equating coefficients one obtains a solution.
Theorem 2.1.4: Let the matrices $Q_i$ and the vectors $q_i$ satisfy the \textit{Riccati equations}: for $i=1,2$ $j\neq i$
\begin{align*}
A'Q_i+A-Q_i^{-1}B'Q_j^{-1}B_jC_j^{-1}B_j'Q_i^{-1}B_jC_j^{-1}B_j'Q_j'&+D_i=0 \\
q_iA-rq_i-(q_iB_j+c_j')C_j^{-1}B_j'Q_j'&-(q_iB_j+c_j')C_j^{-1}B_j'Q_j'&-q_iB_jC_j^{-1}B_j'Q_j'&+d_i=0 
\end{align*}
and the transversality condition
\begin{align*}
x'Q(T)_{1+}q_i=x'(T)F_1e^{rT}-q_i 
\end{align*}
then the strategies $\{u_i^*\}$ given by
\begin{align*}
u_i^*=-x'Q_iB_iC_i^{-1}q_iB_iC_i^{-1}-c_i'C_i^{-1} 
\end{align*}
are a perfect Nash equilibrium. Furthermore, this equilibrium is unique.

While it is not difficult to show that these are the necessary conditions for a solution of the linear-quadratic game, the existence of a solution is quite difficult to establish. The proof is given by Lukes (1971). The uniqueness follows from theorem 2.1.3., since the time horizon is assumed finite, and all the functions involved are analytic.

Theorem 2.1.4 establishes the existence of a unique solution to the linear-quadratic differential game, provided the time horizon is finite. In the case of an infinite time horizon, the following result has been proven by Papavassilopoulos et al. (1979):

Theorem 2.1.5: In the infinite horizon case ($T=\infty$), if the matrices $Q_i$ and the vectors $q_i$ satisfy the Riccati equations, and if the matrix $Z$ given by
\begin{align*}
Z=A-B_1C_1^{-1}B_1'Q_1-B_2C_2^{-1}B_2'Q_2 
\end{align*}
is asymptotically stable, i.e. the eigenvalues of this matrix have negative real parts, then the finite-horizon strategies are also a perfect equilibrium for the infinite horizon case.

This theorem asserts that, provided some conditions are met, there exists a perfect Nash equilibrium for the linear-quadratic game even in the infinite horizon case. However, the uniqueness of this equilibrium remains unresolved.

2.2 Differential Games, State-space Games, and Dynamic Games

The object of this section is to relate differential games to state-space games, and state-space games to general dynamic games. Much of this discussion is based on Fudenberg and Tirole's (1983b) survey of methodology.

The first subsection will examine the relation between state-space games and dynamic games. It will be argued that the state-space assumption is a useful restriction which can be interpreted as requiring strategies to depend on "sufficient statistics."

The second subsection gives an informal discussion of the relation between differential games and state-space games. It will be argued that the equilibria of these two types of game differ because the former have continuous value functions, while the latter rely on discontinuities of the value function as "threats" and "punishments." This leads to an interpretation of differential game equilibria
as a "competitive" equilibrium, the default equilibrium if coordination breaks down.

State-space Games and Dynamic Games

The primary reason for investigating dynamic games, according to Kreps and Spence (1983), is that they focus attention on history as a determinant of structure, conduct, and performance of industries. Intuitively, there are two reasons why history should matter. First, past actions influence the current level of "tangible" variables, such as the current level of capital of the firms. Second, past actions are important in shaping current expectations about the actions and reactions of a firm's rivals.

In general, the relation between past actions and current expectations may be quite complex. Consider the infinitely repeated prisoner's dilemma. One possible equilibrium strategy consists of "collusion" until one player "cheats," and he is subsequently "punished" by his rival, who reverts to the static Nash equilibrium. In this case, any previous cheating leads to subsequent non-cooperation. However, it is also possible that, after some period of time, players "forgive and forget", and attempt to restore cooperation. In that case, the current strategy may depend on how many times the rival has cheated, how long ago he last cheated, etc.
The state-space approach assumes that the current strategy, instead of depending on the entire history of previous play, can be summarized by a "sufficient statistic." For example, Fudenberg and Tirole (1983b) suggest that in the infinitely repeated prisoner's dilemma, a state variable can be introduced which records the number of times either party has cheated. In a Bayesian framework, this could be a sufficient statistic for estimating the probability of cheating in the current period.

The state-space assumption per se does not appear to be very restrictive. It seems quite reasonable that current expectations depend on past behavior only through a few sufficient statistics. The real difficulty lies in developing plausible assumptions about how this summary statistic evolves. If one's rival has cheated once, when can he be trusted again, i.e., what is the probability of cheating again in the following periods? The possible discontinuities in the evolution of this state variable makes the solution of this state-space game difficult.

This is the reason why the state-space approach has been used primarily to model the importance of tangible variables. However, there is no reason why this approach cannot be used to model expectation and reputation effects, provided one is willing to assume that current beliefs depend in a continuous fashion on past actions.
Differential Games and State-space Games

The object of this subsection is an informal discussion of the difference between equilibria with continuous and discontinuous value functions for state-space games. It will be argued that state-space equilibria with discontinuous value functions correspond to "implicit collusion", whereas continuous value functions correspond to "competition."

The discussion will be based on the investment model of Fudenberg and Tirole (1983a), in the case of no discounting. The net revenue \( \pi_i(K_i, K_j) \) of each firm is a function of the levels of capital of the two firms. The price of investment \( I_i \) is normalized to one. If the discount rate were positive, firms would maximize their net present value of profits, 

\[
V_i = \int_0^\infty [\pi_i(K_i, K_j) - I_i] e^{-rt} dt.
\]

With a zero discount rate, this sum would diverge, so instead it is assumed that the firms maximize their steady-state profits, which is given by 

\[
\lim_{t \to \infty} \pi_i(K_i(t), K_j(t)).
\]

No firm may disinvest, and there is an upper bound on the rate of investment, so \( I_i \in [0, \bar{I}] \). There is no depreciation, which implies that the equation of motion for the capital stock is \( \dot{K}_i = I_i \).

This game is a state-space game, and the equation of motion as well as the objective function are continuously differentiable. However, the objective function is linear in the control variable \( I_i \), so the optimal strategies are of the "bang-bang" type, switching discontinuously from zero investment to maximum investment. Hence theorem 2.1.2 cannot
be directly applied to obtain necessary conditions for a 
prefect Nash equilibrium.

One perfect equilibrium of this game is illustrated in 
figure 2.2.1. The reaction function and Stackelberg point of 
firm i are $R_i$ and $S_i$ respectively. In region I, neither 
firm invests; in region II, firm 1 doesn't invest, and firm 
2 invests as fast as possible; in region III, both firms 
invest as fast as possible; regions IV-VI are the symmetric 
equivalents.

![Figure 2.2.1]

Perfect Equilibrium

The perfect equilibrium described by figure 2.2.1 is 
the one originally proposed by Spence (1979). Fudenberg and 
Tirole show that additional perfect equilibria exist. It is 
clear that once the joint investment path has reached $R_1$, 
firm 1 would prefer all investment to cease. Beyond that
level of capital, firm 1 invests only in self-defense.

Figure 2.2.2 shows that, once the investment path reaches $U$, firm 2 prefers to stop investment rather than have joint investment to $R_2$. At any point in the region between $U$ and $R_2$, both firms prefer the stopping of all investment to the alternative of joint investment to $R_2$. Thus one perfect equilibrium strategy is the one in which both firms invest up to $U$, and then stop; at levels of capital above $U$, the strategy of each firm is "invest up to $R_2"." But this strategy is not the only early stopping equilibrium: joint investment up to line $E$, stopping on line $E$, and joint investment beyond line $E$ is also a perfect equilibrium. Hence the region between $U$ and $R_2$ contains an infinite number of perfect equilibria.

---

**Figure 2.2.2**

Early Stopping Equilibria
Now consider the value functions associated with these equilibria. Consider the perfect equilibrium with U as the early stopping surface, and recall that the objective function for firm 1 is \( \lim_{t \to 0} \pi_1(K_1(t), K_2(t)) \). Thus the value of the objective function for the optimal strategy is constant along any joint investment path, and equal to the eventual stopping point. It is clear that this value function is discontinuous: along any joint investment ray through U, the value function takes a downward jump as it crosses U: for levels of capital up to U, the stopping point is on U, whereas for levels of capital above U, the stopping point is \( R_2 \).

This argument shows that all of the early stopping equilibria have discontinuous value functions. In contrast, the equilibrium of figure 2.2.1 has a continuous value function. To avoid cumbersome terminology, it will be referred to as the Spence equilibrium. It will now be argued that the Spence equilibrium can be thought of as a "competitive equilibrium," whereas the early stopping equilibria are "implicitly collusive."

The state-space game was just shown to have an infinite number of perfect equilibria, and the choice between these equilibria corresponds to the choice of a stopping surface. This choice of a stopping surface can be thought of as the result of implicit coordination of expectations, or the result of an explicit, but non-binding contract. Fudenberg
and Tirole note that "if the players cannot achieve this coordination, there is no reason to expect the observed outcome to correspond to any equilibrium."

Fudenberg and Tirole propose one of these many stopping surfaces as "most reasonable." It is illustrated by line $X$ in figure 2.2.3. At any point on $X$, the joint investment path is tangent to firm 2's isoprofit curve. Any amount of joint investment, no matter how small, will make both firms worse off, leading to an investment "truce." Any point below $X$ has the property that firm 2 is willing to undergo joint investment, because it knows that firm 1 will agree to stop on $X$.

---

**Figure 2.2.3**
Fudenberg-Tirole Equilibrium

---

While this equilibrium has much to recommend it, it is not entirely compelling. Suppose the two firms have stopped
investing at point A. If firm 1 stops investing, then it pays for firm 2 to invest a little to point B, and then propose another "truce." Of course, it is not clear why firm 1 should accept a truce at point B, if firm 2 just "cheated" by moving from A to B. Fudenberg and Tirole argue that, for this reason, firm 2 will not try to cheat, and the equilibrium is maintained at A. However, suppose that firm 2 does try to cheat at point A. This precipitates punishment by firm 1, leading to an equilibrium on the reaction function $R_2$.

This argument shows that the Spence equilibrium has particular significance, because it is the "default equilibrium", should coordination about the choice of stopping surface break down. Hence the Spence equilibrium can be thought of as "competitive", in the sense of absence or breakdown of cooperation, whereas the early stopping equilibria involve some amount of cooperation, or "implicit collusion." Indeed, Fudenberg and Tirole note that the "'early stopping' equilibria are reminiscent of the equilibria of supergames....A firm which 'cheats' by investing is punished by a period of joint investment."

This raises an important methodological point. Both "collusive" and "competitive" equilibria are perfect Nash equilibria of a noncooperative game. Whether, and how much collusion can be expected will depend on the likelihood of achieving coordination.
I believe that there is one assumption in the Fudenberg and Tirole model, which makes coordination on their proposed equilibrium particularly likely: it is the irreversibility of investment. At any point above $R_1$, firm 1 never has any incentive to defect from a truce. This is known to both firms, and puts firm 1 in a weak bargaining position. Firm 2 will be able to impose the truce which it most prefers. Viewed another way, firm 2 is the only potential cheater, so it gets to pick its favorite position.

Now suppose that investment were reversible. Then at any point which is not the static Nash equilibrium point, both firms have an incentive to cheat from a "stop/stop" agreement. For example, at a point on $X$, firm 1 would like to reduce investment, if firm 2 doesn't invest. Agreement is much more difficult to reach in this situation, and the default, or "competitive" equilibrium becomes increasingly likely.

The investment game of Fudenberg and Tirole provides an example of the difference between prefect equilibria with continuous and discontinuous value functions. It is not difficult to see that this distinction will generally arise. Along the path of any continuous-value-function perfect equilibrium, it is always possible to propose a "stop/stop" equilibrium at an arbitrary point $A$. The modified strategy --follow the path, stop at $A$, continue on the path for any point strictly beyond $A"--will also be a perfect equilibrium.
Hence, in general, the "competitive" perfect equilibrium is important as the default equilibrium on which the "collusive" equilibria are based.
FOOTNOTES TO CHAPTER 2

1. An analogous theory exists for problems formulated in discrete time, where the equations of motion are given by difference equations.


3. The theory of differential games was first developed by Isaacs (1965) for zero-sum games, with particular emphasis on problems of evasion and pursuit. The class of nonzero-sum games were first studied by Starr and Ho (1969a,b). An excellent reference book on discrete and continuous dynamic games is Basar and Olsder (1982). Much of my presentation of the subject is adapted from this work.

4. Strictly speaking, the information structure \( \eta_i(t) \) generates a sigma-field, and the strategies \( \sigma_i(\eta_i(t),t) \) must be measurable with respect to the corresponding Borel sets. See Basar and Olsder (1982), pp. 210-212 for details.

5. See Basar and Olsder (1982), pp. 278-288 for the derivation of the necessary conditions in both the open-loop and closed-loop case.

6. If discontinuous strategies are allowed, the maximum principle technique can still be applied if the discontinuities are of the "right kind." See Fleming and Rishel (1975), pp. 90-103 for a precise formulation of "admissible discontinuities."

7. See Seierstad and Sydsæter (1977) for a review of sufficiency theorems.

CHAPTER 3

A DIFFERENTIAL GAME MODEL OF DUOPOLY
WITH REVERSIBLE INVESTMENT

This chapter develops a dynamic model of duopoly, based on the assumption that firms may disinvest as well as invest, but cannot adjust their level of capital instantaneously. The duopolists are engaged in a continuous-time differential game in which profit is a quadratic function of the level of capital, and where investment is subject to quadratic adjustment costs. This differential game is solved for the open-loop Nash equilibrium, as well as the perfect closed-loop Nash equilibrium.

Recent advances in game theory have changed the way in which economists view the theory of oligopoly. When traditional static models are replaced by dynamic models, collusion is seen as one of many noncooperative equilibria of a repeated game. However, "competition" is also one of these equilibria. This chapter presents one possible interpretation of the term "competition" in repeated (or "dynamic") games.

There are three reasons for wanting to examine the competitive equilibrium of repeated games. The first is its empirical relevance: many oligopolies fit a "competitive" description better than a collusive one. It could be that the amount of implicit collusion in these
industries is so small as not to be noticeable, or it is possible that the industry is currently in a period of "punishment," or the maximum punishment may be insufficient to sustain collusion. Alternatively, the frequency with which "competition" is observed may simply reflect the difficulty of choosing among the many implicitly collusive equilibria.

The second reason is theoretical. The "competitive equilibrium" is a focal point among the possible noncooperative equilibria of a dynamic game. It is the "default equilibrium" which supports the implicitly collusive equilibria. Hence any supergame model of implicit collusion must be based on a clear understanding of the competitive alternative.

The third reason for studying the competitive equilibria of dynamic games is to see how they differ from static noncooperative equilibria. It would obviously be valuable to know whether the complex machinery associated with supergames or differential games can be left undisturbed when examining oligopolies which are "competing." Related to this point is the confusion concerning the appropriate choice of "conjectural variation."

The model of this chapter is quite similar to that of Spence (1979), although it differs considerably in its focus. Spence studies the optimal investment path in a "new
market", i.e. where firms may grow significantly without generating excess capacity. In his model, the ability to grow is limited by an upper bound on the rate of investment. He assumes that the level of capital, once in place, cannot be reduced: capital cannot be sold, and does not depreciate. In Spence's model the optimal investment path is to grow as fast as possible up to some stopping point, which is not the static Nash equilibrium. The "leading firm" will grow beyond the static Nash level, and, given its commitment to a higher capacity level, it can increase capacity up to a "Stackelberg" position. Thus the first entrant may maintain a larger market share in the long-run.

In a subsequent paper, Fudenberg and Tirole (1983a) show that Spence's equilibrium is not the only perfect equilibrium in the model. They find a continuum of perfect equilibria with an "early stopping" property, which are "reminiscent of the equilibria of supergames." Spence's equilibrium can thus be viewed as the "competitive equilibrium" which supports the "collusive equilibria" of Fudenberg and Tirole.

The present model modifies two of their assumptions. First, investment is allowed to be reversible, but is subject to convex adjustment costs. This assumption is empirically plausible, since capital does not usually have zero resale value. A consequence of this assumption is that preemption is impossible in the long run. This is because an
early entrant no longer has a credible threat of keeping output high, and will generally reduce its output as its rival grows. In the long run, one would expect that the industry equilibrium for two identical firms will be symmetric, and historical asymmetries eventually become unimportant.

The second crucial assumption of the present model is that the strategies are restricted to those which yield continuous value functions. This restriction eliminates the "collusive equilibria" of Fudenberg and Tirole.¹/

The primary focus of the present model is to present a dynamic model of "competition" among oligopolists. Hence the reason for these two assumptions: they keep the dynamic model as close as possible to the static one, which facilitates comparison.

Collusive equilibria are formally ruled out by the restriction to continuous-value-function strategies. However, the assumption of reversible investment also makes the early stopping equilibria of Fudenberg and Tirole less likely. In their model, once firm 1 has grown beyond its reaction curve, it invests only in self-defense. Thus, when firm 2 proposes to stop the investment game, firm 1 will undoubtedly agree and has no incentive to cheat. However, when investment is reversible, firm 1 has an incentive to reduce its level of capital, given its rival's level. This suggests that the coordination necessary to achieve
cooperation will be more difficult to achieve.

The model is formulated as a differential game. However, such games can usually be solved only for the linear-quadratic case. For this reason, the present chapter sacrifices generality to tractability, and specifies the profit function and the adjustment costs to be quadratic. The solution for this specification can be found by numerically solving a simultaneous equation system called the Riccati equations.

Two types of equilibria will be considered. The primary emphasis will be on the perfect Nash equilibrium, which assumes that firms choose optimal "reaction functions. Instead, the open-loop Nash equilibrium assumes that firms can pre-commit themselves to an optimal path.

The main results of this chapter concern the connections between the steady states of these dynamic equilibria, and the equilibria of static models. The following results will be shown: (i) The steady state of the open-loop equilibrium is independent of adjustment costs and equal to the single-period Cournot equilibrium. (ii) The steady state of the perfect equilibrium depends on adjustment costs. In the perfect steady state firms produce more than the Cournot output. (iii) As the adjustment cost of both firms becomes very large, the perfect steady state approaches the Cournot level. (iv) If firm 1's adjustment costs are very large, and firm 2's very small, the perfect
steady state approaches the Stackelberg equilibrium.

The remainder of the chapter is organized as follows. Section 3.1 reviews the dynamic game approach and motivates the emphasis on "competitive" equilibria. In addition, by explaining the difference between static and the dynamic reaction functions, it clarifies the connection between the the dynamic and the static models. Section 3.2 formally presents the model. Section 3.3 develops the necessary conditions for the open-loop equilibrium and the perfect equilibrium, and discusses their uniqueness. Section 3.4 presents numerical examples of both types of equilibria, both for symmetric and asymmetric adjustment costs. Section 3.5 presents the comparative static results, with particular attention to the relation between the steady state of dynamic equilibria and the static equilibria.

3.1 Introduction

This chapter develops a differential game model of duopoly with reversible investment. This section is devoted to providing some background and motivation for the model.

Loosely speaking, differential games differ from general dynamic games in their requirement that the strategies which players follow yield continuous value functions." This requirement eliminates the "implicitly collusive" equilibria typical of supergames, and restricts the equilibrium to one which is "competitive." Since the
focus of dynamic games has been mainly on implicitly collusive ("supergame") equilibria, a dynamic model without collusion requires some justification. The reasons for focusing on "competitive" equilibria will be discussed first.

Next, a quick review of static noncooperative equilibria will be given. This is done for two reasons. First, the static Cournot and associated models are the obvious standard of comparison for any dynamic model of oligopolistic competition. Second, traditional static models have not been insensitive to the fact that oligopolistic competition is dynamic. Indeed, textbook presentations of the Cournot model frequently mention "reaction functions," a term with clear dynamic overtones. This warrants an exploration of the connection between the reaction function of the static models, and the more precise meaning which the term will assume in the model of this chapter.

The Supergame Approach

Scherer (1980, chp.5) presents some of the traditional theories, and introduces them with the following remark:

Economists have developed literally dozens of oligopoly pricing theories.... This proliferation of theories is mirrored by an equally rich array of behavioral patterns actually observed under oligopoly. Casual observation suggests that virtually anything can happen. Some oligopolistic industries appear to maintain prices approximating those a pure monopolist would find most profitable. Others gravitate toward price warfare.
Traditional models view "competition" and "collusion" as two distinct behavioral modes. In the models of competition, such as the Cournot and Bertrand models, firms are assumed to behave noncooperatively, maximizing individual profits given the behavior of their rivals. In contrast, models of collusion assume that firms cooperate to maximize joint profits, either explicitly, by forming a cartel, or through "tacit collusion". This cooperation may enable each firm to achieve profits which are greater than in the noncooperative outcome. However, as the instability of cartels indicates, collusion may be difficult to achieve and sustain.

In this view, the two distinct forms of behavior (collusion and competition) are explained by two fundamentally different behavioral assumptions (cooperative and noncooperative). The choice between the two models is usually made on empirical grounds, although a priori one can examine conditions which make collusion more or less likely.

The assumption of joint profit-maximization underlying the theory of collusion is actually not a "neoclassical" assumption. Neoclassical economics usually assumes individual profit-maximization. Thus, collusion is only reasonable to assume if it is in each firm's self-interest.

More recently, both collusion and competition have been modeled as equilibria of noncooperative games with an infinite time horizon, known as supergames. In the static
Cournot model, firms simultaneously choose output to maximize profits in a single period. The resulting noncooperative Cournot-Nash equilibrium (CNE) is compelling for the single-period model. However, if firms maximize long-term profits, other noncooperative equilibria become possible. Friedman (1971) shows that more "collusive" equilibria can be sustained by the threat of reverting to single-period CNE behavior.

The noncooperative supergame approach integrates the formerly disjoint theories of collusion and Cournot-Nash behavior, but it does not resolve the fundamental ambiguity of oligopoly theory. Collusion and CNE behavior are just two of many equilibria of noncooperative games. As Kreps and Spence (1983) point out, "in general, infinitely repeated games come with an embarassment of riches in terms of the number of possible equilibria." The problem of selecting the "correct model" has been replaced with that of choosing among many noncooperative equilibria.

Although the supergame approach fails to predict a unique equilibrium, it focuses attention to the source of this non-uniqueness. Harsanyi (1964) explains that

...the essential difference between cooperative and non-cooperative games consists only in the fact that in the latter the players are unable to cooperate in achieving a payoff vector outside the set of equilibrium points—however desirable this may be for all of them—but there is no reason why they should not cooperate within the set of equilibrium points.
There is a sense in which the supergame approach has come full circle: whether a collusive or a competitive equilibrium obtains depends on the conditions which facilitate or limit oligopolistic coordination. However, the supergame approach allows for a broader range of equilibria than the traditional view: between the extremes of joint profit-maximization and competition, intermediate amounts of collusion may obtain.\(^2\) However, in the supergame setting, it is not clear what is meant by "competition."

In Friedman's (1971) model, "competition" is single-period Cournot-Nash equilibrium behavior. CNE is the "stick" which supports the "carrot" of collusion. However, Abreu (1982) has shown that CNE is usually not the most severe threat of punishment for symmetric supergames. Thus, CNE is not necessarily the "least collusive" equilibrium.

Abreu's result gives the first two reasons for wanting to examine "competitive oligopoly equilibria." First, it opens the question of an appropriate definition of "competitive oligopoly equilibrium." Second, this equilibrium has special importance, since it supports the collusive equilibria. The third reason is the observation that "competition" is a frequently observed mode of behavior for oligopolies.
Reaction Functions, Conjectural Variations, and Dynamic Games

The traditional textbook exposition of the Cournot model frequently uses the term "reaction function." By this term is meant the optimal quantity $q_i(q_j)$ which firm $i$ would choose, if the quantity chosen by firm $j$ were $q_j$. The Cournot-Nash equilibrium assumes that the firms choose quantities which are a simultaneous solution of these two functions. However, no firm "reacts" to anything, since the Cournot model is a single-period, simultaneous choice model. Thus the term "reaction function" is really a misnomer.\(^\text{3}^\)

Many real-life economic games are repeated games, and thus violate one of the assumptions of the Cournot model. In such a repeated game, intuition suggests that, if firm $i$ changes its quantity, firm $j$ will react as a result. Thus $q_j(t+1)$ will likely depend on $q_i(t)$. Of course, in a single-period model, this time-dependence cannot be captured. The Cournot model assumes (correctly, for a single period framework) that $\frac{\partial q_j}{\partial q_i}=0$.

The realization that firms in a repeated game may react to each other has led to the development of the concept of "conjectural variation." Since $\frac{\partial q_j(t+1)}{\partial q_i(t)}$ is likely to be nonzero in a repeated game, one may try to capture some of this effect in a single-period model by assuming that $\frac{\partial q_j(t)}{\partial q_i(t)}\neq0$, and this term is called the "conjectural variation." However, since $\frac{\partial q_j}{\partial q_i}=0$ in any single-period,
simultaneous game, the choice of conjectural variation is somewhat arbitrary.

A way of choosing a "right" conjectural variation, which would capture the flavor of the dynamic story in a static model, was proposed by Bresnahan (1982). He suggests that the conjectural variation must be consistent with the "actual variation," and terms the resulting equilibrium a "consistent conjectures equilibrium" (CCE).

Bresnahan's consistent conjectures equilibrium works as follows. The profit functions of firm i and firm j are \( p_i(q_i, q_j) \), and \( p_j(q_i, q_j) \) respectively. Firm i determines its optimal quantity by setting \( \frac{dn_i}{dq_i} = 0 \). Firm i conjectures that firm j's reaction will be \( \frac{dq_j}{dq_i} = v_{ij} \neq 0 \). On the basis of this conjecture, firm i's "reaction function" is \( q_i(q_j, v_{ij}) \). Similarly, firm j's reaction function is \( q_j(q_i, v_{ij}) \). Differentiating these functions yields the "actual variations" \( a_{ij} = \frac{dq_i(q_j, v_{ij})}{dq_j} \). The equations which determine the "consistent" variation are simply \( a_{ij} = v_{ij} \).

There are two reasons why one might be skeptical of Bresnahan's consistent conjectures equilibrium. First, it assumes that the approximation \( \frac{dq_j(t+1)}{dq_i(t)} = \frac{dq_j(t)}{dq_i(t)} \) is sensible. However, even a small-epsilon time lag in choosing quantities is likely to produce substantially different results than simultaneous choice. Essentially, it assumes that the "reaction function" is actually not a misnomer. Second, equilibria of repeated games can be quite
different from those of single-period games (e.g. implicit collusion). Thus, one might doubt a priori whether the consistent conjectures equilibrium approximates the equilibrium of the repeated game. Indeed, it will be shown that the CCE does not adequately capture the dynamic effects.

The model of this chapter will consider truly dynamic "best-response" functions, which deserve the name "reaction function." Each firm must choose a "rule for setting output". These rules must be consistent, and the each firm's rule must be optimal given its rival's rule. Thus one might call the resulting equilibrium a "consistent reaction function" equilibrium. It will be shown that this equilibrium differs considerably from the consistent conjectures equilibrium.

One further remark to clarify the difference between the best-response functions used in the present dynamic model, and the traditional reaction function. The static reaction function gives the optimum response to any given output. Each output-reaction pair corresponds to a separate one-dimensional maximization. In contrast, the dynamic reaction function corresponds to one infinite-dimensional maximization, where each firm chooses a reaction function to maximize its objective functional.
3.2 The Model

This section presents a differential game model of duopoly with reversible investment. The instantaneous profit of each firm is assumed to be a function of the levels of capital of the two firms. Each firm may invest or disinvest, but investment is subject to adjustment costs. The objective of each firm is to choose an investment path which maximizes the discounted value of future profits.

The two firms are indexed by \( i=1,2 \). Firm \( i \)'s payoff depends on the investment functions of both firms. Given \( \{I_j(t)\} \), firm \( i \) chooses \( \{I_i(t)\} \) to maximize

\[
V_i = \int_0^\infty \pi_i(K_1, K_2, I_i) e^{-rt} dt
\]

subject to

\[
\dot{K}_i = I_i - \delta K_i, \quad K_i(0) = K_{i0}, \quad i=1,2,
\]

where the conventional notation is used: \( \pi_i \) is the instantaneous profit flow, \( K_i \) is the level of capital, \( I_i \) is the level of gross investment, \( \delta \) is the rate of depreciation, and \( r \) is the discount rate. The firms are assumed to have perfect information about the structure of the game, and the current levels of capital.

The equilibrium concept

As discussed in chapter 2, there are several possible equilibrium concepts for differential games. Both open-loop and perfect Nash equilibria will be considered, with the primary focus on perfect Nash equilibria. The difference
between these two equilibria is the set of admissible strategies for the two firms.

The open-loop strategies correspond to pre-commitment: each firm chooses its entire investment path at the beginning of the game, and cannot modify its path after the start of the game. Each path is required to be optimal given the other firm's path. However, if, due to a random error, the level of capital does not evolve along the equilibrium path, each firm would want to revise its strategy. This is the reason why firms cannot usually credibly pre-commit themselves: they may have an incentive to deviate from the "announced" path.

In a perfect Nash equilibrium, firms choose optimal feedback rules, or "best response functions." A strategy consists of a function \( I_i(K_1,K_2) \) which specifies how much a firm will invest for any given level of \( K_1 \) and \( K_2 \). In contrast to the open-loop equilibrium, this strategy must be optimal for any possible combination of \( K_1 \) and \( K_2 \), not just those along a particular equilibrium path. The formal definition for the investment game is
Definition 3.2.1: The strategies \( (I_1^*(K_1, K_2), I_2^*(K_1, K_2)) \) are a perfect Nash equilibrium if there exist continuous functions \( J_1: \mathbb{R}^2 \rightarrow \mathbb{R} \) such that

\[
J_1(I_1^*, I_2^*) \geq V_1(I_1, I_2^*) \quad \text{and} \quad J_2(I_1^*, I_2^*) > V_2(I_1^*, I_2)
\]

for all continuous feedback rules \( I_1(K_1, K_2) \) and \( I_2(K_1, K_2) \), and for all permissible initial states \( K_1(0), K_2(0) \).

This definition requires that the strategies employed must yield continuous value functions. This eliminates the possibility of implicit collusion, as in the "early stopping equilibria" of Fudenberg and Tirole (1983a). The continuity requirement can be thought of as a restriction to "strictly noncooperative" behavior. The existence of implicit collusion in noncooperative games is certainly an interesting and important aspect of dynamic games. However, the focus of this model is on the "competitive" equilibria of noncooperative supergames.

Assumptions

The distinguishing feature of the model of this section is the assumption that investment is reversible. This contrasts with many recent models, such as Spence (1979), and Dixit (1980), who all assume irreversible investment.\(^4\) When investment is in productive capacity, this assumption need not necessarily hold: in many industries capital has some resale value.
Investment is assumed to be subject to convex adjustment costs. This assumption has been standard in many models of investment, such as Abel (1979), Gould (1968), and Pindyck (1982). These models assume that adjustment costs are a function of gross investment. Instead, the present model assumes that adjustment costs are a function of net investment.

Gould points out that the reason for the conventional specification is mostly technical. In these models, the firm is competitive, and production takes place with constant returns to scale. Letting adjustment costs depend on gross investment introduces a diseconomy of scale, which gives firms a determinate size. Instead, in the present model firms are of determinate size because they have some monopoly power.

Lucas (1967) discusses the empirical merits of specifying adjustment as a function of net investment. In addition to its empirical plausibility, this specification also has a technical advantage. In steady state, the firm incurs no adjustment costs. This facilitates the comparison between static models and the steady state of the dynamic one.

The model just described is too general to yield closed form solutions. As discussed in chapter 2, solutions can be obtained if the problem is of linear-quadratic form. This leads to the following assumption for the instantaneous
profit function:

\[(3.2.3) \quad \pi_i(K_i, K_j, I_i) = [1 - b(K_i + K_j)]K_i - vI_i - \frac{1}{2}c_i(I_i - \delta K_i)^2\]

Profit is given by net revenue minus the cost of investment minus adjustment costs. Just as in Spence (1979) and Fudenberg and Tirole (1983a), net revenue is given in "reduced form" as a function of capital. However, it is necessary to consider whether the quadratic specification is sensible.

Fudenberg and Tirole (1983a) suggest that the instantaneous equilibrium might be Cournot, i.e. the firms choose quantities \(q_i\) subject to the constraint \(q_i \leq k_i\). Alternatively, Kreps and Scheinkman (1983) show that, if capacity is given, Bertrand price competition yields Cournot-like outcomes. In either case, if the demand function is linear, the resulting reduced-form profit function is quadratic if both firms are capacity-constrained, but is linear or constant otherwise. This considerably complicates the maximization problem.

In principle, the problem could be solved by considering separate optimization problems in the capacity-constrained and non-constrained regions, and requiring continuity of the value functions at the boundaries. Instead, the formal solution follows Fudenberg and Tirole in assuming that the firms are in the capacity-constrained region.\(^5\)
An alternative justification for the quadratic specification of the profit function can be obtained by changing the interpretation of the variables. If one views $K_i$ as output, and $I_i$ as the rate of change of output, and lets the rate of depreciation be zero, this model describes a situation with linear demand, where the rate of change of output is subject to adjustment costs. In this case, the quadratic specification makes sense for all values of $K_i$ which yield a nonnegative price.

3.3 The Solution

This section will use the Maximum Principle to solve for both the perfect Nash equilibrium and the open-loop Nash equilibrium. It develops the necessary conditions for each of these equilibria, and shows that there exists a solution for which optimal investment is a linear function of the levels of capital.

In chapter 2, the uniqueness of solutions to linear-quadratic games was discussed. According to Theorem 2.1.3, both the open-loop and the perfect Nash equilibrium of linear-quadratic games with a finite time horizon and analytic strategies are unique, if they exist. For the infinite horizon case, no such result has been proven yet. Although no general proof will be given, the equilibrium will be shown to be unique within the class of linear strategies, at least for some special parameter values.
The Necessary Conditions

Rather than solving the optimization problem (3.2.1)-(3.2.3) directly, it will be useful to consider a change of variables. If \( v \) is the price of a unit of investment, the discount rate is \( r \), and the rate of depreciation is \( \delta \), the cost of capital can be expressed either as \( vI_1 \) or as \( v(r+\delta)K_1 \). This is easily shown by integrating by parts:

\[
\int_0^\infty I_1 e^{-rt} dt = vK_1(0) + \int_0^\infty v(r+\delta)K_1 e^{-rt} dt
\]

The only difference between the two formulations is the extra term \( vK(0) \). This term is only of accounting importance: it depends on whether the value of the firm is taken to include the book value of the initial level of capital. Furthermore, the choice of optimal strategy is unchanged by adding a constant term to the objective function.

Hence the maximization problem becomes

\[
\max V_1 = \int_0^\infty \left\{ \left[ 1 - b(K_1 + K_j) \right] K_1 - v(r+\delta)K_1 - \frac{1}{2}c_1(I-\delta K_1)^2 \right\} e^{-rt} dt
\]

subject to \( \dot{K}_1 = I_1 - \delta K_1 \).

Now let \( \tilde{I} = I - \delta K_1 \), i.e. \( \tilde{I} \) is net investment. The problem can then be written equivalently as

\[
\max V_1 = \int_0^\infty \left\{ \left[ 1 - b(K_1 + K_j) \right] K_1 - v(r+\delta)K_1 - \frac{1}{2}c_1 \tilde{I}_1^2 \right\} e^{-rt} dt
\]

subject to \( \dot{K}_1 = \tilde{I}_1 \).

Henceforth, the tilde will be dropped from the notation, and I will refer to net investment. Also, for brevity let \( A = 1 - v(r+\delta) \).
This differential game can be solved using the Pontryagin maximum principle technique described in the previous chapter. The present-value Hamiltonian is given by

\begin{equation}
H_i = [A-b(K_i+bK_j)]K_i - \frac{1}{2}c_iI_i^2 + \mu_{i1}I_1 + \mu_{i2}I_2
\end{equation}

Then the necessary conditions for a solution are given by

\begin{align}
(3.3.5a) \quad & K_1 = I_1 \\
(3.3.5b) \quad & K_2 = I_2 \\
(3.3.6a) \quad & 0 = \partial H_i/\partial I_1 = -c_1I_1 + \mu_{11} \\
(3.3.6b) \quad & 0 = \partial H_i/\partial I_2 = -c_2I_2 + \mu_{22} \\
(3.3.7a) \quad & r\mu_{11} - \dot{\mu}_{11} = \partial H_i/\partial K_i = [A-2bK_i-bK_j] + \mu_{12}(\partial I_2/\partial K_i) \\
(3.3.7b) \quad & r\mu_{12} - \dot{\mu}_{12} = \partial H_i/\partial K_1 = -bK_1 + \mu_{12}(\partial I_2/\partial K_1) \\
(3.3.7c) \quad & r\mu_{21} - \dot{\mu}_{21} = \partial H_i/\partial K_2 = -bK_2 + \mu_{21}(\partial I_2/\partial K_1) \\
(3.3.7d) \quad & r\mu_{22} - \dot{\mu}_{22} = \partial H_i/\partial K_2 = [A-2bK_2-bK_1] + \mu_{21}(\partial I_1/\partial K_2)
\end{align}

The terms $\partial I_i/\partial K_j$ in equations (3.3.7) are relevant only for the perfect equilibrium, since only then is each player's strategy in pure feedback form. In solving for the open-loop equilibrium, $\partial I_i/\partial K_j = 0$. Eliminating $I_1$ and $I_2$ from (3.3.5) leaves 6 differential equations in the 6 unknowns:

\begin{align}
K_1, K_2, \mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}
\end{align}

Since the problem is linear-quadratic, a closed-form solution can be found. Assume that a solution is given by

\begin{align}
(3.3.8a) \quad & \mu_{11} = \theta_{11}K_1 + \theta_{12}K_2 + \theta_1 \\
(3.3.8b) \quad & \mu_{12} = \theta_{21}K_1 + \theta_{22}K_2 + \theta_2 \\
(3.3.8c) \quad & \mu_{21} = \phi_{11}K_1 + \phi_{12}K_2 + \phi_1 \\
(3.3.8d) \quad & \mu_{22} = \phi_{21}K_1 + \phi_{22}K_2 + \phi_2
\end{align}
If the costate variables are linear in capital, then equations (3.3.6) and (3.3.8) show that investment will also be linear:

\[(3.3.9a) \quad I_1 = (\theta_{11}/c_1)K_1 + (\theta_{12}/c_2)K_2 + \theta_1/c_1\]

\[(3.3.9b) \quad I_2 = (\phi_{21}/c_2)K_1 + (\phi_{22}/c_2)K_2 + \phi_2/c_2\]

Equations (3.3.8) contain 12 unknown parameters \(\{\theta\}\) and \(\{\phi\}\). Substituting this candidate solution into the differential equations (3.3.5)-(3.3.7), one obtains the Riccati equations for the coefficients \(\{\theta\}\) and \(\{\phi\}\). Only the equations for the perfect equilibrium will be given, while those for the open-loop case can be found in appendix C.

**Riccati equations (Perfect equilibrium):**

\[(3.3.10a) \quad \theta_{11}c_1 + \theta_{12}\phi_{21}/c_2 + \theta_{21}\phi_{21}/c_2 - r\theta_{11} = 2b\]

\[(3.3.10b) \quad \theta_{11}c_1 + \theta_{12}\phi_{22}/c_2 + \theta_{22}\phi_{21}/c_2 - r\theta_{12} = b\]

\[(3.3.10c) \quad \theta_{11}c_1 + \theta_{21}\phi_{22}/c_2 + \theta_{22}\phi_{21}/c_2 - r\theta_{21} = b\]

\[(3.3.10d) \quad \theta_{12}\phi_{21}/c_1 + 2\theta_{22}\phi_{22}/c_2 - r\theta_{22} = 0\]

\[(3.3.10e) \quad \phi_{21}\phi_{22}/c_2 + 2\phi_{11}\theta_{11}/c_1 - r\phi_{11} = 0\]

\[(3.3.10f) \quad \phi_{22}\phi_{12}/c_2 + \phi_{12}\theta_{11}/c_1 + \phi_{11}\theta_{12}/c_1 - r\phi_{12} = b\]

\[(3.3.10g) \quad \phi_{22}\phi_{21}/c_2 + \phi_{21}\theta_{11}/c_1 + \phi_{11}\theta_{12}/c_1 - r\phi_{21} = b\]

\[(3.3.10h) \quad \phi_{22}\phi_{22}/c_2 + \phi_{21}\theta_{12}/c_1 + \phi_{12}\theta_{12}/c_1 - r\phi_{22} = 2b\]

\[(3.3.10i) \quad \theta_{11}c_1 + \theta_{12}\phi_{22}/c_2 + \phi_{21}\theta_{22}/c_2 - r\theta_{1} = -A\]

\[(3.3.10j) \quad \theta_{21}c_1 + \theta_{22}\phi_{22}/c_2 + \phi_{22}\theta_{22}/c_2 - r\theta_{2} = 0\]

\[(3.3.10k) \quad \phi_{12}\phi_{22}/c_2 + \phi_{11}\theta_{11}/c_1 + \theta_{11}\phi_{12}/c_1 - r\phi_{1} = 0\]

\[(3.3.10m) \quad \phi_{22}\phi_{22}/c_2 + \phi_{21}\theta_{12}/c_1 + \theta_{12}\phi_{12}/c_1 - r\phi_{2} = -A\]

These 12 simultaneous equations must be satisfied by the parameters of any linear perfect Nash equilibrium.
investment path. Thus, they provide a list of potential perfect Nash equilibria. However, not every solution of the fundamental equations actually corresponds to a Nash equilibrium.

In addition to the Riccati equations, a perfect Nash equilibrium must also satisfy the following transversality conditions:

\[(3.3.11) \lim_{t \to 0^+} \mu_{11} e^{-rt} = \lim_{t \to 0^+} \mu_{12} e^{-rt} = \lim_{t \to 0^+} \mu_{21} e^{-rt} = \lim_{t \to 0^+} \mu_{22} e^{-rt} = 0\]

From equations (3.3.8) we know that each of the \(\mu_{ij}\)'s will converge to a constant if and only if \(K_1\) and \(K_2\) converge to a constant. This suggests that the perfect equilibrium path must converge to a steady state from any initial point \(K_1(0), K_2(0)\).

**Lemma 3.3.1:** Suppose \(I_1(K_1, K_2), I_2(K_1, K_2)\) is a perfect Nash equilibrium. Then the resulting path for the state variables \(K_1, K_2\) must approach an unconstrained steady state, regardless of the initial conditions for \(K_1(0)\) and \(K_2(0)\), i.e.

\[\lim_{t \to 0^+} K_1(t) = K_1^* > 0 \quad \lim_{t \to 0^+} K_2(t) = K_2^* > 0\]

**Proof:** The full proof is rather lengthy, and is given in appendix A. However, the general idea of the proof is simple.

Suppose that an equilibrium path were to terminate with \(K_1 = 0, K_2 = k_2^\#\). Then firm 1 would make zero profits after it reaches this point. We want to show that, no matter what the investment function of firm 2, firm 1 would never want to
stay at this point. Indeed, firm 1 can always choose an investment function which will give it positive profits.

To see this, consider figure 3.3.1, in which in the neighborhood of \( K_1 = 0, \ K_2 = K_2^\# \) we have \( I_2 < 0 \). Then firm 1 can choose an investment path which starts with \( I_1 > 0 \) and then switches to \( I_1 < 0 \), and this path will yield positive profits.

The full proof shows that this can indeed be done for all choices of an investment function \( I_2 \) for firm 2.

---

**Figure 3.3.1**

(Respons of firm 1 to an arbitrary investment function \( I_2 \))

---

Lemma 3.3.1 implies additional restrictions which the solution of the Riccati equations must satisfy.

**Corollary:** Suppose that \( (\theta_{ij}, \phi_{ij}) \) is a solution to the Riccati equations (3.3.10). Then it corresponds to a perfect Nash equilibrium only if

\[
(3.3.12a) \quad (\theta_{11}/c_1) + (\phi_{22}/c_2) < 0 \quad \text{and} \\
(3.3.12b) \quad (\theta_{11}/c_1)(\phi_{22}/c_2) - (\theta_{12}/c_1)(\phi_{21}/c_2) > 0
\]

**Proof:** Consider the equations of motion for \( K_1 \) and \( K_2 \) which
correspond to a given solution \( \theta_{ij}, \phi_{ij} \) of the fundamental equations. Equations (3.3.9) show that the motion of \( K_1 \) and \( K_2 \) is governed by a pair of linear differential equations. The steady state of these equations may be a stable node, an unstable node, or a saddlepoint. ²/²

Suppose that the steady state were an unstable node or a saddlepoint. Then the path would converge to this steady state only for certain special initial conditions (i.e. on the directrix, or at the steady state itself). However, for any other initial conditions \( K_1(0), K_2(0) \), the path corresponding to this solution would eventually violate the transversality conditions. Hence this solution would be a Nash equilibrium only for some particular initial condition, and therefore cannot be a perfect equilibrium.

Now suppose that the steady state corresponds to a stable node. The corresponding path will converge to this steady state for any initial conditions, and is therefore a perfect equilibrium.

It remains to show that the stability requirement is equivalent to equations (3.3.12). Equations (3.3.9) are two simultaneous linear differential equations. Stability requires both eigenvalues of this system of equations to be negative. This requirement is equivalent to the Routh-Hurwitz conditions for stability, which are given in (3.3.12).//
The findings of this section are summarized in:

**Proposition 3.3.1:** If $I_1(K_1, K_2)$, $I_2(K_1, K_2)$ is a perfect Nash equilibrium which is linear in the levels of capital, then

(i) The parameters $\Theta_{ij}, \Theta_{ij}$ must satisfy the Riccati equations (3.3.10).

(ii) The parameters $\Theta_{ij}, \Theta_{ij}$ must satisfy the Routh–Hurwitz stability conditions (3.3.12).

**Uniqueness**

Proposition 3.3.1 gives the necessary conditions which any linear perfect Nash equilibrium must satisfy. By restricting the definition of perfect Nash equilibrium to continuous strategies in feedback form, two sources of nonuniqueness have been eliminated, but there might also be nonlinear perfect Nash equilibria. As discussed in chapter 2, in the finite horizon case it is known that no such nonlinear Nash equilibria exist, and that the linear equilibrium is unique. In the infinite horizon case, the question of uniqueness is unresolved. However, it is conjectured in chapter 2, that even in the infinite horizon case, the perfect equilibrium strategies are unique and linear.

Suppose that the strategy space is restricted to linear strategies. In principle, uniqueness within this restricted class can proven by brute force for any particular choice of
parameters. Any linear Nash equilibrium must satisfy the Riccati equations. Thus, for a given choice of parameters, one can numerically solve the Riccati equations and check each solution for stability. However, for nonlinear equations the number of possible solutions cannot be determined a priori. Different solutions are found numerically by giving different starting values to the Newton-Raphson algorithm. It is possible that some solutions will only be found by given a starting value which is very close to the actual solution. Hence some solutions might "slip through the net" of a gridsearch.

In the case where the adjustment cost of the two firms is equal, the number of possible stable solutions to the Riccati equations can actually be enumerated. This result is proven in the following Lemma.

Lemma 3.3.2: Suppose the two firms have equal adjustment costs \( c_1 = c_2 = c \). Then the stable solution of the Riccati equations also satisfies:

(a) for the perfect Nash equilibrium:

\[
(3.3.13a) \quad (\theta_{11} - rc)\theta_{11} + \theta_{12}^2 = 2bc \\
(3.3.13b) \quad \theta_{12}^3 - \theta_{12}(2\theta_{11} - cr)^2 + bc(2\theta_{11} - rc) = 0
\]

(b) for the open-loop equilibrium:

\[
(3.3.14a) \quad (\theta_{11} - rc)\theta_{11} + \theta_{12}^2 = 2bc \\
(3.3.14b) \quad (2\theta_{11} - rc)\theta_{12} = bc
\]

Proof: The proof is given in appendix B.//
Equations (3.3.13) and (3.3.14) have a maximum number of distinct solutions equal to six and four respectively. Thus, in the symmetric case, a brute force check of uniqueness is possible for any given choice of parameters. I have numerically verified the uniqueness of the linear equilibrium in the symmetric case for a broad range of parameters.

In the asymmetric case, with the caveat that some solutions may have "slipped through the net", I have similarly verified uniqueness of the linear equilibrium for a broad range of parameters. In addition, Theorem 2.1 states the result that linear-quadratic games with a finite horizon have a unique Nash equilibrium, provided the strategies are restricted to be analytic. Hence it is reasonable to conjecture that the equilibria of the differential game are unique. For the remainder of this chapter, it will be assumed that the perfect and the open-loop Nash equilibria are unique.

3.4 Some Numerical Examples

The purpose of this section is to examine the perfect Nash equilibrium and the open-loop Nash equilibrium for some particular numerical choices of parameters. The first example is one in which both firms have identical adjustment costs (c_1 = c_2 = c), whereas in the second example the adjustment costs differ.
The Nash equilibria are found by numerically solving the Riccati equations (3.3.12) and (3.C.1), and selecting the stable root. The numerical solutions were found by using the Newton-Raphson method.  

The Symmetric Case

The first example, in which both firms have identical adjustment costs, will be referred to as the "canonical example." It assumes that the discount rate is \( r=0.05 \), the rate of depreciation is \( \delta=0.1 \), the slope of the demand function is \( b=1 \), and the price of a unit of investment is \( v=1 \). The adjustment cost is \( c=50 \).

These values were chosen after experimenting with several values of the parameters. The choice is motivated by the desire that adjustment costs should "matter". For this choice, the adjustment costs are sufficiently low so that some adjustment takes place, and sufficiently high so that adjustment is not instantaneous.

Recall from Lemma 3.3.2 that in the symmetric case the stable solution of the Riccati equations can be found by considering only two simultaneous equations. For the perfect equilibrium these equations give six potential solutions, while in the open-loop case there are four candidates. The numerical solutions for the canonical example are given in table 3.4.1.
Table 3.4.1
(Solutions of Riccati eqns for \( b=1, v=1, r=0.05, \delta=0.1, c=50 \))

Perfect equilibrium

<table>
<thead>
<tr>
<th>#</th>
<th>(-\theta_{11} = \phi_{22}^*)</th>
<th>(-\theta_{12} = \phi_{21}^*)</th>
<th>(-\theta_{11} = \phi_{22}^*)</th>
<th>(-k^*)</th>
<th>stable</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-8.05</td>
<td>-2.75</td>
<td>3.35</td>
<td>0.310</td>
<td>y</td>
</tr>
<tr>
<td>2</td>
<td>10.55</td>
<td>2.75</td>
<td>-4.25</td>
<td>0.320</td>
<td>n</td>
</tr>
<tr>
<td>3</td>
<td>-4.51</td>
<td>-5.85</td>
<td>4.07</td>
<td>0.392</td>
<td>n</td>
</tr>
<tr>
<td>4</td>
<td>7.01</td>
<td>5.85</td>
<td>-6.42</td>
<td>0.499</td>
<td>n</td>
</tr>
<tr>
<td>5</td>
<td>-0.87</td>
<td>6.97</td>
<td>-1.02</td>
<td>0.168</td>
<td>n</td>
</tr>
<tr>
<td>6</td>
<td>3.37</td>
<td>-6.97</td>
<td>0.28</td>
<td>0.077</td>
<td>n</td>
</tr>
</tbody>
</table>

Open-loop equilibrium

<table>
<thead>
<tr>
<th>#</th>
<th>(-\theta_{11} = \phi_{22}^*)</th>
<th>(-\theta_{12} = \phi_{21}^*)</th>
<th>(-\theta_{11} = \phi_{22}^*)</th>
<th>(-k^*)</th>
<th>stable</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-8.50</td>
<td>-2.57</td>
<td>3.13</td>
<td>0.283</td>
<td>y</td>
</tr>
<tr>
<td>2</td>
<td>11.00</td>
<td>2.57</td>
<td>-3.84</td>
<td>0.283</td>
<td>n</td>
</tr>
<tr>
<td>3</td>
<td>-1.32</td>
<td>-9.75</td>
<td>3.13</td>
<td>0.283</td>
<td>n</td>
</tr>
<tr>
<td>4</td>
<td>3.82</td>
<td>9.75</td>
<td>-3.84</td>
<td>0.283</td>
<td>n</td>
</tr>
</tbody>
</table>

\( k^* = \text{steady-state level of capital} \)

The investment functions are given by equations (3.3.9) and are repeated here for convenience:

\[
I_1 = (\theta_{11}/c_1)K_1 + (\theta_{12}/c_2)K_2 + \theta_1/c_1
\]

\[
I_2 = (\phi_{21}/c_1)K_1 + (\phi_{22}/c_2)K_2 + \phi_2/c_2
\]

The nature of the investment functions for the perfect equilibrium is given in a phase diagram (figure 3.4.1.).
Figure 3.4.1 illustrates some of the qualitative properties of the optimal investment path, which are a direct consequence of the requirement that the steady state must be stable:

(i) Suppose the market for this product has just been created, and the two firms enter simultaneously. Then they will grow simultaneously until they have reached the steady-state level of capacity.
(ii) Suppose that firm 1 has some time to grow before a second firm enters. Then the path starts from $K_1(0) > 0$, $K_2(0) = 0$. The phase diagram shows that, along the equilibrium path, there will be a period during which firm 1 reduces its level of capital.

(iii) If firm 1 is sufficiently large when firm 2 enters, then firm 1 will divest along the entire equilibrium path. But if firm 1 is small when firm 2 enters, then firm 1 will grow initially, and divest only after both firms have grown. Thus the optimal investment of firm 1 involves "overshooting."

The intuition for this overshooting property of the investment path is simple. Consider the case in which firm 1 is allowed to grow undisturbed for a long period. For the moment, neglect the possibility of entry deterrence. Then firm 1 simply grows to its monopoly point. One would expect that the two-firm steady state should involve a smaller output for firm one than the monopolist's output. Hence firm 1 will divest its capacity.

Now consider the case where firm 2 enters before firm 1 has grown to the monopoly output. When firm 2 first enters, firm 1 still desires to increase its output to the monopoly level. But as firm 2 grows, firm 1 will have to adjust to the two-firm steady state. Thus firm 1's capital will increase at first, but decrease later.
The result that in the long run both firms will have equal market share is simply a reflection of the fact that they have identical cost structures. Thus incumbency has no long-run advantages, unless entry can be deterred.

This leads to the question whether it is possible for an incumbent firm to deter entry. It is true that the optimal response to entry is to eventually accommodate the entrant. However, firm 2 will only want to enter if the present value along its optimal path is positive. If the path toward long-run equilibrium is sufficiently painful for firm 2 it may be possible to prevent entry. This means that firm 1 may not want to act as a myopic monopolist in the presence of potential entry. The possibility of entry deterrence, as well as the optimal pre-entry investment is the subject of chapter 5.

The general properties of the phase diagram are just a consequence of the stability requirement for the steady state. Hence the preceding remarks apply to both the perfect equilibrium and the open-loop equilibrium. However, the steady state of the two equilibria, as well as the slope of the isoclines, will be different. The phase diagram for the open-loop equilibrium is given in figure 3.4.2.

The first observation about the difference between the perfect equilibrium and the open-loop equilibrium is that the steady-state level of capital is greater in the former. Thus the perfect steady state is "more aggressive" than the
open-loop steady state.

The intuition for this result derives from the fact that in the perfect equilibrium, both firms follow feedback strategies. The phase diagram shows that an increase in firm 1's output will elicit a decline in firm 2's output. Firm 1's feedback strategy will take this effect into account. This increases the perceived marginal revenue of firm 1, leading to a higher output.

---

**Figure 3.4.2**
(Phase diagram for the open-loop Nash equilibrium)
(b=1,v=1,r=0.05,δ=0.1,c=50)

---

Numerical calculation shows that the steady state of the open-loop path is identical to the static Cournot
equilibrium. This result holds generally, even for the case of asymmetric adjustment costs. The relation between the steady states of the perfect and open-loop some static models will be investigated more thoroughly in the next section.

The Asymmetric Case

In the second example, firm 1 is assumed to have higher adjustment costs than firm 2, with \( c_1 = 100 \) and \( c_2 = 50 \). Again, the solution is found by numerically solving the Riccati equations and selecting the stable root.

It is important to recall that, while a brute force numerical check of uniqueness was possible for particular parameter values in the symmetric case, this approach breaks down in the asymmetric case. This is because it is not possible to determine the maximum number of possible roots in the asymmetric case. The Riccati equations consist of eight nonlinear equations in eight unknowns, which may have anywhere from zero to a large (or infinite) number of solutions. However, a thorough gridsearch for some particular parameter values always resulted in eight roots, of which only one was stable, just like in the symmetric case.

Table 3.4.2 and figure 3.4.3 give the stable solution for the asymmetric case and the corresponding phase diagram.
Table 3.4.2
(Stable solution of the Riccati equations)
\((b=1,v=1,r=0.05,\delta=0.1,c_1=100,c_2=50)\)

**Perfect equilibrium**

<table>
<thead>
<tr>
<th>Firm 1</th>
<th>Firm 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10.36</td>
<td>-3.20</td>
</tr>
<tr>
<td>-3.18</td>
<td>-8.31</td>
</tr>
<tr>
<td>4.33</td>
<td>3.47</td>
</tr>
<tr>
<td>0.328</td>
<td>0.291</td>
</tr>
</tbody>
</table>

**Open-loop equilibrium**

<table>
<thead>
<tr>
<th>Firm 1</th>
<th>Firm 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>-11.22</td>
<td>-2.99</td>
</tr>
<tr>
<td>-2.99</td>
<td>-8.60</td>
</tr>
<tr>
<td>4.03</td>
<td>3.29</td>
</tr>
<tr>
<td>0.283</td>
<td>0.283</td>
</tr>
</tbody>
</table>

\((k_1^*, k_2^* = \text{steady-state levels of capital})\)

The qualitative nature of the equilibrium paths is the same as in the symmetric case, but the slope and intercept of the isoclines has changed.

First, note that the steady state of the open-loop equilibrium is the same as in the symmetric case. Indeed, it corresponds to the static Cournot equilibrium. This result is further discussed in the next section.

The steady state of the perfect equilibrium exhibits a higher level of capital for firm 1 than for firm 2. Schelling's principle (1960) states that a player who can restrict his set of options can gain a strategic advantage. In this case, firm 1 is "more committed" to its current level of capacity and output than its rival.

It is also interesting to note that the intercepts of the isoclines are nearly identical for the two firms. It can
be numerically verified that these intercepts are very close
to the static optimum for a monopolist. This corresponds to
the notion that, when a firm 2 has zero output, firm 1 wants
to produce at the monopoly level.

The reason why the intercept of the isoclines is not
exactly the same as the monopolist's static optimum can be
seen as follows. Equation (3.3.6) showed that, for an
equilibrium path, \( \mu_{11} = c_1 I_1 \) and \( \mu_{22} = c_2 I_2 \). The \( \mu \)'s can be
interpreted as current-value shadow prices. Let \( V_i(K_1, K_2) \)
be the present value of the duopoly game for firm i starting
from initial levels of capital \( K_1, K_2 \). Then \( \partial V_i / \partial K_1 = \mu_{11} \), and
\( \partial V_2 / \partial K_2 = \mu_{22} \). Thus the isocline \( I_1 = 0 \) corresponds to
\( \partial V_1 / \partial K_1 = 0 \). This shows that the \( K_1 \)-intercept of firm 1's
isocline is its "optimal starting point" for the duopoly
game. In general, this point need not be equal to the static
monopoly optimum, but is likely to be close.

3.5 Comparative Statics

This section examines some comparative static results
for the strategic investment model. Particular attention is
given to the relation between the steady state of the
dynamic Nash equilibria, and some static equilibria, such as
Cournot, Stackelberg, and Consistent Conjectures. The
comparative static results in this section are obtained by
solving the model numerically for various choices of
parameters, and comparing the results.
The following results will be shown: (i) The steady state of the open-loop equilibrium is independent of adjustment costs and equal to the single-period Cournot equilibrium. (ii) The steady state of the perfect equilibrium depends on adjustment costs. In the perfect steady state firms produce more than the Cournot output. (iii) As the adjustment cost of both firms becomes very large, the perfect steady state approaches the Cournot level. (iv) If firm 1's adjustment costs are very large, and firm 2's very small, the perfect steady state approaches the Stackelberg equilibrium. (v) If the adjustment costs of both firms are very small, the perfect steady state does not approach the Consistent Conjectures equilibrium.

Adjustment Costs in Open loop Equilibrium

It was noted in the last section that the steady state of the open-loop equilibrium is identical to the Cournot equilibrium. This result, although not proven analytically, holds numerically for all parameter values.

Open-loop equilibria can be thought of as precommitment equilibria. Each firm must choose its entire investment path at some initial time, by considering only the initial level of capital of each firm. It is not allowed to specify a reaction function; rather, it must specify the exact level of investment for each subsequent time. The following argument shows that the Cournot equilibrium must be a steady
state of the dynamic model.

Suppose that at the time when the commitment needs to be made, both firms have exactly the Cournot level of capital, and firm 1 chooses to keep this level of capital forever. Since firm 1 is precommitted to this level of capital, nothing which firm 2 does can affect the level of capital of firm 1. Thus firm 2 will also choose to keep the Cournot level forever, since it yields the greatest profit, given firm 1's level of capital.

When the firms do not start out with the Cournot level of capital, the presence of adjustment costs implies that the level of capital will be changing slowly over time. Now suppose that firm 1's strategy is to approach the Cournot output. By the same argument as just presented, the best response for firm 2 must be to also approach the Cournot level. However, the greater the adjustment costs for firm 2, the slower it will approach this steady state.

Intuition can also be given for the uniqueness of the open-loop equilibrium steady state. Suppose that firm 1 precommits itself to a level of capital $\tilde{K}_1$ forever. Then firm 2 will only want to commit itself to a level $\tilde{K}_2$ forever if $\tilde{K}_2$ is precisely the "Cournot reaction" to $\tilde{K}_1$. But then $\tilde{K}_1$ and $\tilde{K}_2$ can both be optimal only if they correspond to the Cournot equilibrium.

The intuition for the convergence of the open-loop equilibrium to the Cournot equilibrium is simple. In a
precommitment equilibrium, firm 1's output does not affect firm 2's output. Hence the Cournot assumption \( \frac{\partial q_j}{\partial q_i} = 0 \) must hold.

**Adjustment Costs in Perfect Equilibrium**

When firms follow perfect equilibrium strategies, they follow feedback rules. Thus the level of capital of firm 1 will affect the capital of firm 2. It is also reasonable that the speed at which firm 2 can respond to changes in firm 1's capital will influence the strategy followed by firm 1. It will be seen that the steady state of the perfect equilibrium depends on the adjustment cost coefficients \( c_1 \) and \( c_2 \) of the two firms.

Table 3.5.1 illustrates how the level of adjustment costs affects the steady state of the perfect equilibrium, in the case where both firms have equal adjustment costs \( c_1 = c_2 = c \). For reference, it also gives the levels of capital for the corresponding Cournot and Consistent Conjectures equilibria.
Table 3.5.1
Effect of adjustment costs
(b=1, v=1, r=0.05, δ=0.1)

Cournot equilibrium

\[ K^* = 0.2833 \]

Consistent Conjectures equilibrium

\[ K^* = 0.4250 \]

Steady state of perfect equilibrium

<table>
<thead>
<tr>
<th>( c )</th>
<th>( 10^{-4} )</th>
<th>( 10^{-2} )</th>
<th>( 10^{0} )</th>
<th>( 10^{2} )</th>
<th>( 10^{4} )</th>
<th>( 10^{6} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K^* )</td>
<td>0.3151</td>
<td>0.3151</td>
<td>0.3145</td>
<td>0.3083</td>
<td>0.2865</td>
<td>0.2834</td>
</tr>
</tbody>
</table>

The table shows that, as adjustment costs become very large, the steady state of the perfect equilibrium approaches the Cournot level. The intuition for this result is that large adjustment costs are a form of "commitment" to the current level of capital. Suppose that the level of capital for both firms is currently at the Cournot level. Suppose that firm 1 contemplates an increase in its capital. Since firm 2's adjustment costs are very high, it will (almost) not respond to this change, so the Cournot assumption \( \partial q_j / \partial q_i = 0 \) will hold.

Now suppose that adjustment costs are moderate, and suppose again that firm 1 contemplates an increase in capital from the steady state level. Because of the stability of the perfect equilibrium strategies, firm 2 will decrease its level of capital. Hence, for moderate adjustment costs, \( \partial q_j / \partial q_i < 0 \). But this implies that, in
steady state, the marginal revenue of firm 1 is greater than in the Cournot case. Thus firm 1's steady state capital exceeds the Cournot level.

Now consider the case where adjustment costs become very small. This is the case which most closely corresponds to the spirit of Brenahan's (1981) consistent conjectures equilibrium (CCE). For the CCE to be a useful approximation, the steady-state level of capital of the perfect equilibrium should converge to the Consistent Conjectures equilibrium (CCE) of Bresnahan (1981). It was already discussed in section 3.1 that the CCE is subject to some skepticism a priori. Table 3.5.1 shows that the steady state of the perfect equilibrium does not approach the Consistent Conjectures equilibrium. This non-convergence result confirms the original skepticism: the consistent conjectures equilibrium is not a useful equilibrium concept. of both firms become smaller,

One obvious reason why the perfect steady state does not approach the CCE is that in the model, future profits are discounted, while in the latter they are not. However, one might expect convergence in the case where the discount rate is zero. This possibility is investigated in the next subsection, but convergence to CCE will fail to hold even in this case. Before turning to this case, the case of asymmetric adjustment costs is examined.
Table 3.5.2 illustrates how the steady state of the perfect equilibrium changes as the adjustment cost of firm 1 becomes very large, and that of firm 2 very small. For reference, the Stackelberg equilibrium, with firm 1 as the leader, is also given.

<table>
<thead>
<tr>
<th>Table 3.5.2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Effect of asymmetric adjustment costs</strong></td>
</tr>
<tr>
<td><em>(b=1,v=1,r=0.05,δ=0.1)</em></td>
</tr>
<tr>
<td>Stackelberg equilibrium</td>
</tr>
<tr>
<td>( K_1^* = 0.425 )  ( K_2^* = 0.213 )</td>
</tr>
<tr>
<td>Steady state of perfect equilibrium</td>
</tr>
<tr>
<td>( c_1 ) 10 1 0.1</td>
</tr>
<tr>
<td>( c_2 ) 10 100 1000</td>
</tr>
<tr>
<td>( K_1^* ) 0.313 0.405 0.422</td>
</tr>
<tr>
<td>( K_2^* ) 0.313 0.228 0.214</td>
</tr>
</tbody>
</table>

This table shows that the Stackelberg equilibrium is a limiting case of the perfect equilibrium, where firm 1 (with large adjustment costs) is the leader, and firm 2 (with small adjustment costs) is the follower. The intuition for this result is that firm 1 is more "committed" to its level of capital. This is in the spirit of Schelling's principle (1960), that the player who is better able to commit himself will obtain a strategic advantage.

It is important to note that the strategic advantage obtains only in the steady state. If both firms initially have a very small level of capital, then firm 2 will
initially grow faster than firm 1. Hence high adjustment costs are a strategic advantage to a mature firm, but a handicap during its growth period. This issue will be further examined in chapter 5 in the context of entry deterrence.

Discount Rate

A change in the discount rate has two effects: one is through the cost of capital, and the other is a dynamic effect. Recall from equation (3.3.2) that the instantaneous profit in steady state, i.e. when $I_i = \delta K_i$, is given by

\begin{equation}
\pi_i = [1 - b(K_i + K_j)]K_i - \nu(r + \delta)K_i
\end{equation}

Hence an increase in the discount rate is reflected in a higher cost of capital, and consequently the static model predicts that the level of capital will decline.

In the dynamic model, the discount rate also has an effect on the desire to react to changes in the rival's capital. An increase in the discount rate makes current profits more important than future profits, thus lessening the desire to adjust. This effect is analogous to an increase in the adjustment cost coefficient, which was seen to decrease the level of capital. Thus the adjustment effect reinforces the cost of capital effect, and an increase in the discount rate decreases the steady-state level of capital in the perfect equilibrium.
The effect of changes in the discount rate is shown in table 3.5.3. As previously discussed, the perfect equilibrium with large adjustment costs behaves like the Cournot equilibrium. The other polar case is when adjustment costs are very small. The Cournot and Consistent Conjectures equilibria are given: for reference.

<table>
<thead>
<tr>
<th>r</th>
<th>CCE</th>
<th>Cournot</th>
<th>Perfect</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.445</td>
<td>0.296</td>
<td>0.330</td>
</tr>
<tr>
<td>0.02</td>
<td>0.440</td>
<td>0.293</td>
<td>0.326</td>
</tr>
<tr>
<td>0.05</td>
<td>0.425</td>
<td>0.283</td>
<td>0.315</td>
</tr>
<tr>
<td>0.10</td>
<td>0.400</td>
<td>0.267</td>
<td>0.297</td>
</tr>
<tr>
<td>0.20</td>
<td>0.350</td>
<td>0.233</td>
<td>0.260</td>
</tr>
</tbody>
</table>

This table shows that the steady state of the perfect equilibrium does not converge to the Consistent Conjectures equilibrium (CCE) even when the discount rate becomes very small. Hence it appears that the attempt to introduce dynamics via consistent conjectures fails to capture the dynamic effects. Indeed, the Cournot equilibrium is a better proxy for the steady state of the perfect equilibrium than is the CCE.
Lemma 3.3.1 Suppose $I_1(K_1,K_2)$, $I_2(K_1,K_2)$ is a Nash equilibrium. Then the resulting path for the state variables $K_1,K_2$ must approach an unconstrained steady state, i.e.
\[ \lim_{t \to \infty} K_1(t) = K_1^* > 0 \quad \lim_{t \to \infty} K_2(t) = K_2^* > 0 \]

Proof: First, note that a divergent path with $K_i \to \infty$ can never be optimal for firm $i$, since, according to the quadratic specification of the profit function, its profits become infinitely negative. However, the quadratic specification is no longer sensible once a firm reaches a level of capacity where its output is no longer capacity-constrained. Taking this complication into account, a divergent path $K_i \to \infty$ will nevertheless not be optimal, since additional capital has a positive cost, but zero benefit.

We want to show that, in $K_1 - K_2$ space, no Nash equilibrium path can end on one of the boundaries $K_1=0$, $K_2=0$. Without loss of generality, suppose that the equilibrium path hits the $K_2$-axis at time $T$, i.e. $K_1(T)=0$ for some $T>0$. We must show that there exists a time $S>T$ for which $I_1(S)>0$. We will consider two possibilities: at time $T$, firm 2 either (i) has excess capacity, or (ii) is constrained by capacity.

(i) Suppose that at time $T$ firm 2 is not capacity-constrained, i.e. firm 2 has driven firm 1 out of business, but its capacity is in excess of its level of output. Then
at time $T$, there is no incentive for firm 1 to invest, so $I_1(T) = 0$. Firm 2 will want to reduce its capacity, so $I_2(T) < 0$. Thus the only Nash equilibrium path starting at time $T$ will move along the $K_2$-axis toward the origin, so firm 2's level of capacity decreases. Eventually, firm 2 becomes constrained, so case (i) reduces to case (ii).

(ii) Suppose at time $T$ firm 2 is capacity-constrained. With $K_1 = 0$, this implies that firm 2's output must be less than the monopoly output, so the market price must be strictly positive. Then by setting $I_1(T) = 0$ firm 1 can assure itself zero profits. We must show that firm 1 can make positive profits by setting $I_1(T) > 0$.

Recall that $K_1(T) = 0$, and let $K_2(T) = K_2^\#$. If price is strictly positive at $K_2 = K_2^\#$, then it must be positive in a neighborhood $N(0, K_2^\#)$ of this point. If firm 1 can pick a path which lies entirely in this neighborhood, it will make positive profits along this path. There are three cases to be considered:

(a) $I_2 > 0$ throughout $N(0, K_2^\#)$. Then any path starting at $(0, K_2^\#)$ must "head north". Firm 1 can choose $I_1$, and thus control the east-west direction of the path. Let $\epsilon > 0$, and let $I_1(K_1, K_2) > 0$ for $K_2 < K_2^\# + \epsilon$, $I_1(K_1, K_2) < 0$ for $K_2 > K_2^\# + \epsilon$. Then the path first heads northeast and then northwest until $K_1 = 0$. By choosing $\epsilon$ sufficiently small, we can make sure that the path lies entirely within $N(0, K_2^\#)$, so firm 1 makes positive profits.
(b) $I_2 < 0$ throughout $N(0, K_2^*)$. This case is analogous to the previous one. Firm 1 can always construct a path which heads southeast, then southwest, and stays within $N(0, K_2^*)$.

(c) $I_2(0, K_2^*) = 0$. Then the isocline $I_2 = 0$ runs through any neighborhood of $(0, K_2^*)$. Without loss of generality, suppose the isocline has positive slope. Now suppose that $I_2 < 0$ to the south of $I_2 = 0$. Then firm 1 can construct a path which lies in $N(0, K_2^*)$ just like in (b). Suppose instead that $I_2 > 0$ to the south of $I_2 = 0$. Then firm 1 can choose a path which heads northeast with a slope less than the isocline, and terminates at a steady state on the isocline.
APPENDIX B TO CHAPTER 3

The aim of this appendix is to establish Lemma 3.3.2., which claims that, in the case of symmetric adjustment costs, the stable solution of the Riccati equations can be found by solving only two simultaneous equations. The approach is to successively simplify the Riccati equations.

**Lemma 3.8.1:** Suppose the eight parameters \( \Theta_{ij}, \Phi_{ij} \) satisfy the Riccati equations (3.3.10). To each such solution of equations (3.3.10a)-(3.3.10h) corresponds a unique solution \( \Theta_{i}, \Phi_{i} \) of equations (3.3.10i)-(3.3.10m).

**Proof:** Equations (3.3.10i)-(3.3.10m) are four linear equations in the four unknowns \( \Theta_{i}, \Phi_{i} \), with coefficients determined by the parameters \( \Theta_{ij}, \Phi_{ij} \).

Lemma 3.8.1 allows us to reduce the problem of finding twelve unknown parameters to that of finding only eight unknowns. The remaining four unknowns can then be determined by solving a system of four linear equations. Hence the Riccati equations can be reduced to just (3.3.10a-h).

We rewrite the Riccati equations (3.3.10a-h) in the symmetric case:
Riccati Equations (Symmetric case: $c=c_1=c_2$):

(3.B.1a)  $(\theta_{11}^2-cr)\theta_{11} + \theta_{12}\phi_{21} + \theta_{21}\phi_{21} = 2b$

(3.B.1b)  $(\theta_{11}+\phi_{22}-cr)\theta_{12} + \theta_{22}\phi_{21} = b$

(3.B.1c)  $(\theta_{11}+\phi_{22}-cr)\theta_{21} + \theta_{22}\phi_{21} = b$

(3.B.1d)  $(2\phi_{22}-cr)\theta_{22} + \theta_{12}\theta_{21} = 0$

(3.B.1e)  $(2\theta_{11}-cr)\phi_{11} + \phi_{12}\phi_{21} = 0$

(3.B.1f)  $(\theta_{11}+\phi_{22}-cr)\phi_{12} + \theta_{12}\phi_{11} = b$

(3.B.1g)  $(\theta_{11}+\phi_{22}-cr)\phi_{21} + \theta_{12}\phi_{11} = b$

(3.B.1h)  $(\phi_{22}-cr)\phi_{22} + \theta_{12}\phi_{12} + \theta_{12}\phi_{21} = 2b$

**Lemma 3.B.2:** The solutions of the Riccati equations fall into three distinct groups:

(A)  $\theta_{11}+\phi_{22}=cr$

(B)  $\theta_{11}+\phi_{22} \neq cr, \quad \theta_{12}\phi_{21}=(2\theta_{11}-cr)^2$

(C)  $\theta_{11}+\phi_{22} \neq cr, \quad \theta_{12}\phi_{21}=(2\theta_{11}-cr)^2$

\[ \theta_{11}=\phi_{22}, \quad \theta_{22}=\phi_{11}, \quad \theta_{12}=\theta_{21}=\phi_{12}=\phi_{21} \]

Proof: Subtracting (3.B.1c) from (3.B.1b), and (3.B.1g) from (3.B.1f), we get

(3.B.2)  $(\theta_{11}+\phi_{22}-cr)(\theta_{12}-\theta_{21})=0, \quad (\theta_{11}+\phi_{22}-cr)(\phi_{12}-\phi_{21})=0$

Suppose the solution is not in A, i.e. $\theta_{11}+\phi_{22} \neq cr$. Then $\phi_{12}=\theta_{21}$ and $\phi_{12}=\phi_{21}$. Substituting these conditions into (3.B.1a) and (3.B.1h), we obtain $(\theta_{11}-cr)\theta_{11}=(\phi_{22}-cr)\phi_{22}$. Solving this quadratic equation, we find that $\theta_{11}=[Cr+(2\phi_{22}-cr)]/2$. Thus $\theta_{11}=cr-\phi_{22}$ or $\theta_{11}=\phi_{22}$. But the former root violates $\theta_{11}+\phi_{22} \neq cr$. Thus $\theta_{11}=\phi_{22}$.

We have shown that if the solution is not in A, then $\theta_{12}=\theta_{21}$, $\phi_{12}=\phi_{22}$, $\theta_{11}=\phi_{22}$. Using these identities, solve
(3.3.1b) for \( \theta_{22} \) and substitute into (3.3.1d), and solve

(3.3.1g) for \( \phi_{11} \) and substitute into (3.3.1e) to find

(3.3.3a) \[ (2\theta_{11} - cr)b + [\theta_{12} \phi_{21} - (2\theta_{11} - cr)^2] \theta_{12} = 0 \]

(3.3.3b) \[ (2\theta_{11} - cr)b + [\theta_{12} \phi_{21} - (2\theta_{11} - cr)^2] \phi_{21} = 0 \]

Comparing (3.3.3a) and (3.3.3b) we see that

(3.3.4) \[ [\theta_{12} \phi_{21} - (2\theta_{11} - cr)^2] \theta_{12} = [\theta_{12} \phi_{21} - (2\theta_{11} - cr)^2] \phi_{21} \]

which implies that either \((2\theta_{11} - cr)^2 = \theta_{12} \phi_{21} \) or \( \theta_{12} = \phi_{21} \).

Thus if a solution is neither in A nor in B, it must satisfy \( \theta_{11} = \phi_{22}, \theta_{12} = \phi_{21} \). It remains to show that \( \theta_{22} = \phi_{11} \). But this is seen by making all substitutions and comparing (3.3.1d) and (3.3.1e).

Lemma 3.3.3: The solutions in group A and group B cannot correspond to a Nash equilibrium.

Proof: The conditions for stability are (i) \( \theta_{11} + \phi_{22} < 0 \), and

(ii) \( \theta_{11} \phi_{22} - \theta_{12} \phi_{21} > 0 \).

Suppose the solution is in group A. Then \( \theta_{11} + \phi_{22} = s > 0 \), which violates condition (i).

Suppose the solution is in group B. Then \( \theta_{11} = \phi_{22} \), and \( \theta_{12} \phi_{21} = (2\theta_{11} - cr)^2 \), so the solution is stable iff \( \theta_{11}^2 - (2\theta_{11} - cr)^2 > 0 \). But \( \theta_{11}^2 - (2\theta_{11} - cr)^2 = -3\theta_{11}^2 + 4\theta_{11} cr - cr^2 \geq (cr - \theta_{11})(3\theta_{11} - cr) \). This expression is positive iff \( cr > \theta_{11} > cr/3 \). But then \( \theta_{11} = \phi_{22} > 0 \), violating condition (i).

Lemma 3.3.4: There are at most six solutions to the equations in group C. Each of these solutions satisfies the equations

(3.3.5a) \[ (\theta_{11} - cr)\theta_{11} + 2\theta_{12}^2 = 2b \]
(3.8.5b) \[ \theta_{12} - \theta_{12} (2\theta_{11} - cr)^2 + b(2\theta_{11} - cr) = 0 \]

Proof: From Lemma 3.3.2, we know that for each solution in group C we have \( \theta_{11} = \theta_{22} = \theta_{21} = \theta_{12} = \phi_{21} = \phi_{11} \). Thus the Riccati equations reduce to

(3.8.6) \[ (\theta_{11} - cr)\theta_{11} + 2\theta_{12} = 2b \]
(3.8.7) \[ (2\theta_{11} - cr)\theta_{12} + \theta_{12}^2 = b \]
(3.8.8) \[ (2\theta_{11} - cr)\theta_{22} + \theta_{12}^2 = 0 \]

Solving (3.8.8) for \( \theta_{22} \) and substituting into (3.8.7) gives (3.8.5b), while (3.8.6) gives (3.8.5a).

We cannot solve equations (3.8.5a,b) analytically, but we give the graph of these two equations in Figure 3.8.1. Equation (3.8.5a) is the equation of an ellipse with center at \( \theta_{11} = cr/2, \theta_{12} = 0 \), and is equivalent to

(3.8.9) \[ \theta_{12} = \pm \left( b - \frac{1}{2}(\theta_{11} - cr)\theta_{11} \right)^{1/2} \]

Equation (3.8.5b) consists of two "hyperbolas" and one "third-order" curve, which are given by

(3.8.10) \[ \theta_{11} = \left[ cr/2 \right] + \left[ b/(4\theta_{12}) \right] \pm \left[ (b^2 + 4\theta_{12}^4)^{1/2}/(4\theta_{12}) \right] \]

We see from the graph that there can be at most six intersections. //

Lemma 3.8.4 is equivalent to part (a) of Lemma 3.3.2. Part (b) of Lemma 3.3.2 is an analogous statement for the Riccati equations of the open-loop case. The method of proof is analogous to that for the perfect equilibrium, and is omitted.

Note that the stability conditions for group C are given by \( \theta_{11} < 0 \) and \( \theta_{11}^2 - \theta_{12}^2 > 0 \). Thus the stable solutions...
must lie inside the shaded cone. We see that there is only one intersection which lies inside the shaded area, which verifies uniqueness of the linear strategies for this particular choice of parameter values.

Figure 3.B.1
Graph of equations (3.8.5a) and (3.8.5b)
Riccati equations (open-loop equilibrium):

\[(3.1a)\quad \theta_{11}\theta_{11}/c_1 + \theta_{12}\theta_{21}/c_2 - r\theta_{11} = 2b\]

\[(3.1b)\quad \theta_{11}\theta_{12}/c_1 + \theta_{12}\theta_{22}/c_2 - r\theta_{12} = b\]

\[(3.1c)\quad \theta_{11}\theta_{21}/c_1 + \theta_{22}\theta_{21}/c_2 - r\theta_{21} = b\]

\[(3.1d)\quad \theta_{12}\theta_{21}/c_1 + \theta_{22}\theta_{22}/c_2 - r\theta_{22} = 0\]

\[(3.1e)\quad \phi_{21}\phi_{12}/c_2 + \phi_{11}\theta_{11}/c_1 - r\phi_{11} = 0\]

\[(3.1f)\quad \phi_{22}\phi_{12}/c_2 + \phi_{11}\theta_{12}/c_1 - r\phi_{12} = b\]

\[(3.1g)\quad \phi_{22}\phi_{21}/c_2 + \phi_{21}\theta_{11}/c_1 - r\phi_{21} = b\]

\[(3.1h)\quad \phi_{22}\phi_{22}/c_2 + \phi_{21}\theta_{12}/c_1 - r\phi_{22} = 2b\]

\[(3.1i)\quad \theta_{11}\theta_{11}/c_1 + \theta_{12}\theta_{2}/c_2 - r\theta_{1} = -A\]

\[(3.1j)\quad \theta_{21}\theta_{1}/c_1 + \theta_{22}\theta_{2}/c_2 - r\theta_{2} = 0\]

\[(3.1k)\quad \phi_{12}\phi_{2}/c_2 + \phi_{11}\theta_{1}/c_1 - r\phi_{1} = 0\]

\[(3.1m)\quad \phi_{22}\phi_{2}/c_2 + \phi_{21}\theta_{1}/c_1 - r\phi_{2} = -A\]
APPENDIX D TO CHAPTER 3

The following program, written in CBASIC, and implemented on an Osborne Executive, was used to solve the Riccati equations.

```
rem ******************************
rem asymminv.bas
rem solves for as many roots as it can find
rem of a system of nonlinear equations
rem using the Newton-Raphson method
rem the equations are those of the strategic investment
rem model in the asymmetric case
rem Note: root(j%,1)=theta11 for j%th soln
rem root(j%,8)=phi22 for j%th soln
rem const(j%,1)=theta1 for j%th soln
rem const(j%,4)=phi2 for j%th soln
rem
rem ******************************
clear$=chr$(26)
ms%=8:rem maximum number of solutions
nr%=8:rem number of unknown roots
nc%=4:rem number of unknown constant terms
dim a(nr%,nr%),b(nr%),x(nr%),temp(nr%),
   root(ms%,nr%),new.root(nr%),const(ms%,nc%),
   k.st(ms%,2),prof.st(ms%,2),stable$(ms%)
```

```
rem main program
rem
rem gosub 1100:rem input default parameters
10 print "0 - Exit from program"
print "1 - Input parameters"
print "3 - Manually find roots"
print "4 - Print roots on screen"
print "5 - Lineprint roots"
print
input "Enter your choice: ";x%
if x%<0 or x%>20 then goto 10
if x%=0 then stop
on x% gosub 1000,3000,3000,4000,5000
goto 10
```

```
rem closed loop equations
rem
rem ***** compute b(1%)=-f(x) *****
```
\[ b_1 = x(1) * x(1) / c1 + x(2) * x(7) / c2 + x(3) * x(1) / c2 - r * x(1) - 2.0 * b \]
\[ b(1) = -b1 \]
\[ b_2 = x(1) * x(2) / c1 + x(2) * x(8) / c2 + x(4) * x(7) / c2 - r * x(2) - b \]
\[ b(2) = -b2 \]
\[ b_3 = x(1) * x(3) / c1 + x(3) * x(8) / c2 + x(4) * x(7) / c2 - r * x(3) - b \]
\[ b(3) = -b3 \]
\[ b_4 = x(2) * x(3) / c1 + 2.0 * x(4) * x(8) / c2 - r * x(4) \]
\[ b(4) = -b4 \]
\[ b_5 = x(7) * x(6) / c2 + 2.0 * x(5) * x(1) / c1 - r * x(5) \]
\[ b(5) = -b5 \]
\[ b_6 = x(8) * x(6) / c2 + x(6) * x(1) / c1 + x(5) * x(2) / c1 - r * x(6) - b \]
\[ b(6) = -b6 \]
\[ b_7 = x(8) * x(7) / c2 + x(7) * x(1) / c1 + x(5) * x(2) / c1 - r * x(7) - b \]
\[ b(7) = -b7 \]
\[ b_8 = x(8) * x(8) / c2 + x(7) * x(2) / c1 + x(6) * x(2) / c1 - r * x(8) - 2.0 * b \]
\[ b(8) = -b8 \]

return

rem ****** compute a(i%, j%) ******

\[ a(1,1) = 2.0 * x(1) / c1 - r \]
\[ a(2,1) = x(2) / c1 \]
\[ a(3,1) = x(3) / c1 \]
\[ a(4,1) = 0 \]
\[ a(5,1) = 2.0 * x(5) / c1 \]
\[ a(6,1) = x(6) / c1 \]
\[ a(7,1) = x(7) / c1 \]
\[ a(8,1) = 0 \]
\[ a(1,2) = x(7) / c2 \]
\[ a(2,2) = x(1) / c1 + x(8) / c2 - r \]
\[ a(3,2) = 0 \]
\[ a(4,2) = x(3) / c1 \]
\[ a(5,2) = 0 \]
\[ a(6,2) = x(5) / c1 \]
\[ a(7,2) = x(5) / c1 \]
\[ a(8,2) = x(7) / c1 + x(6) / c1 \]
\[ a(1,3) = x(7) / c2 \]
\[ a(2,3) = 0 \]
\[ a(3,3) = x(1) / c1 + x(8) / c2 - r \]
\[ a(4,3) = x(2) / c1 \]
\[ a(5,3) = 0 \]
\[ a(6,3) = 0 \]
\[ a(7,3) = 0 \]
\[ a(8,3) = 0 \]
\[ a(1,4) = 0 \]
\[ a(2,4) = x(7) / c2 \]
\[ a(3,4) = x(7) / c2 \]
\[ a(4,4) = 2.0 * x(8) / c2 - r \]
\[ a(5,4) = 0 \]
\[ a(6,4) = 0 \]
\[ a(7,4) = 0 \]
a(8,4)=0
da(1,5)=0
da(2,5)=0
da(3,5)=0
da(4,5)=0
a(5,5)=2.0*x(1)/c1-r
a(6,5)=x(2)/c1
a(7,5)=x(2)/c1
a(8,5)=0
da(1,6)=0
a(2,6)=0
a(3,6)=0
a(4,6)=0
a(5,6)=x(7)/c2
a(6,6)=x(8)/c2+x(1)/c1-r
a(7,6)=0
a(8,6)=x(2)/c1
a(1,7)=x(2)/c2+x(3)/c2
a(2,7)=x(4)/c2
a(3,7)=x(4)/c2
a(4,7)=0
a(5,7)=x(6)/c2
a(6,7)=0
a(7,7)=x(8)/c2+x(1)/c1-r
a(8,7)=x(2)/c1
a(1,8)=0
a(2,8)=x(2)/c2
a(3,8)=x(3)/c2
a(4,8)=2.0*x(4)/c2
a(5,8)=0
a(6,8)=x(6)/c2
a(7,8)=x(7)/c2
a(8,8)=2.0*x(8)/c2-r
return

rem 190

***** compute a(i%,j%),b(i%) for constant terms *****
a(1,1)=root(n.roots%,1)/c1-r
a(1,2)=root(n.roots%,7)/c2
a(1,3)=0
a(1,4)=root(n.roots%,2)/c2
a(2,1)=root(n.roots%,3)/c1
a(2,2)=root(n.roots%,8)/c2-r
a(2,3)=0
a(2,4)=root(n.roots%,4)/c2
a(3,1)=root(n.roots%,5)/c1
a(3,2)=0
a(3,3)=root(n.roots%,1)/c1-r
a(3,4)=root(n.roots%,6)/c2
a(4,1)=root(n.roots%,7)/c1
a(4,2)=0
a(4,3)=root(n.roots%,2)/c1
a(4,4)=root(n.roots%,8)/c2-r
b(1) = -a
b(2) = 0
b(3) = 0
b(4) = -a
return

rem    *****************************************
rem    open loop equations
rem    *****************************************

rem    **** compute b(i%) = -f(x) ****
200    b1 = x(1)*x(1)/c1 + x(2)*x(7)/c2 - r*x(1) - 2.0*b
b(1) = -b1
b2 = x(1)*x(2)/c1 + x(2)*x(8)/c2 - r*x(2) - b
b(2) = -b2
b3 = x(1)*x(3)/c1 + x(4)*x(7)/c2 - r*x(3) - b
b(3) = -b3
b4 = x(2)*x(3)/c1 + x(4)*x(8)/c2 - r*x(4)
    b(4) = -b4
b5 = x(7)*x(6)/c2 + x(5)*x(1)/c1 - r*x(5)
    b(5) = -b5
b6 = x(8)*x(6)/c2 + x(5)*x(2)/c1 - r*x(6) - b
    b(6) = -b6
b7 = x(8)*x(7)/c2 + x(7)*x(1)/c1 - r*x(7) - b
    b(7) = -b7
b8 = x(8)*x(8)/c2 + x(7)*x(2)/c1 - r*x(8) - 2.0*b
    b(8) = -b8
return

rem    **** compute a(i%, j%) ****
250    a(1, 1) = 2.0*x(1)/c1 - r
a(2, 1) = x(2)/c1
a(3, 1) = x(3)/c1
a(4, 1) = 0
a(5, 1) = x(5)/c1
a(6, 1) = 0
a(7, 1) = x(7)/c1
a(8, 1) = 0
a(1, 2) = x(7)/c2
a(2, 2) = x(1)/c1 + x(8)/c2 - r
a(3, 2) = 0
a(4, 2) = x(3)/c1
a(5, 2) = 0
a(6, 2) = x(5)/c1
a(7, 2) = 0
a(8, 2) = x(7)/c1
a(1, 3) = 0
a(2, 3) = 0
a(3, 3) = x(1)/c1 - r
a(4, 3) = x(2)/c1
a(5, 3) = 0
\begin{verbatim}
a(6,3)=0
a(7,3)=0
a(8,3)=0
a(1,4)=0
a(2,4)=0
a(3,4)=x(7)/c2
a(4,4)=x(8)/c2-r
a(5,4)=0
a(6,4)=0
a(7,4)=0
a(8,4)=0
a(1,5)=0
a(2,5)=0
a(3,5)=0
a(4,5)=0
a(5,5)=x(1)/c1-r
a(6,5)=x(2)/c1
a(7,5)=0
a(8,5)=0
a(1,6)=0
a(2,6)=0
a(3,6)=0
a(4,6)=0
a(5,6)=x(7)/c2
a(6,6)=x(8)/c2-r
a(7,6)=0
a(8,6)=0
a(1,7)=x(2)/c2
a(2,7)=0
a(3,7)=x(4)/c2
a(4,7)=0
a(5,7)=x(6)/c2
a(6,7)=0
a(7,7)=x(8)/c2+x(1)/c1-r
a(8,7)=x(2)/c1
a(1,8)=0
a(2,8)=x(2)/c2
a(3,8)=0
a(4,8)=x(4)/c2
a(5,8)=0
a(6,8)=x(6)/c2
a(7,8)=x(7)/c2
a(8,8)=2.0*x(8)/c2-r
return

rem 290  ***** compute a(i,j),b(i) for constant term *****
a(1,1)=root(n.roots%,1)/c1-r
a(1,2)=0
a(1,3)=0
a(1,4)=root(n.roots%,2)/c2
a(2,1)=root(n.roots%,3)/c1
a(2,2)=-r
\end{verbatim}
a(2,3)=0
a(2,4)=root(n.roots%,4)/c2
a(3,1)=root(n.roots%,5)/c1
a(3,2)=0
a(3,3)=-r
a(3,4)=root(n.roots%,6)/c2
a(4,1)=root(n.roots%,7)/c1
a(4,2)=0
a(4,3)=0
a(4,4)=root(n.roots%,8)/c2-r
b(1)=-a
b(2)=0
b(3)=0
b(4)=-a
return

rem ******************************************************************************
rem  input parameters
rem  ******************************************************************************
1000 print "New parameters:"
   input "r=",line a$
   if a$="" then goto 1010 else r=val(a$)
1010 input "d=",line a$
   if a$="" then goto 1020 else d=val(a$)
1020 input "c1=",line a$
   if a$="" then goto 1025 else c1=val(a$)
1025 input "c2=",line a$
   if a$="" then goto 1030 else c2=val(a$)
1030 input "b=",line a$
   if a$="" then goto 1040 else b=val(a$)
1040 input "v=",line a$
   if a$="" then goto 1050 else v=val(a$)
1050 input "closed-loop (y) or open-loop (n)";line a$
   if a$="" then goto 1060
   if a$="Y" or a$="y" then closed.loop$="y"
   if a$="N" or a$="n" then closed.loop$="n"
1060 a=1.0-v*(r+d)
n.roots%=0
return

rem  ***** default parameters *****
1100 r=.05
d=.1
c1=50.0
c2=50.0
b=1.0
v=1.0
a=1.0-v*(r+d)
closed.loop$="y"
n.roots%=0
return
**** solve for a root from a given starting value ****

n%=nr%
na%=n%
for iter%=1 to 25
  for i%=1 to n%
    temp(i%)=x(i%)
  next i%
next iter%
if closed.loop$="y" then gosub 100:rem
  compute b(i%)=-f(x)
if closed.loop$="n" then gosub 200:rem
  compute b(i%)=-f(x)
norm=0
for i%=1 to n%
  norm=norm+b(i%)#b(i%)
next i%
if norm<=1.0E-15
then
  for i%=1 to n%
    new.root(i%)=x(i%):
  next i%
  return
endif
if closed.loop$="y" then gosub 150:rem
  compute A(i%,j%)=J(x)
if closed.loop$="n" then gosub 250:rem
  compute A(i%,j%)=J(x)
gosub 2.000:rem solve Ax=b
if iflag%=1 then return:rem error return
  if pivoting fails
  for i%=1 to n%
    x(i%)=temp(i%)+x(i%)
  next i%
next iter%
iflag%=1:return:rem error return if no convergence

**** test whether root is new ****

newroot$="y"
for i%=1 to n.roots%
  root.diff=0
  for j%=1 to nr%
    root.diff=root.diff+
      (new.root(j%)-root(i%,j%))*
      (new.root(j%)-root(i%,j%))
  next j%
if root.diff<1.0E-6 then newroot$="n":return
next i%
return

**** compute constant terms ****

n%=nc%
a%=n%
if closed.loop$="y" then gosub 190:rem
  read a(i%,j%)
b(i%) closed-loop
if closed.loop$="n" then gosub 290:rem
read a(i%,j%), b(i%) open-loop

gosub 2.000: rem solve Ax=b
for i%=1 to nc%
    const(n.roots%,i%)=x(i%)
next i%
return

rem 2400
**** compute steady state and determine stability *

n%=2
na%=n%

a(1,1)=root(n.roots%,1)/c1
a(1,2)=root(n.roots%,2)/c1
b(1)=-const(n.roots%,1)/c1
a(2,1)=root(n.roots%,7)/c2
a(2,2)=root(n.roots%,8)/c2
b(2)=-const(n.roots%,4)/c2
gosub 2.000: rem solve Ax=b
p=1.0-b*((x(1)+x(2))
for i%=1 to 2
    k.st(n.roots%,i%)=x(i%)
    prof.st(n.roots%,i%)=(p-v*(r+d))*x(i%)
next i%
if a(1,1)+a(2,2)<0 and a(1,1)*a(2,2)-a(1,2)*a(2,1)>0
    stable$(n.roots%)="y"
else
    stable$(n.roots%)="n"
return

rem 3000
**** find additional roots ****

print clear;
print "Try additional starting values:"
for j%=1 to nr%
    print using ",x(#)=";j%;
    input "",x(j%)
next j%
gosub 2100: rem solve for a root from given starting value
if iflag%=1 then goto 3010: rem no root was found
if n.roots%=0
    then newroot$="y"
    else gosub 2200: rem test whether root is new
if newroot$="n" then goto 3010
n.roots%=n.roots%+1
for j%=1 to nr%
    root(n.roots%,j%)=new.root(j%)
next j%
gosub 2300: rem compute constant terms
gosub 2400: rem compute steady state and stability
return

rem 4000
**** print roots on screen ****

print
for i%=1 to n.roots%
    print using "Root #";i%
    print "-x(1)- -x(2)- -x(3)- -x(4)- -x(5)- -x(6)-
         -x(7)- -x(8)-"
    for j%=1 to nr%
        print using "####";root(i%,j%);
    next j%
    print
    print "-c(1)- -c(2)- -c(3)- -c(4)-"
    for j%=1 to nc%
        print using "####";const(i%,j%);
    next j%
    print
    print "Steady state:",
    print ",-k*1- -k*2- -prf1- -prf2- -stab-"
    k.st(i%,1),k.st(i%,2),prof.st(i%,1),prof.st(i%,2),
    stable$(i%)
    print
    print
    next i%
return

rem ***** lineprint roots *****
5000 lprinter
    if closed.loop$="y" then print "Closed loop"
    else print "Open loop"
    print using "c1=####.#### c2=####.####"
    r=#### d=#### b=#### v=####
    c1,c2,r,d,b,v
    print
gosub 4000:rem print roots
    print
    console
    return

%include a:lineqn
FOOTNOTES TO CHAPTER 3

1. In their survey paper (1983b), Fudenberg and Tirole point out the importance of these discontinuitites in producing their result: "[In] the investment game, the valuation functions are discontinuous in all th "cooperative" equilibria which stop below the reaction curves--one false step by either player triggers retaliation, sending the state to the reaction curves, which is discontinuously worse than early stopping."

2. See Rotemberg and Saloner (1984) for an application of supergames to business cycles.

3. See Friedman (1977, chp.4) for a clear exposition of the relation between the single-period Cournot-Nash equilibrium and the "Cournot reaction function."

4. If investment is in "brand image", as in Schmalensee's 1978 model of ready-to-eat cereals, the "location" of a cereal in "product space" is assumed to be fixed. This is justified because an advertising campaign to change an image is almost as expensive as the creation of a new image, so the salvage value of the investment is very small.

5. A firm with excess capacity will divest independently of the strategy of its rival, until the constrained region is reached again. The possibility of excess capacity will be considered again in the discussion of stability of equilibria.

6. In infinite horizon autonomous problems, cycling is possible only for divergent paths. See Kamien and Schwartz (1981), p.162.

7. Recall that "informational non-uniqueness" is ruled out by requiring the strategies to be of pure feedback form, and that "implicit collusion" equilibria are ruled out by the assumption of analytic strategies.


9. The Newton-Raphson algorithm is an iterative method. It is curious to note that it converges to the stable solution for a wide range of starting values for the algorithm. Indeed, the unstable solutions are found only by doing a careful gridsearch over starting values. This might be interpreted as "corroborating evidence" that the Riccati equations have a unique stable solution.
CHAPTER 4

PRICE COMPETITION IN A DUOPOLY MODEL WITH GOODWILL

This chapter presents a differential game model of price competition in a duopoly. It is assumed that a "goodwill effect" is present, so that a firm which undercuts the price of its rival gains in market share, but only gradually. Each firm chooses its pricing strategy to maximize its net present value of profits. This differential game is solved for both the open-loop and the perfect Nash equilibrium.

The conventional one-period oligopoly model with price-setting firms is that of Bertrand (1883). It assumes a perfect market, in which the firm charging the lowest price instantaneously faces the entire market demand. Given this assumption, each firm has an incentive to undercut its rival until price equals marginal cost.

The assumption that demand responds instantaneously to price differentials is not very realistic. Because of imperfect information, as well as accumulated goodwill, a firm's demand usually declines only gradually if its price is above that of its rival. Hence, in the short run, the firm has some degree of monopoly power and can charge a price above marginal cost. Of course, the temporary increase in profits must be weighed against the future loss in market
share. The aim of this chapter is to find the equilibrium strategies for the two firms when this dynamic effect is present.

The focus of the present model is on price competition, rather than price collusion. As discussed in chapter 2, dynamic games usually have a plethora of perfect Nash equilibria. Those equilibria with discontinuous value functions can be thought of as "collusive," whereas those with continuous value functions are "competitive." The differential gaming model of the present chapter uses optimal control techniques, which assume continuous value functions, and thus looks for "competitive" equilibria.

The analysis is complementary to that of chapter 3. The model of chapter 3 is a quantity-setting model: firms choose capacity, and the market price is determined as a function of total industry output. However, there is some friction in the choice of quantity, given by the adjustment costs incurred when capacity is changed. Instead, this chapter considers price-setting firms, and the friction in price adjustment comes from the slow adjustment of a firm's market share to price differentials.

A model similar to the present one is that of Justman (1983), who also considers a differential game model of oligopoly. He concentrates on intertemporal demand effects: a large cumulative output has a positive effect on the demand for habit-forming goods, but a negative effect on
durable goods. The firms are assumed to set quantities, and he considers the steady states of the open-loop Nash equilibrium.

Two related models will be examined. The first assumes a general functional form for demand, and cannot be solved in closed form. Instead, a phase diagram is used to suggest some properties of a Nash equilibrium. Also, it will be shown that, when the market share responds very slowly to price differentials, the price which each firm charges in steady state will be close to the monopoly price.

The second model assumes a linear demand function. For this special functional form, the equilibrium strategies are shown to have a linear functional form. The coefficients of these strategies are then found numerically, and some examples are presented. It will be shown that, if market share responds very quickly to price differentials, the open-loop Nash equilibrium approaches the Bertrand equilibrium. Surprisingly, the perfect Nash equilibrium has prices between the Bertrand and the monopoly level.

The remainder of this chapter is structured as follows. Section 4.1 presents both the general model, and the linear-quadratic special case. Section 4.2 derives the necessary conditions which must be satisfied by an open-loop and a perfect equilibrium of the general model. The resulting differential equations are investigated using a phase diagram, and some qualitative properties of a Nash
equilibrium are suggested. Section 4.3 derives the necessary conditions and the Riccati equations for the special case. Section 4.4 gives some numerical examples of open-loop and perfect equilibria. Section 4.5 presents some comparative static results. Particular attention is given to the limiting cases when market share responds very quickly or very slowly to price differentials.

4.1 The model

There are two firms, called firm one and two, which compete in a homogeneous good market. At any moment, each firm enjoys monopoly power over its current customers, but if its price is above that of its rival, it will gradually lose its customers. Hence each firm affects the profit of its rival, and the two firms are engaged in a continuous-time differential game, with each firm choosing a pricing strategy which maximizes the net present value of discounted profits.

In an idealized, frictionless market, if two firms sell an identical product, the firm with the lower price will instantaneously capture the entire market demand. However, in real-world markets, when a firm undercuts the price of its rival, demand does tend to shift toward the lower-priced firm, but only gradually.

There are several reasons why one would expect the shift in demand to be gradual. For one, consumers may not
have perfect information about prices, and find out about the lowest price only through advertising, search and word-of-mouth. Second, consumers may be bound by habit or inertia, an effect which can be attributed to the "goodwill" of the firm. In either case, the firm has a "captive audience" in the short run, but not in the long run.

To capture the dynamic effects, the demand structure consists of two parts: a function which describes the demand at any instant, and a differential equation which describes the shift in demand between firms. If a firm charges a price above that of its rival, its level of goodwill will decrease, and the level of goodwill has a positive effect on the firm's demand. This chapter will consider two specifications: one which is fairly general, and can be solved for some qualitative results, the other with simplifying assumptions which permit a closed-form solution.

Consider first the general specification, which can be motivated by the following parable. In a small town, there are two supermarkets. Consumers keep going to the same store, unless they find out that the other store has a lower price. A shift in patronage may occur when two consumers "compare notes," and one discovers that the other is currently getting a lower price. However, the consumer has some "loyalty," and is more likely to switch the larger the price differential. If meetings between consumers are random, the probability of an encounter between consumers
with different loyalties is given by the product of their market shares.

According to this parable, the demand $D_i$ of each firm at any moment depends on its own price $p_i$, as well as its level of goodwill. It assumes that a firm's level of goodwill is directly reflected in its market share $S_i$, where a high level of goodwill corresponds to a market share of $S_i=1$, and very poor goodwill to market share of $S_i=0$. If all consumers shopped at firm $i$, it would face the demand function $D_i(p_i)$, but currently only a fraction $S_i$ of the consumers shop at firm $i$. In the short run, each firm is a monopolist: its demand does not depend on the price of its rival, and is given by

\[(4.1.1) \quad D_i = D_i(p_i) \cdot S_i; \quad S_1 + S_2 = 1; \quad 0 \leq S_i \leq 1; \quad i=1,2\]

However, the firm does not have monopoly power in the long run, since its goodwill (or market share) declines if its price is above that of its rival. The evolution of the market share described by the parable is a diffusion process given by

\[(4.1.2) \quad \dot{S}_i = k(p_j-p_i)S_i(1-S_i) \quad i=1,2 \quad j \neq i\]

The constant $k$ indicates the speed with which market share responds to price differentials: a larger $k$ reflects a greater frequency of encounters, or a greater willingness to switch.

According to this specification, the price of firm 2 affects the demand of firm 1 only indirectly, by influencing
its market share. Of course, one could also consider a model in which the current demand depends on the prices of both firms, as well as the market share, i.e. \( D_i = D_i(p_1, p_j, S_i) \). Such a formulation might result if there are two groups of consumers, with one group receiving price information only by word-of-mouth, while the second watches television ads. This generalization would complicate the analysis without changing the effect which this model attempts to highlight: that of a tradeoff between higher current profits if price is raised, and the resulting decrease in future profits due to the decrease in market share.

The second version of the demand structure sacrifices some degree of generality for the sake of tractability. The easiest simplification of the general model would be to assume that the consumer demand is a linear function of price, leading to \( D_i = S_i N(1 - bp_i) \), where \( N \) is the number of consumers. Alas, this specification cannot be solved in closed form. Instead, it is assumed that

\[
\begin{align}
(4.1.3) \quad D_i &= A + S_i - bp_i; \quad S_1 + S_2 = 0; \quad i=1,2 \\
(4.1.4) \quad S_i &= k(p_j - p_i) \quad i=1,2 \quad j \neq i
\end{align}
\]

In this specification, the level of goodwill \( S_i \) can no longer be identified with a firm's market share, since the equation of motion (4.1.4) allows \( S_i \) to fall outside the bounds \([0, 1]\). Instead, \( S_i \) can be thought of as a "shift factor." When both firms have equal levels of goodwill, their demand function is \( D_i = A - bp_i \). But if one firm
underprices the other, it can steal demand from its rival as consumers shift loyalty.

It is possible for $S_i$ to assume large negative values, which might yield $D_i < 0$ even for $p_i = 0$. The reason for choosing this somewhat unrealistic specification is that it allows the model to be solved in closed form. However, the phase diagram analysis will reveal that this particular specification leads to similar qualitative results as the more general one. This is not surprising, since the general flavor of "goodwill" is captured in both models: shifts in demand are gradual, and in favor of the firm with the lower price.

For either of the two specifications of demand, the two firms are engaged in a continuous-time differential game. Assuming a constant marginal cost $c_i$ for each firm, and with the demand function given by (4.1.1) or (4.1.3), the objective of each firm is to choose a price path $p_i$ which maximizes

$$V_i = \int_0^\infty [D_i(S_i, p_i) \cdot (p_i - c_i)] e^{-rt} dt$$

subject to (4.1.2) or (4.1.4).

According to (4.1.1), the sum of the two market shares must be unity, i.e. $S_i + S_j = 1$. For the second specification, given by (4.1.3), the two shift factors add to zero, i.e. $S_i + S_j = 0$. In either case, there is only a single independent state variable, which will be referred to as $S$, the share of firm 1. Each firm has one control variable, its own price.
The solution of the model is given by a pricing strategy for each firm which yields a Nash equilibrium.

As discussed in chapter 2, one can consider both open-loop Nash equilibria, and perfect Nash equilibria. An open-loop Nash equilibrium is given by a pair of strategies \( \{p_i^*(t)\}, \{p_j^*(t)\} \), which depend on time, but not on the current value of the state variable, such that \( V_i(p_i^*(t), p_j^*(t)) \geq V_i(p_i(t), p_j^*(t)) \) for all price strategies \( p_i(t) \). The restriction to strategies which depend on time alone corresponds to the idea of precommitment, since firms must choose their entire pricing path at the start of the game, but cannot alter the strategy in the light of new information.

A perfect Nash equilibrium is a pair of feedback rules \( \{p_i^*(S), p_j^*(S)\} \), which satisfy the Nash inequality in the space of feedback rules. This equilibrium concept assumes that the firms can continuously observe the market share, and that the pricing strategy must be a Nash equilibrium for all feasible values of \( S \). The next section shows how to solve for both the open-loop and the perfect Nash equilibrium of the general model.

4.2 Solution of the General Model

The general model, alas, is not amenable to a closed-form solution. Nevertheless, it is possible to investigate qualitative properties of the equilibrium strategies. First,
the necessary conditions which must be satisfied by a perfect or an open-loop Nash equilibrium will be derived. Using phase diagram analysis, it will be shown that a possible candidate pair of Nash equilibrium strategies has each firm charging a higher price as its market share increases. Finally, it will be shown that, when market share does not respond to price differentials (i.e. \( k = 0 \)), then the optimal policy for each firm is to charge its monopoly price.

The necessary conditions

As explained in chapter 2, a differential game can be solved by conjecturing a strategy for firm 2 and deriving the implied optimal strategy of firm 1, and vice versa. In a Nash equilibrium, the strategy which each firm conjectures must be consistent with the optimal strategy actually followed by the other firm.

In the case of precommitment (open-loop equilibrium), firm 2's strategy is assumed to be independent of the current state, i.e. \( p_2 = p_2(t) \), whereas in the perfect equilibrium the strategy is given by \( p_2 = p_2(S_1) \). Firm 1 is assumed to know firm 2's strategy, and thus faces a standard one-person optimization problem.

Let all unsubscripted variables refer to firm 1, and let \( \phi(S) \) be firm 2's price given firm 1's share. As a shorthand in notation, let
(4.2.1) \[ R(p) = D(p) \cdot (p-c) \]

Then firm 1 must choose a pricing strategy which maximizes

(4.2.2) \[ \int_0^\infty SR(p)e^{-rt}dt, \quad \text{subject to } S(0) = S_0, \text{ and} \]

(4.2.3) \[ \dot{S} = k[\phi(S) - p]S(1-S) \]

The current-value Hamiltonian for this problem is

(4.2.4) \[ H = SR(p) + \mu k[\phi(S) - p]S(1-S) \]

The necessary conditions which an optimal path must satisfy are given by

(4.2.5) \[ 0 = H_p = SR'(p) - \mu kS(1-S) \Rightarrow \mu k(1-S) = R'(p) \]

(4.2.6) \[ r\mu - \dot{\mu} = H_S = R(p) + \mu k[\phi(S) - p](1-2S) + \mu k\phi'(S)S(1-S) \]

(4.2.7) \[ \mu e^{-rt} \to 0 \text{ as } t \to \infty \]

In the open-loop case, the price of firm 2 does not depend on the share, so \( \phi'(S) = 0 \) in equation (4.2.6). The analysis of the open-loop equilibrium is analogous to that of the perfect equilibrium, and is therefore omitted.

The costate variable \( \mu \) can be interpreted as the current shadow value of incremental market share (or goodwill). The transversality condition given by (4.2.7) requires this shadow value to remain bounded. Hence it is natural to look for a solution which converges to a steady state.

There are three variables which evolve over time: the state variable \( S \), the control variable \( p \), and the costate variable \( \mu \). Equation (4.2.5) can be used to eliminate \( \mu \). Then equations (4.2.3) and (4.2.6) become two autonomous simultaneous differential equations, the solution of which
can be characterized with the use of a phase diagram.

Differentiating equation (4.2.5) yields

\[ \frac{\dot{k}}{k(1-S)} - \frac{\dot{S}}{S} = R''(p) \dot{p} \]  

Combining this equation with (4.2.6), and using (4.2.5) gives equation (4.2.9), which indicates the motion of \( p \). The equation for the open loop case differs from that of the perfect equilibrium in that \( \dot{S}(S) = 0 \). The equation of motion of \( S \) is described by (4.2.3), and is repeated here for convenience.

\[ R''(p) \dot{p} = R'(p)[r - kS' \phi(S)S(1-S) - k\phi(S) - p](1-S)] \]

\[ -R(p)k(1-S) \]

\[ \dot{S} = k[\phi(S) - p]S(1-S) \]  

Phase diagram analysis

Equations (4.2.9) and (4.2.10) describe the evolution of the state variable \( S \), and the control variable \( p \), for a given pricing strategy \( \phi(S) \) of firm 2. This subsection will investigate how the optimal path for firm 1 depends on an arbitrary pricing strategy \( \phi(S) \) followed by firm 2. This will give an indication of the properties of a Nash equilibrium, which requires that the \( \phi(S) \) must be consistent with the pricing strategy of firm 1.

As the easiest possible case, suppose that firm 2's strategy is to set the same price \( p_2 \), regardless of the state, i.e. \( \phi(S) = p_2 \). For this case, the locus of points for which \( \dot{p} = 0 \) and \( \dot{S} = 0 \) are given by
(4.2.11) \[ \dot{p} = 0 \Leftrightarrow R'(p)[r-k(p_2-p)(1-S)] = R(p)k(1-S) \]
\[ \Leftrightarrow r/[k(1-S)] = p_2 + R(p)/R'(p) - p \]

(4.2.12) \[ \dot{S} = 0 \Leftrightarrow p = p_2 \] for $S \in (0, 1)$, or $S = 0$, or $S = 1$

Note that the left-hand side of equation (4.2.11) depends on $S$ alone, and increases as $S$ tends to 1. The right-hand side of (4.2.11) depends on $p$ alone, and its derivative with respect to $p$ is given by $-R\cdot R'/R''$, which is positive, given the usual convexity assumptions. Thus the locus $\dot{p} = 0$ is upward sloping. The locus of points for which $\dot{S} = 0$ is a horizontal line, plus those points where $S = 0$ or $S = 1$. The resulting phase diagram is given in figure 4.2.1.

There is a unique path which converges to the steady state $p = p_2$, $S = S^*$. This path gives the optimal policy for firm 1, since any divergent path would eventually violate the transversality condition. The diagram illustrates that, when firm 2's pricing policy is $\phi(S) = p_2$, firm 1's optimal price increases as its share increases. In other words, a constant price policy for firm 2 leads to an increasing price policy for firm 1.
Now recall that a Nash equilibrium requires that the pricing strategies of the two firms must be consistent. In the preceding case, the constant price policy of firm 2 would have to be optimal, given the increasing price policy of firm 1. This suggests that, as a next step, one should study how a firm responds to an increasing price policy. In other words, suppose next that firm 2's price increases with its market share, i.e. $\phi(S)$ is decreasing in $S$.\(^2\)

The loci for $\dot{p}=0$ and $\dot{S}=0$ are given by

\begin{align*}
(4.2.13) \quad \dot{p} &= 0 \quad \Rightarrow \quad r/[k(1-S)] - \phi'(S)S - \phi(S) = R(p)/R'(p) - p \\
(4.2.14) \quad \dot{S} &= 0 \quad \Rightarrow \quad p = \phi(S) \text{ for } S \in (0,1), \text{ or } S=0, \text{ or } S=1
\end{align*}

As before, the right-hand side of equation (4.2.13) must be increasing in $p$. As long as $\phi(S)$ and $\phi'(S)$ are bounded, the left-hand side is increasing for $S$ sufficiently close to
1. Hence the \( \dot{p} = 0 \) locus will be increasing for \( S \) close to 1, but may be increasing or decreasing for \( S \) close to 0. Figure 4.2.2 illustrates the phase diagram in the case where \( \dot{p} = 0 \) is increasing throughout.

---

**Figure 4.2.2**
Phase diagram for \( \phi(S) \) with \( \phi'(S) < 0 \)

---

For the case drawn in figure 4.2.2, if firm 2's price increases in its market share, firm 1's optimal policy will also increase in its market share. Hence the two pricing policies are consistent in their general shape. Of course, this does not necessarily imply that they indeed are precisely consistent, but it suggests that it might be possible to find a Nash equilibrium in which each firm's price is an increasing function of its market share.
It is important to note that this analysis does not show that there exists a Nash equilibrium with increasing price policies. Neither does it show that there cannot be equilibria with pricing policies which have no such simple form. It is merely suggesting that increasing price policies are a candidate for a Nash equilibrium of the general model.

A limiting case

There is a limiting result which can be obtained from the general model. Suppose that the rate of communication between consumers is zero, i.e. $k=0$. Then the market share of each firm is fixed forever, and is unaffected by the other firm's price. In this limiting case, one would expect that the optimal policy for each firm is simply to charge its monopoly price, independent of whether the firms follow perfect equilibrium strategies or open-loop strategies. This is indeed confirmed by equation (4.2.5), which shows that the optimal price path must have $R'(p)=0$ if $k=0$.

Another limiting case of interest is the one where market share responds very quickly to price differentials, i.e. $k\to\infty$. One would expect that this case should be similar to the Bertrand model, which leads to marginal cost pricing for both firms. However, inspection of the necessary conditions of the general case is not enough to confirm this suspicion. This limiting case will be analyzed in the context of the special functional forms of the next section.
4.3 Solution of the Special Model

This section derives the necessary conditions for the open-loop and perfect Nash equilibrium of the second model, which assumes a linear demand function, and assumes that the rate of change of the market share is proportional to the difference between the prices of the two firms. With these assumptions, the problem is of linear-quadratic form, which allows a closed-form solution: the strategy of each firm is linear in the state variable.

This section derives the Riccati equations which must be satisfied by the coefficients of the Nash equilibrium strategies. Next, it establishes a stability requirement which must be met by the coefficients. Finally, a phase diagram for the second model is given, and compared with that of the more general model of section 4.2.

Derivation of the Riccati equations

The equations of the special model are given by (4.1.3) and (4.1.4), and are repeated here for convenience.

\[(4.3.1)\quad D_i = A + S_i - b p_i \quad i=1,2\]

\[(4.3.2)\quad \dot{S}_i = k (p_j - p_i) \quad i=1,2 \quad j\neq i\]

The necessary conditions which an optimal price strategy must satisfy are obtained using Pontryagin's maximum principle. The current-value Hamiltonians are

\[(4.3.3)\quad H_i = (A + S_i - b p_i)(p_i - c_i) + \mu_i k (p_j - p_i)\]

Thus the necessary conditions are given by
\[ (4.3.4) \quad \frac{\partial H_i}{\partial p_i} = a + S_i - 2bp_i + bc_i - \mu_i k = \mu_i k = S_i - 2bp_i + bc_i \]

\[ (4.3.5) \quad r \mu_i - \dot{\mu}_i = \frac{\partial H_i}{\partial S_i} = p_i - c_i + \mu_i k \left( \frac{\partial p_j}{\partial S_i} \right) \]

For the open-loop equilibrium, the strategy of firm \( j \) depends on time only, so \( \frac{\partial p_j}{\partial S_i} = 0 \) in equation (4.3.5). The derivation of the equilibrium strategies will be presented first for the perfect equilibrium.

To find a solution of equations (4.3.2) and (4.3.5), it is convenient to eliminate \( \mu_i \). Differentiate (4.3.4):

\[ (4.3.6) \quad \dot{\mu}_i k = \dot{S}_i - 2bp_i \]

Combine (4.3.5) and (4.3.4):

\[ (4.3.7) \quad \dot{\mu}_i k = [a + S_i - 2bp_i + bc_i] [r - k \left( \frac{\partial p_j}{\partial S_i} \right) ] - k(p_i - c_i) \]

Equate (4.3.6) and (4.3.7), and substitute for \( \dot{S}_i \):

\[ (4.3.8) \quad \dot{p}_i = \frac{1}{2b} \left( k(p_j - c_i) - [a + S_i - 2bp_i + bc_i] [r - k \left( \frac{\partial p_j}{\partial S_i} \right) ] \right) \]

The solution of equations (4.3.8) and (4.3.2) will be obtained by guessing a linear solution with undetermined coefficients, and showing that it satisfies the differential equations, provided the coefficients satisfy the Riccati equations.

Suppose that firm \( i \) and \( j \) follow pricing strategies given by

\[ (4.3.9) \quad p_i = \alpha_i S_i + \beta_i \]

\[ (4.3.10) \quad p_j = \alpha_j S_j + \beta_j = -\alpha_j S_i + \beta_j \]

Then

\[ (4.3.11) \quad \dot{p}_i = \alpha_i \dot{S}_i = k \alpha_i (p_j - p_i) = k \alpha_i [- (\alpha_i + \alpha_j) S_i + \beta_j - \beta_i] \]

Now derive another expression for \( \dot{p}_i \). From (4.3.10):

\[ (4.3.12) \quad \frac{\partial p_j}{\partial S_i} = -\alpha_j \]
Substitute (4.3.10) and (4.3.12) into (4.3.8):

\[ (4.3.13) \quad \dot{p}_i = \frac{1}{2b} \left\{ [-k\alpha_j - (r+k\alpha_j)(1-2b\alpha_j)]S_i \\
+ [k(\beta_j - c_i) + (2b\beta_i - bc_i - A)(r+k\alpha_j)] \right\} \]

Equating the coefficients of (4.3.13) and (4.3.11) gives the following Riccati equations:

**Riccati equations (Perfect equilibrium):**

\[
(4.3.14a) \quad \alpha_i^2 + 2(\alpha_j + r/2k)\alpha_i - (\alpha_j + r/2k)/b = 0 \quad i=1,2 \quad j \neq i
\]

\[
(4.3.14b) \quad 2b(\alpha_i + \alpha_j + r/k)\beta_i + (1-2b\alpha_i)\beta_j = \\
[1+b(\alpha_j + r/k)]c_i + A(r/k + \alpha_j)
\]

For the open-loop case, the derivation is analogous, except that \( \partial p_j/\partial S_i = 0 \), which results in

**Riccati equations (Open-loop equilibrium):**

\[
(4.3.15a) \quad \alpha_i^2 + (\alpha_j + r/k)\alpha_i - (\alpha_j + r/k)/2b = 0 \quad i=1,2 \quad j \neq i
\]

\[
(4.3.15b) \quad 2b(\alpha_i + r/k)\beta_i + (1-2b\alpha_i)\beta_j = (1+br/k)c_i + Ar/k
\]

**Stability Condition**

The Riccati equations for both the open-loop and the perfect equilibrium will generally have multiple solutions. This subsection will develop a stability condition which imposes an inequality restriction which the solution must satisfy.

Note that (4.3.14a) describes two quadratic equations in the two unknowns \( \alpha_i \) and \( \alpha_j \). This system of equations will generally have four distinct solutions. Once a solution is found, the values for \( \alpha_i \) and \( \alpha_j \) are then used to solve the two linear equations (4.3.14b) for \( \beta_i \) and \( \beta_j \). Thus there
will generally be four quadruples \( \{x_i, \alpha_j, \beta_i, \beta_j\} \) which satisfy the Riccati equations.

The Riccati equations give necessary conditions which any Nash equilibrium must satisfy. However, not every solution of the Riccati equations is necessarily a Nash equilibrium.

In the general model of section 4.2, goodwill was identified with market share, \( S_i + S_j = 1 \), and \( \dot{S}_i = k(p_j - p_i)S_i S_j \). Instead, the present model assumes \( S_i + S_j = 0 \), \( \dot{S}_i = k(p_j - p_i) \), and goodwill is measured by a "shift factor" indicating the number of consumers who prefer one store over the other. The flavor of the two specifications is the same: demand shifts in favor of the low-price firm, but only gradually.

There is a formal difficulty with the present specification. If firm i were to undercut its rival forever, then firm i's demand would eventually be infinite, and firm j's demand would become negative. In reality, there would be an upper bound on \( S_i \), \( S_i \leq A \). If \( S_i = A \), and consequently \( S_j = -A \), then firm i has "stolen" all of firm j's customers, and firm j has zero demand even if \( p_j = 0 \).

However, an equilibrium with \( S_i = A \) would have some bizarre properties. For \( S_i = A \) ever to be reached, the equilibrium strategies of the two firms would have to be such that \( p_j > p_i \) in the neighborhood of \( S_i = A \). When \( S_i = A \) is finally reached, firm i would want to raise its price to the monopoly level. But this would allow firm j to undercut firm
i's price, and customers would shift back to firm j. Thus firm j is like a weed: unless it is constantly cut, it will grow back. A more reasonable account of an equilibrium with $S_i = A$ would require an explicit model of exit.

This argument shows that equilibria for which $S_i$ becomes negative or greater than $A$, although formally admissible, do not have a reasonable economic interpretation. Indeed, the focus of the present chapter is not a model of predation, but rather a model of price competition between two firms. Thus, only those equilibria for which both firms stay in the market will be considered.

The requirement that both firms must stay in the market imposes a restriction on the coefficients of the pricing strategies. Combining (4.3.2) with (4.3.9) and (4.3.10), the market share evolves according to

(4.3.16) \[ \dot{S}_i = k(p_j - p_i) = -k(\alpha_i + \alpha_j)S_i + k(\beta_j - \beta_i) \]

If this equation is unstable, then one of the two firms will eventually drive the other out of business, unless they should happen to start at the (unstable) steady state. Thus the requirement that both firms stay in the market implies that equation (4.3.16) must be stable, which will be true if and only if

(4.3.17) \[ \alpha_i + \alpha_j > 0. \]

This inequality, combined with the Riccati equations, gives the necessary conditions which a Nash equilibrium must satisfy.
Phase diagram analysis

As has been pointed out repeatedly, the special model of this section was chosen for its tractability rather than its realism. This raises the question as to how the results of the special case relate to the more general and realistic specification. The following phase diagram analysis suggests that the qualitative nature of the equilibria is similar.

The phase diagram for the general model was illustrated in figure 4.2.2, under the assumption that firm $j$ increases its price as its market share increases. For the special case this is equivalent to assuming that $\alpha_j > 0$. The corresponding phase diagram will now be derived.

Substituting (4.3.10) into (4.3.2) and (4.3.8) gives the loci for $\dot{p}_i = 0$ and $\dot{S}_i = 0$.

\begin{align*}
\dot{p}_i &= 0 \Rightarrow \\
2b\left(\frac{r}{k+\alpha_j}\right)p_i &= (A+bc_i)\left(\frac{r}{k+\alpha_j}\right) + c_i - \beta_j + \left(\frac{r}{k+2\alpha_j}\right)S_i \\
\dot{S}_i &= 0 \Rightarrow p_i = \beta_j - \alpha_j S_i
\end{align*}

The phase diagram for these equations is illustrated in figure 4.3.1. The convergent path has the property that price for firm $i$ increases as its market share increases, i.e., firm $i$’s optimal response to $\alpha_j > 0$ is a path with $\alpha_j > 0$. This result is similar to the general case illustrated by figure 4.2.2.
Now consider the possibility of an equilibrium with $\alpha_1 < 0$. Then the stability requirement $\alpha_1 + \alpha_j > 0$ implies that $\alpha_j > 0$. But it was just shown that $\alpha_j > 0$ implies $\alpha_1 > 0$. Thus the stability requirement actually implies that each of $\alpha_1$ and $\alpha_j$ must separately be greater than zero. Hence figure 4.3.1 illustrates the only case which satisfies the stability condition.

4.4 Some Numerical Examples

The previous section derived the necessary conditions for the model with linear-quadratic specification. In this section, the Riccati equations will be solved numerically for two particular examples: one symmetric case of identical marginal costs, and one case with different marginal costs.
The Riccati equations (4.3.14) and (4.3.15) are solved numerically by using the Newton-Raphson iterative method. A listing of the program is given in appendix A.

The following results are shown for these examples. In the symmetric case, the steady-state price for both the open-loop and the perfect equilibrium lies between the Bertrand and the monopoly level, with the open-loop price being more "competitive." In the asymmetric case, the firm with the lower marginal cost obtains the larger steady-state demand, and again the open-loop price is below the perfect equilibrium price.

The symmetric case

The first example is for the case where both firms have identical marginal costs. The values of the parameters for this example are: interest rate $r=0.05$, responsiveness of goodwill $k=0.1$, demand curve with slope $b=1$, intercept $A=1$, and marginal cost $c_1=c_2=0.1$. For this choice of parameters, the solutions to the Riccati equations (4.3.14) and (4.3.15) are given in table 4.4.1.
Table 4.4.1
Solutions of Riccati equations
r=0.05, k=0.1, b=1, A=1, c_1=c_2=0.1

<table>
<thead>
<tr>
<th>-α₁</th>
<th>-β₁</th>
<th>-α₂</th>
<th>-β₂</th>
<th>-S*₁</th>
<th>-p*</th>
<th>-stable-</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.384</td>
<td>0.387</td>
<td>0.384</td>
<td>0.387</td>
<td>0.00</td>
<td>0.387</td>
<td>y</td>
</tr>
<tr>
<td>-0.217</td>
<td>0.263</td>
<td>-0.217</td>
<td>0.263</td>
<td>0.00</td>
<td>0.263</td>
<td>n</td>
</tr>
<tr>
<td>-2.096</td>
<td>1.197</td>
<td>0.596</td>
<td>0.713</td>
<td>0.32</td>
<td>0.520</td>
<td>n</td>
</tr>
<tr>
<td>0.596</td>
<td>0.713</td>
<td>-2.096</td>
<td>1.197</td>
<td>-0.32</td>
<td>0.520</td>
<td>n</td>
</tr>
</tbody>
</table>

Open-loop equilibrium

<table>
<thead>
<tr>
<th>-α₁</th>
<th>-β₁</th>
<th>-α₂</th>
<th>-β₂</th>
<th>-S*₁</th>
<th>-p*</th>
<th>-stable-</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.354</td>
<td>0.325</td>
<td>0.354</td>
<td>0.325</td>
<td>0.00</td>
<td>0.325</td>
<td>y</td>
</tr>
<tr>
<td>-0.354</td>
<td>0.325</td>
<td>-0.354</td>
<td>0.325</td>
<td>0.00</td>
<td>0.325</td>
<td>n</td>
</tr>
</tbody>
</table>

S*₁ = steady-state goodwill of firm 1
p* = steady-state price of both firms

Both for the open-loop and the perfect equilibrium there is only one stable solution. The equilibrium strategies of the two firms are depicted in figure 4.4.1 for the perfect Nash equilibrium.

Figure 4.4.1 illustrates the general properties of the perfect equilibrium strategies. As required by the stability condition, the price of each firm increases as its goodwill increases. Note that for sufficiently low levels of goodwill, a firm will charge a price below marginal cost. This result obtains because the shadow value of additional goodwill share is relatively large when the firm has very little goodwill.
The steady-state price of the two firms is between the Bertrand and the monopoly level. The Bertrand price is given by price equal to marginal cost, and equals $p=0.1$. The monopoly price for a firm with goodwill $S=0$ is found by differentiating $(A+S-p)(p-c)$, and equals $p=0.55$. The perfect equilibrium price for this case equals $p=0.38$, which is just about halfway between the monopoly and Bertrand price.

For the open-loop equilibrium, the steady-state price is $p=0.33$, which is also between the Bertrand and the monopoly price. However, this price is below the perfect equilibrium level, and thus more "competitive."

The intuition for this result can be seen as follows. Suppose firm 1 is contemplating a decrease in price from the steady-state level, in order to increase its level of goodwill at the expense of its rival. In the open-loop case,
firm 2 has precommitted itself to its price, and does not respond to firm 1’s price decrease. However, in the perfect equilibrium, the price decrease by firm 1 will lead to a decrease in firm 2’s goodwill, and trigger a decrease in firm 2’s price. Thus, in the open-loop case, firm 1’s price decrease triggers a larger shift in goodwill than in the perfect equilibrium. In other words, there is a greater incentive to undercut the rival’s price in the open-loop case, leading to a more competitive equilibrium price.

The asymmetric case

The asymmetric case examined here is identical to the symmetric case, except that firm 2’s marginal cost is assumed to be $c_2=0.2$, or double firm 1’s marginal cost of $c_1=0.1$. Table 4.4.2 presents the stable solution of the Riccati equations for this case.
Table 4.4.2
Stable solution of Riccati equations
r=0.05, k=0.1, b=1, A=1, c_1=0.1, c_2=0.2

Perfect equilibrium

\[-\alpha_1, -\beta_1, -\alpha_2, -\beta_2, -S_1^*, -p^*, -stable-\]

0.384 0.381 0.384 0.462 0.11 0.421 y

Open-loop equilibrium

\[-\alpha_1, -\beta_1, -\alpha_2, -\beta_2, -S_1^*, -p^*, -stable-\]

0.354 0.309 0.354 0.416 0.15 0.363 y

\[S_1^* = \text{steady-state goodwill of firm 1}\]
\[p^* = \text{steady-state price of both firms}\]

The perfect equilibrium strategies are shown in figure 4.4.2. Again, the stability condition requires the price of each firm to increase with its level of goodwill. Firm 1 has a cost advantage over its rival, and obtains a larger demand in steady state.

Figure 4.4.2
Perfect equilibrium strategies
r=0.05, k=0.1, b=1, A=1, c_1=0.1, c_2=0.2
Comparing the symmetric and the asymmetric case shows that the slope of the price strategies is unaffected by the change in marginal cost. This can also be seen directly from equation (4.3.14a), since $a_i$ and $a_j$ do not depend on $c_i$ and $c_j$. The only effect of the increase in $c_2$ is on the intercept terms $\beta_1$ and $\beta_2$; the former decreases, while the latter increases.

As a result of the shift in the intercepts, the price charged by firm 2 at any given level of goodwill increases. This makes sense, since its marginal cost has increased. On the other hand, firm 1 charges a lower price at any given level of goodwill, even though its marginal cost has not changed. This is because, given that firm 2 now charges higher prices, it has become easier for firm 1 to obtain additional goodwill, which provides an incentive to lower its price.

The shift in the pricing strategies goes in the same direction, increasing the steady-state demand of firm 1, the low-cost producer. This agrees with intuition: since firm 1 has a greater profit margin, it benefits more from increases in goodwill than its rival.

However, the shift of the intercepts goes in opposite directions in terms of the price. The table shows that the net result of these opposing shifts is to increase the steady-state price. This is not surprising, since one would expect the direct effect (the upward shift in firm 2's
pricing strategy) to outweigh the indirect effect (the downward shift of firm 1's pricing strategy).

4.5 Comparative Statics

This section examines some comparative static results for the linear-quadratic pricing model. Particular attention is given to the limiting cases when the responsiveness of goodwill to price differentials becomes either very large or very small.

The following results will be presented: (i) As $k$, the responsiveness of goodwill to price differentials, decreases to zero, the steady-state price approaches the monopoly price, both for the open-loop and the perfect equilibrium. (ii) As $k$ becomes very large, the steady-state price of the perfect equilibrium approaches a price between the monopoly and the Bertrand price. The steady-state price of the open-loop equilibrium approaches the Bertrand level. (iii) If the two firms have different marginal costs, the low-cost firm obtains a greater demand. This advantage increases as $k$ increases. (iv) A decrease in the discount rate $r$ has the same effect as an increase in the responsiveness $k$.

**Goodwill responsiveness: symmetric case**

The effect of an increase in $k$, the responsiveness of goodwill to price differentials, is examined first for the symmetric case, i.e. when both firms have equal marginal
cost.

In the symmetric case, the steady-state level of goodwill is symmetric, i.e. $S_1=S_2=0$. Table 4.5.1 shows the effect of changes in $k$ on the steady-state price, both for the open-loop equilibrium and the perfect equilibrium, when $c_1=c_2=0.1$.

<table>
<thead>
<tr>
<th>Table 4.5.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effect of changes in responsiveness $k$</td>
</tr>
<tr>
<td>$r=0.05, b=1, A=1, c_1=c_2=0.1$</td>
</tr>
<tr>
<td>Bertrand equilibrium</td>
</tr>
<tr>
<td>$p^* = 0.1$</td>
</tr>
<tr>
<td>Monopoly</td>
</tr>
<tr>
<td>$p^* = 0.55$</td>
</tr>
<tr>
<td>Steady state of perfect equilibrium</td>
</tr>
<tr>
<td>$k$</td>
</tr>
<tr>
<td>$p^*$</td>
</tr>
<tr>
<td>Steady state of open-loop equilibrium</td>
</tr>
<tr>
<td>$k$</td>
</tr>
<tr>
<td>$p^*$</td>
</tr>
</tbody>
</table>

First consider the limiting case of $k=0$. Then the steady state price of both the open-loop and the perfect equilibrium is equal to the monopoly price. This confirms the result of section 4.2, where it was proven analytically for the general model. The result is intuitively appealing, since $k=0$ implies that the two firm do not interact, and thus should act as separate monopolists.
The second limiting case is for increases in $k$. Table 4.5.1 shows that the steady-state price of the open-loop equilibrium approaches the Bertrand price. This result also agrees with intuition. When $k \to \infty$, a small price differential yields an instantaneous, massive shift in demand. Thus, if a firm has to precommit itself to a price, the only price which does not give its rival an incentive to undercut is a price equal to marginal cost.

For the perfect equilibrium, an increase in $k$ also leads to a more competitive, lower price. However, the steady-state price approaches a limit above the Bertrand price. In a perfect equilibrium, each firm knows that its rival's price responds to changes in goodwill. Firm 1 knows that, if it cuts its price with the intention of increasing its goodwill, firm 2 will react by decreasing its price. Thus firm 1 has less of an incentive to lower its price, and the price stays above marginal cost.

These results underline the importance of dynamic models of oligopoly. In a simultaneous-choice, single-period game, it is clearly impossible to capture slow adjustment effects, such as the shift in goodwill in the present chapter. However, one might expect that, as the conditions become more similar to the implicit assumptions of the static model, i.e. adjustment becomes almost instantaneous, the equilibrium of the dynamic model should converge to the static equilibrium.
The present analysis shows that there is one important element missing from static models: the possibility of "reaction." It was shown that, if firms precommit themselves to a course of action, and thus do not react to changes in goodwill, then indeed the static equilibrium is reasonable as an approximation to the no-friction dynamic equilibrium. However, when firms follow perfect equilibrium strategies, they do react to changes in the state variable. This will be taken into account when the firms choose their prices, leading to an outcome which differs from the static equilibrium.

**Goodwill responsiveness: asymmetric case**

When the firms have unequal marginal costs, section 4.4 showed that the lower-cost firm obtains the larger market demand. This subsection shows that this result holds for all levels of $k$, and that the advantage increases as $k$ increases. Also, the limiting behavior of the steady-state price as $k \to 0$ and $k \to \infty$, which was just shown for the symmetric case, is seen to extend to the asymmetric case.

Firm 1 is assumed to be the low-cost producer. More specifically, it is assumed that $c_1 = 0.1$ and $c_2 = 0.2$. The effect of changes in $k$ is shown in table 4.5.2.
Table 4.5.2
Effect of changes in responsiveness k
r=0.05, b=1, A=1, c_1=0.1, c_2=0.2

Steady state of perfect equilibrium
\[
\begin{array}{cccccc}
k & 10^{-5} & 0.01 & 0.1 & 1 & 10 & 1000 \\
\hline
p^* & 0.575 & 0.539 & 0.421 & 0.337 & 0.322 & 0.320 \\
S_1^* & 0.050 & 0.059 & 0.107 & 0.178 & 0.197 & 0.200 \\
\end{array}
\]

Steady state of open-loop equilibrium
\[
\begin{array}{cccccc}
k & 10^{-5} & 0.01 & 0.1 & 1 & 10 & 1000 \\
\hline
p^* & 0.575 & 0.536 & 0.363 & 0.189 & 0.154 & 0.150 \\
S_1^* & 0.050 & 0.060 & 0.150 & 1.050 & 10.050 & 1000 \\
\end{array}
\]

First consider the limiting case of k=0. Table 4.5.2 shows that the low-cost firm will have the larger demand in steady state. To see this, suppose that both firms have equal goodwill, S_1=S_2=0, and each firm charges its monopoly price. This price will be lower for the low-cost firm, so the goodwill will shift toward it. Of course, when k is very small, adjustment to steady state will take place only very slowly. In the extreme case of k=0, the goodwill stays constant forever, and the two firms are free to charge two different prices in steady state.

The resulting steady-state price is the monopoly price for each of the two firms, just as in the symmetric case. Recall that the demand function was specified as A+S-bp, so the monopoly price increases as the goodwill increases. If the two firms had equal goodwill, then firm 1's monopoly
price would be below that of firm 2, but this is counterbalanced by the greater goodwill. In fact, this is precisely the reason why the market shares must differ in equilibrium: $\hat{s}_{1}=0$ requires $p_{1}=p_{2}$, yet when $k\to\infty$, both firms must be charging the monopoly price.

Now consider the limiting case of $k\to\infty$ for the open-loop equilibrium. Given the result of the symmetric case, one would expect the price to approach the Bertrand equilibrium. For the case of two different marginal costs, the Bertrand outcome is somewhat indeterminate. Suppose $p=\hat{c}_{2}>\hat{c}_{1}$. Then firm 2 has no incentive to further cut its price, since it would lose money on each unit. However, firm 1 clearly still benefits from a decrease in price: it will obtain a significant increase in demand for a small price drop. Ideally, firm 1 would want to charge a price just $\epsilon$ below $\hat{c}_{2}$, and make $\epsilon$ as small as possible. But for $\epsilon=0$, it would have to share the demand with firm 2. Thus, formally, no equilibrium exists, although its nature is clear: firm 1 undercuts firm 2 and drives it out of business.

Table 4.5.2 shows that, as $k\to\infty$, the steady-state price for the open-loop equilibrium decreases, just as expected. It also shows that, for $k$ sufficiently large, the price eventually falls below firm 2's marginal cost. Thus firm 2 would want to leave the industry, a phenomenon which is not allowed in the present model. This result is again analogous to the the Bertrand model. If one firm has a cost advantage,
and can obtain large increases in demand by cutting its price, it will charge a price which will drive its rival out of business.

Also note that firm 1's goodwill eventually becomes greater than $A=1$ as $k$ increases. This points out the limitations of the linear-quadratic specification: it allows $\hat{S}_1 > 0$, even when $S_1 > A$. Nevertheless, the results, although formally not correct, give an indication of the direction of the effect.

Next, consider the effect of an increase in $k$ on the steady-state price of the perfect equilibrium. Again, as $k$ increases, the price decreases. However, in the limit as $k \to \infty$, the price converges to $p=0.32$, which is greater than firm 2's marginal cost. Thus firm 2 now survives, although its demand is smaller than that of the low-cost form.

The implication of this result is that, when firms follow perfect equilibrium strategies, they take into account their rival's reaction to price decreases, and will thus be less aggressive about lowering price than if the other firm's price were unaffected.

**Discount rate**

The effect of an increase in the discount rate will have an effect exactly analogous to a decrease in the responsiveness coefficient $k$. This follows immediately by inspecting the Riccati equations (4.3.14) and (4.3.15). The
two parameters \( r \) and \( k \) enter these equations only as a ratio \( r/k \), and will therefore have opposite effects.

Consider the intuition for the two limiting cases. Suppose that \( r \to \infty \). Given the results known about \( k \), this leads to an equilibrium with monopoly pricing. The reason for this is that, for a very high discount rate, only present profits matter, so the firm charges the monopoly price corresponding to its current level of goodwill. On the other hand, if \( r \to 0 \), only the long run matters. Hence any temporary decrease in profits from a price decrease is outweighed by the prospect of a larger future goodwill. However, in the perfect equilibrium, firms realize that their price decreases will lead to retaliation, leading to a price above that of the open-loop equilibrium.
APPENDIX A TO CHAPTER 4

The following program, written in CBASIC, and implemented on an Osborne Executive, was used to solve the Riccati equations of this chapter.

```basic
rem ******************************************************
rem reput.bas
rem solves the fundamental equations
rem of the reputation model
rem Let the two strategies be
rem p1=root(i%,1)*s1+root(i%,3)
rem p2=root(i%,2)*s2+root(i%,4)
rem ******************************************************
clear$=chr$(26)
n%=2
na%=n%
dim a(n%,n%),b(n%),x(n%),temp(n%),root(b,4),
new.root(4),
s1.st(8),s2.st(8),p1.st(8),
p2.st(8),stable$(8),tvc$(8)

rem ******************************************************
rem main program
rem ******************************************************
gosub 60:rem read default parameters
n.roots%=0
10 print clear$;
print "0 - Exit from program"
print "1 - Input parameters"
print "2 - Compute and lprint roots"
print "3 - Manual search for a root"
print "4 - Compute and lprint value for given S1(0)"
print input "Enter your choice: ";x%
if x%<0 or x%>20 then goto 10
if x%=0 then stop
on x% gosub 50,8000,500,9000
go to 10

rem ***** input parameters *****
50 print clear$;"Enter <ret> or new value:"
input "r=";line a$
if a$<>"" then r=val(a$)
input "w=";line a$
if a$<>"" then w=val(a$)
input "b=";line a$
if a$<>"" then b=val(a$)
```
input "s1.zero=";line a$
if a$<""> then s1.zero=val(a$):s2.zero=1.0-s1.zero
input "k1=";line a$
if a$<""> then k1=val(a$)
input "k2=";line a$
if a$<""> then k2=val(a$)
return

rem ******* default parameters ******
60 r=.1
w=1.0
b=1.0
s1.zero=1.0
s2.zero=0
k1=0.1
k2=0.1
return

rem ****************************************
rem gridsearch to find the roots of the
rem fundamental equations
rem ****************************************
100 print clear$;
print "Search for all the roots of the fundamental equations"
print "Gridsearch over 4 different starting values"
input "steplength for grid= ";x
start=-x/2.0
finish=x/2.0
for grid1=start to finish step x
for grid2=start to finish step x
print grid1,grid2
x(1)=grid1
x(2)=grid2
gosub 200:rem solve for a root from given starting value
if iflag%=1 then goto 101:rem no root was found
if n.roots%=0
    then newroot$="y"
else gosub 300:rem test whether root is new
    if newroot$="n" then goto 101
    n.roots%=n.roots%+1
    for j%=1 to n%
        root(n.roots%,j%)=new.root(j%)
    next j%
    if n.roots%=4 then return
101 next grid2
next grid1
return
rem 200       ***** solve for a root from a given starting value *
           for iter%=1 to 25
           for i%=1 to n%
             temp(i%)=x(i%)
           next i%
           gosub 1000:rem compute b(i%)=-f(x)
           norm=0
           for i%=1 to n%
             norm=norm+b(i%)*b(i%)
           next i%
           if norm<1.0E-15
             then for i%=1 to n%:
               new.root(i%)=x(i%):
               next i%:
               return
           gosub 1500:rem compute A(i%,j%)=J(x)
           gosub 2000:rem solve Ax=b
           if iflag%=1 then return:rem error return
           if pivoting fails
             then for i%=1 to n%
               x(i%)=temp(i%)+x(i%)
             next i%:
           next iter%:
           iflag%=1:return:rem error return if no convergence

rem 300       ***** test whether root is new *****
           newroot$="y"
           for i%=1 to n.roots%
             root.diff=0
           for j%=1 to n%
             root.diff=root.diff+
               (new.root(j%)-root(i%,j%))*
               (new.root(j%)-root(i%,j%))
           next j%
           if root.diff<1.0E-6 then newroot$="n":return
           next i%
           return

rem 500       ***** find roots manually *****
           print clear;
           print "Try starting values:";
           input "x(1)= "; x(1)
           input "x(2)= "; x(2)
           gosub 200:rem solve for a root from given starting value
           if iflag%=1 then return:rem no root was found
           if n.roots%=0
             then newroot$="y"
           else gosub 300:rem test whether root is new
           if newroot$="n" then return
           n.roots%=n.roots%+1
for j%=1 to n%
  root(n.roots%,j%)=new.root(j%)
next j%
return

rem 1000 ***** compute b(i%) *****
    b(1)=-(x(1)*(x(1)+2.0*x(2)+r/w)-
          (r/w+2.0*x(2))/(2.0*b))
    b(2)=-(x(2)*(x(2)+2.0*x(1)+r/w)-
          (r/w+2.0*x(1))/(2.0*b))
return

rem 1500 ***** compute a(i%,j%) *****
a(1,1)=2.0*(x(1)+x(2))+r/w
a(2,1)=2.0*x(2)-1.0/b
a(1,2)=2.0*x(1)-1.0/b
a(2,2)=2.0*(x(1)+x(2))+r/w
return

rem ******************************************************************************
rem solve linear system Ax=b
rem ******************************************************************************
rem dim a(na%,na%),x(nx%),b(na%)
rem LU=PA i.e. LU give a A with row permutations
rem Thus LUx=PAx=Pb, so d=(U~)(L~)Pb is the
rem desired solution (~=invs)
rem 1) Let x=Pb 2) Solve Lc=x for c 3) Solve Ud=c for d
rem In the computations, both c and d overwrite x
rem input:
rem na%,a(i%,j%),b(i%)
rem output:
rem nx%,x(index%(i%))
2000 nx%=na%
gosub 2200:rem Crout decomposition A=(LD)U
for i%=1 to na%
  x(i%)=b(index%(i%))
next i%
gosub 2100:rem solve Lc=x for c
if iflag%=1 then return
gosub 2150:rem solve Ud=c for d
return

rem 21000 ***** solve L*c=x for c ****************************
rem L is lower triangular
rem answer is returned as x(i%)
2100 iflag%=0
for i%=1 to na%
  sum=0
  if i%=1 then goto 2110
  for j%=1 to i%-1
    sum=sum+a(index%(i%),j%)*x(j%)
  next j%
  x(i%)=sum
if abs(a(index%(i%),i%))<1.0E-30 then
  print "Jacobian is singular":"iflag%=1:return
  x(i%)=(x(i%)-sum)/a(index%(i%),i%)
next i%
return

rem ***** solve Ud=x for d ***********************
rem U is unit upper triangular
rem answer is returned as x(i%)
for i%=na% to 1 step -1
  sum=0
  if i%+1>na% then goto 2160
  for j%=i%+1 to na%
    sum=sum+a(index%(i%),j%)*x(j%)
  next j%
2160
next i%
x(i%)=x(i%)-sum
return

rem ***** Crout decomposition A=(LD)U *****
rem remove line 2215 if no pivoting is desired
rem dim a(na%,na%)
rem LD is lower triangular
rem U is unit upper triangular
rem LD overwrites lower triangle of A
rem U overwrites upper triangle of A
rem input:
rem na%,a(i%,j%)
rem output:
rem a(index%(i%),j%)
2200
  dim index%(na%)
  for i%=1 to na%
    index%(i%)=i%
  next i%
  for k%=1 to na%
    for i%=k% to na%
      sum=0
      if k%<=1 then goto 2210
      for 1%=1 to k%-1
        sum=sum+a(index%(i%),1%)*a(index%(1%),k%)
      next 1%
      a(index%(i%),k%)=a(index%(i%),k%)-sum
  next i%
2210
  gosub 2240:rem determine largest pivot and switch index
  if k%>na% then goto 2230
  for j%=k%+1 to na%
    sum=0
    if k%<=1 then goto 2220
    for 1%=1 to k%-1
      sum=sum+a(index%(k%),1%)*
a(index%(1%),j%)
next 1%
a(index%(k%),j%) =
(a(index%(k%),j%)-sum)/a(index%(k%),k%)
next j%
next k%
return
rem ****************************
determine largest pivot and switch ****************************
max.val=abs(a(index%(k%),k%))
pivot%=k%
if k%>=na% then return
for i%=k%+1 to na%
if abs(a(index%(i%),k%))>max.val then
max.val=abs(a(index%(i%),k%))
pivot%=i%
next i%
switch%=index%(pivot%)
index%(pivot%)=index%(k%)
index%(k%)=switch%
return
rem ****************************
analyze phase diagram ****************************
print clear$;
print "analyzing phase diagram..."
gosub 5100:rem compute constant term for each root
gosub 5200:rem compute steady state mkt share and
            price
gosub 5300:rem determine stability of phase diagram
gosub 5400:rem print roots
return
rem ****************************
compute constant term for each root ****************************
for i%=1 to n.roots%
a11=2.0*b*(r/w+root(i%,1)+root(i%,2))
a12=1.0-2.0*b*root(i%,1)
a21=1.0-2.0*b*root(i%,2)
a22=a11
b1=(r/w+root(i%,2)+1.0)*k1-
   (1.0-2.0*b*root(i%,1))*root(i%,2)
b2=(r/w+root(i%,1)+1.0)*k2-
   (1.0-2.0*b*root(i%,2))*root(i%,1)
det=a11*a22-a12*a21
root(i%,3)=(b1*a22-b2*a12)/det
root(i%,4)=(a11*b2-a21*b1)/det
next i%
return
***** compute steady state market share and price ****

```
for i%=1 to n.roots%
    s1.st(i%)=(root(i%,2)+root(i%,4)-
                root(i%,3))/(root(i%,1)+root(i%,2))
    s2.st(i%)=1.0-s1.st(i%)
    p1.st(i%)=root(i%,1)*s1.st(i%)+root(i%,3)
    p2.st(i%)=root(i%,2)*s2.st(i%)+root(i%,4)
next i%
return
```

***** determine stability of phase diagram *****

```
for i%=1 to n.roots%
    if root(i%,1)+root(i%,2)>0
    then stable$(i%)="y"
    else stable$(i%)="n"
    if root(i%,1)+root(i%,2)>-r/w
    then tvc$(i%)="y"
    else tvc$(i%)="n"
next i%
return
```

***** print roots *****

```
print clear$;
print
"## -x(1) -x(2) -x(3) -x(4) -s1*- -s2*- -p1*- -p2*- -tvc- stb-
for i%=1 to n.roots%
    print using
"## !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! !!!!!! 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r,w,b,k1,k2
print
return

rem
compute values for given starting market share
rem
6500 print clear$;"Computing value..."
horizon=20.0
dt=1.0
s1=s1.zero
s2=1.0-s1.zero
v1=0
v2=0
for t=0 to horizon step dt
    p1=alpha1*s1+beta1
    p2=alpha2*s2+beta2
    p1e1=(s1-p1)*(p1-k1)
    p1e2=(s2-p2)*(p2-k2)
    v1=v1+p1e1*exp(-r*t)*dt
    v2=v2+p1e2*exp(-r*t)*dt
    s1=s1+w*(p2-p1)*dt
    s2=1.0-s1
next t
return
rem
6510 print resulting value ****
print using "alpha2= ###.### beta2= ###.###"; alpha2,beta2
print using "alpha1= ###.### beta1= ###.###"; alpha1,beta1
print using "s1(0)= ###"; s1.zero
print " -t- -s1- -s2- -p1- -p2- -v1- -v2-"
if abs(s1)>9.9 or abs(s2)>9.9 or abs(p1)>9.9 or abs(p2)>9.9 or abs(v1)>9.9 or abs(v2)>9.9 then goto 6520
print using
    "###.### ###.### ###.### ###.### ###.### ###.### ###.###"; t,s1,s2,p1,p2,v1,v2
return

6520 print using
    "###.### ###.### ###.### ###.### ###.### ###.### ###.###"; t,s1,s2,p1,p2,v1,v2
return
rem
7000 lprinter
print
gosub 6510:rem print resulting value
print
console
return

rem 8000  ***** compute and lprint roots *****
gosub 100:rem calculate roots
gosub 10000:rem compute "other roots"
gosub 5000:rem analyze phase diagram
gosub 6000:rem lineprint roots
return

rem 9000  ***** compute and lprint value given S1(0) *****
print clear$;
input "s1(0)=";s1.zero
print "1 - Value for arbitrary alpha2, beta2"
print "2 - Value for a single root"
print "3 - Value for all the roots"
input "Your choice: ";x%
if x%<1
then
   input "alpha2=";alpha2:
   input "beta2=";beta2:
   gosub 9100:compute alpha1 and beta1
gosub 9010:compute and lprint value
return
if x%<2
then
   input "Number of root=";x%
   alpha1=root(x%,1):
   alpha2=root(x%,2):
   beta1=root(x%,3):
   beta2=root(x%,4):
   gosub 9010:compute and lprint value
return
if x%<3
then
   for x%=1 to n.roots%:
   alpha1=root(x%,1):
   alpha2=root(x%,2):
   beta1=root(x%,3):
   beta2=root(x%,4):
   gosub 9010:compute and lprint value
next x%
return
if x%<>1 and x%<>2 and x%<>3 then return

9010  gosub 6500:rem compute value given S1(0)
lprinter
lprinter
print
gosub 6510:rem lprint value
print
console
return

9100  bb=r/w+2.0*alpha2
cc=-(r/w+2.0*alpha2)/(2.0*b)
an1=((-bb+sqr(bb*bb-4.0*cc))/2.0
ans2=\(-bb-\text{sqr}(bb*bb-4.0*cc))\)/2.0
print using "alphai is equal to ***.****" or
###.###
input "Choose 1 or 2!"; x%
if x%=1 then alpha1=ans1 else alpha1=ans2
a11=2.0*b*(r/w+alpha1+alpha2)
a12=1.0-2.0*b*alpha1
a21=1.0-2.0*b*alpha2
a22=a11
b1=(r/w+alpha2+1.0)*k1-(1.0-2.0*b*alpha1)*alpha2
b2=(r/w+alpha1+1.0)*k2-(1.0-2.0*b*alpha2)*alpha1
det=a11*a22-a12*a21
beta1=(b1*a22-b2*a12)/det
return

rem ***** compute "other roots" *****
10000 for i%=1 to n.roots%
alpha2=root(i%,2)
beta2=root(i%,4)
bb=r/w+2.0*alpha2
c=-(r/w+2.0*alpha2)/(2.0*b)
ans1=\(-bb-\text{sqr}(bb*bb-4.0*cc))\)/2.0
ans2=\(-bb-\text{sqr}(bb*bb-4.0*cc))\)/2.0
d1=abs(ans1-root(i%,1))
d2=abs(ans2-root(i%,1))
if d1<d2 then alpha1=ans2
else alpha1=ans1
a11=2.0*b*(r/w+alpha1+alpha2)
a12=1.0-2.0*b*alpha1
a21=1.0-2.0*b*alpha2
a22=a11
b1=(r/w+alpha2+1.0)*k1-(1.0-2.0*b*alpha1)*alpha2
b2=(r/w+alpha1+1.0)*k2-(1.0-2.0*b*alpha2)*alpha1
det=a11*a22-a12*a21
beta1=(b1*a22-b2*a12)/det
root(n.roots%+i%,1)=alpha1
root(n.roots%+i%,2)=alpha2
root(n.roots%+i%,3)=beta1
root(n.roots%+i%,4)=beta2
next i%
n.roots%=n.roots%*2
return
FOOTNOTES TO CHAPTER 4

1. Note that the differential equation (4.1.2) is consistent with the assumption $0 \leq S_i \leq 1$, since $\dot{S}_i = 0$ when $S_i = 0$ or $S_i = 1$. Essentially the same specification was used by Phelps and Winter (1970) in a model of atomistic competition with goodwill effects. Also see Gould (1970) for a discussion of several alternative diffusion mechanisms.

2. Recall that $S_i$ is firm 1's share, so a decrease in $S_i$ corresponds to an increase in firm 2's share.
CHAPTER 5

ENTRY DETERRENCE

This chapter is concerned with an analysis of entry deterrence in the model of reversible strategic investment of chapter 3.

The term "entry deterrence" refers to a simple concept. If there is only one firm in a market, it would like to act as a monopolist to achieve supernormal profits. However, unless the monopolist has a patent or other exclusive right, these profits will attract other firms into the market and erode the monopoly profits. Any action which a monopolist may take to prevent the entry of other firms into his market is a form of entry deterrence.

In the classic limit pricing models of Bain (1949), Modigliani (1958), and Sylos-Labini (1962), the incumbent firm threatens to keep its output at the same level even after another firm enters the market. If the residual demand for the new entrant is insufficient to cover fixed costs, entry is deterred. These models have recently been questioned on the basis of their fundamental assumptions: the "Sylos postulate," which assumes that the incumbent monopolist will maintain his output in the face of entry. However, once entry has occurred, it is usually in the incumbent's self-interest to lower his output, and the treat is thus not credible.
If one insists that threats must be credible to be effective, then one needs to show means by which firms can credibly commit themselves to a particular course of action in the face of entry. One such method, which has recently been studied by Spence (1979) and Dixit (1980), is the use of investment in irreversible capacity. A large level of capacity can commit the incumbent to a large level of post-entry production.

However, when investment is reversible it is no longer credible that a firm will maintain its pre-entry output in the face of entry. Indeed, chapter 3 showed that, in a perfect equilibrium, the incumbent firm accommodates the entrant. The present chapter will show that even when investment is reversible, capital is still a credible entry-deterrent if investment is subject to adjustment costs.

An additional aim of this chapter is to develop the optimal entry-deterring strategy for a monopolist threatened with entry. Entry-deterrence will usually require a departure from myopic profit-maximization. In order to weigh the future benefits of entry-deterrence against losses in current profits, one needs an explicit model of entry.

Section 5.3 develops a probabilistic model of entry along the lines of Kamien and Schwartz (1971). The probability of entry is assumed to increase with the expected profits of the entrant, which in turn are affected by the incumbent's choice of capacity. The incumbent chooses
capital to maximize his expected profits in the face of probabilistic entry. It will be shown that, depending upon the values of certain parameters, the incumbent may choose to deter entry completely, or choose an "entry-retarding" strategy.

This probabilistic model of entry is applied to the model of reversible investment in section 5.4. Given the simplifying assumptions made for the model of reversible investment, it is possible to compute some interesting comparative static results. Of particular interest are the possible tradeoffs between startup costs, adjustment costs, and capacity as entry-deterring tools. The optimal entry-deterring level of capital first rises and then declines as startup costs increase. Also, the incumbent may increase his pre-entry expected profits by raising the industry level of fixed costs, including his own. An increase in adjustment costs will be beneficial to the incumbent if he has eluded entry long enough to grow, but will be detrimental to a monopolist in his early stages of growth.

The remainder of the chapter is organized as follows. Section 5.1 gives a brief introduction to the theory of entry deterrence and to some of the major issues, such as the importance of credible threats and the value of commitments. Section 5.2 discusses some of the literature on entry deterrence, with particular emphasis on those models which use capital as a means of commitment. Section 5.3
modifies the probabilistic model of entry of Kamien and Schwartz (1971) to the case where pre-entry capacity affects the post-entry profits of the incumbent, and where the incumbent’s investment is subject to adjustment costs. Section 5.4 applies this probabilistic model of entry to the model of reversible investment of chapter 3. It is shown that capacity can be an entry-deterring tool even when its commitment value is only partial. Comparative static results are then developed, with particular attention to the tradeoffs between startup costs, adjustment costs, and capacity as entry-deterring tools.

5.1. Overview of current theory

In the textbook exposition of the theory of monopoly, a firm is able to make supranormal profits by virtue of being the only firm in the market. This immediately raises the question as to why the presence of monopoly profits would not attract other firms to enter the industry, thus eventually driving profits down to their normal rate. The theory of entry deterrence is concerned with a firm’s ability to sustain monopoly profits.1/

Innocent versus strategic barriers

When a firm is able to sustain monopoly profits without causing entry, this is attributed to an entry barrier. Salop (1979) distinguishes between two types of entry barriers:
innocent barriers and strategic barriers. "An innocent entry barrier is unintentionally erected as a side effect of innocent profit maximization. In contrast, a strategic entry barrier is purposely erected to reduce the possibility of entry."

Examples of innocent entry barriers are the possession of a patent, or superior technology, or exclusive access to a scarce input. In all these cases, the monopolist has a cost advantage over a potential entrant. However, even if the potential entrant has costs equal to the incumbent, scale economies may provide an innocent entry barrier. In the case of a "natural monopoly," scale economies may be so large that industry profits with two firms are necessarily negative.

One might argue whether the process of obtaining a patent, or of obtaining exclusive access to a scarce input constitute "innocent" behavior. Even the level of scale economies can to some extent be determined by the monopolist through his choice of product quality and production technology. However, the salient characteristic of innocent entry barriers is that, once the cost structure is determined, the monopolist acts in the simple textbook fashion.

In many monopolized industries there are insufficient innocent entry barriers to deter entry. This leads to the possibility that, given his cost structure, the monopolist
might be able to act in a way which will deter entry. This form of strategic behavior, and its resulting strategic entry barriers, have been the major focus of the literature on entry deterrence.

**Optimal strategic behavior**

The possibility of strategic entry-deterring behavior raises two questions. The first is what actions the incumbent firm would have to take in order to deter entry. The second asks whether an entry-deterring strategy is optimal.

The answer to the first question requires a model of the industry equilibrium after entry. The post-entry equilibrium usually depends on some "initial conditions," such as the incumbent's sunk costs or level of capacity. The incumbent is able to determine these initial conditions through his pre-entry behavior. An entry-deterring strategy thus requires the incumbent to set the initial conditions for the post-entry game so that a potential entrant will not find entry profitable.

The second question is concerned with the incumbent's optimal choice of actions in the pre-entry period. The incumbent may choose to act as a myopic monopolist and disregard the possibility of entry, or he may choose an entry-deterring strategy. Since entry-deterring behavior implies a deviation from conventional profit-maximization,
it involves a decrease in profits in order to prevent entry. It is not clear whether an entry-deterring policy with profits constantly below the monopoly level is preferable to a policy of high current profits with lower profits after entry occurs.

To answer the second question, it is necessary to model the link between pre-entry decisions and the process of entry. One possibility is to assume that if any positive profits can be made by the entrant, then entry will occur with certainty, but after some (maybe small) time lapse. In this case the choice of an optimal pre-entry strategy is dichotomous: the incumbent will either act as a myopic monopolist, or it will deter entry, depending upon the net present value of these strategies.

Alternatively, entry can be viewed probabilistically. This is the approach chosen by Kamien and Schwartz (1971). The assumption is that the established firm can affect the probability with which entry occurs. This broadens the scope of possible strategies: in addition to the possibility of allowing or deterring entry, the firm now may follow an entry-retarding strategy. Such an entry-retarding strategy involves a tradeoff in setting the initial conditions: higher profits in the pre-entry period must be weighed against a higher probability of entry.

The model with certain entry is a special case of the probabilistic model. It simply assumes that the probability
of entry jumps from zero to one when the initial conditions are such that the entrant can make positive profits.

The determination of an optimal pre-entry strategy is seen to occur in two steps: first one computes the post-entry profits to the incumbent and the entrant which would result from any given choice of initial conditions (which of course are set by the incumbent in the pre-entry period). Given a model of entry, one can then determine the probability of entry for various choices of the strategic variable. The optimal pre-entry strategy is the one which maximizes expected pre-entry profits for the monopolist.

Credible threats and the value of commitment

The discussion so far has pointed out the link between the post-entry game and the optimal behavior of the monopolist in the pre-entry period. Thus different models of post-entry industry equilibrium will yield different results for the pre-entry behavior of the monopolist.

The early literature on entry deterrence associated with the work of Bain (1949), Modigliani (1958), and Sylos-Labini (1962), assumed that the monopolist would maintain his pre-entry level of output in the post-entry period. This assumption has become known as the Sylos postulate, and is the basis of the limit pricing models. By having a high pre-entry output, the monopolist can reduce the residual demand left for any entrant, and thus deter entry.
The Sylos postulate has recently been questioned by several authors,\(^3\) because the threat of maintaining output in the face of entry is not credible: given that entry has occurred (perhaps by incurring substantial startup costs), the incumbent firm is usually better off by lowering its output.

More generally, once entry has occurred, the original asymmetry between the two firms has disappeared. One would expect the post-entry equilibrium to be no different from any other duopoly equilibrium. Given this view, the post-entry output of the incumbent may be totally unrelated to the pre-entry output. The threat to maintain output is empty: the firm will revert to a duopoly output.

If only credible (e.g. Nash) behavior is to be allowed in the post-entry period, the original limit price theory no longer yields strategic entry barriers. Entry will not be deterred by maintaining high pre-entry output if the entrant believes that the output will be lowered after entry.\(^4\)

However, if the incumbent could make credible his threat to maintain output, then the original limit price theory would indeed provide a strategic entry barrier. One way of doing this might be by signing long-term contracts with his customers. This example is a special case of the general principle developed by Schelling (1960), that, in a gaming situation, it is usually advantageous to a player to be able to limit his choice to a smaller set of options.
If one excludes pre-entry threats which are not credible, then strategic behavior is conditional on the incumbent's ability to credibly commit himself to some post-entry behavior. As will be discussed below, much recent work has focused on investment in irreversible capacity as a means to obtain such commitment. This work will be discussed in section 5.2.

Given the central importance of commitment for the possibility of strategic entry deterrence, one would like to have a sense of how the theory of entry deterrence is modified if commitment is only partial. In the model of chapter 3 of this thesis, the perfect equilibrium strategies in the post-entry game require the incumbent firm to accommodate the entrant by reducing his output. However, due to adjustment costs, the monopolist makes room for the entrant only slowly. This case is intermediate between total commitment and total accommodation. Section 5.4 will show that strategic entry deterrence is still possible when commitments are of this temporary nature.

5.2. Review of some entry deterrence results

The purpose of this section is to review some of the models of entry deterrence, with particular attention to those which focus on investment in irreversible capital. The models which will be discussed are the static models of Spence (1977), and Dixit (1979,1980), as well as the dynamic
model of Spence (1979), and Fudenberg/Tirole (1983). Far from being exhaustive, this review attempts to cover only those papers which are particularly closely related to the analysis of entry deterrence to be developed in section 5.3.

The paper of Spence (1977) departs from the Sylos postulate. He points out that output itself need not be kept at an entry-deterring level. Instead, capacity may serve the same function. Suppose that the incumbent firm will react to entry by expanding its output up to its full capacity. Then it is this level of capacity, and not the pre-entry output, which determines post-entry profits for the entrant. This suggests that "the short-run pricing game is to some extent strategically independent of entry."

In his 1979 paper, Dixit allows the incumbent the choice of whether or not to deter entry, but he keeps the assumption that firm 1 maintains output after entry. He shows that, depending on the fixed costs of the entrant, there are three possible pre-entry cases. In figure 5.2.1, let $x_i$ and $R_i$ be the output and reaction curve of firm $i$. Let $S$ be the Stackelberg point for firm 1, and let $M$ be its monopoly point. Firm 2 has some level of fixed costs, so that firm 2 reacts with $x_2=0$ if $x_1$ is sufficiently large. This implies that $R_2$ has a kink between points A and B. The three cases correspond to different locations of the kink.
If $B > M$, then setting $x_1 = M$ will actually deter entry. The monopoly output is sufficient to deter entry, which therefore is "blockaded." If $B > Z$, then firm 1 prefers to let firm 2 enter: its profits at the Stackelberg point $S$ are greater than those from entry deterrence at $B$. In this case, entry is "ineffectively impeded." If $M < B < Z$, then firm 1 will want to deter entry, since its profits at $B$ are greater than at $S$, and entry will be "effectively impeded."6

This exposition has been formulated as if firm 1 had to choose pre-entry output, but the analysis extends to the choice of pre-entry capacity. The only difference is that when entry is effectively impeded, the firm may find it to its advantage not to produce up to the entry-deterring capacity.
This model is subject to the same criticism as the limit price models: the threat of expanding output to the capacity level is not necessarily credible. It assumes that, even when entry has become irrevocable, the established firm maintains a leadership advantage. In contrast, Dixit’s 1980 paper assumes symmetry in the post-entry period: when entry occurs, a Nash-Cournot equilibrium is reached. However, "the established firm can alter the outcome to its advantage by changing the initial conditions...An irrevocable choice of investment allows it to alter its post-entry marginal cost curve, and thereby the post-entry equilibrium...It can use this privilege to exercise a limited amount of leadership."

Dixit’s presentation is for the case where capacity acts as an upper bound on output. An increase in output beyond capacity requires additional capacity, so the marginal cost has a vertical jump. He shows that this kink in marginal cost introduces a kink in firm 1’s reaction curve, with the location of the kink determined by the level of capacity.

Figure 5.2.2 illustrates the more general case in which capacity enters the cost function by reducing marginal cost. An increase in capacity shifts the reaction curve to the right. By its choice of capacity, firm 1 can pick its reaction curve, and thus can effectively pick its desired equilibrium point on firm 2’s reaction curve.
The conclusions of Dixit's 1979 paper still hold: depending on the level of firm 2's fixed costs, entry may be ineffectively impeded, effectively impeded, or blocked. However, the post-entry actions are now perfectly credible. The pre-entry choice of capacity has committed firm 1 to a post-entry equilibrium which will dissuade firm 2 from entering. Thus Dixit's 1980 paper gives a static model of entry deterrence which does not rely on empty threats.

The model of Spence (1979), which was extended by Fudenberg and Tirole (1983), considers entry deterrence in a dynamic model. Both the incumbent and the entrant have an upper bound on the rate of investment, and investment is irreversible. If firm 1 starts with a positive level of capital, a perfect equilibrium path usually requires firm two to invest as quickly as possible, while firm 1 first
invests, then stops, and finally invests until some terminal surface is reached. However, there are many potential terminal surfaces, all of which are perfect equilibria.

Fudenberg and Tirole single out one of these surfaces as "most reasonable." They narrow down the set of potential stopping points by allowing each firm to propose a non-binding contract to stop investing. In addition, no firm may threaten to refuse a contract which it actually would accept. These assumptions lead to a unique terminal surface and thus to a unique perfect equilibrium.

Now consider the possibility of entry-deterrence. From any given level of capital for firm 1, it is possible (in principle) to calculate the net present value of profits to firm 2 along the perfect equilibrium path. Furthermore, firm 2's value is smaller, the larger firm 1's capital at the time of entry. The general condition for entry deterrence is therefore met: in the pre-entry period, firm 1 can modify the initial conditions of the post-entry game in a way which affects the outcome. Thus, although the post-entry game is dynamic, the pre-entry choice of capital is similar to the static model. In particular, depending on firm 2's startup costs, entry may be blockaded, or firm 1 may choose to deter entry.
5.3 A probabilistic model of entry with adjustment costs

As discussed in section 5.1, the level of capital chosen in the pre-entry period depends on how entry is seen to occur in response to potential positive profits. The view taken here is that entry occurs with a probability which can be affected by the monopolist.\textsuperscript{7}

This approach to entry was first developed by Kamien and Schwartz (1971) to determine the optimal limit price policy in the face of uncertain entry. They assume that the pre-entry price affects the probability of entry, but not the post-entry profits.\textsuperscript{8} However, in the modern view, which emphasizes commitment, entry deterrence hinges on the ability to make post-entry profits depend upon pre-entry actions. Hence I will modify their analysis to allow post-entry profits to depend on pre-entry actions.

An additional complication is introduced if the strategic variable is subject to adjustment costs during the pre-entry period, as it will be in the case of strategic investment. Then the optimal pre-entry policy will not be stationary, i.e. there won't be a single optimal pre-entry level of capital, but rather a growth path starting from when the monopolist first enters the market.

This section develops a probabilistic model of entry where the incumbent's post-entry profits depend upon the level of capital in the pre-entry period, and where investment is subject to adjustment costs in the pre-entry
period. The optimal investment policy cannot be solved for in closed form, but it will be shown that, under some assumptions about second derivatives, there is a unique steady-state level of capital, which does not depend on the starting conditions.

This steady-state level of capital will be the main focus of the comparative static results in section 5.4. It is the level of capital which an optimizing monopolist approaches if, by luck, he eludes entry for a long time. Alternatively, it is the level of pre-entry capital which a monopolist without adjustment costs would choose. This "steady-state level of optimal pre-entry capital" will simply be referred to as the "optimal pre-entry capital", and will be denoted by $\bar{K}$.

The analysis proceeds in three steps. First, the objective function of the monopolist is derived. This requires some care because the post-entry value of firm 1 depends on the level of capital which it has when firm 2 enters. Second, the control problem is formulated and the necessary conditions derived. Finally, it will be shown that the optimal policy converges to a unique steady state.

The objective function

Let $F(t)$ be the probability that entry has occurred by time $t$, where $F(0)=0$, and let $f(t)$ be the corresponding probability density. Then the probability that entry will
occur at time $t$, conditional upon no entry having occurred yet, is called the "hazard rate" at time $t$, and is equal to 
$h = f(t)/(1-F(t))$. The hazard rate is assumed to depend on 
firm 2's value from entry, which in turn depends on firm 1's 
level of capital $K$ when entry occurs. This relation is 
written in reduced form as $h = h(K)$. Hence $F(t)$ evolves 
according to the differential equation 
\[(5.3.1) \quad \dot{F}(t) = f(t) = h(K)(1-F(t)), \quad F(0) = 0.\]

The following assumptions will be imposed on the 
function $h(K)$: (i) the hazard of entry is always 
nonnegative, (ii) an increase in firm 1's level of capital 
lowers the attractiveness of entry to firm 2, and thus 
decreases the hazard of entry, and (iii) the marginal 
"hazard-reducing productivity" of a rise in $K$ decreases with 
$K$. This assumption can be summarized as follows:

**Assumption 5.3.1:** The hazard function $h(K)$ satisfies the 
following assumptions:

(i) $h(K) \geq 0$

(ii) $h'(K) \leq 0$

(iii) $h''(K) \geq 0$

Let $R(K,I)$ be the pre-entry net revenue stream of firm 
1, which depends on its level of capital $K$ and on its level 
of net investment. The equation of motion of the level of 
capital is thus given by 
\[(5.3.2) \quad \dot{K} = I\]
The net revenue stream has three components: the gross revenue generated by a given level of capital, the cost of capital, including depreciation, and adjustment costs.

\[ R(K, l) = \pi(K) - \frac{1}{2} c l^2 \]

The adjustment costs are assumed to be a function of net investment. This may be justified as follows. The notion of adjustment costs is meant to capture the notion that quick changes in the level of capital are more expensive than slow ones. Investment which is aimed at replacing depreciated capital stock is an effort to maintain the status quo. It is the quick change in the net size of the firm which imposes additional costs: having to hire or fire workers rapidly, or having to increase or decrease the size of the plant suddenly. For a discussion of the empirical merits of this formulation, see Lucas (1967).

It is important to note that the definition of I as net investment is only a notational convenience. The formulation does allow for depreciation, which is included with the cost of capital, and "hidden" in \( \pi(K) \).\(^{10}\)

The profit function \( \pi(K) \) is assumed to be strictly concave, increasing up to a maximum value \( K^m \), and decreasing thereafter. This is summarized by

**Assumption 5.3.2**: The profit function \( \pi(K) \) satisfies

(i) \( \pi''(K) < 0 \)

(ii) \( \pi'(K) > 0 \) for \( K < K^m \)

(iii) \( \pi'(K) < 0 \) for \( K > K^m \)
Let $V(K)$ be the net present value of firm 1 when entry occurs. A particular example of such a value function was computed in chapter 3 for the strategic investment model, where $V_1(K_1, K_2)$ gives the value of firm 1, if the two firms have levels of capital $K_1$ and $K_2$ respectively, and in the present notation $V(K) = V_1(K_1, 0)$. This value function is assumed to be concave, and it must be true that the present value of profits which firm 1 obtains at the $K^m$ (the no-entry monopoly level) are greater than the present value of starting a duopoly from this same level $K^m$.

**Assumption 5.3.3:** The value function $V(K)$ satisfies

(i) $V''(K) < 0$

(ii) $\pi(K^m) > rV(K^m)$, where $K^m$ satisfies $\pi'(K)=0$

If entry were certain to occur at time $T$, then the net present value of firm 1 at time $t=0$, i.e. when it first enters the market, would be

\[(5.3.4) \ W(T) = \int_0^T [R(K, I) e^{-r_t} dt + V(K(T))] e^{-r_T} \]

However, the time of entry is uncertain. The expected value of firm 1 at time $t=0$ is a weighted average of the profit which it obtains while it is alone in the market, and the value which it obtains after entry occurs:

\[(5.3.5) \ E_W = \int_0^\infty [\int_0^T R(K, I) e^{-r_t} dt] f(T) dT + \int_0^\infty V(K(T)) e^{-r_T} f(T) dT \]

Changing the order of integration, the first term reduces to

\[(5.3.6) \ \int_0^\infty [\int_0^T f(T) dT] R(K, I) e^{-r_t} dt = \int_0^\infty [1 - F(t)] R(K, I) e^{-r_t} dt \]

Using (5.3.1) to rewrite the second term, (5.3.5) becomes

\[(5.3.7) \ E_W = \int_0^\infty [R(K, I)] + h(K) V(K)] [1 - F(t)] e^{-r_t} dt \]
This expression gives the expected present discounted value of firm 1's profits at time $t=0$, when firm 1 first enters the market. The objective of firm 1 is to maximize (5.3.7), subject to (5.3.1) and (5.3.2).

The necessary conditions

The objective of firm 1 during the pre-entry period is to choose an investment path \{I(t)\}, which maximizes the expected value given by (5.3.7). This is an optimal control problem with investment $I$ as the single control variable. The probability of entry $F$ and the level of capital $K$ are the state variables, with equations of motion given by (5.3.1) and (5.3.2).

This problem can be solved using Pontryagin's maximum principle:

(5.3.8) $H = [\pi(K) - \frac{1}{2}cI^2 + h(K)V(K)](1-F) + \mu I + \lambda h(K)(1-F)$

(5.3.9) $0 = H_I = cI(1-F) + \mu \Rightarrow \mu = cI(1-F)$

(5.3.10) $r\mu - \dot{\lambda} = H_K = [\pi'(K) + h'(K)V(K) + h(K)V'(K) + \lambda h'(K)](1-F)$

(5.3.11) $\lambda - \dot{\lambda} = H_F = [-\pi(K) - \frac{1}{2}cI^2 + h(K)V(K)] - \lambda h(K)$

(5.3.12) $\lambda(t)F(t)e^{-rt} \to 0$ and $\lambda(t)F(t)e^{-rt} \to 0$ as $t \to \infty$

Equation (5.3.8) gives the current value Hamiltonian, and the next four equations give the maximized value of the Hamiltonian, the costate equations, and the transversality conditions. These equations give necessary conditions which must be satisfied by an optimal control.
The steady state

The necessary conditions given by (5.3.1), (5.3.2) and (5.3.8)-(5.3.12) will be shown to have a unique steady state, provided some assumptions are satisfied.

Differentiate (5.3.9) and combine it with (5.3.1):

\[ \dot{\mu} = cI(1-F) - cIn(K)(1-F) \]  

Equating (5.3.10) and (5.3.13) results in (5.3.14), eliminating \( \mu \). Hence the equations of motion for the four unknowns \( I, \lambda, K, F \) are

\[ c\dot{I} = [r + h(K)]cI - [\pi'(K) + h'(K)V(K) + h(K)V'(K) + \lambda h'(K)] \]

\[ \dot{\lambda} = [r + h(K)]\lambda + [\pi(K) - \frac{1}{2}cI^2 + h(K)V(K)] \]

\[ \dot{K} = I \]

\[ \dot{F} = h(K)(1-F) \]

Note from equations (5.3.14)-(5.3.16) that \( I, K, \) and \( \lambda \), are determined independently of \( F \). In steady state, \( \dot{K} = \dot{I} = \dot{\lambda} = 0 \). Thus equations (5.3.14) and (5.3.15) imply

\[ \pi'(K) + h'(K)V(K) + h(K)V'(K) + \lambda h'(K) = 0 \]

\[ (r + h(K))\lambda + \pi(K) + h(K)V(K) = 0 \]

Solving these equations simultaneously yields the expression for the steady-state capital \( \tilde{K} \):

\[ [r + h(\tilde{K})][\pi'(\tilde{K}) + h(\tilde{K})V'(\tilde{K})] - h'(\tilde{K})[\pi(\tilde{K}) - rV(\tilde{K})] = 0 \]

Observe that the steady state is independent of pre-entry adjustment costs, as one would expect. It is also reassuring to note that the expression for \( \tilde{K} \) reduces to that derived by Kamien and Schwartz when \( V'(K) = 0 \). It remains to show that the solution of (5.3.18) is unique.
Proposition 5.3.1: The following conditions are sufficient for the existence of a unique steady-state \( \hat{K} \):

(i) There are precisely two values \( K, \bar{K} \), with \( K < \bar{K} \), which satisfy \( n(K) = rV(K) \). Furthermore, \( n'(\bar{K}) + h(\bar{K})V'(\bar{K}) > 0 \), and \( n'(\bar{K}) + h(\bar{K})V'(\bar{K}) < 0 \).

(ii) \((r+h)(n''+hV'')+h''(m-rV)+2h'(r+h)V'<0\) at \( K=\bar{K} \)

Proof: It must be shown that equation (5.3.18), which defines the steady state, has a unique solution. Denote the left-hand side of (5.3.18) by \( Y(K) \). Then it must be shown that \( Y(\cdot) = 0 \) has a unique solution.

Recall from assumption 5.3.1 that \( h(K) > 0 \). From assumption (i), it is easy to see that \( Y(K) > 0 \), and \( Y(\bar{K}) < 0 \). If it can be shown that \( Y'(K) < 0 \), then the intermediate value theorem implies that there is a unique value \( \hat{K} \), \( \hat{K} \in [K, \bar{K}] \), which solves \( Y(K) = 0 \).

Differentiating \( Y(K) \) yields

(5.3.21) \( Y' = (r+h)(n''+hV'')+2(r+h)h'V' - h''(m-rV) \)

The first term is negative from the concavity of \( n(K) \) and \( V(K) \). The third term is negative since \( (m-rV) \) is positive by assumption 5.3.3(ii). This leaves the second term, which is the product of \( h' \leq 0 \) and \( V' \). The sign of the product is negative if \( V' > 0 \). This is certainly true for \( K = K \), from assumption (i). However, \( V' \) may be negative at \( \bar{K} \). Thus assumption (ii) is necessary to assure that the inequality holds at \( \bar{K} \), and thus over the entire interval \( [K, \bar{K}] \).//
This proposition gives sufficient conditions for uniqueness of $k$, but they are by no means necessary. However, these conditions hold in the model of reversible investment of section 5.4, so uniqueness will be assured.

Given that the steady-state level of capital is unique, it will certainly be independent of the initial conditions of the problem. However, it has yet to be shown that the optimal path actually converges to this steady state. It is shown in appendix A that the sufficient conditions for uniqueness of the steady state also imply that the optimal path must converge to the steady state.

5.4 Entry Deterrence with Reversible Investment

The purpose of this section is to examine entry deterrence in the strategic investment model of chapter 3, in which investment is reversible, but subject to adjustment costs. There are two reasons for this investigation: first, because it provides an indication that strategic entry deterrence is possible even when commitment is only partial. Second, the strategic investment model allows the demonstration of a number of interesting comparative static results, particularly concerning the tradeoffs between startup costs, adjustment costs and preemptive capacity.

As discussed in section 5.1, the ability to deter entry relies on a firm's ability to commit itself to a limited set of options in the post-entry period. Section 5.2 reviewed
some models where such commitment was achieved by investment in productive capacity.

If the level of capacity (denoted by K) can be instantaneously and costlessly adjusted, then the pre-entry choice of K does not restrict the incumbent's choice of capacity for the post-entry period. In contrast, if investment is completely irreversible, the incumbent is committed to having at least the same level of K.

The model of reversible investment with adjustment costs lies between these two polar cases. Firms are able to divest their capacity, but only by incurring convex adjustment costs. It was shown in chapter 3 that, when entry occurs, the perfect equilibrium strategy for the incumbent is to accommodate the entrant, suggesting that entry deterrence might be impossible. However, it will be shown that entry deterrence is possible even though the incumbent accommodates the entrant, provided that adjustment costs are above some threshold level.

The intuition for this result is simple: if adjustment costs are low, the incumbent will divest very quickly, so the outcome of the post-entry game will be almost independent of the initial level of capital, and capital loses its commitment value. In contrast, with high adjustment costs, optimal divestiture will be so slow as to make capital seem almost irreversible. In this case, capital retains full commitment value. The existence of a threshold
level of adjustment costs can be viewed as confirming the robustness of the entry deterrence results, even when commitment is only partial.

The strategic investment model of chapter 3 was formulated by assuming special functional forms. The loss in generality is compensated by the ability to compute comparative static results. Given the linear-quadratic structure of the post-entry game, it is possible to compute the "value functions" for the two firms, which indicate how the net present value of the post-entry game depends on the initial conditions. These value functions can then be incorporated into the probabilistic model of entry of section 5.3, which yields specific predictions as to the optimal pre-entry strategy for a monopolist.

Several interesting comparative static results will be derived. For example, I will show that the optimal pre-entry level of capital first rises and then declines as the level of fixed costs increases. Also, I will show that the incumbent may be able to increase his pre-entry expected value by raising the industry level of fixed costs, including his own. In addition, I will illustrate some of the tradeoffs between fixed costs, adjustment costs, and pre-entry capacity as entry-deterring tools.

The remainder of this section proceeds as follows. In order to apply the probabilistic model of entry of section 5.3, to the model of strategic investment, one must derive
the value functions for the two firms. These value functions are then combined with a specific assumption about the hazard function to give a description of pre-entry optimal behavior. This makes it possible to determine the optimal pre-entry capital $\hat{K}$ for the monopolist. The remainder of the section discusses the comparative static effects of changes in various parameters, such as fixed costs, adjustment costs, the discount rate and the rate of depreciation.

The value functions

The incumbent's ability to deter entry depends upon his ability to affect the initial conditions of the post-entry game. In order to deter entry, firm 1 (the incumbent) must set its level of capital $K_1$ so that the value $V_2$ of entry for firm 2 (the entrant) is negative. The initial capital of firm 2 is $K_2 = 0$.

In the strategic investment model of chapter 3, if entry occurs, the two firms follow perfect equilibrium strategies. These strategies are illustrated in figure 5.4.1. We see that if firm 1 sets its pre-entry level of capital at or above the monopoly level, then it will keep disinvesting until the steady-state level is reached. Thus it accommodates the entry of firm 2. In general, the value of each firm from a particular starting value of $K_1$ is computed by integrating the discounted profits along the equilibrium path.
Because of the linear-quadratic structure of the model, the value functions can actually be computed analytically using the dynamic programming technique.

Let \( V^i(\tilde{K}_1, \tilde{K}_2) \) be the value of the objective function for firm \( i \) starting from an arbitrary time \( t_0 \) with initial levels of capital given by \( \tilde{K}_1, \tilde{K}_2 \). For the model of chapter 3, this value is given by

\[
V^i(\tilde{K}_1, \tilde{K}_2) = \max \int_{t_0}^{\infty} [n_i(K_1, K_2) - \frac{1}{2}c_i^2]e^{-rt}dt \quad \text{w.r.t.} \quad \{I_1(t)\}
\]

subject to \( \dot{K}_1 = I_1, \dot{K}_2 = I_2, K_1(t_0) = \tilde{K}_1, K_2(t_0) = \tilde{K}_2 \).

The dynamic programming equation for this problem is

\[
(5.4.1) \ rV^i = \max \ (n_i - \frac{1}{2}c_i^2 + V^i_1 I_1 + V^i_2 I_2) \quad \text{w.r.t.} \quad \{I_i(t)\}
\]

The partial derivatives \( V^i_{K_j} \), which give the shadow value of starting the game from a higher level of \( K_j \), are equal to the costate variables \( \mu_{ij} \). Equations (3.2.4),
(3.3.4) and (3.3.5) can be used to substitute for \( \pi_i, \mu_{ij} \) and \( I_i \) in equation (5.4.1), which can thus be solved for \( V_i(K_1, K_2) \). Since the following analysis will always assume \( \tilde{K}_2 = 0 \), the notation can be simplified by denoting \( \tilde{K}_1 \) as \( K \), and letting \( V_i(K) = V_i(K, 0) \). Then

\[
(5.4.2) \quad V_1(K) = (1 - bK)K - v(r + \delta)K
\]

\[+ \frac{\pi}{2} \left( \theta_{11}K + \theta_{11} \right)^2/c + \left( \theta_{21}K + \theta_{21} \right) \left( \phi_{21}K + \phi_{21} \right)/c \] / r

\[
(5.4.3) \quad V_2(K) = \left( \frac{\pi}{2} \phi_{21}K + \phi_{21} \right)^2/c + \left( \phi_{11}K + \phi_{11} \right) \left( \theta_{11}K + \theta_{11} \right)/c \] / r

Hence, once the Riccati equations (3.3.11) have been solved numerically for the parameters \( \theta_{ij} \) and \( \phi_{ij} \), the value functions are easily computed.

**Optimal pre-entry capital**

The value functions which have just been derived can now be used in the entry model of section 5.3. Various assumptions can be made about how the hazard rate depends on the post-entry value of firm 2. The special form chosen here is that the hazard rate is proportional to the net present value which firm 2 would obtain if it entered. If the net present value is zero, the hazard is also zero. Let \( F \) be the startup cost incurred by firm 2. Then

\[
(5.4.4) \quad h(K, F) = \sigma (V_2(K) - F), \quad h = 0 \text{ if } V_2(K) < F
\]

Having calculated the value functions, and given the assumption about the hazard rate, the model of section 5.3 can be used to compute the optimal pre-entry level of capital. Recall that this level is the steady state which
the monopolist will reach if he eludes entry for a long time. This optimal pre-entry level is obtained by substituting the value functions and the hazard rate into equation (5.3.18).

Comparative static results could in principle be obtained analytically, but the signs of many terms will generally be ambiguous. Instead, comparative static results are obtained for some specific values of the parameters. Experimentation for a wide range of parameters yielded qualitatively identical results.

**Startup costs**

In the strategic investment model of chapter 3 there are no economies of scale. However, scale economies can be introduced by assuming that firms need to engage in R&D before they can produce the product, which are reflected in startup costs $F$.

The first observation is that the level of capital $K^d$ needed to deter entry falls as the level of startup costs increases. This is illustrated by the dashed line in figure 5.4.2. This is a simple consequence of the fact that $V_2$ decreases in $K$: for higher $F$, $V_2(K)-F$ will equal zero for a smaller value of $K$. 
The second observation is that if there are no startup costs, then firm 1 will never choose to deter entry. At any given combination of $K_1$ and $K_2$, the market price will be the same for both firms. Since the equilibrium path converges to a steady state in which profits are positive, the only way that firm two's value could be negative would be by incurring losses somewhere along the path, i.e. initially. But this would mean that firm 1 would have to keep its pre-entry level of capital at a level where the price is negative. Clearly such a strategy cannot be optimal for firm 1.

Thirdly, there will be some level of startup costs which will be large enough to "blockade entry", i.e. such
that entry is deterred by setting $K^*_1$ at the static monopoly maximum $K^m$.

Having determined how startup costs affect the firm’s absolute ability to deter entry, we now turn to its optimal pre-entry level of capital $\tilde{K}$. The relation between $F$ and $\tilde{K}$ is given in figure 5.4.2. The figure is divided into three regions. In region A, $\tilde{K}$ rises with $F$, then falls in region B, and becomes constant in region C.

In region A entry is ineffectively impeded, and firm one finds it optimal to retard entry, but not to deter it entirely. The figure shows that the optimal entry-retarding value $\tilde{K}$ rises with $F$. In this region, the aim of firm 1 is to reduce the probability $h$ of entry, which is proportional to $V_2(K) - F$. When $F$ is small, this expression can be made small only by setting very large values of $K$ at great cost. However, as $F$ increases, the effectiveness of an increase in $K$ becomes greater. Thus the increasing shape of the curve in region A is due to the fact that an entry-retarding strategy becomes increasingly attractive as its effectiveness increases.

In region B entry is effectively impeded, and firm 1 finds it optimal to deter entry entirely. The negative slope in region B simply reflects the already observed fact that higher levels of $F$ allow the entry-deterring level of capital $K^d$ to be reduced.
Region C is the region in which entry is blockaded. The static monopoly optimum $K^m$ is sufficient to deter entry.

The conclusion is that an entry-deterring monopolist will act as if there were no threat of entry both for high and low levels of startup costs. Only in the range of startup costs for which firm 2 can almost or just be deterred from entering (i.e. for $F$ close to $\hat{F}$ in figure 5.3.2), will strategic behavior differ from myopic behavior. It will be seen that this critical level of $F$ decreases as adjustment costs rise, as the discount rate increases, and as the rate of depreciation decreases. Each of these factors makes entry less attractive for firm 2, so lower fixed costs are sufficient to deter entry.

Now suppose that firm 1 has some influence over the level of startup costs of both firms. For example, suppose the startup costs consist of R&D to produce a high quality product.\(^{12}\) Does it pay for firm 1 to incur startup costs itself, only to reduce the probability of entry by firm 2? The answer is yes.

Figure 5.4.3 shows the expected value EV (net of startup costs) which firm 1 will obtain by setting K optimally. In region A, additional startup costs reduce the expected value, since they add to the cost without significantly affecting the firm's ability to retard entry. However, at the upper end of region A, the decrease in the probability of entry due to an increase in $F$ outweighs the
In region C, the optimal pre-entry \( \tilde{K} \) is unaffected by increases in \( F \), so here added \( F \) produces no benefits and only extra costs, so \( EV \) clearly must decrease.

In region B, the increase in \( F \) initially reduces the amount of \( K \) required to deter entry enough to offset the added cost. However, as \( \tilde{K} \) approaches the monopoly level, the increase in post-entry \( V_1 \) corresponding to the decrease in \( \tilde{K} \) is outweighed by the added direct cost of \( F \). Note that for this particular case, the maximum occurs at the interior of region B, but in general it may be at the boundary.

This result shows that if firm 1 could choose the common level of \( F \), it would set it at a level which will allow it to deter entry totally, but which might be below
the level necessary to blockade entry completely.

Adjustment costs

As the level of adjustment costs increases, it becomes more difficult for the incumbent firm to accommodate entry. This is to its advantage since it increases the commitment value of its level of capital.

The numerical comparisons indeed show that the level of capital necessary to deter entry for a given level of startup costs is reduced when adjustment costs increase. Equivalently, as adjustment costs increase, entry is blockaded at progressively lower levels of startup costs.

The effect of increased adjustment costs \( c \) on the optimal pre-entry capital \( \tilde{K} \) is ambiguous, and depends on the particular level of startup costs. Figure 5.4.4 illustrates the relation between \( \tilde{K} \) and \( F \) for two different levels of adjustment cost \( c \). As \( c \) increases, the entire graph shifts toward the left. This corresponds to the fact that lower levels of \( F \) are now sufficient to retard or deter entry. We see that \( \tilde{K} \) may stay the same (far left and far right), it may increase (near \( \hat{F} \) for the higher level of \( c \)), or decrease (near \( \hat{F} \) for the lower level of \( c \) ).
Now suppose the incumbent could choose the level of $c$. The analysis so far suggests that the firm would want to set $c$ as high as possible. This is because this will allow it to deter or retard entry at lower levels of $F$, which increases its pre-entry expected value. However, this statement is only true if the firm has already reached its entry-deterring level.

Suppose instead that the firm has just entered the market, and is currently a monopolist. Then the firm would prefer its adjustment costs to be low while it is growing. This is true for two reasons. First, higher adjustment costs will make its growth more expensive and slower. This reason holds even if there is no threat of entry. The second reason
is related to the commitment value of capital. Although higher adjustment costs still give a greater degree of commitment, in this case the firm is committed to a low level of capacity. If anything, this will encourage entry. Entry deterrence only works if the firm can commit itself to high level of capacity.

This suggests that the firm should try to maintain flexibility during its growth stage, but should seek ways to limit its flexibility once it has completed its growth.

Open loop versus closed loop

Suppose that immediately after entry occurs, both firms simultaneously sign binding agreements committing them to an investment path from that point on. As discussed in chapter 3, this lessens the amount of competition in the post-entry game. The value of the game from any given starting point is increased for both players. Thus the level of startup costs necessary to blockade entry increases, and entry deterrence becomes more difficult. The graph of $\tilde{K}$ versus $F$ thus shifts to the right.

However, if firm 1 can choose the level of startup costs, then the result is very similar to the closed-loop case, with the difference that entry will now be deterred with higher startup costs.
The discount rate

An increase in the discount rate will lower the post-entry value of both firms. Hence entry will be blockaded at lower values of $F$, and the graph of $\tilde{K}$ versus $F$ shifts to the left. However, the peak of the graph will occur at a lower level of $\tilde{K}$, because now present profits count proportionately more than future profits.

The rate of depreciation

Suppose the rate of depreciation decreases, which lowers the commitment value of capital. One effect is to increase the value of both firms, since now the maintenance cost of capital is cheaper. Thus entry will be blockaded at higher levels of $F$. In addition, both the static monopoly level of capital, and the steady state of the post-entry game will increase. Thus graph of $\tilde{K}$ versus $F$ will shift both up and to the right.
APPENDIX A TO CHAPTER 5

This appendix shows that the sufficient conditions for uniqueness of the steady state also imply that the optimal path for the monopolist converges to a steady state. The differential equations which describe the optimal path are given by (5.3.14)-(5.3.17), and are repeated here for convenience.

(5.A.1) \[ c^I = [r + h(K)]cI - [\pi'(K) + h'(K)V(K) + h(K)V' + \lambda h'(K)] \]

(5.A.2) \[ \lambda = [r + h(K)]\lambda + [\pi(K) - \frac{1}{2}cI^2 + h(K)V(K)] \]

(5.A.3) \[ \dot{\lambda} = I \]

(5.A.4) \[ \dot{F} = h(K)(1 - F) \]

For the optimal path to converge to the steady state, it is necessary to examine the stability of this system of equations. Note that \( I, K, \) and \( \lambda \) are determined independently of \( F \). Furthermore, \( F \) is a cumulative probability, and is therefore bounded by \([0,1]\). Hence the stability of these equations can be established by considering only the first three equations.

Linearizing (5.A.1)-(5.A.3) around the steady state yields:

\[
\begin{bmatrix}
\dot{c}^I \\
\dot{\lambda} \\
\dot{K}
\end{bmatrix} =
\begin{bmatrix}
(r+h) \\
0 \\
1
\end{bmatrix}
+ 
\begin{bmatrix}
-h'/c \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
c^I \\
\lambda \\
K
\end{bmatrix}
+ 
\begin{bmatrix}
Z
\end{bmatrix}
\begin{bmatrix}
I
\end{bmatrix}
\]

(5.A.5)

where

(5.A.6) \[ Z = -[\pi'' + h''V^* + 2h'V' + hV'' + \lambda h'']/c. \]
The stability of the system can be determined from the eigenvalues of this matrix. Denoting the eigenvalues by \( x \), and expanding along the second row, the characteristic polynomial is given by
\[
(r+h+x)(-r-h-x)x-2] = 0
\]
(5.12)

Inspection shows that one root of this polynomial is given by
\[
x_1 = r + h > 0
\]
(5.13)
The other two roots are found from a quadratic equation:
\[
x_2, x_3 = \frac{1}{2} \left( r + h \pm \sqrt{(r+h)^2 + 4(2)} \right)
\]
(5.14)
Note that if \( Z < 0 \), then either both roots will be real and positive, or they will be complex with positive real part. Either of these situations would imply that the system of equations is totally unstable.

On the other hand, if \( Z > 0 \), then one of the roots will be positive, and the other negative. Thus, of the three eigenvalues of the system of equations, two are positive and the other negative, which implies that the steady state constitutes a saddlepoint. Hence \( Z > 0 \) is a necessary and sufficient condition for stability of the steady state.

Now compare the requirement for stability \( (Z > 0) \) with the sufficient condition for uniqueness of the steady state. From proposition 5.3.1, uniqueness of the steady state is assured by having \( Y' < 0 \), where
\[
Y' = (r+h)(n''+hV')+2(r+h)h'V'-h''(n-rV)
\]
(5.15)
It will now be shown that $Y'<0$ if and only if $Z>0$. From equation (5.4.6), $Z>0$ if and only if

$$(5.4.11) \quad n''' + hV'' + 2h'V' + h''V + h''\lambda < 0$$

$$n''' + hV'' + 2h'V' + h''(V + \lambda) < 0$$

From (5.3.19),

$$(5.4.12) \quad V + \lambda = -[n-rV]/(r+h)$$

Thus $Z>0$ if and only if

$$(5.4.13) \quad n''' + hV'' + 2h'V' - h''(n-rV)/(r+h) < 0$$

By dividing (5.4.10) by $(r+h)$, it is seen that $Z>0$ if and only if $Y'/(r+h)<0$, which is equivalent to $Y'<0$.

Thus the optimal path of the monopolist will converge to a steady state precisely under the same conditions under which the steady state is unique.
APPENDIX B TO CHAPTER 5

The following program, written in CBASIC, and implemented on an Osborne Executive, was used to compute the numerical results of chapter 5.

```cbasic
rem symminv.bas
rem solves for as many roots as it can find
rem of a system of nonlinear equations
rem using the Newton-Raphson method
rem the equations are those of the strategic investment
rem model in the symmetric case
rem
rem clear$=chr$(26)
rem n%=4
rem na%=n%
rem
dim a(n%,n%),b(n%),x(n%),temp(n%),root(8,6),
new.root(n%),alpha(b),beta(8),gamma(b),
k.st(8),profit.st(8),stable$(8),k.zero(2),
value(2),inv(2),profit(2),k(2)

rem main program
rem
rem gosub 1100:rem input default parameters
10 print "0 - Exit from program"
print "1 - Input parameters"
print "2 - Automatically find roots"
print "3 - Manually find roots"
print "4 - Print roots on screen"
print "5 - Lineprint roots"
print "7 - Compute values for a range of k1(0)"
print "8 - find pre-entry optimal k1"
print "9 - Find entry-deterring k1(0)"
input "Enter your choice: ";x%
if x%<0 or x%>20 then goto 10
if x%=0 then stop
on x% gosub 1000,2000,3000,4000,5000,7000,7000,
8000,9000
goto 10

rem closed loop equations
rem
```
**** compute \( b(i\%) \) ****

\[
\begin{align*}
b(1) &= -((x(1)-s)*x(1)+x(2)*x(2)+x(2)*x(3)-2.0*t) \\
b(2) &= -((2.0*x(1)-s)*x(2)+x(2)*x(4)-t) \\
b(3) &= -((2.0*x(1)-s)*x(3)+x(2)*x(4)-t) \\
b(4) &= -((2.0*x(1)-s)*x(4)+x(2)*x(3))
\end{align*}
\]

return

**** compute \( a(i\%,j\%) \) ****

\[
\begin{align*}
a(1,1) &= 2.0*x(1)-s \\
a(2,1) &= 2.0*x(2) \\
a(3,1) &= 2.0*x(3) \\
a(4,1) &= 2.0*x(4) \\
a(1,2) &= 2.0*x(2)+x(3) \\
a(2,2) &= 2.0*x(1)+x(4)-s \\
a(3,2) &= x(4) \\
a(4,2) &= x(1) \\
a(1,3) &= x(2) \\
a(2,3) &= 0 \\
a(3,3) &= 2.0*x(1)-s \\
a(4,3) &= x(2) \\
a(1,4) &= 0 \\
a(2,4) &= x(2) \\
a(3,4) &= x(2) \\
a(4,4) &= 2.0*x(1)-s
\end{align*}
\]

return

**** compute constant term for a given root ****

\[
\begin{align*}
det &= (\text{root}(n.roots,1)+\text{root}(n.roots,2)-s)* \\
     & \quad (\text{root}(n.roots,1)-s) - \\
     & \quad (\text{root}(n.roots,3)+\text{root}(n.roots,4))* \\
     & \quad \text{root}(n.roots,2) \\
rt &= (-u)*(\text{root}(n.roots,1)-s) \\
root(n.roots,5) &= rt/det \\
rt &= (\text{root}(n.roots,3)+\text{root}(n.roots,4))*u \\
root(n.roots,6) &= rt/det \\
return
\end{align*}
\]

************ open loop equations ************

**** compute \( b(i\%) \) ****

\[
\begin{align*}
b(1) &= -((x(1)-s)*x(1)+x(2)*x(2)-2.0*t) \\
b(2) &= -((x(1)-s)*x(2)+x(1)*x(2)-t) \\
b(3) &= -((x(1)-s)*x(3)+x(2)*x(4)-t) \\
b(4) &= -((x(1)-s)*x(4)+x(2)*x(3))
\end{align*}
\]

return

**** compute \( a(i\%,j\%) \) ****

\[
\begin{align*}
a(1,1) &= 2.0*x(1)-s \\
a(2,1) &= 2.0*x(2) \\
a(3,1) &= x(3)
\end{align*}
\]
a(4,1)=x(4)
a(1,2)=2.0*x(2)
a(2,2)=2.0*x(1)-s
a(3,2)=x(4)
a(4,2)=x(3)
a(1,3)=0
a(2,3)=0
a(3,3)=x(1)-s
a(4,3)=x(2)
a(1,4)=0
a(2,4)=0
a(3,4)=x(2)
a(4,4)=x(1)-s
return

rem 290 ***** compute constant term for each root *****
z1=u/(root(n.roots%,1)+root(n.roots%,2)-s)
root(n.roots%,5)=z1
root(n.roots%,6)=(root(n.roots%,3)+
    root(n.roots%,4))*z1/s
return

rem 1000 input parameters
rem
print "New parameters:"
input "r=";line a$
if a$="" then goto 1010 else r=val(a$)
1010 input "d=";line a$
if a$="" then goto 1020 else d=val(a$)
1020 input "c=";line a$
if a$="" then goto 1030 else c=val(a$)
1030 input "b=";line a$
if a$="" then goto 1040 else b=val(a$)
1040 input "v=";line a$
if a$="" then goto 1050 else v=val(a$)
1050 input "closed-loop (y) or open-loop (n)";line a$
if a$="" then goto 1060
if a$="y" or a$="Y" then closed.loop$="y"
if a$="N" or a$="n" then closed.loop$="n"
1060 s=c*r
t=c*b
a=1.0-v*(r+d)
u=a*c
n.roots%=0
return

rem 1100 ***** default parameters *****
r=.05
d=.1
c=50.0
b=1.0
v=1.0
s=c*r
t=c*b
a=1.0-v*(r+d)
u=a*c
closed.loop$="y"
n.roots%=0
return

rem
rem find all eight roots of the equations
rem
2000
n.roots%=0
input "step=\$length= ";x
start=-x/2.0
finish=x/2.0
for k1=start to finish step x
for k2=start to finish step x
for k3=start to finish step x
for k4=start to finish step x
print k1,k2,k3,k4
x(1)=k1
x(2)=k2
x(3)=k3
x(4)=k4
gosub 2100:rem solve for a root from given
starting value
if iflag%=1 then goto 2010:rem no root was
found
if n.roots%=0
   then newroot$="y"
else gosub 2200:rem test
   whether root is new
   if newroot$="n" then goto 2010
   n.roots%=n.roots%+1
for j%=1 to 4
   root(n.roots%,j%)=new.root(j%)
next j%
if closed.loop$="y" then gosub 190:rem
   compute constant term
if closed.loop$="n" then gosub 290:rem
   compute constant term
gosub 2400:rem compute steady state
gosub 2500:rem determine stability
rem also add "negative root"
if closed.loop$="y" then gosub 190:rem
   compute constant term
if closed.loop$="n" then gosub 290:rem
    compute constant term
gosub 2400:rem compute steady state
gosub 2500:rem determine stability
if n.roots%=8 then return

2010  next k4
next k3
next k2
next k1

gosub 4000:rem print roots
input "search for another root (Y/N)?";a$
if a$<"Y" and a$<"y" then return
gosub 3000:rem find additional roots
if n.roots%=8 then return
goto 2020

rem  ***** solve for a root from a given starting value *****
rem  2100  for iter%=1 to 25
    for i%=1 to n%
        temp(i%)=x(i%)
    next i%
    if closed.loop$="y" then gosub 100:rem
        compute b(i%)=-f(x)
    if closed.loop$="n" then gosub 200:rem
        compute b(i%)=-f(x)
    norm=0
    for i%=1 to n%
        norm=norm+b(i%)#b(i%)
    next i%
    if norm<=1.0E-15
        then  for i%=1 to n%
            new.root(i%)=x(i%):
        next i%
        return
    if closed.loop$="y" then gosub 150:rem
        compute A(i%,j%)=J(x)
    if closed.loop$="n" then gosub 250:rem
        compute A(i%,j%)=J(x)
gosub 2.000:rem solve Ax=b
if iflag%=1 then return:rem error return
    if pivoting fails
        for i%=1 to n%
            x(i%)=temp(i%)+x(i%)
        next i%
    next iter%
    iflag%=1: return:rem error return if no convergence

rem  ***** test whether root is new *****
rem  2200  newroot$="y"
    for i%=1 to n.roots%
        root.diff=0
        for j%=1 to n%
root.diff=root.diff+
(new.root(j%)-root(i%,j%))*
(new.root(j%)-root(i%,j%))
next j%
if root.diff<1.0E-6 then newroot$="n":return
next i%
return

rem 2400
##### compute steady state level of capital
and profit #####
alpha(n.roots%)=root(n.roots%,1)/c
beta(n.roots%)=root(n.roots%,2)/c
gamma(n.roots%)=root(n.roots%,5)/c
k.st(n.roots%)=-gamma(n.roots%)/
(alpha(n.roots%)+beta(n.roots%))
p=1.0-2.0*b*k.st(n.roots%)
profit.st(n.roots%)=(p-v*(r+d))*k.st(n.roots%)
return

rem 2500
##### determine stability of phase diagram #####
if abs(alpha(n.roots%))>abs(beta(n.roots%))
and (alpha(n.roots%)<0)
then stable$(n.roots%)="y"
else stable$(n.roots%)="n"
return

rem 3000
##### find additional roots #####
print clear;
print "Try additional starting values:"
input "x(1)= ";x(1)
input "x(2)= ";x(2)
input "x(3)= ";x(3)
input "x(4)= ";x(4)
gosub 2100:rem solve for a root from given
starting value
if iflag$=1 then goto 3010:rem no root was found
if n.roots%=0
then newroot$="y"
else gosub 2200:rem test
whether root is new
if newroot$="n" then goto 3010
n.roots%=n.roots%+1
for j%=1 to 4
root(n.roots%,j%)=new.root(j%)
next j%
if closed.loop$="y" then gosub 190:rem compute
constant term
if closed.loop$="n" then gosub 290:rem compute
constant term
gosub 2400:rem compute steady state
gosub 2500:rem determine stability
rem also add "negative root"
 n.roots%=n.roots%+1
 root(n.roots%,1)=s-new.root(1)
 for j%=2 to 4
   root(n.roots%,j%)=-new.root(j%)
 next j%
 if closed.loop$="y" thengosub 190:rem compute
 constant term
 if closed.loop$="n" thengosub 290:rem compute
 constant term
 gosub 2400:rem compute steady state
 gosub 2500:rem determine stability
 3010 return
 rem
 4000 print
 "## -x(1)- -x(2)- -x(3)- -x(4)- -z(1)-
 -z(2)- -k.st- -pf.st- -stb-"
 print
 for i%=1 to n.roots%
   print using
   "## ### ### ### ### ### ### ### ### ### ### ### ### ";
   i%,root(i%,1),root(i%,2),root(i%,3),root(i%,4),root(i%,5),
   root(i%,6),k.st(i%),profit.st(i%),stable$(i%)
 next i%
 return
 rem
 5000 lprinter
 print
 if closed.loop$="y" then print "Closed loop"
   else print "Open loop"
 print using "c=#.#.# r=#.#.#
 d=#.# b=#.#.# v=#.#.#;c,r,d,b,v
 print
 gosub 4000:rem print roots
 print
 console
 return
 rem
 6100 th11=root(root%,1)
 th12=root(root%,2)
 th21=root(root%,3)
 th22=root(root%,4)
 th1=root(root%,5)
 th2=root(root%,6)
 kz1=k.zero(1)
 mu11=th11*kz1+th1
 mu12=th21*kz1+th2
 mu21=th22*kz1+th2
\[
\begin{align*}
\mu_{22} &= \text{th12} \times \text{kz1} + \text{th1} \\
\text{z1} &= \mu_{11} / c \\
\text{z2} &= \mu_{22} / c \\
\text{value}(1) &= ((a-b) \times \text{kz1}) \times \text{kz1} + 0.5 \times \mu_{11} \times \text{z1} + \mu_{12} \times \text{z2}) / \text{r} \\
\text{value}(2) &= (0.5 \times \mu_{22} \times \text{z2} + \mu_{21} \times \text{z1}) / \text{r} \\
\end{align*}
\]

rem 6150

**** compute expected value ****

\[
\begin{align*}
\text{pie1} &= (a-b) \times \text{k.zero}(1) \times \text{k.zero}(1) \\
\text{h} &= \text{sigma} \times \text{value}(2) - \text{f} \\
\text{if h < 0 then h = 0} \\
\text{e.value} &= \text{pie1} + \text{h} \times \text{value}(1) / (\text{r} + \text{h}) - \text{f} \\
\text{pie1.pr} &= \text{a-2.0} \times b \times \text{kz1} \\
\text{v1.pr} &= \text{pie1.pr} + \mu_{11} \times \text{th11} / c + \\
& \quad (\mu_{12} \times \text{th12} + \text{th11} \times \mu_{22}) / \text{c} / \text{r} \\
\text{v2.pr} &= (\mu_{22} \times \text{th12} / c + (\mu_{21} \times \text{th11} + \mu_{11} \times \mu_{22}) / c) / \text{r} \\
\text{if h > 0 then h.pr} &= \text{sigma} \times \text{v2.pr} \text{ else h.pr} = 0 \\
\text{lhs} &= (\text{r} + \text{h}) \times (\text{pie1.pr} + \text{h} \times \text{v1.pr}) - \text{h.pr} \times (\text{pie1} - \text{r} \times \text{value}(1)) \\
\end{align*}
\]

rem 7000

**** automatically compute values ****

\[
\begin{align*}
\text{root}\% &= 1 \\
\text{print} \ "\text{Range for k1.zero}:\" \\
\text{input} \ "\text{k1.start}=\"; \text{k1.start} \\
\text{input} \ "\text{k1.end}=\"; \text{k1.end} \\
\text{input} \ "\text{k1.incr}=\"; \text{k1.incr} \\
\text{input} \ "\text{sigma}=\"; \text{sigma} \\
\text{input} \ "\text{f}=\"; \text{f} \\
\text{lprinter} \\
\text{print} \\
\text{print using} \ "\text{sigma=###.## \ f=###.###}; \text{sigma}, \text{f} \\
\text{print} \ "\text{k1(0) -pie1 - r*v1 - v2 - h - pie1'} - v1' - \\
\text{h'} - \text{ lhs - e.value=}" \\
\text{print} \\
\text{console} \\
\text{for k1=k1.start to k1.end step k1.incr} \\
\text{k.zero}(1) &= k1 \\
\text{k.zero}(2) &= 0 \\
\text{gosub} 6100: \text{rem compute value} \\
\text{gosub} 6150: \text{rem compute e.value} \\
\text{gosub} 7010: \text{rem print} \\
\text{lprinter}; \text{gosub} 7010; \text{console}; \text{rem lprint} \\
\text{next k1} \\
\text{lprinter} \\
\text{print} \\
\text{console} \\
\text{return} \\
\end{align*}
\]

rem 7010

----- print expected value -----

\[
\begin{align*}
\text{print using} \ "\text{### \ .### ** ** ** ** ** ** ** ** ** ** **}}" \\
\end{align*}
\]
## Find pre-entry optimal k

### 8000

**Input:**
- `f.start`
- `f.end`
- `f.incr`
- `sig.start`
- `sig.end`
- `sig.incr`
- `k1.incr`

**Printer output:**
- `-sigma- f- -k1*- -pie1- -v1- -v2- -h- -e.value-`
print using
"****** entry determining level of k1(0) ******"

rem 9000
input "f.start";f.start
input "f.end";f.end
input "f.incr";f.incr
print "range over which to search k1(0):"
input "k1.start";k1.start
input "k1.end";k1.end
sigma=1.0
lprinter
print
"-sigma- -f- -k1(0)- -pie1- -v1- -v2- -h- -e.value-"
print
for f=f.start to f.end step f.incr
root%=1
k1.incr=0.05
for k1=k1.start to k1.end step k1.incr
  k.zero(1)=k1
  k.zero(2)=0
  gosub 6100:rem compute value
  gosub 6150:rem compute e.value
  if value(2)-f<0 then k1.new=k1:goto 9010
next k1

9010 for k1=k1.new-0.04 to k1.new+0.04 step 0.01
  k.zero(1)=k1
  k.zero(2)=0
  gosub 6100:rem compute value
  gosub 6150:rem compute e.value
  if value(2)-f<0 then goto 9020
next k1

9020 print using
"****** entry determining level of k1(0) ******"

rem 9000
input "f.start";f.start
input "f.end";f.end
input "f.incr";f.incr
print "range over which to search k1(0):"
input "k1.start";k1.start
input "k1.end";k1.end
sigma=1.0
lprinter
print
"-sigma- -f- -k1(0)- -pie1- -v1- -v2- -h- -e.value-"
console
next f
lprinter
print
console
return
1. In this section, the entry deterrence problem is described for a single firm, but the analysis extends to a cartel acting jointly to deter entry.

2. For an exhaustive discussion of entry barriers, with emphasis on their application to corporate strategy, see Porter (1980).

3. For example, see Friedman (1979), Salop (1979), Scherer (1980), and Dixit (1980).

4. An alternative interpretation of the limit price theory is given by Milgrom and Roberts (1982). If the potential entrant does not know the cost function of the incumbent, a low price may be a signal of low costs. In this case, pre-entry price gives some indication of post-entry profits to the entrant without resorting to empty threats.

5. This is the assumption of the first of the two models presented in Spence's 1977 paper. In the second model, the post-entry equilibrium assumes marginal cost pricing for both firms, and capital is chosen to affect the cost curve of the incumbent.

6. This classification was developed by Bain (1956) in his classic book on entry deterrence.

7. The analysis of this section assumes a single entrant. Kamien and Schwartz (1971, p.450-1) discuss how the analysis could be modified to account for several entrants. Bernheim (1984) shows that sequential entry could yield some unexpected results.

8. In light of the current emphasis on credible threats and commitment, these two assumptions are actually incompatible. Presumably a high pre-entry price increases the probability of entry because it signals high potential profits to entrants. But if the post-entry price is independent of the pre-entry price, then the pre-entry price is not a reliable signal for post-entry profits.

9. Note, however, that it is not the level of capital which a monopolist with patent protection would want to have when his patent expires, since such a monopolist faces zero probability of entry during the protected period.

10. The discussion surrounding equations (3.3.1)-(3.3.3) in chapter 3 of this thesis shows how several alternative formulations are related: (i) whether I denotes net or gross
investment, (ii) whether the cost of capital is given by \( rK \) or \( I \), and (iii) how to include depreciation in these formulations.

11. This corresponds to equation (14) of Kamien and Schwartz: 
\[
R(p) = \frac{1}{2} (p)(r+h) - h'(p)(\frac{1}{2} (p) - \pi_2),
\]
after noting that the choice variable here is \( K \) instead of \( p \), and that 
\( rV(K) \) corresponds to the after-entry profit stream \( \pi_2 \).

12. The disposable diaper industry is an example of an industry in which a high level of R&D expenditure is needed to match the product quality of the incumbent. For a discussion of entry into this industry, see the Harvard Business School (1981) case on this industry.
LIST OF REFERENCES


