VOID INITIATION IN A CLASS OF COMPRESSIBLE ELASTIC MATERIALS

by

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(1968)

SUBMITTED TO THE DEPARTMENT OF
MECHANICAL ENGINEERING
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF

MASTER OF SCIENCE IN MECHANICAL ENGINEERING

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
November, 1986

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November, 1986

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Submitted to the Department of Mechanical Engineering
on November 30, 1986 in partial fulfillment of the
requirements for the Degree of Master of Science in
Mechanical Engineering

ABSTRACT

This is a study of a bifurcation problem for a solid circular cylinder which is made of a particular class of homogeneous, isotropic, compressible elastic material. The surface of the cylinder is subjected to a purely radial stretch λ (>1) and the cylinder is assumed to be in a state of plane strain. One solution to this problem, for all values of λ, is that of a pure homogeneous stretching in which the cylinder expands radially. However, a second (singular) solution bifurcates from this homogeneous solution at a critical value of the stretch λ ($=\lambda_{cr}$) at which the homogeneous solution becomes unstable. For $\lambda>\lambda_{cr}$, a circular cylindrical cavity forms at the axis of the cylinder.

In this study we have two purposes. The first is to determine an explicit analytical solution to the bifurcation problem for a broad class of <u>compressible</u> elastic materials. The second is to examine the dependence of various physical quantities (such as the critical stress at void initiation) on the constitutive parameters (such as the hardening exponent).

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ACKNOWLEDGEMENTS

I would like to thank Professor Rohan Abeyaratne for his invaluable assistance.

I thank the U.S. National Science Foundation and the U.S. Army Research Office for their financial support.

Finally, I would like to extend a special thanks to my daughter Sevgi Arnold, my husband Dr. Rocky Arnold, and my sister Ruziye Ertan for their encouragement and to Helen Doyle for typing this thesis.

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CHAPTER 1

INTRODUCTION

When subjected to tensile loads, many materials develop internal voids. For example, this has been observed in metals by Tipper (1949), and in elastomers by Yerzley (1939). Typically, under monotonically increasing applied load, the voids first appear, then grow, and eventually coalesce to form cracks.

In order to study this phenomenon in rubber, Gent and Lindley (1958) conducted experiments on internal rupture in rubber. Their test specimens consisted of vulcanized rubber cylinders which were bonded to plane metal end-pieces, and placed in tension. They observed internal voids appear and coalesce along the axis of the cylinder. In order to provide a theoretical explanation for this, they made the ad hoc assumption that voids form when the negative hydrostatic pressure component of the applied stress reaches a critical value. They then calculated the critical value of the negative pressure at which a spherical cavity would grow to be unbounded, i.e. "burst", by using the theory of large elastic deformations. The value they found was in reasonable agreement with their experimental observations.

Hill (1950) had previously carried out a similar analysis in which he studied the growth of a hole from zero initial radius in an elastic-plastic material. McClintock (1968) developed a criterion for fracture in terms of the growth and coalescence of cylindrical holes under any prescribed history of applied principal components of stress and strain that do not rotate relative to the material.

More recently, Ball (1982) carried out a bifurcation analysis of the equations of nonlinear elasticity. His approach was to minimize the energy functional. The crucial step in his analysis was to recognize that if a void is to nucleate, then there will be a certain singularity in the field quantities at the site of nucleation, and therefore the class of admissible functions admitted into the minimization must be broader that used classically. By permitting the deformation field to be less than classically smooth, he was able to generate certain singular solutions of the field equations. Ball (1982) interpreted these solutions in terms of internal rupture, in which a cavity forms in the interior of a solid body which contains no hole in the undeformed state. His analysis <u>automatically</u> predicts the critical load at which the hole appears, and also describes its subsequent growth; it does not require any additional ad hoc assumptions.

In a subsequent study, Abeyaratne and Horgan (1986) showed that such an analysis is more relevant for describing the growth of a pre-existing void of infinitesimal initial size.

Ball (1982)'s analysis consists of two parts. For incompressible elastic materials he carries out an extensive and complete analysis. In addition to proving the existence of these bifurcated solutions involving voids, he goes on to determine them explicitly and to study their stability. However for compressible elastic materials he only proves an existence theorem, and this too only for a limited sub-class of compressible elastic materials. Subsequently Podio-Guidugli, Caffarelli and Virga (1986), and Sivaloganathan (1986) took up this question, but they too confined themselves to estalishing results pertaining to existence for a slightly wider class of materials than Ball.

On the other hand, Abeyaratne and Horgan (1986) determine explicitly the exact deformation and stress fields for such a problem; they also examine the stability issue. However, their analysis is restricted to a special compressible elastic material -- the so-called Blatz-Ko material.

The purpose of the present study is two-fold. Firstly, we determine an explicit analytical solution to this bifurcation problem for a class of compressible elastic materials significantly more general than the Blatz-Ko material. Secondly, we wish to examine the dependence of the various physical quantities (such as the critical stress at void initiation) on the constitutive parameters (such as the hardening exponent).

One of the most general constitutive models for compressible isotropic elastic materials is that given by Ogden (1972). Since the expression he gives is in the form of an infinite series, one can always take a large enough number of terms in order to model any set of available data. Recently, Storakers (1986) showed that even with a few terms (one or two) it is possible to model, quite well, the behavior of compressible rubber-like materials. The constitutive law that we consider is a special case of an Ogden material. It has three arbitrary constitutive parameters, viz. the shear modulus μ, hardening exponent n, and Poisson's ratio ν. For a special choice of these parameters, this constitutive model reduces to the Blatz-Ko description.

We study a bifurcation problem for a solid circular cylinder which is made of this homogeneous, isotropic, compressible elastic material. The surface of the cylinder is subjected to a purely radial stretch λ (>1) and the cylinder is assumed to be in a state of plane strain. One solution to this problem, for all values of λ, is that of a pure homogeneous stretching in which the cylinder expands radially. However a second (singular) solution bifurcates from this homogeneous solution at a critical value of the stretch λ ($=\lambda_{cr}$) at which the homogeneous solution becomes unstable. For $\lambda > \lambda_{cr}$, a circular cylindrical cavity forms at the axis of the cylinder.

In Chapter 2 we present some preliminaries from the nonlinear theory of elasticity. In Chapter 3 we formulate the bifurcation problem by minimizing the appropriate energy functional and deriving the Euler-Lagrange equation and natural boundary condition. In Chapter 4 we describe the class of constitutive relations that we will be considering and study the response of this material in uni-axial tension, simple shear and isotropic extension. In Chapter 5, the problem formulated in Chapter 3 is specialized to the class of materials discussed in Chapter 4. This problem is then solved explicitly. In the first part of Chapter

6 we examine the stability of the solution with the cavity, and then in the latter part we examine how the various results depend upon the Poisson ratio ν (the "compressibility") and the material hardening exponent.

CHAPTER 2

PRELIMINARIES ON FINITE ELASTICITY

In this section we give a brief summary of the fundamental equations of finite elasticity. An extensive treatment of this subject can be found in the treatise by Truesdell and Noll (1965) or in the book by Ogden (1984).

Let $\mathfrak{R}_0$ be the domain in three-space occupied by the interior of a body in its undeformed configuration. A <u>deformation</u> of the body is then described by a sufficiently smooth and invertible transformation

$$\underset{\sim}{x} = \underset{\sim}{\chi}(\underset{\sim}{X}) = \underset{\sim}{X} + \underset{\sim}{u}(\underset{\sim}{X}) \qquad \text{for all } \underset{\sim}{X} \text{ in } \mathfrak{R}_0, \tag{2.1}$$

which maps $\mathfrak{R}_0$ onto a domain $\mathfrak{R}$. Tentatively, we assume that $\underset{\sim}{u} \in C^2\ (\mathfrak{R}_0)$. Here $\underset{\sim}{X}$ is the position vector of a generic material point before deformation, $\underset{\sim}{\chi}(\underset{\sim}{X})$ is its position vector after deformation: $\underset{\sim}{X} \in \mathfrak{R}_0$, $\underset{\sim}{\chi}(\underset{\sim}{X}) \in \mathfrak{R}$. $\underset{\sim}{u}$ is the displacement vector-field associated with the deformation. Thus X_i and x_i are the cartesian material and spatial coordinates, respectively. The deformation-gradient tensor $\underset{\sim}{F}$ is defined by

$$\underset{\sim}{F} = \underset{\sim}{\nabla}\,\underset{\sim}{x} = [\partial x_i / \partial X_j] = \underset{\sim}{I} + \underset{\sim}{\nabla}\,\underset{\sim}{u}, \quad J = \det \underset{\sim}{F} > 0 \text{ on } \mathfrak{R}_0, \tag{2.2}$$

where J is the Jacobian determinant of the mapping (2.1), and $\underset{\sim}{I}$ is the identity tensor.

Since $\underset{\sim}{F}$ is non-singular, according to the polar decomposition theorem, there exists a unique proper orthogonal tensor field $\underset{\sim}{R}$ and unique symmetric positive definite tensor fields $\underset{\sim}{U}$ and $\underset{\sim}{V}$ such that

$$\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{U} = \underset{\sim}{V}\underset{\sim}{R}; \tag{2.3}$$

$\underset{\sim}{R}$ is a measure of the local rotation, while $\underset{\sim}{U}$ and $\underset{\sim}{V}$ is related to the strain. In fact, the Lagrangian strain tensor $\underset{\sim}{E}$ is given by

$$\underset{\sim}{E} = (1/2)\ (\underset{\sim}{U}^2 - \underset{\sim}{I}). \tag{2.4}$$

Let $\underset{\sim}{C}$ and $\underset{\sim}{B}$ stand for the right and left Cauchy-Green deformation measures, respectively, so that

$$\underset{\sim}{C} = \underset{\sim}{F}^T\underset{\sim}{F} = \underset{\sim}{U}^2, \qquad \underset{\sim}{B} = \underset{\sim}{F}\,\underset{\sim}{F}^T = \underset{\sim}{V}^2. \tag{2.5}$$

Both $\underset{\sim}{C}$ and $\underset{\sim}{B}$ are positive definite symmetric tensors, which have the same fundamental scalar invariants I_i and possess common principal values $\lambda_i{}^2$, where $\lambda_i > 0$ are the principal stretches of the deformation at hand. In particular,

$$\begin{aligned} I_1 &= \mathrm{tr}\ \underset{\sim}{C} = \lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2\ , \\ I_2 &= (1/2)[(\mathrm{tr}\ \underset{\sim}{C})^2 - \mathrm{tr}(\underset{\sim}{C}^2)] = \lambda_1{}^2\lambda_2{}^2 + \lambda_2{}^2\lambda_3{}^2 + \lambda_3{}^2\lambda_1{}^2, \\ I_3 &= J^2 = \det\ \underset{\sim}{C} = \lambda_1{}^2\lambda_2{}^2\lambda_3{}^2. \end{aligned} \tag{2.6}$$

The principal strains E_i of the Lagrangian strain tensor are related to the principal stretches λ_i by

$$E_i = (1/2)\ (\lambda_i{}^2-1). \tag{2.7}$$

Let $\underset{\sim}{\tau}(\underset{\sim}{x})$ be the Cauchy stress tensor field (true stress field) accompanying the deformation. Equilibrium, in the absence of body forces, demands that

$$\operatorname{div} \underset{\sim}{\tau} = 0\ ,\quad \underset{\sim}{\tau} = \underset{\sim}{\tau}^T$$

or (2.8)

$$\partial\tau_{ij}/\partial x_j = 0\ ,\quad \tau_{ij} = \tau_{ji} \text{ on } \mathfrak{R}.$$

The component τ_{ij} of the Cauchy stress tensor represents the force per unit current area in the jth direction, acting on a surface currently normal to the ith direction. Next, let $\underset{\sim}{\sigma}(\underset{\sim}{X})$ be the Piola stress tensor field (nominal stress field) corresponding to $\underset{\sim}{\tau}$, whence

$$\underset{\sim}{\sigma} = J\ \underset{\sim}{\tau}\ (\underset{\sim}{F}^{-1})^T,\qquad \underset{\sim}{\tau} = (1/J)\ \underset{\sim}{\sigma}\ \underset{\sim}{F}^T\ , \tag{2.9}$$

where $\underset{\sim}{F}^{-1}$ is the inverse of $\underset{\sim}{F}$. The component σ_{ij} of nominal stress represents the force per unit original area in the jth direction, acting on a surface which originally was normal to the ith direction. Then (2.8) and (2.1) allow the equilibrium equations to be written in the equivalent form

$$\operatorname{div} \underset{\sim}{\sigma} = 0 \quad \text{or} \quad \partial\sigma_{ij}/\partial X_j = 0 \text{ on } \mathfrak{R}_o, \tag{2.10}$$

but $\underset{\sim}{\sigma}$ is, in general, not symmetric. By (2.8), (2.9) we have

$$\underset{\sim}{\sigma}\underset{\sim}{F}^T = \underset{\sim}{F}\underset{\sim}{\sigma}^T.$$

Consider an arbitrary surface S_o in the undeformed region $\mathfrak{R}_o$ which is mapped onto a surface S in the deformed region $\mathfrak{R}$ by means of (2.1). The Cauchy traction $\underset{\sim}{t}$ is given by

$$\underset{\sim}{t} = \underset{\sim}{\tau}\,\underset{\sim}{n} \qquad \text{on } S \tag{2.11}$$

and the Piola traction $\underset{\sim}{s}$ is given by

$$\underset{\sim}{s} = \underset{\sim}{\sigma}\,\underset{\sim}{N} \qquad \text{on } S_o \tag{2.12}$$

where $\underset{\sim}{n}$ and $\underset{\sim}{N}$ denote unit normal vectors on S and S_o, respectively.

We suppose now that the body under consideration is elastic and possesses an elastic potential W, which represents the strain-energy per unit undeformed volume. If the material is in addition homogeneous, W is a function of position on $\mathfrak{R}_o$ exclusively via the components of the deformation-gradient tensor:

$$W = W(\underset{\sim}{F}). \tag{2.13}$$

The constitutive law may be written in terms of the nominal stress tensor as

$$\boldsymbol{\sigma} = \partial W/\partial \mathbf{F} \quad \text{or} \quad \sigma_{ij} = \partial W/\partial F_{ij}. \tag{2.14}$$

In view of (2.9), it follows that alternatively, the constitutive law can be written in terms of the Cauchy stress tensor as

$$\boldsymbol{\tau} = (1/J)\ (\partial W/\partial \mathbf{F})\ \mathbf{F}^T \quad \text{or} \quad \tau_{ij} = (1/J)\ F_{jk}\ \partial W/\partial F_{ik}. \tag{2.15}$$

The dependence of W upon $\mathbf{F}$ is restricted by the principle of material frame indifference according to which, the elastic energy is unaffected by a rigid body motion. Therefore, W must satisfy

$$W(\mathbf{F}) = W(\mathbf{Q}\mathbf{F}),$$

for all proper orthogonal tensors $\mathbf{Q}$ and all non-singular tensors $\mathbf{F}$. This implies that

$$W(\mathbf{F}) = W\ (\mathbf{E}), \tag{2.16}$$

where E is the Lagrangian strain tensor. Thus W is completely determined by $\mathbf{E}$. If the solid is isotropic, the strain-energy density W depends merely on the invariants of $\mathbf{E}$, or equivalently $\mathbf{C}$, whence in this instance

$$W = W(\underset{\sim}{E}) = W(I_1, I_2, I_3) = W(\lambda_1, \lambda_2, \lambda_3). \tag{2.17}$$

when I_i and λ_i were described in (2.6).

From (2.14), (2.5), (2.6), and (2.17) one finds that for an isotropic, elastic material the constitutive law for the Piola stress is

$$\underset{\sim}{\sigma} = J \underset{\sim}{F} (\alpha_1 \underset{\sim}{I} + \alpha_2 \underset{\sim}{C} + \alpha_3 \underset{\sim}{C}^{-1}) , \tag{2.18}$$

where

$$\begin{aligned} \alpha_1 &= (2/J)[\partial W/\partial I_1 + I_1 \, \partial W/\partial I_2], \\ \alpha_2 &= (-2/J)\partial W/\partial I_2, \\ \alpha_3 &= 2J \, \partial W/\partial I_3. \end{aligned} \tag{2.19}$$

On the other hand, the second of (2.9), (2.18) and (2.19) yield the corresponding constitutive law for the Cauchy stress,

$$\underset{\sim}{\tau} = \alpha_1 \underset{\sim}{B} + \alpha_2 \underset{\sim}{B}^2 + \alpha_3 \underset{\sim}{I}. \tag{2.20}$$

Next we consider a <u>pure homogeneous deformation</u> of the form

$$x_i = \lambda_i X_i \text{ (no sum)}, \tag{2.21}$$

in which the λ_i are positive constants. According to (2.2) and (2.5) the λ_i are the principal stretches of such a deformation. From (2.21) together with (2.2) and (2.5)-(2.7), (2.17)-(2.20) one confirms that the stress components associated with such a deformation are

$$\sigma_{ii} = \partial W/\partial \lambda_i \text{ (no sum)} \;, \quad \sigma_{ij} = \sigma_{ji} = 0, \tag{2.22}$$

$$\tau_{ii} = (\lambda_i/\lambda_1\lambda_2\lambda_3)\ \partial W/\partial \lambda_i \text{ (no sum)}, \qquad \tau_{ij} = \tau_{ji} = 0. \tag{2.23}$$

Even though, we considered a pure homogeneous deformation, equation (2.23) is valid for any deformation with τ_{ii} and λ_i being the principal stresses and stretches respectively.

If, for some i and j, we have $\lambda_i > \lambda_j$, then on physical grounds one would expect that $\tau_{ii} > \tau_{jj}$ (no sum). The Baker-Ericksen inequalities, (Baker and Ericksen (1954)), which require

$$(\tau_{ii} - \tau_{jj})(\lambda_i - \lambda_j) > 0 \qquad \text{if } \lambda_i \neq \lambda_j \text{ (no sum)} \tag{2.24}$$

for all pure homogeneous deformations, is a mathematical statement of this requirement.

Now suppose that the domain $\mathfrak{R}$ occupied by the undeformed body is a right cylinder with generators parallel to x_3-axis. Let D_o be the open region of the (X_1,X_2)-plane occupied by the interior of the cross-section of this cylinder. Suppose further that the deformation (2.1) is a plane deformation so that

$$x_\alpha = X_\alpha + u_\alpha(X_1,X_2), \quad x_3 = X_3 \text{ on } \mathfrak{R}. \tag{2.25}$$

For a pure homogeneous plane deformation, the principal stretch $\lambda_3 = 1$, then we have

$$W(\lambda_1,\lambda_2) = W(\lambda_1,\lambda_2,1) \tag{2.26}$$

and

$$\tau_\alpha \equiv \tau_{\alpha\alpha} = (\lambda_\alpha/\lambda_1\lambda_2)\ \partial W/\partial\lambda_\alpha \quad \text{(no sum)}. \tag{2.27}$$

In the bifurcation problem to be considered, the undeformed body is a solid cylinder and thus D_o is simply-connected. Assume the boundary of D_o is subjected to the prescribed displacement

$$\underset{\sim}{u}(\underset{\sim}{X}) = (\lambda - 1)\ \underset{\sim}{X} \quad \text{on } \partial D_o \tag{2.28}$$

where the parameter $\lambda(>1)$ is prescribed and denotes the applied stretch. The analysis of this problem necessarily involves a deformation which is not one to one, and so in order to investigate this, the preceding regularity condition must be relaxed. Thus we allow for the possibility that the mapping (2.1) is one-to-one everywhere on D_o except at a single point X_o. In this event, X_o is assumed to map onto a closed regular curve C, while the simply-connected domain D_o then maps onto a doubly-connected domain D, with C denoting its inner boundary. Thus, in this situation, equation (2.2) is required to hold merely on the domain D_o with X_o deleted, while equation (2.8) holds on the doubly-connected domain D. The variational formulation in the next chapter shows that the inner boundary C must necessarily be traction-free, and so

$$\boldsymbol{\tau}\,\mathbf{n} = \mathbf{0} \quad \text{on} \quad C, \tag{2.29}$$

where $\mathbf{n}$ denotes the unit outward normal vector on C, and $\boldsymbol{\tau}$ is the limiting value of the Cauchy stress (presumed to exist) as a point on C is approached from within D.

CHAPTER 3

FORMULATION OF THE BIFURCATION PROBLEM

Consider a homogeneous, isotropic, compressible, elastic material whose mechanical response is characterized by its elastic potential $W(\lambda_1,\lambda_2)$, in plane strain, and λ_1, λ_2 are the principal stretches of the deformation. A solid circular cylinder of such a material, with undeformed radius A and its cross-section

$$D_o = \{(R,\Theta) \mid 0 \le R < A,\ 0 < \Theta \le 2\pi\},$$

is subjected to a purely radial stretch $\lambda(>1)$ at its surface $R = A$. The resulting deformation is a mapping which takes the point $(X_1,X_2) = (R \cos \Theta, R \sin \Theta)$ to the point $(x_1,x_2) = (r \cos \theta,\ r \sin \theta)$. We assume that the deformation is axisymmetric so that

$$\theta = \Theta \quad \text{and} \quad r = r(R)\ , \quad 0 < R < A\ . \tag{3.1}$$

In order to avoid interpenetration, (2.2) (see also remarks about (2.28)) requires that

$$r'(R) > 0 \quad \text{for} \quad 0 < R < A, \tag{3.2}$$

and

$$r(0+) \ge 0\ . \tag{3.3}$$

Observe that if $r(0+) > 0$ then the deformation (2.1) is not one-to-one at the origin; a cavity of radius $r(0+)$ exists in the deformed state. The prescribed boundary condition requires that

$$r(A) = \lambda A. \tag{3.4}$$

Usually, it is required that the deformation $r(R)$ is twice continuously differentiable on $[0,A]$. For present purposes however, it is essential that one allow for the possibility that $r(R)$ may be singular at the origin. Accordingly, we merely require that

$$r(R) \in C^2((0,A]) \cup C^1([0,A]) \ , \tag{3.5}$$

so that $r''(R)$ need not exist at $R = 0$. Finally, let $\mathcal{A}$ denote the set of all kinematically admissible deformations, i.e. the set of all functions $r(R)$ which satisfy (3.2)-(3.5).

The principal stretches associated with the deformation (3.1) are

$$\lambda_1 = \lambda_R = r'(R), \quad \lambda_2 = \lambda_\theta = \ r(R)/R \ , \quad 0<R<A \ , \tag{3.6}$$

and the corresponding Cauchy stresses (2.27) are

$$\tau_r = (1/\lambda_2)\ \partial W/\partial\lambda_1 \ , \quad \tau_\theta = (1/\lambda_1)\ \partial W/\partial\lambda_2 \tag{3.7}$$

The total stored energy functional associated with any kinematically admissible deformation $\rho(R) \in \mathcal{Q}$ is given by

$$E\{\rho(R)\} = 2\pi \int_{0+}^{A} W(\rho'(R),\ \rho(R)/R\ )\ R\ dR, \tag{3.8}$$

and one seeks a function r(R) in $\mathcal{Q}$ which minimizes E. When the classical methods of the calculus of variations are followed (see Appendix 1), they lead to the Euler-Lagrange equation associated with (3.8) <u>as well as</u> a natural boundary condition at R = 0+ : these are

$$d/dR\ \{R\ \partial W/\partial\lambda_1\} - \partial W/\partial\lambda_2 = 0\ ,\ 0<R<A, \tag{3.9}$$

and

$$r(0+)\ \tau_r(r(0+)\ ) = 0\ , \tag{3.10}$$

respectively. In view of (3.3), the natural boundary condition (3.10) holds if and only if either,

$$r(0+) = 0 \tag{3.11}$$

or

$$\tau_r(r(0+)) = 0\ ,\quad r(0+) > 0\ . \tag{3.12}$$

Note that when (3.12) holds, the deformed configuration involves a cylindrical, traction-free void of radius r(0+) centered at the origin.

The <u>boundary value problem</u> to be solved consists of the nonlinear ordinary differential equation (3.9) subject to the boundary conditions

(3.4) and (3.10). The applied stretch λ (>1) is given, and the solution r(R) is to possess the degree of smoothness described previously.

It may be verified that one solution to this problem, for all values of $\lambda(>1)$, is

$$r(R) = \lambda R \ , \ 0 \leq R \leq A \ . \tag{3.13}$$

We refer to this as the "fundamental solution ": it corresponds to a homogeneous deformation in which the cylinder expands radially. Note that (3.13) satisfies the boundary conditon (3.10) by virtue of having $r(0+) = 0$.

For certain materials, and for certain ranges of values of λ, there exists, in addition a <u>second</u> solution $r(R;\lambda)$ satisfying the boundary condition (3.10) by virtue of having $r(0+;\lambda) > 0$ and $\tau_r(r(0+;\lambda)) = 0$ and so corresponds to a configuration of the body involving a cylindrical void at the origin. In general, it is not possible to determine this solution analytically, though Ball (1982), Podio-Guidugli et al (1986), Sivaloganathan (1986) have established the existence of such solutions in the case of certain compressible materials. Our purpose here is to determine these solutions $r(R;\lambda)$ explicitly, and in closed form, for a certain class of compressible materials, and to study their properties.

CHAPTER 4

A PARTICULAR CLASS OF ELASTIC MATERIALS

Various forms of constitutive equations for compressible rubberlike materials have been proposed in the literature, and new forms are still being investigated. For example, using a combination of theoretical arguments and experimental results, Blatz and Ko (1962) suggested a strain energy function of the form

$$W = (1/2)\mu\, f\, \{J_1 - 3 + ((1-2\nu)/\nu)[J_3{}^{-2\nu/(1-2\nu)} - 1]\}$$
$$+ (1/2)\mu\, (1-f)\{J_2 - 3 + ((1-2\nu)/\nu)[J_3{}^{2\nu/(1-2\nu)} - 1]\} \tag{4.1}$$

where μ, ν, f are constants and

$$J_1 = I_1,\ J_2 = I_2/I_3\ ,\ J_3 = I_3{}^{1/2}\ . \tag{4.2}$$

We note that when $\nu = 1/2$ and the material is incompressible so that $I_3 = 1$, (4.1) formally reduces to the Mooney-Rivlin form, (Mooney (1940)). In case of a certain specific polyurethane rubber, Blatz and Ko chose $f = 0$, $\nu = 1/4$ based on their experiments. In this case (4.1) reduces to

$$W = (1/2)\ \mu\ \{J_2 + 2J_3 - 5\}. \tag{4.3}$$

In term of principal stretches (2.6), in view of (4.2), equation (4.3) becomes

$$W = (1/2)\,\mu(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} + 2\lambda_1\lambda_2\lambda_3 - 5). \tag{4.4}$$

A material characterized by (4.4) is commonly referred to as a "Blatz-Ko material". A detailed discussion of this material may be found in the paper by Knowles and Sternberg (1975).

Another well-known example, is that proposed by Ogden (1972). Generalizing his earlier approach for incompressible materials, he proposed

$$W = \sum_n (\mu_n/\alpha_n)(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3) + F(\lambda_1\lambda_2\lambda_3) \tag{4.5}$$

in which the compressibility is accounted for by the additive function F of $\lambda_1\lambda_2\lambda_3$. In view of the presence of the material parameters α_n, μ_n function $F(\cdot)$, it is possible to fit the form (4.5) to experimental results by taking a sufficiently large number of terms.

The mechanical properties of two porous rubbers of different compressibility have been investigated experimentally by Storakers (1986). Storakers employed the constitutive equation proposed by Hill (1982) which is a special form of the Ogden-model. He evaluated the constitutive parameters such that his experimental data could be represented by this constitutive equation.

We propose to study a particular form of the strain energy density for a compressible homogeneous and isotropic, elastic material, which is special case of the Ogden-model but a generalization of the Blatz-Ko form:

$$W(\lambda_1,\lambda_2,\lambda_3) = (2\mu/n)\{(1/n)(\lambda_1^{-n} + \lambda_2^{-n} + \lambda_3^{-n} - 3) + \alpha(J-1) + \beta(J^{-n/2} - 1)\} \tag{4.6}$$

where

$$\alpha = n/[(2+n)(1-2\nu)], \ \beta = 4[\nu(2+n)-1]/[n(2+n)(1-2\nu)] = (2/n)(\alpha-1) \quad (4.7)$$

and μ, ν, n(>0) are material constants. By linearization of (4.6), it can be shown that μ is the shear modulus and ν is the Poisson's ratio of this material under infinitesimal deformations; n is the hardening exponent of the material. Note that for $n = 2$, $\nu = 1/4$, (4.6) reduces to (4.4).

We consider now certain properties of the material model defined by (4.6). Consider a pure homogeneous deformation of the form (2.21). According to (4.6), (2.22), (2.23) and the last of (2.6), the normal components of $\underset{\sim}{\tau}$ and $\underset{\sim}{\sigma}$ in such a deformation obey the stress-stretch relations

$$\tau_i = J^{-1}\lambda_i\sigma_i = (2\mu/n)[\alpha-\beta(n/2)J^{-((n/2)+1)}-\lambda_i^{-n}\ J^{-1}] \text{ (no sum)}, \qquad (4.8)$$

$$J = \lambda_1\lambda_2\lambda_3,$$

where

$$\tau_i = \tau_{ii}, \ \sigma_i = \sigma_{ii} \text{ (no sum)}, \qquad (4.9)$$

while the corresponding shear stresses vanish. In the undeformed state

$\underset{\sim}{x} = \underset{\sim}{X}$, $\underset{\sim}{F} = \underset{\sim}{I}$ and thus $\underset{\sim}{C} = \underset{\sim}{B} = \underset{\sim}{I}$, $I_1 = I_2 = 3$, $I_3 = J = 1$, $\lambda_i = 1$.

Consequently, W, $\underset{\sim}{\tau}$ and $\underset{\sim}{\sigma}$ vanish in the absence of deformations, as should be the case.

We examine the consequences of (4.8), as far as certain special pure homogeneous deformations are concerned.

For <u>uni-axial tension</u> parallel to the x_1-axis, $\tau_1 = \tau$, $\tau_2 = \tau_3 = 0$, and $\lambda_1 = \lambda$, so that (4.8) gives

$$\lambda_2 = \lambda_3 = \{\alpha\lambda/(1+\beta(n/2)\lambda^{-n/2})\}^{-1/(2+n)} \tag{4.10}$$
$$\tau = (2\mu/n)[\alpha/(1+\beta(n/2)\lambda^{-n/2})]$$
$$.[1-\lambda^{-n(3+n)/(2+n)}\ (\alpha/(1+\beta(n/2)\ \lambda^{-n/2}))^{-n/(2+n)}].$$

The stress-stretch relation (4.10) is plotted in Figure 1 for fixed Poisson's ratio ν and different values of the hardening exponent n while in Figure 2 it is plotted for different values of ν at fixed n. For a certain value of stretch, stress increases with decreasing value of n and increasing value of ν.

For the case of <u>isotropic extension</u> one has

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda \text{ and } \tau_1 = \tau_2 = \tau_3 = \tau$$

and from (4.8), we get

$$\tau = (2\mu/n)\{\alpha - \beta(n/2)\lambda^{-3((n/2)+1)} - \lambda^{-(n+3)}]. \tag{4.11}$$

The stress-stretch relation (4.11) is shown in Figure 3 for fixed Poisson's ratio ν and different values of the hardening exponent n and in Figure 4 for different values of ν at fixed n. For a certain value of stretch λ, stress increases with decreasing value of n and increasing value of ν.

Finally, we consider a homogeneous plane deformation corresponding to a state of <u>simple shear</u>, parallel to the plane $X_3 = 0$. Thus let

$$x_1 = X_1 + \kappa X_2, \ x_2 = X_2, \ x_3 = X_3 \ , \tag{4.12}$$

in which κ is a constant, $\tan^{-1}\kappa$ being the angle of shear. The invariants (2.6) are $I_1 = 3 + \kappa^2$, $I_2 = 3 + \kappa^2$, $I_3 = 1$. Substituting this invariants into (2.6), we obtain the following relations for the principal stretches

$$\lambda_1{}^2 + 1/\lambda_1{}^2 = 2 + \kappa^2 \ , \quad \lambda_2 = 1/\lambda_1, \quad \lambda_3 = 1.$$

The principal stretch λ_1 of this deformation is related to κ through

$$\lambda_1 = \{1+(\kappa^2/2) + \kappa[1+(\kappa^2/4)]^{1/2}\}^{1/2} \tag{4.13}$$

With the aid of (2.20), (4.6), the response of the material in simple shear is found to obey

$$\tau_{12} = \tau_{21} = (\mu/n)[1+(\kappa^2/4)]^{-1/2}\{[1+(\kappa^2/2) - \kappa(1+(\kappa^2/4))^{1/2}]^{-n/2}$$
$$- [1+(\kappa^2/2) + \kappa(1+(\kappa^2/4))^{1/2}]^{-n/2}\} \quad (4.14)$$

$\tau_{3i} = \tau_{i3} = 0$ and τ_{11} , τ_{22} can be calculated.

Note that for $n = 2$, (4.14) reduces to $\tau_{12} = \mu\kappa$. Of course, no matter what the value of n is, for small κ, it reduces to $\tau_{12} = \mu\kappa$ upon linearization. The shear stress-shear strain variation according to equation (4.14) is plotted in Figure 5, τ_{12}/μ vs. κ for different values of the hardening exponent n. For a certain value of shear strain κ, shear stress increases with increasing hardening exponent n.

It can be shown from (4.8), (4.9) that the Baker-Ericksen inequalities (2.24) are satisfied for all pure homogeneous deformations of the particular material under consideration (see Appendix 2 for details).

CHAPTER 5

SOLUTION

Recall that we seek for a <u>second</u> solution $r(R;\lambda)$ of the boundary value problem consisting of the nonlinear differential equation (3.9), the prescribed boundary condition (3.4) and the natural boundary condition (3.12). The differential equation (3.9) may be written in the form

$$R(\partial^2 W/\partial\lambda_1{}^2)r'' + \partial W/\partial\lambda_1 - \partial W/\partial\lambda_2 + (\partial^2 W/\partial\lambda_1\partial\lambda_2)(\lambda_1 - \lambda_2) = 0 \quad (5.1)$$

where $\lambda_1 = r'(R)$, $\lambda_2 = r(R)/R$ are the principal stretches. The order of this second-order differential equation may be reduced by introducing a function $t(R)$ defined by

$$t(R) = \lambda_1/\lambda_2 = R\, r'(R)/r(R)\ (>0), \quad 0 < R \leq A. \quad (5.2)$$

On using (5.2), we may reduce (5.1) to the following first order nonlinear differential equation for $t(R)$:

$$Rt' = (\lambda_2\, \partial^2 W/\partial\lambda_1{}^2)^{-1}\{\partial W/\partial\lambda_2 - \partial W/\partial\lambda_1 + (1-t)\lambda_2\, \partial^2 W/\partial\lambda_1\partial\lambda_2\}$$
$$+ (1-t)t, \quad 0 < R < A. \quad (5.3)$$

In the particular case of the elastic potential (4.6), the right-hand-side of (5.3) may be expressed solely as a function of t,

$$\left.\begin{aligned} &Rt'(R) = F(t(R)), \qquad 0<R<A, \\ &\text{where} \\ &F(t) = [\beta n(n+2)(1-t)t - 4(1 - t^{-(n+1)})t^{(2+(n/2))}] \\ &\qquad .[\beta n(n+2) + 4(n+1)\, t^{-n/2}]^{-1} + (1-t)t, \qquad 0<t<1. \end{aligned}\right\} \quad (5.4)$$

There are two cases to consider. First of all, we observe that $t(R) = 1$ on $0<R<A$ is a solution of (5.4). Hence (5.2) shows that $r(R) = cR$, where c is a constant. In this way, one recovers the homogeneous solution (3.13). Suppose then that $t \neq 1$. It can be shown that without loss of generality, $t(R)$ may be assumed to be less than unity on $0<R<A$ so that from (5.2) and (5.4) it follows that

$$0 < t < 1 , \qquad dt/dR > 0 \qquad \text{for} \quad 0 < R < A . \tag{5.5}$$

The equation (5.4) may now be integrated to get an expression for the undeformed radius R; thus

$$R = A \exp \left(- \int_t^{t_A} [\, 1/F(\xi)]\, d\, \xi \right) , \tag{5.6}$$

where A is the undeformed radius of the circular cylinder and $t_A = t(A)$ is the value of t corresponding to $R = A$. On the other hand (5.2) and (5.4) give

$$(1/r)\, dr/dt = t/F(t) \quad (>0) . \tag{5.7}$$

Integration of (5.7) and using the boundary condition (3.4) yields a corresponding expression for the deformed radius r:

$$r = \lambda A \exp\left(- \int_{t}^{t_A} [\xi/F(\xi)]\, d\xi \right) . \tag{5.8}$$

Observe from (5.5) and (5.7) that the undeformed and deformed radial coordinates (R and r) vary monotonously with t. Therefore, equations (5.6) and (5.8) provide a parametric solution to the differential equation (5.1). The range of the parameter t is

$$0 \le t \le t_A, \tag{5.9}$$

where t_A $(0<t_A<1)$ is yet to be determined.

Now turning to the remaining boundary condition (3.12), first we write the principal stretch λ_θ, by using (5.6) and (5.8), as

$$\lambda_\theta = r/R = \lambda \exp\left(\int_{t}^{t_A} [(1 - \xi)/ F(\xi)] d\xi \right) . \tag{5.10}$$

Then, the principal stretch λ_R, using (5.2), is given by

$$\lambda_R = t\lambda_\theta = t\lambda \exp\left(\int_{t}^{t_A} [(1-\xi)/ F(\xi)]\, d\xi \right) . \tag{5.11}$$

Thus on utilizing (5.2) and $J = t\lambda_\theta^2$, from (4.8) we get an expression for the radial true stress τ_{rr}:

$$\tau_{rr} = (2\mu/n)\{\alpha - \lambda_\theta^{-(2+n)} t^{-(1+n)} (1+\beta(n/2)t^{n/2})\}. \tag{5.12}$$

Finally, in order to satisfy the traction-free boundary condition (3.12) on the cavity surface, we must have $\tau_{rr} \to 0$ as $t \to 0$ or, from (5.12),

$$\lambda_\theta \sim \alpha^{-1/(2+n)} t^{-(1+n)/(2+n)} \quad \text{as} \quad t \to 0 . \tag{5.13}$$

However, the asymptotic behavior of λ_θ as $t \to 0$ may also be determined directly from (5.10). Equating the resulting expression to (5.13) yields

$$\lim_{t \to 0} t^{(1+n)/(2+n)} \lambda \exp \{ \int_t^{t_A} [(1-\xi)/F(\xi)]d\xi \} = \alpha^{-1/(2+n)}. \tag{5.14}$$

In order to evaluate (5.14) it is convenient to introduce the following function f:

$$f(\xi) = (1-\xi)/F(\xi) - (1+n)(2+n)^{-1}(1/\xi) . \tag{5.15}$$

Then, the substitution of (5.15) into (5.14) gives

$$\lim_{t \to 0} t^{(1+n)/(2+n)}\lambda \exp \{ \int_t^{t_A} [(1+n)(2+n)^{-1}(1/\xi)+f(\xi)] \, d\xi = \alpha^{-1/(2+n)}. \tag{5.16}$$

Integrating the first part of (5.16) leads to

$$\lim_{t\to 0} \quad \lambda\, t_A^{(1+n)/(2+n)} \quad \exp\left(\int_t^{t_A} f(\xi)d\xi \right) = \alpha^{-1/(2+n)}, \tag{5.17}$$

which in turn yields,

$$\lambda = \alpha^{-1/(2+n)}\, t_A^{-(1+n)/(2+n)} \exp\left(- \int_0^{t_A} f(\xi)d\xi \right) . \tag{5.18}$$

Thus, _if_ for a prescribed value of the applied stretch $\lambda > 1$, (5.18) can be solved for a number t_A in the range $0 < t_A < 1$, _then_ (5.6),(5.8) with $0 \le t \le t_A$ is a _solution to the boundary-value problem_ at hand. This solution involves an internal cavity.

To verify that (5.18) can indeed be solved for an appropriate value of t_A, we simply observe that the auxiliary function G(t) defined by

$$G(t) = \alpha^{-1/(2+n)}\, t^{-(1+n)/(2+n)} \exp\left(- \int_0^{t} f(\xi)d\xi \right)$$

$$\text{for } 0 \le t \le 1 \quad , \tag{5.19}$$

associated with the right hand side of (5.18), can be shown to be monotonously decreasing (see Appendix 3) and

$$G(0) = \infty,\ G(1) = \alpha^{-1/(2+n)} \exp\left(- \int_0^1 f(\xi)d\xi \right) > 0. \tag{5.20}$$

Therefore, (5.18) <u>can</u> be solved, in fact, uniquely, for a root t_A in the range $0 < t_A < 1$, provided that $\lambda > G(1)$, i.e.

$$\lambda > \lambda_{cr} = \alpha^{-1/(2+n)} \exp\left(- \int_0^1 f(\xi)\, d\xi \right) . \tag{5.21}$$

<u>Thus whenever the prescribed stretch</u> λ <u>is greater than</u> λ_{cr}, <u>the existence of a bifurcated solution involving a cavity is guaranteed and this solution is given by</u> (5.6),(5.8),(5.9),(5.18).

CHAPTER 6

DISCUSSION

For all values of the applied stretch $\lambda>1$, one solution to the problem at hand is $r(R)=\lambda R$, which describes a uniform enlargement of the cylinder. For $\lambda>\lambda_{cr}$, we have in addition, a second solution $r(R;\lambda)$ describing a deformation involving an internal cavity. In this section we discuss the properties of this second bifurcated solution. However, before doing this, since for $\lambda>\lambda_{cr}$ we have two solutions, we first examine their relative stability.

6.1 Comparison of energies

Since for values of $\lambda > \lambda_{cr}$ we obtain two solutions, it is natural to compare their energies (at the same value of prescribed stretch).

Let $W(\lambda_1, \lambda_2)$ denote the plane strain elastic potential of a general isotropic compressible elastic solid. The cylinder is subjected to a radial deformation so the principal stretches are given by (3.6). Note that

$$d\lambda_\theta/dR = (1/R)(\lambda_R-\lambda_\theta) \quad . \tag{6.1}$$

For our purposes, by using (6.1), we may write the equilibrium equation (3.9) in the equivalent form

$$d/dR \{ R^2 [W - (\lambda_R - \lambda_\theta) \ \partial W/\partial\lambda_1] \} - 2 R W = 0,$$
$$\text{on } 0 < R < A. \qquad (6.2)$$

The equivalence of (6.2) and (3.9) may be verified by direct expansion. Equation (6.2) is in fact a specialization to radial solutions of a general conservation law given by Green (1973).

Integration of (6.2) over the interval $\epsilon < R < A$ gives

$$\int_\epsilon^A 2\pi R \ W \ dR = [\pi R^2 (W - (\lambda_R - \lambda_\theta) \ \partial W/\partial\lambda_1)]_{R=\epsilon}^{R=A} . \qquad (6.3)$$

Here ϵ is any number in the range $0 < \epsilon < A$. The lower limit of the term on the right hand side of (6.3) vanishes as $\epsilon \to 0+$, and the left-hand-side becomes the total elastic energy of the cylinder, E. Thus we get

$$E = \pi A^2 [W(r'(A),\lambda) - (r'(A)-\lambda)\partial W(r'(A),\lambda)/\partial\lambda_1] \qquad (6.4)$$

where $\lambda = r(A)/A$ is the prescribed stretch at the outer boundary. For the homogeneous solution we have $r'(A) = \lambda$, while for the bifurcated solution we have $r'(A) = \lambda t_A$. Thus (6.4) gives

$$E_h = \pi A^2 W(\lambda,\lambda) \qquad (6.5)$$

and

$$E_b = \pi A^2 [W(\lambda t_A,\lambda) - \lambda(t_A - 1) \ \partial W(\lambda t_A,\lambda)/\partial\lambda_1], \qquad (6.6)$$

where E_h and E_b denote the total elastic energies associated with the homogeneous solution and the bifurcated solution, respectively.

The difference between these energies is

$$E_b - E_h = \pi A^2 [W(\lambda t_A, \lambda) - W(\lambda, \lambda) - \lambda(t_A - 1)\, \partial W(\lambda t_A, \lambda)/\partial\lambda_1] \quad (6.7)$$

which for the particular material (4.6) at hand specializes to

$$\begin{aligned} E_b - E_h = - \pi A^2 (2\mu/n^2)\lambda^{-n}\, t_A^{-(1+n)}\, \{\alpha\, t_A^{n/2}[(2t_A^{1+(n/2)} + n) \\ - (2+n)t_A] + (1 - t_A^{n/2})\,[n(1-t_A) - t_A(1-t_A^{n/2})]\}. \end{aligned} \quad (6.8)$$

Since $0<t_A<1$, it follows from a detailed but straightforward calculation (see Appendix 4) that

$$E_b < E_h. \quad (6.9)$$

Consequently, the energy of the bifurcated solution (whenever it exists) is strictly less than that of the homogeneous solution corresponding to the same value of λ. Thus, one does expect a cavity to appear when the applied stretch λ exceeds the critical value λ_{cr}.

6.2 Results

For $\lambda > \lambda_{cr}$ we have obtained a stable, bifurcated solution involving an internal cylindrical hole. This is given by (5.6), (5.8), (5.9), (5.18). The radius of the cavity in the deformed configuration, b=r(0+), is given by (5.8) as

$$b/A = \lambda \exp \{ -\int_0^{t_A} (\xi/F(\xi))d\xi \}, \qquad (6.10)$$

where t_A is obtained by solving (5.18). It may be confirmed from (6.10) that b increases with decreasing t_A. Furthermore it follows from (5.18) and the monotonous decreasing character of G(t) in (5.19) that t_A decreases with increasing λ. Thus the cavity radius b increases monotonously as the prescribed stretch λ is increased. Note that as $\lambda \to \infty$ (and so from (5.18), as $t_A \to 0$) it follows from (6.10) that $b \to \infty$. The variation of the cavity radius with prescribed stretch as described by (5.18), (6.10) is shown in Figures 6 and 7; Figure 6 shows this for fixed Poisson's ratio and different values of the hardening exponent, while Figure 7 displays this for different values of Poisson's ratio.

We turn next to examine the critical value of stretch λ_{cr} at void initiation. This is given by (5.21). Graphs of λ_{cr} versus the hardening exponent n, and λ_{cr} versus Poisson's ratio ν are drawn in Figures 8 and 9 respectively. We note that the value of λ_{cr} decreases with increasing n, and also with increasing ν. Although $\lambda_{cr} \to 1$ when $\nu \to 1/2$, the critical stress increases as ν increases (as we will see next) since the material tends to become incompressible in this limit.

We next consider the radial stress τ_{rr} at the outer boundary. This stress, $\tau = \tau_{rr}$ at R = A, may be calculated from (5.12), (5.10) and (5.18):

$$\tau = (2\mu/n)\,\alpha\{1 - (1 + \beta(n/2)\tau_A{}^{n/2})\,\exp[(2+n)\int_0^{t_A} f(\xi)d\xi]\}. \quad (6.11)$$

The variation of the radial stress at the outer boundary boundary τ with the prescribed stretch λ is plotted in Figures 10 and 11. First τ increases with increasing prescribed stretch λ, (during this stage the hole has not as yet developed,) then, after the void initiates at $\lambda = \lambda_{cr}$, τ decreases with increasing λ ($> \lambda_{cr}$).

The critical value of radial stress τ_{cr} (τ for $\lambda = \lambda_{cr}$) decreases with increasing hardening exponent n, and also with decreasing Poisson's ratio ν; this is shown in Figures 12 and 13 respectively.

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APPENDIX 1

This is a derivation of the equilibrium equation (3.9) and the natural boundary condition (3.10).

Let $\mathcal{Q}$ be the set of kinematically admissible functions defined in Chapter 3, and for any $\rho \in \mathcal{Q}$, set

$$f(R,\rho,\rho') = R\ W(\rho',\rho/R) \ ; \tag{A1.1}$$

then the energy functional to be minimized becomes

$$E(\rho) = 2\pi \int_{0+}^{A} f(R,\rho,\rho')\ dR. \tag{A1.2}$$

Suppose that r(R) is the minimizing function, and choose any continuously differentiable function $\eta(R)$ which vanishes at $R = A$. Then for any constant ϵ the function $\rho(R;\epsilon) = r(R) + \epsilon\ \eta(R)$ is admissible and (A1.2) leads to

$$E(\epsilon) = E(r+\epsilon\eta) = 2\pi \int_{0+}^{A} f(R,\ r+\epsilon\eta,\ r'+\epsilon\eta')\ dR. \tag{A1.3}$$

Since $E(r(R)) \leq E(\rho(R))$, it follows that $E(\epsilon)$ takes on its minimum value when $\epsilon = 0$. But this is possible only if

$$dE(\epsilon)/d\epsilon = 0 \quad \text{when } \epsilon = 0. \tag{A1.4}$$

If we denote the integrand in (A1.3) by f,

$$f = f(R,\ r+\epsilon\eta,\ r'+\epsilon\eta'), \tag{A1.5}$$

and notice that

$$df/d\epsilon = \partial f/\partial\rho \quad \eta + \partial f/\partial\rho' \ \eta' \tag{A1.6}$$

we obtain from (A1.3) the result

$$dE(\epsilon)/d\epsilon = 2\pi \int_{0+}^{A} (\partial f/\partial\rho \ \eta + \partial f/\partial\rho' \quad d\eta/dR)\ dR \tag{A1.7}$$

by differentiating under the integral sign. Finally, when $\epsilon \to 0$, the necessary condition (A1.4) takes the form

$$\int_{0+}^{A} (\partial f/\partial r \ \eta + \partial f/\partial r' \quad d\eta/dR)\ dR = 0. \tag{A1.8}$$

where the derivatives of f are now evaluated at the function r(R). Next, integrating the second term in (A1.8) by parts, we get

$$\int_{0+}^{A} [\partial f/\partial r \ \eta - d(\partial f/\partial r')/dR \ \eta]\ dR \ + [\partial f/\partial r' \ \eta(R)]_{0+}^{A} = 0. \tag{A1.9}$$

The left-hand member of (A1.9) must vanish for all permissible variations $\eta(R)$. Thus it must vanish, in particular, for all variations which are zero at both ends. For all such η's the integrated term is zero. Therefore, (A1.9) becomes

$$\int_{0+}^{A} [\partial f/\partial r - d/dR\ (\partial f/\partial r')]\ \eta\ dR = 0. \tag{A1.10}$$

Since $\eta(R)$ is arbitrary, we conclude that

$$\partial f/\partial r - d/dR\ \ (\partial f/\partial r') = 0 \quad \text{over} \quad (0,A). \tag{A1.11}$$

This gives the equilibrium equation (3.9).

It now follows that the second term must itself vanish for all permissible η's. Since r(A) is specified, $\eta(A)$ vanishes. But r(0+) is not preassigned, therefore $\eta(0+)$ is completely arbitrary and we conclude that its coefficient must vanish, yielding the natural boundary condition

$$\partial f/\partial r' = R\ \partial W/\partial \lambda_1 = R\ \lambda_2 \tau_r = r\ \tau_r = 0,$$

or,

$$r(0+)\ \tau_r(r(0+)) = 0. \tag{A1.12}$$

APPENDIX 2

In this appendix we verify that the particular constitutive law (4.6) satisfies the Baker-Ericksen inequalities. According to these inequalities we must have

$$(\tau_i - \tau_j)(\lambda_i - \lambda_j) > 0, \qquad (\lambda_i \neq \lambda_j), \qquad \text{(A2.1)}^*$$

for all pure homogeneous deformations. For the material (4.6), this condition is equivalent to

$$(2\mu/n)J^{-1}(\lambda_j{}^{-n} - \lambda_i{}^{-n})(\lambda_i - \lambda_j) > 0, \qquad (\lambda_i \neq \lambda_j). \qquad \text{(A2.2)}^*$$

by (4.8). Since $\mu>0$, $n>0$ and $J>0$ it is seen that (A2.2) is equivalent to

$$(\lambda_i{}^{-n} - \lambda_j{}^{-n})(\lambda_i - \lambda_j) < 0, \qquad (\lambda_i \neq \lambda_j). \qquad \text{(A2.3)}^*$$

Since $n>0$, (A2.3) holds.

* No sum on i or j.

APPENDIX 3

In Chapter 5 we claimed that $G(t)$ was a monotonously decreasing function of t for $0 \leq t \leq 1$, and also that $G(0)=\infty$, $G(1)>0$. To show this we first recall equation (5.19)

$$G(t) = \alpha^{-1/(2+n)} \quad t^{-(1+n)/(2+n)} \exp \{ - \int_0^t f(\xi)\, d\xi \}, \tag{A3.1}$$

where $f(\xi)$ is given by (5.15). From this,

$$G(0) = \lim_{\epsilon \to 0} G(\epsilon)$$

$$= \alpha^{-1/(2+n)} \epsilon^{-(1+n)/(2+n)} \exp \{- \int_0^\epsilon f(\xi) d\xi \} = \infty . \tag{A3.2}$$

Also,

$$G(1) = \alpha^{-1/(2+n)} \exp \{- \int_0^1 f(\xi) d\xi \} > 0 . \tag{A3.3}$$

Next, to show the monotonousness of $G(t)$, we examine its derivative:

$$G'(t) = -\alpha^{-1/(2+n)} \quad t^{-(1+n)/(2+n)} [t^{-1} + f(t)].$$

$$.\exp \{ - \int_0^t f(\xi) d\xi \}. \tag{A3.4}$$

It can be verified that

$$G'(0) = -\infty \quad , \qquad G'(1) < 0 \quad . \tag{A3.5}$$

Thus, if we can show that $t^{-1} + f(t) > 0$ for $0<t<1$, then $G'(t)<0$ for $0<t<1$. Now we look at the numerator N and the denominator D of $[t^{-1} + f(t)]$ separately. From (5.15) and (5.4),

$$N = -4(2+n)^2 + 4t^{1+n} + 4(1+n)(3+n)t + \gamma(4+n)(1-t)t^{n/2}, \tag{A3.6}$$

where $\gamma = -n(2+n)\beta$ and the maximum value of γ is 4. Replace γ with $\gamma_{max} = 4$; we get

$$N \le 4 \{-(2+n)^2 + t^{1+n} + (1+n)(3+n)t + (4+n)(1-t)t^{n/2}\}. \tag{A3.7}$$

By regrouping the terms, we get

$$N \le -4\{(3+n)(1-t)(1-t^{n/2}) + (1-t^{n/2}) + n(3+n)(1-t) + t^{1+n/2}(1-t^{n/2})\}. \tag{A3.8}$$

Since $n>0$, we can see from (A3.8) and (A3.7) that

$$N \le 0 \quad \text{for} \quad 0<t<1 \; . \tag{A3.9}$$

Now we examine the denominator D:

$$D = t(2+n)[-4(2+n) + 4\, t^{1+n} + 4(1+n)t + 2\gamma(1-t)t^{n/2}]. \tag{A3.10}$$

By replacing γ with 4 in (A3.10), we get

$$D \le 4t(2+n)[-(2+n) + t^{1+n} + (1+n)t + 2(1-t)t^{n/2}]. \tag{A3.11}$$

By regrouping (A3.11), we get

$$D \le -4t(2+n)[n(1-t) + (2-t)(1-t^{n/2} + 4t^{1+(n/2)}(1-t^{n/2})]. \tag{A3.12}$$

Since $n>0$, we observe that $D\le 0$ for $0<t<1$. Therefore, $[t^{-1} + f(t)] = N/D > 0$ for $0<t<1$. Thus from (A3.4),

$$G'(t) < 0 \quad \text{for} \quad 0<t<1 \quad . \tag{A3.13}$$

APPENDIX 4

The inequality (6.9) of Chapter 6 will be verified here, i.e. we will show that $E_b - E_h < 0$. Equation (6.8) can be written as

$$E_b - E_h = - K\,[L + M], \tag{A4.1}$$

where

$$K = \pi a^2 (2\mu/n^2) \lambda^{-n} t_A^{-(1+n)} , \tag{A4.2}$$

$$L = \alpha t_A^{n/2}\, g(t_A) \quad , \quad g(t_A) = (2t_A^{1+(n/2)} + n) - (2+n) t_A, \tag{A4.3}$$

$$M = (1-t_A^{n/2})\, h(t_A) \ , \quad h(t_A) = n(1-t_A) - t_A(1-t_A^{n/2}). \tag{A4.4}$$

Clearly K is positive. We will show that L and M, or equivalently $g(t_A)$ and $h(t_A)$, are also positive. Recall that $0 < t_A < 1$. We observe that

$$g(0) = h(0) = n \ (>0) \ , \quad g(1) = h(1) = 0. \tag{A4.5}$$

The derivatives of $g(t_A)$ and $h(t_A)$ are

$$dg(t_A)/dt_A = -(2+n)(1-t_A^{n/2}) \quad < 0, \tag{A4.6}$$

$$dh(t_A)/dt_A = -[n\{1-(1/2)t_A\} + (1-t_A^{n/2})] < 0, \tag{A4.7}$$

where we have used $0<t_A<1$ in establishing the inequalities here. Hence g and h are monotonously decreasing function of t_A and so by (A4.5), $g(t_A)$ and $h(t_A)$ are positive. Since K, L and M are positive, it follows that $E_b - E_h < 0$.

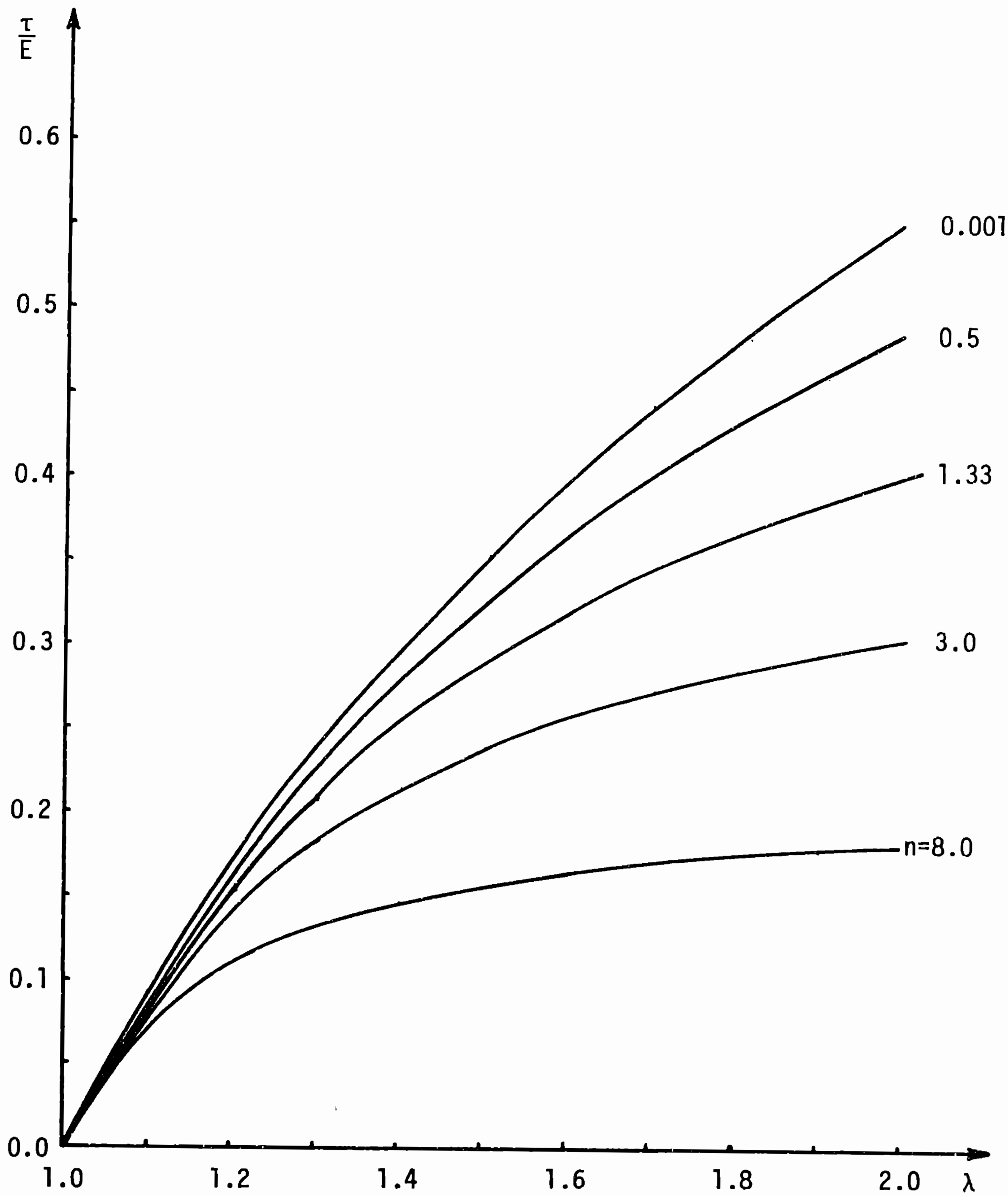

Fig. 1. Stress-stretch curves in uniaxial tension for different values of hardening exponent n, Poisson's ratio $\nu = 0.3$.

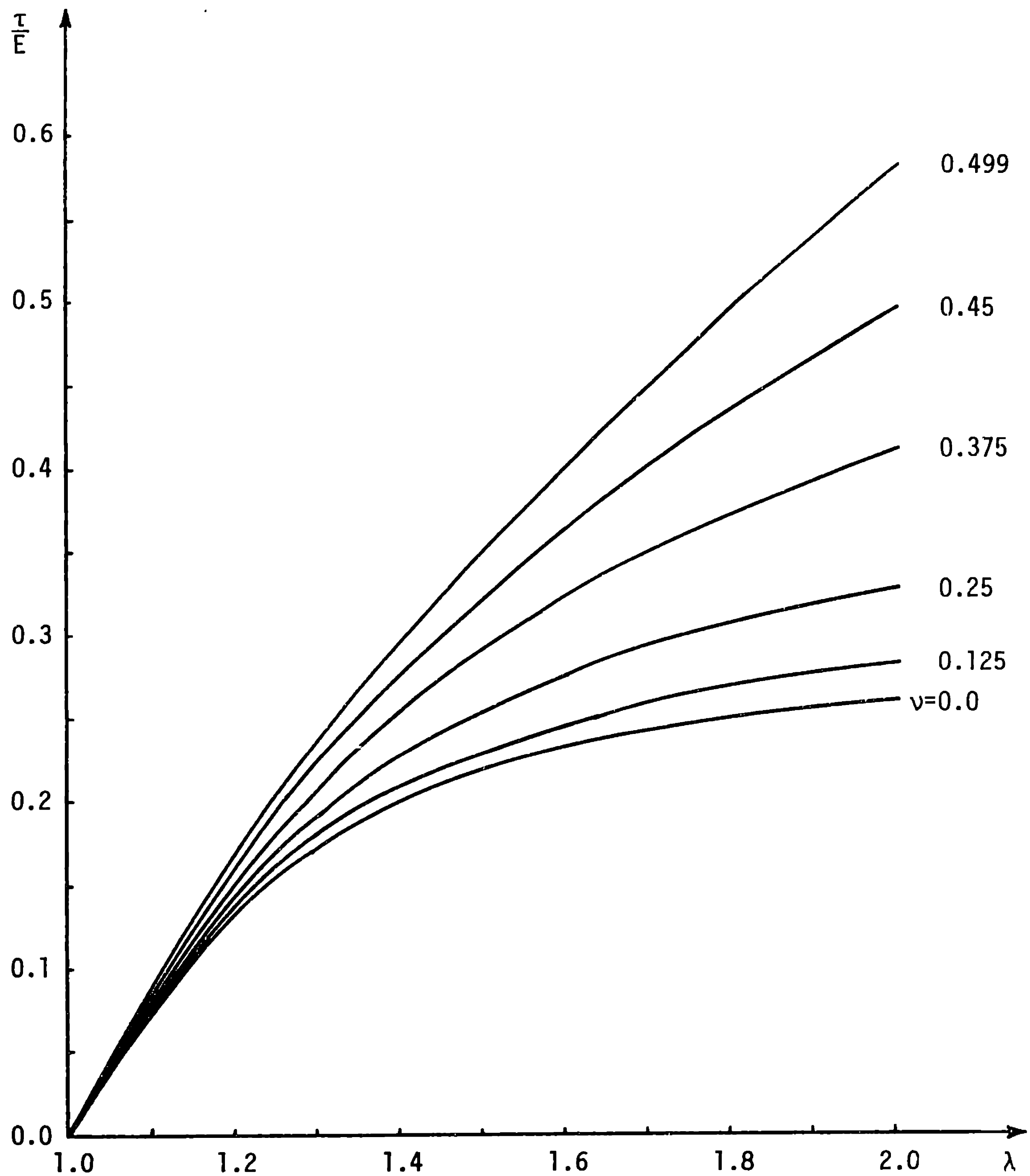

Fig. 2. Stress-stretch curves in uniaxial tension for different values of Poisson's ratio ν, hardening exponent $n = 2$.

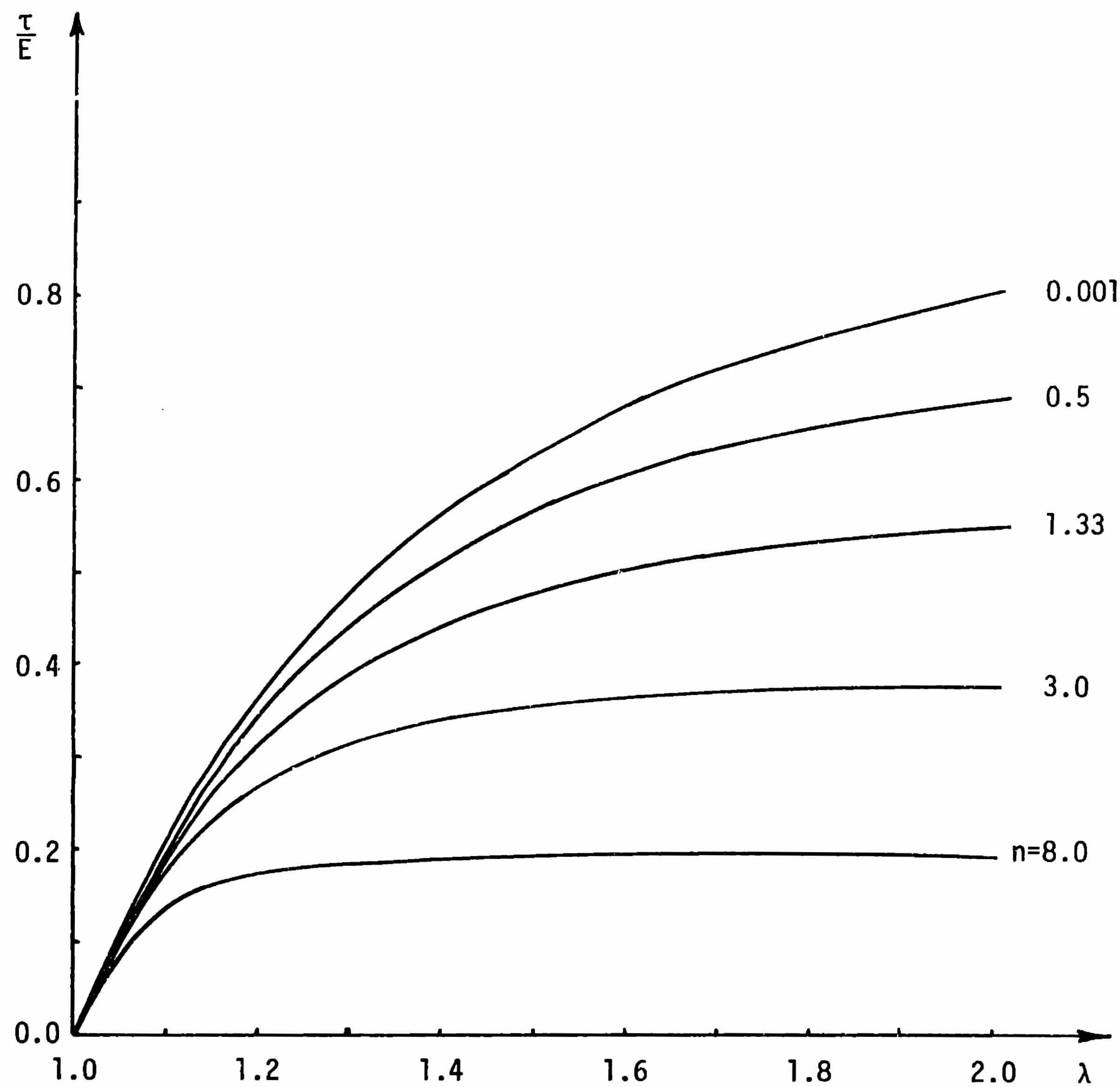

Fig. 3. Stress-stretch curves in isotropic extension for different values of hardening exponent n, Poisson's ratio $\nu = 0.3$.

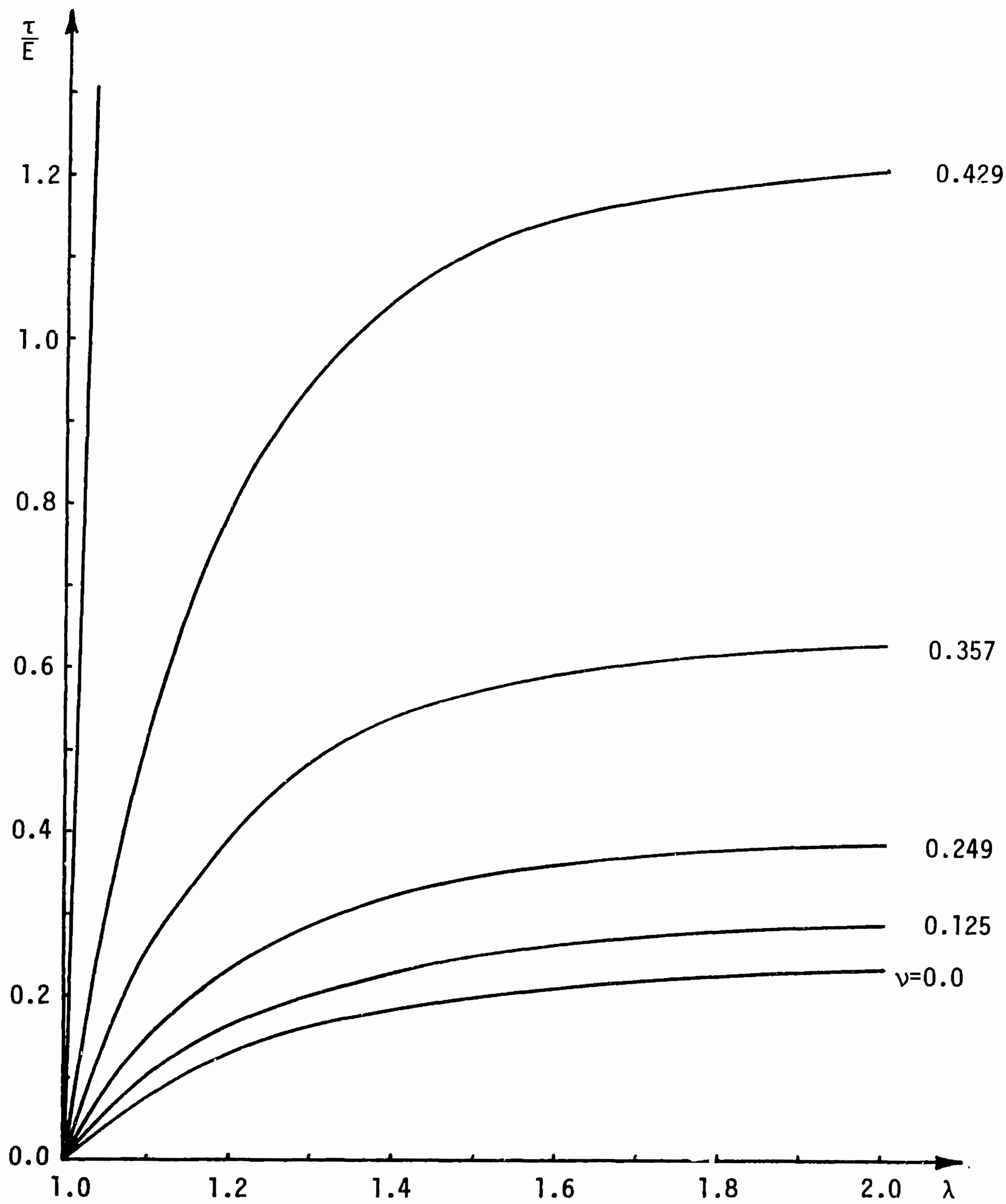

Fig. 4. Stress-stretch curves in isotropic extension for different values of Poisson's ratio ν, hardening exponent $n = 2$.

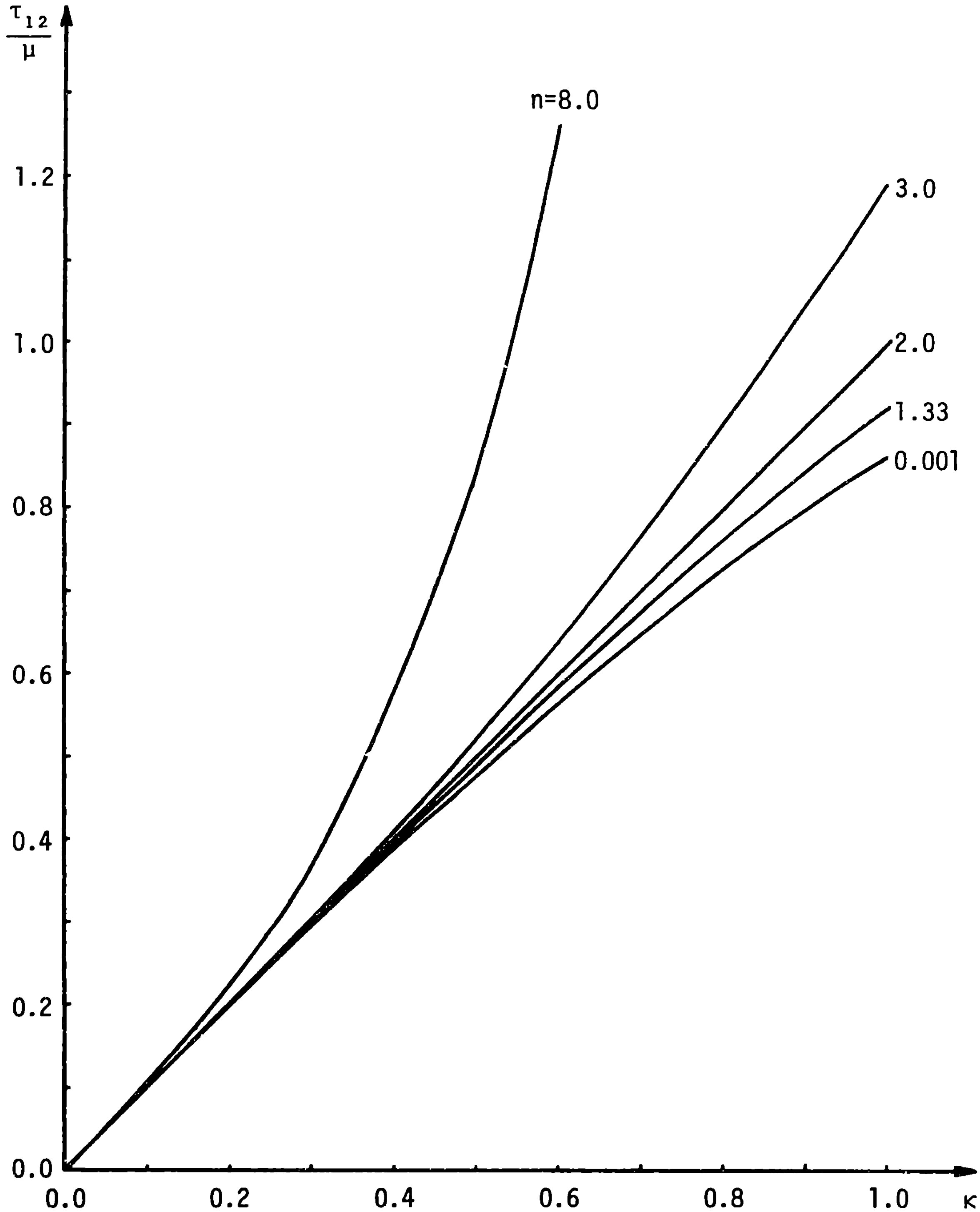

Fig. 5. Stress-strain curves in simple shear for different values of hardening exponent n .

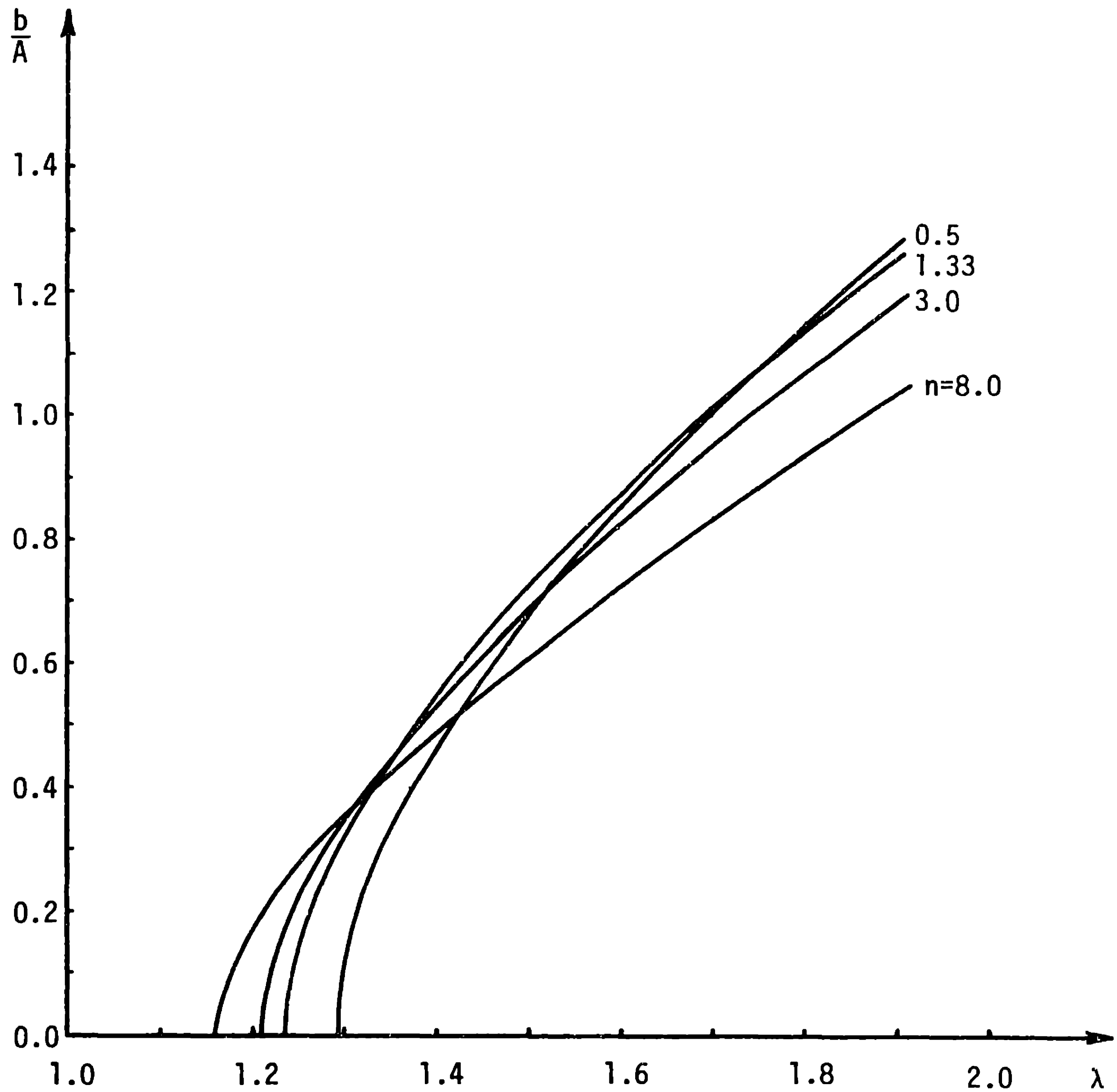

Fig. 6. Variation of the deformed cavity radius b with prescribed stretch λ for different values of hardening exponent n, Poisson's ratio ν = 0.3 .

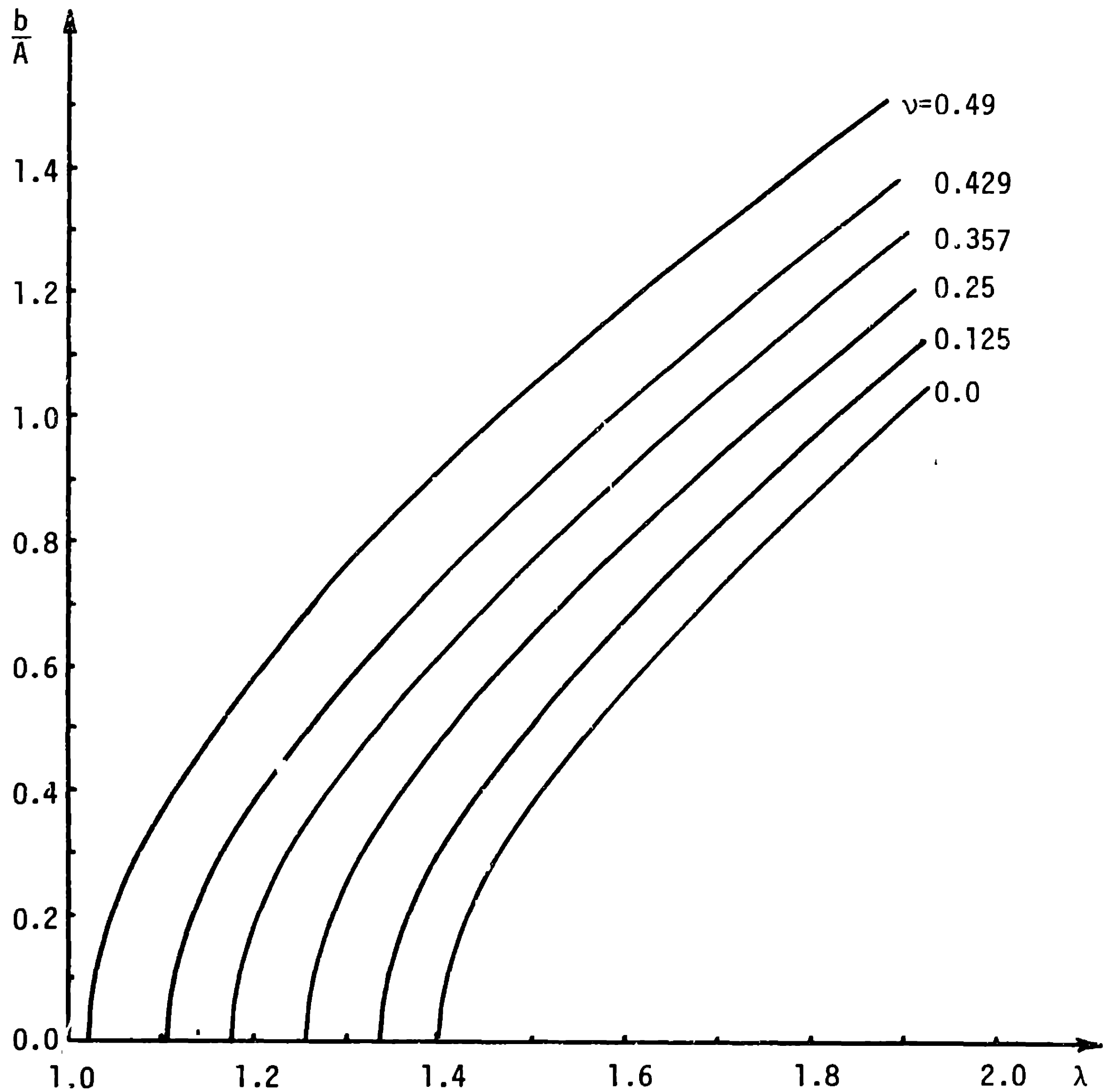

Fig. 7. Variation of the deformed cavity radius b with prescribed stretch λ for different values of Poisson's ratio ν, hardening exponent $n = 2$.

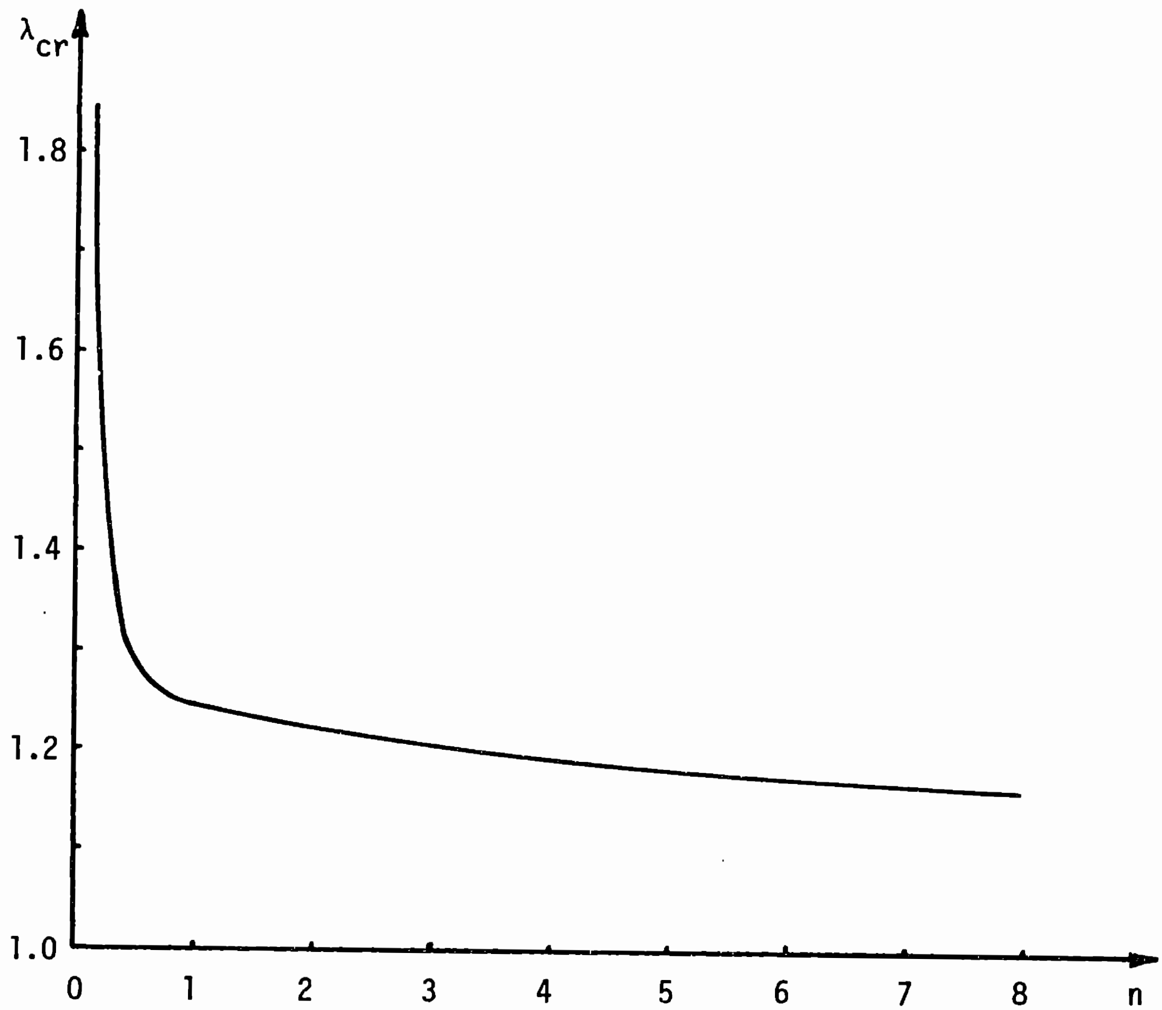

Fig. 8. Variation of critical stretch λ_{cr} with the hardening exponent n, Poisson's ratio $\nu = 0.3$.

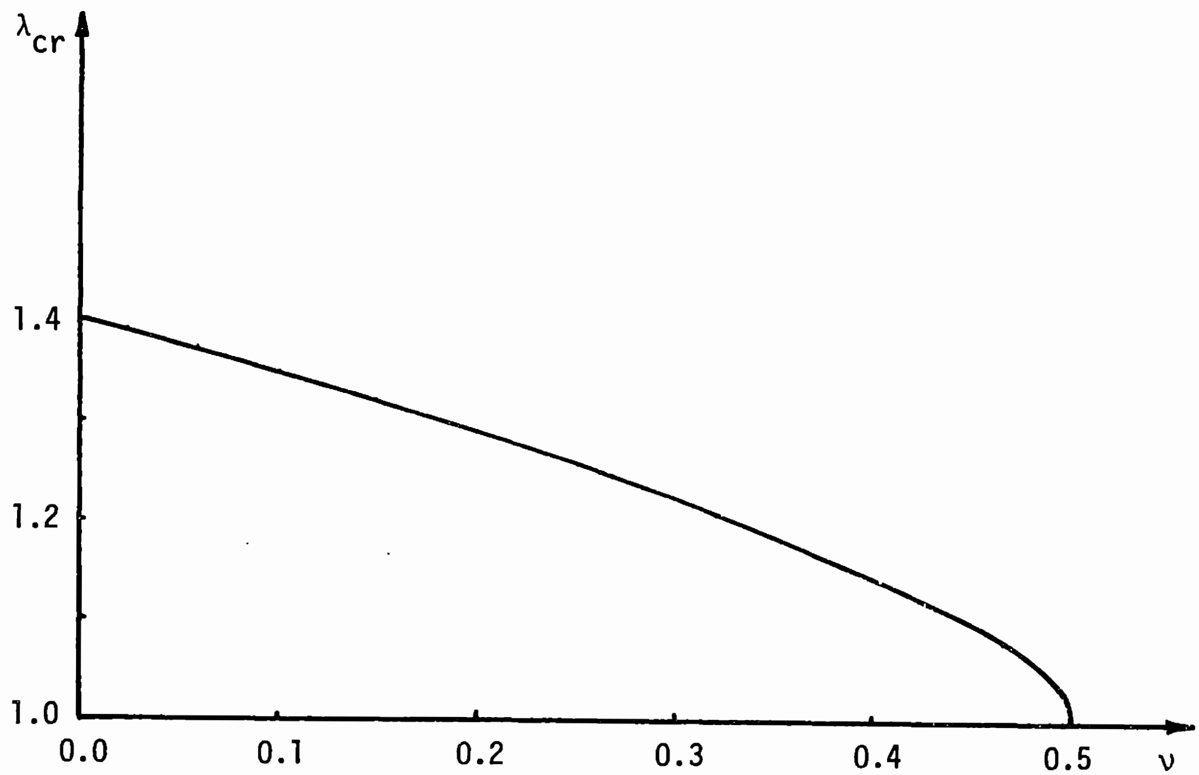

Fig. 9. Variation of critical stretch λ_{cr} with the Poisson's ratio ν, hardening exponent $n = 2$.

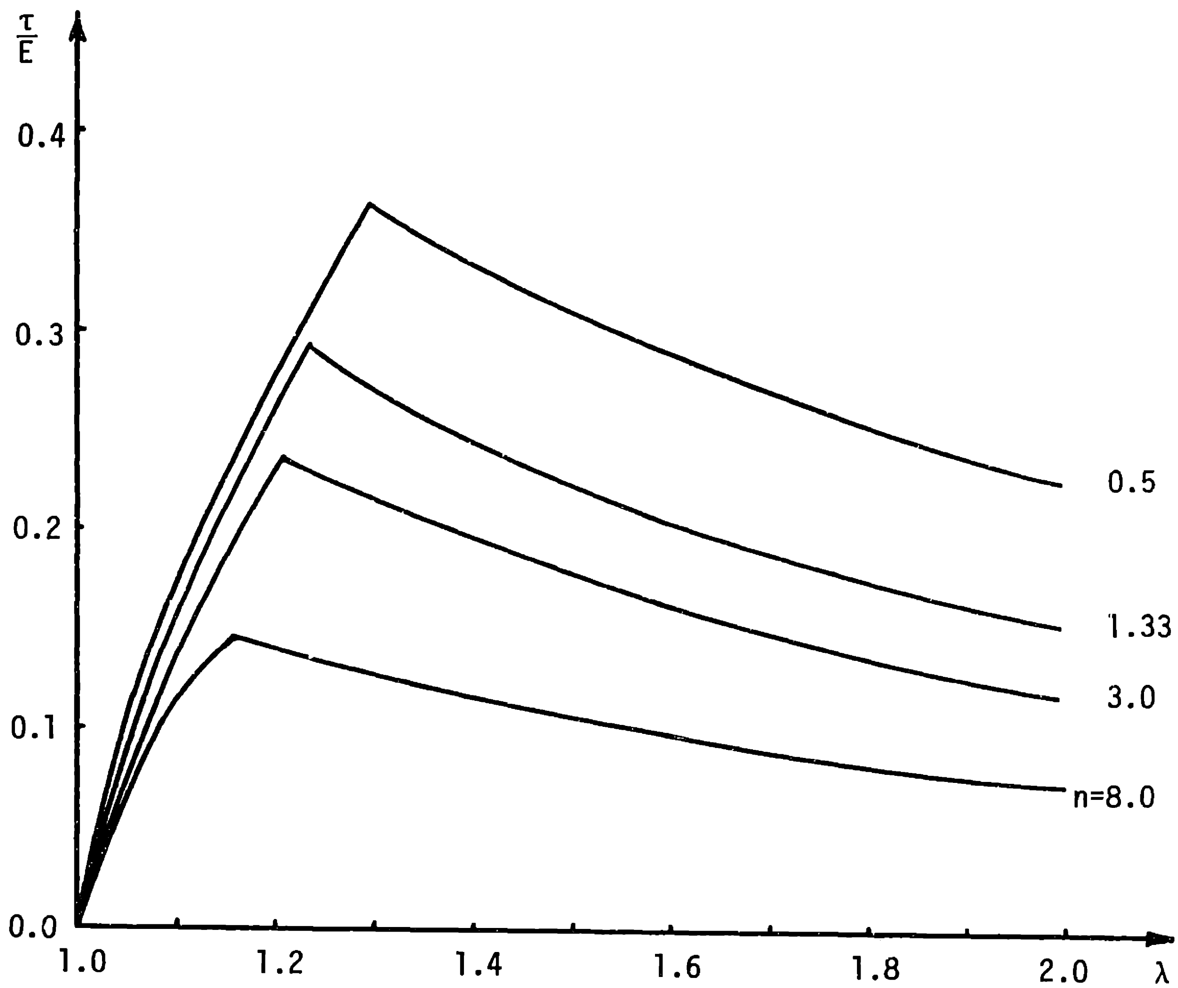

Fig. 10. Radial stress at outer boundary τ vs. prescribed stretch λ for different values of hardening exponent n, Poisson's ratio $\nu = 0.3$.

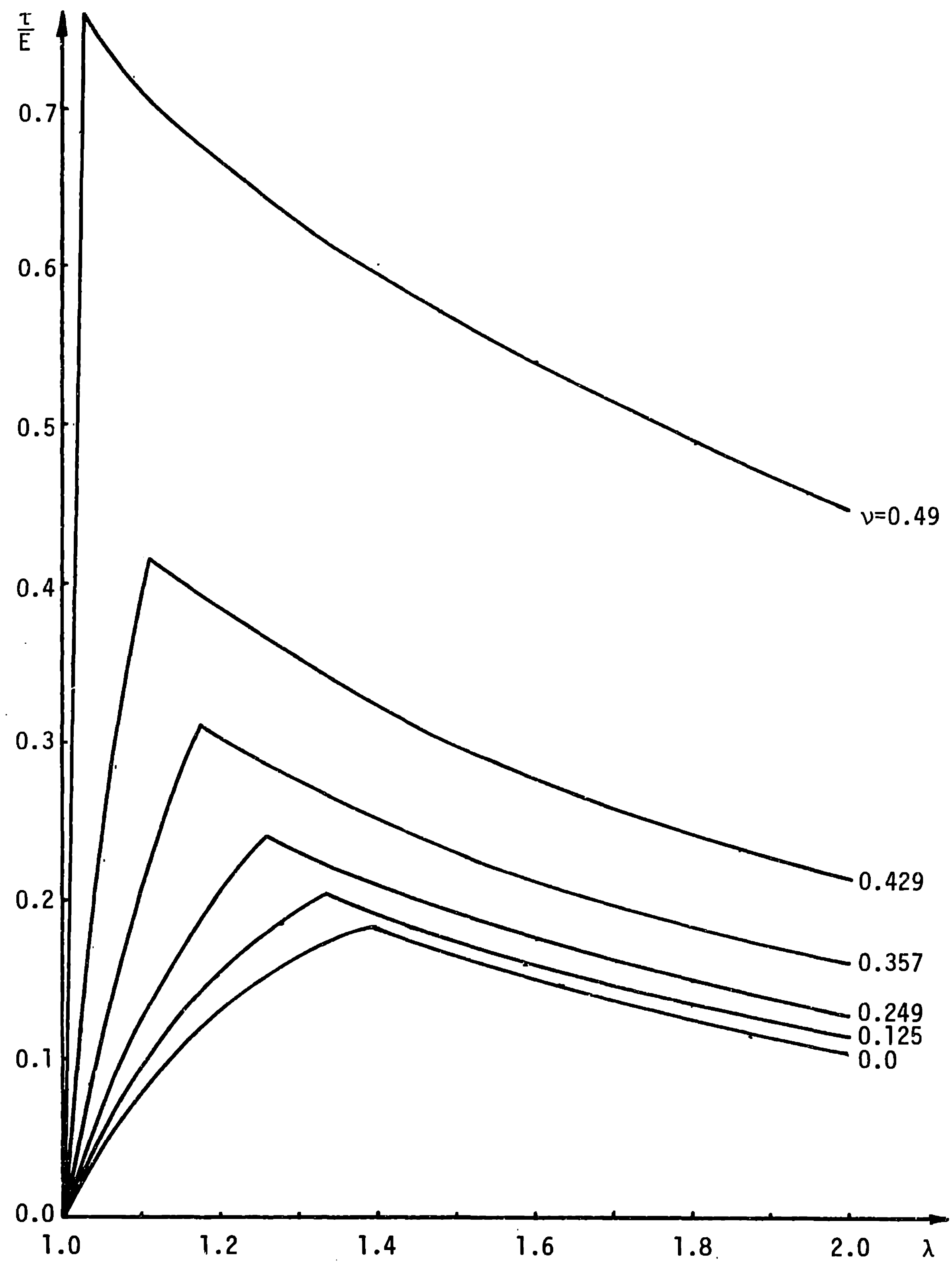

Fig. 11. Radial stress at outer boundary τ vs. prescribed stretch λ for different values of Poisson's ratio ν, hardening exponent $n = 2$.

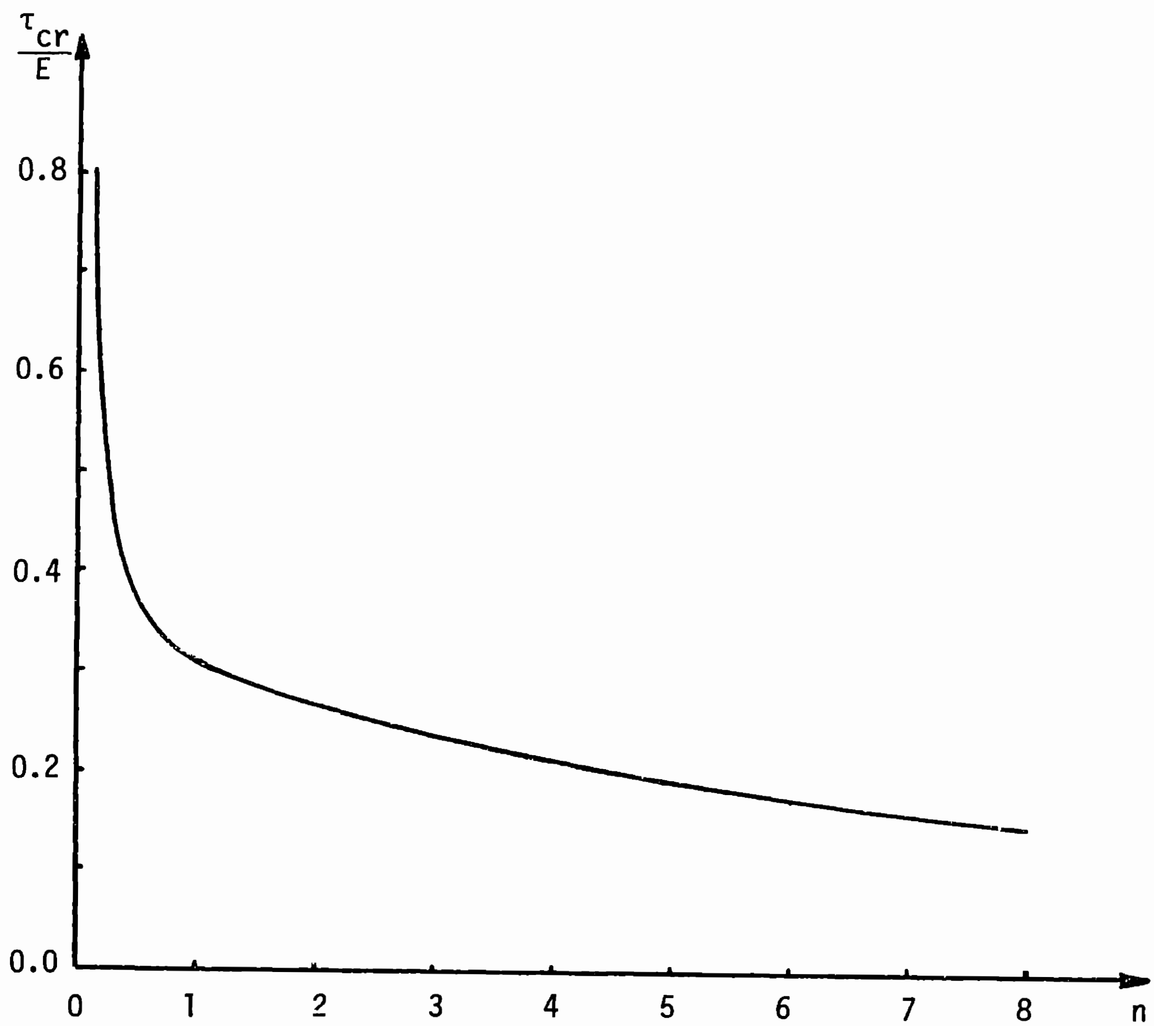

Fig. 12. Variation of critical stress τ_{cr} with the hardening exponent n, Poisson's ratio ν = 0.3 .

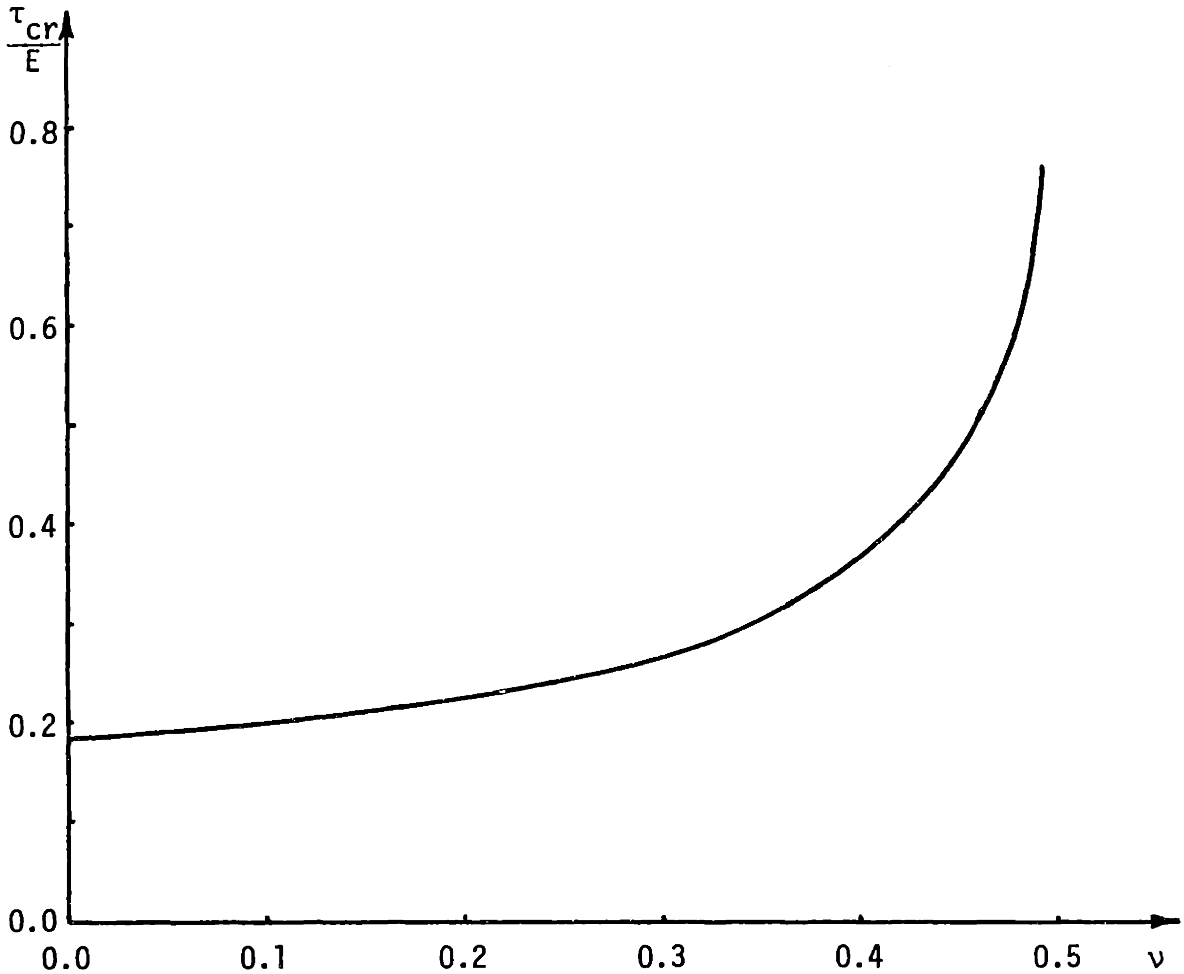

Fig. 13. Variation of critical stress τ_{cr} with the Poisson's ratio ν, hardening exponent $n = 2$.