COMMODITY BONDS:
A FINANCING ALTERNATIVE

by

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COMMODITY BONDS:  
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ABSTRACT

The purpose of this thesis is to present an analysis of commodity bonds, i.e., bonds where coupon and/or principal payments are linked to the price of a reference commodity bundle.

The thesis uses an intertemporal continuous-time model with stochastic investment and consumption opportunities in analyzing the demand for commodity bonds. Individuals who are relatively more risk averse than the market are found to have a greater demand for commodity bonds. When the market is relatively more risk averse than unity, individuals who are affected more than the average by changes in commodity prices are also found to have a greater demand for these bonds. Furthermore, demand is found to be greater across all individuals for those commodities with negative consumption betas, whenever the market and commodity prices become more volatile. Interest in these bonds should also be greater with commodity-exporting developing countries. The former are found to be a more effective means of hedging commodity price and quantity risk.

The thesis also determines a pricing model for commodity bonds with an option feature. It extends a previous analysis and includes the convenience yield of holding and storing the commodity. The effect of neglecting the convenience yield is shown to result in major price differences for commodities with a low convenience yield and enormous difference for oil bonds.

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CHAPTER I

INTRODUCTION

In 1863, the Confederate States of America were fighting a very costly war against the United States of America. Their principal asset was the cotton produced by the southern region. Such characteristics motivated their issuance of bonds payable in bales of cotton. This is the first known case of a corporation or government issuing bonds whose payoff is linked to the price of a given commodity.

A conventional bond makes semi-annual coupon payments determined by a coupon rate and pays out the principal amount at maturity. The nominal return to the investor is known but with the inflation rate being uncertain over time, the real return is uncertain. As a result, in periods of high level of inflation, investors have sought ways to link their investments to real assets. Commodity bonds (while also being attractive investments) are an attempt to fulfill that particular need.

Commodity bonds exist in two different varieties, those of a forward type and those of an option type. For the commodity bonds of a forward type, the coupon and/or principal payments are linked to a stated quantity of a commodity. A conventional $1000 face, 8% coupon bond would make annual payments of $80, while a similar silver bond of
a forward type would make, for instance, coupon payments equal to the current monetary value of three ounces of silver. The payoffs to these bonds reveal them to be similar to a money-fixed bond and a portfolio of forward contracts. Each quantity coupon payment is analogous to a forward contract; however, there is one major difference. In a forward contract, no money initially changes hands, and the contract is settled at maturity. Here the investor who holds the long side of the contract, has already fulfilled his obligation by buying the bond. Commodity bonds of the forward type seem, therefore, not to offer new payoffs structure beyond those already achieved with forward contracts and standard bonds. Their only advantage lies in the fact that forward contracts are negotiated between the two parties and are not easily traded. A market for commodity bonds would be liquid. However, a quest for liquidity does not seem sufficient to justify the introduction of forward type bonds and such a factor can explain why commodity bonds of the option type have been more prevalent.

A commodity bond of the option type makes normal coupon payments just as conventional bonds do, but upon maturity the holder of these bonds has the option of receiving the bond's face value or the monetary equivalent of a stated quantity of the commodity. For instance, the investor has the option of receiving the $1000 face value or the current value of 34 barrels of oil. Since these bonds include an option feature, the coupon rate is generally lower than the prevailing market rates. The advantage to the issuer is thus the
ability to face lower interest charges while accepting to share the benefits of an appreciation in the price of the commodity by writing a call option on the commodity. Such issuers would typically be governments or corporations having ready access to the underlying commodity and seeking a better hedge of their liabilities with their assets. The advantage to the investor is the ability to take a liquid, portable, and divisible position in a commodity, benefit from a price hike, and simultaneously receive a guaranteed minimum return on his investment through the fixed coupon payments. Commodity bonds of an option type are a unique financial instrument since commodity options exist for only a small number of commodities. Furthermore, when they do exist, commodity options are of short maturity and do not allow the holder to take bets on variations in long-term commodity price trends. These commodity bonds do allow such bets to be made and would seem to be popular with investors when commodity prices are depressed and/or expected to rise in the near future.

The remainder of the thesis analyzes the various aspects and characteristics of commodity bonds. Chapter 2 begins with a description of commodity bonds that have been issued in the financial markets. Close attention is given to the environment in which these bonds were issued and the rationale for doing so. The chapter also presents the terms of the prospectus for each commodity bond, thus enabling a better understanding of the terms under which these bonds were issued.
Chapter 3 analyzes the demand for commodity bonds. O'Hara (1984) uses a two-period, dynamic programming model for the analysis of commodity bonds of the forward type. She shows that although commodity bonds protect against relative price changes, they dampen changes in real wealth. Since changes in real income affect consumption, commodity bonds would provide insurance against risks to future consumption. This chapter uses a continuous time model with stochastic investment opportunities and consumption opportunities. It begins with the simple case of one consumption good and determines the conditions under which these bonds are desirable and to whom. The chapter also analyzes the effect of endogenous and exogenous variables on the demand for commodity bonds. A generalization to the multi-good case is then made.

Chapter 4 determines the prices of the commodity bonds of the option type. An earlier analysis by Schwartz (1982) assumes that spot commodities' prices behave like asset prices. This chapter further his analysis and incorporates the convenience yield required by investors for holding and storing the commodity in the pricing of the bonds, in an environment of uncertain interest rates. The chapter analyzes the effect of not having the option feature in the commodity price at maturity but rather as an average of the commodity prices before maturity and determines the price of the bonds when they are denominated in a currency other than the underlying commodity.
Chapter 5 analyzes the desirability of commodity bonds from the perspective of commodity-producing developing countries. An analysis of their commodity price exposure is first made as well as a description of the means available to them to hedge away that exposure. The advantages of commodity bonds over International Commodity Agreements, futures markets, and countertrade, are then presented.
CHAPTER II

PREVIOUS EXPERIENCES WITH COMMODITY BONDS

The highly inflationary environment that prevailed in the late 1970s sent many investors searching for ways to tie their capital to physical assets rather than paper money and, more particularly, to those commodities whose prices had varied tremendously. As a result, bonds linked to the price of silver, gold, or oil have been found to be popular with some money managers and investors in the late 1970s. Before we proceed with a brief description of these bonds, we must, however, mention that these instruments are by no means a novelty. In 1945, the French government's Caisse Nationale d'Energie issued a bond indexed on the price of electricity in order to pay for the nationalization of utilities. On top of their 3% coupon, investors were paid additional income from a fund that received 1% of the gross utility revenues in France. The new versions of indexed bonds are, however, more creative and have proved to be not only a hedge against inflation but also an attractive investment.

A. SILVER BONDS

At the start of 1979, silver bullion was priced at $6 an ounce, then peaked at $50 an ounce in January 1980, before dropping to $33 an ounce in February 1980. These wide price fluctuations may have played
a major role in the February 4, 1980, announcement of the Sunshine Mining Company to issue commodity bonds and try to hedge variations in working capital.

The Sunshine Mining Company is the largest U.S. silver producer, the sole operator and principal owner of the largest silver mine in the United States. In 1979 alone, 3,513,300 troy ounces of silver were produced from the Sunshine Mine. The company raised at least $25,000,000 by selling $1,000 certificates of 8-1/2% silver-indexed bonds due April 15, 1995. The bonds trade on the New York Stock Exchange, make semi-annual coupon payments on April 15 and October 15, and promised principal payments are the greater of $1,000 or the market value of 50 ounces of silver. However, under the terms of the issue, the bonds are callable at the indexed principal amount on or after April 15, 1985, if such amount is $2,000 or more for a period of 30 consecutive days. In each year subsequent to April 15, 1982, the company calls for redemption 7% of the adjusted original issue.

Holders whose bonds have been called have the right to elect not to have their bonds redeemed. The face amount of bonds called but not redeemed is added to the amount called for redemption in subsequent years. In the event of default, the trustees are entitled by a security agreement, to receive:

..."3.627% of the annual mining production of the Sunshine Mine, limited in all events, however, to (i) a number of ounces of silver per year, determined at the date of acceleration, sufficient to supply in equal annual installments from such date through April 15, 1995, the total number of ounces of silver specified in all bonds outstanding at the date of acceleration, and, in any
event, (ii) not more than 50 ounces of silver per $1,000 face amount of outstanding bonds."

The company is not restricted from the creation of senior indebtedness, but must maintain qualified reserves equal to at least 400% of the aggregate amount of silver required by all outstanding silver-backed and silver-related securities.

With this offering, investors have the possibility to invest in silver, receive a return on their investment through coupon payments, and simultaneously have a chance for substantial capital gains if silver prices rise. The main advantage to the company is the ability to raise capital at a much lower rate than through conventional channels and simultaneously hedge its operating exposure.

B. **GOLD BONDS**

1. The "Giscards"

Gold bonds have been more widespread than silver bonds and one of the more recent tentatives to issue such instruments was made by the French government. In 1973, France tried to persuade peasants to unveil gold coins hidden under their mattresses and lend them to the government. Valery Giscard d'Estaing, then Finance Minister, decided to have France issue gold-indexed bonds. The "Giscard", as this bond is commonly referred to, carries a 7% nominal coupon rate and a redemption value indexed to the price of a 1 kg gold bar. The French
government raised 6.5 billion francs with these bonds. However, the issue contained a safeguard clause stating that interest payments and the final redemption price would automatically be linked to the price of a 1 kg gold bar if the French franc ever lost its fixed value against gold and other currencies. Unfortunately for France, the French franc was pressured out of the European currency market in 1977 and floated. Furthermore, under the Jamaica Agreement, the International Monetary Fund formally abolished gold parities for currencies in 1978. These two events triggered the safeguard clause and in 1980, the French government had to pay bondholders 393 francs in interest per bond and not 70 francs, as was originally anticipated. The "Giscard", which were selling at par in 1975, increased to over 7,000 French francs in the past decade.

2. Refinemet Gold Bonds

The French gold bonds previously described guaranteed the investor a coupon payment of at least 7% of the face value and exposed him/her to gold price risk only through the principal repayment. The Refinemet Gold bonds differ in the sense that they fully expose the investor to gold price changes; both principal and interest are linked to the market price of gold.

Refinemet International Company is a Delaware Corporation whose activities include the reclamation and refining of precious metals, the trading and financing of precious and other metals, and the
manufacturing of precious metal reclamation equipment. In January 1981, the company issued 3.25% gold indexed bonds due February 1, 1996, in an aggregate principal amount equal to the market price of 100,000 ounces of gold. Under the terms of the issue, interest payments are made annually on February 1 at the rate of 3.25% of the market price of gold amount to which each bond is indexed. The interest is payable, in dollars, "on the basis of the average market price of gold for the 10 trading days ending on the fifth business day prior to the interest payment date", as stated by the prospectus summary. The bonds are payable at maturity, or upon earlier redemption, in dollars, on the basis of the average market price of gold for the 10 trading days ending on the fifth business day prior to maturity or redemption. A holder of these bonds also has the possibility, if he so desires, to aggregate interest payments due on the same date in round lots of 100 ounces and receive the interest in gold bullion. However, payments in gold bullion are made only at designated agencies in London and Zurich.

3. **Echo Bay Gold Warrants**

Another version of gold-linked bonds was issued in January 1981, by Echo Bay Mines Ltd., a subsidiary of IU International Corporation. Although not gold bonds per se, the company issued 1,550,000 units each consisting of one $3.00 cumulative redeemable voting preferred share together with 4 gold purchase warrants. Although the promised cash flows generated from the cumulative preferred shares resemble
those from a perpetuity, the gold purchase warrants bring a unique feature to this issue. Each unit was sold at $50, of which the allocation was $25 for the price of the preferred share and $25 for prepayment on account of the exercise price. The cumulative preferential dividends of $3.00 per share are payable semi-annually on the last day of June and December. The shares entitle their holders to one vote per share and to special rights should the dividends on redemptions be in arrears of more than two years. The gold warrants are transferrable separately from the preferred shares after a date, not set at the time of issue, but not later than December 30, 1983. The gold warrants give their holders the right to purchase 0.0706 of a troy ounce of gold from Echo Bay at a price of $595 per troy ounce. Of the four warrants, one is exercisable on January 31, 1986, one on January 31, 1987, one on January 31, 1988, and one on January 31, 1989. Upon exercise of the warrant, the holder is required to pay the balance of the exercise price, excluding the prepayment, and a 1% delivery fee payable at the time of delivery of the gold. Therefore, upon exercise, there will be physical delivery of gold. If the holder chooses not to exercise and the market price of gold exceeds $626 per troy ounce on the exercise date, the holder may at any time within two years following such exercise date, tender the gold warrant for cancellation and Echo Bay will, within 90 days after the tender, pay the holder the difference between the market price of the gold which would have been delivered and the purchase price net of prepayment, plus a service fee. In this case, the holder gets the direct payoff of a call option. If, however, the holder chooses not to exercise and
the market price of gold is below $625 per troy ounce on the exercise
date, the holder may tender the gold warrant for cancellation in
return for the prepayment. If such turns out to be the case, Echo Bay
would thus have received an interest-free loan of over $36,000,000.

As complex as this offer already appears, it includes an
additional clause. The prospectus summary states that:

"If by any exercise date, the Lupin Gold project
has not achieved commercial production, the gold
purchase warrants exercisable on that date shall
not be exercisable for purchase of gold unless
Echo Bay is producing gold at an annualized rate
of 60,000 troy ounces."

This raises an important feature that has already been encountered:
the need to ensure that a sufficient amount of the underlying
commodity will be readily available during the life of the issue,
feature that cannot be guaranteed for cash crops. Such a feature can
reduce the risk of default since the underlying commodity can
eventually be used to repay the loan.

C. OIL BONDS

Gold and silver have always been linked to monetary instruments in
the banking system. Hence the existence of gold and silver bonds is
perhaps not surprising. However, after the two oil shocks of the
1970s, we have seen the appearance of oil-linked bonds. In January
1981, the Reagan administration seriously considered issuing such
bonds in order to finance the strategic petroleum reserve. The
two most successful examples of oil-linked bonds are the Mexican "Petrobonos" and the Petro-Lewis bonds.

1. **Mexican Petrobonos**

Petrobonos are Mexican oil bonds issued in bearer form by a trust fund set up by the National Financiera S.A. (NAFINSA), the government-owned development bank. They are registered with the Comision Nacional de Valores, the equivalent of the American SEC, and are listed on the Mexican Stock Exchange. The bonds were introduced by the government of President Lopez-Portillo as a device to bring back money when a 45% devaluation of the peso sent billions across the border to the U.S. in 1976. As a result, most issues were tailored to appeal to local Mexican investors who had invested abroad. The first issue, in April 1977, raised 2 billion pesos and most of the funds were used to finance Mexican oil development.

The bonds were issued in denominations of 1,000 pesos and multiples thereof. They had a maturity of 3 years and carried a coupon rate of 12.65823% per annum, payable quarterly in pesos. However, the coupon was subject to a 21% Mexican withholding tax, resulting in a net coupon of 10%. This tax was applicable to both Mexican and foreign investors. Furthermore, each bond was backed by a specific number of barrels of API-33 light crude, notionally purchased at the time of issue by the trustees of bondholders. Upon maturity of an issue, the Petrobonos are redeemed by NAFINSA at a price equal to
the peso value of the underlying oil quantity less the full amount of interest paid out over the life of the bonds. Therefore, because international oil prices are set in dollars, the Petrobonos also include currency risk for non-Mexican investors. Redemption is carried out at the official repayment price, calculated on the basis of the average oil export price for the 25 days preceding the maturity date. The special feature of deducting coupon payments from the principal payments make the Petrobonos a low-interest loan to the Mexican government in return for capital gains if oil prices rise. The Mexican government has made five successful issues of Petrobonos.

2. Petro-Lewis Oil Notes

The second offering of oil-linked bonds was made by the Petro-Lewis Corporation, a Denver-based company involved in oil and gas exploration, production, and property acquisition activities. In April 1981, the company issued $20,000,000 worth of 9% guaranteed oil-indexed notes due in 1986. The notes are similar in structure to the Mexican Petrobonos but have a maturity of 5 years rather than 3. The notes pay a nominal interest of 9% on an annual basis. For each $1,000 of principal, the notes pay at maturity principal plus accrued nominal interest and contingent interest. Contingent interest is defined as the increase over $668.96 of (i) the average crude oil price of 18-1/2 barrels of crude oil for the 3 months ending February 28, 1986, or (ii) if greater, the highest average crude oil price of 18-1/2 barrels of crude oil, up to a maximum of $1,258 or $68 per
barrel, for any calendar quarter through the quarter ending December 31, 1985. Crude oil price means the arithmetic average of the official or government posted selling prices for the following seven types of crude oil:

- Saudi Arabian light 34°
- Kuwaiti 31°
- Nigerian Bony 37°
- Venezuelan Tia Juana 26°
- Indonesian Minas 34°
- UK Forties 36.5°, and
- Mexican Isthmus 34°

The $668.96 base price is the average crude oil price of 18-1/2 barrels of crude oil for the calendar quarter ended March 31, 1981, based on an average crude oil price of $36.16 per barrel. Furthermore, from August 15, 1983 through September 14, 1983, holders of the notes had the option to elect an early maturity date of November 1, 1983 and receive $1,182 for each $1,000 principal amount. The Petro-lewis notes differ from the Petrobonos in that principal repayments include a call option on oil prices and protect the holders from a slide in oil prices. This is not the case with the Petrobonos where repayment of principal is fully linked to an oil bundle.
CHAPTER III

THE DEMAND FOR COMMODITY BONDS

One of the most important models in finance theory, the Sharpe-Lintner Capital Asset Pricing Model (CAPM) (Sharpe, 1964), is based on a single-period model with very restrictive assumptions. Although the model has been widely criticized and widely tested by the academic community, it is still extensively used in the non-academic world. This chapter derives the demand functions for commodity bonds using a continuous-time intertemporal model similar to the one derived by Merton (1971, 1973). The model applies a dynamic programming technique to the consumption-portfolio problem for a household whose income is generated by capital gains on investments in tradeable assets. The derivations are done in nominal terms. The chapter begins with a one-consumption good version of this model and continues with a multi-good extension.

A. THE ONE-CONSUMPTION GOOD CASE

1. Assumptions

The general assumptions retained for this analysis are similar to the ones made in Merton (1973). Households are assumed to behave as price takers in a perfectly competitive market and trading always takes place at equilibrium prices. Households can buy and sell as
much of an asset as they want at market prices and may short-sell any asset with full use of the proceeds. It is further assumed that households hold wealth in the form of risky assets and an instantaneously riskless asset for which the borrowing and lending rates are equal. All assets are assumed to be perfectly divisible and have limited liability. Households can trade continuously and face no transactions costs or taxes. Finally, asset prices are assumed to be stationary and log-normally distributed.

Most of the assumptions made are the standard assumptions required to make a perfect market. These have been widely discussed in the finance literature and are mainly retained for the sake of simplicity, while doing no damage to the analysis. Nevertheless, Fama argues in Cootner (1964) that stock and commodity price changes follow a stable Pareto distribution with infinite second moments. It is important to note, however, that nothing has been said about the homogeneity of households' expectations as is required in the derivation of the CAPM and similar models.

2. Asset Returns

In this section there is a single consumption good, the commodity, whose price is assumed to be generated by an Ito process:

$$\frac{dP_1}{P_1} = \alpha_1 dt + s_1 dz_1$$

(3.1)

where $\alpha_1$ is the expected percentage change in the commodity's price per unit time and $\sigma^2_1$ is the instantaneous variance per
unit time. The instantaneously riskless rate of interest is assumed to follow the following differential equation:

\[ dr = \alpha_r dt + \sigma_r dz_r \]  \hspace{1cm} (3.2)

Each individual can hold three assets in his portfolio: A commodity bond, equity, and a default-free bond. The value of the commodity bond depends only on the price of the commodity, the rate of interest, and time to maturity.

\[ Q_1 = Q_1(P_1, r, T) \]  \hspace{1cm} (3.3)

The value of the default free bond, \( Q_3 \), depends only on the rate of interest and time to maturity.

\[ Q_3 = Q_3(r, T) \]  \hspace{1cm} (3.4)

The remaining asset, in the form of equity, is also assumed to be generated by an Ito process:

\[ \frac{dQ_2}{Q_2} = R_2 dt + \sigma_2 dz_2 \]  \hspace{1cm} (3.5)

Ito processes, while continuous, are not differentiable and we must arm ourselves with a tool in order to manipulate functions which involve Ito processes. A thorough description of Ito processes is given in Ito and McKean (1964) but we will limit ourselves to a statement of Ito's lemma.
**Lemma:** Let $F = F(X_1, \ldots, X_n, t)$ be a function at least twice differentiable where the $X_i$'s are generated by Ito processes, then its differential is given by:

$$
\begin{align*}
\frac{dF}{dt} &= \sum_{i=1}^{n} \frac{\partial F}{\partial X_i} dX_i + \frac{\partial F}{\partial t} dt + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 F}{\partial X_i \partial X_j} dX_i dX_j
\end{align*}
$$

where the product $dX_i dX_j$ is defined by the rule

$$
\begin{align*}
\frac{dz_i}{dt} &= \rho_{ij} dt \\
\frac{dz_i}{dt} &= 0 \quad i = 1, \ldots, n
\end{align*}
$$

and $\rho_{ij}$ is the correlation coefficient between the Gauss-Weiner processes $dz_i$ and $dz_j$.

Now, equipped with Ito's lemma, we can determine the percentage change in the commodity bond's price as well as that of the default-free bond.

$$
\begin{align*}
\frac{dQ_1}{dP_1} &= \frac{\partial Q_1}{\partial r} dr + \frac{\partial Q_1}{\partial T} dT + \frac{1}{2} \frac{\partial^2 Q_1}{\partial r^2} dr^2 + \frac{1}{2} \frac{\partial^2 Q_1}{\partial P_1 \partial r} dP_1 dr
\end{align*}
$$

However, $T$ is defined as time to maturity, so that $dT = -dt$.

Furthermore, as an application of the multiplication rule given in Ito's Lemma, the following products are obtained:
\[ dP_1 = P_1^2 \left( \begin{array}{c} \alpha_1 \ dt + s_1 \ dz_1 \end{array} \right)^2 = P_1^2 e_1^2 dt \]

\[ dr^2 = \left( \alpha_r dt + \sigma_r dz_r \right)^2 = e_r^2 dt \]

\[ dP_1 \ dr = P_1 \left[ \sigma_1 \ dt + s_1 \ dz_1 \right] \left[ \alpha_r dt + \sigma_r dz_r \right] = P_1 \ e_1 \sigma_r \rho_1 \ dt \]

With these new expressions, equation (6) becomes:

\[ \frac{dQ_1}{Q_1} = \left( \begin{array}{c} \alpha_1 \frac{P_1}{Q_1} \frac{\partial Q_1}{\partial P_1} + \frac{\alpha_r}{Q_1} \frac{\partial Q_1}{\partial r} + \frac{1}{2} \frac{P_1^2 s_1^2}{Q_1} \frac{\partial^2 Q_1}{\partial P_1^2} + \frac{1}{2} \frac{\alpha_r^2}{Q_1} \frac{\partial^2 Q_1}{\partial r^2} \\
+ \frac{P_1 s_1 \sigma_r}{Q_1} \rho_1 \frac{\partial^2 Q_1}{\partial P_1 \partial r} - \frac{1}{Q_1} \frac{\partial Q_1}{\partial T} \end{array} \right) dt \\
+ \left[ s_1 \frac{P_1}{Q_1} \frac{\partial Q_1}{\partial P_1} \ dz_1 + \frac{\sigma_r}{Q_1} \frac{\partial Q_1}{\partial r} \ dz_r \right] \tag{3.8} \]

Similar derivations for the default-free bond, yield:

\[ dQ_3 = \frac{\partial Q_3}{\partial r} dr + \frac{\partial Q_3}{\partial T} dT + \frac{1}{2} \frac{\partial^2 Q_3}{\partial r^2} dr^2 \tag{3.9} \]

and

\[ \frac{dQ_3}{Q_3} = \left( \begin{array}{c} \frac{\alpha_r}{Q_3} \frac{\partial Q_3}{\partial r} + \frac{1}{2} \frac{\sigma_r^2}{Q_3} \frac{\partial^2 Q_3}{\partial r^2} - \frac{1}{Q_3} \frac{\partial Q_3}{\partial T} \end{array} \right) dt \\
+ \frac{\sigma_r}{Q_3} \frac{\partial Q_3}{\partial r} dz_r \tag{3.10} \]

We further assume that interest rates are non-stochastic so that \( \sigma_r = 0 \).

We define \( e_1 = \frac{P_1}{Q_1} \frac{\partial Q_1}{\partial P_1} \) the commodity bond's price elasticity. As a result of these new specifications, asset returns are now fully expressed by:
\[ \frac{dQ_1}{Q_1} = \left[ \alpha_1 e_1 + \frac{\alpha}{Q_1} \frac{\partial Q_1}{\partial r} + \frac{1}{2} \left( \frac{p_{11}}{Q_1} + \frac{2}{Q_1} \left( \frac{\partial Q_1}{\partial p_{11}} \right) \right) \right] dt + \sigma_1 dz_1 \]

\[ \frac{dQ_1}{Q_1} = R_1 dt + \sigma_1 dz_1 \]

\[ \frac{dQ_2}{Q_2} = R_2 dt + \sigma_2 dz_2 \]

\[ \frac{dQ_3}{Q_3} = \left( \frac{\alpha_r}{Q_3} \frac{\partial Q_3}{\partial r} - \frac{1}{Q_3} \frac{\partial Q_3}{\partial Q} \right) dt = R_3 dt = R_f dt \]  

(3.11)

3. Budget Equation

In order to derive the individual's budget equation, the framework followed is one in which all the wealth is held in the assets, income is generated by capital gains, and the individual must reduce his asset holdings in order to consume. A discrete model with time periods of length \( h \) is first developed before we derive the continuous model by taking \( h \) to 0.

Let \( W(t) \) and \( Q_1(t) \) be wealth and asset prices at the beginning of period \( t \).

\( N_1(t) \) represents the number of shares of asset 1 held at date \( t \).
Thus,

\[ W(t) = \sum_{i=1}^{3} N_i(t) Q_i(t) \]  \hspace{1cm} (3.12)

In order to consume between dates \( t \) and \( t+h \), the individual must reduce asset holdings at date \( t \). All of that consumption is in the form of the commodity, according to our model, and we note \( C(t) \) the rate of consumption per unit time.

Consumption is thus given by:

\[ \sum_{i=1}^{3} [N_i(t) - N_i(t+h)] Q_i(t) = P_i(t).C(t).h \]  \hspace{1cm} (3.13)

Thus,

\[ W(t+h) - W(t) = \sum_{i=1}^{3} N_i(t+h) Q_i(t+h) - \sum_{i=1}^{3} N_i(t) Q_i(t) \]

\[ = \sum_{i=1}^{3} N_i(t+h) [Q_i(t+h) - Q_i(t)] - P_i(t).C(t).h \]

and

\[ \frac{W(t+h) - W(t)}{h} = \sum_{i=1}^{3} N_i(t+h) \frac{Q_i(t+h) - Q_i(t)}{h} - P_i(t).C(t) \]  \hspace{1cm} (3.14)

As \( h \) goes to 0, we obtain the continuous version of this equation:

\[ dW(t) = \sum_{i=1}^{3} N_i(t) dQ_i(t) - P_i(t).C(t).dt \]  \hspace{1cm} (3.15)

We now introduce \( \omega_i \), the proportion of the portfolio held in asset \( i \).

\[ \omega_i(t) = \frac{N_i(t) Q_i(t)}{W(t)} \]  \hspace{1cm} (3.16)
Equation (15) becomes

\[ dW = \sum_{i=1}^{3} \omega_i W \frac{dQ_i}{Q_i} - P_1C dt \]  

(3.17)

Using equation (11) and \( \omega_3 = 1 - \sum_{i=1}^{2} \omega_i \), we obtain our budget equation:

\[ dW = \sum_{i=1}^{2} \omega_i W(R_i - R_f) dt + (WR_f - P_1C) dt + \sum_{i=1}^{2} \omega_i W \sigma_i dz_i \]  

(3.18)

4. **Maximization Problem**

Each individual is faced with the problem of choosing a portfolio and consumption pattern that will maximize the expected value of a time-additive von Neumann-Morgenstern utility function and bequest function.

The problem is formulated as:

\[ \max_{(c, \omega)} \int_0^T \left[ U(C(t), t) dt + B(W(T), T) \right] \]  

(3.19)

subject to equation (18) and \( W(0) = W_0 \).

The utility function \( U \) is assumed to be strictly concave in \( C \) and the bequest function is strictly concave in \( W \). The general technique used to solve these types of problems is that of dynamic programming which is well described in Dreyfus (1965).
We begin by introducing a function $J$ defined by:

$$ J(W,P_1,t) = \max_{(c,\omega_1)} \mathbb{E}_t[\int_0^T U[c(\tau),\tau]d\tau + R(W(T),T)] $$

(3.20)

and define:

$$ \phi(\omega,c; W,P_1,t) = U[c(t),t] + L[J] $$

(3.21)

where $L$ is the Dynkin operator over the variables $W$ and $P_1$ for a given set of controls $\omega$ and $C$ defined by:

$$ L[J] = \frac{\partial J}{\partial t} + \left[ \sum_{i=1}^{3} \omega_i R_i W - P_1 C \right] \frac{\partial J}{\partial W} + \left[ \frac{1}{2} \right] \sum_{i=1}^{3} P_1 \frac{\partial J}{\partial P_1} + \frac{1}{2} \sum_{i=1}^{3} \frac{\partial^2 J}{\partial P_1^2} $$

$$ + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \omega_i \omega_j \omega_k \frac{\partial^2 J}{\partial W^2} + \sum_{i=1}^{3} P_1 W \omega_j W \frac{\partial^2 J}{\partial P_1 \partial W} $$

(22)

$\Omega = (\sigma_{ij})_{i,j}$ is the covariance matrix of asset returns and $V = (V_{1j})_j$ is the covariance vector of asset returns with the commodity price.

Merton (1969) has shown that under our stated conditions, optimal sets of controls do exist and are found by differentiating the Lagrangian

$$ L = \phi + \lambda \left[ 1 - \sum_{i=1}^{3} \omega_i \right] $$

(3.23)

The first-order conditions give us:

$$ 0 = U_c - P_1 J_W $$

$$ 0 = \omega_1 R_1 W + \frac{1}{2} \sum_{i=1}^{3} \sigma_{11} \omega_1 \omega_1 + \lambda \sum_{i=1}^{3} P_1 V_{11} J_W $$

$$ 0 = \omega_2 R_2 W + \frac{1}{2} \sum_{i=1}^{3} \sigma_{12} \omega_1 \omega_1 + \lambda \sum_{i=1}^{3} P_1 V_{12} J_W $$
\[ 0 = -\lambda + R_3 J_{WW} + \omega_3 J_{WW} \left( \sum_{i=1}^{3} \sigma_{3i} \omega_i \right) + WP_1 V_{13} J_{WW} \]

\[ 0 = 1 - \sum_{i=1}^{3} \omega_i \]

where we note \( F_x = \frac{\partial F}{\partial x} \).

However, the terms of the covariance matrices defined by:

\[ \sigma_{ij} = \rho_{ij} \sigma_i \sigma_j \]

and the condition \( \sigma_3 = 0 \) imply that \( \sigma_{3i} = 0 \) for all \( i \).

For a similar reason, \( V_{ij} = \rho_{ij} S_i \sigma_j \) implies \( V_{13} = 0 \). The first-order conditions thus become:

\[ U_c = P_1 J_{WW} \quad (3.24) \]

\[ 0 = (R_1 - R_f) J_{WW} + \omega_3 J_{WW} \left( \sum_{i=1}^{2} \sigma_{1i} \omega_i \right) + P_1 J_{WW} V_{11} \quad (3.25) \]

\[ 0 = (R_2 - R_f) J_{WW} + \omega_3 J_{WW} \left( \sum_{i=1}^{2} \sigma_{2i} \omega_i \right) + P_1 J_{WW} V_{12} \quad (3.26) \]

\[ \lambda = R_f J_{WW} \quad (3.27) \]

\[ 1 = \sum_{i=1}^{3} \omega_i \quad (3.28) \]

5. Demand Functions

The analysis just described the portfolio choice problem faced by an individual and enables us to derive the optimal asset portfolio.
Equation (24) states that at the optimum, the marginal utility of consumption must equate the marginal utility of wealth. In matrix form, equations (25) and (26) can be re-written:

\[
\begin{pmatrix}
\sigma_{11} & \sigma_{12} \\
\sigma_{12} & \sigma_{22}
\end{pmatrix}
\begin{pmatrix}
\omega_1^W \\
\omega_2^W
\end{pmatrix}
= \frac{-J_W}{J_{WW}}
\begin{pmatrix}
R_1 - R_f \\
R_2 - R_f
\end{pmatrix}
- \frac{p_1 J_{1W}}{J_{WW}}
\begin{pmatrix}
V_{1,1} \\
V_{1,2}
\end{pmatrix}
\] (3.29)

The various terms of the variance-covariance matrix are defined by:

\[
\sigma_{ij} = \rho_{ij} \sigma_i \sigma_j
\] (3.30)

\(\rho\) denotes the correlation coefficient between the commodity and the equity asset, so that

\[
\sigma_{12} = \rho \sigma_1 \sigma_2
\] (3.31)

\[
\sigma_{11} = \sigma_1^2
\] (3.32)

\[
\sigma_{22} = \sigma_2^2
\] (3.33)

We furthermore assume that \(\rho^2 \neq 1\) so that the variance-covariance matrix is non-singular. We are now able to derive the demand functions for the various assets by matrix inversion on equation (29).

\[
\begin{pmatrix}
\omega_1^W \\
\omega_2^W
\end{pmatrix}
= \frac{-J_W}{J_{WW}}
\begin{pmatrix}
\sigma_2^2 - \rho \sigma_1 \sigma_2 \\
-\rho \sigma_1 \sigma_2 & \sigma_1^2
\end{pmatrix}
\begin{pmatrix}
R_1 - R_f \\
R_2 - R_f
\end{pmatrix}
- \frac{p_1 J_{1W}}{J_{WW}}
\begin{pmatrix}
\frac{a_1}{\sigma_1} \sigma_2^2 \\
\frac{a_1}{\sigma_1} \rho \sigma_1 \sigma_2
\end{pmatrix}
\] (3.34)
and \( \omega_3W = W - \omega_1W - \omega_2W \) \hspace{1cm} (3.35)

After simplifications, these equations are written:

\[ \omega_1W = \frac{-J_w}{J_{I_1}} \left[ \frac{(R_1 - R_f)}{\sigma_1^2} - \rho \frac{(R_2 - R_f)}{\sigma_1 \sigma_2} \right] - \frac{p_1 J_{I_1}}{J_{I_1}} \frac{g_1}{\sigma_1} \] \hspace{1cm} (3.36)

\[ \omega_2W = \frac{-J_w}{J_{I_1}} \left[ \frac{(R_2 - R_f)}{\sigma_2^2} - \rho \frac{(R_2 - R_f)}{\sigma_2} \right] \] \hspace{1cm} (3.37)

\[ \omega_3W = W - \omega_1W - \omega_2W \] \hspace{1cm} (3.38)

In order to express the last term of equation (36) as a function of the other variables, Fischer (1975) points out that the consumption decision by the individual is guided by commodity price changes relative to his wealth, so that consumption is a function of real wealth.

\[ C = C(W/P_1, t) \] \hspace{1cm} (3.39)

By differentiating this equation relative to \( W \) and to \( P_1 \), we obtain:

\[ \frac{\partial C}{\partial W} = C_1 \frac{\partial (W/P_1)}{\partial W} = C_1 \frac{1}{P_1} \] \hspace{1cm} (3.40)

\[ \frac{\partial C}{\partial P_1} = C_1 \frac{\partial (W/P_1)}{\partial P_1} = -C_1 \frac{W}{P_1^2} \] \hspace{1cm} (3.41)

and thus the following relationship holds:

\[ \frac{\partial C}{\partial W} = \frac{-P_1}{W} \frac{\partial C}{\partial P_1} \] \hspace{1cm} (3.42)
Furthermore, when we differentiate equation (24) relative to $W$ and to $P_1$, it yields:

$$U_{cc} \frac{\partial C}{\partial P_1} = J_W + P_1 J_{1W}$$

(3.43)

$$U_{cc} \frac{\partial C}{\partial W} = P_1 J_{WW} = -U_{cc} \frac{P_1}{W} \frac{\partial C}{\partial P_1} = -\frac{P_1}{W} (J_W + P_1 J_{1W})$$

(3.44)

As a result,

$$\frac{-P_1 J_{1W}}{J_{WW}} = W + \frac{J_W}{J_{WW}}$$

(3.45)

The individual's absolute risk tolerance $T$ is defined as $T \equiv \frac{-U_C}{U_{cc}}$

We note that $T > 0$ because more is better ($U_C > 0$) and the utility function is assumed to be concave in $C$ or, which is the same, the first units of consumption are worth more to the individual than the subsequent units.

We define $MT = \frac{-J_W}{J_{WW}}$ as the individual's "modified" risk tolerance.

$$\frac{-J_W}{J_{WW}} = \frac{-U_C}{U_{cc} \frac{\partial C}{\partial W}} = \frac{T}{\frac{\partial C}{\partial W}} = \frac{T}{C_W}$$

(3.46)

High values of this variable are obtained for high values of $T$ and/or low values of the marginal propensity to consume. We can expect $C_W$ to be a decreasing function of $W$. There could conceivably exist individuals for whom $C_W = 0$. These individuals have so much wealth that any increase in that wealth would not induce them to consume more. At the other extreme, very poor individuals would consume all of their additional gains and we would have for them: $C_W = 1$, for at least $W$ lower than a certain subsistence level.
With these simplifications, the demand equations for each individual $k$ can be written:

$$\omega_1^k W^k = M^k T^k \left[ \frac{(R_1 - R_F)}{(1-\rho)\sigma_1^2} - \rho \frac{(R_2 - R_F)}{(1-\rho)\sigma_2} \right] + \frac{1}{\epsilon_1} (W^k - M^k T^k) \quad (3.47)$$

$$\omega_2^k W^k = M^k T^k \left[ \frac{(R_2 - R_F)}{(1-\rho)\sigma_2^2} - \rho \frac{(R_1 - R_F)}{(1-\rho)\sigma_2} \right] \quad (3.48)$$

$$\omega_3^k W^k = W^k - \omega_1^k W^k - \omega_2^k W^k \quad (3.49)$$

A look at the demand equations reveals that the demand for the equity asset is only comprised of a speculative component and is equal to the demand for a risky asset by a single period mean-variance maximizing investor. However, the demand for the commodity bond is also comprised of a similar speculative component and a hedging component. $A$ and $B$ denoting the bracket terms in equations (47) and (48), these equations become:

$$\omega_1^k W^k = M^k T^k A + (W^k - M^k T^k)/\epsilon_1 \quad (3.50)$$

$$\omega_2^k W^k = M^k T^k B \quad (3.51)$$

We define $M^M = \sum_k M^k$ as the market's modified risk tolerance. By aggregating equations (50) and (51), we obtain:
\[ \omega^M_1 = M^T A + (M - M^N)/e_1 \]  
\[ \omega^M_2 = M^T B \]  
(3.52)  
(3.53)

Substituting the values of \( A \) and \( B \) obtained from these equations back into equations (50) and (51), we are able to derive another expression for the demand functions:

\[ \omega^k = \omega^M_1 M^T + \frac{\omega^k}{\alpha_1} \left[ 1 - \frac{MT^k / W^k}{MT^M / M} \right] \]  
(3.54)

\[ \omega^k_2 = \omega^M_2 M \frac{MT^k}{MT^M} \]  
(3.55)

6. The Determinants of the Demand for Commodity Bonds

We now consider the case where the equity asset is the market portfolio before the introduction of commodity bonds. Modern portfolio theory tells us that investors would hold the market portfolio, levered up or down according to their aversion to risk. Two approaches can be taken here. We can analyze the change in the individual's portfolio mix after the introduction of a positive amount of commodity bonds or we can analyze and determine who would want to issue or hold such bonds, if they didn't exist.

The latter approach will be taken here, as we are trying to justify the introduction of such bonds. In this case, we must have \( \omega^M_1 = 0 \), and equation (54) becomes:
\[ \omega_k^k w^k = \frac{w^k}{e_1} \left[ 1 - \frac{MT_k^k}{MT^M/M} \right] \]  

Thus, if \( \frac{MT_k^k}{w^k} < \frac{MT^M}{M} \), we have \( \omega_k^k w^k > 0 \). This is a result that we would intuitively expect: when individual \( k \) has a lower relative modified risk tolerance than the market or, which is the same, a higher relative modified risk aversion, he would have a positive demand for commodity bonds.

However, a comparison of equations (56) and (50) reveals that even in this case, the demand for commodity bonds is not limited to the hedging component. Samuelson (1985) points out that sellers of such bonds will be those least averse to price risk as they are bribed to take on some of the irreducible variability by an appropriate market-clearing premium. For such individuals, their attitude towards commodity bonds will be guided by their speculative demand. Samuelson also notes that if the supply of commodity bonds were to come only from individuals willing to take a little more risk for a premium, the market for commodity bonds would not be viable. This market must also be driven by a commercial function with the involvement of major players, big corporations or governments, seeking to hedge the variations in their production costs or revenues.

We now take a closer look at the determinants of the demand for commodity bonds. From equation (47) it can be seen that it depends on the required rates of return for the three assets, the correlation
between the commodity and the market, their respective volatilities, and the individual's modified risk tolerance.

From this point on, \( D^k_1 \) will denote individual k's demand for asset i:

\[
D^k_1 = \omega^k_1 \omega^k
\]

(3.57)

In taking the partial derivative of equation (47) with respect to \( R_1 \), we obtain:

\[
\frac{\partial D^k_1}{\partial R_1} = \frac{\rho MT^k}{(1-\rho^2)\sigma^2_1} > 0
\]

(3.58)

This is a general property of most demand functions as they are decreasing with respect to the price. Equation (58) tells us that if \( R_1 \) increases or, which is the same, the price of the commodity bond goes down, the demand for it will go up.

Before taking the partial of equation (47) with respect to \( R_2 \), we note that whenever \( \rho > 0 \), the market serves as a hedge against inflation in the sense that the value of this asset goes up at the same time that investors need it the most: when commodity prices go up.

\[
\frac{\partial D^k_1}{\partial R_2} = \frac{-\rho MT^k}{(1-\rho^2)\sigma_1 \sigma_2}
\]

(3.59)
Therefore, if $\rho > 0$, it implies $\frac{\partial D^k}{\partial R_2} < 0$. In other words, whenever $R_2$ decreases or $Q2$ goes up, the demand for commodity bonds will also go up. Thus, when $\rho > 0$, commodity bonds and the market act as substitutes.

$$\frac{\partial D^k}{\partial R_f} = \frac{MT^k}{(1-\rho^2)\sigma_2^2 \sigma_2 \sigma_1} (\rho \sigma_1 - \sigma_2)$$  (3.60)

We introduce the commodity bond's market beta, defined as:

$$\beta_1 = \frac{\text{cov}(\tilde{e}_1, \tilde{e}_2)}{\text{var}(\tilde{e}_2)} = \frac{\rho \sigma_1}{\sigma_2}$$  (3.61)

and notice that when $\beta_1 > 1$, it implies $\rho \sigma_1 - \sigma_2 > 0$.

Equation (60) indicates that when default-free bond prices go down so that $R_f$ goes up, there will be a greater demand for commodity bonds when their market beta is greater than 1. Remembering that $\sigma_1 = e_1 \sigma_1$, the commodity bonds' market beta is $e_1$ times the commodity's market beta.

$$\frac{\partial D^k}{\partial \sigma_1} = \frac{MT^k}{(1-\rho^2)\sigma_2} \left[ \rho \frac{R_2 - R_f}{\sigma_2} - 2 \frac{R_2 - R_f}{\sigma_1} \right]$$  (3.62)

By using a continuous time framework, Breeden (1979) derived an intertemporal pricing relationship that must hold at each instant in time:

$$R_1 - R_f = \frac{\beta_{1C}}{\beta_{2C}} (R_2 - R_f)$$  (3.63)
where $\beta_{1c}$ is the consumption-beta for asset 1, defined by:

$$\beta_{1c} = \frac{\text{cov}(\varepsilon_1, \text{dln } C)}{\text{var (d ln C)}}$$  \hspace{1cm} (3.64)$$

With the use of these relationships, equation (62) becomes:

$$\frac{\partial D^k_1}{\partial \sigma_1} = \frac{MT^k_1 (R_2 - R_f)}{(1-\rho^2) \sigma_1^2} \left[ \frac{\rho}{\sigma_2} - \frac{2}{\sigma_1} \beta_{1c} \right]$$  \hspace{1cm} (3.65)$$

We are still assuming $\rho > 0$. A similar analysis can be made for $\rho < 0$. In general, we would expect $\beta_{2c} > 0$ as individuals would increase their rate of consumption when the market is going up. However, the sign of $\beta_{1c}$ is ambiguous. If commodity prices and consumption are negatively correlated, which is what would happen if higher commodity prices induced individuals to reduce their rate of consumption, equation (65) indicates a greater demand for commodity bonds when commodity prices become more volatile. Such would be the case as long as

$$\beta_{1c} < \rho \beta_{2c} \sigma_1 \sigma_2$$  \hspace{1cm} (3.66)$$

Furthermore, we notice that:

$$\frac{\partial D^k_1}{\partial \sigma_2} = \frac{\rho MT^k_1 (R_2 - R_f)}{(1-\rho^2) \sigma_1 \sigma_2}$$  \hspace{1cm} (3.67)$$
When $\rho$ is positive, it has been shown that the market and commodity bonds act as substitutes. As the market becomes more volatile, it is a less accurate hedge against price changes. This increases the demand for commodity bonds.

$$
\frac{\partial D^k_1}{\partial \rho} = \frac{MT^k}{\sigma_1(1-\rho^2)^2} \left[ 2\rho \frac{(R_1-R_f)}{\sigma_1} - (1+\rho^2) \frac{(R_2-R_f)}{\sigma_2} \right]
$$

(3.68)

In using Breeden's intertemporal pricing relationship, equation (68) becomes:

$$
\frac{\partial D^k_1}{\partial \rho} = \frac{MT^k(R_2-R_f)}{\sigma_1(1-\rho^2)^2} \left[ 2\rho \frac{\beta_{1C}}{\beta_{1C}} - \frac{(1-\rho^2)}{\sigma_2} \right]
$$

(3.68)

With the above-stated assumptions of $\rho > 0$ and $\beta_{1C} < \rho \beta_{2C} \frac{\sigma_1}{2\sigma_2}$, we notice that $-\frac{\partial D^k_1}{\partial \rho} > 0$. An intuitive explanation lies in the fact that when the commodity's correlation with the market decreases, the market becomes a less desirable hedging tool. Furthermore, it can easily be shown that when $\rho$ decreases, the variance of the total portfolio decreases due to the inclusion of commodity bonds. As a result, the latter become more attractive.

The previous analysis derived in the case of a single-good economy is also valid when relative commodity prices are fixed and individuals consume the same consumption bundle. In that case, the commodity bond described would be a CPI-bond. However, these assumptions are very
restrictive since commodity prices are known to fluctuate somewhat independently, and individuals to have differing tastes. The next section thus extends our analysis to the case of a multi-good economy with stochastic consumption opportunities.

B. THE MULTI-GOOD CASE

The model presented in this section is a multi-good extension of our previous analysis. There have been very few attempts to extend Merton's intertemporal asset pricing model and incorporate the case of many consumption goods. Long (1974) took such an approach but only in the case of a discrete-time economy. A satisfactory extension was made by Breeden (1979, 1984) in the derivation of a consumption asset pricing model and in the examination of the allocational roles of futures markets in a multi-good and multi-period economy and by Cox, Ingersoll and Ross (1985).

1. The Model

All the assumptions made in the single-good case are repeated in this section for a description of our economy. We now examine the case where there are $m$ consumption goods among which $l$ are commodity goods with $l \leq m$. The price dynamics for these goods are assumed to be generated by Ito processes.

$$\frac{dP_j}{P_j} = a_j dt + s_j dx_j \quad j = 1, \ldots, m$$

(3.70)

where $a_j$ and $s_j$ are constant.
There are \( n \) assets with returns that are also assumed to be generated by Ito processes. The first \( \ell \) assets are commodity bonds with \( \ell \leq m \leq n \).

\[
\frac{dQ_i}{Q_i} = R_i dt + \sigma_i dz_i \\
1 = 1, \ldots, n
\]

(3.71)

The default-free bond's return is given by

\[
\frac{dQ_{n+1}}{Q_{n+1}} = R_{n+1} dt = R_f dt
\]

(3.72)

Along the same lines as the single-good case, each commodity bond is a function of its own commodity price, the interest rate, and time to maturity.

\[
Q_i = Q_i(P_i, r, T_i) \\
i = 1, \ldots, \ell
\]

(3.73)

As an application of Ito's lemma, it is easy to see that \( dx_j = dz_j \) for \( j = 1, \ldots, \ell \) and that each commodity bond is perfectly correlated with its own commodity price.

We denote \( C^k_j \) the rate of consumption of good \( j \) by individual \( k \) by \( C^k_j \) and define \( e^k = \sum_{j=1}^{m} p_j C^k_j \) as the individual's rate of nominal expenditures. An analysis similar to that of the previous section shows that individual \( k \)'s budget constraint is given by

\[
dW^k = \sum_{i=1}^{n} \omega^k_i (R_i - R_f) W^k dt + (W^k R_f - e^k) dt + \sum_{i=1}^{n} \omega^k_i W^k \sigma^k_i dz_i
\]

(3.74)
or in matrix form
\[ dW^k = \omega^k (R_a - R_f)W^k dt + (W^k R_f - e^k)dt + W^k \sigma_a dz_a \tag{3.75} \]

where
- \( \omega^k \) is the portfolio weights vector for individual \( k \);
- \( R_a \) the assets return vector;
- \( \sigma_a \) the \( n \times n \) diagonal matrix of assets standard derivations; and
- \( dz_a \) the Gauss-Weiner processes vector.

At each instant, individual \( k \) is assumed to maximize a time additive von Neumann-Morgenstern utility function given by,
\[ E_o \left[ \int_0^T u^k(c^k, t) dt + B^k[W(T^k), T^k] \right] \]

where \( c^k \) denotes the rate of consumption vector for individual \( k \):
\[ c^k = (c^k_j)_j \]. Let \( u^k(e^k, P', t) = \text{Max } u^k(c^k, t) \) describe individual \( k \)'s indirect utility function for consumption expenditures and \( P' \) the transpose of the consumption good price vector. The dynamic programming methodology described in the previous section, yields the following first-order conditions:
\[ u^k_e(e^k, P', t) = J^k_W(W^k, S', t) \tag{3.76} \]

\[ \omega^k W^k = \frac{-J^k}{J^k_W} W^{\frac{-1}{a}}(R_a - R_f) - v^{\frac{-1}{a}} \frac{J^k_S}{J^k_W} \tag{3.77} \]

where \( S' \) is the transpose of the state variables vector (that is, variables that describe the investment, income, and consumption
opportunity sets), \( V_{aa} \) is the \( nxn \) variance-covariance matrix of asset returns and \( V_{as} \) the \( nxm \) covariance matrix of asset returns with the state variables.

2. **The Demand Functions**

From this point onward, the commodity prices are chosen to be the state variables. By differentiating equation (3.76) with respect to \( W \), we obtain:

\[
U_k^k (e^k, P', t) \cdot e_W^k = J_{WW}^k (W^k, S', t)
\]

Equations (3.76) and (3.78) combined, yield:

\[
-\frac{J_k^k}{J_{WW}^k} \cdot \frac{-U_k^k}{U_{ee}^k} = M T_k^k
\]

The last term in equation (3.77) denoted \( H_k^k = \frac{-J_k^k}{J_{WW}^k} \) was shown by Merton (1973) to represent individual \( k \)'s hedging demands against adverse changes in the consumption-investment opportunity set.

Equation (3.77) thus becomes:

\[
\omega_k W_k \approx M T_k V_{aa}^{-1}(R_k - R_f) + V_{aa}^{-1} H_k
\]

by aggregating across all individuals, we obtain:
\[ \omega^M_W = MT^M V^{-1}_{aa} (R^M_{a} - R^M_{a}) + V^{-1}_{aa} \omega^M_s \quad (3.81) \]

where \( H^M_S = \sum_k H^k_S \). Equations (3.80) and (3.81) together give us a new expression for the asset demand functions:

\[ \omega^k_W = \frac{MT^k \cdot M}{MT^M} \omega^M + V^{-1}_{aa} \omega^M_s \left( H^k_s - \frac{MT^k \cdot H^M_S}{MT^M} \right) \quad (3.82) \]

With an argument similar to that of the previous section, the net demand for any commodity bond across the market should be zero, thereby yielding \( \omega^M_1 = 0 \) for \( i=1, \ldots, \ell \). Furthermore, Breeden (1979) has shown that \( V^{-1}_{aa} \omega^M_s \) has, for columns, the portfolio of assets most highly correlated with the state variables, those state variables being here the commodity prices. Hence column \( j \) gives the portfolio that has the maximum correlation with state variable \( P_j \).

As an application of Ito's lemma, we saw earlier that this price is perfectly correlated with commodity bond \( Q_j \). Thus \( V^{-1}_{aa} \omega^M_s = \begin{bmatrix} \Gamma \end{bmatrix}_0 \), where \( \Gamma \) is an \( \ell \times \ell \) diagonal matrix that can be normalized to unity by proper scaling of state variables. With these new results, equation (3.82) can be re-written:

\[ \omega^k_W = H^k_1 - \frac{MT^k \cdot H^M_1}{MT^M} \quad i=1, \ldots, \ell \quad (3.83) \]

and

\[ \omega^k_W = \frac{MT^k \cdot M}{MT^M} \omega^M_1 \quad i=\ell+1, \ldots, n \quad (3.84) \]
These equations are similar to equations (3.55) and (3.56) derived in the single-consumption good case. In order to obtain the exact link between the two sets of equations, we must express \( H^k_i \) in terms of known parameters. To that effect, we introduce the additional assumption that individuals have time-additive isoelastic utility functions. Under this condition, Dieffenbach (1976) has shown that individual k's vector of percentage compensating variations in wealth for changes in the state variables is not a function of k's wealth level, or:

\[
\frac{J^k_i}{W^k_j W^k} = -\lambda^k_i
\]  

(3.85)

where \( \lambda^k_i \) does not depend on \( W^k \) but does, in general, depend on the \( P_j \)s. By differentiating with respect to \( W^k \), we obtain:

\[
J^k_{1W} W^k J^k_j W^k - J^k_i \left[ W^k_j W^k_j W^k \right] = 0
\]  

(3.86)

Replacing \( J^k_i \) with its expression from equation (3.85), the condition becomes:

\[
J^k_{1W} = -\lambda^k_i \left[ J^k_j W^k + W^k_j W^k \right]
\]  

(3.87)

or

\[
H^k_i \frac{J^k_j}{W^k_i} = \lambda^k_i \left( W^k - MT^k \right)
\]  

(3.88)

By aggregating across individuals, we obtain:
\[ H_i^M = \lambda_i^M . M - \lambda_i^{M, T} \ MT^M \]  \hspace{1cm} (3.89)

where \( \lambda_i^M = \frac{1}{M} \sum_k \lambda_i^k \ W^k \) and \( \lambda_i^{M, T} = \frac{1}{MT^M} \sum_k \lambda_i^k \ MT^k \). Therefore, the

Therefore, the demand for commodity bonds is given by

\[ \omega_i^k = \lambda_i^{k, W} - \lambda_i^{k, MT} - \frac{MT^k}{MT^M} \lambda_i^M + MT^k \lambda_i^{M, T} \]  \hspace{1cm} (3.90)

By rearranging these terms, we can derive a more useful expression for the demand for commodity bonds.

\[ \omega_i^k = \lambda_i^{k, W} [1 - \frac{MT^k}{MT^M}] + MT^k (\frac{M}{MT^M} - 1)(\lambda_i^M - \lambda_i^H) \]

\[ + MT^k (\lambda_i^{M, T} - \lambda_i^M) \] \hspace{1cm} (3.91)

The demand for commodity bonds is thus comprised of three terms, the first of which is similar to the expression obtained in equation (3.56). The reason for this is that if all individuals consumed the same bundle in the same quantities, \( \lambda_i^k \) would be equal across individuals and \( \lambda_i^k = \lambda_i \) implies both \( \lambda_i^M = \lambda_i \) and \( \lambda_i^{M, T} = \lambda_i \) so the last two terms of equation (3.91) cancel out. Furthermore, when all individuals consume the same bundle, it can be considered as one consumption good and we find the same results as in the previous section. This first term examined has a positive contribution to the demand for commodity bonds whenever individual \( k \) has a lower relative risk tolerance than the market or, which is the same, a higher relative risk aversion. The second term reveals that when the market is relatively more risk averse than unity, individual \( k \) would have an additional demand for commodity bond \( i \) when he is affected
more than the average individual by changes in that commodity's price. This would tend to be the case of commodities for which individual $k$ has a very inelastic demand. The sign of the last term is the same for all individuals and does not play a major role in the analysis of the demand for commodity bonds.
CHAPTER IV

THE PRICING OF COMMODITY BONDS

The model for valuing commodity bonds of the option type is based on the general option pricing methodology pioneered by Black and Scholes (1973) in their classic article, and which was extended by Merton (1973a). In order to derive a pricing formula for European call options, they showed that the payoff to a call option on a stock can be replicated by holding a continuously revised, self-financed portfolio comprised of stocks and risk-free bonds. This derivation thus relies on an arbitrage condition as sure profits can be made whenever the option price deviates from its replication value given by the Black-Scholes formula. In the derivation of these formulas, no assumptions are required on investors' risk preferences. The price is solely given by the cost of replicating the dynamic portfolio.

The methodology has been extended to the pricing of securities whose payoffs depend on the prices of one or more securities, not necessarily shares of stock. A few examples are given in Dunn and McDonnell (1981), Cox and Rubinstein (1984), and Mason and Merton (1985). Schwartz (1982) has used this methodology to price commodity-linked bonds issued by corporations as the former were previously defined. However, certain assumptions made in his analysis are questionable in the real world. His article assumes that the equities
of the firms issuing the commodity bonds are traded continuously in frictionless markets. Such an assumption may not hold when pricing commodity bonds issued by government agencies of developing countries. Other key assumptions are the absence of carrying costs to the commodity (for example, evaporation, obsolescence, insurance, and storage), and that the commodity is held for speculative purposes like a stock. These last assumptions are not robust and Schwartz himself mentioned that they are not likely to hold for oil. Ingersoll (1982) has noted that the rate of return required to hold the commodity should be greater than the expected rate of return on the commodity by an amount equal to the convenience yield. The convenience yield can be high for such a commodity as oil. As a result, Ingersoll concludes that the analysis made by Schwartz can be used if the term \( rP_B \) in equation (12) is replaced by \( (rP-\delta)B_p \), where \( \delta \) is the convenience yield. However, inspection of this term reveals it not to be homogenous: \( rP \) is stated in a currency—for instance, dollars—whereas \( \delta \) is stated in percentage terms.

The price of the commodity bonds is given by the replication value and we must, therefore, be very careful in deriving the exact cost of replicating a dynamic portfolio of bonds and the underlying commodity. This chapter will adopt such a methodology in first deriving the price of standard commodity bonds. A second part will include specific features of the commodity bonds as stated in Chapter II and thus determine the price of those features.
A. The Pricing of Standard Commodity Bonds

1. General Model

A standard commodity bond is the one described in the first chapter. It is a straight bond with an option feature on the spot price of the commodity. The assumptions made here are similar to those made in the previous chapter and are those necessary for the functioning of perfect capital markets. The interest rate for instantaneous riskless borrowing and lending is generated by a stationary Markov process given by:

$$dr = \alpha_r \, dt + \sigma_r \, dz_r$$  \hspace{1cm} (4.1)

For the moment, no other specifications are given for the functional forms of $\alpha_r$ and $\sigma_r$ as functions of $r$.

The commodity price is assumed to follow the Ito process given by the equation

$$\frac{dP}{P} = \alpha_p \, dt + \sigma_p \, dz_p$$  \hspace{1cm} (4.2)

$dz_r$ and $dz_p$ are Gauss-Weiner processes with

$$E(dz_r) = E(dz_p) = 0; \quad dz_r^2 = dz_p^2 = dt; \quad \text{and} \quad dz_r \, dz_p = \rho \, dt$$  \hspace{1cm} (4.3)
The rate of return required by investors to hold the commodity is given by
\[ \alpha_p = \alpha_p + \delta \] (4.4)
where \( \delta \) is the convenience yield of holding the commodity.

Therefore, the total return from holding the asset is given by
\[ \frac{dP}{P} + \delta dt = (\alpha_p + \delta) dt + \sigma_p \, dz \] (4.5)

This is equivalent to saying that the commodity is paying out a continuous dividend at a constant rate of \( \delta \). In that case, the relevant price for the option holder who does not receive the dividend is:
\[ P^* = P \, e^{-\delta \tau} \] (4.6)

where \( \tau \) is time to maturity \( \tau = T-t \)

McDonald and Siegel (1984) note that \( P^* \) can be interpreted as the price paid today for delivery of the underlying commodity at time \( T \) or, which is the same, the present value of the forward price:
\[ P^* = N \, e^{-r \tau} \] (4.7)

As a result of Ito's lemma, we obtain:
\[ dp^* = \frac{\partial p^*}{\partial \tau} (-\delta \, dt) + \frac{\partial p^*}{\partial p} \, dp + \frac{1}{2} \frac{\partial^2 p^*}{\partial p^2} \] (4.8)
which can be re-written

$$dP* = \delta P* dt + e^{-\delta \tau} dp$$

$$dP* = \delta P* dt + e^{-\delta \tau} [p\alpha dt + p\sigma dz_p]$$

$$dP* = (\alpha + \delta)P* dt + pe^{-\delta \tau} \sigma dz_p$$

$$dP*/P* = (\alpha + \delta) dt + \sigma dz_p$$  \hspace{1cm} (4.9)$$

which is the same as equation (4.5).

The price of a default-free bond is assumed to depend only on the interest rate and time to maturity.

$$G = G(r, \tau)$$  \hspace{1cm} (4.10)$$

It follows from Ito's lemma that:

$$dG = -\frac{\partial G}{\partial \tau} dt + \frac{\partial G}{\partial r} dr + \frac{1}{2} \frac{\partial^2 G}{\partial r^2} dr^2$$

$$= \left[ \frac{1}{2} \sigma^2 \frac{G}{r} + \alpha \frac{G}{r} - G_{\tau} \right] dt + \sigma \frac{G}{r} dz_r$$
Thus,

\[
\frac{dG}{G} = \frac{1}{G} \left[ \frac{1}{2} \sigma_r^2 C_{rr} + \alpha_r C_r - C_\tau \right] dt + \frac{\sigma_r G_r}{G} dz_r
\]

(4.11)

\[
\frac{dG}{G} = \alpha_G dt + \sigma_G d\tau
\]

with

\[
\alpha_G = \frac{1}{G} \left[ \frac{1}{2} \sigma_r^2 C_{rr} + \alpha_r C_r - C_\tau \right]
\]

and

\[
\sigma_G = \frac{\sigma_r G_r}{G} \quad d\tau = dz_r
\]

Similarly, the price of a commodity bond is assumed to depend only on the commodity price, the interest rate, and time to maturity.

\[
B = B(P, r, \tau)
\]

(4.12)

An application of Ito’s Lemma yields:

\[
\begin{align*}
\frac{dB}{B} &= \frac{-\partial B}{\partial \tau} dt + \frac{\partial B}{\partial p} dp + \frac{\partial B}{\partial r} dr + \frac{1}{2} \frac{\partial^2 B}{\partial p^2} dp^2 \\
&+ \frac{1}{2} \frac{\partial^2 B}{\partial r^2} dr^2 + \frac{\partial^2 B}{\partial p \partial r} dPdr
\end{align*}
\]

(4.13)

By substituting the expressions of \(dp\) and \(dr\) given by equations (4.1) and (4.2), we can write:

\[
\begin{align*}
\frac{dB}{B} &= \frac{1}{B^2} \left[ \frac{1}{2} P^2 \sigma_B^2 \right] + \frac{1}{2} \sigma_B^2 p^2 + \frac{1}{2} \sigma_B^2 r^2 + \sigma_B^2 \sigma_B P + \alpha_B B + \alpha_B B - B_\tau \right] dt \\
&+ \frac{PB \sigma_p \sigma_r}{B} dz_p + \frac{B \sigma_r}{B} dz_r
\end{align*}
\]

(4.14)

where \(\sigma_{pr} = \rho \sigma_p \sigma_r\).
The insight of the Black-Scholes methodology lies in the discovery that the payoff to a contingent claim can be replicated by holding a continuously revised portfolio of bonds and the underlying security. In that case, to avoid arbitrage, the return from holding either the claim or the portfolio should be the same

$$\frac{dB + Cdt}{B} = X_1(P,t)rdt + X_2(P,t)\frac{dG}{G} + X_3(P,t)\frac{dP^*}{P^*}$$  \hspace{1cm} (4.15)$$

where

$$X_1 + X_2 + X_3 = 1$$  \hspace{1cm} (4.16)$$

and $P^*$ is the relevant speculative price for the investor that has to hold and store the commodity.

Implicit in the left-hand side of equation (4.15) is the assumption that coupon payments are made continuously. Although coupon payments are generally made annually or semi-annually, most bonds are traded with interest that accrues daily. Thus the model given by equation (4.15) is just a convenient way of approximating the way in which fixed-rate securities are actually traded.

By equating, respectively, the coefficients of $dz_p$ and $dz_r$ in equation (4.15), we obtain

$$\frac{PB}{B} \sigma_p = X_3 \sigma_p$$  \hspace{1cm} (4.17)$$

and

$$\frac{B \sigma_r}{B} = \frac{C \sigma_r}{G}$$  \hspace{1cm} (4.18)$$
This yields the following relation:

\[ X_2 = \frac{B_r G}{G_r B} ; \quad X_3 = \frac{PB}{B} \quad \text{and} \quad X_1 = 1 - X_2 - X_3 \quad (4.19) \]

By equating the coefficient of \( dt \) in equation (4.15) and substituting for \( X_1, X_2, \) and \( X_3 \) from equation (4.19), we obtain the partial differential equation governing the value of commodity bonds at every point in time.

\[
\frac{1}{2} \sigma_p^2 B_{pp} + \frac{1}{2} \sigma_r^2 B_{rr} + \rho \sigma_p \sigma_r B_{pr} + (r-\delta)PB + (\alpha - \lambda \sigma_r)B - B_r - rB + C = 0 \quad (4.20)
\]

where \( \lambda = \frac{\sigma_C}{\sigma_G} \) is the market price of interest rate risk. Brennan and Schwartz (1980) have shown that \( \lambda \) will be the same for all interest-dependent securities.

The first thing to notice is that the \( rPB_p \) term in the Schwartz derivation is replaced by \( (r-\delta)PB \) and not by \( (rP-\delta)B \) as was stated by Ingersoll (1982). Furthermore, this equation and, therefore, the price of the commodity bond is independent of the rate of growth of the commodity price \( \alpha_p \). This was a main conclusion in the derivation of option prices. The latter were found not to depend on the rate of growth of the underlying security but rather, among other terms, on its volatility.

At maturity, the holder receives the principal amount \( F \) plus the excess, if any, of the commodity's value over the exercise value.
Thus, at maturity

$$B(P,r,0) = F + \text{Max}[0, P-E]$$  \hspace{1cm} (4.21)

Another condition reflects the fact that the value of an interest-dependent security goes to zero as the interest rate approaches infinity

$$\lim_{r \to \infty} B(P,r,T) = 0$$  \hspace{1cm} (4.22)

Since the term $rPB_p$ in the Schwartz derivation is replaced by $(r-\delta)PB_p$, we can be tempted to use the results of the Schwartz calculations and substitute $r$ by $r-\delta$. However, $r$ appears twice in the partial differential equation and the second term $-rB$ is unchanged and not changed to $-(r-\delta)B$. Readers familiar with the option methodology can derive these results by using a dividend-paying stock analogy. We will, therefore, continue our analysis and derive the prices of commodity bonds.

2. **Price Derivation**

We first assume a constant interest rate so that:

$$\sigma_r = \sigma_r = \sigma_{pr} = 0$$  \hspace{1cm} (4.23)
Equation (4.20) becomes:

\[ \frac{1}{2} \sigma^2 \partial^2 p_{pp} + (r - \delta)p_B - B_T - rB + C = 0 \]  

(4.24)

We define the new function \( D \) by

\[ D(p^*, \tau) = D(pe^{-\delta \tau}, \tau) = B(p, \tau) - (C/r)[1 - e^{-r \tau}] - Fe^{-r \tau} \]  

(4.25)

when \( \tau = 0 \), it verifies

\[ D(p, 0) = B(p, 0) - F = \text{Max}[0, P-E] \]  

(4.26)

Furthermore, by taking the derivatives of equation (4.25) with respect to \( P \) and \( \tau \), we obtain the following relation:

\[ e^{-\delta \tau}D_1 = B_p \]  

(4.27)

\[ e^{-2\delta \tau}D_{11} = B_{pp} \]  

(4.28)

\[ -\delta pe^{-\delta \tau}D_1 + D_2 = B_T - Ce^{-r \tau} + rFe^{-r \tau} \]  

(4.29)

\[ rD = rB - C + Ce^{-r \tau} - rFe^{-r \tau} \]  

(4.30)

By substituting these relations into equation (4.24), we have that the function \( D \) satisfies the following partial differential equation:

\[ \frac{1}{2} (pe^{-\delta \tau})^2 \sigma^2 p_{11} + r(pe^{-\delta \tau})D_1 - D_2 - rD = 0 \]  

(4.31)
This is the same equation derived by Black and Scholes (1973) for the pricing of European call options, where \( P^* = P e^{-\delta \tau} \) is the proper speculative price. Combined with the boundary condition at maturity given by equation (4.26), we can derive an exact solution for \( D \).

\[
D(P e^{-\delta \tau}, \tau) = P e^{-\delta \tau} N(d_1) - E e^{-r\tau} N(d_2) \tag{4.32}
\]

with

\[
d_1 = \frac{\ln(P/E) + (r-\delta)\tau}{\sigma \sqrt{\tau}} + \frac{1}{2} \frac{\sigma}{p} \sqrt{\tau} \tag{4.33}
\]

and

\[
d_2 = \frac{\ln(P/E) + (r-\delta)\tau}{\sigma \sqrt{\tau}} - \frac{1}{2} \frac{\sigma}{p} \sqrt{\tau} \tag{4.34}
\]

The price of a commodity bond, assuming a constant interest rate and a constant convenience yield is thus given by:

\[
B(P, \tau) = (C/r)[1-e^{-r\tau}] + Pe^{-r\tau} + P e^{-\delta \tau} N(d_1) - E e^{-r\tau} N(d_2) \tag{4.35}
\]

and where \( d_1 \) and \( d_2 \) are given above.

In the event that we assume interest rates to be stochastic, it is the general equation (4.20) that has to be solved with the boundary conditions given by equations (4.21) and (4.22). A similar partial differential equation has been solved by Cox, Ingersoll, and Ross (1985) when the bond is a pure discount one. A satisfactory closed form solution for a coupon bond has not been found. The general
procedure for solving equations similar to equation (4.20) thus relies on numerical techniques which are well described in Smith (1978). However, the computational costs of solving this partial differential equation with two space variables and one time variable, are relatively high. Such concern has led Brennan and Schwartz (1980), in their analysis of convertible bonds, to show that within a reasonable range of short-term interest rate fluctuation, between 5% and 20%, the errors resulting from the constant interest rate model relative to the stochastic interest rate model are less than 1.6% of the bond's value. As a result and for practical purposes, they conclude that the constant interest rate model can be used without great loss in the valuation of bonds. We will, therefore, limit our analysis to the constant interest rate case.

3. Numerical Analysis

The general pricing equation to be used is given by (4.35). We will use this equation to price standard commodity bonds with a face value of \( F = \$1,000 \). Of great interest to our analysis will be the effect of not including the convenience yield \( \delta \), as was the case in the Schwartz derivation. In order to obtain the convenience yield, equations (4.6) and (4.7) give us a relation between the forward price \( N \) and the spot price \( P \) of the commodity

\[
N = Pe^{(r-\delta)\tau}
\]  

(4.36)
Thus we can infer

\[ r - \delta = \frac{1}{\tau} \ln \left( \frac{N}{P} \right) \]  

(4.37)

Futures prices for various commodities will be used as proxies for forward prices. Price data is obtained from the settling prices as they are listed in the Wall Street Journal. The reference commodity bundle for each commodity is chosen in such a way that its initial value is close to the bond's face value. The interest rate is 8.5%, which is where T-bill rates now stand. The coupon rate is chosen to be 4%. The bonds that we will evaluate are assumed to have a maturity of 5 years. Finally, in order to derive the bond's price, an estimate of the commodity's volatility must be obtained. We use the adjusted sample variance derived from monthly data on commodity prices during the past five years.

The results given in Tables 4.1 through 4.5 reveal that for commodities with a convenience yield under 2%—commodities such as silver, gold, and aluminum—the pricing error from the Schwartz model is relatively large, or under 7.2% of the bond's value. For copper, where the convenience was estimated to be 7.6%, the error term can be enormous: for example, we estimate it to be 25.83% of the bond's value when the reference commodity bundle is initially equal to the face value of the bond. We notice from the tables that the error term increases with the initial value of the commodity bundle. The largest difference comes from the valuation of oil bonds. The convenience
yield is an extremely high 36.6%, which makes the option feature worthless. The values of $d_1$ and $d_2$ necessary for the calculation of the option feature are all under $-3$. As a result, we have both $N(d_1) = 0$ and $N(d_2) = 0$ so that the oil bonds are priced like conventional coupon bonds.
Table 4.1

Silver

\[ \delta = 0.5\% \quad \sigma_p = 45.7\% \]

\[ c = 4\% \quad r = 8.5\% \quad F = 1,000 \quad \tau = 5 \]

<table>
<thead>
<tr>
<th>Commodity Bundle Value</th>
<th>Straight Bond</th>
<th>Schwartz Valuation</th>
<th>Adjusted Valuation</th>
<th>Pricing Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>816.7</td>
<td>1338.4</td>
<td>1316.8</td>
<td>1.64%</td>
</tr>
<tr>
<td>800</td>
<td>816.7</td>
<td>1178.3</td>
<td>1164.5</td>
<td>1.18%</td>
</tr>
<tr>
<td>500</td>
<td>816.7</td>
<td>971.8</td>
<td>964.5</td>
<td>0.76%</td>
</tr>
</tbody>
</table>
Table 4.2

Gold

$\delta = 1.3\% \quad \sigma_p = 22.8\%$

$c = 4\% \quad r = 8.5\% \quad F = 1,000 \quad \tau = 5$

<table>
<thead>
<tr>
<th>Commodity Bundle Value</th>
<th>Straight Bond</th>
<th>Schwartz Valuation</th>
<th>Adjusted Valuation</th>
<th>Pricing Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>816.7</td>
<td>1208.7</td>
<td>1155.5</td>
<td>4.6%</td>
</tr>
<tr>
<td>800</td>
<td>816.7</td>
<td>1047.2</td>
<td>1009.5</td>
<td>3.73%</td>
</tr>
<tr>
<td>500</td>
<td>816.7</td>
<td>821.2</td>
<td>859.6</td>
<td>1.35%</td>
</tr>
</tbody>
</table>
Table 4.3

Aluminum

$\delta = 1.9\% \quad \sigma_p = 17.7\%$

c = 4\% \quad r = 8.5\% \quad F = 1,000 \quad \tau = 5$

<table>
<thead>
<tr>
<th>Commodity Bundle Value</th>
<th>Straight Bond</th>
<th>Schwartz Valuation</th>
<th>Adjusted Valuation</th>
<th>Pricing Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>816.7</td>
<td>1184.8</td>
<td>1105.2</td>
<td>7.2%</td>
</tr>
<tr>
<td>800</td>
<td>816.7</td>
<td>1019.1</td>
<td>966.8</td>
<td>5.41%</td>
</tr>
<tr>
<td>500</td>
<td>816.7</td>
<td>848.8</td>
<td>837.3</td>
<td>1.37%</td>
</tr>
</tbody>
</table>
Table 4.4

Copper

$\delta = 7.6\% \quad \sigma_p = 19.7\%

c = 4\% \quad r = 8.5\% \quad F = 1,000 \quad \tau = 5$

<table>
<thead>
<tr>
<th>Commodity Bundle Value</th>
<th>Straight Bond</th>
<th>Schwartz Valuation</th>
<th>Adjusted Valuation</th>
<th>Pricing Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>816.7</td>
<td>1193.9</td>
<td>948.8</td>
<td>25.83%</td>
</tr>
<tr>
<td>800</td>
<td>816.7</td>
<td>1029.2</td>
<td>876.2</td>
<td>17.46%</td>
</tr>
<tr>
<td>500</td>
<td>816.7</td>
<td>857.9</td>
<td>823.1</td>
<td>4.23%</td>
</tr>
</tbody>
</table>
Table 4.5

Crude Oil

\[ \delta = 36.6\% \quad \sigma_p = 15.5\% \]

\[ c = 4\% \quad r = 8.5\% \quad F = 1,000 \quad \tau = 5 \]

<table>
<thead>
<tr>
<th>Commodity Bundle Value</th>
<th>Straight Bond</th>
<th>Schwartz Valuation</th>
<th>Adjusted Valuation</th>
<th>Pricing Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>816.7</td>
<td>1178.2</td>
<td>816.7</td>
<td>44.26%</td>
</tr>
<tr>
<td>800</td>
<td>816.7</td>
<td>1004.5</td>
<td>816.7</td>
<td>22.99%</td>
</tr>
<tr>
<td>500</td>
<td>816.7</td>
<td>841.9</td>
<td>816.7</td>
<td>3.08%</td>
</tr>
</tbody>
</table>
B. Inclusion of Special Features

The previous analysis derived the prices of standard commodity bonds, i.e., straight coupon bonds where the holder has the option at maturity to receive the larger between the spot price of the commodity bundle and the face amount. Most of the bonds previously issued are not that simple. They include additional features that we will now evaluate.

1. Option Feature on an Average of Commodity Prices

Commodity bonds are issued by entities which are major players in the market for the underlying commodity. One concern holders of those bonds may have is the ability of those entities to manipulate prices on the maturity date and make the option feature worthless. As a result, in most instances, the option feature at maturity involves not the spot price of the commodity but rather an average of commodity prices taken from a given period of time preceding the maturity date.

\[ \bar{P} = \frac{1}{n} \sum_{t=T-n+1}^{T} P_t \]  

(4.39)

The commodity price is generated by an Ito process given by equation (4.2), with \( \alpha_p \) and \( \sigma_p \) constant, so that

\[ dP_t = \alpha_p P_t dt + \sigma_p P_t dz \]

By summing over \( t \) and dividing by \( n \), we obtain
\[ d\tilde{P} = \alpha_p \tilde{P} \, dt + \sigma_p \tilde{P} \, dz_p \]

or

\[ \frac{d\tilde{P}}{\tilde{P}} = \alpha_p \, dt + \sigma_p \, dz_p \]

(4.40)

The average price follows a similar process with the same expected rate of return. The results of our previous analysis can thus be used as we substitute \( P \) by \( \tilde{P} \), when \( n \) is small relative to the life of the issue.

The value of the commodity bond is given by:

\[ B(\tilde{P}, \tau) = \frac{C}{r} \left[ 1 - e^{-r\tau} \right] + \frac{F e^{-r\tau}}{(1-e^{-r\tau})} \tilde{P} - \tilde{P} e^{-\delta \tau} N(\tilde{d}_1) - \tilde{P} e^{-\tau} N(\tilde{d}_2) \]

(4.41)

where

\[ \tilde{d}_1 = \frac{\ln(\tilde{P}/F) + (r-\delta)\tau}{\sigma_p \sqrt{\tau}} + \frac{1}{2} \sigma_p \sqrt{\tau} \]

(4.42)

\[ \tilde{d}_2 = \frac{\ln(\tilde{P}/F) + (r-\delta)\tau}{\sigma_p \sqrt{\tau}} - \frac{1}{2} \sigma_p \sqrt{\tau} \]

(4.43)

2. Stochastic Principal Amount

A large number of commodities are priced in U.S. dollars, whereas the debt outstanding in the world is not entirely denominated in dollars. This raises the question of pricing commodity bonds when the principal amount of the bond and the commodity are denominated in different currencies. For the sake of simplicity, we will assume that the commodity is priced in dollars; that the bond is priced in the foreign currency; and we will adopt the view of the U.S. investor. We
will face both commodity price risk and and currency risk. Let $x$ be the exchange rate between the currencies, i.e., dollars per unit of the foreign currency. If $F^f$ represents the bond's face value in terms of the foreign currency, we have $F = xF^f$, with $x$ random and generated by an Ito process:

$$\frac{dx}{x} = \alpha_x dt + \sigma_x dz_x$$  \hspace{1cm} (4.44)$$

so that

$$\frac{dF}{F} = \alpha_x dt + \sigma_x dz_x$$  \hspace{1cm} (4.45)$$

At maturity, the holder of the bond receives

$$\text{Max}[P,F] = F + \text{Max}[P-F,0]$$  \hspace{1cm} (4.46)$$

where both $P$ and $F$ are risky.

Stulz (1982) has priced a European call option on the maximum of two risky assets, where the payoff at maturity is given by

$$\text{Max}\{\text{Max}(P,F) - K,0\}$$  \hspace{1cm} (4.47)$$

If we choose the exercise price $K$ to be zero, his analysis gives us directly the price of our commodity bond when $F$ is risky. Using our previous notations, and the notations in Stulz (1982), the price of a commodity bond is:
\[ B(P,F,\tau) = D(\text{Pe}^{-\delta \tau}, 0, \tau) + D(F, 0, \tau) - M(\text{Pe}^{-\delta \tau}, F, 0, \tau) + C/r \left(1 - e^{-r\tau}\right) \]

\[ = \text{Pe}^{-\delta \tau} + F - \left[\text{Pe}^{-\delta \tau} - \text{Pe}^{-\delta \tau} N(d_1) + FN(d_2)\right] + C/r \left(1 - e^{-r\tau}\right) \]

\[ = F + \text{Pe}^{-\delta \tau} N(d_1) - FN(d_2) + C/r \left[1 - e^{-r\tau}\right] \quad (4.48) \]

where

\[ d_1 = \frac{\ln(P/F) - \delta \tau}{\sigma \sqrt{\tau}} + \frac{1}{2} \sigma \sqrt{\tau} \quad (4.49) \]

\[ d_2 = \frac{\ln(P/F) - \delta \tau}{\sigma \sqrt{\tau}} - \frac{1}{2} \sigma \sqrt{\tau} \quad (4.50) \]

\[ \sigma^2 = \frac{\sigma^2}{\rho_{px}^2} + \frac{\sigma^2}{\rho_{px}^2} - 2\rho_{px} \sigma \rho_{px} \sigma \quad (4.51) \]

\( \rho_{px} \) is the correlation coefficient between the commodity price and the exchange rate.

The same result can be obtained by using equation (4.46) and the analysis by Fischer (1978) when he derives the price of a European call option where the exercise price is uncertain. Using his notation, a hedge security can be found in the currency itself so that \( r_h = \alpha x \). The call option is thus found to be

\[ D(P,F,\tau) = \text{Pe}^{-\delta \tau} N(d_1) - F N(d_2) \]

where \( d_1 \) and \( d_2 \) were previously defined.
The commodity bond is thus worth

$$B(P,F,\tau) = C/r \left[1-e^{-\frac{\tau}{r}} \right] + F + D(P,F,\tau)$$  \hspace{1cm} (4.52)

which is the same as equation (4.48).
CHAPTER V

COMMODITY BONDS FROM AN LDC PERSPECTIVE

A. THE NEED TO HEDGE

The economic recession of 1981-82 reduced the demand for many primary commodities, thus depressing both trade and prices of those commodities. Such a phenomenon has been widely observed in previous recessions. In effect, Figure 1 indicates that the current dollar weighted index of 33 non-fuel commodity prices started climbing in March 1983 as the U.S. was pulling out of the recession. However, that same figure reveals a drop in the index after June 1984, a drop which can be attributed to many factors.

The 1983-84 recovery was driven by the U.S. as the other OECD countries were slow to come out of the recession and developing countries had installed harsh austerity measures in order to cope with their external debt problems. The commodity price index rose by 15% in the first five quarters of the recovery while the American Gross Domestic Product (GDP) increased by only 14% during the same period. However, it is argued that the index should rise slower than the GDP, as was the case in the previous recoveries, since excess capacity and stocks accumulated during the recession will first be drawn on before any upward demand pressure is felt on prices. The rapid rise in
Figure 1

Weighted Index of Commodity Prices
Current U.S. Dollars (1977 = 100)
prices can thus be attributed to isolated effects from the supply side of the market. Indeed, the U.S. introduced a payment-in-kind acreage reduction program which reduced the 1983 acreage by 80 million acres. This was followed by a severe drought which further reduced the U.S. grain supply. Similar droughts and civil unrest in South America, West Africa, and Asia also contributed to the rise in the index above that expected. Thus, the 1984 drop in prices results partly from a downward market revision resulting from the 1983 overshooting.

The commodity price index is stated in current dollars and the dollar was very strong after June 1984. Although the exact impact of an appreciation of the dollar on commodity prices is difficult to measure, the appreciation will tend to put downward pressure on the dollar price of those commodities. For instance, most commodities are denominated in dollars and for a commodity importing country, an appreciation of the dollar relative to the local currency is equivalent to a higher local price for that commodity. This will be translated by downward pressure on the demand for that commodity, for the same dollar price, and thus downward pressure on the dollar price of that commodity. For a commodity exporting country, an appreciation of the dollar is equivalent to a higher local selling price and results in an upward pressure on the supply for that commodity. This also tends to depress the dollar price of that commodity.

Interest rates also have adverse effects on commodity prices. High interest rates tend to depress commodity prices directly by
increasing the cost of storage as defined by Brennan (1958) and indirectly by reducing the expectation of economic growth. Figure 2 reveals the high volatility of current dollar commodity prices over the past 35 years. This volatility is exacerbated by factors on which LDCs and industrial producers have no influence, factors such as droughts, volatile exchange rates, and volatile interest rates. The interest rate effect is double for most developing countries since it is precisely when their interest bill is highest that their earnings from commodity exports will be lowest due to depressed commodity prices, other things being equal. The outstanding LDC debt is not entirely dollar-denominated, but interest rates do tend to move together. The volatile nature of export earnings can be seen from Figure 3 which plots in constant dollars the purchasing power index of primary commodities exported by developing countries in terms of imported manufacturers. The effect is greater for those developing countries listed in Table 1 for which exports of primary commodities account for more than 50% of their total export earnings.

These facts reveal a serious need for developing countries to protect themselves from such volatility and to hedge it. Rolfo (1980) has argued for the use of futures markets by countries subject to both price and quantity risk and Lessard (1977,1980) has stressed the need for developing countries to shift commodity price risk to the financial markets. Before we analyze the various means available to developing countries to achieve these tasks, we will first give an appropriate definition of hedging, since this simple concept is very often misunderstood.
Figure 2

Weighted Index of 33 Commodity Prices
(Excluding Energy)
Current U.S. Dollars (1977 = 100)
Figure 3.

Purchasing Power Index of 33 Primary Commodities
Exported by Developing Countries
in Terms of Imported Manufactures
Constant U.S. Dollars (1977 = 100)
<table>
<thead>
<tr>
<th>Country</th>
<th>Commodity</th>
<th>% of Exports</th>
</tr>
</thead>
<tbody>
<tr>
<td>Belize</td>
<td>sugar</td>
<td>54.4</td>
</tr>
<tr>
<td>Colombia</td>
<td>coffee</td>
<td>51.9</td>
</tr>
<tr>
<td>Cuba</td>
<td>sugar</td>
<td>76.4</td>
</tr>
<tr>
<td>Ecuador</td>
<td>petroleum</td>
<td>56.2</td>
</tr>
<tr>
<td>El Salvador</td>
<td>coffee</td>
<td>67.6</td>
</tr>
<tr>
<td>Mexico</td>
<td>petroleum</td>
<td>69.6</td>
</tr>
<tr>
<td>Venezuela</td>
<td>petroleum</td>
<td>66.5</td>
</tr>
<tr>
<td>Algeria</td>
<td>petroleum</td>
<td>64.2</td>
</tr>
<tr>
<td>Angola</td>
<td>petroleum</td>
<td>71.8</td>
</tr>
<tr>
<td>Barundi</td>
<td>coffee</td>
<td>88.7</td>
</tr>
<tr>
<td>Congo</td>
<td>petroleum</td>
<td>85.1</td>
</tr>
<tr>
<td>Egypt</td>
<td>petroleum</td>
<td>63.5</td>
</tr>
<tr>
<td>Ethiopia</td>
<td>coffee</td>
<td>62.1</td>
</tr>
<tr>
<td>Gabon</td>
<td>petroleum</td>
<td>70.3</td>
</tr>
<tr>
<td>Liberia</td>
<td>iron ore</td>
<td>52.0</td>
</tr>
<tr>
<td>Mauritania</td>
<td>iron ore</td>
<td>70.0</td>
</tr>
<tr>
<td>Mauritius</td>
<td>sugar</td>
<td>63.2</td>
</tr>
<tr>
<td>Nigeria</td>
<td>petroleum</td>
<td>94.3</td>
</tr>
<tr>
<td>Rwanda</td>
<td>coffee</td>
<td>61.2</td>
</tr>
<tr>
<td>Uganda</td>
<td>coffee</td>
<td>93.3</td>
</tr>
<tr>
<td>Zambia</td>
<td>copper</td>
<td>91.7</td>
</tr>
<tr>
<td>Indonesia</td>
<td>petroleum</td>
<td>53.2</td>
</tr>
<tr>
<td>Iran</td>
<td>petroleum</td>
<td>83.6</td>
</tr>
<tr>
<td>Iraq</td>
<td>petroleum</td>
<td>99.1</td>
</tr>
<tr>
<td>Syria</td>
<td>petroleum</td>
<td>58.7</td>
</tr>
<tr>
<td>Fiji</td>
<td>sugar</td>
<td>51.5</td>
</tr>
</tbody>
</table>

Source: World Bank
The *Dictionary of Modern Economics* (1983) defines "hedging" as "an action taken by a buyer or seller to protect his income against a rise in prices in the future." Similarly, a World Bank Glossary on financial terminology (1984) defines a hedge as "a security acquired to cover possible loss on speculative investments." The two definitions given here are representative of some of the misconceptions on hedging. In both instances, hedging is defined as a means to cover losses from a position held in the cash market. It is, however, not emphasized that a price has to be paid for such protection. The fact is that the main purpose of hedging is to reduce price risk or reduce the volatility in revenues resulting from that risk. A Merrill Lynch brochure (1978) gives a more accurate definition and describes hedgers as investors "trying to assure themselves reasonable profit margins by dampening price volatility. They are willing to pass up the chance of greater rewards in order to face less risk." It is surprising how such a simple concept can still be so misinterpreted. Various discussions with government officials and highly ranked managers of certain international organizations reveal that hedging is considered to be a speculative action, when the converse happens to be true. It is precisely when you do not hedge and are willing to bear the full risk in hopes of higher returns that you act as a speculator. These managers display what Lessard (1980) calls an asymmetry in their incentives. If the treasurer of a corporation or the finance minister of a country enters into a futures contract, for example, and prices turn out to be favorable, resulting in a loss on the futures side, he may be held responsible for that loss. If, however, prices turned
out to be unfavorable, it would be seen as an act of God, something on which the economic entity had no influence. Little consideration will be given to the fact that the entity bought insurance against price variations and will have to pay a price for such protection. As we now examine the various means available to developing countries for hedging, we can already have a sense of the forms of action most likely to be taken.

B. THE MEANS AVAILABLE TO HEDGE

1. International Commodity Agreements (ICAs)

ICAs are by far the most important course of action on which developing countries rely. This concept has been used for over a century as a means of adjusting commodity prices. Agreements exist for such commodities as cocoa, coffee, natural rubber, olive oil, sugar, and tin. The stated principal objective of the various ICAs is generally the stabilization of prices, which is what hedging is all about. However, as a result of the asymmetry in their incentives, producers are more concerned about how low prices will fall and consumers about how high they will rise. The emphasis placed on price stabilization is rather an inducement to the importing countries to enter these agreements. The producing countries' main interests lie in devising price support systems, which are achieved mainly through the allocation of export quotas or the installment of buffer stocks. In both cases, prices are allowed to fluctuate freely within a
predetermined range. When prices fall through the floor, export quotas will be applied or the buffer-stock manager will enter the market and purchase sufficient amounts of the commodity. Either action will tend to raise the price of the commodity and bring it back within the range. When prices hit the ceiling, export quotas will be loosened or the buffer-stock manager will sell the commodity in the spot market in order to drive the price down in the range. The price range may be fixed through the life of the agreement or can be reset at various periods in time.

The effectiveness of the commodity agreements in which buffer stocks exist has been impaired by the limited availability of funds. In theory, a buffer stock is viable as long as prices fall through the floor as often as they hit the ceiling, and for reasonable lengths of time. In such a case, the amount of funds generated from a sale of the commodity due to high prices will offset the amount needed to buy back the commodity when prices are low. But, as we have seen, prices do not fluctuate nicely around a given range. They tend to deviate considerably and for a long time from their previous level. Furthermore, that deviation has, on average, been on the downside during the 1980s, thus requiring an even greater amount of funds to support commodity prices. As a result, whenever the buffer-stock managers are limited in the amount of funds they have access to, the system will be ineffective, as the recent tin crisis shows. In order to get around this constraint, the United Nations Conference for Trade and Development (UNCTAD) adopted in June 1980, the creation of a
Common Fund for commodities. The fund was intended to "facilitate the conclusion and functioning of International Commodity Agreements, particularly concerning commodities of special interest to developing countries." This was to be achieved by pooling the funds from the various ICAs which would be associated with the common fund and have this pool available to finance buffer stocks of various ICAs. The common fund would come into effect when it is ratified by at least 90 member states and when half the funds necessary for the second account of the fund, or $280 million, is collected. This second account should enable the undertaking of development initiatives linked to an improvement in the production and marketing of commodity products. While the second condition has been met, the number of states to have signed the accord still falls short of 90. As a result, price stabilization through buffer stocks remains, in most cases, a theoretical concept.

Export quotas are the primary economic instrument used by the coffee agreement, the sugar agreement, and most commodity cartels. The system, as it operates within the coffee agreement, sets a global quota for exports from member exporting countries to member importing countries. Member countries' exports to non-member importers are not controlled. Within the global quota, export quotas are allocated to each exporting country. The system has worked well for the coffee agreement but not for the sugar agreement where the problem seems to be one of inducing members to reduce the supply in order to maintain prices. Pindyck (1983) notes that for a cartel to be successful,
there should be a relatively inelastic demand for the commodity, the supply functions of those producers that are not cartel members should also be inelastic, and there should prevail a mechanism by which cartel discipline can be maintained. In many instances, where the three necessary conditions were not met, developing countries have tried to implement a price support system through the use of export quotas or cartel pricing. The motive presumably is the greater control they can thus exact and an avoidance of the financial markets for which there seems to be a complete distrust. The ineffectiveness of most commodity agreements, among other matters, should make a linkage with the financial markets a goal to be sought.

2. Futures Markets

The futures exchanges have been used for over a century by producers seeking to shift price risk through an appropriate market mechanism. Futures contracts are very often confused with forward contracts. Although the two are very similar in nature, there are important differences between them. An investor who takes a long position in a forward contract commits himself to buying a stated quantity of an asset at a price set today (the forward price) on a stated date in the future (the maturity date). The value of the contract is initially zero and changes as the spot price changes. If the spot price at the maturity date is higher than the initial forward price, the investor has a gain since he must buy at a price lower than the prevailing market price. Thus, in a forward contract, no money
changes hands until the maturity date. With a futures contract, no money also changes hands initially and the value of the contract is also zero when the contract is initiated. However, as futures prices change, the party against whom prices have moved must deliver the difference in the prices to the other party so that the value of a futures contract is always zero. In order to facilitate this continuous transfer of funds, both parties must make margin deposits in interest-bearing accounts. Funds will be drawn from or added to these accounts as prices change. Margin is generally 5 to 10% of the futures price, for a given contract.

It is widely believed that margin is the only investment to be made when entering a futures contract. However, many investors have been wiped out when the margin constituted the maximum they could invest. If prices move tremendously, funds needed can surpass the initial margin so that new investments will have to be made. Breeden (1981) shows that an amount equivalent to five times margin should be set aside by a prudent investor. Despite these differences between forward and futures contracts, Cox, Ingersoli, and Ross (1981) have shown that forward prices and futures prices will be equal if interest rates are non-stochastic.

The main advantage of futures markets is the insurance they offer against price risk. By entering in various futures contracts, a producer can lock in the price at which the commodity will be sold in the future. Price discovery is another benefit, in the sense that
economists, forecasters, and investors can derive information on the market's expectations about the future. Tesler (1981) argues against these two benefits as explaining the existence of futures markets, but rather argues that such markets exist because they provide an "invisible hand" for trading. Opponents of futures markets argue that they enable sophisticated speculators to manipulate prices to the detriment of outsiders who are less well informed. Such concern in the U.S. has led to tight regulation of the U.S. futures markets. In addition to self-regulating processes installed by most exchanges, the Commodity Futures Trading Commission, a federal regulatory agency oversees the activities of American futures exchanges. The CFTC has strict laws against investors taking huge positions in these markets for fear of having them corner the market or manipulating prices like the Hunt brothers tried to do with silver in 1981.

As a result, commodity producing LDCs cannot hedge the entirety of their output through this channel when they are a big player in the world market for that commodity. As we saw earlier, these countries are the ones that would benefit most from diversifying and hedging their risks. These laws are made to protect the small investors and cannot be considered to be unfair. The recent tin crisis in the London Metal Exchange, where laws are not as strict, reveals the danger of letting one player take huge positions in a commodity. The crisis erupted on October 24, 1985, when the manager of the International Tin Council's price-support stockpile was unable to make good on his commitment to buy 68,000 metric tons of tin for future
delivery. There were no exchange requirements for margin payments, no clearinghouse to spot overs extended positions, and no position or price limits set by the Exchange. The London Metal Exchange provides the world's main pricing mechanism for aluminum, copper, lead, nickel, tin, and zinc. Prices of these metals have also fallen since the crisis. Traders worry that if LME member firms are not paid by the ITC, they will have to reduce their holdings in the other metals, thus causing prices to fall. This crisis can only increase the developing countries' distrust of futures markets.

3. Countertrade

Countertrade is defined as a financing scheme in which the settlements are made in the form of physical goods or commodities rather than money. One variant of a countertrade agreement has existed for centuries and is called barter. In such an instance, one party exchanges a stated quantity of a good or commodity for a stated quantity of another good. The transactions take place almost simultaneously or, in most instances, within a year. The success of a barter agreement relies heavily in the matching of interests of both parties. Such a constraint has caused our modern economies to move away from barter to monetary economies. A justification for barter in our modern days has often been given in the foreign exchange shortage many developing countries are facing. This argument is unfounded since the very earnings generated by the sale of one commodity can be used to buy the other commodity assuming, of course, that the
transaction takes place quasi-simultaneously and that the exchange rate does not make any drastic jumps in the interim. Furthermore, in order to set the terms of the agreement, both parties will refer to the monetary value of their commodities.

A second form of countertrade is the "buyback arrangement" where one party typically imports production facilities and agrees to deliver at some future date a stated amount of the output. Such agreements have typically involved the financing of processing plants in developing countries. Unlike the barter agreement, the two transactions are separated in time. The producer can lock in today the earnings from the future delivery of the output, which in most cases is not a primary commodity. Although this agreement does include an insurance against the volatility of the future price of the output, it is nevertheless very project-specific. Neither of the two variants of the countertrade examined offer the producer protection against the risk in future commodity prices.

The third variant, called a "counterpurchase agreement", can serve as a hedge against commodity price risk. The arrangement typically has one party importing certain goods or commodities and committing itself to export at an agreed date, a specified amount of a commodity. A close look at this transaction reveals it to be a combination of spot purchase and short forward for the first party. Therefore, the main advantages offered by this form of countertrade will be the same as those offered by forwards contracts. As we saw
earlier, these contracts offer similar services to futures contracts but while futures contracts generally have maturities of less than two years, forward contracts are by no means restricted in their maturity. Furthermore, while futures contracts do not exist for all commodities, forward contracts can be created for any asset. However, there is no market for forward contracts and the terms of the contracts have to be negotiated between the two parties. This means loss of efficiency due to lack of information, most probably from the part of the developing country.

C. THE ADVANTAGES OF COMMODITY BONDS

The main arguments given previously show the importance of having developing countries rely on a market mechanism in order to hedge away their exposure rather than on negotiations which is what International Commodity Agreements and Forward Contracts boil down to. We have shown commodity bonds to be some combination of straight bonds and commodity options, and futures contracts to be among the most efficient means available for a producer to hedge commodity price risk. As a result, the advantages that commodity bonds have over futures contracts will at least incorporate those that commodity options have over futures contracts.

There are regulatory limits on how much a futures price can move in a single day. As a result, futures prices cannot move quickly to equilibrium to incorporate new information. Commodity options are not
subject to such limits and can move rapidly in equilibrium if, for instance, OPEC announces a change in policy. Futures contracts require a margin deposit and most investors lose their deposit when they are not able to meet their margin calls on time. By issuing commodity bonds, developing countries can eliminate the risk associated with margin calls. Futures contracts have a maturity of less than a year and exist for a limited amount of commodities. By issuing commodity bonds, developing countries can create a larger maturity hedge for any commodity. This is most relevant for commodities subject to the hazards of the business cycle, for instance, raw materials. Commodity bonds include an option which increases the value of the bond and enables the issuer to borrow below prevailing market rates.

Furthermore, developing countries have learned the sad reality that bankers lend to the rich. When the rich become poor, bankers will keep on lending to them if it is in their interest to do so. This stresses the need for developing countries to be more linked into the world financial system. Countries that have done so have been able to re-schedule their debts more easily than isolated countries whose insolvency would not impair the world financial system and that have relied mainly on a few international institutions. By issuing commodity bonds, countries have access to the world financial system where securities are priced more efficiently than through negotiations.
CHAPTER VI

CONCLUSION

A general description of the characteristics of commodity bonds has been given in this thesis. An analysis of the demand for commodity bonds was made in an economic environment where both stochastic investment and stochastic consumption opportunities exist. The analysis was carried through by using Merton's (1971) continuous-time model in our multigood setting. It has been shown that individuals who are relatively more risk averse than the market will have a greater demand for commodity bonds. Furthermore, when the market is relatively more risk averse than unity, individuals who are affected more than the average by changes in the commodity price will also have a greater demand for commodity bonds. Such will also be the case of companies or governments whose income relies greatly on commodity prices. To those individuals/governments, commodity bonds will serve a hedging purpose and reduce income variability resulting from volatility in commodity prices. However, Samuelson (1985) points out that the viability of a commodity bonds market cannot be guaranteed by simply letting risk-prone speculators issue commodity bonds to risk-averse hedgers. The market must be guided by a commercial function and involve major players, corporations, or governments.
The role to be played by commodity-producing LDCs is, therefore, crucial. Aside from fulfilling their hedging purpose, commodity bonds can offer them many more advantages than possible with International Commodity Agreements, Futures markets or Countertrade. Commodity bonds will link them to the financial markets and give them access to the Eurobond market. Furthermore, commodity bonds of the option type will enable them to borrow at rates much lower than those prevailing in the markets. Our previous analysis has revealed that an optimal time to issue these bonds would be at the bottom of a recession. At that time, interest rates and financing costs are relatively higher and commodity prices tend to be depressed. Commodity bonds would appeal to investors at that time since the latter would expect the economy to pick up and with it commodity prices. This would give more value to the option feature of commodity bonds.

The prices at which these bonds should sell in the market were determined by using the option methodology pioneered by Black and Scholes (1973). Our analysis extends the study by Schwartz (1982) and includes the convenience yield of holding and storing the commodity as well as other features not considered in his article. However, the consideration of default risk was not made in this thesis. When dealing with developing countries, such risk can be a crucial element in the pricing of financial instruments. Nevertheless, that risk is greatly reduced when the issuer holds a sufficient amount of the commodity available. Such covenants exist in some of the bonds described in Chapter 2. When commodity prices jump up, the stock of commodity can be used to repay the loan.
Budd (1983) notes that issuers without natural access to the underlying commodity take an enormous risk by issuing commodity bonds. France will learn this the hard way in 1988 when it will have to repay an estimated FF60 billion on an original issue of FF6.5 billion gold "Giscards". These considerations would tend to limit the use of commodity bonds to hard commodities or raw materials, commodities that can be stored and available over the life of the issue, commodities for which there is no quantity risk. Commodity bonds can still be issued for soft commodities but default risk will be discounted in their prices. With inflation far from its double-digit highs of the early 1980s, it remains to be seen whether investors will acknowledge the investment role to be played by commodity bonds as opposed to their initial role as hedges against inflation.
REFERENCES


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