LONGITUDINAL AND TORSIONAL VIBRATION OF DRILL STRINGS

BY

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by

Khaja Z. Khan

Submitted to the Department of Ocean Engineering on May 9, 1986 in partial fulfillment of the requirements for the Degree of Master of Science in Ocean Engineering

ABSTRACT

The purpose of this study was to develop a set of analytical and numerical models of the linear longitudinal and torsional vibration response of drill strings.

Longitudinal and torsional natural frequencies and mode shapes of the drill string are estimated for different boundary conditions at the bit and at the top. Appendix A deals with the derivation of the partial differential equation and shows the implementation of a computer program to get the natural frequencies and mode shapes for different boundary conditions. Longitudinal and torsional transfer functions are calculated between the difficult to measure downhole properties such as dynamic bit displacement and the dynamic fluctuations in WOB and also between easily measured mechanical variables on the surface such as axial force, torque or acceleration and the downhole properties. Appendix B deals with the development of the theory and derivation of these transfer functions.

The results have relevance to the basic understanding of the forces developed at the bit and at the surface and their effect on the drilling efficiency.

Thesis Supervisor: J. Kim Vandiver

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This work is dedicated to my late brother, Asif, to whom I owe so much.
NOMENCLATURE

A Cross-sectional area of the drill string, sq.ft.

C Speed of propagation of longitudinal waves in the material of the drill string, fps

C\_θ Speed of propagation of torsional (shear) waves, fps

E Modulus of elasticity, lbs/sq.ft.

F Axial force in the string

G Shear modulus, lbs/sq.ft.

h\_a Length of segment in the BHA, ft.

h\_b Length of segment in the drill pipe, ft.

J Polar moment of inertia of the string, ft\(^4\)

K Spring constant of drawworks, lbs/ft.

K\_θ Torsional spring constant of rotary table, lb-ft/rad

K\_B Spring constant at the bottom end, lbs/ft.

K\_B\_θ Torsional spring constant at the bottom end, lb-ft/rad

L Length of the drill string, ft.

M Mass of swivel and travelling block, lb-sec\(^2\)/ft.

M\_θ Mass moment of inertia of rotary drive, lb-ft-sec\(^2\)

N Number of segments for discretization of the drill string

R Viscous damping constant, lb-sec/ft\(^2\)

R\_θ Torsional damping constant, lbs-sec/rad

T Torque in the string, lb-ft.

t Time, sec

u Longitudinal displacement of any point in drill string, ft.

w Circular frequency, rad/sec

θ Amplitude of vibration in angular displacement of any point in drill string, rad
\( x \) Axial location coordinate

\( \rho \) Mass per unit length of drill string, \( \text{lb-sec}^2/\text{ft}^2 \)

\( \rho_\theta \) Mass polar moment of inertia per unit length of drill string, \( \text{lb-sec}^2 \)

\( \zeta \) Modal damping factor
INTRODUCTION

Vibrations of drill strings have long been of interest to the petroleum production industry because of the damage they cause to the surface drilling equipment and to the drill string itself. In fact the effects of axial vibration can even be observed at the surface in the form of Kelly bouncing and whipping of the draw works cables. Torsional vibrations are normally not seen from the rig floor because the rotary table drive tends to "fix" the vibrational angular motion at the surface. Nonetheless, large dynamic torque can be generated at the top.

The goal of this thesis is to study the longitudinal and torsional vibration response of drill strings. It essentially consists of two main sections. The first section deals with the eigenvalue solution of the drill strings wherein the natural frequencies and mode shapes are estimated for different boundary conditions. The latter section deals with the calculation of longitudinal and torsional transfer functions between arbitrarily chosen points on the drill string. The study is limited to torsional and longitudinal vibrations only and lateral motions of the string (due to bending, buckling, whirling, whipping, etc.) are neglected. For the sake of simplicity, the effect of different types of friction, such as fluid, rubbing, and material, which act along the string, is approximated by viscous damping. Furthermore, it is assumed that the torsional and longitudinal motions considered are independent.

Longitudinal and torsional vibration of bars are governed by the well-known wave equation. Appendix A is devoted to the derivation of the governing partial differential equation and the implementation of a computer program that gives the eigenvalues and eigenvectors for different
boundary conditions. The program LONTOR is based on the finite difference method which readily applies to any type of boundary condition. The program essentially generates the finite difference matrix and uses a standard eigensolver to get the natural frequencies and the mode shapes.

Longitudinal and torsional transfer functions are calculated between the difficult to measure downhole properties such as dynamic bit displacement and the dynamic fluctuations in weight on bit (WOB) and also between easily measured mechanical variables on the surface such as axial force, torque or acceleration and the downhole properties. An analytical method called the mobility method of vibration analysis proposed by Paslay [1] is found to be most versatile and pertinent to the problem addressed in this study and, hence, is adopted. Other authors [2-5] have adopted different approaches to the solution of vibration response but none of them attempt to find the relationship between the downhole properties and the surface variables as a function of rotary speed (excitation frequency).

The mobility method is useful when the steady state motion, which results from external forces varying sinusoidally with time, is known for subsystems which are to be mechanically connected in various ways to form the given system. The drill string is modelled to include the effects of the mass and stiffness of draw works and the derrick. The actual lengths and diameters of the drill pipe, drill collars, and Kelly have been retained in the model and are treated as subsystems. The vibration characteristics of each of these subsystems are separately determined and they are then connected analytically by requiring that the boundary conditions on each subsystem be satisfied. This leads to the desired solution. This method has particular merit when it is desired to combine a
simple longitudinal model of the drill pipe and derrick to a coupled axial-bending model of the bottom hole assembly.
MODELLING OF DRILL STRING

A typical drill string consists of the Bottom Hole assembly (BHA),
drill pipe, and Kelly. The upper end of the Kelly is attached to a swivel.
The drill string can be raised and lowered by means of a travelling block
which attaches to an elastic wire cable. The BHA consists of a number
drill collars of circular cross section. The drill collars have more or
less the same sectional properties and hence, can be assumed as a single
unit with uniform cross section. The BHA constitute roughly about 1/10th
to 1/12th of the entire drill string length and the rest is accounted for
by the drill pipe. The drill pipe consists of a large number of pipe
sections, each about 30-35 feet long. The drill pipe sections are rigidly
joined together by heavy couplings called tool joints. The tool joints
have area and area moment of inertia 4 to 6 times that of the drill pipe.
However, Bradbury and Wilhurt [6] showed that the tool joints have
negligible effect on the longitudinal and torsional vibrations. Therefore,
the drill pipe is also assumed as a long cylinder having uniform cross
section.

A lot of uncertainty exists regarding the modelling of end boundary
conditions of the drill string. Some authors have modelled the top
boundary condition as being fixed and ignoring the effect of mass of the
swivel and travelling block and the stiffness of the elastic wire cable and
derrick. A realistic model should include the effect of these to
accurately predict the forces and motions developed at the top. Its
importance comes into sharper focus when we try to relate the BHA
properties like displacement and forces to those at the surface. There is,
however, less uncertainty regarding the boundary condition at the lower end
of the string. The cutting action of the bit gives rise to bit motions which are periodic in nature. This is accompanied by forces being developed at the bit. Therefore, at the bit, a displacement and a force is assumed to be acting. A schematic drawing of an idealized drill string used to calculate the transfer functions is shown in Figure (1).

For the eigenvalue solution, the drill string is modelled for three combinations of end boundary conditions as follows:

1. fixed at top and bottom
2. fixed at top and free at bottom
3. mass-spring at top and spring at bottom

It is immediately obvious that the last case is the most realistic one as it accounts for the mass and stiffness of the draw works and derrick at the top. Also, a spring at the bit provides a means of varying the bottom end condition from fixed to free or to any intermediate value by choosing an appropriate value of the spring constant. The first two cases are included to compare the natural frequencies for different end conditions. Moreover, the first two cases provide a means for testing the accuracy of the computer program LONTOR. The program was tested by choosing a uniform rod, the natural frequencies and mode shapes of which are known in advance from the well known wave equation for uniform rods. The results were found to be in excellent agreement with each other.
DISCUSSION OF RESULTS

The natural frequencies and mode shapes of the drill string are obtained for different end conditions. The method adopted here is the finite-difference which easily applies to any type of boundary condition. The implementation of the computer program LONTOR to calculate the natural frequencies and mode shapes is discussed in Appendix A.

The transfer functions between the BHA properties and also between the BHA and surface properties have been derived in Appendix B. The procedure is based on the mobility method of vibration analysis.

Longitudinal and torsional vibrations in drill strings are exactly analogous and, if only the symbols are changed, the same equations govern both. Hence the results for torsional vibrations can be obtained by simply replacing the areas by polar moment of inertias, mass by mass moment of inertia and longitudinal stiffness by torsional stiffness. However, the top end condition for torsional vibration will be different from the longitudinal case. For longitudinal vibration, the mass $M$ and spring constant $K$ represent the mass and stiffness of swivel, travelling block and the drawworks. However for torsional vibrations, the rotary table tends to 'fix' the angular motion of the drill string. Thus, for torsional vibrations, the portion of the drill string above the rotary table is ignored, that is, the Kelly and Mass and stiffness provided by swivel, travelling block and drawworks, $M$ and $K$ are replaced by the mass moment of inertia $M_\theta$ and torsional stiffness $K_\theta$ of the rotary table.

For a sample calculation, a drill string of length 6850 feet is chosen. The length of BHA (consisting of a number of drill collars) is assumed to be 750 feet and the rest is accounted for by the drill pipe and
Kelly. A summary of the geometric properties and other input parameters used for sample calculations are listed in Table I.

1. **Eigenvalue Solution**
   
a. **Longitudinal Vibration**

   The natural frequencies and mode shapes for the drill string are obtained for three combinations of end conditions as follows:
   
i. drill string fixed at both ends
   
   ii. drill string fixed at top and free at bottom
   
   iii. Mass-spring at the top and a spring at bottom

   Table II summarizes the first eight natural frequencies for the above end condition. Case (iii) is obviously the most realistic. Moreover it provides a means of varying the bottom end condition from fixed to free or to any intermediate value. A high value of $1 \times 10^{10}$ lbs/ft. for the spring constant at the bottom was chosen in order to give a fixed end condition at the bottom. Hence the natural frequencies of case (iii) correspond very closely to mass-spring at top and fixed at bottom boundary condition. The inclusion of a mass and spring at the top has the effect of softening the top boundary condition with the result that the natural frequencies for case (iii) are lower than case (i). As expected, case (ii) gives the lowest first natural frequency.

   Figures 2, 3 and 4 show the first two modes of longitudinal vibration for the three cases under consideration. As can be seen from the figures, there is a change in the amplitude of the wave form at the intersection of the BHA and the drill pipe. This can be explained as follows: at the interface of the BHA and drill pipe there is a sudden change of section
from $A_1$ to $A_2$. Since the drill collar and drill pipe are connected rigidly, the intersection point experiences the same displacement and force. But the drill pipe section being smaller by a factor of about 4.5 will have to undergo a large strain to maintain the same force as on the BHA side. Thus there will be an increase in the strain in the drill pipe manifested by an abrupt change in the slope of the curve of the mode shape at the intersection point.

b. **Torsional Vibration**

Table III shows the first eight natural frequencies of torsional vibration for the three end conditions under consideration. It should be noted here that the length of the drill string is reduced to 6800 feet since the Kelly is ignored. Moreover the cross-sectional areas are replaced by the polar moment of inertias. Further, the rotary table is assumed to provide a mass moment of inertia and torsional stiffness of 1200 lb-sec-ft$^2$ and $1.0 \times 10^6$ lbs-ft/rad respectively.

Figures 5, 6 and 7 show the first two torsional mode shapes for the three end conditions. The high value of the mass moment of inertia and torsional stiffness of the rotary table essentially gives a fixed end condition at the top. This is evident from Figure 7 where the angular displacements at the top are almost zero. Except for this the torsional mode shapes are similar to the longitudinal ones.

2. **Transfer Functions**

a. **Longitudinal Vibration**

The transfer functions have been derived by breaking up the
composite drill string (Figure 1) into four independent subsystems as shown in Figure B.1. Thus the original dimensions of the drill collar, drill pipes, and Kelly have been retained. Also the top boundary condition idealized as a Mass-spring system is modelled as an independent subsystem whose vibration characteristics have been determined separately in Appendix B. The mass spring values used in the example were obtained from reference 2. Finally, the transfer functions are derived by combining these subsystems to represent the composite drill string. The transfer functions derived express the relationship between the following properties:

a. Bit-Bit
   1. Bit acceleration and weight on bit (WOB)

b. Bit-Top of Kelly
   2. Bit acceleration and force at top of Kelly
   3. Acceleration at top of Kelly and WOB
   4. Acceleration at top of Kelly and bit acceleration
   5. Force at top of Kelly and WOB

c. Bit-Lower end of Kelly
   6. Bit acceleration and force at lower end of Kelly
   7. Acceleration at lower end of Kelly and WOB
   8. Acceleration at lower end of Kelly and bit acceleration
   9. Force at lower end of Kelly and WOB

Transfer functions 2 to 5 provide a means of relating the BHA properties such as dynamic bit acceleration and the dynamic fluctuations in WOB (or
TOB) to the surface variables such as axial force, torque or acceleration. A very convenient location for making measurements at the surface however would be at the lower end of the Kelly. Thus, transfer functions 6 to 9 relate the BHA properties to the corresponding ones at the lower end of the Kelly. All example transfer functions were computed using the data in Table I.

Figures 8 to 16 show the longitudinal transfer functions plotted as a function of the exciting frequency. Figure 8 shows the relationship between the bit acceleration and the dynamic fluctuations in WOB. The point which is immediately obvious from this transfer function is the importance of the drill collars to the vibration response of the overall drill string. The multiple peaks indicate resonance of the overall drill string. The envelope of the frequency response curve indicates the influence of drill collar length.

Figures 11 and 15 are the transfer functions of acceleration at the top end and lower end of the Kelly with respect to the bit acceleration respectively. The peak acceleration at the Kelly in both the cases occur at about 2.5 Hz which corresponds to a rotary speed of 50 rpm using a tricone bit. The same phenomenon is noticed in the transfer function relating the forces at the top to the bit force as seen in Figures 12 and 16. However, using Daering's simplified equation [3] to predict the natural frequency of the fundamental longitudinal mode, we get a value of 5.6 Hz which corresponds to a rotary speed of 112 rpm. But from figures 11, 12, 15 and 16 it is clear that resonance occurs at a significantly lower frequency than the one predicted by Daering's simplified formula. This fact is also supported by field measurements taken on drill strings.
and reported by Wolf and Zackenhous [7].

Figures 11 and 15 also indicate that peak accelerations at the top are developed which are about 1.5 times the bit acceleration. But the peak forces developed at the top as indicated in Figures 12 and 16 are about 2.5 times the bit force (WOB). These values are damping dependent. In this case 1% damping was assumed.

b. **Torsional Vibration**

For torsional vibrations the Kelly is ignored as discussed earlier and the drill string consisting of the BHA and the drill pipe only is considered. However the mass moment of inertia and the torsional stiffness provided by the rotary table at the top is included. The example uses very crude values and better values are needed. The transfer functions calculated for torsional vibration relate the BHA properties to those at the top of the drill pipe. Figures 17 to 21 show these transfer functions. Figure 17 shows the relationship between the bit tangential acceleration and the torque developed at the bit (TOB). It is similar to the longitudinal case (Figure 8) except that at higher frequencies the smaller peaks get completely damped out and we see only the envelope under the frequency response curve.

Figure 20 relates the tangential acceleration at the top to that at the bit. As expected, the torsional motion seen at the top is very small due to the large mass moment of inertia and stiffness of the rotary drive which essentially fixes the drill pipe at the top. At higher frequencies even these small motions get completely damped out. Figure 21 indicates that the torques are developed at the top which are as high as 2.3 times
the torque at the bit (TOB).
CONCLUSION

A procedure for estimating the natural frequencies and mode shapes for different end boundary conditions have been developed. Transfer functions which relate the difficult to measure downhole properties like the dynamic fluctuations of the WOB, TOB and axial and tangential accelerations to easy to measure similar properties at the surface have been developed.

The results indicate that large motions and forces are developed at the top at certain frequencies. The maximum forces predicted at the top are about 2.5 times those at the bit for 1% model damping. Longitudinal motions of 1.5 times the bit motion are developed at the surface, with 1% damping. A direct consequence of this may be reduced drilling efficiency and rough running. The driller can ensure smooth running either by drilling at a lower rotary speed or beyond the speed at which motions are developed. However, smooth running topside does not guarantee the best operating conditions down hole. These results suggest that predicted transfer functions and measured forces and motions at the top can be used to compute forces and motions at the bit. The transfer functions derived are for a simple longitudinal and torsional model of the drill string where the bending effects are completely ignored. Therefore a further study should model the drill string for coupled axial-bending effects.
<table>
<thead>
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<th></th>
<th>O.D. (inches)</th>
<th>I.D. (inches)</th>
<th>Area (feet **2)</th>
<th>Length (feet)</th>
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<tbody>
<tr>
<td>BHA</td>
<td>6.25</td>
<td>2.64</td>
<td>0.1749</td>
<td>750.0</td>
</tr>
<tr>
<td>Drill Pipe</td>
<td>4.64</td>
<td>3.82</td>
<td>0.0378</td>
<td>6050.0</td>
</tr>
<tr>
<td>Kelly</td>
<td>6.50</td>
<td>3.0</td>
<td>0.1814</td>
<td>50.0</td>
</tr>
</tbody>
</table>

\[ M = 20,000 \text{ lbs} = 612.12 \text{ slugs} \]
\[ K = 6,400,000 \text{ lbs/ft.} \]
\[ K_B = 1.0 \times 10^{10} \text{ lbs/ft.} \]
\[ M_\theta = 1,200 \text{ lbs-sec-ft}^2 \]
\[ K_\theta = 1.0 \times 10^6 \text{ lbs-ft/rad} \]
\[ K_{B\theta} = 1.0 \times 10^{10} \text{ lbs-ft/rad} \]
\[ C = 16,860 \text{ ft/sec.} \]
\[ C_\theta = 10,650 \text{ ft/sec.} \]
\[ E = 4.32 \times 10^9 \text{ lbs/sec}^2 \]
\[ \zeta = 0.01 \]
<table>
<thead>
<tr>
<th>Natural Frequency Number</th>
<th>Natural Frequencies at Hz</th>
<th>Mass-spring at Top Spring at Bottom</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>1.343</td>
<td>0.467</td>
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<td>2</td>
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<td>1.566</td>
</tr>
<tr>
<td>3</td>
<td>3.957</td>
<td>2.817</td>
</tr>
<tr>
<td>4</td>
<td>5.086</td>
<td>4.115</td>
</tr>
<tr>
<td>5</td>
<td>6.00</td>
<td>5.425</td>
</tr>
<tr>
<td>6</td>
<td>7.08</td>
<td>6.736</td>
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<td>7</td>
<td>8.34</td>
<td>8.038</td>
</tr>
<tr>
<td>8</td>
<td>9.647</td>
<td>9.321</td>
</tr>
<tr>
<td>Natural Frequency Number</td>
<td>Natural Frequencies at Hz</td>
<td></td>
</tr>
<tr>
<td>--------------------------</td>
<td>--------------------------</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Fixed at Top and Bottom</td>
<td>Fixed at Top and Free at Bottom</td>
</tr>
<tr>
<td>1</td>
<td>0.861</td>
<td>0.275</td>
</tr>
<tr>
<td>2</td>
<td>1.714</td>
<td>0.972</td>
</tr>
<tr>
<td>3</td>
<td>2.540</td>
<td>1.779</td>
</tr>
<tr>
<td>4</td>
<td>3.255</td>
<td>2.612</td>
</tr>
<tr>
<td>5</td>
<td>3.790</td>
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</tr>
<tr>
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<tr>
<td>7</td>
<td>5.297</td>
<td>5.124</td>
</tr>
<tr>
<td>8</td>
<td>6.970</td>
<td>5.947</td>
</tr>
</tbody>
</table>
FIG. 1 SCHEMATIC OF DRILL STRING
FIG. 2 Longitudinal Mode Shapes for Fixed-Fixed case
FIG. 3 Longitudinal Mode Shapes for Fixed-Free case
FIG. 4 Longitudinal Mode Shapes for Mass-Spring at Top Spring at Bottom
FIG. 5  Torsional Mode Shape for Fixed-Fixed case
FIG. 6 Torsional Mode Shapes for Fixed-Free case
FIG. 7 Torsional Mode Shapes for Mass Spring at Top Spring at Bottom
FIG. 8  XFER FUNC OF BIT AX AND WOB
FIG. 9 XFER FUNC OF BIT AX AND FORCE AT UPPER END OF KELLY
FIG. 10  XFER FUNC OF AX AT UPPER END OF KELLY AND WOB
FIG. 11  XFER FUNC OF AX AT UPPER END
OF KELLY AND BIT AX
FIG. 13 XFER FUNC. OF BIT AX AND FORCE AT LOWER END OF KELLY
FIG. 15  XFER FUNC OF AX AT LOWER END OF KELLY BIT AX
FIG. 16  XFER FUNC OF FORCE AT LOWER END OF KELLY AND WOB
FIG. 17 XFER FUNC OF BIT AX AND TOB
FIG. 18 XFER FUNC OF BIT AX AND TORQUE AT TOP
BIBLIOGRAPHY


APPENDIX A

DERIVATION OF WAVE EQUATION AND IMPLEMENTATION OF COMPUTER

A.1 Equation of Motion

Figure A.1(a) shows the forces acting on a differential element of the drill string along with the coordinate system used in the analysis. A function of both x and t, u represents the longitudinal displacement of a point along the string at any given time. The damped, forced equation of motion is obtained by summing the forces that act on a differential drill string element. These forces include an internal force associated with material stress \((AE \frac{\partial^2 u}{\partial x^2})\), a viscous damping force \((R \frac{\partial u}{\partial t})\), an inertial force \((\rho \frac{\partial^2 u}{\partial t^2})\) and a distributed exciting force per unit length \((F(x,t))\).

Summing these forces and using the appropriate sign gives

\[
\rho \frac{\partial^2 u}{\partial t^2} + R \frac{\partial u}{\partial t} - AE \frac{\partial^2 u}{\partial x^2} = F(x,t)
\]

(A.1)

where \(R\) is a viscous damping constant and \(\rho\) is the mass per unit length of the string. The free, undamped equation for eigenvalue analysis is obtained by dropping the viscous and exciting forces from A.1:

\[
AE \frac{\partial^2 u}{\partial x^2} - \rho \frac{\partial^2 u}{\partial t^2} = 0
\]

(A.2)

since separation of variables works, the longitudinal displacement \(u(x,t)\) can be written as

\[
u(x,t) = u(x) T(t)
\]

(A.3)

substituting A.3 in A.2 gives

\[
AE \frac{d^2 u(x)}{dx^2} T(t) - u(x) \rho \frac{d^2 T(t)}{dt^2} = 0
\]
Let,
\[
\frac{AE}{\rho} \frac{1}{u(x)} \frac{d^2 u(x)}{dx^2} = \frac{1}{T(t)} \frac{d^2 T(t)}{dt^2} = -w^2
\]

Finally,
\[
\frac{AE}{\rho} \frac{d^2 u}{dx^2} + w^2 u = 0 \quad (A.4)
\]
\[
\frac{AE}{\rho} \frac{d^2 T}{dt^2} + w^2 T = 0 \quad (A.5)
\]
equation A.4 has no dependence on time and can be solved for the chosen boundary conditions.

A.2 Boundary Conditions

Boundary conditions at the ends of the string considered are shown in Figure 2 and are as follows:

1. fixed at top and bottom
2. fixed at top and free at bottom
3. mass-spring at top and spring at bottom

Case 1 The B.C. for this case are
\[
u(x=0) = u(x+L) = 0 \quad (A.6)
\]

Case 2 The B.C. for this case are
\[
\frac{3u}{3x} \bigg|_{x=0} = 0 \quad (A.7)
\]
\[
u(x=L=0) \quad (A.8)
\]
Case 3 Forces act at the top of the drill string and are therefore considered part of drill string boundary conditions. These forces shown in Figure A.1(b) are: inertial force of the swivel and travelling block \((M \frac{\partial^2 u}{\partial t^2})\), force associated with the pipe stress \((AE \frac{\partial u}{\partial x})\) and force acting through the stranded wire cable \((KU)\). Summation of these forces gives an equilibrium operation that serves as the boundary condition for the top end of the drill string. This equation is

\[
AE \frac{\partial u}{\partial x} + Ku + M \frac{\partial^2 u}{\partial t^2} = 0 \quad \text{at } x=L \tag{A.9}
\]

and at the bottom,

\[
AE \frac{\partial u}{\partial x} + K_u u = 0 \quad \text{at } x=0 \tag{A.10}
\]

**Finite-Difference Formulation**

The equation to solve is the one given by A.4 with the end conditions for different cases given by A.6 to A.9. The method to be implemented is the finite-difference method, which is a numerical method that readily applies to any type of boundary value problem. It consists in breaking up the drill string into \(N\) segments of length \(h\) (Figure A.2). Making use of the finite-difference equations,

\[
\left( \frac{du}{dx} \right)_j = \frac{1}{2h} \left[ -u_{j-1} + u_{j+1} \right]
\]

\[
\left( \frac{d^2 u}{dx^2} \right)_j = \frac{1}{h^2} \left[ u_{j-1} - 2u_j + u_{j+1} \right]
\]

we can come up with a set of linear equations and solve for \(w\) and \(u\), the eigen values and eigen vectors.
Finite-Difference Matrix

a. Drill string fixed at top and bottom

Unknowns: \( N-1 \) unknowns \( u_1, u_2, \ldots, u_{N-1} \)

Equations: Applying A.4 at \( j=1, \ldots, N-1 \) provides \( N-1 \) equations involving \( N+1 \) unknowns. Using equation (A.6), we get \( u_0 = u_N = 0 \), giving us a set of \( N-1 \) equations with \( N-1 \) unknowns.

Thus, applying the finite-difference equations to (A.4) we obtain:

\[
\frac{A_j E_j}{\rho_j} \underbrace{\frac{N^2}{2} [u_{j-1} - 2u_j + u_{j+1}]} + \frac{w_j^2 u_j}{} = 0
\]

But

\[
\frac{A_j E_j}{\rho_j} = c^2
\]

\[
\frac{c^2}{h^2} [-u_{j-1} + w_j - u_{j+1}] = \frac{w_j^2 u_j}{} \quad \text{at } 2 \leq j \leq N-2
\]

\[
\frac{c^2}{h_a^2} [2u_1 - u_2] = \frac{w_1^2 u_1}{} \quad \text{at } j=1
\]

\[
\frac{c^2}{h_b^2} [-u_{N-2} + 2u_{N-1}] = \frac{w_{N-1}^2 u_{N-1}}{} \quad \text{at } j=N-1
\]

Interface Condition

Due to the difference in the cross-sectional area between the drill collar and drill pipe section, a discontinuity exists at the interface of the drill collar and drill pipe. The force condition at this point is

\[
(\text{EA})_a \frac{\partial u}{\partial x} \bigg|_{\text{D.C.}} = (\text{EA})_b \frac{\partial u}{\partial x} \bigg|_{\text{D.P.}} \quad \text{Figure (A.2)}
\]

To account for this discontinuity, two fictitious points \( U_L \) and \( U_U \) on
either side of the interface point are considered. The differential equation and the force condition is applied at the interface point \( m \) and the equations are written in terms of these fictitious points. Finally, \( U_U \) and \( U_L \) are eliminated and the finite-difference equation written for the interface point in terms of the real points.

Applying the differential equation at the interface point gives

\[
(\text{EA})_a \frac{u_{m-1} - 2u_m + u}{h_a^2} + \rho_a u_m m^2 = 0 \quad (A.11)
\]

\[
(\text{EA})_b \frac{u_{m-1} - 2u_m + u}{h_b^2} + \rho_b u_m w^2 = 0 \quad (A.12)
\]

The force condition gives

\[
(\text{EA})_a \frac{u_m - u_{m-1}}{2h_a} = (\text{EA})_b \frac{u_{m+1} - u_L}{2h_b} \quad (A.13)
\]

Solving equation A.11 and A.12 for \( U_U \) and \( U_L \), we get

\[
u_u = u_m [2 - \frac{h_a}{c^2} w^2] - u_{m-1}
\]

\[
u_L = u_m [2 - \frac{h_b}{c^2} w^2] - u_{m+1}
\]

Using this in equation A.13 yields

\[
\frac{c^2}{(A/A_a + A/A_b) h_a h_b} [-2 A/A_a u_m + 2(A/A_a + A/A_b) u_m - 2 A/A_b u_{m+1}] = w^2 u_m
\]

\[
\frac{c^2}{h} [-2 C_{m1} u_{m-1} + 2 C_{m2} u_m = 2 C_{m3} u_{m+1}] = w^2 u_m
\]

where

\[
C_{m1} = \frac{h^2 A_a}{h_a (A/A_a + A/A_b)}
\]

\[
C_{m2} = \frac{h^2}{h_a h_b} \frac{(A/A_a + A/A_b)}{(A/A_a + A/A_b)}
\]

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\[ C_{m3} = \frac{h^2 A}{h_b (A_a h_a + A_b h_b)} \]

Thus, we get a tridiagonal matrix. Expressing it in the form of a standard eigenvalue problem,

\[
\begin{bmatrix}
2 & -1 & & & & \\
-1 & 2 & -1 & & & \\
& -2C_{M1} & 2C_{M2} & -2C_{M3} & & \\
& & 1 & 2 & -1 & \\
& & & -1 & 2 & \\
\end{bmatrix}
\begin{bmatrix}
\hat{u}_1 \\
\hat{u}_2 \\
\vdots \\
\hat{u}_N-2 \\
\hat{u}_N-1 \\
\end{bmatrix} = \lambda \begin{bmatrix}
\hat{u}_1 \\
\hat{u}_2 \\
\vdots \\
\hat{u}_N-2 \\
\hat{u}_N-1 \\
\end{bmatrix}
\]

we can solve for the eigenvalues and eigenvectors.

b. **Drill string fixed at top and free at bottom**

For this case, the number of unknowns are \( N \) and applying A.4 at \( j=0, \ldots, N-1 \) provides \( N \) equations involving \( N+2 \) unknowns. Boundary condition (A.8) gives \( \hat{u}_N = 0 \) and equation (A.7) gives us \( \hat{u}_N = \hat{u}_1 \). Following the same procedure as in previous case, we get the following tridiagonal matrix which can be used to evaluate the eigenvalues and eigenvectors for this case.

\[
\begin{bmatrix}
2 & -2 & & & & \\
-1 & 2 & -1 & & & \\
& -2C_{M1} & 2C_{M2} & -2C_{M3} & & \\
& & 1 & 2 & -1 & \\
& & & -1 & 2 & \\
\end{bmatrix}
\begin{bmatrix}
\hat{u}_0 \\
\hat{u}_1 \\
\vdots \\
\hat{u}_m \\
\hat{u}_{N-2} \\
\hat{u}_{N-1} \\
\end{bmatrix} = \lambda \begin{bmatrix}
\hat{u}_0 \\
\hat{u}_1 \\
\vdots \\
\hat{u}_m \\
\hat{u}_{N-2} \\
\hat{u}_{N-1} \\
\end{bmatrix}
\]
c. Mass-spring at top and spring at bottom

The boundary condition at the top as given by equation (A.9) can be used to express \( U_{N+1} \) in terms of \( U_N \) and \( U_{N-1} \).

\[
\frac{AE}{M} \frac{\partial^2 u}{\partial x^2} + Ku + M \frac{\partial^2 u}{\partial t^2} = 0 \tag{A.9}
\]

Using separation of variables to remove the time dependence yields

\[
\frac{AE}{M} \frac{du}{dx} + \left( \frac{K}{M} - w^2 \right) u = 0
\]

Applying the finite-difference equation at \( j=N \), we get

\[
u_{N+1} = u_{N-1} - \frac{2h}{A_bE_b} (K - MW^2) u_N \tag{A.14}
\]

Similarly the boundary condition at the bottom, equation (A.10) is used to express \( U_{-1} \) in terms of \( U_0 \) and \( u_1 \). That is,

\[
u_{-1} = u_1 + \frac{2K_Bh}{E_Aa} u_0 \tag{A.15}
\]

Thus, the number of unknowns in this case are \( N+1 \) and (A.4) provides \( N+1 \) equations involving \( N+3 \) unknowns. Using equations (A.11) and (A.12) we finally get a set of \( N+1 \) equations with \( N+1 \) unknowns. Now, applying the finite difference equation to (A.4), we get

\[
\frac{c^2}{h^2} (-u_{j-1} + 2u_j - u_{j+1}) = w^2 u_j \quad \text{at } j \ 1 \leq j \leq N-1
\]

\[
\frac{c^2}{h^2} \left[ 2 \left( 1 - \frac{K_B h}{A_E a} \right) u_0 - 2u_1 \right] = w^2 u_0 \quad \text{at } j=0
\]

\[
\frac{c^2}{h^2} \left[ -2u_{N-1} + 2 \left( 1 + \frac{K_B h}{A_E a} \right) u_N \right] = w^2 u_N \quad \text{at } j=N
\]
The finite difference matrix resulting from the above equations, which again is a tridiagonal, is as shown.

\[
\frac{c^2}{h^2} \begin{bmatrix}
 2C_0 & -2 & & & & \\
 -1 & 2 & -1 & -2CM_1 & -2CM_2 & -2CM_3 \\
 & -1 & 2 & -1 & & \\
 0 & -2CN_1 & 2CN_2 & & & \\
 & & & & & \\
 & & & & & \\
\end{bmatrix} \begin{bmatrix}
 u_0 \\
 u_1 \\
 \vdots \\
 \ddots \\
 u_{N-1} \\
 u_N \\
\end{bmatrix} = \omega^2 \begin{bmatrix}
 u_0 \\
 u_1 \\
 \vdots \\
 \ddots \\
 u_{N-1} \\
 u_N \\
\end{bmatrix}
\]

where

\[C_0 = (1 - \frac{K_B h}{A_a E})\]

\[CN_1 = \frac{1}{(1 + \frac{2MC^2}{A_b E h})}\]

and

\[CN_2 = CN_1 (1 + \frac{Kh}{A_b E})\]
FIG. A.2 DISCRETIZATION OF STRING
FIG. A.3 MODELING OF BOUNDARY CONDITIONS
B.1 Mobility Equations for Subsystems

For a mechanical system which possesses linear elastic and viscous properties and has mass elements which do not lead to gyroscopic forces, the steady-state displacement amplitude resulting from a force system, all of whose components vary sinusoidally with time, can be determined by a procedure known as the mobility method [8]. The procedure consists of separating the system into subsystems whose vibration properties can be conveniently determined with arbitrary force amplitudes applied to each point at which the subsystem connects to the remainder of the system. The subsystems are then mechanically joined by satisfying two interconnection requirements; the force and motion requirements. The force requirement is essentially a statement of dynamic equilibrium or balance of forces on an isolated subsystem. The motion requirement is a statement of a restriction imposed by the geometric constraints; i.e., a compatibility requirement.

Now, applying the above procedure to a drill string, Figure (B.1) shows how the composite drill string is isolated into four subsystems satisfying the force and displacement requirement. In order to find, say, the relationship between the bit displacement and the bit force, it is necessary to find the mobility $M_{44}'(W)$, that is, the ratio of longitudinal displacement at $x_4$ to the exciting force at $x_4$ in terms of the mobilities of the subsystems. But first, writing down the mobility equations

For the Mass-spring system,

$$x_1 = -M_{11}(w)f_1$$

(B.1)
Kelly

\[ x_2 = \tilde{M}_{22}(w)f_2 + \tilde{M}_{21}(w)f_1 \]  
(B.2)

\[ x_1 = \tilde{M}_{11}(w)f_1 - \tilde{M}_{12}(w)f_2 \]  
(B.3)

Drill Pipe

\[ x_3 = \bar{M}_{33}(w)f_3 + \bar{M}_{32}(w)f_2 \]  
(B.4)

\[ x_2 = \bar{M}_{22}(w)f_2 - \bar{M}_{23}(w)f_3 \]  
(B.5)

Drill Collar

\[ x_4 = M_{44}(w)f_4 + M_{43}(w)f_3 \]  
(B.6)

\[ x_3 = M_{33}(w)f_3 + M_{34}(w)f_4 \]  
(B.7)

Since no gyroscopic forces are developed,

\[ M_{44}(w) = M_{44}(w) ; \quad M_{43}(w) = M_{34}(w) \]

\[ \bar{M}_{33}(w) = \bar{M}_{22}(w) ; \quad \bar{M}_{32}(w) = \bar{M}_{23}(w) \]

\[ \tilde{M}_{22}(w) = \tilde{M}_{11}(w) ; \quad \tilde{M}_{21}(w) = \tilde{M}_{12}(w) \]

The driving-point mobilities and transfer mobilities of the subsystems have been derived by Paslay [1] and they are reproduced here.

Driving point mobilities of the subsystems,

\[ M_{xx} = \frac{1}{\rho A_x L_x} \frac{1}{w^2} + \frac{2}{\rho A_x L_x} \sum_{n=1}^{\infty} \frac{1}{[\frac{n \pi c}{L_x}]^2 - w^2 + i2\zeta w[n \pi c]} \]  
(B.I)

Transfer mobilities of the subsystem,

\[ M_{xy} = \frac{1}{\rho A_x L_x} \frac{1}{w^2} + \frac{2}{\rho A_x L_x} \sum_{n=1}^{\infty} \frac{(-1)^n}{[\frac{n \pi c}{L_x}]^2 - w^2 + i2\zeta w[n \pi c]} \]  
(B.II)

where \( x \) and \( y \) vary from 1 to 4.

The mobility of the spring-Mass subsystem is

\[ \tilde{M}_{11} = \frac{1}{K - Mw^2} \]
B.2 Transfer Function Equations

Transfer functions relating the mechanical properties like
displacements and forces at the bit and also between the bit and the top of
the drill string can be derived by systematic elimination and substitution
among the mobility equations (B.1) to (B.7).

Substituting equation (B.1) into equation (B.3)

\[ f_1 = \frac{\bar{M}_{12}}{\bar{M}_{11} + \bar{M}_{11}} f_2 \equiv F_{12} f_2 \quad (B.8) \]

Subtracting equation (B.4) from equation (B.7) gives

\[ f_2 = \frac{(M_{33} + \bar{M}_{33}) f_3 + M_{34} f_4}{\bar{M}_{32}} \quad (B.9) \]

Subtracting equation (B.2) from equation (B.5) and substituting the value
of \( f_1 \) from equation (B.8) gives

\[ f_2 = \frac{\bar{M}_{23}}{[\bar{M}_{22} + \bar{M}_{22} - F_{12} \bar{M}_{21}]} f_3 \equiv F_{23} f_3 \quad (B.10) \]

Substituting equation (B.10) into equation (B.9) yields

\[ f_3 = \frac{M_{34}}{F_{23} \bar{M}_{32} - (M_{33} + \bar{M}_{23})} f_4 \equiv F_{34} f_4 \quad (B.11) \]

Finally, substituting (B.11) into (B.6) gives the relationship between the
bit displacement and bit force.

\[ x_4 = (M_{44} + \bar{M}_{34} \bar{M}_{44}) f_4 \equiv M_{44}' f_4 \quad (B.12) \]
Transfer Functions between Bit and Lower End of Kelly

Substituting (B.11) into (B.6) gives

\[ x_4 = \left( \frac{M_{44}}{F_{34}} + M_{34} \right) f_3 \equiv M'_{43} f_3 \]  \hspace{1cm} (B.13)

Substituting the value of \( f_3 \) from (B.10) into (B.13),

\[ x_4 = \frac{M'_{43}}{F_{23}} f_2 \equiv M'_{42} f_2 \]  \hspace{1cm} (B.14)

Eliminating \( f_2 \) from equation (B.5),

\[ x_2 = \left( \frac{M_{22}}{F_{23}} - M_{23} \right) F_3 \]

\[ x_2 = \left( \frac{M_{22}}{F_{23}} - M_{23} \right) F_{34} f_4 \equiv M'_{24} f_4 \]  \hspace{1cm} (B.15)

Dividing (B.15) by (B.12), we get

\[ \frac{x_2}{x_4} = \frac{\left( x_2 / f_4 \right)}{\left( x_4 / f_4 \right)} = \frac{M'_{24}}{M'_{44}} \]  \hspace{1cm} (B.16)

Dividing (B.12) by (B.14), we get

\[ \frac{f_2}{f_4} = \frac{\left( x_4 / f_4 \right)}{\left( x_4 / f_2 \right)} = \frac{M'_{44}}{M'_{42}} \]  \hspace{1cm} (B.17)

Transfer Functions between Bit and Upper End of Kelly

Eliminating \( f_2 \) from (B.14) using (B.8), we get

\[ x_4 = \frac{M'_{42}}{F_{12}} f_1 \equiv M'_{41} f_1 \]  \hspace{1cm} (B.18)

By a series of elimination and substitution of \( f_1, f_2, \) and \( f_3 \) starting with (B.1), we get

\[ x_1 = -\hat{M}_{11} f_1 \]

\[ x_1 = -\hat{M}_{11} F_{12} f_2 \]  \hspace{1cm} from (B.8)
\[ x_1 = -M_{11}F_{12}F_{23}f_3 \quad \text{from (B.10)} \]
\[ x_1 = -M_{11}F_{12}F_{23}F_{34}f_4 \quad \text{from (B.11)} \]
\[ x_1 = M_{14}f_4 \quad \text{(B.19)} \]

Dividing (B.19) by (B.12), we get
\[ \frac{x_1}{x_4} = \frac{x_1/f_4}{x_4/f_4} = \frac{M_{14}}{M_{44}} \quad \text{(B.20)} \]

Dividing (B.12) by (B.18), we get
\[ \frac{f_1}{f_4} = \frac{x_4/f_4}{x_4/f_1} = \frac{M_{44}}{M_{41}} \quad \text{(B.21)} \]

**Torsional Transfer Functions**

The same results as derived above can be applied directly to obtain the torsional transfer functions by making the following substitutions:

- the axial force, \( f \) in the longitudinal case is replaced by the moment about the axis of the string \( T \), and the axial displacement, \( u \) is replaced by the angular rotation about the axis, \( \theta \).

The driving point and transfer mobilities for the longitudinal case are given by equation B.I and B.II. The equivalent mobilities for torsional vibration are:

**Driving point mobilities of the subsystems,**

\[ M_{xx} = \frac{1}{\rho \theta J x x} + \frac{2}{\rho \theta J x x} \sum_{n=1}^{\infty} \frac{\frac{n\pi C_\theta}{L_x} \frac{1}{L_x^2}}{\frac{n\pi C_\theta}{L_x}^2 - \omega^2 + i2\zeta \omega \frac{n\pi C_\theta}{L_x}} \quad \text{(B.III)} \]

**Transfer mobilities of the subsystem,**
\[ M_{xy} = \frac{1}{\rho \, J_x \, L_x \, \omega^2} + \frac{2}{\rho \, J_x \, L_x} \sum_{n=1}^{\infty} \frac{(-1)^n}{n \pi C_\theta \, \left[ \frac{L_x}{L_x} \right]^2 - \omega^2 + i 2 \zeta \omega \left\{ \frac{n \pi C_\theta \, L_x}{L_x} \right\}} \]  

(B.IV)

Thus, in the torsional case, the cross sectional areas, A are replaced by the polar moment of inertias, J of the string subsystems, the mass per unit length \( \rho \) is replaced by the mass polar moment of inertia per unit length, \( \rho \theta \) of string and the speed of propagation of longitudinal waves, C given by \( \sqrt{EA/\rho} \) is replaced by the speed of torsional waves, \( C_\theta \) given by \( \sqrt{GJ/\rho \theta} \) where G is the shear modulus.
FIG. B.1  SUBSYSTEMS OF THE DRILL STRING