Phase Truncation Effects in Direct Digital Frequency Synthesis

by

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Abstract

A model of an ideal direct digital frequency synthesizer is developed, and then modified to include phase truncation. An exact expression for the output spectrum of the phase-truncated synthesizer is derived. This spectrum is found to contain the desired frequency and $2Q' - 1$ spurious components, where $Q'$ is the effective number of truncated phase bits. An exact expression for the total noise power due to phase truncation is also derived; the noise power is found to depend primarily on the number of locations in the sine ROM.

The effect of frequency multiplication via full-wave rectifier frequency doublers is also explored. Experimental results are presented which indicate that frequency multiplication by a factor of $n$ on the synthesizer output increases phase truncation noise power by a factor of $n^2$; this is found to be very similar to the effect of dividing the number of ROM locations by $n$. It is found that some DDS noise components not caused by phase truncation also increase with frequency multiplication at the same rate.

It is also found that when the synthesizer output is observed for only a short time, phase truncation can restrict output frequency resolution. However, in the light of the results mentioned above, it is found that this effect is not likely to be important in practice.

Thesis Supervisor: Robert S. Kennedy
Title: Professor of Electrical Engineering
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Chapter One
Introduction and Overview

1.1 Frequency Synthesis Techniques
Currently, there are three basic frequency synthesis techniques in wide use:

- indirect synthesis (phase-locked loops and similar circuits),
- direct analog synthesis (selective mixing and division of fixed-frequency sinusoids),
- direct digital synthesis (digital table look-up of sinusoid values).

Each of these techniques has advantages, disadvantages, and tradeoffs which are fully discussed in [1]. This thesis discusses direct digital synthesis, first described in [2].

The main advantages of direct digital synthesis are rapid frequency switching, arbitrarily fine frequency resolution, and ease of implementation (it is possible to implement a Direct Digital Synthesizer (DDS) on one LSI integrated circuit [3]). The main disadvantages of direct digital synthesis are narrow output bandwidth and difficulty in achieving high spectral purity. This thesis investigates the effect of phase truncation (defined in §1.3.3) on these advantages and disadvantages.

1.2 Ideal Direct Digital Synthesizer
In order to introduce the basic concepts of direct digital synthesis, this section describes and analyzes an idealized Direct Digital Synthesizer (DDS).

1.2.1 General Description
Figure 1.1 shows an idealized DDS. The phase accumulator calculates the phase of the desired signal. This phase is used to select digital sine wave values stored in a Read-Only Memory (ROM). These values are converted to analog in an digital-to-analog (D/A) converter. The analog signal is then low-pass filtered to remove the alias frequencies.
Figure 1.1
Block Diagram of an ideal Direct Digital Synthesizer
1.2.2 Detailed Description and Analysis

This subsection provides a detailed description of the ideal DDS model, derives the Nyquist sampling theorem result for the D/A output signal, and shows that the ideal DDS output is a pure sinusoid of the desired frequency.

Using the frequency command \( f_0 \) (the desired output frequency), the phase accumulator generates samples of a linearly increasing phase \( \phi(t) \) with derivative \( d\phi/dt = 2\pi f_0 \equiv \omega_0 \). The period between samples of \( \phi(t) \) is \( T_c \), where \( T_c = 1/f_c \) and \( f_c \) is the DDS clock frequency. Each sample of \( \phi(t) \) is applied to a ROM containing values of a cosine wave, to obtain an infinite-precision digital representation of a sample of \( \cos(\phi(t)) \). This digital representation is converted to analog by a D/A converter. In this ideal case, the D/A converter is considered to produce an impulse of area \( T_c \cos(\phi(t)) \) for each sample of \( \phi(t) \) produced by the phase accumulator (the motivation for modeling the D/A output as a series of impulses is discussed in §1.3.1). Call this D/A output signal \( x_a(t) \). This signal is plotted in Figure 1.2a. If we let \( \phi(t) = 0 \) for \( t = 0 \), then \( \phi(t) = \omega_0 t \), and the signal \( x_a(t) \) is given by

\[
x_a(t) = \sum_{k=-\infty}^{\infty} T_c \cos(\omega_0 t) \delta(t - kT_c).
\]  

(1.1)

If we define the impulse train

\[
\delta_{T_c}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_c)
\]

then

\[
x_a(t) = T_c \cos(\omega_0 t) \delta_{T_c}(t).
\]

Using \( \mathcal{F}[\cdot] \) to represent the Fourier transform (see Appendix), we replace multiplication in the time domain with convolution in the frequency domain:

\[
\mathcal{F}[x_a(t)] = \frac{T_c}{2\pi} \mathcal{F}[\cos(\omega_0 t)] \ast \mathcal{F}[\delta_{T_c}(t)].
\]

(1.2)
The transform of an impulse train is another impulse train:

$$\mathcal{F}[\delta_T(t)] = \frac{2\pi}{T_c} \sum_{k=-\infty}^{\infty} \delta(t - k\omega_c)$$  \hspace{1cm} \left(\omega_c = \frac{2\pi}{T_c}\right)$$

and the transform of a cosine is two impulses:

$$\mathcal{F}[\cos(\omega_0 t)] = \pi \left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right).$$

We can now perform the convolution from equation (1.2):

$$X_a(\omega) \equiv \mathcal{F}[x_a(t)] = \pi \sum_{k=-\infty}^{\infty} \delta(\omega - (k\omega_c - \omega_0)) + \delta(\omega - (k\omega_c + \omega_0)).$$  \hspace{1cm} (1.3)

This is the well-known Nyquist sampling theorem result, with the desired cosine wave at \(\omega = \pm \omega_0\) and with alias components around multiples of the sampling frequency \(\omega_c\). This spectrum is plotted in Figure 1.3a.

The D/A output signal \(x_a(t)\) passes through an ideal lowpass filter with cutoff frequency \(f_c/2\) (the Nyquist frequency). The filtered result is the DDS output signal \(x(t)\). If we constrain \(f_0\) to be less than \(f_c/2\), then the filter removes all alias components, yielding \(X(\omega) = \pi \left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)\), and \(x(t) = \cos(\omega_0 t)\), the desired output signal.

### 1.3 Practical DDS Design

This section describes the ways in which practical DDS implementations differ from the ideal DDS described in §1.2.

#### 1.3.1 Impulses, pulses, and staircases

The most obvious difficulty in implementing the ideal DDS described above is the production of impulses at the D/A output. Actual D/A converters do not produce impulses; they produce staircases. The majority of DDS implementations use the
waveform $x_s(t)$ shown in Figure 1.2b, a staircase function with “steps” of duration $T_c$. Examination of Figures 1.2a and 1.2b shows that $x_s(t)$ can be written as the convolution

$$x_s(t) = x_a(t) * w_{T_c}(t)$$

where $x_a(t)$ is the ideal signal defined in equation (1.1) and the unit-area window function $w_{T_c}(t)$ is defined as

$$w_{T_c}(t) = \begin{cases} 
1/T_c, & \text{if } |t| < T_c/2; \\
0, & \text{otherwise}.
\end{cases}$$

Taking the Fourier transform, we obtain

$$X_s(\omega) \equiv \mathcal{F}[x_s(t)] = \mathcal{F}[x_a(t)]\mathcal{F}[w_{T_c}(t)].$$

The transform of a rectangular window is a $\sin(x)/x$ function:

$$\mathcal{F}[w_{T_c}(t)] = \frac{\sin(\omega T_c/2)}{\omega T_c/2}.$$ 

We now have the transform of $x_s(t)$:

$$X_s(\omega) = X_a(\omega) \frac{\sin(\omega T_c/2)}{\omega T_c/2}. \quad (1.4)$$

We see that $X_s(\omega)$ is equal to $X_a(\omega)$ multiplied by a $\sin(x)/x$ envelope. This spectrum is plotted in Figure 1.3b. Passing this through an ideal lowpass filter, we obtain a DDS output signal of

$$\frac{\sin(\omega_0 T_c/2)}{\omega_0 T_c/2} \cos(\omega_0 t).$$

Although most DDS implementations use the staircase function described above, some designs use the pulsed waveform $x_p(t)$ shown in Figure 1.2c, in order to reduce D/A converter noise and/or to permit the use of higher-frequency portions of the aliased spectrum (using a bandpass filter instead of a lowpass filter at the output) [4].
Figure 1.2
DDS Waveforms before output filtering
a.) Ideal (impulsive) signal spectrum $X_a(f)$

b.) Staircase signal spectrum $|X_s(f)|$

c.) Pulsed signal spectrum $|X_p(f)|$

Figure 1.3
DDS Spectra before output filtering
The output consists of pulses of duration $T_p$, and is the convolution of $x_a(t)$ with the window function $w_{T_p}(t)$, defined as

\[
    w_{T_p}(t) = \begin{cases} 
    1/T_p, & \text{if } |t| < T_p/2; \\
    0, & \text{otherwise.}
\end{cases}
\]

This yields the spectrum

\[
    X_p(\omega) = X_a(\omega) \frac{\sin(\omega T_p/2)}{\omega T_p/2}.
\]  

(1.5)

This spectrum is plotted in Figure 1.3c. Note that the ideal signal $x_a(t)$ is given by

\[
    x_a(t) = \lim_{T_p \to 0} x_p(t).
\]

To preserve generality and to simplify the analysis, this thesis assumes the ideal $x_a(t)$ case unless otherwise indicated. One should use the relations given in equation (1.4) or (1.5) before applying such results to actual DDS designs.

### 1.3.2 Phase and Frequency Quantization

In an actual DDS, the phase $\phi(t)$ from the phase accumulator is not represented by a continuous value, but by a finite-length integer. The phase is therefore quantized.

The phase accumulator is usually a binary adder and register, producing a binary number representing a value between 0 and $2\pi$ (the accumulator actually produces $\phi(t)$ modulo $2\pi$). If the accumulator is $N$ bits wide, then the phase quantum is $2\pi/2^N$.

Phase quantization does not produce any errors if one defines the desired frequency $f_0$ to be appropriately quantized. Recall that $\phi(t) = 2\pi f_0 t$, and that samples of $\phi(t)$ are produced at intervals of $T_c$. If the frequency quantum (also called frequency resolution) is defined as $1/(2^N T_c) = f_c/2^N$, so that $f_0$ is an exact multiple of $f_c/2^N$, then the output of the phase accumulator, although quantized, is exactly the desired phase. Note that if fine frequency resolution (i.e., small frequency quantum) is desired, $N$ must be large.
1.3.3 Phase Truncation

Figure 1.1 implies that the phase $\phi(t)$ passes unchanged from the phase accumulator to the ROM look-up table. This is not the case in most DDS designs. Recall from the previous subsection that the phase accumulator width $N$ must be large to achieve good frequency resolution. If $R$ is the number of bits entering the ROM, then the ROM must contain $2^R$ locations (there are techniques to reduce the number of ROM locations, but the number of locations is still proportional to $2^R$). The desired frequency resolution may dictate an accumulator width of 32 or more bits, corresponding to over 4 billion locations in the ROM! Clearly, this is an unworkable approach with present ROM technology.

The solution to this problem is phase truncation. The $R$ high-order bits of $\phi(t)$ are passed to the ROM, and the $N - R$ low-order bits are truncated (i.e., discarded). This permits arbitrarily fine frequency resolution ($N$ can be as large as desired) with reasonable ROM sizes, at the expense of spectral purity. The effect of phase truncation is the main topic of this thesis.

1.3.4 Output Filtering

Of course, the ideal lowpass filter required at the output of the ideal DDS is impossible to implement, so there will be aliased frequency components in the DDS output. The filter requirement sets a more stringent upper limit on the output frequency $f_0$ than the Nyquist frequency $f_c/2$. The output frequency is often limited to no more than $f_c/4$, allowing a filter transition band from $f_c/4$ to $3f_c/4$. Higher values of $f_0$ require progressively more complex (higher-order) output filters.

We will assume that a perfect output filter is used. Design of the DDS output filter is important, but outside the scope of this thesis.
1.3.5 Amplitude Quantization and D/A Converter Noise

The sinusoid samples stored in ROM have limited precision, so they can take only a finite number of values, i.e., the output amplitude is quantized. This introduces errors proportional to the size of the amplitude quantum.

Real D/A converters introduce errors of their own. The most important D/A error is "glitch" noise, i.e., spurious pulses during output transitions. (The pulsed waveform described in §1.3.1 attempts to remove glitch noise by zeroing the waveform during D/A transitions.) Amplitude quantization and D/A noise are important effects, but they are not in the primary scope of this thesis. They are discussed in §3.2.2 in connection with frequency multiplication.

1.4 Overview of Results

Chapter 2 derives an exact expression for the spectrum of the phase-truncated signal (eqn. 2.10). It shows that for certain values of output frequency $f_0$ (called "principal frequencies" in §2.1.1), phase truncation produces no error. The principal frequencies are those which would have been available if phase truncation had not been used. When $f_0$ is not at a principal frequency, the DDS output spectrum contains the desired signal and $2^{Q'} - 1$ spurious components, where $Q'$ is the effective number of truncated phase bits. For certain output frequencies (see §2.3.2), the spurious components take the form of sidebands around the desired component, with the level of the largest sideband approximately $1/2^R$ times that of the desired frequency, where $R$ is the number of phase bits entering the ROM look-up table. An exact expression for the total power in the spurious components caused by phase truncation is also derived (eqn. 2.14). For $2^R \gg 1$, this total power is $\approx \pi^2 2^{-2R}/3$.

Chapter 3 examines the effect of frequency multiplication via full-wave rectifier
frequency doublers. It first examines the effect on a pure sinusoid carrier which is amplitude- or phase-modulated by another sinusoid. It is found that the AM sidebands pass unaffected through the doubler, while the PM sidebands are doubled in amplitude (quadrupled in power). By experimental observation, it is found that the spurious sidebands due to phase truncation behave exactly like the sinusoid-modulation PM sidebands; this effect is similar to the effect of halving the number of sinc ROM locations. It is also found that the noise components mentioned in §1.3.5 behave partially like AM sidebands and partially like PM sidebands.

Chapter 4 examines the effect of phase truncation when the DDS output is observed for only a short time. It is found that under these conditions, DDS output frequency may be restricted, but that this effect is not likely to be noticeable in practical systems.
Chapter Two

Spectrum of the Phase-Truncated Signal

This chapter derives expressions for Fourier transform of the phase-truncated DDS output signal. It also examines the total noise power in the signal, relative to the desired frequency component.

Except where otherwise indicated, this chapter analyzes the ideal DDS described in §1.2, with two modifications: phase and output frequency are quantized (as described in §1.3.2), and phase truncation (as described in §1.3.3) is used. The reader is again reminded to apply the corrections given in equation (1.4) or (1.5), as appropriate, before using these ideal results in actual practice.

The Appendix lists the notations, variable names, and abbreviations used in this thesis. The reader is urged to consult the Appendix if uncertain about the meaning of a formula.

2.1 Derivation of the phase-truncated DDS spectrum

We now consider the DDS signal produced with the truncated phase value described in §1.3.3. We will derive the transform of the signal before output filtering: the filtered output spectrum can be obtained by application of the output filter characteristic to the spectrum derived here. For the ideal output filter, this would mean considering only the spectrum between $-f_c/2$ and $f_c/2$.

2.1.1 Phase truncation error signal $\epsilon(n)$

Recall that samples of the ideal phase $\phi(t)$ are produced at intervals of $T_c$. Let $\Phi(n)$ denote the sample of $\phi(t)$ produced at time $t = nT_c$, as represented in quantized binary form in the phase accumulator: $\Phi(n) = (2^N/2\pi)\phi(nT_c)$, where both $n$ and $\Phi(n)$ are
integers. Substituting in equation (1.1) and using \( n = t/T_c \) and \( \phi(t) = \omega_0 t \), we have the ideal signal \( x_a(t) \) given by

\[
x_a(t) = \sum_{k=-\infty}^{\infty} T_c \cos \left( \frac{2\pi}{2^N} \Phi(t/T_c) \right) \delta(t - kT_c).
\] (2.1)

In the phase-truncated synthesizer, the \( Q \) least significant bits in the binary representation of \( \Phi(n) \) are discarded, yielding the truncated phase value \( \Phi_t(n) \):

\[
\Phi_t(n) = \Phi(n) - (\Phi(n) \mod 2^Q).
\]

This truncated phase is used to look up a sinusoid value: \( \cos((2\pi/2^N)\Phi_t(t)) \). This value is applied to an ideal D/A converter, yielding the impulsive truncated-phase signal \( x_t(t) \):

\[
x_t(t) = \sum_{k=-\infty}^{\infty} T_c \cos \left( \frac{2\pi}{2^N} \Phi_t(t/T_c) \right) \delta(t - kT_c).
\] (2.2)

The only difference between equations (2.1) and (2.2) above is in the truncated phase \( \Phi_t(t) \). We can rewrite the truncated phase as

\[
\Phi_t(n) = \Phi(n) - \varepsilon(n), \quad \varepsilon(n) = (\Phi(n) \mod 2^Q),
\]

where \( \varepsilon(n) \) represents the error due to truncation of phase sample \( n \). We see that the phase-truncated signal \( x_t(t) \) is the ideal signal \( x_a(t) \) phase-modulated by the error signal \( \varepsilon(n) \).

2.1.2 Properties of the error signal \( \varepsilon(n) \)

Between successive phase samples, the phase \( \Phi(n) \) is incremented by the integer value \( \Delta_0 \), where

\[
\Delta_0 = \Phi(n) - \Phi(n - 1) = \frac{2^N}{2\pi} \omega_0 T_c = 2^N f_0/f_c.
\]
If we let $\Phi(0) = 0$, then $\Phi(n) = n\Delta_0$, and $\varepsilon(n) = (n\Delta_0 \mod 2^Q)$. If we let $d_0 = (\Delta_0 \mod 2^Q)$, then $\Delta_0 - d_0$ is divisible by $2^Q$, and we have

$$
\varepsilon(n) = (n\Delta_0 \mod 2^Q) = (n(\Delta_0 - d_0) + nd_0) \mod 2^Q = nd_0 \mod 2^Q.
$$

We see that the truncation error signal $\varepsilon(n)$ depends only on $n$, $d_0$, and $Q$. When $\Delta_0$ is represented in binary form, $d_0$ corresponds to the value in the $Q$ low-order bits: the same bits which are truncated from $\Phi(n)$ to produce $\Phi_t(n)$. Figure 2.1 illustrates the binary representations of $\Phi(n)$, $\Phi_t(n)$, $\Delta_0$, and $d_0$.

First, we note that if $d_0 = 0$, then the phase truncation error $\varepsilon(n)$ is zero for all $n$. From the above equations, and defining $R = N - Q$, we have

$$
d_0 = (\Delta_0 \mod 2^Q) = ((2^R f_0/f_c) \mod 2^Q) = ((2^Q 2^R f_0/f_c) \mod 2^Q).
$$

We see that $d_0 = 0$ will be zero if and only if $2^R f_0/f_c$ is an integer; this will be true for output frequencies $f_0$ which are exact multiples of $f_c/2^R$. We shall call these values of $f_0$ (for which $d_0 = 0$) principal frequencies. When $f_0$ is at a principal frequency, there is no error due to phase truncation, and the phase-truncated signal $\varepsilon_t(t)$ is exactly equal to the ideal signal $\varepsilon_n(t)$. The set of principal frequencies corresponds to the set of output frequencies available in the ideal, non-truncated synthesizer (see §1.3.2). The other output frequencies (between principal frequencies) are made possible by the improved frequency resolution achieved using phase truncation (see §1.3.3).

We now examine the properties of $\varepsilon(t)$ when $d_0 \neq 0$ ($f_0$ not at a principal frequency). From equation (2.3), we see that $\varepsilon(n)$ is periodic with period no greater than $2^Q$:

$$
\varepsilon(n + 2^Q) = ((d_0 n + d_0 2^Q) \mod 2^Q) = (d_0 n \mod 2^Q) = \varepsilon(n).
$$
Figure 2.1
Binary Representations of Φ(n) and related variables
The actual period of \( \varepsilon(n) \) will depend on the common factors (if any) in \( d_0 \) and \( 2^Q \). Let us remove any common factors, yielding integer values \( p \) and \( q \) such that

\[
\frac{p}{q} = \frac{d_0}{2^Q} \quad \text{p, q relatively prime.} \tag{2.4}
\]

We now rewrite \( \varepsilon(n) \) in terms of \( p \) and \( q \):

\[
\varepsilon(n) = (d_0 n \mod 2^Q) \\
= d_0 n - 2^Q \lfloor nd_0/2^Q \rfloor \\
= \frac{2^Qpn}{q} - 2^Q \lfloor np/q \rfloor \\
= \frac{2^Q}{q} (np \mod q). \tag{2.5}
\]

Since \( p \) and \( q \) are relatively prime, the period of \( (np \mod q) \) is \( q \), and therefore \( q \) is the period of \( \varepsilon(n) \).

### 2.1.3 Decimation of \( \varepsilon(n) \) and \( x_t(t) \)

In order to arrive at the spectrum \( X_t(\omega) \), we shall decompose the signals \( \varepsilon(n) \) and \( x_t(t) \) into subsets which are more well-behaved. These subsets will be recombined in §2.1.5.

Let us consider a subset of the points of \( \varepsilon(n) \) generated by selecting every \( q \) points: those points at which \( (n \mod q) = m \), where \( m \) is an integer such that \( 0 \leq m < q \). Define this subset as the function \( \varepsilon_{(m)}(n) \):

\[
\varepsilon_{(m)}(n) = \begin{cases} 
\varepsilon(n), & \text{if } (n \mod q) = m; \\
0, & \text{otherwise.}
\end{cases}
\]

Note that when \( (n \mod q) = m \),

\[
\varepsilon_{(m)}(n) = \frac{2^Q}{q} (np \mod q) = \frac{2^Q}{q} (p(n \mod q) \mod q) = \frac{2^Q}{q} (pm \mod q).
\]

We see that \( \varepsilon_{(m)}(n) \) is a much simpler function than \( \varepsilon(n) \); \( \varepsilon_{(m)}(n) \) is a periodic train of pulses of value \( 2^Q(pm \mod q)/q \). Figure 2.2 shows a typical \( \varepsilon(n) \) signal and corresponding \( \varepsilon_{(m)}(n) \) for different values of \( m \).
Figure 2.2
Typical error signal $\varepsilon(n)$ and subsets of $\varepsilon(n)$
Having taken a subset of $\varepsilon(n)$, let us now consider the corresponding subset of $x_t(t)$, which we shall call $x_{(m)}(t)$:

$$x_{(m)}(t) = \begin{cases} x_t(t), & \text{if } (t/T_c \mod q) = m; \\ 0, & \text{otherwise}. \end{cases} \quad (2.6)$$

Recalling that $n = t/T_c$, we notice that $x_{(m)}(t)$ involves only those values of $\varepsilon(n)$ contained in $\varepsilon_{(m)}(n)$, and is nonzero only when $\varepsilon_{(m)}(n)$ is nonzero. Combining the above with equation (2.2), and substituting $qk + m$ in place of $k$, we have

$$x_{(m)}(t) = \sum_{k=-\infty}^{\infty} T_c \cos \left( \frac{2\pi}{2N} (\Phi(t) - \varepsilon(qk + m)) \right) \delta(t - (qk + m)T_c)$$

$$= \sum_{k=-\infty}^{\infty} T_c \cos \left( \omega_0 t - \frac{2\pi}{2N} \left( pm \mod q \right) \right) \delta(t - (qk + m)T_c)$$

$$= \sum_{k=-\infty}^{\infty} T_c \cos \left( \omega_0 t - \frac{2\pi}{q2^R} (pm \mod q) \right) \delta(t - (qk + m)T_c).$$

Defining the phase offset $\theta_m$,

$$\theta_m = \frac{2\pi}{q2^R} (pm \mod q),$$

we have

$$x_{(m)}(t) = \sum_{k=-\infty}^{\infty} T_c \cos (\omega_0 t - \theta_m) \delta(t - (qk + m)T_c). \quad (2.7)$$

Notice that $x_{(m)}(t)$ has a fixed phase offset $\theta_m$, instead of the varying phase error $\varepsilon(t/T_c)$ contained in $x_t(t)$.

### 2.1.4 Spectrum of the signal subset $x_{(m)}(t)$

Let us define an impulse train $\delta_{(m)}(t)$:

$$\delta_{(m)}(t) = \sum_{k=-\infty}^{\infty} \delta(t - (qk + m)T_c).$$

From equation (2.7), we see that

$$x_{(m)}(t) = T_c \cos (\omega_0 t - \theta_m) \delta_{(m)}(t).$$
Taking the Fourier transform, we have

\[ X_{(m)}(\omega) \equiv \mathcal{F}[x_{(m)}(t)] = \frac{T_c}{2\pi} \mathcal{F}[\cos(\omega_0 t - \theta_m)] * \mathcal{F}[\delta_{(m)}(t)]. \]

The transform of \( \delta_{(m)}(t) \) is a phase-shifted impulse train:

\[
\mathcal{F}[\delta_{(m)}(t)] = \frac{2\pi}{qT_c} e^{j\omega_m T_c} \sum_{k=-\infty}^{\infty} \delta(\omega - (k\omega_c/q))
= \frac{2\pi}{qT_c} \sum_{k=-\infty}^{\infty} e^{j2\pi km/q} \delta(\omega - (k\omega_c/q)),
\]

and, noting that we can change a phase offset into a time shift

\[ \cos(\omega_0 t - \theta_m) = \cos\left(\omega_0(t - \theta_m/\omega_0)\right), \]

we obtain the transform

\[
\mathcal{F}[\cos(\omega_0 t - \theta_m)] = \pi e^{-j\omega_m/\omega_0} \left( \delta(\omega - \omega_0) + \delta(\omega + \omega_0) \right)
= \pi \left( e^{-j\theta_m} \delta(\omega - \omega_0) + e^{j\theta_m} \delta(\omega + \omega_0) \right).
\]

We now have the transform of \( x_{(m)}(t) \):

\[
X_{(m)}(\omega) = \frac{\pi}{q} \sum_{k=-\infty}^{\infty} e^{j2\pi km/q} \left( e^{-j\theta_m} \delta(\omega - ((k\omega_c/q) + \omega_0)) + e^{j\theta_m} \delta(\omega - ((k\omega_c/q) - \omega_0)) \right) \tag{2.8}
\]

We see that \( x_{(m)}(t) \) is nothing more than a phase-shifted sinusoid, sampled at \( qT_c \) intervals (compare equations 1.3 and 2.8).

### 2.1.5 The phase-truncated spectrum \( X_t(\omega) \)

From equations (2.2) and (2.6), we see that the signal \( x_t(t) \) can be represented as the sum of all the subsets \( x_{(m)}(t) \):

\[ x_t(t) = \sum_{m=0}^{q-1} x_{(m)}(t). \]
The Fourier transform is a linear operation, so the spectrum $X_t(\omega)$ is given by the sum of the spectra $X_{(m)}(\omega)$:

$$X_t(\omega) = \sum_{m=0}^{q-1} X_{(m)}(\omega)$$

$$= \frac{\pi}{q} \sum_{m=0}^{q-1} \sum_{k=-\infty}^{\infty} e^{j2\pi km/q} \left( e^{-j\theta_m} \delta(\omega - \left( (k\omega_c/q) + \omega_0 \right)) + e^{j\theta_m} \delta(\omega - \left( (k\omega_c/q) - \omega_0 \right)) \right) \tag{2.9}$$

Noting that the position of the frequency impulses in equation (2.9) does not depend on $m$, we rewrite the spectrum as

$$X_t(\omega) = \pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - \left( (k\omega_c/q) + \omega_0 \right)) + b_k \delta(\omega - \left( (k\omega_c/q) - \omega_0 \right))$$

where

$$a_k = \frac{1}{q} \sum_{m=0}^{q-1} e^{j(2\pi km/q - \theta_m)} \quad \text{and} \quad b_k = \frac{1}{q} \sum_{m=0}^{q-1} e^{j(2\pi km/q + \theta_m)}.$$

We now substitute the definition of $\theta_m$ into the expression for $a_k$:

$$a_k = \frac{1}{q} \sum_{m=0}^{q-1} \exp \left( \frac{j2\pi km}{q} - \frac{j2\pi (pm \mod q)}{q2^R} \right)$$

$$= \frac{1}{q} \sum_{m=0}^{q-1} \exp \left( \frac{j2\pi}{q} (km - 2^{-R}(pm \mod q)) \right)$$

After performing a similar substitution on $b_k$, we obtain our final expression for $X_t(\omega)$:

$$X_t(\omega) = \pi \sum_{k=-\infty}^{\infty} a_k \delta \left( \omega - \left( \frac{k\omega_c}{q} + \omega_0 \right) \right) + b_k \delta \left( \omega - \left( \frac{k\omega_c}{q} - \omega_0 \right) \right) \tag{2.10}$$

$$a_k = \frac{1}{q} \sum_{m=0}^{q-1} e^{j2\pi(km-2^{-R}(pm \mod q))/q} \tag{2.10a}$$

$$b_k = \frac{1}{q} \sum_{m=0}^{q-1} e^{j2\pi(km+2^{-R}(pm \mod q))/q} \tag{2.10b}$$
2.2 Interpretation of the spectral result

Equation (2.10) is an exact expression for the Fourier transform of the phase-truncated DDS signal $x_t(t)$. In this section, we will interpret this expression and make some observations about the spectrum $X_t(\omega)$.

2.2.1 Comparison with the non-truncated DDS

As indicated in §2.1.2, when $f_0$ is at a principal frequency (a multiple of $f_c/2^R$) phase truncation has no effect, so the phase-truncated DDS produces exactly the same output as the ideal DDS. At a principal frequency, $d_0 = 0$, so $p = 0$, $q = 1$, $a_k = b_k = 1$, and we see that $X_t(\omega)$ in equation (2.10) is equivalent to the ideal spectrum $X_a(\omega)$ in equation (1.3). [This serves as a cross-check of our $X_t(\omega)$ result.]

When $f_0$ is not at a principal frequency, $d_0 \neq 0$, $q \neq 1$, and the form of $X_t(\omega)$ is different from that of $X_a(\omega)$. However, because both $a_k$ and $b_k$ are periodic in $k$, with period $q$ (e.g., $a_k = a_{k+q} = a_{k+2q} = \ldots$), $X_t(\omega)$ is still periodic in frequency with period $\omega_c$. Because of this periodicity, and the fact that $x_t(t)$ is a real function (hence $X_t(\omega) = X_t^*(-\omega)$), the frequency range $0 \leq \omega \leq \omega_c/2$ characterizes the entire spectrum $X_t(\omega)$. In this frequency range, $X_a(\omega)$ has only one frequency component (at $\omega = \omega_0$), while $X_t(\omega)$ has $q$ components: the desired component at $\omega_0$ and $q - 1$ spurious components caused by phase truncation.

2.2.2 Interpretation of the values $p$ and $q$

The values $p$ and $q$ appear prominently in equation (2.10), but their physical meaning is not immediately apparent. The value $q$ can be interpreted as $q = 2^{Q'}$, where $Q'$ is the effective number of truncated phase bits. Figure 2.3 shows $p$ and $q$ for various values of the phase increment $\Delta_0$. If the least significant bit (LSB) of the binary representation of $\Delta_0$ is nonzero (meaning that $f_0$ is an odd multiple of $f_c/2^N$), then $Q' = Q$ and $q = 2^Q$. If the LSB of $\Delta_0$ is zero, then $Q' < Q$ and $q < 2^Q$. The difference $Q - Q'$
Phase Increment $\triangle_o$ with $f_0$ at a principal frequency ($p=0$, $Q'=0$, $q=1$):

<table>
<thead>
<tr>
<th>MSB</th>
<th>LSB</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 1 0 0 0 1 0 0 1 1 0 1 0</td>
<td>0 0 0 0 0 0 0 0 0 0 0 0 0 0 0</td>
</tr>
</tbody>
</table>

Phase Increment $\triangle_o$ with LSB nonzero ($Q'=Q$, $q=2^Q$):

$$p=01111001001001000001_2$$

<table>
<thead>
<tr>
<th>MSB</th>
<th>LSB</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 1 0 0 1 1 1 1 0 0 0 0 0 0 1 1 1</td>
<td>1 0 1 0 0 1 0 1 0 1 0 0 0 0 0 0 0 0</td>
</tr>
</tbody>
</table>

Phase Increment $\triangle_o$ with $0<Q'<Q$ ($q=2^{Q'}$):

$$p=1010111_2$$

<table>
<thead>
<tr>
<th>MSB</th>
<th>LSB</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 1 1 0 0 0 1 1 1 0 1 1 1 1 1 1 1</td>
<td>1 0 1 0 1 0 1 1 1 0 0 0 0 0 0 0 0 0</td>
</tr>
</tbody>
</table>

**Figure 2.3**

Binary representation of $\Delta_o$ and related variables
is the number of truncated bit positions to the right of the lowest-order nonzero bit in $\Delta_0$. If the $Q$ low-order bits of $\Delta_0$ are zero, then $Q' = 0$ and $f_0$ is at a principal frequency.

We call $Q'$ the effective number of truncated bits because if the $Q - Q'$ LSB's of the phase increment $\Delta_0$ are zeroes, the $Q - Q'$ LSB's of $\Phi(n)$ are always zero, so truncating them produces no error. Although $Q$ bits of $\Phi(n)$ are truncated (set to zero), only $Q'$ of those bits can take nonzero values.

### 2.3 Evaluation of spectral coefficients

In this section, we will evaluate the spectral coefficients $a_k$ and $b_k$ in equations (2.10a,b). Although the coefficients are simply sums of exponentials, the exponents contain modulo functions which complicate the evaluation.

Note that $a_k$ is the complex conjugate of $b_{-k}$. [Remember that $X_1(\omega)$ is the spectrum of a real-valued signal.] Therefore we shall evaluate only the coefficients $a_k$, since the coefficients $b_k$ can be derived from them by the relation $b_k = a_{-k}^*$.

#### 2.3.1 Coefficient of the desired frequency component

First, we will evaluate the coefficient of the desired frequency component in $X_1(\omega)$ at $\omega = \omega_0$. From equation (2.10), we see that this coefficient is $a_0$:

$$a_0 = \frac{1}{q} \sum_{m=0}^{q-1} e^{-j2\pi2^{-n}(pm \mod q)/q}.$$

Note that the summand above depends only on $(pm \mod q)$. By definition $p$ and $q$ are relatively prime, so as $m$ varies from 0 to $q - 1$, $(pm \mod q)$ will take on each of the integer values from 0 to $q - 1$ exactly once. Therefore, we can replace $(pm \mod q)$ with $m$ in the above expression, changing the order of summation but not changing
the result:

\[ a_0 = \frac{1}{q} \sum_{m=0}^{q-1} e^{-j2\pi 2^{-R}m/q}. \]

This is a simple sum of a geometric series, reducible with the formula \( \sum_{m=0}^{n-1} r^m = (r^n - 1)/(r - 1) \) from [5]. Therefore, we have

\[ a_0 = \frac{e^{-j2\pi 2^{-R}} - 1}{q (e^{-j2\pi 2^{-R}}/q - 1)}. \]

Using the relation \( \sin(x) = \frac{1}{2j}(e^{jx} - e^{-jx}) \), we see that \( e^{j2x} - 1 = 2j \sin(x)e^{jx} \), and we have

\[ a_0 = \frac{\sin(-\pi 2^{-R})e^{-j\pi 2^{-R}}}{q \sin(-\pi 2^{-R}/q)e^{-j\pi 2^{-R}/q}} \]

\[ = e^{-j\pi 2^{-R}(q-1)/q} \sin(\pi 2^{-R})/q \sin(\pi 2^{-R}/q). \] (2.11)

Note that at a principal frequency (where \( q = 1 \), \( a_0 = 1 \) as predicted for the ideal signal.

For most DDS designs, \( 2^R \gg \pi \), and we can use the approximations \( \sin(\pi 2^{-R}) \approx \pi 2^{-R} \) and \( \sin(\pi 2^{-R}/q) \approx \pi 2^{-R}/q \). Applying this to the magnitude of \( a_0 \) in equation (2.11) yields \( |a_0| \approx 1 \). Therefore, if the value of \( R \) is reasonably large, the power loss in the desired frequency component due to phase truncation is very small. [However, this small loss is quite useful in computing the total noise power (see §2.4).]

2.3.2 Coefficients for \( p = 1 \) and \( p = q - 1 \)

In the special case where \( p = 1 \), evaluation of \( a_k \) and \( b_k \) is quite straightforward. With \( p = 1 \), the expression \((pm \mod q)\) in equation (2.10a) reduces to just \( m \), yielding

\[ a_k|_{p=1} = \frac{1}{q} \sum_{m=0}^{q-1} e^{j2\pi(k - 2^{-R})m/q}. \]

Using the same method as in §2.3.1, we obtain

\[ a_k|_{p=1} = e^{j\pi(k - 2^{-R})(q-1)/q} \frac{\sin(\pi(k - 2^{-R}))}{q \sin(\pi(k - 2^{-R})/q)}. \]
Using the identities \( \sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta) \), and \( \cos(\pi k) = (-1)^k \), we have

\[
a_k|_{p=1} = e^{i\pi(k-2^{-R})(q-1)/q} \frac{(-1)^k \sin(-\pi 2^{-R})}{q \sin(\pi(k - 2^{-R})/q)}
= e^{-i\pi(2^{-R}+(k-2^{-R})/q)} \frac{\sin(-\pi 2^{-R})}{q \sin(\pi(k - 2^{-R})/q)}.
\]

(2.12)

If \( 2^R \gg 1 \) and \( k \ll q \), we can approximate

\[
\sin(\pi 2^{-R}) \approx \pi 2^{-R} \quad \text{and} \quad \sin(\pi(k - 2^{-R})/q) \approx \pi k/q \quad \text{for} \quad k \neq 0.
\]

This yields \( |a_k| \approx 2^{-R}/k \). Therefore, for \( p = 1 \) and large \( q \), the magnitude of the first spurious sideband (at \( \omega = \omega_0 \pm \omega_c/q \)) is lower than the desired component by a factor of \( \approx 2^R \), and succeeding sidebands go down as \( 1/k \). As \( k \) increases, the \( 1/k \) approximation breaks down, and the smallest coefficient is at \( k = q/2 \), where \( |a_{q/2}| \approx \pi 2^{-R}/q \). As \( k \) continues to increase, \( a_k \) increases until \( k = q \), where \( |a_q| \approx 1 \) again (at the alias frequency \( \omega_0 + \omega_c \)). This can be seen in Figure 2.4, which shows a phase-truncated spectrum where \( p = 1 \).

When \( p = q - 1 \) we can also evaluate \( a_k \) easily. Note that \((q-1)m \mod q) = q - m \) for \( 1 \leq m \leq q \). Note also that the summand in equation (2.10a) is periodic in \( m \) with period \( q \). We can therefore change the limits of the summation and replace \((pm \mod q)\) with \((q - m)\), yielding

\[
a_k|_{p=q-1} = \frac{1}{q} \sum_{m=1}^{q} e^{i2\pi(km-2^{-R}(q-m))/q}.
\]

Using the same methods as before, we obtain

\[
a_k|_{p=q-1} = e^{-i\pi(2^{-R}-(k+2^{-R})/q)} \frac{\sin(\pi 2^{-R})}{q \sin(\pi(k + 2^{-R})/q)}.
\]

(2.13)

Comparison of equations (2.12) and (2.13) shows that the approximations derived in the previous paragraph for \( p = 1 \) will apply equally well when \( p = q - 1 \).
solid lines indicate components with $a_k$ coefficients
dashed lines indicate components with $b_k$ coefficients

**Figure 2.4**
Typical phase-truncated spectrum $X_t(f)$ with $p = 1$
2.3.3 Coefficients for arbitrary $p$

The difficulty in evaluating $a_k$ is due to the modulo function $(pm \mod q)$ and the term $km$ in the exponent of the summand in equation (2.10a). The presence of the $km$ term prevents us from eliminating the modulo function by re-ordering the summation as we did in §2.3.1. However, by performing a suitable transformation on $k$, we will be able to re-order the summation, yielding a result very similar to that derived for $p = 1$ in §2.3.2.

Let us define a new variable $k'$ as a function of $k$ and $p$: $k'$ is an integer such that $0 \leq k' \leq q - 1$ and $(pk' \mod q) = (k \mod q)$. Since $p$ and $q$ are relatively prime, this defines a one-to-one mapping from $k$ to $k'$ for $0 \leq k \leq q - 1$.

Since the function $e^{iz}$ is periodic in $x$ with period $2\pi$, we can replace $km$ with $(km \mod q)$ in equation (2.10a), yielding

$$a_k = \frac{1}{q} \sum_{m=0}^{q-1} e^{j2\pi((km \mod q) - 2^{-R}(pm \mod q))/q}$$

Let us manipulate the $(km \mod q)$ term:

$$(km \mod q) = (m(k \mod q) \mod q) = (m(pk' \mod q) \mod q) = (k'(pm \mod q) \mod q).$$

We can now substitute for $(km \mod q)$ in the expression for $a_k$:

$$a_k = \frac{1}{q} \sum_{m=0}^{q-1} e^{j2\pi((k'(pm \mod q) \mod q) - 2^{-R}(pm \mod q))/q}$$

$$= \frac{1}{q} \sum_{m=0}^{q-1} e^{j2\pi(k'(pm \mod q) - 2^{-R}(pm \mod q))/q}$$

$$= \frac{1}{q} \sum_{m=0}^{q-1} e^{j2\pi(k' - 2^{-R})(pm \mod q)/q}$$

We can now re-order the summation, eliminating the modulo function as in §2.3.1:

$$a_k = \frac{1}{q} \sum_{m=0}^{q-1} e^{j2\pi(k' - 2^{-R})m/q}.$$
This is almost the same sum as in §2.3.2, yielding our final result:

\[ a_k = e^{-j\pi(2^{-R} + (k'-2^{-R})/q)} \frac{\sin(-\pi 2^{-R})}{q \sin(\pi(k' - 2^{-R})/q)} \]  

(2.14)

where \( k' \) is an integer such that \( 0 \leq k' \leq q - 1 \) and \((pk' \mod q) = (k \mod q)\).

Note that equation (2.14) is identical to equation (2.12), with \( k' \) in place of \( k \). [When \( p = 1, \) \( k' \) is equivalent to \( k \).] Therefore, the set of spurious component amplitudes will be independent of \( p \). The value of \( p \) only affects the frequency at which a particular amplitude is found (i.e., the value of \( p \) only affects the mapping from \( k \) to \( k' \) in equation 2.13).

Because of the \( k \to k' \) mapping defined above, the placement of the phase truncation noise components is complicated when \( pk' > q \). However, for small \( k' \) (i.e, \( k' < q/p \)), we see that the components will be grouped as sidebands around \( \omega_0 \), at frequencies \( \omega_0 \pm \omega_c k' p/q \). Because of the similarity between equations (2.12) and (2.14), the approximations derived in §2.3.2 for \( p = 1 \) still hold for arbitrary \( p \), if we replace \( k \) with \( k' \). When \( 2^R \gg 1 \), the largest spurious component is of amplitude \( \approx 2^{-R} \), and succeeding sidebands go down as \( \approx 1/k' \). From the definition of \( k' \), we see that the largest component (\( k' = 1 \)) always occurs when \( k = p \), at frequency \( \omega_0 \pm \omega_c p/q \).

2.4 Total Noise Power

In this section, we shall show that the total average power in \( x_t(t) \) is equal to that in \( x_a(t) \), and that therefore the total power in all the spurious sidebands produced by phase truncation must be equal to the power loss in the desired component: \( 1 - |a_0|^2 \).

2.4.1 Definitions

This section deals with the total noise power in the phase-truncated signal; i.e., the power in all undesired spectral components. We define \( P \) as the power in the phase
truncated signal \( x_t(t) \), relative to the power in the ideal signal \( x_a(t) \). Since the spectra of both signals are periodic with period \( \omega_c \), we need only consider the power ratio in an \( \omega_c \)-wide spectral segment, yielding:

\[
P = \frac{1}{2} \sum_{k=0}^{q-1} |a_k|^2 + |b_k|^2.
\]

Since \( |a_k| = |b_{-k}| = |b_{q-k}| \), we have \( P = \sum_{k=0}^{q-1} |a_k|^2 \). We define \( P_n \) as the fraction of the power in \( x_t(t) \) which is due to noise: \( P_n = (P - |a_0|^2)/P \).

[Note that although these idealized signals are impulsive, and therefore have infinite power, we are dealing here with power ratios, and in this case the "infinites" (i.e., \( \delta(t) \), \( \delta^2(t) \)) cancel out.]

2.4.2 Evaluation of the power ratio \( P \)

In evaluating \( P \), we first consider the case when \( f_0 \) is at a principal frequency, i.e., when the phase truncation error is zero. In this case, \( q = 1 \), \( a_0 = 1 \) and we see directly that \( P = 1 \) when \( f_0 \) is at a principal frequency.

Next, we consider the case when \( f_0 \) is not at a principal frequency. The signal \( x_t(t) \) consists of impulses of area \( \cos((2\pi/2^N)\Phi_t(n)) \), where \( \Phi_t(n) \) is the truncated version of \( \Phi(n) = n\Delta_0 \). Of course, the effective phase is \( ((2\pi/2^N)\Phi_t(n)) \mod 2\pi \), so the effective value of \( \Phi(n) \) is \( (\Phi(n) \mod 2^N) \); call this effective value \( \Phi'(n) \). Since \( f_0 \) is not at a principal frequency, we know that \( \Delta_0 \) is an odd multiple of \( 2^{Q'-Q'} \), and therefore \( d' = \Delta_0/2^{Q'-Q'} \) is an odd integer. Therefore, we have

\[
\Phi'(n) = (\Delta_0 n \mod 2^N) = 2^{Q'-Q'}(d'n \mod 2^{R+Q'}). 
\]

Since \( d' \) is odd, \( 2^{R+Q'} \) and \( d' \) must be relatively prime, and therefore \( \Phi'(n) \) is periodic with period \( 2^{R+Q'} \). Also, as \( n \) takes integer values from 0 to \( 2^{R+Q'} - 1 \), the function \( 2^{Q'-Q}\Phi'(n) \) will take on each of the integer values from 0 to \( 2^{R+Q'} - 1 \) exactly once.
The truncated phase \(2^{-Q} \Phi'_t(n) = [2^{-Q} \Phi'(n)]\) will also be periodic with period \(2^{R+Q'}\), and will take on each of the integer values from 0 to \(2^R - 1\) exactly \(2^{Q'}\) times during one period. Therefore, during any \(2^{R+Q'}T_c\) period, \(z_t(t)\) consists of \(2^{R+Q'}\) impulses: \(2^{Q'}\) impulses of area \(\cos(k(2\pi/2^R))\) for each integer value of \(k\) from 0 to \(2^R - 1\).

Let us compare this to the result when \(f_0 = f_c/2^R\), the lowest principal frequency. In this case, \(\Delta_0 = 2^Q\), and \(\Phi_t(n) = \Phi(n) = n2^Q\). The phase \(\Phi(n)\) is periodic with period \(2^R\), and during any \(2^R T_c\) period, \(z_t(t)\) consists of \(2^R\) impulses: one impulse of area \(\cos(k(2\pi/2^R))\) for each integer value of \(k\) from 0 to \(2^R - 1\).

From Parseval’s theorem, we know that the average power in a periodic signal \(x_t(t)\) depends only on the integral \(\int x_t^2(t)dt\) over one period. [Since the signals we are discussing are impulsive, the integral \(\int x_t^2(t)dt\) does not actually exist, but recall that the impulse is defined as the limit of a finite function as the function duration goes to zero: the integral \(\int x_t^2(t)dt\) is infinite in this limit but the ratio of two such integrals is finite.] From the previous two paragraphs, we know that during any \(2^{R+Q'}T_c\) period, the phase-truncated signal contains exactly the same components as the ideal principal-frequency signal considered above. Therefore, the power in the two signals must be equal, and we have \(P = 1\) for the general case.

### 2.4.3 Noise power and SNR

From the above, we know that \(P = 1\), and therefore \(P_n\), the noise power in the phase-truncated signal, is given by \(P_n = (P - |a_0|^2)/P = 1 - |a_0|^2\), or

\[
P_n = 1 - \frac{\sin^2(\pi 2^{-R})}{q^2 \sin^2(\pi 2^{-R}/q)}.
\]

We define the signal-to-noise ratio (SNR) as the ratio of the power in the desired signal component to the power in the spurious signal components:

\[
\text{SNR} = \frac{|a_0|^2}{P_n} = \frac{|a_0|^2}{1 - |a_0|^2} = \frac{1}{P_n} - 1.
\]
Note that for any reasonably large SNR, we have \( \text{SNR} \approx 1/P_n \).

2.5 Comparison with previously published results

In previously published analyses of DDS phase truncation, the actual structure of the phase-truncated spectrum has not been investigated; instead, authors have only produced estimates of the total noise power due to phase truncation. The usual approach has been to assume that the phase truncation error (called \( \epsilon(n) \) in this thesis) is a uniformly distributed random variable independent of the true phase \( \phi(t) \). Of course, this is not exactly valid, but we shall see that it produces results which agree with a first-order approximation to the exact result in equation (2.15).

Using the assumption stated above, Hosking in [6] produces an "RMS phase noise" estimate of \( \pi 2^{-R}/\sqrt{3} \) (in our notation). Squaring this produces the estimate \( P_n \approx \pi^2 2^{-2R}/3 \). In [4], Cole uses a similar assumption and produces \( P_n \approx \pi^2 2^{-2R}/3 \), but this is due to a mathematical error; when this error is corrected, Cole's result agrees with Hosking's.

The assumption that \( \epsilon(n) \) is uniformly distributed is true in the limit as \( q \) approaches infinity, so we should compare \( \lim_{q \to \infty} P_n \) to the approximation mentioned above. Assuming that \( 2^R \gg \pi \), we use the approximation \( \sin^2(x) \approx x^2 - x^4/3 \), and we have

\[
\lim_{q \to \infty} P_n = 1 - \frac{\sin^2(\pi 2^{-R})}{\pi^2 2^{-2R}} 
\approx \frac{\pi^2 2^{-2R} - \pi^2 2^{-2R} + \pi^2 4^{-4R}/3}{\pi^2 2^{-2R}} = \frac{\pi^2 2^{-2R}}{3}. \tag{2.17}
\]

We see that this agrees with the previously published result. When we evaluate the derivative \( dP_n/dq \), we see that \( P_n \) is an increasing function of \( q \), and therefore the limit in equation (2.17) can be a useful upper bound on \( P_n \).
Chapter Three
Frequency Multiplication

As mentioned in §1.1, one disadvantage of direct digital frequency synthesis is narrow output bandwidth: DC to less than $f_c/2$ (as noted in §1.3.4, $f_0$ is often constrained to be $\leq f_c/4$). One technique for increasing bandwidth is to perform frequency multiplication on the DDS output signal. This provides a dramatic increase in bandwidth, but can reduce the output signal-to-noise ratio considerably. This chapter examines the effect of frequency multiplication on the DDS output spectrum.

3.1 The Full-wave Rectifier Frequency Doubler

The full-wave rectifier frequency doubler is quite simple: it consists of a full-wave rectifier (i.e., an absolute value operation) followed by a bandpass filter. The operation of the doubler is easily seen from the Fourier series expansion of a full-wave-rectified sinusoid:

$$|\sin(\omega t)| = \frac{2}{\pi} - \sum_{k=2,4,6\ldots} \frac{4}{\pi(n^2 - 1)} \cos(k\omega t).$$

The bandpass filter removes the DC term and the $4\omega$ and higher harmonics, leaving only the $-\frac{4}{3\pi} \cos(2\omega t)$ frequency-doubled term.

An important feature of a frequency doubler is its effect on amplitude-modulated (AM) signals and phase-modulated (PM) signals. In particular, we are interested in the case where the modulating signal is unwanted noise, such as the DDS phase error $\varepsilon(n)$ or the amplitude quantization error described in §1.3.5.

We can rewrite the absolute value function $|z|$ (i.e., full-wave rectification) as $|z| = z(|x|/x)$. The value $|z|/x$ is just the signum function $\text{sgn}(x)$, so $|z| = x \text{sgn}(x)$. The signum function corresponds to an ideal limiter: its value is either 1 or $-1$, depending
on the sign of the input. We should therefore expect the frequency doubler to perform better with AM noise (which is suppressed by a limiter) than with PM noise (which is unaffected by a limiter).

In this section, we shall study the case where the modulating signal is a pure sinusoid. In §3.2, we shall consider the case where the modulating signal is DDS noise (phase truncation error and other error sources).

3.1.1 Amplitude-modulated input signal

Let us consider the amplitude-modulated signal

\[ x_{AM}(t) = \sin(\omega_0 t)(1 + \alpha \cos(\omega_m t)), \]

where \( \omega_0 \) is the carrier frequency, \( \omega_m \) is the modulation frequency and \( \alpha \) is the modulation index (it is assumed that \( 0 \leq \alpha < 1 \)). By trigonometric identity,

\[ x_{AM}(t) = \sin(\omega_0 t) + \frac{\alpha}{2} \sin((\omega_0 + \omega_m)t) + \frac{\alpha}{2} \sin((\omega_0 - \omega_m)t). \] (3.1)

Of course, this is the carrier at \( \omega_0 \) and two AM sidebands at \( \omega_0 \pm \omega_m \) with amplitude \( \alpha/2 \) relative to the carrier. As mentioned above, a limiter suppresses amplitude modulation, so \( \text{sgn}(x_{AM}(t)) \) is just a square wave of frequency \( \omega_0 \), whose Fourier series expansion is:

\[ \text{sgn}(x_{AM}(t)) = \text{sgn}(\sin(\omega_0 t)) = \frac{4}{\pi} \sum_{k=1,3,5,...} \frac{1}{k} \sin(k\omega_0 t). \] (3.2)

Since \( |x_{AM}(t)| = x_{AM}(t) \text{sgn}(x_{AM}(t)) \), we can find the output of the frequency doubler by multiplying equations (3.1) and (3.2) and omitting components of frequency \( \leq \omega_m \) or \( \geq 4\omega_0 - \omega_m \), which will be eliminated by the bandpass filter. This yields

\[ -\frac{4}{3\pi} \left( \cos(2\omega_0 t) + \frac{\alpha}{2} \cos((2\omega_0 + \omega_m)t) + \frac{\alpha}{2} \cos((2\omega_0 - \omega_m)t) \right) \]

at the output of the frequency doubler. Comparing this with equation (3.1), we see that the AM sidebands have exactly the same frequency offset and amplitude relative
to the carrier as in the input signal $x_{AM}(t)$: the original AM spectrum has merely been translated up in frequency.

### 3.1.2 Phase-modulated input signal

We now consider the phase-modulated (or frequency-modulated: FM and PM are equivalent when the modulating signal is a sinusoidal) signal

$$x_{PM}(t) = \sin(\omega_0 t + \beta \sin(\omega_m t)),$$

where $\beta$ is the maximum phase deviation in radians, and it is assumed that $0 \leq \beta \ll 1$ (we are interested in the behavior of low-level noise components, so the assumption of a small phase deviation is appropriate). Since $\beta \ll 1$, we can use the standard narrowband FM approximation (see [7]):

$$x_{PM}(t) \approx \sin(\omega_0 t) + \frac{\beta}{2} \sin((\omega_0 + \omega_m) t) - \frac{\beta}{2} \sin((\omega_0 - \omega_m) t).$$  \hspace{1cm} (3.3)

Note that this is quite similar to equation (3.1), except for the sign of the lower sideband.

As mentioned above, narrowband phase modulation passes through a limiter relatively unaffected, so we can approximate $\text{sgn}(x_{PM}(t))$ as:

$$\text{sgn}(x_{PM}(t)) \approx \frac{4}{\pi} \sum_{k=1,3,5...} \frac{1}{k} \left( \sin(k\omega_0 t) + \frac{\beta}{2} \sin((k\omega_0 + \omega_m) t) - \frac{\beta}{2} \sin((k\omega_0 - \omega_m) t) \right).$$

Multiplying this by equation (3.3), neglecting terms proportional to $\beta^2$, and omitting terms of frequency $\leq \omega_m$ or $\geq 4\omega_0 - \omega_m$, we have the approximation

$$-\frac{4}{3\pi} \left( \cos(2\omega_0 t) + \beta \cos((2\omega_0 + \omega_m) t) - \beta \cos((2\omega_0 - \omega_m) t) \right)$$

for the output of the frequency doubler. Comparing this with equation (3.3), we see that the sidebands have the same frequency offset from the carrier, but the amplitude of the sidebands relative to the carrier has doubled.

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3.1.3 Conclusions for sinusoid-modulated signals

From the above, we can predict the effect of frequency-doubling a sinusoid of frequency $\omega_0$ which is amplitude- or phase-modulated by another sinusoid, of frequency $\omega_m$. For amplitude modulation, we predict that the doubler output will contain the frequency-doubled carrier wave at $2\omega_0$ and AM sidebands at frequencies $2\omega_0 \pm \omega_m$, at the same amplitude (relative to the carrier) as in the original signal. For phase modulation, we predict the same result, except that the relative sideband amplitude will be twice as high as in the original signal.

3.1.4 Experimental Verification

To verify these conclusions, the effect of frequency-doubling an AM signal and a PM signal was observed experimentally. An 8.125 MHz carrier was modulated by a 50 kHz sinusoid. The AM modulation index was 1%, and the PM phase deviation was 0.01 radian. In both cases we predict sidebands of $20 \log_{10}(0.005) = -46$ dB power relative to the carrier. Figures 3.1 and 3.2 show the modulated signal spectra.

Figures 3.3 and 3.4 show the result of passing these signals through two frequency doublers in series. We see that the sidebands remain at the same frequency offset from the carrier, as predicted. The AM sideband power level has increased by 1.4 dB, while the PM sideband power has increased by 12 dB, relative to the carrier. The PM result corresponds to a quadrupling of the relative sideband amplitude, exactly as predicted. [Note also that small sidebands at $4\omega_0 \pm 2\omega_m$ have appeared; these most likely correspond to terms which were neglected in the approximation in §3.1.2.] The increase in the AM sideband level is quite small by comparison, and is probably due to the non-ideal performance of the diode bridges used to implement full-wave rectifiers. [In the transition region (i.e., input near zero), the diode bridge will behave like a squaring operation, rather than an absolute value; a squaring operation will double
Figure 3.1
Spectrum of AM signal with modulation index of 1%
Figure 3.2
Spectrum of PM signal with maximum deviation of 0.01 radian
the amplitude of either AM or PM sidebands.] Thus, the experimental results are in general agreement with the analytical predictions.

### 3.2 Frequency multiplication of the DDS output signal

We now consider the effect of applying the output of a DDS to a frequency doubler. Since the phase truncation error signal $e(n)$ is a phase-modulating signal on the DDS output, the results of the previous section lead us to suspect that the spurious components caused by phase truncation will be doubled in amplitude (relative to the desired component) at the output of the frequency doubler. Also of interest is the effect on amplitude quantization error and D/A converter noise. Experimental observations of these effects were performed; this section describes the results.

#### 3.2.1 Phase truncation noise

Figure 3.5 shows part of the spectrum of a DDS output signal at a principal frequency. As we recall from §2.1.2, there is no phase truncation error at a principal frequency, and we see that the spectrum is quite clean. Figure 3.6 shows the output spectrum from the same DDS, but not at a principal frequency. We see the spurious components exactly as predicted by equations (2.10) and (2.12). [This provides experimental confirmation of our analysis in Chapter 2.]

Figure 3.7 shows the same signal after passing through two frequency doublers in series. Note that each of the phase truncation sidebands has remained at the same frequency offset from the main component, but has risen in amplitude by 12 dB. This is exactly the same behavior as was observed in the sinusoid-modulated PM signal in §3.1. Figure 3.8 shows the same DDS signal after being frequency-doubled 6 times. We see that the phase truncation sidebands are still at the same frequency offset, and have risen 36 dB over the non-multiplied signal: again an increase of 6 dB per doubler. Figure 3.9 shows the signal after passing through 10 doublers. From a 6 dB/doubler
Figure 3.3
Frequency-quadrupled AM signal spectrum
Figure 3.4
Frequency-quadrupled PM signal spectrum
DDS Parameters:

$f_c=30 \text{ MHz}, f_o=8.4375 \text{ MHz}, R=10, Q'=0, q=1, p=0$

**Figure 3.5**

DDS output signal at a principal frequency
DDS Parameters for Figures 3.6 through 3.12:
$f_c=30$ MHz, $f_0=8.43750179$ MHz, $R=10$, $Q'=14$, $q=16384$, $p=1$,
$f_c/q=1831$ Hz, 8 data bits entering D/A converter

**Figure 3.6**
Phase-truncated DDS output signal with $R=10$
sideband level increase, we would predict the two closest sidebands to be at the same level as the main component. Instead, our analysis breaks down at this point, and the main component has disappeared, merging with the sidebands.

3.2.2 Amplitude quantization & D/A converter error
As mentioned in §1.3.5, amplitude quantization and D/A converter error are noise components in the DDS output signal. At first glance, it is not obvious whether this noise will behave like AM modulation or like PM modulation when frequency-doubled. Figure 3.10 shows the same spectrum as Figure 3.6, but with an expanded frequency scale. We see two large sidebands due to phase truncation, and many smaller components due to amplitude quantization and D/A converter error. Figure 3.11 shows this signal after passing through two frequency doublers. We see that some frequency components have risen by 12 dB, like the PM sidebands in §3.1.2, while others have remained at the same level, like the AM sidebands in §3.1.1. Figure 3.12 shows the same DDS output signal after passing through 6 frequency doublers. We see that the PM-type sidebands continue to rise by 6 dB/doubler, while the AM-type sidebands are now no longer observable (although they are probably still there, at the same low level).

3.2.3 ROM size and frequency multiplication
As shown in §2.5, when $2^R \gg 1$ and $q \gg 1$, the total noise power due to phase truncation is approximately proportional to $2^{-2R}$. Combined with the results of this chapter, this suggests that one frequency doubling will have the same effect on phase truncation sidebands as reducing R by 1 (i.e., halving the number of ROM locations). Figure 3.13 shows a DDS output signal with R=8. Comparing this with Figure 3.7, a frequency-quadrupled DDS signal with R=10, we see that the noise due to phase truncation appears exactly the same, so the ROM size vs. frequency multiplication tradeoff appears to be valid.
Figure 3.7
Phase-truncated DDS output signal, frequency-quadrupled
Figure 3.8
Phase-truncated DDS output signal, frequency-multiplied by 64
Figure 3.9
Phase-truncated DDS output signal, frequency-multiplied by 1024
Figure 3.10
Amplitude quantization & D/A noise
Figure 3.11
Amplitude quantization & D/A noise, frequency-quadrupled
Figure 3.12
Amplitude quantization & D/A noise, frequency-multiplied by 64
DDS Parameters:
\[ f_c = 30 \text{ MHz}, \quad f_0 = 8.43750715 \text{ MHz}, \quad R = 8, \quad Q' = 14, \quad q = 16384, \quad p = 1, \]
\[ f_c/q = 1831 \text{ Hz}, \quad 8 \text{ data bits entering D/A converter} \]

**Figure 3.13**
Phase-truncated DDS signal with \( R = 8 \)
Chapter Four

Short-term Frequency Resolution

This chapter describes an effect which appears when the output of a DDS is observed for only a short time. In some cases, phase truncation can effectively limit the output frequency resolution. [Recall from §1.3.3 that, in general, frequency resolution can be made as fine as desired.]

When the research for this thesis was begun, it was thought that this might be an important effect. The research has shown that this effect, although interesting and quite real, is not likely to be important in actual practice.

4.1 Example of the effect

Consider a phase-truncated DDS whose output is observed for only a short time: from \( t = 0 \) to \( t = T \). During this period, the DDS produces \( Tf_c \) samples of the output signal; define \( n_T = Tf_c \). If \( d_0 \), the truncated portion of \( \Delta_0 \), is such that \( 0 < d_0 < 2^Q/n_T \), then the DDS output is identical to that produced when \( d_0 = 0 \). This is because the largest value produced in the truncated portion of \( \Phi(n) \) is \( n_T d_0 \), and if this value is less than \( 2^Q \), it will be truncated and have no effect on the output. Similarly, when \( 2^Q - (2^Q/n_T) < d_0 < 2^Q - 1 \), the output is identical to that produced when \( d_0 = 2^Q - 1 \). Recalling that \( d_0 = 0 \) if and only if \( f_0 \) is at a principal frequency, we see that this effectively limits the frequency resolution in the vicinity of a principal frequency to \( (f_c/2^N)(2^Q/n_T) = 1/(2^R T) \). We shall call this the truncation effect.

4.2 Natural limit on frequency resolution

Observing the DDS output signal only when \( 0 \leq t \leq T \) is equivalent to multiplying it
by a window \( w_T(t) \), where

\[
w_T(t) = \begin{cases} 
1, & \text{if } 0 < t < T; \\
0, & \text{otherwise.}
\end{cases}
\]

Of course, this is the same as convolving the DDS output spectrum with:

\[
\mathcal{F}[w_T(t)] = W_T(f) = e^{-i\pi f T} \frac{\sin(\pi f T)}{\pi f T}.
\]

The width of the main lobe of \( W_T(f) \) is \( 2/T \), so this convolution performs a moving spectral average about \( 2/T \) wide. This puts a natural limit on observable frequency resolution of about \( 1/T \).

### 4.3 Frequency Multiplication

Since the truncation effect limits output frequency resolution to \( 1/(2^R T) \), and the natural limit on observable frequency resolution is \( 1/T \), we see that the truncation effect will only be important if the output signal is frequency-multiplied by a factor of \( 2^R \) or more. But our results from §2.3.3 show that there will be phase truncation sidebands of relative amplitude \( \approx 2^{-R} \), and §3.2 shows that frequency multiplication increases the effective phase truncation sideband level by the multiplication factor. Therefore, if the DDS signal is frequency-multiplied by \( 2^R \), there will be spurious sidebands at the same level as the desired component. Figure 3.9 shows that the desired component may completely disappear in this case, merging with the sidebands. Clearly, such a high frequency multiplication factor is not feasible. Furthermore, the signal-to-noise ratio at the output will usually restrict the frequency multiplication factor even further; with frequency multiplication by \( 2^R \), the output SNR is less than 1, i.e., the spurious components are stronger than the desired signal! Therefore, the output frequency resolution will indeed be restricted by phase truncation when the signal is observed for only a short time, but it will be restricted even more by the short observation time itself.
Chapter Five
Conclusions

We have developed a model of an ideal phase-truncated synthesizer, and have
derived equations to relate the ideal synthesizer to actual synthesizers (equations 1.4
and 1.5). We have derived an exact expression for the output spectrum of the ideal
phase-truncated synthesizer (eqn. 2.10). We have found that when $f_0$ is at a prin-
cipal frequency (i.e., a multiple of $f_c/2^R$) there is no error due to phase truncation.
[The principal frequencies correspond to the output frequencies that would be avail-
able if phase truncation were not used.] When $f_0$ is not at a principal frequency, the
output spectrum contains the desired frequency component and $q - 1$ spurious com-
ponents, where $q = 2^{Q'}$, and $Q'$ is the effective number of truncated phase bits. For
output frequencies where $p = 1$ or $p = q - 1$, the spurious components form side-
bands around the desired components, spaced $f_c/q$ apart, which (for $2^R \gg 1$ and
$k' \ll q$) are of magnitude $\approx 2^{-R}/k'$, where $k'$ is the sideband number, i.e., the side-
band amplitudes are: $\approx 2^{-R}, \frac{1}{2}2^{-R}, \frac{1}{3}2^{-R} \ldots$ (see §2.3.2). For other values of $p$, the
spurious components have exactly the same levels, but their placement in the spec-
trum is more complex (see §2.3.3); some of the spurious components (those for which
$k' < q/p$) form sidebands around $f_0$, spaced $f_c p/q$ apart, i.e., at $f_0 \pm k' f_c p/q$. For
all values of $p$, the largest spurious components are at $f_0 \pm f_c p/q$ and are of magni-
tude $\sin(\pi 2^{-R})/(q \sin(\pi (1 \mp 2^{-R})/q)) \approx 2^{-R}$; the smallest spurious components are at
$f_0 \pm f_c/2$, and are of magnitude $\sin(\pi 2^{-R})/(q \sin(\pi (q/2 \mp 2^{-R})/q)) \approx \pi 2^{-R}/q$.

We have derived an exact expression for the total power in the spurious components
caused by phase truncation (eqn. 2.15), which depends only on $q$ and $R$. The limit
\[ \lim_{q \to \infty} P_n \] provides a useful upper bound on this power (see §2.5). For $2^R \gg 1$, this
limit is $\approx \pi^2 2^{-2R}/3$, and the output SNR is $\approx 2^R (3/\pi^2) \approx 6R - 5$ dB. [For $R=1$,
\[ \lim_{q \to \infty} \text{SNR} = -1.7 \text{ dB}. \]

When a pure sinusoid-modulated AM signal passes through a full-wave rectifier frequency doubler, the doubler output contains the frequency-doubled carrier wave, and AM sidebands at the same amplitude and frequency offset (relative to the carrier) as in the original signal. For a PM signal, we obtain the same result, except that the relative sideband amplitude is twice as high as in the original signal. When we pass a DDS output signal through frequency doublers, the spurious frequency components due to phase truncation behave exactly like PM sidebands: their amplitude (relative to the main component) doubles. The spurious components due to amplitude quantization and D/A error produce mixed results: some components behave like AM sidebands, and some behave like PM sidebands. For \( 2^R \gg 1 \), passing through one frequency doubler has the same effect on phase truncation sidebands as halving the number of ROM locations.

When a DDS output signal is observed for only a short time, phase truncation can restrict the effective output frequency resolution. However, this is unlikely to be an important effect in an actual system, because the short observation time will restrict the observable frequency resolution even further.
Appendix

Notation and Abbreviations

A.1 Symbols and Mathematical Operations

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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</thead>
<tbody>
<tr>
<td>§</td>
<td>section number</td>
</tr>
<tr>
<td>( \mathcal{F}[x(t)] )</td>
<td>Fourier transform of ( x(t) ): ( \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt )</td>
</tr>
<tr>
<td>([x])</td>
<td>floor function (integer part of ( x )): ([x]) is the largest integer ( \leq x )</td>
</tr>
<tr>
<td>( x \mod y )</td>
<td>modulo function: ( x \mod y = x - (y[x/y]) )</td>
</tr>
<tr>
<td>( \text{sgn}(x) )</td>
<td>signum function (sign of ( x )): ( \text{sgn}(x) = x/</td>
</tr>
<tr>
<td>( X^* )</td>
<td>complex conjugate of ( X ): ( X^* = \text{Re}{X} - \text{Im}{X} )</td>
</tr>
<tr>
<td>( x</td>
<td>_{y=z} )</td>
</tr>
</tbody>
</table>

A.2 Functions and Variables

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta(t) )</td>
<td>the unit-area impulse function (the Dirac delta function)</td>
</tr>
<tr>
<td>( \delta_{\tau}(t), \delta_{(m)}(t) )</td>
<td>impulse trains, i.e. infinite series of regularly spaced impulses</td>
</tr>
<tr>
<td>( \Delta_0 )</td>
<td>binary phase increment. The phase ( \Phi(n) ) is incremented by ( \Delta_0 ) between samples: ( \Delta_0 = 2^N f_0/f_c )</td>
</tr>
<tr>
<td>( \epsilon(n) )</td>
<td>phase truncation error: ( \epsilon(n) = (\Phi(n) \mod 2^Q) )</td>
</tr>
<tr>
<td>( \omega_c, \omega_0 )</td>
<td>angular frequencies: ( \omega_c = 2\pi f_c ), ( \omega_0 = 2\pi f_0 )</td>
</tr>
<tr>
<td>( \phi(t) )</td>
<td>phase of the desired sinusoid: ( \phi(t) = \omega_0 t )</td>
</tr>
<tr>
<td>( \Phi(n) )</td>
<td>samples of ( \phi(t) ) as represented in integer form in the phase accumulator: ( \Phi(n) = (2^N/2\pi)\phi(nT_c) )</td>
</tr>
<tr>
<td>( \Phi_t(n) )</td>
<td>truncated phase samples: ( \Phi_t(n) = \Phi(n) - \epsilon(n) )</td>
</tr>
<tr>
<td>( a_k, b_k )</td>
<td>coefficients of spectral components in ( X_t(\omega) ). ( a_k ) is the complex conjugate of ( b_{-k} ). The general solution for ( a_k ) is in eqn. (2.14)</td>
</tr>
<tr>
<td>( d_0 )</td>
<td>the part of ( \Delta_0 ) which corresponds to the truncated part of ( \Phi_t(n) ): ( d_0 = (\Delta_0 \mod 2^Q) )</td>
</tr>
<tr>
<td>( f_c )</td>
<td>DDS clock frequency: ( f_c = 1/T_c )</td>
</tr>
</tbody>
</table>
\( f_0 \) frequency of the desired sinusoid. \( f_0 \) must be a multiple of \( f_c/2^N \) (see §1.3.4)

\( k \) summation index variable

\( m \) signal subset number: \( x_t(t) \) is divided into \( q \) subsets \( x_{(m)}(t) \), with \( 0 \leq m < q \)

\( n \) sample number, denoting the sample produced at time \( nT_c \): \( n = t/T_c \)

\( N \) width (in bits) of the phase accumulator. The accumulator output can take \( 2^N \) different values.

\( p, q \) \( p \) and \( q \) are relatively prime integers, such that \( (p/q) = (d_0/2^Q) \). The meaning of \( p \) and \( q \) is discussed in §2.2.2

\( P \) ratio of the power in \( x_t(t) \) to the power in \( x_a(t) \)

\( P_n \) the fraction of the power in \( x_t(t) \) contained in spurious components

\( Q \) number of truncated phase bits: \( Q = N - R \)

\( Q' \) effective number of truncated phase bits: \( Q' = \log_2(q) \)

\( R \) number of phase bits entering the ROM look-up table. The table contains \( 2^R \) entries.

\( T_c \) time period between samples produced by the DDS: \( T_c = 1/f_c \)

\( T_p \) pulse width in the pulsed waveform \( x_p(t) \)

\( w_{T_c}(t), w_{T_p}(t) \) unit-area window functions (pulses) of duration \( T_c \) and \( T_p \), respectively

\( x(t) \) the desired output signal: \( x(t) = \cos(\omega_0 t) \)

\( x_a(t) \) impulsive waveform produced by the ideal DDS (see eqn. 1.1)

\( x_p(t), x_s(t) \) pulsed and staircase signals, respectively, produced by practical DDS's. See §1.3.1 and figure 1.2

\( x_t(t) \) impulsive waveform produced by the phase-truncated DDS (see eqn. 2.2)

\( X(\omega) \) Fourier transform of \( x(t) \)
A.3 Abbreviations

AM       amplitude modulation
D/A      digital-to-analog (converter)
DDS      direct digital synthesizer
LSB      least significant bit (in the binary representation)
LSI      large scale integration
MSB      most significant bit (in the binary representation)
PM       phase modulation
ROM      read-only memory
SNR      signal-to-noise ratio
References


