CALCULATIONS OF VISCOELASTIC FLOW IN A JOURNAL BEARING

by

Antony N. Beris

Diploma, National Technical University of Athens, Greece, 1980

Submitted to the Department of Chemical Engineering in Partial Fulfillment of the Requirements of the Degree of DOCTOR OF PHILOSOPHY at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY September 1985 © Massachusetts Institute of Technology, 1985

Signature of Author ___________________________ Department of Chemical Engineering

Certified by ___________________________ Robert C. Armstrong
Thesis Supervisor

Certified by ___________________________ Robert A. Brown
Thesis Supervisor

Accepted by ___________________________ Chairman, Departmental Committee on Graduate Studies

FEB 20 1986

LIBRARIES Archives
CALCULATIONS OF VISCOELASTIC FLOW IN A JOURNAL BEARING

by

ANTONY N. BERIS

Submitted to the Department of Chemical Engineering on September 20, 1985 in partial fulfillment of the requirements for the Degree of Doctor of Philosophy in Chemical Engineering

ABSTRACT

This thesis investigates the rheological behavior of model viscoelastic fluids in a nearly viscometric flow. The flow between eccentric rotating cylinders—journal bearing flow—is investigated for a viscoelastic fluid obeying the Giesekus constitutive equation. This flow problem is used to study the causes of failure of convergence of the Galerkin finite element method in the simulation of viscoelastic flows when the elasticity in the flow, as measured by the Deborah number (De), becomes important. The eccentric cylinder geometry is particularly advantageous for this study because the flow can be varied systematically from a simple steady shear flow for concentric cylinders with a small gap, to a complex flow with recirculation as the eccentricity ($\varepsilon$) of the two cylinders is increased. In the particular case of the upper-convected Maxwell model (UCM) this failure arises because of the existence of a limit point with respect to De in the solution family. This limit point is a numerical artifact caused by non-linear interactions between high-frequency eigensolutions, supported by the system of the discretized equations arising from the Galerkin finite element formulation, and the approximation to the solution of the continuum problem. A spectral-finite-element method is developed which is devoid of these numerical instabilities.

Analytical methods are used to study the effect of systematically varying the parameters of the constitutive equation in the predictions for stress and velocity fields in the journal bearing. Singular perturbation methods are used to derive expansions valid for small gaps between the cylinders, but for all Deborah numbers. Results for Newtonian, second-order, Crimnale-Ericksen-Filbey, upper-convected Maxwell, Oldroyd-B, and White-Metzner constitutive equations separate the effects of elasticity, memory, and shear thinning on the development of the large stress gradients that hinder the numerical solutions with these models in more complicated geometries. Boundary layers next to the inner cylinder are predicted for the UCM fluid with width $-\text{De}$; adding a retardation term to form the Oldroyd-B model lessens this dependence to $O(\sqrt{\text{De}})$. Extrapolation of the results at eccentricities high enough for a flow separation to occur show the size and location of the separation region to be highly dependent on the constitutive equation used.

Calculations are performed using the Galerkin/finite-element method at both low ($\varepsilon=0.1$) and moderate ($\varepsilon=0.4$) eccentricities and demonstrate both the De limit and some numerical instabilities which eventually degrade the quality of the solution. Results are given for five viscoelastic constitutive equations each derived as a limit of the Giesekus model. Comparisons with exact results for Newtonian, second-order, and corotational
Maxwell-like fluids set the accuracy of the calculations as a function of eccentricity and Deborah number. Computer-implemented perturbation methods are used to demonstrate bifurcation and turning points in De for an upper-convected Maxwell fluid. Depending on the mesh size and the eccentricity used, one to three bifurcation points are found before a limit point in De is reached. The bifurcation points are considered to be numerical artifacts because their locations are strongly mesh dependent and the associated bifurcating solutions oscillate with frequency determined by the nodes spacing in the mesh used. The locations of the limit points (De ≈ 3.6 and 0.93 for ε = 0.1 and 0.4, respectively) are moderately stable to extensive mesh refinement but the solution quality is poor due to the presence of high frequency azimuthal oscillations—especially at ε = 0.4. Similar solution pathology is demonstrated for the three-constant Oldroyd-B model. No limiting value of De is found for calculations with the Leonov-like version of the Giesekus fluid at ε = 0.1 when a medium size mesh is used. Calculations at ε = 0.1 with a finer mesh or at ε = 0.4 with all the meshes used fail when irregular oscillations destroy the quality of the solution.

The behavior of the recirculation region of the flow and the value of the load on the inner cylinder are very sensitive to the value of the mobility parameter α used in the Giesekus model. The recirculation disappears at low values of De except when the mobility parameter α was so small that the shear viscosity is almost constant over the range of shear rates in the calculations. The recirculation persists over the entire range of accessible De for the UCM fluid, the limit of α = 0 of the Giesekus model. The behavior of the recirculation is coupled directly to the shear viscosity by calculations with an inelastic fluid with the same viscosity predicted by the Giesekus model.

Calculations with the UCM model at a non-zero Reynolds number such that the system of governing equations changes mathematical type—a pair of equations switching from elliptic to hyperbolic—show enhanced oscillations in the solutions. This finding indicates the close connection between the numerical instabilities and the hyperbolic-like nature of the equation set.

A new spectral/finite-element method is presented for simulating the two-dimensional flow of a viscoelastic fluid in the journal bearing. The flow boundaries are represented as coordinate surfaces in a bipolar coordinate system. The streamfunction and components of stress are approximated by finite-element basis functions in the radial direction and Fourier components in the azimuthal one. Flows for the upper convected Maxwell model (UCM) are produced up to high values of elasticity (De = 100), substantially extending the finite-element calculations. For slightly eccentric cylinders the results converge to a unique solution as the refinement of the approximation is increased. At high Deborah numbers, the stress profiles develop boundary layers closely matching the asymptotic analysis predictions. Calculations for high values of eccentricity show that fluid elasticity decreases the size of the flow recirculation.

Thesis Supervisors: Robert C. Armstrong

Robert A. Brown

Titles: Professor of Chemical Engineering
ACKNOWLEDGEMENTS

I wish to acknowledge the support, guidance and help received by my thesis advisors Professors Robert A. Brown and Robert C. Armstrong. The research enthusiasm of the first and the deep physical intuition of the second will undoubtedly influence me throughout all my career. I would also like to thank the members of my thesis committee: Profs. Brady, Brenner and Patera for their constructive suggestions.

This research was financially supported by the Office of Naval Research, the National Science Foundation and The Information Processing Center at MIT. The support of the above organisations is gratefully acknowledged. Special thanks are due to Bobby Burke at IPS and Bob Bivins at Los Alamos National Laboratory for their technical assistance, and to Gloria Collver-Jacobson for her expert typing of Chapter 3 and the Appendices of this thesis.

This thesis could not have been accomplished without the help of numerous friends from both inside and out of MIT. Special mention should be made to John Tsikoiannis with whom life at MIT would have been a lot less pleasant than it ended up being. I am also thankful to John Congalides, Michael Stoukides, John Tsamopoulos and Bob Barat for their help and pleasant company, John Lawler, Susan Muller and the rest in room 66-246 for their enlightening discussions, Miguel Bibbo for the excellent graphics codes that he provided me, and George Prassos and his family in New York for offering me hospitality at numerous instances.

Finally, I dedicate this thesis to my parents whose sacrifices on my behalf and their constant encouragement made this thesis possible.
CONTENTS

1. Introduction 8
   1.1 Outstanding Problems in Viscoelasticity 8
   1.2 Viscoelastic Fluid Flow in a Journal Bearing 13

2. Rheological Considerations and Governing Equations 19
   2.1 Material Functions 19
   2.2 Constitutive Equation 22
   2.3 Governing Equations and Boundary Conditions 32
      2.3.1 Formulation of the equations in cylindrical coordinates 34
      2.3.2 Formulation of the equations in bipolar coordinates 35

3. Perturbation Analysis 39
   3.1 Regular Perturbation in Deborah Number 39
   3.2 Domain Perturbation 43
      3.2.1 Formulation of domain perturbation equations 44
      3.2.2 Second-order fluid 52
      3.2.3 Upper-convected Maxwell fluid 58
         3.2.3a Expansion for small De 59
         3.2.3b Expansion for high values of De 60
         3.2.3c Exact solution for all values of De 61
      3.2.4 Oldroyd-B fluid 62
      3.2.5 White-Metzner model 64
      3.2.6 Criminale-Briggs-Filbey model 66
      3.2.7 Comparison of velocity and stress fields 68
      3.2.8 Torque and loads 104
      3.2.9 Analysis of flow separation 109
      3.2.10 Discussion of domain perturbation results 111

4. Finite Element Calculations 114
   4.1 Formulation of the Finite Element Method 114
   4.2 Comparison with Known Flow and Stress Fields 122
      4.2.1 Concentric cylinders 122
      4.2.2 Eccentric cylinders: Newtonian fluid 133
      4.2.3 Eccentric cylinders: Second-order fluid 133
   4.3 Small Eccentricities 140
      4.3.1 Upper-convected Maxwell model 140
      4.3.2 Oldroyd-B fluid model 149
      4.3.3 Leonov-like model 152
      4.3.4 Leonov-like with retardation model 152
      4.3.5 Inelastic Leonov-like model 156
      4.3.6 Corotational Maxwell-like model 160
      4.3.7 Giesekus fluid: $\alpha = 3/4$ 160
   4.4 Moderate Eccentricities 160
      4.4.1 Upper-convected Maxwell model 160
      4.4.2 Giesekus fluid with retardation time: $\lambda_1 > 0$, $\lambda_2 > 0$, $0 < \alpha < 0.5$ 160
      4.4.3 Inelastic Leonov-like model 171

5. Analysis of Finite Element Results 175
   5.1 Comparison of the Loads on the Inner Cylinder 175
5.2 Numerical Instabilities and Change of Type of the Equations 177
  5.2.1 Change of type of the Equations 178
  5.2.2 Finite-element calculations with non-zero inertia 182
  5.2.3 Upwind finite-elements 186
5.3 Use of Smoothing Techniques 190
5.4 Conclusions from the Finite-Element Calculations 193

  6.1 Introduction to the Spectral Methods 195
  6.2 Formulation of the Spectral/Finite-Element Method 196
  6.3 Comparison with the Exact Newtonian Solution 199
  6.4 Upper-Convected Maxwell Model 200
     6.4.1 Small eccentricities 200
     6.4.2 Moderate eccentricities 215
     6.4.3 High eccentricities 228
  6.5 Leonov-like with Retardation Model 236
  6.6 Comparison of the Loads on the Inner Cylinder 242
     6.6.1 Upper-convected Maxwell model 243
     6.6.2 Leonov-like model with retardation time 249
  6.7 Discussion of the Spectral/Finite-Element Results 249

7. Conclusions 252

References 254

List of Symbols 263

Appendix A Components of the Residual Equations in Cylindrical Coordinates 268


Appendix C Correspondence between the Domain Perturbation and the Phan-Thien and Tanner (1981) solution for the CEF Equation 272

Appendix D Cylindrical Couette Flow of the Leonov-like with Retardation Model 275

Appendix E Cylindrical Couette Flow of Giesekus Fluid: $0 \leq \alpha \leq 1$, $\lambda_1 > 0$, $\lambda_2 = 0$. 277

Appendix F Analysis of an one-element Finite-Element Solution of the Cylindrical Couette Flow of the Corotational-Maxwell-like Fluid 279

Appendix G Linear Stability Analysis of Cylindrical Couette Flow of an Upper-Convected Maxwell Fluid towards 2-Dimensional Azimuthal Disturbances 282

Appendix H Hyperbolicity Region in Cylindrical Couette Flow for an Upper-Convected Maxwell Fluid 285
Appendix I  Streamline-Upwind Finite-Element Method as Proposed by Hughes (1983)  287

Appendix J  Components of the Residual Equations in Bipolar Coordinates  289

Appendix K  Useful Integral Formulas for the Evaluation of the Load in Bipolar Coordinates  291
1. INTRODUCTION

The unique mechanical properties shared by polymeric materials have led to their widespread use, ranging from packaging to aircraft manufacturing. Polymer chemistry has indeed revolutionized the area of material science in the last few decades, providing a wealth of new materials with very diverse mechanical properties. For a new polymer to have a potential for commercial applications, it is essential that it is easily and economically processed. Because most polymer processing is carried out in the fluid state, it is important to understand polymer fluid flow behavior.

1.1 Outstanding Problems in Viscoelasticity

It is well known that polymer solutions and melts exhibit viscoelastic fluid behavior, having a number of characteristic properties lying between those seen in elastic solids and viscous fluids. Figure 1.1, taken from a recent article by Bird and Curtiss (1984), shows the most preeminent viscoelastic properties. The challenge to the engineering scientist is first to quantitatively understand viscoelastic flow phenomena and then to use this knowledge to design more efficient processing equipment. Unfortunately, neither of the above objectives has been accomplished, although substantial progress has been achieved in the past two decades.

Most of the progress has been made in understanding viscoelasticity. The search was mainly directed towards deriving mathematical models capable of predicting the observed viscoelastic phenomena. The most general functional expression for the stress in a viscoelastic material—under the assumption of a simple fluid—is useful only if the flow is sufficiently slow or the fluids under consideration only slightly elastic. Under these conditions, the functional expression for the stress can be expanded in a series—the hierarchical equations of Rivlin and Ericksen (1955) or Coleman and Noll (1961)—and can be used with confidence, in the sense that any behavior can be described under the before-mentioned operating conditions.

However, the relative ease with which unambiguous rheological equations of state are used in slow-flow problems is not carried over in more
Figure 1.1 Characteristic viscoelastic properties (from Bird and Curtiss 1984). The ten experiments sketched here show how the behavior of polymeric liquids is qualitatively different from that of Newtonian liquids. 

a) a polymeric liquid climbs a rotating rod; b) rises above a rotating disc; c) moves radially inward along a rotating disc; d) recoils in a tube when the pump is turned off; e) swells when it emerges from a tube; f) siphons across a gap; g) develops a slight convex surface when flowing down a trough; h) develops a vortex when the tube diameter decreases for slow flow; i) moves towards a transversely oscillating cylinder along the line of oscillation; j) causes falling spheres to grow further apart.
general flow conditions. The equations that are generated by the simple fluid concept become quickly intractable as the complexity of the flow increases. Homogeneous shear—viscometric—flows can be described in the most general case by three material functions of the shear rate; see Chapter 2. The work of Pipkin and Owen (1967) indicates, though, that thirteen independent kernel functions are needed to describe even first-order departures from viscometric flows. Approximate equations of state describing the dependence of the stress tensor on the strain rate and—in contrast to the Newtonian case—on the previous strain history of a material point in the fluid, have to be employed if progress is to be made. These can be chosen among the several promising constitutive equations which have been proposed (Bird et al. 1977a).

Unfortunately, even the simplest of the proposed models is complicated enough to essentially preclude any closed-form solutions except in the cases of homogeneous shear and extensional flows. Furthermore, approximations widely used in Newtonian calculations as the lubrication approximation have been proved to be inapplicable (Langlois 1963) in viscoelastic flows.

As a consequence, most previous work has been limited to comparing the predictions of various models in homogeneous flows (usually in homogeneous shearing or viscometric flows). Agreement of the predictions of the models with experimental data in viscometric flows, though, does not necessarily imply agreement in a more complicated flow. Moreover, it is still an open question in what class of flows (if there is one) agreement between theory and experiments would be sufficient to guarantee agreement in any other flow. Thus, we need to investigate more complicated flows, which requires the use of numerical methods. Unfortunately, both finite element and finite difference methods have failed to converge when applied to viscoelastic flow simulations in complicated geometries when the elasticity in the flow became important (Mendelson et al. 1982; Crochet and Walters 1983).

Fluid elasticity can be characterized quantitatively by the Deborah number, $De$, which is defined as the ratio of a characteristic time constant of the fluid, $\lambda$, to a characteristic time of the flow. As the Deborah number increases from zero to large values, the material behavior changes from that of a Newtonian fluid to that corresponding to an elastic solid.
Loss of numerical solution occurs when De is near unity.

The failure of the numerical calculations when elastic forces become important has not been definitively linked to either a peculiarity in the fluid model or to a defect in the numerical scheme (Mendelson et al. 1982; Crochet and Walters 1983; Holstein 1981; Tanner 1982; Yeh et al. 1984). For a fluid which follows the upper-convected Maxwell model, it has been shown (Kim-E 1984; Yeh 1984) that the loss of convergence in the family of numerical solutions is due to the presence of a limit point in the curve describing the solution family with respect to the Deborah number. This limit point might be an inherent feature of the model or a numerical artifact, present only in the solution of the discretized equations. Unfortunately, the presence of the corner singularity in the sudden contraction geometry predominantly affected the performance of the finite element method in that geometry (Yeh 1984) and left the question of the nature of the limit point unanswered.

Unresolved difficulties with the numerical computations of viscoelastic fluid flows in complex geometries have lead to this thesis to investigate the rheological behavior of model viscoelastic fluids in a nearly viscometric flow. The thesis focuses on the calculation of the flow of a viscoelastic fluid in a journal bearing, two long eccentric cylinders the inner one of which is rotating about its axis. The underlying rationale is that analysing a number of different models in a geometry which can be smoothly changed from a viscometric to a complex flow will allow a systematic investigation of the effects of the boundary shape, flow type and the constitutive equation on the computations. When coupled with careful experiment the outcome should be a true test of the applicability of a range of continuum models.

The principal constitutive equation that is used in this work is the GieseKus model (GieseKus 1982). This model was selected because of several features. First, it can be derived from a molecular picture for concentrated polymer solutions; nevertheless, it is relatively simple, because it is written in differential form which can be easily incorporated in the numerical calculations. Second, it yields a range of different shear and extensional flow behavior by suitably selecting the value of its three parameters. Two extreme cases of this model are considered mostly: The upper convected Maxwell (UCM) and the Leonov-like
with a small retardation time (LER). A number of other cases of the Gieseus model and some additional models are also used, occasionally, for comparison purposes. The constitutive equations are discussed further in Chapter 2.

1.2 Viscoelastic Fluid Flow in a Journal Bearing

The journal bearing consists of two nested, parallel cylinders with axes separated by a distance \( e \), as shown in Figure 1.2. The radii of the inner and outer cylinders are \( R_0 \) and \( R \), respectively, and the inner one rotates at linear velocity \( V = \Omega R_0 \), where \( \Omega \) is the angular velocity. The configuration of the cylinders is completely determined by two parameters, the dimensionless eccentricity,

\[
\varepsilon = e/\delta,
\]

which is written in terms of the average gap \( \delta = R - R_0 \) between the cylinders, and the dimensionless gap thickness

\[
\mu = \delta/R_0 = (R - R_0)/R_0.
\]

The equations governing the flow of a viscoelastic fluid in the journal bearing and the appropriate boundary conditions, expressed in the cylindrical coordinate system shown in Figure 1.2, are described in Section 2.3.1.

The journal bearing geometry offers a number of advantages for a comprehensive study of viscoelastic flows. First, by varying the eccentricity between zero and unity, the flow field can be deformed from a simple shearing motion to a complex flow with flow separation at high eccentricities. Second, in contrast to the sudden contraction (Yeh et al. 1984) or driven cavity (Mendelson et al. 1982) flow systems, the journal bearing geometry has no geometrical corners or singularities along the boundaries of the flow. Third, whereas the approximate inlet and outlet conditions on velocity (and possibly on stress) must be specified in all open flow problems, these conditions are replaced in the journal bearing by exact periodicity constraints on these variables.

An additional advantage of studying the journal bearing geometry
Figure 1.2 Journal bearing geometry. The dashed curve (- - -) indicates the position for which azimuthal profiles are given in subsequent figures; the dotted curve (........) gives the location for radial profiles.
is that several closed formed solutions exist, which can be compared with the results of the numerical simulations. In the concentric configuration, for all the models used here, the exact solution is either known from the literature or can be easily calculated. For the Newtonian and the second order fluid models, the exact solution for arbitrary \( \epsilon \) and \( \mu \) is also known (Wannier 1950; Kamal 1966; Ballal and Rivlin 1976) and is unique, according to the Tanner-Giesekus-Huilgol theorem (Tanner 1966; Huilgol 1973). Finally, for some of the models, an asymptotic solution can be obtained, through perturbation analysis, valid for small \( \epsilon \) and \( \mu \) (Davies and Walters 1973; Phan-Thien and Tanner 1981); see also Chapter 3. These studies are classified in Table 1.1 according to the constitutive equation used. In the same Table, the method used and the restrictions of the analysis are also indicated.

Restricting the analysis to small gaps (\( \mu << 1 \)) reduces the equations of motion to the classical lubrication approximation (Reynolds 1886) for a Newtonian fluid. For some viscoelastic constitutive equations explicit in stress, the lubrication equations, valid for small gaps but for any eccentricity, can also be formulated, and solved, with the position \( r_1(\theta) \) of the outer cylinder given by the expression consistent to \( O(\mu) \) as

\[
r_1(\theta) = R_0 + \delta (1 + \epsilon \cos \theta).
\]

(1-3)

This approach was originally employed for Newtonian fluids (Sommerfeld 1904) and has been adapted for second- and third-order fluid models (Reiner et al. 1969; Davies and Walters 1973). More complicated constitutive models are analysed by expanding the shape of the gap between the cylinders for small eccentricities. This approach was employed by Davies and Walters (1973) for a four constant Oldroyd model and by Phan-Thien and Tanner (1981) for the Criminale-Ericksen-Filbey constitutive relation. These authors used a regular perturbation in \( \epsilon \) using expression (1.3) to describe the shape of the outer cylinder and applied boundary conditions at a fictitious boundary positioned at the mean radial location \( R_0 + \delta \). The more formal approach of domain perturbation as formulated by Joseph and Sturges (1975) is used in Chapter 3 of this thesis to derive asymptotic solutions for the upper-convected Maxwell (UCM), Oldroyd-B
Table 1.1 Summary of Previous Calculations for the Two-Dimensional Flow Between Eccentric Rotating Cylinders

<table>
<thead>
<tr>
<th>Constitutive Equation</th>
<th>Restrictions</th>
<th>Reference</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Newtonian</td>
<td>Small gap, small eccentricity, no inertia.</td>
<td>Reynolds, 1886</td>
<td>A. Lubrication approximation.</td>
</tr>
<tr>
<td></td>
<td>Small gap, no inertia.</td>
<td>Sommerfeld, 1904</td>
<td>B. A, applied to the journal bearing.</td>
</tr>
<tr>
<td></td>
<td>Small gap, small inertia.</td>
<td>DiPrima and Stuart, 1972</td>
<td>C. Same as B but included effect of small inertia.</td>
</tr>
<tr>
<td></td>
<td>No inertia.</td>
<td>Joukowski and Tschaplygin, 1904</td>
<td>D. Exact solution in bipolar coordinates.</td>
</tr>
<tr>
<td></td>
<td>No inertia.</td>
<td>Wannier, 1950</td>
<td>E. Exact solution in modified bipolar coordinates.</td>
</tr>
<tr>
<td></td>
<td>Small inertia.</td>
<td>Kamal, 1966</td>
<td>F. Low Reynolds number correction to exact solution.</td>
</tr>
<tr>
<td>Inelastic</td>
<td>Power law fluid; small gap, no inertia.</td>
<td>Tanner, 1963</td>
<td>Same as B.</td>
</tr>
<tr>
<td></td>
<td>Second-order fluid; small gap, no inertia.</td>
<td>Reiner et al., 1969</td>
<td>Same as B.</td>
</tr>
<tr>
<td></td>
<td>Second-order fluid; no inertia.</td>
<td>Davies and Walters, 1973</td>
<td>Same as D.</td>
</tr>
<tr>
<td></td>
<td>Third-order fluid; small gap, no inertia.</td>
<td>Ballal and Rivlin, 1976</td>
<td>Same as B.</td>
</tr>
<tr>
<td></td>
<td>Four constant Oldroyd; small gap, small eccentricity, no inertia.</td>
<td>Davies and Walters, 1973</td>
<td>Same as B.</td>
</tr>
<tr>
<td></td>
<td>Criminale-Ericksen Filbey (CEF) fluid; small gap, small eccentricity, no inertia.</td>
<td>Davies and Walters, 1973</td>
<td>H. Lubrication equations valid for small gap plus regular perturbation in eccentricity same as H.</td>
</tr>
</tbody>
</table>
and White-Metzner models.

Numerical methods have not been previously used to solve the viscoelastic fluid flow problem in a journal bearing. Both finite-difference and finite element methods have been used to simulate viscoelastic fluid flows in other geometries. Among these two methods, the finite element is the most extensively used due to its distinctive advantage of accommodating irregularly shaped boundaries with little additional effort.

A finite-element method is used in this thesis to calculate the viscoelastic flow in the journal bearing. The method and the results for the Giesekeus model at both low (ε=0.1) and moderate (ε=0.4) eccentricities are presented in Chapter 4. The analysis of the results is given in Chapter 5.

The bipolar coordinate system shown in Figure 1.3 is the natural one for the journal bearing flow problem, since there the surfaces of the inner and the outer cylinder, which represent the flow boundaries, are coordinate surfaces. Furthermore, the exact analytical solution for the stream function and for the components of the velocity for a Newtonian fluid has in this coordinate system only a few non-zero Fourier modes in the angular-like coordinate θ. To exploit these particular characteristics of the journal bearing problem a spectral/finite-element method using the bipolar coordinate system of Figure 1.3 has been developed. The analysis of the method and the obtained results are presented in Chapter 6. Finally, a discussion of the conclusions drawn is presented in Chapter 7.
Figure 1.3 Bipolar coordinate system for two eccentric cylinders of radii $R_0$ and $R$ with the centers displaced by a distance $e$. When $R_0$, $R$ and $e$ are given (or equivalently $R_0$, $e$ and $\mu$ defined from equations 1-1, and 1-2), the parameter $a$ is obtained from

$$a = R_0 \frac{\sqrt{1-e^2}}{\epsilon} \left( 1 + \mu + \frac{1 - e^2}{4} \mu^2 \right)^{1/2}$$

and the values $\xi_1$ and $\xi_2$ defining the boundaries of the eccentric annular region are given by

$$\xi_1 = \text{arcsinh}(a/R_0); \quad \xi_2 = \text{arcsinh}(a/R)$$

The transformation between the bipolar $(\xi, \theta)$ and the cartesian $(x, y)$ coordinates is given by

$$x = a \frac{\sinh \xi}{\cosh \xi + \cos \theta}, \quad \xi = \text{arctanh} \frac{2ax}{a^2 + (x^2 + y^2)},$$

$$y = a \frac{\sin \theta}{\cosh \xi + \cos \theta}, \quad \theta = \arctan \frac{2ay}{a^2 - (x^2 + y^2)}.$$
2. RHEOLOGICAL CONSIDERATIONS AND GOVERNING EQUATIONS

Viscoelastic fluids are rheologically characterized by their material functions, such as viscosity and normal stress coefficients, which are obtained through a series of standard rheological tests. A complete discussion of steady and unsteady material functions, and the tests used to obtain them, is given by Bird et al. (1977a). A constitutive equation model for a particular viscoelastic material should at least reproduce the experimentally measured viscometric material functions. This is no measure of how good the predictions of such an equation will be in an arbitrary flow; this point is addressed in this thesis.

In the next subsection, an outline of the most commonly used material functions is offered together with a description of the most common experimental observations.

2.1 Material Functions

Three of the most often-used material functions are the viscosity \( \eta \), and the first and the second normal stress coefficients, \( \psi_1 \) and \( \psi_2 \), respectively. These are defined for a simple steady shear flow (see Figure 2.1) as

\[
\eta = \frac{\tau_{12}}{\dot{\gamma}},
\]

\[
\psi_1 = \frac{N_1}{\dot{\gamma}^2} = \frac{(\tau_{11} - \tau_{22})}{\dot{\gamma}^2},
\]

\[
\psi_2 = \frac{N_2}{\dot{\gamma}^2} = \frac{(\tau_{22} - \tau_{33})}{\dot{\gamma}^2},
\]

where \( \tau \) is the extra stress tensor, \( N_1 \) and \( N_2 \) the first and second normal stress differences, and \( \dot{\gamma} \) the magnitude of the rate of strain tensor

\[
\dot{\gamma} = \sqrt{\frac{\tau : \tau}{2}}.
\]

The rate of strain tensor \( \dot{\gamma} \) is defined in terms of the velocity \( \nabla \) as

\[
\dot{\gamma} = \nabla v + v \nabla^T,
\]

and coincides with the shear stress for a simple shear flow.
Figure 2.1 Simple steady shear flow experiment.

\[ v_z = \dot{\epsilon} z \]
\[ v_r = -\left(\dot{\epsilon}/2\right) r \]

Figure 2.2 Steady elongational flow experiment.
Each of these functions exhibits the same qualitative behavior as the shear rate increases. For low values of \( \dot{\gamma} \), each material function is constant and equal to its zero-shear-rate value, denoted by the subscript 0. At high shear rates, the behavior is modelled by a decreasing function with power-law dependence on the shear rate, with particular values of power-law index, \( n \). This is the well known shear-thinning phenomenon. The low and high shear rate asymptotes are separated by a transition region. The second normal stress coefficient, \( \psi_2 \), is generally an order of magnitude smaller than and of opposite sign from \( \psi_1 \). Since \( \psi_2 \) is also very difficult to measure, the matching of this material function is rarely considered in selecting a constitutive equation.

It is sometimes more useful to refer to the stress-ratio \( N_1/\tau \), defined as

\[
N_1/\tau = \psi_1 \dot{\gamma}/\eta ,
\]

where \( \tau \) is the extra stress magnitude, \( \tau = \sqrt{(1:1)/2} \) (\( 1:1 \) in simple shear flow), instead of the first normal stress coefficient \( \psi_1 \). The stress-ratio is a dimensionless rather than dimensional number, which can also serve as an objective measure of the importance of elasticity in a given flow. Larger values of stress-ratio, correspond to more elastic flow (for a Newtonian fluid it is zero, since \( \psi_1=0 \)). Furthermore, the definition of the stress-ratio is free from the ambiguity that exists in the definition of the Deborah number for cases of a fluid with a spectrum of characteristic relaxation times (Tanner 1976). Half the stress-ratio is used in the literature under the name of "recoverable shear", \( S_R \), name whose use will be avoided here since it is quite misleading.

The elongational or Trouton viscosity \( \tilde{\eta} \) in a steady simple uniaxial elongational flow (see Figure 2.2) is defined as

\[
\tilde{\eta}(\dot{\varepsilon}) = (\tau_{zz} - \tau_{rr})/\dot{\varepsilon} ,
\]

where \( \dot{\varepsilon} \) is the elongation rate. Although elongational behavior is very important in determining the polymer rheology in many industrial applications, as in fiber spinning, it is very difficult to measure. Actually,
elongational data exist only for polymer melts, not for solutions. For polymer solutions only rough estimates of the elongational viscosity have been made from experiments with flows of strong elongational character as the ductless siphon (Chao and Williams 1983) and the stretching flow generated by the spin-line rheometer (Jackson et al. 1984). For low density polyethylene (LDPE) melts (Laun and Munstedt 1978) $\tilde{n}(\dot{\epsilon})$ is observed to be constant for low $\dot{\epsilon}$ (satisfying the Trouton's relation, $\tilde{n}_0 = 3n_0$), to increase to a maximum at moderate $\dot{\epsilon}$, and then to decrease in a power-law fashion at higher $\dot{\epsilon}$.

2.2 Constitutive Equations

Starting with the simple models of Kramers (1946), Rouse (1953), Zimm (1956) and others, in the last half century a large number of ever more sophisticated models of dilute polymer solutions have been developed, the latest of which can describe most of the observed phenomena for small deformations where linear theory holds (Ferry 1970). An extension of this work to concentrated polymer solutions and polymer melts has proven to be extremely difficult. In fact, the first successful models, the network models (Green and Tobolsky 1946; Lodge 1977; Lodge et al. 1982; James and Segalman 1977; Phan Thien and Tanner 1977), were derived as variations on theories for describing rubber elasticity (Treloar 1958).

The reptation models of Doi and Edwards (1978a and 1978b) have been developed in a parallel effort. A fundamental assumption in these models is that the polymer chains are only able to move by sliding alongside one another in a reptation manner. The recent model developed by Curtiss and Bird (1981) using kinetic theory, incorporates the idea that the interaction of closely-packed polymer chains is manifested in the determination of the mobility of the individual structural elements. The mobility, $m$, is defined, as in small Reynolds hydrodynamics, as the proportionality factor between the velocity $\mathbf{v}$ of a particle and the force $\mathbf{F}$ exerted by the surrounding fluid

$$\mathbf{v} = m \cdot \mathbf{F}.$$ \hspace{2cm} (2-7)

The mobility tensor is anisotropic in general and depends on the actual
configuration of the individual elements of the polymer chain.

The Curtiss-Bird model gives qualitatively correct predictions at least for simple shear and elongational flows, the only cases—except the work of Malkus and Bernstein which is mentioned in the following—to which it has been applied so far (Bird et al. 1982a and 1982b; Saab et al. 1982). The major drawback to the application of this model in more complicated flows is its complexity, since the stress tensor is given as the sum of two complicated integral expressions in terms of the deformation history of a fluid element as measured by the Cauchy strain tensor (Bird et al. 1977a p.430) and the rate of strain tensor \( \dot{\gamma} \).

Currie (1982) showed that the strain dependence of the constitutive equation derived by the Curtiss-Bird model is completely determined by a single potential function which depends on the first and the second invariants of the Cauchy strain tensor. Currie (1982) also provided a good approximation for the potential function with which it is determined much easier than by using the exact expression. Using this approximation, and a special numerical finite element technique that allowed to accurately calculate particle paths, Malkus and Bernstein (1984) simulated the flow of viscoelastic fluids over slots. At large \( \text{De} \) the computations did not diverge but required a large computation time, due to the required particle tracking, which limited their investigation to \( \text{De} \approx 2 \).

Working along the same lines, Gieseokus (1982) derived a substantially simpler model by considering the tensorial mobility as dependent on the mean rather than on the actual configuration of the polymer molecule under consideration (Gieseokus 1984). The configuration dependence of the stress was then eliminated and a differential constitutive equation was obtained. A more detailed derivation of the Gieseokus model follows.

Assume that a polymer molecule consists of beads of the \( i \)-th kind, \( i=1,...,N \). A first hypothesis is that the motion of the \( i \)-th bead, described by the position vector \( \mathbf{r}_i \), is related to the driving force \( \mathbf{f}_i \) in the form of a generalized Stokes' law by means of a tensorial drag coefficient \( \xi_i \):

\[
\mathbf{f}_i = \xi_i \cdot (\mathbf{r}_i)'(1) = (\mathbf{r}_i)'(1) \cdot \xi_i \. \tag{2-8}
\]
where the subscript \((1)\) denotes the upper convected time derivative defined as

\[
\dot{r}_i(1) = \frac{\text{Dr}}{\text{Dt}} - \nabla v \cdot \dot{r}_i - \dot{r}_i \cdot \nabla v.
\]  \hspace{1cm} (2-9)

The macroscopic excess stresses associated with the \(i\)-th bead are then given (see Bird et al. 1977b) by the expectation value (designated by brackets):

\[
\Sigma_i = -n_i < r_i f_i > = -n_i < r_i(\dot{r}_i)^{(1)} \cdot S_i > ,
\]  \hspace{1cm} (2-10)

where \(n_i\) is the number of beads of the \(i\)-th kind per unit volume.

A second hypothesis is that \(\Sigma\) does not depend on the actual configuration of the respective bead, but only on the mean configuration so that it can be taken out of the brackets:

\[
\zeta^{-1} \cdot S_i + S_i \cdot \zeta^{-1} = -n_i ( < r_i f_i > )^{(1)} .
\]  \hspace{1cm} (2-11)

Let an isotropic equilibrium configuration \(R_i\) which satisfies the relation

\[
< R_i R_i > = 1/3 < R_i \cdot R_i > \delta ,
\]  \hspace{1cm} (2-12)

where \(\delta\) is the unit tensor, be associated with the deformed configuration. These two tensors may then be connected by the mapping with a "configuration tensor" \(b_{ij}\) as

\[
< r_i r_i > = b_{ij} \cdot < R_i R_i > = 1/3 < R_i \cdot R_i > b_{ij} .
\]  \hspace{1cm} (2-13)

In the following, assume for simplicity that only one kind of beads exists (one relaxation model) in which case the subscript \(i\) can be dropped. Also, let the normalized \(\zeta^{-1}\) to be denoted by the relative mobility tensor \(\beta\):

\[
\zeta^{-1} = \beta Z^{-1} ,
\]  \hspace{1cm} (2-14)

with equilibrium value, \(\beta_0 = \delta\). Then, eq. (2-11) becomes
\[
\frac{1}{2} \left( \hat{b} \cdot \dot{\Sigma} + \Sigma \cdot \dot{\hat{b}} \right) + n^0 (\dot{b})(1) = \dot{\gamma},
\]
(2-15)

where \( n^0 \) is the equilibrium viscosity,

\[
n^0 = (1/6) n \langle R_i \cdot R_j \rangle > Z,
\]
(2-16)

\( n \) being the total number of beads per unit volume and the overbar denoting the average over all the beads of a molecule.

The classical theory of the "recoverable strain" suggests that the configuration tensor \( \hat{b} \) should be considered as a basic quantity with the extra stress tensor \( \Sigma \) and the relative mobility tensor \( \hat{\beta} \) dependent on \( \hat{b} \) alone. The exact form of dependence should have been derived from the structure of the polymer fluid, but this is not feasible at present. Instead, Giesekeus made two last assumptions. He assumed a linear neo-Hookean constitutive equation

\[
\Sigma = G \left( \hat{b} - \hat{\delta} \right),
\]
(2-17)

relating \( \Sigma \) and \( \hat{b} \), and a linear relationship between the relative mobility tensor \( \hat{\beta} \) and \( \hat{b} \)

\[
\hat{\beta} = \hat{\delta} + \alpha \left( \hat{b} - \hat{\delta} \right), \quad 0 < \alpha < 1.
\]
(2-18)

Then, relationship (2-15) becomes

\[
\left[ \hat{\delta} + \alpha \left( \hat{b} - \hat{\delta} \right) \right] \cdot \left( \hat{b} - \hat{\delta} \right) + \lambda(\hat{b})(1) = \dot{\gamma},
\]
(2-19)

or equivalently,

\[
\left[ \hat{\delta} + \alpha \lambda_1 / n^0 \right] \cdot \Sigma + \lambda_1 S(1) = n^0 \dot{\gamma},
\]
(2-20)

where \( \alpha \) (0\( < \alpha < 1 \)) is a dimensionless number called the mobility factor, since it characterizes the anisotropy of the mobility tensor (\( \alpha = 0 \) corresponding to isotropic mobility), and where

\[
\lambda_1 = n^0 / G,
\]
(2-21)
is the relaxation time.

Adding a retardation term to (2-20) is equivalent to considering the total extra stress $\tau$ as consisting of a viscoelastic part $\overline{S}$ obeying (2-20) plus a viscous contribution $\eta_3 \dot{\gamma}$:

$$\tau = \overline{S} + \eta_3 \dot{\gamma}, \quad (2-22)$$

where $\eta_3$ is referred to as a solvent viscosity since, in the case of polymer solutions, eq. (2-22) can be interpreted as assigning the total stress $\tau$ as the sum of a polymer ($\overline{S}$) and a solvent ($\eta_3 \dot{\gamma}$) contribution.

Equivalently, $\overline{S}$ can be eliminated between (2-20) and (2-22) to produce a differential equation for the total extra stress tensor $\tau$

$$(\delta + A \lambda_1 \eta_0 \overline{I}) \tau + \lambda_1 \tau(1) = \eta_0 (\dot{\gamma} + \lambda_2 \dot{\gamma}(1) - A(\lambda_2 / \lambda_1) \lambda_2 \dot{\gamma}^2) + A \lambda_2 (\overline{I} \cdot \dot{\gamma} + \dot{\gamma} \cdot \overline{I}), \quad (2-23)$$

where $A$ is a modified mobility factor

$$A = \omega / (1 - \lambda_2 / \lambda_1); \quad (2-24)$$

$\lambda_1$ and $\lambda_2$ are the relaxation and retardation times, respectively, and $\eta_0$ is the zero-shear rate viscosity defined as

$$\eta_0 = \eta^0 + \eta_3. \quad (2-25)$$

The retardation time $\lambda_2$ is related to the solvent viscosity through

$$\lambda_2 = \lambda_1 \eta_3 / (\eta^0 + \eta_3). \quad (2-26)$$

It is the association of the retardation time with the solvent viscosity that explains the usefulness of introducing it in modelling polymer solution behavior (Jackson et al. 1984).

A number of well-known differential models are obtained as particular limits of the Giesekus model, as is shown in Table 2.1. The Newtonian (NEW), upper-convected Maxwell (UCM), second-order fluid (SOF) and Oldroyd-B (OLD) models are obtained exactly. The Leonov-like (LEL) and corotational
<table>
<thead>
<tr>
<th>Constitutive Equation</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Newtonian (NEW)</td>
<td>$0, \lambda$</td>
<td>$0, \lambda$</td>
<td>$-$</td>
</tr>
<tr>
<td>Upper Convected Maxwell (UCM)</td>
<td>$\lambda_1 &gt; 0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>Leonov-like (LEL)</td>
<td>$\lambda_1 &gt; 0$</td>
<td>$0$</td>
<td>$1/2$</td>
</tr>
<tr>
<td>Corotational Maxwell-like (CML)</td>
<td>$\lambda_1 &gt; 0$</td>
<td>$0$</td>
<td>$1$</td>
</tr>
<tr>
<td>Second Order Fluid (SOF)</td>
<td>$0$</td>
<td>$\lambda_2 &lt; 0$</td>
<td>$-$</td>
</tr>
<tr>
<td>Oldroyd-B (OLD)</td>
<td>$\lambda_1 &gt; 0$</td>
<td>$\lambda_2 &gt; 0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
Maxwell-like (CML) fluids duplicate the Leonov (Leonov et al. 1976) and corotational Maxwell models, respectively, only in steady simple shear.

Of particular interest are two cases: \((\alpha=0, \lambda_2=0)\) which corresponds to the UCM model, and \((\alpha=0.5, \lambda_2=0)\) which gives the LEL model. The UCM model is the most commonly used model in numerical calculations since it is relatively simple and still able to predict qualitatively correct viscoelastic behavior in steady shear flow. Unfortunately it has a number of well-known shortcomings such as a constant viscosity, a zero second normal stress coefficient, and an elongational viscosity that becomes infinite at a finite elongational rate. On the other hand the LEL model predicts a shear thinning viscosity and an elongational viscosity which is always finite. Nevertheless, the viscosity at large shear rates decreases inversely proportionally to the shear rate (the limiting behavior for a physically acceptable model) and the elongational viscosity changes only from the Trouton value \(3\eta\) at \(\dot{\varepsilon}=0\) to \(4\eta\) as \(\dot{\varepsilon} \to \infty\). The viscometric material functions for the UCM, LEL and the other models are summarized in Figure 2.3. Shear thinning of the viscosity and the first normal stress difference is represented by both the CML and LEL models. The CML model predicts a decrease in shear stress with shear rate beyond \(\dot{\gamma}(\lambda_1-\lambda_2)=1\), and so is unrealistic. The Giesekus model gives a second normal stress coefficient \(\psi_2\) which ranges between zero and \(-1/2\psi_1\) depending on the values of \(\alpha, \lambda_1\) and \(\lambda_2\). The UCM, OLD and SOF limits of the Giesekus model all have \(\psi_2=0\). The elongational viscosity of the SOF fluid increases linearly with increasing strain rate. The steady elongational viscosities of the UCM and OLD are unbounded at a dimensionless extension rate \(\lambda_1 \dot{\varepsilon}\) of 1/2.

The UCM (\(\alpha=0\)) and LEL (\(\alpha=1/2\)) models are used most in this study, as representing the two extreme cases of the Giesekus model without the model becoming aphysical. However, it is also of interest to see how the Giesekus model predictions change for intermediate values of \(\alpha\); this is shown in Figure 2.4. In this study, a small retardation time of \(\lambda_2 = \lambda_1/1000\) is also used when \(\alpha=0\) in order to alleviate the excessive shear thinning that occurs for \(\alpha>0\). For \(\alpha=0.5\) adding \(\lambda_2\) gives the Leonov-like with retardation (LER) model. As can be seen in Figure 2.4, introducing \(\lambda_2\) causes a maximum in the stress-ratio at \(\dot{\gamma} \sim \lambda_1/\lambda_2\). Therefore,
Figure 2.3 Viscoelastic functions for the six limits of the Giesekus model listed in Table 2.1. All quantities are dimensional. The ordinates are as shown except in the elongational viscosity plot for the Oldroyd-B model. The elongational viscosity for that model is plotted against $\varepsilon \lambda_1$, i.e., $\eta$ for OLD becomes infinite at $\varepsilon \lambda = 1/2$ for all $\lambda_2 = \lambda_1$. 
Figure 2.4 Material functions predicted by the Giesekeus model with different values of the mobility parameter $\alpha$. (a)-(c) Steady shear flow properties. $N_1$ is the first normal stress difference $\tau_{xx} - \tau_{yy}$, and in (c) one-half the stress ratio $(\tau_{xx} - \tau_{yy})/\tau_{yx}$ is presented. (d) Elongational viscosity.
it is essential that the ratio $\lambda_2/\lambda_1$ is small enough for large values of stress-ratio to be obtained. Also Figure 2.4 shows that a substantially large, yet finite, value for the elongational viscosity can be reached at intermediate values of $\alpha$. In fact $\eta$ approaches $(2/\alpha)\eta$ asymptotically at high $\dot{\gamma}$.

The effect on the velocity field of the shear thinning predicted by the LER model is examined by finite element calculations with a generalized Newtonian fluid with the same viscosity as the LER model. The extra stress tensor for this fluid is written as

$$\mathbf{I} = \eta(\dot{\gamma})\dot{\gamma}$$

(2-27)

and the viscosity is given by

$$\eta = \eta^0 \int \frac{1}{2 (\lambda \dot{\gamma})^2} \left((1 + 4(\lambda \dot{\gamma})^2)^{1/2} - 1\right) \mathbf{I} + 0.001 \eta^0$$

(2-28)

where $\lambda^{-1}$ is a characteristic shear rate that separates the regions of constant viscosity ($\eta_0$) and shear thinning and $\dot{\gamma} = \sqrt{\dot{\gamma}^2 - \dot{\gamma}^*(2)}$ is the shear rate. When eqs. 2.27 and 28 are put in dimensionless form, $\lambda$ can be used to form a Deborah number analogous to Eq. 4.9; this 'inelastic' Deborah number is identical to the Carreau number used in the generalized Newtonian fluid calculations reported by Kim-E et al. (1983). Increasing $\lambda$ has the same effect on viscosity as increasing the relaxation time $\lambda_1$ in the Deborah number in the viscoelastic calculations with the LER model. The first term of eq. (2.28) is exactly the viscosity predicted by the Giesekus model for $\alpha = 1/2$ (LEL model). The second term is the additive Newtonian part associated with the retardation time $\lambda_2$, which we have taken to be 0.001 $\lambda_1$.

Two other models, unrelated to Giesekus model, have been used in this thesis. The Criminale-Ericksen-Filbey (CEF) model (Criminale et al. 1958) is the generalization of the SOF that accounts correctly for shear-rate dependence of the viscosity and normal stress coefficients:

$$\mathbf{I} = \eta(\dot{\gamma})(\dot{\gamma} - \lambda(\dot{\gamma})\dot{\gamma}(1))$$

(2-29)
where \( \eta(\dot{\gamma}) = m\dot{\gamma}^{n-1} \), and \( \lambda(\dot{\gamma}) = \eta(\dot{\gamma})/G_0 \) were used. Since the CEF model is strictly valid only for steady-state shear flow, it seems appropriate to apply this model to the journal bearing flow only when the effect of the elastic memory of the fluid is small. Phan-Thien and Tanner (1981) have done this for small eccentricities; the reported results in Section 3.2.2 are a special case of their analysis.

Similarly, the White-Metzner (WM) model (White and Metzner 1963) is a generalization of the UCM fluid that accounts for shear-rate dependence of the viscosity and relaxation time:

\[
\dot{i} + \lambda(\dot{\gamma})_w(1) = \eta(\dot{\gamma})_w, \tag{2-30}
\]

where \( \eta(\dot{\gamma}), \lambda(\dot{\gamma}) \) are the same functions as in the CEF model (Eq. 2-29). Then the viscometric functions predicted by these two models are the same. The power-law model is the most general form of the viscosity that is needed to describe the flow in a journal bearing in the limit of small gap and eccentricity. The Taylor series expansion in eccentricity about the uniform shear flow (\( \mu << 1 \)) of any more general viscosity function \( \eta(\dot{\gamma}) \), e.g. the Carreau model (Bird et al. 1977a), gives the same series to order \( \epsilon \) as for the power-law fluid with the constants \( m \) and \( n \) related to the base shear rate \( \dot{\gamma}_b \equiv V/6 \) and the local slope \( d\eta(\dot{\gamma}_b)/d\dot{\gamma} \) (for more details see Section 3.2.1).

2.3 Governing Equations and Boundary Conditions

The governing equations are the equations of mass and momentum conservation, and one constitutive equation, for example, the Giesekus model (2-23). Inertia is neglected at this stage of investigation because viscoelastic flows are in general characterized by small Reynolds numbers, and because Kim-E et al. (1983) have shown that inertia alone can not produce the peculiar effects associated with viscoelastic flows—at least in the flow through a sudden contraction. Under the assumptions of an incompressible, steady, creeping flow, the conservation equations are written in compact tensor form as

\[
\nabla \cdot \underline{\mathbf{v}} = 0, \tag{2-31}
\]
\[ \nabla \cdot \mathbf{v} - \nabla p = 0 \quad . \]  

(2-32)

Equations (2-23) and (2-31,32) are solved for the flow inside the journal bearing (see Figure 1.2) as induced by the steady rotation of the inner cylinder with angular velocity \( \Omega \), while the outer cylinder is held stationary. The length of the journal bearing is considered to be so large that the flow is assumed to be planar two-dimensional. This is certainly true for the Newtonian case because of the assumption of the creeping flow. Of course, due to the nonlinearities of the constitutive equation, the flow may lose its stability and become three-dimensional or (and) time periodic. The present formulation can not account for these last two cases.

The boundary conditions on the velocity used for this flow are the no-slip and no penetration conditions at the surfaces of the inner and outer cylinders,

\[ \mathbf{v} = V_0 \mathbf{e}_\theta \quad , \]  

(2-33)

at the surface of the inner cylinder, and

\[ \mathbf{v} = 0 \quad , \]  

(2-34)

at the surface of the outer cylinder, where \( \mathbf{e}_\theta \) is the tangential to the inner cylinder unit vector.

The appropriate coordinate system for the journal bearing geometry is the bipolar coordinate system (Bird et al. 1977a) in which the above equations have been solved analytically for a Newtonian fluid (Wannier 1950). Nevertheless, a cylindrical coordinate system with respect to the axis of the inner cylinder is also used because complicated constitutive equations are written much more easily in this representation. The drawback of this choice is that the surface of the outer cylinder is represented by a function of the radial distance \( r \) with respect to the angular position \( \theta \),

\[ r_1(\theta) = \epsilon \delta \cos \theta + \left[ R^2 - (\epsilon \delta \sin \theta)^2 \right]^{1/2} \quad , \]  

(2-35)

instead of as a coordinate surface. This simplifies for small \( \epsilon \) as
\[ r_1(\theta) = R + \delta \epsilon \cos \theta + O(\epsilon^2), \quad (2-36) \]

where \( r_1(\theta) \) denotes the radial position of the outer surface at \( \theta \) angular position. This expression makes the use of perturbation methods feasible, as explained in more detail in the next Chapter.

2.3.1 Formulation of the Equations in Cylindrical Coordinates

In a cylindrical coordinate system, the \( z \)-axis of which coincides with the axis of the inner cylinder (see Figure 1.2), the components of the conservation equations (Equations 2-31 and 2-32), and of the constitutive equation (Equations 2-20 and 2-22) can be written, for a two-dimensional flow, as

\[ \frac{1}{r} \frac{\partial}{\partial r}(rv_r) + \frac{1}{r} \frac{\partial}{\partial \theta} v_\theta = 0, \quad (2-37) \]

\[ - \frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r}(rS_{rr}) + \frac{1}{r} \frac{\partial}{\partial \theta} S_{r\theta} - \frac{1}{r} S_{\theta \theta} + \eta_g \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r}(rv_r) \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} v_r - \frac{2}{r^2} \frac{\partial}{\partial \theta} v_\theta \right] = 0, \quad (2-38) \]

\[ - \frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 S_{r\theta}) + \frac{1}{r} \frac{\partial}{\partial \theta} S_{\theta \theta} + \eta_g \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r}(rv_\theta) \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} v_\theta + \frac{2}{r^2} \frac{\partial}{\partial \theta} v_r \right] = 0, \quad (2-39) \]

\[ S_{rr} + \lambda_1 \left\{ v_r \frac{\partial S_{rr}}{\partial r} + \frac{1}{r} v_\theta \frac{\partial S_{r\theta}}{\partial \theta} - 2 \left[ S_{rr} \frac{\partial}{\partial r} v_r + \frac{1}{r} S_{r\theta} \frac{\partial}{\partial \theta} v_r \right] \right. \\
+ \alpha \left[ S_{rr}^2 + S_{r\theta}^2 \right] \right\} = 2n^0 \frac{\partial}{\partial r} v_r, \quad (2-40) \]

\[ S_{r\theta} + \lambda_1 \left\{ v_r \frac{\partial S_{r\theta}}{\partial r} + \frac{1}{r} v_\theta \frac{\partial S_{r\theta}}{\partial \theta} + \left( \frac{1}{r} v_\theta - \frac{\partial}{\partial r} v_\theta \right) S_{rr} - \left( \frac{1}{r} \frac{\partial}{\partial r} v_r \right) S_{\theta \theta} + \alpha S_{r\theta} \left( S_{rr} + S_{r\theta} \right) \right\} \\
= n^0 \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} v_r \right) + \frac{1}{r} \frac{\partial}{\partial \theta} v_\theta \right], \quad (2-41) \]

\[ S_{\theta \theta} + \lambda_1 \left\{ v_r \frac{\partial S_{\theta \theta}}{\partial r} + \frac{1}{r} v_\theta \frac{\partial S_{\theta \theta}}{\partial \theta} + 2 \left( \frac{1}{r} v_\theta - \frac{\partial}{\partial r} v_\theta \right) S_{r\theta} \right\} \]
\[-2 \left( \frac{1}{r} \frac{\partial}{\partial \theta} v_\theta + \frac{1}{r r} v_r \right) S_{\theta \theta} + \alpha \left( S_{\theta \theta}^2 + S_{r \theta}^2 \right) \]
\[= 2 \eta \left( \frac{1}{r} \frac{\partial}{\partial \theta} v_\theta + \frac{1}{r r} v_r \right). \tag{2-42} \]

Equation 2-37 is the continuity equation formulated for an incompressible fluid, Equations 2-38 and 2-39 are the radial and azimuthal components of the momentum equation, and Equations 2-40, 2-41 and 2-42 are the three non-zero components (radial, shear and azimuthal) of the Giesekeus constitutive equation. With \(v_\theta\) and \(v_r\), the azimuthal and radial components of the velocity are denoted respectively, and \(S_{\theta \theta}, S_{rr}, \) and \(S_{r \theta}\) are the azimuthal, radial normal stress and shear stress component of the viscoelastic part of the extra stress tensor (see Equation 2-22).

The boundary conditions (Equations 2-33 and 2-34) are simply expressed as

\[v_\theta = V, \quad v_r = 0, \quad r = R_0, \tag{2-43} \]
\[v_\theta = 0, \quad v_r = 0, \quad r = r_1(\theta), \tag{2-44} \]

with \(r_1(\theta)\) given by Equation 2-35.

2.3.2 Formulation of the Equations in Bipolar Coordinates

In the bipolar coordinate system showed in Figure 1.3 let the velocity components \(v_\theta\) and \(v_\xi\) be expressed—for an incompressible fluid—in terms of the stream function \(\psi\) as

\[v_\theta = -\frac{\xi}{a} \frac{\partial \psi}{\partial \xi}, \quad v_\xi = \frac{\psi}{a} \frac{\partial \psi}{\partial \theta}, \tag{2-45} \]

where

\[\xi = \cosh \xi + \cos \theta, \tag{2-46} \]

and \(a\) is the parameter defined in the caption of figure 1.3. Let also the non-zero components of the extra stress tensor for a two-dimensional flow be represented as \(\tau_{\xi \xi}, \tau_{\xi \theta}, \) and \(\tau_{\theta \theta}. \) Then the components of the momentum equations (2-32) in bipolar coordinates are
\[- \frac{X}{a} \frac{\partial \rho}{\partial \xi} + \frac{X}{a} \left[ \frac{\partial}{\partial \xi} \xi_\theta + \frac{\partial}{\partial \theta' \theta} \right] + \frac{1}{a} \left( \tau_{\theta' \theta'} - \tau_{\xi \xi} \right) \sin \theta + \frac{2}{a} \tau_{\xi \theta} \sin \theta = 0, \tag{2-47} \]

\[- \frac{X}{a} \frac{\partial \rho}{\partial \theta} + \frac{X}{a} \left[ \frac{\partial}{\partial \xi} \xi_\theta + \frac{\partial}{\partial \theta' \theta} \right] + \frac{1}{a} \left( \tau_{\theta' \theta'} - \tau_{\xi \xi} \right) \sin \theta - \frac{2}{a} \tau_{\xi \theta} \sin \xi _\theta = 0. \tag{2-48} \]

If the pressure is eliminated from the momentum equations 2-47 and 2-48 we get

\[X \left( \frac{\partial^2}{\partial \theta^2} - \frac{\partial^2}{\partial \xi^2} + 1 \right) \left( \frac{\tau_{\xi \theta}}{X} \right) + X \frac{\partial^2}{\partial \xi \partial \theta} \left( \frac{\tau_{\xi \xi} - \tau_{\theta \theta}}{X} \right) = 0. \tag{2-49} \]

Dividing by \(X \eta^n V\) and using the dimensionless modified extra stress tensor \(t^*\):

\[t^* = \frac{a}{\eta^n V X} \frac{1}{X} \xi_\theta, \tag{2-50} \]

equation 2-49 simplifies to

\[\left( \frac{\partial^2}{\partial \theta^2} - \frac{\partial^2}{\partial \xi^2} + 1 \right) t^*_{\xi \theta} + \frac{\partial^2}{\partial \xi \partial \theta} \left( t^*_{\xi \xi} - t^*_{\theta \theta} \right) = 0. \tag{2-51} \]

Furthermore, let the dimensionless modified viscoelastic stress tensor \(T^*\) be defined as

\[T^* = \frac{a}{\eta^n V X} \frac{1}{X} \xi_\theta, \tag{2-52} \]

and the dimensionless modified stream function \(\psi^*\) as

\[\psi^* = \psi X \frac{V}{a}. \tag{2-53} \]

The components of \(t^*\) tensor can be readily expressed in terms of the components of \(T^*\) tensor by using the relationship 2-22 between the \(T^*\) and \(t^*\).
and \( S \) tensors (in the following the superscripts \( * \) will be omitted for the sake of clarity):

\[
\begin{align*}
\tau_{\xi\xi} &= T_{\xi\xi} + 2n^* \psi_{\xi\theta}, \\
\tau_{\xi\theta} &= T_{\xi\theta} + n^* (\psi + \psi_{\theta\theta} - \psi_{\xi\xi}), \\
\tau_{\theta\theta} &= T_{\theta\theta} - 2n^* \psi_{\xi\theta},
\end{align*}
\] (2-54a, 2-54b, 2-54c)

where \( \psi_{\xi\theta} = \partial^2 \psi / \partial \xi \partial \theta \) and similarly for the other subscripted \( \psi \) terms.

Then Equation 2-51 can be written as

\[
\left( \frac{\partial^2}{\partial \theta^2} - \frac{\partial^2}{\partial \xi^2} + 1 \right) T_{\xi\theta} + \frac{\partial^2}{\partial \xi \partial \theta} (T_{\xi\xi} - T_{\theta\theta}) + n^* [\psi_{\xi\xi\xi\xi} + \psi_{\theta\theta\theta\theta} + 2 \psi_{\xi\xi\theta} - 2 \psi_{\xi\xi} + \psi_{\theta\theta} + \psi] = 0. \tag{2-55}
\]

Substitutions (2-52) and (2-53) turn out to be particularly useful in obtaining simple expressions for the components of the Gieseus equation (Equation 2-22) in bipolar coordinates:

\[
\begin{align*}
T_{\xi\xi} + \text{De'} \{ (X\psi_{\theta} + \psi_{\theta}) \frac{\partial}{\partial \xi} T_{\xi\xi} + (-X\psi_{\xi} + \psi_{\xi}) \frac{\partial}{\partial \theta} T_{\xi\xi} \\
+ T_{\xi\xi}(\psi_{\theta} \sinh \xi + \psi_{\xi} \sin \xi - 2X \psi_{\xi\theta}) - 2 T_{\xi\theta}(X \psi_{\xi\theta} + \psi_{\cos \theta}) \\
+ \alpha X (T_{\xi\xi}^2 + T_{\xi\theta}^2) \} &= 2 \psi_{\xi\theta}, \tag{2-56}
\end{align*}
\]

\[
\begin{align*}
T_{\xi\theta} + \text{De'} \{ (X\psi_{\theta} + \psi_{\theta}) \frac{\partial}{\partial \xi} T_{\xi\theta} + (-X\psi_{\xi} + \psi_{\xi}) \frac{\partial}{\partial \theta} T_{\xi\theta} \\
+ T_{\xi\theta}(\psi_{\theta} \sinh \xi + \psi_{\xi} \sin \xi) - T_{\theta\theta}(X \psi_{\theta\theta} + \psi_{\cos \theta}) \\
- T_{\xi\xi}(-X \psi_{\xi\xi} + \psi_{\cosh \xi}) \\
+ \alpha X T_{\xi\theta}(T_{\xi\xi} + T_{\theta\theta}) \} &= \psi + \psi_{\theta\theta} - \psi_{\xi\xi}, \tag{2-57}
\end{align*}
\]

\[
\begin{align*}
T_{\theta\theta} + \text{De'} \{ (X\psi_{\theta} + \psi_{\theta}) \frac{\partial}{\partial \xi} T_{\theta\theta} + (-X\psi_{\xi} + \psi_{\xi}) \frac{\partial}{\partial \theta} T_{\theta\theta} \\
+ T_{\theta\theta}(\psi_{\theta} \sinh \xi + \psi_{\xi} \sin \xi + 2X \psi_{\xi\theta}) - 2 T_{\xi\theta}(-X \psi_{\xi\theta} + \psi_{\cos \theta}) \\
+ \alpha X (T_{\theta\theta}^2 + T_{\xi\theta}^2) \} &= -2 \psi_{\xi\theta}, \tag{2-58}
\end{align*}
\]
where $De'$ is a modified Deborah number defined as
\[ De' = \frac{R_0}{a} De = \frac{\xi}{\sqrt{1-\epsilon^2}} \left( 1 + \mu + \frac{1-\epsilon^2}{4} \mu^2 \right)^{-1/2} De. \] (2-59)

The boundary conditions (Equations 2-33 and 2-34) can be expressed in terms of the modified stream function $\psi$ using (2-45) and (2-53) as
\[ \psi = Q^*X, \quad \psi_\xi = 1 + Q^* \sinh \xi, \quad \xi = \xi_1, \] (2-60)
\[ \psi = 0, \quad \psi_\xi = 0, \quad \xi = \xi_2, \] (2-61)

where $\xi_1$ and $\xi_2$ are the $\xi$ coordinates of the inner and outer cylinder respectively (see Figure 1.3), and $Q^*$ is the flowrate per unit length (made dimensionless by $aV$).

The flowrate $Q^*$ is unknown a priori, therefore one more equation is needed in order to determine it. This is generated by requiring the pressure to be $2\pi$ periodic. Since the pressure is recovered by integrating over $\theta$ the $\theta$-momentum equation (Equation 2-48), $2\pi$ periodicity is assured when the closed integral of the $\theta$-momentum equation around the inner cylinder at any position $\xi$, say at $\xi=\xi_1$, is zero:
\[ \int_0^{2\pi} \left( \frac{\partial}{\partial \xi} t_{\xi \theta} + \frac{\partial}{\partial \theta} t_{\theta \theta} \right) X - t_{\xi \theta} \sin \theta - t_{\theta \xi} \sinh \xi \right) d\theta = 0. \] (2-62)

Equation 2-62 can be readily expressed in terms of the components of the $T$ tensor using the relationships provided by Equation 2-54.

Equations 2-55 to 2-58 together with the boundary conditions 2-60 to 2-62 represent a complete mathematical formulation of the flow problem in bipolar coordinates in terms of the modified stream function $\psi$ and the three components of the modified viscoelastic stress tensor $T$. Notice that by using the modified instead of the actual components of the stress and the stream function the only nonlinear terms which appear are quadratic, a feature particularly advantageous for the application of spectral methods (see Chapter 6).
3. PERTURBATION ANALYSIS

Perturbation techniques constitute a powerful tool for the analysis of non-linear partial differential equations (Bender and Orszag 1978; Nayfeh 1981; Van Dyke 1975). The only prerequisite for their application is the existence of a parameter(s) for a particular value of which the solution of the equations is known. The key idea of the perturbation methods is to expand the solution for an arbitrary set of parameters values as a Taylor series around the known solution, and substituting this series in the original equation recover a set of equations, one for each order in the Taylor expansion, which are easier to solve than the original equation. In most of the cases, calculation of the first order correction term in the series is sufficient to reveal the qualitative features of the solution for a substantial range of parameter values around the corresponding ones to the known solution.

Perturbation methods are applicable (in principle) for the study of the viscoelastic flow in the journal bearing since two parameters exist, the Deborah number, De, and the eccentricity, ε, for a particular value of which an analytical solution of the problem is available. At De = 0 the original problem degenerates to Newtonian flow in a journal bearing, and at De = 0 to viscoelastic Couette flow. Two different perturbation expansions are therefore possible, one in De and the other in ε, and they are analysed in the Sections 3.1 and 3.2 respectively.

3.1 Regular Perturbation in Deborah Number

The Deborah number characterizing the importance of elasticity in a given flow, is a characteristic dimensionless number of primary importance in viscoelastic flows. It appears in both differential and integral models, in the most general and the simplest of the constitutive equations. In most of the cases, at the limit of De = 0 the Newtonian constitutive equation is recovered (in some cases, this limit corresponds to an inelastic model). This fact has led towards the development of regular perturbation expansions in De, irrespective of the particular geometry used. Starting from the most general formulation for a "simple" fluid in this way the different "order" fluid models have been derived (Rivlin and Ericksen
1955; Coleman and Noll 1961), each one corresponding to a higher order approximation. Similarly, starting from the upper convected Maxwell model, it is easy to show that the zero order approximation corresponds to a Newtonian fluid, the first order to a particular case of the second order fluid (eq. (2-23) with \( \alpha = \lambda_1 = 0 \)), the second order approximation to a particular case of the third order fluid, and so on.

In the journal bearing flow problem a regular expansion in \( \text{De} \) (or rather in \( \text{De}' \)) has been attempted for the UCM model. Since this flow is two-dimensional and confined, the Newtonian flow field is a unique solution for the second order fluid (Giesekeus 1963; Tanner 1966; Huilgol 1973). Therefore, to calculate a non-zero correction to the Newtonian flow field the solution to the perturbation equations had to be calculated at least up to the second order.

The equations were formulated in the bipolar coordinate system shown in Figure 1.3 in terms of the modified stream function \( \psi^* \), and extra stress tensor \( t^*_e \) defined by the eqs. 2-53 and 2-50 respectively. The same section contains the equations set for the UCM model as well, if we take the parameters \( n^* \) and \( \alpha \) equal to zero in the momentum (Eq. 2-55) and the constitutive equations (2-56) to (2-58). Also notice that for the UCM model the two modified tensors \( t^*_e \) and \( T^*_e \) defined in section 2.3.2 coincide. The boundary conditions correspond to eqs. (2-60) to (2-62). Then, \( \psi^* \) and \( t^*_e \) were substituted by their expansions in \( \text{De} \) (in the following the superscripts * will be omitted for the sake of clarity) as

\[
\begin{align*}
\psi & = \psi^0 + \psi^1 \text{De} + \psi^2 \text{De}^2 + \ldots \\
t & = t^0 + t^1 \text{De} + t^2 \text{De}^2 + \ldots
\end{align*}
\]

The equation for zero-order solution \( \psi^0 \), corresponding to the Newtonian (modified) stream function, is obtained by letting \( \text{De}=0 \) in the equations 2-55 to 2-58 and is equivalent to the biharmonic equation for the original stream function \( \psi^0 \)

\[
\nabla^4 \psi^0 = B \psi^0 = \psi^0_{\xi\xi\xi\xi} + 2 \psi^0_{\xi\xi\theta\theta} + 2 \psi^0_{\xi\xi\xi\theta} - 2 \psi^0_{\xi\xi\theta\theta} + 2 \psi^0_{\theta\theta} + \psi^0 = 0, \quad (3-2)
\]

where \( B \) is defined as the operator which being applied to \( \psi^0 \) produces
Equation 3-2.

Equation 3-2 justifies the use of the modified instead of the original stream function since the biharmonic operator \( \nabla^4 \) which corresponds to a linear equation with variable coefficients, is exchanged with the B operator corresponding to a linear equation with constant coefficients. This transformation of the stream function, obtained through the Equation 2-53, is attributed by Kamal (1966) to Abel (1963).

Separation of variables easily shows that the general solution of eq. (3-2) consists of a series of products of a hyperbolic (or polynomial) function in \( \xi \) and a trigonometric function in \( \theta \). Equation 3-2 has been solved by Kamal (1966), subject to the particular boundary conditions dictated by eqs. (2-60) tc (2-62), and the solution is given as

\[
\psi = A_0 \sinh \xi + B_0 \cosh \xi + \left[ D_0 \sinh \xi + E_0 \cosh \xi \right] \xi + Q^e
\]

\[
\quad \left[ A_1 + B_1 \xi + D_1 \sinh(2\xi) + E_1 \cosh(2\xi) \right] \cos \theta , \quad (3-3)
\]

where the values of the constants \( Q, A_0, B_0, D_0, E_0, A_1, B_1, D_1 \) and \( E_1 \) are given in Appendix B.

Starting from this solution the procedure for calculating higher-order terms is mathematically straightforward: The Newtonian stress \( t^0 \) corresponding to \( \psi^0 \) is calculated and the constitutive equations (2-56) to (2-58) allow the first-order correction to the stress field to be expressed as

\[
t^1_{\xi\xi} = t^0_{\xi\xi}(1) + 2 \psi^1_{\xi\theta} \quad , \quad (3-4a)
\]
\[
t^1_{\xi\theta} = t^0_{\xi\theta}(1) + (\psi^1 + \psi^1_{\theta\theta} - \psi^1_{\xi\xi}) \quad , \quad (3-4b)
\]
\[
t^1_{\theta\theta} = t^0_{\theta\theta}(1) - 2 \psi^1_{\xi\theta} \quad , \quad (3-4c)
\]

where \( \psi^1 \) is the first order correction to \( \psi \). Then, eqs. (3-4) are substituted in the momentum equation (2-55) and an equation for \( \psi^1 \) is recovered by collecting the terms that are proportional to \( De \). This result has the form

\[
B \psi^1 = f_1(\psi^0) \quad , \quad (3-5)
\]

where \( f_1 \) is some function of \( \psi^0 \) and its derivatives. Then \( \psi^1 \) is determined
by solving eq. (3-5) subject to the boundary conditions

$$\psi^1 - \psi^1 = 0, \quad \xi = \xi_1, \xi_2.$$  (3-6)

As mentioned before, in this problem the first-order correction is zero. Indeed, when the calculations outlined above were carried out using the symbolic manipulator MACSYMA (MACSYMA 1977; Pavelle et al. 1981) \( f_1 \) in eq. (3-5) turned out to be zero.

The same procedure (calculate the \( \xi^1 \) stress from Equation 3-4, then express the \( \xi^2 \) correction in terms of \( \psi^2 \) and lower order terms and then put the final expression to the momentum equation) generates a similar to eq. (3-6) equation for \( \psi^2 \). Similarly, equations of the form

$$B \psi^n = f_n(\psi^{n-1}, \psi^{n-2}, \ldots \psi^0),$$  (3-7)

are formally derived.

Unfortunately this scheme although conceptually straightforward turned out to be computationally infeasible, even with the help of MACSYMA, because of the proliferation of the terms composing the function \( f_n \) appearing in the right-hand-side of eq. (3-7). Characteristically, \( f_2 \) consisted of more than a thousand terms! Each term was a product of a hyperbolic function in \( \xi \) and a trigonometric function in \( \theta \). Since, in general, a particular solution to each one of these terms is a sum of several similar terms, the final solution to just the second order correction \( \psi^2 \) could easily be the sum of several thousand terms, which was impracticable to calculate.

However, some qualitative remarks can still be made just from the structure of the equations and the form of the zero order solutions. First, from the formulation of the problem in bipolar coordinates, the dimensionless number that appears is \( De' \) rather than \( De \). Since, \( De' = \varepsilon De/\sqrt{1-\varepsilon^2} \) for small \( \varepsilon \), for the same changes in \( De \) more drastic changes are expected at high than at low eccentricity values. Second, the Newtonian solution (3-3) consists of one Fourier mode in the azimuthal direction. It is easy to show that, with the exception of the first order correction which turns out to be zero, each higher-order correction to the stream function will involve two more Fourier modes. The conclusion drawn for the addition
of hyperbolic terms in the \( \xi \) direction is similar. The general conclusion is therefore that, for small enough \( \text{De}' \) values, the solution is well represented by a few Fourier modes and would not develop radial boundary layers which require higher-order hyperbolic terms for their description.

Unfortunately, this analysis can not tell us how small \( \text{De} \) must be for these conclusions to be correct. In that respect, even if the second order term was calculated the analysis would have been of much lesser value than the perturbation in \( \epsilon \), carried out in the next Section, which is not limited to small \( \text{De} \) values.

3.2 Domain Perturbation in Eccentricity

In this section, results of perturbation analyses are given for the second-order fluid (SOF), Criminale-Eriksen-Fibley (CEF), upper-convected Maxwell (UCM) and White-Metzner (WM) models. These equations were picked to separate the effects of elasticity, memory and shear thinning on the development of the velocity and stress fields. The results are formally valid for all levels of elasticity, as measured by a Deborah number \( \text{De} \) defined as the ratio of the relaxation time \( \lambda \) of the fluid over a characteristic time for the flow. In the journal bearing the characteristic time for the flow is proportional to \( R_0/V \); therefore, the Deborah number is defined as \( \text{De} = \lambda V/R_0 \).

Many other authors have studied the field mechanics of a journal bearing for small gaps and eccentricities. These studies are classified in Table 1.1 according to the constitutive equation and the method of solution used. Restricting the analysis to small gaps (\( \mu \ll 1 \)) reduces the equation of motion to the classical lubrication approximation (Reynolds 1886). We take this approach here, but formalize it by using singular perturbation techniques to organize the ordering for both the equations of motion and constitutive relations. As shown below, this formal procedure considerably simplifies the constitutive equations.

The formulation of the mathematical problem is given in the next subsection along with the details of the five constitutive equations. The forms of the velocity and stress fields are developed in subsections 3.2.2 to 3.2.6 and compared in subsection 3.2.7.
3.2.1 Formulation of domain perturbation equations

The creeping flow forms of the equations of motion and continuity in cylindrical coordinates are

$$\frac{\partial}{\partial r} (\sigma_{rr}) + \frac{1}{r} \frac{\partial}{\partial \theta} (\sigma_{r\theta}) + \frac{1}{r} (\sigma_{rr} - \sigma_{\theta\theta}) = 0, \quad (3-8)$$

$$\frac{\partial}{\partial r} (\sigma_{r\theta}) + \frac{\partial}{\partial \theta} (\sigma_{\theta\theta}) + \sigma_{r\theta} = 0, \quad (3-9)$$

$$\frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r} = 0, \quad (3-10)$$

where \((u, v)\) are the radial and azimuthal components of the velocity and \(\sigma_{rr}, \sigma_{r\theta}, \text{ etc.}\) are the components of the total stress tensor which is defined in terms of the isotropic pressure \(p(r, \theta)\) and the extra stress tensor \(\tau\) as

$$\tau = (\sigma + \delta p). \quad (3-11)$$

When the flow is two-dimensional with no axial velocity symmetry considerations demand that \(\tau_{rZ} = \tau_{Z\theta} = 0\). All constitutive equations used here also give \(\tau_{ZZ} = 0\).

The no-slip boundary conditions on the two cylindrical surfaces for the cylindrical coordinate system shown in Figure 1.2 are

$$u = 0, \quad v = R_0 \Omega = V, \quad r = R_0, \quad (3-12)$$

$$u = 0, \quad v = 0, \quad r = R + \delta \varepsilon \cos \theta + O(\varepsilon^2). \quad (3-13)$$

The only boundary conditions necessary in the angular coordinate are that each component of velocity and stress be periodic with a period of \(2\pi\).

The complete problem is composed of solving eqs. (3-8) to (3-13) together with a particular constitutive equation for the extra stress. It can be compactly written as

$$R(X, r, \theta) = 0, \quad (3-14)$$
where $\tilde{\chi} = \{ u, v, p, \tau_\rho, \tau_\theta, \tau_\phi \}$.

The five equations used here are listed in Table 3.1 along with viscosities $\eta$ and first-normal stress coefficients $\Psi_1$ that correspond to the models. In each model, $\eta_0$ is the zero-shear-rate viscosity; $G_0$ is the shear modulus; $\lambda_0 = \eta_0/G_0$ is the zero-shear-rate relaxation time; and $\dot{\gamma} = \nabla u + \nabla v^\top$ is the rate-of-strain tensor.

To simplify the analysis and to facilitate comparison among results for the four rheological models, we take the second normal stress coefficient $\Psi_2$ equal zero in each model and set the first normal stress coefficient $\Psi_1$ equal to $2\eta\lambda$ where $\lambda$ is a shear-rate-dependent relaxation time and $\eta$ is a shear-rate-dependent viscosity.

The second-order fluid (see Table 3.1) is the simplest constitutive equation that exhibits normal stresses. It is interesting as a model fluid for comparison to numerical results because the velocity field for the two-dimensional creeping flow with solid boundaries is the same as the Newtonian fluid (Giesekus 1963; Tanner 1966) and is unique (Huligol 1973). The SOF model is particularly advantageous for studying the journal bearing, because an exact solution for arbitrary $\epsilon$ and $\eta$ is known (Joukowsky and Tschaplygin 1904) and has been the basis of several previous studies (Ballal and Rivlin 1976; Rivlin 1979). The Criminale-Erickson-Fibley model (Criminale et al. 1958; Bird et al. 1977a) is the generalization of the SOF that accounts correctly for the shear-rate-dependence of the velocity and normal stress coefficient; we use it with

$$\eta = \eta(\dot{\gamma}), \quad \lambda = \eta(\dot{\gamma})/G_0,$$  \hspace{1cm} (3-15)

where $\dot{\gamma} = \sqrt{\frac{1}{2}} II_d = \sqrt{\frac{1}{2}} (\dot{\gamma} : \dot{\gamma})$ is the magnitude of the rate of strain tensor and $II_d$ is the second invariant of the rate-of-strain tensor. Since the CEF model is strictly valid only for steady-state-shear flow, it seems appropriate to apply this model to the journal bearing flow when the effect of the elastic memory of the fluid is small. Phan-Thien and Tanner (1981) have done this for small eccentricities; our results in subsection 3.2.2 are a special case of their analysis.

The UCM model combines fluid elasticity with memory and has been a
<table>
<thead>
<tr>
<th>Model</th>
<th>Constitutive Equations</th>
<th>Viscosity $\eta$</th>
<th>Primary Normal Stress Coefficient $\psi_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Second Order Fluid (SOF)</td>
<td>$\tau = \eta_o \dot{\gamma} - \eta_o \lambda_o \dot{\gamma}(1)$</td>
<td>$\eta_o$</td>
<td>$2\eta_o \lambda_o = 2\eta_o G_o$</td>
</tr>
<tr>
<td>Criminale-Ericksen-Filbey (CEF)</td>
<td>$\tau = \eta \dot{\gamma} - \frac{n^2}{G_o} \dot{\gamma}(1)$</td>
<td>$n = n(\dot{\gamma})$</td>
<td>$2n^2(\dot{\gamma})/G_o$</td>
</tr>
<tr>
<td>Upper-Convected Maxwell (UCM)</td>
<td>$\tau + \lambda_o \tau(1) = \eta_o \dot{\gamma}$</td>
<td>$\eta_o$</td>
<td>$2\eta_o \lambda_o = 2\eta_o G$</td>
</tr>
<tr>
<td>Oldroyd-B Fluid (OLD)</td>
<td>$\tau + \lambda_o \tau(1) = \eta_o (\dot{\gamma} + \lambda_2 \dot{\gamma}(1))$</td>
<td>$\eta_o$</td>
<td>$2\eta_o (\lambda_o - \lambda_2)$</td>
</tr>
<tr>
<td>White-Metzner (WM)</td>
<td>$\tau + \frac{n}{G_o} \tau(1) = n \dot{\gamma}$</td>
<td>$n = n(\dot{\gamma})$</td>
<td>$2n^2(\dot{\gamma})/G_o$</td>
</tr>
</tbody>
</table>

* All models are chosen to give the secondary normal stress coefficient $\psi_2 = 0$. The model parameters are also adjusted to give identical viscometric functions for the second order fluid and upper-convected Maxwell model and for the CEF and White-Metzner equations.
standard constitutive equation for evaluating numerical algorithms. As shown in Mendelson et al., (1982), the UCM constitutive equation reduces to the SOF model in the limit of small elasticity, i.e. small Deborah number; because of this limit, the results of Fix and Paslay (1967) for a corotational Maxwell model in the limit of small De are essentially identical to those for the SOF model. The Oldroyd-B fluid (OLD) model can be seen as a combination of SOF and UCM models. It is most closely related to the UCM model and another way to describe its stress tensor is to consider it as the sum of a viscoelastic stress tensor $\mathbf{S}_{\text{vp}}$, obeying a UCM constitutive equation plus the addition of a viscous contribution $N_b \dot{\gamma}$. The retardation time $\lambda_2$ is directly related to the solvent viscosity through Equation (2-26). Because of this physical explanation this model is considered appropriate for the description of the rheological behavior of polymer solutions (Prilutski et al. 1983). The White-Metzner (WM) model (White and Metzner 1963) is a generalization of the UCM fluid that accounts for shear-rate dependence of the viscosity and relaxation time; see eq. (3-15). We take the viscosity $\eta(\dot{\gamma})$ in both the CEF and WM constitutive equations to be the power-law relationship

$$\eta(\dot{\gamma}) = m\dot{\gamma}^{n-1}, \quad (3-16)$$

where $n$ is the power law index. Then the viscometric functions predicted by these two models are the same.

The power law model is the most general form of the viscosity that is need to describe the flow in a journal bearing with small eccentricity and small gap. To see this consider the more general Carreau (Bird et al. 1977a) model for viscosity

$$\eta(\dot{\gamma}) = \eta_0 \left[1 + (\Lambda \dot{\gamma})^2\right]^{N-1 \over 2}, \quad (3-17)$$

where $\Lambda$ is a time constant and $N$ is a power-law index. The reciprocal of $\Lambda$ marks the shear rate where the viscosity begins to decrease from its zero-shear-rate value $\eta_0$. As shown in subsection 3.2.2 the flow for concentric cylinders with small gap ($\epsilon = 0$, $\mu \ll 1$) is viscometric with shear rate $\dot{\gamma} = \dot{\gamma}_0 = V/\delta$, and thus the viscosity is constant throughout the gap and equal to $\eta_0$. If the Carreau model eq. (3-17) is expanded
for a flow field only slightly perturbed from this uniform shear flow, the series for the viscosity function that results is the same as the corresponding series to order \( \epsilon \) for a power-law fluid with the constants \( m \) and \( n \) given by

\[
m = \frac{n_o [1 + (\Lambda \dot{\gamma}_o)^2]^{\frac{N-1}{2}}}{(\dot{\gamma}_o)^{n-1}},
\]

\[
n = \frac{[1 + N (\Lambda \dot{\gamma}_o)^2]}{[1 + (\Lambda \dot{\gamma}_o)^2]}.
\]

For shear rates \( \dot{\gamma}_o \ll 1/\Lambda \), eqs. (3-16) and (3-18) give

\[
n = m \dot{\gamma}_o n^{-1} = n_o, \ n = 1,
\]

as the zero-shear-rate viscosity of the Carreau model. At high shear rates \( \dot{\gamma}_o \gg 1/\Lambda \) eqs. (3-16) and (3-18) reduce to

\[
n = n_o (\Lambda \dot{\gamma}_o)^{N-1}, \ n = N.
\]

Moreover, the dimensionless shear-rate \( \Lambda \dot{\gamma}_o \) in the Carreau model is

\[
\Lambda \dot{\gamma}_o = \frac{AV}{\delta} = \frac{AV}{R_o} \frac{1}{\mu} = \frac{De}{\mu},
\]

so that if the Deborah number is near unity, the shear-rate is \( O(1/\mu) \). Therefore for the journal bearing flow, the Carreau model reduces to the power-law relationship (3-20) whenever \( \text{De} = O(1) \) or greater.

Analytical solutions to eqs. (3-8) to (3-13) with the constitutive equations in table 3.1 are found by expanding the stress and the velocity fields in the eccentricity \( \epsilon \) and the dimensionless gap-width \( \mu \). The expansion in \( \epsilon \) is accomplished formally by perturbing the domain about the concentric state using the domain perturbation technique developed in Joseph and Sturges, (1975). The problem for eccentric cylinders is first
transformed to a coordinate system \((R_0 \leq \xi \leq R, 0 \leq \nu \leq 2\pi)\) where the cylinders are concentric by the mapping

\[
r = r(\xi; \epsilon, \theta) = R_0 + (\xi - R_0) \frac{r_1(\theta) - R_0}{R - R_0},
\]

where \(r_1(\theta)\), the radial position of the outer cylinder is defined by eq.(2-35). Equation 3-22 can be expanded for small \(\epsilon\) as

\[
r = \xi + \epsilon[(\xi - R_0) \cos \theta] + O(\epsilon^2).
\]

The variables are then formally expanded in \(\epsilon\) in the transformed domain \((\xi, \theta)\) as

\[
\begin{bmatrix}
u(\xi, \theta) \\
\rho(\xi, \theta) \\
\tau_{ij}(\xi, \theta)
\end{bmatrix} = \sum_{k=0}^{\infty} \begin{bmatrix}
u^{[k]}(\xi, \theta) \\
\rho^{[k]}(\xi, \theta) \\
\tau_{ij}^{[k]}(\xi, \theta)
\end{bmatrix} \epsilon^k.
\]

Because the two radial coordinates \(\xi\) and \(r\) are equal at \(\epsilon = 0\), the velocity, pressure and stress written in the eccentric frame \((r, \theta)\) are simply related to variables expressed in the coordinates of the concentric problem as

\[
\begin{bmatrix}
u(r, \theta) \\
\rho(r, \theta) \\
\tau_{ij}(r, \theta)
\end{bmatrix} = \begin{bmatrix}
u^{[0]}(\xi, \theta) \\
\rho^{[0]}(\xi, \theta) \\
\tau_{ij}^{[0]}(\xi, \theta)
\end{bmatrix} + \epsilon \begin{bmatrix}
u^{[1]}(\xi, \theta) \\
\rho^{[1]}(\xi, \theta) \\
\tau_{ij}^{[1]}(\xi, \theta)
\end{bmatrix} + \frac{-\epsilon(\xi - R_0) \nu \cos \theta}{\delta} + O(\epsilon^2).
\]

The extra term in \(\nu(r, \theta)\) arises from application of the chain rule for the total derivative of the angular velocity with \(\epsilon\) and the fact that \(\nu^{[0]}(\xi, \theta)\) is not zero. The boundary conditions at each order of \(\epsilon\) are
developed by substituting eqs. (3-25) into eqs. (3-12) and (3-13) and collecting terms of equal order in $\varepsilon$. The conditions for the lowest two orders are

$$u^{[0]}(R_0, \theta) = u^{[0]}(R, \theta) = v^{[0]}(R, \theta) = 0, \quad v^{[0]}(R_0, \theta) = V, \quad (3-26a)$$

$$u^{[1]}(R_0, \theta) = u^{[1]}(R, \theta) = v^{[1]}(R, \theta) = 0, \quad v^{[1]}(R, \theta) = V \cos \theta. \quad (3-26b)$$

To generate the perturbation equations of order $n$, the $n$-th derivative of the original equations (3-14) with respect to $\varepsilon$ has to be taken and then take its value at $\varepsilon = 0$ and equate it to zero:

$$R^{[n]} = \left[ \frac{d^n}{d\varepsilon^n} \left[R(x, r, \theta)\right]\right]_{\varepsilon=0} = 0. \quad (3-27)$$

In taking the total derivative with respect to $\varepsilon$ in (3-27) it would have seemed to consist of two terms: one involving the dependence of $R$ to $\varepsilon$ through the variable $x$ and the other through the implicit dependence of $r$ (see eq. (3-22)). As Sturges and Joseph (1970) first pointed out, the second term is always zero due to the coincidence at $\varepsilon = 0$ of the transformed and the original domain.

Therefore, the zero and first order equations can be formally written as zero order:

$$R(x^{[0]}, \xi, \theta) = 0, \quad (3-28)$$

first order:

$$\left[ \frac{\partial R}{\partial x} \right]_{x^{[0]}, \xi, \theta} \cdot x^{[1]} = 0, \quad (3-29)$$

where

$$\left[ \frac{\partial R}{\partial x} \right]_{x^{[0]}, \xi, \theta}$$

is the Jacobian matrix with $ij$ element, $\frac{\partial R_i}{\partial x_j}$.
equal to the derivative of the i residual with respect to the j variable evaluated with the zero order approximation \( \{x^{[0]}_i, \xi, \theta\} \).

To make the computations feasible, at each order in \( \varepsilon \) of (3-24) the variables are sought in terms of a singular expansion in the dimensionless gap-width \( \mu \); that is

\[
\begin{bmatrix}
  u[k] \\
v[k] \\
p[k] \\
\tau_{ij}[k]
\end{bmatrix} = \begin{bmatrix}
  q[k]_{u} a[k]_u \\
q[k]_{v} a[k]_v \\
q[k]_{p} a[k]_p \\
q[k]_{ij} a[k]_{ij}
\end{bmatrix} + O \begin{bmatrix}
  (a[k]_{u,1}) \\
  (a[k]_{v,1}) \\
  (a[k]_{p,1}) \\
  (a[k]_{ij,1})
\end{bmatrix},
\]

(3-30)

where the singular nature of each variable is given by the fact that most of the integers \( a[k]_u, a[k]_v, a[k]_p \) and \( a[k]_{ij} \) are negative. Only the most singular term has been retained for each variable, although the method presented here can be generalized to higher-order (albeit difficult to calculate) approximations in \( \mu \).

Langlois (1963) and Pearson (1967) have presented similar expansion methods for analysis of fluid-film lubrication. Langlois found that the stress fields in a region governed by such a singular perturbation could not in general be matched to stresses at prescribed inflow and outflow boundaries. The angular periodicity of the journal bearing removes this difficulty.

The analysis for the four constitutive equations are presented next. In each case, the system of equations for the variables \( (u^{[1]}, v^{[1]}, p^{[1]}, \tau_{ij}^{[1]}) \) is reduced to a single linear fourth-order equation for the complex streamfunction \( \delta F(\xi)e^{i\theta} \) which is related to \( u^{[1]} \) and \( v^{[1]} \) by

\[
u^{[1]} = \text{Re}[iF(\xi)e^{i\theta}\delta]/\xi,
\]

(3-31)
\[ v^{[1]} = \text{Re}[-F'e^{i\theta}] , \]

where \( F'(\xi) = \frac{dF}{d\xi} \) and \( \text{Re}(\cdot) \) stands for the real part of the argument. The boundary conditions for the \( F(\xi) \) at the inner and outer cylinders are

\[ F(R_o) = F(R) = F'(R_o) = 0 , \quad F'(R) = -V/\delta . \quad (3-32) \]

The stresses and pressure are given by

\[ \tau^{[1]} = \text{Re}[\sigma(\xi)e^{i\theta}] , \quad (3-33) \]

\[ p^{[1]} = \text{Re}[\pi(\xi)e^{i\theta}] . \quad (3-34) \]

3.2.2 Second-order fluid

The mathematical arguments that lead to perturbation solutions are similar for each of the four constitutive equations. We present the details for the second-order fluid model, since its particularly simple mathematical form makes the analysis most transparent.

The velocity and stress fields for flow between concentric cylinders \((\varepsilon = 0)\) are independent of the azimuthal angle and are governed by the equation set

constitutive equation

\[ \tau^{[0]}_{rr} = 2\eta_o \frac{du^{[0]}}{d\xi} - 2\lambda_o \eta_o [u^{[0]} \frac{d^2u^{[0]}}{d\xi^2} - 2 (\frac{du^{[0]}}{d\xi})^2] , \quad (3-35) \]

\[ \tau^{[0]}_{r\theta} = \eta_o \left[ \frac{dv^{[0]}}{d\xi} - \frac{v^{[0]}}{\xi} \right] - \lambda_o \eta_o [u^{[0]} \frac{d}{d\xi} (\frac{dv^{[0]}}{d\xi} - \frac{v^{[0]}}{\xi}) + \frac{3v^{[0]}}{\xi} \frac{du^{[0]}}{d\xi} + \frac{v^{[0]}u^{[0]}}{\xi^2} - \frac{3du^{[0]}}{d\xi} \frac{dv^{[0]}}{d\xi} - \frac{u^{[0]}}{\xi} \frac{dv^{[0]}}{d\xi} ] , \quad (3-36) \]
\[
\tau_{\theta\theta}^{[0]} = \frac{2\eta_o [u^{[0]}]}{\xi} - 2\lambda_0 \eta_o \left[ u^{[0]} \frac{d}{d\xi} \left( \frac{u^{[0]}}{\xi} \right) + \frac{2v^{[0]}}{\xi} \frac{dv^{[0]}}{d\xi} \right] - (\frac{v^{[0]}}{\xi})^2 - 2(\frac{u^{[0]}}{\xi})^2 - (\frac{dv^{[0]}}{d\xi})^2,
\]

(3-37)

continuity
\[
\frac{d}{d\xi} [u^{[0]}] + \frac{u^{[0]}}{\xi} = 0,
\]

(3-38)

equation of motion
\[
\frac{d}{d\xi} (\tau^{[0]}_{rr} - \rho^{[0]}) + \frac{1}{\xi} (\tau_{rr}^{[0]} - \tau_{\theta\theta}^{[0]}) = 0
\]

(3-39)
\[
\frac{d}{d\xi} (\xi \tau_{r\theta}^{[0]}) + \tau_{r\theta}^{[0]} = 0
\]

(3-40)

with the boundary conditions
\[
u^{[0]}(R_o) = u^{[0]}(R) = v^{[0]}(R) = 0, \quad v^{[0]}(R_o) = V.
\]

(3-41)

The boundary conditions (3-41) and the continuity equation (3-38) imply that the radial velocity is zero, i.e. \( u^{[0]}(r) = 0 \), which simplifies eqs. (3-35) to (3-40) considerably.

The singular expansion in the gap parameter \( \mu \) is introduced by changing to the stretched radial coordinate \( \xi \)
\[
\xi = R_o (1 + \xi \mu).
\]

(3-42)

Substituting this new variable into eqs. (3-35) to (3-40) along with the expansions (3-30) of the field variables and collecting the terms in \( \mu \) gives
\[
\tau_{rr}^{[0]} = 0,
\]

(3-43)
\[
\tau_{r\theta}^{[0]} \mu^{[0]} \mu^{[0]} = \frac{\eta_o}{R_o \mu} \frac{d\theta^{[0]} \theta^{[0]} \mu}{d\xi} \mu^{[0]},
\]

(3-44)
\[
\dot{\theta}^0_{\theta \theta} \mu a^0_{\theta \theta} = \frac{2\lambda}{R_0^2} \left( a^0_\nu \right)^2 \left( \frac{d\theta^0_{\theta \theta}}{d\zeta} \right)^2 ,
\]

(3-45)

\[
- \frac{d\theta^0_{\theta \theta}}{d\zeta} a^0_\rho + \mu a^0_{rr} \frac{d}{d\zeta} (\theta^0_{rr})
\]

+ \mu \left[ \dot{\theta}^0_{rr} a^0_{rr} - \dot{\theta}^0_{\theta \theta} a^0_{\theta \theta} \right] = 0 ,
\]

(3-46)

\[
\frac{d}{d\tau} \theta^0_{r \theta} = 0 .
\]

(3-47)

Variables with a caret (e.g. \( \dot{\theta}^0 \)) in these expressions and throughout the paper have their radial dependence expressed as a function of the stretched coordinate \( \zeta \).

The appropriate boundary conditions are

\[
\theta^0_{\mu \nu} = V , \quad \zeta = 0 ,
\]

(3-48)

\[
\theta^0_{\mu \nu} = 0 , \quad \zeta = 1 .
\]

The correct scaling factors for each of the variables are calculated by balancing the powers of \( \mu \) that appear in eqs. (3-43) to (3-48). The boundary conditions (3-48) imply that \( a^0_\nu = 0 \) and eq. (3-47) forces \( \dot{\theta}^0_{r \theta} \) to be a constant. Equation (3-44) is solved with the boundary conditions (3-48) to yield \( a^0_{r \theta} = -1 \) and the homogeneous shear flow

\[
\theta^0(\zeta) = V(1-\zeta) .
\]

(3-49)

The remaining components of the constitutive equation are solved with (3-49) to give the components of stress. The entire solution for the case of concentric cylinders is
\[ u^{[0]} = 0 \quad , \quad v^{[0]} = V(1-\zeta) + O(\mu) \quad , \quad p^{[0]} = p_o , \quad (3-50) \]

\[ \tau_{r\theta}^{[0]} = -\frac{\eta_o V}{R_o \mu} + O(1) \quad , \quad \tau_{\theta\theta}^{[0]} = \frac{2D\eta_o V}{\mu^2 R_o} + O(\frac{1}{\mu}) \quad , \quad \tau_{rr}^{[0]} = 0 , \quad (3-51) \]

where \( De = \lambda_o V/R_o \) is the Deborah number for the shear flow and \( p_o \) is a constant reference pressure.

The equations governing the corrections to the field variables \((u^{[1]}, v^{[1]}, p^{[1]}, \tau_{ij}^{[1]})\) for small eccentricity are found by substituting eq. (3-24) into eqs. (3-8) to (3-10) and SOF (Table 3.1) and collecting terms of order \( \varepsilon \). The resulting equations are

\[ \tau_{rr}^{[1]} = 2\eta_o \frac{\partial u^{[0]}}{\partial \xi} - 2\lambda_o \eta_o \varepsilon \frac{v^{[0]}}{\xi} \frac{\partial^2 u^{[1]}}{\partial \theta \partial \xi} - \frac{1}{\xi} \frac{\partial u^{[1]}}{\partial \theta} \left( \frac{\partial v^{[0]}}{\partial \xi} - \frac{v^{[0]}}{\xi} \right) \quad , \quad (3-52) \]

\[ \tau_{r\theta}^{[1]} = \eta_o \left( \frac{\partial v^{[1]}}{\partial \xi} + \frac{1}{\xi} \frac{\partial u^{[1]}}{\partial \theta} - \frac{v^{[1]}}{\xi} \right) + \lambda_o \eta_o \frac{u^{[1]}}{\xi} \left( \frac{\partial^2 v^{[0]}}{\partial \xi^2} - \frac{v^{[0]}}{\xi} \right) + \frac{3v^{[0]} u^{[1]}}{\xi^2} \]

\[ -3 \frac{\partial v^{[0]}}{\partial \xi} \frac{\partial u^{[1]}}{\partial \xi} - \frac{\partial v^{[0]}}{\partial \xi} u^{[1]} + \frac{1}{\xi^2} \frac{\partial v^{[0]}}{\partial \theta} - \frac{1}{\xi} \frac{\partial v^{[1]}}{\partial \theta} \frac{\partial v^{[0]}}{\partial \xi} \quad , \quad (3-53) \]

\[ \tau_{\theta\theta}^{[1]} = 2\eta_o \left( \frac{1}{\xi} \frac{\partial v^{[1]}}{\partial \theta} + \frac{u^{[1]}}{\xi} \right) - 2\lambda_o \eta_o \left( \frac{v^{[0]}}{\xi} \frac{\partial}{\partial \theta} \left( \frac{1}{\xi} \frac{\partial v^{[1]}}{\partial \theta} + \frac{u^{[1]}}{\xi} \right) \right) \]

\[ - \frac{2v^{[0]} \partial v^{[1]}}{\xi} + \frac{2v^{[1]} \partial v^{[0]}}{\xi} - \frac{2v^{[0]} u^{[1]}}{\xi^2} - 2 \frac{\partial v^{[0]}}{\partial \xi} \frac{\partial v^{[1]}}{\partial \xi} \]

\[ + \frac{v^{[0]} \frac{\partial u^{[1]}}{\partial \theta} - \frac{1}{\xi} \frac{\partial v^{[0]}}{\partial \theta} \frac{\partial u^{[1]}}{\partial \theta}}{\xi^2} \quad , \quad (3-54) \]

\[ \xi \frac{\partial u^{[1]}}{\partial \xi} + \frac{\partial v^{[1]}}{\partial \theta} + u^{[1]} = 0 \quad , \quad (3-55) \]
\[-\frac{\partial \mathbf{p}^{[1]}_{rr}}{\partial \zeta} + \frac{\partial}{\partial \zeta} \mathbf{I}_{rr}^{[1]} + \frac{1}{\zeta} \frac{\partial \mathbf{I}_{r\theta}^{[1]}}{\partial \theta} + \frac{1}{\zeta} \left( \mathbf{I}_{rr}^{[1]} - \mathbf{I}_{r\theta}^{[1]} \right) = 0 \quad , \tag{3-56}\]

\[-\frac{\partial \mathbf{p}^{[1]}_r}{\partial \theta} + \frac{\partial}{\partial \zeta} \left( \xi \mathbf{I}_{r}^{[1]} \right) + \frac{\partial \mathbf{I}_{r\theta}^{[1]}}{\partial \theta} + \mathbf{I}_{rr}^{[1]} = 0 \quad . \tag{3-57}\]

The boundary conditions on the corrections to the velocity are

\[u^{[1]}(R_0, \theta) = u^{[1]}(R, \theta) = v^{[1]}(R_0, \theta) = 0 \quad , \quad v^{[1]}(R, \theta) = \zeta \cos \theta \quad , \tag{3-58}\]

along with periodicity in the azimuthal direction.

The complex streamfunction defined by eq. (3-31) is introduced into eqs. (3-52) to (3-58) to satisfy the periodicity conditions. In order to develop the singular perturbation for small gaps, the radial function \(F(\xi)\) is written as \(F(\xi) = \hat{F}(\xi) + O(\mu^{1/2})\). The boundary condition on \(F(\xi)\) in eq. (3-32) are independent of \(\mu\), so \(a_F = 0\) and \(F(\xi)\) is expressed as

\[F(\xi) = \hat{F}(\xi) + O(\mu) \quad . \tag{3-59}\]

Substituting eq. (3-59) into the eqs. (3-52) to (3-58) and expressing the stress and pressure according the eqs. (3-33) and (3-34) gives

\[\hat{S}_{rr}^{[1]} = \frac{2n_0}{R_0} \left[ 1\hat{F}' + (1-\zeta)\hat{F}' + \text{De} \hat{F} \right] \quad , \tag{3-60}\]

\[\hat{S}_{r\theta}^{[1]} = \frac{-n_0}{R_0} \left[ \hat{F}'' - 1\hat{F}'(1-\zeta)\text{De} + 21\hat{F}'\text{De} \right] \quad , \tag{3-61}\]

\[\hat{S}_{\theta\theta}^{[1]} = \frac{4n_0}{R_0} \frac{1}{2} \text{De} \hat{F}'' \quad , \tag{3-62}\]

\[\left( \frac{\partial \mathbf{g}^{[1]}_{\mu}}{\partial \zeta} \right)_{\theta \theta} = -\mu \mathbf{S}_{\theta \theta}^{[1]} \quad , \tag{3-63}\]
\[ \hat{\pi}^{[1]}_{\nu} = -\frac{1}{\mu} \left( \frac{\partial}{\partial \zeta} S_{R\theta} \right) a_{\nu}^{[1]} + \hat{S}_{\theta\theta} a_{\theta\nu}^{[1]}, \quad (3-64) \]

where \( \hat{F}' = d\hat{F}/d\zeta \). In eqs. (3-60) to (3-64), the components of the complex stress tensor \( S \) in eq. (3-33) have been written in terms of the stretched radial coordinate \( \zeta \) as

\[ S_{ij}(\zeta) = \hat{S}_{ij}(\zeta) \mu^{\alpha_{ij}} + O(\mu^{\alpha_{ij} + 1}). \quad (3-65) \]

Matching the powers of \( \mu \) in eqs. (3-60) to (3-64) gives the scale factors

\[ a_{rr}^{[1]} = 0, \quad a_{\theta\theta}^{[1]} = a_{\rho}^{[1]} = -2, \quad a_{r\theta}^{[1]} = -1. \quad (3-66) \]

Then the radial momentum equation (3-63) reduces simply to

\[ \frac{\partial \hat{\pi}}{\partial \zeta} = 0, \quad (3-67) \]

implying that the correction to the pressure is a constant \( \hat{\pi}_0 \) across the gap. With this information, the azimuthal-momentum equation (3-64) can be differentiated with respect to \( \zeta \) to yield

\[ \hat{F}(iv) = 0, \quad (3-68) \]

as the governing equation for the streamfunction.

The function \( \hat{F}(\zeta) \) that satisfies (3-68) and the boundary conditions (3-32) is

\[ \hat{F} = \nu(1 - \zeta)\zeta^2 \quad (3-69) \]

which is independent of \( \nu \) as expected by the existence proof of Tanner (1966) and Giesekus (1963). The components of stress \( \{\hat{S}_{ij}^{[1]}\} \) and the correction to pressure \( \hat{\pi} \) are each recovered by substituting (3-69) into eqs. (3-60) to (3-64).
3.2.3 Upper-convected Maxwell fluid

The analysis of the flow for the UCM fluid follows directly the method used above for the SOF constitutive model. The components of the UCM constitutive equation in cylindrical coordinates \((\xi, \theta)\) are

\[
\frac{\partial u}{\partial \xi} = \frac{1}{2 \eta_o} \tau_{rr} + \frac{1}{G_o} \left( \frac{u}{2 \xi} \frac{\partial \tau_{rr}}{\partial \xi} + \frac{v}{2 \xi} \frac{\partial \tau_{r\theta}}{\partial \xi} \right) - \frac{\tau_{r\theta}}{\xi} \frac{\partial u}{\partial \theta} - \tau_{rr} \frac{\partial u}{\partial \xi}, \quad (3-70)
\]

\[
\frac{\partial u}{\partial \theta} + \xi^2 \frac{\partial }{\partial \xi} \left( \frac{v}{\xi} \right) = \frac{\xi}{\eta_o} \tau_{r\theta} + \frac{1}{G_o} \left( \frac{u \xi}{\partial \xi} \frac{\partial \tau_{r\theta}}{\partial \xi} + \frac{v}{\partial \theta} \frac{\partial \tau_{r\theta}}{\partial \xi} \right) + v(\tau_{rr} - \tau_{r\theta}) - \tau_{rr} \xi \frac{\partial v}{\partial \xi} - \tau_{r\theta} \left( \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \xi} \right), \quad (3-71)
\]

\[
\frac{1}{\xi} \frac{\partial v}{\partial \xi} + \frac{u}{\xi} = \frac{1}{2 \eta_o} \tau_{r\theta} + \frac{1}{G_o} \left( \frac{u \xi}{\partial \xi} \frac{\partial \tau_{r\theta}}{\partial \xi} + \frac{v}{\partial \theta} \frac{\partial \tau_{r\theta}}{\partial \xi} \right) - \tau_{r\theta} \left( \frac{\partial v}{\partial \theta} + \frac{\partial u}{\partial \xi} \right), \quad (3-72)
\]

The solution for concentric cylinders of eqs. (3-70) to (3-72), (3-8) to (3-10) and boundary conditions (3-12) to (3-13) with \(r = \xi\) is identical to eqs. (3-49) to (3-51) for the second-order fluid with the Deborah number \(De\) defined as

\[
De = \lambda_o V/R_o = \eta_o V/G_o R_o. \quad (3-73)
\]

The constitutive equations relating the first-order corrections \((u^{[1]}, v^{[1]}, p^{[1]}, \tau_{ij}^{[1]}))\) are written in terms of the radial dependence of the complex stream-function eq. (3-59) as

\[
\hat{F}' + FDe = \frac{R_o}{2 \eta_o} \left( 1 + iDe(1-\zeta) \right) S_{rr} a^{[1]} \quad \hat{a}_{rr} \mu, \quad (3-74)
\]

\[
-F'' = \mu \frac{R_o}{\eta_o} \left( 1 + iDe(1-\zeta) \right) S_{r\theta} \hat{a}_{r\theta} + \frac{R_o}{\eta_o} De S_{rr} \hat{a}_{rr} + 2De \hat{F}, \quad (3-75)
\]
\[ 0 = \mu^2 \frac{R}{2\eta_o} (1 + i\text{De}(1-\zeta) ) \hat{a}_{\theta\theta}^{[1]} + \frac{R}{\eta_o} \hat{a}_{r\theta}^{[1]} + 1 + 2i\text{De} \hat{F}' , \quad (3-76) \]

where \( \hat{F}' = d\hat{F}/d\zeta \) and only the lowest order terms in \( \mu \) have been retained. The appropriate equations of motion are identical to eqs. (3-63) to (3-64). Balancing the powers of \( \zeta \) in eqs. (3-74) to (3-76) and (3-63) to (3-64) gives the same scale factors \( (\hat{a}_{rr}^{[1]}, \hat{a}_{\theta\theta}^{[1]}, \hat{a}_{r\theta}^{[1]}, \hat{a}_{r\theta}^{[1]}) \) as for the SOF model (cf. eq. (3-66)). By following the procedure used in Section 3.1, a single linear equation governing \( \hat{F}(\zeta) \) is found:

\[ [-2\text{De}(1-\zeta) + i(1-\text{De}^2(1-\zeta)^2)]\hat{F}'(1\zeta) + [-2\text{De}^2(1-\zeta)^2 + 2i\text{De}(1-\zeta)]\hat{F}'' + 2i\text{De}^2(1-\zeta)^2 \hat{F}''' + 4i\text{De}^4(1-\zeta)\hat{F}' + 4i\text{De}^4\hat{F} = 0 . \quad (3-77) \]

The boundary conditions for (3.77) are eqs. (3.32) rewritten in the stretched variable \( \zeta \) as

\[ \hat{F}(0) = \hat{F}(1) = \hat{F}'(0) , \quad \hat{F}'(1) = -V . \quad (3-78) \]

Solutions to eqs. (3-77) and (3-78) are found here by three separate procedures; perturbation expansions are calculated that are valid only at low and high Deborah number and a solution is calculated that is exact for all values of De. The appropriate forms of the stress components \( \{\hat{S}_{ij}\} \) and pressure \( \hat{p} \) are recovered from eqs. (3-74) to (3-76) and eq. (3-64), respectively, for each solution of \( \hat{F} \).

3.2.3a Expansion for small De

The solution to eqs. (3-77) and (3-78) is calculated as a Taylor series in \( \text{De} \) as

\[ \hat{F}(\zeta) = \sum_{k=0}^{\infty} \text{De}^k \hat{F}^{(k)}(\zeta) , \quad (3-79) \]
by standard methods in regular perturbation analysis (Bender and Orszag 1975). The result up to the sixth-order term in $De$ is

$$\frac{\hat{F}}{V} = (y - 2y^2 + y^3) + \frac{De^2}{10} (y^5 - 3y^3 + 2y^2) + \frac{De^3}{30} (-y^6 + 4y^3 - 3y^2)$$

$$+ \frac{De^4}{10} \left( -\frac{y^7}{21} - \frac{3}{10} y^5 + \frac{239}{210} y^3 - \frac{83}{105} y^2 \right)$$

$$+ iDe^5 \left( -\frac{y^8}{140} + \frac{y^6}{100} + \frac{y^5}{75} - \frac{39}{1050} y^3 + \frac{22}{1050} y^2 \right)$$

$$+ De^6 \left( -\frac{61}{7560} y^9 + \frac{1}{700} y^7 + \frac{1}{225} y^6 + \frac{239}{21000} y^5 \right)$$

$$- \frac{1952}{756000} y^4 - \frac{4992}{756000} y^3 + O(De^7) \right) , \quad (3-80)$$

where $y = 1 - \zeta$. The series (3-80) proceeds in integer powers of $De$ after skipping the $O(De)$ term. This is expected because at this order the UCM model reduces to the SOF equation (see Mendelson et al., 1982) for which the flow field is the same as for a Newtonian fluid. Black and Denn (1976), used a similar perturbation method for a UCM fluid flowing in a planar converging section and reached the same result regarding the structure of the expansion.

3.2.3b Expansion for high values of $De$

By defining a new small parameter $\chi = 1/De$, the differential equation (3-77) is rearranged to yield

$$\left[ -2\chi(1-\zeta) - i(1-\zeta^2) + i\chi^2 \right] \chi^2 \hat{F}(1) + \left[ -2(1-\zeta)^2 + 21\chi(1-\zeta) \right] \chi \hat{F}'''$$

$$+ 21(1-\zeta)^2 \hat{F}'' + 4i(1-\zeta) \hat{F}' + 4i \hat{F} = 0 \right) . \quad (3-81)$$

The appropriate boundary conditions are (3-78). A solution to eqs. (3-81) and (3-78) by a regular expansion in $\chi$ gives $\hat{F}(\zeta)$ to be of the form

$$\hat{F}_0(\zeta) = V(1-\zeta) + (1-\zeta)^2 \sum_{n=0}^{\infty} b_n \chi^n , \quad (3-82)$$
where \( b_0 \) is determined form the boundary conditions and the coefficients \( (b_1, b_2 \ldots) \) are determined by the higher-order problems in \( \chi \). The form (3-82) cannot satisfy simultaneously the boundary conditions \( \hat{F}_0(0) = \hat{F}_0(0) = 0 \) for \( \chi = 0 \) and implies the development of a boundary layer near \( \zeta = 0 \) in this limit of \( \chi \).

We analyze this case by standard singular perturbation analysis (Bender and Orszag, 1975). It turns out that the appropriate scaling of the inner problem is \( \zeta/\chi \) implying that the size of the boundary layer is \( O(\chi) \) or \( O(\text{De}^{-1}) \). Equation (3-82) is retained as an outer-solution (valid far from \( \zeta = 0 \)) with the \( b_k \) determined by matching with the inner-solution. The form of \( \hat{F}(\zeta) \) that is uniformly valid up to \( O(\chi^3) \) is found to be

\[
\frac{\hat{F}(\zeta)}{V} = \chi(1-\zeta) + \frac{\chi}{1-1} \left( \frac{2\chi}{(1-1)} + 1 \right) e^{(1-1)\chi/\chi} - (1-\zeta)^2 + O(\chi^3) \quad (3-83)
\]

3.2.3c Exact solution for all values of \( \text{De} \)

The equation (3-77) and boundary conditions (3-78) are re-written for nonzero values of \( \text{De} \) in terms of the new variable \( t = \text{De}(1-\zeta) \) and streamfunction \( \hat{G}(t) = \hat{F}(\zeta)/V \) to yield

\[
(-2t + i(1-t^2))\hat{G} - (-2t^2 + 2it)\hat{G}'' + 2it^2\hat{G}'' + 4it\hat{G}' + 4i\hat{G} = 0 \quad (3-84)
\]

\[
\hat{G}(0) = \hat{G}(\text{De}) = \hat{G}'(\text{De}) = 0 \quad , \quad \hat{G}'(0) = 1/\text{De} \quad ,
\]

where \( \hat{G}' = d\hat{G}/dt \). The general solution to equation (3-84) is

\[
\hat{G}(t) = Ae^{(1-1)t} + Be^{-(1+1)t} + Ct^2 + Dt \quad ,
\]

where the constants \( (A,B,C,D) \) depend on the boundary conditions. For the boundary conditions in eq. (3-85) these are

\[
A = \text{De}^{-1} \left\{ (1 - 1 - \frac{2}{\text{De}})e^{(1-1)\text{De}} + (1 + 1 + \frac{2}{\text{De}})e^{-(1+1)\text{De}} + 2 \right\}^{-1},
\]

\[
B = -A \quad ,
\]
3.2.4 Oldroyd-B fluid

The flow and stress fields for this model at \( \varepsilon = 0 \) (concentric cylinders) are identical to the form of SOF and UCM fluids, eqs. (3-49) to (3-51), with the Deborah number substituted by a modified Deborah number \( \text{De}' \)

\[
\text{De}' = \frac{\left(\lambda_0 - \lambda_2\right) V}{R_0}
\]  

Equations for the \( O(\varepsilon) \) corrections to the stress \( \hat{S}_{ij}(\zeta) \) are developed analogously to eqs. (3-74) to (3-76) and are

\[
\hat{S}_{rr}(\zeta) = \frac{2\eta_0}{R_0} \frac{\left[iF'(1 + i\text{De}^*(1-\zeta)) + \hat{F} \text{De}'\right]}{\left[1 + i\text{De} (1-\zeta)\right]},
\]  

\[
\hat{S}_{r\theta}(\zeta) = -\frac{\eta_0}{R_0} \left[ \frac{F''(1 + i\text{De}^*(1-\zeta)) - 2i\text{De}^* F' + 2\text{De}' \text{De}F}{\left(1 + i\text{De} (1-\zeta)\right)} \right]
\]  

\[
+ 2 \text{De} \frac{\left[1F'(1 + i\text{De}^*(1-\zeta)) + \hat{F} \text{De}'\right]}{\left[1 + i\text{De} (1-\zeta)\right]^2},
\]  

\[
\hat{S}_{\theta \theta}(\zeta) = \frac{2\eta_0}{R_0} \left[ \frac{(\text{De} - 2\text{De}^*) F'' - 2i\text{De}' \text{De}F'}{\left[1 + i\text{De}(1-\zeta)\right]} \right]
\]  

\[
+ \text{De} \frac{F''(1 + i\text{De}^*(1-\zeta)) - 2i\text{De}^* F' + 2\text{De}' \text{De}F}{\left[1 + i\text{De}(1-\zeta)\right]^2}
\]  

\[
+ 2\text{De}^2 \frac{1F'(1 + i\text{De}^*(1-\zeta)) + \text{De}' F}{\left[1 + i\text{De}(1-\zeta)\right]^3},
\]

where

\[
\text{De} = \frac{\lambda_0 V}{R_0}, \quad \text{De}^* = \frac{\lambda_2 V}{R_0}
\]
The momentum equations again reduce to
\[
\hat{\pi} = -i \frac{d}{dc} \hat{S}_{r\theta} + \hat{S}_{\theta\theta}, \quad \frac{d\hat{\mu}}{dc} = 0, \quad (3-92)
\]
and these combined with eqs. (3-89) to (3-91) give
\[
\frac{(1 + i\lambda^*)}{(1-\lambda^*)} \left[ -2t + i(1-t^2) \right] \hat{G}^{(iv)} +
2(t^2 - it)\hat{G}^{'''} + 2it^2\hat{G}^{''} - 4it\hat{G}^{'} + 4i\hat{G} = 0, \quad (3-93)
\]
where \( t = De(1-\zeta), \ \lambda^* = \lambda_2/\lambda_0 \) and \( \hat{G}(t) = \hat{F}(z)/V \). The boundary conditions for eq. (3-93) are the same as eq. (3-85).

A Taylor series solution in \( De \) can be constructed to eqs. (3-93) and (3-85) by using the same techniques applied to subsection 3.2.3a; however the validity of the results is restricted to Deborah numbers below unity and the series is of little use.

Instead, an expansion to large \( De \), following the same procedure as the one outlined in 3.2.3b is feasible and yields a uniformly valid solution for \( \hat{F}(\zeta) \)
\[
\hat{F}(\zeta) = (1-\zeta)\zeta + \frac{\sqrt{\chi}(1+1)}{2b} \left( 1 + \sqrt{\chi} (1+1)(b + \frac{3}{4b}) \right) \cdot
\left[ e^{(1-1) b\zeta/\sqrt{\chi}} - (1-\zeta)^2 \right] +
\]
\[
e^{(1-1) b\zeta/\sqrt{\chi}} \left[ - \frac{\zeta^2}{2} + \frac{\sqrt{\chi} (1+1)(4b^2 - 1)}{4b} \zeta \right] + O(\chi^{3/2}), \quad (3-94)
\]
where
\[
\chi = \frac{1}{De} \quad \text{and} \quad b = \left[ (1 - \lambda^{*}/\lambda^{*}) \right]^{1/2}.
\]

Comparing the solution (3-94) with the equivalent expression for the UCM fluid (3-83) notice the qualitative change on the width of the boundary
layer next to the inner cylinder \((\zeta = 0)\) from \(O(\frac{1}{\text{De}})\) to \(O(\frac{1}{\sqrt{\text{De}}})\) .

To cover intermediate values of \(\text{De}\), the complete problem (3-93) and (3-85) was solved.

Expanding the solution in a Taylor series in \(t\), \(\hat{G}(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k\) leads to \(a_0 = 0\) and \(a_1 = 1/\text{De}\). The coefficients \(a_2\) and \(a_3\) are undetermined and recursion relations relate every higher-order coefficient to the four previous ones. The solution is written in compact form by choosing \(\hat{G}_2(t)\) and \(\hat{G}_3(t)\) to be the two linearly independent series solutions that satisfy the conditions

\[
\begin{align*}
a_0 &= 0, \quad a_1 = 0 \quad a_2 = 1, \quad a_3 = 0, \\
a_0 &= 0, \quad a_1 = 0, \quad a_2 = 0, \quad a_3 = 1,
\end{align*}
\tag{3-95a}
\tag{3-95b}
\]

respectively; then

\[
\hat{G}(t) = \frac{t}{\text{De}} + a_2 \hat{G}_2(t) + (-ia_3) \hat{G}_3(t),
\tag{3-96}
\]

where \(a_2\) and \(a_3\) are determined by the boundary conditions at \(t = \text{De}\).

The series forms of \(\hat{G}_2(t)\) and \(\hat{G}_3(t)\) converged so slowly that they were of no practical use for Deborah numbers greater than 4. Velocity fields for larger values of \(\text{De}\) were calculated by solving eq. (3-93) numerically as a system of eight ordinary differential equations with initial conditions given by eqs., (3-95a and 3-95b) for the two functions \(\hat{G}_2(t)\) and \(\hat{G}_3(t)\) respectively. These calculations were carried out using a fourth-order Adams-Moulton technique (Gear 1971) in double precision arithmetic on a Data General MV4000.

3.2.5 White-Metzner fluid

The components of the WM constitutive equation are identical to those for the UCM model eqs. (3-70) to (3-72) with the zero-shear-rate viscosity \(n_0\) replaced by eq. (3-16). The flow and stress fields for concentric cylinders \((\epsilon = 0)\) are identical to the form for the second-order fluid eqs. (3-49) to (3-51) with the Deborah number redefined as...
\[
\text{De} = \frac{n \[0\]_V}{G \frac{R}{O}} = \frac{m(V/\delta)^{n-1}V}{G \frac{R}{O}},
\]
(3-97)

where \(n \[0\]\) is the viscosity of the homogeneous shear flow evaluated at
the shear rate \(V/R\), and with \(n_o\) in eqs. (3-51) replaced by \(n \[0\]\).

Equations for the \(\zeta\)-order corrections to the stress \(\hat{S}_{ij}(\zeta)\) are
developed analogously to eqs. (3-74) to (3-76) and are

\[
\hat{S}_{rr}(\zeta) = \frac{2n \[0\]}{R_o} \frac{(\hat{F}' + \hat{F}\text{De})}{[1 + i\text{De}(1-\zeta)]},
\]
(3-98)

\[
\hat{S}_{r\theta}(\zeta) = \frac{n \[0\]}{R_o} \left[ \frac{n\hat{F}'' + 2\text{De}^2\hat{F}}{[1 + i\text{De}(1-\zeta)]} + \frac{2\text{De}(\hat{F}' + \hat{F}\text{De})}{[1 + i\text{De}(1-\zeta)]^2} \right],
\]
(3-99)

\[
\hat{S}_{\theta\theta}(\zeta) = \frac{2n \[0\}\text{De}}{R_o} \left[ \frac{n\hat{F}'' - 2\text{De}\hat{F}'}{[1 + i\text{De}(1-\zeta)]} + \frac{\hat{F}'' + 2\text{De}^2\hat{F}}{[1 + i\text{De}(1-\zeta)]^2} + \frac{2\text{De}(\hat{F}' + \hat{F}\text{De})}{[1 + i\text{De}(1-\zeta)]^3} \right],
\]
(3-100)

The momentum equations again reduce to

\[
\hat{\tau} = -\frac{\partial}{\partial \zeta} \hat{S}_{r\theta} + \hat{S}_{\theta\theta}, \quad \frac{\partial \hat{\tau}}{\partial \zeta} = 0,
\]
(3-101)

and these can be combined with eqs. (3-98) to (3-100) to give

\[
\begin{align*}
&n[-2t + i(1-t^2)]\hat{G}^{(1\nu)} - [2(n-1) - 2nt^2 + (4n-2)it]\hat{G}'''
+ [2(1-n)t - 4(1-n)i + 2it^2]\hat{G}'' - 4it\hat{G}' + 4i\hat{G} = 0,
\end{align*}
\]
(3-102)

where \(t = \text{De}(1-\zeta)\) and \(\hat{G}(t) = \hat{F}(\zeta)/V\). The boundary conditions for eq. (3-102)
are the same as eq. (3-85).

A Taylor series solution in \(\text{De}\) can be constructed to eqs. (3-102) and
(3-85) by using the same techniques applied in subsection 3.2.3a; however,
the validity of the result is restricted to Deborah numbers below unity
and we do not carry out this calculation. When an expansion for large \(\text{De}\)
is attempted as in subsection 3.2.3b a new boundary layer near the outer
cylinder (\(\zeta = 1\)) is uncovered in which all terms in eq. (3-102) are of
equal importance; for this reason we chose instead to solve the complete problem (3-102) and (3-85) instead of the large De limiting behavior.

Expanding the solution in a Taylor series in t, as in eq. (3-95), leads again to \( a_0 = 0 \) and \( a_1 = 1/De \). The coefficients \( a_2 \) and \( a_3 \) are undetermined and recursion relations relate every higher-order coefficient to the four previous ones. The solution is written in compact form by choosing \( \hat{G}_2(t) \) and \( \hat{G}_3(t) \) to be the two linearly independent series solutions that satisfy the conditions

\[
\begin{align*}
    a_0 &= 0, \\
    a_1 &= 0, \\
    a_2 &= 1, \\
    a_3 &= 0,
\end{align*}
\]

(3-103)

\[
\begin{align*}
    a_0 &= 0, \\
    a_1 &= 0, \\
    a_2 &= 0, \\
    a_3 &= 1,
\end{align*}
\]

respectively; then

\[
\hat{G}(t) = \frac{t}{De} + a_2 \hat{G}_2(t) + (-ia_3) \hat{G}_3(t),
\]

(3-104)

where \( a_2 \) and \( a_3 \) are determined by the boundary conditions at \( t = De \), as in eqs. (3-96).

Unfortunately, the series forms of \( \hat{G}_2(t) \) and \( \hat{G}_3(t) \) converged so slowly that they were of no practical use for Deborah numbers greater than 0.5. Velocity fields for larger values of \( De \) were calculated by solving eq. (3-102) numerically as a system of eight ordinary differential equations with initial conditions given by eq. (3-103) for the two functions \( \hat{G}_2(t) \) and \( \hat{G}_3(t) \). These calculations were carried out using a fourth-order Adams-Moulton technique (Gear 1971), in double precision arithmetic on a Data General Eclipse MV4000. The solution of the form (3-104) seemed to involve almost exact cancellation of \( \hat{G}_2(t) \) by \( \hat{G}_3(t) \), which caused loss of accuracy for \( De > 11 \) at \( n = 0.3 \).

3.2.6 Criminale-Ericksen-Filbey fluid

The components of the CEF constitutive equation are identical to those for the SOF model eqs. (3-35) to (3-37) with the zero-shear-rate viscosity \( \eta_0 \) replaced by eq. (3-16) and the zero-shear-rate relaxation time \( \lambda_0 \) replaced by \( \eta^0/G_0 = (m/G_0)^{\frac{\nu}{\lambda_0}} \). The flow and stress fields for concentric cylinders (\( \epsilon=0 \)) are identical to eqs. (3-49) to (3-51) with the Deborah number defined by eq. (3-97) and with \( \eta \) replaced by \( \eta[0] \). The
equations governing the $\varepsilon$-order corrections to the components of stress are

$$
\hat{S}_{rr}(\zeta) = \frac{2n[0]}{R_0} \left[ i\hat{F}' + De(1-\zeta)\hat{F}' + De\hat{F} \right], \quad (3-105)
$$

$$
\hat{S}_{r\theta}(\zeta) = \frac{-n[0]}{R_0} \left[ n\hat{F}'' - i\hat{F}''(1-\zeta)De + 2i\hat{F}'De \right], \quad (3-106)
$$

$$
\hat{S}_{\theta\theta}(\zeta) = \frac{4n[0]}{R_0} De\hat{F}''n, \quad (3-107)
$$

and the scale factors are the same as for the second-order fluid; see eq. (3-66). The constant correction to the pressure $\hat{\pi}$ is calculated from eq. (3-101) using stresses form eqs. (3-105) to (3-107).

The radial part of the streamfunction $\hat{\phi}(\zeta) = \hat{G}(t)\nu$ is governed by the Euler equation written in terms of $t = De(1-\zeta)$.

$$
\hat{\phi}^{(iv)}(ni+t) - 4(n-1)\hat{\phi}'' = 0, \quad (3-108)
$$

Eq. (3-76) has the solution satisfying the boundary conditions (3-59)

$$
\hat{G}(t) = A(ni+t)^{4n-1} + Bt^{2/2} + Ct + D, \quad (3-109)
$$

where the constants $(A,B,C,D)$ are

$$
A = -De/2d, \\
B = -(ni+De)^{4n-1} - (ni)^{4n-1} - (4n-1)(ni+De)^{4n-2}De/dDe, \quad (3-110)
$$

$$
C = [1 - ADe(ni)^{4n-2}(4n-1)]/De, \\
D = -A(ni)^{4n-1},
$$

with

$$
d = De[(ni+De)^{4n-1} - (ni)^{4n-1}] - De^2(4n-1)[(ni+De)^{4n-2} + (3-111)
$$
The exact solution eqs. (3-109) to (3-111) is readily converted into the form of a special case of the result given by Phan-Thien and Tanner (1981), once several typographical errors in their publication are taken into account; the details are given in Appendix C. Also discussed in Appendix C are the special limits of the exact solution needed for the power-law constants \( n \) of 1/2, 1/3, and 1/4. These limits follow directly from results in Phan-Thien and Tanner (1981).

### 3.2.7 Comparison of velocity and stress fields

The velocity and stress fields predicted for the five constitutive equations are compared by plotting the first-order corrections \( u^{[1]}, v^{[1]}, p^{[1]}, \tau_{ij}^{[1]} \). When these are calculated from eqs. (3-31), (3-33) and (3-34) each can be represented by a purely radial function and a phase angle that depends on the radial coordinate as

\[
\begin{align*}
u^{[1]}(\xi, \theta) &= |\delta F(\xi)| \cos[\theta + \phi_v(\xi)] , \\
v^{[1]}(\xi, \theta) &= |\delta F'(\xi)| \cos[\theta + \phi_v(\xi)] , \\
\tau_{ij}^{[1]}(\xi, \theta) &= |S_{ij}(\xi)| \cos[\theta + \phi_{ij}(\xi)] , \\
p^{[1]}(\xi, \theta) &= |\pi_0| \cos[\theta + \phi_p] .
\end{align*}
\]

The complete velocity and stress fields are recovered by substituting these forms into the expansion eq. (3-25). Because \( u^{[0]} \) and \( \tau_{rr}^{[0]} \) are zero and \( \tau_{r\theta}^{[0]} \) and \( \tau_{\theta\theta}^{[0]} \) are constants the first-order corrections eqs. (4-1) contain all the radial and azimuthal structure in these variables. The angular velocity \( v(\xi, \theta) \) has radial variation from both the concentric solution \( v^{[0]} \) and the correction \( v^{[1]} \) and so the total velocity \( v(\xi, \theta) \) is plotted here. An eccentricity of 0.4 has been used for computing \( v(\xi, \theta) \) to amplify the differences between the five constitutive models. The corrections \( (u^{[1]}, p^{[1]}, \tau_{ij}^{[1]} \) and velocity \( v \) are put in the dimensionless forms suggested by the perturbation analysis and listed in Table 3.2. Each
TABLE 3.2 Scales used for Dimensionless Corrections to the Velocities and Stresses and for the Complete Azimuthal Velocity.

<table>
<thead>
<tr>
<th>VARIABLE</th>
<th>DIMENSIONLESS FORM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radial Velocity</td>
<td>$u^* = \frac{u}{V\mu}$</td>
</tr>
<tr>
<td>Tangential Velocity</td>
<td>$v^* = \frac{v}{V}$</td>
</tr>
<tr>
<td>Radial Normal Stress</td>
<td>$T_{rr}^* = \frac{T_{[1]} R_o}{2n[0]_V}$</td>
</tr>
<tr>
<td>Tangential Normal Stress</td>
<td>$T_{\theta\theta}^* = \frac{T_{[1]} \mu R_o}{2n[0]_V De}$</td>
</tr>
<tr>
<td>Shear Stress</td>
<td>$T_{r\theta}^* = \frac{T_{[1]} R_o \mu}{n[0]_V}$</td>
</tr>
<tr>
<td>Pressure</td>
<td>$\pi_o^* = \frac{\pi_o \mu^2 R_o}{n[1]_V}$</td>
</tr>
</tbody>
</table>
variable is plotted as a function of the dimensionless radial coordinate \( \zeta = (R-R_0)/\delta \).

Radial profiles of the correction \( u^* \) to the radial velocity for the Maxwell fluid, based on eqs. (3-86 to 3-87), are shown on Figure 3.1 for Deborah numbers between 0 and 64 with \( \theta = 3\pi/2 \). The profile for \( De=0 \) corresponds to the profile for either a Newtonian or a second-order fluid. The radial velocity for the UCM fluid changed little up to Deborah numbers near unity, but evolved rapidly for \( De \) between unity and sixteen with the maximum value of \( u^* \) shifting from \( \zeta = 2/3 \) for \( De=0 \) to \( \zeta = 1/2 \) at large \( De \). Only small changes in the shape of the profile were evident for values of \( De \) beyond 16 and the asymptotic expansion eq. (3-83) developed for high Deborah numbers approximated these profiles to within better than one percent. Although the overall shape of the profiles at high \( De \) appeared quite benign, the singular behavior of the derivative \( \partial u^*/\partial \zeta \) at the inner cylinder \( (\zeta = 0) \) was obvious from Figure 3.1 and the fact that this derivative must be zero at \( \zeta = 0 \). This singularity has a pronounced effect on the stress field.

Radial profiles for the correction \( u^* \) to the radial velocity for the Oldroyd-B fluid are shown in Figure 3.2 for the same Deborah number, \( De=8 \), but for 5 different values of \( \lambda^* \). Clearly, increasing \( \lambda^* \) corresponds to a shift from quadratic Maxwell-like \( (\lambda^* = 0) \) towards the cubic Newtonian profile (recovered on the limit \( \lambda^* = 1 \)). Nevertheless, the profile at \( \lambda^* = 0.8 \) corresponds more to a quadratic one shifted toward the outer wall \( \zeta = 1 \), with a corresponding increase in the width of the boundary layer at the inner cylinder surface \( (\zeta = 0) \) in accordance with the predictions of the asymptotic analysis (see Section 3.2.4).

Adding shear thinning to the constitutive equation via the White-Metzner model had little effect on the velocity correction \( u^*(\zeta, 0) \), as shown by Figure 3.3 for \( n=0.3 \). The magnitudes and locations in \( \zeta \) of the maxima of this correction were essentially the same for both UCM and WM models. The changes in velocity field of the CEF fluid were qualitatively different with increasing \( De \) from those of the WM model. As shown by the profiles in Figure 3.4, the magnitude of the correction \( u^* \) decreased with increasing \( De \) and the location in \( \zeta \) of the maximum moved toward the outer cylinder. No boundary-layers in \( u^* \) appeared near the inner cylinder \( (\zeta = 0) \); however, a region of rapid change in \( u^* \) was evident at \( \zeta = 1 \). Again
Figure 3.1 Radial profiles of the correction to the radial velocity $u^*$ for UCM model with $\theta = 3\pi/2$ and $0 \leq De \leq 64$. 
Figure 3.2 Radial profiles of the correction to the radial velocity $u^*$ for OLD model with $\theta=3\pi/2$, $De=8$ and $0 \leq \lambda^* \leq 0.8$. 
Figure 3.3 Radial profiles of the correction to the radial velocity $u_\star$ for WM model with $\theta=3\pi/2$, $n=0.3$, and $0 \leq De \leq 8$. 
Figure 3.4 Radial profiles of the correction to the radial velocity $u^*$ for CEF model with $\theta=3\pi/2$, $n=0.3$ and $0 \leq De \leq 64$. 
this region of rapid variation caused large changes in the stress field, as described below.

The angular variation in the components of velocity for the five constitutive equations are best examined by plotting contours of \( u^* \) and \( v^* \) for the entire flow, \( 0 \leq \theta \leq 2\pi, \ 0 \leq \zeta \leq 1 \). Contours of the radial velocity correction \( u^*(\zeta, \theta) \) are shown in Figure 3.5 for the UCM and WM models for a range of \( \text{De} \) and with \( n=0.3 \) and \( n=0.5 \) for the WM model. The results for the Newtonian and SOF constitutive relations were almost identical with the contours for \( \text{De}=0.01 \), Figure 3.5a. Here fluid moved inward in the contracting part of the geometry \((0 \leq \theta \leq \pi) \) and outward between \( \pi \leq \theta \leq 2\pi \). The radial velocity vanished identically at \( \theta=0 \) and \( \theta=2\pi \). Adding memory in the UCM model shifted the flow downstream, especially near the inner cylinder where the local Deborah number is largest, as shown by Figures 3.5a-5d.

Adding a retardation time to the UCM model significantly changed the velocity field by shifting it more downstream as Figure 3.6 shows, where contours of \( u^* \) are given for the oldroyd-B fluid at \( \text{De}=8 \) and for 5 different values of \( \lambda^* \). Shear thinning acting alone has no effect on the velocity field for slightly eccentric cylinders, as shown in Figures 3.5a, 3.5e, and 3.5i. However, the shear thinning of the relaxation time \( \tilde{\gamma} \) for values of Deborah number as low as unity appreciably changed the velocity field.

The tangential velocity field \( v^*(\zeta, \theta) \) for the UCM and WM fluids are represented by the contours of streamfunction shown as Figure 3.7 for \( \varepsilon = 0.4 \). The contours for the Newtonian and second-order fluids in Figure 3.7a exhibited flow separation from the outer cylinder symmetric about the point of widest gap, \( \theta = 0 \). As discussed in section 3.2.9, this was expected from the exact solution for the Newtonian fluid and gave an indication of the range of validity, in terms of \( \varepsilon \), of the perturbation solution. The region of flow separation disappeared and the velocity field close to the inner cylinder distorted as the Deborah number was increased in the convected Maxwell model. When shear thinning was included the region of flow separation was flattened but did not disappear.

The contours of azimuthal and radial velocity for the CEF model are shown on Figure 3.8. The contours of \( u^* \) (Figures 3.8a-3.8d) are perturbed towards the outer cylinder, as expected from Figure 3.4, and shifted
Figure 3.5 Contours of the correction to the radial velocity $u^*(\zeta, \theta)$ for the upper-convected Maxwell and White-Metzner ($n=0.3$ and $n=0.5$) models for De of 0.01, 1, 4, and 10. The contours are equally spaced between the maximum and minimum values.
Figure 3.5 (continued)
Figure 3.5 (continued)
Figure 3.6 Contours of the correction to the radial velocity \( u^*(\zeta, \theta) \) for the Oldroyd-B model for \( De=8 \) and for \( \lambda^* = 0, 0.1, 0.2, 0.4 \) and 0.8. The contours are equally spaced between the maximum and minimum values.
Figure 3.7 Contours of the tangential velocity $v^*(\zeta, \theta)$ for the UCM and WM ($n=0.3$) fluids with $\varepsilon=0.4$ and $De=0.1$, 1, 4, and 10. The contours are equally spaced between the maximum and minimum values.
Figure 3.7 (continued)
Figure 3.8 Contours of the correction to the radial velocity $u^*$ and the total azimuthal velocity $v^*$ for the CEF model with $\varepsilon=0.4$, $n=0.3$ and Deborah numbers of 0.01, 1, 4, and 10. The predictions for the SOF model are indistinguishable from the CEF results for De=0.01.
Figure 3.8 (continued)
slightly upstream near the inner cylinder with increased De. Unlike the UCM predictions, the CEF model shows flow separation at the outer cylinder developing with increasing De.

The corrections to the radial normal stress $\tau^{*}_{rr}$ are shown on Figures 3.9-11 for the UCM, SOF, WM, CEF models. In each case, $\tau^{*}_{rr}$ included a term that reduced to the Newtonian contribution for small De (set $\epsilon = 0$ in eq. (3.34) and so was not proportional to De. The results for De = 0.01 were identical for the SOF and UCM models and were the same as the Newtonian contribution. The contribution for the elastic stress had altered $\tau^{*}_{rr}$ slightly for De = 1.0 and was large for high De; the difference in the stress maxima between Deborah numbers of four and ten (see Figures 3.9c and 3.9d) was roughly proportional to De, indicating that the elastic contribution was dominant. For higher values of De, the contours of $\tau^{*}_{rr}$ for the SOF model were identical to those shown in Figure 3.9d.

The effects of memory and shear thinning in the elastic stress $\tau^{*}_{rr}$ of the UCM and WM fluids are clearly shown in Figure 3.10. Near the inner cylinder, the base velocity convects the stress $\tau^{*}_{rr}$ downstream. Near the outer cylinder, the locations and magnitudes of the stress peaks are unaffected by either De or n. The stress peaks do broaden with increasing De for all values of n. For the CEF fluid (see Fig. 3.11a-11d), there is no appreciable convection of $\tau^{*}_{rr}$ downstream because this model has no memory. No steep gradients were found in $\tau^{*}_{rr}$ as De was increased for either the SOF, UCM, or WM models. In each model, the effect of elasticity was proportional to $\text{De} \hat{\phi}(\xi)$ which did not develop boundary layers with increasing De. The elastic part of the radial stress for the CEF model had terms proportional to both $\text{De} \hat{\phi}(\xi)$ and $\text{De} \hat{\phi}'(\xi)$ (see eq. (3-105)), and developed steep gradients near the outer cylinder for high De; this is shown in Figures 3.11a-11d for no = 0.3.

The corrections to the azimuthal normal stress $\tau^{*}_{\theta\theta}$ are shown on Figures 3.11 and 3.12 for the CEF, UCM, and WM models. In each of these models, as well as in the result for the SOF, $\hat{S}_{\theta\theta}$ was homogeneous in De and so $\tau^{*}_{\theta\theta}$ has been scaled with De; see Table 3.2. The scaled correction $\tau^{*}_{\theta\theta}$ was independent of changes in De for the SOF model and contours of $\tau^{*}_{\theta\theta}$ for this case were identical to those for the UCM model at De = 0.01, which is shown as Figure 3.12a.

Increasing De for the UCM fluid caused the contours of $\tau^{*}_{\theta\theta}$ to shift
Figure 3.9 Contours of the dimensionless radial normal stress $T_{rr}^*$ and the correction to the dimensionless shear stress $T_{r6}^*$ for the SOF constitutive equation with Deborah numbers of 0.01, 1, 4, and 10.
Figure 3.10 Contours of the dimensionless radial normal stress $t_{rr}^*$ for the UCM and WM models with the same parameter values given for Figure 3.5.
Figure 3.10 (continued)
Figure 3.10 (continued)
Figure 3.11 Contours of $\tau_{rr}^*$, $\tau_{\theta\theta}^*$, and $\tau_{r\theta}^*$ for the CEF model with $n=0.3$ and Deborah numbers of 0.01, 1, 4, and 10.
Figure 3.11 (continued)
Figure 3.11 (continued)
Figure 3.12  Contours of the correction to the azimuthal normal stress, $\tau_{\phi\phi}$ for the UCM and WM with the same parameter values given for Figure 3.5.
Figure 3.12 (continued)
downstream and steep radial gradients to develop near both cylinders. The steep gradients near the inner cylinder are caused by the singularity in \( \hat{F}'(\zeta) \) that developed at high \( \text{De} \) (see section 3.2.3b). The more intense boundary layer in \( \tau_{\theta\theta}^* \) near the outer cylinder results from the form of \( \hat{S}_{\theta\theta}(\zeta) \) in eq. (3.68) with \( n = 1 \) and did not lead to changes in the velocity field.

The introduction of a retardation time in the OLD model, decreased considerably the steepness of the boundary layer in \( \tau_{\theta\theta}^* \) seen in the UCM fluid, especially next to the inner cylinder. This can be seen in Figures 3-13 and 3-14a where the contours and radial profiles of \( \tau_{\theta\theta}^* \) are plotted for the same value of \( \text{De} \) but for four different values of \( \lambda^* \). Again, the smoothing occurring next to the inner cylinder was expected because of the substantial decrease of the velocity boundary layer occurring in the same place. The influence of the retardation time on the other components of the stress \( \tau_{rr}^* \) and \( \tau_{r\theta}^* \) was similar, as Figures 3-14b and 3-14c show.

The introduction of shear thinning in the WM model did not change the spatial structure of \( \tau_{\theta\theta}^* \), but only decreased the maximum values that occurred along the outer cylinder. The azimuthal normal stress \( \tau_{\theta\theta}^* \) for the CEF model, shown on Figures 3.11e-11h, was entirely different from the fields based on the UCM equation. As Deborah number was increased, \( \tau_{\theta\theta}^* \) became nearly zero over the entire flow, except close to the outer cylinder. Mathematically, this reduction was caused by the decrease in \( \hat{F}(\zeta) \) with \( \text{De} \) shown in Figure 3.4. Near the outer cylinder, the boundary-layer in \( \hat{F}'(\zeta) \) caused rapid increased in \( \tau_{\theta\theta}^* \) to its value along the wall.

The corrections to the shear stress \( \tau_{r\theta}^* \) for Deborah numbers of 0.01, 1, 4, and 10 are shown in Figures 3.9, 3.11, and 3.15 for the SOF, CEF, and WM models, respectively. The correction for the SOF constitutive equation was composed of viscous and elastic contributions [see eq. (3.35)] and the contours of \( \tau_{r\theta}^* \) evolved between limits for high and low \( \text{De} \), as shown in figures 3.9e-3.9h. The shear stress for the CEF model had the same behavior as the tangential normal stress discussed above; at high Deborah number, \( \tau_{r\theta}^* \) approached zero throughout the bulk of the flow with a boundary-layer near the outer cylinder.

The corrections to the shear stress for the UCM and WM fluids are shown in Figure 3.15. At \( \text{De} = 0.01 \), the contours were identical to those for the SOF model with the magnitude scaled with the viscosity at \( \dot{\gamma}_0 \).
Figure 3.13 Contours of the correction to the azimuthal normal stress \( \tau_{\theta\theta} \) for the Oldroyd-B model for \( De=8 \) and \( \lambda^* = 0, 0.1, 0.4 \) and 0.8.
Figure 3.14 Radial profiles of the correction to the stress, $\tau^*$ for OLD model with $\theta=0$, $De=\theta$ and $0 \leq \lambda^* \leq 0.4$. (a) Dimensionless radial normal stress $\tau_r^*$, (b) correction to the dimensionless shear stress $\tau_{r\theta}^*$ and (c) correction to the dimensionless tangential normal stress $\tau_{\theta\theta}^*$. 
Figure 3.14 (continued)
Figure 3.14 (continued)
Figure 3.15 Contours of the correction to the shear stress, $\tau_{r\theta}$ for the UCM and WM with the same parameter values given for Figure 3.5.
Figure 3.15 (continued)
Figure 3.15 (continued)
Increasing Deborah number moved the stress peaks near the outer cylinder upstream nearly 90°. Near the inner cylinder, the upstream shift is close to 270° for the UCM fluid and greater for the WM model. Shear thinning with high De moves the peaks of $\tau_{r0}^*$ off the outer cylinder.

The corrections to the pressure fields for the four constitutive equations can be represented by the changes with De of the magnitude $|\tau_{0}|$ and phase angle $\phi_p$ in eq. (3-112). This magnitude is plotted in the dimensionless form $|\tau_{0}^*|$ (see Table 3.2) in Figure 3.16 as a function of De for all four constitutive equations. The phase angles are given in Figure 3.17. The results for the SOF and UCM models are given as the limits $n=1$ for the CEF and WM models, respectively. The differences in pressure in Figure 3.16 at De = 0 reflect the differences in viscosity of the fluids with the three values of $n$. Changing the power-law index $n$, gave qualitatively different results for the two types of models at high De. The WM model predicted magnitudes $|\tau_{0}^*|$ that decreased only slightly with decreasing $n$, still increasing asymptotically proportionally with De as De $\rightarrow \infty$. However, the values calculated from the CEF equation decreased dramatically as $n$ was reduced to 0.3. As discussed next, this difference resulted in quantitatively different predictions for bearing loads.

3.2.8 Torque and Loads

The torque per unit length on the inner cylinder is given by

$$ L = \int_{0}^{2\pi} \left[ \tau_{r0}^* \right]_{r=R_0} R_0^2 \, d\theta , $$

which is calculated by substituting in the expansion eq. (3-25) for a particular constitutive equation. Each of the $\varepsilon$-order corrections varies sinusoidally with $\theta$ (cf. eq. (3-112)) and thus contributes nothing to the integral in eq. (3-113), which reduces to the torque for concentric cylinders:

$$ L = -2\pi \eta (\dot{\gamma}_0) V R / \mu . $$

The $x$- and $y$-components of the load on the inner cylinder per unit
Figure 3.16 The dependence on Deborah number of the magnitude of the pressure $|\tau_0|$ predicted for the four constitutive models.
Figure 3.17 The dependence on De of the phase shift $\phi_p$ of the pressure field for the four constitutive equations.
length (see Figure 1.2) are given by the formulas

\[ F_x = \int_0^{2\pi} \left[ -p + \tau_{rr} \cos \theta - \tau_{r\theta} \sin \theta \right]_{r=R_o}^{R} d\theta, \quad (3-115) \]

\[ F_y = \int_0^{2\pi} \left[ -p + \tau_{rr} \sin \theta - \tau_{r\theta} \cos \theta \right]_{r=R_o}^{R} d\theta. \quad (3-116) \]

Because each term in eqs. (3-115 to 3-116) is multiplied by either \( \sin \theta \) or \( \cos \theta \), the constant stresses form the concentric solution will not contribute to the loads. Also, an order-of-magnitude analysis shows that the pressure contributions to \( F_x \) and \( F_y \) are well approximated by

\[ F_x = -\varepsilon R_o \int_0^{2\pi} p[1](r, \theta) \cos \theta d\theta = -\varepsilon \pi R_o |\pi_0| \cos \phi_p, \quad (3-117) \]

\[ F_y = -\varepsilon R_o \int_0^{2\pi} p[1](r, \theta) \sin \theta d\theta = -\varepsilon \pi R_o |\pi_0| \sin \phi_p. \quad (3-118) \]

Hence the magnitude of the loads is proportional to the magnitude of the pressure \(|\pi_0|\) plotted in Figure 3.16 and the direction of the loads \( \theta \) is related to the phase shift \( \phi_p \) of the pressure \( \pi_0 \) plotted in Figure 3.17 through

\[ \theta = \pi - \phi_p. \quad (3-119) \]

In Table 3.3, the torques and loads derived here for the SOF constitutive equation are compared to the approximate result of Davies and Walters, (1973) and to the exact calculation in Ballal and Rivlin (1976). The analysis of Davies and Walters follows the approach of Sommerfeld (1904) for a Newtonian fluid and is the exact solution for cylinders with narrow gap and the outer surface expressed as \( r = R_o + (R-R_o)(1+\varepsilon \cos \theta) \).

The approach of domain perturbation used here gives results for the torque and loads that differ by less than 20% from the exact values, even at eccentricities as high \( \varepsilon = 0.4 \). These results establish the range of validity for the calculations with the SOF model. The accuracy of the
TABLE 3.3 Comparison of Torques and Loads Predicted For The Second-Order Fluid to Results of The Exact Solution [15] and The Expansion Based on the Deformed Cylinder [14]

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>DOMAIN PERTURBATION</th>
<th>EXPANSION ON DEFORMED CYLINDER</th>
<th>EXACT SOLUTION</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DIMENSIONLESS TORQUE $L/\eta_o VR_o$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>-56.55</td>
<td>-56.55</td>
<td>-66.15</td>
</tr>
<tr>
<td>0.1</td>
<td>-56.55</td>
<td>-57.68</td>
<td>-67.27</td>
</tr>
<tr>
<td>0.2</td>
<td>-56.55</td>
<td>-61.11</td>
<td>-70.69</td>
</tr>
<tr>
<td>0.3</td>
<td>-56.55</td>
<td>-66.94</td>
<td>-76.55</td>
</tr>
<tr>
<td>0.4</td>
<td>-56.55</td>
<td>-75.41</td>
<td>-85.07</td>
</tr>
<tr>
<td></td>
<td>DIMENSIONLESS FORCE IN X-DIRECTION $F_x/\eta_o VDe$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>101.79</td>
<td>102.18</td>
<td>102.02</td>
</tr>
<tr>
<td>0.2</td>
<td>203.58</td>
<td>207.07</td>
<td>207.11</td>
</tr>
<tr>
<td>0.3</td>
<td>305.36</td>
<td>319.15</td>
<td>320.22</td>
</tr>
<tr>
<td>0.4</td>
<td>407.15</td>
<td>447.24</td>
<td>450.67</td>
</tr>
<tr>
<td></td>
<td>DIMENSIONLESS FORCE IN Y-DIRECTION $F_y/\eta_o V$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>-152.68</td>
<td>-152.69</td>
<td>-160.99</td>
</tr>
<tr>
<td>0.2</td>
<td>-305.36</td>
<td>-305.55</td>
<td>-322.22</td>
</tr>
<tr>
<td>0.3</td>
<td>-458.04</td>
<td>-459.48</td>
<td>-484.56</td>
</tr>
<tr>
<td>0.4</td>
<td>-610.73</td>
<td>-617.00</td>
<td>-650.52</td>
</tr>
</tbody>
</table>
results for the other four constitutive equations has not bee rigorously established.

3.2.9 Analysis of Flow Separation

As pointed out above, the formation and structure of a region of separated flow on the outer cylinder is a sensitive indicator of the particular constitutive equation used in the analysis. The location of the stagnation points on the outer cylinder and the minimum eccentricity for flow separation can both be calculated in closed form and so are used here in this capacity.

Flow separation starts at the outer boundary at a point \( \theta_{\text{min}} \) where

\[
\frac{\partial v}{\partial \zeta} = 0 \text{ at } \zeta = 1
\]  

(3-120)

Applying this criterion to the form of the azimuthal velocity given by eq. (3-31) yields the value of eccentricity for a particular choice of \( \theta \)-separation.

\[
\epsilon_s(\theta) = -V\left[ \left( \text{Re}(\hat{F}''(1)) + V \right) \cos \theta - \text{Im}(\hat{F}''(1)) \sin \theta \right]^{-1}
\]  

(3-121)

where \( \text{Im}(\cdot) \) denotes the imaginary part of the argument. The minimum of \( \epsilon = \epsilon_s \) for separation occurs when \( d\epsilon_s / d\theta = 0 \) or at

\[
\theta = \theta_{\text{min}} = \tan^{-1} \left[ -\text{Im}(\hat{F}''(1)) / (\text{Re}(\hat{F}''(1)) + V) \right]
\]  

(3-122)

Substituting eq. (3-122) into eq. (3-121) yields the minimum eccentricity \( \epsilon_{\text{min}} \), which is plotted in Figure 3.18 as a function of \( De \) for the four constitutive equations with \( \mu = 0.09 \). The result for zero Deborah number, \( \epsilon_{\text{min}} = 0.333 \), is within 10% of the result for the full solution of the Newtonian fluid for \( \mu \) less than 0.1. This is another indication of the extended range of validity of the perturbation results.

The behavior with Deborah number of the minimum eccentricities predicted by the CEF and WM models are qualitatively different. The \( \epsilon_{\text{min}} \) calculated for the UCM model increased rapidly with increasing values of \( De \); part of
Figure 3.18 The dependence on De of the minimum eccentricity $\epsilon_{\text{min}}$ for the separation predicted by the four constitutive equations.
the calculations for these cases are shown on Figure 3.18 as a dashed curve to indicate the alrge error in the domain approximation inherent to such large eccentricities. Adding shear thinning in the White-Metzner fluid creates a shallow maximum in $\epsilon_{\min}$ at a value of $De$ that decreases with increasing power-law index. Past this maximum, $\epsilon_{\min}$ drops to a limiting value for high $De$. The results for the CEF fluid do not show the maximum in $\epsilon_{\min}$ for any value of $n$: $\epsilon_{\min}$ decreases with increasing $De$ to a limiting value for high $De$ which is considerably below the corresponding limit for the White-Metzner fluid.

As seen by comparing Figures 3.7 and 3.8, the shapes of the regions of separated flow differed for the WM and CEF models at high $De$. The separation curve for the WM fluid flattened near the outer cylinder and shifted downstream of $\theta = 0$. For the CEF model, this region grew into the flow and along the outer cylinder with increasing $De$.

The azimuthal angle $\theta_{\min}$ for the inception of flow separation at $\epsilon_{\min}$ is plotted on Figure 3.19 for the same models shown on Figure 3.18. When shear thinning was present $\theta_{\min}$ increased with increasing $De$. The separation point for the UCM fluid moved upstream as $De$ was increased; this trend is demonstrated in Figure 3.8.

3.2.10 Discussion of domain perturbation results

Even in the simple geometry of the journal bearing, the velocity and stress fields that develop with increasing Deborah number are extremely complex. Steep gradients in the components of stress develop at Deborah numbers as low as four for the UCM, CEF, and WM constitutive equations. These gradients are similar to those found to limit numerical calculations at even lower values of $De$ in more complicated flows (Mendelson et al. 1982). The perturbation analysis of the journal bearing proves conclusively that those steep stress gradients are built into the constitutive equations and are not a result of singularities caused by corners in the flow or of improper treatment of inflow boundaries. These features, along with the exact solution that is known for a second-order fluid (Ballal and Rivlin 1976), make the journal bearing flow an ideal test problem for comparing the performance of numerical methods and contrasting the results of different constitutive equations. Numerical calculations for this geometry
Figure 3.19  The dependence on De of the azimuthal angle for the flow separation at the minimum eccentricity predicted by the four constitutive equations.
are discussed in sections 4-6.

The contrasts between predictions for the UCM and WM constitutive equations that include fluid memory and the SOF and CEF model that do not, are striking. As shown in Section 3.2.7, the structure of the velocity and stress fields are entirely different for all but small values of \( \text{De} \). The differences are well illustrated by comparing the magnitude of the pressure \( |\tau_0^\#| \) for the CEF and WM models with the same viscosity and normal stress functions so that both models give identical results for the small gap, concentric cylinders. Even for the modest conditions \( \text{De}=2 \) and \( n = 0.3 \), the predictions of these two models differ by almost 100 percent. At a Deborah number of unity the predictions for \( |\tau_0^\#| \) are still 30 percent apart.

These differences point to the difficulties of interpreting experimentally determined loads on the inner cylinder in terms of the viscometric functions for moderate and high Deborah numbers. Clearly the interrelation between the size and distribution of the loads and the fluid material functions is very dependent on the model used for \( \tau_0^\# \). The application of the CEF model for the measurement of primary normal stress coefficients proposed by Phan-Thien and Tanner (Phan-Thien and Tanner 1981), is only justified for values of \( \text{De} \) less than unity where the pressure variation caused by elasticity is small.

Normal stress determinations based on load measurements for moderate Deborah numbers (1<\( \text{De} < 10 \)) are likely to be more interesting, but to do this the sensitivity of the predictions to the rheological model must be established. The formal perturbation analysis presented here can be extended to other, more accurate constitutive equations in differential form and possibly to some integral models. The prediction of load measurements and flow separation at high elasticity are stringent tests for a constitutive model; the comparison of experimental results to calculations for a variety of models may lead to new insights into complex viscoelastic flows.
4. FINITE ELEMENT CALCULATIONS

4.1 Formulation of the Finite Element Method

A finite element program was developed to investigate the performance of numerical methods in the simulation of the viscoelastic flow in the journal bearing. The mixed Galerkin finite element formulation (Crochet 1982) was used to solve the conservation (2-31,32) and constitutive equations (2-23), subject to the no-slip boundary conditions (2-33,34), for a 2-dimensional planar flow defined in the cylindrical coordinate system shown in Figure 1.2.

The flow domain is divided into quadrilateral elements by drawing lines corresponding to fixed values of the azimuthal angle $\theta$ and the normalized radial coordinate $\zeta$

$$\zeta = \frac{r - R_0}{r_1(\theta) - R_0},$$  \hspace{1cm} (4-1)

where $r_1(\theta)$ is the radial position of the outer cylinder surface given by expression (2-35). The individual elements are mapped isoparametrically to the unit squares defined in terms of $\xi, \eta$ ( $-1 \leq \xi \leq 1$ and $-1 \leq \eta \leq 1$; see Figure 4.1 ) using biquadratic Lagrangian polynomials $\{\phi^k(\xi, \eta)\}$ (Prenter, 1975) with

$$r = \sum_{k=1}^{9} r_k \phi^k(\xi, \eta), \hspace{0.5cm} \theta = \sum_{k=1}^{9} \theta_k \phi^k(\xi, \eta),$$  \hspace{1cm} (4-2)

where $(r_k, \theta_k)$ are the $(r, \theta)$ positions of the k-th node of the element.

The biquadratic Lagrangian polynomials $\{\phi^k(\xi, \eta)\}$ are associated with the nodes of the element and they are formed from as tensor products of the one-dimensional quadratic polynomials $\phi^i(\xi)$ (see Figure 4.2) as

$$\phi^k(\xi, \eta) = \phi^i(\xi)\phi^j(\eta),$$  \hspace{1cm} (4-3)

where $k = 3(i-1)+j$ and $i, j$ (1 $\leq i, j \leq 3$) denote the order of the $k$ node in the $\xi$ and $\eta$ axis respectively (see Figure 4.1). The same functions $\{\phi^k(\xi, \eta)\}$
Figure 4.1 Quadrilateral element used in finite element method, shown in the local coordinates $\xi$, $\eta$.

Figure 4.2 Quadratic basis functions.
are also used to approximate the two velocity components \((v_r, v_\theta)\) and the three non-zero elastic stress components \((s_{rr}, s_{r\theta}, s_{\theta\theta})\) within each element. Similarly, bilinear Lagrangian polynomials \(\psi^l(\xi, \eta)\), associated with the four vertex nodes of the element, are used to approximate the pressure \(p\). These approximations—which are referred to collectively as QQL (quadratic-quadratic-linear)—can be written as

\[
\begin{align*}
&v_r(r, \theta) & u_i \\
v_\theta(r, \theta) & v_i \\
s_{rr}(r, \theta) & = \sum_{i=1}^{N} s_{rr,i} \psi^i(r, \theta), \quad p(r, \theta) = \sum_{i=1}^{M} p_i \psi^i(r, \theta), \quad (4-4) \\
s_{r\theta}(r, \theta) & s_{\theta\theta,i} \\
s_{\theta\theta}(r, \theta) & s_{r\theta,i}
\end{align*}
\]

where \(\mathbf{s}\) is the elastic stress tensor defined as

\[
\mathbf{s} = \mathbf{S} - n^0 \mathbf{\hat{Y}} = \mathbf{1} - (n^0 n_3) \mathbf{\hat{y}}, \quad (4-5)
\]

and where \(M\) and \(N\) are the number of the nodes and vortices, respectively, of the finite element mesh used.

The elastic stress \(\mathbf{s}\) rather than the extra stress \(\mathbf{\hat{Y}}\) was used as a primary variable because this splitting procedure was found to produce a smaller approximation error in the solution (Mendelson et al., 1982; Van Schartingen and Crochet, 1985). The QQL mixed formulation was chosen based on the findings of previous work by Yeh et al. (1984) and Kim-E (1984) where it was found to be the most efficient among the several tested.

The coefficients in (4-4) are determined by equating to zero the Galerkin weighted residual integrals formed by taking the weighted average of the equations (2-30,31) and (2-23) after the composite expression for the extra stress tensor \(\mathbf{\hat{Y}}\) is substituted into them:

\[
\int (\nabla \cdot \mathbf{\hat{Y}}^*) \psi^l \, dV = 0 \quad , \quad l=1,M \quad (4-6a)
\]

\[
\int e_1 \cdot (\nabla \cdot (\mathbf{s}^* + (1 + n^*) \mathbf{\hat{y}}^*) - \nabla p^*) \phi^k \, dV = 0 \quad , \quad k=1,N \quad (4-6b)
\]
where $\mathbf{e}_1$ or $\mathbf{e}_j$ are the unit vectors in the cylindrical coordinate system used ($i,j = r$ or $\theta$), and the superscript * denotes dimensionless quantities. The dimensionless form of equation (4-6) has been obtained by scaling stress with $\eta_0 \Omega$, distance with $R_0$, velocity with $V = \Omega R_0$, and time with $\Omega^{-1}$. The dimensionless relaxation $\lambda_1^*$ and retardation $\lambda_2^*$ times are defined as

$$\lambda_1^* = \lambda_1 \frac{V}{R_0} = \lambda_1 \Omega, \quad N = 1,2$$

and the dimensionless "solvent viscosity" $\eta^*$ is

$$\eta^* = \eta_0 / \eta^0$$

The Deborah number is defined as the dimensionless relaxation time

$$De = \lambda_1^* = \lambda_1 \frac{V}{R_0} = \lambda_1 \Omega$$

With the formulation (4-6c) of the Giesekus constitutive equation and for $\alpha \neq 0$, $\lambda_2^*$ has to be zero and any retardation effect be incorporated into the model through a nonzero $\eta^*$. On the other hand, for $\alpha = 0$ the retardation term could be included either explicitly, through a non-zero $\lambda_2^*$, or implicitly, through a non-zero $\eta^*$.

In the expressions (4-6), the basis functions $\psi_1(\xi, \eta)$ and $\phi_k(\xi, \eta)$ used to form the finite element approximations for the pressure, velocity and stress, are also used as weighting functions of the continuity, momentum and constitutive equations, respectively. This is the standard Galerkin procedure and, for self-adjoint problems, guarantees that the numerical approximation is the best fit to the solution of the corresponding continuum problem (Barrett and Morton 1984). The weak forms of the Galerkin weighted residuals are obtained from equations (4-6) by applying Green's theorem to selective terms in (4-6b, c) so that the final integral expressions do not involve second or higher order derivatives—see Appendix A for the expression of the final equations in a cylindrical coordinate system. The closed nature of the flow is represented by forcing each
variable in the expansion (4-4) to be periodic on the interval \(0 \leq \phi \leq 2\pi\).

In a Petrov-Galerkin formulation weighting functions different from the basis functions are chosen to minimize the error for a non-self-adjoint problem. Only the Petrov-Galerkin formulation for the convection-diffusion problem (Hughes and Brooks 1982) has been explored in detail. Neither the Galerkin or Petrov-Galerkin formulations is a-priori guaranteed to produce a best fit to the solution of a viscoelastic flow problem. The Galerkin finite-element formulation is used in the results described here. The application of the streamline/upwind/Petrov-Galerkin technique proposed by Hughes and Tezduyar (1984) is discussed in Section 5.2.3.

Substituting the pressure, velocity and stress components by the expansions (4-4) into the weak forms of the Galerkin weighted residuals and numerically evaluating the integrals using 4 X 4 Gaussian quadrature (Carnahan et al. 1969), gives a system of \((5N + M)\) nonlinear algebraic equations involving the unknown coefficients \(x_{\uparrow} = (u, v, s_{rr}, s_{\theta\theta}, s_{r\theta}, p)^{\uparrow}\) and is compactly written as

\[
R(x; b) = 0, \quad (4-10)
\]

where \(b_{\uparrow} = (\lambda_1, \lambda_2, \alpha)\).

The set of equations (4-10) is solved by Newton's method, which beginning with an initial approximation \(x_{\uparrow}^{(0)}\) forms successive iterates as

\[
x_{\uparrow}^{(k+1)} = x_{\uparrow}^{(k)} + \delta x_{\uparrow}^{(k+1)}, \quad (4-11a)
\]

where \(\delta x_{\uparrow}^{(k+1)}\) is the solution of

\[
J(x_{\uparrow}^{(k)})\delta x_{\uparrow}^{(k+1)} = -R(x_{\uparrow}^{(k)}), \quad (4-11b)
\]

and \(J_{ij}(x_{\uparrow}^{(k)}) = (\partial R_i/\partial x_j)\) is a component of the Jacobian matrix evaluated at the last iterate \(x_{\uparrow}^{(k)}\). Equations (4-11) are solved by Gaussian elimination using the frontal algorithm of Hood (1976) which accounts for the arrow structure of the Jacobian matrix that results from the periodicity constraints. The Newton iterations were terminated when the maximum
component of the correction vector \( \Delta k^{(k)} \) was less than \( 10^{-5} \).

Newton's method guarantees quadratic convergence for a first iterate close enough to the exact solution (Johnson and Riess 1982). It offers several other advantages. First, by calculating the change of the solution with respect to a parameter change, it allows the prediction of a good initial guess using the solution at a nearby point in the parameter space (first order continuation). Second, by tracing the changes of the sign of the determinant of the Jacobian, it allows the detection of bifurcation points, where one solution family meets another; see Figure 4.3a. Finally, several extensions of the Newton's method are possible allowing the detection and calculation of limit points where the solution family curves back in itself; see Figure 4.3b.

For given eccentricity and gap width, families of flows are computed by varying the parameters \((b_1, b_2, b_3) = (\lambda_1, \lambda_2, \alpha)\) with initial approximations for the Newton iterations constructed by continuation methods (Keller 1977; Brown et al. 1980) according to the formula

\[
X(b^0 + \Delta b) = X(b^0) + \sum_{j=1}^{3} \left[ \frac{\partial x}{\partial b_j} \right]_{b^0} (\Delta b_j),
\]  

(4-12)

where \( b^0 \) is the nearest set of parameters at which a solution is known and \( \Delta b \) is the desired change in the parameters. The tangent vectors to the solution surface in parameter space \( \left( \frac{\partial x}{\partial b_j} \right)_{b^0} \) evaluated at the known solution are calculated from the linear equation set

\[
J(x(b^0)) \cdot (\Delta b_{b^0}) = -\left( \frac{\partial R}{\partial b} \right)_{b^0},
\]  

(4-13)

which is solved simultaneously with the last Newton iterate computed from eq. (4-11b).

The uniqueness and existence of the solution through parameter space, and hence with increasing Deborah number, is described by the solvability of eqs. (4-13) at any particular set of parameter values. The turning of a solution curve with respect to changing a parameter and the bifurcation to multiple solutions of (4-10) are marked by a singular Jacobian matrix. Simple bifurcation points are easily detected
Figure 4.3 Bifurcation and limit points shown qualitatively in the dependence of a solution variable $x_k$ on the parameter $p_1$. a) Bifurcation point where one solution family intersects with another; b) limit point in the solution family shown qualitatively in the dependence of a solution variable $x_k$ on the parameter $p_1$; c) dependence of the limit point position with respect to the parameter $p_1$ on another parameter $p_2$ - qualitative description.
by checking for a change in sign of the determinant of the Jacobian matrix. As described in several references (Keller 1977; Ungar and Brown 1982; Yamaguchi et al. 1984), a limit point in a parameter $b_j$ amounts to a vertical tangent vector ($\partial \mathbf{x} / \partial b_j$) and the loss of solution to eq. (4-13). To coax the calculations around limit points we introduce an arc length $s$ parameterization for the solution family defined by the additional residual equation relating it to the parameter $b_j$ in which the limit point occurs. This equation is

$$R_{N_t+1} = (s-s^0)^2 - \| \mathbf{x}(s) - \mathbf{x}(s^0) \|_2^2 - \| b_j(s) - b_j(s^0) \|_2^2,$$  \hspace{1cm} (4-14)

where $N_t = 5N + M$, $\| \mathbf{x} \|_2$ is the $R_2$-norm of $\mathbf{x}$, and $s^0$ is the value of the arc length at the closest known member of the family. The augmented equations (4-10) and (4-14) are solved by Newton's method and analytic continuation in arc length is used to generate first approximations. A modified version of Abbott's chord method (1978) is used to trace limit points as a second parameter is varied.

The stability of a steady viscoelastic flow in finite element representation is closely coupled to the occurrence of bifurcation and limit points as parameters are varied. To show this we follow the development of Yamaguchi et al. (1984) and first note that the full time dependent versions of the momentum and constitutive equations are reduced by the Galerkin finite element method to a set of nonlinear ordinary differential equations of the form

$$M \cdot \frac{d\mathbf{x}}{dt} = R(\mathbf{x}; \mathbf{b}),$$  \hspace{1cm} (4-15)

where the coefficients $\{x_l\}$ in expansions (4-4) are time-dependent. A steady solution $x_0$ of eqs. (4-15) of course satisfies the residual equations (4-10). The stability of this steady state to infinitesimal perturbations written in the same finite element bases is examined by considering perturbations to $x_0$ of the form

$$\mathbf{x}(t) = x_0 + \mathbf{x} \exp(\mathbf{qt}),$$  \hspace{1cm} (4-16)

where $\mathbf{x}$ has small magnitude. Substituting eq. (4.16) into (4.15) and
retaining only terms linear in $\dot{x}$ gives the generalized eigenvalue problem

$$\lambda M \dot{x} = J(x_0) x,$$

(4-17)

where $J(x_0)$ is the Jacobian matrix evaluated at the steady solution. Clearly, the stability of $x_0$ changes only at values of the parameters where either a real eigenvalue $\lambda_1$ or the real part of a complex conjugate pair of eigenvalues changes sign. The first case corresponds to either a simple bifurcation or a limit point and the second to bifurcation to a family of time periodic solutions that branch from the steady solutions. These ideas are used in Section 5.2 to project the behavior of time-dependent methods for computing the steady flows given here.

Four uniform finite element meshes have been used in the calculations presented in the next sections except where otherwise mentioned. The distribution of elements in the radial and azimuthal directions and the total number of unknowns for each discretization are listed in Table 4.1.

4.2 Comparison with Known Flow and Stress Fields

In this section, the exact solutions for each of the constitutive equations in Table 2.1 in the limit of zero eccentricity (cylindrical Couette flow) and the exact solutions for every value of eccentricity available for the Newtonian and SOF models are compared to finite-element calculations. The existence of limit and bifurcation points is also tested for each of these models. The convergence of the finite-element results for the values of the field variables and for the presence and location of critical points with increasing mesh refinement is a sensitive test of the accuracy of the numerical scheme.

4.2.1 Concentric Cylinders ( $\epsilon = 0$ )

For concentric cylinders the solutions for the velocity and stress fields for the SOF, UCM and OLM models are all the same provided that these solutions are written in terms of the Deborah number defined by eq. (4-9). We have concentrated the numerical calculations on the UCM
<table>
<thead>
<tr>
<th>NAME</th>
<th>ELEMENT DISTRIBUTION</th>
<th>Number of Unknowns</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\theta$ Direction</td>
<td>$r$ Direction</td>
</tr>
<tr>
<td>M1</td>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>M2</td>
<td>20</td>
<td>5</td>
</tr>
<tr>
<td>M3</td>
<td>30</td>
<td>8</td>
</tr>
<tr>
<td>M4</td>
<td>40</td>
<td>10</td>
</tr>
</tbody>
</table>
model with the mesh M2 listed in Table 4.1. The maximum error in the field variables for these calculations is listed in Table 4.2 for De between 0 and 7.25. Only for the elastic component of the shear stress \( s_{\theta\theta} \) did the relative error exceed a few parts per thousand of the exact solution. The three percent error in \( s_{\theta\theta} \) (relative to \( r_{\theta\theta} \)) was due to a less accurate approximation to the derivative \( \partial v_\theta / \partial r \), which was the only non-zero component of the velocity gradient tensor.

Calculations for the UCM model revealed four bifurcation points in the range \( 0 \leq De \leq 7.25 \); these occurred at Deborah numbers of approximately 2.16, 2.75 ± 0.25, 5.70 and 6.32. The structure of each new solution to eq. (4-10) that branches from these critical points was examined by computing the eigenvector which correspond to the zero eigenvalue of \( J \). As described in Yamaguchi et al. (1984), this vector is a first approximation to the difference between the known solution \( \mathbf{x}_0 \) and the bifurcating one. Eigenvectors computed for the first, third and fourth critical points had azimuthal oscillations with the same frequency (or half the frequency for the first vector) as the nodal points in the finite element mesh. This result seemed to indicate that these bifurcations were related to a "numerical instability" caused by the finite element discretization and were not properties of the original conservation and constitute equations—see also Section 5.2 for a thorough discussion of this important observation.

The Leonov-like model is set apart from the other models in Table 2.1 by its ability to represent reasonably shear thinning of the first normal stress difference and viscosity (see Figure 2.3). The shear thinning viscosity of this model, with an effective power-law slope asymptotically approaching -1, caused a steep boundary-layer in the azimuthal velocity even when the cylinders were concentric. A boundary layer in the azimuthal velocity was developed near the inner cylinder for Deborah numbers ( \( De = \lambda_1 \) since \( \lambda_2 = 0 \) ) as small as unity. The exact form of this velocity profile (see Appendix D) is compared in Figure 4.4 to calculations with the mesh M2. By \( De = 2 \) the finite-element results deviated substantially from the exact profile. The inaccuracy in the finite element description of \( ( \partial v_\theta / \partial r ) \) caused worse approximation error to the steeper boundary layer predicted for the azimuthal normal stress \( s_{\theta\theta} \), as demonstrated for \( De = 1 \) in Figure 4.5. Redistributing
Table 4.2 Maximum Relative Errors in Finite Element Calculations for Concentric Cylinders with UCM Model. Mesh M2 was used.

<table>
<thead>
<tr>
<th>Variable ( De )</th>
<th>( v_0 )</th>
<th>( S_{\theta\theta} )</th>
<th>( p )</th>
<th>( v_r^* )</th>
<th>( S_{rr}^* )</th>
<th>( S_{r\theta}^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (NEW)</td>
<td>&lt;10(^{-5})</td>
<td>0</td>
<td>&lt;10(^{-15})</td>
<td>&lt;10(^{-20})</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2.0</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
<td>10(^{-5})</td>
<td>3;10(^{-5})</td>
<td>0.04</td>
</tr>
<tr>
<td>5.75</td>
<td>0.0017</td>
<td>0.007</td>
<td>0.001</td>
<td>3;10(^{-5})</td>
<td>0.001</td>
<td>0.3</td>
</tr>
<tr>
<td>7.25</td>
<td>0.002</td>
<td>0.008</td>
<td>0.001</td>
<td>2.4;10(^{-5})</td>
<td>0.001</td>
<td>0.33</td>
</tr>
</tbody>
</table>

*Since the exact solution for these variables is zero, the maximum absolute error is reported in these last three columns.*
Figure 4.4 Dimensionless azimuthal velocity profiles $v_\theta^* v_\theta/V$ for the Leonov-like fluid ($\alpha = 1/2$) between concentric cylinders for Deborah numbers of 0.1, 0.5, 1 and 2. Finite element predictions are shown by solid points, and exact results by solid lines. All calculations were done with M2 and $\mu = 0.1$. 
Figure 4.5 Dimensionless azimuthal normal stress $\tau_\theta = \tau_\theta / (\eta \Omega)$ for the Leonov-like fluid between concentric cylinders for $De = 1$ and $\mu = 0.1$. Results for the uniform mesh M2 (o), a graded version of the same grid with smaller elements near the inner cylinder (o) and the exact solution (—) are shown.
the elements radially to be closer to the inner cylinder can, of course, alleviate the problems in approximating these boundary layers. Results are shown in Figure 4.5 for a graded version of M2 that are in good agreement with the exact solution. Nevertheless, the computational power that has to be spend in order to resolve the boundary layer is substantial and increases sharply with increasing Deborah number.

Introducing a retardation time into the Leonov-like model lessens the shear thinning of the fluid and alleviates these steep gradients, thereby permitting calculations to much higher values of De with a given mesh. We use

\[ \lambda_2 = 0.001 \lambda_1 \quad (4-18) \]

as the value of the retardation time in all our calculations (i.e., model). The azimuthal velocity profiles computed for concentric cylinders (ε=0) with the Leonov-like fluid with the small retardation time (LER; \( \lambda_1 > 0, \lambda_2 = 0.001 \lambda_1, \alpha=1/2 \)) are shown in Figure 4.6 for Deborah numbers ranging from 0.1 to 8.0 along with the exact solution (see Appendix D) for these cases. The radially graded version of M4 used in these calculations was clearly sufficient for reproducing the exact velocity and stress profiles for all values of Deborah number.

The corotational Maxwell-like fluid corresponds to a mobility factor \( \alpha = 1 \). As is well known for this model the viscosity decreases too fast with increasing shear rate with the result that the shear stress shows a maximum when plotted versus shear rate. Shear rate is thus a double valued function of shear stress, and this is manifested in Couette flow (ε = 0) by a region \( \text{De}_c^{(1)} \leq \text{De} \leq \text{De}_c^{(2)} \) in which no simple shearing solution exists—see Appendix E. The two disjoint solution families on either side of this region are shown on Figure 4.7 for \( \mu = 0.1 \) as plots of the first normal stress difference \( N_1 \) and the shear stress \( T_{rr} \) on the surface of the inner cylinder. Both critical points (\( \text{De}_c^{(1)} = 0.0633 \) and \( \text{De}_c^{(2)} = 0.15 \)) mark ends to the solution families.

Finite element calculations for the corotational Maxwell-like fluid with M2 are shown on Figure 4.7 by the points on the first family and are in good agreement with the exact result. The terminal point \( \text{De}_c^{(1)} \) was replaced in the finite element representation by a limit point at
Figure 4.6 Dimensionless azimuthal velocity profiles $v_\theta^* = v_\theta/\Omega R_0$ for the Leonov-like fluid with a retardation time $\lambda_2 = 10^{-3}\lambda_1$ between concentric cylinders for Deborah numbers between 0.1 and 8.0. Finite element predictions are shown by solid points, and the exact results by solid lines. All calculations were performed with mesh M^4 and $\nu = 0.1$. 
Figure 4.7 Dimensionless shear stress \( \tau^* = \tau_{r\theta} / (\eta \dot{\alpha}) \) and first normal stress difference \( N_1 = \tau^*_{r\theta} - \tau^*_{rr} \) on the surface of the inner cylinder for the corotational Maxwell-like model between concentric cylinders (\( \epsilon = 0 \)) with \( \mu = 0.1 \) and the uniform M2 mesh. Finite element results are shown by the points (o) and exact solutions by the solid (physical solutions) and dashed (aphysical solutions) curves.
$De_1 = 0.065$. Algebraic equations with only quadratic non-linearities, as result from the finite element representation of the Giesekus constitutive equation, can only yield smooth solution curves at a limit point (see Chapter 2 in Iooss and Joseph, 1980). We were unable to calculate on the reverse part of the physical solution family for the finite element approximation probably because of extremely rapid variations in the solution with small changes in arc length. The algebraic representation of the terminus is a limit point with a small radius of curvature. However, using only one element, it is possible to find an analytic solution to the discretized problem—see Appendix F—which exhibits a limit point behavior.

All Giesekus fluids with $1/2 < \alpha \leq 1$ have viscosity functions that show excessive shear thinning beyond a critical shear rate and so have disjoint solutions in Couette flow like that demonstrated above for the CML fluid—see Appendix E. These solutions are illustrated in Figure 4.8 for $\alpha = 3/4$ and $\mu = 0.1$. Here the physically realistic solution (the solid curves on Figure 4.8) exists up to $De = De_c = 0.141$ and the second a-physical family (the dashed curves on Figure 4.8) begins infinitesimally close to $De = 0$ and exists for all large Deborah numbers. As for the Leonov-like fluid ( $\alpha = 1/2$), the Giesekus model at $\alpha = 3/4$ with $\lambda_2 = 0$ predicted the formation of a steep velocity gradient $\partial v_\theta / \partial r$ near the inner cylinder even for Deborah numbers as low as 0.1.

Finite element calculations with M2 were good approximations to the exact result for $De \leq 0.1$, but failed to reproduce the steep gradients at the inner cylinder for Deborah numbers near the terminus. As a consequence the numerical calculations failed to recognize the end of the solution family and produced "ghost" solutions up to Deborah numbers near three. As indicated on Figure 4.8, these "ghost" solutions were unrelated to those in the second family of exact solutions but appeared rather as an extension of the first solution family beyond the terminus. Calculations with a uniform mesh of 4 azimuthal and 20 radial elements did resolve the velocity gradients and predicted a limit point at $De = 0.145$ in good agreement with the exact value for the terminus. These results emphasize that failure to approximate steep gradients well can cause the numerical method to miss critical points as well as introduce artificial ones as was seen for the UCM model.
Figure 4.8  Same as Figure 4.7 but for the Giesekus model with $\alpha = 3/4$. The dotted curve shows the "ghost" solutions given by the finite element calculations.
4.2.2 Eccentric Cylinders (ε ≠ 0): Newtonian Fluid (Giesekus Model; α = λ₁ = λ₂ = 0)

Finite element calculations with the mesh M2 were compared to the exact flow field for a Newtonian fluid (Wannier 1950; Kamal 1966) for eccentricities of 0.1, 0.3, and 0.5. The decomposition of the stress tensor given by eq. (4.5) guarantees that the elastic contributions are zero, so we concentrate only on the accuracy of the velocity field.

The radial variation of the azimuthal velocity at the widest gap between the cylinders (θ = 0) is shown in Figure 4.9a as a function of the normalized radial coordinate ζ = (r-R₀)/[δ(1+εcosθ)]. The existence of the recirculation near the outer wall which begins at ε = 1/3 is denoted by the negative velocities for ε = 0.5. The agreement between the exact result and the finite element calculation is good for all three values of ε. Similar accuracy for the azimuthal velocity was found for all viscoelastic calculations without shear thinning; hence, neither plots of v₉(r,θ) nor stream function, which is dominated by v₉ for small eccentricities, are useful variables for determining accuracy.

Radial profiles of v₉(ζ,-π/2) are displayed as Figure 4.9b and show approximation error for high eccentricities; at ε = 0.5 the maximum deviation was approximately ten percent of the exact result.

To verify the convergence of the numerical solution to the exact one with mesh size, radial and azimuthal profiles of the radial velocity computed for a Newtonian fluid (De = 0) are compared in Figure 4.10 to the exact solution at ε=0.4. The convergence of the solution to the exact value with increasing mesh is apparent. The maximum relative error in the profiles for v₉ was 4.5 percent for mesh M2 and 1.2 percent for mesh M4 for 0.1|v₉|Sv₉|Sv₉|max. The error was found to decrease quadratically with decreasing mesh size h.

4.2.3 Eccentric Cylinders (ε = 0): Second-Order Fluid (Giesekus Model; λ₁ < 0, α = λ₂ = 0)

Calculations with mesh M2, at ε=0.1, were carried out for Deborah numbers up to 29 where the computations were stopped because of excessive approximation error in the velocity and stress fields. No bifurcation
Figure 4.9 Radial profiles of (a) dimensionless azimuthal velocity $v_R(\zeta, 0)$, and (b) dimensionless radial velocity $v_T(\zeta, -\pi/2)$ for a Newtonian fluid between eccentric cylinders. Finite element calculations with mesh M2 (o) and the exact solution (——) are shown for eccentricities of 0.1, 0.3 and 0.5 all with $\mu = 0.1$. 
Figure 4.10 Radial (a) and azimuthal (b) variation of the dimensionless radial velocity \( v_r^* = v_r/\Omega R_0 \) for the Newtonian fluid with \( \varepsilon = 0.4 \). Finite element calculations with meshes M2(V) and M4(0) are compared with the exact solution (---).
points were found in this range of \( De \), so in this respect the existence and uniqueness theorems of Giesekus (1963)-Tanner (1966) and Huligol (1973) were obeyed by the finite element approximation. Radial and azimuthal profiles of \( \nu_r(\zeta,r) \) and \( s_{\theta\theta}(\zeta,\theta) \) are shown on Fig. 4.11a-4.11d for Deborah numbers of 1, 15, and 23.6. The finite element calculations are faithful to the exact results up to \( De = 15 \) where small radial and azimuthal oscillations began in each variable. By \( De = 23.6 \) the azimuthal oscillations were so large that finite element results had little resemblance to the exact solution. The oscillations in the velocity and stress field seen here for the SOF model are similar to those in a planar contraction observed in Mendelson et al. (1982). The gradually varying gap of a journal bearing with low eccentricity has yielded accurate solutions up to \( De = 10 \) \( (We = 1001) \) whereas the high extension rates caused by the abrupt contraction limit calculations to \( De \) somewhat less than unity with a similar finite element discretization.

Finite-element calculations at \( \epsilon = 0.4 \) were carried out for both the finite-element meshes M2 and M4. For the coarser discretization M2, the calculations were terminated by a limit point at \( De_C = 2.4 \) beyond which no steady-state solution to the discrete equations was found. This limit point must be an artifact of the numerical approximation and is in contrast to the unique solution computed up to very high values of \( De \) \( (De = 29) \) when the eccentricity was small \( (\epsilon = 0.1) \). The value of \( De \) at the limit point for the SOF model was very sensitive to the finite element mesh. Calculations with the mesh M4 were terminated by a limit point at \( De_C = 3.05 \). This large sensitivity of the value of \( De_C \) to the finite-element approximation is indicative of numerically induced limit points reported elsewhere (Schroder and Keller 1983; Chang and Brown 1984).

The velocity and stress fields computed near \( De = De_1 \) had slight oscillations. These oscillations were not as severe as those computed at the lower value of \( \epsilon \) but at much higher \( De \) where there was no indication of a fictitious limit point. Sample azimuthal profiles of the radial velocity are shown in Figure 4.12; small oscillations developed where \( \nu_r^* \) had its extreme values for \( De > 1 \). The dimensionless shear stress \( \tau_{\theta\phi}^* \) along the inner cylinder and the elastic part of the polymer contribution to the azimuthal normal stress, \( s_{\theta\theta} = S_{\theta\theta} - \eta_{BU} \dot{\gamma}^* \) at \( \zeta = 0.7 \) are shown.
Figure 4.11 Comparison of finite element results and the exact solution for a second-order fluid between eccentric cylinders obtained with $\epsilon = 0.1$, $\mu = 0.1$ and mesh M2. (a) Dimensionless radial velocity $v^*(\xi, -\pi/2)$ across gap; (b) dimensionless radial velocity $v^*(0.7, \theta)$ as a function of angle $\theta$; (c) dimensionless elastic part of azimuthal normal stress $s^*_{\theta\theta}(\xi, -\pi/2)/De$ across gap; (d) azimuthal profile of $s^*_v(0.7, \theta)/De$. 

137
Figure 4.12 Azimuthal variation of the dimensionless radial velocity $v_r^* = v_r/R_0$ for the SOF model with $\varepsilon = 0.4$, as computed with mesh M4. Calculations are for four values of De.
in Figure 4.13 for De = 3.05. The azimuthal normal stress had no oscillations, whereas the shear stress had oscillations in the same regions as $v_r$.

4.3 Small Eccentricities

4.3.1 Upper Convected Maxwell (UCM) Model (Giesekus Model; $a = 0$, $\eta^* = 0$, $\lambda_1 > 0$)

Calculations were performed for the UCM model using all four meshes listed in Table 4.1. For each mesh the maximum Deborah number attainable was set by a limit point $De_1$ where the family of solutions to eq. (4-10) reversed directions and continued toward lower values of $De$. For $0 \leq De \leq De_1$ one or two bifurcation points were found for each mesh. The locations of the limit and bifurcation points for the four meshes are summarized in Figure 4.14 where the error bars denote the accuracy to which the bifurcation points were determined using bisection based on the criterion for a change in sign of the determinant of the Jacobian matrix.

Two distinctly different types of behavior were observed. First, the location of the limit point for meshes M2, M3 and M4 increased slowly with increasing mesh; for a change from 2000 to 8000 degrees of freedom $De_1$ increased only 10%. Second, the locations of the bifurcation points were much more sensitive to the finite element mesh, with both values decreasing substantially with increasing number of elements from M3 to M4. The null vector for the first bifurcation point was computed for mesh M2 and corresponded to new solutions with azimuthal oscillations of half the frequency of the nodal spacing. It is believed that the same behavior would have been found at the second bifurcation point. The bifurcating solutions thus have the same form and occur at nearly the same values of $De$ as the concentric case. Again these modes seemed indicative of a numerical instability.

The finite element results for the UCM fluid have been compared with our perturbation solution of Section 3.2.3. Half the maximum pressure difference on the inner cylinder (the amplitude of the pressure oscillation for the perturbation results) predicted for the four finite element
Figure 4.13 Azimuthal variation of the dimensionless shear stress \( \tau^*_{\theta} = \tau_{\theta \theta}/\eta \Omega \) along the inner cylinder \( (b) \) and the elastic component of the dimensionless azimuthal normal stress \( s^*_{\theta \theta} = (\tau_{\theta \theta} - \eta \gamma_{\theta \theta})/\eta \Omega \) at \( \zeta = 0.7 \) for the SOF model with \( \epsilon = 0.4 \), as computed with mesh M4. Calculations are for \( De = 3.05 \).
Figure 4.14 Dependence on mesh of Deborah numbers for bifurcation and limit points for the UCM fluid with $\varepsilon = 0.1$ and $\nu = 0.1$. Results for each of the four meshes in Table 4.1 are shown.
meshes is compared to the perturbation results in Figure 4.15. This pressure difference is proportional to the loads on the inner cylinder for small \( \mu \) and \( \varepsilon \) (see Section 3.2.8). The finite element results were in good agreement up to the limit point for each mesh. The displacement of the numerical solutions for \( \mu = 0.1 \) from the perturbation results valid only for infinitesimal \( \mu \) was expected.

Radial and azimuthal profiles of the radial velocity for the UCM model are shown as Figure 4.16 for the three meshes M1, M2, and M4 and \( \text{De}=2 \). Clearly, the results computed with the smallest mesh M1 had little to do with the solution of the original flow problem, whereas the calculations with M2 and M4 were in qualitative agreement with the perturbation solution. For both these latter meshes, the radial velocity had azimuthal oscillations with the same frequency as the mesh and magnitude a few percent of the maximum velocity. The oscillations for the two meshes were not in phase with one another, which led to a quantitative difference in the predictions of the magnitude of \( v_r(r, \theta) \) for a specific angular location (cf. results for M2 and M4 on Figure 4.16a).

The first appearance of the azimuthal oscillations was clearly coupled to the bifurcation points located along the family of steady solutions. For the meshes M2 and M4, \( \text{De}=2 \) was reasonably close to one of these bifurcation points and the numerical solution to eqs. (4-10) was distorted by some multiple of the null vector corresponding to this close bifurcation. The oscillations in the solution caused by the proximity to the bifurcation point disappeared by \( \text{De}=2.5 \) (cf. curve for \( \text{De}=3 \) in Figure 4.17). However, the magnitude of the oscillations rapidly increases as we follow the solution in the upper branch, beyond the limit point, as Figure 4.17 shows.

The solutions to the UCM model for \( \text{De}=3.5 \) again had azimuthal oscillations as shown in Figure 4.18. Even though the azimuthal oscillations persisted in the radial velocity, the overall agreement between the perturbation and finite element results remained good for both the velocity (Figures 4.18a and 4.18b) and stress (Figures 4.18c and 4.18d) fields. The close agreement for the elastic stress \( a_{\theta \theta} \) is shown on Fig. 12c which also demonstrates the beginning of a steep stress gradient at the outer wall \( \zeta = 1 \), just as predicted by the perturbation result. The perturbation analysis also predicted the downstream shift of the
Figure 4.15  Half the maximum pressure difference on the inner cylinder for finite element (points) and perturbation (solid line) results with the UCM fluid, $\varepsilon = 0.1$, and $\mu = 0.1$. Results for each of the four meshes are shown and $\Delta p$ is made dimensionless by $\eta_0 V/R_0$. 
Figure 4.16 Comparison of finite element results with three different meshes for the CEM model for $\text{De} = 2.0$, $\varepsilon = 0.1$ and $\mu = 0.1$. (a) dimensionless radial velocity $v(\theta, r, z) = \frac{v(r, \theta, z)}{c_0}$ as a function of angular coordinate $\theta$ and radial coordinate $r$. (b) Dimensionless radial velocity $v(\theta, r, z) = \frac{v(r, \theta, z)}{c_0}$ as a function of angular coordinate $\theta$ and radial coordinate $r$. (c) Dimensionless radial velocity $v(\theta, r, z) = \frac{v(r, \theta, z)}{c_0}$ as a function of angular coordinate $\theta$ and radial coordinate $r$. (d) Dimensionless radial velocity $v(\theta, r, z) = \frac{v(r, \theta, z)}{c_0}$ as a function of angular coordinate $\theta$ and radial coordinate $r$. (e) Dimensionless radial velocity $v(\theta, r, z) = \frac{v(r, \theta, z)}{c_0}$ as a function of angular coordinate $\theta$ and radial coordinate $r$. (f) Dimensionless radial velocity $v(\theta, r, z) = \frac{v(r, \theta, z)}{c_0}$ as a function of angular coordinate $\theta$ and radial coordinate $r$.
Figure 4.17 Azimuthal profiles for the dimensionless radial velocity $v_r(0.7, \theta)$ for UCM fluid, $\nu=0.1$, $\varepsilon=0.1$ for three values of $De$ before and after the limit point (see insert).
Figure 4.18 Comparison of finite element and perturbation results for UCM model with De = 3.5, ε = 0.1 and µ = 0.1. Finite element calculations used mesh M4. (a) Dimensionless radial velocity \( v_\xi^*(\zeta, -\pi/2) \) across gap; (b) dimensionless radial velocity \( v_\theta^*(0.7, \theta) \) as a function of angle; (c) dimensionless elastic part of azimuthal stress \( s_{\theta\xi}^*(\zeta, -\pi/2)/De \) across gap; (d) azimuthal profile of \( s_{\theta\xi}^*(0.7, \theta)/De \).
velocity field caused by elasticity as demonstrated by comparison with the Newtonian result on Figure 4.18b.

Figures 4.18a-4.18d are indicative of the quality of the finite element solution with M4 up to the limit point De₁ = 3.62. These results indicated that the limit point represented an abrupt departure of the true solution for the UCM model from the perturbation approximation and might not be a numerical artifact of this extremely fine grid. It was disconcerting however that a numerically converged value of De₁ was not obtained for these meshes; between M2 and M4 De₁ increased almost linearly with increasing number of unknowns.

4.3.2 Oldroyd-B Fluid Model (Giesekeus model; α = 0, λᵢ, η⁺ > 0)

Previous authors (Crochet and Keunings 1982; Crochet and Walters 1983) have suggested that adding the retardation time in the OLD model enhanced the range of convergence in De for numerical calculations. We have used Abbott's algorithm to trace the limit point described for the UCM fluid in the new parameter

\[ L = \frac{\lambda_2/\lambda_1}{1 - \lambda_2/\lambda_1} \]  \hspace{1cm} (4-19)

This parameter is the ratio η⁺ of the solvent (ηₘ) and UCM (η) contributions to the viscosity when the extra stress tensor for the OLD model \( \mathbf{\tau} \) is decomposed in a viscoelastic \( \mathbf{\tau} \) plus a Newtonian contribution \( \eta_m \) (see Equations 4.8, 2.26).

As shown in Figure 4.19 for mesh M2, introducing the retardation time resulted in only a slight increase in the limiting value of \( \lambda_1 \) and a decrease in De₁ for \( \lambda_2/\lambda_1 \) up to 0.3. The azimuthal oscillations in \( v_r(r, \theta) \) present for the SOF and UCM models were accentuated in the OLD model at moderate values of \( L \), as shown in Figure 4.20. This is not unexpected since the constitutive equation for the Oldroyd model is a sum of terms that are identical (except for the sign of \( \lambda_2 \)) to the UCM and SOF models. The difficulties associated with obtaining accurate approximations to these terms was emphasized by calculations with large \( \lambda_1 \) and \( \lambda_2/\lambda_1 \) near unity (\( L \gg 1 \)). In this limit the flow is Newtonian. However, calculations rely on exact cancellation of terms
Figure 4.19 Limiting values of $\lambda_1$ and $\text{De} = \lambda_1 - \lambda_2$ computed for the ODL fluid with $\epsilon = 0.1$, $\mu = 0.1$ and mesh M2 for different values of $\lambda_2/\lambda_1$.
Figure 4.20 Azimuthal variation of the dimensionless radial velocity $v_r(0.7, \theta)$ for the OLD fluid with $\epsilon = 0.1$, $\mu = 0.1$ and several values of $L$ (Equation 4.19). Calculations were all done for $\lambda_1 = 3.0$ and with mesh M2.
proportional to $I_{1}(1)$ and $\dot{I}_{1}(1)$ the constitutive equation and show the extreme oscillations displayed in Figure 4.20 for smaller L.

4.3.3 Leonov-Like Model (Giesekus Model; $\alpha = 1/2$, $\lambda_{i} > 0$, $\eta^{*} = 0$)

The development of the velocity boundary layer near the inner cylinder with the LEL model, described in Section 4.2.1, limited reliable finite-element calculations with this model to Deborah numbers less than unity. Calculations with a graded version of M2 mesh for $\mu = 0.1$ and $\epsilon = 0.1$ were performed up to $De = 1$ and showed no signs of either bifurcation or limit points. Calculations were possible to much higher values of $De$ but were less accurate because of the high values of $\partial v_{\theta}/\partial r$ next to the inner cylinder. Radial and azimuthal dependence of the radial velocity are shown on Figure 4.21 for $De = 1.0$. These solutions had no oscillations. The shear thinning caused a marked decrease in $v_{r}(r,\theta)$ from the Newtonian result which is also shown on Figure 4.21.

The graded finite element mesh was needed for accurate prediction of the absence of critical points for the LEL fluid. Calculations with the uniform version of mesh M2 produced velocity fields which underpredicted the radial gradient $\partial v_{\theta}/\partial r$ and had no oscillations; however, these calculations predicted a bifurcation point at $De = De_{c} = 0.46$. As described above, this critical point disappeared when the graded version of M2 was used.

4.3.4 Leonov-Like Model with a non-zero Retardation Time (Giesekus Model; $\alpha = 0.5$, $\lambda_{i} > 0$, $\eta^{*} = 0.001$)

Calculations of the motion of the LER fluid between slightly eccentric cylinders ($\epsilon = 0.1$ and $\epsilon = 0.2$) were performed using the graded version of mesh M2 introduced for the calculations with the LEL model shown in Figure 4.5. Smooth velocity and stress fields were computed up to $De = 15$ for both eccentricities. Sample profiles of the azimuthal and radial velocities are shown in Figure 4.22 for three values of $De$. There are no oscillations in any of the profiles. It seemed that the calculations could be continued to almost indefinitely high values of $De$. But, these results could not be confirmed by calculations with
Figure 4.21 Comparison of radial velocity field for the Leonov-like and Newtonian fluids for $De = 1.0$, $\varepsilon = 0.1$, and $\nu = 0.1$. A graded version of the mesh M2 was used. (a) Radial variation of $v_\zeta(\zeta, -\pi/2)$ across gap; (b) dimensionless radial velocity $v_\theta(0.7, \theta)$ as a function of azimuthal angle.
Figure 4.22 Azimuthal and radial variation of the dimensionless angular and radial velocity components for the LER model with $\epsilon = 0.1$, $\mu = 0.1$, and Deborah numbers of 0.25, 1.48, and 15.5. The dimensionless velocity components are $v_\theta^* = v_\theta/\Omega R_0$ and $v_r^* = v_r/\Omega R_0$. Calculations were done with a graded version of mesh M2.
Figure 4.22 (continued)
the fine mesh M4 because of the onset of azimuthal oscillations at $De = 0.1$ which prevented calculations at high $De$. Isayev and Upadhyay (1984) reported qualitatively similar results for finite element calculations for $S_R < 1$ using the Leonov model with two time constants and a retardation time corresponding to $\eta^* = 0.01$.

4.3.5 Inelastic Leonov-Like Model

A central question in this research was to determine whether experimentally measurable differences in the velocity and stress fields can be predicted and used to distinguish the applicability of various constitutive models for elastic fluid behavior. To investigate which of the observed phenomena calculated with the LER fluid were due to the elasticity inherent in that model and which due to the shear thinning exhibited by this model, the calculations with the LER model were repeated using the inelastic model described by equations 2.27 and 2.28. The inelastic constitutive equation described by these equations, has the same viscosity function as the LER model. As expected, more distinguishable elastic effects from the LER model were found at high Deborah numbers. The azimuthal variation of $v_r$ and $v_\theta$ for eccentricities of 0.1 and 0.2 are shown in Figure 4.23 for sample values of $De$ computed with both the LER model and the inelastic model. The maximum in azimuthal velocity clearly shifted downstream (to higher $\theta$) and decreased for both eccentricities, as shown in Figures 4.23a and 4.23c. The change in the maximum value of $v_\theta$ was almost ten percent for $\epsilon = 0.2$. Large relative differences occurred in the radial component. This shift of the velocity field must be a result of the elastic character of the fluid, since the equation set for the generalized Newtonian fluid in Stokes flow will allow a symmetry plane about $\theta = 0^\circ$ and must have a unique solution since a full variational principle exits (Bird 1960).

4.3.6 Corotational Maxwell-Like Model (Giesekus Model; $\alpha = 1$, $\eta^* = 0$)

The loss of existence of steady solution which occurs for the CML fluid between concentric cylinders carried over to the eccentric configu-
Figure 4.23 Comparison of velocity fields predicted for the inelastic constitutive equation and the LER model. Results are for $\varepsilon = 0.1$, $De = 11.5$: (a) $v_\theta^*(0.7,\theta)$ and (b) $v_r^*(0.7,\theta)$; $\varepsilon = 0.2$, $De = 7.57$: (c) $v_\theta^*(0.7,\theta)$ and (d) $v_r^*(0.7,\theta)$. The dimensionless velocity $v^* = \nu/\Omega R_o$. 
Table 4.3 Finite Element Predictions of Relaxation Time $\lambda_1$ at the Limit Point for the CML Model with $\mu = 0.1$ and Mesh M2.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>0</th>
<th>0.1</th>
<th>0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>0.065</td>
<td>0.065</td>
<td>0.028</td>
</tr>
</tbody>
</table>

(0.0633)*

* EXACT ANALYTICAL VALUE
ration and limited finite element calculations to values of De below De₁ for the concentric case. The variation of the Deborah number limit for this model with eccentricity is tabulated in Table 4.3. At ε = 0.4 this limit has been reduced to De₁ = 0.028 where the flow is almost Newtonian and exhibits a strong separation near the outer wall close to θ = 0.

4.3.7 Giesekus Fluid; α = 3/4, λ₁ > 0, ηₖ = 0

The steep velocity gradients at small Deborah numbers present in calculations with this fluid for concentric cylinders also caused difficulties when the cylinders were eccentric. Calculations with the uniform version of the mesh M2 at ε = 0.1 predicted a limit point at De = 2.145 and a bifurcation point at De = 0.17. Calculations with the graded version of M2 did not predict the terminus of the solution family expected for the Giesekus fluid with α = 3/4. The "ghost" solutions beyond the expected limiting value can only be removed by using finite element meshes with more refined radial approximations. The calculations discussed in Section 4.2.1 indicate that a mesh with 10 radial elements (like M4) would probably be sufficient.

4.4 Moderate Eccentricities

Increasing the eccentricity to ε = 0.4 introduced recirculation in the velocity field for the Newtonian fluid thereby complicating the kinematics of the motion significantly. Calculations were performed for the SOF, UCM, and Giesekus models for eccentricity ε = 0.4, gap u = 0.1, and the full range of accessible values of De using both finite-element meshes M2 and M4. The results for the SOF fluid were presented in Section 4.2.3. The results for the other viscoelastic constitutive equations are described below.

4.4.1 Upper-Convected Maxwell Model (Giesekus Model; α = λ₂ = 0, λ₁ > 0)
The calculations for the UCM model with \( \epsilon = 0.4 \) were terminated by a limit point at \( \text{De} = \text{De}_L = 0.93 \) for mesh M4. As was the case for the calculations with the lower eccentricity discussed in the previous Section, the velocity and stress fields were polluted with very regular oscillations in both the azimuthal and radial directions. The magnitude of the oscillations grew with increasing De as demonstrated by the profiles of \( v_r \) and \( v_\theta \) for \( \xi = 0.7, 0 \leq \theta \leq 2\pi \), shown as Figure 4.24 for Deborah numbers of 0.2, 0.6, and 0.9314. Although the oscillations in the radial velocity are substantial this component of velocity is approximately 100 times smaller than the tangential velocity, so the composite velocity fields are deceptively smooth when viewed only in terms of the stream function. This point is made clear in the stream function contours plotted in Figure 4.25 for the Newtonian and UCM models. Even at the limiting values of De the streamlines had not changed substantially from the Newtonian result.

The oscillations in the velocity and stress fields in the UCM calculations at \( \epsilon = 0.4 \) shed doubt on the reliability of the limit point predicted from the calculations. We could only test the sensitivity of this prediction by decreasing the accuracy of the approximation to the mesh M2. A limit point was predicted at \( \text{De} = \text{De}_L = 0.99 \), compared to the value 0.93 predicted for M4. Azimuthal and radial oscillations were again present in the solutions near the limit point and had the spatial frequency of the mesh.

At least three changes in sign of the determinant of the Jacobian matrix occurred for the discrete set of equations resulting from mesh M4 and the UCM model. These singular points indicated the presence of three bifurcation points in the solution the algebraic equations. The bifurcation points were located in the regions of \( \text{De}, 0.2 < \text{De}_C^1 < 0.6, 0.6 < \text{De}_C^2 < 0.7, \) and \( 0.7 < \text{De}_C < 0.72 \). The points \( \{\text{De}_C^i\} \) were not located precisely, because the previous results in Section 4.3.1 with the simpler problem for lower eccentricity showed that these values were extremely sensitive to the finite element mesh and that the bifurcating solutions had large amplitude oscillations. Both results point to the bifurcation points as artifacts of a numerical instability in Galerkin's method for solution of the viscoelastic equations. This point is pursued in Section 5.2.
Figure 4.24  Azimuthal variation at \( \xi = 0.7 \) of the dimensionless azimuthal \( \nabla_\theta = \nabla_\theta / \Omega R_0 \) (a, c, e) and dimensionless radial \( \nabla_r = \nabla_r / \Omega R_0 \) (b, d, f) velocity components for the UCM model with \( \epsilon = 0.4 \), \( \mu = 0.1 \), and Deborah numbers of 0.2, 0.6, and 0.9314. Calculations were done with mesh M4.
Figure 4.24 (continued)
Figure 4.24 (continued)
Figure 4.25 Streamlines predicted for calculations with the Newtonian, SOF, UCM, and LER models using the mesh M4. The size of the recirculation region has been enhanced by stretching the gap width by 10. Figure (b) for the UCM model corresponds to the limit point value of De.
4.4.2 Gieseekus Fluid with Retardation Time; \( \lambda_1 > 0, \lambda_2 > 0, 0 < \omega \alpha \tau/2 \)

The behavior of the solution to the general Gieseekus fluid with retardation was distinctly different from the results described above for the UCM model with respect to both the form of the numerical instability and the existence of limit points. This is shown by first considering calculations with the mobility parameter set to \( \alpha = 1/2 \), giving the LER model. The effect of varying \( \alpha \) is considered later.

For mesh M4 and \( \varepsilon = 0.4 \), smooth velocity and stress fields were obtained with the LER model up to \( \text{De} = 0.5 \). Spatially irregular, large amplitude oscillations developed in the converging portion of the gap for all components of the solution at higher values of \( \text{De} \). The suddenness of the development of these oscillations is demonstrated by the plots shown in Figure 4.26 of \( v_\theta \) and \( v_r \) as functions of azimuthal angle for Deborah numbers of 0.5 and 0.59. The oscillations in the solution were much more severe than those in the UCM calculations (cf. Figures 4.24 and 4.26). Calculations with the LER model beyond \( \text{De} = 0.59 \) were considered worthless, but could be continued by using small steps in \( \text{De} \). At least one bifurcation point existed along the family of flows between \( 0 < \text{De} < 0.59 \) for the LER fluid calculated with mesh M4 and was located in the region \( 0.2 < \text{De}_c < 0.35 \). Calculation with the coarser mesh M2, exhibited a limit point at \( \text{De}_1 = 0.35 \). The sensitivity of the results with mesh refinement indicate the fictitious nature of the limit point.

One of the interesting features of the calculations with the LER model is the disappearance of the flow recirculation with increasing \( \text{De} \) for \( \varepsilon = 0.4 \), as demonstrated by the streamlines plotted in Figure 4.25 for Deborah numbers of 0.1 and 0.5. Azimuthal velocity profiles for \( \theta = \pi/2 \) are shown in Figure 4.27 for values of \( \text{De} \) between 0 and 0.59. The recirculation present at \( \text{De} = 0 \) disappears for \( \text{De} = 0.05 \).

The sensitivity of the change in the velocity field to the value of \( \alpha \) in the Gieseekus model was explored by varying \( \alpha \) and examining the form of the recirculation region. The recirculation vanished with increasing \( \text{De} \) for all values of \( \alpha \) tested; the critical values of \( \text{De} \) for the loss of the recirculation are listed in Table 4.4.

The result for the UCM model, that no change in the velocity field was observed up to the limiting value of \( \text{De} \), is consistent with the
Figure 4.26 Azimuthal variation of the dimensionless azimuthal \( v_\theta^* = v_\theta/\Omega R_0 \) and dimensionless radial \( v_r^* = v_r/\Omega R_0 \) velocity components for the LER model with \( \epsilon = 0.4 \), \( \mu = 0.1 \), and Deborah numbers of 0.5 and 0.59. Calculations were done with mesh M4.
Figure 4.27 Dimensionless azimuthal velocity $v_\theta^* = v_\theta/\Omega R_0$ as a function of radial position for the LER model and several values of $De$ at $\theta = 0^\circ$. Results are for $\epsilon = 0.4$ and $\mu = 0.1$. Note the small reverse flow for $De = 0.0$. 

LER, $\epsilon = 0.4$  
M4 MESH
Table 4.4 Dependence of the critical Deborah number for flow separation on the mobility parameter in the Giesekus model.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$D_e^{\text{crit}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (UCM)</td>
<td>—</td>
</tr>
<tr>
<td>0.01</td>
<td>0.4</td>
</tr>
<tr>
<td>0.05</td>
<td>0.1</td>
</tr>
<tr>
<td>0.5</td>
<td>0.05</td>
</tr>
</tbody>
</table>
trend of sustained flow separation for decreasing \( \alpha \). The cause for the change in the recirculation with \( \alpha \) is discussed in the next Subsection 4.4.3.

The calculations with \( \alpha \) different from 1/2 and zero were not continued to values of \( \text{De} \) high enough that the failure of the calculations by either the formation of a limit point or by excessive oscillations in the solution could be detected. The results reported in Section 4.3 for the LER model with the lower eccentricity \( \epsilon = 0.1 \) suggest that calculations with the Giesekus model can be continued to higher values of \( \text{De} \) than the value corresponding to the limit point for the UCM fluid if numerical instabilities do not destroy solution accuracy.

4.4.3 Inelastic Leonov-Like Model

The decrease of the size of recirculation region with increasing \( \text{De} \) reported in the last Subsection 4.4.2 is a feature that can be measured experimentally (Lawler et al., 1985) and therefore of particular value in deciding which model better describes the rheological behavior of the fluid under investigation. The calculations were repeated with \( \epsilon = 0.4 \) and mesh M4 with the inelastic constitutive equation described by Equations 2-27 and 2-28 with the same viscosity function as the LER model to determine whether the disappearance of the recirculation is truly an elastic phenomenon. Azimuthal and radial velocity profiles computed for this fluid with (inelastic) \( \text{De} = 0.5 \) are plotted in Figure 4.28 and compared with the results for the LER model using the same Deborah number. The radial profile of \( v_\theta \) at the widest portion of the gap (\( \delta = 0 \); Figure 4.28a) is almost identical for both models indicating that the behavior of the recirculation region with changing \( \alpha \) and \( \text{De} \) is a function primarily of the shear thinning viscosity and cannot be used to distinguish the performance of viscoelastic models in the journal bearing flow. The radial and azimuthal velocity profiles for \( \text{De} = 0.5 \) with the LER model differed only slightly from the results for the inelastic model at other locations, as shown by Figures 4.28b - 4.28d.

More distinguishable elastic effects are expected to be found with the LER model at higher Deborah numbers, as was the case at lower eccentricities reported in Subsection 4.3.5, but at eccentricity \( \epsilon = 0.4 \) the
Figure 4.28 Comparison of velocity fields predicted for the inelastic constitutive equation (2.12) for $\lambda = 0.5$ and the LER model for $De = 0.5$ with $\varepsilon = 0.4$ and mesh M4. Results are for (a) $v_0^*(\zeta, 0)$, (b) $v_0^*(0.7, \theta)$, (c) $v_\theta^*(\zeta, -\pi/2)$, and (d) $v_\phi^*(0.7, \theta)$. The dimensionless velocity $v^*$ is given by $v^* = v/\Omega R_0$. 
Figure 4.28 (continued)
finite-element calculations with the LER model were limited to $De \leq 0.59$. 
5. ANALYSIS OF FINITE-ELEMENT RESULTS

5.1 Comparison of the Loads on the Inner Cylinder

The magnitude and direction of the force per unit length on the inner cylinder caused by the flow were used in Section 3.2.8 to compare models that were variations on the SOF and UCM fluids with shear thinning viscosity. We write this force in dimensionless form as

\[ F = F_x \delta_x + F_y \delta_y, \]  

where \( \phi = \tan^{-1}(F_y/F_x) \) is the angle at which the maximum force occurs and \( F = |F| \) is the magnitude of the force. The components of the force, evaluated using expressions (3-115) and (3-116), are made dimensionless with \( nV \). As discussed in Section 3.2.8, the magnitude of the force in primarily composed of the pressure on the cylinder when the gap is as small as \( \mu = 0.1 \). The dimensionless magnitude of the force on the inner cylinder is plotted in Figure 5.1 for both the UCM and LER fluids with eccentricities of 0.1 and 0.4 along with the loads predicted for the inelastic models with the corresponding viscosity. The shear thinning introduced in the LER model caused a decrease in pressure and load. Similar results were predicted for the inelastic fluid with the viscosity given by eq. (2-28). The predictions for the inelastic and elastic models differed substantially only at very high values of De that could be computed only for low eccentricities with the LER model. The pressure increased slightly for the UCM fluid, as predicted previously by the asymptotic results from Section 3.2.3. The loads predicted by the finite element calculations and by the perturbation analysis agreed well up to the limiting value of De for the numerical calculations.

The angle for the load \( \phi \) calculated for eccentricities of 0.1 and 0.4 are shown on Fig. 5.1b. Increasing the Deborah number for the UCM model caused the angle to shift to less than -90° which would tend to center the inner cylinder. The dependence of the angle on De also agreed well with the asymptotic predictions. The maximum force shifted to higher values of \( \phi \) for the LER model which would tend to push the inner cylinder toward the small part of the gap. This result was qualita-
Figure 5.1 Dependence on De of (a) The magnitude of the dimensionless force exerted by the fluid on the inner cylinder and (b) the direction in which the force acts. Results are compared for eccentricities of 0.1 and 0.4 for the Newtonian, inelastic, UCM, and LER models with μ = 0.1. Note that the load scale in (a) is logarithmic so that load changes of several orders of magnitude, due to the shear thinning, can be accommodated.
tively similar to the perturbation calculations for a White-Metzner fluid with small power law index \( n = 0.3 \) and moderate \( 0 < \text{De} < 1 \) Deborah number. The angle is \( \phi = -90^\circ \) and the force is in the \( y \)-direction for the Newtonian fluid \( (\text{De} = 0) \). The qualitative difference in the shift of the phase angle predicted by the the LER and UCM models is totally an effect of the differences in the elastic responses of the models. The calculations with the inelastic fluid with the viscosity given by eq. (2-28) or with a Newtonian fluid show no change in the phase angle from \(-90^\circ\) because of the symmetry of the velocity field.

5.2 Numerical Instabilities and Change of Type of the Equations

One of the major results of this research is the demonstration of the sensitivity of both the numerical behavior and also the shapes of the predicted velocity and stress fields to the choice of constitutive equation in the same complex flow. An important question was whether information about the impact of the choice of the constitutive equation on the velocity and stress fields could be isolated from the calculations containing numerical artifacts.

Calculations with the UCM model for either concentric or eccentric cylinder flow revealed several branch points to other families of solutions to the algebraic equation set. These new solutions had high frequency azimuthal oscillations and seemed to be numerical artifacts of the finite-element discretization. Even so, these are important numerical results because they point to the inherent mathematical instability in this discrete version of the viscoelastic problem beyond the lowest critical value of \( \text{De} \). The stability arguments based on eq. (4.15) indicate that the numerical description of Couette flow is unstable beyond this critical point and hence cannot be computed by any temporal marching scheme or by time-like iteration schemes, such as successive substitution. Newton's method is superior in this respect, because of its ability to calculate both stable and unstable steady solutions (Yeh et al. 1984; Ungar and Brown 1982; Yamaguchi et al. 1984).

For \( \epsilon=0.1 \), bifurcations between the primary family for the UCM fluid and two new families containing solutions with rapid azimuthal oscillations were located for the three finest meshes. The movement
of these critical points with mesh refinement indicates that the finite element solutions are not free from artifacts even at $De=1$, $\epsilon=0.1$. The linearization of the partial differential equations, as represented by the Jacobian matrix, has not converged with mesh. Also, solutions at higher values of $De$ are distorted with azimuthal oscillations that may originate with the null vectors that correspond to the bifurcating solutions. At least 3 bifurcating solutions were also detected at $\epsilon=0.4$ up to $De=0.94$.

The apparent independence of the presence of bifurcations on the value of the eccentricity, prompted the examination of the stability of the cylindrical Couette flow of the UCM fluid against azimuthal disturbances. A linear stability analysis, described in Appendix G, showed that in the limit of small gap and for small assumed azimuthal frequencies, no bifurcations to steady state solutions exist; they are all Hopf bifurcations.

The UCM model also exhibited a limiting value of $De$ in the journal bearing for either $\epsilon=0.1$ or $\epsilon=0.4$, but the LER model did not. Both calculations were affected by oscillations in the stress and velocity fields that were accompanied by the presence of the fictitious bifurcation points mentioned above. The oscillations in the UCM fluid results were uniform around the gap, whereas those for the LER fluid were confined to the converging portion of the gap. Oscillations also were present in the results for the second-order fluid model and caused a fictitious limiting value of $De$.

It seems clear that numerical instabilities introduced by using the Galerkin finite element method for solution of viscoelastic flow problems hamper the solution for all these constitutive equations and pollute the correct behavior for any of the models.

5.2.1 Change of type of the equations

Recently, Joseph (1984) suggested a change of type in the equations from elliptic to hyperbolic as a possible cause for the failure of the numerical calculations in the simulation of viscoelastic flows. He associated the difficulties in the calculations—as well as many effects in the flows of viscoelastic fluids—with wave propagation, the nonlinear
smoothing of shocks, and the "shock up" of smooth solutions. These effects usually accompany the appearance of real characteristics and a change of type in the governing equations and are seen at the sonic transition in gas dynamics (Meyer 1982).

Initially Joseph et al. (1985) treated the problem of hyperbolicity and change of type for viscoelastic flows in general. They showed that for a two-dimensional flow, the set of the six non-trivially zero equations (continuity, two momentum and three constitutive equations) has always two elliptic and two hyperbolic equations, whereas the remaining pair can switch from hyperbolic to elliptic depending on the values of Reynolds and Deborah numbers. Joseph et al. (1985) introduced the notion that in many models (and in all models on motions perturbing uniform ones) the vorticity is the variable which changes from elliptic to hyperbolic type. Later, Yoo et al. (1985) tried to explain the striking results of Metzner et al. (1969) for flow through a sudden contraction with an analysis of the vorticity characteristics obtained by perturbing irrotational sink flow for an upper convected Maxwell model.

For the cylindrical Couette flow, it can be shown (see Appendix H) that the region where the switching of the pair of the equations occur is next to the inner cylinder or between the two cylinders depending on the values of Re, De and μ as Figure 5.2 shows. Notice, that there for each value of μ a threshold value Re\text{min} below which this change of type can not occur for any value of the Deborah number. The value of Re\text{min} decreases with increasing μ, as the Equation (H-6) indicates, with 4 as the limiting minimum value at μ = ∞ (flow around a cylinder imbeded in an infinite sea of viscoelastic fluid).

In the above, and in general, all these analyses, a nonzero inertia is necessary in order for a change of type to occur. For example, using the UCM model it has been shown (Joseph 1984) that for creeping flow, the criterion for a change of type to occur is

$$\det \left| \tau + n^0/\lambda_1 \delta \right| < 0 \quad (5-2)$$

which is impossible if the equations are solved exactly, since then it can be shown (Joseph 1984; Marchal et al. 1984) that the matrix \(\tau + n\delta/\lambda_1\) is always positive definite. This criterion can be violated,
Figure 5.2  Change of type of equations in cylindrical Couette flow. Limits in De (De_{max}, De_{min}) at each value of the normalized wall distance \( \zeta = (r-R_0)/R_0 \) inside which four out of six equations are hyperbolic. For 5 Re values: Re=5, 10, 50, 500, 1000: a) with \( \mu=0.1 \), b) \( \mu=1 \). and c) \( \mu=10 \).
Figure 5.2 (continued)
however, by an approximate numerical solution, perhaps locally in a part of the flow. Then, a change of type in the equation and amplification of the error can result.

Indeed, for the numerical calculations with the UCM fluid reported in Subsection 4.4.1, it has been observed that for $\varepsilon=0.4$, using mesh M2, violation of Equation (5-2) occurs just after the first bifurcation point. The violation is observed first in the two elements next to the outer cylinder, then, as $\text{De}$ increases, eq. (5-2) is violated in neighboring elements. Violations of eq. (5-2) were also observed to occur in the flow of the UCM fluid around a sphere confined in a small region surrounding the forward stagnation point (Marchal et al. 1984).

5.2.2 Finite-element calculations with non-zero inertia

The connection of the observed oscillations to the change of type of the equations becomes more clear by comparing the obtained solution profiles at zero-inertia to those obtained with non-zero inertia, when a change of type can indeed occur and has not to be associated with approximation error.

Calculations were carried out for $\mu = 0.1$ and various values of $\text{De}$, at $\text{Re} = 1000$ which is higher than the corresponding $\text{Re}_{\text{min}}$, still lower than the corresponding $\text{Re} = \text{Re}_T$ for the Taylor instability for a Newtonian fluid to take place (Chandrasekhar 1961; Drazin and Reid 1981). Two finite-element meshes were used, M2 and M4. The results for the highest $\text{De}$ calculated ($\text{De}=1.14$) were drastically different from those for the other two. As seen in Figures 5.3, 5.4 and 5.5, whereas there is convergence with mesh refinement to a relatively smooth solution at $\text{De}=0$ (Newtonian) and $\text{De}=0.22$, the opposite occurs at $\text{De}=1.14$: the small azimuthal oscillations present in the solution with the M2 mesh, instead to decrease, get considerably stronger in the solution with the finer mesh M4. This result, indicates clearly an instability of the numerical scheme, the Fourier mode corresponding to the mesh spacing being the excited mode. Still, the oscillations are very regular, quite similar to those observed with the inertialess calculations (see Figure 4.17). Furthermore, as seen from Figure 5.2a at $\text{De}=1.14$ 65% of the annulus between the two cylinders has four out of six equations
Figure 5.3 Radial (a), and azimuthal (b) variation of the dimensionless radial $v_r^{*} = v_r / \theta R_0$ velocity components for the UCM model with $Re = 1000$, $\epsilon = 0.1$, $\mu = 0.1$, at $De = 0$ (Newtonian). Calculations were done with mesh M2 (results presented by triangles $V$ ) and mesh M4 (results presented by continuous line).
Figure 5.4  Radial (a), and azimuthal (b) variation of the dimensionless radial $v_r^* = v_r/UR_0$ velocity components for the UCM model with $Re = 1000$, $\varepsilon = 0.1$, $\mu = 0.1$, at $De = 0.22$. Calculations were done with mesh M2 (results presented by triangles $\triangle$) and mesh M4 (results presented by continuous line).
Figure 5.5 Radial (a), and azimuthal (b) variation of the dimensionless radial \( v_r^* = v_r/R_0 \) velocity components for the UCM model with \( Re = 1000 \), \( \epsilon = 0.1 \), \( \mu = 0.1 \), at \( De = 1.14 \). Calculations were done with mesh M2 (results presented by triangles \( \triangledown \)) and mesh M4 (results presented by continuous line).
hyperbolic. Therefore, the oscillations seen in Figure 5.5b at Re=1000, De=1.14 can most safely be attributed to the hyperbolic character of the equations and by comparison one could attribute the oscillations seen at Re=0 to the same reason. However, one must be very cautious; simply because the equations have an enhanced hyperbolic character does not necessarily imply that the finite-element solutions will present oscillations. As seen from Figure 5.2a, at De=0.22 and Re=1000, in 85% of the flow domain four equations are hyperbolic. Still, as Figure 5.4b shows, the finite-element solution converges with mesh refinement as nicely as in the Newtonian case.

As a concluding remark, it seems that the change of type is a contributing factor to the appearance of fictitious bifurcations and numerical oscillations when a regular Galerkin finite-element method is used. The method is not totally useless however: except that it might still produce good solutions for some mesh sizes and parameter values, because of the regularity of the oscillations the smoothed solutions might still contain valid information—see Section 5.3.

5.2.3 Upwind finite-elements

It is possible that the difficulties encountered in the numerical calculations and the oscillations in the solutions are caused by the improper discretization of the continuum equations. Indeed, as mentioned in Section 4.1, the Galerkin formulation used in the finite element method, produces an optimal approximation to the solution only if the problem is self-adjoint (Barrett and Morton 1984). A modification on the formulation is needed in order to properly discretize non-self-adjoint problems.

This need has been already recognized when the Galerkin finite element method was used to solve the advection-diffusion equation

\[ \nabla^2 \phi - \text{Pe} \, \nabla \cdot \nabla \phi = 0 \tag{5-3} \]

where \( \phi \) is the transferred quantity (for example temperature or concentration); Pe = VL/k is the Peclet number with V the velocity, \( \nabla \) the gradient, L a characteristic length of the problem, and k the diffusivity;
and \( \mathbf{u} \) a unit vector tangent to th. (constant) velocity (\( V = \mathbf{u} U \)). The convective term in eq. (5-3) destroys the self-adjointness of the equation and causes the Galerkin finite element solution to oscillate if the elemental Peclet number

\[
\text{Pe'} = \text{Pe} \frac{h}{L}, \tag{5-4}
\]

is bigger than order one. In eq. (5-4) \( h \) denotes a characteristic length of the element.

This behavior was first explained by Christie et al. (1976) who analyzed the discretized equations derived from the application of the Galerkin finite element method on the advection-diffusion equation (5-3). They showed that, in the one-dimensional case and when bilinear basis functions are used, the discretized equations are identical with those obtained from a central finite difference approximation and they admit a closed-form solution which is oscillatory for \( \text{Pe'} > 2 \).

This oscillatory behavior had already been observed in the central finite difference approximations of hyperbolic equations, such as the supersonic regime of the transonic flow equations. The use of upwind difference approximations in the supersonic region by Murman and Cole (1971) eliminated the oscillations and has become standard practice in transonic flow calculations (Jameson 1975). Using upwind finite differences in the advection-diffusion equation removes the oscillations too, but, for moderate Peclet numbers, this technique leads to excessively overdifferent solutions. It is possible however, because of the simplicity of the one-dimensional advection diffusion problem, to obtain an exact nodal solution for this problem by using a mixed (central/upwind) finite difference approximation with the wright proportions of the two different approximations (Christie et al. 1976).

In the last decade, several modifications of the classical Galerkin finite element formulation have been proposed aiming at incorporating the "upwind" and mixed finite difference formulations into the finite element method. All these methods discretize the one-dimensional advection-convection equation optimally, i.e. the solution of the discrete problem assumes at the nodal positions the values of the exact solution of the continuum problem. These "upwind" methods are distinguished from one
another in the particular way in which they modify the standard Galerkin formulation.

In particular, it has been proposed to use different weighting and basis functions: for example, for linear basis functions Christie et al. (1976) proposed piecewise linear weighting functions; Heinrich et al. (1977) proposed quadratic functions; and Barrett and Morton (1984) exponential forms. Similarly, Heinrich and Zienkiewicz (1977) proposed cubic weighting functions to be used with quadratic basis functions. Alternatively, a general method which calculates the appropriate weighting functions, given a choice of the basis functions, has been proposed by Brooks and Hughes (1982). This method, which is called streamline-upwind/Petrov-Galerkin, offers the additional advantage of an easier implementation in multidimensional problems. Other "upwind" methods include the use of special points in implementing the Gaussian quadrature, when evaluating numerically the finite element integrals (Hughes 1978), and the addition of a balancing diffusion term in the original equation (Kelly 1980). In the one-dimensional advection-diffusion problem the last method translates into solving the equation

\[(k + k') \frac{\partial^2 \phi}{\partial x^2} - \nu \frac{\partial \phi}{\partial x} = 0 , \quad (5-5)\]

where \(k'\) is the balancing diffusivity

\[k' = 1/2 \, \alpha' \, |V| \, h , \quad (5-6)\]

and \(\alpha'\) is a multiplicative factor depending on the elemental Peclet number \(Pe'\)

\[\alpha' = \coth(Pe'/2) - 2/Pe' . \quad (5-7)\]

The above mentioned "upwind" methods have been originally developed for the one-dimensional advection diffusion problem, where their predictions coincide with the exact solution for that problem. However, the performance of these methods in more general or multidimensional problems is different and depends crucially on the particular way by which the application
of the original method was extended to handle the more general problem. For example, to solve the multidimensional advection diffusion equation with the balancing diffusion approach the original method developed in (5-6) can be extended in two different ways: the balancing diffusion term can be defined depending on a scalar

$$k' = 1/2 \alpha' |\mathbf{v}| h,$$

or a tensor diffusivity

$$k' = 1/2 \alpha'/|\mathbf{v}| \nabla \nabla h.$$  \hspace{1cm} (5-9)

It has been found (Brooks and Hughes 1982) that the second approach by defining a "streamwise" diffusivity gives much better results than the first which produces solutions suffering from excessive "crosswind" diffusion (i.e. in a direction normal to the streamlines). Even so, when source terms are present in the equation, the addition of an extra diffusion term changes the equation significantly and distorts the solution unless the source term is weighted differently in the residual equations.

The best method to solve the multidimensional advection-diffusion equation is the streamline-upwind/Petrov-Galerkin (SU/PG) proposed by Brooks and Hughes (1982). Their method changes the standard Galerkin one only by modifying the weighting functions and one recent formulation of it, by Tezduyar and Hughes (1983) is presented in Appendix I. Unfortunately, since in the viscoelastic flow case not all the equations are hyperbolic ($A_i$, $i=1,n$ in (1-9) are complex diagonal matrices) and no vectors $F_i$ exist for which relationships (1-8) hold, it is not clear whether this method would lead to an accurate solution should it had been used to solve viscoelastic flow equations. At the moment, and for a system of nonlinear equations of mixed type—as the viscoelastic flow equations are—no numerical scheme exists with available proof of convergence. Most probably, the streamline-upwind/Petrov-Galerkin method represents the best candidate for an efficient numerical scheme for a system of equations close to fully hyperbolic i.e. for the viscoelastic flow equations, at high Deborah and Reynolds numbers.
5.3 Use of Smoothing Techniques

Further evidence supporting the claim that the oscillations are produced by an improper discretization of the equations came from the high regularity of the oscillations observed in the UCM model solutions. Because the oscillations are so regular there, they can be removed completely by applying the final solution to a simple smoothing filtering.

For a one-dimensional problem, the smoothed, filtered, values are calculated from the unsmoothed ones as

\[ v_i' = \frac{(v_{i-1} + 2v_i + v_{i+1})}{4}, \tag{5-10} \]

with a prime denoting the smoothed value.

The smoothing introduced by eq. (5-10) is a particular case of the Shuman filtering (Orszag and Gottlieb 1979)

\[ v_i' = v_i + \beta (\alpha_1 (v_{i+1} - v_i) + \alpha_2 (v_i - v_{i-1})), \tag{5-11} \]

where \( \alpha_1 \) and \( \alpha_2 \) are chosen such that the last term approximates the second derivative of \( v \) in the grid \( x_i \) and \( \beta \) an adjustable parameter. In a uniform grid with nodes separated by distance \( \Delta x \) and for \( \beta = (\Delta x/2)^2 \) eq. (5-11) reduces to eq. (5-10).

When the processing described by the two-dimensional equivalent to eq. (5-10) was applied to the finite element solution for the UCM model calculated at \( \Delta e=0.6 \) and \( \epsilon=0.4 \), a posteriori, the result was completely smoothed; compare the smoothed and unsmoothed three-dimensional profiles of the radial velocity and the azimuthal normal stress in Figures 5.6 and 5.7. The filtering of a numerical solution a posteriori has also been applied by Orszag and Gottlieb (1979) to the solution of the one-dimensional compressible flow problem, obtained with the application of spectral methods. Smoothing techniques have also been applied to viscoelastic flow equations (Davies 1984) but in a different manner essentially wiping out all the viscoelastic contributions to the momentum equations.

The success of the smoothing suggests the triggering of an oscillatory eigenmode of the discretized set of the equations as a possible cause
Figure 5.6 Three-dimensional profiles of the radial velocity $v_r/V$ in the transformed coordinate frame $(\zeta, \theta)$ for the UCM model at $\epsilon=0.4$, $\mu=0.1$ with mesh M2. (a) unsmoothed solution, and (b) smoothed solution.
Figure 5.7 Three-dimensional profiles of the azimuthal normal component of the elastic part of the stress $\sigma_{\theta\theta}/(\eta V/R_0)$, in the transformed coordinate frame ($\zeta, \theta$), for the UCM model at $\epsilon=0.4$, $\mu=0.1$ with mesh M2. (a) unsmoothed solution, and (b) smoothed solution.
for the oscillations. This eigensolution primarily is giving rise to fictitious bifurcation points at specific values of the problem parameters where the problem becomes numerically singular. Still, because of the existent nonlinearities, the eigensolution partially coexists and superimposes to the regular solution which corresponds to the solution of the continuum problem. In most of the cases were these extraneous eigensolutions develop (c.f. in hyperbolic problems) it happens that is highly oscillatory. When this eigensolution happens to be highly regular too, it can be removed, simply by filtering out the higher Fourier modes from the resulting polluted solution, which is what the application of the Shuman filtering achieves.

In regards to the obtained smoothed solution after the application of the smoothing suggested by the use of the Equation (5-10), one would expect an improvement in the presentability of the numerical solution but not necessarily in its reliability. The actual numerical solution still has oscillations which due to the existant nonlinearities might have distorted it irreversibly. Nevertheless, it is worthwhile to note here that in the case of the compressible flow problem this seems not to be the case and the smoothed results are very close to the available exact solution (Orszag and Gottlieb 1979).

5.4 Conclusions from the Finite-Element Calculations

At this stage it is crucial to differentiate between numerical artifacts introduced by the presence of these instabilities and solution pathology caused by the particular choice of constitutive equation. If the arguments presented in Sections are not convincing enough, suffice to compare the behavior of the results for the meshes M2 and M4. The sensitiviy of the locations in De of the bifurcation points for the UCM and LER models and the fictitious limit point for the SOF and LER models are exactly the behavior expected for critical points introduced by the numerical approximation and it is claimed that this behavior should be ignored when studying the results.

Only the limit point value of De calculated for the UCM model was relatively constant with mesh refinement. Still because of the before mentioned numerical instabilities and the nonlinear nature of the equations,
the finite element results alone are not conclusive about whether it is a real feature of the continuum problem or an artifact of the numerical method.

The reason of the numerical instabilities is not known with certainty. It is suspected that the system of the discretized equations generated by the finite-element method allows the existence of eigensolutions with high frequency azimuthal variations. These fictitious solutions are triggered at high De and are superposed to the approximate solution of the continuum equations. Some of the action that might be taken to remedy this problem was outlined in this Chapter and involves the use of numerical filtering (smoothing) of the solution and (or) the use of upwind weighting functions in the finite-element formulation. Still, other solutions might be the penalization of the incompressibility constraint (Bercovier and Engelman 1979; Beris et al. 1985) or the use of a spectral method which takes into account the spacial periodicity of the journal bearing problem. This last procedure has been pursued here and the analysis of the method and the obtained results are presented in the next Chapter.
6. SPECTRAL/FINITE-ELEMENT METHOD

6.1 Introduction to Spectral Methods

Spectral methods involve representation of the solution to a problem as a truncated series of known functions of the independent variables. They have recently (Gottlieb and Orszag 1977) emerged as a viable alternative to finite-difference and finite-element methods for the numerical solution of partial differential equations. Spectral methods have proved particularly useful in numerical fluid dynamics especially in the study of turbulence and transition, numerical whether prediction, and ocean dynamics.

The most important application of spectral methods involves the use of Fourier series in order to solve problems with periodic boundary conditions. A truncated Fourier series approximation of partial differential equations subject to periodic boundary conditions is known to converge exponentially with the number of included modes (Gottlieb and Orszag 1977). When no periodic boundary conditions apply in the problem, an expansion in Chebyshev polynomials has been shown (Gottlieb and Orszag 1977) to keep the convergence characteristics of the Fourier series expansion. Another advantage of spectral methods is their ability to treat hyperbolic equations like the wave equation without introducing an artificial dispersion.

Therefore, a consistent application of spectral methods in the flow in the journal bearing would have required the expansion of the variables in terms of a Fourier series in the azimuthal direction and a Chebyshev series in the radial direction. However, provided that the Newton's method is to be used to solve for the steady state solution, this procedure produces a set of nonlinear equations with a full Jacobian matrix which makes the problem computationally intractable.

To alleviate that problem a regular finite element representation in terms of a series of local basis functions in the radial direction is used instead of the Chebyshev approximation. This mixed spectral/finite-element method, (S/FEM), combines the computational efficiency of the finite element method with the superior approximation power of the Fourier spectral method in the azimuthal direction. Furthermore, Newton's method
can be used with all its advantages, including computer-implemented nonlinear analysis, as outlined in Section 4.1.

A similar to the S/FEM mixed spectral method, but with finite differences instead of finite elements, was used by Caponi et al. (1982) in the solution of the flow over a periodic wave surface. The obtained results were in good agreement with available analytical solutions for small amplitude wavy surfaces. A mixed spectral/finite-difference method, with the spectral expansion taken in terms of Gegenbauer polynomials, was also used by Dennis and Singh (1978) and Dennis et al. (1980).

Any spectral method also allows the use of efficient smoothing (filtering) techniques as the application of a low-pass filter (Orszag and Gottlieb 1979; Gottlieb et al. 1981). An example of a low pass filter is one which multiply the coefficient of the \( k \)-th Fourier mode by a function \( f(k) \)

\[
f(k) = \begin{cases} 
1 & k < k_0 \\
\exp(-\alpha(k-k_0)^4) & k > k_0 
\end{cases},
\]

where \( k \) is a spectral (wavenumber) index, \( k_0 \) is typically 1/2-1/3 the maximum wavenumber and \( \alpha \) is a constant so that the highest modes are dumped by a factor of roughly \( e^{-1} \) to \( e^{-10} \). The low-pass filter given by Eq. (6-1), especially useful for the discontinuous solutions of hyperbolic problems, has been used by Orszag and Gottlieb (1979) in the solution of the compressible Euler equations using spectral methods.

6.2 Formulation of the Spectral/Finite-Element Method

The spectral/finite-element method is formulated in the natural coordinate system for the problem, the bipolar coordinate system. This choice of coordinate system, alleviates one possible reason for the poor performance of the finite-element method described in Section 4. At high eccentricity, the solution failure to converge might be attributed to the fact that the outer cylinder boundary cannot be represented exactly with the isoparametric mapping in the cylindrical coordinate system, but is rather substituted by a piecewise quadratic line, which has therefore
$C_0$ continuity. This might be considered as a singular boundary, introducing the same sort of difficulties as seen in the sudden contraction to be caused by the singular corner (Yeh 1984). These singularities are avoided if a bipolar coordinate system (Bird et al. 1977a) is used, since the surfaces of the cylinders are represented there by coordinate surfaces (see Fig. 1.3).

The S/FEM approximates each dependent variable $v$ by combining a finite element expansion in polynomial basis functions in the radial-like coordinate $\xi$ with a truncated Fourier series expansion in the angular-like coordinate $\theta$ of a bipolar coordinate system (see Fig. 1.3):

$$u_B \sum_{k=1}^{N} \left[ u_k \cos(k\theta) + v_k' \sin(k\theta) \right], \quad (6-2)$$

where

$$\begin{align*}
\begin{bmatrix} v_k \\ v_k' \end{bmatrix} &= \begin{bmatrix} v_k(\xi) \\ v_k'(\xi) \end{bmatrix} = \sum_{i=1}^{M} \begin{bmatrix} v_{ki} \\ v_{ki}' \end{bmatrix} \phi_{i}(\xi), \quad (6-3)
\end{align*}$$

for $N$ Fourier modes, and $M$ finite-element basis functions.

In the S/FEM we use as primary variables the modified stream function $\psi^\ast$ and the modified viscoelastic stress tensor $T^\ast$ non-zero components, $T^\ast_{\xi\xi}$, $T^\ast_{\xi\theta}$, and $T^\ast_{\theta\theta}$. The modified stream function $\psi^\ast$, and the modified viscoelastic stress tensor $T^\ast$ are defined by eqs. (2-52) and (2-53), in terms of the stream function $\psi$, and the viscoelastic stress tensor $S$, respectively. We prefered to use the stream function instead of the velocity components as primary variables so that the incompressibility condition (2-31) is satisfied exactly.

We used the so modified variables for two reasons. First, because the exact Newtonian solution expressed in terms of these variables involves only one (except the constant term) Fourier mode in the azimuthal direction, this choice of dependent variables was therefore expected to allow an accurate solution to be obtained for all values of eccentricity, at least for small values of elasticity (small $De$). Second, the equations of motion and the constitutive equations expressed in terms of the modified variables, eqs. (2-55) - (2-58) involve only products of $\theta$-dependent
functions which simplifies greatly the calculation of the residual equations as shown below.

The residual equations are formed, according to the Galerkin procedure already outlined in Chapter 4, by integrating each of the eqs. (2-55) - (2-58) with each one of the functions that have been used to approximate the dependent variables. In the S/FEM these functions are products of a regular finite-element basis function $\phi^i(\xi)$ times $\sin(k\theta)$ or $\cos(k\theta)$. Hermite cubic basis functions $X^i$ (Strang and Fix 1973) and Lagrangian quadratic $\phi^i$ (see Figure 4.2) were used to approximate the modified stream function $\psi^*$ and $\tau^*$ stress components, respectively. The Hermite cubic functions were used in order to be able to satisfy exactly the boundary conditions (2-60) and (2-61) which specify the values of both the modified stream function $\psi^*$ and its first radial derivative $\psi^*_r$ at the surfaces of the two cylinders. These conditions specify the coefficients of the Hermite functions which are attached to the two surfaces up to a constant $Q^*$, the flowrate per unit length. This constant is in turn determined from eq. (2-62) evaluated at the surface of the inner cylinder.

The residual equations were calculated in the weak formulation obtained by selectively integrating by parts some of the terms of the integral equations. This integration-by-parts was performed in order to eliminate the presence of derivatives higher than third order for the stream function or higher than second-order for the stress components. The absence of these and higher derivatives is necessary for the integrability of the Galerkin residual equations due to the $C^1$ and $C^0$ continuity of the Hermite cubic $X^i$ and Lagrangian quadratic $\phi^i$ basis functions that were used to approximate $\psi^*$ and the $\tau^*$ components, respectively. The final form of the equations is given in Appendix J.

Each one of the dependent variables appearing in the residual equations was substituted by a series similar to the form given by eqs. (6-2) and (6-3). The resulting double integrals in $\theta$ and $\xi$ were then evaluated in two steps. First the integration over $\theta$ was performed exactly using the Fourier summation formulae. The exact evaluation of the integrals over $\theta$ was possible because of the special structure of the residual equations (J-1) - (J-4) having only quadratic nonlinearities and only multiplicative terms (this was one of the reasons of using the modified versions of $\psi$ and $\tau$). Even so, the resulting equations are horrendous,
requiring over 50 pages of computer code.

Then the integration over $\xi$ was performed numerically using 4 point Gaussian quadrature (Carnahan et al. 1969) which was used for the evaluation of the finite-element integral Equations as well (see Section 4.1). Because of the presence of hyperbolic functions of $\xi$ in the integrands this integration was not exact as was the case with the finite-element calculations. However, it was sufficiently accurate, since change from 4 to 8 Gauss point left the final answer of sample calculations unaltered.

The above procedure led to a set of nonlinear algebraic equations for the $2N+1$ Fourier coefficients corresponding to each one of the $M$ basis functions used for each variable. If $K$ radial elements were used this corresponds to $2(K+1)$ Hermite basis functions for the stream function and to $(2K+1)$ Lagrangian quadratic for each one of the 3 stress components. This corresponds to a system of $(8K+5)(2N+1)$ nonlinear equations with a bandwidth of $13(2N+1)$ which was solved using Newton's method and a modified version of Hood's algorithm (Hood, 1976) in exactly the same way as explained for the finite-element case (Section 4.1).

Once the solution (i.e. the values of the coefficients $v_{1k}$ corresponding to $\psi^*$ and the components of $T^*$) is calculated, the velocities and the pressure are computed in a post-processing step. The velocities are directly recovered from the modified stream function and its derivatives as

$$\frac{V}{V} = \frac{\partial}{\partial \theta} \psi + \frac{\psi}{X} \sin(\theta), \quad (6-4)$$

$$\frac{V}{V} = -\frac{\partial}{\partial \xi} \psi + \frac{\psi}{X} \sinh(\xi), \quad (6-5)$$

where $X = \cos(\theta) + \cosh(\xi)$.

The pressure is recovered by integrating the $\theta$-momentum, eq. (2-48), and $\xi$-momentum, (eq.(2-47), equations from a reference position ($\xi_0$, $\theta_0$) where it is set arbitrarily equal to 0. This position was taken in this work to be at the surface of the inner cylinder, ($\xi_0=\xi_1$) and for $\theta=\theta_0=0$. The $\theta$-integration is performed analytically whereas the $\xi$-integration is performed numerically using Gaussian quadrature.

6.3 Comparison with the Exact Newtonian Solution
The exact solution for a Newtonian fluid involves only the first term of a Fourier series expansion in \( \theta \) in the bipolar coordinate system (Kamal 1966) and, for the primary variables used here, this choice of coordinate system and primary variables gives accurate solutions for all values of eccentricity for a Newtonian fluid. Using only one radial element the loads on the inner cylinder were within only 0.6\% of the exact value for eccentricity \( \varepsilon = 0.1 \). Using 40 uniformly spaced radial elements gave the loads with a relative error of no more than 0.008\% for \( 0.1 \leq \varepsilon \leq 0.8 \). Similar good agreement was obtained for the flowrate and the pressure variation, indicated in Table 6.1.

The availability of the exact solution for a Newtonian fluid was more helpful in debugging the code rather than in validating the method, since no interaction between different Fourier modes takes place. This is the reason that the method was not set up for the SOF model for which the Newtonian stream function is a unique solution (with different stress values). Rather, we used the known stress values for the SOF model to compare them against the values generated from the code's equations when the values of the exact Newtonian stream function was substituted into the constitutive equations. The values were identical, indicating the correctness of the constitutive equations.

6.4 Upper-Convected Maxwell Model

The UCM model, eqs. (2-56) - (2-58) for \( \alpha = 0 \), \( n_0 = 0 \), was used with \( \mu = 0.1 \) and for various values of the eccentricity parameter \( \varepsilon = \), \( 0.1 \leq \varepsilon \leq 0.8 \). The numerical solutions were transformed to the cylindrical coordinate system of Figure 1.2 where various azimuthal and radial profiles were generated, mainly along the broken and dotted lines indicated in Figure 1.2. These results are discussed separately for each eccentricity value used in the following subsections.

6.4.1 Small eccentricities (\( \varepsilon = 0.1 \))

All four different discretizations shown in Table 6.2 were used in calculating the solution at \( \varepsilon = 0.1 \). Smooth solutions were obtained up to \( \text{De} = 100 \). At that low eccentricity, the azimuthal profiles of
Table 6.1  Comparison of the magnitude of the loads per unit length $| F^* |$, $| F |/(\eta V)$, the flowrate per unit length $Q^* - Q/(R_0 V)$, and the pressure variation around the inner cylinder surface (defined as 1/2 the maximum pressure difference $\Delta p^* - \Delta p/(\eta V)$) of the S/FEM predictions for different number of radial elements with the exact result and the finite element predictions obtained with mesh M4.

| NUMBER OF RADIAL ELEMENTS | FLOWRATE PER UNIT LENGTH $Q^*$ | LOAD PER UNIT LENGTH $| F^* |$ | PRESSURE VARIATION $\Delta p^*$ |
|---------------------------|-------------------------------|-----------------------------|-------------------------------|
| 1                         | 0.48362                       | 196.657                     | 61.006                        |
| 2                         | 0.48402                       | 197.290                     | 61.204                        |
| 7                         | 0.48424                       | 197.571                     | 61.291                        |
| 20                        | 0.48439                       | 197.710                     | 61.333                        |
| FINITE ELEMENTS MESH M4   | ----                          | 197.746                     | 61.471                        |
| EXACT VALUES              | 0.48443                       | 197.748                     | 61.346                        |
Table 6.2 Spectral/Finite-Element Discritizations Used in Calculations

<table>
<thead>
<tr>
<th>NAME</th>
<th>NUMBER OF ELEMENTS IN $\xi$ DIRECTION</th>
<th>NUMBER OF FOURIER MODES (MAX k)</th>
<th>NUMBER OF unknowns</th>
</tr>
</thead>
<tbody>
<tr>
<td>N1</td>
<td>10</td>
<td>4</td>
<td>765</td>
</tr>
<tr>
<td>N1G</td>
<td>20 *</td>
<td>4</td>
<td>1485</td>
</tr>
<tr>
<td>N2</td>
<td>20</td>
<td>7</td>
<td>2475</td>
</tr>
<tr>
<td>N4</td>
<td>50</td>
<td>15</td>
<td>12555</td>
</tr>
<tr>
<td>N5</td>
<td>200</td>
<td>7</td>
<td>24075</td>
</tr>
</tbody>
</table>

* NOTE. The 20 radial elements of mesh N1 were graded towards the surface of the inner cylinder. The relative positions $(\xi_2 - \xi_1)/(\xi_1)$ of the nodes were as follows. 0, 0.0025, 0.005, 0.0085, 0.012, 0.016, 0.02, 0.025, 0.03, 0.0365, 0.043, 0.0515, 0.06, 0.07, 0.08, 0.09025, 0.105, 0.12, 0.135, 0.1525, 0.17, 0.1925, 0.215, 0.2425, 0.27, 0.305, 0.34, 0.38, 0.42, 0.465, 0.51, 0.555, 0.6, 0.65, 0.7, 0.75, 0.8, 0.85, 0.9, 0.95, 1.
the variables were almost sinusoidal, as predicted from the domain perturbation analysis, and could be reproduced easily with as low as four Fourier modes. Accurate calculation of the radial dependence of the stress variables was more difficult because of the development of boundary layers with increasing De (also predicted from the perturbation analysis). These steep gradients were presumably also responsible for the onset of numerical instabilities, in the form of radial oscillations and one spurious bifurcation seen at De≈90, when the finest discretization N5 was used.

Fig. 6.1 shows that convergence with mesh refinement is achieved with N2 discretization for De<5. As seen from Figs. 6.2 and 6.3, for De=10, the finest discretization N5 is needed to achieve convergence, although qualitatively the results do not change. Especially in the azimuthal direction, the only difference in the results for the "dominant" variables \( \nu_\theta \) and \( \tau_{\theta\theta} \), obtained with different discretizations, was a parallel transposition of the solution profiles as seen in Figs. 6.2a and 6.2e. However, the changes in the radial profiles of the stresses were more important since increasing the radial elements from 10 (N1) to 200 (N5) significantly improved the description of the steep gradients in the solution (see Figs. 6.3c through 6.3e). These gradients, which started forming at De values as low as three (compare Figs. 4.1c and 6.1e) accentuated as De increases (see Figs 6.3c through 6.3e). At De=95 the boundary layer in \( \tau_{\theta\theta} \) was so steep that was not resolved even with the finest mesh used, N5 (see Figs 6.4b through 6.4e).

Poor resolution caused the occurrence of radial oscillations in the azimuthal normal stress \( \tau_{\theta\theta} \) (see Fig. 6.4d and in more detail Fig. 6.4e). These oscillations are assumed to be responsible for the only bifurcation point seen in the calculations, which occurred for De between 90 and 95 and mesh N5. This bifurcation was not observed with discretization N1 up to De=100 and the steep stress gradients were totally missed because of the coarseness of the mesh.

Figures 6.1a, 6.3b and 6.4a show the characteristic transition from a cubic to a parabolic profile as De increases from 0 to \( \infty \), predicted from the domain perturbation analysis. The parabola is not perfect. It has a small boundary layer next to the inner wall, which although easily seen in Fig. 6.1a for De=5 becomes more difficult to see as De
Figure 6.1 Radial profiles for the UCM fluid at $\theta = \pi/2$ for $\epsilon = 0.1$ and $De = 0, 1, 2, 3,$ and $5$ obtained with the spectral/finite element method and meshes N1 (broken line) and N2 (continuous line). (a) Dimensionless radial velocity $V_r^* = V_r/V_s$, (b) dimensionless radial normal stress $\tau_{rr}^* = \tau_{rr}/(\eta_0 \Omega)$, and (c) dimensionless azimuthal normal stress $\tau_{\theta \theta}^* = \tau_{\theta \theta}/(\eta_0 \Omega)$. 
Figure 6.1 (continued)
Figure 6.2 Azimuthal profiles for the UCM fluid at $\xi=0.7$ for $\epsilon=0.1$ and $De=10$ obtained with the spectral/finite element method using four different discretizations. (a) Dimensionless azimuthal velocity $v_\theta^*=v_\theta/V$, (b) dimensionless radial velocity $v_r^*=v_r/V$, (c) dimensionless radial normal stress $\tau_{rr}^* = \tau_{rr}/(\eta_0 \Omega)$, (d) dimensionless shear stress $\tau_{r\theta}^* = \tau_{r\theta}/(\eta_0 \Omega)$, (e) dimensionless azimuthal normal stress $\tau_{\theta\theta}^* = \tau_{\theta\theta}/(\eta_0 \Omega)$ and (f) determinant of $\tau^* + 1/De\delta$. 
Figure 6.2 (continued)
Figure 6.3 Radial profiles for the UCM fluid at $\theta = \pi/2$ for $r=0.1$ and $De=10$ obtained with the spectral/finite element method using four different discretizations. (a) Dimensionless azimuthal velocity $v_\theta^* = v_\theta/V$, (b) dimensionless radial velocity $v_r^* = v_r/V$, (c) dimensionless radial normal stress $\tau_{rr}^* = \tau_{rr}/(n_0 \Omega)$, (d) dimensionless shear stress $\tau_{\theta r}^* = \tau_{\theta r}/(n_0 \Omega)$, (e) dimensionless azimuthal normal stress $\tau_{\theta \theta}^* = \tau_{\theta \theta}/(n_0 \Omega)$ and (f) determinant of $\tau_\star^*/De^\delta$. 
Figure 6.4 Radial profiles for the UCM fluid at $\theta=-\pi/2$ for $\varepsilon=0.1$ and $De=95$ obtained with the spectral/finite element method using discretization $N5$. (a) Dimensionless radial velocity $V_r^*=V_r/V$, (b) dimensionless radial normal stress $\tau_{rr}^*=\tau_{rr}/(\eta_0 \Omega)$, (c) and (d) dimensionless azimuthal normal stress $\tau_{\theta\theta}^*=\tau_{\theta\theta}/(\eta_0 \Omega)$, (e) dimensionless shear stress $\tau_{\theta r}^*=\tau_{\theta r}/(\eta_0 \Omega)$ and (f) determinant of $\tau^*+1/De\delta$. 
Figure 6.4 (continued)
increases because of the shrinkage of the boundary layer. The presence of the boundary layer is noticeable by the dramatic development of the steep gradients in the $\tau_{\theta\theta}$ radial profiles next to the inner wall (see Fig.6.4d). The explanation of the steep stress gradients next to the outer wall is more subtle. This feature, as for all the previous mentioned above, was first seen in the domain perturbation analysis (see Section 3.2). There, it was explained as caused by the convective terms in the stress equations.

Another quantity, which is presented here for the first time, is the evaluation of the determinant $I^*+1/De_{\theta}$. As explained in Section 5.2.1 this quantity is associated with a change-of-type of the equations. The set of 6 equations (continuity, momentum and constitutive) has two real characteristics when $\text{det}(I^*+1/De_{\theta})$ is positive and four when it is negative. Although, for inertialess flows it has been shown that it should be always positive (Marchal et al. 1984), in numerical solutions it has been observed to become negative in some points of the flow (Marchal et al. 1984; see also Section 5.2.1). Furthermore, since this violation occurs usually just before the numerical method fails, it has been associated with the loss-of-convergence and it has been proposed (Crochet 1984) as a measure of the validity of the numerical solution. As seen from Figs. 6.2f, 6.3f and 6.4f the determinant of $I^*+1/De_{\theta}$ was always positive in the spectral/finite element calculations, even with the most coarse discretization N1. This is another indication of the reliability and self-consistency of the calculations. The fact that the calculations never failed to converge might be related with the observation that the before mentioned criterion was never violated.

6.4.2 Moderate eccentricities ($\varepsilon=0.4$)

At $\varepsilon=0.4$, where a separation (recirculation) region develops in the wide part of the journal bearing, it is the azimuthal dependence of the variables that is most difficult to reproduce by the truncated Fourier series. The reason is that in contrast to the recirculation region where the stresses have very small values, in the rest of the flow they assume (especially the azimuthal normal stress, $\tau_{\theta\theta}$) very big values because of the steep azimuthal velocity gradients imposed
by more steep variations in the flow geometry at \( \varepsilon = 0.4 \) as compared to \( \varepsilon = 0.1 \). Therefore, in the bipolar coordinate formulation of the problem, and for \( \xi \) values close to \( \xi = \xi_2 \) (outer boundary) the azimuthal profiles present steep variations from the part that lies inside the recirculation region to the part that lies outside. In order for these profiles to be accurately reproduced by a truncated Fourier series, a substantial number of modes needs to be included. As the magnitude of \( i_{00} \) increases roughly proportionally to with \( De \), the demand on Fourier modes increases rapidly as \( De \) is increased. 7 Fourier modes have been found adequate to represent the solution up to \( De = 4.4 \) and 15 up to \( De = 6 \) which is lower than the values achieved at \( \varepsilon = 0.1 \) but still almost an order-of-magnitude higher than the limit point value \( De = 0.93 \) obtained with the finite element method (see Section 4.4.1).

Figure 6.5 shows the evolution of the radial profiles of the variables with \( De \) for \( De \leq 4 \). In Fig. 6.5a the transition from cubic to parabolic of the radial velocity profiles can again be seen. Figs 6.5b through 6.5d show the development of a boundary layer next to the outer wall in the stress profiles. Of particular interest is the evolution of the separation region with \( De \) shown in Fig. 6.6 where the azimuthal and radial extent of the separation region can be judged from the azimuthal and radial profiles of the azimuthal velocity, the region of negative azimuthal velocities corresponding to the returning half of the separation region. As Fig. 6.6 shows, the separation region shrinks uniformly in both radial and angular extent as \( De \) increases from 0 to 4.

Figure 6.7 shows several azimuthal profiles for the azimuthal normal stress \( i_{00} \) taken at different \( \xi \) positions for 5 different values of \( De \), for \( De \leq 4 \). As the recirculation region is approached and as \( De \) increases the profiles exhibit steeper azimuthal gradients. The steepest stress gradients occur next to the outer wall as Fig. 6.7c shows. At \( De = 4 \) these can be marginally resolved with the 7 Fourier modes of N2 discretization.

To calculate at higher \( De \) the finer N4 discretization with 15 Fourier modes was used. Solution was obtained up to \( De = 6 \) before encountering difficulties similar to those seen with N2 discretization at \( De = 4 \). In Figs 6.8 and 6.9 the solutions obtained with the two different discretizations are compared with each other at \( De = 4 \) and with the solution
Figure 6.5 Radial profiles for the VCM fluid at $e=m/2$ for $c=0.4$ and $D=0$.
Figure 6.6. Profiles for the dimensionless azimuthal velocity $v_{\theta} = v/\omega$ for $c=0.4$ and $\theta=0$, 1, 2, 3, 4, obtained with the spectral/finite element method with the UCM model, using discretization $N_{2}$.

(a) Azimuthal profile at $r=0.99$.

(b) Normalized radial distance $\zeta$. 

(0° 90° $\theta_{A}$)
Figure 6.7: Azimuthal profiles for the dimensionless azimuthal normal stress \( \theta (\theta - \theta_0)(\theta - \theta_0) \) for \( \theta = 0.4 \) and \( \theta = 0.9 \). Obtained with the spectral finite element method with the UCM model, using discretization N2.

(a) Azimuthal profile at \( \eta = 0.09 \).
(b) Azimuthal profile at \( \eta = 0.7 \).
(c) Azimuthal profile at \( \eta = 0.99 \).
Figure 6.7 (continued)
Figure 6.8 Azimuthal profiles for the UCM fluid at $\zeta=0.7$ for $\varepsilon=0.4$ and $\text{De}=4$ and 6 obtained with the spectral/finite element method using two discretizations N2 and N4. (a) Dimensionless azimuthal velocity $v_\theta^*=v_\theta/V$, (b) Dimensionless radial velocity $v_r^*=v_r/V$, (c) dimensionless radial normal stress $\tau_r^* = \tau_r/(\eta_0 \Omega)$, (d) dimensionless shear stress $\tau_\theta^* = \tau_\theta/(\eta_0 \Omega)$, (e) dimensionless azimuthal normal stress $\tau_\phi^* = \tau_\phi/(\eta_0 \Omega)$, and (f) determinant of $\tau^*+1/\text{De}\delta$. 
Figure 6.9 Radial profiles for the UCM fluid at $\theta = -\pi/2$ ($\theta = 0$ for $v_\theta^*$) for $\varepsilon = 0.4$ and $De = 4$ and 6 obtained with the spectral/finite element method using two discretizations $N2$ and $N4$. (a) Dimensionless azimuthal velocity $v_\theta^* = v_\theta/V$, (b) Dimensionless radial velocity $v_r^* = v_r/V$, (c) dimensionless radial normal stress $\tau_{rr}^* = \tau_{rr}/(\eta_0\Omega)$, (d) dimensionless shear stress $\tau_{r\theta}^* = \tau_{r\theta}/(\eta_0\Omega)$, (e) dimensionless azimuthal normal stress $\tau_{\theta\theta}^* = \tau_{\theta\theta}/(\eta_0\Omega)$ and (f) determinant of $\tau^* + 1/De6$. 
Figure 6.9 (continued)
obtained at $D_e=6$ with discretization $N4$. The two solutions at $D_e=4$ are almost indistinguishable. The only place where the improvement of the finer discretization $N4$ is apparent is in the azimuthal profile of the determinant of $I^{n+1}/D_e$ shown in Fig. 6.8f where the wiggles of the profile corresponding to $N2$ discretization disappear when $N4$ was used.

The wiggles reappear in the solution for $D_e=6$ calculated with $N4$ indicating that more refinement is necessary before a further increase in $D_e$ is attempted. As Figs 6.9b through 6.9d show, at $D_e=6$ the stress gradients next to the outer wall have increased substantially so that a refinement in the radial direction is also warranted for calculations at higher $D_e$.

6.4.3 High eccentricities ($\varepsilon=0.8$)

The difficulties in approximating the azimuthal dependence of the variables rapidly increase as the eccentricity is increased beyond $\varepsilon=0.4$. For $\varepsilon=0.8$ even 15 Fourier modes are inadequate to represent the steep changes in the azimuthal normal stress at $D_e=0.85$ with the consequence of destroying the quality of the azimuthal representation of the other variables too (see Fig. 6.10). Clearly, the demand on Fourier modes increases sharply as the eccentricity approaches unity and makes the computations very time consuming.

It is worthwhile to note that one iteration using mesh $N4$, with 15 Fourier modes, takes 125 cpu seconds on a Cray X-MP supercomputer. It is estimated an increase to 25 Fourier modes would result in 5-fold increase in cpu time requirements, the computational time being proportional to the cube of the number of Fourier modes. However, the cpu time increases only linearly with the number of radial elements making the use of very fine (radially) meshes as $N5$ entirely feasible. For a comparison between different computers capabilities Table 6.3 compares the speed and cost of the different computers which were used in this thesis.

The recirculation region at $\varepsilon=0.8$ extends for $\theta=-\pi/2$ and causes the radial profile of $v_r$ at $\theta=-\pi/2$ to be inverted (negative $v_r$—see Fig. 6.11b) as compared to similar profiles taken at $\varepsilon=0.1$ and 0.4 (see Figs. 6.1a and 6.5a). Also the recirculation region is responsible
Figure 6.10 Azimuthal profiles for the UCM fluid at $\zeta=0.7$ for $\varepsilon=0.8$ and $\text{De}=0$, 0.35 and 0.85 obtained with the spectral/finite element method using the discretization $N4$. (a) Dimensionless azimuthal velocity $v_\theta^*=v_\theta/V$, (b) dimensionless radial velocity $v_r^*=v_r/V$, (c) dimensionless radial normal stress $\tau_{rr}^*=\tau_{rr}/(\eta_0 \Omega)$, (d) dimensionless shear stress $\tau_{r\theta}^*=\tau_{r\theta}/(\eta_0 \Omega)$, (e) dimensionless azimuthal normal stress $\tau_{\theta\theta}^*=\tau_{\theta\theta}/(\eta_0 \Omega)$ and (f) determinant of $\tau^*+1/\text{De}\delta$. 
Figure 6.10 (continued)
Figure 6.10 (continued)
<table>
<thead>
<tr>
<th>Computers</th>
<th>Usage</th>
<th>Relative Speed (Multics = 1)</th>
<th>Cost for the equivalent of 1 h Multics Calculation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Texas Instruments TI-59</td>
<td>Calculator, exact solution at concentric cylinders</td>
<td>$10^{-4}$</td>
<td>0</td>
</tr>
<tr>
<td>PC-AT IBM Personal Computer</td>
<td>Word Processing Plotting (Graphics) communications with remote computers</td>
<td>0.05</td>
<td>0</td>
</tr>
<tr>
<td>Data General MV 4000</td>
<td>Plotting (Graphics) Integration of ODE's</td>
<td>0.17</td>
<td>$25</td>
</tr>
<tr>
<td>Data General MV 8000</td>
<td>Numerical Simulation</td>
<td>0.5</td>
<td>$8</td>
</tr>
<tr>
<td>Multics Honeywell 6180 at MIT</td>
<td>Numerical Simulation</td>
<td>1</td>
<td>$8 (night shift)</td>
</tr>
<tr>
<td>Cray - 1S at Los Alamos Scientific Laboratory</td>
<td>Numerical Simulation</td>
<td>60</td>
<td>$5 (night shift)</td>
</tr>
<tr>
<td>Cray - 1 with COS 1.12 (CYBER front end) at University of Minn</td>
<td>Numerical Simulation</td>
<td>90</td>
<td>$12</td>
</tr>
<tr>
<td>Cray X - MP (VAX front end) at Naval Research Laboratory</td>
<td>NumericalSimulation</td>
<td>120</td>
<td>$7</td>
</tr>
</tbody>
</table>
Figure 6.11 Radial profiles for the UCM fluid at $\theta=-\pi/2$ for $\varepsilon=0.8$ and $\text{De}=0$, 0.35 and 0.85 obtained with the spectral/finite element method using the discretization N4. (a) Dimensionless azimuthal velocity $v_\theta^* = v_\theta/V$, (b) dimensionless radial velocity $v_r^* = v_r/V$, (c) dimensionless radial normal stress $\tau_{rr}^* = \tau_{rr}/(\eta_0 \Omega)$, (d) dimensionless shear stress $\tau_{r\theta}^* = \tau_{r\theta}/(\eta_0 \Omega)$, (e) dimensionless azimuthal normal stress $\tau_{\theta\theta}^* = \tau_{\theta\theta}/(\eta_0 \Omega)$ and (f) determinant of $\tau^*+1/\text{De}$. 

233
for the abrupt decline in the stress values in the azimuthal (see Figs 6.10d and 6.10e) and radial profiles (see Fig. 6.11e). As seen in Fig. 6.11a the recirculation region again shrinks as De increases, but the calculations did not advance in high enough De values to see an appreciable difference.

6.5 Leonov-like with Retardation Model

Calculations with the LER model failed to produce results that converged with mesh refinement at ε=0.1 except at low De (De≤0.06), where the results duplicated equivalent finite-element calculations. Use of a graded towards the inner wall distribution of 20 radial elements with 4 Fourier modes (N1G mesh in Table 6.2) allowed solutions to be calculated up to high De values (calculations advanced up to De=7 with no convergence problems). However, use of N5 mesh resulted in a singular Jacobian at De=0.06 with no apparent reason: the calculated solution was completely smooth. It is of interest to report here the results obtained with the N1G mesh, because the solution exhibited a dramatic change around De=0.5 which might hinder the causes for failure of both the finite-element and the spectral/finite element methods when the finest meshes were used.

The solution obtained with mesh N1G, exhibited very small azimuthal variations for all the variables, except in a small region around De=0.5 where the pressure rearranged itself its maximum being moved from θ=90° to θ=200° as Fig. 6.12 shows. This rearrangement of the pressure translated into an abrupt transition of the direction of the loads on the inner cylinder as discussed in Section 6.6.2. Because of these mild azimuthal variations the solution appeared insensitive in azimuthal refinement; when three additional Fourier modes were used the results remained the same.

The radial variations of the variables were quite mild too, except of the velocities and the azimuthal normal stress which exhibited a boundary layer next to the inner cylinder surface (see Fig. 6.13j) which is attributed to the shear thinning properties of the LER model (see Section 2.2). The transition region near De=0.5 coincided with the steepest gradients for vθ and τθθ shown in Figs. 6.13a and 6.13e. Comparison with corresponding finite element calculations obtained with intermediate
Figure 6.12 Azimuthal profiles for the LER fluid at $\zeta=0$ for $\epsilon=0.1$ and $De=0.29$, $0.5$, $1.5$ and $7$ obtained with the spectral/finite element method using the discretization N1G. (a) Dimensionless shear stress $\tau^*_\theta=\tau_{\theta}/(\eta_0\Omega)$, (b) dimensionless azimuthal normal stress $\tau^*_\theta=\tau_{\theta}/(\eta_0\Omega)$, (c) dimensionless pressure $p^*=p/(\eta_0\Omega)$, and (d) determinant of $\tau^*+1/De\delta$. 

237
Figure 6.13 Radial profiles for the LER fluid at $\theta = \pi/2$ for $\varepsilon = 0.1$ and $D = 0.29, 0.5, 1.5$ and $7$ obtained with the spectral/finite element method using the discretization N1G. (a) Dimensionless azimuthal velocity $v_\phi^* = v_\phi/V$, (b) dimensionless radial velocity $v_r^* = v_r/V$, (c) dimensionless radial normal stress $r_{rr}^* = r_{rr}/(\eta_0\Omega)$, (d) dimensionless shear stress $r_{r\theta}^* = r_{r\theta}/(\eta_0\Omega)$, (e) dimensionless azimuthal normal stress $\tau_{\theta\theta}^* = \tau_{\theta\theta}/(\eta_0\Omega)$ and (f) determinant of $\tau^* + 1/De$. 
Figure 6.13 (continued)
mesh M2 (see Section 5.3.4) showed that the predictions of the two methods start diverging themselves right before the transition region \((\text{De}=0.29)\) in the S/FEM, which is totally absent from the finite-element calculations. The divergence of the results is most evident in the calculation of the loads discussed below. The disagreement between the two methods is thought to be due to the inability of the finite-element mesh M2 with only 5 radial elements to resolve well the steep gradients on the velocity and the stress. However, the issue is not definitely settled given the fact that both methods failed to converge for \(\text{De} \) higher than 0.11 when finer meshes were used.

An additional point of interest is the almost (within 1%) constancy of the determinant \(1^*+1/\text{De}^*\) in the whole flow region (see Figs 6.12c and 6.13f). Also the determinant remained always positive, with decreasing value as \(\text{De} \) was increased. Whether this is a feature of creeping flows with the LER model, as happens with the UCM model, is not known since a proof similar to the one holding for the UCM fluid (Marchal and Crochet 1984) is not available.

6.6 Comparison of the Loads on the Inner Cylinder

The \(x\) and \(y\) components of the load on the inner cylinder per unit length (see Figure 1.2), made dimensionless by \(\eta_0 V\), are given in terms of the dimensionless pressure \(p^*=p_0a/\eta_0 V\), and the components of the dimensionless modified extra stress \(T^*\) in the bipolar coordinate system shown in Figure 1.3 (with the superscripts * omitted for simplicity) as

\[
F_x = \int_0^{2\pi} \left\{ \frac{\sin \theta \sinh \xi}{X} T_{\xi \theta} + \frac{(1 + \cos \theta \cosh \xi)}{X} \left( T_{\xi \xi} - \frac{P}{X} \right) \right\}_\xi d\theta, (6-6)
\]

\[
F_y = \int_0^{2\pi} \left\{ \frac{(1 + \cos \theta \cosh \xi)}{X} T_{\xi \theta} - \frac{\sin \theta \sinh \xi}{X} \left( T_{\xi \xi} - \frac{P}{X} \right) \right\}_\xi d\theta, (6-7)
\]

where \(X=\cos \theta + \cosh \xi\), \(\phi\) is the directional angle of the load, \(\phi=\tan^{-1}(F_y/F_x)\), and \(F=|F|\) is the magnitude of the load. After the the pressure and the stress components have been substituted by their truncated Fourier expansions, eqs.(6-2) and (6-3), the resulting azimuthal integrals can
be evaluated exactly using the formulas provided in Appendix K, obtaining a truncated Fourier series expression for $F$.

6.6.1 Upper convected Maxwell fluid

The results with the UCM model and all the methods used at $\varepsilon=0.1$ are summarized in Fig. 6.14. Here predictions of the magnitude and the direction of the loads on the inner cylinder from the perturbation analysis and the numerical methods, using different mesh sizes, are compared as a function of the Deborah number. Figures 6.14a and 6.14c, show that the predictions of the perturbation analysis are within 10% from the results obtained with the finest discretization (N5) of the spectral/finite-element method up to $De=100$. Figures 6.14b and 6.14d show that the FEM and S/FEM methods converge to the same solution with mesh refinement before the limit point predicted by the finite element method.

As Figs 6.14a and 6.14b show, all the reported solutions predict the magnitude of the loads after a small decrease at $De=1$ to increase approximately proportionally with $De$ as predicted from the perturbation analysis (see 3.2.7). The most reliable answer for the direction of the loads $\phi$ seems to come from the domain perturbation analysis, predicting the direction to monotonically decrease with increasing $De$, from $-90^\circ$ corresponding to the Newtonian limit, to $0^\circ$ which is the limit at infinite coming from the asymptotic analysis presented in Section 3.2.2c. (see Figs. 6.14c and 6.14d). Deviations from this behavior that are observed at high $De$ values with the S/FEM seem to be caused by approximation error since the finer the discretization is used the smaller the observed deviations from the perturbation analysis predictions are (see Fig. 6.14c).

The predictions from all the methods used, for the magnitude and the direction of the loads with the UCM model at $\varepsilon=0.4$ are compared in Fig. 6.15 as a function of $De$ for $0\leq De \leq 5$. The results for the S/FEM for the two discretizations N2 and N4 are practically identical up to $De=4$ and again coincide with the FEM results for $De$ values before the limit point reached by the FEM solution ($De=0.93$). The perturbation analysis results while qualitatively correct, deviate significantly from the converged S/FEM results at $De=4$ especially in the prediction
Figure 6.14 Comparison of the magnitude and direction of the load per unit length on the inner cylinder for UCM fluid with $\varepsilon=0.1$ and $\mu=0.1$ between results obtained with the spectral/finite element, finite element and domain perturbation methods. The load per unit length has been made dimensionless with respect to $nV$. (a) Dependence of the magnitude of the loads on $De$, $0<De<100$, (b) detail of (a) in the $0<De<5$ region, (c) dependence of the direction of the loads on $De$, $0<De<100$, (d) detail of (c) in the $0<De<5$ region.
Figure 6.14 (continued)
Figure 6.14 (continued)
Figure 6.14 (continued)
Figure 6.15 Comparison of the magnitude and direction of the load per unit length on the inner cylinder for UCM fluid with \( \varepsilon=0.4 \) and \( \mu=0.1 \) between results obtained with the spectral/finite element, finite element and domain perturbation methods. The load per unit length has been made dimensionless with respect to \( nV \). (a) Dependence of the magnitude of the loads on \( De, 0<De<5 \), (b) dependence of the direction of the loads on \( De, 0<De<5 \).
of the magnitude of the loads which they underpredict by 50%. This is not surprising giving the magnitude of ε (ε=0.4) which is too large for the domain perturbation analysis to be expected to give quantitatively correct answers.

6.6.2 Leonov-like model with retardation time

The predictions from the FEM and S/FEM methods for the magnitude and the direction of the loads with the LER model at ε=0.4 are compared in Fig. 6.15 as a function of De for 0<De≤5. The relatively coarse meshes M2 and N1G were used in the FEM and S/FEM respectively since the finer meshes did not converge beyond De=0.1 up to which value their predictions were almost identical with that for the coarser meshes. The results for the S/FEM show an abrupt transition (see also Section 6.5) occurring at De=0.29 through 0.5 during which the magnitude of the loads increase and the direction of the loads changes from -100° to 20°. Both methods agree very well for De≤0.1 and show the same asymptotic behavior for large De values.

6.7 Discussion of the Spectral/Finite-Element Results

The spectral/finite-element method substantially extended previous finite-element calculations with the UCM model to higher De values. The improvement was felt more at ε=0.1 where a 30-fold gain in De was achieved; at ε=0.4 the range of results was extended 7-fold. Calculations at high ε remain a challenge for De≥1. The method’s greatest advantage is that it is stable against azimuthal oscillations. The resolution in the azimuthal direction is only limited by the available computational power which dictates how many Fourier modes can be included in the calculations. Because of the structure of the set of resulting algebraic equations the computational work increases proportionally to the cube of the included number of Fourier modes but only proportionally to the number of radial elements. This is the reason of the high success of the method at low and moderate values of ε where a few Fourier modes are adequate to describe the azimuthal variation of the variables, still, the methods allows by using a large number of radial elements (see mesh
Figure 6.16 Comparison of the magnitude and direction of the load per unit length on the inner cylinder for the LER fluid with $\varepsilon=0.1$ and $\nu=0.1$ between results obtained with the spectral/finite element and finite element methods. The load per unit length has been made dimensionless with respect to $n\nu$. (a) Dependence of the magnitude of the loads on $De$, $0 \leq De \leq 5$, (b) dependence of the direction of the loads on $De$, $0 \leq De \leq 5$. 
N5) to resolve steep radial boundary layers developing at high De values. On the other hand, the method using a finite element representation in the radial direction, becomes unstable at very high De values developing radial oscillations in the stresses.

The key results of the S/FEM calculations with the UCM fluid were (a) all the methods converge to a unique solution for small De values, and (b) the finite element method, although very accurate even with the intermediate mesh M2, at De<3, can not reach higher De values without developing a limit point which therefore must be attributed to the instabilities seen previously at lower De in the form of azimuthal oscillations and spurious bifurcating solutions. These instabilities are also present in the spectral/finite element method because of its mixed nature, but, at least for small and moderate eccentricities, develop for much larger De values and in the radial direction (see Fig. 24).

The main result of the application of the S/FEM with the LER model was the failure of the method to converge with mesh refinement even at ε=0.1 and whereas the results obtained with the coarse mesh N1G were, in general, quite benign. The FEM results also exhibited instabilities when the finest mesh N4 was used. Furthermore, results obtained with the coarser meshes with the two methods strongly disagreed for De>0.2. These results shed doubt on the reliability of the obtained solutions using the LER model, even when they do not exhibit any sort of oscillations and wiggles. It remains for future work to establish the causes of this discrepancy.
7. CONCLUSIONS

The major conclusions concerning the use of numerical methods in the calculation of viscoelastic flows with the UCM model are (a) all the methods converge to a unique solution for small De values, (b) the finite element method is the most accurate method giving reliable results even with the intermediate mesh M2 for small De values, but cannot reach higher De values without developing a limit point which is therefore attributed to the instabilities seen previously at lower De in the form of azimuthal oscillations and spurious bifurcating solutions, (c) the S/FEM method is a numerically stable method and powerful enough to give accurate solutions when not many Fourier modes are necessary to describe the azimuthal dependence of the variables. Because of its finite-element component it also develops instabilities, in the radial direction, but, at least for small and moderate eccentricities, they develop for much larger De values.

The above represents the major conclusion of this thesis, partially fulfilling its primary objective: the detection and cure of the cause of the failure of the numerical simulations of highly elastic viscoelastic flows. It is a partial fulfilment because the spectral/finite element method does not represent the ultimate numerical method. Although clearly better than the regular finite element method it has two serious drawbacks: First, it is applicable only to spatially periodic problems, and second—which is more important—by using finite element basis functions in one direction, it becomes unstable at high De. Nevertheless, it represents a step towards a more accurate evaluation of the derivatives and elimination of dispersion errors. If we therefore accept that the success of the spectral/finite element method is due to the accurate representation of the azimuthal derivatives by the Fourier series, we can easily—at least in principle—extend the method to a fully spectral one, and expect to achieve better results. Of course, this would imply a change from a steady-state to a time-integration approach since for reasons already explained, the Newton method used in this thesis to find the steady-state solution is intractable with the spectral methods because the Jacobian matrix loses its banded structure.

Another conclusion was that the perturbation methods yield results
valid well beyond any reasonable expectation. This is a conclusion easily reached after just a quick look at Figure 6.14, from which we see the perturbation results are valid up to very high De values, although the analysis was done on the assumption that De was of order one in magnitude. Given all the difficulties and the cost associated with the numerical results the perturbation solutions offer the best return on investment.

Other conclusions, drawn from the problem investigated here, are:

Even in near viscometric flows bounded by a smooth boundary steep gradients develop in the stresses near the walls as De increases for at least some of the models used here (UCM). Moreover, sometimes these regions with high gradients are not located next to the boundaries (as in the case of steep azimuthal changes in \( T_{0\theta} \) at high \( \epsilon \)) which helps to explain the difficulty of the calculations. These features are highly model dependent and show the danger of extrapolating viscometric results even to nearly viscometric situations.

The calculations with the LEH model showed that convergence with mesh refinement is the best criterion for testing the reliability of a numerical solution and not the smoothness of the obtained solution. The failure to obtain a converged solution with either FEM or S/FEM at the low eccentricity of \( \epsilon = 0.1 \) is disturbing, and a matter which warrants future investigation before using the LEH model in any other flow problems. The use of a different finite element basis function for the stress approximations might improve the performance of the S/FEM although the problem might be in the LEH model and not with the numerical methods.
REFERENCES


Abel, V.V. 1963 The integration of harmonic and bi-harmonic equations in curvilinear coordinates. Translation from Inzhenerny Zhurnal, No 1, USSR.


Lawler, J.V. 1984 Personal communication.


LIST OF SYMBOLS

Symbols appearing only once in one section are generally not included in this list. The equation, figure, table or section references give the locations where the symbols are first used or where they are defined.

Italic and Roman Symbols

a Parameter in bipolar coordinates Fig. 1.3
A Modified "mobility factor" in Giesekus model Eq. 2-24
a' Multiplicative factor in upwind finite-element methods Eq. 5-7
b Parameter vector Eq. 4-10
b Configuration tensor Eq. 2-13
De Deborah number, = \lambda/t (general definition) Sec. 1-1
= \lambda t \Omega (for the journal bearing flow) Eq. 4-9
De' Modified Deborah number in bipolar coordinates Eq. 5-59
e Distance between the axes of the cylinders Fig. 1.2
e_1 Unit vector Eq. 4-6
f Driving force Eq. 2-8
F Complex stream function Eq. 3-31
F Force per unit length on inner cylinder Eq. 5-1
G Modulus of elasticity Eq. 2-17
G Modified complex stream function Eq. 3-84
h Mesh spacing Eq. 5-4
J Jacobian matrix Eq. 4-11
k Diffusivity Sec. 5.2.3
k' Balancing diffusivity Eq. 5-6
L Characteristic length Eq. 5-4
m Constant of proportionality in power-law viscosity model Eq. 2-29
\textit{m} Mobility tensor Eq. 2-7
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>Mass matrix</td>
<td>Eq. 4-15</td>
</tr>
<tr>
<td>n</td>
<td>Power law index in power-law viscosity model</td>
<td>Sec. 2.1</td>
</tr>
<tr>
<td>n</td>
<td>Number density of beads in Giesekeus model</td>
<td>Eq. 2-16</td>
</tr>
<tr>
<td>N₁</td>
<td>First normal stress difference</td>
<td>Eq. 2-2a</td>
</tr>
<tr>
<td>N₂</td>
<td>Second normal stress difference</td>
<td>Eq. 2-2b</td>
</tr>
<tr>
<td>p</td>
<td>Pressure</td>
<td>Eq. 2-32</td>
</tr>
<tr>
<td>Pe</td>
<td>Peclet number, (-VL/k)</td>
<td>Eq. 5-3</td>
</tr>
<tr>
<td>Pe'</td>
<td>Elemental Peclet number, (-Vh/k)</td>
<td>Eq. 5-4</td>
</tr>
<tr>
<td>Q</td>
<td>Flowrate per unit length in journal bearing</td>
<td>Eq. 2-61</td>
</tr>
<tr>
<td>r</td>
<td>Radial coordinate in cylindrical coordinates</td>
<td>Fig. 1.2</td>
</tr>
<tr>
<td>r₁(θ)</td>
<td>Radial position of the outer cylinder</td>
<td>Eq. 1-3</td>
</tr>
<tr>
<td>R</td>
<td>Radius of inner cylinder</td>
<td>Fig. 1.2</td>
</tr>
<tr>
<td>R₀</td>
<td>Radius of outer cylinder</td>
<td>Fig. 1.2</td>
</tr>
<tr>
<td>R(x,b)</td>
<td>Residual vector</td>
<td>Eq. 4-10</td>
</tr>
<tr>
<td>R₁</td>
<td>Equilibrium configuration of beads of 1-kind</td>
<td>Eq. 2-12</td>
</tr>
<tr>
<td>s</td>
<td>Elastic stress tensor</td>
<td>Eq. 4-5</td>
</tr>
<tr>
<td>S</td>
<td>Viscoelastic extra stress (except for Chapter 3)</td>
<td>Eqs. 2-10,22</td>
</tr>
<tr>
<td>S</td>
<td>Complex extra stress (only for Chapter 3)</td>
<td>Eqs. 3-33</td>
</tr>
<tr>
<td>Sᵣ</td>
<td>Recoverable shear, (-N₁/(2t))</td>
<td>Fig. 2-3</td>
</tr>
<tr>
<td>t</td>
<td>Time</td>
<td>Sec. 1.1</td>
</tr>
<tr>
<td>t</td>
<td>Modified extra stress tensor in bipolar coordinates</td>
<td>Eq. 2-50</td>
</tr>
<tr>
<td>T</td>
<td>Modified viscoelastic stress tensor in bipolar coordinates</td>
<td>Eq. 2-52</td>
</tr>
<tr>
<td>U</td>
<td>Potential function</td>
<td>Sec. 2.2</td>
</tr>
<tr>
<td>u</td>
<td>Radial velocity</td>
<td>Eq. 3-10</td>
</tr>
<tr>
<td>v</td>
<td>Azimuthal velocity</td>
<td>Eq. 3-10</td>
</tr>
<tr>
<td>v</td>
<td>Velocity vector</td>
<td>Eq. 2-4</td>
</tr>
<tr>
<td>V</td>
<td>Linear velocity at the surface of inner cylinder</td>
<td>Sec. 1.2</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition and Context</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>------------------------</td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>Cartesian coordinate</td>
<td>Fig. 1.3</td>
</tr>
<tr>
<td>X</td>
<td>Vector of unknowns in finite-element formulation</td>
<td>Eq. 4-10</td>
</tr>
<tr>
<td>X</td>
<td>Parameter in bipolar coordinates</td>
<td>Eq. 2-46</td>
</tr>
<tr>
<td>y</td>
<td>Cartesian coordinate</td>
<td>Fig. 1.3</td>
</tr>
</tbody>
</table>

**Greek Symbols**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition and Context</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>Parameter in Giesekus equation (mobility factor)</td>
</tr>
<tr>
<td>( \beta )</td>
<td>Relative mobility tensor</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>Magnitude of rate of strain tensor</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>Rate of strain tensor</td>
</tr>
<tr>
<td>( \delta )</td>
<td>Average gap width, ( = R-R_0 )</td>
</tr>
<tr>
<td>( \delta )</td>
<td>Unit tensor</td>
</tr>
<tr>
<td>( \epsilon )</td>
<td>Eccentricity parameter</td>
</tr>
<tr>
<td>( \zeta )</td>
<td>Normalized radial coordinate</td>
</tr>
<tr>
<td>( \zeta )</td>
<td>Tensorial drag coefficient</td>
</tr>
<tr>
<td>( \eta )</td>
<td>Viscosity</td>
</tr>
<tr>
<td>( \eta_T )</td>
<td>Trouton viscosity</td>
</tr>
<tr>
<td>( \eta_s )</td>
<td>Solvent viscosity</td>
</tr>
<tr>
<td>( \eta^* )</td>
<td>Dimensionless solvent viscosity</td>
</tr>
<tr>
<td>( \theta )</td>
<td>Angular coordinate in cylindrical ((r,\theta)) or bipolar ((\xi,\theta)) coordinate system</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>Characteristic time constant for the fluid</td>
</tr>
<tr>
<td>( \lambda_1 )</td>
<td>Relaxation time</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>Retardation time</td>
</tr>
<tr>
<td>( \mu )</td>
<td>Dimensionless gap thickness</td>
</tr>
<tr>
<td>( \xi )</td>
<td>Transformed radial coordinate in domain perturbation</td>
</tr>
<tr>
<td>( \pi )</td>
<td>3.14159...</td>
</tr>
<tr>
<td>( \pi_0 )</td>
<td>Complex pressure</td>
</tr>
</tbody>
</table>
\( \sigma \) Stress tensor  
\( \sigma_1 \) Eigenvalue  
\( \tau \) Magnitude of the extra stress tensor  
\( \tau \) Extra stress tensor  
\( \phi \) Angular direction of \( F \)  
\( \phi^1 \) Lagrangian quadratic basis function  
\( \phi^k \) Lagrangian biquadratic basis function  
\( \psi \) Modified stream function in bipolar coordinates  
\( \psi_1 \) First normal stress coefficient  
\( \psi_2 \) Second normal stress coefficient  
\( \psi \) Stream function  
\( \psi^1 \) Lagrangian bilinear basis function  
\( \Omega \) Angular velocity of the inner cylinder

Mathematical Operations

\( \nabla \) "Del" operator  
\( \nabla^2 \) Biharmonic operator  
\( \frac{D}{Dt} \) Substantial or material derivative  
\( (1) \) Upper convected derivative

Superscripts

\( \dagger \) Transpose of a matrix or a tensor  
\( ^0 \) Equilibrium or reference value  
\( * \) Dimensionless quantity  
\( [k] \) \( k \)-th order in domain perturbation expansion

Subscripts

\( i \) Index of bead in a chain
o  Zero shear stress value  
(1) Upper convected derivative

Abbreviations
CEF  Criminale-Ericksen-Filbey model  
CML  Corotational Maxwell-like model  
FEM  finite-element method  
LEL  Leonov-like model  
LER  Leonov-like with retardation model  
OLD  Oldroyd-B fluid  
S/FEM  Spectral/finite-element method  
SOF  Second order fluid  
UCM  Upper convected Maxwell model  
WM  White-Metzner model
APPENDIX A
Components of the Residual Equations
in Cylindrical Coordinates

In a cylindrical coordinate system, the z-axis of which coincides with the axis of the inner cylinder (see Figure 1.2), the components of the Galerkin residual equations for a two-dimensional planar flow, corresponding to the week formulation of the integral equations 4-6, are expressed in dimensional form as

\[ \int \psi^1 \left[ \frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{1}{r} \frac{\partial}{\partial \theta} v_\theta \right] r \, dr \, d\theta = 0 , \quad (A-1) \]

\[ \int \phi^k \left[ -\frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} (rs_{rr}) + \frac{1}{r} \frac{\partial}{\partial \theta} s_{r\theta} - \frac{1}{r} s_{\theta\theta} - (1+n^*) \left\{ \frac{2}{r^2} \frac{\partial}{\partial \theta} v_\theta 
\right. \right.
\]
\[ \left. + \frac{1}{r^2} \frac{\partial}{\partial r} (rv_r) \right] r \, dr \, d\theta - (1+n^*) \int \frac{\partial \phi^k}{\partial r} \frac{\partial}{\partial r} (rv_r) \, dr \, d\theta 
\]
\[ - (1+n^*) \int \frac{1}{r} \frac{\partial \phi^k}{\partial \theta} \frac{\partial}{\partial \theta} v_\theta \, dr \, d\theta = 0 , \quad (A-2) \]

\[ \int \phi^k \left[ -\frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 s_{r\theta}) + \frac{1}{r} \frac{\partial}{\partial \theta} s_{\theta\theta} + (1+n^*) \left\{ \frac{2}{r^2} \frac{\partial}{\partial \theta} v_\theta 
\right. \right.
\]
\[ \left. + \frac{1}{r^2} \frac{\partial}{\partial r} (rv_r) \right] r \, dr \, d\theta - (1+n^*) \int \frac{\partial \phi^k}{\partial r} \frac{\partial}{\partial r} (rv_r) \, dr \, d\theta 
\]
\[ - (1+n^*) \int \frac{1}{r} \frac{\partial \phi^k}{\partial \theta} \frac{\partial}{\partial \theta} v_\theta \, dr \, d\theta = 0 , \quad (A-3) \]

\[ \int \phi^k \left[ s_{rr} + \lambda^* \left( v_r \frac{\partial}{\partial r} s_{rr} + \frac{1}{r} v_\theta \frac{\partial}{\partial \theta} s_{r\theta} - 2 (s_{rr} \frac{\partial}{\partial r} v_r + \frac{1}{r} s_{r\theta} \frac{\partial}{\partial \theta} v_\theta) \right) 
\]
\[ + \alpha \left[ \left( s_{rr} + 2 \frac{\partial}{\partial r} v_r \right)^2 + \left( s_{r\theta} + \frac{\partial}{\partial r} v_\theta \right)^2 + 2 \frac{\partial}{\partial r} v_r \frac{\partial}{\partial \theta} v_\theta \right] \right) 
\]
\[ - (\lambda^*_1 + \lambda^*_2 \left( \frac{2}{r} \frac{\partial}{\partial r} v_r \left[ \frac{\partial}{\partial r} (r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta} v_\theta \right] + 4 \left( \frac{\partial}{\partial r} v_r \right)^2 \right) \right] r \, dr \, d\theta - 
\]
\[ 2(\lambda^*_1 + \lambda^*_2) \int \frac{\partial \phi^k}{\partial r} v_r \frac{\partial}{\partial r} v_r \, dr \, d\theta - 
\]
\[ 2(\lambda^*_1 + \lambda^*_2) \int \frac{\partial \phi^k}{\partial \theta} v_\theta \frac{\partial}{\partial \theta} v_\theta \, dr \, d\theta = 0 , \quad (A-4) \]
\[ \int J^k \{ s_{r\theta} + \lambda^*_1 \left[ v_r \frac{\partial}{\partial r} s_{r\theta} + \frac{1}{r} v_\theta \frac{\partial}{\partial \theta} s_{r\theta} + \left( \frac{1}{r} v_\theta - \frac{\partial}{\partial r} v_r \right) s_{rr} - \right. \\
\left. \frac{1}{r} \left( \frac{\partial}{\partial \theta} v_r \right) s_{r\theta} + \alpha \left( s_{rr} + s_{\theta\theta} \right) \left( s_{r\theta} + r \frac{\partial}{\partial r} \left( \frac{1}{r} v_\theta \right) + \frac{1}{r} \frac{\partial}{\partial \theta} v_r \right) \right] \} r dr d\theta \\
\left( \lambda^*_1 + \lambda^*_2 \right) \left[ \frac{3}{r} (\frac{\partial}{\partial r} v_r + \frac{\partial}{\partial \theta} v_\theta) - 2 \frac{1}{r^2} (\frac{\partial}{\partial \theta} v_\theta + v_r \frac{\partial}{\partial r} v_r) \right] r dr d\theta - \\
\left( \lambda^*_1 + \lambda^*_2 \right) \int \frac{\partial}{\partial \theta} k v_\theta \left( r \frac{\partial}{\partial r} \left( \frac{1}{r} v_\theta \right) + \frac{1}{r} \frac{\partial}{\partial \theta} v_r \right) r dr d\theta = 0 \ , \quad (A-5) \]

\[ \int J^k \{ s_{r\theta} + \lambda^*_1 \left[ v_r \frac{\partial}{\partial r} s_{r\theta} + \frac{1}{r} v_\theta \frac{\partial}{\partial \theta} s_{r\theta} - 2 s_{r\theta} \right] \frac{\partial}{\partial r} \left( \frac{1}{r} v_\theta \right) - \\
2 s_{r\theta} \frac{1}{r} \left( \frac{\partial}{\partial \theta} v_\theta + v_r \right) + \alpha \left( s_{r\theta} + \frac{2}{r} \left( \frac{\partial}{\partial \theta} v_\theta + v_r \right) \right)^2 + \\
\left[ s_{r\theta} + r \frac{\partial}{\partial r} \left( \frac{1}{r} v_\theta \right) + \frac{1}{r} \frac{\partial}{\partial \theta} v_r \right] \} - 2(\lambda^*_1 + \lambda^*_2) \cdot \\
\left[ r \frac{\partial}{\partial r} \left( \frac{1}{r} v_\theta \right) r \frac{\partial}{\partial r} \left( \frac{1}{r} v_\theta \right) + \frac{1}{r} \frac{\partial}{\partial \theta} v_r \right] - 4 \frac{1}{r^2} \left( \frac{\partial}{\partial \theta} v_\theta + v_r \right)^2 \right] r dr d\theta - \\
- 2(\lambda^*_1 + \lambda^*_2) \int \frac{\partial}{\partial r} k v_\theta \left( \frac{\partial}{\partial \theta} v_\theta + v_r \right) r dr d\theta - \\
2(\lambda^*_1 + \lambda^*_2) \int \frac{\partial}{\partial \theta} k v_\theta \frac{1}{r} \left( \frac{\partial}{\partial \theta} v_\theta + v_r \right) r dr d\theta = 0 \ . \quad (A-6) \]

The equations are written in terms of the dimensionless variables (with the superscript * omitted for the sake of convenience): \( v_\theta \) and \( v_r \) represent the azimuthal and radial components of the velocity respectively, and \( s_{r\theta} \), \( s_{rr} \), and \( s_{r\theta} \) represent the azimuthal normal stress, radial normal stress and shear stress components of the elastic part of the stress (see Equation 4-5).
APPENDIX B

Solution for the Flow of a Newtonian Fluid
in a Journal Bearing
by Kamal (1966)

The modified stream function \( \psi^* \), solution of the equation 3-2, subject to the boundary conditions specified by the Equations 2-60 to 2-62, as reported by Kamal (1966)—corrected for typos and adjusted to the bipolar coordinate system used here—is given as

\[
\psi^* = A_0 \sinh \xi + B_0 \cosh \xi + \left( D_0 \sinh \xi + E_0 \cosh \xi \right) \xi + Q^* \\
\left[ A_1 + B_1 \xi + D_1 \sinh(2\xi) + E_1 \cosh(2\xi) \right] \cos \phi,
\]

where

\[
A_0 = \left( Q H_0 - H_0 \right) / F_1 \\
B_0 = \left( Q H_7 - H_7 \right) / F_1 \\
D_0 = \sinh^2(\xi_1 - \xi_2) \sinh(\xi_2) / F_2 \\
E_0 = Q \cosh(\xi_1 - \xi_2) / F_3 \\
A_1 = -Q \left[ 2 \xi_1 \cosh(\xi_1 - \xi_2) - \sinh(\xi_1 - \xi_2) \right] / (2F_3) \\
B_1 = E_0 \\
D_1 = Q \cosh(\xi_1 + \xi_2) / (2F_3) \\
E_1 = Q \sinh(\xi_1 + \xi_2) / (2F_3) \\
Q^* = \frac{F_3 H_2}{F_2} \\
F_1 = (\xi_1 - \xi_2)^2 \sinh^2(\xi_1 - \xi_2) \\
F_2 = \sinh(\xi_1 - \xi_2) \left[ 2 \sinh \xi_1 \sinh \xi_2 \sinh(\xi_1 - \xi_2) - (\xi_1 - \xi_2)(\sinh^2 \xi_1 - \sinh^2 \xi_2) \right] \\
F_3 = -\sinh(\xi_1 - \xi_2) + (\xi_1 - \xi_2) \cosh(\xi_1 - \xi_2) \\
H_2 = (\xi_1 - \xi_2) \sinh \xi_1 - \sinh \xi_2 \sinh(\xi_1 - \xi_2) \\
H_4 = -\xi_2 \sinh \xi_2 \sinh(\xi_1 - \xi_2) + \xi_2 (\xi_1 - \xi_2) \sinh \xi_1
\]
\[
H_6 = \xi_2 (\xi_1 - \xi_2) \cosh \xi_1 - \xi_1 \cosh \xi_2 \sinh (\xi_1 - \xi_2) \\
H_7 = \xi_1 (\xi_1 - \xi_2) - \cosh \xi_2 \sinh \xi_1 \sinh (\xi_1 - \xi_2) + \xi_1 \sinh \xi_1 \cosh \xi_1 - \xi_1 \sinh \xi_2 \cosh \xi_2 \\
H_8 = -\cosh \xi_1 \cosh \xi_2 \sinh (\xi_1 - \xi_2) - \xi_1 \cosh \xi_2 + \xi_2 \cosh \xi_1 
\]

and where \( \xi_1 \) and \( \xi_2 \) define the coordinates of the inner and outer cylinder surfaces respectively and are given in the caption of Figure 1.3.
APPENDIX C
Correspondence between the Domain Perturbation and the Phan-Thien and Tanner (1981) solution for the CEF Equation.

Phan-Thien and Tanner (1981), henceforth PTT, solved the small eccentricity journal bearing problem for the general constitutive CEF model:

\[ I = n A^2 + (v_1 + v_2) A^2 - \frac{1}{2} v_1 B \]  \hspace{1cm} (C-1)

where \( A \) and \( B \) are the first two Rivlin-Erickson tensors,

\[ A = \dot{Y} \]  \hspace{1cm} (C-2)
\[ B = \frac{D}{dt} \dot{Y} + \dot{v} \nabla^T \dot{Y} - \dot{Y} \dot{v} \]  \hspace{1cm} (C-3)

\( U \) is the velocity vector, \( \dot{Y} \) the rate of strain tensor and \( n, v_1, v_2 \) are material functions; all are functions of the shear rate \( \dot{\gamma} = \sqrt{\frac{1}{2} \dot{Y} \dot{Y}^T} \).

Following exactly the same approach as the domain perturbation method outlined in Chapter 3. Although in a much less formal manner, they were able to arrive at a fourth order equation (Equation (28) of PTT) for the complex strain function \( \hat{F}(\xi)(=f(z) \text{ in PTT notation}) \) which, corrected for some typos, can be rewritten in the notation used in Chapter 3 as

\[ [1 + N - \frac{1}{2} 16(1-\xi)] \hat{F}^{(iv)} - iM \hat{F}^{(iv)} = 0 \]  \hspace{1cm} (C-4)

where

\[ N = \left( \frac{d \ln n}{d \ln Y} \right)_{\dot{Y}=\hat{Y}_0} \quad M = \left( \frac{d \ln v_1}{d \ln Y} \right)_{\dot{Y}=\hat{Y}_0} \]

\[ \theta = \frac{\delta v_1 \dot{Y}_0}{n R_0} = \frac{v_1 V}{n R_0} \]  \hspace{1cm} (C-5)

Their solution, (Equations (30) - (34) of PTT), again corrected for typos, can be rewritten in the notation used in Chapter 3, (for \( M = -1/2, -1 \) and \(-3/2\)), as
\[ f = a[1 + N - \frac{1}{2}\iota(1 - \xi)]^{2M+3} + \frac{1}{2} \beta \xi^2 + \gamma \xi + \delta \]  
(C-6)

where

\[ a = \left( (1 + N) \frac{1}{2} 1^0 \right)^{2M+2} \left( 2(1 + N) + \frac{1}{2} 1^0(2M + 1) \right) \]

\[ - (1 + N)^{2M+2} \left( 2(1 + N) - \frac{1}{2} 1^0(2M + 3) \right)^{-1} \cdot \]

\[ \beta = 2a((1 + N - \frac{1}{2} 1^0)^{2M+2}[1 + N + (M + 1)1^0] - (1 + N)^{2M+3}) \]

\[ \gamma = -\frac{1}{2} 1^0a(2M + 3)(1 + N - \frac{1}{2} 1^0)^{2M+2} \]

\[ \delta = a(1 + N - \frac{1}{2} 1^0)^{2M+3} \]  
(C-8)

Using the identity,

\[ \dot{Y}(1) = B - 2\gamma Y \cdot Y \]  
(C-9)

the CEF model used in Chapter 3 can be written as

\[ \tau = \eta \dot{Y} + \left( \frac{2n^2}{G_o} \right) \eta Y \cdot Y - \frac{1}{2} \left( \frac{2n^2}{G_o} \right) B \]  
(C-10)

which, by comparison with eq. (C-1) shows that it is a particular case of the model used in PTT with

\[ \nu_1 = \frac{2n^2}{G_o} \]  
(C-11)

\[ \nu_2 = 0 \]  
(C-12)

\[ n = m \eta^{-n-1} \]  
(C-13)

Indeed, using from formula (C-5)

\[ N = n - 1 \]

\[ M = 2n - 2 \]

\[ \iota = \frac{2n}{G_o} \frac{V}{R_o} = 2De \]  
(C-14)
equation (C-4) becomes

\[ [n - i \delta (1 - \xi)] \tilde{F}^{(1v)} - i(n - 1) \delta \tilde{F}^{(1v)} = 0 \]  

(C-15)

which, being divided by \(i \delta^n\) reduces to equation (3-108).

Similarly, it is easy to show that the solution given by (C-6), (C-7) under the substitutions (C-14) also reduces to solution (3-109), (3-110). For the particular limits \(n = 1/4, 1/2\) and \(3/4\) (corresponding to \(\mu = -3/2, -1\) and \(-1/2\)) the solution of (3-109) becomes:

a. \(n = \frac{1}{4}\) :

\[ G = A_1 \cdot \ln(\frac{1}{4} + t) + B_1 t^2 + C_1 t + D_2 \]  

(C-16)

b. \(n = \frac{1}{2}\) :

\[ G = A_2 \cdot (\frac{1}{2} + t) \ln(\frac{1}{2} + t) + B_2 t^2 + C_2 t + D_2 \]  

(C-17)

c. \(n = \frac{3}{4}\) :

\[ G = A_3 \cdot (\frac{3}{4} + t)^2 \ln(\frac{3}{4} + t) + B_3 t^2 + C_3 t + D_3 \]  

(C-18)

where the constants \(A_1, B_1, C_1, D_1, i = 1, 2\) and 3 are determined from the boundary conditions (3-85).
APPENDIX D
Cylindrical Couette Flow of the Leonov-like
with Retardation (LER) Model

In the cylindrical Couette flow, the momentum equations require

$$
\tau_{r\theta} = -\frac{A}{r^2},
$$

(D-1)

where $\tau_{r\theta}$ is the extra shear stress and $A$ is a constant. Equation (D-1), can be written in dimensionless form as

$$
\tau^*_{r\theta} = -\frac{Y}{(r^*)^2},
$$

(D-2)

where $n_0\lambda_2$, $R_0$ were used to non-dimensionalize the extra stress, $\tau$, and radial distance $r$ respectively, and $Y$ is another constant.

In terms of the viscoelastic part of the extra stress tensor $S$ made dimensionless with respect to $n_0\lambda_1$, equation (D-2) can be rewritten as:

$$
S_{r\theta} = -\frac{Y}{(r^*)^2} - \eta^* X,
$$

(D-3)

where

$$
X = De r^* \frac{\partial \Omega^*}{\partial r^*},
$$

(D-4)

$$
De = \frac{\lambda_1}{R_0},
$$

(D-5)

and $\Omega^*$ is the angular velocity ($\Omega = \Omega_0/r$) made dimensionless with respect to $V/R_0$.

Equation (D-3) can be used together with the Giesekus constitutive equation (2-20) to obtain an equation for $X$. In the special case of $\alpha = 1/2$ (LER model) this procedure generates a cubic equation for $X$:
\[(\eta^*)^2 x^3 + \left(\frac{-2Yn^*}{(r^*)^2}\right)x^2 - \left(1 + \eta^* - \frac{Y^2}{(r^*)^4}\right)x - \frac{Y}{(r^*)^2} = 0 \]. \quad (D-6)

For a given Y equation (D-6) generates at least one real value for \(X = X(Y)\). To find the value of Y, the boundary conditions

\[\Omega^* = 1 \quad , \quad r^* = 1 \quad \quad \quad \quad \quad \quad \quad \quad (D-7a)\]

\[\Omega^* = 0 \quad , \quad r^* = 1 + \mu \quad \quad \quad \quad \quad \quad \quad \quad (D-7b)\]

are necessary.

Using the definition for X (eq. D-4) and the boundary conditions (D-7) an integral equation is obtained

\[D_e = - \int_{1}^{1+\mu} \frac{X(Y,r)}{r^*} \, dr^* \quad , \quad (D-8)\]

with \(X(Y,r)\) given as solution of (D-6).

For a given value of Y, equation (D-8) determines the value of \(D_e\), or alternatively for given value of \(D_e\), equation (D-8) can be solved (using the secant method) for Y. Since there is development of steep gradients next to the inner wall \((r^* - 1)\) special care must be taken to evaluate the integral appearing in (D-8) numerically. The strategy that was followed in the calculations in this thesis, consisted of dividing the domain of integration into several subdomains, so that in each of them X would vary by at most, say 20%, and then use Gaussian quadrature (Carnahan et al., 1969) to perform the numerical integration in each one of the subdomains. The fact that (D-6) might generate more than one real value for X did not create a problem: Simply, through the domain, as \(r^*\) changes, a continuous value of X has to be chosen by consistently using the same — in order — root of (D-6). If this approach is followed it turns out that the solution to the equation (D-8) always exists and is unique.
APPENDIX E

Cylindrical Couette Flow of Giesekus Fluid:

\[ 0 \leq \alpha \leq 1, \lambda_1 > 0, \lambda_2 = 0 \]

Proceeding along the same lines followed in Appendix D but using no retardation time \( \eta^* = \lambda_2 = 0 \) it is possible to find an explicit expression for \( X \) (eq. D-4) for the general Giesekus model (2-20):

\[
X = 2\alpha Y \cdot \frac{(r^*)^2 + k \cdot (2\alpha - 1) \cdot \sqrt{(r^*)^4 - 4\alpha^2 Y^2}}{(2\alpha - 1)(r^*)^2 + k \cdot \sqrt{(r^*)^4 - 4\alpha^2 Y^2}} \tag{E-1}
\]

where \( k \) can be either +1 or -1 depending in which branch of the solution you want to calculate (for \( k = +1 \), the solution continuing smoothly from the Newtonian case, is physical, the other one is aphysical).

If equation (E-1) is substituted into

\[
\Omega^*(r^*) = \frac{1}{De} \int \frac{X}{r^*} \, dr^* \tag{E-2}
\]

and after some lengthy algebraic manipulations, it is possible to derive an explicit formula for \( \Omega^*(r^*) \):

\[
\Omega^*(r^*) = C + \frac{1}{De} \left\{ \frac{\sqrt{g}}{4} \ln\left(\frac{-\mu}{\lambda} \right) + A^2 \frac{N \cdot Y \alpha}{\mu \Lambda g} \right. \\
+ k \cdot \frac{A}{2} \tan^{-1}\left(\frac{K}{2Y \alpha}\right) + k \cdot \frac{AY \alpha}{\mu \Lambda g} \\
- k \frac{\sqrt{g}}{4} \ln\left(\frac{K - Z \alpha}{K + Z \alpha} \right) \right\} \tag{E-3}
\]

where

\[ g = 4\alpha(1 - \alpha) \]
\[ A = (1 - 2\alpha) \]
\[ N = (r^*)^2 \]
\[ Z = \frac{2Ya}{\sqrt{g}} , \]

\[ K = \sqrt{N^2 - 4Y^2a^2} , \quad 0 < 2Ya < 1 \]

\[ \Lambda = N - Z \]

\[ \mu = N + Z \]

(E-4)

and C, Y (0 < 2Ya < 1) are two constants determined from the boundary conditions (D-7). Again, choosing \( k = \pm 1 \) the physical or aphysical solution branches can be calculated.
APPENDIX F

Analysis of a one-element Finite-element Solution of the Cylindrical Couette Flow of the Corotational Maxwell-like Fluid

A one-element formulation of the \( \theta \)-momentum equation in cylindrical coordinates was used in order to numerically calculate the cylindrical Couette flow behavior (no \( \theta \)-dependence) when the CML constitutive equation is used.

For a CML fluid the dimensionless shear stress \( \tau^*_{r\theta} \) given for a simple shear flow (Giesekus 1982) as

\[
\tau^*_{r\theta} = \frac{r^* \frac{d}{dr^*} \Omega^*}{1 + \left( \text{De} \ r^* \frac{d}{dr^*} \Omega^* \right)^2}
\]

where \( r^* \), \( \Omega^* \) are the dimensionless radial coordinate and angular velocity respectively and \( \text{De} = \lambda_0 V/R_0 \). The inner radius \( R_0 \), the angular velocity of the inner cylinder \( \Omega_0 = V/R_0 \) and the mean shear stress \( \eta_0 \Omega \) were used to non-dimensionalize \( r \), \( \Omega \) and \( \tau_{r\theta} \) respectively. In the following, the \( \ast \) will be omitted from the dimensionless variables for simplicity.

When eq. (F-1) is used to substitute \( \tau_{r\theta} \) in the \( \theta \)-momentum equation (no \( \theta \)-dependence is assumed)

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \tau_{r\theta} \right) = \Omega, \quad (F-2)
\]

the following equation is generated

\[
3\Omega_r + r\Omega_{rr} - \frac{6\text{De}^2}{3} \left[ - \frac{3\Omega_r}{r^3} + \frac{3\Omega_{r\theta} \Omega_r}{r^2} \right] = 0 \quad (F-3)
\]

where \( \Omega_r = \frac{d\Omega}{dr} \) etc.
Assuming a representation for $\omega$ in terms of Lagrangian quadratic basis functions $\phi_i$ (see Figure 4.2) a weak Galerkin formulation of (F-3) leads to integral equations of the form:

$$\int_{1}^{1+\mu} \frac{d\phi_i}{dr} \left[ 1 - \frac{De^2}{3} r^2 (\Omega_r)^2 \right] \Omega_r r^2 dr = 0 .$$  \hspace{1cm} (F-4)

Using only one element - 3 quadratic functions $\phi^{-1}, \phi^0, \phi^1$ - and the boundary conditions (D-7) for $\Omega$ leaves only one undetermined coefficient $c$:

$$\Omega = \phi^{-1} + C \cdot \phi^0$$  \hspace{1cm} (F-5)

which can be found from eq. (F-4) for $i = 0$, when expression (F-5) is substituted for $\Omega$.

Equ. (F-4) for $i = 0$ was evaluated using a 5-point Gaussian quadrature (Abramowitz and Stegun 1965) and generates a cubic equation for $c$ which for $\mu = 0.1$ is

$$20805 \cdot De^2 C^3 - 29973 \cdot De^2 C^2 + ( -58.907 + 20843 \cdot De^2 ) C$$

$$+ ( 27.353 - 5606.2 \cdot De^2 ) = 0 .$$  \hspace{1cm} (F-6)

As a cubic equation, eq. F-6 could have for different values of $De$, 1 or 3 real solutions. Having evaluated $C$ from (F-6), the value of $\tau_{r0}$ on the wall of the inner cylinder can be easily calculated from eq. (F-1) and the results are presented with the continuous line in Fig. F.1 against the exact solution (solid circles) presented in Appendix D.

The exact solution has two termination points (shown with the open circles in Fig. F.1) of the physical ($De < 0.0633$) and the aphysical ($De > 0.15$) branches of the solution. As seen from Fig. F.1, the termination point of the physical branch of the solution at $De = 0.0633$ is approximated by the finite element method with a limit point at $De = 0.084$. Notice, that when the solution exists, it is approximated very well, even by this highly simplified one element representation.
Figure F.1 Dependence of the dimensionless value of the shear stress at the inner wall $\tau_{r,0}$, with respect to the Deborah number, for the CML fluid. The results obtained with a one-element finite-element method are shown with the continuous line and those corresponding to the exact solution with circles. Open circles denote the termination points of the exact solution.
APPENDIX G

Linear Stability Analysis of Cylindrical Couette Flow of an Upper-Convected Maxwell Fluid Towards 2-Dimensional Azimuthal Disturbances

The linear stability analysis is performed by perturbing the conservation and constitutive equations (2-37) - (2-42) for the UCM fluid ($n_3 = 0$, $\alpha = 0$) with a 2-dimensional azimuthal disturbance. If $\mathbf{X} = \{v_\theta, v_r, r_{rr}, r_{r\theta}, r_{\theta\theta}, p\}$ represents the unknown variables, and $\mathbf{X}_0$ denotes their values for the Couette steady state solution a small disturbance $\mathbf{X}_k(r, \theta)$ is assumed to be superposed to $\mathbf{X}_0$ of the form.

$$\mathbf{X}_k = \text{Re}\{\mathbf{X}_k(r) e^{i k \theta} + a_k t\} \quad , \quad (G-1)$$

$$\mathbf{X} = \mathbf{X}_0 + \mathbf{X}_k \quad , \quad (G-2)$$

where $\mathbf{X}_k(r)$ is a complex (in general) variable assumed to be of much smaller magnitude than the base solution $\mathbf{X}_0$, and Re denotes the real part.

Substituting expression (G-2) in the time-dependent form of the eqs. (2-37) - (2-42) and applying regular perturbation procedures (Bender and Orzag 1978) around the base solution $\mathbf{X}_0$, a set of six linear equations for $\mathbf{X}(r)$ can be obtained. The equation for obtaining $\mu$ and $k$ are complicated and have to be solved numerically. But in the limit of $\mu \rightarrow 0$ and for $k = 0(1)$ turns out that the equations are exactly the same as those derived in the first order of the domain perturbation, eqs. (3-74) - (3-76) with the transformed domain coinciding with the concentric domain, and the variable $Z$ replacing the variable $t$ in eq. (3-84), where

$$Z = -i \Sigma + K (1 - \zeta) \quad , \quad (G-3)$$

with

$$\Sigma = a_k De \quad , \quad K = k De \quad . \quad (G-4)$$
The general solution for the complex stream function is therefore given by equation (3-86) with \( z \) in place of \( t \):

\[
G(z) = \beta_1 z + \beta_2 z^2 + \beta_3 e^{(1-i)z} + \beta_4 e^{-(-1+i)z}
\] (G-5)

The boundary conditions that apply are homogeneous:

\[
G(z) = G'(z) = 0 \quad , \quad z = -i\Sigma + K \quad , \quad -i\Sigma
\] (G-5)

therefore, for a non-trivial solution to exist, the system of equations for the unknown coefficients \( \beta_i \), \( i = 1, 4 \) derived when (G-5) is substituted in (G-6) must be singular. This is formally expressed as

\[
\begin{vmatrix}
S & S^2 & e^{(1-i)S} & e^{-(-1+i)S} \\
S + K & (S + K)^2 & e^{(1-i)(S+K)} & e^{-(-1+i)(S+K)} \\
1 & 2S & (1-i)e^{(1-i)S} & -(1+i)e^{-(-1+i)S} \\
1 & 2(S+K) & (1-i)e^{(1-i)(S+K)} & -(1+i)e^{-(-1+i)(S+K)}
\end{vmatrix} = 0
\] (G-7)

with \( S \neq -i\Sigma \), and \( \text{det} \neq \text{determinant} \).

Equation (G-7) can be rearranged as a quadratic equation for the (complex) eigenvalue \( \Sigma \):

\[
at^2 + 2b\Sigma + C = 0
\] (G-8)

with \( a, b, c \) being complicated functions of \( K \).

First, we looked for the possibility of bifurcations which, corresponding to \( \Sigma = 0 \), would require \( C(k) = 0 \). This equation though, had no real solution for \( K \). Then equation (G-8) was solved for various \( K \) values and revealed an infinity of \( k \) values for which \( \Sigma \) values could exist (\( \text{Re}(\Sigma) = 0 \)). Taking into account the definition \( k\text{De} \), this means that for each \( k \) value there are an infinite number of \( \text{De} \) values for which Hopf bifurcations to time periodic 2-d azimuthal disturbances are predicted. The lowest \( K \) value corresponding to a zero real part in \( \Sigma \) was
K-1 which means that at least for \( k = 1 \) when the perturbation is expected to be valid, the flow bifurcates to a time periodic state as easily as for \( \text{De} = 1 \). Of course, the higher the value of \( k \) for which the perturbation solution is valid, the lower the \( \text{De} \) for the first Hopf bifurcation.
APPENDIX H

Hperbolicity Region in Cylindrical Couette
Flow for an Upper-Convected Maxwell Fluid

For a cylindrical Couette flow of a UCM fluid Joseph's criterion
(Joseph et al., 1985) for four real characteristics is expressed as

\[ \Sigma(q) = \frac{G^2 \text{ReDe}}{q} \left\{ \left( 1 - \frac{q}{\Theta(\mu)^2} \right)^2 - \frac{4 \text{De}}{q \text{Re}} \right\} - 1 > 0 \quad , \]  \hspace{1cm} (H-1)

where

\[ G = \frac{(1 + \mu)^2}{\mu(2 + \mu)} \quad , \]  \hspace{1cm} (H-2)
\[ q = (r^*)^2 \quad , \]  \hspace{1cm} (H-3)

with \( r^* \) the dimensionless radial coordinate.

Since

\[ \Sigma(q) \leq \frac{G^2 \text{ReDe}}{q} \left( 1 - \frac{4 \text{De}}{q \text{Re}} \right) - 1 = S(q) \quad . \]  \hspace{1cm} (H-4)

and \( \max S(q) = \left( \frac{\text{Re}}{4G} \right)^2 - 1 \)  \hspace{1cm} (H-5)

it turns out that the minimum \( \text{Re} \) for the extra pair of real characteristics to appear is

\[ \text{Re}_{\text{min}} = 4G = \frac{4(1 + \mu)^2}{\mu(2 + \mu)} \quad . \]  \hspace{1cm} (H-6)

The characteristic surface separating the region of 4 real from that of 2 real characteristics is defined by the locus \( q = (r^*)^2 \) satisfying

\[ \Sigma(q) = 0 \quad . \]  \hspace{1cm} (H-7)

This surface is either attached to the inner cylinder (when \( \Sigma(1) > 0 \)) or
might be located in the space between the two cylinders.

The necessary and sufficient conditions for this latter case to occur are

(a) \( \Sigma(1) < 0 + \text{ReDe} < 4\text{De}^2 \alpha^2 + 1 \), \hspace{1cm} (H-8)

(b) \( \frac{\partial \Sigma}{\partial q} |_{q=1} > 0 \) \{ \mu > \sqrt{3} - 1 \) and \( \text{ReDe} > \frac{26(\delta + 2)}{6^2 + 26 - 2} \), \hspace{1cm} (H-9)

(c) the equation (H-7):

\[
\text{ReDe} q^3 - (2\text{ReDe}(1 + \mu)^2 + \mu^2(2 + \mu)^2)q^2 + \\
\quad \quad \quad + \text{ReDe}(1 + \mu)^4 q - 4\text{De}^2(1 + \mu)^4 = 0
\] \hspace{1cm} (H-7a)

which is cubic in \( q \) must have 3 real roots.

This last requirement is imposed because the intermediate hyperbolic zone cannot be attached to the outer cylinder since \( \Sigma(1 + \mu)^2 \) < 0. After some tedious algebraic manipulations, (c) and be expressed in the form of a double inequality:

\[
\frac{\mu^2(2 + \mu)^2}{108A^2} \left\{ (2A + 1)(A^2 - 8A - 2) - 2(A^2 + 4A + 1)^{3/2} \right\} < \text{De}^2 \hspace{1cm} (H-10a)
\]

and

\[
\text{De}^2 < \frac{\mu^2(2 + \mu)^2}{108A^2} \left\{ (2A + 1)(A^2 - 8A - 2) + \zeta(A^2 + 4A + 1)^{3/2} \right\} \hspace{1cm} (H-10b)
\]

where

\[
A = \frac{\text{ReDe}(1 + \mu)^2}{\mu^2(2 + \mu)^2}
\] \hspace{1cm} (H-11)

In calculating the characteristic surface (in \( r^*, \text{De} \) space) for given \( \text{Re} \) it is easier, instead of solving (H-10a) for \( q \) (given \( \text{De} \)), to solve for \( \text{De}_{\text{max}}, \text{De}_{\text{min}} \) at a given location \( r^* \) since (H-10a) is a quadratic equation with respect to \( \text{De} \).
APPENDIX I
Streamline-Upwind/Finite-Element Method
as Proposed by
Tezduyar and Hughes (1983)

The best method to solve the multidimensional advection-diffusion equation is the streamline-upwind/Petrov-Galerkin (SU/PG) proposed by Brooks and Hughes (1982). Their method changes the standard Galerkin one only by modifying the weighting functions. Recently, Tezduyar and Hughes (1983) generalized the SU/PG method for first order hyperbolic systems of the form

\[
\frac{\partial \mathbf{U}}{\partial t} + \sum_{i=1}^{n} A_i \frac{\partial \mathbf{U}}{\partial x_i} + \mathbf{G} = 0 ,
\]  

(1-1)

where \( \mathbf{U} \) is the solution vector, \( n \) are the space dimensions,

\[
A_i = A_i(\mathbf{U}, x, t), \quad \mathbf{G} = G(\mathbf{U}, x, t).
\]  

(1-2)

More specifically, if \( \mathbf{W} \) is the vector of the basis functions used to approximate the transferred quantity \( \mathbf{U} \) (which is a vector in the general case) the corresponding weighting function \( \Omega \) is:

\[
\Omega = \mathbf{W} + \sum_{j=1}^{n} T_j \cdot \frac{\partial \Omega}{\partial x_j} ,
\]  

(1-3)

where

\[
T_j = \tau_j A_j^\dagger ,
\]  

(1-4)

\[
\tau_j = \{1/2 \text{ or } 1/\sqrt{15}\} \ h_j/\rho_j ,
\]  

(1-5)

\[
h_j = [h_c^2 + h_n^2]^{1/2} ,
\]  

(1-6)

\[
\rho_j = \max_{k < m} |\lambda_k(A_j)| ,
\]  

(1-7)
where $h_\xi$ and $h_\eta$ are the elemental characteristic lengths in the $\xi$, $\eta$ directions correspondingly (see Figure 4.1), and $\lambda_k(A_j)$ the $k$-th eigenvalue of $A_j$ matrix.

Hughes and Tezduyar (1984) applied this method to the compressible Euler equations where the matrices $A_i$ are derived from vectors $F_i$ as

$$A_i = \partial F_i / \partial U,$$  \hspace{1cm} (I-8)

and there exist a matrix $S$ so that the matrices $A_i$ can be diagonalized using the similarity transformation

$$S^{-1} A_i S = \Lambda_i,$$  \hspace{1cm} (I-9)

where $\Lambda_i$ is a real diagonal matrix. Under these conditions and for the one-dimensional case ($n=1$) it turns out that the proposed method leads to equations very similar to those obtained under the Lax-Wendroff finite difference method (Richtmyer and Morton 1967).
APPENDIX J

Components of the Residual Equations in Bipolar Coordinates

The momentum equation (2-51) gives rise to

\[
\int \psi^1 \left[ \left( \frac{3^2}{\partial \dot{\theta}} + 1 \right) T_{\xi \theta} + \frac{3^2}{\partial \dot{\xi} \dot{\theta}} \left( T_{\xi \xi} - T_{\theta \theta} \right) + \right.
\]

\[
\left. n^* \left[ \psi_{\theta \dot{\theta}} \psi_{\dot{\xi} \dot{\theta}} + 2 \psi_{\xi \xi} \psi_{\theta \theta} - 2 \psi_{\xi \xi} \psi_{\theta \theta} + \psi \right] \right] d\xi d\theta
\]

\[
\int \frac{\partial}{\partial \xi} \psi^1 \frac{\partial}{\partial \xi} T_{\xi \theta} d\xi d\theta + \int \frac{\partial^2}{\partial \xi^2} \psi \frac{\partial^2}{\partial \xi^2} d\xi d\theta = 0 . \quad (J-1)
\]

where \( \psi^1 \) represents the product of a Hermite cubic basis function \( \chi^1(\xi) \) times a Fourier function (cos\((k\theta)\) or sin\((k\theta)\)).

The residual equations arising from the integration of the components of the Giesekus equation in bipolar coordinates, eqs. (2-56) - (2-58) are:

\[
\int \psi^1 \left[ T_{\xi \xi} + D \left[ \left( \psi_{\theta} + \psi_{\xi \xi} \right) \frac{\partial}{\partial \xi} T_{\xi \xi} + \left( -\psi_{\xi \xi} + \psi_{\xi \xi} \right) \frac{\partial}{\partial \theta} T_{\theta \theta} \right. \right.
\]

\[
+ \left. T_{\xi \xi} \left( \psi_{\theta} \sinh \xi + \psi_{\xi \xi} \sin \theta - 2 \psi_{\xi \xi} \psi_{\theta \theta} \right) \right] d\xi d\theta = 0 , \quad (J-2)
\]

\[
\int \psi^1 \left[ T_{\xi \theta} + D \left[ \left( \psi_{\theta} + \psi_{\xi \xi} \right) \frac{\partial}{\partial \xi} T_{\xi \theta} + \left( -\psi_{\xi \xi} + \psi_{\xi \xi} \right) \frac{\partial}{\partial \theta} T_{\theta \theta} \right. \right.
\]

\[
+ \left. T_{\xi \theta} \left( \psi_{\theta} \sinh \xi + \psi_{\xi \xi} \sin \theta \right) \right) - T_{\xi \xi} \left( \psi_{\theta \theta} + \psi_{\theta \theta} \right) \right] d\xi d\theta = 0 , \quad (J-3)
\]

\[
\int \psi^1 \left[ T_{\theta \theta} + D \left[ \left( \psi_{\theta} + \psi_{\xi \xi} \right) \frac{\partial}{\partial \xi} T_{\theta \theta} + \left( -\psi_{\xi \xi} + \psi_{\xi \xi} \right) \frac{\partial}{\partial \theta} T_{\theta \theta} \right. \right.
\]

\[
+ \left. T_{\xi \theta} \right] d\xi d\theta = 0 . \quad (J-4)
\]
\[ + T_{\psi_0} (\psi_0 \sin \xi + \psi_1 \sin \theta + 2 \chi \psi_{\xi_0}) - 2 T_{\xi_0} (\xi \psi_{\xi_0} + \psi \cos \theta) \]
\[ + \alpha \left( T_{\theta\theta}^2 + T_{\xi\theta}^2 \right) + 2 \psi_{\xi\theta} = 0. \]  

where \( \text{De}' \) is a modified Deborah number defined by eq. (2-59) and where \( \psi_1 \) represents the product of a Lagrangian quadratic basis function \( \psi_1(\xi) \) times a Fourier function (\( \cos(k\theta) \) or \( \sin(k\theta) \)).
APPENDIX K

Useful Integral Formulas for the Evaluation of the load in Bipolar Cooridnates

The following integral formulas are useful in evaluating the integral expressions arising from eqs. (6-6) and (6-7) after the pressure and stress components have been substituted by their respective truncated Fourier series used to represent them in the spectral/finite-element method.

\[ K_n = \frac{2\pi}{1 + a \cos \theta} \left[ \frac{\cos(n\theta)}{(1 + a^2)^{1/2}} \right]_{0}^{2\pi} = \frac{2\pi}{(1 + a^2)^{1/2}} \left( \frac{(1 + a^2)^{1/2} - 1}{a} \right)^n, \quad (K-1) \]

\[ M_n = \frac{2\pi}{(1 + a^2)^{1/2}} \left( \frac{(1 + a^2)^{1/2} - 1}{a} \right)^n \left( 1 + n \left( \frac{1 + a^2}{a} \right)^{1/2} \right), \quad (K-2) \]

where \( a \) is a constant, \(-1 < a \leq 1\) and \( n \) a non-negative integer. Expression (K-1) corresponds to formula 3.613 (p.366) of Gradshteyn and Ryzhik (1965). Expression (K-2) was derived through repeated use of integration by parts, using expression (K-1) and the value of \( M_0 \) corresponding to formula 3.645 (p.379) of Gradshteyn and Ryzhik (1965).