A FAILURE DETECTION ALGORITHM FOR LINEAR DYNAMIC SYSTEMS

by

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ABSTRACT

A new failure detection and isolation (FDI) algorithm for linear
dynamic systems, the Orthogonal Series Generalized Likelihood Ratio
(OSGLR) Test, is presented. The failure hypothesis for this test is
that the failure mode of the component under test can be represented by
a truncated series expansion. This formulation of the failure detection
problem leads to a failure detection algorithm with several desirable
properties. First, the truncated series expansion can represent a large
class of failure modes. Therefore, the test is robust to failure mode
uncertainty, in that it can accurately detect and isolate many different
failure modes. Second, because the unknown coefficients of the orthog-
onal series enter the system equations linearly, the coefficients may be
estimated using a linear estimation scheme. This greatly reduces the
amount of computation required as compared to other GLR tests. Finally,
the false-alarm rate for the continuous-time OSGLR test can be
asymptotically approximated. This cannot be done for most other FDI
algorithms.

Simulation results are presented, and the performance of the
OSGLR test is compared to that of the GLR test of Willsky and Jones
[57]. For actuator failures, the performance of the OSGLR test is
comparable to or superior to the performance of the GLR test. In
addition, the OSGLR algorithm requires significantly less computation
than the GLR algorithm.

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CHAPTER 1

INTRODUCTION

1.1 Background

In recent years, there has been increasing interest in the problem of failure detection in dynamic systems. In the aerospace field, this interest has been spurred by the need for highly reliable flight control and navigation systems. For example, in a fly-by-wire (FBW) control system, there is no mechanical connection between the pilot and the control surfaces. Therefore, the flight control system is "flight-critical," meaning that if the flight control system fails, then the aircraft will be lost. The reliability of the flight control system must therefore approach that of the airframe. The accepted reliability requirement for a commercial transport is a probability of failure of $10^{-9}$ for a 10 hour flight, or a failure rate of $10^{-10}$ failures per hour [47]. The failure rate of control system components is typically orders of magnitude larger than this. Therefore, redundant systems must be used to achieve satisfactory levels of reliability. For example, both the Space Shuttle and the F-16 have FBW control systems with redundant computers and inertial sensors [47].

In order to manage the resources of a redundant system, some sort of automatic failure detection is required. Automatic failure detection may be accomplished in a number of ways. In self-test, or built-in test, components are monitored by test equipment to determine if a failure has occurred. Unfortunately, the coverage (probability of detection) provided by self-test is generally low, so that reliable failure detection usually cannot be accomplished using self-test alone.
The most straightforward means of automatic failure detection is the use of direct redundancy, in which two or more identical sensors are used to measure the same quantity. If two sensors are used, failure detection can be accomplished simply by comparing the outputs of the sensors. If three or more sensors are used, then the failure may be isolated as well, using a simple voting scheme. Direct redundancy has limitations, however. First, the replication of sensors may impose a penalty, in terms of cost, weight, power, and volume. Furthermore, direct redundancy cannot be used to detect certain types of failures, such as actuator failures.

When direct redundancy is not practical, analytic redundancy must be used. Analytic redundancy involves the comparison of the outputs of dissimilar sensors, using the dynamics of the system to determine what relationship should exist among these outputs. Inasmuch as the system dynamics depend on the actuators, analytic redundancy may be also used to detect and isolate actuator failures. Analytic redundancy may also be used to reduce the number of sensors required to achieve a given level of reliability, at the expense of increased computation and software complexity.

1.2 Problem Description

In this thesis, we will be concerned with the problem of detecting and isolating failures in linear dynamic systems, both in continuous time and discrete time. In discrete time, the state equation and measurement equation are given by

\[
\begin{align*}
\mathbf{x}(k+1) &= \Phi(k)\mathbf{x}(k) + B(k)\mathbf{u}(k) + \mathbf{w}(k) \\
\mathbf{y}(k) &= C(k)\mathbf{x}(k) + D(k)\mathbf{u}(k) + \mathbf{v}(k)
\end{align*}
\]  

(1-1) (1-2)

where \( \mathbf{x}(k) \) is the state vector (at time \( k \)), \( \Phi(k) \) is the state transition matrix, \( \mathbf{u}(k) \) is a known input vector, \( \mathbf{y}(k) \) is a vector of measurements, and \( B(k), C(k), \) and \( D(k) \) are known matrices. \( \mathbf{w}(k) \) and \( \mathbf{v}(k) \) are independent, zero-mean, white Gaussian sequences with covariances given by
\[ E[w(k)w^T(j)] = Q(k)\delta_{kj} \]  \hspace{1cm} (1-3)

\[ E[v(k)v^T(j)] = R(k)\delta_{kj} \]  \hspace{1cm} (1-4)

where \( \delta_{kj} \) is the Kronecker delta. A similar description can be given in the continuous-time case.

Equations (1-1) through (1-4) describe the system in its normal or unfailed mode of operation. When a failure occurs, the failure can manifest itself in any number of ways. If the plant dynamics change, the matrices \( \Phi(k) \) and/or \( B(k) \) may change. If an actuator fails, the sequence \( u(k) \) will be different from the commanded input sequence. If a sensor fails, then a single element of \( y(k) \) will be different from that given by Eq. (1-2).

A number of failure detection and isolation (FDI) algorithms for dynamic systems have been proposed, such as the detection filter of Beard [3] and Jones [35], generalized likelihood ratio (GLR) methods [40, 57], and multiple model (MM) methods [60]. A survey of FDI algorithms has been made by Willsky [58]. More recently, Walker [55] has made a survey of FDI techniques developed since the time of the survey by Willsky. The recent paper by Willsky [60], which emphasizes GLR and MM methods, also has an extensive bibliography. Because of the availability of these references, a comprehensive review of FDI algorithms will not be given here. Instead, several algorithms will be discussed to illustrate some of the difficulties involved.

An algorithm based on the concept of the generalized likelihood ratio (GLR) test has been studied by McAulay and Denlinger [40], Willsky and Jones [57], and others. The GLR algorithm has been applied successfully to a number of problems, including the detection of incidents on freeways [59] and the detection of arrhythmias in electrocardiograms [28]. It is assumed that the failed system can be modeled as

\[ x(k+1) = \Phi(k)x(k) + B(k)u(k) + w(k) + b_1(k)n(k, \theta)v \]  \hspace{1cm} (1-5)
\[ y(k) = C(k)x(k) + D(k)u(k) + v(k) + d_i(k)n(k, \theta) \nu \] (1-6)

where the index \( i \) denotes the type of the failure, \( n(k, \theta) \) is the mode shape of the failure, and \( \nu \) is the unknown magnitude of the failure. \( \theta \) represents the time of the failure. For example, if the failure is assumed to be a bias failure in one of the actuators, then \( b_i(k) \) will be the column of the matrix \( B(k) \) corresponding to that actuator, \( n(k, \theta) \) will be a unit step function that begins at time \( k = \theta \), and \( d_i(k) \) will be the zero vector.

The GLR test proceeds in the following way: First, the measurement sequence \( y(k) \) is filtered, using a Kalman filter [26, Ch. 4], [34, Ch. 7] based on Eqs. (1-1) through (1-4), to produce a residual sequence \( \hat{y}(k) \). For each possible time of failure, \( \theta \), the effect of a unit failure on the residual is computed. Each of these failure signatures, \( g_i(k, \theta) \), is then correlated with the residual sequence to produce an estimate of \( \nu \), \( \hat{\nu}(k, \theta, i) \). This estimate is then used to form a decision function, \( \ell(k, \theta, i) \). If \( \ell(k, \theta, i) \) exceeds a threshold for any \( \theta \) or \( i \), then a failure is detected. The failure is isolated to the type of failure \( i \) and time of failure \( \theta \) which maximize \( \ell(k, \theta, i) \). (The exact details of the GLR algorithm may be found in Chapter 2 or Reference [57].)

Note that for each possible time of failure, \( \theta \), a matched filter must be implemented to perform the correlation operation. If the failure can occur at any instant, then a complete GLR implementation would require a bank of matched filters that grows linearly with time. To prevent the number of filters from growing indefinitely, the time of failure is often restricted, say, to \( k - N < \theta < k \). Even so, the computational burden of the GLR algorithm can be quite high if \( N \) is large.

This problem is typical of methods where a transition at an unknown time is hypothesized. For example, the multiple model method requires a linearly growing bank of Kalman filters, rather than matched filters [60]. The reason for this complexity is that \( \theta \) enters into the
system model non-linearly. Hence, non-linear estimation is required to estimate $\theta$.

Another problem with algorithms of this type is that they are designed with a specific mode of failure in mind. If a component fails in an unanticipated way, the algorithm may have unpredictable results. The algorithm may fail to detect the failure, or worse yet, it may detect the failure but isolate the failure to the wrong component. In general, it is desirable for the algorithm to be robust, in the sense that it performs well regardless of the exact mode of failure. Robustness to failure mode uncertainty will be a primary issue in this thesis.

(A related problem is the development of FDI algorithms which are robust to dynamic model uncertainty [16, 38, 43]. This problem is not dealt with in this thesis.)

An FDI algorithm that is robust to failure mode uncertainty was developed by Beard [3] and Jones [35], and is known as the detection filter. The detection filter is an observer whose gain matrix is chosen so that each failure type causes the residuals to lie along a single direction or in a plane that corresponds to the type of the failure, regardless of the mode of the failure. In this way, failures may be isolated simply by determining in which direction or plane the residuals lie.

Unfortunately, the detection filter has a number of limitations that make it unsuitable for many applications. First, the theory of Beard [3] and Jones [35] is limited to time-invariant systems. Detection filters have not been developed for time-varying systems. Second, the detection filter design process breaks down for some time-invariant systems, even though it should be possible to isolate the different types of failures in these systems. For example, it is not difficult to construct a system where the elimination of a sensor makes it possible to design a detection filter, even though a detection filter cannot be designed for the original system. Finally, the design process is difficult, as there is not any straightforward way to relate the choice of free parameters in the gain matrix to the performance of the algorithm.
1.3 Thesis Objectives and Overview

The goal of this thesis is to develop a failure detection and isolation algorithm for linear dynamic systems which is robust in the sense that the algorithm is insensitive to the modes of failure of the components being tested. In addition, the algorithm should be computationally efficient.

In Chapter 2, a brief review is given of hypothesis testing in general, and failure detection in particular. The conclusions drawn from the discussion there motivate the development of the FDI algorithm presented in this thesis.

In Chapter 3, the Orthogonal Series Generalized Likelihood Ratio (OSGLR) test is developed. The underlying assumption of the OSGLR test is that the failure modes can be represented by a truncated orthogonal series expansion. If this assumption is at least approximately satisfied, then the resulting test will be insensitive to the exact mode of the failure. Because the unknown coefficients in the orthogonal series enter the system description linearly, the resulting test is computationally efficient. The algorithm is developed for both continuous-time and discrete-time linear dynamic systems.

In Chapter 4, the performance of the OSGLR algorithm is analyzed. An approximation to the false-alarm rate of the test is developed, using methods of asymptotic analysis. In addition, the evaluation of the probability of detection is discussed.

In Chapter 5, implementation issues are discussed. In particular, the details of choosing the basis functions used in the orthogonal series representation and finding a realization for the basis functions are discussed.

In Chapter 6, two measures are proposed which indicate whether or not the failure of one type of component is distinguishable from that of a different type. The measures may be used to predict when a FDI algorithm will be unable to correctly isolate a failure. Alternatively, they may be used to determine whether the inability of a particular FDI
algorithm to isolate a failure is a defect in the algorithm or an intrinsic property of the dynamic system.

In Chapter 7, the OSGLR algorithm is applied to the problem of detecting actuator failures in a transport aircraft. Simulation results are given for a C-130 aircraft.

Finally, a review of the major contributions of this thesis is presented in Chapter 8. Possible extensions of this thesis and areas for future research are suggested.
CHAPTER 2

HYPOTHESIS TESTING

In this chapter, a brief review of hypothesis testing is given, especially as it relates to the problem of failure detection. In Section 2.1, the problem of hypothesis testing with a fixed-length data sample is discussed. In particular, the likelihood ratio and generalized likelihood ratio tests are described. The failure detection and isolation algorithm presented in this thesis is derived as a generalized likelihood ratio test.

In Section 2.2, sequential tests are discussed, including the Sequential Probability Ratio Test (SPRT) of Wald [52] and Bayesian tests [7, 17, 21], which are optimal tests. It is seen that in general, these tests cannot be applied to the problem of failure detection in dynamic systems. As a result, suboptimal rules must be developed.

In Section 2.3, the problem of failure detection in linear dynamic systems is discussed. Two algorithms based on the generalized likelihood ratio test are examined in detail. These algorithms have limitations: they are computationally complex, and they may not be robust to failure mode uncertainty. These problems motivate the FDI algorithm described in Chapter 3.

2.1 Fixed-Data Tests

In this section, we consider the problem of binary hypothesis testing with a fixed-length data sample. The hypothesis testing problem is binary because we are trying to decide which of two hypotheses, $H_0$ or $H_1$, is true. (The extension to multiple hypotheses, or $m$-ary
hypothesis testing, is straightforward. See, for example, [51, Sec. 2.3].) The fixed-length data sample may be a single scalar or single vector of observations, or it may be a continuous-time or discrete-time waveform over a fixed interval. For the present discussion, we will assume a single scalar observation, \( z \).

The observation \( z \) depends on whether \( H_0 \) or \( H_1 \) is true. The goal of the hypothesis testing problem is to design a decision rule which takes the observation \( z \) and determines whether \( H_0 \) or \( H_1 \) is true. Any decision rule will have the effect of dividing the domain of \( z \), \( Z \), into two regions, \( Z_0 \) and \( Z_1 \). If \( z \in Z_0 \), then the decision rule declares that \( H_0 \) is true; if \( z \in Z_1 \), then the decision rule declares that \( H_1 \) is true.

In any practical situation, \( z \) will be a noisy measurement. Hence, there will be a chance that a decision error will be made. In the binary hypothesis testing case, two types of errors are possible. A **Type I error**, or false alarm, occurs when \( H_0 \) is true but the decision rule declares that \( H_1 \) is true. A **Type II error**, or missed alarm, occurs when \( H_1 \) is true but the decision rule declares that \( H_0 \) is true. Corresponding to these two types of errors are the conditional probabilities \( P_{FA} \) (probability of false alarm) and \( P_M \) (probability of missed alarm), defined by

\[
P_{FA} = \Pr(z \in Z_1 | H_0) \quad (2-1)
\]

\[
P_M = \Pr(z \in Z_0 | H_1) \quad (2-2)
\]

2.1.1 **Simple Hypotheses**

In the simplest case, the probability density of \( z \) depends only on whether \( H_0 \) and \( H_1 \) is true. That is, the conditional probability densities \( p(z|H_0) \) and \( p(z|H_1) \) are given. In this case, the hypotheses are said to be simple.
Because the probability density of $z$ is known under each hypothesis, the decision error probabilities may be written as

$$P_{FA} = \int_{Z_1} p(z \mid H_0) \, dz$$  \hspace{1cm} (2-3)

$$P_M = \int_{Z_0} p(z \mid H_1) \, dz$$  \hspace{1cm} (2-4)

We would like to choose the decision regions $Z_0$ and $Z_1$ so that $P_{FA}$ and $P_M$ are minimized. However, the goal of minimizing $P_{FA}$ and the goal of minimizing $P_M$ are in conflict. For example, we could take $Z_0$ to be $Z$, and $Z_1$ to be the empty set. Then

$$P_{FA} = 0$$  \hspace{1cm} (2-5)

but

$$P_M = 1$$  \hspace{1cm} (2-6)

Such a decision rule would be useless for any practical system. Therefore a trade-off exists between $P_{FA}$ and $P_M$. A number of criteria exist for determining how best to determine the regions $Z_0$ and $Z_1$. (See, for example, [51, Sec. 2.2].) In this section, only two criteria will be discussed. Essentially all criteria give a decision rule with the same form.

The first criterion that will be discussed is the Bayes Criterion. First, it is assumed that a cost can be assigned to each of the decision errors. $C_{10}$ is the cost of a false alarm, and $C_{01}$ is the cost of a missed alarm. In general, costs could also be assigned to making the correct decisions, which would be denoted $C_{00}$ and $C_{11}$. However, the results are similar whether or not these costs are included in the criterion. For the sake of simplicity, these costs will not be included here.
Second, it is assumed that the a priori probabilities of the hypotheses, \( P_0 \) and \( P_1 \), are known, where

\[
P_0 = \Pr(H_0) \quad (2.9)
\]

\[
P_1 = \Pr(H_1) \quad (2.9)
\]

The decision regions are chosen to minimize the Bayes risk, which is defined as the expected value of the cost, namely

\[
R = C_{10} \Pr(\text{false alarm}) + C_{01} \Pr(\text{missed alarm}) \quad (2.9)
\]

The first probability in Eq. (2.9) may be expanded to give

\[
\Pr(\text{false alarm}) = \Pr(z \epsilon Z_1 \text{ and } H_0) \\
= \Pr(z \epsilon Z_1 \mid H_0) \Pr(H_0) \\
= P_{FA} P_0 \quad (2.10)
\]

Similarly,

\[
\Pr(\text{missed alarm}) = P_{M} P_1 \quad (2.11)
\]

Therefore, the Bayes risk can be expressed as

\[
R = C_{10} P_0 \int_{Z_1} p(z \mid H_0) \, dz + C_{01} P_1 \int_{Z_0} p(z \mid H_1) \, dz \quad (2.12)
\]

where we have used Eqs. (2.3), (2.4), and (2.9) through (2.11). Equation (2.12) may be rewritten as
\[ R = \int_{Z_1} (C_{10} p(z|H_0) - C_{01} p(z|H_1)) \, dz + C_{01} p_1 \]

(2-13)

To minimize the Bayes risk, the region \( Z_1 \) must be chosen so that \( z \in Z_1 \) if the integrand of the above expression is negative, and so that \( z \in Z_0 \) if the integrand is positive. Therefore, the optimum decision rule is given by

\[
\text{decide } H_1 \quad \begin{array}{c}
C_{01} p_1 p(z|H_1) \\
\geq \\
\text{decide } H_0
\end{array} \quad \begin{array}{c}
C_{10} p_0 p(z|H_0)
\end{array} \quad (2-14)
\]

This decision rule may be more conveniently written as

\[
\Lambda(z) = \frac{p(z|H_1)}{p(z|H_0)} \quad \text{decide } H_1 \quad \begin{array}{c}
\geq \\
\text{decide } H_0
\end{array} \quad \frac{C_{10} p_0}{C_{01} p_1} = T \quad (2-15)
\]

where \( \Lambda(z) \) is the likelihood ratio (LR), and \( T \) is the decision threshold. This decision rule is the likelihood ratio test (LRT).

Note that the decision rule does not specify what decision is made when \( \Lambda(z) = T \). In this case, the integrand in Eq. (2-13) is zero, and the Bayes risk is unchanged whether the point \( z \) is included in \( Z_0 \) or \( Z_1 \). However, the error probabilities, \( P_{FA} \) and \( P_{M} \), will depend on how the regions are chosen, if the likelihood ratio equals the threshold with nonzero probability.

In many circumstances, a Bayes test is unfeasible. For example, the designer of the test may be unable or unwilling to assign costs to the decision errors. Alternately, it may not be possible to determine the a priori probabilities of the hypotheses. A useful criterion in such a case is the Neyman-Pearson Criterion: Find the decision rule which minimizes \( P_M \), subject to the constraint that
\[ P_{FA} \leq \alpha \quad (2-16) \]

where \( \alpha \) is the maximum permissible false alarm probability. It can be shown that the optimum decision rule for this criterion is also a likelihood ratio test of the form

\[
\begin{align*}
&\text{decide } H_1 \\
&\Lambda(z) > T \\
&\text{decide } H_0
\end{align*} \quad (2-17)
\]

where the threshold \( T \) is chosen so that

\[ P_{FA} = \alpha \quad (2-18) \]

For both of the criteria discussed above, the resulting decision rule turned out to be the likelihood ratio test. Other criteria yield the LRT as well.

2.1.2 Composite Hypotheses

The situation is more complicated when the hypotheses include unknown parameters. In this case, the conditional probability densities of \( z \) under the two hypotheses become

\[
H_0: p(z \mid H_0, \theta_0) \quad (2-19)
\]

\[
H_1: p(z \mid H_1, \theta_1) \quad (2-20)
\]

where \( \theta_0 \) and \( \theta_1 \) are vectors of unknown parameters. Such hypotheses are said to be composite. Failure detection almost always involves composite hypotheses, because the time of the failure is unknown. In
addition, there may be other unknowns, such as the magnitude of a failure.

If the probability densities of $\theta_0$ and $\theta_1$ are known, then the composite hypotheses may be converted to simple hypotheses via

$$p(z|H_0) = \int_{\theta_0} p(z|H_0, \theta_0) p(\theta_0|H_0) \, d\theta_0$$  \hspace{1cm} (2-21)$$

$$p(z|H_1) = \int_{\theta_1} p(z|H_1, \theta_1) p(\theta_1|H_1) \, d\theta_1$$  \hspace{1cm} (2-22)$$

where $\theta_0$ and $\theta_1$ are the domains of $\theta$ and $\theta_1$, respectively. If the criterion for selecting the decision rule considers only the decision errors, then a LRT will be the optimum decision rule. However, if the criterion assigns weights to the decision errors that are functions of and $\theta_0$ and $\theta_1$, then a more complicated test will result.

In general, the distributions of the vectors $\theta_0$ and $\theta_1$ are not known. This greatly complicates the problem of determining the best decision rule, because the decision error probabilities will depend on the actual values of $\theta_0$ and $\theta_1$, which are unknown.

For most problems in failure detection, the conditional probability density of $z$ under $H_0$ (the no-failure hypothesis) does not depend on any unknown parameters. That is, $p(z|H_0)$ is given. In this case, $P_{FA}$ can be determined for any given decision rule, although $P_M$ cannot. Therefore, a Bayesian criterion makes no sense for this type of problem. However, a Neyman-Pearson criterion may make sense in some cases, as will be seen below.

Clearly, no decision rule for this problem can perform better than a (hypothetical) decision rule that assumes knowledge of the value of $\theta_1$. Of course, such a decision rule cannot be implemented, unless it turns out that the resulting test does not depend on $\theta_1$. If the test does not depend on $\theta_1$, it is said to be a uniformly most powerful test, because it performs as well as any other test with the same false-alarm probability, regardless of the value of $\theta_1$. 

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When a uniformly most powerful test does not exist, some other approach must be taken. If the vector $\theta_1$ were known, then the best test would be the likelihood ratio test

$$
\Lambda(z, \theta_{-1}) = \frac{p(z \mid H_1, \theta_1)}{p(z \mid H_0)} \begin{cases} 1 & \text{decide } H_1 \\ 0 & \text{decide } H_0 \end{cases} \quad (2-23)
$$

A reasonable approach is to use the test above with an estimate of $\theta_1$ in place of $\theta_1$. Because we have no \textit{a priori} information about $\theta_1$, an appropriate estimate to use is the maximum likelihood estimate, defined by

$$
\hat{\theta}_1 = \arg \max_{\theta_{-1}} p(z \mid H_1, \theta_{-1}) \quad (2-24)
$$

Comparing Eqs. (2-23) and (2-24), we can define the generalized likelihood ratio (GLR) by

$$
\Lambda(z) = \Lambda(z, \hat{\theta}_1) = \max_{\theta_{-1}} \frac{p(z \mid H_1, \theta_1)}{p(z \mid H_0)} \quad (2-25)
$$

The generalized likelihood ratio test (GLRT) is then given by

$$
\Lambda(z) \begin{cases} 1 & \text{decide } H_1 \\ 0 & \text{decide } H_0 \end{cases} \quad (2-26)
$$

where the threshold $T$ is chosen to meet the false-alarm requirements.

2.2 Sequential Tests

Sequential tests differ from fixed-data tests in that the number of data taken, and hence the time at which the test terminates, is not
known *a priori*. Instead, the data themselves determine when the test will terminate. By so doing, it may be possible to reduce the amount of data required to reach a decision, on average, for specified decision error probabilities. Failure detection algorithms are almost always sequential tests, because it is desirable to detect a failure as soon after it has occurred as possible so that corrective action may be taken promptly.

The first sequential hypothesis test was the Sequential Probability Ratio Test (SPRT), developed by Wald [52]. Suppose that a sequence of independent random variables \( \{z(1), z(2), \ldots \} \) is observed. The conditional probability density of each \( z(i) \) is \( p(z \mid H_0) \) or \( p(z \mid H_1) \), depending on which hypothesis is correct. If the number of data taken were known in advance, then the optimum test would be the likelihood ratio test, given by

\[
\Lambda(k) = \frac{p(z(1) \mid H_1) p(z(2) \mid H_1) \cdots p(z(k) \mid H_1)}{p(z(1) \mid H_0) p(z(2) \mid H_0) \cdots p(z(k) \mid H_0)} \begin{cases} \text{decide } H_1 & \text{if } \Lambda(k) > \xi_T \\ \text{decide } H_0 & \text{if } \Lambda(k) < \xi_T \end{cases}
\]

where \( k \) is the number of data observed. The SPRT uses the likelihood ratio \( \Lambda(k) \) in a slightly different way, as follows:

- If \( \Lambda(k) > b \), decide \( H_1 \)
- If \( \Lambda(k) < a \), decide \( H_0 \)
- If \( a < \Lambda(k) < b \), take another sample

It is sometimes more convenient to work with the log likelihood ratio (LLR), especially if the data are Gaussian. Then the test statistic is given by

\[
\ell(k) = \ln \Lambda(k)
\]

\( \ell(k) \) may be generated recursively via the difference equation
\[ \ell(k) = \ell(k-1) + \ln \frac{p(z(k) \mid H_1)}{p(z(k) \mid H_0)}, \quad k = 1, 2, \ldots \quad (2-29) \]

where

\[ \ell(0) = 0 \quad (2-30) \]

The SPRT decision rule then becomes

If \( \ell(k) > B \), decide \( H_1 \)
If \( \ell(k) < A \), decide \( H_0 \)
If \( A < \ell(k) < B \), take another sample

where

\[ A = \ln a \quad (2-31) \]
\[ B = \ln b \quad (2-32) \]

The time at which the SPRT terminates is a random variable, because it depends on the data. This random variable is called the decisive sample number (DSN). The mean value of the DSN is known as the average sample number (ASN), and it gives an indication of how much delay there will be in making a decision. Wald and Wolfowitz [53] showed that the SPRT is optimal in the following sense: The SPRT produces an ASN which is no larger than the ASN produced by any sequential test that yields the same or smaller error probabilities.

The SPRT is not suitable for most problems in failure detection, because a failure is a transition from one hypothesis to the other. In the SPRT formulation, there is no such transition. Also, most sequential tests designed for failure detection never make the decision that \( H_0 \) is true, because a failure may occur at any time. Instead, the test continues taking data until an alarm is sounded, indicating that a
failure ($H_1$) has occurred. The goals of the test are to minimize the number of false alarms and to minimize the delay between a failure and the alarm.

As in the fixed-length data sample case, the failure detection problem may be formulated in a Bayesian framework. Chow [16] and Chow and Willsky [17] describe the Bayesian approach to the design of sequential FDI decision rules. This formulation of the FDI problem is an example of a Bayes Sequential Decision Problem (BSDP), described by Blackwell and Girshick [7] and Ferguson [21]. The goal of the BSDP is to find the Bayes Sequential Decision Rule (BSDR), which is the decision rule that minimizes the Bayes risk.

The specification of the Bayes risk is somewhat more complicated for sequential tests than for fixed-data tests. In addition to the costs associated with making the wrong decision (false alarm or missed detection), there are costs associated with delays in making a decision. Furthermore, the description of the hypotheses is more complex. Besides the a priori probabilities of each hypothesis, the distribution of any parameters associated with the failure hypothesis, such as the onset time of the failure or the magnitude of the failure, must be specified.

Generally, it is quite difficult to determine cost functions that relate decision errors to degradation in system performance in a meaningful way. Furthermore, the a priori distribution of the onset time of the failure may be poorly defined, due to lack of data about the system. In these cases, a Bayesian test cannot be developed. The cost functions and probability distributions may be regarded as design parameters of the test. In this case, however, the test cannot be considered to be optimal in any sense.

Even if the Bayes risk can be specified in a meaningful way, the determination of the BSDR may be quite difficult. Although in principle the solution may be found using dynamic programming, in practice the solution is too complex to determine except for the simplest problems [17]. Therefore, the Bayes approach, although appealing, is not practical for most sequential failure detection problems.
When an optimal test cannot be found, a suboptimal test must be developed. One approach that may be used is to derive the test as a fixed-data test. All of the data up until the current time, \( k \), are used to form the likelihood ratio or, more often, the generalized likelihood ratio, \( \Lambda(k) \). The fixed-data test would then be

\[
\text{decide } H_1 \\
\Lambda(k) \gtrless T \\
\text{decide } H_0
\]  

(2-33)

This test is modified to become a sequential test by eliminating the decision that \( H_0 \) is true, so that the test becomes

\[
\text{decide } H_1 \\
\Lambda(k) \gtrless T \\
\text{take another sample}
\]  

(2-34)

If the false alarm rate of this test can be evaluated, then the threshold \( T \) can be determined to achieve the desired false-alarm rate. Such a test has the flavor of a Neyman-Pearson test, although of course the test is not optimal in any sense.

For example, consider the problem of detecting a bias that occurs at an unknown time in a white Gaussian sequence. The two hypotheses are given by

\[
H_0: \ y(k) = w(k) 
\]  

(2-35)

\[
H_1: \ y(k) = w(k) + b \cdot u(k-\theta)
\]  

(2-36)

where \( w(k) \) is a zero-mean, white Gaussian sequence with variance \( \sigma^2 \), \( b \) is a bias of known magnitude, and \( \theta \) is the time of the failure. \( u(k-\theta) \) is the unit step function, so that
\[ u(k-\theta) = \begin{cases} 
0, & k < \theta \\
1, & k \geq \theta 
\end{cases} \quad (2-37) \]

The hypotheses are composite, because the time of failure is unknown. Therefore, the appropriate test to use is the GLR test. The LLR is given by

\[ l(k, \theta) = \frac{b}{\sigma^2} \sum_{i=\theta}^{k} \left[ y(i) - \frac{b}{2} \right] \quad (2-38) \]

Hence, the GLR test statistic is given by

\[ l(k) = \max_{\theta} l(k, \theta) \quad (2-39) \]

\( l(k) \) may be generated recursively by the difference equation

\[ l(k) = \max \left( l(k-1) + \frac{b}{\sigma^2} (y(k) - \frac{b}{2}), 0 \right) \quad (2-40) \]

with initial condition

\[ l(0) = 0 \quad (2-41) \]

The resulting sequential test is then given by

\[ \text{decide } H_1 \quad \text{if} \quad l(k) \geq z \quad \text{T} \quad (2-42) \]

\[ \text{take another sample} \]

The test described above is the cumulative sum test of Page [41, 42], which was later developed by Chien [13] and Chien and Adams [14] as
the Modified SPRT (MSPRT). It is interesting to note that neither Page
nor Chien derived the test as a GLR test, although Page [41] was aware
that the test, as described above, was a GLR test. (However, the termi-
nology "generalized likelihood ratio test" did not exist at the time of
Page's work.)

In this particular problem, the resulting GLR test is quite sim-
ple, because a simple expression could be found for the GLR test statistic. However, this is the exception, rather than the rule. Suppose
that the magnitude of the failure, as well as the time of the failure,
is unknown. Then the LLR is given by

\[
l(k, \theta, b) = \frac{b}{\sigma^2} \sum_{i=0}^{k} \left( y(i) - \frac{b}{2} \right) (2-43)
\]

The GLR test statistic is given by

\[
l(k) = \max_{\theta, b} l(k, \theta, b) \quad (2-44)
\]

If the maximization over b is performed holding \( \theta \) fixed, we have that

\[
b(k, \theta) = \frac{\sum_{i=0}^{k} y(i)}{k - \theta + 1} \quad (2-45)
\]

Hence, the GLR test statistic is given by

\[
l(k) = \max_{\theta} \frac{1}{2\sigma^2(k - \theta + 1)} \left[ \sum_{i=0}^{k} y(i) \right]^2 \quad (2-46)
\]

No simple formula for \( l(k) \), such as in Eq. (2-40), can be found for this
problem. Hence, in order to implement the GLR test described for this
problem, it is necessary to keep track of the summation
\[ \sum_{i=0}^{k} y(i) \]

for all values of \( \theta \). This complexity is due to the fact that \( \theta \) enters the problem nonlinearly. Therefore, nonlinear estimation is required to estimate \( \theta \). This problem will occur in general whenever the time of failure, \( \theta \), is one of the unknown parameters in the (composite) failure hypothesis. On the other hand, the estimation of a parameter that enters the hypothesis linearly, such as \( b \), is relatively easy. These two observations hold for linear dynamic systems as well, and are important to the choice of the failure hypotheses of the test proposed in Chapter 3.

2.3 **Linear Dynamic Systems**

In this thesis, we will be concerned with the detection of failures in linear dynamic systems, both in continuous time and discrete time. For the present discussion, we will be concerned with only discrete-time systems. In the normal mode of operation (H_0), the state dynamics and measurement equation are given by

\[ x(k+1) = \Phi(k)x(k) + B(k)u(k) + w(k) + g(k) \] (2-47)

\[ y(k) = C(k)x(k) + D(k)u(k) + v(k) + h(k) \] (2-48)

where

- \( x(k) \) is an n-dimensional state vector
- \( \Phi(k) \) is an \( n \times n \) state transition
- \( u(k) \) is a (known) q-dimensional input vector
- \( B(k) \) is an \( n \times q \) input matrix
- \( g(k) \) is a (known) n-dimensional bias vector
\( \mathbf{y}(k) \) is an \( m \)-dimensional vector of measurements

\( \mathbf{C}(k) \) is an \( m \times n \) state measurement vector

\( \mathbf{D}(k) \) is an \( m \times q \) input measurement vector

\( \mathbf{h}(k) \) is a (known) \( m \)-dimensional bias vector

\( \mathbf{w}(k) \) and \( \mathbf{v}(k) \) are independent, zero-mean, white Gaussian sequences with covariances given by

\[
\mathbf{E}[\mathbf{w}(k)\mathbf{w}^T(j)] = \mathbf{Q}(k)\delta_{kj} \tag{2-49}
\]

\[
\mathbf{E}[\mathbf{v}(k)\mathbf{v}^T(j)] = \mathbf{R}(k)\delta_{kj} \tag{2-50}
\]

where \( \delta_{kj} \) is the Kronecker delta.

The failure hypotheses depend on the nature of the failures expected. For example, we could easily model actuator failures, sensor failures, dynamics changes, or changes in the noise statistics. As was seen in the previous section, it is generally not possible to determine the optimal sequential test for a dynamic system, or even to define a meaningful optimality criterion. Hence, suboptimal failure detection and isolation algorithms must be developed.

In this section, two such tests are described. These tests are based on the approach described in the previous section. That is, the tests are derived as generalized likelihood ratio tests over a fixed observation interval, and then modified to account for the sequential nature of the failure detection problem. The first of these tests, known as the GLR test, was developed by McAulay and Denlinger [40] and Willsky and Jones [57]. (This name is somewhat confusing, as it refers to a specific generalized likelihood ratio test, whereas other GLR tests are possible.) The other test of this type described in this section is the Multiple Model (MM) Method [60].
2.3.1 The GLR Test for Dynamic Systems

The details of the GLR test may be found in [57]. The presentation here differs slightly from the presentation in [57], in order to be consistent with the derivation of the algorithm presented in Chapter 3.

The GLR failure hypothesis is that

\[
\underline{x}(k+1) = \Phi(k)\underline{x}(k) + B(k)\underline{u}(k) + \underline{w}(k) + \underline{g}(k) + b(k)n(k,\theta)\nu \quad (2-51)
\]

\[
y(k) = C(k)\underline{x}(k) + D(k)\underline{u}(k) + y(k) + h(k) + d(k)n(k,\theta)\nu \quad (2-52)
\]

where \( b(k) \) and \( d(k) \) are known vectors that depend on the type of the failure. For example, if an actuator failure is modeled, \( b(k) \) will be the column of the matrix \( B(k) \) corresponding to that actuator, and \( d(k) \) will be the corresponding column of \( D(k) \). If a sensor failure is modeled, \( b(k) \) will be the zero vector, and \( d(k) \) will be given by

\[
d_i(k) = \delta_{ij}, \quad i = 1, 2, \ldots, m \quad (2-53)
\]

where \( j \) is the index corresponding to the failed sensor. \( n(k,\theta) \) is the mode shape, or simply mode, of the failure, which occurs at time \( \theta \). Generally, we have that

\[
n(k,\theta) = 0, \quad k < \theta \quad (2-54)
\]

For example, if a bias failure is assumed, then

\[
n(k,\theta) = \begin{cases} 
0 & k < \theta \\
1 & k \geq \theta 
\end{cases} \quad (2-55)
\]

Finally, \( \nu \) is the magnitude of the failure.
For the analysis that follows, the deterministic input and bias terms in the state and measurement equations that are common to both hypotheses may be eliminated, due to the linearity of the equations. Therefore, the no-failure hypothesis \(H_0\) and the failure hypothesis \(H_1\) are represented by

\[
H_0: \quad \hat{x}(k+1) = \Phi(k)\hat{x}(k) + \bar{w}(k) \tag{2-56}
\]

\[
y(k) = C(k)\hat{x}(k) + \bar{v}(k) \tag{2-57}
\]

\[
H_1: \quad \hat{x}(k+1) = \Phi(k)\hat{x}(k) + \bar{w}(k) + b(k)n(k, \theta)v \tag{2-58}
\]

\[
y(k) = C(k)\hat{x}(k) + \bar{v}(k) + d(k)n(k, \theta)v \tag{2-59}
\]

Suppose the data \(y(k)\) are observed over the observation interval

\[
k_0 \leq k \leq k_f \tag{2-60}
\]

For a given time of failure, \(\theta\), and magnitude of failure, \(v\), the LR is given by

\[
\Lambda(k_f, \theta, v) = \frac{p(y(k_0), y(k_0+1), \ldots, y(k_f) \mid H_1, \theta, v)}{p(y(k_0), y(k_0+1), \ldots, y(k_f) \mid H_0)} \tag{2-61}
\]

Because the \(y(k)\) are not independent from time step to time step, the evaluation of the conditional probabilities is difficult. To evaluate the LR, a Kalman filter is implemented, based on the normal mode \(H_0\) system. The filter equations are

\[
\hat{x}^-(k) = \Phi(k)\hat{x}^+(k) \tag{2-62}
\]
\[ \hat{x}^+(k) = \hat{x}^-(k) + K(k)\gamma(k) \]  
\[ \gamma(k) = y(k) - C(k)\hat{x}^-(k) \]  
where \( K(k) \) is the Kalman gain matrix, and \( \gamma(k) \) is the residual, given by

The Kalman gain matrix is given by

\[ K(k) = P^-(k)C^T(k)M^{-1}(k) \]  
where \( P^-(k) \) is the covariance of the estimation error

\[ e^-(k) = x(k) - \hat{x}^-(k) \]  
and \( M(k) \) is the covariance of \( \gamma(k) \), given by

\[ M(k) = C(k)P^-(k)C^T(k) + R(k) \]

The covariance is propagated by

\[ P^-(k+1) = \Phi(k)P^+(k)\Phi^T(k) + Q(k) \]  
\[ P^+(k) = [I - K(k)C(k)]P^-(k) \]

The LR may then be written in terms of the residual sequence \( \gamma(k) \) rather than the measurement sequence. Because the residual sequence is (conditionally) a white Gaussian sequence, the LR is easier to determine in terms of \( \gamma(k) \) than in terms of \( y(k) \).

Due to the linearity of the state equation and the filter equations, the residual may be expressed under each hypothesis as
\[ H_0: \quad \gamma(k) = \gamma_0(k) \quad (2-70) \]

\[ H_1: \quad \gamma(k) = \gamma_0(k) + g(k, \theta) \nu \quad (2-71) \]

where \( \gamma_0(k) \) is a zero-mean, white Gaussian sequence with covariance \( \Phi(k) \). \( g(k, \theta) \) is the failure signature of a failure occurring at time \( \theta \). \( g(k, \theta) \) is given by

\[ g(k, \theta) = C(k)f(k, \theta) + \Omega(k)n(k, \theta) \quad (2-72) \]

where \( f(k, \theta) \) is the influence of the failure mode \( n(k, \theta) \) on the state estimation error. \( f(k, \theta) \) may be generated recursively by

\[ f(k+1, \theta) = \Phi(k)[I - K(k)C(k)]f(k, \theta) + [h(k) - \Phi(k)K(k)d(k)]n(k, \theta) \quad (2-73) \]

with the initial condition

\[ f(k, \theta) = 0 \quad (2-74) \]

It can be seen therefore that

\[ f(k, \theta) = 0, \quad k < \theta \quad (2-75) \]

The LR is given by

\[ \Lambda(k, \theta, \nu) = \frac{p(\gamma(k_0), \gamma(k_0+1), \ldots, \gamma(k_f) \mid H_1, \theta, \nu)}{p(\gamma(k_0), \gamma(k_0+1), \ldots, \gamma(k_f) \mid H_0)} \quad (2-76) \]

Because the residual sequence is (conditionally) Gaussian and white, the LLR ratio has a particularly simple form:
\[
\ell(k_F, \theta, \nu) = \nu \chi(k_F, \theta) - \frac{1}{2} \nu^2 S(k_F, \theta) \quad (2-77)
\]

where

\[
\chi(k_F, \theta) = \sum_{k=0}^{k_F} \bar{q}^T(k, \theta) M^{-1}(k) \gamma(k) \quad (2-78)
\]

\[
S(k_F, \theta) = \sum_{k=0}^{k_F} \bar{q}^T(k, \theta) M^{-1}(k) \bar{q}(k) \quad (2-79)
\]

Now, the generalized likelihood ratio is given by

\[
\ell(k_F) = \max_{\hat{\theta}, \hat{\nu}} \ell(k_F, \hat{\theta}, \hat{\nu}) \quad (2-80)
\]

Performing the maximization over \(\hat{\nu}\) first, we have that

\[
\hat{\nu}(k_F, \theta) = \frac{\chi(k_F, \theta)}{S(k_F, \theta)} \quad (2-81)
\]

Hence, the GLR test statistic is given by

\[
\ell(k_F) = \max_{\hat{\theta}} \frac{1}{2} \chi^2(k_F, \hat{\theta}) \quad \frac{S(k_F, \theta)}{S(k_F, \theta)} \quad (2-82)
\]

As a matter of convenience, the GLR decision function is defined by

\[
DF(k_F) = 2 \ell(k_F) = \max_{\hat{\theta}} \frac{\chi^2(k_F, \hat{\theta})}{S(k_F, \theta)} \quad (2-83)
\]

A failure is detected when the decision function exceeds the detection threshold.

A closed-form solution to the above maximization does not exist in general. Therefore, in order to implement the GLR, the statistic

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\( \lambda(k_f, \theta) \) must be computed for all possible times of failure, \( \theta \). As a result, a bank of matched filters that grows linearly with time is required. To avoid this unlimited growth in the amount of computation, the assumed time of failure may be restricted, say, to be in the range \( k_f - N < \theta \leq k_f \). Even so, the amount of computation required to implement the GLR can be quite large, especially if \( N \) is large.

### 2.3.2 Multiple Model Method

In its simplest form, the Multiple Model (MM) Method [60] resembles the GLR test described in the previous section. Rather than assume that the failure is additive, as in Eqs. (2-47) and (2-48), it is assumed that the system transitions to another system model of the form

\[
\bar{x}(k+1) = \Phi_1(k)\bar{x}(k) + B_1(k)u(k) + \bar{w}_1(k) + \bar{q}_1(k) \tag{2-84}
\]

\[
y(k) = C_1(k)\bar{x}(k) + D_1(k)u(k) + \bar{v}_1(k) + \bar{h}_1(k) \tag{2-85}
\]

where \( \bar{w}_1(k) \) and \( \bar{v}_1(k) \) are independent, zero-mean, white Gaussian sequences with covariances \( \bar{Q}_1(k) \) and \( \bar{R}_1(k) \), respectively. Such a model can represent some kinds of failures which cannot be represented in the GLR test. For example, a change in the dynamics of the system may manifest itself as a change in the state transition matrix, \( \Phi(k) \).

The no-failure hypothesis (\( H_0 \)) is simply that no transition occurs. The failure hypothesis (\( H_1 \)) is that a transition occurs at time \( \theta \). That is, Eqs. (2-47) and (2-48) are valid for \( k < \theta \), and Eqs. (2-84) and (2-85) are valid for \( k \geq \theta \). The failure detection test is similar to the GLR test, in that the test statistic is the generalized log likelihood ratio. However, the calculation of the LLR is not as straightforward as in the GLR test, because the Kalman filter based on \( H_0 \) does not produce a white residual sequence when \( H_1 \) is true.
Therefore, a single Kalman filter cannot be used to implement the MM test. Instead, a separate Kalman filter must be implemented for each $\theta$. Hence, the MM method requires a bank of Kalman filters that grows linearly with time.

2.3.3 Summary

The complexity of the GLR test and of the MM method is not a given property of generalized likelihood ratio tests. Rather, the complexity arises from the need to estimate the random onset time of the failure, $\theta$. Because $\theta$ affects the measurements non-linearly, the estimation of $\theta$ requires a non-linear estimation scheme. In particular, this is the reason for the growing bank of matched filters or Kalman filters.

For most FDI problems, the goal is to quickly detect and isolate failures, not to estimate the time of failure. Therefore, the failure time $\theta$ is included in the failure hypothesis only to allow the failure mode to be accurately represented, not because it is an important parameter. If a failure hypothesis can accurately model system failures without an unknown parameter such as the time of failure, then that hypothesis might be used to develop an FDI algorithm that is more computationally efficient than the GLR test or the MM method.

Another problem with both the GLR test and the MM method is that both are designed with specific modes of failure in mind. If a component of the system fails in an unmodeled way, then the algorithms may have unpredictable results. One possibility is that the algorithms may fail to detect the failure, or that the detection will be greatly delayed. An even worse possibility is that the failure will be detected, but isolated to the wrong component. In general, it is desirable that an FDI algorithm be robust to failure mode uncertainty.

One method for achieving a robust FDI algorithm is to include among the failure hypotheses all possible failure modes. However, this approach is not practical for two reasons. First, for each failure mode (i.e., each failure hypothesis), a bank of filters (either matched filters or Kalman filters) must be implemented to calculate the test
statistic. This can increase the amount of computation required significantly. Second, it is usually not possible to anticipate all of the possible failure modes.

A conclusion that can be drawn from the above discussion is that the characteristics of an FDI algorithm that is based on the generalized likelihood ratio test are a function of the failure hypotheses. This is trivially true, since the hypotheses uniquely determine the test, except for implementation details. Therefore, in order to develop a test with desired properties, the failure hypotheses must be selected appropriately. This approach is taken in the next chapter, where the failure hypotheses are selected to yield a computationally efficient, robust FDI algorithm.
CHAPTER 3

THE ORTHOGONAL SERIES GLR TEST

In Chapter 2, it was seen that is is generally not possible to determine the optimal FDI algorithm for a dynamic system, or even to specify what criterion should be optimized. Therefore, one must usually be content with a test developed by some other means. One useful approach is to treat the sequential testing problem as a fixed-length data testing problem. Several algorithms have been developed in this way, including Multiple Model (MM) Methods [60] and the Generalized Likelihood Ratio (GLR) Test [40, 57]. Although these tests are quite powerful, they tend to be computationally burdensome. This complexity arises from the need to estimate the random onset time of the failure, which generally requires a nonlinear estimation structure. In addition, these methods may not be robust to failure mode uncertainty.

One FDI algorithm that is robust to failure mode uncertainty is the detection filter of Beard [3] and Jones [35]. Unfortunately, the applicability of the detection filter is limited by a number of factors. One restriction is that the theory is limited to linear time-invariant systems. Also, the detection filter design process breaks down for systems where two or more failure types are not "output separable," even though it should be possible to detect and isolate failures for some such systems. Finally, little guidance exists on how to choose the free parameters in the design process, such as the filter eigenvalues.

In this chapter, an algorithm that addresses these problems, the **Orthogonal Series Generalized Likelihood Ratio (OSGLR) Test**, is proposed
As suggested by its name, the OSGLR test is indeed a GLR test. The hypothesis upon which the test is based is that the failure modes can be represented as truncated orthogonal series of time functions. Because such a series can represent a broad class of failure modes, the test should be robust to failure mode uncertainty. The test is not as computationally complex as other GLR methods, because the time of failure does not enter the failure hypothesis explicitly. The only unknowns in the failure hypothesis are the coefficients of the terms in the orthogonal series. Because they enter the state and measurement equations linearly, these unknowns can be estimated by relatively simple linear schemes.

For the most part, we will deal only with continuous-time systems in this chapter. Because the discrete-time case is similar to the continuous-time case, and because it is somewhat more cumbersome notationally, it will be presented in Appendix A. In practice, the discrete-time case is likely to be more useful, because it is more amenable to computer implementation.

3.1 OSGLR Hypotheses

We are interested in detecting failures in linear dynamic systems, which under normal conditions are modeled by

\[
\frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) + w(t) \tag{3-1}
\]

\[y(t) = C(t)x(t) + D(t)u(t) + v(t) \tag{3-2}\]

where \(x(t)\) is an \(n\)-dimensional state vector, \(u(t)\) is a \(q\)-dimensional input vector, and \(y(t)\) is an \(m\)-dimensional measurement vector. \(A(t), B(t), C(t), \text{ and } D(t)\) are known matrices with appropriate dimensions. \(w(t)\) and \(v(t)\) are independent, zero-mean, Gaussian processes with autocorrelation functions given by

\[
E[w(t_1)w^T(t_2)] = Q(t_1)\delta(t_1-t_2) \tag{3-3}
\]
\[ E[y(t_1)\dot{y}(t_2)] = R(t_1)\delta(t_1-t_2) \quad (3-4) \]

When a failure occurs, either the state dynamics change or the measurement equation changes. For example, if the \( i \)th actuator fails, the actual input vector, \( u_a(t) \), differs from the commanded input, \( u(t) \), as follows:

\[ u_a(t) = u(t) + e_{qi}f(t) \quad (3-5) \]

where \( e_{qi} \) is a \( q \)-dimensional unit vector in the \( i \)th coordinate direction, i.e., the elements of \( e_{qi} \) are given by

\[ (e_{qi})_j = \delta_{ij}, \quad j = 1, 2, \ldots, q \quad (3-6) \]

where \( \delta_{ij} \) is the Kronecker delta. \( f(t) \) is the mode shape of the failure. For example, if the failure is a bias shift, then \( f(t) \) is a step function of some magnitude at the time of the failure. Thus, the state dynamics and measurement equations become

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t) + b_i(t)f(t) + w(t) \quad (3-7) \]

\[ y(t) = C(t)x(t) + D(t)u(t) + d_i(t)f(t) + v(t) \quad (3-8) \]

where \( b_i(t) \) and \( d_i(t) \) are the \( i \)th columns of \( B(t) \) and \( D(t) \), respectively.

Similarly, a failure in the \( i \)th sensor can generally be represented in the measurement equation as

\[ y(t) = C(t)x(t) + D(t)u(t) + e_{mi}(t)f(t) + v(t) \quad (3-9) \]
As in the case of an actuator failure, \( f(t) \) depends on the mode of the failure. For example, if the output of the \( i \)th sensor is fixed at zero (except for the additive noise), then

\[
f(t) = -c_i^T(t)x(t) - d_i^T(t)u(t)
\]  

(3-10)

where in this case \( d_i^T(t) \) is the \( i \)th row of \( D(t) \), and where \( c_i^T(t) \) is the \( i \)th row of \( C(t) \).

The situation is more complicated for changes in the plant dynamics. For example, the state dynamics equation may become

\[
\frac{dx(t)}{dt} = [A(t) + \Delta A(t)]x(t) + B(t)u(t) + w(t)
\]  

(3-11)

where \( A(t) + \Delta A(t) \) is the new state dynamics matrix. It often turns out that Eq. (3-11) can be represented in the form

\[
\frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) + h(t)f(t) + w(t)
\]  

(3-12)

where \( h(t) \) and \( f(t) \) are such that

\[
h(t)f(t) = \Delta A(t)x(t)
\]  

(3-13)

Such a representation is not possible for arbitrary \( \Delta A(t) \). For such a representation to exist, \( \Delta A(t) \) must have rank 1. Then \( \Delta A(t) \) can be represented as

\[
\Delta A(t) = h(t)\alpha^T(t)
\]  

(3-14)

where \( h(t) \) spans the column space of \( \Delta A(t) \), and \( \alpha(t) \) is a vector of coefficients. Then \( f(t) \) must be given by
\[ f(t) = \alpha^T(t)x(t) \]  \hspace{1cm} (3-15)

For many types of dynamics changes, \( \Delta A(t) \) can be represented as in Eq. (3-14). For example, if the dynamics change affects only a single element of the state vector, say the \( i^{th} \), then only the \( i^{th} \) row of \( \Delta A(t) \) is nonzero. Hence,

\[ b(t) = e_{ni} \]  \hspace{1cm} (3-16)

and

\[ a_j(t) = \Delta A_{ij}(t) \]  \hspace{1cm} (3-17)

On the other hand, if the dynamics change is such that the effect of only the \( i^{th} \) element of the state vector on the state equation changes, then only the \( i^{th} \) column of \( \Delta A(t) \) is nonzero. Then

\[ b_j(t) = \Delta A_{ji}(t) \]  \hspace{1cm} (3-18)

and

\[ a(t) = e_{ni} \]  \hspace{1cm} (3-19)

Even when the dynamics change does not fall into one of the two categories above, it is likely that \( \Delta A(t) \) can be decomposed as in Eq. (3-14). For example, consider the spring-mass system shown in Figure 3-1. The state equation for this system is given by
Figure 3-1. Spring-mass system.
\[
\frac{d}{dt} \begin{bmatrix}
  x_1(t) \\
  v_1(t) \\
  x_2(t) \\
  v_2(t)
\end{bmatrix} = \begin{bmatrix}
  0 & 1 & 0 & 0 \\
  -\frac{K}{M_1} & 0 & \frac{K}{M_1} & 0 \\
  0 & 0 & 0 & 1 \\
  \frac{K}{M_2} & 0 & -\frac{K}{M_2} & 0
\end{bmatrix} \begin{bmatrix}
  x_1(t) \\
  v_1(t) \\
  x_2(t) \\
  v_2(t)
\end{bmatrix} + \begin{bmatrix}
  0 \\
  1 \\
  0 \\
  0
\end{bmatrix} u(t)
\]

(3-20)

If the spring stiffness were to suddenly change, then the change in the state dynamics matrix would be

\[
\Delta A(t) = \begin{bmatrix}
  0 & 0 & 0 & 0 \\
  -\frac{\Delta K}{M_1} & 0 & \frac{\Delta K}{M_1} & 0 \\
  0 & 0 & 0 & 0 \\
  \frac{\Delta K}{M_2} & 0 & -\frac{\Delta K}{M_2} & 0
\end{bmatrix}
\]

(3-21)

which has rank 1. Therefore, we have that

\[
b(t) = \begin{bmatrix}
  0 \\
  -\frac{\Delta K}{M_1} \\
  0 \\
  +\frac{\Delta K}{M_2}
\end{bmatrix}
\]

(3-22)

and

\[
\alpha^T(t) = (1 \ 0 \ -1 \ 0)
\]

(3-23)
This result is typical of dynamics changes in which the failure affects the different elements of the state vector proportionally. This proportionality is expressed by the vector $b(t)$.

Thus, for most types of failures (actuator failures, sensor failures, and some types of dynamics changes), the effect of the failure can be captured in the state and measurement equations as

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) + b(t)f(t) + w(t) \quad (3-24)$$

$$y(t) = C(t)x(t) + D(t)u(t) + d(t)f(t) + v(t) \quad (3-25)$$

The vectors $b(t)$ and $d(t)$ are known ahead of time for each type of failure, whereas the mode of the failure, $f(t)$, is generally unknown a priori.

In the analysis that follows, the terms due to the input $u(t)$ in the state and measurement equations can be neglected, due to the linearity of the equations and the fact that $B(t)$, $D(t)$, and $u(t)$ are known. It is important to remember, however, that these terms must be included in any implementation of the OSGLR algorithm. Specifically, these terms must be included in the Kalman filter that estimates $x(t)$. (See Section 3.2.)

For the time being, the discussion will be confined to the binary hypothesis testing case. That is, we will assume that we are only trying to detect a single failure type, rather than detect and isolate from a set of failure types. Later, the results will be extended to include the isolation problem.

The OSGLR test will be derived in the manner suggested in Section 2.2. That is, the fixed-length data test for data observed over the interval

$$t_0 \leq t \leq t_f$$
will be found. This fixed data test will then be suitably modified to form a sequential test.

Therefore, the no-failure ($H_0$) and failure ($H_1$) hypotheses are given by

$$H_0: \quad \frac{dx(t)}{dt} = A(t)x(t) + w(t) \quad (3-26)$$

$$y(t) = C(t)x(t) + v(t) \quad (3-27)$$

$$H_1: \quad \frac{dx(t)}{dt} = A(t)x(t) + w(t) + b(t)f(t) \quad (3-28)$$

$$y(t) = C(t)x(t) + v(t) + d(t)f(t) \quad (3-29).$$

Because the failure mode shape $f(t)$ has not been specified, the hypothesis $H_1$ is not complete. In order that the test be robust to failure mode uncertainty, it would be desirable to allow $f(t)$ to be completely arbitrary. However, this assumption does not lead to a well-posed problem. Therefore, some further assumptions must be made.

The approach that will be taken here is to represent the mode shape $f(t)$ by a truncated series expansion with unknown coefficients. The motivation is that if the basis functions of the expansion are chosen properly, it should be possible to approximately represent a rich class of failure modes. Therefore, it is assumed that $f(t)$ can be expressed as

$$f(t) = \sum_{i=1}^{p} a_{1i} \phi_{1i}(t) \quad (3-30)$$

where $p$ is the number of basis functions, the $a_{1i}$ are unknown coefficients, and the $\phi_{1i}(t)$ are the basis functions. Equation (3-30) can be expressed more conveniently in vector form as
\[ f(t) = \phi_1^T(t) \alpha_1 \quad (3-31) \]

The subscript "1" indicates that this is an intermediate representation. Ultimately, we will be interested in a representation of the form

\[ f(t) = \psi^T(t_f - t) \alpha(t_f) \quad (3-32) \]

The reason for converting the representation of \( f(t) \) to that in Eq. (3-32) will become clear later. It cannot be overemphasized that the representation of \( f(t) \) in Eq. (3-31) is meant to be an approximation. There is no reason to believe that an actual failure mode will have this particular form.

Two important features of these hypotheses are that the failure hypothesis does not include a parameter representing the time of failure, and that the unknown parameters enter into the problem linearly. Hence, nonlinear estimation will not be required. As will be seen, this significantly reduces the amount of computation required relative to other GLR methods.

3.2 Derivation of the OSGLR Algorithm

The OSGLR test will be derived in several steps. First, the test will be derived based on the representation of \( f(t) \) given by Eq. (3-31). Next, the test will be converted to a form that corresponds to a second intermediate representation of \( f(t) \). A special case of this representation is the representation in Eq. (3-32). Finally, the test statistic will be converted from an integral representation to a differential equation representation.

To determine the form of the test, we proceed as follows. A Kalman filter [26, Ch. 4; 34, Ch. 7], based on the unfailed system statistics (\( H_0 \)), is used to generate the residual process \( y(t) \). The filter equations are
\[
\frac{d\hat{x}(t)}{dt} = A(t)\hat{x}(t) + K(t)\gamma(t)
\]  \hspace{1cm} (3-33)

\[
\hat{x}(t_0) = 0
\]  \hspace{1cm} (3-34)

\[
\gamma(t) = \gamma(t) - C(t)\hat{x}(t)
\]  \hspace{1cm} (3-35)

\[
K(t) = P(t)C^T(t)R^{-1}(t)
\]  \hspace{1cm} (3-36)

\[
\frac{dP(t)}{dt} = A(t)P(t) + P(t)A^T(t) + Q(t) - P(t)C^T(t)R^{-1}(t)C(t)P(t)
\]  \hspace{1cm} (3-37)

\[
P(t_0) = P_0
\]  \hspace{1cm} (3-38)

where \(\hat{x}(t)\) is the estimate of \(x(t)\), \(P(t)\) is the covariance of the estimation error, and \(K(t)\) is the Kalman gain matrix. The process \(\gamma(t)\) contains exactly the same information as \(\gamma(t)\), because each can be determined unambiguously from the other. However, \(\gamma(t)\) is easier to work with than \(\gamma(t)\) because it is a white noise process, whereas \(\gamma(t)\) is correlated in time.

By the linearity of the Kalman filter and the system equations, the residual can be decomposed as

\[
\gamma(t) = \gamma_0(t) + \gamma_1(t)
\]  \hspace{1cm} (3-39)

where \(\gamma_0(t)\) is the residual process that results under \(H_0\), and \(\gamma_1(t)\) is the part of the residual due to the failure. Again, due to the linearity of the filter and the system, \(\gamma_1(t)\) can be expressed as
\[ \gamma_1(t) = G_1(t)a_1 \]  

(3-40)

where the matrix \( G_1(t) \) represents the influence of the vector of coefficients \( a_1 \) on the residuals, and remains to be determined. Therefore, we can rewrite the two hypotheses as

\[ H_0: \gamma(t) = \gamma_0(t), \quad t_0 \leq t \leq t_f \]  

(3-41)

\[ H_1: \gamma(t) = \gamma_0(t) + G_1(t)a_1, \quad t_0 \leq t \leq t_f \]  

(3-42)

where \( \gamma_0(t) \) is a zero-mean, Gaussian process with the autocorrelation function

\[ E[\gamma_0(t_1)\gamma_0^T(t_2)] = R(t_1)\delta(t_1-t_2) \]  

(3-43)

Hence, the problem of deciding between \( H_0 \) and \( H_1 \) has been reduced to the problem of deciding whether or not a bias signal is present in the white residual process.

It can be shown [51, Sec. 4.2] that the information vector, defined by

\[ \chi(t_f) = \int_{t_0}^{t_f} G_1^T(t)R^{-1}(t)\gamma(t)dt \]  

(3-44)

is a sufficient statistic, i.e., that it contains all the information contained in the residual process regarding the hypotheses. Hence, it can be used in place of the entire time history of the residuals to determine whether a failure has occurred. Now, \( \chi_1(t_f) \) is a Gaussian random vector, because \( \gamma(t) \) is a Gaussian random process and \( \chi_1(t_f) \) is a linear transformation of \( \gamma(t) \). Hence, its probability density is
completely specified by its mean and covariance. Under $H_0$, the mean of $\bar{X}_1(t_f)$ is given by

$$E[\bar{X}_1(t_f) | H_0] = E \left[ \int_{t_0}^{t_f} G_1^T(t) R^{-1}(t) Y(t) dt | H_0 \right]$$ \hspace{1cm} (3-45)$$

Interchanging the order of the expectation and the integration, we have that

$$E[\bar{X}_1(t_f) | H_0] = \int_{t_0}^{t_f} G_1^T(t) R^{-1}(t) E[Y(t) | H_0] dt$$ \hspace{1cm} (3-46)$$

But

$$E[Y(t) | H_0] = 0$$ \hspace{1cm} (3-47)$$

Therefore,

$$E[\bar{X}_1(t_f) | H_0] = 0$$ \hspace{1cm} (3-48)$$

The covariance matrix of $\bar{X}_1(t_f)$, known as the information matrix, is given by

$$S_1(t_f) = E[\bar{X}_1(t_f) \bar{X}_1^T(t_f) | H_0]$$

$$= E \left[ \int_{t_0}^{t_f} G_1^T(t) R^{-1}(t) Y(t) dt \right] \left[ \int_{t_0}^{t_f} G_1^T(t) R^{-1}(t) Y(t) dt \right]^T \left| H_0 \right]$$ \hspace{1cm} (3-49)$$

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The product of two integrals in the above equation can be expressed as a double integration by introducing the dummy variables of integration \( t_1 \) and \( t_2 \). Interchanging the order of the expectation and the integration yields

\[
S_1(t_f) = \int_{t_0}^{t_f} \int_{t_0}^{t_f} G_1^T(t_1) R^{-1}(t_1) E[Y(t_1) Y^T(t_2) | H_0] R^{-1}(t_2) G_1(t_2) dt_1 dt_2
\]

(3-50)

Using Eqs. (3-41) and (3-43), this reduces to

\[
S_1(t_f) = \int_{t_0}^{t_f} \int_{t_0}^{t_f} G_1^T(t_1) R^{-1}(t_1) R(t_1) \delta(t_1-t_2) R^{-1}(t_2) G_1(t_2) dt_1 dt_2
\]

(3-51)

Integrating over either \( t_1 \) or \( t_2 \) and simplifying the result gives

\[
S_1(t_f) = \int_{t_0}^{t_f} G_1^T(t) R^{-1}(t) G_1(t) dt
\]

(3-52)

Under \( H_1 \), \( X_1(t_f) \) has the same covariance, but its mean is given by

\[
E[X_1(t_f) | H_1] = \int_{t_0}^{t_f} G_1^T(t) R^{-1}(t) E[Y(t) | H_1] dt
\]

(3-53)

Using Eq. (3-42) to evaluate \( E[Y(t) | H_1] \), we have that

\[
E[X_1(t_f) | H_1] = \int_{t_0}^{t_f} G_1^T(t) R^{-1}(t) G_1(t) a_1 dt
\]

(3-54)
Thus,

\[ E[\bar{x}_1(t_f) | H_1] = S_1(t_f) a_1 \]  \hspace{1cm} (3-55)

Therefore, the two hypotheses can be rewritten once again as

\[ H_0: \bar{x}_1(t_f) \sim N(0, S_1(t_f)) \]  \hspace{1cm} (3-56)

\[ H_1: \bar{x}_1(t_f) \sim N(S_1(t_f)a_1, S_1(t_f)) \]  \hspace{1cm} (3-57)

The problem of deciding between \( H_0 \) and \( H_1 \) has now been reduced to the problem of deciding whether the Gaussian random vector \( \bar{x}_1(t_f) \) has zero mean or nonzero mean. Because the hypothesis \( H_1 \) is composite, an appropriate test to use is the generalized likelihood ratio (GLR) test. The first step is to determine the likelihood ratio (LR), defined by

\[ \Lambda(t_f, a_1) = \frac{p(\bar{x}_1(t_f) | H_1, a_1)}{p(\bar{x}_1(t_f) | H_0)} \]  \hspace{1cm} (3-58)

The probability densities are given by

\[ p(\bar{x}_1(t_f) | H_0) = \frac{1}{(2\pi)^{P/2}|S_1(t_f)|^{1/2}} \exp \left\{ -\frac{1}{2} \bar{x}_1^T(t_f) S_1^{-1}(t_f) \bar{x}_1(t_f) \right\} \]  \hspace{1cm} (3-59)

\[ p(\bar{x}_1(t_f) | H_1, a_1) = \frac{1}{(2\pi)^{P/2}|S_1(t_f)|^{1/2}} \]

\[ \cdot \exp \left\{ -\frac{1}{2} \left[ \bar{x}_1(t_f) - S_1(t_f)a_1 \right]^T S_1^{-1}(t_f) \left[ \bar{x}_1(t_f) - S_1(t_f)a_1 \right] \right\} \]  \hspace{1cm} (3-60)
Therefore, the LR is given by

\[ \Lambda(t_f, \hat{a}_1) = \exp \left\{ \hat{a}_1^T \chi_1(t_f) - \frac{1}{2} \hat{a}_1^T S_1(t_f) \hat{a}_1 \right\} \]  \hspace{1cm} (3-61)

It will be more convenient to work with the log likelihood ratio (LLR), defined by

\[ \ell(t_f, \hat{a}_1) = \ln \Lambda(t_f, \hat{a}_1) = \hat{a}_1^T \chi_1(t_f) - \frac{1}{2} \hat{a}_1^T S_1(t_f) \hat{a}_1 \]  \hspace{1cm} (3-62)

The GLR test statistic is given by

\[ \ell(t_f) = \max_{\hat{a}_1(t_f)} \ell(t_f, \hat{a}_1(t_f)) \]  \hspace{1cm} (3-63)

The maximization may be performed by taking the gradient of the right side of Eq. (3-62) with respect to \( \hat{a}_1 \) and setting the result to zero. The result is

\[ \hat{a}_1(t_f) = S_1^{-1}(t_f) \chi_1(t_f) \]  \hspace{1cm} (3-64)

Substituting this value back into the LLR gives the GLR test statistic

\[ \ell(t_f) = \frac{1}{2} \chi_1^T(t_f) S_1^{-1}(t_f) \chi_1(t_f) \]  \hspace{1cm} (3-65)

As a matter of convenience, the GLR decision function is defined by

\[ DF(t_f) = 2\ell(t_f) = \chi_1^T(t_f) S_1^{-1}(t_f) \chi_1(t_f) \]  \hspace{1cm} (3-66)

Then the GLR test is given by

54
The threshold is written as $T^2$ because the decision function is a positive definite form. Therefore, a negative threshold would yield a trivial test that always decides that $H_1$ is true. Also, writing the threshold as $T^2$ rather than $T$ will simplify the results of later chapters.

To complete the derivation of the test, we must determine $G_1(t)$. Recall that $\gamma(t)$ is given by

$$\gamma(t) = y(t) - C(t)\hat{x}(t)$$  \hspace{1cm} (3-68)

Expanding $y(t)$ by using Eqs. (3-29) and (3-31), we have that

$$\gamma(t) = C(t)x(t) + \bar{v}(t) + d(t) \phi_{1}^{T}(t) a_{1} - C(t)\hat{x}(t)$$  \hspace{1cm} (3-69)

If the state estimation error is defined as

$$\bar{e}(t) = x(t) - \hat{x}(t)$$  \hspace{1cm} (3-70)

then $\gamma(t)$ is given by

$$\gamma(t) = C(t)e(t) + \bar{v}(t) + d(t) \phi_{1}^{T}(t) a_{1}$$  \hspace{1cm} (3-71)

Now, by the linearity of the filter and state equations, we can express $\bar{e}(t)$ as the sum of two parts:

$$\bar{e}(t) = \bar{e}_0(t) + \bar{e}_1(t)$$  \hspace{1cm} (3-72)

55
where \( e_0(t) \) is the estimation error that results under \( H_0 \), and \( e_1(t) \) is the part of the estimation error due to \( a_1 \). As with \( y_1(t) \), \( e_1(t) \) can be expressed as a linear function of \( a_1 \):

\[
e_1(t) = F_1(t)a_1
\]  

(3-73)

where the matrix \( F_1(t) \) represents the influence of the vector of coefficients \( a_1 \) on the state estimation error. Therefore, we can write

\[
y(t) = C(t)e_0(t) + v(t) + [C(t)F_1(t) + d(t)\phi_1^T(t)]a_1
\]  

(3-74)

The first two terms in Eq. (3-74) may be identified as \( y_0(t) \), the residual under \( H_0 \). Hence, the remainder of the equation is \( y_1(t) \), so that

\[
y_1(t) = [C(t)F_1(t) + d(t)\phi_1^T(t)]a_1
\]  

(3-75)

Therefore, we have that

\[
G_1(t) = C(t)F_1(t) + d(t)\phi_1^T(t)
\]  

(3-76)

Finally, we must determine \( F_1(t) \). To do this, we must determine how \( a_1 \) affects \( e_1(t) \). The differential equation for \( e(t) \) may be found from the differential equations for \( x(t) \) and \( \hat{x}(t) \):

\[
\frac{de(t)}{dt} = \frac{dx(t)}{dt} - \frac{\hat{dx}(t)}{dt} = A(t)x(t) + w(t) + b(t)\phi_1^T(t)a_1 - A(t)\hat{x}(t) - K(t)y(t)
\]  

(3-77)

Expanding \( y(t) \) as in Eq. (3-71) and collecting terms yields
\[
\frac{de(t)}{dt} = [A(t) - K(t)C(t)]e(t) + [b(t) - K(t)d(t)]^T \phi_1(t) a_1 \\
+ w(t) - K(t)v(t)
\]  
(3-78)

Recall that \( e_1(t) \) is the part of the estimation error due to \( a_1 \). The differential equation for \( e_1 \) is therefore

\[
\frac{d}{dt} e_1(t) = [A(t) - K(t)C(t)]e_1(t) + [b(t) - K(t)d(t)]^T \phi_1(t) a_1
\]  
(3-79)

with initial condition

\[
e_1(t_0) = 0
\]  
(3-80)

But we know that \( e_1(t) = F_1(t)a_1 \). Therefore,

\[
\frac{dF_1(t)}{dt} a_1 = [A(t) - K(t)C(t)]F_1(t)a_1 + [b(t) - K(t)d(t)]^T \phi_1(t) a_1
\]  
(3-81)

Because Eq. (3-81) is true for all \( a_1 \), we must have that

\[
\frac{dF_1(t)}{dt} = [A(t) - K(t)C(t)]F_1(t) + [b(t) - K(t)d(t)]^T \phi_1(t)
\]  
(3-82)

Also, because of Eq. (3-80),

\[
F_1(t_0) = 0
\]  
(3-83)

Equations (3-76), (3-82), and (3-83) specify \( G_1(t) \), as required.

The above equations specify the OSGLR test for the failure as represented by Eq. (3-31). However, this representation has two
weaknesses. First, because the basis functions $\phi_{1i}(t)$ were chosen arbitrarily, they may be highly correlated. For example, suppose that the basis functions are given by

$$\phi_{1i}(t) = e^{(i-1/2)t}, \quad i = 1, 2, \ldots, p$$

(3-84)

over the observation interval

$$t_0 = -\infty < t < t_f = 0$$

(3-85)

Then the correlation matrix, defined by

$$[P_{\phi}]_{ij} = \int_{t_0}^{t_f} \phi_{1i}(t)\phi_{1j}(t)dt$$

(3-86)

is given by

$$[P_{\phi}]_{ij} = \int_{-\infty}^{0} e^{(i+j-1)t}dt = \frac{1}{i+j-1}$$

(3-87)

For example, if $p$ is 3, then $P_{\phi}$ is given by

$$P_{\phi} = \begin{bmatrix}
1 & 1/2 & 1/3 \\
1/2 & 1/3 & 1/4 \\
1/3 & 1/4 & 1/5
\end{bmatrix}$$

(3-88)

This matrix may be recognized as a Hilbert matrix [48, p. 34], which is the classic example of an ill-conditioned matrix. Thus, using such a basis causes the problem to be numerically ill-conditioned.
(The preceding example is not as contrived as it may appear. In fact, one of the bases proposed in Section 5.1 is very similar to the one used in this example.)

The second problem is that the basis functions are defined relative to an absolute time scale, rather than with respect to the terminal time, \( t_f \). For a number of reasons, it is desirable to define the basis functions relative to the time \( t_f \), in which case the basis functions are \( \phi_i(t_f-t) \) rather than \( \phi_{i1}(t) \). For one thing, if the transformed basis functions are functions of \( t_f-t \), then they will have the same shape on the time scale defined relative to the end of the observation interval, \( t_f \). Also, if the system is time-invariant, the OSGLR equations will then be time-invariant in steady state.

Both of these problems can be remedied in the following way: A new set of basis functions will be defined that is the original set of basis functions orthogonalized over the interval \([t_0,t_f]\). This will eliminate the first of the problems discussed above. The second problem may be solved by judicious choice of the original set of basis functions.

Suppose the vector of basis functions \( \phi_1(t) \) is transformed by an invertible linear transformation \( \Gamma(t) \), so that

\[
\phi_2(t,t_1) = \Gamma(t_1)\phi_1(t) \tag{3-89}
\]

Ultimately, the time of the transformation, \( t_1 \), will be chosen as the endpoint of the observation interval, \( t_f \). The dummy variable \( t_1 \) is used at this point for clarity. Note that the transformation \( \Gamma(t_f) \) is intended to orthogonalize the vector of basis function \( \phi(t) \) over the interval \([t_0,t_f]\), although it is not necessary that it do so. Recall that

\[
f(t) = \phi_1^T(t)a_1 \tag{3-90}
\]
Solving Eq. (3-89) for $\phi_{11}^T(t)$ yields

$$f(t) = \phi_{22}^T(t, t_1) \Gamma^{-T}(t_1) a_1$$  \hspace{1cm} (3-91)

Therefore, $f(t)$ can be represented by

$$f(t) = \phi_{22}^T(t, t_1) a(t_1)$$  \hspace{1cm} (3-92)

where

$$a(t_1) = \Gamma^{-T}(t_1) a_1$$  \hspace{1cm} (3-93)

Because the representation of $f(t)$ in Eq. (3-92) is analogous to the representation of $f(t)$ in Eq. (3-90), the information vector and matrix for this representation of $f(t)$ can be determined directly from the results obtained earlier:

$$X_2(t_f, t_1) = \int_{t_0}^{t_f} G_2^T(t, t_1) R^{-1}(t) \gamma(t) dt$$  \hspace{1cm} (3-94)

$$S_2(t_f, t_1) = \int_{t_0}^{t_f} G_2^T(t, t_1) R^{-1}(t) G_2(t, t_1) dt$$  \hspace{1cm} (3-95)

where

$$G_2(t, t_1) = C(t) F_2(t, t_1) + d(t) \phi_{22}^T(t, t_1)$$  \hspace{1cm} (3-96)

$$\frac{d}{dt} F_2(t, t_1) = [A(t) - K(t) C(t)] F_2(t, t_1) + [b(t) - K(t) d(t)] \phi_{22}^T(t, t_1)$$  \hspace{1cm} (3-97)
\[ F_{2}(t_0, t_1) = 0 \] (3-98)

Of course, the failure hypothesis has not changed, because the set of basis functions \( \{ \phi_{21}(t, t_1) \} \) span the same space of possible failure modes as does the set of basis functions \( \{ \phi_{11}(t) \} \), because \( \Gamma(t_1) \) is invertible. Therefore, it should be possible to relate the information vectors, the information matrices, and influence matrices for the two representations. To do this, consider \( e_1(t) \), the estimation error due to the failure. \( e_1(t) \) may be represented as

\[ e_1(t) = F_1(t)a_1 \] (3-99)

or

\[ e_1(t) = F_2(t, t_1)a(t_1) \] (3-100)

But \( a_1 \) and \( a(t_1) \) are related by Eq. (3-93). Therefore,

\[ e_1(t) = F_2(t, t_1)\Gamma^{-T}(t_1)a_1 \] (3-101)

Comparing Eq. (3-99) and Eq. (3-101), it must be that \( F_1(t) \) and \( F_2(t, t_1) \) are related by

\[ F_2(t, t_1) = F_1(t)\Gamma^T(t_1) \] (3-102)

Similarly,

\[ G_2(t, t_1) = G_1(t)\Gamma^T(t_1) \] (3-103)
Substituting this expression for $G_2(t,t_1)$ into Eqs. (3-94) and (3-95) yields

$$x_2(t_f,t_1) = \Gamma(t_1)x_1(t_f) \quad (3-104)$$

$$s_2(t_f,t_1) = \Gamma(t_1)s_1(t_f)\Gamma^T(t_1) \quad (3-105)$$

As indicated earlier, we are interested in the case where $t_1 = t_f$, because the transformation $\Gamma(t_1)$ was intended to produce desirable properties in the transformed basis functions when $t_1$ corresponds to the end of the observation interval, $t_f$. For this special case,

$$x_2(t_f,t_f) = \Gamma(t_f)x_1(t_f) \quad (3-106)$$

$$s_2(t_f,t_f) = \Gamma(t_f)s_1(t_f)\Gamma^T(t_f) \quad (3-107)$$

$x_1(t_f)$ and $s_1(t_f)$ have integral representations, as given in Eq. (3-94) and Eq. (3-95), respectively. $x_1(t_f)$ and $s_1(t_f)$ may also be determined from the differential equations

$$\frac{d}{dt_f}x_1(t_f) = g_{1}^{T}(t_f)R^{-1}(t_f)y(t_f) \quad (3-108)$$

$$\frac{d}{dt_f}s_1(t_f) = g_{1}^{T}(t_f)R^{-1}(t_f)g_{1}(t_f) \quad (3-109)$$

with initial conditions

$$x_1(t_0) = 0 \quad (3-110)$$
\[ S_0(t) = 0 \]  

(3-111)

Differential equations for \( X_2(t', t_f) \) and \( S_2(t', t_f) \) will be derived below.

Before proceeding with this derivation, however, we will digress to define a useful matrix that will appear in the differential equations. Consider taking the derivative with respect to \( t_1 \) of \( a_1 \), which is given by

\[ \dot{a}_1 = \Gamma^T(t_1) a(t_1) \]  

(3-112)

Because \( a_1 \) is independent of \( t_1 \), the derivative of the left side of Eq. (3-112) must be zero. Therefore,

\[ 0 = \left[ \frac{d}{dt_1} \Gamma^T(t_1) \right] a(t_1) + \Gamma^T(t_1) \frac{d}{dt_1} a(t_1) \]  

(3-113)

Solving Eq. (3-113) for \( \frac{da_1(t_1)}{dt_1} \) yields

\[ \frac{d}{dt_1} a(t_1) = -\Gamma^T(t_1) \frac{d\Gamma^T(t_1)}{dt_1} a(t_1) \]  

(3-114)

Hence, if the matrix \( A_a(t_1) \) is defined by

\[ A_a(t_1) \overset{\Delta}{=} -\Gamma^T(t_1) \frac{d\Gamma^T(t_1)}{dt_1} \]  

(3-115)

then the differential equation for \( a(t_1) \) is given by

\[ \frac{d}{dt_1} a(t_1) = A_a(t_1) a(t_1) \]  

(3-116)
Not surprisingly, the matrix \( A_a(t_f) \) will appear in the differential equations for \( \chi_2(t_f, t_f) \) and \( S_2(t_f, t_f) \).

To determine the differential equation for \( \chi_2(t_f, t_f) \), the derivative of both sides of Eq. (3-104) is taken with respect to \( t_f \), which gives

\[
\frac{d}{dt_f} \chi_2(t_f, t_f) = \frac{d\Gamma(t_f)}{dt_f} \chi_1(t_f) + \Gamma(t_f) \frac{d\chi_1(t_f)}{dt_f} \quad (3-117)
\]

The first term on the right side of this equation may be expressed in terms of \( \chi_2(t_f, t_f) \) by use of Eq. (3-104). The derivative in the second term is given by Eq. (3-108). Therefore,

\[
\frac{d}{dt_f} \chi_2(t_f, t_f) = \frac{d\Gamma(t_f)}{dt_f} \Gamma^{-1}(t_f) \chi_2(t_f, t_f)
\]

\[+ \Gamma(t_f) G_1^T(t_f) R^{-1}(t_f) \gamma(t_f) \quad (3-118)\]

The expression which multiplies \( \chi_2(t_f) \) may be recognized as \(-A_a^T(t_f)\), whereas the expression which multiplies \( R^{-1}(t_f) \gamma(t_f) \) is \( G_2(t_f, t_f) \). Therefore, the differential equation for \( \chi_2(t_f, t_f) \) is

\[
\frac{d}{dt_f} \chi_2(t_f, t_f) = -A_a^T(t_f) \chi_2(t_f, t_f) + G_2^T(t_f, t_f) R^{-1}(t_f) \gamma(t_f) \quad (3-119)
\]

with initial condition

\[
\chi_2(t_0, t_0) = 0 \quad (3-120)
\]
Similarly, the differential equation for $S_2(t_f, t_f)$ is given by

$$\frac{d}{dt_f} S_2(t_f, t_f) = -A^T(t_f)S_2(t_f, t_f) - S_2(t_f, t_f)A(t_f)$$

$$+ C^T(t_f, t_f)R^{-1}(t_f)C_2(t_f, t_f)$$

(3-121)

with initial condition

$$S_2(t_0, t_0) = 0$$

(3-122)

The above differential equations depend on $R(t_f)$ and $\gamma(t_f)$ from the Kalman filter, $A_a(t_f)$, which is given, and $G_2(t_f, t_f)$. From Eq. (3-96),

$$G_2(t_f, t_f) = C(t_f)F_2(t_f, t_f) + d(t_f)\phi^T_2(t_f, t_f)$$

(3-123)

where $C(t_f)$, $d(t_f)$, and $\phi_2(t_f, t_f)$ are given. So all that remains is to determine $F_2(t_f, t_f)$. From Eq. (3-102),

$$F_2(t_f, t_f) = F_1(t_f)\Gamma^T(t_f)$$

(3-124)

A differential equation for $F_2(t_f, t_f)$ may be obtained from this equation in the same way the differential equations for $\tilde{x}_2(t_f, t_f)$ and $S_2(t_f, t_f)$ were found. The result is

$$\frac{d}{dt_f} F_2(t_f, t_f) = [A(t_f) - K(t_f)C(t_f)]F_2(t_f, t_f)$$

$$+ [b(t_f) - K(t_f)d(t_f)]\phi^T_2(t_f, t_f) - F_2(t_f, t_f)A(t_f)$$

(3-125)
with initial condition

\[ F_2(t_0, t_0) = 0 \]  \hspace{1cm} (3-126)

Now, suppose that the additional requirement is imposed that \( \phi_2(t, t_f) \) be shift-invariant, that is, that \( \phi_2(t, t_f) \) can be represented as

\[ \phi_2(t, t_f) = \phi(t_f - t) \]  \hspace{1cm} (3-127)

This requirement is imposed so that the transformed basis functions will have the same form relative to the end of the observation interval. By Eq. (3-89), \( \phi(t_f - t) \) can then be expressed in terms of the original basis functions as

\[ \phi(t_f - t) = \Gamma(t_f) \phi_1(t) \]  \hspace{1cm} (3-128)

Taking the derivative of Eq. (3-128) with respect to \( t_f \) yields

\[ \phi'(t_f - t) = \frac{d \Gamma(t_f)}{dt_f} \phi_1(t) \]  \hspace{1cm} (3-129)

where \( \phi'(*) \) denotes the derivative of \( \phi(*) \) with respect to its argument. By Eq. (3-128),

\[ \phi'(t_f - t) = \frac{d \Gamma(t_f)}{dt_f} \Gamma^{-1}(t_f) \phi(t_f - t) \]  \hspace{1cm} (3-130)

The term multiplying \( \phi(t_f - t) \) in this equation may be recognized as \( -A_T a(t_f) \). Therefore,

\[ \phi'(t_f - t) = -A_T a(t_f) \phi(t_f - t) \]  \hspace{1cm} (3-131)
Because the left side of Eq. (3-131) is a function only of $t_f - t$, the right side must have the same property. This requires that $A_a$ be a constant matrix. By virtue of Eq. (3-131), the matrix $A_\phi$ will be defined by

$$A_\phi = -A^T_a$$

(3-132)

Therefore,

$$\frac{d}{d\tau} \phi(\tau) = A_\phi \phi(\tau)$$

(3-133)

where $\tau$ is defined by

$$\tau = t_f - t$$

(3-134)

$\tau$ may be thought of as defining a relative time scale that runs backward from the end of the observation interval, $t_f$. In some ways, the relative time $\tau$ is more natural for the failure detection problem than the absolute time $t$ because it represents time relative to the current time ($t_f$), rather than relative to some arbitrary fixed initial time.

Note that the basis functions affect the OSGLR equations only through $A_a(t_f)$ and $\phi^2(t_f,t_f)$. Therefore, if $\phi^2(t,t_f)$ is required to be shift-invariant, the OSGLR equations will be functions only of the system matrices, $A_\phi$, and $\phi(0)$. In other words, there is no need to specify the underlying basis functions $\phi^1(t)$ or the transformation $\Gamma(t_f)$. Since there is little motivation for using a basis $\phi^2(t,t_f)$ which is not shift-invariant, it will be assumed throughout the rest of this thesis that $\phi^2(t,t_f)$ is shift-invariant.

Furthermore, the matrices $F_2(t,t_f)$, $G_2(t,t_f)$, and $S_2(t,t_f)$, and the vector $\chi_2(t,t_f)$ need to be calculated only for $t = t_f$. Therefore, to simplify the notation, only one time argument will be used for these matrices, i.e.
\[
F(t) \triangleq F_{2}(t, t) \quad (3-135)
\]
\[
G(t) \triangleq G_{2}(t, t) \quad (3-136)
\]
\[
S(t) \triangleq S_{2}(t, t) \quad (3-137)
\]
\[
\chi(t) \triangleq \chi_{2}(t, t) \quad (3-138)
\]

The OSGLR equations for continuous-time systems are summarized in Table 3-1.

Table 3-1. Summary of continuous-time OSGLR equations.

<table>
<thead>
<tr>
<th>Estimation error influence matrix propagation:</th>
</tr>
</thead>
</table>
| \[
\frac{dF(t)}{dt} = [A(t) - K(t)C(t)]F(t) + [b(t) - K(t)d(t)]\phi^{T}(0) + F(t)\phi^{T} 
\]
| \[
F(t_{0}) = 0
\]

Residual influence matrix:
\[
G(t) = C(t)F(t) + d(t)\phi^{T}(0)
\]

Information vector propagation:
\[
\frac{d\chi(t)}{dt} = \phi^{T}\chi(t) + G^{T}(t)R^{-1}(t)\gamma(t)
\]
\[
\chi(t_{0}) = 0
\]

Information matrix propagation:
\[
\frac{dS(t)}{dt} = \phi^{T}S(t) + S(t)\phi^{T} + G^{T}(t)R^{-1}(t)G(t)
\]
\[
S(t_{0}) = 0
\]

Decision Function:
\[
DF(t) = \chi^{T}(t)S^{-1}(t)\chi(t)
\]

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3.3 **Multiple Hypotheses**

In Sections 3.1 and 3.2, the OSGLR test was derived as a binary hypothesis test. That is, there was a single failure hypothesis. In the general case, there are a number of failure hypotheses, say \( r \), corresponding to the different components of the system which may fail.

The extension of the OSGLR test from a binary hypotheses test to a multiple hypothesis test is relatively straightforward. Each failure type is assigned an index, \( i \). Corresponding to each failure type are the vectors \( b(t) \) and \( d(t) \) which represent the effect of the failure on the state dynamics and measurement equation. In addition, a matrix \( A_{\Phi_i} \) and vector \( \Phi_i(0) \) are specified which generate a vector of basis functions, \( \Phi_i(t) \), for the failure mode of the \( i \)th component. An OSGLR detector must then be implemented for each failure type \( i \). In particular, the influence matrices \( F_i(t) \) and \( G_i(t) \), the information matrix \( S(t) \), and the information vector \( x_i(t) \) must each be propagated in time. Finally, the decision function \( DF_i(t) \) is calculated from the information vector and the information matrix.

A number of strategies may be used to detect and isolate failures. The simplest strategy is to simply declare as failed the first component \( i \) whose decision function, \( DF_i \), crosses the detection threshold \( T^2 \).

In many systems, however, it may be difficult to distinguish between two types of failures. (See Chapter 6.) In such systems, the probability of incorrect isolation of a failure may be unacceptably high. One way to deal with this problem is to use an isolation threshold. That is, a failure is declared when any one of the decision functions crosses the threshold \( T^2 \). However, the failure is not isolated to the \( i \)th component until

\[
DF_i - DF_j > T^2_i, \quad j = 1, 2, \ldots, r \\
\text{for } j \neq i
\]

(3-139)
where $T^2_I$ is the isolation threshold. The difference on the left side of Eq. (3-139) is denoted by

$$DF_{ij} = DF_i - DF_j$$  \hspace{1cm} (3-140)

where $DF_{ij}$ is the isolation function for deciding between the hypotheses $H_i$ and $H_j$. ($DF_i$ may be referred to as the detection decision function to avoid confusion.) It is easily verified that $DF_{ij}$ is twice the natural logarithm of the generalized likelihood ratio for deciding between $H_i$ and $H_j$. This strategy should reduce the probability of incorrect isolation, at the expense of longer time to isolate a failure.

3.4 Failure Accommodation

After a failure has been detected, action must be taken by the FDI system to accommodate for the failure. Accommodation involves two distinct actions. First, the failed component, which is usually a sensor or an actuator, must be physically isolated from the system so that it can do no more harm. For a sensor failure this involves removing the sensor from any control law calculations so that its effect cannot propagate through the system. For an actuator failure, the actuator must be neutralized by eliminating its power source, or by some other means. This aspect of accommodation is problem specific, and will not be discussed further here.

The other action the FDI system must take is to prepare to continue performing failure detection. A number of bookkeeping operations must be performed, such as changing the system models to account for the loss of the failed component, and reinitializing the information vector and information matrix to zero for each of the remaining components. It is less obvious how the Kalman filter should be updated to account for the effect of the failure. This issue will be the subject of the remainder of this section.
In Section 3.2 the state estimate produced by the Kalman filter was denoted by \( \hat{x}(t) \). More properly, the estimate could be denoted by \( \hat{x}(t \mid t, H_0) \), indicating that the estimate is of \( x(t) \), given the data up to time \( t \), and given that \( H_0 \) is true. Similarly, the covariance of the state estimation error, which was denoted by \( P(t) \), could be denoted by \( P(t \mid t, H_0) \). The conditional probability of \( x(t) \) is completely specified by these two quantities, so that

\[
p(x(t \mid t, H_0) = \mathcal{N}(\hat{x}(t \mid t, H_0), P(t \mid t, H_0)) \tag{3-141}
\]

Now, suppose that a failure has been declared, so that, say, \( H_1 \) is believed to be true. The Kalman filter must be updated to account for the failure, so that monitoring of the other components can continue. That is, the state estimate and covariance must be set to

\[
\hat{x}(t) = \hat{x}(t \mid t, H_1) \tag{3-142}
\]

\[
P(t) = P(t \mid t, H_1) \tag{3-143}
\]

This notation presupposes that \( x(t) \), conditioned on the data up to time \( t \) and \( H_1 \), is Gaussian. However, this is not strictly true. Consider expressing \( p(x(t \mid t, H_1) \) as

\[
p(x(t \mid t, H_1) = \int p(x(t \mid t, H_1, a(t)) \ p(a(t) \mid t, H_1) \ da(t) \tag{3-144}
\]

Certainly, \( p(x(t \mid t, H_1, a(t)) \) can be determined simply by implementing the Kalman filter equations for \( H_1 \), assuming that \( a \) is given. The result is that

\[
p(x(t \mid t, H_1, a(t)) = \mathcal{N}(\hat{x}(t \mid t, H_0) + F(t)a(t), P(t \mid t, H_0)) \tag{3-145}
\]
However, the probability \( p(a(t) \mid t, H_1) \) in Eq. (3-144) is nonsensical, because \( a(t) \) is a nonrandom parameter. Therefore, it is not strictly correct to specify a probability density for \( x(t) \) conditioned on \( H_1 \).

The general problem of estimating nonrandom parameters is quite involved, and will not be discussed here. When the data are Gaussian and the measurements are linear, as in this problem, the results are somewhat easier to interpret. We are interested in the optimal estimate of \( x(t) \), in the sense that \( \hat{x}(t \mid t, H_1) \) is the minimum mean square error (mmse), unbiased estimator of \( x(t) \). As discussed by Schewepe [46, Sec. 6.4], such an estimator may be obtained by treating \( a(t) \) as if it were a Gaussian random variable, whose a priori mean is zero and a priori covariance is infinite. This approach correctly gives the distribution of the estimation errors, which are random variables. Hence, for all intents and purposes, we can treat \( a(t) \) as if it were a random variable.

With this understanding, we can proceed to determine \( \hat{x}(t \mid t, H_1) \) and \( P(t \mid t, H_1) \). The state estimate is given by

\[
\hat{x}(t \mid t, H_1) = E[x(t) \mid H_1]
\]  
(3-146)

Recall from Section 3.2 that

\[
x(t) = \hat{x}(t \mid t, H_0) + e(t)
\]

\[
= \hat{x}(t \mid t, H_0) + e_0(t) + F(t) a(t)
\]  
(3-147)

where \( e(t) \) is the estimation error, \( e_0(t) \) is the part of the estimation that results under \( H_0 \), and \( F(t) \) is the estimation error influence matrix. Hence,

\[
\hat{x}(t \mid t, H_1) = \hat{x}(t \mid t, H_0) + E[e_0(t) \mid t, H_1] + F(t) E[a(t) \mid t, H_1]
\]  
(3-148)
But

$$E[e_0(t) | t, H_1] = 0$$

(3-149)

Also, we can identify

$$E[a(t) | t, H_1] = \hat{a}(t)$$

(3-150)

Therefore,

$$\hat{x}(t | t, H_1) = \hat{x}(t | t, H_0) + F(t) \hat{a}(t)$$

(3-151)

To determine the covariance of the estimation error, we must first determine the estimation error

$$e(t | t, H_1) = x(t) - \hat{x}(t | t, H_1)$$

(3-152)

Using Eq. (3-151), we have that

$$e(t | t, H_1) = x(t) - \hat{x}(t | t, H_0) - F(t) \hat{a}(t)$$

(3-153)

Using Eq. (3-147), we have that

$$e(t | t, H_1) = e_0(t) + F(t)[a(t) - \hat{a}(t)]$$

$$= e_0(t) + F(t) \tilde{a}(t)$$

(3-154)

where

$$\tilde{a}(t) = a(t) - \hat{a}(t)$$

(3-155)
Then the covariance is given by

\[
P(t \mid t, H_1) = \mathbb{E}[\tilde{e}(t \mid t, H_1)\tilde{e}^T(t \mid t, H_1) \mid t, H_1]
\]

\[
= \mathbb{E}[\tilde{e}_0(t)\tilde{e}_0^T(t) \mid t, H_1] + F(t)\mathbb{E}[\tilde{a}(t)\tilde{a}^T(t) \mid t, H_1] F^T(t)
\]

\[
+ F(t)\mathbb{E}[\tilde{a}(t)\tilde{e}_0^T(t) \mid t, H_1] + \mathbb{E}[\tilde{e}_0(t)\tilde{a}^T(t) \mid t, H_1] F^T(t)
\]

(3-156)

The first two terms of the above equation are easily identified. \( \tilde{e}_0(t) \) was defined as the estimation error in \( \hat{x}(t \mid t, H_0) \), less the estimation error due to \( \tilde{a}(t) \). Hence,

\[
\mathbb{E}[\tilde{e}_0(t)\tilde{e}_0^T(t) \mid t, H_1] = P(t \mid t, H_0)
\]

(3-157)

The second term in Eq. (3-156) is just the covariance of the coefficient estimation error, weighted by \( F(t) \). Therefore,

\[
F(t)\mathbb{E}[\tilde{a}(t)\tilde{a}^T(t)] F^T(t) = F(t)P_\tilde{a}(t) F^T(t) = F(t)S^{-1}(t) F^T(t)
\]

(3-158)

To evaluate the third and fourth terms of Eq. (3-156), we must find an expression for \( \tilde{a}(t) \). Recall that

\[
\hat{a}(t) = S^{-1}(t)\hat{x}(t)
\]

(3-159)

But

\[
\hat{x}(t) = S(t)a(t) + \int_{t_0}^{t} G_2^T(\tau, t)R^{-1}(\tau)Y(\tau)d\tau
\]

(3-160)
Therefore,
\[
\tilde{a}(t) = \int_{t_0}^{t} s^{-1}(t) G_2^T(\tau, t) R^{-1}(\tau) \gamma_0(\tau) d\tau \tag{3-161}
\]

Recall that \( \gamma_0(t) \) is the Kalman filter residual that would occur if \( \tilde{a}(t) \) were zero, i.e., the residual that occurs under \( H_0 \). Therefore,
\[
\mathbb{E}[\tilde{a}(t)e_{0}^T(t) \mid t, H_1] = \mathbb{E}[\int_{t_0}^{t} s^{-1}(t) G_2^T(\tau, t) R^{-1}(\tau) \gamma(\tau) d\tau e_{0}^T(t) \mid t, H_0] \tag{3-162}
\]

Moving the expectation operator inside the integral yields
\[
\mathbb{E}[\tilde{a}(t)e_{0}^T(t) \mid t, H_1] = \int_{t_0}^{t} s^{-1}(t) G_2^T(\tau, t) R^{-1}(\tau) \mathbb{E}[\gamma(\tau)e_{0}^T(t) \mid t, H_0] d\tau \tag{3-163}
\]

But by the orthogonal projection theorem [34, p. 202],
\[
\mathbb{E}[\gamma(\tau)e_{0}^T(t) \mid t, H_0] = 0, \quad \tau \leq t \tag{3-164}
\]

Therefore, the integral in Eq. (3-163) is zero, and the last two terms of Eq. (3-156) are zero. Hence,
\[
P(t \mid t, H_1) = P(t \mid t, H_0) + P(t) P_a(t) F^T(t) \tag{3-165}
\]

Therefore, to accommodate for a failure declared at time \( t \), the Kalman filter estimate and covariance should be updated as follows:
\[ \hat{x}(t^+) = \hat{x}(t^-) + F(t)\hat{\alpha}(t) \quad (3-166) \]

\[ p(t^+) = p(t^-) + F(t)P_a(t)F^T(t) \quad (3-167) \]

where \( t^- \) and \( t^+ \) are the times just prior to \( t \) and just after \( t \), respectively. A similar approach to accommodation was suggested by Willsky and Jones [57] for use with their GLR algorithm.

3.5 Related Topics

In this section, the relationship of the OSGLR algorithm to earlier work in the fields of estimation and failure detection is discussed.

In Section 3.4, the estimate \( \hat{x}(t|\tau_H^1) \) was related to the estimate \( \hat{x}(t|\tau_H^0) \) by

\[ \hat{x}(t|\tau_H^1) = \hat{x}(t|\tau_H^0) + F(t)\hat{\alpha}(t) \]

Based on the interpretation that \( \hat{\alpha}(t) \) may be considered to be a Gaussian random variable, both conditioned on the measurements and unconditionally, it is possible to determine \( \hat{x}(t|\tau_H^1) \) and \( \hat{\alpha}(t) \) directly, using a single Kalman filter. That is, \( x(t) \) and \( a(t) \) may be considered to be parts of an augmented state vector defined by

\[ z(t) = \begin{bmatrix} x(t) \\ a(t) \end{bmatrix} \quad (3-168) \]

The state equation for \( z(t) \) is then given by

\[ \frac{d}{dt} \begin{bmatrix} x(t) \\ a(t) \end{bmatrix} = \begin{bmatrix} A(t) & b(t)\Phi^T(0) \\ 0 & A_a \end{bmatrix} \begin{bmatrix} x(t) \\ a(t) \end{bmatrix} + \begin{bmatrix} w(t) \\ 0 \end{bmatrix} \quad (3-169) \]

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(Cf. Eqs. (3-28), (3-32), and (3-116).) The measurement equation is given by

\[ y(t) = [C(t) \ d(t) \ \Phi^T(0)] \begin{bmatrix} x(t) \\ a(t) \end{bmatrix} + v(t) \]  

(3-170)

(Cf. Eqs. (3-29) and (3-32).) The initial covariance of the augmented state vector is given by

\[ P_z(t_0) = \begin{bmatrix} P_0 & 0 \\ 0 & \infty I \end{bmatrix} \]  

(3-171)

The state estimate \( \hat{z}(t|t) \) is then given by

\[ \hat{z}(t|t) = \begin{bmatrix} \hat{x}(t|t, H_1) \\ \hat{a}(t) \end{bmatrix} \]  

(3-172)

Of course, this approach does yield the same state estimates as obtained in Section 3.4. Friedland [23] first developed the so-called "separated-bias estimation" equations for the case where the bias (in this case \( a(t) \)) is a constant, i.e.,

\[ A_a = 0 \]  

(3-173)

Friedland [23] also suggested that the extension to the case where the bias is time-varying, i.e.

\[ A_a \neq 0 \]  

(3-174)
would not be difficult. Tacker and Lee [50] provided the extension of Friedland's work to include this case. Bierman [6] showed that this result followed directly from Friedland's earlier work if the time-varying bias is expressed in the form

\[
\underline{a}(t) = \phi_a(t, t_0) \underline{a}(t_0)
\]

(3-175)

where \( \phi_a(t, t_0) \) is the transition matrix for \( \underline{a}(t) \). Equation (3-175) is similar to Eq. (3-93), where \( \underline{a}(t) \) was expressed as

\[
\underline{a}(t) = \Gamma^{-T} \underline{a}(t_1)
\]

(3-176)

A later derivation of Friedland's results by Ignagni [33] derived the separated-bias equations in a manner similar to that used in Section 3.4. Ignagni first found the "perfectly known bias estimator for \( \underline{x} \)," which is the same as \( \hat{x}(t | t, H_1, \underline{a}(t)) \), and related it to the "bias-free estimator of \( \underline{x} \)," as in Eq. (3-145). He then determined the "adjusted estimate of \( \underline{x} \)," which is \( \tilde{x}(t | t, H_1) \), from the bias-free estimate of \( \underline{x} \) and estimate of the bias, as in Eq. (3-151). The only difference between the results of [33] and the results of Section 3.4 (aside from the fact that he dealt only with the discrete-time problem) is that the estimate of the bias in [33] is found using a Kalman filter in normal form, rather than in information form, as is done here. In this respect, the results presented here most resemble Friedland's [24], who showed that the bias estimation is more readily implemented in information form. Finally, Chang and Dunn [12] recognized that the failure accommodation scheme suggested by Willsky and Jones [57] is directly related to separated-bias estimation. For a complete review of separated-bias estimation and related topics, see the survey paper by Friedland [22].

In a different vein, Chow [16] and Chow and Willsky [17] suggested that an approximate solution to the Bayesian formulation of the failure detection problem could be obtained as follows: The residual sequence
\( z(k) \) (which may be generated by a Kalman filter or by some other method) is used to generate the test statistics \( z(k) \) via the difference equation

\[
    z(k+1) = \tilde{A}z(k) + \tilde{B}r(k+1)
\]  

(3-177)

The domain of \( z(k) \) is then divided into decision regions so as to minimize the Bayes risk. The resulting test is called a Markov approximation to the optimal Bayes test.

The (discrete-time) OSGLR test is very similar in character to this approach. The OSGLR sufficient statistic \( \chi(k) \) is generated by a difference equation similar to Eq. (3-177):

\[
    \chi(k+1) = \Phi_{\phi} \chi(k) + G^T(k+1)M^{-1}(k+1)\gamma(k+1)
\]  

(3-178)

There are two major differences between these two approaches. First, the decision regions of the OSGLR test are not chosen to minimize a Bayesian cost function, but rather are determined by the statistics of \( \chi(k) \) under \( H_0 \). Second, the matrices \( \tilde{A} \) and \( \tilde{B} \) of the Markov approximation are chosen in an ad hoc manner, whereas the OSGLR matrices \( \Phi_{\phi}(k) \) and \( G(k) \) are derived in a logical fashion from the series of time functions which represent the failure modes. Clearly, a synthesis of the two approaches is possible. That is, the OSGLR algorithm could be used to generate the statistics \( \chi(k) \), but the decision regions could be chosen to minimize the Bayes risk.
The performance of a failure detection and isolation (FDI) test depends on three types of events: false alarms, the detection of failures, and the (correct or incorrect) isolation of failures. A complete probabilistic description of these events, together with the distribution of failures, is required in general to determine the performance of a fault-tolerant system.

Unfortunately, it is generally quite difficult to evaluate the performance of an FDI test analytically. In principle, the performance of an FDI test could be determined by Monte Carlo simulation. However, the probability of false alarm, missed detection, or incorrect isolation is very small for an effective FDI test. The amount of simulation required to estimate these probabilities accurately by Monte Carlo methods is therefore prohibitive.

In this chapter, a partial solution to the problem of evaluating the performance of the OSGLR test is given. In Section 4.1, the false-alarm performance of the OSGLR test is considered. In particular, an asymptotic expression for the steady-state false-alarm rate of the continuous-time OSGLR test is derived. Based on this analysis, an asymptotic bound is derived for the steady-state false-alarm rate of the discrete-time OSGLR test, and the conditions under which this bound is valid are discussed. In Section 4.2, a simple bound for the probability of detection is given.
4.1 False-Alarm Performance

The problem of determining the false-alarm performance of the OSGLR test is an example of a first-passage problem [8, 19, 30, 49]. First-passage problems are generally quite difficult to solve, and analytic solutions exist only for a few special cases. In this section, methods of asymptotic analysis are used to approximate the steady-state false-alarm rate of the continuous-time OSGLR test. The discrete-time case is also considered.

4.1.1 Problem Formulation

In Chapter 3, the OSGLR sufficient statistic \( \chi(t) \) was shown to be the state of a linear system driven by white noise. Specifically, the differential equation for \( \chi(t) \) when no failures are present is given by:

\[
\frac{d\chi(t)}{dt} = A_\phi \chi(t) + G^T(t)R^{-1}(t)\chi(t)
\]  

(4-1)

where \( \chi(t) \) is a zero-mean, Gaussian white noise process with intensity \( R(t) \). A failure is declared when the OSGLR test statistic exceeds a threshold:

\[
DF(t) = \chi(t)^T S^{-1}(t) \chi(t) > T^2
\]  

(4-2)

\[
declare a failure \quad \chi(t)^T S^{-1}(t) \chi(t) > T^2
\]

continue sampling

The time of false alarm, \( t_{FA} \), is defined to be the earliest time at which the decision function exceeds the threshold (when no failures are present). More rigorously,

\[
t_{FA} = \inf \{ t : DF(t) > T^2 \}
\]  

(4-3)
where "inf" denotes the infimum over the indicated set. If the probability distribution of $t_{FA}$ can be determined, then the false-alarm performance can be completely characterized.

The probability distribution of $t_{FA}$ will be denoted by

$$P_{FA}(t) = Pr(t_{FA} < t)$$  \hspace{1cm} (4-4)

The probability that no false alarm has occurred at or prior to time $t$ is then

$$P_{NA}(t) = Pr(t_{FA} > t) = 1 - P_{FA}(t)$$  \hspace{1cm} (4-5)

The **instantaneous false-alarm rate** at time $t$ is defined by

$$\lambda_{FA}(t) = -\frac{1}{P_{NA}(t)} \frac{d}{dt} P_{NA}(t) = -\frac{d}{dt} \ln P_{NA}(t)$$  \hspace{1cm} (4-6)

It is this false-alarm rate that we wish to determine.

It should be noted at this point that in some studies, the notation $P_{FA}(t)$ has been used to denote the instantaneous probability that the test statistic $DF(t)$ exceeds the threshold, i.e.,

$$P_{FA}(t) = Pr(DF(t) > T)$$  \hspace{1cm} (4-7)

Clearly, this probability is different from that defined in Eq. (4-4). When defined by Eq. (4-4), $P_{FA}(t)$ denotes the probability that a false alarm has occurred prior to time $t$. That is, $P_{FA}(t)$ is the probability of the event that $DF(t) > T$ for some $t < t$. When defined by Eq. (4-7), $P_{FA}(t)$ denotes only the probability that the decision function exceeds the threshold at time $t$, regardless of what has happened before.
This is not the event in which we are interested. Therefore, Eq. (4-7) is of little or no value in the false-alarm performance analysis of a failure detection test. Henceforth, \( P_{fa}(t) \) will be defined by Eq. (4-4).

To determine the distribution of false alarms, the probability density of the state \( \chi(t) \) must be propagated forward in time. The evolution of the probability density \( p(\chi, t) \) of \( \chi(t) \) is governed by the **Fokker-Planck equation**, which is also known as **Kolmogorov's forward equation** [34, Sec. 4.9], [49, Ch. 4]. For this problem, the Fokker-Planck equation is given by

\[
\frac{\partial}{\partial t} p(\chi, t) = \frac{1}{2} \text{tr}[Q_\chi(t)p_{XX}(\chi, t)] - \chi A T \phi p(\chi, t) - \text{tr}[A_v]p(\chi, t)
\]

(4-8)

where

\[
Q_\chi(t) = g^T(t)R^{-1}(t)G(t)
\]

(4-9)

The notation \( \text{tr}[\cdot] \) denotes the trace of the matrix \( [\cdot] \). \( p(\chi, t) \) and \( p_{XX}(\chi, t) \) are the vector and matrix of partial derivatives of \( p(\chi, t) \), respectively, defined by

\[
[p_{\chi}(\chi, t)]_i = \frac{\partial}{\partial \chi_i} p(\chi, t), \quad i = 1, \ldots, p
\]

(4-10)

and

\[
[p_{XX}(\chi, t)]_{ij} = \frac{\partial^2}{\partial \chi_i \partial \chi_j} p(\chi, t), \quad i = 1, \ldots, p
\]

(4-11)
It will be more convenient to work with Eq. (4-8) in the form

\[
\frac{3}{\partial t} p(\chi, t) = V^T_x Q(t) V_\chi p(\chi, t)/2 - \chi^T A^T_\phi V_\chi p(\chi, t) - \text{tr}[A_\phi] p(\chi, t)
\]  

(4-12)

where \( V_\chi \) is the gradient operator, defined by

\[
V_\chi = \begin{bmatrix}
\frac{\partial}{\partial \chi_1} \\
\vdots \\
\frac{\partial}{\partial \chi_p}
\end{bmatrix}
\]

In [49, Sec. 4.6], it is shown that the appropriate boundary condition for this type of problem is that

\[
p(\chi, t) = 0, \quad \chi \in B(t)
\]  

(4-13)

where \( B(t) \) is the boundary, defined by

\[
B(t) = \{ \chi : \chi^T S^{-1}(t) \chi = T^2 \}
\]  

(4-14)

The initial condition for this problem is that \( p(\chi, t) \) is prescribed for \( t = t_0 \), so that

\[
p(\chi, t_0) = p_0(\chi)
\]  

(4-15)

where \( p_0(\chi) \) is the initial probability density of \( \chi \).
At this point, it should be noted that \( p(\chi, t) \) is not a true probability density function because its integral, over the domain of \( \chi \), is not unity. This is because the density \( p(\chi, t) \) represents only those members of the ensemble of all time histories \( \chi(t) \) which do not cross the decision threshold before time \( t \). Said another way, some of the probability density "leaks out" at the boundary. This leakage represents the false alarms in which we are interested. Therefore, the probability that a false alarm has not yet occurred at time \( t \) is simply

\[
P_{\text{NA}}(t) = \int_{D(t)} p(\chi, t) \, d\chi \quad (4-16)
\]

where \( D(t) \) is the domain of \( \chi \), given by

\[
D(t) = \{ \chi : \chi^T S^{-1}(t) \chi \leq T^2 \} \quad (4-17)
\]

First-passage problems are generally extremely difficult to solve, and the problem formulated here is no exception. The difficulty lies in the fact that to determine the false-alarm rate a partial differential equation must be solved. To make the problem tractable, two simplifying assumptions will be made. First, it will be assumed that the system describing the OSGLR test is time-invariant, that is, that \( Q(\chi) \) and \( S(t) \) are constant matrices. The condition that \( Q(\chi) \) is constant essentially means that the state dynamics and measurement equation are time-invariant, and that the Kalman filter for the plant has reached steady state. This will ensure that \( R(t) \) is a constant matrix. To ensure that \( G(t) \) and \( S(t) \) are constant, the OSGLR test must have run long enough for the propagation of the estimation error influence matrix, \( F(t) \), and the information matrix, \( S(t) \), to reach steady state. (See Table 3-1.) Note that this requires that the Lyapunov equation

\[
A S + S A^T + Q = 0 \quad (4-18)
\]

be satisfied.
The second assumption is that we are interested in the steady-state false-alarm rate, rather than in the complete history of the false-alarm rate. The steady-state false-alarm rate is defined by

$$\lambda_{ss} = \lim_{t \to \infty} \lambda_{FA}(t)$$  \hspace{1cm} (4-19)

The steady-state false-alarm rate is of interest because it determines the probability distribution of $t_{FA}$ after the initial transients in the test die out. To determine the steady-state false-alarm rate, consider solving Eq. (4-12) by separation of variables [31, p. 444], subject to the first simplifying assumption above. Then Eq. (4-12) may be written as

$$\frac{\partial p(\chi,t)}{\partial t} = L_\chi p(\chi,t)$$  \hspace{1cm} (4-20)

where $L_\chi$ is a linear differential operator that is not a function of $t$. We search for solutions of the form

$$p(\chi,t) = f(\chi)h(t)$$  \hspace{1cm} (4-21)

Then Eq. (4-20) becomes

$$f(\chi) \frac{\partial}{\partial t} h(t) = h(t)L_\chi f(\chi)$$  \hspace{1cm} (4-22)

Dividing both sides of Eq. (4-22) by $f(\chi)h(t)$ yields

$$\frac{1}{h(t)} \frac{\partial}{\partial t} h(t) = \frac{1}{f(\chi)} L_\chi f(\chi)$$  \hspace{1cm} (4-23)

Because the left side of Eq. (4-23) is a function only of $t$, and the right side of Eq. (4-23) is a function only of $\chi$, both sides must be equal to the same constant, say $-\lambda$. Hence, we have that
\[ \frac{dh(t)}{dt} + \lambda h(t) = 0 \quad (4-24) \]

and

\[ L \frac{d}{dx} f(x) + \lambda f(x) = 0 \quad (4-25) \]

In order for \( p(\chi, t) \) to satisfy the homogeneous boundary condition of Eq. (4-13), we must have that

\[ f(\chi) = 0 , \quad \chi \in B \quad (4-26) \]

where the boundary \( B \) is independent of time, because \( S \) is constant. Non-trivial solutions to Eq. (4-25) subject to the boundary condition Eq. (4-26) exist only when \( \lambda \) is an eigenvalue of the differential equation. If \( \lambda_i \) is an eigenvalue, and \( f_i(x) \) is the corresponding eigenfunction, then \( \exp(-\lambda_i t)f_i(x) \) is a solution to Eq. (4-20). By the linearity of Eq. (4-20) and the homogeneity of the boundary condition (4-26), any function of the form

\[ p(\chi, t) = \sum_{i=1}^{\infty} a_i e^{-\lambda_i (t-t_0)} f_i(\chi) \quad (4-27) \]

is a solution to Eq. (4-20) and satisfies the boundary condition (4-13). In order for \( p(\chi, t) \) to satisfy the initial condition (4-15), there must exist \( a_i \) such that

\[ p(\chi, t_0) = p_0(\chi) = \sum_{i=1}^{\infty} a_i f_i(\chi) \quad (4-28) \]

Whether such \( a_i \) exist for all \( p_0(\chi) \) depends on whether the \( f_i(\chi) \) form a complete basis. It will be assumed without proof that this is true for this problem.
The important point is that as \( t \) becomes large (i.e., long after the initial time), the solution is dominated by the term corresponding to the smallest eigenvalue, because that term decays more slowly than any other term. Then we have that

\[
p(\chi, t) = a_1 e^{-\lambda_1 (t-t_0)} f_1(\chi), \quad t >> t_0
\]  

(4-29)

where \( \lambda_1 \) is the smallest eigenvalue. That is, the probability density \( p(\chi, t) \) reaches a steady-state shape, \( f_1(\chi) \), which dissipates exponentially. The probability that no alarm has occurred is then

\[
P_{NA}(t) = \int_D p(\chi, t) \, d\chi
\]

\[
x = a_1 \int_D f_1(\chi) \, d\chi
\]

(4-30)

(4-31)

Then

\[
\lambda_{FA}(t) = -\frac{d}{dt} \ln P_{NA}(t)
\]

\[
= \lambda_1, \quad t >> t_0
\]

(4-32)

Therefore, the steady-state false alarm rate can be identified as \( \lambda \), the smallest eigenvalue of

\[
\frac{1}{2} \chi^T Q \chi \nabla f(\chi) - \chi^T A \nabla f(\chi) + (\lambda - \text{tr}[A]) f(\chi) = 0
\]

(4-33)

subject to the boundary condition (4-26).
Finally, note that we have implicitly assumed that all the eigenvalues $\lambda_i$ are positive. If some of the eigenvalues are negative, then the corresponding terms in the summation of Eq. (4-27) will not decay. Rather, these terms will increase without bound. But this is not possible, since the resulting probability density would be unbounded. The conclusion is that the eigenvalues must be positive, as was assumed. This is easily proved in the scalar case by converting the eigenvalue problem to a proper Sturm-Liouville problem [31, p. 205]. (See Section 4.1.4) A similar result holds for the vector case.

4.1.2 WKB Approximation of the Smallest Eigenvalue

Unfortunately, Eq. (4-33) is still a partial differential equation, except for the case where $\chi$ is a scalar. In that case, Eq. (4-33) is an ordinary differential equation whose solution can be expressed in terms of parabolic cylinder functions. In the more general case, closed-form solutions for the eigenvalues do not exist. For low-order problems, it may be possible to determine the eigenvalues by solving the partial differential equation numerically. However, this approach is not practical for any $\chi$ with dimension larger than about two. Therefore, some other approach must be used to determine the eigenvalue.

The approach that will be developed here is to find an asymptotic approximation to the smallest eigenvalue, using WKB theory [5, Ch. 10]. (The theory is named after Wentzel, Kramers, and Brillouin.) WKB theory is a technique for asymptotically solving differential equations that is often used in mathematical physics. The approximation is asymptotic because the accuracy of the approximation improves as the threshold, $T$, increases toward infinity. Because the threshold for an FDI test is usually chosen to be large to prevent false alarms, the asymptotic approximation will be of practical importance.

4.1.2.1 Scalar Case

When $\chi$ is a scalar, Eq. (4-33) becomes an ordinary differential equation:
\[
\frac{1}{2} Q \chi \frac{d^2}{d\chi^2} f(\chi) - \chi A_\phi \frac{d}{d\chi} f(\chi) + (\lambda - A_\phi) f(\chi) = 0 \quad (4-34)
\]

Because (the \(1\times1\) matrix) \(A_\phi\) is stable, it can be represented in the form

\[
A_\phi = -\frac{1}{\tau} \quad (4-35)
\]

where \(\tau\) is a (positive) time constant. Furthermore, the variance of \(\chi, S\), satisfies the Lyapunov equation

\[
Q_\chi + 2A_\phi S = 0 \quad (4-36)
\]

Therefore,

\[
Q_\chi = \frac{2S}{\tau} \quad (4-37)
\]

Substituting the expressions for \(A_\phi\) and \(Q_\chi\) in Eqs. (4-35) and (4-37) into Eq. (4-34), and multiplying the result by \(\tau\) gives

\[
S \frac{d^2}{d\chi^2} f(\chi) + \chi \frac{d}{d\chi} f(\chi) + (\lambda \tau + 1) f(\chi) = 0 \quad (4-38)
\]

The appropriate boundary condition for this equation, as determined from Eqs. (4-26) and (4-14), is

\[
f(\pm \sqrt{S} \, \tau) = 0 \quad (4-39)
\]

The effect of the threshold \(T^2\) can be made more apparent by transforming the differential equation (4-38), which is valid for \(\chi\) in the interval \([-\sqrt{ST}, \sqrt{ST}]\), to one that is valid over an interval that is independent of \(T\). To this end, we define \(z\) by
\[ \chi = \sqrt{S} \tau \]  

(4-40)

and the function \( g(z) \) by

\[ g(z) = f(\chi) = f(\sqrt{S} \tau) \]  

(4-41)

Then the differential equation for \( g(z) \) is given by

\[ \frac{1}{T^2} \frac{d^2}{dz^2} g(z) + z \frac{d}{dz} g(z) + (\lambda \tau + 1) g(z) \]  

(4-42)

The boundary condition is that

\[ g(\tau) = 0 \]  

(4-43)

Notice that the highest derivative of \( g(z) \) is Eq. (4-42) is multiplied by \( 1/T^2 \). Assuming that the threshold is large, relative to unity, then \( 1/T^2 \) is small. We can write Eq. (4-42) as

\[ \varepsilon \frac{d^2}{dz^2} g(z) + z \frac{d}{dz} g(z) + (\lambda \tau + 1) g(z) \]  

(4-44)

where

\[ \varepsilon = 1/T^2 \]  

(4-45)

The presence of the small parameter \( \varepsilon \) multiplying the highest derivative of \( g(z) \) in Eq. (4-44) means that the problem of determining \( \lambda \) is a singular perturbation problem (4-44). There are a number of methods available to find approximate solutions to Eq. (4-44), including boundary-layer theory, multiple scale analysis, and WKB theory [5]. Although all three of these techniques can be applied to Eq. (4-44), only WKB theory generalizes easily to the case where \( \chi \) is a vector rather than a scalar. Therefore, only WKB theory will be presented here.
A minor complication at this point is that it is not known how small \( \lambda \) is, relative to \( \varepsilon \). Indeed, we are trying to determine \( \lambda \) as a function of \( \varepsilon \). However, it will be necessary to have some idea of the size of \( \lambda \) in the asymptotic analysis that follows. It turns out that \( \lambda \) is asymptotically small as \( \varepsilon \to 0 \). To demonstrate this, we will assume that \( \lambda \) is asymptotically small, and then show that this assumption is consistent.

The basic idea of WKB theory is to assume that the solution of Eq. (4-44) has the asymptotic expansion

\[
g(z) = \exp \left( \frac{1}{\delta} S_0(z) + S_1(z) + \ldots \right), \quad (\varepsilon \to 0) \tag{4-46}
\]

(Henceforth, asymptotic expressions should be interpreted as being valid in the limit \( \varepsilon \to 0 \), even if the limit is not explicitly stated.) Here \( \delta \) is a small parameter that represents how rapidly the solution varies. To determine \( \delta \), we substitute the asymptotic expression for \( g(z) \) in Eq. (4-46) into Eq. (4-44). After cancelling the common exponential factor, the result is

\[
\varepsilon \left( \frac{1}{\delta^2} (S'_0(z))^2 + \frac{1}{\delta} S''_0(z) + \frac{2}{\delta} S'_0(z)S'_1(z) + (S'_1(z))^2 + S''_1(z) + \ldots \right)
\]

\[
+ z \left( \frac{1}{\delta} S'_0(z) + S'_1(z) + \ldots \right) + (\lambda t + 1) = 0 \tag{4-47}
\]

Assuming that \( S'_0(z), S''_0(z), S'_1(z), S''_1(z) \), etc., are all of order \( 1 \) \( (O(1)) \), then the largest terms in Eq. (4-47) are \( O(\varepsilon/\delta^2) \), \( O(1/\varepsilon) \), and \( O(1) \). In order for Eq. (4-47) to be satisfied to highest order, at least two of these terms must be of the same order. However, by hypothesis \( \delta \) is a small parameter, meaning that

\[
\delta \ll 1 \tag{4-48}
\]
Hence,

\[
\frac{1}{\delta} \gg 1 \tag{4-49}
\]

and the \(O(1)\) terms must be small compared to the \(O(\epsilon/\delta^2)\) and \(O(1/\delta)\) terms. Hence, we must have that

\[
O(\epsilon/\delta^2) = O(1/\delta) \tag{4-50}
\]

and therefore,

\[
O(\delta) = O(\epsilon) \tag{4-51}
\]

The simplest choice for \(\delta\) is therefore

\[
\delta = \epsilon \tag{4-52}
\]

This is the distinguished limit [5, pp. 435-437, p. 486] for this problem, meaning the choice of \(\delta\) for which it is possible to obtain an asymptotic expansion for \(g(z)\) as in Eq. (4-46).

Therefore, we look for solutions of the form

\[
g(z) \approx \exp \left( \frac{1}{\epsilon} S_0(z) + S_1(z) + \ldots \right) \tag{4-53}
\]

Substituting this expression for \(g(z)\) in Eq. (4-44) and collecting terms in common powers of \(\epsilon\) yields

\[
\frac{1}{\epsilon} \left( (S_0'(z))^2 + zS_0(z) \right) + \left[ S_0''(z) + zS_0'(z) + 1 \right] + \ldots + \lambda \tau = 0 \tag{4-54}
\]
In order for this equation be satisfied to \( O(1/\varepsilon) \), we must have that
\[
(S_0'(z))^2 + zS_0'(z) = 0 \quad (4-55)
\]

Similarly, if Eq. (4-54) is to be satisfied to \( O(1) \), then we must have that
\[
S_0''(z) + 2S_0'(z)S_1'(z) + zS_1'(z) + 1 = 0 \quad (4-56)
\]

This process may be continued for successively higher powers of \( \varepsilon \). However, it will be sufficient for our purposes to determine only \( S_0(z) \) and \( S_1(z) \). Therefore, it will not be necessary to continue this process further.

Before proceeding to solve Eqs. (4-55) and (4-56), it should be noted that these equations have limited ranges of validity. The reason is that we implicitly assumed that \( z \) is \( O(1) \). Hence, in the neighborhood of \( z = 0 \), the dominant balance argument used to determine \( \delta \) is no longer valid. Therefore, a separate solution must be found in this region that will be asymptotically matched to the solution for \( z \neq 0 \).

At this point, we are ready to determine \( S_0(z) \) and \( S_1(z) \). Consider first Eq. (4-55), which is a quadratic in \( S_0'(z) \), and therefore has two solutions:
\[
S_0'(z) = -z \quad (4-57)
\]

and
\[
S_0'(z) = 0 \quad (4-58)
\]

Consider first the case represented by Eq. (4-57). Its solution is
\[ S_0(z) = -\frac{z^2}{2} + c_1 \quad (4-59) \]

where \( c_1 \) is an arbitrary constant. To determine \( S_1(z) \), we use Eq. (4-56), which in this case becomes

\[ zS'_1(z) = 0 \quad (4-60) \]

The solution to this equation is simply

\[ S_1(z) = c_2 \quad (4-61) \]

where \( c_2 \) is an arbitrary constant. At this point, we have that

\[ g(z) = \exp \left( \frac{1}{\epsilon} \frac{z^2}{2} + c_3 \right) \quad (4-62) \]

where

\[ c_3 = \frac{1}{\epsilon} c_1 + c_2 \quad (4-63) \]

Because the constants \( c_1 \) and \( c_2 \) simply scale the solution, they may be set to zero with no loss in generality. So consider the function \( g(z) = \exp(-z^2/2\epsilon) \). If the differential operator \( L_z(\epsilon) \) is defined by

\[ L_z(\epsilon) = \epsilon \frac{d^2}{dz^2} + z \frac{d}{dz} + 1 \quad (4-64) \]

then the differential equation which we are trying to solve (Eq. (4-44)) may be represented by

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\[ L_z(\varepsilon)g(z) + \lambda \tau g(z) = 0 \quad (4-65) \]

But it may be verified that

\[ L_z(\varepsilon) e^{-z^2/\varepsilon} = 0 \quad (4-66) \]

That is, \( \exp(-z^2/2\varepsilon) \) almost satisfies the differential Eq. (4-44), except for the term \( \lambda \tau g(z) \). But \( \lambda \tau \) was assumed to be exponentially small. Therefore, as \( \varepsilon \to 0 \), we have that

\[ g(z) = e^{-z^2/\varepsilon} \quad (4-67) \]

This (approximate) solution is not unexpected. It simply represents the Gaussian density that \( \chi \) has as the threshold \( T \to \infty \).

However, the solution (4-67) does not satisfy the boundary condition of Eq. (4-43). Hence, Eq. (4-67) can be a valid approximation to \( q(z) \) only in the interior of the interval \([-1,1]\). Said another way, Eq. (4-67) is not a uniformly valid approximation to \( g(z) \) over \([-1,1]\).

To determine a uniformly valid approximation, we must also investigate the other solution to Eq. (4-55), i.e.,

\[ S_0'(z) = 0 \quad (4-68) \]

The solution to this equation is simply

\[ S_0(z) = c_4 \quad (4-69) \]

where \( c_4 \) is a constant. Then Eq. (4-56) becomes

\[ zS_1'(z) + 1 = 0 \quad (4-70) \]
for which the solution is

\[ S_1'(z) = -\frac{1}{z} \quad (4-71) \]

Note that \( S_1(z) \) must have a singularity at \( z=0 \), because \( S_1'(0) \) is infinite. We will return to this point shortly. For the present, we will consider Eq. (4-71) only for \( z > 0 \). The solution of Eq. (4-71) is then given by

\[ S_1(z) = -\ln z + c_5, \quad z > 0 \quad (4-72) \]

Therefore, a solution to the differential equation is (approximately) given by

\[ g(z) = \exp \left( \frac{1}{\varepsilon} S_0(z) + S_1(z) \right) \]

\[ = \frac{1}{z} \exp \left( \frac{c_4}{\varepsilon} + c_5 \right), \quad z > 0 \quad (4-73) \]

Now, consider the function \( 1/z \). Using the differential operator of Eq. (4-64), we have that

\[ (L_z(\varepsilon) + \lambda \tau) \frac{1}{z} = \frac{\varepsilon}{z^3} - \frac{1}{z} + \frac{1}{z} + \lambda \tau \frac{1}{z} \quad (4-74) \]

Hence, the function \( 1/z \) satisfies the differential equation to \( O(\varepsilon/z) \). But the largest term on the right side of Eq. (4-74) is \( O(1/z) \). Thus, \( 1/z \) is a valid approximation so long as

\[ \frac{1}{z} \gg \frac{\varepsilon}{\varepsilon^3} \quad (4-75) \]

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In other words, $1/z$ is a valid approximation, so long as

$$z^2 \gg \varepsilon \quad (4-76)$$

Hence, Eq. (4-73) is an asymptotic approximation to a solution of the differential equation (4-44) over a limited range of validity. Thus, the fact that $S_0(z)$ is singular at the origin is not important, since the solution is not valid in the neighborhood of the origin anyway.

Now, Eq. (4-44) is a linear, homogeneous differential equation. Therefore, any linear combination of solutions is also a solution. So in order to satisfy the boundary condition

$$g(1) = 0 \quad (4-77)$$

we take the difference of the solutions (4-67) and (4-73) to obtain

$$g(z) = \exp \left( -\frac{z^2}{2\varepsilon} \right) - \frac{1}{z} \exp \left( -\frac{c}{\varepsilon} + c_5 \right) \quad (4-78)$$

Then

$$g(1) = \exp \left( -\frac{1}{2\varepsilon} \right) - \exp \left( -\frac{c}{\varepsilon} + c_5 \right) \quad (4-79)$$

In order to satisfy the boundary condition, we must have that

$$\exp \left( -\frac{c}{\varepsilon} + c_5 \right) = \exp \left( -\frac{1}{2\varepsilon} \right) \quad (4-80)$$

Therefore,

$$g(z) = \exp \left( -\frac{z^2}{2\varepsilon} \right) - \frac{1}{z} \exp \left( -\frac{1}{2\varepsilon} \right), \quad (\varepsilon > 0, \ z > 0, \ z^2 \gg \varepsilon) \quad (4-81)$$
is a valid approximation over the indicated range of validity. Similarly,

\[ g(z) = \exp\left(-\frac{z^2}{2\varepsilon}\right) + \frac{1}{z} \exp\left(-\frac{1}{2\varepsilon}\right), \quad (\varepsilon \to 0, \ z < 0, \ z^2 \gg \varepsilon) \quad (4-82) \]

Note that in the interior of the interval \([-1,1]\), the second term of Eqs. (4-81) and (4-82) is small compared to the first. To demonstrate this, consider the ratio of the two terms:

\[
\frac{\frac{1}{z} e^{-1/2\varepsilon}}{e^{-z^2/2\varepsilon}} = \frac{1}{z} e^{-\frac{(1-z^2)}{2\varepsilon}} \quad (4-83)
\]

This ratio is exponentially small when

\[
\frac{1 - z^2}{2\varepsilon} \gg 1 \quad \text{and} \quad z^2 \gg \varepsilon \quad (4-84)
\]

Therefore, the second term may be neglected when

\[
1 - z^2 \gg \varepsilon \quad (4-85)
\]

So the solutions given by Eqs. (4-81) and (4-82) agree with the solution given by Eq. (4-67) in the interior of the interval \([-1,1]\). Thus, the solution may be pieced together over the different regions of validity. This process is known as asymptotic matching. Therefore, the solution to Eq. (4-44) may be approximated by
\[
g(z) = \begin{cases} 
-\frac{z^2}{2\varepsilon} + \frac{1}{z} - \frac{1}{2\varepsilon} & (z < 0, z^2 \gg \varepsilon) \\
-\frac{z}{2\varepsilon} & (1-z^2 \gg \varepsilon) \\
-\frac{z}{2\varepsilon} - \frac{1}{z} - \frac{1}{2\varepsilon} & (z > 0, z^2 \gg \varepsilon)
\end{cases}
\] (4-86a)

Now that we have determined an asymptotic approximation to \(g(z)\), we wish to determine the eigenvalue \(\lambda\). However, the eigenvalue cannot be determined from the analysis above, because the eigenvalue is exponentially small. As a result, \(\lambda\) was ignored in the derivation and therefore cannot be determined directly from the analysis. However, the instantaneous false-alarm rate can be related to the shape of the probability density. To determine the false-alarm rate from \(g(z)\), Eq. (4-44) is integrated over the interval \([-1,1]\), so that

\[
\varepsilon \int_{-1}^{1} \frac{d^2}{dz^2} g(z) dz + \int_{-1}^{1} (z \frac{d}{dz} g(z) + g(z)) dz + \lambda \tau \int_{-1}^{1} g(z) dz = 0
\]

(4-87)

The integrand of the second term in the above equation can be written as

\[
z \frac{d}{dz} g(z) + g(z) = \frac{d}{dz} [zg(z)]
\]

(4-88)

Therefore, Eq. (4-87) reduces to

\[
\varepsilon \left. \frac{d}{dz} g(z) \right|_{z=-1}^{1} + zg(z) \left|_{z=-1}^{1} + \lambda \tau \int_{-1}^{1} g(z) dz = 0
\]

(4-89)
But \( g(z) = 0 \) at \( z = \pm 1 \). Therefore,

\[
\lambda \tau = - \varepsilon \frac{[g'(1) - g'(-1)]}{\int_{-1}^{1} g(z) \, dz}
\]

(4-90)

So an asymptotic expression for \( \lambda \tau \) may be found by determining an asymptotic expression for \( g'(\pm 1) \) and \( \int g(z) \, dz \).

To determine an asymptotic expression for \( g'(1) \), Eq. (4-86c) may be used, because \( 1 >> \varepsilon \). Therefore,

\[
g'(1) = \frac{d}{dz} \left[ e^{-z^2/2\varepsilon} - \frac{1}{z} e^{-1/2\varepsilon} \right] \bigg|_{z=1}
\]

\[
= \left[ -\frac{z}{\varepsilon} e^{-z^2/2\varepsilon} + \frac{1}{z^2} e^{-1/2\varepsilon} \right] \bigg|_{z=1}
\]

\[
= -\frac{1}{\varepsilon} e^{-1/2\varepsilon} (1 - \varepsilon)
\]

(4-91)

Similarly,

\[
g'(-1) = \frac{1}{\varepsilon} e^{-1/2\varepsilon} (1 - \varepsilon)
\]

(4-92)

The asymptotic evaluation of \( \int g(z) \, dz \) is slightly more difficult because the asymptotic representation of \( g(z) \) has three regions of validity. As a first step, note that \( g(z) \) is symmetric, so that

\[
\int_{-1}^{1} g(z) \, dz = 2 \int_{0}^{1} g(z) \, dz
\]

(4-93)

The next step is to break up this integral into two integrals, so that each is over a region where only a single asymptotic representation of
the integrand is required. This may be done by choosing any point in the interior of \([0,1]\), say \(a\). Then

\[
\begin{align*}
\int_0^1 g(z) \, dz &= \int_0^a g(z) \, dz + \int_a^1 g(z) \, dz \\
&= \int_0^a g(z) \, dz + \int_0^1 g(z) \, dz \\
&= \int_0^a g(z) \, dz + \int_0^1 g(z) \, dz.
\end{align*}
\] (4-94)

Over the interval \([0,a]\), Eq. (4-86b) is valid, whereas over the interval \([a,1]\), Eq. (4-86c) is valid. Therefore,

\[
\int_0^1 g(z) \, dz = \int_0^a \frac{2e^{-z^2/2\varepsilon}}{z} \, dz + \int_a^1 \left( e^{-z^2/2\varepsilon} - \frac{1}{z} e^{-1/2\varepsilon} \right) \, dz.
\] (4-95)

Collecting terms in \(\int e^{-z^2/2\varepsilon} \, dz\), we have that

\[
\int_0^1 g(z) \, dz = \int_0^a e^{-z^2/2\varepsilon} \, dz + \int_a^1 \frac{1}{z} e^{-1/2\varepsilon} \, dz.
\] (4-96)

The first integral on the right side of Eq. (4-96) may be expanded to give

\[
\begin{align*}
\int_0^1 g(z) \, dz &= \int_0^a e^{-z^2/2\varepsilon} \, dz + \int_a^1 e^{-z^2/2\varepsilon} \, dz + \int_a^1 e^{-z^2/2\varepsilon} \, dz + \int_a^1 \frac{1}{z} e^{-1/2\varepsilon} \, dz \\
&= \int_0^a e^{-z^2/2\varepsilon} \, dz + \int_0^1 e^{-z^2/2\varepsilon} \, dz + \int_a^1 e^{-z^2/2\varepsilon} \, dz + \int_a^1 \frac{1}{z} e^{-1/2\varepsilon} \, dz.
\end{align*}
\] (4-97)

The first and last integrals on the right side of this expression are given by

\[
\int_0^\infty e^{-z^2/2\varepsilon} \, dz = \sqrt{\frac{\pi \varepsilon}{2}}.
\] (4-98)
\[
\int_{\ln a}^{1} \frac{1}{z} e^{-1/2\varepsilon} \, dz = -(\ln a) \, e^{-1/2\varepsilon}
\]  

(4-99)

The second of these two is exponentially small as \(\varepsilon \to 0\). Also, the second term of Eq. (4-97) is exponentially small, because

\[
\int_{1}^{\infty} e^{-z^2/2\varepsilon} \, dz < \int_{1}^{\infty} e^{-z/2\varepsilon} \, dz
\]

\[
= 2\varepsilon \, e^{-1/2\varepsilon}
\]  

(4-100)

Therefore,

\[
\int_{-1}^{1} g(z) \, dz \sim \sqrt{2\pi\varepsilon}
\]  

(4-101)

Finally, Eqs. (4-91), (4-92), and (4-101) can be substituted into Eq. (4-90) to give an asymptotic expression for \(\lambda\tau\):

\[
\lambda\tau \sim \frac{2}{\sqrt{2\pi}} \frac{e^{-1/2\varepsilon}}{\sqrt{\varepsilon}} (1 - \varepsilon), \quad (\varepsilon \to 0)
\]  

(4-102)

Replacing \(\varepsilon \) by \(1/T^2\) in this equation yields

\[
\lambda\tau \sim \frac{2T}{\sqrt{2\pi}} e^{-T^2/2} \left(1 - \frac{1}{T^2}\right), \quad (T \to \infty)
\]  

(4-103)

Since \(\tau\) is given, this is the desired result for the steady-state false-alarm rate \(\lambda\).

4.1.2.2 Vector Case

The determination of an asymptotic approximation to the steady-state false-alarm rate for the case where \(\chi(t)\) is a vector proceeds
exactly as in the scalar case, although some aspects of the solution are slightly more difficult. Recall that the eigenvalue problem is to find the smallest eigenvalue \( \lambda \) of

\[
\frac{1}{2} \nabla^T \Omega \nabla f(\chi) - \chi^T A^T \phi \chi f(\chi) + (\lambda - \text{tr}[A]) f(\chi) = 0 \quad (4-104)
\]

subject to the boundary condition

\[
f(\chi) = 0, \quad \chi \in \{ \chi : \chi^T S^{-1} \chi = T^2 \} \quad (4-105)
\]

where \( S \) is the solution of the Lyapunov equation

\[
A^T \phi S + S A^T \phi + \Omega \chi = 0 \quad (4-106)
\]

It will be assumed that

\[
S = I \quad (4-107)
\]

This results in no loss in generality, because the coordinate system can always be transformed by a linear transformation into one where this is true. Therefore, the results can be transformed also. However, this will not be necessary, because the result depends on an invariant of the matrix \( A \). Therefore, the boundary condition becomes

\[
f(\chi) = 0, \quad \chi \in \{ \chi : \chi^T \chi = T^2 \} \quad (4-108)
\]

Also,

\[
A^T \phi + A^T \phi + \Omega \chi = 0 \quad (4-109)
\]
This last equation will be used sever 1 times in the following
derivation.

To begin, the domain of the differential equation is normalized
from a hypersphere of radius T to a hypersphere of radius 1. As in the
scalar case, this makes the effect of the threshold T more apparent.
This is done by the change of variables

\[ x = Tz \]  \hspace{1cm} (4-110)

The function \( g(z) \) is defined by

\[ g(z) = f(Tz) \]  \hspace{1cm} (4-111)

Then Eq. (4-104) becomes

\[ \frac{1}{2} \frac{1}{T^2} \nabla^T \nabla g(z) - z A^T \nabla g(z) + (\lambda - \text{tr}[A]) g(z) = 0 \]  \hspace{1cm} (4-112)

(In order to simplify the notation, the subscript "z" will be omitted
from the gradient operator \( \nabla_z \) henceforth.) As in the scalar case, the
small parameter \( \varepsilon \) is identified as \( \varepsilon = 1/T^2 \). Then the differential
Eq. (4-112) becomes

\[ \frac{\varepsilon}{2} \nabla^T \nabla g(z) - z A^T \nabla g(z) + (\lambda - \text{tr}[A]) g(z) = 0 \]  \hspace{1cm} (4-113)

subject to the boundary condition

\[ g(z) = 0, \quad z \in B = \{ z : \frac{z^T z}{\varepsilon} = 1 \} \]  \hspace{1cm} (4-114)

Before proceeding further, we note that, as in the scalar case,
the function \( \exp(-z^T z/\varepsilon) \) satisfies the differential equation (4-113)
except for exponentially small errors. Therefore, we will determine an
asymptotic solution to Eq. (4-113) that satisfies the boundary condition (4-114) which is valid for \( \frac{z^{0}}{\varepsilon} \gg \varepsilon \), and which matches the solution \( \exp\left( -z^{0}/\varepsilon \right) \) away from the boundary.

It is assumed that the solution has the asymptotic approximation

\[
g(z) = \exp\left( \frac{1}{\varepsilon} S_{0}(z) + S_{1}(z) \right)
\]  \hspace{1cm} (4-115)

Substituting Eq. (4-115) into Eq. (4-113) and matching orders in \( \varepsilon \) yields, after cancelling the common exponential,

\[
\frac{1}{\varepsilon} \left\{ \frac{1}{2} \left( \nabla S_{0}(z) \right)^{T} Q_{\chi} \nabla S_{0}(z) - z^{T} A_{\phi} \nabla S_{0}(z) \right\} \\
+ \left\{ \frac{1}{2} \nabla_{\chi}^{T} \nabla S_{0}(z) + \left( \nabla S_{0}(z) \right)^{T} Q_{\chi} \left( \nabla S_{1}(z) \right) - z^{T} A_{\phi} \nabla S_{1}(z) \right\} \\
+ O(\varepsilon) + \lambda = 0
\]  \hspace{1cm} (4-116)

As in the scalar case, it is assumed that smallest eigenvalue \( \lambda \) is exponentially small. Therefore, to \( O(1/\varepsilon) \) and \( O(1) \), respectively, we have that

\[
\frac{1}{2} \left( \nabla S_{0}(z) \right)^{T} Q_{\chi} \nabla S_{0}(z) - z^{T} A_{\phi} \nabla S_{0}(z) = 0
\]  \hspace{1cm} (4-117)

and

\[
\frac{1}{2} \nabla_{\chi}^{T} \nabla S_{0}(z) + \left( \nabla S_{0}(z) \right)^{T} Q_{\chi} \left( \nabla S_{1}(z) \right) - z^{T} A_{\phi} \nabla S_{1}(z) - \text{tr}[A_{\phi}] = 0
\]  \hspace{1cm} (4-118)

It will be shown that the asymptotic solution to the partial differential Eq. (4-113) actually requires two terms of the form of Eq. (4-115). More specifically, the asymptotic solution will have the form

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\begin{equation}
g(z) = g_1(z) - g_2(z) \tag{4-119}
\end{equation}

where

\begin{equation}
g_i(z) = \exp \left( \frac{1}{\epsilon} S_{0i}(z) + S_{1i}(z) \right), \quad i = 1, 2 \tag{4-120}
\end{equation}

$S_{0i}(z)$ and $S_{1i}(z)$ satisfy the PDE's (4-117) and (4-118), respectively. In order that $g(z)$ satisfy the boundary condition (4-114), we require that the $S_{01}(z)$ and $S_{11}$ satisfy

\begin{equation}
S_{01}(z) = S_{02}(z) \tag{4-121}
\end{equation}

and

\begin{equation}
S_{11}(z) = S_{12}(z) \tag{4-122}
\end{equation}

At this point it is possible to find the solution to the WKB problem posed above by guessing at a solution and showing that it satisfies all the appropriate equations. However, this approach is rather sterile, and does not lead to an understanding of the solution. Furthermore, the WKB solution is valid only when the domain of the differential equation satisfies certain constraints. Therefore, we will digress at this point to examine the partial differential equations (4-117) and (4-118) in detail.

The first observation is that the solution to the PDE (4-117) is equivalent to solving an optimal control problem. To show this, consider the following optimization problem:

\begin{equation}
\min_{\omega_z(*)} J = \int_{-\infty}^{0} \frac{1}{2} \omega_z(t) Q^{-1} \omega_z(t) dt \tag{4-123}
\end{equation}
subject to

$$\frac{dz(t)}{dt} = A_z z(t) + w_z(t) \tag{4-124}$$

$$z(-\infty) = 0 \tag{4-125}$$

where \(z(0)\) is given. This problem may be solved using dynamic programming [4], [10, Sec. 4.2]. The usual approach in dynamic programming is to embed the optimization problem in a more general optimization problem, namely,

$$V(z,t) = \min_{w_z(t)} \int_0^T \frac{1}{2} w_z(t)^T Q^{-1} w_z(t) dt \tag{4-126}$$

subject to the state equation (4-124), \(z(0)\) given, and

$$z(t) = z \tag{4-127}$$

The optimal value function \(V(z,t)\) satisfies a partial differential equation which gives the optimal cost-to-go as a function of the initial time and initial condition.

For our problem, it turns out that we would like to find the optimal cost as a function of the final value of \(z(t)\), rather than as a function of the initial value. In order to make use of the results of dynamic programming, the time variable \(\tau\) will be defined by

$$\tau = -t \tag{4-128}$$

Then the optimization problem (4-123) becomes

$$\min_{w_z(t)} J = \int_0^\infty \frac{1}{2} w_z(-\tau_1)^T Q^{-1} w_z(-\tau_1) d\tau_1 \tag{4-129}$$

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subject to the state equation

\[
\frac{dz(-\tau)}{d\tau} = -A_z z(-\tau) - w_z(-\tau)
\]  

(4-130)

The boundary condition is the same as that given in Eq. (4-125). The optimal value function is then

\[
V(z, \tau) = \min_{w_{z}(\cdot)} \int_{\tau}^{\infty} \frac{1}{2} w_{z}(\tau) Q^{-1}_{\chi} w_{z}(\tau) d\tau
\]

subject to the state equation (4-130), and

\[
\underline{z}(-\infty) = 0
\]

(4-132)

and

\[
\underline{z}(-\tau) = \underline{z}
\]

(4-133)

Because the system is time-invariant, and because of the infinite time horizon, the optimal value function should be independent of \( \tau \), i.e.,

\[
V(z, \tau) = V(\underline{z})
\]

(4-134)

The optimal value function may be determined by solving the Hamilton-Jacobi-Bellman (HJB) equation [10, Sec. 4.2]. For this problem, the HJB equation is

\[
\min_{w_z} \left\{ \frac{1}{2} w_z^T Q^{-1}_z w_z + [V(z)]^T \left[ -A_z z - w_z \right] \right\} = 0
\]

(4-135)

Performing the indicated minimization yields
\[ w_z = Q \nabla V(z) \]  \hspace{1cm} (4-136)

Substituting this expression for \( w_z \) back into the HJB equation (Eq. (4-135)) yields the HJB partial differential equation (HJB-PDE) for this problem:

\[ \frac{1}{2} [\nabla V(z)]^T Q \chi [\nabla V(z)] + z^T A \phi \nabla V(z) = 0 \]  \hspace{1cm} (4-137)

Comparison of the HJB-PDE and the partial differential equation for \( S_0(z) \) (Eq. (4-117)) reveals a strong similarity. If we identify

\[ S_0(z) = -V(z) \]  \hspace{1cm} (4-138)

then it can be seen that the two equations are the same.

The above observation may be interpreted in the following way:

Given that \( z(t) \) has a given value at the final time \( t = t_f \), then the solution of the optimal control problem posed above determines the most likely (i.e., minimum energy) trajectory of \( z(t) \) for the time prior to \( t_f \). In particular, the solution to the optimal control problem becomes:

\[ w_z(t) = Q \nabla V(z) = -Q \chi S_0(z) \]  \hspace{1cm} (4-139)

This yields, as the most likely trajectory for \( z(t) \), the trajectory given by

\[ \frac{dz(t)}{dt} = A \phi z(t) - Q \chi S_0(z) \]  \hspace{1cm} (4-140)

subject to \( z(t_f) \) given.
Now, consider the partial differential equation for \( S_1(z) \) (Eq. (4-118)). This equation may be rewritten as

\[
\left[ A_\phi^* z - Q \nabla S_0(z) \right]^T \nabla S_1(z) = \frac{1}{2} \nabla^T Q \nabla S_0(z) - \text{tr}[A_\phi] \tag{4-141}
\]

If \( z \) is parameterized by \( t \) as in the discussion above, then the term in brackets on the left side of Eq. (4-141) may be identified as \( dz/dt \), and hence the left side of Eq. (4-141) is \( dS_1/dt \). Therefore, we have that

\[
\frac{d}{dt} S_1(z(t)) = \frac{1}{2} \nabla^T Q \nabla S_0(z(t)) - \text{tr}[A_\phi] \tag{4-142}
\]

That is, the optimal trajectories \( z(t) \) are characteristics [31, p. 388] of the partial differential equation (4-141). \( S_1(z(t)) \) may be determined simply by integrating both sides of Eq. (4-142), so that

\[
S_1(z(t)) = \int_{t_0}^{t} \left( \frac{1}{2} \nabla^T Q \nabla S_0(z(t)) - \text{tr}[A_\phi] \right) dt + S_1(z(t_0)) \tag{4-143}
\]

Earlier we identified the optimal trajectory for our optimization problem as the most likely trajectory of the state \( z(t) \). The next term in the WKB expansion, \( S_1(z) \), could therefore be expected to depend only on the characteristics of \( S_0(z) \) along the optimal trajectory. Indeed, this is the case, as shown above.

With this background, the complete solution to the WKB equations can be given. One of the solutions to Eq. (4-117) is given by

\[
S_{01}(z) = -\frac{1}{2} z^T z \tag{4-144}
\]

This may be verified by direct substitution. As noted earlier, the function
\[ g_1(z) = e^{-z T z / 2 \varepsilon} \quad (4-145) \]

satisfies the differential equation (4-113), except for the exponentially small error term \( \lambda g_1(z) \). Therefore,

\[ S_{11}(z) = 0 \quad (4-146) \]

\[ S_{21}(z) = 0 \quad (4-147) \]

This term is the part of the asymptotic expansion for \( g(z) \) for \( z T z \gg \varepsilon \) that matches the asymptotic expansion for \( g(z) \) in the interior of the domain D.

Before proceeding with the solution for \( g_2(z) \), it is worth noting the trajectories that are implied by the solution (4-145). The gradient of \( S_{01}(z) \) is given by

\[ \nabla S_{01}(z) = -z \quad (4-148) \]

Hence by Eq. (4-140), the optimal state trajectories \( z_1(t) \) satisfy

\[ \frac{dz_1(t)}{dt} = \left[ A \phi + Q \chi \right] z_1(t) \quad (4-149) \]

But by Eq. (4-109),

\[ A \phi + Q \chi = -A^T \phi \quad (4-150) \]
Therefore, the optimal trajectories obey

\[ \frac{dz_1(t)}{dt} = -\lambda_1^T z_1(t) \]  \hspace{1cm} (4-151)

Now, the matrix \( A_1 \) is a stable matrix. (See Chapter 5.) Therefore,

\[ \lim_{t \to \infty} z_1(t) = 0 \]  \hspace{1cm} (4-152)

Hence, the function \( S_{01}(z) \) can be interpreted as the negative of the minimum cost for trajectories which reach the point \( z \) from the origin, given an infinite amount of time.

Now consider the second solution to Eq. (4-117), \( S_{02}(z) \). Recall that the boundary condition is

\[ S_{02}(z) = S_{01}(z) , \quad z \in B \]  \hspace{1cm} (4-153)

where \( B \) is the boundary that bounds the domain of the partial differential equation. A little reflection leads to the following interpretation: \( S_{02}(z) \) is simply the negative of the minimum cost incurred by trajectories which reach the point \( z \) from the origin, subject to the constraint that each trajectory passes through the boundary at least once.

For this problem, the solution \( S_{02}(z) \) is particularly simple. Recall that \( S_{01}(z) \) is given by

\[ S_{01}(z) = -\frac{1}{2} z^T z \]  \hspace{1cm} (4-154)

The boundary is defined by the surface

\[ B = \{ z : z^T z = 1 \} \]  \hspace{1cm} (4-155)
Therefore,

\[ S_{01}(z) = -\frac{1}{2}, \quad z \in B \]  

(4-156)

Hence, the function

\[ S_{02}(z) = -\frac{1}{2} \]  

(4-157)

satisfies the partial differential equation (4-118) and the boundary condition (4-153). Therefore, the optimal trajectories \( z_2(t) \) satisfy

\[ \frac{dz_2(t)}{dt} = A_2 z_2(t) \]  

(4-158)

where once again we have used Eq. (4-140). Again, this solution has a simple interpretation. The minimum cost to reach any point on the boundary from the origin is the same, namely, 1/2. Therefore, the optimal trajectory which reaches a given point \( z \) by way of the boundary is the trajectory which \( \text{reaches} \) the point on the boundary (via an optimal trajectory \( z_1(t) \)), such that the system \( \text{coasts} \) to the desired point \( z \) from that point on the boundary. Since coasting incurs no cost, the optimal cost is 1/2 for all \( z \). For a more general boundary, the solution \( S_{02}(z) \) will, of course, be more complicated.

Figure 4-1 shows typical optimal trajectories \( z_1(t) \) and \( z_2(t) \) which end at the same point, \( z \). The dashed line indicates the optimal trajectory from the origin to the point on the boundary where the trajectory \( z_2(t) \) begins. As discussed above, this trajectory is implied by the interpretation of the solution \( S_{02}(z) \).

Given this understanding of the solution, it can be seen that there exists a case where the WKB solution will break down. In particular, suppose there is a point \( z \in D \) such that the trajectory \( z_1(t) \)
leading to that point passes outside the domain of the differential equation, D. (See Figure 4-2.) Then the trajectory \( z_1(t) \) also satisfies the definition of \( z_2(t) \). That is, \( z_1(t) \) is the minimum cost trajectory between origin and \( z \). But \( z_1(t) \) also passes through the boundary. Therefore, \( z_1(t) \) is also the optimal trajectory between the origin and \( z \), subject to the constraint that the trajectory passes through the boundary—the constraint is satisfied coincidentally. But then along the portion of the trajectory \( z_1(t) \) between the boundary and \( z \),

\[
S_{01}(z) = S_{02}(z) \tag{4-159}
\]

\[
S_{11}(z) = S_{12}(z) \tag{4-160}
\]

\[
\vdots
\]

\[
\vdots
\]

Therefore,

\[
g_1(z) = g_2(z) \tag{4-161}
\]

for any such point \( z \). By Eq. (4-119), the result is:

\[
g(z) \rightarrow 0 \tag{4-162}
\]

But this result is nonsensical unless \( g(z) = 0 \) for all values of \( \varepsilon \). This will only be the case if it is not possible for the system

\[
\frac{dz(t)}{dt} = A_s z(t) + w(t) \tag{4-163}
\]

to reach the point \( z \) from the origin without crossing the boundary using a finite amount of energy. Although such a result may be possible if \( Q_X \) is singular, in general this will not be the case.

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Figure 4-1. Typical optimal trajectories $z_1(t)$ and $z_2(t)$.

Figure 4-2. Case for which the WKB solution breaks down.
Thus, the WKB solution breaks down whenever there is a trajectory \( \bar{z}_1(t) \) that leaves and reenters the domain \( D \). This will occur if, at some point \( \bar{z}_1(t) \) on the boundary,

\[
\frac{dz_1^T(t)}{dt} \cdot \bar{n} < 0, \quad \bar{z}_1(t) \in B
\]  \hspace{1cm} (4-164)

where \( \bar{n} \) is the outward unit vector normal to the boundary. Recall that

\[
\frac{dz_1(t)}{dt} = - A^T_\phi \bar{z}_1(t)
\]  \hspace{1cm} (4-165)

Hence,

\[
\frac{dz_1^T(t)}{dt} \cdot \bar{n} = - z_1^T(t) A^\phi \bar{n}
\]  \hspace{1cm} (4-166)

Therefore, the WKB solution is valid only if

\[
z^T A^\phi \bar{n} < 0, \quad \text{for all } z \in B
\]  \hspace{1cm} (4-167)

For the problem at hand, the vector normal to the surface is given by

\[
\bar{n} = \bar{z}, \quad \bar{z} \in B
\]  \hspace{1cm} (4-168)

Therefore,

\[
z^T A^\phi \bar{n} = z^T A^\phi \bar{z} = - \frac{1}{2} z^T Q_\chi \bar{z}, \quad z \in B
\]  \hspace{1cm} (4-169)

But \( Q_\chi \) is positive semidefinite. Therefore, condition (4-167) holds and the WKB solution is valid for this problem.
At this point, we are almost ready to determine the asymptotic false-alarm rate. First, note that the partial differential equation (4-113) may be represented as

\[-\lambda g(z) = \nabla^T \left( \frac{\epsilon}{2} \nabla g(z) - \lambda \phi g(z) \right) \] (4-170)

Integrating both sides of Eq. (4-170) over the domain \( D \) yields

\[-\lambda \int_D g(z) dz = \int_D \nabla^T \left( \frac{\epsilon}{2} \nabla g(z) - \lambda \phi g(z) \right) dz \] (4-171)

Using the divergence theorem [31, Sec. 6.13] to express the second integral above as a surface integral, we have that

\[-\lambda \int_D g(z) dz = \int_B \left( \frac{\epsilon}{2} \nabla g(z) - \lambda \phi g(z) \right)^T n \, ds \] (4-172)

where \( ds \) is the element of area on the boundary. Recall that \( g(z) \) is zero on \( B \). Using this fact and Eq. (4-168) and solving Eq. (4-172) for \( \lambda \) yields

\[ \lambda = \frac{-\frac{\epsilon}{2} \int_B \left[ \nabla g(z) \right]^T z \, ds}{\int_D g(z) dz} \] (4-173)

Thus, the asymptotic evaluation of \( \lambda \) can be performed by asymptotically evaluating the two integrals in Eq. (4-173).

We begin with the evaluation of the integral in the denominator of Eq. (4-173). Because the integral is over \( D \), an asymptotic expansion for \( g(z) \) is required that is valid over the entire domain of \( z \). The solution described above is valid for \( z^T z \gg \epsilon \). Note that in the interior of \( D \), \( g_2(z) \) is exponentially small compared to \( g_1(z) \), because
\[ S_{02} (z) < S_{01} (z) \quad (4-174) \]

But recall that \( g_1 (z) \) satisfies the PDE (4-113) except for exponentially small errors. Therefore, the asymptotic expression for \( g(z) \) is given by

\[
g(z) \sim \begin{cases} 
  g_1 (z), \quad (1 - \frac{T}{z} z) \gg \epsilon \\
  g_1 (z) + g_2 (z), \quad (\frac{T}{z} z) \gg \epsilon 
\end{cases} \quad (4-175a)
\]

\[
g(z) \sim \begin{cases} 
  g_1 (z), \quad (1 - \frac{T}{z} z) \gg \epsilon \\
  g_1 (z) + g_2 (z), \quad (\frac{T}{z} z) \gg \epsilon 
\end{cases} \quad (4-175b)
\]

Of course, the regions of validity of the two expressions above overlap. Therefore, the integral in the denominator of Eq. (4-173) can be represented by

\[
\int_{D} g(z) dz = \int_{D_1} g_1 (z) dz - \int_{D_2} (g_1 (z) - g_2 (z)) dz \quad (4-176)
\]

where the regions \( D_1 \) and \( D_2 \) are disjoint, and

\[
D = D_1 \cup D_2 \quad (4-177)
\]

The regions \( D_1 \) and \( D_2 \) are selected so that

\[
z \in D_1 + 1 - \frac{T}{z} z \gg \epsilon \quad (4-178)
\]

\[
z \in D_2 + \frac{T}{z} z \gg \epsilon \quad (4-179)
\]

For example, we could take

\[
D_1 = \{ z : 0 \leq \frac{T}{z} z \leq a \} \quad (4-180)
\]
\[ D_2 = \{ z : a < \mathbf{z}^T \mathbf{z} \leq 1 \} \]  

(4-181)

where \( a \in (0,1) \). Then Eq. (4-176) may be expressed as

\[
\int_{D} g(z) dz - \int_{D} g_1(z) dz - \int_{D_2} g_2(z) dz 
\]  

(4-182)

The second integral on the right side of Eq. (4-182) is exponentially small, and hence can be ignored. Therefore,

\[
\int_{D} g(z) dz - \int_{D} e^{-\frac{z^T z}{\varepsilon}} dz 
\]  

(4-183)

As in the scalar case, the domain of the integration can be extended to include the entire \( p \)-dimensional Euclidean space, with only exponentially small errors occurring as a result. Therefore, the integral in the denominator of Eq. (4-173) is approximated with exponentially small errors by

\[
\int_{D} g(z) dz \sim (2\pi\varepsilon)^{p/2} 
\]  

(4-184)

Next, consider the integral in the numerator of Eq. (4-173). The gradient of \( g(z) \) on the boundary is given by

\[
\nabla g(z) = \nabla g_1(z) - \nabla g_2(z) 
\]  

(4-185)

\( \nabla g_1(z) \) is given by

\[
\nabla g_1(z) = -\frac{z}{\varepsilon} e^{-\frac{z^T z}{\varepsilon}} 
\]  

(4-186)
\( \nabla g_2(z) \) is given by

\[
\nabla g_2(z) = \nabla \exp \left( -\frac{1}{2} \varepsilon + S_{12}(z) \right) \\
= (\nabla S_{12}(z)) \exp \left( -\frac{1}{2} \varepsilon + S_{12}(z) \right)
\]

(4-187)

On the boundary,

\[
\frac{z}{\bar{z}}^T \frac{z}{\bar{z}} = 1, \quad z \in \mathbb{B}
\]

(4-188)

and

\[
S_{12}(z) = 0, \quad z \in \mathbb{B}
\]

(4-189)

Therefore,

\[
\nabla g(z) = \left\{ -\frac{z}{\varepsilon} - \nabla S_{12}(z) \right\} e^{-1/2\varepsilon}, \quad z \in \mathbb{B}
\]

(4-190)

We have yet to determine \( S_{12}(z) \). For our purposes, it will suffice to determine only \( \nabla S_{12}(z) \) for \( z \in \mathbb{B} \). This may be accomplished as follows: We know that on the boundary, \( S_{12}(z) = 0 \). Therefore, the gradient of \( S_{12}(z) \) must be normal to the boundary. In this case, the normal direction is simply along \( z \). Therefore,

\[
\nabla S_{12}(z) = a(z)z, \quad z \in \mathbb{B}
\]

(4-191)

where \( a(z) \) is some unknown function of \( z \) to be determined. By Eqs. (4-141) and (4-191), \( a(z) \) must satisfy

\[
(A_0 - Q_X \nabla S_{02}(z))^T za(z) = \frac{1}{2} \nabla^T Q_X \nabla S_{02}(z) - \text{tr}[A_0], \quad z \in \mathbb{B}
\]

(4-192)
But $S_{02}(z)$ is a constant, so that

$$\nabla S_{02}(z) = 0 \quad (4-193)$$

Therefore,

$$z^T A^T \phi z a(z) = -\text{tr}[A^T \phi], \quad z \in B \quad (4-194)$$

Hence,

$$\nabla S_{12}(z) = \frac{-\text{tr}[A^T \phi]}{z^T A^T \phi z} z, \quad z \in B \quad (4-195)$$

Thus, by Eqs. (4-190) and (4-195), the gradient of $g(z)$ on the boundary is given by

$$\nabla g(z) = -\frac{z}{\varepsilon} + \frac{\text{tr}[A^T \phi]}{z^T A^T \phi z} e^{-1/2\varepsilon}, \quad z \in B \quad (4-196)$$

Therefore, the integral in the numerator of Eq. (4-173) is (approximately) given by

$$\int_B [Q^T \nabla g(z)]^T z d\sigma = -e^{-1/2\varepsilon} \int_B \left[ \frac{z^T}{\varepsilon} - \frac{\text{tr}[A^T \phi]}{z^T A^T \phi z} \right] Q^T z d\sigma \quad (4-197)$$

We begin the evaluation of this integral by calculating the contribution of the first term in the brackets, namely the integral

$$\int_B \frac{z^T Q^T z}{\varepsilon} d\sigma \quad (4-198)$$
Now, $Q_X$ is a symmetric, positive semidefinite matrix. Therefore, it may be represented in the form

$$Q_X = \sum_{i=1}^{P} \mu_i e_i e_i^T$$  \hspace{1cm} (4-199)

where the $\mu_i$ are the eigenvalues of $Q_X$, and the $e_i$ are the corresponding (unit) eigenvectors. Then

$$\int_{B} z^T Q_X z \, d\sigma = \sum_{i=1}^{P} \mu_i \int_{B} (z^T e_i)^2 \, d\sigma$$  \hspace{1cm} (4-200)

Due to the symmetry of the hypersphere, each of the integrals in the summation above has the same value. Therefore,

$$\int_{B} (z^T e_i)^2 \, d\sigma = \frac{1}{p} \sum_{j=1}^{P} \int_{B} z^T e_j e_j^T z \, d\sigma$$

$$= \frac{1}{P} \int_{B} z \left[ \sum_{j=1}^{P} e_j e_j^T \right] z \, d\sigma$$  \hspace{1cm} (4-201)

But

$$\sum_{j=1}^{P} e_j e_j^T = I$$  \hspace{1cm} (4-202)

Therefore,

$$\int_{B} (z^T e_i)^2 \, d\sigma = \frac{1}{P} \int_{B} z^T z \, d\sigma$$

$$= \frac{1}{P} \int_{B} \, d\sigma$$  \hspace{1cm} (4-203)
Using this last result and Eq. (4-200), we have that

\[
\int_{B} \frac{z^{T} Q \chi T z}{z} d\sigma = \sum_{i=1}^{p} \mu_{i} \frac{1}{p} \int_{B} d\sigma \tag{4-204}
\]

But

\[
\sum_{i=1}^{p} \mu_{i} = \text{tr}[Q_{\chi}] = -2 \text{tr}[A_{\phi}] \tag{4-205}
\]

Therefore,

\[
\int_{B} \frac{z^{T} Q \chi T z}{z} d\sigma = \frac{-2}{p} \text{tr}[A_{\phi}] \int_{B} d\sigma \tag{4-206}
\]

The evaluation of the contribution of the second term in the integral of Eq. (4-197) is not as involved as the evaluation of the contribution of the first term. The integral to be evaluated is

\[
\int_{B} \frac{z^{T} Q \chi T z}{z} d\sigma \tag{4-207}
\]

But

\[
z^{T} Q_{\chi} z = -2z^{T} A_{\phi} z \tag{4-208}
\]

Therefore,

\[
\int_{B} \frac{z^{T} Q_{\chi} T z}{z} d\sigma = -2 \int_{B} d\sigma \tag{4-209}
\]
Combining the results of Eq. (4-197), (4-206), and (4-209) we have that

\[
\int_B [Q \chi_v g(z)]^T z \, d\sigma = 2 \text{tr}[A_\phi] e^{-1/2\varepsilon} \left\{ \frac{1}{p\varepsilon} - 1 \right\} \int_B d\sigma \tag{4-210}
\]

The integral \(\int d\sigma\) is simply the surface area of a \(p\)-dimensional unit hypersphere, which is given by

\[
\int_B d\sigma = \frac{p\pi^{p/2}}{\Gamma\left(\frac{p}{2} + 1\right)} \tag{4-211}
\]

where \(\Gamma(\cdot)\) denotes the gamma function. Therefore,

\[
\int_B [Q \chi_v g(z)]^T z \, d\sigma = 2 \text{tr}[A_\phi] \frac{\pi^{p/2}}{\Gamma\left(\frac{p}{2} + 1\right)} e^{-1/2\varepsilon} \left\{ \frac{1}{p\varepsilon} - p \right\} \tag{4-212}
\]

Finally, the expressions in Eq. (4-184) and (4-212) are substituted into Eq. (4-173) to obtain

\[
\lambda = -\frac{\text{tr}[A_\phi]}{\Gamma\left(\frac{p}{2} + 1\right)(2\varepsilon)^{p/2}} e^{-1/2\varepsilon} \left\{ 1 - p\varepsilon \right\} \tag{4-213}
\]

Because the denominator of Eq. (4-173) was evaluated with only exponentially small errors, the dominant error in this approximation is due to the next term in the asymptotic expression for \(\chi_v g(z)\). This error is due to the next term in the exponent of \(g_2(z)\), namely \(S_{22}(z)\). Therefore, the approximation of \(\chi_v g(z)\) (Eq. (4-196)) has an error which is \(O(\varepsilon^{-1/2\varepsilon})\). Because \(\chi_v g(z)\) is \(O\left(\frac{1}{\varepsilon} e^{-1/2\varepsilon}\right)\), the relative error in the asymptotic approximation \(O(\varepsilon^2)\). That is,
\[ \lambda = \frac{-\text{tr}[A_{\phi}]}{\Gamma \left( \frac{P}{2} + 1 \right)(2\varepsilon)^{P/2}} \ e^{-1/2\varepsilon} \left\{ 1 - p\varepsilon + O(\varepsilon^2) \right\} \] (4-214)

Recall that \( \varepsilon = 1/T^2 \). Therefore, in terms of the threshold \( T \),

\[ \lambda = \frac{-\text{tr}[A_{\phi}] T^P}{\Gamma \left( \frac{P}{2} + 1 \right) 2^{P/2}} \ e^{-T^2/2} \left\{ 1 - \frac{P}{T^2} + O\left( \frac{1}{T^4} \right) \right\}, \quad T \to \infty \] (4-215)

which is the desired result.

4.1.3 Discrete-Time Systems

Thus far, we have considered the evaluation of false-alarm performance for the continuous-time OSGLR test. In this section, the false-alarm performance of the discrete-time OSGLR test is considered.

The analysis of the discrete-time case proceeds analogously to the continuous-time case. As in the continuous-time case, it is necessary to assume that the system is time-invariant in order to obtain results. Therefore, we will begin by assuming that the system is time-invariant, and that the Kalman filter and OSGLR equations have reached steady state. Then the difference equation for the OSGLR sufficient statistic \( \chi(k) \) is given by

\[ \chi(k+1) = \Phi \chi(k) + G^T M^{-1} \gamma(k+1) \] (4-216)

where \( \gamma(k) \) is a zero-mean, white, Gaussian residual sequence with covariance \( M \). A failure is declared when the detection function exceeds the detection threshold:
\[ D_F(k) = \chi^T(k)S^{-1}\chi(k) \gtrless T^2 \]  
\[ \text{declare a failure} \]
\[ \text{continue sampling} \]

S is the covariance of \( \chi \), and satisfies the discrete-time version of the Lyapunov equation:

\[ S = \phi S\phi^T + Q_\chi \]  
\[ (4-218) \]

where

\[ Q_\chi = G^T \Lambda^{-1} G \]

A false alarm occurs when the decision function exceeds the threshold (in the absence of a failure). As in the continuous-time case, the time of false alarm, \( k_{FA} \), is defined by

\[ k_{FA} = \inf \{ k : D_F(k) \geq T^2 \} \]  
\[ (4-219) \]

Ideally, we would like to determine the distribution of \( k_{FA} \), which is denoted by

\[ P_{FA}(k) = \Pr(k_{FA} < k) \]  
\[ (4-220) \]

Then the probability that a false alarm has not occurred by time \( k \) is given by

\[ P_{NA}(k) = \Pr(k_{FA} > k) \]
\[ = 1 - P_{FA}(k) \]  
\[ (4-221) \]
In principle, it is possible to determine the distribution of the false-alarm time by propagating the probability density of $\chi(k)$, $p(\chi,k)$, forward in time. Whereas in the continuous-time case the propagation of the probability density was expressed as a partial differential equation (the Fokker-Planck equation), in the discrete-time case the propagation of the density is expressed by an integral:

$$p(\chi,k+1) = \int_D \frac{1}{(2\pi)^{D/2}|Q_X|^{1/2}} \exp \left\{ -\frac{1}{2} (\chi - \phi u)^T Q_X^{-1} (\chi - \phi u) \right\} p(u,k) \, du,$$

$$\chi \in D \tag{4-222}$$

where the integral is over the region

$$D = \{ u : u^T S^{-1} u \leq T^2 \} \tag{4-223}$$

Then $P_{NA}(k)$ is given by

$$P_{NA}(k) = \int_D p(\chi,k) \, d\chi \tag{4-224}$$

As in the continuous-time case, we may solve for $p(\chi,k)$ by separation of variables. We assume solutions of the form

$$p(\chi,k) = f(\chi)h(k) \tag{4-225}$$

Then Eq. (4-222) becomes

$$f(\chi)h(k+1) = \int_D L(\chi,u)f(u)du \cdot h(k) \tag{4-226}$$

where $L(\chi,u)$ is the kernel of the integral in Eq. (4-222). Dividing both sides of Eq. (4-226) by $f(\chi)h(k)$, we have that
\[
\frac{h(k+1)}{h(k)} = \frac{\int_D L(x,u)f(u)du}{f(x)}
\] (4-227)

Because the left side of this equation is a function only of \( k \), and the right side is a function only of \( x \), each side must be equal to the same constant. Therefore,

\[ h(k+1) = \mu h(k) \] (4-228)

and

\[ \int_D L(x,u)f(u)du = \mu f(x) \] (4-229)

Equation (4-229) is an integral equation [32, Ch. 3]. Nontrivial solutions to Eq. (4-229) exist only when \( \mu \) is an eigenvalue of the integral equation. The general solution to Eq. (4-229) may then be expressed as

\[ p(x,k) = \sum_{i=1}^{\infty} c_i \mu_i^k f_i(x) \] (4-230)

where each \( \mu_i \) is an eigenvalue of the integral equation, and \( f_i(x) \) is the corresponding eigenfunction.

For large \( k \), the summation in Eq. (4-230) is dominated by the term with the largest eigenvalue. Therefore,

\[ p(x,k) \approx c_1 \mu_1^k f_1(x), \quad k >> k_0 \] (4-231)

where \( \mu_1 \) is the eigenvalue with the largest magnitude. Thus,

\[ P_{NA}(k) = c \mu_1^k, \quad k >> k_0 \] (4-232)
where

\[
    c = c_1 \int_D f_1(\chi) \, d\chi
\]  

(4-233)

Note that Eq. (4-232) implies that

\[
    \frac{P_{NA}(k)}{P_{NA}(k-1)} = \mu_i, \quad k > k_0
\]  

(4-234)

Therefore, \( \mu_1 \) may be identified as the steady-state probability that no alarm occurs at time \( k \), given that none has occurred prior to time \( k \). Hence, \( 1-\mu_1 \) is the probability of a false alarm at time \( k \), given that none has occurred prior to time \( k \). Thus, the problem of finding the (steady-state) false-alarm performance of the OSGLR test requires finding the largest eigenvalue of the integral equation

\[
    \mu f(\chi) = \int_D \frac{1}{(2\pi)^{P/2} |Q_{\chi}|^{1/2}} \exp \left\{ -\frac{1}{2} (\chi - \phi \mu)^T Q_{\chi}^{-1} (\chi - \phi \mu) \right\} f(u) du
\]  

(4-235)

Unfortunately, the solution of the above eigenvalue problem is as difficult as the solution of the eigenvalue problem for the steady-state false-alarm rate in the continuous-time case (Eq. (4-33)). Therefore, we would like to find an asymptotic expression for \( \mu \), as was done in the continuous-time case for \( \lambda \). Unfortunately, it was not possible to determine an asymptotic expression for \( \mu \).

However, it may be possible to use the results of Section 4.1.2 to bound the false-alarm performance of the discrete-time OSGLR test. To do this, we will find a continuous-time system that, when sampled, produces a discrete-time process with the same statistical properties as \( \chi(k) \). We can then relate the false-alarm rate of the discrete-time system to that.
of the continuous-time system, for which the false-alarm rate has been determined.

To begin, note that the asymptotic false-alarm rate derived in the previous section applies to any continuous-time linear system driven by white noise. That is, suppose we have the system

\[
\frac{d\mathbf{x}_c(t)}{dt} = \mathbf{A}_c \mathbf{x}_c(t) + \mathbf{w}_c(t)
\]  

(4-236)

where \( \mathbf{x}_c(t) \) has dimension \( p \), \( \mathbf{A}_c \) is a stable matrix, and \( \mathbf{w}_c(t) \) is a zero-mean, white Gaussian process with intensity \( \mathbf{Q}_c \). (The subscript "c" denotes "continuous-time.") The steady-state covariance of \( \mathbf{x}_c(t) \) is given by \( \mathbf{S}_c \), where \( \mathbf{S}_c \) satisfies the Lyapunov equation

\[
\mathbf{A}_c \mathbf{S}_c + \mathbf{S}_c \mathbf{A}_c^T + \mathbf{Q}_c = \mathbf{0}
\]  

(4-237)

If we define \( t_{PA} \) as the first-passage time of \( \mathbf{x}_c(t) \) across the boundary

\[
\mathbf{x}_c^T(t) \mathbf{S}_c^{-1} \mathbf{x}_c(t) = T^2
\]  

(4-238)

then the results of Section 4.1.2 apply directly. In particular, the steady-state false-alarm rate is (asymptotically) given by

\[
\lambda_c = \frac{-\text{tr}[\mathbf{A}_c] T^p}{\Gamma(\frac{p}{2} + 1) 2^{p/2}} e^{-T^2/2} \left\{ 1 - \frac{p}{T^2} + O\left(\frac{1}{T^4}\right) \right\}, \ (T \to \infty)
\]  

(4-239)

Also, for large \( t \), the probability that \( \mathbf{x}_c(t) \) has not crossed the boundary is given by
\[ p_{NA}^c(t) = C_c e^{-\lambda t}, \quad t >> t_0 \]  \hspace{1cm} (4-240)

where the coefficient \( C_c \) depends on the initial probability distribution of \( X_c(t) \).

Now, suppose that the process \( X_c(t) \) is sampled with sampling period \( \Delta t \), so that

\[ X_d(k) = X_c(k\Delta t) \]  \hspace{1cm} (4-241)

(The subscript "d" denotes "discrete-time.") The state equation for \( X_d(k) \) is given by

\[ X_d(k+1) = A_d X_d(k) + w_d(k) \]  \hspace{1cm} (4-242)

where

\[ A_d = e^{A \Delta t} \]  \hspace{1cm} (4-243)

and

\[ w_d(k) = \int_{k\Delta t}^{(k+1)\Delta t} \exp\{A_c[(k+1)\Delta t - t]\} w_c(t) \, dt \]  \hspace{1cm} (4-244)

\( w_d(k) \) is a zero-mean, white Gaussian sequence with covariance

\[ Q_d = \int_0^{\Delta t} \exp(A_c^T) Q_c \exp(A_c^T) \, d\tau \]  \hspace{1cm} (4-245)

The steady-state covariance of \( X_d(k) \), \( S_d \), satisfies
\[ S_d = A_d S_d A_d^T + Q_d \quad (4-246) \]

Note that because \( x_d(k) \) is obtained by sampling \( x_c(t) \), it is possible to directly relate (some of) the statistical properties of \( x_c(t) \) to those of \( x_d(k) \). In particular, the covariances of \( x_c(t) \) and \( x_d(k) \) are the same:

\[ S_d = S_c \quad (4-247) \]

Suppose that the matrices \( A_c \) and \( Q_c \) are chosen so that the resulting discrete-time process \( x_d(k) \) has the same statistical description as \( x(k) \), i.e., so that

\[ \phi = A_d = e^{A_c \Delta t} \quad (4-248) \]

and

\[ Q_x = Q_d \quad (4-249) \]

Then for the purposes of determining the statistical properties of \( x(k) \), we can treat \( x(k) \) as if it were generated by sampling the continuous-time process of Eq. (4-236), so that

\[ x(k) = x_c(k\Delta t) \quad (4-250) \]

The validity of the assumption that there exist matrices \( A_c \) and \( Q_c \) such that Eqs. (4-248) and (4-249) are satisfied will be examined later.

Now, consider the false-alarm times \( t_{FA} \) and \( k_{FA} \). For any realization of the process \( x_c(t) \) (and hence of \( x(k) \)), we must have that

\[ t_{FA} \leq k_{FA} \Delta t \quad (4-251) \]
This follows from the fact that if $\chi(k)$ lies outside the boundary, then so must $\chi_0(k\Delta t)$—they are the same vector. Therefore,

$$p_{NA}^C(k\Delta t) \leq p_{NA}^d(k)$$  \hspace{1cm} (4-252)

where the superscripts "c" and "d" have obvious meanings. Now, for $k$ large,

$$p_{NA}^C(k\Delta t) = C_c e^{-\lambda k\Delta t}, \hspace{0.5cm} k >> k_0$$  \hspace{1cm} (4-253)

and

$$p_{NA}^d(k) = C_d \mu^k, \hspace{0.5cm} k >> k_0$$  \hspace{1cm} (4-254)

where the coefficients $C_c$ and $C_d$ depend on the initial probability distribution of $\chi_0(t)$ (and hence of $\chi(k)$). Combining these last three equations, we have that for large $k$

$$C_c(e^{-\lambda \Delta t}) \leq C_d \mu^k, \hspace{0.5cm} k >> k_0$$  \hspace{1cm} (4-255)

In order for this inequality to hold for arbitrarily large $k$, we must have that

$$e^{-\lambda \Delta t} \leq \mu$$  \hspace{1cm} (4-256)

Solving for $\lambda_c$ yields

$$\lambda_c \geq \frac{-\ln \mu}{\Delta t}$$  \hspace{1cm} (4-257)
The term on the right side of Eq. (4-257) is defined to be the false-alarm rate for the discrete-time OSGLR test:

\[
\lambda_d \Delta = -\ln \mu \quad \frac{\Delta t}{(4-258)}
\]

Thus, the false-alarm rate for the discrete-time test is bounded from above by

\[
\lambda_d \leq \lambda_c \quad (4-259)
\]

where an asymptotic expression for \(\lambda_c\) is given in Eq. (4-239).

The asymptotic expression for \(\lambda_c\) depends on the dimension of \(X_c(t)\), \(p\); the threshold, \(T\); and the trace of \(A_c\). Of these, \(p\) and \(T\) are given. Therefore, we have only to relate \(\text{tr}[A_c]\) to \(\phi\) to determine the bound on the discrete-time false-alarm rate. Now, the trace of \(A_c\) may be expressed as

\[
\text{tr}[A_c] = \sum_{i=1}^{p} \lambda_i \quad (4-260)
\]

where the \(\lambda_i\) are the eigenvalues of \(A_c\). Recall that \(\phi\) is given by

\[
\phi = e^{A \Delta t} \quad (4-261)
\]

It can be shown [25, Ch. 5] that the eigenvalues of \(\phi\), \(\mu_i\), are given by

\[
\mu_i = e^{\lambda_i \Delta t}, \quad i = 1, 2, \ldots, p \quad (4-262)
\]
Therefore,

\[ \lambda_i = \frac{\ln \mu_i}{\Delta t}, \quad i = 1, 2, \ldots, p \]  \hspace{1cm} (4-263)

Therefore, \( \text{tr}[A_c] \) is given by

\[
\text{tr}[A_c] = \frac{1}{\Delta t} \sum_{i=1}^{p} \frac{\ln \mu_i}{\Delta t} \\
= \frac{1}{\Delta t} \ln \prod_{i=1}^{p} \mu_i \\
= \frac{1}{\Delta t} \ln (\det \phi_\phi) \hspace{1cm} (4-264)
\]

because

\[
\prod_{i=1}^{p} \mu_i = \det \phi_\phi \hspace{1cm} (4-265)
\]

Thus, \( \lambda_d \) is (asymptotically) bounded by

\[
\lambda_d \leq \frac{-\ln(\det \phi_\phi) T^p}{\Delta t \Gamma(\frac{p}{2} + 1) 2^{p/2}} e^{-T^2/2 \left\{ 1 - \frac{p}{T^2} + O\left(\frac{1}{T^4}\right) \right\}}, \quad T \to \infty \hspace{1cm} (4-266)
\]

We return now to the question of whether a matrix \( A_c \) exists such that Eq. \( (4-248) \) is satisfied. Formally, we may take the natural logarithm of Eq. \( (4-248) \) to obtain

\[
A_c \Delta t = \ln \phi_\phi \hspace{1cm} (4-267)
\]
It can be shown [25, pp. 239-241] that if $\Phi_\phi$ has a nonzero determinant, then $\Phi_\phi$ has a logarithm. However, the logarithm may be a complex matrix, even if $\Phi_\phi$ is real. Below, we briefly outline under what conditions $\ln \Phi_\phi$ is a real matrix.

To begin, suppose that $\Phi_\phi$ has elementary divisors

$$
(\mu - \mu_1)^{p_1}, (\mu - \mu_2)^{p_2}, \ldots, (\mu - \mu_u)^{p_u}
$$

where

$$
p_1 + p_2 + \ldots + p_u = p
$$

(See [25] for the definition of elementary divisors.) $\Phi_\phi$ may be expressed as

$$
\Phi_\phi = \tilde{\Phi} U^{-1}
$$

where $\tilde{\Phi}$ is a matrix in Jordan form, with blocks corresponding to the elementary divisors. That is, $\tilde{\Phi}$ is given by

$$
\tilde{\Phi} = \text{diag} \{J_1, J_2, \ldots, J_u\}
$$

where each $J_i$ is a $p_i \times p_i$ Jordan block:

$$
J_i =
\begin{bmatrix}
\mu_i & 1 & 0 & \cdots & 0 \\
0 & \mu_i & 1 & \cdots & 0 \\
0 & 0 & \mu_i & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \mu_i
\end{bmatrix}
$$

(4-272)
The columns of $U$ are the eigenvectors and generalized eigenvectors of $\Phi \phi$. The general expression for the logarithm of $\Phi \phi$ is given by [25, p. 241]

$$\ln \Phi \phi = UX_\phi \text{ diag } \{\ln J_1, \ln J_2, \ldots, \ln J_u\} X_\phi^{-1} U^{-1} \quad (4-273)$$

where $X_\phi$ is any matrix that commutes with $\Phi$. The logarithm of each Jordan block is given by

$$\ln J_i = \begin{bmatrix}
\ln \mu_i & 1/u_i & -1/2u_i^2 & 1/3u_i^3 & \cdots \\
0 & \ln \mu_i & 1/u_i & -1/2u_i^2 & \cdots \\
0 & 0 & \ln \mu_i & 1/u_i & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \quad (4-274)$$

Consider first the case where the elementary divisors are co-prime, i.e.,

$$u_i \neq u_j \quad (4-275)$$

In that case, the factors $X_\phi$ and $X_\phi^{-1}$ may be omitted from the expression (4-273) [25, p. 241]. Then $\ln \Phi \phi$ may be expressed as

$$\ln \Phi \phi = U \text{ diag } \{\ln J_1, \ln J_2, \ldots, \ln J_u\} V \quad (4-276)$$

where

$$V = U^{-1} \quad (4-277)$$
If $U$ is partitioned by columns, and $V$ is partitioned by rows, with partitions corresponding to the partitions of $\phi$, then $\ln \phi$ can be expressed as

$$
\ln \phi = \sum_{i=1}^{\tilde{I}} U_i \ln J_i V_i
$$

(4-278)

Now, consider the contribution to the sum in Eq. (4-278) from each block $J_i$. There are three cases to consider:

1. $u_i$ real, positive. In this case, $\ln u_i$ is real if we use the principle value of the logarithm. Therefore, the matrix $\ln J_i$ is a real matrix. Furthermore, $U_i$ is real, because the eigenvectors of a real matrix corresponding to a real eigenvalue are real. Similarly, $V_i$ is real, because the rows of $V_i$ are the left eigenvectors of $\phi_i$. Therefore, the contribution to the sum of Eq. (4-278) due to a real, positive eigenvalue is real.

2. $u_i$ complex. Because $\phi_i$ is real, $\mu_i^*$ (the complex conjugate of $u_i$) must also be an eigenvalue, so that

$$
\mu_{i+1} = \mu_i^*
$$

(4-279)

Furthermore, $\mu_{i+1}$ must have the same multiplicity as $\mu_i$. The left and right eigenvectors corresponding to the two eigenvalues are related by

$$
U_{i+1} = U_i^*
$$

(4-280)

$$
V_{i+1} = V_i^*
$$

(4-281)

If $\ln J_i$ and $\ln J_{i+1}$ are chosen so that

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\[ \ln \mu_{i+1} = (\ln \mu_i)^* \quad (4-282) \]

then

\[ \ln J_{i+1} = (\ln J_i)^* \quad (4-283) \]

Therefore, by Eqs. (4-280), (4-281), and (4-283),

\[ U_{i+1} \ln J_{i+1} V_{i+1} = (U_i \ln J_i V_i)^* \quad (4-284) \]

Hence, the sum of the contributions to \( \ln \phi \) due to \( \mu_i \) and \( \mu_{i+1} \) is real.

3. \( \mu_i \) real, negative. In this case, the corresponding eigenvalue of \( \ln \phi \) is given by

\[ \lambda_i = \ln \mu_i = \ln (-\mu_i) + (2n + 1)\pi i \quad (4-285) \]

where \( n \) is any integer. That is, \( \lambda_i \) is complex. Now, if \( \ln \phi \) is real, all complex eigenvalues of \( \ln \phi \) must occur in complex-conjugate pairs. But that does not happen in this case.

The conclusion is that if the elementary divisors of \( \phi \) are co-prime, then we can find a real logarithm of \( \phi \) if and only if there are no zero or negative real eigenvalues.

The situation is somewhat more complicated if the elementary divisors are not co-prime. The discussion above concerning positive and complex eigenvalues still holds. However, the objection to negative eigenvalues may not be valid. For example, suppose that there are two negative eigenvalues such that

\[ \mu_i = \mu_{i+1} < 0 \quad (4-286) \]
Then we can choose

\[ \lambda_i = \ln (-\mu_i) + \pi i \]  
(4-287)

\[ \lambda_{i+1} = \ln (-\mu_{i+1}) - \pi i \]  
(4-288)

That is, \( \lambda_i \) and \( \lambda_{i+1} \) are complex conjugates. Without going through the details, the conclusion is the following: A real matrix \( A_c \) that is a solution to

\[ \phi_c = e^{A_c \Delta t} \]  
(4-289)

can be found if and only if

\[ \det \phi_c \neq 0 \]  
(4-290)

and all the elementary divisors of \( \phi_c \) corresponding to negative eigenvalues occur in pairs of two.

In this thesis, the OSGLR basis functions are chosen so that all the eigenvalues of \( \phi_c \) are positive. Hence, there exists a real matrix \( A_c \) such that Eq. (4-289) is satisfied.

Finally, it must be determined whether a real matrix \( Q_c \) exists such that Eq. (4-249) is satisfied. Using Eqs. (4-237) and (4-247), we have that

\[ Q_c = -A_c S_d - S_d A_c^T \]  
(4-291)

Hence, \( Q_c \) may be obtained directly from \( A_c \) and \( S_d \), the covariance of \( \chi(k) \).

It is clear from Eq. (4-291) that \( Q_c \) is symmetric. However, there is no guarantee the \( Q_c \) is positive definite, or even semidefinite.
If $Q_c$ is indefinite, then it cannot be a valid noise intensity matrix. If this is the case, no continuous-time process exists that, when sampled, produces a sequence with the same statistics as $\chi(k)$. In this case, the bound in Eq. (4-266) is not valid. Therefore, the applicability of the bound must be determined for each case by determining whether $Q_c$ is positive semidefinite or not.

There is one condition for which we expect the results of this section to be applicable to the discrete-time case. The continuous-time OSGLR test may be considered to be the result of a limiting process applied to the discrete-time OSGLR test. In the limit as $\Delta t \to 0$, the false-alarm rate of the continuous-time OSGLR test is the same as that of the discrete-time case. That is,

$$\lim_{\Delta t \to 0} \lambda_d = \lambda_c \tag{4-292}$$

Therefore, if $\Delta t$ is small, relative to the time constants of the system and of $\Lambda_c$, we expect that

$$\lambda_d \approx \lambda_c \tag{4-293}$$

In this case, the results for the false-alarm rate of the continuous-time OSGLR test may be applied directly to the discrete-time case, in which case Eq. (4-266) holds (approximately) with equality.

4.1.4 Comparison to Numerical Results

In this subsection, the asymptotic results obtained earlier in this section are compared to results obtained numerically. As a practical matter, numerical results can be obtained only when $\chi(t)$ (or $\chi(k)$ in the discrete-time case) is a scalar ($p = 1$).

First consider the eigenvalue problem for determining the steady-state false-alarm rate of the continuous-time OSGLR test when $p = 1$. The differential equation and boundary conditions are given in Eqs. (4-38) and (4-39), which are repeated here for convenience:
\[
S \frac{d^2}{d\chi^2} f(\chi) + \chi \frac{d}{d\chi} f(\chi) + (\lambda \tau + 1) f(\chi) = 0 \quad (4-294)
\]

\[
f(\pm \sqrt{S} \tau) = 0 \quad (4-295)
\]

where \( S \) is the (unconditional) covariance of \( \chi \), and \( \tau \) is the threshold. \( \chi \) can always be normalized so that

\[
S = 1 \quad (4-296)
\]

Therefore, we will consider the eigenvalue problem

\[
\frac{d^2}{d\chi^2} f(\chi) + \chi \frac{d}{d\chi} f(\chi) + (\lambda \tau + 1) f(\chi) = 0 \quad (4-297)
\]

\[
f(\pm \tau) = 0 \quad (4-298)
\]

This eigenvalue problem may be converted to a proper Sturm-Liouville problem [31, p. 205] by introducing the function \( u(\chi) \), defined by

\[
f(\chi) = e^{-\chi^2/2} u(\chi) \quad (4-299)
\]

Substituting this expression for \( f(\chi) \) into Eq. (4-297) yields the differential equation

\[
\frac{d^2}{d\chi^2} u(\chi) - \chi \frac{d}{d\chi} u(\chi) + \lambda \tau u(\chi) = 0 \quad (4-300)
\]

subject to the boundary conditions

\[
u(\pm \tau) = 0 \quad (4-301)
\]
This eigenvalue problem is a proper Sturm-Liouville problem, which ensures that all the eigenvalues are positive. (This was one of the assumptions of Section 4.1.1.) Also, if the eigenvalues are ordered by magnitude, then each successive eigenfunction has one more zero crossing than the previous eigenfunction [44, p. 421]. That is, the first eigenfunction has no zero crossings, the second has one zero crossing, etc. Furthermore, it is easily verified that each eigenfunction must be even or odd. Therefore, the first eigenfunction must be even, with no zero crossings.

Given this information, it is possible to determine what threshold \( T \) is implied by a given eigenvalue \( \lambda T \). (Ideally, we would like to find \( \lambda T \) as a function of \( T \). However, finding \( T \) as a function of \( \lambda T \) will suffice for our purposes.) The threshold is obtained as follows: Since the first eigenfunction is even, we have that

\[
    u'(0) = 0 \tag{4-302}
\]

Because an eigenfunction may be scaled arbitrarily, we may take

\[
    u(1) = 1 \tag{4-303}
\]

The differential equation (4-300) is then integrated using the above initial conditions. Because the first eigenfunction has no zero crossings, the first point \( \chi \) at which \( u(\chi) = 0 \) must correspond to the threshold.

The above method was used to determine the threshold for a wide range of false-alarm rates (eigenvalues). A fourth-order Runge-Kutta integration [31, Sec. 3.5] with a step size of 0.02 was used. For each value of the threshold, the one-term and two-term asymptotic approximations were computed. The asymptotic approximations are given by

\[
    (\lambda T)_{1} = \sqrt{\frac{2}{\pi}} T e^{-\frac{2}{\pi}} \tag{4-304}
\]
\[(\lambda T)_2 = \sqrt{\frac{\pi}{2}} T e^{-T^2/2} \left(1 - \frac{1}{T^2}\right) \quad (4-305)\]

where \((\lambda T)_1\) and \((\lambda T)_2\) are the one-term and two-term asymptotic approximations, respectively. Figure 4-3 shows the relative error in these approximations as a function of the threshold, T. (The relative error is defined by the error in the approximation divided by the actual value of the eigenvalue.) As might be expected, the two-term approximation is significantly better than the one-term approximation. The error in the two-term approximation is less than 1% for T larger than 4.0. Even the one-term approximation is accurate to within 5% for thresholds larger than 5.0.

For practical purposes, determining the false-alarm rate to within 1% is probably adequate. Because of the exponential dependence of the false-alarm rate on the threshold, the false-alarm rate is extremely sensitive to small changes in the threshold. The false-alarm rate exhibits the same sensitivity to uncertainty in the statistics of the measurement and process noise of the system, because the threshold is normalized in terms of the variance of \(\chi\), which depends on these statistics. These statistics are often poorly known. Furthermore, the false-alarm analysis assumes that the noise in the system is Gaussian, which is often not the case. Therefore, the errors in modeling will often contribute more to the error in determining the false-alarm rate than does the asymptotic analysis itself. In any event, the error in the asymptotic expression is small, and is acceptable for most purposes.

Now consider the discrete-time false-alarm rate analysis of Section 4.1.3. A bound on the discrete-time false-alarm rate was given in Eq. (4-266). Although the bound is not valid for all cases, it is easily verified that the bound is valid when \(\chi(k)\) is a scalar. Below, we briefly investigate how tight the bound is for this case.

The discrete-time false-alarm rate was defined by Eq. (4-258). In the scalar case, \(\mu\) is the largest eigenvalue of
Figure 4-3. Relative error of the asymptotic expression for the false-alarm rate for the scalar case.
\[
\mu f(\chi) = \int_{-\sqrt{S} T}^{+\sqrt{S} T} \frac{1}{\sqrt{2\pi} \sqrt{Q_\chi}} \exp \left\{ -\frac{1}{2} \frac{(\chi - \phi u)^2}{Q_\chi} \right\} f(u) \, du
\] (4-306)

where \( S \) is the (unconditional) variance of \( \chi(k) \), and satisfies

\[
S = \phi^2 S + Q_\chi
\] (4-307)

The integral in Eq. (4-306) may be converted to one whose limits are \( \pm 1 \). The resulting integral equation is

\[
\mu g(z) = \int_{-1}^{1} \frac{T}{\sqrt{2\pi(1 - \phi^2)}} \exp \left\{ -\frac{T^2}{4} \frac{(z - \phi u)^2}{1 - \phi^2} \right\} g(u) \, du
\]

\[
= \int_{-1}^{1} L(z,u) g(u) \, du
\] (4-308)

where the kernel \( L(z,u) \) is implicitly a function of \( T \) and \( \phi \). The eigenvalue problem may be solved numerically in the following way [32, Sec. 3.15]: The integral in Eq. (4-308) may be approximated by numerical integration, using a method such as the trapezoidal rule or Simpson’s rule. The integral equation (4-308) may then be approximated by

\[
\mu g(z) = \sum_{i=1}^{N} D_i L(z,u_i) g(u_i)
\] (4-309)

where the points \( u_i \) are the sample points used in the integration, and the coefficients \( D_i \) depend on the integration rule used. In particular,
\[ \mu g(u_j) = \sum_{i=1}^{N} D_i L(u_j, u_i) g(u_i), \quad i = 1, 2, \ldots, N \quad (4-310) \]

This equation may be represented in matrix form by

\[ \mu g = Lg \quad (4-311) \]

where the vector \( g \) and the matrix \( L \) have obvious definitions. Thus, the largest eigenvalue of the integral equation (4-308) is (approximately) the largest eigenvalue of \( L \).

This method was used to determine the discrete-time false-alarm rate \( \lambda_d \) for several values of \( T \) and \( \phi \). The results are shown in Table 4-1. The results are normalized by the time constant \( \tau_C \), defined by

\[ \phi = e^{-\Delta t/\tau_C} \quad (4-312) \]

Also shown in the table is the bound on the false-alarm rate, \( \lambda_C \), normalized by \( \tau_C \).

Table 4-1. False-alarm rate of the discrete-time OSGLR test.

| Threshold T | \( \lambda_C \tau_C \) | \( \lambda_d \tau_C \) \\ | \( \Delta t/\tau_C = 0.1 \) | \( \Delta t/\tau_C = 1.0 \) |
|---|---|---|
| 3 | \( 2.395 \times 10^{-2} \) | \( 1.162 \times 10^{-2} \) | \( 2.630 \times 10^{-3} \) |
| 4 | \( 9.931 \times 10^{-4} \) | \( 3.717 \times 10^{-4} \) | \( 6.309 \times 10^{-5} \) |
| 5 | \( 1.421 \times 10^{-5} \) | \( 4.257 \times 10^{-6} \) | \( 5.883 \times 10^{-7} \) |
For the shorter sampling time ($\Delta t/\tau_C = 0.1$), $\lambda_d$ is somewhat smaller than $\lambda_C$, by a factor of 2.1 to 3.3, depending on the threshold. For $\Delta t/\tau_C = 1.0$, $\lambda_d$ is significantly smaller than $\lambda_C$. For $T = 5$, $\lambda_d$ is 24 times smaller than $\lambda_C$. Thus, for small $\Delta t$ ($\Delta t/\tau_C \ll 1.0$), the bound is not too bad. For larger $\Delta t$, the bound is very conservative.

4.2 Detection Performance

In this section, we briefly discuss a method for bounding the detection performance of the OSCLR test. The sufficient statistic $\mathbf{x}(t)$ continues to obey the differential equation (4-1), which is repeated here for convenience:

$$
\frac{d\mathbf{x}(t)}{dt} = A(t)\mathbf{x}(t) + C(t)R^{-1}(t)y(t)
$$  

(4-313)

In this case, however, $y(t)$ is a Gaussian white noise process with intensity $R(t)$ and mean $\mathbf{m}(t)$. $\mathbf{m}(t)$ is given by

$$
\mathbf{m}(t) = C(t)e(t) + d(t)f(t)
$$  

(4-314)

where $C(t)$ is the measurement matrix of the system, e(t) is the mean estimation error, $d(t)$ is a vector that depends on the failure type, and $f(t)$ is the failure mode shape. The mean estimation error satisfies the differential equation

$$
\frac{de(t)}{dt} = [A(t) - K(t)C(t)]e(t) + [b(t) - K(t)d(t)]f(t)
$$  

(4-315)

with initial condition

$$
e(t_0) = 0
$$  

(4-316)
where $A(t)$ is the system dynamics matrix, $K(t)$ is the Kalman filter gain matrix, $b(t)$ is a vector that depends on the failure type, and $t_0$ is the initial time. (See Chapter 3 for details.) Note that in contrast to the analysis of Chapter 3, here we allow the mode shape $f(t)$ to be arbitrary.

The time of detection, $t_D$, is defined in the same way as was the time of false alarm (Eq. (4-3)). That is,

$$t_D \overset{A}{=} \inf \left\{ t : DF(t) \geq T^2 \right\}$$

(3-317)

where $DF(t)$ is the OSGLR detection decision function. The distribution of the time of failure is given by

$$P_D(t) = P(t_D \leq t)$$

(4-318)

The probability that a failure has not been detected by time $t$ is denoted by

$$P_M(t) = 1 - P_D(t)$$

(4-319)

Although in some cases the detection performance of an FDI test may be characterized by some average of the detection time, such as the mean time to detection (MTTD), in general the complete distribution of the time of detection is required. For example, the evaluation of fault-tolerant systems by the use of semi-Markov models [54] requires that the distribution of the detection time be given for any given failure mode.

Unfortunately, the analysis of the false-alarm performance developed in Section 4.1 cannot be applied to the problem of determining detection performance. For one thing, the analysis presented there was an analysis of the steady-state behavior of the test in the absence of failures. We are interested not in the steady-state behavior of the
test, but in the transient behavior of the test immediately following the failure.

Furthermore, a general analysis of the detection performance of the OSGLR test must be at least as complex as the analysis of the false-alarm performance, since the false-alarm problem is a special case of the detection problem (i.e., \( f(t) = 0 \)). The general problem is quite difficult.

However, it is possible to bound the detection performance, in the following way. We note that if

\[
DF(t) \geq T^2
\]  \hspace{1cm} (4-320)

then

\[
t_D \leq t
\]  \hspace{1cm} (4-321)

by Eq. (4-317). Therefore,

\[
P_D(t) = \Pr(t_D \leq t)
\]

\[
\geq \Pr(DF(t) \geq T^2)
\]  \hspace{1cm} (4-322)

Therefore, by Eq. (4-319),

\[
P_M(t) \leq 1 - \Pr(DF(t) \geq T^2)
\]  \hspace{1cm} (4-323)

Thus, \( P_D(t) \) and \( P_M(t) \) can be bounded from below and above, respectively.

Although the above probabilities are written as unconditional probabilities, it would be more appropriate to condition the probabilities on the event that no false alarm has occurred up until time \( \theta \), the
time of the failure. We will return to this point later. For the time
being, we will continue to use the unconditional probabilities defined
above.

Recall that the OSGLR sufficient statistic \( \bar{\chi}(t) \) is a Gaussian
random vector with covariance \( S(t) \), and the OSGLR decision function is
given by

\[
DF(t) = \bar{\chi}^T(t)S^{-1}(t)\bar{\chi}(t)
\]  \hspace{1cm} (4-324)

Therefore, \( DF(t) \) is a noncentral chi-squared random variable [1, p. 113],
with \( p \) degrees of freedom. The noncentrality parameter is given by

\[
\delta^2(t) = \bar{\chi}^T(t)S^{-1}(t)\bar{\chi}(t)
\]  \hspace{1cm} (4-325)

where \( \bar{\chi}(t) \) is the mean value of \( \chi(t) \). \( \bar{\chi}(t) \) satisfies the differential
equation

\[
\frac{d\bar{\chi}(t)}{dt} = A\bar{\chi}(t) + G^T(t)R^{-1}(t) m(t)
\]  \hspace{1cm} (4-326)

with initial condition

\[
\bar{\chi}(t) = 0
\]  \hspace{1cm} (4-327)

If we denote the distribution function of a noncentral chi-squared random
variable \( x \) with \( p \) degrees of freedom and noncentrality parameter \( \delta^2 \) by
\( F(x; \delta^2, p) \), then the detection probabilities are bounded by

\[
P_D(t) \geq 1 - F(T^2; \delta^2(t), p)
\]  \hspace{1cm} (4-328)
\[ P_M(t) \leq F(T^2; \delta^2(t), p) \quad (4-329) \]

The bounds in Eqs. (4-328) and (4-329) may be evaluated by numerical methods, or by tables of the chi-squared distribution, as given in [29].

Now, suppose that we condition the probability of detection on the event that no false alarms occur before the time of the failure. I.e., we want to determine

\[ P_D(t \mid \text{no false alarm prior to } \theta) \]

where \( \theta \) is the time of the failure. We saw in Section 4.1 that the conditional density of \( X(t) \) given that no false alarm has occurred is nearly Gaussian, except for a narrow region near the boundary, and of course outside the boundary. (This was shown for time-invariant systems, but it should also be true for time-varying systems.) It was also shown there that the contribution to the probability density from these regions is exponentially small. Therefore, the bounds given in Eqs. (4-328) and (4-329) are correct, at least to exponentially small errors. Thus, the bounds will be meaningful far out into the tails of the distribution of the detection time.

Also, note that the bounds are conservative, in the sense that they specify the worst possible detection performance. Thus, using the bounds to predict the overall performance of a fault-tolerant system will also produce a conservative result. This is necessary if the results of such an analysis are used to demonstrate that a fault-tolerant system meets specified requirements.
CHAPTER 5

IMPLEMENTATION CONSIDERATIONS

In this chapter, some of the issues involved in the actual implementation of the OSGLR test are discussed. The discussion is limited to continuous-time systems, although the extension to discrete-time systems is straightforward.

In Section 5.1, two sets of orthonormal functions, the Legendre and Laguerre functions, are discussed. In Section 5.2, a simple method for finding realizations for these functions is described. Finally, the issues involved in selecting the time scale and number of orthonormal functions to use in the OSGLR failure hypotheses are discussed.

5.1 Orthonormal Functions

In Chapter 3, it was shown that the vector of basis functions, \( \mathbf{\phi}(\tau) \), satisfies the differential equation

\[
\frac{d}{d\tau} \mathbf{\phi}(\tau) = A_\phi \mathbf{\phi}(\tau)
\]  

(5-1)

Hence, specification of \( A_\phi \) and \( \mathbf{\phi}(0) \) determines the basis functions completely.

Note that the choice of \( A_\phi \) and \( \mathbf{\phi}(0) \) is not completely arbitrary. There are two restrictions on the choice of \( A_\phi \) and \( \mathbf{\phi}(0) \). First, \( A_\phi \) and \( \mathbf{\phi}(0) \) must generate a set of linearly independent functions. Second, the matrix \( A_\phi \) must be stable, so that the basis vector \( \mathbf{\phi}(\tau) \) is bounded as \( \tau \to \infty \). Despite these restrictions, there is enormous latitude in the choice of the basis functions.
In order to simplify the choice of $A_\phi$ and $\phi(0)$, the basis functions $\phi_i(\tau)$ will be limited to one of two sets of functions: the Legendre functions and the Laguerre functions. These sets of basis functions have been used in the field of communications for time series analysis [56], network synthesis [37, Ch. 19] and representation of autocorrelation and crosscorrelation functions [37, Ch. 18]. Each set of basis functions forms a complete, orthonormal basis over the interval $[0, \infty)$. The bases are orthonormal because

$$\int_0^\infty \phi_i(\tau)\phi_j(\tau) \, d\tau = \delta_{ij} \quad (5-2)$$

where $\delta_{ij}$ is the Kronecker delta. The bases are complete because the series representation of a function $f(\tau)$, given by

$$f(\tau) = \sum_{i=1}^{\infty} a_i \phi_i(\tau) \quad (5-3)$$

where

$$a_i = \int_0^\infty f(\tau)\phi_i(\tau) \, d\tau \quad (5-4)$$

converges in the mean to $f(\tau)$ if $f(\tau)$ is piecewise continuous and square-integrable. That is,

$$\lim_{p \to \infty} \int_0^\infty \left| f(\tau) - \sum_{i=1}^{p} a_i \phi_i(\tau) \right|^2 \, d\tau = 0 \quad (5-5)$$

Thus, there is some justification for using these basis sets, because it should be possible to represent most of the energy in any reasonably well-behaved failure mode by using enough terms in the series expansion.
This argument is necessarily vague, because the OSGLR failure hypothesis is based on a heuristic argument. Nevertheless, these bases are acceptable, as will be seen in Chapter 7.

5.1.1 Legendre Functions

The Legendre functions are obtained by orthonormalizing the set of functions

\[ \{ \psi_n(\tau) = e^{-(n+\frac{1}{2})\kappa \tau}, \ n = 0, 1, 2, \ldots \} \] (5-6)

over the interval \((0, \infty)\). The resulting functions are denoted by \(v_n(\tau)\). The orthonormalization may be performed using a Gram-Schmidt procedure [48, Sec. 3.3] or by frequency domain techniques [37, Ch. 18]. The result is

\[ v_0(\tau) = \sqrt{k} e^{-\left(\frac{k}{2}\right)\tau} \]

\[ v_1(\tau) = \sqrt{3k} \left( -1 + 2e^{-k\tau} \right) e^{-\left(\frac{k}{2}\right)\tau} \]

\[ v_2(\tau) = \sqrt{5k} \left( 1 - 6e^{-k\tau} + 6e^{-2k\tau} \right) e^{-\left(\frac{k}{2}\right)\tau} \]

\[ \vdots \]

\[ v_n(\tau) = \sqrt{k(2n + 1)} P_n(2e^{-k\tau} - 1) e^{-\left(\frac{k}{2}\right)\tau} \]

\[ \vdots \]

where \(P_n(*)\) is the \(n^{th}\) Legendre polynomial.

5.1.2 Laguerre Functions

The Laguerre functions are obtained by orthonormalizing the set of functions
\[ \psi_n(\tau) = \frac{(kt)^n e^{-kt}}{n!}, \quad n = 0, 1, 2, \ldots \]  \quad (5-7) \]

over the interval \([0, \infty)\). The Laguerre functions, denoted by \(l_n(\tau)\), are given by

\[ l_0(\tau) = \sqrt{2k} e^{-kt} \]
\[ l_1(\tau) = \sqrt{2k} (2kt - 1)e^{-kt} \]
\[ l_2(\tau) = \sqrt{2k} (2k^2 \tau^2 - 4kt + 1)e^{-kt} \]
\[ \cdots \]
\[ l_n(\tau) = \sqrt{2k} \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} \frac{(2kt)^m}{m!} e^{-kt} \]
\[ \cdots \]

Both the Legendre functions and the Laguerre functions contain the parameter \(k\), which allows the time constant of the functions to be varied. Therefore, the selection of the OSGLR basis functions will entail three steps:

1. Selection of either Legendre or Laguerre functions as the basis.
2. Selection of the parameter \(k\).
3. Selection of the number of basis functions \(p\) in the truncated set.

The details of this process considered in Section 5.3.

5.2 Realization of the Basis Functions

Once it has been determined which set of basis functions to use, a matrix \(A_\phi\) and an initial condition \(\phi(0)\) must be found that generates
that basis. Although it would appear that this is a difficult task, it will now be shown that it is remarkably easy to do.

First, we will consider the structure of the matrix $A_\phi$. We will first deal with the Legendre functions. Recall that the vector of basis functions $\phi(\tau)$ was obtained by orthonormalizing the set of functions

$$
\{\psi_i(\tau) = e^{-\frac{1}{2}(i-\frac{1}{2})k\tau}, \ i = 1, 2, \ldots, p\} \quad (5-8)
$$

The $\psi_i(\tau)$ satisfy the differential equation

$$
\frac{d}{d\tau} \psi_i(\tau) = -(i - \frac{1}{2})k \psi_i(\tau), \ i = 1, 2, \ldots, p \quad (5-9)
$$

with initial condition

$$
\psi_i(0) = 1, \ i = 1, 2, \ldots, p \quad (5-10)
$$

Equation (5-9) may be expressed in vector form as

$$
\frac{d}{d\tau} \psi(\tau) = A_\psi \psi(\tau) \quad (5-11)
$$

where

$$
A_\psi = \text{diag} \left( \frac{k}{2}, \frac{3k}{2}, \ldots (p - 1/2)k \right) \quad (5-12)
$$

Similarly, the Laguerre functions are obtained by orthonormalizing the set of functions

$$
\{\psi_i(\tau) = \frac{(kt)^{i-1} e^{-kt}}{(i-1)!}, \ i = 1, 2, \ldots, p\} \quad (5-13)
$$
The functions \( \psi_i(\tau) \) satisfy the differential equation

\[
\frac{d}{d\tau} \psi_i(\tau) = \begin{cases} 
-k\psi_i(\tau), & i = 1 \\
-k\psi_1(\tau) + k\psi_{i-1}(\tau), & i = 2, 3, \ldots, p
\end{cases}
\] (5-14)

with initial conditions

\[
\psi_i(0) = \begin{cases} 
1, & i=1 \\
0, & i = 2, 3, \ldots, p
\end{cases}
\] (5-15)

In vector form, we have that

\[
\frac{d}{d\tau} \psi(\tau) = A\psi(\tau)
\] (5-16)

where

\[
A = \begin{bmatrix}
-k & 0 & 0 & \cdots & 0 & 0 \\
k & -k & 0 & 0 & 0 \\
0 & k & -k & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & k & -k \\
0 & 0 & 0 & \cdots & k & -k
\end{bmatrix}
\] (5-17)

An important point here is that for both types of basis functions, \( A\psi \) is lower triangular.

Now, the functions \( \hat{\phi}(\tau) \) are obtained by a linear transformation of the vector \( \psi(\tau) \), so that

\[
\hat{\phi}(\tau) = \Gamma \psi(\tau)
\] (5-18)
Then

$$\frac{d}{d\tau} \phi(\tau) = \Gamma \frac{d}{d\tau} \psi(\tau) \quad (5-19)$$

By Eq. (5-11) or Eq. (5-16), we have that

$$\frac{d}{d\tau} \phi(\tau) = \Gamma A_\psi \psi(\tau) \quad (5-20)$$

Finally, solving Eq. (5-18) for $\psi(\tau)$ and substituting the result into Eq. (5-20) gives the result that

$$\frac{d}{d\tau} \phi(\tau) = \Gamma A_\psi \Gamma^{-1} \psi(\tau) \quad (5-21)$$

Therefore, $A_\phi$ may be identified as

$$A_\phi = \Gamma A_\psi \Gamma^{-1} \quad (5-22)$$

Note that the orthonormalization process was assumed to be a Gram-Schmidt procedure, so that

$$\phi_1(\tau) = \Gamma_{11} \psi_1(\tau)$$

$$\phi_2(\tau) = \Gamma_{21} \psi_1(\tau) + \Gamma_{22} \psi_2(\tau)$$

$$\cdots$$

$$\phi_p(\tau) = \sum_{i=1}^{p} \Gamma_{pi} \psi_i(\tau)$$
In other words, the transformation matrix, $\Gamma$, is lower triangular. Because the inverse of a lower triangular matrix is also lower triangular, it follows that $\Gamma^{-1}$ is lower triangular. Therefore, $A_\psi$, which is the product of three lower triangular matrices, is also lower triangular.

Furthermore, the diagonal elements of $A_\phi$ and $A_\psi$ are the same, that is,

$$[A_\phi]_{ii} = [A_\psi]_{ii}, \quad i = 1, 2, \ldots, p \quad (5-23)$$

To show this, note that $[A_\phi]_{ii}$ is represented by the double summation

$$[A_\phi]_{ii} = \sum_{k=1}^{p} \sum_{l=1}^{p} [\Gamma]_{ik} [A_\psi]_{kl} [\Gamma^{-1}]_{li} \quad (5-24)$$

Because each of the matrices is lower triangular, each term in the summation is nonzero only if

$$i \geq k \quad \text{and} \quad k \geq l \quad \text{and} \quad l \geq i \quad (5-25)$$

Hence, the only nonzero term in the sum occurs when $i = k = l$, so that

$$[A_\phi]_{ii} = [\Gamma]_{ii} [A_\psi]_{ii} [\Gamma^{-1}]_{ii} \quad (5-26)$$

Because $\Gamma$ is lower triangular,

$$[\Gamma^{-1}]_{ii} = 1/[\Gamma]_{ii} \quad (5-27)$$

and Eq. (5-23) then follows directly from Eq. (5-26).
To determine the remaining elements of $A_\phi$ and $\phi(0)$, we must impose the orthonormality condition

$$
\int_0^\infty \phi_i(\tau)\phi_j(\tau) \, d\tau = \delta_{ij} \tag{5-28}
$$

To this end, define the correlation matrix

$$
P = \int_0^\infty \phi(\tau)\phi^T(\tau) \, d\tau \tag{5-29}
$$

Then

$$
A_\phi P + PA_\phi^T = \int_0^\infty [A_\phi \phi(\tau)\phi^T(\tau) + \phi(\tau)\phi^T(\tau)A_\phi^T] \, d\tau
$$

$$
= \int_0^\infty \frac{d}{d\tau} [\phi(\tau)\phi^T(\tau)] \, d\tau
$$

$$
= \phi(\infty)\phi^T(\infty) - \phi(0)\phi^T(0) \tag{5-30}
$$

Because $A_\phi$ is asymptotically stable, it follows that

$$
\phi(\infty) = 0 \tag{5-31}
$$

Therefore, the correlation matrix $P$ satisfies the Lyapunov equation

$$
A_\phi P + PA_\phi^T + \phi(0)\phi^T(0) = 0 \tag{5-32}
$$

The orthonormality condition (Eq. (5-28)) requires that

$$
P = I \tag{5-33}
$$
Therefore, the remaining elements of $A_\phi$ and $\phi(0)$ must be chosen so that

$$A_\phi + A_\phi^T + \phi(0)\phi^T(0) = 0 \quad (5-34)$$

Because $A_\phi$ is lower triangular, this last equation may be solved directly. We already know that

$$[A_\phi]_{ii} = [A_\psi]_{ii} \quad (5-35)$$

Therefore,

$$\phi_i(0) = \sqrt{-2[A_\psi]_{ii}} \quad (5-36)$$

Then the elements of $A_\phi$ below the diagonal are given by

$$[A_\phi]_{ij} = -\phi_i(0)\phi_j(0)$$

$$= -\sqrt{-2[A_\psi]_{ii}} \sqrt{-2[A_\psi]_{ij}} \quad (i > j) \quad (5-37)$$

Therefore, for the Legendre functions,

$$[A_\phi]_{ij} = \begin{cases} 
0 & i < j \\
-(i - 1/2)k & i = j \\
-2\sqrt{(i - 1/2)(j - 1/2)} k & i > j 
\end{cases} \quad (5-38)$$

and

$$\phi_i(0) = \sqrt{(2i - 1)} k \quad (5-39)$$
For the Laguerre functions,

\[
[A_{\phi}]_{ij} = \begin{cases} 
0, & i < j \\
-k, & 1 = j \\
-2k, & 1 > j 
\end{cases} \tag{5-40}
\]

and

\[
\phi_i(0) = \sqrt{2}k \tag{5-41}
\]

(Note that in this case for \( i \) even, the function \( \phi_i(\tau) \) is the negative of the Laguerre function \( \ell_{i-1}(\tau) \).)

5.3 Choice of Basis Functions

In this section, we briefly discuss how to choose the OSGLR basis functions. Three things must be specified:

1. The form of the basis functions (either Laguerre functions or Legendre functions).
2. The parameter \( k \) which specifies the time scale of the functions.
3. The number of basis functions, \( p \).

Recall that the OSGLR failure hypotheses are intended to approximately represent many different failure modes. Therefore, for any given system, the choice of basis functions will not be obvious. The best choice for the basis functions will probably depend on a number of factors that are problem-specific. Nevertheless, we will present below a broad outline of the issues involved.

First, the choice of whether to use Laguerre functions or Legendre functions is probably not very important. Of course, one could perform the analysis for both types of basis functions, and determine which choice gives the better results. Therefore, we will assume that one of these two types has been selected, perhaps arbitrarily.

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Next, we note that the performance of the OSGLR test depends not only on the choice of the basis functions, but also on the detection threshold, $T_2^2$, and the isolation threshold, $T_1^2$. Quite often, the detection threshold is selected to achieve a given false-alarm rate. Thus, the choice of a set of basis functions will imply a certain threshold. However, much of the analysis below assumes that the threshold is given. Therefore, the process of determining the basis functions and thresholds will be iterative, like most design procedures. This process should converge quickly, because the threshold is not very sensitive to the free parameters (p and k) of the basis functions or to the false-alarm rate. Figure 5-1 and Figure 5-2 show the false-alarm rates of the OSGLR test (as determined by the analysis in Section 4.1) for the Legendre and Laguerre functions, respectively. Note that for a large range of the number of basis functions (ranging from one to ten) and over many orders of magnitude of the false-alarm rate, the threshold varies only slightly. Therefore, the threshold will not vary much from iteration to iteration, even if the set of basis functions is changed significantly.

One of the motivations for the OSGLR test was to design a test that is robust to failure mode uncertainty. However, the basis functions cannot be chosen without some knowledge of the failure modes, as will be seen below. Therefore, it will be assumed that, for design purposes, a representative set of failure modes can be specified. In particular, we will assume that we are designing the OSGLR detector for the $i^{th}$ failure type, with failure mode $f_i(t)$. In general, the design process would be such that the OSGLR test works well for a set of failure modes, rather than for a single failure mode.

In the formulation of the OSGLR hypotheses, we argued that if the possible failure modes of the system are well represented by the (truncated) orthogonal series expansion, then the test should be robust. More precisely, the important point is that the resulting mean process in the residuals of the Kalman filter (i.e., the failure signature) can be approximately represented by the OSGLR hypotheses.

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Figure 5-1. False-alarm rate of the OSGLR test with Legendre functions as the basis functions.
Figure 5-2. False-alarm rate of the OSGLR test with Laguerre functions as the basis functions.
Recall from Section 4.2 that the OSGLR decision function $DF_i(t)$ is a noncentral chi-squared random variable. Therefore,

$$E[DF_i(t) | H_i] = \delta_{i/i}(t) + p_i$$  \hspace{1cm} (5-42)

where $p_i$ is the dimension of $X_i(t)$. The noncentrality parameter $\delta_{i/i}^2(t)$ is given by

$$\delta_{i/i}^2(t) = X_{i/i}^T(t)S_{i/i}^{-1}(t)X_{i/i}(t)$$  \hspace{1cm} (5-43)

where $X_{i/i}(t)$ is the mean value of $X_i(t)$. The noncentrality parameter may be interpreted as the squared magnitude of the projection of the mean of the Kalman filter residual, $\bar{m}_i(t)$, onto the subspace of possible failure signatures generated by the OSGLR failure hypothesis $H_i$. (See Section 6.3.) As such, the noncentrality parameter is bounded by

$$\delta_{i/i}^2(t) \leq d^2(t)$$  \hspace{1cm} (5-44)

where $d^2(t)$ is the signal-to-noise ratio of the failure signature $\bar{m}_i(t)$, defined by

$$d^2(t) = \int_t^0 \bar{m}_i^T(t') R^{-1}(t') \bar{m}_i(t') \, dt'$$  \hspace{1cm} (5-45)

The equality in Eq. (5-44) holds only when the OSGLR failure hypothesis can represent the failure mode $f(t)$ exactly.

In order that detection occur as soon as possible after a failure has occurred, we would like $\delta_{i/i}^2(t)$ to be as large as possible. Since $\delta_{i/i}^2(t)$ is bounded by $d^2$, this implies that

$$\delta_{i/i}^2(t) \simeq d^2(t)$$  \hspace{1cm} (5-46)
is desirable. Suppose that the OSGLR basis functions are chosen so that Eq. (5-46) is true. Detection is not likely to occur until \( d^2(t) \) is comparable in size to the threshold, \( T^2 \). Similarly, detection will almost certainly have occurred by the time \( d^2(t) \) is much larger than \( T^2 \). Therefore, we can confine our attention to the time interval

\[
t_1 < t < t_2
\]

where \( t_1 \) and \( t_2 \) are such that

\[
T - d(t_1) \gg 1
\]

and

\[
d(t_2) - T \gg 1
\]

\( |T - d(t)| \) is the distance between \( \Lambda_i(t) \) and the nearest point on the decision boundary, assuming that \( \delta^2_{i/i}(t) = d^2(t) \). (If an isolation threshold is used, similar considerations might require that this interval be enlarged to include the likely isolation times.) The general conclusion is this: When we require that the truncated series of the OSGLR failure hypothesis be able to approximately represent the possible failure modes of the system, we mean that we desire

\[
\delta^2_{i/i}(t) = d^2(t), \quad t_1 \leq t \leq t_2
\]

for each possible failure mode, where the interval \( [t_1, t_2] \) represents the interval over which detection (and isolation) of the failure is likely to occur.

How closely \( \delta^2_{i/i}(t) \) must approximate \( d^2(t) \) depends on a number of factors. If we are interested only in detection, then it may not be
necessary that $\delta_{i/i}^2(t)$ closely approximate $d^2(t)$. On the other hand, if we are interested in isolating the failure to one of several types, it may be important that $\delta_{i/i}^2(t)$ approximate $d^2(t)$ closely. This is especially true if there is another type of failure, say the $j^{th}$, that is nearly indistinguishable from the failure type under consideration. (See Chapter 6.)

Within the guidelines set forth earlier in this section, we have only two ways to achieve the goal in Eq. (5-44):

1. Selection of the parameter $k$.
2. Selection of the number of basis functions, $p$.

The parameter $k$ determines the time scale of the Legendre or Laguerre functions. As a rule of thumb, $k$ may be selected so that $1/k$ corresponds to a characteristic time scale, such as the arithmetic or geometric mean of $t_1$ and $t_2$. In any event, some trial and error will probably be involved in determining $k$.

Note that for a fixed value of $k$, increasing the number of basis functions $p$ will increase $\delta_{i/i}^2(t)$. Because the bases are complete, $\delta_{i/i}^2(t)$ can be made arbitrarily close to $d^2(t)$ by choosing $p$ sufficiently large. However, $p$ cannot be made arbitrarily large for two reasons. First, increasing $p$ increases the false-alarm rate for a given threshold. In other words, increasing $p$ increases the required threshold for a given maximum false-alarm rate. Second, increasing the number of basis functions increases the amount of computation required to perform the OSGLR test. Therefore, $p$ should be just large enough to satisfy Eq. (5-49).

At this point, two observations are in order. The first is that because the failure signature is affected by both the plant and the Kalman filter, the number of basis functions needed to adequately represent a given failure mode depends not only on the failure mode itself, but also on the system. In particular, actuator failures are more easily represented by the OSGLR failure hypothesis than are sensor failures. The reason for this is the following: The transfer function from an
actuator input to the Kalman filter residual is generally low-pass in nature. Roughly speaking, the truncated terms of the series expansion are high-frequency terms. Therefore, these terms contribute little to the energy in the failure signature for actuator failures. Thus, the energy represented by the OSGLR failure hypothesis ($\delta_{1/1}^2$) closely approximates the energy in the failure signature ($\delta^2$). This idea is very much like the filter hypothesis of describing function theory [27, Ch. 3].

On the other hand, the transfer function from a sensor to the Kalman filter residuals is generally high-pass in nature. As a result, the high-frequency content of the failure mode is accentuated. Therefore, it may be quite difficult to represent the failure mode of a sensor adequately using the series expansion of the OSGLR test.

The other observation is that the number of basis functions which can be used may be limited by numerical considerations. Recall that one of the reasons cited in Chapter 3 for using an orthonormal basis was to ensure that the problem was well-conditioned. However, even though the basis functions are orthonormal, the columns of the residual influence matrix $G_2(t,t_e)$, which form the basis of possible failure signatures, are not necessarily orthonormal. In fact, they may be highly correlated. In effect, the system dynamics and the Kalman filter destroy the conditioning that was imposed. As a result, it may be necessary to limit the number of basis functions.
CHAPTER 6

DISTINGUISHABILITY OF FAILURES

In some systems, the failure of one component may closely mimic the effects of the failure of another component. In such systems, it may be impossible to distinguish between different types of failures, regardless of the FDI algorithm used. Therefore, quantitative measures would be desirable to indicate to what extent the failures of different components are distinguishable. The measures would serve two purposes: First, they could be used to alert the system designer that a change in the system is needed to achieve fault tolerance. Such a change could be, for example, the addition of more sensors or the change in location of an actuator. Second, the measures may be used to determine whether the inability of a particular FDI algorithm to isolate failures is a deficiency of the algorithm or a property of the system.

In this chapter, two measures of distinguishability are proposed. The first of these is interpreted as the distance between two failure hypotheses, assuming that one failure mode is fixed and that the other failure mode is allowed to take on its worst-case value. Based on a geometric interpretation of this distance, a second measure of distinguishability is defined, which is interpreted as the angle between the hypotheses.

6.1 Distance Measure

The distinguishability measure presented below will measure how closely the failure of one component, say component j, can mimic the failure of another component, component i, when the failure mode of
component i is given. If the failure mode of component j is thought of as an unknown control, and "how closely" is interpreted as implying a cost function, then this formulation could be expected to lead to some sort of optimal control problem. As shown below, this is indeed the case.

Chow [15] used a similar approach in his study of the GLR test. However, he constrained the failure mode of instrument j to be a member of some limited class of failure modes, e.g., step biases of unknown time and magnitude. This approach neglects the fact that there may be a failure mode of component j outside of the class which mimics the failure mode of component i more closely than does any failure mode within the class.

We will formulate the problem as follows: First, it is assumed that the observation interval \([t_0, t_f]\) is fixed. Next, the failure mode of component i is specified to be some function, \(f_i(t)\). For the time being, it is assumed that the failure mode \(f_j(t)\) is specified as well. Note that for either failure, the onset time of the failure, \(\theta\), may be anywhere in the interval \([t_0, t_f]\). Then the problem of determining which component has failed is a binary hypothesis testing problem, with hypotheses

\[
H_i: \quad \frac{dx(t)}{dt} = A(t)x(t) + w(t) + b_i(t)f_i(t) \quad (6-1)
\]

\[
y(t) = C(t)x(t) + v(t) + d_i(t)f_i(t) \quad (6-2)
\]

\[
H_j: \quad \frac{dx(t)}{dt} = A(t)x(t) + w(t) + b_j(t)f_j(t) \quad (6-3)
\]

\[
y(t) = C(t)x(t) + v(t) + d_j(t)f_j(t) \quad (6-4)
\]

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where the usual definitions from Chapter 3 apply here. Note that because the failure modes \( f_i(t) \) and \( f_j(t) \) are assumed known, these hypotheses are simple, rather than composite.

As was done in Chapter 3, the data \( \{ y(t), t_0 \leq t \leq t_f \} \) will be reduced to a sufficient statistic which contains all the information about the two hypotheses. To begin, the data are filtered using a Kalman filter based on the unfailed \( (H_0) \) system equations

\[
H_0: \quad \frac{dx(t)}{dt} = A(t)x(t) + w(t) \tag{5-5}
\]

\[
y(t) = C(t)x(t) + v(t) \tag{5-6}
\]

The hypotheses \( H_i \) and \( H_j \) can then be written as

\[
H_i: \quad y(t) = y_0(t) + \overline{m}_i(t) \tag{6-7}
\]

\[
H_j: \quad y(t) = y_0(t) + \overline{m}_j(t) \tag{6-8}
\]

where \( y(t) \) is the Kalman filter residual, and \( y_0(t) \) is a zero-mean, white Gaussian process with intensity \( R(t) \). \( \overline{m}_i(t) \) and \( \overline{m}_j(t) \) are the means in the residual \( y(t) \) under \( H_i \) and \( H_j \), respectively. \( \overline{m}_i(t) \)
and \( \overline{m}_j(t) \) are given by

\[
\overline{m}_\ell(t) = C(t)e_\ell(t) + d_\ell(t)f_\ell(t) \tag{6-9}
\]

where the subscript \( \ell \) can be either \( i \) or \( j \). \( e_\ell(t) \) is the mean value of the estimation error under \( H_\ell \). The differential equation for \( e_\ell(t) \) is

\[
\frac{d e_\ell(t)}{dt} = [A(t) - K(t)C(t)]e_\ell(t) + [b_\ell(t) - K(t)d_\ell(t)]f_\ell(t) \tag{6-10}
\]
with initial condition

\[ e_2(t_0) = 0 \]  \hspace{1cm} (6-11)

The sufficient statistic for this problem is given by

\[ \chi = \int_{t_0}^{t_f} [m_j(t) - \underline{m}_i(t)]^T R^{-1}(t) \underline{y}(t) \, dt \]  \hspace{1cm} (6-12)

Because \( \underline{y}(t) \) is a Gaussian random process, \( \chi \) is a Gaussian random variable. The mean of \( \chi \) under \( H_i \) is given by

\[ \chi_i = E[\chi | H_i] \]

\[ = E \left[ \int_{t_0}^{t_f} [m_j(t) - \underline{m}_i(t)]^T R^{-1}(t) \underline{y}(t) \, dt | H_i \right] \]  \hspace{1cm} (6-13)

Moving the expectation operator inside the integral yields

\[ \chi_i = \int_{t_0}^{t_f} [m_j(t) - \underline{m}_i(t)]^T R^{-1}(t) E[\underline{y}(t) | H_i] \, dt \]  \hspace{1cm} (6-14)

By Eq. (6-7),

\[ E[\underline{y}(t) | H_i] = \underline{m}_i(t) \]  \hspace{1cm} (6-15)

Therefore,

\[ \chi_i = \int_{t_0}^{t_f} [m_j(t) - \underline{m}_i(t)]^T R^{-1}(t) \underline{m}_i(t) \, dt \]  \hspace{1cm} (6-16)
Similarly, the mean of \( \chi \) under \( H_j \) is given by

\[
\chi_j = \mathbb{E}[\chi | H_j] = \int_{t_0}^{t_f} (m_j(t) - \overline{m}_j(t))^T R^{-1}(t) m_j(t) \, dt \tag{6-17}
\]

The variance of \( \chi \) under \( H_i \) is given by

\[
S = \mathbb{E}[(\chi - \chi_i)^2 | H_i] \tag{6-18}
\]

The term \( (\chi - \chi_i) \) is given by

\[
\chi - \chi_i = \int_{t_0}^{t_f} (m_j(t) - \overline{m}_j(t))^T R^{-1}(t) [\overline{y}(t) - \overline{m}_i(t)] \, dt \tag{6-19}
\]

Under \( H_i \),

\[
\overline{y}(t) - \overline{m}_i(t) = \overline{y}_0(t) \tag{6-20}
\]

Using the last three equations, we have that

\[
S = \mathbb{E} \left[ \left( \int_{t_0}^{t_f} (m_j(t) - \overline{m}_j(t))^T R^{-1} \overline{y}_0(t) \, dt \right)^2 \right] \tag{6-21}
\]

where the conditioning on \( H_i \) has been suppressed because \( \overline{y}_0(t) \) is independent of the hypotheses. The integral term which is squared in the above equation may be converted to a double integral by the introduction of dummy variables of integration. The result is that
\[ S = \int_{t_0}^{t_f} \int_{t_0}^{t_f} [\bar{m}_j(t_1) - \bar{m}_i(t_1)]^T R^{-1}(t_1) E[\gamma_0^T(t_1)\gamma_0^T(t_2)] R^{-1}(t_2) \]
\[ \cdot [\bar{m}_j(t_2) - \bar{m}_i(t_2)] dt_1 dt_2 \]  \hspace{1cm} (6-22)

Recall that

\[ E[\gamma_0(t_1)\gamma_0^T(t_2)] = \delta(t_1 - t_2) R(t_1) \]

Therefore,

\[ S = \int_{t_0}^{t_f} \int_{t_0}^{t_f} [\bar{m}_j(t_1) - \bar{m}_i(t_1)]^T R^{-1}(t_2) \delta(t_1 - t_2) \]
\[ \cdot [\bar{m}_j(t_2) - \bar{m}_i(t_2)] dt_1 dt_2 \]  \hspace{1cm} (6-23)

Integrating over \( t_1 \) or \( t_2 \) yields the final expression for the variance

\[ S = \int_{t_0}^{t_f} [\bar{m}_j(t) - \bar{m}_i(t)]^T R^{-1}(t)[\bar{m}_j(t) - \bar{m}_i(t)] dt \]  \hspace{1cm} (6-24)

The variance of \( \chi \) under \( H_j \) is also given by \( S \).

Thus, the problem of deciding whether \( H_i \) or \( H_j \) is true has been reduced to that of deciding whether the Gaussian random variable \( \chi \) has mean \( \chi_i \) or \( \chi_j \). The parameter that determines the performance that can be achieved under these circumstances is the signal-to-noise ratio, defined by

\[ d^2 = \frac{(\chi_j - \chi_i)^2}{S} \]  \hspace{1cm} (6-25)
But

\[ x_j - x_i = s \tag{6-26} \]

by Eqs. (6-16), (6-17), and (6-24). Therefore,

\[ d^2 = s \tag{6-27} \]

So the distance between the hypotheses is defined to be

\[ d = s^{1/2} \tag{6-28} \]

The distance \( d \) is simply the number of standard deviations that separate the conditional means \( x_1 \) and \( x_2 \).

Up to this point, it has been assumed that both failure modes \( f_i(t) \) and \( f_j(t) \) are known. However, what we really want is to find the failure mode \( f_j(t) \) that most closely mimics the failure mode \( f_i(t) \). In other words, we want to find the distance between the hypotheses for the worst-case \( f_j(t) \). Therefore, the distinguishability measure \( \Delta_{ij}(f_i(\cdot), t_f) \) is defined by

\[ \Delta_{ij}(f_i(\cdot), t_f) = \min_{f_j(t)} \min_{t_0 \leq t \leq t_f} d \tag{6-29} \]

or alternatively,

\[ \Delta_{ij}^2(f_i(\cdot), t_f) = \min_{f_j(t)} \int_{t_0}^{t_f} [m_j(t) - m_i(t)]^T R^{-1}(t)[m_j(t) - m_i(t)] \, dt \tag{6-30} \]
where $m_j(t)$ and $m_i(t)$ are defined by Eqs. (6-9), (6-10), and (6-11).

The problem of determining $\Delta^2_{ij}$ has two interpretations. The obvious interpretation is as an optimal control problem. Specifically, the system to be controlled is

$$\frac{de(t)}{dt} = A_{KF}(t)e(t) + b_{KF}(t)f_j(t) \quad (6-31)$$

where

$$A_{KF}(t) = A(t) - K(t)C(t) \quad (6-32)$$

$$b_{KF}(t) = b_j(t) - K(t)d_j(t) \quad (6-33)$$

The output of the system is given by

$$m_j(t) = C(t)e(t) + d_j(t)f_j(t) \quad (6-34)$$

Hence, the problem may be recognized as being equivalent to the optimal linear quadratic tracking problem, discussed in a number of references (e.g., [2, 36]). In this case, the reference to be tracked is $m_i(t)$.

A less obvious interpretation is that the cost function to be minimized is the same as that which is minimized to solve the optimal least-squares filtering problem [46, Sec. 7.3]. In this case, $f_j(t)$ is interpreted as infinite variance noise driving the state $e(t)$, rather than as a control input. $m_i(t)$ is then interpreted as a measurement, i.e.

$$m_i(t) = C e(t) + d_j f_j(t) + v_j(t) \quad (6-35)$$

where $v_j(t)$ is a white noise process with intensity $R(t)$.  

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Similarly, in discrete time, the distinguishability measure
\( \Delta_{ij}(f_i(\cdot), k_f) \) is defined by

\[
\Delta^2_{ij}(f_i(\cdot), k_f) = \min_{\substack{f_j(k) \quad k = k_0 \\
\quad 0 \leq k < k_f}} \left\{ \sum_{k=0}^{k_f} \left[ m_j(k) - m_i(k) \right] \right\}^T M^{-1}(k) \left[ m_j(k) - m_i(k) \right]
\]

(6-36)

where \( m_i(k) \) and \( m_j(k) \) are the conditional means of the Kalman filter
residual \( \gamma(k) \) under \( H_i \) and \( H_j \), respectively, and \( M(k) \) is the
covariance of \( \gamma(k) \). \( m_i(k) \) and \( m_j(k) \) are given by

\[
m_k(k) = C(k)e_k(k) + d_k(k)f_k(k)
\]

(6-37)

where \( k \) can be either \( i \) or \( j \). \( e_k(k) \) is the mean estimation error
under \( H_k \), given by

\[
e_{k+1}(k+1) = \phi(k)[I - K(k)C(k)]e_k(k) + [b_{k+1}(k) - \phi(k)K(k)d_k(k)]f_k(k)
\]

(6-38)

with initial condition

\[
e_k(k_0) = 0
\]

(6-39)

where \( K(k) \) is the Kalman gain matrix for the normal-mode filter.

The interpretations of the optimization given in the
continuous-time case hold in the discrete-time case also. That is, the
optimization problem (6-36) may be interpreted as a discrete-time
optimal tracking problem, where the system to be controlled is

\[
e(k+1) = \phi_{KF}(k)e(k) + b_{KF}(k)f_j(k)
\]

(6-40)
where

\[ \phi_{kF}(k) = \phi(k)[I - K(k)C(k)] \]  \hspace{0.5cm} (6-41)  

\[ b_{kF}(k) = b_j(k) - \phi(k)K(k)d_j(k) \]  \hspace{0.5cm} (6-42)  

The output of the system is given by

\[ m_j(k) = C(k)e(k) + d_j(k)f_j(k) \]  \hspace{0.5cm} (6-43)  

The reference to be tracked is of course \( m_i(k) \). Alternatively, the optimization problem (6-36) may be interpreted as a least-squares filtering problem. In this case, \( m_i(k) \) is interpreted as a measurement, so that

\[ m_i(k) = C(k)e(k) + d_j(k)f_j(k) + v_j(k) \]  \hspace{0.5cm} (6-44)  

where \( v_j(k) \) is measurement noise with covariance \( M(k) \).

The optimization problems posed above may be solved using either of the interpretations given, although the form of the solution is different for each interpretation. Of course, both yield the same value for \( \Delta_{ij} \). Appendix B gives the derivation of the solution for \( \Delta_{ij} \), using both interpretations, in the discrete-time case.

If \( \Delta_{ij} \) is small, then there is a failure mode \( f_j(t) \) such that it will be difficult to distinguish between \( H_i \) and \( H_j \). Just how small \( \Delta_{ij} \) must be before a problem arises is difficult to say because of the assumptions implicit in the definition of \( \Delta_{ij} \). Specifically, these assumptions are that

1. The failure modes \( f_i(t) \) and \( f_j(t) \) are known a priori, and the isolation test is designed for them.
2. The test is a fixed interval test, rather than a sequential test.
(3) No other types of errors can occur, such as a missed isolation or missed detection.

All of these assumptions are violated for a sequential FDI test. Nevertheless, some idea of isolation performance may be obtained by considering the performance of the test for which the above conditions hold.

Therefore, consider the problem of determining whether \( \chi \) has mean \( \chi_i \) or \( \chi_j \). Assuming that \( \chi_i > \chi_j \), the test is given by

\[
\text{decide } H_i \quad \text{if } \chi \quad > \quad T
\]

\[
\text{decide } H_j \quad \text{if } \chi \quad < \quad T
\]

(If \( \chi_i < \chi_j \), the inequalities in the above test are reversed.) The probability of deciding \( H_j \) when \( H_i \) is true is given by

\[
P_{ij} = \Pr[\chi < T \mid H_i]
\]

Under \( H_i \),

\[
\chi \sim N(\chi_i, S)
\]

Therefore,

\[
P_{ij} = \int_{-\infty}^{T} \frac{1}{\sqrt{2\pi S}} \exp \left\{ -\frac{1}{2} \frac{(\chi - \chi_j)^2}{S} \right\} \, d\chi
\]

\[
= \text{erfc} \left( \frac{\chi_i - T}{\sqrt{S}} \right)
\]

where \( \text{erfc}(\cdot) \) is the complementary error function, defined by

\[
\text{erfc}(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-u^2/2} \, du
\]
(Note that this definition of the complementary error function is not the usual definition. The definition used here is taken from [51, p. 37].) Similarly,

\[ P_{ij} = \text{erfc} \left( \frac{T - X_j}{\sqrt{S}} \right) \]  \hspace{1cm} (6-50)

If the threshold is picked so that the misclassification probabilities are equal, then

\[ \frac{X_i - T}{\sqrt{S}} = \frac{T - X_j}{\sqrt{S}} \]  \hspace{1cm} (6-51)

so that

\[ T = \frac{X_i + X_j}{2} \]  \hspace{1cm} (6-52)

Then the misclassification probabilities are given by

\[ P_{ij} = P_{ji} = \text{erfc} \left( \frac{X_i - X_j}{2\sqrt{S}} \right) \]  \hspace{1cm} (6-53)

But

\[ X_i - X_j = S = \Delta_{ij}^2 \]  \hspace{1cm} (6-54)

Therefore,

\[ P_{ij} = P_{ji} = \text{erfc} \left( \frac{\Delta_{ij}}{2} \right) \]  \hspace{1cm} (6-55)

If the threshold is different than that given by Eq. (6-52), one of the misclassification probabilities will be larger than that given by Eq. (6-55). Therefore,
\[
\max (P_{ij}, P_{ji}) \geq \text{erfc}\left(\frac{\Delta_{ij}}{2}\right)
\]  

(6-56)

This result may be related to the sequential FDI test by noting that each violation of the assumptions discussed above degrades the isolation performance in some way:

1. No test can perform better than a test that has a priori knowledge of the failure modes \(f_i(t)\) and \(f_j(t)\).
2. A sequential test may not use all the data available before time \(t_f\) to make a decision.
3. In addition to the misclassification errors, the test may fail to detect a failure at all, or may detect but not isolate the failure by time \(t_f\).

Therefore, Eq. (6-56) bounds the achievable isolation performance, in the following sense. For either \(H_i\) or \(H_j\) (but not necessarily both), the probability that a failure (with mode shape \(f_i(t)\) or \(f_j(t)\), respectively) will not be correctly detected and isolated by time \(t_f\) is at least as large as \(\text{erfc}(\Delta_{ij}/2)\). Thus, \(\Delta_{ij}\) may be used as a guide to determine when it will be difficult to isolate certain types of failures.

6.2 Geometric Interpretation

In this section, a geometric interpretation of the distance measure \(\Delta_{ij}\) will be provided. Based on that interpretation, a relative measure of distinguishability will be defined.

Consider the continuous-time case. The set of square-integrable, \(m\)-dimensional vector functions on the interval \([t_0, t_f]\) is denoted by \(L^m_{2}[t_0, t_f]\), or simply \(L^m_2\). \(L^m_2\) is a complete, infinite-dimensional vector space. A valid inner product for \(L^m_2\) is defined by

\[
\langle m_1(\cdot), m_2(\cdot) \rangle = \int_{t_0}^{t_f} m_1^T(t) \cdot R^{-1}(t) \cdot m_2(t) \, dt \quad (6-57)
\]
where \( R(t) \) is a symmetric, positive definite matrix. If the norm for \( L_2^m \) is defined by

\[
\left\| m(\cdot) \right\| = \sqrt{\langle m(\cdot), m(\cdot) \rangle}^{1/2}
\]

(5-58)

then \( L_2^m \) is a Hilbert space.

Given this background, the problem of determining \( \Delta_{ij} \) may be succinctly stated as

\[
\Delta_{ij} = \min_{m_j \in V_j} \left\| m_j - m_i \right\|
\]

(6-59)

where \( V_j \) is the subspace of \( L_2^m \) which is the set of all \( m_j(\cdot) \) that can be generated by Eqs. (6-9), (6-10), and (6-11). By the orthogonal projection theorem [39, p. 5], the minimizing \( m_j(\cdot) \), denoted by \( m_j^*(\cdot) \), is the unique \( m_j^*(\cdot) \in V_j \) such that

\[
\langle m_i(\cdot) - m_j^*(\cdot), m_j(\cdot) \rangle = 0 \quad \text{for all } m_j(\cdot) \in V_j
\]

(6-60)

In other words, \( m_j^*(\cdot) \) is the projection of \( m_i(\cdot) \) onto \( V_j \), and the representation error

\[
m_i(\cdot) - m_j^*(\cdot)
\]

(6-61)

is orthogonal to the subspace \( V_j \). In particular,

\[
\langle m_i(\cdot) - m_j^*(\cdot), m_j(\cdot) \rangle = 0
\]

(6-62)

That is, the representation error is orthogonal to \( m_j^*(\cdot) \). This situation is represented graphically in Figure 6-1. In the figure, the vector space \( L_2^m \) is depicted as being spanned by \( e_1, e_2, e_3 \), whereas the subspace \( V_j \) is spanned by \( e_1 \) and \( e_2 \). \( \Delta_{ij} \) is simply the length of the vector \( m_i(\cdot) - m_j^*(\cdot) \).
Figure 6-1. Geometric interpretation of the orthogonal projection theorem.
Based on this geometric interpretation, a relative measure of distinguishability, \( \alpha_{ij}(f_i(\cdot), t_f) \), will be defined as the angle between \( \mathbf{m}_i(\cdot) \) and \( \mathbf{m}_j^*(\cdot) \). Because the representation error is orthogonal to the projection, \( \alpha_{ij} \) is given by

\[
\alpha_{ij} = \sin^{-1} \left( \frac{||\mathbf{m}_i(\cdot) - \mathbf{m}_j^*(\cdot)||}{||\mathbf{m}_i(\cdot)||} \right) \tag{6-63}
\]

But by definition,

\[
\Delta_{ij} = ||\mathbf{m}_i(\cdot) - \mathbf{m}_j^*(\cdot)|| \tag{6-64}
\]

Therefore,

\[
\alpha_{ij} = \sin^{-1} \left( \frac{\Delta_{ij}}{||\mathbf{m}_i(\cdot)||} \right) \tag{6-65}
\]

As a practical matter, both \( \alpha_{ij} \) and \( \Delta_{ij} \) are useful measures of the distinguishability of failures. As noted earlier, if \( \Delta_{ij}/2 \) is small, say on the order of unity or less, then it will not be possible to isolate a failure of component \( i \) or \( j \) (with failure mode \( f_i(\cdot) \) or \( f_j(\cdot) \)) reliably. If \( \Delta_{ij}/2 \) is small, but \( \alpha_{ij} \) is large (say, greater than 20 deg), the problem is that the energy in the failure signature \( \mathbf{m}_i(\cdot) \) is small, not that the two failure types are similar.

On the other hand, if \( \Delta_{ij}/2 \) is large, but \( \alpha_{ij} \) is small (say, less than 5 deg), then isolation should be possible, given knowledge of \( f_i(\cdot) \) and \( f_j(\cdot) \), and given that the system model is accurate. However, the isolation performance is likely to be sensitive to failure mode uncertainty, modeling uncertainty, and of course to the actual (sequential) FDI test used.
6.3 **Relationship to the OSGLR Algorithm**

In Section 6.1, the mean in the Kalman filter residual under $H_j$, $m_j(t)$, was produced by an input, $f_j(t)$, that was allowed to be arbitrary. Suppose instead that the input is required to take the form of a truncated series expansion, as in the OSGLR hypotheses. Then $f_j(t)$ is given by

$$f_j(t) = \phi_j^T(t_f - t) a_j(t_f)$$  \hspace{1cm} (6-66)

The resulting mean in the Kalman filter residual is given by

$$m_j(t) = G_{2j}(t,t_f) a_j(t_f)$$  \hspace{1cm} (6-67)

where the influence matrix $G_{2j}(t,t_f)$ was defined in Chapter 3. For a given value of $a_j(t_f)$, the distance $d$ is given by

$$d = \left| m_i(\cdot) - G_{2j}(\cdot,t_f) a_j(t_f) \right|$$  \hspace{1cm} (6-68)

If it is assumed that the subspace $V_j$ of possible $m_j(\cdot)$ generated by the the truncated series expansion approximates the subspace $V_j$ generated by allowing $f_j(\cdot)$ to be arbitrary, then $\Delta_{ij}$ may be approximated by

$$\Delta_{ij} = \min_{a^*(t_f)} \left| m_i(\cdot) - G_{2j}(\cdot,t_f) a^*(t_f) \right|$$  \hspace{1cm} (6-69)

The subspace generated by the series expansion is spanned by the columns of $G_{2j}(\cdot,t_f)$. The orthogonal projection theorem requires that the representation error

$$m_i(\cdot) - G_{2j}(\cdot,t_f) a_j(t_f)$$  \hspace{1cm} (6-70)
be orthogonal to the columns of $\mathbf{G}_{2j}(\cdot, t_f)$. Therefore,

$$\int_{t_0}^{t_f} \mathbf{G}_{2j}(t, t_f) \mathbf{R}^{-1}(t) \mathbf{G}_{2j}(t, t_f) \, dt \mathbf{a}^*_j(t_f)$$

$$= \int_{t_0}^{t_f} \mathbf{G}_{2j}^T(t, t_f) \mathbf{R}^{-1}(t) \mathbf{m}_1(t) \, dt$$  \hspace{1cm} (6-71)

The integral on the left side of this equation may be recognized as $S_j(t_f)$, the information matrix for the hypothesis $H_j$. The integral on the right is the mean of the information vector, $\mathbf{X}_j(t_f)$, given that $H_i$ is true, which may be denoted $\mathbf{X}_{j/i}(t_f)$. Therefore, $a^*_j(t_f)$ is given by

$$a^*_j(t_f) = S_j^{-1}(t_f) \mathbf{X}_{j/i}(t_f) \hspace{1cm} (6-72)$$

By Eqs. (6-69) and (6-72), $\Lambda^2_{ij}$ is (approximately) given by

$$\Lambda^2_{ij} = \left| \left| \mathbf{m}_i(\cdot) - \mathbf{m}_j^*(\cdot) \right| \right|^2 \hspace{1cm} (6-73)$$

where

$$\mathbf{m}_j^*(\cdot) = \mathbf{G}_{2j}(\cdot, t_f) S_j^{-1}(t_f) \mathbf{X}_{j/i}(t_f) \hspace{1cm} (6-74)$$

Note that the Pythagorean theorem is valid for pre-Hilbert spaces [39, pg. 49], as is therefore true for Hilbert spaces. Therefore, the right side of Eq. (6-73) can be expressed as

$$\left| \left| \mathbf{m}_i(\cdot) - \mathbf{m}_j^*(\cdot) \right| \right|^2 = \left| \left| \mathbf{m}_i(\cdot) \right| \right|^2 - \left| \left| \mathbf{m}_j^*(\cdot) \right| \right|^2 \hspace{1cm} (6-75)$$
Combining Eqs. (6-73) and (6-75) yields

\[ \Delta_{ij}^2 = \left| \left| m_1(\cdot) \right| \right|^2 - \left| \left| m_j(\cdot) \right| \right|^2 \]  \hspace{1cm} (6-76)

The norm of \( m_j(\cdot) \) is given by

\[ \left| \left| m_j(\cdot) \right| \right|^2 = \int_{t_0}^{t_f} m_j^T(t)R^{-1}(t)m_j(t) \, dt \]  \hspace{1cm} (6-77)

Using Eq. (6-75), we have that

\[ \left| \left| m_j(\cdot) \right| \right|^2 = a_j^T(t_f) \int_{t_0}^{t_f} G_{2j}(t,t_f)R^{-1}(t)G_{2j}(t,t_f) \, dt \left| a_j^T(t_f) \right| \]  \hspace{1cm} (6-78)

Once again, the integral above is the information matrix, \( S_j(t_f) \).

Using the above equation and the expression for \( a_j^*(t_f) \) in Eq. (6-72), we have that

\[ \left| \left| m_j(\cdot) \right| \right|^2 = \overline{A_{j/1}(t_f)} S_{j}^{-1}(t_f) \overline{A_{j/1}(t_f)} \]  \hspace{1cm} (6-79)

The above expression may be recognized as the noncentrality parameter of the decision function \( DF_j(t_f) \), given that \( H_1 \) is true. Thus,

\[ \left| \left| m_j(\cdot) \right| \right|^2 = \delta_{j/1}^2(t_f) \]  \hspace{1cm} (6-80)

Therefore,

\[ \Delta_{ij}^2 = \left| \left| m_1(\cdot) \right| \right|^2 - \delta_{j/1}^2(t_f) \]  \hspace{1cm} (6-81)
As a special case of the above result, we have that for \( j = i \)

\[
\Delta_{ii}^{2} = \left| \mathbf{m}_{i}^{}(\cdot) \right|^{2} - \delta_{i/i}^{2}(t_{f}) \tag{6-82}
\]

But clearly

\[
\Delta_{ii}^{2} = 0 \tag{6-83}
\]

Therefore,

\[
\left| \mathbf{m}_{i}^{}(\cdot) \right|^{2} = \delta_{i/i}^{2}(t_{f}) \tag{6-84}
\]

Thus,

\[
\Delta_{ij}^{2} = \delta_{i/i}^{2}(t_{f}) - \delta_{j/i}^{2}(t_{f}) \tag{6-85}
\]

Now, consider the OSGLR isolation decision function, \( \text{DF}_{ij}(t_{f}) \),

given by

\[
\text{DF}_{ij}(t_{f}) = \text{DF}_{i}(t_{f}) - \text{DF}_{j}(t_{f}) \tag{6-86}
\]

The expected value of this decision function, given that \( H_{i} \) is true, is given by

\[
E[\text{DF}_{ij}(t_{f})|H_{i}] = E[\text{DF}_{i}(t_{f})|H_{i}] - E[\text{DF}_{j}(t_{f})|H_{i}] \tag{6-87}
\]

Now, \( \text{DF}_{i}(t_{f}) \) and \( \text{DF}_{j}(t_{f}) \) are noncentral chi-squared random variables with noncentrality parameters \( \delta_{i/i}^{2}(t_{f}) \) and \( \delta_{j/i}^{2}(t_{f}) \), with \( p_{i} \) and \( p_{j} \) degrees of freedom, respectively. \( p_{i} \) and \( p_{j} \) are the number of terms in the series expansion for \( f_{i}(t) \) and \( f_{j}(t) \), respectively. Therefore,
\[ E[D_{ij}(t_f)|H_i] = \delta^2_{i/i}(t_f) - \delta^2_{j/i} + p_i - p_j \] (6-88)

Assuming that \( p_i \) and \( p_j \) are equal, and using Eq. (6-85), we have that

\[ E[D_{ij}(t_f)|H_i] = \Delta^2_{ij}(f_i(\cdot), t_f) \] (6-89)

Therefore, if \( \Delta^2_{ij} \) is small compared to the isolation threshold \( T^2_I \) (if there is one) then isolation is not likely to occur. If there is no isolation threshold, then an incorrect isolation is likely, because the signal-to-noise ratio is small.
CHAPTER 7

APPLICATION TO A TRANSPORT AIRCRAFT

In this chapter, the OSGLR test is applied to the problem of detecting failures of the flight control surfaces of a transport aircraft. Simulation results are used to compare the performance of the OSGLR test to the GLR test of Willsky and Jones [57].

The aircraft model used in this study is the C-130 transport. The C-130 is a medium- to long-range, high-wing, turboprop transport. The particular version of the C-130 used in this study has short takeoff and landing capability provided by trailing-edge doubled-slotted flaps. A description of the C-130 aircraft may be found in [45]. Much of the description of the C-130 and of the nonlinear simulation presented here is taken from [9].

This chapter is organized as follows: Section 7.1 briefly describes the nonlinear simulation of the C-130 aircraft. Section 7.2 details the steps used to generate the linear system model required to implement the OSGLR test and the GLR test. Also included in this section is a description of the method used to implement the linear simulation. In Section 7.3, the issue of distinguishability of failures is discussed. Finally, simulation results are presented in Section 7.4 which demonstrate the advantages and limitations of the OSGLR test.

7.1 **Nonlinear Simulation**

The nonlinear simulation describes the dynamics of the C-130 aircraft using six degree-of-freedom equations of motion. The aerodynamic
forces and moments are described by one-, two-, and three-dimensional tables. Actuator dynamics are also incorporated into the simulation. Also included in the simulation is a flight control system with an auto-pilot.

The surfaces available for control of the aircraft are the elevator, ailerons, rudder, and flaps (Table 7-1). The simulation allows for independent motion of the left and right ailerons and left and right flaps so that failures of these surfaces may be simulated.

Table 7-1. C-130 control surfaces.

<table>
<thead>
<tr>
<th>Input</th>
<th>Control Surface</th>
<th>Symbol</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>elevator</td>
<td>$\delta_E$</td>
<td>deg</td>
</tr>
<tr>
<td>2</td>
<td>right aileron</td>
<td>$\delta_{AR}$</td>
<td>deg</td>
</tr>
<tr>
<td>3</td>
<td>left aileron</td>
<td>$\delta_{AL}$</td>
<td>deg</td>
</tr>
<tr>
<td>4</td>
<td>rudder</td>
<td>$\delta_R$</td>
<td>deg</td>
</tr>
<tr>
<td>5</td>
<td>right flap</td>
<td>$\delta_{FR}$</td>
<td>percent</td>
</tr>
<tr>
<td>6</td>
<td>left flap</td>
<td>$\delta_{FL}$</td>
<td>percent</td>
</tr>
</tbody>
</table>

The measurements are available (or sampled) at a rate of 50 Hz. To simulate sensor noise, independent, zero-mean Gaussian random numbers were added to the sensor outputs, with standard deviations as given in Table 7-2. (The standard deviation of the rate-of-climb measurement is probably too low to be realistic. However, the results presented here do not seem to be sensitive to this sensor.)

Finally, turbulence is included in the simulation. The turbulence model will be discussed in detail in Section 7.2.3.

7.2 Linear Model

In order to implement the OSGLR or GLR test, a linear, discrete-time, state-space model of the system is required. This section describes the development of such a model. The numerical values of the model are given in Appendix C.
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Table 7-2. C-130 sensors.

<table>
<thead>
<tr>
<th>Sensor</th>
<th>Measured Variable</th>
<th>Units</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>airspeed</td>
<td>ft/s</td>
<td>11.0</td>
</tr>
<tr>
<td>2</td>
<td>lateral acceleration</td>
<td>ft/s²</td>
<td>0.98425</td>
</tr>
<tr>
<td>3</td>
<td>normal acceleration</td>
<td>ft/s²</td>
<td>0.98425</td>
</tr>
<tr>
<td>4</td>
<td>roll rate</td>
<td>rad/s</td>
<td>0.0024</td>
</tr>
<tr>
<td>5</td>
<td>pitch rate</td>
<td>rad/s</td>
<td>0.0007</td>
</tr>
<tr>
<td>6</td>
<td>yaw rate</td>
<td>rad/s</td>
<td>0.0007</td>
</tr>
<tr>
<td>7</td>
<td>bank angle</td>
<td>rad</td>
<td>0.01</td>
</tr>
<tr>
<td>8</td>
<td>elevation</td>
<td>rad</td>
<td>0.01</td>
</tr>
<tr>
<td>9</td>
<td>heading</td>
<td>rad</td>
<td>0.01</td>
</tr>
<tr>
<td>10</td>
<td>rate-of-climb</td>
<td>ft/s</td>
<td>0.25</td>
</tr>
<tr>
<td>11</td>
<td>barometric altitude</td>
<td>ft</td>
<td>10.0</td>
</tr>
</tbody>
</table>

7.2.1 Linearization of the Aircraft Dynamics

The first step in determining a linearized model is to identify the state vector of the system. The state vector used in this study does not correspond to any of the axis systems commonly used in the study of aircraft dynamics, such as body axes or stability axes [20]. Instead, a hybrid axis system was used that corresponds more closely to the sensors on the aircraft. The ten states of the aircraft are shown in Table 7-3.

The nonlinear simulation integrates the differential equation

\[
\frac{dx_-P(t)}{dt} = f(x_-P(t), u(t), y_t(t))
\]  

(7-1)

where \(x_-P(t)\) is the vector of plant (i.e., aircraft) states, \(u(t)\) is the vector of inputs (i.e., actuator positions), and \(y_t(t)\) is the vector of turbulence velocities. The vector of measurements is given by
Table 7-3. States of the C-130 linearized model.

<table>
<thead>
<tr>
<th>State</th>
<th>Description</th>
<th>Symbol</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>aircraft velocity</td>
<td>( V )</td>
<td>ft/s</td>
</tr>
<tr>
<td>2</td>
<td>angle of attack</td>
<td>( \alpha )</td>
<td>rad</td>
</tr>
<tr>
<td>3</td>
<td>sideslip angle</td>
<td>( \beta )</td>
<td>rad</td>
</tr>
<tr>
<td>4</td>
<td>roll rate</td>
<td>( p )</td>
<td>rad/s</td>
</tr>
<tr>
<td>5</td>
<td>pitch rate</td>
<td>( q )</td>
<td>rad/s</td>
</tr>
<tr>
<td>6</td>
<td>yaw rate</td>
<td>( r )</td>
<td>rad/s</td>
</tr>
<tr>
<td>7</td>
<td>bank angle</td>
<td>( \phi )</td>
<td>rad</td>
</tr>
<tr>
<td>8</td>
<td>elevation</td>
<td>( \theta )</td>
<td>rad</td>
</tr>
<tr>
<td>9</td>
<td>heading</td>
<td>( \psi )</td>
<td>rad</td>
</tr>
<tr>
<td>10</td>
<td>altitude</td>
<td>( h )</td>
<td>ft</td>
</tr>
</tbody>
</table>

\[
y(t) = g(x_0(t), u(t), y(t)) + v(t) \quad (7-2)
\]

where \( v(t) \) is additive sensor noise.

To develop a linearized system model, the nonlinear functions \( f \) and \( g \) are expanded in a Taylor series about a nominal point, \((x_{p_0}, u_0, y_{t_0})\). The nominal point used in this study corresponds to cruising flight at an altitude of 1000 ft, with an airspeed of 150 knots. The nominal operating point is specified in Table C-1 of Appendix C. Because the turbulence has zero mean, we have that

\[
y(t) = 0 \quad (7-3)
\]

The vector \( x_{p_0} \) and \( u_0 \) are chosen so that the aircraft is approximately in equilibrium at the desired operating condition, so that
\[ f(x_p, u_0, y_t) = 0 \] (7-4)

The differential Eq. (7-1) can then be approximated by retaining terms of the Taylor series up to first order, so that

\[ \frac{dx_p(t)}{dt} = A_p (x_p(t) - x_{p0}) + B_{p1} (u(t) - u_0) + B_{p2} y_t(t) + \dot{x}_{p0} \] (7-5)

where

\[ A_p \triangleq \frac{\partial f}{\partial x_p}(x_{p0}, u_0, y_{t0}) \] (7-6)

\[ B_{p1} \triangleq \frac{\partial f}{\partial u}(x_{p0}, u_0, y_{t0}) \] (7-7)

\[ B_{p2} \triangleq \frac{\partial f}{\partial y_t}(x_{p0}, u_0, y_{t0}) \] (7-8)

\[ \dot{x}_{p0} \triangleq f(x_{p0}, u_0, y_{t0}) \] (7-9)

Similarly, the measurement Eq. (7-2) can be approximated by

\[ y(t) = C_p (x_p(t) - x_{p0}) + D_{p1} (u(t) - u_0) + D_{p2} y_t(t) + \zeta_0 + \eta(t) \] (7-10)
where

\[ C_p \overset{\Delta}{=} \frac{\partial g}{\partial x_p} (x_p, u_0, y_{t_0}) \]  \hspace{1cm} (7-11)

\[ D_{p1} \overset{\Delta}{=} \frac{\partial g}{\partial u} (x_p, u_0, y_{t_0}) \]  \hspace{1cm} (7-12)

\[ D_{p2} \overset{\Delta}{=} \frac{\partial g}{\partial y_t} (x_p, u_0, y_{t_0}) \]  \hspace{1cm} (7-13)

\[ y_0 \overset{\Delta}{=} g (x_p, u_0, y_{t_0}) \]  \hspace{1cm} (7-14)

As defined above, the determination of the linearized system matrices requires the evaluation of partial derivatives. This was not possible to do analytically, because the simulation includes look-up tables rather than analytic expressions to describe some of the system dynamics. Therefore, the system matrices were found by approximating the partial derivatives by a finite difference operator. The matrices of the linearized model are given in Table C-2 of Appendix C.

7.2.2 Discretization of the System

Because the measurements are available at discrete points in time, a discrete-time model of the system is required. The system was discretized [26, Ch. 3] to obtain a model of the form

\[ x_p(t_{k+1}) = \Phi_p (x_p(t_k) - x_{p0}) + B_{p1} (u(t_k) - u_0) + B_{p2} y(t_k) + \Delta x_p \]  \hspace{1cm} (7-15)
where the times $t_k$ correspond to the times at which measurements are available. Because the measurements are sampled at a 50 Hz rate, the time increment is

$$\Delta t = t_{k+1} - t_k = 0.02 \text{ s}$$  \hfill (7-16)

An implicit assumption in the discretization is that $\underline{u}(t)$ and $\underline{v}_t(t)$ are constant over the time increment, so that

$$\underline{u}(t) = \underline{u}(t_k), \quad t_k \leq t < t_{k+1}$$  \hfill (7-17)

$$\underline{v}_t(t) = \underline{v}_t(t_k), \quad t_k \leq t < t_{k+1}$$  \hfill (7-18)

This is not precisely true. However, Eqs. (7-17) and (7-18) are a good approximation, because of the short (0.02 s) sampling interval, relative to the time constants of the aircraft dynamics and to the correlation time of the turbulence. The matrices of the discrete-time state equation are given in Table C-3 of Appendix C.

When no confusion can result, the time argument $t_k$ will be replaced by the time index $k$. Hence, Eqs. (7-15) and (7-10) become

$$\underline{x}_{p}^{(k+1)} = \underline{\phi}_{p}^{(k)} \underline{x}_{p}^{(k)} + \underline{B}_{p1d} \underline{u}^{(k)} - \underline{u}_0^{(k)} + \underline{B}_{p2d} \underline{v}_t^{(k)} + \Delta x_{p0}$$  \hfill (7-19)

$$\underline{y}_{p}^{(k)} = \underline{C}_{p}^{(k)} \underline{x}_{p}^{(k)} - \underline{x}_{p0}^{(k)} + \underline{D}_{p1} \underline{u}^{(k)} - \underline{u}_0^{(k)} + \underline{D}_{p2} \underline{v}_t^{(k)} + \underline{y}_{p0}$$  \hfill (7-20)

7.2.3 Turbulence Model

The turbulence model used in the simulation is the Dryden model of military specification MIL-F-8785B [11]. The changes in airspeed due to turbulence in the longitudinal, lateral, and vertical axes are denoted by
$u_g, v_g, w_g$, respectively. These three velocities are the elements of the turbulence vector, $y_t$, mentioned earlier:

$$y_t \sim A \begin{bmatrix} u_g \\ v_g \\ w_g \end{bmatrix}$$  \hspace{1cm} (7-21)

Each element of $y_t$ is assumed to be an independent, zero-mean Gaussian process, which is defined by its power spectral density:

$$\phi_{u_g}(\omega) = \sigma_u^2 \frac{2L_u}{\pi V} \frac{1}{1 + \left(\frac{L_u}{V} \omega\right)^2}$$  \hspace{1cm} (7-22)

$$\phi_{v_g}(\omega) = \sigma_v^2 \frac{L_v}{\pi V} \frac{1 + 3(\frac{L_v}{V} \omega)^2}{(1 + (\frac{L_v}{V} \omega)^2)^2}$$  \hspace{1cm} (7-23)

$$\phi_{w_g}(\omega) = \sigma_w^2 \frac{L_w}{\pi V} \frac{1 + 3(\frac{L_w}{V} \omega)^2}{(1 + (\frac{L_w}{V} \omega)^2)^2}$$  \hspace{1cm} (7-24)

where $V$ is the airspeed of the aircraft. $\sigma_u^2$, $\sigma_v^2$, and $\sigma_w^2$ are the variances of the turbulence in each of the axes. $L_u$, $L_v$, and $L_w$ are scale lengths that specify the distance over which the turbulence is correlated. The scales for clear air turbulence are determined by the height of the aircraft above the ground, $h$, as follows:

If $h > 1750$ ft, then

$$L_u = L_v = L_w = 1750 \text{ ft}$$
If \( h < 1750 \text{ ft} \), then

\[
L_w = h
\]

\[
L_u = L_v = (1750 \text{ ft})^{2/3} h^{1/3}
\]

For this study, the altitude was assumed to be

\[
h = 1000 \text{ ft}
\] (7-25)

Therefore,

\[
L_w = 1000 \text{ ft}
\] (7-26)

\[
L_u = 1452 \text{ ft}
\] (7-27)

\[
L_v = 1452 \text{ ft}
\] (7-28)

The variances of the turbulence are related by

\[
\frac{\sigma_u^2}{L_u} = \frac{\sigma_v^2}{L_v} = \frac{\sigma_w^2}{L_w}
\] (7-29)

MIL-F-8785B specifies the maximum level of the clear air turbulence as

\[
\sigma_w = 6.5 \text{ ft/s}
\] (7-30)

for an altitude of 1000 ft. Therefore,

\[
\sigma_u = 7.833 \text{ ft/s}
\] (7-31)

\[
\sigma_v = 7.833 \text{ ft/s}
\] (7-32)
In order to implement the Kalman filter for the aircraft states, a state-space model of the turbulence is required. The first step is to find a shaping filter (in the frequency domain) that, when driven by white noise, produces a random process with the desired spectral density. For example, suppose $w_u(t)$ is a Gaussian white noise process such that

$$E[w_u(t)w_u(t+\tau)] = \delta(\tau)\sigma^2_u$$

(7-33)

Then

$$\phi_{w_u}(\omega) = \frac{\sigma^2_u}{\pi}$$

(7-34)

We want to find a transfer function, $T_{ug}(s)$, such that

$$T_{ug}(j\omega)T_{ug}^*(j\omega)\phi_{w_u}(\omega) = \phi_{w_u}(\omega)$$

(7-35)

such that $T_{ug}(s)$ is a stable transfer function. It is easily verified that $T_{ug}(s)$ is given by

$$T_{ug}(s) = \sqrt{\frac{2L_u}{v}} \frac{1}{1 + \frac{L_u}{v}s}$$

(7-36)

Similarly, if

$$\phi_{w_v}(\omega) = \frac{\sigma^2_v}{\pi}$$

(7-37)
\[ \phi_w(\omega) = \frac{\sigma_w^2}{\pi} \]  

(7-38)

then

\[ T_{v_1}(s) = \sqrt{L_v} \frac{1 + \frac{\sqrt{3} L_v}{V} s}{\left(1 + \frac{L_v}{V} s\right)^2} \]  

(7-39)

\[ T_{w_1}(s) = \sqrt{L_w} \frac{1 + \frac{\sqrt{3} L_w}{V} s}{\left(1 + \frac{L_w}{V} s\right)^2} \]  

(7-40)

The next step is to convert the transfer functions to state-space representations of the form

\[ \frac{dx_i}{dt} = A_i x_i + b_i w_i \]  

(7-41)

\[ y_i = c_i^T x_i \]  

(7-42)

where the subscript \( i \) can be \( u, v, \) or \( w, \) depending on which transfer function is being represented. However, the triple \( (A_i, b_i, c_i^T) \) is not unique. To determine \( A_i, b_i, \) and \( c_i, \) the transfer functions were represented in phase-variable canonical form [36, p. 82]. Thus,

\[
A_v = \begin{bmatrix}
0 & 1 \\
-(V/L_v)^2 & -2V/L_v
\end{bmatrix}
\]  

(7-43)
\[
\begin{bmatrix}
    b_u \\
    b_v \\
    1
\end{bmatrix} = \begin{bmatrix}
    0 \\
    1
\end{bmatrix}
\]

(7-44)

\[
c_v = [(V/L_v)^{3/2} \sqrt{3(V/L_v)^{1/2}}]
\]

(7-45)

and similarly for \( A_w, b_w, \) and \( c_w \). \( A_u, b_u, \) and \( c_u \) are given by

\[
A_u = [-V/L_u]
\]

(7-46)

\[
b_u = [1]
\]

(7-47)

\[
c_u = [(2V/L_u)^{1/2}]
\]

(7-48)

The three turbulence models are then augmented to form a single state-space description of the turbulence:

\[
\frac{dx_t(t)}{dt} = A_t x_t(t) + B_t w(t)
\]

(7-49)

\[
y_t(t) = C_t x_t(t)
\]

(7-50)

where

\[
x_t^T(t) = [x_u^T(t) \ x_v^T(t) \ x_w^T(t)]
\]

(7-51)

\[
A_t = \text{block diag } (A_u, A_v, A_w)
\]

(7-52)

\[
B_t = \text{block diag } (b_u, b_v, b_w)
\]

(7-53)
\[
\begin{align*}
\mathbf{w}_t^T(t) &= \begin{bmatrix} \mathbf{w}^T_u(t) & \mathbf{w}^T_v(t) & \mathbf{w}^T_w(t) \end{bmatrix} \\
\mathbf{C}_t &= \text{block diag } \begin{bmatrix} \mathbf{C}_u & \mathbf{C}_v & \mathbf{C}_w \end{bmatrix}
\end{align*}
\] (7-54)

(7-55)

As with the plant model, the turbulence model is discretized to correspond to the 50 Hz sampling rate. The resulting discrete-time turbulence model is given by

\[
\begin{align*}
\mathbf{x}_t(k+1) &= \phi_t \mathbf{x}_t(k) + \mathbf{w}_{td}(k) \\
\mathbf{y}_t(k) &= \mathbf{C}_t \mathbf{x}_t(k)
\end{align*}
\] (7-56)
(7-57)

where \( \mathbf{w}_{td}(k) \) is a zero-mean, white Gaussian sequence with covariance \( \mathbf{Q}_t \). The matrices of the discrete-time turbulence model are given in Table C-4 of Appendix C.

7.2.4 State Augmentation

To obtain a single state-space representation, the aircraft state vector, \( \mathbf{x}_p \), and the turbulence state vector, \( \mathbf{x}_t \), are augmented to form a single state vector

\[
\begin{bmatrix}
\mathbf{x}_p(k) \\
\mathbf{x}_t(k)
\end{bmatrix}
\] (7-58)

The state dynamics and measurement equations follow directly from Eqs. (7-19), (7-20), (7-56) and, (7-57):

\[
\begin{align*}
\mathbf{x}(k+1) &= \phi \mathbf{x}(k) - \mathbf{x}_0 + \mathbf{B} \mathbf{s}(k) - \mathbf{s}_0 + \Gamma \mathbf{w}_{td}(k) + \Delta \mathbf{x}_0
\end{align*}
\] (7-59)
\[ y(k) = C(x(k) - x_0) + D(u(k) - u_0) + y(k) + y_0 \]  

(7-60)

where

\[ \phi = \begin{bmatrix} \phi_p & B_{p2d} C_t \\ 0 & \phi_t \end{bmatrix} \]  

(7-61)

\[ B = \begin{bmatrix} B_{p1d} \\ 0 \end{bmatrix} \]  

(7-62)

\[ C = \begin{bmatrix} C_p & D_{p2} C_t \end{bmatrix} \]  

(7-63)

\[ D = \begin{bmatrix} D_{p1} & 0 \end{bmatrix} \]  

(7-64)

\[ \Gamma = \begin{bmatrix} 0 \\ I \end{bmatrix} \]  

(7-65)

\[ x_0 = \begin{bmatrix} x_{p0} \\ 0 \end{bmatrix} \]  

(7-66)

\[ \Delta x_0 = \begin{bmatrix} \Delta x_{p0} \\ 0 \end{bmatrix} \]  

(7-67)

The partitions of the above matrices follow from the definition of the augmented state vector.

Note that it would have been more accurate to augment the state vector in continuous-time, and then to perform the discretization.
However, it was more convenient to develop the augmented, discrete-time system as presented here. Because of the high sampling rate, the errors produced by this method are small.

Finally, the matrices of the steady-state Kalman filter were found, using the control system software package CTRL-C [18]. The Kalman filter is based on the state equation (7-59), the measurement equation (7-60), the covariance matrix $Q_t$, and the covariance matrix of the measurement noise, $R$, as determined from Table 7-2. The Kalman gain matrix $K$, the estimation error covariance $P^-$, and the residual covariance $M$ are given in Table C-5 of Appendix C.

7.2.5 Linear Simulation

Most of the simulation results presented in Section 7.4 are based on a linear simulation, rather than on the nonlinear simulation. In this section, the method used to generate those results is discussed.

The most straightforward method of simulating the linear system is to simply implement the state equation (7-59) and the measurement equation (7-60), using a Gaussian random number generator to produce the sequences $w_t(k)$ and $v(k)$ with the appropriate statistics. (Of course, Eqs. (7-59) and (7-60) must be modified to simulate a failure.) The measurement sequence $y(k)$ is then used by the Kalman filter to produce the residual sequence $\tilde{y}(k)$.

However, note that all that is required by the OSGLR and GLR tests is the residual sequence, $\tilde{y}(k)$. The residual sequence can be expressed as

$$\tilde{y}(k) = y_0(k) + \tilde{m}(k) \quad (7-68)$$

where $y_0(k)$ is a zero-mean, white, Gaussian sequence with covariance $M$. The vector $\tilde{m}(k)$ is given by

$$\tilde{m}(k) = C\tilde{e}(k) + d(k)f(k) \quad (7-69)$$

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where \( e(k) \) is the mean estimation error, \( d(k) \) is determined by the type of failure being simulated, and \( f(k) \) is the failure mode shape. \( e(k) \) satisfies the difference equation

\[
e(k+1) = \Phi[I - KC]e(k) + [b - \Phi d]f(k)
\] (7-70)

where again \( b \) is determined by the type of failure being simulated. The initial condition is that

\[
e(k_0) = 0
\] (7-71)

where \( k_0 \) is the initial time. Equations (7-68) through (7-71) were used to generate the Kalman filter residuals for the linear simulations. This method is more efficient than implementing a Kalman filter and a linear simulation, and produces residuals with the same statistics.

7.3 Distinguishability of Failures

Before proceeding to the simulation results, we will use the distinguishability measures defined in Chapter 6 to determine to what degree the failures of the C-130 control surfaces are distinguishable from one another.

Recall from Chapter 6 that the distance between two hypotheses \( H_i \) and \( H_j \) (\( \Delta_{ij} \)) and the angle between two hypotheses (\( \alpha_{ij} \)) are functions of the (assumed) failure mode of \( H_i \), \( f_i(t) \). For the purposes of this section, only one type of failure mode was assumed, namely, a step bias failure occurring at time \( \theta \). Note that because the system is time-invariant and the Kalman filter is (assumed to be) operating in steady state, the time origin may be shifted arbitrarily. Therefore, the distinguishability measures are functions of \( t - \theta \), the length of time since the onset of the failure.

Figure 7-1 shows the distinguishability measures \( \Delta_{ij} \) and \( \alpha_{ij} \) for a 1 deg bias failure of the elevator. (In this section and in Section 7.4, the subscripts of the distinguishability measures and of the
Figure 7-1. Distinguishability measures for a unit (1 deg) bias failure of the elevator.
decision functions correspond to the control surfaces under consideration, as listed in Table 7-1.) The five $a_{1j}$'s are greater than 35 deg for 10 s after the failure, and are larger than 50 deg soon after the failure. This indicates that on a relative basis, an elevator bias failure is easily distinguishable from the failure of any other control surface. On an absolute basis, all the $\Delta_{1j}$'s are larger than 21 by time $t - \theta = 1$ s, which is quite large. Therefore, we expect that for a well-designed FDI test, failure detection and isolation of a 1 deg bias failure of the elevator should take less than 1 s.

Figure 7-2 shows the distinguishability measures for a 1 deg bias failure of the rudder. Note that all the $a_{4j}$'s are greater than 72 deg for the entire time of the plot. That is, the failure signature of a rudder failure is nearly perpendicular to any failure signature that can be generated by any other control surface failure. On the other hand, the $\Delta_{4j}$'s are relatively small. At $t - \theta = 2$ s, the five $\Delta_{4j}$'s are about 4.0, which is small. This is simply a reflection of the fact that the signal-to-noise ratio ($d^2$) of the rudder failure (when tested against the hypothesis that no failure has occurred) is small. The conclusion is that if a rudder failure is large enough to be detected, then it will be easily distinguishable from failures of the other control surfaces.

Figure 7-3 shows the distinguishability measures for a 1 deg bias failure of the right aileron. This figure is quite different from the previous two. In particular, the distinguishability measures associated with the left aileron, the right flap, and the left flap (corresponding to the subscripts 3, 5, and 6, respectively) are small. Except for the first 1.0 s following the failure, $a_{26}$ is less than 15 deg, $a_{25}$ is less than 10 deg, and $a_{23}$ is less than 5 deg. $a_{23}$, $a_{25}$, and $a_{26}$ all decrease as the time after failure, $t - \theta$, increases. Therefore, a failure of the right aileron is not very distinguishable from failures of another wing control surface (flap or aileron) based on the relative measure $a_{2j}$.

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Figure 7-2. Distinguishability measures for a unit (1 deg) bias failure of the rudder.
Figure 7-3. Distinguishability measures for a unit (1 deg) bias failure of the right aileron.
The failure that most closely resembles the right aileron failure is, not surprisingly, a failure of the left aileron. As indicated by the distance measure $\Delta_{23}$, a failure of the right aileron is barely distinguishable from a failure of the left aileron. Even after 10 s, $\Delta_{23}$ is only 4.6, which is too small to ensure reliable isolation, even by a well-designed FDI test. Of course, if the failure were larger than 1 deg, $\Delta_{23}$ would scale proportionally. Nevertheless, these results indicate that aileron failures will be difficult to isolate. This will be clearly evident in the simulation results (Section 7.4).

Figure 7-4 shows the distinguishability measures for a 1 percent bias failure of the right flap. This figure is qualitatively similar to Figure 7-3. That is, the distinguishability measures associated with the other three wing control surfaces (in this case, the left flap, and the right and left ailerons) are small. In this case, the failure that most resembles the right flap failure is a failure of the right aileron. However, the failure of the right flap is somewhat more distinguishable (as measured by $\alpha_{52}$) than was the failure of the right aileron (as measured by $\alpha_{23}$). 10 s after the onset of a failure, $\alpha_{52} = 2.98$ deg, whereas $\alpha_{23} = 1.63$ deg.

The results presented in this section may be explained in terms of the aircraft's dynamics. The elevator's primary effect is to produce a moment about the pitch axis. The other control surfaces do not produce significant moments about the pitch axis. Therefore, an elevator failure is easily distinguished from failures of the other control surfaces. Similarly, the rudder's primary effect is to produce a yawing moment, whereas none of the other control surfaces produce significant moments about the yaw axis. Therefore, rudder failures are easily distinguished from other failures.

On the other hand, the primary effect of the ailerons is to produce a moment about the roll axis. Furthermore, even though their purpose is to produce lift, flaps produce significant rolling moments when operated differentially. In that regard, they behave very much like ailerons. Therefore, it is not surprising that a failure of one of these
Figure 7-4. Distinguishability measures for a unit (1 percent) bias failure of the right flap.
four control surfaces is not very distinguishable from failures of the other three.

7.4 Simulation Results

In this section, the performance of the OSGLR test and the GLR test are compared, using both the linear and the nonlinear simulation of the C-130 aircraft. For the most part, we will be concerned with control surface failures, although the results for one sensor failure are also presented.

The basis functions of the OSGLR failure hypotheses were determined by trial and error, so that good performance was achieved for many different failure modes. For each failure hypothesis, six basis functions were used in the truncated series expansion. The basis functions are the discrete-time equivalent of the Laguerre functions, with time constant

$$\tau = 3.0 \text{s}$$ (7-72)

(See Chapter 5.) That is, the basis functions are obtained by orthogonalizing the (discrete-time) functions

$$\psi_i(k) = k^{i-1}z^k, \quad i = 1, 2, \ldots, 6$$ (7-73)

where

$$z = e^{-\Delta t/\tau}$$ (7-74)

The assumed failure mode of each GLR failure hypothesis is a step function (or bias) of unknown magnitude occurring at time $\theta$. The failure onset time $\theta$ is constrained to the data window $t - t_w < \theta < t$, where $t_w$ is the length of the window. The GLR test was implemented using two different data windows: a 2 s (100 sample) window and a 5 s (250 sample)
window. Most of the results presented in this section are for the 2 s data window.

The simulation results can be divided into two categories: those based on the linearized models and those based on the nonlinear simulation. For the most part, the simulation results presented are based on the linearized models. The reasons for this are twofold. First, the linear simulations demonstrate the characteristics of the OSGLR test unobscured by nonlinear effects. Second, the linear simulation requires considerably less computation than the nonlinear simulation. Therefore, linear models are used to demonstrate the important features of the OSGLR test. The nonlinear simulation is used to show the effects of nonlinearities and to generate test cases that are not easily generated using the linear simulation.

In order to determine the performance of the algorithms, a detection threshold must be set for each. Because no performance specifications have been given, the selection of the threshold is somewhat arbitrary. We will set the OSGLR threshold so that the resulting false-alarm rate of each OSGLR detector is $10^{-4}$ per hour or less. Using the results for discrete-time systems of Chapter 4, we have that

$$T^2 = 56.86 \quad (7-75)$$

As noted earlier, the false-alarm rate is very sensitive to the selection of the threshold. As a result, the threshold is not very sensitive to the specification of the false-alarm rate. For example, if we require that the false-alarm rate be decreased to $10^{-6}$ per hour, then the threshold must be increased to only $T^2 = 67.10$. Thus, the results presented here will not be sensitive to the exact value of the false-alarm rate specification.

It is somewhat more difficult to determine the threshold for the GLR test. In order to compare the GLR and OSGLR tests on a fair basis, we should select the threshold for the GLR test so that each GLR detector has the same false-alarm rate as the OSGLR detectors. Unfortunately, no
analytic expression for the false-alarm rate of the GLR test exists. The false-alarm rate could be determined in principle by Monte Carlo simulation. However, the amount of simulation that would be required would be enormous, because of the very small rate at which false alarms occur. Therefore, we will simply set the GLR detection threshold to the same value as the OSGLR detection threshold.

7.4.1 Linear Simulation with No Failure

For this case, the linear simulation was used to generate Kalman filter residuals for the unfailed system. The duration of the simulation was 50 s.

Figure 7-5 shows the OSGLR detection decision functions for the six actuators for this simulation. Several features in the figure are noteworthy. First, because each of the decision functions is a central chi-squared random variable with six degrees of freedom, the mean value of each decision function should be 6. This fact appears to be verified by the figure.

Second, some of the decision functions have peak values that approach 20. This is far below the detection threshold $T^2 = 56.86$, as would be expected for a simulation of such short duration.

The third noteworthy feature of Figure 7-5 is that the detection decision functions corresponding to the ailerons and flaps ($DF^2$, $DF^3$, $DF^5$, and $DF^6$) show a striking similarity. This is not unexpected, as we saw in the previous section that failures of a given control surface on the wings (aileron or flap) is not easily distinguishable from the failure of any other control surface on the wing. This was due to the fact that these four surfaces have similar effects on the dynamics of the aircraft. Therefore, the OSGLR detectors for these surfaces are similar. Hence, the OSGLR decision functions for these surfaces are similar, even when there is no failure present. This is an effect that will be apparent in all of the simulations, whether a wing control surface has failed or not.

Figure 7-6 shows the GLR detection decision functions for the same simulation. For this case, the data window for the GLR test was 2 s long.
Figure 7-5. OSGLR detection decision functions for the linear simulation with no failures. (Sheet 1 of 2)
Figure 7-5. OSGLR detection decision functions for the linear simulation with no failures. (Sheet 2 of 2)
Figure 7-6. GLR detection decision functions for the linear simulation with no failures and a 2 s (100 sample) data window. (Sheet 1 of 2)
Figure 7-6. GLR detection decision functions for the linear simulation with no failures and a 2 s (100 sample) data window. (Sheet 2 of 2)
(100 samples). The general character of the GLR decision functions is somewhat different than that of the OSGLR decision functions. The decision functions seem to be somewhat noisier and to have a smaller mean value.

However, there are also some similarities between Figures 7-6 and 7-5. First, the GLR decision functions for the wing control surfaces \((DF_2', DF_3', DF_5', \text{ and } DF_6')\) are all similar, although the similarity is not as great as was the case for the OSGLR decision functions. Second, the peak values of some of the GLR decision functions are close to 20, as was the case with the OSGLR algorithm. From this we may conclude that the assumption that the GLR and OSGLR detection thresholds are the same is not unreasonable.

7.4.2 Elevator Bias Failure

In this case, a -1 deg bias of the elevator was simulated, using the linear simulation. The failure occurred at time \(t = 10\) s of a 50 s simulation.

Figure 7-7 shows the OSGLR detection decision functions for the six control surfaces. The decision functions generally have the characteristics that we expect. Immediately following the onset of the failure, the decision function corresponding to the elevator, \(DF_1\), increases rapidly, indicating a failure of the elevator. To a lesser extent, the other decision functions increase as well, although they are always much less than \(DF_1\). (Note the vertical scale of Figure 7-7.) For the detection threshold selected, detection occurs at \(t = 10.22\) s, 0.22 s after the onset of the failure.

Several other features of the figure are noteworthy. First, the four decision functions \(DF_2', DF_3', DF_5', \text{ and } DF_6'\) are very nearly equal. The reason for this behavior is given in Section 7.4.1. Second, note that the elevator failure is easily isolated, because \(DF_1\) is significantly larger than the other decision functions. This is in agreement with the results of Section 7.3, where we concluded that elevator failures are easily distinguishable from other control surface failures.
Figure 7-7. OSGLR detection decision functions for the linear simulation with a -1 deg bias failure of the elevator at time $t = 10$ s. (Sheet 1 of 2)
Figure 7-7. OSGLR detection decision functions for the linear simulation with a -1 deg bias failure of the elevator at time $t = 10$ s. (Sheet 2 of 2) Note the expanded time scale in the vicinity of $t = 10$ s.
Figure 7-8 shows the GLR detection decision functions for the same simulation, using a 2 s (100 sample) data window. For the 2 s period immediately following the failure, the GLR detection functions closely resemble the OSGLR decision functions. The elevator decision function increases rapidly, crossing the detection threshold at $t = 10.12$ s.

It is not surprising that the GLR detection time is less than that of the OSGLR test, because the GLR hypotheses can represent the failure mode exactly, whereas the OSGLR hypotheses can only approximate the failure mode. Also, the time scale of the basis functions is $\tau = 3$ s, which is considerably longer than the time required to detect the failure. If the time scale is reduced to $\tau = 0.5$ s, then the detection time for the OSGLR test is the same as for the GLR test. However, it was felt that the longer time scale was desirable to allow for failures that take longer to detect. Furthermore, the detection time of 0.22 s is probably acceptable.

Despite the good performance of the GLR test, the algorithm does display some undesirable characteristics. Note that at $t = 12$ s (2 s after the onset of the failure), the decision functions suddenly level off. This is due, of course, to the finite data window of the GLR test. The GLR algorithm accumulates data for only the length of the data window, which is 2 s long in this case. After that time, information about the failure is lost.

Furthermore, note that $D F_{5}$ exceeds $D F_{1}$ after $t = 33.2$ s. Had the threshold been larger, or the failure been smaller, the GLR test could have isolated the failure to the wrong component, namely, the right flap. The reason for this behavior is again related to the data window. After $t = 12$ s, the actual failure (a step failure at $t = 10$ s) is not one of the failures considered by the GLR test. Therefore, the behavior of the algorithm is unpredictable after $t = 12$ s.

The OSGLR test does not have the undesirable characteristics of the GLR test discussed above. The OSGLR hypotheses can represent the step failure, at least approximately, over a long time period, even though the approximation is somewhat inaccurate for a very short time.
Figure 7-8. GLR detection decision functions for the linear simulation with a -1 deg bias failure of the elevator at time $t = 10$ s. A 2 s (100 sample) data window was used.
period. Over a long time period, the OSGLR test continues to accumulate information about the failure. As a result, the OSGLR test does not display the characteristics of the GLR test which are associated with the finite data window.

7.4.3 Rudder Bias Failure

In this case, a 2 deg bias of the rudder was simulated, using the linear simulation. The failure occurs at time $t = 10$ s of a 50 s simulation.

Figure 7-9 shows the resulting OSGLR detection decision functions. In many ways, this simulation resembles the simulation presented in Section 7.4.2. The decision function corresponding to the failed component, $DF_4$, increases rapidly following the onset of the failure. The other decision functions increase also, but much more slowly than $DF_4$. The failure is detected at $t = 13.06$ s, 3.06 s after the beginning of the failure.

Once again, this result confirms the conclusion of Section 7.3 that rudder failures are easily distinguishable from the other actuator failures. This is demonstrated by the extremely large difference between $DF_4$ and the other decision functions.

Figure 7-10 shows the GLR detection decision functions for this simulation, using a 2 s (100 sample) data window. After the onset of the failure, the rudder decision function, $DF_4$, increases quickly, indicating a failure of the rudder. The other five decision functions increase little, if any. The failure is detected at $t = 11.80$ s, when $DF_4$ exceeds the detection threshold. Shortly thereafter, at $t = 12$ s, $DF_4$ abruptly changes character. At this time, $DF_4$ levels off, except for wide fluctuations due to noise. Again, this behavior is attributable to the finite data window of the GLR test. Note that the detection threshold been only slightly larger, say, $T^2 = 75$, then the detection time would have been greatly increased, from 1.08 s to 7.08 s.

Note that the detection time for the OSGLR test is somewhat longer than for the GLR test. The reason for this is as follows. The step
Figure 7-9. OSGLR detection decision functions for the linear simulation with a 2 deg bias failure of the rudder at time $t = 10$ s.
Figure 7-10. GLR detection decision functions for the linear simulation with a 2 deg bias failure of the rudder at time $t = 10$ s. A 2 s (100 sample) data window was used.
failure in the rudder causes rapid changes in the mean values of some of the Kalman filter residuals (Figure 7-11). Specifically, the first six components and the tenth component of the mean residual vector are nearly discontinuous immediately following the onset of the failure at t = 10 s. The OSGLR hypotheses are unable to accurately represent the discontinuities in the residuals. Therefore, some of the energy of the failure signature cannot be used by the OSGLR algorithm to detect the failure. On the other hand, the GLR hypotheses can represent the failure signature exactly within the data window.

To illustrate this point, Figure 7-12 shows the time histories of the signal-to-noise ratio $d^2$ in the residual process up until time t and the noncentrality parameter of $D_{F4}$, $\delta_4^2$. Ideally, $\delta_4^2$ approximates $d^2$. (See Section 5.3.) In this case, however, there is a significant difference between $d^2$ and $\delta_4^2$ following the failure, due to the inability of the OSGLR hypotheses to represent the failure mode exactly. This accounts for the increased detection time for the OSGLR test, relative to the GLR test.

Notwithstanding the above discussion, the overall performance of the OSGLR test for this case is generally good. Although the detection time is slightly longer for the OSGLR test than for the GLR test, the OSGLR algorithm appears to be more robust than the GLR algorithm.

7.4.4 Right Aileron Bias Failure

In this case, the linear simulation was used to simulate a 1 deg bias failure of the right aileron. The failure occurred at time t = 10 s of a 50 s simulation.

Figure 7-13 shows the OSGLR detection decision functions for this simulation. Immediately following the failure, these four decision functions corresponding to the wing control surfaces ($D_{F2}$, $D_{F3}$, $D_{F5'}$, and $D_{F6}$) begin to rise steadily. To the scale of the plot, these four decision functions cannot be distinguished. The elevator decision function ($D_{F1}$) also rises steadily following the failure. On the plot, $D_{F1}$ appears to be close to the four decision functions of the ailerons and
Figure 7-11. Mean values of the Kalman filter residuals for a 2 deg bias failure of the rudder at $t = 10$ s. (Sheet 1 of 2)
Figure 7-11. Mean values of the Kalman filter residuals for a 2 deg bias failure of the rudder at $t = 10$ s. (Sheet 2 of 2)
Figure 7-12. Noncentrality parameter of the OSGLR detection decision function $D_{F4}$ for a 2 deg bias failure of the rudder at $t = 10$ s.
Figure 7-13. OSGLR detection decision functions for the linear simulation with a 1 deg bias failure of the right aileron at time $t = 10$ s.
flaps. On an absolute scale, however, this difference is large. Finally, the rudder decision function (DF₄) also increases somewhat, although not nearly so much as the other five decision functions. For the detection threshold given, the detection occurs 0.56 s after the onset of the failure.

In order to determine which of the four detection decision functions of the wing control surfaces is largest following the failure, (some of) the OSGLR isolation decision functions are plotted in Figure 7-14. Recall that the isolation decision function DFᵢⱼ is given by

$$DFᵢⱼ = DFᵢ - DFⱼ$$

(7-76)

Shown in the figure are DF₂₃, DF₂₅, and DF₂₆. Following the failure, all three of these isolation decision functions are positive, correctly indicating that the right aileron is the surface that has failed. Also, except for a brief period immediately following the failure,

$$DF₂₆ > DF₂₅ > DF₂₃$$

(7-77)

These results confirm one of the conclusions of Section 7.3, namely, that it is relatively easy to distinguish between failures of the right aileron and the elevator or rudder, but progressively more difficult to distinguish a right aileron failure from the failure of the left flap, the right flap, and the left aileron.

Note that because the failure of the right aileron is barely distinguishable from failures of the other three wing control surfaces, it would be wise to use an isolation threshold for this system to prevent incorrect isolations. We have not attempted to determine an isolation threshold for this study. However, it is clear that an isolation threshold that is large enough to be effective at preventing incorrect isolations will cause a significant delay in the isolation of the failure, perhaps 10 s or more.
Figure 7-14. OSGLR isolation decision functions for the linear simulation with a 1 deg bias failure of the right aileron at time $t = 10$ s.
Figure 7-15 shows the GLR detection decision functions for this simulation. (Note the difference in scale from Figure 7-13.) In many respects, Figure 7-15 resembles Figure 7-13. Immediately following the failure, the aileron and flap decision functions increase rapidly. At \( t = 12 \) s, however, the rate of increase of these decision functions slows, due to the finite data window. To a lesser degree, the elevator and rudder decision functions increase also. Detection occurs 0.48 s after the failure.

Figure 7-16 shows the GLR isolation decision functions \( DF_{23}, DF_{25}, \) and \( DF_{26} \). (Cf. Figure 7-14.) Note that the GLR isolation decision functions are somewhat smaller than the OSGLR isolation decision functions. Once again, this is because the finite data window limits the amount of information that can be accumulated about the failure. Also note that even as late as 7.5 s after the failure, \( DF_{23} \) is occasionally negative, indicating that the failure is in the left aileron, rather than the right aileron. Even at \( t = 26 \) s, 16 s after the failure, \( DF_{23} \) is close to zero. Thus, the behavior of the OSGLR test seems to be more robust for this case.

7.4.5 Rudder Ramp Failure

The failures simulated thus far have all been step bias failures. This puts the GLR test at an advantage, since the GLR test assumes a bias failure mode, whereas the OSGLR test assumes a more general failure mode shape. To see how these tests perform with a different failure mode, a ramp failure of the rudder was simulated, using the linear simulation. The ramp begins at \( t = 10 \) s, and the ramp increases at a rate of 0.1 deg/s.

Figure 7-17 shows the resulting OSGLR decision functions. The decision functions change very little until about \( t = 15 \) s. At that time, the rudder decision function begins to increase, albeit slowly at first. As the magnitude of the failure increases, \( DF_4 \) increases more rapidly. \( DF_4 \) crosses the detection threshold at \( t = 23.08 \) s, so that the time to detection is 13.08 s. Meanwhile, the other five decision
Figure 7-15. GLR detection decision functions for the linear simulation with a 1 deg bias failure of the right aileron at time $t = 10$ s. A 2 s (100 sample) data window was used.
Figure 7-16. GLR isolation decision functions for the linear simulation with a 1 deg bias failure of the right aileron at time $t = 10$ s. A 2 s (100 sample) data window was used.
Figure 7-17. OSGLR detection decision functions for the linear simulation with a 1 deg/s ramp failure of the rudder at time $t = 10$ s.
functions increase only slightly until about \( t = 24 \, \text{s} \), at which time these decision functions begin to increase. Thus, this failure is easily detected using the OSGLR test.

Figure 7-18 shows the GLR detection decision functions for this simulation, using a 2 s (100 sample) data window. This figure is similar to Figure 7-17, except that the decision functions are generally smaller than for the OSGLR test. Also, \( DF_4 \) seems to be quite a bit noisier. The detection time for the GLR test is 18.88 s, which is significantly longer than for the OSGLR test. The major reason for this is that the data window is too short to allow enough data to be accumulated to detect the failure.

To demonstrate this last point, the GLR data window was increased to 5 s (250 samples) (Figure 7-19). In this case, \( DF_4 \) increases more rapidly than with the 2 s data window. The detection time for this case is 14.90 s, which compares favorably with the OSGLR test, although the OSGLR test still performs slightly better than the GLR test in this case.

7.4.6 Nonlinear Simulation with No Failure

This simulation is similar to that of Section 7.4.1, except that the nonlinear simulation was used instead of the linear simulation. In this case, it was necessary to implement the Kalman filter in order to generate the residuals.

Figure 7-20 shows the OSGLR detection decision functions for the nonlinear simulation with no failures. Ideally, the decision functions should resemble those of Figure 7-5. However, the decision functions in this case are quite different. Specifically, the decision functions for the wing control surfaces (\( DF_2 \), \( DF_3 \), \( DF_5 \), and \( DF_6 \)) have peak values of approximately 500, which is a factor of 25 larger than the peaks seen in Figure 7-5. This effect is due to the differences between the nonlinear model and the linearized model generated from it. It was determined that the greatest source of error is due to mismodeling of the aerodynamic moments about the roll axis. This produces a bias in the estimate of roll rate, which is small compared to the standard deviation of the
Figure 7-18. GLR detection decision functions for the linear simulation with a 0.1 deg/s ramp failure of the rudder at time $t = 10$ s. A 2 s (100 sample) data window was used.
Figure 7-19. GLR detection decision functions for the linear simulation with a 0.1 deg/s ramp failure of the rudder at time \( t = 10 \) s. A 5 s (250 sample) data window was used.
Figure 7-20. OSGLR detection decision functions for the nonlinear simulation with no failures. (Sheet 1 of 2)
Figure 7-20. OSGLR detection decision functions for the nonlinear simulation with no failures. (Sheet 2 of 2)
estimation error. However, this bias is integrated by the Kalman filter to produce a very large bias in the estimate of the bank angle. This in turn causes the residual \( \gamma_7(t) \) (the residual associated with the bank angle measurement) to exhibit a significant bias. Because the four wing control surfaces primarily affect the roll axis, the decision functions associated with these surfaces are the ones most affected by this modeling error. The decision function associated with the elevator is also affected, because elevator deflections cause a moment about the roll axis, due to coupling between the longitudinal and the lateral dynamics.

Figure 7-21 shows the GLR detection decision functions for the same simulation. Again, a 2 s (100 sample point) data window was used. As was the case with the OSGLR algorithm, the GLR decision functions are larger for the nonlinear simulation than for the linear simulation. In this case, the decision functions corresponding to the control surfaces on the wing reach peak values of approximately 35. This is much lower than the peak values of the OSGLR decision functions. There are two reasons for this. First, the GLR data window is relatively short. The energy in the residuals due to the biases in the estimation error is correlated over a time period much longer than 2 s. Hence, we might expect that a GLR detector with a longer data window would produce much larger decision functions. In fact, this is the case, as shown by Figure 7-22, which shows the GLR detection decision functions using a 5 s (250 sample) data window. (Note the difference in scale from Figure 7-21.) In this case, the decision functions corresponding to the control surfaces on the wings have peak values of approximately 75, which is about twice as large as the peak values using a 2 s window.

The other reason that the GLR decision functions are smaller than the OSGLR decision functions is more subtle. Essentially, each GLR or OSGLR detector finds the failure input \( f(t) \) which generates a mean process in the residuals that most closely matches the observed residual process. However, the GLR algorithm considers (in this case) only step failures, whereas the OSGLR algorithm considers more general failure modes. Hence, the OSGLR algorithm can find among its hypotheses a failure input time history that matches the observed residuals more closely
Figure 7-21. GLR detection decision functions for the nonlinear simulation with no failures and a 2 s (100 sample) data window. (Sheet 1 of 2)
Figure 7-21. GLR detection decision functions for the nonlinear simulation with no failures and a 2 s (100 sample) data window. (Sheet 2 of 2)
Figure 7-22. GLR detection decision functions for the nonlinear simulation with no failures and a 5 s (250 sample) data window. (Sheet 1 of 2)
Figure 7-22. GLR detection decision functions for the nonlinear simulation with no failures and a 5 s (250 sample) data window. (Sheet 2 of 2)
than does any of the step failures considered by the GLR algorithm. Thus, the same property of the OSGLR algorithm that makes it robust to failure mode uncertainty also makes it more sensitive to modeling errors.

7.4.7 Stuck Elevator

In this case, the nonlinear simulation was used to simulate a stuck elevator. Such a failure might be caused, for example, by a jam of a mechanical linkage. The failure occurred at time $t = 10$ s of a 50 s simulation. Figure 7-23 shows the resulting error in the elevator position, i.e., the difference between the actual elevator position and the commanded elevator position. Note that the history of the error does not fit into any easily characterized category, such as a bias or ramp failure.

Figure 7-24 shows the resulting OSGLR detection decision functions. The OSGLR test performs quite well in this case, despite the complexity of the failure input. Approximately 1 s after the onset of the failure, the elevator decision function, $DF_1$, increases rapidly, indicating a failure. The other five decision functions also increase, but they are always significantly smaller than $DF_1$. For the threshold selected earlier, the failure is detected at $t = 11.94$ s. (Note, however, that this threshold would not be used unless the problem of modeling errors had been addressed.)

Figure 7-25 shows the GLR detection decision functions for this case, using a 2 s (100 sample) data window. For the 3 s immediately following the onset of the failure, the GLR decision functions resemble the OSGLR decision functions. (Cf. Figure 7-24.) $DF_1$ increases rapidly, indicating a failure of the elevator. For the threshold selected, the detection of the failure occurs at $t = 12.14$ s, 2.14 s after the onset of the failure. This is not significantly different from the OSGLR test.

However, this result is somewhat misleading. Note that at about $t = 13$ s, $DF_1$ begins to decrease for the GLR test, until at $t = 15$ s it is at about the same value that it had before the failure. For a brief time, the four decision functions corresponding to the control surfaces
Figure 7-23. Error in elevator deflection due to stuck elevator.
Figure 7-24. OSGLR detection decision functions for the nonlinear simulation with a stuck elevator at time \( t = 10 \) s.
(Sheet 1 of 2)
Figure 7-24. OSGLR detection decision functions for the nonlinear simulation with a stuck elevator at time $t = 10$ s. Note the expanded time scale in the vicinity of $t = 10$ s.
Figure 7-25. GLR detection decision functions for the nonlinear simulation with a stuck elevator at time $t = 10$ s. A 2 s (100 sample) data window was used.
on the wing exceed $DF_1$. At $t = 16$ s, $DF_1$ again increases until $t = 18$ s, where once again $DF_1$ decreases slightly. At about $t = 19$ s, $DF_5$ exceeds $DF_1$ slightly. Shortly thereafter, $DF_1$ begins to increase steadily and is always greater than the other decision functions.

Also, note that if the detection threshold is increased (e.g., to account for modeling errors), then the decision time is likely to increase significantly. For example, if the threshold is set to $T^2 = 200$, then the time of the detection would be $t = 22.52$ s, 12.52 s after the failure. On the other hand, the time to detection for the OSGLR test for this threshold is only 6.82 s.

This behavior is caused by two separate effects. The first is the relatively short (2 s) data window. Thus, when the failure input subsides in the vicinity of $t = 15$ s (Figure 7-23), the GLR decision functions also decrease.

The other reason for this behavior is that the actual failure mode does not agree with any of the hypothesized failure modes of the GLR algorithm. As a result, the results of the test are unpredictable. In particular, $DF_1$ is sometimes less than the other decision functions. As a result, we see that the GLR test is not robust. On the other hand, the failure mode can be represented, at least approximately, by the OSGLR failure hypotheses. As a result, the OSGLR test is robust to failure mode uncertainty, and does not have the undesirable properties of the GLR test displayed in Figure 7-25.

Finally, note that increasing the length of the GLR data window to 5 s (250 samples) improves the performance of the GLR test only slightly (Figure 7-26). Immediately following the failure, the behavior of the decision functions is the same as for the GLR test with a 2 s data window. Following time $t = 12.5$ s, however, $DF_1$ declines only slightly, whereas $DF_5$, $DF_2$, $DF_3$, and $DF_6$ continue to increase. At about $t = 17.5$ s, these four decision functions all exceed $DF_1$. It is not until $t = 19$ s that $DF_1$ again is the maximum decision function. With the threshold $T^2 = 56.86$, the detection time is 2.14 s, which compares favorably with the OSGLR test. However, for the threshold $T^2 = 200,$

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Figure 7-26. GLR detection decision functions for the nonlinear simulation with a stuck elevator at time t = 10 s. A 5 s (250 sample) data window was used.
the detection time is 9.79 s, which is somewhat longer than the detection
time for the OSGLR test with that threshold.

7.4.8 Yaw-Rate Sensor Bias Failure

In Section 5.3, it was suggested that the OSGLR test may not be well-suited to the problem of detecting sensor failures. To illustrate the difficulties involved, the linear simulation was used to simulate a 0.02 rad/s bias failure of the yaw-rate sensor. The bias failure occurs at \( t = 1 \) s of a 5 s simulation.

Figure 7-27 shows the noncentrality parameter, \( \delta_{12}^2 \), of the yaw-rate sensor detection decision function, \( DF_{12} \). Recall that

\[
E[DF_{12}] = \delta_{12}^2 + p
\]  

(7-78)

where \( p \) is the number of basis functions, which in this case is six. The noncentrality parameter is shown rather than the decision function so that the results are not obscured by noise. Also shown in Figure 7-27 is the signal-to-noise ratio, \( d^2 \), of the resulting failure signature. The noncentrality parameter is bounded by

\[
\delta_{12}^2 \leq d^2
\]  

(7-79)

where the equality holds only if the failure mode can be represented exactly by the truncated series expansion of the OSGLR failure hypothesis. Thus, the size of the noncentrality parameter relative to the signal-to-noise ratio gives an indication of how well the failure mode is represented by the truncated series expansion. In particular, it shows how much of the energy in the failure signature cannot be represented. In the figure, \( d^2 \) increases sharply after the failure, and then levels off, although it continues to increase at a moderate rate thereafter. On the other hand, \( \delta_{12}^2 \) is quite small compared to \( d^2 \) immediately following
Figure 7-27. Noncentrality parameter of the OSGLR decision function $DF_{12}$ for a 0.02 rad/s bias failure of the yaw-rate sensor at $t = 1$ s.
the failure. At about \( t = 1.5 \) s, \( \delta_{12}^2 \) begins to increase moderately. After this time, the difference between \( d^2 \) and \( \delta_{12}^2 \) is nearly constant.

This behavior is easily explained. Figure 7-28 shows the mean values of the residuals due to the failure. Immediately following the failure, most of the energy in the failure signature is due to the yaw-rate sensor residual, \( \gamma_6 \). The Kalman filter is a high-pass filter from measurements to residuals. Thus, the step bias failure of the yaw-rate sensor produces a spike in \( \gamma_6 \) that is less than 0.2 s wide. This spike accounts for the large initial jump in the signal-to-noise ratio. As the estimation error is affected by the failed sensor, the other residuals are affected also, although the failure signatures in these residuals are generally smoother than in \( \gamma_6 \). The crux of the problem is that the spike in \( \gamma_6 \) is not well represented by the OSGLR failure hypotheses. Roughly speaking, the energy in the \( \gamma_6 \) residual is concentrated at high frequencies. On the other hand, the other residuals are primarily characterized by low-frequencies, so that they are well-represented by the basis functions. Thus, the OSGLR test ignores the spike in the \( \gamma_6 \) residual because it cannot represent it. This accounts for the difference between \( d^2 \) and \( \delta_{12}^2 \).

Since the failure mode cannot be approximated well using the OSGLR failure hypotheses, the OSGLR test may not be robust to failure mode uncertainty in this case. Specifically, there are two problems: 

1. Because \( \delta_{12}^2 \) is so much smaller than \( d^2 \), detection may be significantly delayed compared to a test that has \textit{a priori} knowledge of the failure mode (such as a GLR test designed specifically for bias failures).

2. It is possible that one of the other decision functions will exhibit a larger response to the failure than does \( DF_{12} \), so that an incorrect isolation is likely. These problems are not unique to the OSGLR test. Any test, including the GLR test, that depends on knowledge of the failure mode will be susceptible to these problems. The problem is more acute for sensor failures than for actuator failures, because high-frequency errors in representing the failure mode of an actuator are
Figure 7-28. Mean values of the Kalman filter residuals for a 0.02 rad/s bias of the yaw-rate sensor at t = 1 s. (Sheet 1 of 2)
Figure 7-28. Mean values of the Kalman filter residuals for a 0.02 rad/s bias of the yaw-rate sensor at t = 1 s. (Sheet 2 of 2)
filtered out by the plant dynamics and the Kalman filter. This is be-
cause the transfer function from inputs to residuals is generally low-
pass. On the other hand, small, high-frequency errors in representing
the failure mode of a sensor are accentuated by the Kalman filter.

In general, it is not possible to deal with the first problem
above. To take advantage of the energy available for failure detection,
knowledge of the shape of the failure signature is required. However, it
may be possible to deal with the second problem above. Two approaches
are possible: (1) Use direct redundancy instead of analytic redundancy;
or (2) Use an FDI algorithm that is inherently robust to failure mode
uncertainty, such as the detection filter of Beard [3] and Jones [35].

7.4.9 Summary

As measured by the time required to detect a failure, it would
appear that the performance of the OSGLR test is roughly comparable to
that of the GLR test. Table 7-4 summarizes the detection times of the
two tests for the control surface failures simulated. Generally, the
performance of the GLR tests is better when the failure mode simulated is
the same as one of those hypothesized in the GLR failure hypotheses. For
other failure modes, the OSGLR test performs better. Nevertheless, the
detection times are comparable for all of the failures.

However, the results presented in Table 7-4 are somewhat mislead-
ing. In almost all of the test cases, the GLR test exhibited nonrobust
behavior. This is caused by two features of the GLR test. First, in
order to make the GLR test computationally feasible, it is necessary to
restrict the hypothesized time of failure by using a data window. As a
result, if detection does not occur when the time of the failure is
within the data window, then information about the failure is lost.
Second, the GLR test assumes a particular failure mode. Consequently,
the behavior of the test is not predictable when a different failure mode
occurs.
Table 7-4. Summary of the detection performance of the OSGLR and GLR tests.

<table>
<thead>
<tr>
<th>Failure Mode Simulated</th>
<th>Time to Detect Failure (s)</th>
<th>GLR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>OSGLR</td>
<td>2 s Data Window</td>
</tr>
<tr>
<td>-1.0 deg Elevator Bias</td>
<td>0.22</td>
<td>0.12</td>
</tr>
<tr>
<td>2.0 deg Rudder Bias</td>
<td>3.06</td>
<td>1.80</td>
</tr>
<tr>
<td>1.0 deg Right Aileron Bias</td>
<td>0.56</td>
<td>0.48</td>
</tr>
<tr>
<td>0.1 deg/s Rudder Ramp</td>
<td>13.08</td>
<td>18.88</td>
</tr>
<tr>
<td>Stuck Elevator</td>
<td>1.94</td>
<td>2.14</td>
</tr>
</tbody>
</table>

On the other hand, the OSGLR test does not exhibit these undesirable characteristics. The test can continue to accumulate information about a failure for as long as the truncated series expansion of the OSGLR failure hypothesis can adequately represent the failure. Also, the series expansion can represent, at least approximately, many different failure modes. Therefore, the OSGLR test is robust to failure mode uncertainty, whereas the GLR test is not.

Furthermore, the OSGLR test requires far less computation than does the GLR test, at least in this case. Table 7-5 summarizes the computational requirements of the two tests. Each number in the table is the ratio of the CPU time required to run a particular test to the length of the simulation for which the test was implemented. The tests were implemented in FORTRAN on a Digital Equipment Corporation VAX 11/780. For each test, six detectors were implemented, corresponding to the six
control surfaces. The GLR test requires 18.5 or 46.2 times more computation than the OSGLR test, depending on whether a 2 s or 5 s data window is used for the GLR test.

Table 7-5. Computational requirements of the OSGLR and GLR algorithms.

<table>
<thead>
<tr>
<th>Ratio of CPU Time to Simulation Time</th>
<th>OSGLR</th>
<th>GLR</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 Basis Functions</td>
<td>2 s Data Window</td>
<td>5 s Data Window</td>
</tr>
<tr>
<td>0.686</td>
<td>12.66</td>
<td>31.66</td>
</tr>
</tbody>
</table>

One disadvantage of the OSGLR test is that it appears to be more sensitive to modeling errors than the GLR test. The reasons for this were explained in Section 7.4.6.

Finally, it appears that the OSGLR test may be unsuitable for detecting and isolating sensor failures, as discussed in Section 7.4.8. However, the same may be said of the GLR test, if the failure mode is not known in advance.
CHAPTER 8

SUMMARY AND RECOMMENDATIONS

8.1 Summary of Thesis

In this thesis, a new failure detection and isolation (FDI) algorithm for linear dynamic systems was developed, which is called the Orthogonal Series Generalized Likelihood Ratio (OSGLR) Test. A primary motivation for the development of the OSGLR test is the fact that other FDI tests, such as the GLR test of Willsky and Jones [57], are computationally burdensome, and may not be robust to failure mode uncertainty.

In Chapter 2, a review of hypothesis testing was given, with special attention paid to the failure detection problem. As explained there, the difficulty with the GLR test and other similar FDI tests can be attributed to the characterization of the failure hypotheses. The GLR failure hypothesis is that a given failure mode occurs at some unknown time. The estimation of the unknown failure time, \( \theta \), generally requires an exhaustive search over all possible failure times, which requires a significant amount of computation. Furthermore, if the actual failure mode is different from the hypothesized failure mode, the results of the test may be unpredictable. In other words, the test is generally not robust to failure mode uncertainty.

The OSGLR test, presented in Chapter 3, is designed to address the problems discussed above. In the characterization of the OSGLR failure hypotheses, the failure modes are represented by a truncated series expansion. Because a broad class of failure modes can be represented in this way, at least approximately, the resulting test can be expected to
be robust to failure mode uncertainty. Furthermore, the unknown coefficients of the series expansion are easily estimated using a linear estimation structure. Therefore, the resulting test is computationally efficient.

In Chapter 4, the performance of the OSGLR test is analyzed. The bulk of the chapter is devoted to the problem of determining the false-alarm performance of the test. Although it was not possible to solve the false-alarm problem for the OSGLR test exactly, it is possible to determine an asymptotic expression for the steady-state false-alarm rate of the continuous-time OSGLR test, which is valid as the threshold becomes large. Based on this result, an asymptotic bound on the steady-state false-alarm rate of the discrete-time OSGLR test is derived. Conditions under which the bound is valid are discussed in Chapter 4. Results are presented which compare the asymptotic approximations to results obtained numerically. In the continuous-time case, the approximation is quite good. In the discrete-time case, the bound is conservative, especially for long sampling intervals. Also in Chapter 4, a simple bound on the detection performance of the OSGLR test is derived.

In Chapter 5, possible basis functions for the OSGLR basis functions are examined. In addition, guidelines for choosing the basis functions are given.

One problem that may be encountered when designing an FDI test for a particular system is that it is difficult to distinguish between different types of failures (i.e., failures of different components). In such systems, it may be difficult for an FDI test to reliably isolate a failure to the correct failure type. In Chapter 6, two measures of distinguishability are developed. The first of these may be interpreted as the distance between two different failure hypotheses. Based on a geometric interpretation of this distance, a second measure of distinguishability is defined, which can be interpreted as the angle between the hypotheses.
Finally, in Chapter 7, the performance of the OSGLR test is compared to that of the GLR test, using linear and nonlinear simulations of a C-130 aircraft.

8.2 Conclusions and Contributions

The primary contribution of this thesis is the presentation of a new failure detection and isolation algorithm for linear dynamic systems, the OSGLR test. As described above, the test is computationally efficient. Furthermore, the test is robust to failure mode uncertainty, at least for actuator failures.

Unfortunately, the OSGLR test does not seem to be well-suited for the problem of detecting and isolating sensor failures. The problem is that the Kalman filter used to generate the measurement residuals accentuates the high-frequency content of the failure mode, so that small changes in the failure mode can produce large changes in the failure signature (i.e., the mean in the residuals). Note, however, that this problem is not unique to the OSGLR test, because any test based on Kalman filter residuals (such as the GLR test) is susceptible to this problem.

Another important contribution is the determination of the steady-state false-alarm of the OSGLR test, using methods of asymptotic analysis. In general, it is quite difficult to determine the false-alarm performance of an FDI test, due to the sequential nature of the failure detection problem. As such, the application of WKB theory [5, Ch. 10] to the problem of determining the false-alarm rate, as described in Chapter 4, is of theoretical interest. Perhaps more importantly, the asymptotic expressions may be used to demonstrate that the OSGLR test meets the false-alarm requirements of a system, at least under idealized conditions.

A third contribution of this thesis is the demonstration of the advantages and limitations of the OSGLR test relative to the more traditional GLR test of Willsky and Jones, when applied to a realistic system.
Simulation results demonstrate that the performance of the OSGLR test is comparable to that of the GLR test, when the performance is measured by the time required to detect a failure. However, the GLR test exhibits nonrobust behavior, which is caused by two features of the GLR test: (1) the finite data window of the GLR test, and (2) the fact that the GLR test assumes a specific failure mode, when other failure modes are possible. On the other hand, the OSGLR performed well for a variety of failure modes.

One conclusion drawn from the simulation results is that the OSGLR test is more sensitive to modeling errors than the GLR test. The reason is that OSGLR test can represent more general failure modes than the GLR test. As a result, the OSGLR can better approximate the modeling errors as a failure than can the GLR test. Hence, the same property of the OSGLR test that makes it robust to failure mode uncertainty also makes it sensitive to modeling errors.

Finally, the distinguishability measures defined in Chapter 6 are an important contribution. These measures may be used to predict whether it is possible to reliably isolate a failure to the failed component. An important feature of the measures is that they are not defined in terms of a particular FDI test. As a result, they are applicable to any system, regardless of the FDI test used.

8.3 Suggestions for Further Work

The results presented in this thesis suggest problems which might be addressed in future research. These problems are outlined below:

(1) A primary motivation for the OSGLR test was to develop an FDI test that is robust to failure mode uncertainty. A related problem is to develop FDI tests that are robust to modeling errors [16, 38, 43]. To be genuinely robust, an FDI test must address both of these problems. One approach would be to incorporate some of the ideas for developing tests that are robust to modeling errors into the OSGLR framework.
In Chapter 4, an asymptotic expression for the false-alarm rate of the continuous-time OSGLR test was derived. The analysis was limited to time-invariant systems operating in steady state. It would be desirable to extend this result to include the time-varying case.

Also in Chapter 4, the false-alarm rate for the discrete-time OSGLR test was bounded, using results from the continuous-time case. The bound is conservative, in that it overstates the false-alarm rate. Also, the bound is not valid under certain conditions. As a practical matter, the OSGLR test must be implemented in discrete time. Therefore, we would like to have an analysis that accurately predicts the false-alarm rate for the discrete time case.

One of the reasons cited in Chapter 3 for using an orthonormal set of basis functions is to ensure that the algorithm is numerically well-conditioned. However, even though the basis functions that represent the failure mode are orthonormal, the resulting basis functions of the failure signature may be highly correlated. In effect, the system and the Kalman filter destroy the conditioning that was imposed on the basis functions. As a result, the number of basis functions that may be used is limited by numerical considerations. This problem should be resolved.

As mentioned previously, the OSGLR test is not particularly well-suited to the problem of detecting sensor failures. Further work is needed to develop an FDI test for sensors that is robust to failure mode uncertainty.

It was not possible to determine a method for evaluating the isolation performance of the OSGLR test. Such a method should be developed, if possible.
In Chapter 4, a simple lower bound on the detection performance of the OSGLR test was given. It would be desirable to develop a more precise method for evaluating detection performance, perhaps by extending the WKB analysis of the false-alarm performance.
APPENDIX A

DERIVATION OF THE DISCRETE-TIME OSGLR ALGORITHM

In this appendix, the discrete-time version of the OSGLR algorithm will be derived. The derivation is very similar to the derivation of the continuous-time algorithm given in Chapter 3.

The OSGLR hypotheses for the unfailed (H₀) and failed (H₁) system are given by

\[
H_0: \quad x(k+1) = \Phi(k)x(k) + w(k), \quad k_0 \leq k \leq k_f \quad (A-1a)
\]
\[
y(k) = C(k)x(k) + v(k), \quad k_0 \leq k \leq k_f \quad (A-1b)
\]

\[
H_1: \quad x(k+1) = \Phi(k)x(k) + w(k) + b(k)f(k), \quad k_0 \leq k \leq k_f \quad (A-2a)
\]
\[
y(k) = C(k)x(k) + v(k) + d(k)f(k), \quad k_0 \leq k \leq k_f \quad (A-2b)
\]

where

- \(x(k)\) is an \(n\)-dimensional state vector
- \(y(k)\) is an \(m\)-dimensional measurement vector
- \(\Phi(k)\) is an \(n\times n\) state transition matrix
- \(C(k)\) is an \(m\times n\) measurement matrix
- \(b(k)\) is a known \(n\)-dimensional vector
- \(d(k)\) is a known \(m\)-dimensional vector
- \(f(k)\) is the failure input
\( \underline{w}(k) \) and \( \underline{v}(k) \) are zero-mean, independent, white Gaussian sequences with covariances given by

\[
E[\underline{w}(k)\underline{w}^T(j)] = Q(k)\delta_{ij} \quad \text{(A-3)}
\]

\[
E[\underline{v}(k)\underline{v}^T(j)] = R(k)\delta_{ij} \quad \text{(A-4)}
\]

where \( \delta_{kj} \) is the Kronecker delta. For both hypotheses, the initial state \( \underline{x}(k_0) \) is a zero-mean Gaussian random vector, with covariance given by

\[
E[\underline{x}(k_0)\underline{x}^T(k_0)] = P_0 \quad \text{(A-5)}
\]

The vectors \( \underline{b}(k) \) and \( \underline{d}(k) \) represent the effect of the failure on the state dynamics and the measurement, respectively. The failure input is not arbitrary, but rather is assumed to be represented by the summation

\[
f(k) = \sum_{i=1}^{P} a_{1i} \phi_{1i}(k) = \underline{\varphi}_1(k)\underline{a}_1 \quad \text{(A-6)}
\]

As a first step toward finding the generalized likelihood ratio (GLR) test for these hypotheses, we must determine the log-likelihood ratio (LLR) for the hypotheses for a given value of \( a_1 \):

\[
l(k_f, a_1) = \ln \frac{p(y(k_0), y(k_0+1), \ldots, y(k_f) \mid H_1, a_1)}{p(y(k_0), y(k_0+1), \ldots, y(k_f) \mid H_0)} \quad \text{(A-7)}
\]

Because the \( y(k) \) are correlated in time, through the state dynamics, it is quite difficult to determine the joint probability density function of the measurements. Therefore, the LLR will not be determined by Eq. (A-7). Instead, the sequence \( y(k) \) will be orthogonalized by filtering
the data using a Kalman filter. The resulting residual sequence, \( y(k) \), is white, i.e., the residual vector at a given time is statistically independent of all the other residual vector at any other time. Therefore, the LLR may be written as

\[
\ell(k_f, a_1) = \ln \frac{p(y(k_0), y(k_0+1), \ldots, y(k_f) \mid H_1, a_1)}{p(y(k_0), y(k_0+1), \ldots, y(k_f) \mid H_0)}
\]

\[
= \sum_{k=k_0}^{k_f} \ln \frac{p(y(k) \mid H_1, a_1)}{p(y(k) \mid H_0)}
\]

(A-8)

The filter equations, based on \( H_0 \), are

\[
\hat{x}^-(k+1) = \phi(k)\hat{x}^+(k) \quad (A-9a)
\]

\[
\hat{x}^+(k) = \hat{x}^-(k) + K(k)y(k) \quad (A-9b)
\]

\[
y(k) = y(k) - C(k)\hat{x}^-(k) \quad (A-9c)
\]

\[
K(k) = P^-(k)C^T(k)M^{-1}(k) \quad (A-9d)
\]

\[
M(k) = C(k)P^-(k)C^T(k) + R(k) \quad (A-9e)
\]

\[
P^-(k+1) = \phi(k)P^+(k)\phi^T(k) + Q(k) \quad (A-9f)
\]

\[
P^+(k) = [I - K(k)C(k)]P^-(k) \quad (A-9g)
\]

The initial condition of the filter is that
\[ \hat{X}(k_0) = 0 \]  \hspace{1cm} \text{(A-10a)}

\[ P^-(k_0) = P_0 \]  \hspace{1cm} \text{(A-10b)}

Under hypothesis \( H_0 \), the residual sequence \( \gamma(k) \) is a zero-mean, white Gaussian sequence with covariance \( M(k) \). Under hypothesis \( H_1 \), the residual sequence can be decomposed as

\[ \gamma(k) = \gamma_0(k) + \gamma_1(k) \]  \hspace{1cm} \text{(A-11)}

where \( \gamma_0(k) \) is the residual sequence that would result under \( H_0 \), and \( \gamma_1(k) \) is the part of the residual due to \( a_1 \). By the linearity of the Kalman filter and state equations, \( \gamma_1(k) \) may be expressed as

\[ \gamma_1(k) = G_1(k)a_1 \]  \hspace{1cm} \text{(A-12)}

where the influence matrix \( G_1(k) \) remains to be determined. Note that \( \gamma_1(k) \) is deterministic once \( G_1(k) \) has been specified. Therefore,

\[ p(\gamma(k) \mid H_0) = \frac{1}{(2\pi)^{P/2}|M(k)|^{1/2}} \exp\left\{ -\frac{1}{2} \gamma \text{T}(k)M^{-1}(k)\gamma(k) \right\} \]  \hspace{1cm} \text{(A-13)}

\[ p(\gamma(k) \mid H_1, a_1) = \frac{1}{(2\pi)^{P/2}|M(k)|^{1/2}} \exp\left\{ -\frac{1}{2} (\gamma(k) - G_1(k)a_1) \text{T}M^{-1}(k)(\gamma(k) - G_1(k)a_1) \right\} \]  \hspace{1cm} \text{(A-14)}

Therefore, the LLR is given by

\[ \ell(k_f, a_1) = \sum_{k=k_0}^{k_f} \left\{ a_1 \text{T}G_1^T(k)M^{-1}(k)\gamma(k) - \frac{1}{2} a_1 \text{T}G_1^T(k)M^{-1}(k)G_1(k)a_1 \right\} \]  \hspace{1cm} \text{(A-15)}

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If we identify the information vector $\chi_1(k_f)$ and the information matrix $S_1(k_f)$ as

$$\chi_1(k_f) = \sum_{k=k_0}^{k_f} \gamma_1(k) G_1^T(k) M^{-1}(k) \gamma_1(k)$$  \hspace{1cm} (A-16)

$$S_1(k_f) = \sum_{k=k_0}^{k_f} G_1^T(k) M^{-1}(k) G_1(k)$$  \hspace{1cm} (A-17)

then the LLR can be expressed as

$$\ell(k_f, a_1) = a_1^T \chi_1(k_f) - \frac{1}{2} a_1^T S_1(k_f) a_1$$  \hspace{1cm} (A-18)

Note that $\chi_1(k_f)$ is a sufficient statistic, just as $\chi_1(t_f)$ was in the continuous-time case. In this case, however, the sufficient statistic is obtained as a byproduct of determining the likelihood ratio. In the continuous-time case, the likelihood ratio could not be found without first determining the sufficient statistic.

The GLR test statistic is defined by

$$\ell(k_f) = \max_{\hat{a}_1} \ell(k_f, \hat{a}_1)$$  \hspace{1cm} (A-19)

Taking the derivative of the right side of Eq. (4-18) with respect to $a_1$ and setting the result to zero yields the value $\hat{a}_1$ that maximizes the LLR. The result is

$$\chi_1(k_f) - S_1(k_f) \hat{a}_1 = 0$$  \hspace{1cm} (A-20)
Therefore, the maximum likelihood estimate of \( \hat{a}_{1} \) is given by

\[
\hat{a}_{1} = S^{-1}(k_f)\chi_1(k_f)
\]  
(A-21)

where it is assumed that \( S^{-1}(k_f) \) exists. The resulting GLR test statistic is then

\[
\lambda(k_f) = \frac{1}{2} \chi_1^T(k_f)S^{-1}(k_f)\chi_1(k_f)
\]  
(A-22)

The GLR test is given by

\[
DF(k_f) = \chi_1^T(k_f)S^{-1}(k_f)\chi_1(k_f) \begin{cases} < 1 & \text{decide } H_1 \\ \geq 1 & \text{decide } H_0 \end{cases}
\]  
(A-23)

where the decision function \( DF(k_f) \) has been defined so as to eliminate the factor of 1/2 from the GLR test statistic (Eq. (A-22)).

The influence matrix \( G_1(k) \) is determined in the same way that \( G_1(t) \) was determined in the continuous-time case. The result is that

\[
G_1(k) = C(k)F_1(k) + d(k)\hat{x}_1^T(k)
\]  
(A-24)

The matrix \( F_1(k) \) represents the influence of \( \hat{a}_{1} \) on the state estimation error, which is defined by

\[
e(k) = \chi(k) - \hat{x}_{-}(k)
\]  
(A-25)

Under \( H_1 \), \( e(k) \) can be decomposed as

\[
e(k) = e_{-0}(k) + F_1(k)\hat{a}_{1}
\]  
(A-26)
where \( e_0(k) \) is the estimation error that would result under \( H_0 \). \( F_1(k) \) satisfies the difference equation

\[
F_1(k+1) = \Phi(k)(I - K(k)C(k))F_1(k) + [b(k) - \Phi(k)K(k)d(k)]\phi_1^T(k)
\]

(A-27)

with initial condition

\[
F_1(k_0) = 0
\]

(A-28)

As in the continuous-time case, the basis vector \( \phi_1(k) \) will be transformed by a time-dependent, invertible transformation \( \Gamma(k_1) \), so that

\[
\phi_2(k, k_1) = \Gamma(k_1)\phi_1(k)
\]

(A-29)

Then the failure input \( f(k) \) can be represented as

\[
f(k) = \phi_2^T(k)\bar{a}_1
\]

\[
= \phi_2^T(k, k_1)\Gamma^{-T}(k_1)\bar{a}_1
\]

\[
= \phi_2^T(k, k_1)a(k_1)
\]

(A-30)

where

\[
\bar{a}(k_1) \overset{\Delta}{=} \Gamma^{-T}(k_1)\bar{a}_1
\]

(A-31)

A difference equation for \( \bar{a}(k_1) \) can be derived as

\[
\bar{a}(k_1+1) = \Gamma^{-T}(k_1+1)\bar{a}_1
\]

\[
= \Gamma^{-T}(k_1+1)\Gamma^T(k_1)a(k_1)
\]
\[
\phi_a(k_1) = \phi_a(k_1) = \phi(k_1) = \phi(k_1)
\]

\[
\phi_a(k_1) = \Gamma - \Gamma^T(k_1+1) \Gamma^T(k_1)
\]

(A-33)

The matrix \( \phi_a(k_1) \) will appear frequently in the remainder of this appendix.

Now, the information vector and information matrix corresponding to the representation of \( f(k) \) in the last line of Eq. (A-30) can be determined directly, since the form of this representation is identical to that given in Eq. (A-6). The result is

\[
\chi_2(k_f, k_1) = \frac{1}{k_f} \sum_{k=k_0}^{k_f} G^T_2(k, k_1) M^{-1}(k) \gamma(k)
\]

(A-34)

\[
\gamma_2(k_f, k_1) = \frac{1}{k_f} \sum_{k=k_0}^{k_f} G^T_2(k, k_1) M^{-1}(k) G_2(k, k_1)
\]

(A-35)

where the influence matrix \( G_2(k, k_1) \) is given by

\[
G_2(k, k_1) = C(k) F_2(k, k_1) + d(k) \phi^T_2(k, k_1)
\]

(A-36)

The estimation error influence matrix \( F_2(k, k_1) \) can be determined by a difference equation similar to Eq. (A-27). However, it will not be necessary to do so. Note that we must have, for any \( a(k_1) \), that

\[
F_2(k, k_1) a(k_1) = F_1(k) a_{-1}
\]

\[
= F_1(k) \Gamma^T(k_1) a(k_1)
\]

(A-37)
Therefore,

$$F_2(k, k_1) = F_1(k)\Gamma(k_1)$$  \hspace{1cm} (A-38)

Similarly,

$$G_2(k, k_1) = G_1(k)\Gamma(k_1)$$  \hspace{1cm} (A-39)

Therefore,

$$\chi_2(k_f, k_1) = \Gamma(k_1)\chi_1(k_f)$$  \hspace{1cm} (A-40)

$$S_2(k_f, k_1) = \Gamma(k_1)S_1(k_f)\Gamma^T(k_1)$$  \hspace{1cm} (A-41)

Of course, the value of the decision function is unchanged by the transformation:

$$DF(k_f) = \chi_1^T(k_f)S_1^{-1}(k_f)\chi_1(k_f)$$

$$= \chi_2^T(k_f, k_1)S_2^{-1}(k_f)\chi_2(k_f, k_1)$$  \hspace{1cm} (A-42)

Finally, we shall be interested only in the case where

$$k_1 = k_f$$

Hence, we will define

$$\chi(k_f) \overset{\Delta}{=} \chi_2(k_f, k_f)$$  \hspace{1cm} (A-43)
\[ S(k_f) \equiv S_2(k_f, k_f) \quad (A-44) \]
\[ G(k_f) \equiv G_2(k_f, k_f) \quad (A-45) \]
\[ F(k_f) \equiv F_2(k_f, k_f) \quad (A-46) \]

The decision function is given by
\[ DF(k_f) = X^T(k_f) S^{-1}(k_f) \chi(k_f) \quad (A-47) \]

It is straightforward to develop difference equations for \( \chi(k) \), \( S(k) \), and \( F(k) \) that will generate \( \chi(k+1) \), \( S(k+1) \), and \( F(k+1) \):
\[ \chi(k+1) = \Gamma(k+1) \chi(k+1) \]
\[ = \Gamma(k+1) \{ \chi(k) + G_1^T(k+1) M^{-1}(k+1) \chi(k+1) \} \]
\[ = \Gamma(k+1) \Gamma^{-1}(k) \chi(k) + G_1^T(k+1) M^{-1}(k+1) \chi(k+1) \]
\[ = \phi_a^{-T}(k) \chi(k) + G^T(k+1) M^{-1}(k+1) \chi(k+1) \quad (A-48) \]

Similarly,
\[ S(k+1) = \Gamma(k+1) S_1(k+1) \Gamma^T(k+1) \]
\[ = \Gamma(k+1) \{ S_1(k) + G_1^T(k+1) M^{-1}(k+1) S_1(k+1) \} \Gamma^T(k+1) \]
\[ = \phi_a^{-T}(k) S(k) \phi_a^{-1}(k) + G^T(k+1) M^{-1}(k+1) S(k+1) G(k+1) \quad (A-49) \]
\[ F(k+1) = F_1(k+1) \Gamma^T(k+1) \]
\[ = \{ \phi(k) [I - K(k) C(k)] F_1(k) + \frac{b(k)}{k} - \phi(k) K(k) d(k) \} \phi_a^T(k) \Gamma^T(k+1) \]
\[ = \{ \phi(k) [I - K(k) C(k)] F(k) + \frac{b(k)}{k} - \phi(k) K(k) d(k) \} \phi_a^T(k+1) \phi_a^{-1}(k) \quad (A-50) \]
The initial conditions are given by

\[ \chi(k_0) = G^T(k_0)M^{-1}(k_0)\chi(k_0) \]  
(A-51)

\[ S(k_0) = G^T(k_0)M^{-1}(k_0)G(k_0) \]  
(A-52)

\[ F(k_0) = 0 \]  
(A-53)

G(k) is given by

\[ G(k) = C(k)F(k) + \mathcal{d}(k)\phi^T_{\phi_2}(k,k) \]  
(A-54)

As in the continuous-time case, the requirement will now be imposed that the vector of basis functions, \( \phi_{\phi_2}(k,k_f) \), be shift-invariant, i.e.,

\[ \phi_{\phi_2}(k,k_f) = \phi(k_f-k) \]  
(A-55)

Combining this with Eq. (A-29), we have that

\[ \phi(k_f-k) = \Gamma(k_f)\phi_1(k) \]  
(A-56)

We also must have that

\[ \phi(k_f+1-k) = \Gamma(k_f+1)\phi_1(k) \]

\[ = \Gamma(k_f+1)\Gamma^{-1}(k_f)\Gamma(k_f)\phi_1(k) \]

\[ = \phi^T_{\phi} \Gamma(k_f)\phi(k_f-k) \]  
(A-57)
Because the left side of Eq. (A-57) is a function only of \( k_f - k \), the right side must be also. Therefore, \( \Phi_a(k_f) \) must be a constant matrix. Therefore, \( \Phi \) satisfies the difference equation

\[
\Phi(i+1) = \Phi \Phi(i) \tag{A-58}
\]

where

\[
\Phi = \Phi_a^{-T} \tag{A-59}
\]

and \( i \) is a dummy index defined by

\[
i = k_f - k \tag{A-60}
\]

Therefore, Eqs. (A-47) through (A-54) describe the discrete-time OSGLR algorithm, if the substitutions

\[
\Phi_2(k,k) \rightarrow \Phi(0) \tag{A-61}
\]

\[
\Phi_a^{-T}(k) \rightarrow \Phi \tag{A-62}
\]

are made. The discrete-time OSGLR equations are summarized in Table A-1.
Table A-1. Summary of discrete-time OSGLR equations.

State estimation error influence matrix propagation:

\[ F(k+1) = \phi(k)[I - K(k)C(k)]F(k) + [b(k) - \phi(k)K(k)d(k)]\phi^T(0)\phi^T \]

\[ F(k_0) = 0 \]

Residual influence matrix:

\[ G(k) = C(k)F(k) + d(k)\phi^T(0) \]

Information vector propagation:

\[ \chi(k+1) = \phi\chi(k) + G^T(k+1)M^{-1}(k+1)\gamma(k+1) \]

\[ \chi(k_0) = G^T(k_0)M^{-1}(k_0)\gamma(k_0) \]

Information matrix propagation:

\[ S(k+1) = \phi S(k)\phi^T + G^T(k+1)M^{-1}(k+1)G(k+1) \]

\[ S(k_0) = G^T(k_0)M^{-1}(k_0)G(k_0) \]

Decision Function:

\[ DF(k) = \chi^T(k)S^{-1}(k)\chi(k) \]
APPENDIX B

SOLUTION OF AN OPTIMIZATION PROBLEM

The problem to solve, as posed in Section 6.1, is the following: Find the cost \( \Delta_{ij}^2 \) corresponding to the solution of the optimization problem

\[
\Delta_{ij}^2 = \min_{f_j(k)} \sum_{k=k_0}^{k_f} \left[ (C(k)e(k) + d_j(k)f_j(k) - m_l(k))^T M^{-1}(k) \right. \\
\left. \frac{\partial}{\partial f_j(k)} (C(k)e(k) + d_j(k)f_j(k) - m_l(k)) \right] \quad (B-1)
\]

where the state \( e(k) \) satisfies the state equation

\[
e(k+1) = \Phi_{KF}(k)e(k) + h_{KF}(k)f_j(k) \\
e(k_0) = 0 \quad (B-2)
\]

This optimization problem may be interpreted in two ways. The obvious interpretation is as an optimal control problem. The problem may also be interpreted as an optimal filtering problem. The first interpretation leads to a backward recursion for the solution; the second leads to a forward recursion. Of course, both methods give the same value for \( \Delta_{ij} \).
B.1 Backward Recursion

The optimization problem is very much like the optimal tracking problem discussed in a number of references (e.g., [36]), except for the presence of cross-weighting terms between the state $\underline{e}(k)$ and the control $f_j(k)$ in Eq. (B-1). Because the solution to this particular problem does not seem to be readily available elsewhere, the solution will be derived here.

There are two approaches that could be used to solve this problem: the variational approach [10] and dynamic programming [4]. Both approach of course lead to the same control law. However, dynamic programming yields as a byproduct the cost $\Delta_{ij}^2$, which corresponds to that control law. Because we are more interested in the cost itself rather than the actual control used to achieve that cost, dynamic programming is the approach that will be used here.

To begin, the optimal value function $V(\underline{e}(n), n)$ is defined by

$$V(\underline{e}(n), n) = \min_{f_j(k)} \sum_{k=n}^{k_f} \left[ C(k)\underline{e}(k) + \frac{d_j(k)f_j(k)}{f_j(k)} - m_j(k) \right] M^{-1}(k) \cdot \left[ C(k)\underline{e}(k) + \frac{d_j(k)f_j(k)}{f_j(k)} - m_j(k) \right]$$

(B-4)

where $\underline{e}(k)$ satisfies the state Eq. (B-2) for $k \geq n$ with initial condition $\underline{e}(n)$. Then clearly,

$$\Delta_{ij}^2 = V(0, k_0)$$

(B-5)

So if we can solve for the optimal value function for each $\underline{e}(n)$ and $n$, we can solve as a special case the originally stated problem.

To find the optimal value function, we use the basic recursion of dynamic programming to work backward from the final time. The recursion for this problem is
\[ V(\underline{e}(n), n) = \min_{f_j(n)} \left\{ \left[ (C(n)\underline{e}(n) + d_j(n)f_j(n) - \underline{m}_j(n)) \right]^T M^{-1}(n) \cdot \left[ (C(n)\underline{e}(n) + d_j(n)f_j(n) - \underline{m}_j(n)) \right] + V(\underline{e}(n+1), n+1) \right\} \] (B-6)

where \( \underline{e}(n+1) \) is given by

\[ \underline{e}(n+1) = \phi_{KF}(n)\underline{e}(n) + b_{-KF}(n)f_j(n) \] (B-7)

It will be shown that the optimal value function is a quadratic polynomial in the state \( \underline{e}(n) \). This will be proved by induction, by demonstrating that

1. If \( V(\underline{e}(n+1), n+1) \) is a quadratic polynomial in \( \underline{e}(n+1) \), then \( V(\underline{e}(n), n) \) is a quadratic polynomial in \( \underline{e}(n) \).
2. \( V(\underline{e}(k_f), k_f) \) is a quadratic polynomial in \( \underline{e}(k_f) \).

To demonstrate the first point, we assume that

\[ V(\underline{e}(n+1), n+1) = \underline{e}(n+1)^T P(n+1)\underline{e}(n+1) + 2\underline{g}(n+1)^T \underline{e}(n+1) + h(n+1) \] (B-8)

Then

\[ V(\underline{e}(n), n) = \min_{f_j(n)} \left\{ \left[ (C(n)\underline{e}(n) + d_j(n)f_j(n) - \underline{m}_j(n)) \right]^T M^{-1}(n) \cdot \left[ (C(n)\underline{e}(n) + d_j(n)f_j(n) - \underline{m}_j(n)) \right] + \underline{e}(n+1)^T P(n+1)\underline{e}(n+1) + 2\underline{g}(n+1)^T \underline{e}(n+1) + h(n+1) \right\} \] (B-9)
Using the state equation (B-7), we have that

\[
V(e(n), n) = \min_{f_j(n)} \left\{ \left[ (C(n)e(n) + d_j(n)\phi_f(n) - m_1(n)) \right]^{T} M^{-1}(n) \cdot \left[ (C(n)e(n) + d_j(n)\phi_f(n) - m_1(n)) \right] \\
+ \left[ \phi_{KF}(n)e(n) + b_{KF}(n)f_j(n) \right]^{T} P(n+1) \left[ \phi_{KF}(n)e(n) + b_{KF}(n)f_j(n) \right] \\
+ 2d_j(n)g(n+1) \left[ \phi_{KF}(n)e(n) + b_{KF}(n)f_j(n) \right] + h(n+1) \right\}
\]

(B-10)

To find the \( f_j(n) \) which minimizes the above expression, the term in brackets is differentiated with respect to \( f_j(n) \) and set to zero. The result is, after cancelling the common factor of 2,

\[
\frac{d_j^{T}(n)M^{-1}(n)[C(n)e(n) + d_j(n)f_j(n) - m_1(n)]}{
+ \frac{b_{KF}^{T}(n)P(n+1)[\phi_{KF}(n)e(n) + b_{KF}(n)f_j(n)] + b_{KF}^{T}(n)g(n+1)}{}} = 0
\]

(B-11)

Hence, the optimal control at time \( n \) is given by

\[
f_j(n) = -K_1(n)e(n) + K_2(n)m_1(n) - K_3(n)g(n+1)
\]

(B-12)

where

\[
K_1(n) = \left[ \frac{d_j^{T}(n)M^{-1}(n)d_j(n) + b_{KF}^{T}(n)P(n+1)b_{KF}(n)}{}} \right]^{-1} \cdot \left[ \frac{d_j^{T}(n)M^{-1}(n)C(n) + b_{KF}^{T}(n)P(n+1)\phi_{KF}(n)}{}} \right]
\]

(B-13)
\[ K_2(n) = \left[ d_j^T(n)M^{-1}(n)d_j(n) + b_{KF}^T(n)P(n+1)b_{KF}(n) \right]^{-1} d_j^T(n)M^{-1}(n) \] (B-14)

\[ K_3(n) = \left[ d_j^T(n)M^{-1}(n)d_j(n) + b_{KF}^T(n)P(n+1)b_{KF}(n) \right]^{-1} b_{KF}^T(n) \] (B-15)

Substituting this expression for \( f_j(n) \) back into Eq. (B-10) yields the optimal value function for time \( n \):

\[ V(\underline{e}(n),n) = \]

\[ \left[ \begin{bmatrix} C(n) - d_j(n)K_1(n)\underline{e}(n) + d_j(n)K_2(n) - I \end{bmatrix} M_1(n) - d_j(n)K_3(n)g(n+1) \right]^T \]

\[ - \cdot \left[ \begin{bmatrix} C(n) - d_j(n)K_1(n)\underline{e}(n) + d_j(n)K_2(n) - I \end{bmatrix} M_1(n) - d_j(n)K_3(n)g(n+1) \right] \]

\[ + \left[ \begin{bmatrix} \Phi_{KF}(n) - b_{KF}(n)K_1(n)\underline{e}(n) + b_{KF}(n)K_2(n)m_1(n) - b_{KF}(n)K_3(n)g(n+1) \end{bmatrix} \right]^T \]

\[ - \cdot \left[ \begin{bmatrix} \Phi_{KF}(n) - b_{KF}(n)K_1(n)\underline{e}(n) + b_{KF}(n)K_2(n)m_1(n) - b_{KF}(n)K_3(n)g(n+1) \end{bmatrix} \right] \]

\[ + 2\underline{g}(n+1) \]

\[ + \left[ \begin{bmatrix} \Phi_{KF}(n) - b_{KF}(n)K_1(n)\underline{e}(n) + b_{KF}(n)K_2(n)m_1(n) - b_{KF}(n)K_3(n)g(n+1) \end{bmatrix} \right]^T \]

\[ + h(n+1) \] (B-16)

Clearly, \( V(\underline{e}(n),n) \) is a quadratic in \( \underline{e}(n) \). Therefore, we can find \( P(n) \), \( g(n) \), and \( h(n) \) by identifying terms in powers of \( \underline{e}(n) \). The results are:
\[ P(n) = \left[ C(n) - \frac{d_j(n)K_1(n)}{M^{-1}(n)} \right]^T M^{-1}(n) \left[ C(n) - \frac{d_j(n)K_1(n)}{M^{-1}(n)} \right] \]
\[ + \left[ \Phi_{TF}(n) - \frac{b_{KF}(n)K_1(n)}{M^{-1}(n)} \right]^T P(n+1) \left[ \Phi_{TF}(n) - \frac{b_{KF}(n)K_1(n)}{M^{-1}(n)} \right] \]
\[ = \Phi_{KF}^T(n) P(n+1) \Phi_{KF}(n) + C^T(n) M^{-1}(n) C(n) \]
\[ - \left[ C^T(n) M^{-1}(n) \frac{d_j(n)}{M^{-1}(n)} + \Phi_{KF}^T(n) P(n+1) \frac{b_{KF}(n)}{M^{-1}(n)} \right] \]
\[ \cdot \left[ \frac{d_j(n) M^{-1}(n)}{M^{-1}(n)} + \frac{b_{KF}(n) P(n+1) b_{KF}(n)}{M^{-1}(n)} \right]^{-1} \]
\[ \cdot \left[ C^T(n) M^{-1}(n) \frac{d_j(n)}{M^{-1}(n)} + \Phi_{KF}^T(n) P(n+1) \frac{b_{KF}(n)}{M^{-1}(n)} \right]^T \]  \hspace{1cm} \text{(B-17)}

\[ q(n) = \left[ C(n) - \frac{d_j(n)K_1(n)}{M^{-1}(n)} \right]^T M^{-1}(n) \]
\[ \cdot \left[ \left( \frac{d_j(n)K_2(n) - I}{m_1(n)} - \frac{d_j(n)K_3(n)q(n+1)}{M^{-1}(n)} \right) \right] \]
\[ + \left[ \Phi_{KF}(n) - \frac{b_{KF}(n)K_1(n)}{M^{-1}(n)} \right]^T P(n+1) \]
\[ \cdot \left[ \frac{b_{KF}(n)K_2(n) - I}{m_1(n)} - \frac{b_{KF}(n)K_3(n)q(n+1)}{M^{-1}(n)} \right] \]
\[ + \left[ \Phi_{KF}(n) - \frac{b_{KF}(n)K_1(n)}{M^{-1}(n)} \right]^T q(n+1) \]
\[ = \left\{ \Phi_{KF}^T(n) - \left[ C^T(n) M^{-1}(n) \frac{d_j(n)}{M^{-1}(n)} + \Phi_{KF}^T(n) P(n+1) \frac{b_{KF}(n)}{M^{-1}(n)} \right] \right\} \]
\[ \cdot q(n+1) \]
\[ + \left\{ \left[ C^T(n) M^{-1}(n) \frac{d_j(n)}{M^{-1}(n)} + \Phi_{KF}^T(n) P(n+1) \frac{b_{KF}(n)}{M^{-1}(n)} \right] \right\} \]
\[ \cdot \left[ \frac{d_j(n) M^{-1}(n)}{M^{-1}(n)} + \frac{b_{KF}(n) P(n+1) b_{KF}(n)}{M^{-1}(n)} \right]^{-1} \]
\[ \cdot \left[ C^T(n) M^{-1}(n) \frac{d_j(n)}{M^{-1}(n)} + \Phi_{KF}^T(n) P(n+1) \frac{b_{KF}(n)}{M^{-1}(n)} \right]^T \]
\[ - C^T(n) M^{-1}(n) \right\} m_1(n) \]  \hspace{1cm} \text{(B-18)
\begin{align*}
h(n) &= h(n+1) + \left[ \begin{bmatrix} d_j(n)K_2(n) - I \end{bmatrix} & m_i(n) - d_j(n)K_3(n)g(n+1) \right]^T M^{-1}(n) \\
&\quad \cdot \left[ \begin{bmatrix} d_j(n)K_2(n) - I \end{bmatrix} m_i(n) - d_j(n)K_3(n)g(n+1) \right] \\
&\quad + \left[ b_{KF}(n)K_2(n)m_i(n) - b_{KF}(n)K_3(n)g(n+1) \right]^T P(n+1) \\
&\quad \cdot \left[ b_{KF}(n)K_2(n)m_i(n) - b_{KF}(n)K_3(n)g(n+1) \right] \\
&\quad + 2g_T(n+1) \left[ b_{KF}(n)K_2(n)m_i(n) - b_{KF}(n)K_3(n)g(n+1) \right] \\
&= h(n+1) \\
&\quad + m_i^T(n) \left[ M^{-1}(n) - M^{-1}(n)d_j(n)[d_j^T(n)M^{-1}(n)d_j(n) + b_{KF}^T(n)P(n+1)b_{KF}(n)]^{-1} \\
&\quad \cdot d_j^T(n)M^{-1}(n) \right] m_i(n) \\
&\quad - g_T(n+1) b_{KF}^T(n) \left[ d_j^T(n)M^{-1}(n)d_j(n) + b_{KF}^T(n)P(n+1)b_{KF}(n) \right]^{-1} \\
&\quad \cdot b_{KF}(n)g(n+1) \\
&\quad + 2m_i^T(n)M^{-1}(n)d_j(n) \left[ d_j^T(n)M^{-1}(n)d_j(n) + b_{KF}^T(n)P(n+1)b_{KF}(n) \right]^{-1} \\
&\quad \cdot b_{KF}(n)g(n+1) \\
\end{align*}

(B-19)

Finally, we must show that the optimal value function at time \( k_f \) is a quadratic polynomial in \( e(k_f) \). The optimal value function at \( n = k_f \) is given by

\begin{align*}
V(e(k_f), k_f) &= \min_{j \in \mathcal{J}(k_f)} \left[ C(k_f)e(k_f) + d_j(k_f)f_j(k_f) - m_i(k_f) \right]^T M^{-1}(k_f) \\
&\quad \cdot \left[ C(k_f)e(k_f) + d_j(k_f)f_j(k_f) - m_i(k_f) \right] \\
\end{align*}

(B-20)

There are two cases to consider. If \( d_j(k_f) \) is the zero vector, then there is no minimization to perform, and the optimal value function at \( k_f \) is simply
\[
V(e(k_f), k_f) = [C(k_f)e(k_f) - \underline{m}_1(k_f)]^T M^{-1}(k_f) [C(k_f)e(k_f) - \underline{m}_1(k_f)] \\
(B-21)
\]

Therefore, the optimal value function is a quadratic in \(e(k_f)\), with

\[
P(k_f) = C^T(k_f)M^{-1}(k_f)C(k_f) \\
(B-22)
\]

\[
g(k_f) = -C^T(k_f)M^{-1}(k_f)\underline{m}_1(k_f) \\
(B-23)
\]

\[
h(k_f) = \underline{m}_1^T(k_f)M^{-1}(k_f)\underline{m}_1(k_f) \\
(B-24)
\]

If \(d_j(k_f)\) is nonzero, then the minimization indicated must indeed be performed. Differentiating the term in brackets in Eq. (B-20) with respect to \(f_j(k_f)\) and setting the result to zero yields

\[
2d_j^T(k_f)M^{-1}(k_f)[C(k_f)e(k_f) + d_j(k_f)f_j(k_f) - \underline{m}_1(k_f)] = 0 \\
(B-25)
\]

Therefore, the required control is simply

\[
f_j(k_f) = -[d_j^T(k_f)M^{-1}(k_f)d_j(k_f)]^{-1}[d_j^T(k_f)M^{-1}(k_f)[C(k_f)e(k_f) - \underline{m}_1(k_f)]] \\
(B-26)
\]

Substituting this expression for \(f_j(k_f)\) back into Eq. (B-20) yields the optimal value function at time \(k_f\):
\[
V(e(k_f), k_f) = \{ [I - \frac{d_j}{d_j}(k_f)] [\frac{d_j^T}{d_j}(k_f) M^{-1}(k_f) \frac{d_j}{d_j}(k_f)]^{-1} \frac{d_j}{d_j}(k_f) M^{-1}(k_f) \} \\
\cdot [C(k_f) e(k_f) - \frac{m_i}{m_i}(k_f)]^T M^{-1}(k_f) \\
\cdot \{ [I - \frac{d_j}{d_j}(k_f)] [\frac{d_j^T}{d_j}(k_f) M^{-1}(k_f) \frac{d_j}{d_j}(k_f)]^{-1} \frac{d_j}{d_j}(k_f) M^{-1}(k_f) \} \\
\cdot [C(k_f) e(k_f) - \frac{m_i}{m_i}(k_f)] \\
= [C(k_f) e(k_f) - \frac{m_i}{m_i}(k_f)]^T \\
\cdot [M^{-1}(k_f) - M^{-1}(k_f) \frac{d_j}{d_j}(k_f) [\frac{d_j^T}{d_j}(k_f) M^{-1}(k_f) \frac{d_j}{d_j}(k_f)]^{-1} \frac{d_j^T}{d_j}(k_f) M^{-1}(k_f)] \\
\cdot [C(k_f) e(k_f) - \frac{m_i}{m_i}(k_f)] \\
\tag{B-27}
\]

which is a quadratic in \( e(k_f) \). For this case,

\[
P(k_f) = C^T(k_f) [M^{-1}(k_f) - M^{-1}(k_f) \frac{d_j}{d_j}(k_f) [\frac{d_j^T}{d_j}(k_f) M^{-1}(k_f) \frac{d_j}{d_j}(k_f)]^{-1} \\
\cdot \frac{d_j^T}{d_j}(k_f) M^{-1}(k_f)] C(k_f) \\
\tag{B-28}
\]

\[
g(k_f) = -C^T(k_f) [M^{-1}(k_f) - M^{-1}(k_f) \frac{d_j}{d_j}(k_f) [\frac{d_j^T}{d_j}(k_f) M^{-1}(k_f) \frac{d_j}{d_j}(k_f)]^{-1} \\
\cdot \frac{d_j^T}{d_j}(k_f) M^{-1}(k_f)] m_i(k_f) \\
\tag{B-29}
\]

\[
h(k_f) = m_i^T(k_f) [M^{-1}(k_f) - M^{-1}(k_f) \frac{d_j}{d_j}(k_f) [\frac{d_j^T}{d_j}(k_f) M^{-1}(k_f) \frac{d_j}{d_j}(k_f)]^{-1} \\
\cdot \frac{d_j^T}{d_j}(k_f) M^{-1}(k_f)] m_i(k_f) \\
\tag{B-30}
\]

In summary, \( \Delta^2_{ij} \) may be calculated as follows: \( P(k_f) \), \( g(k_f) \), and \( h(k_f) \) are initialized using Eqs. (B-22) through (B-24) or (B-28) through (B-30), depending on whether \( d_j(k_f) \) is zero or nonzero, respectively. Then \( P(n) \), \( g(n) \), and \( h(n) \) are propagated backward from time \( n = k_f \) to time \( n = k_0 \), using Eqs. (B-17) through (B-19). Finally, \( \Delta^2_{ij} \) is given by
\[ \Delta^2_{ij} = V(\overline{0}, k_0) \]
\[ = h(k_0) \]  
(B-31)

B.2 Forward Recursion

In some cases, it would be desirable to solve the optimization problem of Eq. (B-1) by a forward recursion, rather than by a backward recursion, as developed in Section B.1. For example, suppose that we want to see how the distinguishability measure \( \Delta_{ij}(f_i(\cdot), k_f) \) evolves after a failure. That is, suppose a failure occurs at time \( \theta \). Then we want to determine \( \Delta_{ij}(f_i(\cdot), \theta) \), \( \Delta_{ij}(f_i(\cdot), \theta+1) \), etc. With the backward recursion, the entire optimization problem must be solved for each value of \( k_f \). With a forward recursion, the results of the optimization to determine \( \Delta_{ij}(f_i(\cdot), k) \) can be used to determine \( \Delta_{ij}(f_i(\cdot), k+1) \) in a single step. Thus, a forward recursion is much more efficient than a backward recursion.

As it turns out, the optimization problem of Eq. (B-1) has the same form as the weighted least-squares formulation of the Kalman filter [46, Sec. 6.5]. In this case, \( f_j(k) \) corresponds to the process noise, and \( m_i(k) \) corresponds to the measurement. Therefore, the solution of the optimal filtering problem gives the solution to the problem stated here. The resulting cost is given by

\[ \Delta^2_{ij} = \sum_{k=k_0}^{k_f} Y_e^T(k) M_e^{-1}(k) Y_e(k) \]  
(B-32)

where \( Y_e(k) \) is the residual of the Kalman filter for estimating \( e(k) \), and \( M_e(k) \) is the covariance of \( Y_e(k) \). (Note that because the problem is deterministic, the word "covariance" is not really appropriate. However, this slight inconsistency in nomenclature should not be confusing.)

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A slight difficulty with the above approach is that there is no weight on \( f_j(k) \) in the cost function, except through the state and measurement equations. In other words, the covariance of \( f_j(k) \) is infinite. Therefore, the Kalman gain matrix must be determined by a limiting process, the details of which are omitted here.

There are two cases to consider, depending on whether or not \( d_j(k) \) is zero. These two cases correspond to the two different sets of initial conditions in the backward recursion. The results are summarized in the next two subsections.

B.2.1 Zero \( d_j(k) \)

Suppose that we have the estimate \( \hat{e}^+(k) \) of \( e(k) \) and covariance matrix \( P^+_e(k) \). The state estimate is propagated via

\[
\hat{e}^-(k+1) = \Phi_{KF}(k)\hat{e}^+(k)
\]  \hspace{1cm} (B-33)

and the estimation error covariance is propagated via

\[
P^-_e(k+1) = \Phi_{KF}(k)P_e(k)\Phi_{KF}(k)^T
\]  \hspace{1cm} (B-34)

(Note that the covariance \( P^-_e(k+1) \) does not include the effect of \( f_j(k) \). If it did, then \( P^-_e(k+1) \) would be infinite in the direction of \( b_{KF}(k) \). Thus, the definition of \( P^-_e(k+1) \) differs slightly from the usual definition.)

The measurement residual at time \( k \) is given by

\[
\gamma_e(k) = m^i(k) - C(k)\hat{e}^-(k)
\]  \hspace{1cm} (B-35)
If \( f_j(k-1) \) were zero, then the covariance of \( \gamma_e(k) \) would be given by

\[
M_1(k) = C(k)P_e^-(k)C^T(k) + M(k)
\] (B-36)

Because \( f_j(k-1) \) has infinite variance, the covariance of \( \gamma_e(k) \) is infinite. However, the inverse of the covariance is finite, and is given by

\[
M_{e}^{-1}(k) = M_{1}^{-1}(k)
- M_{1}^{-1}(k)C(k)b_{KF}^T(k-1)[b_{KF}^T(k-1)C^T(k)M_{1}^{-1}(k)C(k)b_{KF}(k-1)]^{-1}
\cdot b_{KF}^T(k-1)C^T(k)M_{1}^{-1}(k)
\] (B-37)

That is, \( M_{e}^{-1}(k) \) is \( M_{1}^{-1}(k) \), less any component along the direction \( C(k)b_{KF}(k-1) \). The estimate of \( \gamma_e(k) \) is updated by

\[
\hat{\gamma}_e^+(k) = \hat{\gamma}_e^-(k) + K_e(k)\gamma_e(k)
\] (B-38)

where the Kalman gain matrix is given by

\[
K_e(k) = P_e^-(k)C^T(k)M_{e}^{-1}(k) + b_{KF}^T(k-1)[b_{KF}^T(k-1)C^T(k)M_{1}^{-1}(k)C(k)b_{KF}(k-1)]^{-1}
\cdot b_{KF}^T(k-1)C^T(k)M_{1}^{-1}(k)
\] (B-39)

The two terms in the Kalman gain matrix have natural interpretations. The first term is simply the gain matrix for a filter with residual covariance \( M_{e}^{-1}(k) \) and estimation error covariance \( P_e^-(k) \). The second term,
when multiplied by $y_e(k)$, is $b_{k\gamma}(k-1)$ times the estimate of $f_j(k-1)$.

Finally, the covariance is updated by

$$P_e^+(k) = [I - K_e(k)C(k)]P_e^-(k)[I - K_e(k)C(k)]^T + K_e(k)M(k)K_e^T(k)$$  \hspace{1cm} (B-40)

Because the initial state $e(k_0)$ is known (Eq. (B-3)), we have that

$$\hat{e}^-_e(k_0) = \hat{e}^+_e(k_0) = 0$$  \hspace{1cm} (B-41)

$$P_e^-(k_0) = P_e^+(k_0) = 0$$  \hspace{1cm} (B-42)

Also,

$$M_e^{-1}(k_0) = M_e^{-1}(k_0)$$  \hspace{1cm} (B-43)

B.2.2 Nonzero $d_j(k)$

The results for this case may be obtained by augmenting the state vector to include $f_j(k)$ and using the results of the previous subsection. The results are summarized below.

First, suppose that we have the estimates $\hat{e}_e^+(k)$ and $\hat{f}_j^+(k)$. Then the state estimate is propagated by

$$\hat{e}^-_e(k+1) = \phi_{k\gamma}(k)\hat{e}^+_e(k) + b_{k\gamma}(k)\hat{f}^+_j(k)$$  \hspace{1cm} (B-44)
The covariance of the estimation error is propagated by

$$
P_e^-(k+1) = \Phi_{KP}(k)P_e^+(k)\Phi_{KP}^T(k) + b_{KP}(k)b_{KP}^T(k) + \Phi_{KF}(k)P_{ef}(k)b_{KF}^T(k) + [\Phi_{KF}(k)P_{ef}(k)b_{KF}(k)]^T \tag{B-45}
$$

The residual is given by

$$
\gamma_e(k) = \bar{m}_j(k) - C(k)\hat{e}^-(k) \tag{B-46}
$$

If $f_j(k)$ were zero, then the covariance of $\gamma_e(k)$ would be given by

$$
M_1(k) = C(k)P_e^-(k)C^T(k) + M(k) \tag{B-47}
$$

Because $f_j(k)$ has infinite variance, we have that

$$
M_e^{-1}(k) = M_1^{-1}(k) - M_1^{-1}(k)d_j(k)[d_j^T(k)M_1^{-1}(k)d_j(k)]^{-1}d_j^T(k)M_1^{-1}(k) \tag{B-48}
$$

The estimate of $\hat{e}(k)$ is updated by

$$
\hat{e}^+(k) = \hat{e}^-(k) + K_e(k)\gamma_e(k) \tag{B-49}
$$

where

$$
K_e(k) = P_e^-(k)C^T(k)M_e^{-1}(k) \tag{B-50}
$$

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The estimate of \( \hat{f}_j(k) \) is given by

\[
\hat{f}_j^+(k) = K_f(k) \gamma_e(k) \tag{B-51}
\]

where

\[
K_f(k) = \left[ d^T(k)M_1^{-1}(k)d(k) \right]^{-1} d^T(k)M_1^{-1}(k) \tag{B-52}
\]

Finally, the covariance is updated by

\[
P_e^+(k) = [I - K_e(k)C(k)]P_e^-(k)[I - K_e(k)C(k)]^T + K_e(k)M(k)K_e^T(k) \tag{B-53}
\]

\[
P_f^+(k) = \left[ d_j^T(k)M_1^{-1}(k)d_j(k) \right]^{-1} \tag{B-54}
\]

\[
P_{ef}^+(k) = -P_e^-(k)C(k)K_f^T(k) \tag{B-55}
\]

Because the initial state \( \hat{e}(k_0) \) is known, we have that

\[
\hat{e}^-(k_0) = \hat{e}^+(k_0) = 0 \tag{B-56}
\]

\[
P_e^-(k_0) = P_e^+(k_0) = 0 \tag{B-57}
\]

Therefore,

\[
M_1(k_0) = M(k_0) \tag{B-58}
\]

\( M_e(k_0) \) is determined by Eq. (B-48).
APPENDIX C

C-130 LINEAR MODELS

This Appendix contains the linear models of the C-130 aircraft described in Chapter 7.

- Table C-1 presents the nominal operating point about which the system is linearized. The nominal operating point corresponds to the aircraft cruising at an altitude of 1000 ft with an airspeed of 150 knots.

- Table C-2 presents the linearized system matrices in continuous time. The system equations are

\[
\frac{dx_p(t)}{dt} = A_p(x_p(t) - x_{p0}) + B_{p1}(u(t) - u_0) + B_{p2}y(t) + \dot{x}_{p0}
\]

\[
y(t) = C_p(x_p(t) - x_{p0}) + D_{p1}(u(t) - u_0) + D_{p2}y(t) + y_0
\]

- Table C-3 presents the matrices of the discrete-time state equation. The state equation is given by

\[
x_p(k+1) = \phi_p(x_p(k) - x_{p0}) + B_{p1}(u(k) - u_0) + B_{p2}y(k) + \Delta x
\]
• Table C-4 presents the matrices of the turbulence model. The turbulence model is

\[ x_t(k+1) = \Phi_t x_t(k) + \gamma_{td}(k) \]

\[ y_t(k) = C_t x_t(k) \]

where \( \gamma_{td}(k) \) is a zero-mean, white Gaussian sequence with covariance \( Q_t \). The turbulence level for this model is

\[ \sigma_w = 6.5 \text{ ft/s} \]

• Table C-5 presents the Kalman filter matrices, including the gain matrix (K), the estimation error covariance (P^-), and the residual covariance (M).
### Table C-1. Nominal operating point.

\[
\begin{align*}
\mathbf{x}_{p_0} &= \begin{bmatrix}
2.531715 \times 10^2 \\
1.010324 \times 10^1 \\
9.99999999 \times 10^8 \\
9.99999999 \times 10^8 \\
9.99999999 \times 10^8 \\
-3.729332 \times 10^{-3} \\
1.819324 \times 10^{-1} \\
9.99999999 \times 10^8 \\
1.000000 \times 10^3 
\end{bmatrix} \\
\mathbf{y}_0 &= \begin{bmatrix}
2.531715 \times 10^2 \\
1.193584 \times 10^1 \\
-3.200513 \times 10^1 \\
0.000000 \times 10^0 \\
0.000000 \times 10^0 \\
-3.729332 \times 10^0 \\
1.819324 \times 10^{-1} \\
0.000000 \times 10^0 \\
1.750290 \times 10^{-4} \\
1.000000 \times 10^3 
\end{bmatrix} \\
\mathbf{u}_0 &= \begin{bmatrix}
8.163345 \times 10^{-1} \\
-1.932319 \times 10^0 \\
1.932319 \times 10^0 \\
-1.101288 \times 10^0 \\
0.000000 \times 10^0 \\
0.000000 \times 10^0 
\end{bmatrix} \\
\dot{\mathbf{x}}_{p_0} &= \begin{bmatrix}
8.078525 \times 10^{-6} \\
7.791139 \times 10^{-9} \\
8.425501 \times 10^{-11} \\
-1.275621 \times 10^{-5} \\
3.587018 \times 10^{-6} \\
-7.783488 \times 10^{-7} \\
0.000000 \times 10^0 \\
0.000000 \times 10^0 \\
1.750290 \times 10^{-4} 
\end{bmatrix} 
\end{align*}
\]

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Table C-2. Continuous-time linear aircraft model.

**STATE DYNAMICS MATRIX, \( \mathbf{A}_p \):**

**Columns 1 through 5:**

\[
\begin{array}{ccc}
-3.284008E-02 & 1.687600E+01 & 4.687153E-03 \\
4.952755E-06 & -1.542769E-25 & -2.059644E-01 \\
1.723216E-04 & -5.659418E-02 & -2.918714E-01 \\
1.322017E-04 & -1.344157E+00 & -5.161389E-09 \\
9.850753E-05 & -3.393291E-03 & 7.238354E-01 \\
0.000000E+00 & 0.000000E+00 & 0.000000E+00 \\
0.000000E+00 & 0.000000E+00 & 0.000000E+00 \\
0.000000E+00 & 0.000000E+00 & 0.000000E+00 \\
6.913456E-07 & -2.531686E+02 & 9.393395E-01 \\
\end{array}
\]

**Columns 6 through 10:**

\[
\begin{array}{cccc}
0.000000E+00 & 1.203851E-02 & -3.216927E+01 & 0.000000E+00 \\
0.000000E+00 & 4.698465E-04 & 8.948782E-08 & 0.000000E+00 \\
-9.949836E-01 & 1.254163E-01 & 4.779452E-05 & 0.000000E+00 \\
1.392612E-01 & 0.000000E+00 & 0.000000E+00 & 0.000000E+00 \\
0.000000E+00 & -2.511998E-04 & -4.000219E-08 & 0.000000E+00 \\
-1.670165E-01 & 0.000000E+00 & 0.000000E+00 & 0.000000E+00 \\
1.813769E-01 & 0.000000E+00 & 0.000000E+00 & 0.000000E+00 \\
3.729324E-03 & 0.000000E+00 & 0.000000E+00 & 0.000000E+00 \\
1.008511E+00 & 0.000000E+00 & 0.000000E+00 & 0.000000E+00 \\
0.000000E+00 & -9.474237E-02 & 2.531703E+02 & 0.000000E+00 \\
\end{array}
\]

**ACTUATOR INPUT MATRIX, \( \mathbf{B}_{p1} \):**

**Columns 1 through 5:**

\[
\begin{array}{ccccc}
-1.486683E-02 & 5.492291E-03 & -5.492714E-03 & 1.004757E-02 & -6.310199E-02 \\
-1.293354E-03 & -6.563469E-04 & -6.563341E-04 & 6.124513E-09 & -1.316076E-03 \\
0.000000E+00 & 0.000000E+00 & 0.000000E+00 & 0.000000E+00 & 0.000000E+00 \\
0.000000E+00 & -1.564789E-02 & 1.574993E-02 & 3.315156E-03 & -7.893338E-03 \\
-5.356416E-02 & 5.899375E-04 & 5.899368E-04 & -3.280669E-09 & 7.693349E-04 \\
0.000000E+00 & -1.228423E-03 & 6.428465E-04 & -7.823915E-03 & 1.039242E-03 \\
0.000000E+00 & 0.000000E+00 & 0.000000E+00 & 0.000000E+00 & 0.000000E+00 \\
0.000000E+00 & 0.000000E+00 & 0.000000E+00 & 0.000000E+00 & 0.000000E+00 \\
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0.000000E+00 & 0.000000E+00 & 0.000000E+00 & 0.000000E+00 & 0.000000E+00 \\
\end{array}
\]

**Column 6:**

\[
\begin{array}{cccc}
-6.310199E-02 \\
-1.316076E-03 \\
0.000000E+00 \\
7.693349E-04 \\
-1.039242E-03 \\
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0.000000E+00 \\
0.000000E+00 \\
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\end{array}
\]

303
Table C-2. Continuous-time linear aircraft model. (Cont.)

**TURBULENCE INPUT MATRIX, $B_{p2}$:**

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Table C-2. Continuous-time linear aircraft model. (Cont.)

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**TURBULENCE MEASUREMENT MATRIX, \( D_{p2} \):**

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305
Table C-3. Discrete-time linear aircraft model.

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ACTUATOR INPUT MATRIX, $B_{1d}$:

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Table C-3. Discrete-time linear aircraft model. (Cont.)

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BIAS VECTOR, $\Delta x_0$:

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Table C-4. Discrete-time turbulence model.

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| Table C-5. Kalman filter matrices. |

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| 1.801214E-07 | -2.157785E-07 | -6.782983E-07 | 1.315928E-03 | -4.380887E-03 |
| 5.080371E-07 | 2.502673E-05 | -3.391386E-07 | 3.273212E-03 | 2.775623E-02 |
| 2.531766E-06 | 1.282145E-06 | 3.250736E-05 | -2.361287E-03 | 1.806336E-01 |
| 1.163617E-07 | 6.128248E-05 | -2.800875E-07 | 1.262316E-03 | 3.539467E-03 |
| 2.183719E-06 | -1.679327E-07 | -1.558088E-06 | 1.786068E-04 | 3.369504E-02 |
| 3.697426E-08 | 4.371301E-09 | -3.986929E-07 | 1.983428E-04 | 1.658519E-02 |
| 2.169548E-09 | 6.972365E-07 | 1.334234E-08 | 1.300534E-04 | -1.199443E-04 |
| 1.867565E-06 | -4.187801E-06 | 8.557281E-05 | 3.674710E-03 | 5.591839E-02 |
| 4.833695E-05 | -1.765794E-02 | 1.621893E-01 | -2.101914E+01 | 1.897346E+03 |
| 2.391898E-04 | -2.148379E-02 | -1.687853E-03 | 1.574075E+03 | 6.953642E+00 |
| 2.945212E-04 | 7.354289E-01 | -7.255576E-04 | 2.846259E+01 | 1.153308E+01 |
| 3.548288E-04 | -2.908231E-04 | 1.294579E-02 | 1.875295E-01 | 3.468357E+01 |
| 3.295285E-03 | 1.218417E-03 | 6.242813E-01 | 1.828013E+00 | 2.828539E+02 |

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| 8.821583E-02 | 1.628741E-05 | 1.424555E-05 | 7.491076E-06 | 2.966525E-05 |
| 3.593467E-03 | 1.646649E-05 | 8.007499E-05 | -5.877278E-07 | -5.804387E-05 |
| 2.277585E-01 | -2.169328E-05 | 9.749863E-07 | 8.532772E-08 | 2.393236E-06 |
| -4.427199E-03 | 1.497517E-04 | 1.983721E-04 | 1.680137E-04 | 1.129181E-06 |
| 1.989685E-04 | 1.983721E-04 | 1.374932E-03 | -1.223541E-03 | -7.915767E-06 |
| 2.859618E-03 | 1.517487E-02 | -1.850651E-03 | -1.369800E-03 | 1.715626E-02 |
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| 8.296568E+02 | 2.878980E-02 | 5.657662E-02 | -1.388227E-01 | 1.348499E-02 |
| 8.984466E+01 | 5.719538E+01 | 1.513996E+00 | -6.359362E-02 | -6.368597E-02 |
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| -1.144798E-05 | -4.909787E-08 | 1.772827E-08 | 2.116633E-10 | 2.748901E-10 |
| 1.480121E-11 | 1.517497E-08 | -1.858861E-09 | -1.369800E-09 | -4.955454E-04 |
| -4.913790E-04 | -1.501174E-05 | 8.970266E-06 | 5.859458E-06 | 5.189955E-05 |
| 9.15243E-07 | -9.15243E-07 |
Table C-5. Kalman filter matrices. (Cont.)

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310
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Table C-5. Kalman filter matrices. (Cont.)
List of References


BIography

Steven Ray Hall was born in West Palm Beach, Florida, on July 6, 1959, and grew up in cities throughout central Florida. He graduated from St. Petersburg Senior High School in 1977. Since then, he has attended the Massachusetts Institute of Technology, where he was awarded the degrees of Bachelor of Science in Aeronautics and Astronautics and Master of Science in Aeronautics and Astronautics in June 1980 and February 1982, respectively. Since June 1980, he has been affiliated with The Charles Stark Draper Laboratory, Inc. as a Draper Fellow and as a Summer Staff employee.

Mr. Hall is a member of the Tau Beta Pi honorary society. He was a Fannie and John Hertz Foundation Fellow from September 1983 to June 1985.

Mr. Hall has lived in Malden, Massachusetts, since 1982. He is married to the former Karen Fuller of Walled Lake, Michigan.