ON LONG NONLINEAR INTERNAL WAVES OVER BOTTOM TOPOGRAPHY

by

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Submitted to the Department of Civil Engineering on December 19, 1984 in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Hydrodynamics.

ABSTRACT

Long nonlinear internal waves appear to be a ubiquitous feature of many of the world's marginal seas, straits and continental shelves. In the absence of dissipation, their propagation in regions of shallow uniform depth is well described by the Korteweg-de Vries (KdV) equation; however, on the continental shelf friction and topography may be comparable to, or dominate, dispersion and nonlinearity.

This thesis examines, through a combined theoretical and experimental program, the question of wave evolution and stability over bottom topography. The KdV evolution equation is formulated to include continuous stratification, slowly varying depth, boundary dissipation and higher order (cubic) nonlinearity, which may dominate quadratic nonlinearity under certain circumstances. The laboratory experiments, employing a salt stratified system, were conducted for single and multiple rank-ordered solitary waves incident on simple slope/shelf topography. Very good agreement between theory and experiment was obtained for a range of stratifications, topography and wave amplitudes; significant errors resulted if dissipation or cubic nonlinearity was neglected in the numerical results. Weak shearing and strong overturning wave instabilities were observed in some experiments. In certain cases an instability of the lowest mode wave led to the generation of a second-mode solitary wave. The application of the results to the prediction and interpretation of field data is discussed.

Thesis Supervisor: Professor W. K. Melville
Title: Associate Professor of Civil Engineering
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CHAPTER I

INTRODUCTION

1.1 Oceanic Observations and Background

Numerous measurements of internal waves in the coastal zones have been made though relatively little is known about their evolution in these regions where rapid changes in currents and topography are common. In contrast, the evolution of the internal-wave field in the deep ocean, away from boundary influences, is better understood with significant advances having been made recently (Garret and Munk 1979). The major difference between these two cases is the presence of topography; therefore to help form an overall picture of the generation, propagation and dissipation of internal waves in the oceans one would need to improve understanding of internal wave behavior in the presence of topography.

Some of the more dramatic measurements of internal waves in regions where topography is expected to play a major role are from the Andaman Sea (Northeast Indian Ocean). Perry and Schimke (1965) measured groups of internal waves with wave heights up to 80 meters and wave lengths up to 2000 meters traveling on the main pycnocline (500 meters deep) in water 1500 meters deep. Osborne and Burch (1980), in a subsequent field study in the Andaman Sea, concluded that these waves were generated near the Andaman and Nicobar Island chains and were propagating towards the Sumatra coastline, hundreds of
kilometers away. Apel and Holbrook (1983) reported similar large waves in the Sulu Sea where wave heights were 90m, wavelengths were 2-16km and crest lengths were over 300km.

Halpern (1971) and Haury, Briscoe and Orr (1979) measured smaller scale internal wave groups in the Massachusetts Bay. Wave heights were from 10 to 20 meters and wave lengths were 300 meters, in water 80 meters deep. The waves were generated on the Stellenwagen Bank and were propagating towards the Massachusetts coastline. Similar phenomena have been measured in lakes (Thorpe 1971; Hunkins and Fleigel 1973). Again the waves could be characterized as large amplitude long waves.

In addition to field measurements, airborn and satellite images of the sea surface show that internal waves are prevalent in many of the world's coastal regions. Among the first images are those presented by Apel et al. (1975) which show the surface signatures of large internal wave groups off the New York bight and the southwest coast of Africa. Fu and Holt (1982) present an assortment of images made with the SEASAT synthetic aperture radar (SAR). They report that "internal waves constitute a major element in the wealth of information contained in the SEASAT SAR imagery." Furthermore, they state that most of these waves are found in coastal regions in separate groups propagating towards the shore. Figure 1.1 is an example of such imagery taken in the Sulu Sea. From the curvature of the crests one can deduce that the waves are propagating towards the upper left of the image, a region of decreasing depth.
Figure 1.1. Satellite image of large internal waves in the Sulu Sea (from Apel et al. 1983).
There are several qualitative and quantitative explanations
for the generation of these wave groups available in the literature
along with field observations (Lee and Beardsley 1974; Maxworthy 1979;
Farmer and Smith 1980; Chereskin 1983). Though differing in detail
all involve the interaction of stratified tidal flow with topographi-
cal features such as the continental shelf break, submarine canyons
or sills. In the simplest model stratified tidal flow over some topo-
 graphical feature results in a downstream disturbance, such as an
internal hydraulic jump or lee waves. As the tide turns this dis-
turbance is able to move over the topographical feature and eventu-
ally evolve into a wave group. The possibility of nonlinear upstream
effects leading to generation is suggested by the work of Baines
(1984). The exact mechanism is probably site specific, depending on
the ambient stratification, tidal flow and topography. In lakes wind
forcing is the generation mechanism (Thorpe et al. 1972).

The fate of these waves away from the generating region is
not well understood, although many interesting possibilities exist.
Since these waves are generally observed to propagate into regions of
decreasing depth (Fu and Holt 1982) dissipation, reflection and sta-
bility may all be expected to influence wave evolution. In the case
of the Andaman Sea, wave groups moving towards the Sumatra coast might
either be reflected or become unstable and break as they reach shore.
Figure 1.2 from the Sulu Sea illustrates this point. A wave group
(wave crests appear as light bands) is propagating directly towards
Palawan Island, only 30km away. These waves have traveled

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several hundred kilometers and remained coherent; however, their fate in the increasingly shallow water is uncertain. Other satellite images from similar situations display little evidence of reflection. Regardless of reflection one would expect that stability and dissipation (by boundary shear or breaking events) are important in determining the fate of these waves as they propagate into shallow water.

Haury, Briscoe and Orr (1979) observed overturning instabilities in the Massachusetts Bay measurements in a region of slowly changing depth. The exact mechanism was not identified although a shear instability of the type described by Thorpe (1978) was suggested. Along similar lines Sandstrom and Elliott (1984) conducted a field study of the internal waves on the Scotian shelf and found that large amplitude waves, presumably generated at the shelf break, were completely dissipated within a distance 10 - 30 km inshore of the shelf break. Estimates of the local Richardson number gave values less than 25 leading to the conclusion that shear instability was a probable candidate for the dissipation. In addition they concluded that boundary and interfacial shear was significant, though not large enough to explain all the observed dissipation.

The evolution and stability of these waves over variable topography is an important topic, one which has consequences for the coastal environment. Haury, Briscoe and Orr (1979) and Pingree and Mardell (1981) discussed the effects of wave stability and breaking on vertical mixing. Internal wave breaking may be an effective mechanism for mixing nutrient rich water from the bottom to the biologically
active upper layer (Sandstrom and Elliott 1984). Additionally, instabilities of these waves may play a part in the transfer of tidal energy to the high frequency portion of the energy spectrum. The measurements of Haury, Briscoe and Orr demonstrated that wave energy, originally tidal energy, could be transferred to the scale of turbulence by breaking. This conclusion is certainly speculative but it is known that places such as the Andaman Sea and Gulf of California, where large scale tidally generated waves are observed, are regions of significant tidal dissipation (Miller 1966). Waves incident on the coast have also been suggested as a mechanism for coastal seiche excitement at Palawan Island in the Sulu Sea and Puerto Rico (Giese et al. 1982).

Given the prevalence of internal waves in the coastal regions and the possible environmental effects understanding of wave evolution over bottom topography is an important topic. The objective of this thesis is to help resolve this question through a combined theoretical and experimental study of long nonlinear internal wave evolution over simple slope/shelf topography. The goal is to identify the basic governing dynamics and test appropriate evolution equations. The ultimate goal is an improved predictive capability.

1.2 Aspects of Modeling

Nearly all previous attempts to describe the evolution of long internal waves have employed theories which incorporate weak non-linearity and dispersion. The best known of these is the
Korteweg-de Vries (KdV) theory for long waves in shallow water. Lee and Beardsley (1974) used KdV theory to describe the initial evolution (in constant depth) of the waves observed in Massachusetts Bay and Osborne and Burch (1980) used it to analyze the Andaman Sea observations. KdV theory assumes a balance between nonlinearity and dispersion parameterized by two nondimensional variables

\[ \alpha = a/D \quad (1.1) \]

and

\[ \beta = (D/\ell)^2 \quad (1.2) \]

where \( a \) is a wave amplitude scale, \( D \) is the water depth, and \( \ell \) is a wave length scale. For this balance to occur the nonlinear parameter \( \alpha \) and the dispersive parameter \( \beta \) must be the same order of magnitude and both small: \( \beta = O(\alpha) \ll 1 \). From the Massachusetts Bay data \( \alpha = (10m/80m) = 0.13 \) and \( \beta = (80m/300m)^2 = 0.07 \); therefore, KdV theory appears to be applicable.

KdV theory may not be appropriate for all cases. From the Andaman Sea data \( \alpha = (80m/1500m) = 0.053 \) and \( \beta = (1500m/2000m)^2 = 0.56 \) which violates the balance \( \beta = O(\alpha) \) and the long wave assumption \( \beta \ll 1 \). The Sulu Sea data of Apel and Holbrook (1984) also fall into this category.

There are theories which allow for intermediate or deep water and still consider slightly nonlinear-dispersive long waves. For an infinitely deep stratified fluid Benjamin (1967) and Ono (1975)
derived an equation (denoted BO) which governs waves that are long with respect to one of the fluid layers or with respect to the depth of the density variation (continuous stratification) when both layers are deep. Joseph (1977) and Kubota, Ko and Dobbs (1978) derived an intermediate depth equation (denoted JKKD). Again the waves are long with respect to one of the layers or the depth of density variation. For these theories the dispersive parameter is given by \( \beta = h_s/\xi \), where \( h_s \) = depth of the shallow layer of density variation. Balance of nonlinearity and dispersion requires \( \beta = O(\alpha) \ll 1 \), with the added restriction \( H/\xi \gg 1 \) for the BO theory and \( H/\xi = O(1) \) for the JKKD theory, where \( H \) is the depth of the deeper layer. The essential distinction between the three theories lies in the treatment of dispersion. More dispersion is allowed in the deep and intermediate depth theories than in the KdV theory (i.e. \( \beta_{KdV} = (D/\xi)^2 \) and \( \beta_{BO, JKKD} = h_s/\xi \)).

A characteristic of these KdV-type equations is a class of isolated permanent wave solutions termed solitary waves (KdV: Whitham 1974, §13.12; JKKD: Joseph 1977; BO: Benjamin 1967). The behavior of an initial disturbance is qualitatively similar in all three. The disturbance can result in a train of nonlinear waves which separate into a group of rank-ordered solitary waves; the largest in front, followed by a dispersive tail. Inverse scattering theory can be used to solve this initial value problem analytically for the KdV (Gardner et al. 1974) and BO (Fokas and Ablowitz 1983) equations.
Each theory has been developed for a specific situation; however, Kubota, Ko and Dobbs (1978) have shown that their equation collapses to the KdV/BO equation in the shallow/deep water limit implying that it is uniformly valid. Chen and Lee (1979) showed that this is not the case since the solitary wave solutions do not collapse consistently. There is no smooth transition for changing depth. Segur and Hammack (1982) also point out that the scaling assumptions of the JKKD and KdV theories are not consistent (definition of $\beta$) and consequently they expect the JKKD theory to be slightly singular in the KdV limit. Their experiments for a two-layer constant depth system support the use of the KdV theory even when it might appear to be inappropriate. The KdV equation had a much larger range of validity than the JKKD equation. Wave characteristics were well predicted even when $\beta \approx 10\alpha$.

In addition to violation of the requirements on the dispersive parameter $\beta$ some measurements show wave amplitudes large enough to question the assumption of weak nonlinearity. Sandstrom and Elliott (1984) measured waves where $\alpha = (50m/150m) = 0.33$. In order to explain observed wave length and phase speed they employed the theory of Geer and Grimshaw (1983) which extends the KdV solutions for solitary waves to second order in wave amplitude. In such extreme cases it would appear that higher order theory must be used, but Koop and Butler (1981) found that the KdV theory does very well in predicting wave shape, speed, etc. for $\alpha \leq 0.2$.  

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Many observations are qualitatively similar to the theoretical results described above lending strength to the argument that nonlinear-dispersive theory is appropriate. Sandstrom and Elliott (1984) obtained an acoustic image of an internal wave which agrees remarkably in shape to the KdV solitary wave solution. Rank-ordering in wave length (Fu and Holt 1982) and amplitude (Hunkins and Fliegel 1973; Osborne and Burch 1980; Haury et al. 1978) are characteristic of wave evolution described by nonlinear-dispersive theory. Rank-ordering in wavelength can be explained by linear dispersion, however, amplitude rank-ordering is a nonlinear feature.

Though nonlinear-dispersive long wave theory is suggested for describing wave evolution it may not be necessary for the entire propagation process. Criteria for the application of KdV theory to the leading surface waves from an isolated disturbance have been developed by Hammack and Segur (1978). Their work is easily extended to include the internal wave situation. The results, in the context of a two-layer system, are based upon the parameters

\[ \bar{V}_o = \frac{3}{2} \frac{(d_+ - d_-)}{(d_+ d_-)^{3/2}} L_0 a_o \]

and

\[ U_o = \frac{L_0}{(d_+ d_-)^{1/2}} |V_o| \]
where $d_+$ is the upper layer depth, $d_-$ is the lower layer depth, $a_o$ is the amplitude scale for initial disturbance and $L_o$ is length scale for initial disturbance. $V_o$ is a scale for the volume of the initial disturbance and $U_o$ is an Ursell number for the disturbance.

For a rectangular initial disturbance the analysis depends on the sign of $V_o$. If $V_o > 0$ solitary waves will asymptotically develop and if $V_o < 0$ no solitary waves are formed. Instead a train of nonlinear oscillatory waves results. Usually $d_+ < d_-$ and $a_o < 0$ giving $V_o > 0$. For this regime if $U_o \gg 1$ KdV theory is necessary immediately and solitary waves emerge from the disturbance after a time given by

$$t_s = 6 \frac{U_o^2}{V^3} \left( \frac{d_+ + d_-}{g \Delta \rho / \rho} \right)^{1/2} \quad (1.3)$$

where $g = \text{gravitational acceleration}$ and $\Delta \rho / \rho = \text{density difference}$ between the upper and lower layers normalized by an average density.

If $U_o = O(1)$ linear-nondispersive theory is sufficient until

$$t = 2 \left( \frac{d_+ + d_-}{g \Delta \rho / \rho} \right)^{1/2} \frac{U_o}{V^2} \quad (1.4)$$

after which nonlinear-dispersive theory applies. Sorting of solitary waves is given by (1.3). For $U_o \ll 1$ linear-dispersive theory is adequate until
\[ t = 2 \left( \frac{d_+ + d_-}{g \Delta \rho \rho} \right)^{1/2} \frac{1}{V_0^3} \]

after which nonlinear-dispersive theory is necessary. Solitary wave sorting is again given by (1.3). Hammack and Segur derived similar criteria for \( V_0 < 0 \), but they will not be discussed.

There are some practical difficulties in applying this analysis. The rectangular initial data requirement appears to be limiting. If the initial data is not highly oscillatory the results are the same as above except when \( U_0 \gg 1 \). In this regime linear-nondispersive theory is adequate for a time given by (1.4) after which nonlinear-dispersive theory should be used. It should also be noted that the analysis assumes no breaking (i.e., nonlinear-nondispersive theory is never allowed to become asymptotic). It is possible that breaking, particularly near the generation region, could occur thereby altering the initial scales \( a_0 \) and \( L_0 \).

With these considerations we take the Massachusetts Bay data as an example. There \( a_0 = -10 \text{m}, d_+ = 30 \text{m}, d_- = 50 \text{m} \) and \( L_0 = 1000 \text{m} \) giving \( V_0 = 5.1 \) and \( U_0 = 132 \). Nonlinear-dispersive theory is necessary after \( t \approx 10 \text{ min} \) (\( \Delta \rho / \rho = 0.002 \)) or almost immediately. The sorting time for solitary waves is \( \sim 5 \times 10^4 \text{ sec} \). If the waves are moving at roughly the linear long wave phase speed,

\[ c^2 = (g \frac{\Delta \rho}{\rho} \frac{d_+ + d_-}{d_+ + d_-}) \]

(1.5)
this time translates to a distance of 30km. From the Andaman Sea data
\(d_+ = 500m, d_- = 1000m, a_o = -80m\) and \(L_o = 25km\), the scale of the
topographical features in the Andaman and Nicobar Island chains.
These give \(V_o = 4.2\) and \(U_o = 148\). Nonlinear-dispersive theory
becomes necessary after \(t = 77\text{min} (\Delta \rho / \rho = 0.002)\) or a distance of
about 12km (the scale of the initial disturbance). The emergence time
for solitary waves is \(t_s = 4.9 \times 10^5\) sec which translates to a
distance of about 1200km.

From this analysis it appears that nonlinear-dispersive
time should be useful in modeling the evolution of long internal
waves in coastal regions. Also note that the characterization of the
wave trains in terms of solitary waves, an asymptotic state, is not
always correct. For Massachusetts Bay the sorting distance is less
than the scale of the bay but for the Andaman Sea this distance is
larger than the scale of the sea (\(700\text{km}\) in the wave propagation
direction). In either case we might expect changing depth and dissi-
pation to interfere with wave evolution.

Important nondimensional parameters associated with topo-
graphy and dissipation are respectively

\[
\lambda = \frac{\delta}{L}
\]  
(1.6)

and

\[
\gamma = \frac{\delta}{D} = \frac{(u_0/c)^{1/2}}{D}
\]  
(1.7)
where $L$ is a length scale of the depth variation, $\delta$ is the scale height of boundary layer height, $c$ is linear long wave phase speed and $\nu$ is a representative kinematic viscosity (eddy viscosity for the field, molecular for the laboratory). From the Massachusetts Bay data $L \sim 10$km giving $\lambda = (300\text{m}/10000\text{m}) = 0.03$. Changes in the depth relative to initial scale depth $D$ may be $O(1)$ but the change over one wave length is small: $\lambda = O(\beta)$, implying that topographic influences are comparable to nonlinearity and dispersion. If $\lambda \gg \alpha, \beta$ then topographic influences are dominant.

The dissipation parameter $\gamma$ can also be estimated assuming a representative value for the eddy viscosity $\nu_e = 1 - 10 \text{ cm}^2/\text{s}$. In the Massachusetts Bay $c = 50 \text{ cm/s}$ giving $\gamma = 0.003 - 0.013$. This dissipation is confined primarily to a small region near the boundary. This is consistent with Leblond (1966), who investigated viscous effects on internal waves. Since $\gamma < |\alpha|$ viscosity is not expected to dominate nonlinearity, but over the long distances these waves are observed to propagate the cumulative effect of damping may be comparable to those of nonlinearity, dispersion and topography.
1.3 Problem Definition

Based upon observed wave characteristics, Segur and Hammack's (1982) conclusion that the KdV assumptions need only be satisfied marginally and the modeling analysis of the preceding section we might attempt to describe internal wave evolution in the coastal regions using KdV theory. It is not clear, however, that in regions of slowly changing depth KdV theory will always be adequate. In particular there may be some parametric regions in which the theory fails either because of scaling considerations or instabilities.

There have been no studies in which the KdV equation for internal waves over variable topography has been tested. Madsen and Mei (1969) found good agreement between experiments and theory, similar in assumption to KdV theory, for long finite amplitude surface waves over slope/shelf topography. However, the extrapolation of this conclusion to internal waves is not straightforward due to possible interfacial instabilities and the behavior of the KdV equation for internal waves.

Liu et al. (1984) compared field measurements from the Sulu Sea with predictions from the intermediate depth JKKD theory including the effects of radial spreading, wave damping and variable depth. Taking measured displacements at one location as initial conditions they obtained qualitative agreement in observed wave amplitudes over regions of essentially constant depth. Although they concluded that all three effects were important, they did not directly compare
measured and predicted wave profiles for either constant or variable depth regions. As a result, any conclusions regarding the ability of this theory to predict the evolution of a wave packet under these or different physical conditions needs further testing. No conclusions regarding KdV theory can be drawn.

This thesis is an attempt to address this problem and determine the validity of the KdV theory for variable depth by comparison of theoretical models with laboratory experiments over slope/shelf topography. The role of wave damping on evolution is considered. Additionally, the occurrence and mechanisms of flow instabilities, not modeled by any weakly nonlinear theory, are discussed. The outcome is an improved understanding of topographic influences on internal wave evolution and stability and a modeling capability to predict the fate of these waves in coastal regions.

Laboratory measurements using slope/shelf geometry and salt stratification were chosen for testing the theory for several reasons. The most obvious one is the difficulty and expense of acquiring good field data under a set of conditions broad enough to fully test the theory. The presence of mean currents or variations in background density could be included in the theory but these influences might only obscure the analysis of topographic changes. Therefore laboratory experiments were chosen because of repeatability and the wide range which the relevant parameters could be varied. The slope/shelf geometry was chosen as the simplest model of the continental shelf region. Salt stratification, rather than an immiscible two-layer
system, was chosen so as not to suppress any flow instabilities and mixing which might occur.

The most significant problem encountered with laboratory experiments is scaling of results up to field scale. To do this effectively the important nondimensional parameters must be of comparable magnitude in both laboratory and field situations. In addition to the parameters $\alpha, \beta, \lambda,$ and $\gamma$ introduced earlier other important parameters are

$$\varepsilon = \frac{h_p}{D}$$  \hspace{1cm} (1.8)

$$\sigma = \frac{2(\rho_+ - \rho_-)}{\rho_+ + \rho_-} = \frac{\Delta \rho}{\rho}$$  \hspace{1cm} (1.9)

and

$$R_{bl} = \frac{|\alpha|c\delta}{\nu}$$  \hspace{1cm} (1.10)

where $h_p$ is a scale height of the pycnocline and $\rho_+ / \rho_-$ is the upper/lower layer density. Here $\varepsilon$ is the interface parameter. If $\varepsilon \ll 1$ the system is nearly two-layered. $\sigma$ is the Boussinesq parameter. The Reynolds number for the boundary layer flow $R_{bl}$ is composed of the fluid velocity scale $|\alpha|c$, the boundary layer height $\delta$ and the viscosity.

Typical values of these parameters for the ocean and the laboratory are shown in Table 1.1.
### a) Physical Scales

<table>
<thead>
<tr>
<th>Location</th>
<th>(a_o)</th>
<th>(\ell)</th>
<th>(D)</th>
<th>(d_+)</th>
<th>(L)</th>
<th>(c)</th>
<th>(\sigma = \frac{\Delta p}{\rho})</th>
<th>(h_\rho)</th>
<th>(\nu)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Andaman Sea</td>
<td>-80 m</td>
<td>2000 m</td>
<td>1500 m</td>
<td>500 m</td>
<td>250 km</td>
<td>2 cm/s</td>
<td>0.002</td>
<td>500 m</td>
<td>1 - 10 cm²/s (eddy viscosity)</td>
</tr>
<tr>
<td>Mass. Bay</td>
<td>-10 m</td>
<td>300 m</td>
<td>80 m</td>
<td>30 m</td>
<td>10 km</td>
<td>0.5 m/s</td>
<td>0.002</td>
<td>30 m</td>
<td>1 - 10 cm²/s (eddy viscosity)</td>
</tr>
<tr>
<td>Laboratory</td>
<td>-2 cm</td>
<td>100 cm</td>
<td>30 cm</td>
<td>10 cm</td>
<td>700 cm</td>
<td>15 cm/s</td>
<td>0.002</td>
<td>2 cm</td>
<td>0.01 cm²/s (molecular viscosity)</td>
</tr>
</tbody>
</table>

### b) Nondimensional Scales

<table>
<thead>
<tr>
<th>Location</th>
<th>(\alpha = \frac{a_o}{D})</th>
<th>(\beta = \left(\frac{\ell}{L}\right)^2)</th>
<th>(\lambda = \frac{\ell}{L})</th>
<th>(\gamma = \frac{\delta}{D} = \frac{(\nu \ell/c)^{1/2}}{D})</th>
<th>(R_{b\ell} = \frac{\alpha \cdot c \cdot \delta}{\nu})</th>
<th>(\varepsilon = \frac{h_\rho}{D})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Andaman Sea</td>
<td>-0.053</td>
<td>0.56</td>
<td>0.008</td>
<td>2 - 7 \cdot 10^{-4}</td>
<td>100 - 300</td>
<td>0.33</td>
</tr>
<tr>
<td>Mass. Bay</td>
<td>-0.13</td>
<td>0.07</td>
<td>0.03</td>
<td>3 - 10 \cdot 10^{-3}</td>
<td>50 - 160</td>
<td>0.38</td>
</tr>
<tr>
<td>Laboratory</td>
<td>-0.067</td>
<td>0.09</td>
<td>0.14</td>
<td>9 \cdot 10^{-3}</td>
<td>26</td>
<td>0.07</td>
</tr>
</tbody>
</table>

Table 1.1. Representative Scales in the Ocean and Laboratory
Similarity between the field and laboratory is preserved over all the parameters if a representative value of the eddy viscosity is used for the oceanic conditions. Thus, to apply the conclusions of this work to the field the eddy viscosity must be specified, preferably from model calibrations with field data.

The experiments and comparisons focus primarily on the problem of an internal solitary wave propagating from a deep section up a slope to a shallower region of constant depth. Although field observations generally show groups of nonlinear waves, the usefulness of KdV theory is best tested in the simple case of a single incident wave. Some studies of multiple incoming waves will be discussed.

The experiments were conducted around the 'turning-point' problem. A solitary wave in a two-layer system is a wave of depression/elevation for $d_+/d_- < 1$. If a wave of depression propagates up a slope through a point where $d_+ = d_-$ (turning point) and onto a shelf where $d_+ > d_-$ it can no longer exist as a solitary wave of depression and must either reverse polarity or break up into an oscillatory wave group (Miles 1980). This behavior is due to the coefficient of the quadratic nonlinear term in the KdV equation (Eqn. (2.77) in Ch. 2) which changes signs upon passing through the turning point. In the region of the turning point higher order cubic nonlinearity becomes important (Miles 1980). This situation does not occur in the surface wave case. Investigation of this topographically induced scattering should provide a good test of the KdV theory for internal waves over bottom topography.
1.4 Thesis Outline

In Chapter Two the KdV equation including continuous stratification, variable topography, cubic nonlinearity and dissipation is derived. A description of the numerical procedure used for the solution is given.

Chapter Three contains a summary of theoretical work on the inviscid two-layer turning-point problem. The influence of boundary layer damping on the inviscid conclusions is discussed.

Chapter Four describes the experimental set-up and procedure. Comparison of the experimental results with predictions of the model equations, with and without dissipation and for two-layer or continuous stratification models are described. The importance of dissipation, continuous stratification, cubic nonlinearity and reflection are discussed. Good agreement between theory and experiment is found over a wide range of parameters.

An investigation into the conditions leading to wave instability are discussed in Chapter Five. Both shearing and strong overturning instabilities are observed in the experiments. Strong instabilities are found to result in generation of higher mode internal waves and considerable vertical mixing. An assessment of the role of wave instabilities on dissipation is presented.

Chapter Six discusses and summarizes the conclusions of the thesis.
CHAPTER II
EQUATION DEVELOPMENT

2.1 Introduction

We wish to obtain an evolution equation for long nonlinear internal waves over slowly varying topography. The resulting KdV equation should also include continuous stratification, dissipation and cubic nonlinearity. Since eventual application of the model to coastal ocean conditions is anticipated continuous stratification is included as a more realistic model. Two-layer theory is available as a specific case of the more general continuous stratification theory, however, the usefulness of the simpler two-layer model in wave evolution on the continental shelf needs testing. The data from the Sulu Sea (Liu et al. 1984) demonstrated the importance of dissipation on wave evolution indicating the necessity of considering damping. The cubic nonlinearity is included as the first correction when the coefficient of the quadratic nonlinear term becomes small. In the context of a two-layer system this occurs in the neighborhood of equal layer depths \( d_+ = d_- \). The coefficients of the continuous stratification equation are given by integrals of the vertical structure function over the water column (Benney 1966), rather than algebraic relationships as in the two-layer model. They may also become small or zero under certain circumstances (Long 1956; Gear and Grimshaw 1983).
Of particular interest are situations where the main pycnocline is near the mid-depth level of the water column; a situation likely to be encountered somewhere on the continental shelf as the waves propagate shoreward.

In Chapter 1 the important nondimensional parameters (Eqns. (1.1), (1.2), (1.6), (1.7), (1.8) and (1.9))

\[ \alpha = \frac{a}{D}, \quad \beta = \frac{d^2}{L}, \quad \lambda = \frac{L}{L}, \quad \gamma = \frac{\gamma}{D}, \quad \varepsilon = \frac{h\gamma}{D} \]

(2.1a,b,c,d,e)

were introduced. Table 1.1 showed representative values of these parameters for both coastal ocean and laboratory situations which support the following assumptions used in the derivation of the governing KdV equation. The nonlinear \( \alpha \) and dispersive \( \beta \) parameters are assumed to be in balance \( \beta = O(\alpha) \ll 1 \). The slope of the bottom topography is slow \( \lambda = O(\beta) \), however, the depth change relative to a reference depth \( D \) may be \( O(1) \). Boundary damping is not dominant \( \gamma \ll O(\alpha) \), but is important in wave evolution over distances. The Boussinesq parameter is small \( \sigma \ll 1 \). The interface parameter \( \varepsilon \) is assumed to be \( O(\alpha) \), although this is not necessary for application to the field conditions.
Benney (1966) derived the KdV equation for a constant depth, continuously stratified and inviscid system. Djordjevic and Redekopp (1978) considered slowly varying topography but for the specific case of weak exponential density profile. Koop and Butler (1981) added a model of boundary layer damping to the result of Benney. The cubic nonlinear convection has been examined by Miles (1979) and Gear and Grimshaw (1983) for a steady, continuously stratified system. However, no general equation incorporating all of these influences has been studied. The remainder of this chapter presents a systematic derivation of the general evolution equation. Specialization of this equation for a two-layer system including interfacial shear (details in Appendix 2) is given. Lastly, the numerical technique used for the solution is described.

2.2 KdV Evolution Equation

Keeping the scales described above in mind we consider the system in Figure 2.1. The free surface is assumed to be a rigid no-slip boundary. In reality the stress at the free surface is much less than at a no-slip boundary (Longuet-Higgins 1953). The assumption of no-slip at \( z = 0 \) is only made to facilitate the determination of the side wall boundary layer dissipation (§2.4), important for comparisons with experiments. The no-slip upper boundary assumption will then be removed. The rigid lid approximation is valid provided \( \sigma \ll 1 \). The presence of thin boundary layers
Figure 2.1. Definition sketch for the continuously stratified system.
at \( z = 0 \) and \( z = -h \) allows the problem to first be constructed as an outer inviscid problem which is then corrected by boundary layer analysis at \( z = 0 \) and \( z = -h \). As shown later this correction determines the outer vertical velocity boundary conditions. The derivation procedure is straightforward and follows closely the work of Kakutani and Matsuuchi (1975) and Mei (1983, §11.9) for long, weakly nonlinear surface waves.

The governing equations for the inviscid region are (hydrostatic component removed)

\[
\begin{align*}
    u_x + w_z &= 0 \\  \\
    \rho_t + u\rho_x + w\rho_z + \bar{\rho}_z &= 0 \\  \\
    (\bar{\rho}(z)+\rho)(u_t+uu_x+ww_z) &= -p_x \\  \\
    (\bar{\rho}(z)+\rho)(w_t+uw_x+ww_z) &= -p_z - g\rho
\end{align*}
\]  

(2.2)  
(2.3)  
(2.4)  
(2.5)

where \( \bar{\rho}(z) \) is the background density distribution, \( \rho \) is the wave induced density perturbation, \( (u,w) \) are the wave induced horizontal and vertical velocities, \( g \) is the acceleration of gravity, \( t \) is time, \( x \) is the horizontal coordinate and \( z \) is the vertical coordinate, positive upwards. Equations (2.2) - (2.5) are nondimensionalized by
\[ u = \alpha uu', \quad w = \alpha w' = \alpha^{3/2} u' \]

\[ x = x'/\ell, \quad z = z'/D, \quad t = t'/\ell/u, \]

\[ \bar{\rho}(z) = \rho_0 \bar{\rho}^I(z), \quad \rho = \sigma \rho_0 \rho', \quad p = \alpha \rho_0 U^2 p' \]

where \( D/\ell = \alpha^{1/2} \) and \( U^2 = gD \). After dropping the primes

\[ u_x + w_z = 0 \quad (2.6) \]

\[ \rho_t + \alpha u \rho_x + \alpha w \rho_z + \left( \frac{\alpha}{\sigma} \right) \omega \bar{\rho} = 0 \quad (2.7) \]

\[ (\bar{\rho} + \sigma \rho)(u_t + \alpha uu_x + \alpha wu_z) = -p_x \quad (2.8) \]

\[ \alpha(\bar{\rho} + \sigma \rho)(w_t + \alpha uw_x + \alpha ww_z) = -p_z - \rho \quad . \quad (2.9) \]

Note that the perturbation density has been expanded as \( \sigma \rho \). If
the interface parameter \( \epsilon = O(1) \) then the density should be
expanded as \( \alpha \sigma \rho \) and \( U^2 = \sigma gD \). The rigid lid assumption requires
\( \sigma \ll 1 \) or equivalently \( \sigma = O(\alpha) \); allowing \( \sigma \) to be replaced by
\( \alpha \) in (2.6) - (2.9).

Considering rightward propagating disturbances only the
phase variable
\[ s = \int_{x_0}^{x} \frac{dx}{c(x)} - t , \quad (2.10) \]

and slow horizontal coordinate \( X = \alpha x \) are defined. Here \( c(x) \) is the linear longwave phase speed which is unknown at this point. The choice of \( s \) allows construction of an evolution equation in which the coefficients are functions of the slow variable only.

Introducing these new coordinates along with the expansions

\[
\begin{pmatrix}
  u \\
  w \\
  \rho \\
  p
\end{pmatrix} = \begin{pmatrix}
  u^{(0)} \\
  w^{(0)} \\
  \rho^{(0)} \\
  p^{(0)}
\end{pmatrix} + \alpha \begin{pmatrix}
  u^{(1)} \\
  w^{(1)} \\
  \rho^{(1)} \\
  p^{(1)}
\end{pmatrix} + O(\alpha^2)
\]

into (2.6) - (2.9) results in a set of equations for each order in \( \alpha \):

\[ 0(\alpha^0) \]
\[ \frac{1}{c} \frac{\partial u^{(0)}}{\partial s} + \frac{\partial w^{(0)}}{\partial z} = 0 \]

\[ - \frac{\partial p^{(0)}}{\partial s} + \rho z w^{(0)} = 0 \quad (2.11a,b,c,d) \]

\[ - \rho \frac{\partial u^{(0)}}{\partial s} + \frac{1}{c} \frac{\partial p^{(0)}}{\partial s} = 0 \]

\[ - \frac{\partial p^{(0)}}{\partial z} - \rho(0) = 0 \]

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\( 0(\alpha) \ \frac{1}{c} \frac{\partial u^{(1)}}{\partial s} + \frac{\partial w^{(1)}}{\partial z} = - \frac{\partial u^{(0)}}{\partial x} \)

\[- \frac{\partial \rho^{(1)}}{\partial s} + \rho_{z} w^{(1)} = - \frac{1}{c} u^{(0)} \frac{\partial \rho^{(0)}}{\partial s} - w^{(0)} \frac{\partial \rho^{(0)}}{\partial z} \]

\[- \frac{\rho \, \partial u^{(1)}}{\partial s} + \frac{1}{c} \frac{\partial p^{(1)}}{\partial s} = \rho^{(0)} \frac{\partial u^{(0)}}{\partial s} - \frac{\rho}{c} u^{(0)} \frac{\partial u^{(0)}}{\partial \xi} - \rho w^{(0)} \frac{\partial u^{(0)}}{\partial z} \]

\[- \frac{\partial p^{(1)}}{\partial z} - \rho^{(1)} = - \frac{\partial w^{(0)}}{\partial s} \]

(2.12a, b, c, d)

Eliminating \( u^{(n)}, \rho^{(n)} \) and \( p^{(n)} \) gives a single equation for \( w^{(n)} \):

\[ 0(\alpha) \ L \ (w^{(0)}) = 0 \] (2.13)

\[ 0(\alpha) \ L \ (w^{(1)}) = F_{1} \] (2.14)

where

\[ L = \left( \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} - \frac{\rho_{z}/c^{2}}{\rho} \right) \right) \] (2.15)
and

\[
F_1 = - (\rho \frac{\partial u^{(0)}}{\partial x})_z - \frac{1}{c} (\rho \frac{\partial p^{(0)}}{\partial x})_z + \frac{1}{c^2} \rho \frac{\partial^2 w^{(0)}}{\partial s^2} \\
+ \frac{1}{c} \left( \rho^{(0)} \frac{\partial u^{(0)}}{\partial s} - \frac{\partial}{\partial c} u^{(0)} \frac{\partial u^{(0)}}{\partial s} - \rho \frac{\partial w^{(0)}}{\partial s} \frac{\partial u^{(0)}}{\partial z} \right)_z \\
+ \frac{1}{c^2} u^{(0)} \frac{\partial \rho^{(0)}}{\partial s} + \frac{1}{c^2} w^{(0)} \frac{\partial \rho^{(0)}}{\partial z} .
\]  

(2.16)

The problems for \( w^{(0)} \) and \( w^{(1)} \) are complete once the boundary conditions are specified. Consider first the \( 0(\alpha^0) \) problem (2.13). The inviscid boundary conditions are

\[
w^{(0)} = 0 \text{ at } z = 0, -h(x)
\]  

(2.17)

provided that the slope is small (i.e., \( h = h(\lambda x), \lambda = 0(\alpha) \)). By the same reasoning the influence of a thin boundary layers \( (\gamma \leq 0(\alpha)) \) should not enter this \( 0(1) \) problem, thus (2.17) serves as the boundary conditions for (2.13). This assumption will be shown correct in the boundary layer analysis below.

The \( 0(\alpha^0) \) problem is then solved by assuming the separable solution \( w^{(0)} = -\phi(z) A_s(X,s) \). Substitution into (2.13) and (2.17) determines the vertical structure function \( \phi(z) \) for mode \( n \) and the corresponding linear long wave speed \( c_n \) from the
eigenvalue problem

\[ L(\phi_n) = \frac{\partial}{\partial z} \left( \rho \frac{\partial \phi_n}{\partial z} \right) - \left( \frac{\rho_z}{c_n^2} \right) \phi_n(z) = 0 \] (2.18)

\[ \phi_n(0) = \phi_n(-h) = 0 \] . (2.19)

Since the boundary condition (2.19) is applied at \( z = -h(X) \) both \( c_n \) and \( \phi_n \) vary with \( X \). Only the lowest mode (\( n=1 \)) is likely to be significant so the subscript \( n \) will be dropped with the understanding that mode 1 interval waves are being considered.

Solutions satisfying (2.11a,b,c,d) are

\[ w'(0) = -\phi A_S \]

\[ u(0) = c \phi'A \] (2.20a,b,c,d)

\[ p(0) = -\rho c^2 \phi'A \]

\[ \rho'(0) = -\rho_z \phi A \]

where \( A(X,s) \) is the amplitude function and \( (\ )' = d(\ )/dz \).

The equation governing the amplitude evolution is found from consideration of the \( O(\alpha) \) problem. Substituting (2.20a,b,c,d) and (2.16) into (2.14) gives
\begin{align*}
L(w^{(1)}) &= -\frac{\partial}{\partial x} (\bar{\rho} c \phi' A)' - \frac{1}{c} \frac{\partial}{\partial x} (\bar{\rho} c^2 \phi' A)'
+ AA_s \{ (\bar{\rho} \phi'')^2 - (\bar{\rho}_z \phi')^2 - (\bar{\rho} c^2)'^2 + \frac{1}{c^2} \bar{\rho}_{zz} \phi^2 \}
+ \frac{\rho_0}{c^2} \phi \ A_{ss} .
\end{align*}

Multiplying (2.21) by \( \phi \) and integrating in \( z \) from \(-h\) to \( 0 \) (solvability condition for \( w^{(1)} \): Mei 1983, §2.4) gives, after using (2.18), (2.19) and suitable integration by parts,

\begin{align*}
\left[ -\rho_0 \phi' w^{(1)} \right]_{-h}^{0} &= 3 \ c_X A \int_{-h}^{0} \bar{\rho} \phi'^2 \ dz \\
&+ c \ A \int_{-h}^{0} \frac{\partial}{\partial x} (\bar{\rho} \phi'^2) \ dz \\
&+ 2c \ A_X \int_{-h}^{0} \bar{\rho} \phi'^2 \ dz \\
&+ 3 \ AA_s \int_{-h}^{0} \bar{\rho} \phi'^3 \ dz \\
&+ \frac{1}{c^2} AA_{ss} \int_{-h}^{0} \bar{\rho} \phi^2 \ dz ,
\end{align*}

(2.22)
where note is made that \( c = c(X) \) and \( \phi = \phi(z; X) \).

Further progress cannot be made until \( w^{(1)} \) at \( z = 0 \) and \( z = -h \) are specified. For the inviscid problem

\[
w^{(1)}(1) = 0 \quad \text{at } z = 0
\]

and

\[
w^{(1)} = -u^{(0)} \frac{dh}{dX} \quad \text{at } z = -h(X).
\]

However, with damping the boundary layers at \( z = 0 \) and \( z = -h \) must be analysed to determine the boundary conditions on \( w^{(1)} \).

2.2.1 Boundary Layer Analysis

The boundary conditions for the inviscid problem are found by considering corrections for the viscous boundary layers at \( z = 0 \) and \( z = -h(x) \) and matching the results with the outer inviscid problem. The analysis is essentially identical to the work of Kakutani and Matsuuchi (1975) and Mei (1983, §11.9) for long surface waves. In what follows only the sloping boundary at \( z = -h(x) \) is described, the procedure for the horizontal boundary at \( z = 0 \) (assumed to be a no-slip boundary) is virtually identical.

Consider Figure 2.2. The boundary layer coordinate system \((x_b, z_b)\) is rotated an angle \( \theta(x) \) from the outer coordinate
Figure 2.2. Definition sketch for bottom boundary layer problem.
system \((x,z)\). Since \(\gamma = \delta/D \leq O(\alpha)\) a stretched vertical coordinate 
\(\hat{z} = z_b/\gamma\) is defined. Now within the boundary layer the dependent 
variables are assumed to be given by the sum of the outer contribution, 
now denoted \((\ )_o\), and a boundary layer correction \((\ )_c\):

\[
\begin{align*}
    u_{b\hat{z}} &= u_o \cos \theta + w_o \sin \theta + u_c \\
    w_{b\hat{z}} &= w_o \cos \theta - u_o \sin \theta + w_c \\
    p_{b\hat{z}} &= p_o + p_c
\end{align*}
\] (2.23a,b,c)

where the boundary layer and correction quantities are functions 
of \((x_b, z, t)\) and the outer quantities can be written as functions 
of \((x_b, z_b, t)\) through

\[
\begin{align*}
    x &= x_b \cos \theta - z_b \sin \theta \\
    z &= z_b \cos \theta - x_b \sin \theta
\end{align*}
\] (2.24a,b)

Since \(\gamma \ll 1\) the density is considered constant within the 
boundary layer and therefore does not appear in (2.23). This is 
easily satisfied if the interface height is small \((\epsilon \ll 1)\) and 
the bottom boundary does not intersect the interface region. 
This last assumption also avoids the problem of boundary 
streaming induced by the combined presence of a sloping bottom
and vertical density gradient (Phillips 1970). Matching with the outer inviscid solutions requires

\[ (\ )_c \to 0 \text{ as } \hat{z} \to \infty \]  \hspace{1cm} (2.25a,b)

and

\[ u_{b\hat{z}}' \cdot w_{b\hat{z}}' = 0 \text{ at } \hat{z} = 0 \text{ (z=-h)} . \]

Recalling the assumption of small bottom slope, it follows that

\[ \tan \theta = -\frac{\partial h}{\partial x} = -\alpha \frac{\partial h}{\partial x} . \]

Therefore,

\[ \cos \theta = 1 + O(\alpha^2) , \]

\[ \sin \theta = -\alpha \frac{dh}{dx} + O(\alpha^3) , \]

and (2.23a,b) become

\[ u_{b\hat{z}} = u_o - \alpha \frac{dh}{dx} w_o + u_c + O(\alpha^2) \]

and

\[ w_{b\hat{z}} = w_o + \alpha \frac{dh}{dx} u_o + w_c + O(\alpha^2) \]  \hspace{1cm} (2.26a,b)

Similarly (2.25a,b) give

\[ x = x_b + O(\alpha) \]

\[ z = z_b + O(\alpha) . \]  \hspace{1cm} (2.27a,b)
The governing equations for the boundary layer
quantities are, using the same nondimensionalization as the
inviscid problem and dropping the \( bl \) subscript,

\[
\alpha^2 \frac{\partial u}{\partial X_b} + \alpha \frac{1}{c} \frac{\partial u}{\partial s_b} + \left( \frac{\alpha}{\gamma} \right) \frac{\partial w}{\partial z} = 0
\]  

(2.28)

\[- \frac{\partial u}{\partial s_b} + \alpha u \left[ \frac{\partial u}{\partial X_b} + \frac{1}{c} \frac{\partial u}{\partial s_b} \right] + \left( \frac{\alpha}{\gamma} \right) w \frac{\partial u}{\partial z} =
\]

\[- \frac{1}{\rho(-h)} \left( \frac{\partial u}{\partial X_b} + \frac{1}{c} \frac{\partial u}{\partial s_b} \right) + \frac{1}{R} \left\{ (\alpha \frac{\partial}{\partial X_b} + \frac{1}{c} \frac{\partial}{\partial s_b} ) u^2 + \frac{1}{\gamma^2} \frac{\partial^2 u}{\partial z^2} \right\}
\]

(2.29)

\[
\alpha \left( - \frac{\partial w}{\partial s_b} + \alpha u \left[ \frac{\partial w}{\partial X_b} + \frac{1}{c} \frac{\partial w}{\partial s_b} \right] + \left( \frac{\alpha}{\gamma} \right) w \frac{\partial w}{\partial z} \right) =
\]

\[- \frac{1}{\rho(-h)} \frac{1}{\gamma} \frac{\partial p}{\partial z} + \frac{1}{R} \left\{ (\alpha \frac{\partial}{\partial X_b} + \frac{1}{c} \frac{\partial}{\partial s_b} ) w^2 + \frac{1}{\gamma^2} \frac{\partial^2 w}{\partial z^2} \right\}
\]

(2.30)

where \( X_b \) and \( s_b \) are the slow space and the phase
variables in the boundary layer coordinate system. Here
\( R = (U_D/v) \) is a Reynolds number and \( v \) is the kinematic
viscosity at \( z = -h \). Recall that

\[
\gamma = \frac{\delta}{D} = \left( \frac{v_D}{U} \right) \frac{1}{2} \frac{1}{D} = \left( \frac{v}{U_D} \right) \frac{1}{2} \frac{\delta}{D}
\]

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or
\[
\frac{UD}{\nu} = \gamma^{-2} \alpha^{-1/2}.
\]

Thus in (2.28) – (2.30) \( M \) may be replaced by
\[
M = \gamma^{-2} \alpha^{-1/2} \tilde{M}^* \tag{2.31}
\]

where \( \tilde{M}^* \) is an \( O(1) \) quantity.

Substitution of (2.23c) and (2.26a,b) into (2.28) – (2.30), expanding the boundary layer corrections as
\[
\begin{pmatrix}
  u \\
  w \\
  p_c
\end{pmatrix}
= \begin{pmatrix}
  u^{(0)} \\
  w^{(0)} \\
  p_c^{(0)}
\end{pmatrix}
+ \alpha \begin{pmatrix}
  u^{(1)} \\
  w^{(1)} \\
  p_c^{(1)}
\end{pmatrix}
+ O(\alpha^2),
\]

and using identical expansions for the outer variables gives, from (2.28),

\[
0(\alpha^0) \quad \gamma \left( \frac{\partial}{\partial \xi} \right) \frac{\partial}{\partial z} w_c^{(0)} = 0 \tag{2.32a}
\]

\[
0(\alpha) \quad \frac{1}{c} \frac{\partial}{\partial s} (u_o^{(0)} + u_c^{(0)}) + \gamma \left( \frac{\partial}{\partial \xi} \right) \frac{\partial}{\partial z} (w_c^{(1)})
\]
\[+
\frac{\partial}{\partial z} w_o^{(0)} = 0 \tag{2.32b}
\]
from (2.29),

\[ 0(a^0) - \frac{3}{\partial s} (u_o^{(0)} + u_c^{(0)}) = - \frac{1}{\rho(-h)} \frac{1}{c} \frac{3}{\partial s} (p_o^{(0)} + p_c^{(0)}) + \frac{1}{\mathbb{R}_-} \frac{3}{\partial z^2} (u_c^{(0)}) \]  

(2.33)

and from (2.30),

\[ 0(a^0) \frac{\partial}{\partial \gamma} (\frac{\partial}{\partial z} \gamma) \cdot (p_c^{(0)}) = 0 \]  

(2.34)

Equations (2.27a,b) have been used to write all quantities as functions of \( X \) and \( s \) to the present approximation rather than \( X_b \) and \( s_b \). Here \((\gamma/\alpha)\) is considered an \( O(1) \) quantity and the fact

\[ \frac{3}{\partial z} \gamma \cdot (\gamma)^o = \gamma \frac{3}{\partial z} (\gamma)^o \]

is used.

From (2.32a)

\[ \frac{3}{\partial z} w_c^{(0)} = 0 \]

or \[ w_c^{(0)} = G(X,s) \].
From (2.25a) \( w_c \to 0 \) as \( \hat{z} \to \infty \) thus \( w_c^{(0)} = 0 \). From (2.26b) and the counterpart analysis for the \( z = 0 \) boundary layer the \( O(\alpha^0) \) inviscid boundary conditions (2.17) are confirmed:

\[
w^{(0)}(z=0) = w^{(0)}(z=-h) = 0.
\]

In a similar fashion (2.34) gives \( p_c^{(0)} = 0 \). Putting this result in (2.33) along with the outer \( O(\alpha^0) \) x-momentum equation (2.11c) results in

\[
\Re - \frac{\partial u_c^{(0)}}{\partial s} + \frac{\partial^2}{\partial z^2} u_c^{(0)} = 0
\]

(2.35)

which is subject to

\[
u_c^{(0)} \to 0 \quad \text{as} \quad \hat{z} \to \infty
\]

(2.36a,b)

and

\[
u_c^{(0)} = -u_o^{(0)} \quad \text{at} \quad \hat{z} = 0 \ (z=-h).
\]

Introducing the Fourier transform of a function \( f(s) \)

\[
\hat{f}(k) = \int_{-\infty}^{\infty} e^{-iks} f(s) ds
\]

(2.37)

and the inverse

\[
f(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iks} \hat{f}(k) dk
\]

(2.38)
and transforming (2.35) and (2.36a,b) gives

$$-ik \mathbb{R}^* u^c(0) + \frac{\partial^2}{\partial \hat{z}^2} u^c(0) = 0$$  \hspace{1cm} (2.39)

subject to $u^c(0) \to 0$ as $\hat{z} \to \infty$ and $u^c(0) = -u^o(0)$ at $\hat{z} = 0$.

The bounded solution to this equation is, after inverting the transform,

$$ u^c(0) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} u^o(0) \left|_{z=-h} \right. e^{-\sigma \hat{z}} e^{iks} dk $$  \hspace{1cm} (2.40)

where $\sigma = (-ik \mathbb{R}^*)^{1/2}$.

Now from the $O(\alpha)$ continuity equation (2.32b) and using the $O(\alpha^0)$ inviscid continuity equation (2.11a)

$$ \frac{\partial \omega^c(1)}{\partial \hat{z}} = - \left( \frac{\chi}{\alpha} \right) \frac{1}{c} \frac{\partial u^c(0)}{\partial s} $$

or

$$ \frac{\partial \omega^c(1)}{\partial \hat{z}} = \left( \frac{\chi}{\alpha} \right) \frac{\phi'(-h)}{2\pi} \int_{-\infty}^{\infty} \hat{u} (ik) e^{-\sigma \hat{z}} e^{iks} dk $$

where (2.20b) has been used to replace $u^o(0)$ ($z=-h$). Integrating once gives
\[ w_c^{(1)} = \left( \frac{\gamma}{\alpha} \right) \frac{\phi'(-h)}{2\pi} \int_{-\infty}^{\infty} \hat{A} \left( \frac{ik}{\sigma} \right) (1-e^{-\sigma z})e^{iks} \, dk + b(X,s). \tag{2.41} \]

Requiring that \( w_c^{(1)} \to 0 \) as \( \hat{z} \to \infty \) gives

\[ b(X,s) = -\left( \frac{\gamma}{\alpha} \right) \frac{\phi'(-h)}{2\pi} \int_{-\infty}^{\infty} \hat{A} \left( \frac{ik}{\sigma} \right) e^{iks} \, dk. \]

Recalling that \( \sigma = (-ik \mathbb{R}_-)^{1/2} \) and letting \( \hat{z} = 0 \) in (2.41) leads to

\[ w_c^{(1)} (\hat{z}=0) = -\frac{\phi'(-h)}{2\pi (\mathbb{R}_-)^{1/2}} \left( \frac{\gamma}{\alpha} \right) \int_{-\infty}^{\infty} \hat{A}(-1+i \text{sgn} \, k) |k|^{1/2} e^{iks} \, dk. \]

Therefore, the boundary condition for \( w_o^{(1)} (z=-h) \) is, using (2.25b) and (2.26b),

\[ w_o^{(1)} (z=-h) = -u_o^{(0)} |_{z=-h} \frac{dh}{dX} - w_c^{(1)} (z=0) \]

or

\[ w_o^{(1)} (z=-h) = -c\phi'(-h)A \frac{dh}{dX} \]

\[ + \frac{\phi'(-h)}{2\pi (\mathbb{R}_-)^{1/2}} \left( \frac{\gamma}{\alpha} \right) \int_{-\infty}^{\infty} \hat{A}(-1+i \text{sgn} \, k) |k|^{1/2} e^{iks} \, dk \tag{2.42} \]

A similar analysis at \( z = 0 \) gives
\[ w_o(1) = - \frac{\phi'(0)}{2\pi (2 R_+^* )^{1/2}} (\gamma/\alpha) \int_{-\infty}^{\infty} \hat{A}(-1+\text{sgn } k)|k|^{1/2} e^{iks} \, dk \quad (2.43) \]

where \( R_+^* = \gamma^2 \alpha^{1/2} (UD/v_+^*) \) and \( v_+^* \) is the kinematic viscosity at \( z = 0 \).

2.2.2 Completion of the 0(\alpha) Problem

At this point we return to (2.22) and specify the left-hand side from (2.42) and (2.43) in the preceding boundary layer analysis. From Leibnitz's rule the second term on the right-hand side of (2.22) can be rewritten as

\[
cA \int_{-h}^{0} \frac{\partial}{\partial x} (\bar{\rho} \phi'^2) \, dz = cA \frac{\partial}{\partial x} \int_{-h}^{0} \bar{\rho} \phi'^2 \, dz - cA (\bar{\rho} \phi'^2) \bigg|_{-h}^{0} \frac{dh}{dx}.
\]

Introducing this equation along with (2.42) and (2.43) into (2.22) gives the evolution equation for \( A(X,s) \),

\[
(3c_X I_0^* + cI_{0X}^*) A + 2cI_0^* A_X + 3I_1 A A_s + \frac{1}{2} \frac{I_2}{2} A_{sss} = \]

\[
\frac{1}{2\pi} (\gamma/\alpha) \left\{ \frac{\bar{\rho}(0) \phi'^2(0)}{(2 R_+^* )^{1/2}} + \frac{\bar{\rho}(-h) \phi'^2(-h)}{(2 R_-^* )^{1/2}} \right\} \int_{-\infty}^{\infty} \hat{A}(-1+\text{sgn } k)|k|^{1/2} e^{iks} \, dk \quad (2.44)\]

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where

\[ I_0 = \int_{-h}^{0} \frac{\rho \phi_r}{2} \, dz \]

\[ I_1 = \int_{-h}^{0} \frac{\rho \phi_r}{3} \, dz \] \hspace{1cm} (2.45a, b, c)

and

\[ I_2 = \int_{-h}^{0} \rho \phi_r^2 \, dz \].

Note that the integrals (2.45a, b, c) and the coefficient of the dissipative term all vary with \( X \).

The integral operator in the dissipative term can be transformed using the convolution theorem to give

\[
\int_{-\infty}^{\infty} \hat{\Lambda} \left(-1+i \text{sgn} k\right) |k|^{1/2} e^{ik s} \, dk =
\]

\[
\sqrt{2\pi} \int_{-\infty}^{\infty} \frac{\partial \Lambda}{\partial s'} \frac{1-\text{sgn}(s-s')}{|s-s'|^{1/2}} \, ds'.
\] \hspace{1cm} (2.46)

Therefore, the evolution equation can also be written as
\[ A \frac{3}{2X} \left( \frac{3}{2} \ln c + \frac{1}{2} \ln I_0 \right) + A_x + \frac{3}{2c} \frac{I_1}{I_0} AA_s \]

\[ + \frac{1}{2c^2} \frac{I_2}{I_0} A_{sss} = \frac{1}{4\sqrt{\pi I_0 c}} \left( \frac{\partial (0) \phi' (0)}{\partial x} \right) \left( \frac{\partial (0) \phi' (0)}{\partial x} \right) \]

\[ + \frac{\partial (-h) \phi' (-h)}{\partial x} \]

\[ \cdot \int_{-\infty}^{\infty} \frac{3A}{s-s'} \frac{\text{sgn}(s-s')}{|s-s'|^{1/2}} \, ds' \quad (2.47) \]

We note here that the effect of viscosity in (2.47) causes not only dissipation, through the term \(-\text{sgn}(s-s')/|s-s'|^{1/2}\) in the integral, but also dispersion through the term \(|s-s'|^{-1/2}\)

(Kakutani and Matsuuchi 1975).

2.2.3 **Cubic Nonlinear Correction**

Under certain circumstances it is possible for the coefficient of the quadratic nonlinear term in (2.47), \(3I_1/2C_0\), to become small. Cases in point are uniform or weak exponential stratification under the Boussinesq assumption \(\sigma \ll 1\) (Long 1956) or when a vertically sheared current is added to the theory with uniform stratification (Gear and Grimshaw 1983). Of more importance to the present study, which deals primarily with sharper stratifications (\(\varepsilon = h/D = 0(\alpha)\)), are situations when the two layers are nearly of equal depths. With waves propagating from deep to shallow water, or vice versa, it is possible that such a

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situation, described earlier as the turning point problem, will be encountered. When the coefficient becomes $O(\alpha)$ the order of the entire quadratic term becomes $O(\alpha^2)$ and cubic nonlinearity $O(\alpha^2)$ may balance or dominate quadratic nonlinearity in some parametric region (Miles 1979). Here the KdV balance between nonlinearity and dispersion is lost. In the context of a two-layer system the appropriate balance is

$$\beta = 0 \left( \frac{|d_+ - d_-|}{D} \right) \alpha.$$ 

If $|d_+ - d_-|/D = 0(\alpha)$ then $\beta = 0(\alpha^2)$ and cubic nonlinearity, which limits the maximum attainable wave height (Long 1956), should be incorporated into the theory to improve the predictive capability.

Starting from Long's (1953) equation for steady waves in a constant depth fluid, invoking the balance $\beta = 0(\alpha^2)$, Miles (1979) and Gear and Grimshaw (1983) found the first correction to the KdV equation (2.47) to be $-(3I_3/cI_0)A^2 A_s$ where

$$I_3 = \int_{-h}^{0} \frac{1}{\beta \phi} \phi^4 \, dz.$$  \hspace{1cm} (2.48)

In the derivation of the evolution equation (2.47) the effects of nonlinearity dispersion, depth variation and dissipation all entered the equation separately. Similarly we expect the influence of cubic nonlinearity to enter separately when $\beta = 0(\alpha^2)$. The
cubic nonlinearity can be added directly to (2.47) to give the extended KdV (EKdV) equation

\[ A \frac{\partial}{\partial x} \left( \frac{3}{2} \ln c + \frac{1}{2} \ln I_0 \right) + A_x + \frac{3}{2c} \frac{I_1}{I_0} AA_s \]

\[ - \frac{3}{c} \frac{I_3}{I_0} A^2 A_s + \frac{1}{2c} \frac{I_2}{I_0} A_{sss} = \]

\[ \frac{1}{4\sqrt{\pi}c} I_0^{\chi(0)} \int_{\mathbb{R}_+^{1/2}}^{\infty} \frac{\rho(-h)\phi_1^2(0) + \rho(-h)\phi_1(-h)^2}{\mathbb{R}_-^{1/2}} \left[ s - s' \right]^{-1/2} ds' \]

(2.49)

Formal attempts to derive the unsteady equation using three different perturbation methods have been unsuccessful. All three methods give identical results, but are not consistent with Miles or Gear and Grimshaw in the steady limit. Furthermore, (2.49) collapses to the proper two-layer version (see §2.3) of the cubic coefficient given by Kakutani and Yamasaki (1978) and Helfrich, Melville and Miles (1984), in which the derivations started from the assumption of unsteady two-layer flow. Details of the unsteady continuous stratification derivations and the differences from the steady and two layer results are discussed in Appendix 1.
The relative importance of quadratic to cubic nonlinearity is shown by Miles (1979) to be given by

$$\tilde{\alpha} = \frac{I_1}{2 I_3} \quad .$$  \hspace{1cm} (2.50)

Cubic nonlinearity is negligible if $|\tilde{\alpha}| > |\alpha|$ and important for $\tilde{\alpha} = O(\alpha)$. For the two-layer stratification (§2.3)

$$|\tilde{\alpha}| = \frac{1}{2} \frac{(d-d^2)(1-2d)}{(1-d)^3 + d^3} \quad .$$  \hspace{1cm} (2.51)

where $d = d_+/D$ is the nondimensional depth of the lower layer.

Equation (2.51) is plotted in figure 2.3. Typically $|\alpha| = 0.03-0.07$ which from Figure 2.3 shows that cubic nonlinearity is important for $0.3 < d < 0.7$. This is a significant region around the equal depths implying that cubic nonlinearity may be extremely important for waves over bottom topography which pass through the turning point. This is especially true for slow topographic variations since the waves will spend considerable time where cubic nonlinearity is comparable to quadratic nonlinearity. Note that $|\tilde{\alpha}| \to 0$ as $d \to 1$. This is misleading since KdV theory for internal waves becomes invalid as one of the layer depths nears the total depth. Intermediate on deep water theory becomes appropriate.

It might be argued that additional higher order terms should be included along with the cubic correction in (2.49).
Figure 2.3. Plot of equation (2.51).
Koop and Butler (1981) have calculated all these second order terms for the two-layer constant depth case. Formally these additional terms are \(0(\alpha \beta, \beta^2)\). But when cubic nonlinearity is comparable to quadratic nonlinearity \(\beta = 0(\alpha^2)\) and these additional terms are \(0(\alpha, \alpha^2)\) relative to cubic nonlinearity. On the other hand if the cubic term is retained in simulations outside the region where \(\tilde{\alpha} = 0(\alpha)\), say on a shelf after the turning point region, formally one should also include all the higher order terms. This higher equation would then be valid over a distance \(O(\alpha^{-2})\), whereas the KdV equation is valid over a distance \(O(\alpha^{-1})\) or equivalently \(O(\lambda^{-1})\). Thus carrying the cubic term only in the KdV equation should introduce a very slight error over distances \(O(\lambda^{-1})\) when \(|\tilde{\alpha}| \gg |\alpha|\) and a significant error only over distances \(O(\lambda^{-2})\). The experimental measurements (Chapter 4) with which (2.49) is tested over propagation distances on the order of 2-3L where L is the length of the slope, therefore higher order effects, other than the cubic nonlinearity, should not be significant.

2.2.4 Renormalization and Solitary Wave Solutions

Analysis and numerical solution of (2.49) is most convenient if the equation is renormalized. First returning to dimensional variables and invoking the Boussinesq approximation
\((\rho(z) = \rho_o)\) gives

\[ A \frac{\partial}{\partial x} \left( \frac{3}{2} \ln c + \frac{1}{2} \ln I_0 \right) + A_x + \frac{3}{2c} \frac{I_1}{I_0} A_{ss} = \]

\[ - \frac{3}{c} \frac{I_3}{I_0} A_{ss} + \frac{1}{2c} \frac{I_2}{I_0} A_{sss} = \]

\[
\frac{\rho_o \sqrt{v}}{4\pi} \frac{1}{c} I_0 \left\{ \phi''(0) + \phi''(-h) \right\} \int_{-\infty}^{\infty} \frac{\partial A}{\partial s'} \frac{1 - \text{sgn}(s-s')}{|s-s'|^{1/2}} \, ds'. \quad (2.52)
\]

The viscosity has also been replaced by an average value. The difference in viscosity between fresh and sea water is less than 3% at 20°C (Myers, Holm and McAllister 1969, p2-4). The corresponding eigenvalue problem for \(\phi(z)\) and \(c\), (2.18) and (2.19), becomes

\[
\phi'' - \left( \frac{g}{c^2} \right) \frac{\rho}{\rho_o} \phi = 0
\]

\[ (2.53a,b) \]

\[ \phi(0) = \phi(-h) = 0 \quad . \]

Now nondimensionalize the integrals \(I_j(j=0,1,2,3)\) by \(\rho_o, D\), introduce the dimensionless variables

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\[ \sigma = \frac{c_0}{\xi} s, \quad \xi = \frac{x}{L}, \quad \tau = \frac{\beta}{\lambda} \int_{\xi_0}^{\xi} \frac{1}{c^3} \frac{I_2}{I_0} \, d\xi \]

\[ c = c/c_0, \quad \zeta = c^{3/2} \int_0^{1/2} A/a, \quad J_0 = I_0/I_{00} \]

\[ (2.54a,b,d,c,d,e,f) \]

where \( c_0 \) and \( I_{00} \) are reference values of \( c \) and \( I_0 \) at \( \xi_0 \), and transform (2.52) to

\[ \zeta_t + 12[U_1 \zeta - 2U_2 \zeta^2] \zeta_\sigma + \zeta_{\sigma\sigma\sigma} = \]

\[ U_3 \int_{-\infty}^{\infty} \frac{\partial \zeta}{\partial \sigma'} \frac{1-\text{sgn}(\sigma-\sigma')}{|\sigma-\sigma'|^{1/2}} \, d\sigma' \]  \hspace{1cm} (2.55)

where

\[ U_1 = \frac{1}{4} \frac{a}{\beta} \left( \frac{c}{J_0} \right)^{1/2} \frac{I_1}{I_2} \] \hspace{1cm} (2.56)

\[ U_2 = \frac{1}{4} \frac{a^2}{\beta} \left( c J_0 \right)^{-1} \frac{I_3}{I_2} \] \hspace{1cm} (2.57)

\[ U_3 = \frac{\nu b}{c_0} \frac{1}{D^2} \frac{1}{(4\pi)^{1/2}} \frac{c^2}{I_2} \{ \phi'(0)^2 + \phi'(-h)^2 \} \] \hspace{1cm} (2.58)
are measures of quadratic nonlinearity, cubic nonlinearity and
dissipation, respectively, relative to dispersion. When
$\lambda \gg \alpha, \beta, \gamma$ (2.55) reduces to $\zeta = \text{constant or}$

$$A \propto c^{-3/2} I_0^{-1/2} \quad (2.59)$$

which can be characterized as a generalization of Green's law for
continuously stratified fluids.

When dissipation is neglected and the depth is
constant ($c=1, I_0=1$) (2.55) possesses the solitary wave solution
(Miles 1979)

$$\zeta = (\cosh^2 \theta - \mu \sinh^2 \theta)^{-1} \quad (2.60)$$

when

$$U_1 = 1 + \mu$$

$$U_2 = \mu \quad (0 < \mu < 1) \quad (2.61a,b)$$

and where

$$\theta = \sigma - 4\tau \quad . \quad (2.62)$$

For a given depth $D$, wave amplitude $a$ and stratification $\bar{\rho}(z)$
the length scale $\lambda$ is found from (2.56), (2.57) and (2.61a,b),
\[ \beta = \frac{1}{4} \frac{\alpha}{I_2} (I_1 - \alpha I_3) \]  

(2.63)

after which \( \mu \) is recovered from (2.61). The dimensionless
(by c) nonlinear phase speed \( c_{n1} \) of the solitary wave is

\[ c_{n1} = 1 + \frac{1}{2} \frac{\alpha}{I_o} (I_1 - \alpha I_3) . \]  

(2.64)

For a conservative system \( (U_3 = 0) \) (2.55) possesses

the integral invariants

\[ M = \int_{-\infty}^{\infty} \zeta \, d\sigma = \text{constant} \]  

(2.65a)

and

\[ E = \int_{-\infty}^{\infty} \zeta^2 \, d\sigma = \text{constant} \]  

(2.65b)

provided \( \zeta \uparrow 0 \) as \( |\sigma| \uparrow \infty \). For constant depth (2.65a) is a

statement of mass conservation and (2.65b) is a statement of energy

conservation. If the depth is variable \( M \) is not physical mass

but a "pseudo" mass as a consequence of the unidirectional

propagation assumption (Miles 1979). Conservation of physical

mass requires consideration of a trailing shelf and reflected wave

(Miles 1979; Knickerbocker and Newell 1980). Conservation of \( E \)

derpresents true energy conservation. The constant depth

conservative form of the EKdV equation (2.55) may be transformed
to KdV equation and solved exactly for a given initial condition by inverse scattering theory (see §5 of Helfrich, Melville and Miles (1984) in Appendix 3).

An adiabatic approximation for the decay of a solitary wave solution of (2.55) with \( U_2 = 0 \) (KdV equation) in constant depth can be constructed as follows. Multiply (2.55) by \( \zeta \) and integrate over \( \sigma \) to get

\[
\frac{\partial}{\partial t} \int_\infty^\infty \zeta^2 \, d\sigma = \left( \frac{2}{\pi} \right)^{1/2} u_3 \int_\infty^\infty \zeta \left| k \right|^{1/2} \left( -1 + i \, \text{sgn} \, k \right) e^{ik\sigma} \, dk \, d\sigma
\]

where (2.46) has been used. Inverting the order of integration on the right-hand side, noting that

\[
\hat{\zeta}(-k) = \hat{\zeta}(k)^* = \int_{-\infty}^{\infty} \zeta \, e^{-ik\sigma} \, d\sigma
\]

where \((\ )^* = \text{complex conjugate and } \zeta(\sigma, \tau) \text{ real, gives}

\[
\frac{\partial}{\partial t} \int_{-\infty}^{\infty} \zeta^2 \, d\phi = - \left( \frac{2}{\pi} \right)^{1/2} u_3 \int_{-\infty}^{\infty} \left| \hat{\zeta} \right|^2 \left| k \right|^{1/2} \, dk.
\]

(2.66)

The contribution from \( i \, \text{sgn} \, k \) to (2.66) is zero because the integrand is odd. This term does, however, cause dispersion.

The slowly varying solitary wave is from (2.60) with \( \mu = 0 \)

\[
\zeta = A(\tau) \, \text{sech}^2 \left( A\, u_1 \right)^{1/2} (\sigma - 4 \int A \, d\tau)
\]

(2.67)

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which has the Fourier transform

$$\hat{\zeta}(k) = \frac{\pi k}{U_1} \text{cosech} \left( \frac{\pi k}{2(U_1)^{1/2}} \right).$$  \hspace{1cm} (2.68)

Introducing (2.67) and (2.68) into (2.64) gives

$$\frac{3}{\partial \tau} \left( \frac{U_1^{3/2}}{U_1^{1/2}} \right) \left( \int_{-\infty}^{\infty} \text{sech}^4 q \, dq \right) = - (2\pi)^{3/2} \frac{U_3}{|U_1|^2} \left( \int_{0}^{\infty} \frac{k^{5/2}}{\sinh^2 \left( \frac{\pi k}{2(U_1)^{1/2}} \right)} \, dk \right).$$  \hspace{1cm} (2.69)

The integral on the left-hand side is $4/3$ and the right-hand side is (Gradshteyn and Ryzhik 1980 §3.527)

$$\int_{0}^{\infty} \frac{k^{5/2}}{\sinh^2 \left( \frac{\pi k}{2(U_1)^{1/2}} \right)} \, dk = - 4 \frac{(U_1)^{7/4}}{\pi^{7/2}} \Gamma(7/2) \zeta(5/2)$$

where $\Gamma(x)$ is the gamma function and $\zeta(x)$ is the zeta function.

Thus (2.69) becomes

$$\frac{dA}{d\tau} = - \frac{8}{\sqrt{2}} \frac{1}{\pi^2} \frac{U_1^{1/4}}{U_3} \Gamma(7/2) \zeta(5/2) \Lambda^{5/4}$$

and after integrating from 0 to $\tau$

$$\frac{A(\tau)}{A_0} = \left( 1 + C(U_1 A_0)^{1/4} U_3 \right)^{-4} \hspace{1cm} (2.70)$$
where $A_0 = A(\tau = 0)$ and

$$C = \frac{2^{1/2}}{\pi} \Gamma(7/2) \zeta(5/2) = 0.64 \quad .$$

Equation (2.70) is consistent with the results of Ott and Sudan (1970) and Mei (1983 §11.9). An adiabatic approximation for the extended equation could not be done analytically.

2.3 Two-Layer Model

For long interval waves the most energetic mode is typically the first mode. Therefore, it is common practice to study internal wave motions through the simpler two-layer system which retains only the dominant first mode. The primary differences between the continuous stratification and the two-layer models are:

1. an increase in the linear phase speed in the corresponding two-layer model and

2. a strong shear layer at the interface in two-layer model.

In what follows the evolution equation (2.55) is specialized for a two-layer system (Figure 2.4) and interfacial shear is included.

Testing of the resulting two-layer model and the continuous stratification model with experimental data to determine the usefulness
Figure 2.4. Definition sketch for two-layer system.
of the simpler two-layer model in situations involving topographic scattering and therefore significant dispersion is described in Chapter 4.

From (2.53a,b) the vertical structure function must satisfy

\[ \phi''_\pm = 0 \quad (2.72) \]

subject to

\[ \phi_- = 0 \quad \text{at } z = -d_- \]

\[ \phi_+ = 0 \quad \text{at } z = d_+ \quad (2.73a,b,c) \]

\[ \phi_+ = \phi_- \quad \text{at } z = 0 \]

where the +/- subscript refers to the upper/lower layer, respectively. The solutions to (2.72) and (2.73a,b,c) are

\[ \phi_+ = -\left(\frac{z-d_+}{d_+}\right) \quad (2.74a,b) \]

\[ \phi_- = \frac{z}{d_-} \]

The linear phase speed is found by integrating (2.53a) across the interface.
\[ \phi'(0^+) - \phi'(0^-) - g/c^2 \left( \frac{\rho_+ - \rho_-}{\rho_0} \right) \phi(0) = 0 \]

and using (2.74a,b)

\[ c^2 = g \frac{\Delta \rho}{\rho_0} \frac{d_+ d_-}{d_+ + d_-} . \quad (2.75) \]

From (2.45a,b,c), (2.48) and (2.74a,b)

\[ I_0 = \rho_0 \left( \frac{d_+ - d_-}{d_+ d_-} \right) \]

\[ I_1 = \rho_0 \left( \frac{d_+^2 - d_-^2}{d_+^2 + d_-^2} \right) \]

\[ I_2 = \frac{1}{3} \rho_0 \left( d_+ + d_- \right) \quad (2.76a,b,c,d) \]

and

\[ I_3 = \rho_0 \left( \frac{d_+^3 + d_-^3}{d_+^3 + d_-^3} \right) . \]

Noting that \( d_- = d_-(x) \) the dimensional governing equation (2.52) becomes

\[ \frac{1}{2} \frac{c_x}{c} A + A_x + \frac{3}{2c} \left( \frac{d_+ - d_-}{d_+ d_-} \right) AA_s - \frac{3}{2} \frac{1}{d_+^2 d_-^2} \left( \frac{d_+^3 + d_-^3}{d_+^3 + d_-^3} \right) A^2 A_s + \frac{d_+ + d_-}{6c} \quad \text{Ass} = \]

\[ \frac{\sqrt{V}}{4m^{1/2} c} \left( \frac{d_+ - d_-}{d_+ + d_-} \right) \left( \frac{1}{d_+^2} + \frac{1}{d_-^2} \right) \left\{ \frac{\partial A}{\partial s} \right\} \left[ \frac{i - \text{sgn}(s-s')}{s-s'} \right]^{1/2} ds' \quad (2.77) \]
where \( A(x,s) \) is the interfacial displacement. Nondimensionalizing
the layer depths \( d_\pm \) by the total depth \( D \) at the reference location
\( x_o \),

\[
\begin{align*}
d_+ &= (1-d)D \\
d_- &= f(x) \cdot D
\end{align*}
\]

(2.78a,b)

where \( f(x_o) = d \), introducing (2.54a,b,c,d,e) and noting that
\( \zeta = c^{1/2} A/a \) (2.77) reduces to (2.55) with

\[
U_1 = \frac{3}{4} \frac{\alpha}{B} c \frac{3^{1/2}}{(1-d-f(\xi))} \frac{(1-d-f(\xi))}{(1-d)^2 f(\xi)^2}
\]

\[
U_2 = \frac{3}{4} \frac{\alpha^2}{B} c \frac{(1-d)^3 + t^3(\xi)}{(1-d)^3 f(\xi)^3 (1-d+f(\xi))}
\]

(2.79a,b,c)

\[
U_3 = \frac{3c^2}{2\pi} 1/2 (\nu \xi) c_o 1/2 \frac{1}{B} \frac{1}{(1-d+f(\xi))} \left\{ \frac{1}{(1-d)^2} + \frac{1}{f(\xi)^2} \right\}
\]

and

\[
c^2 = \frac{f(\xi)}{d(1-d+f(\xi))}
\]

(2.80)

In the two-layer situation viscous boundary layers on either
side of the interface will develop. The influence of this inter-
facial damping on the decay of a solitary wave was examined by
Leone, Segur and Hammack (1982). They developed an adiabatic approximation for the decay of a single solitary wave. We would like to incorporate this effect in to the general evolution equation (2.55) for the two-layer case. The interfacial boundary layer analysis is essentially identical to the bottom boundary layer problem in §2.2.1. For analytical tractability the interface is assumed to be flat for the boundary layer analysis. If this assumption is not made a fully nonlinear analysis results because the boundary layer scale $\gamma$ is comparable to $\alpha$ the wave amplitude scale. Thus the kinematic boundary condition applied at $z = \alpha A$ in the inviscid problem is applied at $\tilde{z} = \frac{\alpha}{\gamma} A \approx A$ ($A=0(1)$ in the present nondimensionalization). The condition is nonlinear at the lowest order. If the boundary is assumed flat when solving for the boundary layer correction, equations identical to (2.35) govern the boundary layer correction on either side of the interface. The problem is completed by requiring that the shear stress be continuous across the interface, as opposed to the no-slip condition at the bottom. Details of the analysis are given in Appendix 2. The result is to change (2.79c) to

$$U_3 = \frac{3c^2}{2\pi^{1/2}} \left( \frac{v}{c} \right)^{1/2} \frac{1}{D\beta} \frac{1}{(1-d+f)} \left[ \frac{1}{(1-d)^2} + \frac{1}{f^2} + \frac{1}{2} \left( \frac{(1-d)+f}{(1-d)^2f^2} \right) \right]$$

(2.81)

The third term in the braces is the interfacial correction.
2.4 **Side Wall Boundary Layers**

In order to compare predictions of the extended KdV equation with laboratory data dissipation along the tank side walls must be included in the theory. The correction is rather simple, however, since the boundary layers along the walls will be identical to the top and bottom boundary layers by virtue of the fact that horizontal velocity is vertically uniform in each layer except near the interface. Thus (2.56) is modified to

\[
U_3 = \frac{1}{2\pi^{1/2}} \left( \frac{\nu_b}{\nu_c^*} \right)^{1/2} \frac{1}{\nu_b} \frac{2}{I_2} \left\{ (2 \frac{d_{\pm}}{w}) \phi^{i2}(0) + (1 + \frac{2d}{w}) \phi^{i2}(-h) \right\} \tag{2.82}
\]

where \( w \) is the tank width. The factors \( 2d_{\pm}/w \) account for two side wall boundary layers of length \( d_{\pm}/w \) relative to the upper/lower boundary layer. Note that the boundary layer at \( z = 0 \) has been removed and just the upper layer side wall dissipation appears in (2.82). The corresponding two-layer model (2.81) is altered in the same manner.

Equation (2.82) is a good approximation for continuous stratification conditions provided that the interfacial height is small compared to the total depth (\( \epsilon = O(\alpha) \)). For \( \epsilon = O(1) \) the horizontal velocity is not vertically uniform over most of the depth and (2.82) would overestimate the side wall influence.
2.5. **Numerical Methods**

Equation (2.55) was solved using the explicit pseudo-spectral method of Fornberg and Whitham (1978). The method evaluates the $\sigma$-derivatives of $\zeta(\sigma, \tau)$ in Fourier space, and steps forward in $\tau$ using a leapfrog procedure. For the constant-coefficient KdV equation

$$u_{\tau} + uu_{\sigma} + u_{\sigma\sigma\sigma} = 0$$  \hspace{1cm} (2.83)

the method gives

$$u_{j}^{m+1} - u_{j}^{m-1} + 2i \Delta\tau \ u_{j}^{m} \ F^{-1}\{k \ F(u_{j}^{m})\}$$

$$- 2i \ F^{-1}\{\sin(k^3 \Delta\tau) \ F(u_{j}^{m})\} = 0,$$  \hspace{1cm} (2.84)

where

$$u_{j}^{m} = u(j \Delta\sigma, m\Delta\tau),$$

$$k = \frac{2\pi}{N\Delta\sigma} \ \nu \ \ (\nu=0, \pm1, \pm2, \cdots, \pm\frac{N}{2})$$

and $N$ is the number grid points in the periodic $\sigma$-domain. Here $F$ and $F^{-1}$ represent the forward and inverse discrete Fourier transforms respectively,
\[ \hat{u}_\nu = F(u) = \sum_{j=0}^{N-1} u_j e^{-(2\pi i j \nu / N)} \]  
(2.85a)

and

\[ u_j = F^{-1}(\hat{u}) = \frac{1}{N} \sum_{\nu} \hat{u}_\nu e^{(2\pi i j \nu / N)} \]  
(2.85b)

where only one half of the contributions at \( \nu = \pm \frac{N}{2} \) are included in (2.85b).

When the same procedure is extended to use on (2.55), we obtain

\[ \zeta_{j}^{m+1} - \zeta_{j}^{m-1} + 24 i \Delta \tau (\frac{\mu_{j}^{m}}{1 - 2 \mu_{j}^{m}}) \hat{\zeta}_{j}^{m} F^{-1} \{ k \hat{\zeta}^{m} \} \]

\[ -2i F^{-1} \{ \sin(k \Delta \tau) \hat{\zeta}^{m} \} \]

\[ - (2(2\pi)^{1/2} \mu_{j}^{m}) F^{-1 \{ (-1 + i \text{ sgn } k) |k|^{1/2} \hat{\zeta}^{m} \} = 0 \]  
(2.86)

where (2.46) has been used. Note that the method easily handles the dissipative operator.

A numerical stability criterion for (2.86) may be approximated by considering the conservative linear equation

\[ \zeta_{t} + B \zeta + \zeta_{\sigma \sigma} = 0 \]  
(2.87)
with $B$ constant. From Fornberg and Whitham (1978) (2.87) has the stability condition

$$|± \sin[\left(\frac{\pi}{\Delta \sigma}\right)^3 \Delta \tau] - \frac{\pi}{\Delta \sigma} B \Delta \tau| < 1 \quad .$$

(2.88)

This is generalized for (2.86) with $U_3 = 0$ by setting

$$B = 12(U_1 \zeta - 2U_2 \zeta^2) \text{ or } B \lesssim 12(U_1 - 2U_2)$$

since $|\zeta| < 1$. The maximum of $|U_1 - 2U_2|$ occurs on the shelf when $U_1 < 0$ so that (2.86) becomes

$$|\sin[\left(\frac{\pi}{\Delta \sigma}\right)^3 \Delta \tau] - \frac{\pi}{\Delta \sigma} 12(U_1 - 2U_2)_{\text{shelf}}| < 1 \quad .$$

(2.89)

The condition (2.89) may be very restrictive, since it is determined from $(U_1 - 2U_2)_{\text{shelf}}$, which in some cases may be an order of magnitude larger than $U_1 - 2U_2$ prior to the slope. To alleviate this problem and optimize the time step $\Delta \tau$, a dummy advective term $c_a \zeta$ with $c_a$ constant, was added and subtracted in (2.86). The numerical approximation then becomes

$$\zeta_{j}^{m+1} - \zeta_{j}^{m-1} + 2i \Delta \tau[12(U_1 \zeta_j^m - 2U_2 \zeta_j^m) + c_a]F^{-1}\{\zeta_j^m\}$$

$$- 2i F^{-1}\{\sin(k^3 \Delta \tau + c_a k \Delta \tau)\zeta_j^m\}$$

$$- 2(c_a / \pi)^{1/2} U_3^m F^{-1}\{(-1 + i \text{ sgn } k)|k|^{1/2}\zeta_j^m\} = 0 \quad ,$$

(2.90)
and the stability condition (2.87) is modified to

$$|\pm \sin[\left(\frac{n}{\Delta \sigma}\right)^3 \Delta \tau + \left(\frac{n}{\Delta \sigma}\right) c_a \Delta \tau] - \left(\frac{n}{\Delta \sigma}\right)\Delta \tau[12(U_1 - 2U_2) + c_a]| < 1.$$ (2.91)

The constant $c_a$ is chosen so that $\Delta \tau$ is maximized and (2.91) is satisfied during the entire run. Equation (2.91) is for the inviscid ($U_3 = 0$) model, however, the presence of the dissipative term in (2.91) only helps guarantee stability.

Implementation of the numerical method was straightforward. All computations were done using a periodic domain (by virtue of Fourier transforms) of length $N\Delta \sigma$, where $N = 256$ typically. The discrete Fourier transforms were computed with the efficient Fast Fourier Transform (FFT) algorithm of Cooley, Lewis and Welch (1969). After every 40 steps in $\tau$ the solution was smoothed by setting

$$\zeta_j^{m-1}/2 = \frac{1}{2} (\zeta_j^m + \zeta_j^{m-1})$$

and

$$\zeta_j^{m-3}/2 = \frac{1}{2} (\zeta_j^{m-1} + \zeta_j^{m-2})$$

and restarting the solution procedure at $\tau$ step $(m-1/2)$ as suggested by Fornberg and Whitham. This served to suppress
high wavenumber artificial disturbances (saw-toothing).

For continuous stratification runs the vertical structure eigenvalue problem (2.53a,b) first had to be solved at several locations in order to determine the phase speed \( c \) and the integrals \( I_j(j=0,1,2,3) \) and therefore \( U_1, U_2 \) and \( U_3 \), as functions of \( \xi \). For a given density profile, either experimental data or an analytical expression, this was accomplished using 4th order Runge-Kutta integration and the shooting method (Dahlquist and Björck 1974, §8.4.2). At a given location \( (\xi) \) two estimates of the eigenvalue \( c \) were computed from the two-layer model \( (c_{t1} \text{ and } 0.8 c_{t1}) \) and (2.53a) was integrated from \( z = - h \) subject to

\[
\phi(h) = 0
\]

and

\[
\phi'(-h) = 1 \text{ (nondimensional by } D) \]

for each value of \( c \) to give two estimates of \( \phi(0) \). The secant method

\[
c_{n+1} = c_n - \phi_n(0) \frac{(c_n - c_{n-1})}{\phi_n(0) + \phi(0)}
\]

was then used to give a new estimate of \( c \), where the subscripts refer to the estimate level, not the mode number. Equation (2.53a) was then integrated to give a new estimate of \( \phi(z) \).
The iteration was continued until consecutive values of \( c \) differed by \( 10^{-6} \). The corresponding \( \phi(z) \) was checked to insure \( \phi(0) \leq 10^{-6} \) and that the first mode structure (no zero crossings) had been computed. Next \( \phi(z) \) was normalized to have a maximum value of 1 and the integrals \( I_j \) were computed using Simpson's Rule. The procedure was then repeated at several locations and tabulated values of \( c, I_0, I_1, I_2 \) and \( I_3 \) versus \( \xi \) were stored.

The integration variable \( \tau \) in (2.90) was related to \( \xi \) by (2.55c). Tabulated values of \( \tau \) and corresponding \( \xi \) were calculated and stored. Thus at a given step \( \tau \) in the program solution \( \xi \) was found, and therefore \( I_j (j=0,1,2,3) \) and the coefficients \( U_j (j=1,2,3) \) were computed from (2.57) - (2.59).

The procedure for the two-layer model was simpler. The coefficients could be calculated directly from (2.79a,b,c) given the depth at any location \( \xi \).

Appendix 3 contains a copy of the numerical code for continuous stratification with dissipation. The two-layer model code is a simplified version of this program.

The numerical method was checked for accuracy in

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Figure 2.5. Comparison of adiabatic approximation (——) with the numerical solution (O) for the decay of KdV solitary wave for $U_1 = 1.0, U_3 = 0.4$. 
several ways. Runs were first made for the conservative constant depth version of (2.55) with the solitary wave solution (2.60) as the initial condition. The calculations showed the initial wave propagated without change in form over 20 - 50 characteristic wave widths and at the correct phase speed ($\sigma/\tau = 4$) within detectable error. Additionally the quantities (2.65a,b) were calculated and were conserved to within 0.01%. For the inviscid variable depth runs discussed in Chapter 3 (2.65a,b) were conserved to within 1% in the worst cases. Typically they changed less than 0.1%.

The code with dissipation was tested by comparing the analytical adiabatic decay of a KdV solitary wave given by (2.70) with numerical calculations for constant depth (without cubic nonlinearity). Figure 2.5 shows an example result. The error between the theory and numerical results is small (< 2%). This difference is probably attributable to the dispersive effects of the dissipation which are lost when making the adiabatic approximation.

Given these results the numerical method can be expected to give accurate results in the general variable depth dissipative runs.
CHAPTER III

TURNING POINT PROBLEM

3.1 Introduction

As an introduction to the behavior of long internal waves over variable topography a numerical study of solitary waves in a two-layer system incident on a smooth transition between regions of constant depth was undertaken. The problem is of particular interest because internal solitary waves are waves of elevation/depression for \( d_+ > d_- \). If the transition is such that \( d_+ - d_- \) changes sign the incoming wave can no longer exist in its original form and must either reverse polarity or scatter into a dispersive packet (Miles 1980). The resolution of this question may have importance in field applications. For example the scattering of incoming waves into dispersive packets may cause the internal wave energy spectrum to shift to higher frequencies.

If the length of the transition is short, \( \lambda \gg \alpha \), the incoming wave must scatter into a dispersive packet (Djordjevic and Redekopp 1978; Miles 1980). The incoming wave varies in the transition region according to Green's Law and arrives in the final constant depth region having only changed magnitude and length, not sign. The inverse scattering problem or the shelf admits no solitary waves of reversed polarity.
When \( \lambda \leq 0(\alpha) \) Green's Law is no longer valid, non-linearity and dispersion, along with topography, affect wave solution in the transition region. Knickerbocker and Newell (1980) studied a model KdV equation and determined that for very slow slopes a single solitary wave of reversed polarity followed by a dispersive tail would emerge past the transition region. However, in some region near the point of equal layer depths (the turning point)

\[ |d_+ - d_-|/D = 0(\alpha) \]

In such a region cubic nonlinearity becomes comparable to and may dominate quadratic nonlinearity. The role of cubic nonlinearity has not been addressed adequately, therefore, the conclusion of Knickerbocker and Newell need further testing. In fact much of their work was done for very slow slopes \( (\lambda = 0(\alpha^{3-4})) \) where the effect of cubic nonlinearity is enhanced because the waves are in the turning point region for a long time.

The problem of single and multiple solitary waves moving from deep to shallow or shallow to deep water was addressed by Helfrich, Melville and Miles (1984), a copy of which comprises Appendix 4. A synopsis of the results is described in §3.2.\textsuperscript{1}

\textsuperscript{1}The original problem formulation and initial numerical work were done by the last two authors. The present author resolved numerical stability problems and implemented the inverse scattering solutions.
In §3.3 wave propagation over bump and tough topography is studied. In §3.4 viscous dissipation is added to the evolution equation and the results of §3.1 are re-examined. Section 3.5 discusses the laboratory and oceanic implications of the results.

3.2 Inviscid Turning Point Problem

We consider an inviscid two-layer system in which the slope length \( L \) is long compared to the wave length \( \lambda \) and \( \lambda = O(\alpha) \). From Chapter 2, the governing equation (2.55) including cubic nonlinearity, is

\[
\xi_T + 12(U_1 \xi - 2U_2 \xi^2)\xi_\sigma + \xi_{\sigma\sigma\sigma} = 0 \quad (3.1)
\]

where from (2.79a,b)

\[
U_1 = \frac{3}{4} \frac{\alpha}{\beta} c^{3/2} \frac{(1-d-f(\xi))}{(1-d)^2 f(\xi)^2} \quad (3.2)
\]

and

\[
U_2 = \frac{3}{4} \frac{\alpha^2}{\beta} c \frac{(1-d)^3 + f(\xi)^3}{(1-d)^3 f(\xi)^3 (1-d+f(\xi))} \quad . \quad (3.3)
\]

To investigate waves travelling from deep to shallow water a cosine shaped transition was used:
\[ d_\perp = dD \quad (\xi = \frac{x}{L} < 0) \quad (3.4a) \]

\[ = f(\xi)D = D \left\{ d + \frac{1}{2}(d-d_\perp)(\cos \pi \xi - 1) \right\} \quad (0 < \xi < 1) \quad (3.4b) \]

\[ = d_\perp D \quad (\xi > 1). \quad (3.4c) \]

Numerical solutions of (3.1) with the transition (3.4) and single incident wave for various values of \( d, d_\perp, \alpha \) and \( \lambda \) showed that multiple solitary waves of reversed polarity, followed by a dispersive tail, may emerge on the shelf (\( \xi > 1 \)). The number of transmitted solitary waves on the shelf increases as \( \lambda \) decreases for fixed \( \alpha, d \) and \( d_\perp \).

Figure 3.1 illustrates a case when multiple solitary waves of reversed polarity emerge (\( \lambda \ll \alpha \)). The first two waves at \( \xi = 1.59 \) are moving faster than the linear phase speed and the first wave has separated from the front of the scattered packet. Inverse scattering solutions on the shelf (see below) show that 12 transmitted solitary waves eventually emerge.

Rather than continue numerical solutions for long distances on the shelf, numerical solutions on the slope were coupled with inverse scattering transform (IST) solutions on the shelf to determine the number and amplitudes of transmitted solitary waves which asymptotically emerge. More precisely (3.1)
Figure 3.1. Evolution of a single wave of depression over a transition of decreasing depth for \((a, \lambda, d, d_1) = (-0.0667, 0.0041, 0.6, 0.15)\). Profiles of \(\eta\) are shown at \(\xi = 0, 0.66, 0.96, 1.59\). Note the separation of the leading waves from the scattered packet. IST shows 12 waves of reversed polarity emerging in this case.
was solved numerically until $\xi > 1$. The solution at this point was then used as the initial condition for the IST solution. Details of the procedure and the accuracy are described in §5 of Helfrich, Melville and Miles (Appendix 4). With this coupled technique increased ranges of the parameters $\alpha$ and $\lambda$ could be studied quickly and economically.

Figure 3.2 shows the influence of $\lambda$ on $N$, the number of transmitted waves, for various values of the incident wave amplitude $\alpha = a_0/D$ with $d = 0.6$ and $d_1 = 0.2$. $N$ increases as $\lambda$ decreases for a given $\alpha$. Figure 3.3 shows the regions of transition from $N = 0,1$ and $N = 1,2$ as a function of $\lambda$ and $\alpha$. The transition from $N = 0,1$ occurs when $\lambda = \alpha$. For a smaller depth change the transition $N = 0,1$ occurs for increased $\alpha$ at the same $\lambda$ (figures 7 and 8 in Appendix 4).

These results contrast with the findings of Knickerbocker and Newell (1980) who found only one transmitted wave. They also found that the amplitude of the transmitted wave $\alpha_T$ approached the asymptote $\alpha_T = -\frac{4}{9} \alpha$, independent of the transition length. Figure 3.4 shows the dependence of the first transmitted wave $\alpha_T^{(1)}$ versus $\alpha$ for several values of $\lambda$. Contrary to the findings of Knickerbocker and Newell the transmitted amplitude does not approach an asymptote but instead increases as $\lambda$ decreases.
Figure 3.2. Number $N$ of transmitted solitary waves versus $\lambda$ for fixed values of $d_0 = 0.6$ and $d_1 = 0.2$, and incident amplitudes $\alpha = -0.0833$ (---□---); $-0.0667$ (---○---); $-0.05$ (---△---); $-0.0333$ (---+---); $-0.0167$ (---×---).

Figure 3.3. Incident-wave amplitudes $\alpha$ at transitions between $N = 0,1$ and $N = 1,2$ versus the slope-length parameter $\lambda$, obtained from the data in figure 3.2. ○, $N = 0$; ●, 1; □, 2.
Figure 3.4. Amplitude $\alpha_T$ of first transmitted solitary wave versus the amplitude $\alpha$, of the incident wave for $d = 0.6$, $d_1 = 0.2$. $\square$, $\lambda = 0.0041$; $\circ$, $\lambda = 0.0082$; $\times$, $\lambda = 0.0123$; $\times$, $\lambda = 0.0225$; $\times$, $\lambda = 0.041$. 
The differences between the conclusions here and the findings of Knickerbocker and Newell are not completely attributable to the cubic term incorporated in (3.1). Runs without the cubic term showed only slight quantitative differences in the asymptotic results than those including the term. Differences in wave shapes near the turning point were present, however.

Since oceanic observations generally show groups of rank-ordered waves the effect of interaction during scattering on the transmission of solitary waves of reversed polarity was investigated. Figure 3.5 shows a case of two rank-ordered solitary waves. Comparison of the superposition of two runs, each with one of the waves in figure 3.5, shows that the interaction between the waves is significant. However, waves of reversed polarity still emerge from the front of the packet.

Several runs were made with a solitary wave propagating from shallow to deep water in which a turning point was encountered in the transition region. Even when \( \lambda \ll \alpha \) no waves of reversed polarity were transmitted.

3.3 Bump and Trough Topography

In addition to transitions from deep to shallow or vice versa internal waves in the coastal regions may be scattered by gradual bumps or troughs encountered in the bottom
Figure 3.5. Evolution of a pair of rank-ordered solitary waves over a transition of decreasing depth for \((a_1, a_2, \lambda, d, d_1) = (-0.083, -0.0667, 0.015, 0.6, 0.2)\). Profiles are shown at \(\xi = 0, 0.59, 0.93, 1.28\). As in figure 3.1, the lead waves are separating from the scattered packet.
topography. In the Massachusetts Bay, for example, there are several bumps in the sea floor between the generation site at Stellenwagen Bank and the coastline near Plymouth, Massachusetts where the waves are eventually dissipated. Thus the evolution of solitary waves over such features is of practical interest.

Consider the topography in which the bottom layer depth is given by

\[ d_- = d D \quad (\xi < 0 \text{ and } \xi > 1) \]  
\[ d_- = D\{d + \frac{1}{2}(d-d_1) (\cos 2\pi \xi - 1)\} \quad (0 < \xi < 1) \]  

The topography (3.5) is a bump / trough for \( d_1 \geq d \). In all the results below the incident wave is negative (\( d > 0.5 \)). When \( d_1 > (1-d) \) the incoming wave does not pass through a turning point and when \( d_1 < (1-d) \) a turning point is encountered on either slope of the bump.

Figure 3.6 illustrates the evolution of a solitary wave over a bump where \( \lambda \approx \alpha \) where no turning point is encountered. Upon passing over the topography the wave develops a weak dispersive tail but retains its general amplitude and shape. IST results give one transmitted wave with amplitude \( a_T/a_0 = 0.99 \). The initial wave is only slightly changed.

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Figure 3.6. Evolution of a single wave of depression incident on a cosine bump for $(a, \lambda, d, d_1) = (-0.0667, 0.025, 0.6, 0.5)$. No turning point is encountered. Profiles for $\xi = 0, 0.75, 1.0, 1.50$. IST shows one transmitted solitary wave: $a_T/a_0 = 0.99$. 
Figure 3.7. Same as Figure 3.6 for \((a, \lambda, d, d', c) = (-0.0667, 0.025, 0.6, 0.3)\). Two turning points are encountered. Profiles for \(\xi = 0, 0.6, 1.0, 1.50\). IST shows two transmitted solitary waves: \(a_T/a_0 = 0.90, 0.14\).
Figure 3.7 illustrates the effect of a larger bump in which two turning points are encountered. All other parameters \((\alpha_0, \lambda, d)\) are unchanged from figure 3.6. Significant scattering of the original wave is observed, however, it does not completely disintegrate. IST solutions show that two solitary waves emerge asymptotically with amplitudes \(a_T/a_0 = 0.90, 0.14\).

The effect of a trough is illustrated in figure 3.8. A short shelf behind the wave has developed by \(\xi = 1.0\) which eventually disperses. The original wave has been changed only slightly. IST gives \(N = 1\) with \(a_T/a_0 = 0.99\).

3.4 Influence of Damping

In this section the effect of viscous damping on the inviscid results of §3.2 is investigated. Allowing for damping at the bottom and interface only the evolution equation (3.1) becomes (equation (2.55) in Chapter II)

\[
\zeta_T + 12(U_1 \zeta - 2U_2 \tau^2) \zeta_\sigma + \zeta_{\sigma\sigma} = U_3 \int_{-\infty}^{\infty} \frac{\partial \zeta}{\partial \sigma'} \frac{1 - \text{sgn}(\sigma - \sigma')}{|\sigma - \sigma'|^{1/2}} \, d\sigma
\]

(3.6)

where \(U_1\) and \(U_2\) are given by (3.2) and (3.3) and from (2.81) in Chapter 2

\[
U_3 = \frac{3}{2} \frac{c^2}{\pi^{1/2}} \left(\frac{\nu_l}{c_0}\right)^{1/2} \frac{1}{\bar{D}} \frac{1}{(1-d+f(\xi))^2} \left\{ \frac{1}{f(\xi)^2} + \frac{1}{2} \frac{(1-d+f(\xi))^2}{(1-d)^2 f(\xi)^2} \right\}
\]

(3.7)
Figure 3.8. Evolution of a single wave of depression incident on a cosine trough for \((a, \lambda, d, d_1) = (-0.067, 0.025, 0.6, 0.8)\). Profiles at \(\xi = 0, 0.5, 1.0, 1.50\). IST shows one transmitted solitary wave: \(a_T/a_0 = 0.99\).
Figure 3.9 shows the results of a run with a single wave incident on a cosine transition (3.4) from deep to shallow water. The run parameters are \( (\alpha_0, \lambda, \delta/D, d_1) = (-0.0667, 0.0081, 0.0083, 0.6, 0.15) \). There is significant damping of the scattered wave packet and at \( \xi = 1.03 \) the group is nearly completely dissipated. This same set of parameters without damping gave three transmitted solitary waves.

The importance of viscosity in the polarity reversal problem can be judged from the ratio \( \gamma/\lambda \). When \( \gamma \geq \lambda \) viscous dissipation is strong enough to eliminate many transmitted solitary waves. For slow slopes (\( \lambda \ll |\alpha| \)) the viscosity acts over such a long time that a significant portion of the incoming wave energy is dissipated. Insufficient energy is left to generate solitary waves on the shelf. When \( \lambda \approx |\alpha| \) viscosity does not cause significant dissipation on the slope. However, transmitted waves in the absence of damping are very weak (see figure 3.4), hence even slight damping on the slope can eliminate them.

If viscosity is very weak and \( \gamma \ll \lambda \) the effects of topography are much more important than viscosity. Slightly damped solitary waves will be transmitted when \( \lambda \leq \alpha \).

3.5 Laboratory and Oceanic Implications

Applying the results of §3.2 to the laboratory or ocean depends upon the value of \( \gamma = \delta/D \). A length for significant
Figure 3.9. Viscous damping of a single solitary wave of depression incident over a transition of decreasing depth for \((\alpha, \lambda, \delta/D, d, d_1) = (-0.0667, 0.0082, 0.0083, 0.6, 0.15)\).
Profiles at \(\xi = 0, 0.37, 0.58, 1.03\).
damping can be estimated from the adiabatic damping of a single solitary wave (§2.3). For a decay to one half the amplitude equation (2.69) gives

$$\tau_{1/2} = 0.3 \frac{u^{-1}}{3}$$

(where $u_1 = 1$, $f(x) = d$, $c = 1$ and $N_0 = 1$ for constant depth). For a two-layer system equation (3.7) gives (after converting to dimensional variables)

$$x_{1/2} = 2.16 \frac{D \delta^{-1}}{d_+ \left( \frac{d_+}{d_-} + \frac{1}{2} \frac{D^2}{d_+ d_-} \right)^{-1}}$$  \hspace{1cm} (3.8)

where $\tau = \frac{d_+ d_- x}{(6 \delta^3)}$ has been used.

For typical laboratory conditions $(v, c_o, \lambda, D, d_+, d_-) = (0.01 \text{ cm/s}, 15 \text{ cm/s}, 100 \text{ cm}, 30 \text{ cm}, 10 \text{ cm}, 20 \text{ cm})$ which give $x_{1/2} \approx 40 \text{ m}$. Actually this scale should be less, $x_{1/2} \approx 25 \text{ m}$, since side walls (tank width = 38 cm) are not included in (3.8). Thus for a slope with $L > 25 \text{ m}$ damping can be expected to eliminate any transmitted waves. For $L < 25 \text{ m}$ we estimate $\lambda > 0.04$ which for $|\alpha| = 0.07$ is in the range of marginal transmission without damping. Detection of transmitted solitary waves on a laboratory appears unlikely.

Table 3.1 illustrates some scales typical of observations on the eastern United States continental shelf.
## (a) Observed Quantities

<table>
<thead>
<tr>
<th>Location</th>
<th>$\alpha_o$</th>
<th>$\ell$</th>
<th>$\nu_e$</th>
<th>$c$</th>
<th>$L$</th>
<th>$D$</th>
<th>$d_+$</th>
<th>$d_-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Massachusetts Bay</td>
<td>-10 m</td>
<td>300 m</td>
<td>1-10 cm$^2$/s</td>
<td>0.5 m/s</td>
<td>10km</td>
<td>30 m</td>
<td>30 m</td>
<td>50 m</td>
</tr>
<tr>
<td>Andaman Sea</td>
<td>-80 m</td>
<td>2000 m</td>
<td>1-10 cm$^2$/s</td>
<td>2 m/s</td>
<td>250 km</td>
<td>1500 m</td>
<td>500 m</td>
<td>1000 m</td>
</tr>
</tbody>
</table>

## (b) Calculated Scales

| Location         | $\delta$    | $x_{1/2}$ | $x_{1/2}/L$ | $\lambda$ | $|\alpha|$ |
|------------------|--------------|-----------|-------------|------------|----------|
| Massachusetts Bay| 0.25 - 0.8 m | 40 - 80 km| 4 - 8       | 0.03       | 0.13     |
| Andaman Sea      | 0.3 - 1 m    | 4600 - 8600 km | 18 - 33    | 0.008     | 0.04     |

Table 3.1 Typical Oceanic Scales for the Analysis of Damping on the Turning Point Problem. (a) Observed Quantities and (b) Calculated Scales.
(Massachusetts Bay) and larger waves such as those in the Andaman or Sulu Seas. Accurate identification of an effective eddy viscosity is difficult so a range from 1 to 10 cm$^2$/s was examined. For both examples Table 3.1 shows that $x_{1/2} > L$ and $\lambda < |\alpha|$. Given the proper depth change (i.e., turning point) these estimates indicate that transmitted solitary waves are possible for both the extremely large scale waves and the smaller waves observed in shallower regions.

Note that the solitary wave sorting distances (18 km in Massachusetts Bay and 1200 km in the Andaman Sea) calculated in Chapter I are less than or equal to the damping scale. Emerging solitary waves will be damped, but not dominated by dissipation. This also shows that damping should not dominate the sorting of waves from the initial generating disturbance.
CHAPTER IV

COMPARISONS OF THEORY AND EXPERIMENT

4.1 Introduction

In Chapter 1 field measurements and scaling arguments were used to identify KdV theory as the appropriate choice for describing internal wave evolution in the coastal regions. It was also pointed out that the theory is untested for variable topography. In this chapter, a series of laboratory experiments studying the propagation of single and rank-ordered solitary waves from deep to shallow water over slope/shelf topography are described. The experiments are designed to test the KdV theory developed in Chapter 2 and identify the parametric regions of validity. Under what conditions the theory gives reliable results is crucial information for oceanic analysis or prediction.

The relevant physical parameters for consideration are incident wave amplitude, water depth, slope length, depth change and stratification. These physical parameters can be arranged into the nondimensional parameters $\alpha$, $\lambda$, $\varepsilon$ and $\sigma$ (Equations (1.1), (1.6), (1.8) and (1.9)), which were introduced in Chapter 1. A final nondimensional parameter is

$$r = \frac{d-s}{d_+} \quad (4.1)$$
where $d_s$ is the scale depth of lower layer on the shelf. It is related to the total depth change and the "strength" of the turning point. If $r > 1$, no turning point is encountered on the slope. For $r < 1$, the "strength" of the turning point increases as $r$ decreases.

To systematically vary all of these parameters would involve an excessive number of experiments and is not necessary in order to judge the accuracy of the theory. Since the aim is to understand wave evolution over topography, the parameters $\lambda$ and $r$ are important, along with $\alpha$ (nonlinearity). Recall that the theory assumes $\lambda = O(\alpha)$. One thing we would like to determine, is how restrictive is this assumption. For example, as $\lambda$ increases, at what point does the neglect of wave reflection in the KdV theory become a source of significant error?

The stratification parameters $\varepsilon$ and $\sigma$, which help determine phase speed, are important to wave evolution through their influence on wave stability. For the sole purpose of testing the theory we would like to avoid instabilities and breaking events. Their occurrence, however, helps define situations when the theory may or may not be applied with confidence. In the larger context of understanding the basic dynamics which govern internal wave evolution on the continental shelf, they are of interest. Thus, the experiments were conducted in a salt stratified fluid rather than a system of two immiscible fluids. Chapter 5 discusses the occurrences and mechanisms of observed instabilities.
All experiments with topography were designed so the incident waves would encounter a turning point. If the theory adequately predicts the scattering process in these situations it could be expected to give good results in cases without a turning point, when the topographic influence is reduced.

The experiments were conducted with $\varepsilon$ small and nearly constant. The height of the topography and the upper layer depth were the same in all runs. The lower layer depth was changed by adjusting the total depth. Variations in slope length and incident wave amplitude were considered, as were three relative density differences.

Table 4.1 lists the parameter ranges examined. The limiting factor in the choice of values was tank length. A shelf length of at least 1.5-2 L was desired so that wave profiles could be measured a distance 0(L) past the shelfbreak and at the same time avoid interference at the measuring station from waves reflected from the tank end. Combined with a wave generating area of about 5 m before the slope and a tank length of 24 m (see §4.2.1), this limited L to 7 m. The rise of the slope $\Delta h$ was chosen so that $\Delta h/L \ll 1$.

Incident wave amplitudes were chosen such that $\lambda \geq |\alpha|$. The maximum wave height attainable was limited because of the difficulty of generating a clean solitary wave without a significant dispersive tail as the amplitude increased (see §4.2.2, below). Incident waves with amplitudes less than 1 cm were dissipated very quickly over the 24 m tank.

In §4.2, the experimental set-up, procedure and data analysis techniques are described.
### a) Dimensional Parameters

<table>
<thead>
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<th>$a_o$ (cm)</th>
<th>D (cm)</th>
<th>L (cm)</th>
<th>$d_-$ (cm)</th>
<th>$d_+$ (cm)</th>
<th>$\Delta h$ (cm)</th>
<th>$h_p$ (cm)</th>
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<td>35.0</td>
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<td>1.7 - 3.0</td>
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<td>701</td>
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### b) Nondimensional Parameters

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<th>$\tau$</th>
<th>$\sigma$</th>
<th>$\epsilon$</th>
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<td>0.0</td>
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<td>0.59</td>
<td>0.035</td>
<td>0.73</td>
<td>0.053</td>
</tr>
</tbody>
</table>

Table 4.1. Ranges of Parameters Examined Experimentally.
Comparisons of the two-layer theory with experiments are described in §4.3. Tests were conducted for constant depth and simple slope/shelf topography. In §4.4 the continuous stratification theory is studied. The importance of cubic nonlinearity is also assessed.

4.2 Experimental Set-up and Procedure

4.2.1 Wave Tank and Topography

All experiments were conducted in a glass-walled wave tank 0.6 m high, 0.38 m wide and 24 m long. The slope/shelf topography was constructed with a false bottom. Figure 4.1 shows a sketch of the tank.

The false bottom consisted of plate glass sections, either 0.76 or 1.52 m long, held in the tank by a frame (Figure 4.2). Two bars of pvc, each lined on one side with neoprene rubber, were pressed against the side walls by pvc pipes with end caps. The caps were adjustable so that the frame was held in place by friction. All seams (between glass and frame, and frame and tank) were sealed with silicon rubber.

The use of the modular glass false bottom had several advantages over other options such as plywood sheet. The modular structure facilitated changes in topography. Secondly, the use of plate glass resulted in a smooth boundary identical to the side walls. This is an important consideration because the side wall dissipation is treated as an extension of the bottom boundary layer in the theoretical devel-
Figure 4.1. Wave tank and false bottom set-up.
Figure 4.2. False bottom section.
opment (§7.4). The design also kept variations in water depth small (lateral variations were ±0.1 cm and horizontal variations were ±0.1-0.2 cm).

4.2.2 Wave Generator

Interfacial waves were generated using a flap-type wave generator (Thorpe 1978). The system consisted of a long (1.2 m) flat plexiglass sheet, hinged to an airfoil section fixed in the interface (Figure 4.3). A neoprene wiper was installed around the edges of the flap and the airfoil, reducing communication from above and below the interface. Vertical motion of the flap caused flow past the airfoil section, opposite directions in each layer, resulting in wave generation. The airfoil section reduced mixing at the front of the wavemaker and prevented fluid from the upper layer from being pulled underneath the wavemaker (and vice versa).

Accurate and repeatable movement of the flap was possible using a stepper motor connected to a threaded rod. The motor was controlled by a microprocessor (DEC Falcon SBC) with a real time clock (rtc). Each digital pulse from the microprocessor resulted in an angular displacement (0.9 or 1.8°) of the motor shaft and, therefore, a fixed vertical displacement of the free end of the flap. Timed pulse streams gave a known displacement versus time, which was easily repeatable. The vertical increments were very small (0.00254 cm at the connection of the rod and flap for a 1.8° angular displacement), so that the motion of the flap was effectively continuous.
Figure 4.3. Wave Generator.
Timing information (number of overflows of the real time clock between successive steps) for a particular sequence of paddle movements was generated on a separate DEC LSI 11/23 microcomputer, then downloaded, along with a control program, to the microprocessor memory. The control program started the clock operating at a selectable frequency and, after waiting for a queue from the main data acquisition program (§4.2.6), generated a pulse when the proper number of clock overflows occurred, after which a new count of overflows was initiated. A copy of the assembly language program, MOVMT.MAC, is given in Appendix 5.

Single and multiple solitary wave generation was accomplished by moving the flap such that mass flux in each layer approximated that of a wave or waves passing the tip of the airfoil. Flap movement was determined as follows (after Goring 1978).

Assuming a rigid lid and perfect seal around the flap edges, the vertically averaged flow \( u_+ \) in the upper layer is given by

\[
u_+ = - \frac{1}{\omega d} \frac{dV}{dt} \tag{4.2}\]

where \( w \) is the tank width and \( V \) is the volume of water in the control section above the flap. Thus,

\[
V = w L (d_+ - \frac{1}{2} \xi) \tag{4.3}
\]
where \( L \) is the length of the flap and \( \xi \) is the vertical position of the flap end, positive upwards. Substituting (4.3) into (4.2), gives

\[
\frac{d\xi}{dt} = \frac{2u_+d_+}{L}. \tag{4.4}
\]

For a single solitary wave in a two layer system, the interfacial displacement at a fixed location is

\[
A = A_0 \, \text{sech}^2 \left[ \frac{c}{\xi_0} (t_o - t) \right] \tag{4.5}
\]

where \( A_0 \) is the wave amplitude, \( t_0 \) is a time delay, \( c \) is the linear phase speed (2.75) and

\[
\xi_0^{-2} = \frac{3}{4} \left( \frac{d_+ - d_-}{d_+} \right) A_0. \tag{4.6}
\]

To leading order the upper layer velocity is given by

\[
u_+ = - \frac{c}{d_+} A = - A_0 \frac{c}{d_+} \text{sech}^2 \left[ \frac{c}{\xi_0} (t_o - t) \right]. \tag{4.7}
\]

Substitution into (4.4) gives

\[
\frac{d\xi}{dt} = -2 A_0 \frac{c}{L} \text{sech}^2 \left[ \frac{c}{L} (t_o - t) \right]
\]

and integration from \( t = 0 \) to \( t \), gives
\[ \xi(t) = -\frac{2}{L} \frac{A_0}{\xi_o} (\tanh \left( \frac{c}{\xi_o} t_o \right) - \tanh \left[ \frac{c}{\xi_o} (t_o - t) \right]). \quad (4.7) \]

Defining the total flap stroke as

\[ S_v = -\frac{4}{L} \frac{A_0}{\xi_o}, \quad (4.8) \]

and noting that as \( t \to \infty \), \( \xi \to S_v \), we find that \( \tanh \left( \frac{c}{\xi_o} t_o \right) = 1 \).

This is realistically unattainable, so we choose \( t_o = 3.8 \xi_o / c \),

which gives \( \tanh \left( ct_o / \xi_o \right) = 0.999 \). The final equation for flap movement for generation of a single solitary wave is

\[ \frac{\xi(t)}{S_v} = \frac{1}{2} (0.999 - \tanh \left[ 3.8 \left( 1 - \frac{t}{t_o} \right) \right]). \quad (4.9) \]

As described above, the flap is moved in incremental steps \( (\Delta \xi) \) by the stepper motor. The procedure is to solve \((4.9)\) (for a given set of conditions \( d_+ \), \( d_- \), \( A_0 \) and \( \Delta \rho/\rho \)) to generate a table of times (actually number of clock overflows given the rtc frequency) between successive flap increments. A copy of the Fortran program PULTIM.FOR used to generate this timing information is given in Appendix 5.

The procedure for generation of two solitary waves is identical to above, except \((4.5)\) is replaced by
\[ A = A_0 \, \text{sech}^2 \left[ \frac{c}{L_0} \left( t_o - t \right) \right] + A_1 \, \text{sech}^2 \left[ \frac{c}{L_1} \left( t_o + t_d - t \right) \right] \quad (4.10) \]

where \( A_1 \) is the amplitude of the second wave, \( L_1 \) is given from (4.6) with \( A_1 \), and \( t_d \) is a time delay between wave crests.

Figure 4.4 shows two examples of waves generated with the wavemaker. Generation of large waves \(|a| > 0.06\) resulted in mixing at the wavemaker and generation of a weak dispersive tail.

4.2.3 **Conductivity Probes**

Precision Measurement Engineering micro scale conductivity instruments (model 106) were used to measure background density profiles and interfacial displacements (see §4.2.8, below). Figure 4.5 illustrates the straight and L-shaped probes used.

The probe output voltage \( v \) was converted to density by inverting

\[ v = a_0 + a_1 (\rho - 1)^{1/2} + a_2 (\rho - 1). \quad (4.11) \]

Here \( a_0, a_1 \) and \( a_2 \) are coefficients found by least squares fitting of output voltage versus known densities of saltwater solutions (found using a hydrometer). The calibrations were done with seven saltwater baths over a range of densities from 1.0 gm/cm\(^3\) (fresh water) to a density slightly greater than the saltwater used in the
Figure 4.4. Example of typical solitary waves generated by the wavemaker system. (a) $\alpha = -0.043$ and (b) $\alpha = -0.068$ with $(D, d, c_{o}, \sigma) = (33.5 \text{ cm}, 11 \text{ cm}, 15.8 \text{ cm}, 0.0356)$. 
Figure 4.5. Conductivity probes.

(a) Straight Probe
(b) L-Shaped Probe
Figure 4.6. Probe positioning assembly.
experiment. Calibrations for each probe were performed at the beginning of each experiment. Temperatures of the calibration baths were kept equal to the fill water in the tank to avoid thermal effects. The calibrations were extremely stable over the 3 to 5 hours of experimental runs. Errors of less than 2% in computed values of ρ - 1 were found.

Figure 4.6 shows the probe positioning system. The use of computer controlled stepper motors allowed accurate profiling of the static density distribution. The number of steps the motor had taken could be converted directly into probe position relative a reference location. A copy of the Fortran program PRCALM.FOR used for simultaneous profiling of up to 4 probes is given in Appendix 5.

4.2.4 Laser Doppler Velocimeter

A 2 watt argon laser (Lexel model 95-2) with backscatter optics and a counter type signal processor (TSI 1990 series) was used to measure fluid velocities. Velocity data from the signal processor was acquired digitally by the data acquisition system (§4.2.6).
4.2.5 Flow Visualization

Wave evolution and breaking events were visualized by dyeing one layer or using the shadowgraph technique. Photographs (medium format Hasselblad camera) and movies (16 mm) were taken of breaking events and later analyzed.

4.2.6 Data Acquisition

A DEC LSI 11/23 microcomputer was used for data acquisition and control. The computer was equipped with a Data Translation real time clock (DT2769), 8 channel analog to digital converter (DT2782) and two digital I/O boards (DT2768). Accessing board functions was simplified using the Data Translation program package DTLIB.

All data acquisition during the experiments was done with a 10 Hz data rate. This rate was much faster than the characteristic period of the scatter waves (< 0.2 Hz). The main data acquisition program IWDATI.FOR (up to 8 analog channels) is listed in Appendix 5.

4.2.7 Experimental Procedure

One day prior to an experimental run, salt water was mixed to the desired density (commercial table salt, Culinox 999) in a mixing vat and pumped into the tank. The saltwater and the fresh water for the upper layer (in holding tank) were allowed to sit overnight to reach room temperature and degas.
The next day fresh water was slowly spread over the salt water using floating diffusers at a rate which minimized mixing. The filling usually took 4-5 hours and resulted in a density profile with a maximum slope thickness $h_0$ of 1.5-2 cm. Figure 4.7 shows a typical initial profile.

Just before the completion of filling, the conductivity probes were calibrated (§4.2.3), attached to the profiling mechanisms and positioned along the tank. One probe was located at the base of the slope ($x_0 = \xi_0 = 0$) to measure the incident wave profiles. Second and third probes were usually located 2 and 6.5 m past the shelf break. A fourth probe was located at varying positions along the tank, depending on the data required.

The typical procedure was to fill the tank with 25.5 cm of salt water (in deepest section) and 11 cm of fresh water, and make two runs with the same amplitude incident wave. Conductivity profiles were taken before the first and after the second runs. Next, 1.5 cm of salt water was slowly drained from the lower layer and the procedure repeated. Again, 1.5 cm was drained and two trials run. In this fashion, all three values of $r$ (see Table 4.1), could be examined with all other parameters ($\alpha$, $\lambda$, $\sigma$ and $\varepsilon$) kept nearly constant.

Figure 4.7 displays a typical example of the variation of the density profile from the initial condition to after the second and sixth runs of the day. The variation is very slight, $h_0$ changed from about 1.8 cm to 2.5 cm ($\varepsilon = 0.05-0.075$). This example was for a case with wave breaking. Without breaking, the variation is slightly less.
Figure 4.7. Typical density profiles during course of an experiment. Initial (---), after two runs (-----) and after six runs and two drains (----).
4.2.8 **Interfacial Displacement Calculations**

Interfacial displacements were calculated assuming lowest mode motion. The conductivity probes were positioned within the interfacial region prior to a run and measured conductivities were converted to equivalent vertical displacements of the static conductivity (density) profile. The Fortran program VOLDIP.FOR used for these calculations is listed in Appendix 5.

A test of the accuracy of these measurements was made by comparing velocities calculated using the measured displacement records with velocities measured with the LDV system. From (2.20b) the calculated velocity at the laser measuring depth \( z_v \) is

\[
u(t) = c \phi'(z_v) A(t)\]

where \( A(t) \) is the measured interfacial displacement. Here \( c \) and \( \phi'(z) \) are found by solving the eigenvalue problem (2.53, b) using the measured static density profile.

Figure 4.8 shows an example comparison for a scattered wave packet on the shelf. Agreement between the measured fluid velocities and the calculation from the probe data is very good, indicating that the conductivity probes and lowest mode analysis technique give reliable estimates of interfacial displacements. From analysis of other comparisons, uncertainty in displacement calculation was found to be less than 10% of the measured wave height.

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Figure 4.8. Comparison of velocity measured with LDV (———) and velocity calculated from probe data (———). Velocity data from lower layer 2 m past the shelf break.
A possible source of error was due to slow draining of fluid near the probe tip. This problem is most pronounced with the straight conductivity probes (Figure 4.5a). As the interface drops, a boundary layer of salty water is left on the glass tip which distorts the measured conductivity record as it drains down the tip. This distortion is reduced by using the L-shaped probes (Figure 4.5b). They are positioned in the interface with the glass shaft perpendicular to the \((u, w)\) velocity directions, thereby avoiding drainage problems. Some mixing occurs as the water flows around the shaft; however, this mixing is confined primarily to the region of the thicker stainless steel section, about 5 cm away from the electrodes.

Figure 4.9 shows an example comparing the responses of the two probe types to the same wave packet. The probes were at the same vertical level, separated laterally by 1 cm. Differences are noticeable in the decreasing segments of the signal and in the wave troughs when draining is effecting the straight probe. This difference is typical, being within the 10% uncertainty level.

Since just two L-shaped probes were available (only one in the early experiments), use of the straight probes could not be avoided. Generally, the L-shaped probes were used in the shelf break region because of the rapid changes in interfacial displacement.
Figure 4.9. Comparison of straight (-----) and L-shaped (-----) probes. Data taken 2 m past shelf break.
4.3 **Two-Layer Model Comparison**

4.3.1 **Constant Depth**

Several experiments were conducted to examine the damping of solitary waves and test the boundary layer dissipation model developed in Chapter 2. Leone, Segur and Hammack (1982) examined the adiabatic approximation (2.70) for a two-layer system but their results were inconclusive. Significant discrepancies between the theory and experiments were found. They attributed the difference to residual kinetic vorticity in the boundary layers due to a fast propagating surface wave. The presence of a significant freely propagating surface wave resulted from their use of a vertical piston wave generator. The flap type wave generator (§4.2) used in the present study did not cause generation of a measurable surface mode wave. Thus, the problem reported by Leone et al. was avoided.

From (2.70), (2.79a), (2.31) and including the side wall boundary layers (§2.4), the adiabatic decay of a KdV solitary wave in a two-layer system is given by

\[
\frac{A}{A_0} = (1 + B \frac{x}{D})^{-4}
\]  

(4.12)

where
\[ B = 0.64 \frac{3}{2\sqrt{\pi}} \left( \frac{\nu}{\xi} \right)^{1/2} \left( \frac{D}{L} \right) \frac{1}{d_+ d_-} \frac{1}{(d_+ + d_-)} \]

\[ \left[ \left( 1 + \frac{2d_-}{w} \right) d_+^2 + \frac{2d_+ d_-^2}{w} + \frac{1}{2} (d_+ + d_-)^2 \right], \quad (4.13) \]

c is given by (2.75) and \( b \) is given by (4.6).

Figure 4.10 shows a comparison of (4.12) with measured solitary wave decay for three values of \( \sigma \) and two initial wave amplitudes. In this and all subsequent comparisons with experiments, \( \nu \) was taken to be 0.01 cm\(^2\)/s, the value for fresh water at 20 °C. Data from the probe nearest to the wavemaker was used as the initial amplitude. The waves were allowed to reflect from the tank wall (16.6 m from the first probe) so that a larger propagation distance could be examined. The reflection, equivalent to head-on collision of two solitary waves, causes a slight phase shift which is very small compared to the total propagation distance and is ignored in Figure 4.10.

The comparisons show that the adiabatic approximation gives a good estimate if interfacial dissipation is included. Examination of the terms in (4.13) shows that interfacial dissipation (last term in brackets) accounts for 52% of the total dissipation (\( d_+ = 11 \) cm, \( d_- = 22.5 \) cm and \( w = 38 \) cm).
Figure 4.10. Displacement measurements versus adiabatic approximation for decay of KdV solitary wave over uniform depth (two-layer model). For (a) $\Delta \rho/\rho = 0.0165$, (b) $\Delta \rho/\rho = 0.0355$ and (c) $\Delta \rho/\rho = 0.0516$. $(D, d_{+}) = (33.5\text{ cm}, 11\text{ cm})$. 

$\left(\cdots\right)$ theory with interfacial damping for small $\square$ initial wave, \left(\leftarrow\rightarrow\right)$ theory for large $\bigcirc$ initial wave and $\leftarrow\rightarrow$ theory without interfacial damping for small initial wave.
The amplitude decay results suggest that the dissipation operator in the evolution equation (2.55) properly models the effects of the boundary damping. A further test is to compare measured wave profiles with those predicted by the theoretical model ((2.55) with $U_1$, $U_2$ and $U_3$ given by (2.79a, b) and (2.81) including side wall dissipation.

Figure 4.11 shows two comparisons for $\alpha = -0.043$ and $-0.068$. The abscissa is $-\beta^{-1/2} \sigma = -c_0 \ s/D$ where $\sigma$ is given by (2.54a) and $s$ is the phase variable (2.10). Nondimensional time increases to the right. The wave profiles at the first probe ($x/D = 0$) were used as the initial condition in the numerical solution. The comparisons are from locations before wave reflection.

Consistent with the adiabatic analysis, the amplitudes are predicted very well; however, errors in phase are present. The error, about $+4\%$ of travel time to the last probe, is due primarily to the difference in linear phase speeds between the two-layer approximation and the continuous stratification model. For the conditions of Figure 4.11, the two-layer phase speed is from (2.75) $16.0 \ \text{cm/s}$. Using the measured density profiles, phase speed found from the eigenvalue problem (2.53a, b) is $15.5 \ \text{cm/s}$ in each example. The two models give a difference in arrival time of $3.2\%$ of the total travel time.

4.3.2 **Variable Topography**

In this section the theoretical model for the two-layer system is compared with experimental data for slope/shelf topography.
Figure 4.11. Two-layer model comparisons for uniform depth. \((L, d_+, c_0, \sigma) = (33.5 \text{ cm}, 11 \text{ cm}, 16.0 \text{ cm/s}, 0.0355)\) (a) \(\alpha = -0.043\) and (b) \(\alpha = -0.068\).
The measured displacements at the slope base are used as the initial conditions in the numerical computations. The topographic variation is given by

\[ d_\text{e} = d \frac{X}{L} < 0 \]  \hspace{1cm} (4.14a)

\[ = f(\xi) D = [d - (d - d_1) \xi] D \quad (0 < \xi < 1) \]  \hspace{1cm} (4.14b)

\[ d_\text{s} = d_1 D \quad (\xi > 1) \]  \hspace{1cm} (4.14c)

where \( L \) is the slope length used in the experiments.

Figure 4.12 shows example comparisons for two incident wave amplitudes (\( \alpha = -0.040, -0.058 \)) with the same physical set-up: \( (L, D, d_\text{s}, c_0, \sigma) = (488 \text{ cm}, 36.5 \text{ cm}, 8.0 \text{ cm}, 16.4 \text{ cm/s}, 0.0356) \). For this and all subsequent comparisons over slope/shelf topography, the nondimensional displacement \( \eta = A/a_0 \) is plotted against \( -\lambda \sigma = -c_0 \text{ s/L} \). Nondimensional time increases to the right.

The theory captures the qualitative aspects of the scattered wave packets and is accurate in wave amplitude predictions. However, as expected from the constant depth comparisons, errors in wave phase are evident. In particular, the time for the leading trough to pass is not well modeled in either example. Comparisons for other parameter values give the same discrepancies.

Again, the error is largely attributable to the difference in phase speeds between the continuous stratification model and the corresponding two-layer model. Table 4.2 lists phase speeds for the
Figure 4.12. Two-layer model comparisons for slope/shelf topography. 
\[(L, D, d_s, c_0, \sigma) = (488 \text{ cm}, 36.5 \text{ cm}, 8 \text{ cm}, 16.4 \text{ cm/s}, 0.0356)\]. 
(a) \((\alpha, \lambda, r) = (-0.040, 0.15, 0.73)\) and 
(b) \((\alpha, \lambda, r) = (-0.058, 0.13, 0.73)\).
Figure 4.12b. See Figure 4.12a.
<table>
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<th>Model</th>
<th>Deep $\xi &lt; 0$</th>
<th>Shelf $\xi &gt; 1$</th>
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<tr>
<td>continuous stratification</td>
<td>15.5 cm/s</td>
<td>11.9 cm/s</td>
</tr>
<tr>
<td>two-layer</td>
<td>16.0 cm/s</td>
<td>12.7 cm/s</td>
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</tbody>
</table>

Table 4.2. Linear phase speeds in deep section and on shelf for conditions of Figure 4.12.
conditions of Figure 4.12a. The two-layer model overpredicts phase speed by only 3% for $\xi < 0$, but for $\xi > 1$, the error is 7%.

Although these values seem small, their cumulative effect on the dispersion of the scattered waves over distances of 2–3 L is significant. As will be shown in the next section, the continuous stratification model greatly improves upon the two-layer results.

4.4 Continuous Stratification Model Comparisons

4.4.1 Constant Depth

The analysis of §4.3.1 showed that interfacial shear must be included if wave damping is to be modeled accurately. Comparison of the adiabatic approximation for continuous stratification with the data of Figure 4.10 requires an ad hoc inclusion of the two-layer interfacial damping if good agreement is to be attained. The coefficient $U_3$ (2.82) must be modified to (nondimensional)

$$U_3 = \frac{1}{2} \pi^{1/2} \left( \frac{v_t}{c} \right)^{1/2} \frac{1}{D \beta} \frac{e^2}{I_2} \cdot$$

$$\left[ \left( \frac{d_+}{w} \right)^2 \phi^2(0) + \left( 1 + \frac{2d_-}{w} \right) \phi^2(-h) + \frac{D^2}{2} \left( \frac{d_+ + d_-(\xi)}{d_+ d_-(\xi)} \right)^2 \right] \quad (4.15)$$

where the third term in the brackets is the two-layer model interfacial dissipation. The continuous model predictions are then vir-
tually identical to the two-layer predictions for the conditions of Figure 4.10.

The necessity of including interfacial dissipation in either model to obtain good agreement is somewhat surprising in view of the presumably weaker shear layer in the continuously stratified experiments than in a true two-layer system. The rate of energy dissipation per unit width in the interfacial shear layer is given by

$$\frac{dE}{dt} = \mu \int_{-h}^{0} \left( \frac{\partial u}{\partial z} \right)^2 dz. \quad (4.16)$$

The vertical extent of the shear layer in the continuously stratified system scales by $h_\rho$. For the two-layer system the height is $2\delta$. Both cases will have nearly the same velocity change across the interface. Therefore, the ratio of continuous to two-layer interfacial dissipation scales as $2\delta/h_\rho$. For typical experimental conditions this ratio is about 0.25, illustrating the much stronger dissipation in a true two-layer system. More precise calculations of (4.16) only confirm this estimate.

Koop and Butler (1981) were also forced to make an ad hoc approximation to account for the observed damping in their experiments conducted in a salt stratified system. Rather than use the two-layer model they treated the interface as a no-slip boundary. This method gives an even greater (~10%) interfacial effect than the model used here.
Viscous dissipation at the free surface due to surface contamination is the most likely explanation for our need to use the two-layer interfacial dissipation model. Mei (1983, §8.3) states that surface contamination of the free surface may lead to boundary layer damping equivalent to a no-slip boundary. Visual observation during experiments using dye streaks confirmed the presence of a surface boundary layer. Table 4.3 examines the relative magnitudes of the terms in brackets in (4.13) for (a) no free surface shear and full interfacial shear and (b) no-slip free surface and 0.25 times the interfacial shear. Calculations are for typical conditions using the two-layer approximation ($\phi'(0) = -D/d_+$ and $\phi'(-h) = D/d_-$. Both situations give nearly the same total value implying that overestimation of the interfacial shear may be compensating dissipation neglect due to surface contamination.

Using the two-layer interfacial dissipation model we are now in a position to reexamine the evolution of solitary waves over constant depth. We first note that in the continuous stratification model ((2.55) with $U_1$, $U_2$ and $U_3$ given by (2.56), (2.57) and (4.15), respectively) the displacement measured at the probes is usually not the maximum displacement $A(x,t)$. Instead, the streamline displacement at the probe depth $z_p$ is given by

$$\eta(x,t) = A(x,t) \phi(z_p)$$  \hspace{1cm} (4.17)

In all comparisons with the continuous model $\eta$, nondimensionalized by the initial wave amplitude $a_0$, is plotted against $-c_0 s/L$. 

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<table>
<thead>
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<th>Case</th>
<th>Upper Layer</th>
<th>Bottom Layer</th>
<th>Interface</th>
<th>Total</th>
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<td>(b)</td>
<td>14.64</td>
<td>4.89</td>
<td>2.57</td>
<td>22.05</td>
</tr>
</tbody>
</table>

Table 4.3. Terms in (4.15) (two-layer approximation) for \((D, d_4, d_-, w) = (33.5 \text{ cm}, 11 \text{ cm}, 22.5 \text{ cm}, 38 \text{ cm})\). Calculations are for (a) no free surface shear and full interfacial shear and (b) free surface shear and 0.25 times interfacial shear.
Figure 4.13 shows comparisons with the same experimental data as in Figure 4.11. The agreement is excellent in both cases. The slight discrepancy is within the error of the probes. Note that not only amplitude but arrival time is predicted accurately.

4.4.2 Variable Topography

Consider first Figure 4.14 where the experimental runs shown in Figure 4.12 are compared with the continuous stratification model. The agreement is extremely good. The most notable error is at \( \xi = 1.0 \). We will return to this in §4.2.2.2. It is clear from Figure 4.14 that the continuous stratification model is much better than the two-layer model. This was anticipated by the comparisons of phase speeds in Table 4.2.

4.4.2.1 Variation of Experiment Parameters

At this point we are ready to examine in detail the influence of the parameters \( \alpha, \lambda, r \) and \( \sigma \). For the most part the comparisons are insensitive to \( \sigma \), so the data for \( \sigma = 0.035 \) will be examined in detail and selected examples from the other values illustrating differences will be discussed.

Consider first, runs for small \( \alpha = -0.040 \) and \( r = 0.73 \) \( (d_m = 8.0) \), the weakest depth change. Figure 4.15 is for the longest slope (\( \lambda = 701 \) cm) and shows good agreement. Figure 4.16 is for nearly identical conditions with the shortest slope (\( \lambda = 244 \) cm). The agreement is excellent at both measuring stations, even though
Figure 4.13. Continuous stratification model comparisons for uniform depth. Same experiments as Figure 4.11. (a) $\alpha = -0.043$ and (b) $\alpha = -0.068$. 
Figure 4.14. Continuous stratification model comparisons for slope/shelf topography. Same experiments as Figure 4.12. (L, D, d_s, c_0, ω) = (488 cm, 36.5 cm, 8 cm, 16.0 cm/s, 0.0356).

(a) (α, λ, r) = (-0.040, 0.15, 0.73) and (b) (α, λ, r) = (-0.058, 0.13, 0.73).
Figure 4.14b. See Figure 4.14a.
Figure 4.15. \((a, \lambda, r) = (-0.041, 0.12, 0.73)\). \((L, D, d_s, c_o, \sigma) = (701 \text{ cm}, 36.5 \text{ cm}, 8 \text{ cm}, 15.9 \text{ cm/s, } 0.0335)\).
Figure 4.16. \((a, \lambda, \tau) = (-0.039, 0.33, 0.73)\). \((L, D, d, c_o, \sigma) = (244 \text{ cm}, 36.5 \text{ cm}, 8 \text{ cm}, 15.8 \text{ cm/s}, 0.0355)\).
\( \lambda = 10 \alpha \). Note that a small amplitude shelf is present behind the incident wave at \( \xi = 0 \). The weak reflection does not noticeably affect the comparison. Figure 4.14a shows the intermediate slope length (\( \lambda = 488 \text{ cm} \)) for these conditions.

We next consider the same water depth (\( r = 0.73 \)) and slope lengths for the larger amplitude waves (\( \alpha = -0.063 \)) in which nonlinearity is more important. Figures 4.17, 4.14b and 4.18 display the results for the three slopes. The comparisons for all slope lengths are very good, but not as accurate as for the small amplitude waves. Slight errors are present in the arrival time of the first crest at the shelf measuring stations. The difference, however, is only 2-5% of the total travel time from \( \xi = 0 \). The largest discrepancy, in all three cases, is in the prediction of the tail of the packet. The leading trough and two or three crests are captured, but amplitudes and frequencies of the tail predicted poorly in some cases (Figure 4.18, in particular).

Next, consider experiments for the shallowest lower layer on the shelf \( r = 0.45 \) (\( d_{-\theta} = 5.0 \text{ cm} \)). In these runs the "strength" of the turning point and subsequent scattering is increased from the previous comparisons. We first examine the small amplitude waves (\( \alpha = -0.04 \)) which exhibited no strong breaking events. Figures 4.19, 4.20 and 4.21 display the comparisons for the long (\( \lambda = 0.12 \)), intermediate (\( \lambda = 0.17 \)) and short (\( \lambda = 0.31 \)) slopes, respectively. Again, the agreement is extremely good in all cases. With the exception of the comparison at \( \xi = 1.0 \) in Figure 4.20, the only significant problem
Figure 4.17. \((a, \lambda, \tau) = (-0.063, 0.10, 0.73)\). \((L, D, d_s, c_0, \sigma) = (701 \text{ cm}, 36.5 \text{ cm}, 8 \text{ cm}, 16.0 \text{ cm/s}, 0.0359)\).
Figure 4.18. \((a, \lambda, r) = (-0.065, 0.27, 0.73)\). \((L, D, d_s, c_o, \sigma) = (244 \text{ cm}, 36.5 \text{ cm}, 8 \text{ cm}, 15.9 \text{ cm/s}, 0.0360)\).
Figure 4.19. \((a, \lambda, r) = (-0.045, 0.12, 0.45)\). \((L, D, d_0, c_0, \sigma) = (701 \text{ cm}, 33.5 \text{ cm}, 5 \text{ cm}, 15.5 \text{ cm/s}, 0.0350)\).
Figure 4.20. \((a, \lambda, \tau) = (-0.043, 0.17, 0.45)\). \((L, D, d_S, c_0, \sigma) = (488 \text{ cm, } 33.5 \text{ cm, } 5 \text{ cm, } 15.5 \text{ cm/s, } 0.0356)\)
Figure 4.21. \((a, \lambda, \tau) = (-0.048, 0.31, 0.45)\). \((L, D, d_s, c_0, \sigma) = (244 \text{ cm}, 33.5 \text{ cm}, 5 \text{ cm}, 15.3 \text{ cm/s}, 0.0355)\).
is a small error in the arrival time (<3% of total travel time) at the furthest measuring station in Figures 4.19 and 4.20. The data for the short slope, Figure 4.21, shows the presence of a significant reflected shelf behind the incident wave, but again, neglect of this feature in the theory still gives very good results.

To this point only experimental runs without strong wave instabilities have been considered. Figures 4.22, 4.23 and 4.24 show comparisons for $\lambda = 0.10, 0.15$ and 0.28 cm, respectively, $r = 0.45$ and $\alpha = -0.062$. In all cases wave breaking, accompanied by significant vertical mixing and generation of a second mode solitary wave, occurred in a region ±$\xi$ from the shelf break. Details of the breaking events and second mode generation are discussed in Chapter 5; here, we just want to display comparisons of the theory with the data. The results for all three slope lengths are surprisingly good even though the weakly nonlinear theory cannot model any wave instabilities.

That the theory does poorly in the breaking (shelf break) region is evident from the comparison data $\xi = 1.0$ in Figure 4.23. (The flat crest results because the interface was raised above the probe level.) Evidence that the probe measurement gives a reasonable estimation of displacement in this example is shown in Figure 4.25 where the LDV measurement is compared with velocity computed using probe displacement data (see §4.2.8). Until the probe signal drops out the comparison is extremely good. The numerical predictions past the breaking region recover and comparisons at $\xi = 1.41$ and 2.33 show only errors in phase. The amplitude and frequency of the leading trough and first several crests are well modeled.
Figure 4.22. \((\alpha, \lambda, r) = (-0.069, 0.10, 0.45)\). \((L, D, dS, c_0, \sigma) = (701 \text{ cm}, 33.5 \text{ cm}, 5 \text{ cm}, 15.5 \text{ cm/s}, 0.0359)\).
Figure 4.23. \((a, \lambda, \tau) = (-0.063, 0.15, 0.45)\). \((L, D, d_{-5}, c_0, \sigma) = (488 \text{ cm}, 33.5 \text{ cm}, 5 \text{ cm}, 15.5 \text{ cm/s}, 0.0356)\).
Figure 4.24. \((a, \lambda, \tau) = (-0.070, 0.28, 0.45)\). \((L, D, d_0, c_0, \sigma) = (244 \text{ cm}, 33.5 \text{ cm}, 5 \text{ cm}, 15.4 \text{ cm/s}, 0.0360)\).
Figure 4.25. LDV velocity data measurement 2 cm above bottom at shelf break ($\xi = 1.0$) versus velocity computed from probe data for the experiment in Figure 4.23.
Comparisons for the intermediate depth change $r = 0.59$ ($d_{-s} = 6.5 \text{ cm}$) are consistent with the results to this point and will not be discussed.

Experiments for $\sigma = 0.053$ also showed no new features and were modeled with the same accuracy as the $\sigma = 0.035$ data. Figure 4.26 shows a sample result for $(\alpha, \lambda, r) = (-0.042, 0.16, 0.73)$. The only exception was a slightly poorer modeling capability when wave breaking occurs. Figure 4.27 illustrates one for case $(\alpha, \lambda, r) = (-0.072, 0.10, 0.45)$. In general, the predicted frequencies of the first few crests are about 20% too high.

Results for $\sigma = 0.017$ were qualitatively the same as found for $\sigma = 0.035$. Errors in arrival time were largest in these runs and increased as $r$ decreased. Figures 4.28 and 4.29 for the long slope, $\alpha = -0.04$ and $r = 0.73$, show the worst case results. The amplitudes and frequencies are very good, but the error in phase is from $-4$ to $-7\%$ of the travel time at $\xi = 1.29$ and $-7$ to $-10\%$ at $\xi = 1.93$. This error is systematic and appears in other comparisons with $\sigma = 0.017$.

4.4.2.2 Wave Evolution on the Slope

Several comparisons in the previous section showed significant errors in the model calculations at the shelf break. To examine this problem in more detail a series of runs were conducted to measure wave displacements on and just after the slope.

Only 4 conductivity probes were available for the experiment. One was located at $\xi = 0$, two others were positioned 2 and 6.5 m past the shelf break and the fourth was moved to specific locations.
Figure 4.26. \((\alpha, \lambda, \tau) = (-0.042, 0.16, 0.73)\). \((L, D, d_\text{s}, c_0, \sigma) = (488 \text{ cm}, 36.5 \text{ cm}, 8 \text{ cm}, 19.1 \text{ cm/s}, 0.0525)\).
Figure 4.27. \((a, \lambda, \tau) = (-0.072, 0.10, 0.45)\). \((L, D, d_s, c_o, \sigma) = (701 \, \text{cm}, 33.5 \, \text{cm}, 5 \, \text{cm}, 18.9 \, \text{cm/s}, 0.0539)\).
Figure 4.28. $(\alpha, \lambda, r) = (-0.036, 0.13, 0.73)$. $(L, D, d_s, c_0, \sigma) = (701 \text{ cm}, 36.5 \text{ cm}, 8 \text{ cm}, 10.9 \text{ cm/s}, 0.0165)$. 
Figure 4.29. \((a, \lambda, r) = (-0.046, 0.12, 0.45)\). \((L, D, d_S, c_0, \sigma) = (701 \text{ cm}, 33.5 \text{ cm}, 5 \text{ cm}, 10.6 \text{ cm/s}, 0.0165)\).
on the slope for each run. The repeatability feature of the wave maker permitted generation of the same incident wave each trial, whereby a composite picture of wave evolution on the slope was constructed.

Figure 4.30 compares the incident wave in the first run with the incident wave from the last (fifth) trial. The agreement is extremely good, especially in amplitude and shape, with only a very small phase error (< 0.5 sec). The density profile changed slightly over the course of five runs, but no breaking occurred and the variation was less than that shown in Figure 4.7 (six runs with tank draining and breaking).

Figure 4.31 is a composite picture of wave evolution on the slope for \((\alpha, \lambda, r, \sigma) = (-0.055, 0.15, 0.59, 0.0336)\). Data at \(\xi = 0.39 - 1.20\) were taken in the five successive runs and data at \(\xi = 0, 1.41\) and \(2.33\) are from the first run. The agreement is very good everywhere on the slope. The error in amplitude is localized near \(\xi = 1.0\) and is probably due to the abrupt slope change at the shelf break. This violates the slow variation assumption of the theoretical model. The fluid flowing off the shelf must undergo a significant vertical acceleration, in conflict with the approximation of long wave theory.

4.4.4.3 **Importance of the Cubic Term**

Figure 4.32 shows a sample comparison of the KdV model, the EKdV model and experimental data for \((\alpha, \lambda, r, \sigma) = (-0.043, 0.12, 0.59, 0.0355)\). The extended KdV equation gives significantly better
Figure 4.30. Comparison of first (——) and fifth (— —) incident waves ($\xi = 0$) for same set-up. Amplitudes normalized by $a_0$, the amplitude of the wave in trial 1. $(a_0, D, d_+, \sigma) = (-2.0 \text{ cm}, 33.5 \text{ cm}, 11 \text{ cm}, 0.0336)$. 

\[ \left( \frac{c_0}{D} t \right) \]
Figure 4.31. Composite wave evolution on slope. \((a, \lambda, \tau) = (-0.05, 0.15, 0.59)\). \((L, D, d, c_0, \sigma) = (488 \text{ cm}, 35 \text{ cm}, 6.5 \text{ cm}, 15.3 \text{ cm/s}, 0.0336)\).
Figure 4.32. Comparison of continuous stratification model with (---) and without (-----) cubic nonlinearity for \((a, \lambda, r) = (-0.043, 0.12, 0.59)\). \((L, D, d_{-s}, c_0, \sigma) = (701 \text{ cm}, 35 \text{ cm}, 6.5 \text{ cm}, 15.8 \text{ cm/s}, 0.0355)\).
Figure 4.33. Two rank-ordered waves. \((a_1, s_2, \lambda, \tau) = (-0.049, -0.040, 0.16, 0.73)\). \((L, D, d_S, c_0, \sigma) = (488 \text{ cm}, 36.5 \text{ cm}, 8 \text{ cm}, 15.8 \text{ cm/s}, 0.0349)\).
Figure 4.34. Two rank-ordered waves. \((a_1, a_2, \lambda, r) = (-0.055, -0.044, 0.15, 0.59)\). \((L, D, d_s, c_o, \sigma) = (488 \text{ cm}, 35 \text{ cm}, 6.5 \text{ cm}, 15.5 \text{ cm/s}, 0.0349)\).
Figure 4.35. Two rank-ordered waves. \((a_1, a_2, \lambda, r) = (-0.060, -0.049, 0.15, 0.45)\). \((L, D, d_{-5}, c_o, \sigma) = (488 \text{ cm}, 33.5 \text{ cm}, 5 \text{ cm}, 15.5 \text{ cm/s}, 0.0349)\).
results, especially near the shelf break, demonstrating the influence of cubic nonlinearity on long internal waves propagating through a turning point.

4.4.4.4 Multiple Incident Waves

In this section we examine the evolution of two rank-ordered solitary waves incident on slope/shelf topography. Rank-ordered groups are the most common observation so we wish to see if the excellent agreement between theory and experiment for a single incident wave also applies to multiple waves.

Comparisons between theory and experiment for two waves $\alpha_1$ and $\alpha_2$ incident on the intermediate length slope ($\lambda = 0.15$) for all three values of $r$ are illustrated in Figures 4.33 - 4.35. Again, the overall agreement is quite good. The largest errors, aside from measurements at $\xi = 1.0$, are in the middle of the scattered packets at $\xi = 2.33$. In this region, the scattered tail of the first wave is interacting with front of the second wave. This disagreement is not too surprising in view of the sometimes poor comparisons in the tails for single incident waves. The numerical experiment for multiple incidents described in Chapter 3 demonstrated the importance of nonlinear interaction between scattered wave packets. Thus, if the model does not predict the tail of the first packet very well the interaction will not be modeled accurately. In spite of this shortcoming, the agreement is satisfactory and improves as $r$ increases.
5.1 Introduction

In Chapter 4 the occurrence of wave breaking at the shelf break was noted. Such strong nonlinear events are of interest not only for their effects on the modeling in Chapter 4, which was surprisingly small, but also because of their importance in horizontal and vertical mixing. Of particular interest is the potential impact of such events on the coastal environment.

In this chapter, observed instabilities are detailed. Kinematics of the instabilities are discussed and empirical breaking criteria (regions of instability in parameter space) are defined. The generation of higher (2nd) mode waves from an instability of the lowest mode is shown to occur under certain circumstances. The role of wave breaking on dissipation is examined.
5.2 Wave Instabilities

5.2.1 Description and Observations

Essentially two of instabilities were observed to occur in the shelf break region. They can be described as a weak shearing instability and a strong overturning instability. Additionally, shearing of the transmitted wave crests just past the shelf break may occur.

Figure 5.1 shows sketches illustrating the beginning of a typical breaking event, which in all cases is localized to a region \( \pm \) from the shelf break. In Figure 5.1(a) the leading face of the incident wave (propagating to the right) has elongated and moved on the shelf. The rear face has steepened due to nonlinearity. The flow in the lower layer at the shelf break is off-shelf. In Figure 5.1(b) the velocity in the lower layer has reached maximum (in the off-shelf direction) and appears to confront control section caused by the expansion of the lower-layer depth just off the shelf. An opposing flow under the crest of the first scattered wave is present leading to a significant horizontal gradient of x-velocity and a resulting vertical velocity under the developing crest. Mixing occurs along the forward and rear faces of the crest.

As the wave moves further on-shelf (Figure 5.1(c)) the mixing becomes localized on the rear face of the first crest and in some cases causes a sharp corner to develop in the trough. A flow similar to a jet is ejected down the slope (Figure 5.1(d)) and overturning may follow.
Figure 5.1. Beginning of typical wave instability at shelf break.
The difference between the weak and strong instabilities is primarily in the amount of mixing and the occurrence of overturning in the latter case, as will be illustrated below with photographs. The controlling factors in the process are primarily incident wave amplitude and the initial depth of the lower layer on the shelf. Other parameters, such as the relative density difference and slope length, play a weaker role and do not change the basic description given above.

Figure 5.2 shows a series of shadowgraph images taken at the shelf break during a strong overturning event. The scale markings are spaced 1 cm apart. The clock reads 10 sec per revolution. Parameters for this run were \((\alpha, \lambda, r, \sigma) = (-0.067, 0.29, 0.45, 0.0530)\) and \((L, D, d_\perp) = (244 \text{ cm}, 33.5 \text{ cm}, 11 \text{ cm})\). Note the sharp corner in frame (6) and the subsequent flow of mixed fluid down the slope in frames (7) through (10). In frames (10) through (12), the large region of overturning and mixing is moving onto the shelf. In frame (13), the mixed fluid appears similar to a turbulent density intrusion. Frames (16) through (24) show remnants of the turbulent mixing.

The nearly horizontal striations present in the last 10 frames are decaying internal waves generated during the mixing event (Pearson and Linden 1983). Viscosity slowly damps these features. Their horizontal extent is less than \(\pm \) from the shelf break. The head of the turbulent intrusion (frame (13)) eventually propagates away from the mixing and forms a second mode solitary internal wave (Davis and Acrivos 1967). These waves will be discussed more in §5.2.3.
Figure 5.2. Sequence of shadowgraphs showing a strong overturning event at the shelf break for \((\alpha, \lambda, r, \sigma) = (-0.067, 0.29, 0.45, 0.0530)\) and \((L, D, d_S, c_0) = (244 \text{ cm}, 33.5 \text{ cm}, 5 \text{ cm}, 18.7 \text{ cm/s})\). Clock in lower center of each frame has a 10 sec/rev. face. Approximate time between frames is 0.9 sec.
Figure 5.3 shows photos at the shelf break for a breaking event under the same conditions as Figure 5.2 except $\Delta \rho/\rho = 0.0172$. Note the similarity between these photos and Figure 5.2. The features of the breaking events are nearly identical regardless of the density differences. The only significant change is a decrease in the extent of mixing and a slower evolution as $\Delta \rho/\rho$ decreases.

When the $\lambda$ decreases (slope length increases) the breaking becomes less violent but still occurs in the same sequence of events described above.

A typical nonoverturning instability is illustrated in Figure 5.4. The run parameters are $(a, \lambda, r, \sigma) = (-0.031, 0.32, 0.45, 0.0355)$ and $(L, D, d_\perp) = (244 \text{ cm}, 33.5 \text{ cm}, 11 \text{ cm})$. As the rear of the first wave passes the shelf break, a dislocation of the interface develops (frame (7)). Weaker mixing is evident a short time later (frames (9) through (12)). Of particular interest in these photos is the apparent higher modal structure at the shelf break without strong overturning. Note the splitting of the light band just off-shelf in frame (3). Each band of light/dark corresponds to a section of the interface where $n_{zz} < 0$ (or $\rho_{zz} < 0$ for the salt stratification used) where $n$ is the refractive index (Nowbray 1967). Lowest mode motion will show only one light band below a dark band. In frames (5) through (9), a weak higher mode structure is present just to the right of the shelf break. In frames (7) and (8), the displacement of the top of the dark band is out of phase with the displacement of the light band at the bottom of the interface.
Figure 5.3. Sequence of shadowgraphs showing strong overturning at shelf break. Same conditions as Figure 5.2, except $\sigma = 0.017$. 
Figure 5.4. Non-overturning instability at shelf break. \((a, \lambda, \tau, \sigma) = (-0.031, 0.32, 0.45, 0.0355)\) and \((L, D, d-S, c_o) = (244 \text{ cm}, 33.5 \text{ cm}, 11 \text{ cm}, 15.3 \text{ cm/s})\).

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Figure 5.5 shows an example of a shearing instability of the crest of the first transmitted wave just past the shelf. Run parameters were \((\alpha, \lambda, r, \sigma) = (-0.066, 0.15, 0.45, 0.0346)\) and \((L, D, d_+) = (488 \text{ cm}, 33.5 \text{ cm}, 11 \text{ cm})\).

5.2.2 Velocities in the Breaking Region

The observations and shadowgraphs of the strong breaking events suggest the presence of a flow control just off-shelf and interfacial shearing instabilities as the mechanisms for the breaking events. Therefore, to further help define the instabilities, fluid velocities in the shelf break region were measured. The LDV measurement system described in §4.2.4 was employed. Since the main features of the breaking events were not strongly dependent on slope length or density difference, the velocity measurements were conducted with the intermediate slope length and density difference: \((L, \sigma) = (488 \text{ cm}, 0.035)\).

Figure 5.6 displays the lower layer horizontal velocity 20 cm before, at, and 20 cm past the shelf break during a strong breaking event. This figure is a composite of three experimental runs with the parameters \((\alpha, \lambda, r, \sigma) = (-0.063, 0.15, 0.45, 0.035)\) and \((L, D, d_+, c_0, c_s) = (488 \text{ cm}, 33.5 \text{ cm}, 11 \text{ cm}, 15.5 \text{ cm/s}, 9.8 \text{ cm/s})\). These conditions are the same as shown in Figure 4.24. The maximum on-shelf velocity exceeds the linear long wave phase speed on the shelf \(c_s\) (from continuous stratification model) just before and at the shelf break. These measurements suggest that the breaking may be due to a
Figure 5.5. Shearing instability of first transmitted crest just past the shelf break for \((\alpha, \lambda, r, \sigma) = (-0.066, 0.15, 0.45, 0.0346)\) and \((L, D, d_s, c_0) = (488 \text{ cm}, 33.5 \text{ cm}, 11 \text{ cm}, 15.3 \text{ cm/s})\).
Figure 5.6. Lower layer velocities (2 cm above bottom) in shelf break region during a strong breaking event. \((a, \lambda, r, j) = (-0.063, 0.15, 0.45, 0.035)\) and \((L, D, d_{-B}, c_0, c_s) = (488 \text{ cm}, 33.5 \text{ cm}, 5 \text{ cm}, 15.5 \text{ cm/s}, 9.8 \text{ cm/s})\).
kinematic instability which occurs over a distance - D and time scale - D/c_s; however, the local nonlinear phase speed is larger than the recorded velocities.

Note also that the horizontal gradient in x-velocity is very large. When the velocity at the shelf break is a minimum, the velocity 20 cm off the shelf is nearly zero. The local vertical velocity can be estimated from the continuity equation

\[ w = - \int_{-h}^{z} \frac{\partial u}{\partial x} \, dz = - \Delta u \frac{d_s}{\Delta x}. \]

Using \( \Delta u = -0.7c_s \), \( \Delta x = 20 \) cm and \( d_s = 5 \) cm, we find \( w = 0.18c_s \).

During the time scale of breaking \( D/c_s \), the interface would move vertically about 0.18D or 6 cm for the given conditions. The displacement is \( O(1) \) relative to the local lower layer depth. The resulting overturning is not surprising.

Instability resulting from the Richardson number

\[ R_i = - \frac{g}{\rho} \frac{\partial \rho}{\partial z} \left( \frac{\partial u}{\partial z} \right)^2 \]  \hspace{1cm} (5.1)

falling below 0.25 is also a possibility. From (2.20a) we have

\[ R_i = - \frac{g}{\rho} \frac{\partial \rho}{\partial z} \left( c (\phi''(z) A)^2 \right. \]  \hspace{1cm} (5.2)
This criteria is based on a linear analysis for steady stratified shear flows and is probably not directly applicable to transient nonlinear wave phenomenon. With this in mind, Table 5.2 lists the lowest values of (5.2) calculated from measured data at $\xi = 1.0$ for the breaking situation discussed in the preceding two paragraphs. Neither location is below 0.25, but at the first crest $R_i = 0.34$, which may be sufficient for instability. Figure 5.2 does show stable flow at the trough and mixing under the first crest.

The lower layer velocity at $\xi = 1.0$ during a weak event without overturning is shown in Figure 5.7. The on-shelf velocity is always less than $c_s$. The minimum calculated Richardson number (5.2) is 1.0 and occurs at the leading trough. Refering to the photographs in Figure 5.4 we see that the slight mixing is due to a dislocation of the interface. Water flowing off-shelf is apparently accelerated in the negative $z$-direction resulting in a drawdown of the interface. The same process undoubtedly occurs in the stronger events and contributes to the mixing.

The use of a smooth transition at the shelf break, rather than a sharp corner, would probably reduce this feature. Dislocation was present, though, in experiments with the long slope ($L = 701$ cm, $\lambda = 0.11$) even though the transition was essentially smooth. Clearly, this dislocation is an important feature of the mixing mechanisms. Velocity measurements in the vicinity of the dislocation were attempted but were not successful because of refraction of the laser beams by local gradients in the density (index of refraction).
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<td>0.73</td>
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Table 5.1. Values of the Richardson number at the shelf break during a strong breaking event. $(\alpha, \lambda, \tau, \sigma) = (-0.063, 0.15, 0.45, 0.035)$ and $(L, D, d_\perp, c_0, c_s) = (488 \text{ cm}, 35.5 \text{ cm}, 11 \text{ cm}, 15.5 \text{ cm/s}, 9.8 \text{ cm/s})$. 
Figure 5.7. Lower layer velocity (2 cm above bottom) at shelf break during a non-overturning event. \((a, \lambda, r, \sigma) = (-0.062, 0.15, 0.59, 0.0356)\) and \((L, D, d_S, c_O, c_S) = (488 \text{ cm}, 35 \text{ cm}, 6.5 \text{ cm}, 15.8 \text{ cm/s}, 10.9 \text{ cm/s})\).
5.2.3 Second-Mode Wave Generation

The occurrence of a strong overturning event was usually accompanied by the generation of a second-mode solitary wave of the type described by Davis and Acrivos (1967). These waves are characterized by an isolated bulge of the pycnocline and are long with respect to the interface thickness, but short compared to the total depth. Evolution in a quiescent fluid is described by the theory of Benjamin (1967) and Ono (1975) discussed in Chapter 1.

Shadowgraph images in Figure 5.8 show the initial development and subsequent second-mode solitary wave (frame (3)) for a typical case where \((a, \lambda, r, \sigma) = (-0.062, 0.15, 0.45, 0.035)\) and \((L, D, d) = (488 \text{ cm}, 33.5 \text{ cm}, 11 \text{ cm})\). The displacement records from two conductivity probes located 2 m past the shelf and separated vertically by 1 cm are shown in Figure 5.9 for a run with nearly identical parameters as above. The calculated displacements are not representative of the amplitude of the bulge because the data processing procedure assumes lowest mode motion. The second-mode nature of the motion is, however, corroborated.

Davis and Acrivos (1967) found that these waves could be generated with relative ease by simply disturbing the interface. Kao and Pao (1979) and Maxworthy (1980) investigated the general problem of second-mode solitary wave generation by the gravitational collapse of a mixed region. In view of these results, second-mode generation in the present context is not surprising. Of particular interest is the identification of a situation in which an instability of a lower mode wave leads to higher mode generation.
Figure 5.8. Shadowgraphs of second-mode solitary wave generation 35 cm past shelf break. \((a, \lambda, r, \sigma) = (-0.060, 0.15, 0.45, 0.035)\) and \((L, D, d_s) = (488 \text{ cm}, 33.5 \text{ cm}, 5 \text{ cm})\).
Figure 5.9. Probe displacement records showing second-mode wave. Measurements taken at $\xi = 1.41$ for $(\alpha, \lambda, \tau, \sigma) = (-0.064, 0.15, 0.45, 0.0346)$ and $(L, D, d_s, c_o, c_s) = (488 \text{ cm, } 33.5 \text{ cm, } 5 \text{ cm, } 15.3 \text{ cm/s, } 9.8 \text{ cm/s})$. 

\[
\eta = \frac{A}{a_o}
\]
The second-mode solitary waves propagated onto the shelf a distance $O(10D)$ before being damped to an unobservable amplitude. No second-mode solitary waves were observed to propagate off-shelf.

5.2.4 Parametric Regions of Instabilities

Visual observations and photographs were used to identify the presence and type (shearing and overturning) of instabilities for the complete set of experimental runs. The objective was to obtain an empirical criteria with which to predict such events in the coastal regions. In particular, we would like to determine when a second-mode solitary wave will be generated.

The important physical variables which were examined in the experiments and are expected to play a role in the wave breaking are $a_0$, $d_s$, $L$ and $\Delta \rho/\rho$. With the introduction of the wave length scale $\lambda$ these may be arranged into the nondimensional parameters $-a_0/d_s$, $\lambda = \ell/L$ and $\sigma = \Delta \rho/\rho$. Figure 5.10 shows the regions of instabilities for $\sigma = 0.035$. Second-mode waves and significant vertical mixing occur for $(-a_0/d_s) > 0.35-0.4$. Dependence on slope length is quite weak but does increase as $\lambda$ decreases. For a fixed $\lambda$, the transition from clean transmission to strong overturning is a relatively small region of weaker shearing. Figures 5.11 and 5.12 are for $\sigma = 0.017$ and 0.053, respectively. Fewer experiments were run with these density differences, so the transition region is not well defined.
Figure 5.10. Regions of instabilities for $\Delta \rho/\rho = 0.035$.
- over-turning and second-mode generation;
- shearing;
- clean transmission.
Figure 5.11. Regions of instabilities for $\Delta \rho / \rho = 0.017$.
- ● overturning and second-mode generation;
- □ shearing;
- △ clean transmission.
Figure 5.12. Regions of instabilities for $\Delta \rho / \rho = 0.053$.
- ● overturning and second-mode generation;
- □ shearing;
- △ clean transmission.
Figure 5.13. Composite of instability regions.

- - - - - $\Delta \rho / \rho = 0.017$;
- - - - - $\Delta \rho / \rho = 0.035$;
- - - - - $\Delta \rho / \rho = 0.053$. 

Figure 5.13 is a composite of the transition region for all values of $\sigma$. In general, the results are not strongly dependent on $\sigma$ or $\lambda$. The tendency is for instability to occur at lower incident amplitudes as the slope length decreases. Empirical criteria for wave instabilities at the shelf break are listed in Table 5.2. All of the data used to determine these conditions were for relatively thin interfaces where $\varepsilon_s = h \rho / D_s = 0.1$ and $D_s$ is the total depth on the shelf. A few experiments in the initial stages of this work were done with $\varepsilon_s = 0.3$. The thicker interface tended to initiate the shearing instability at slightly lower values of $(-a_0/d_s)$.

5.3 Wave Dissipation Due to Breaking

In addition to the prediction of breaking events, the amount of incident energy lost from the lowest mode is of interest. Some of this energy is radiated as second or higher mode waves, the remainder contributes to local mixing in the shelf break region. In this section, estimates of the fraction of incident lowest-mode wave energy lost by breaking are made, as well as the fraction of energy lost to viscous stress.

The total wave energy flux through a vertical section between time $t_1$ and $t_2$, or the work $W$ on the vertical section, is given by

$$W = \int_{t_1}^{t_2} \int_{-h}^{0} (pu + \frac{1}{2} \rho u^3) \, dz \, dt \quad (5.3)$$
<table>
<thead>
<tr>
<th>Instability</th>
<th>$-\frac{a_o}{d_s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>None</td>
<td>$&lt; 0.3$</td>
</tr>
<tr>
<td>Shearing$^3$</td>
<td>$0.3 &lt; \left(-\frac{a_o}{d_s}\right) &lt; 0.4$</td>
</tr>
<tr>
<td>Mixing and Second-Mode</td>
<td>$&gt; 0.4$</td>
</tr>
</tbody>
</table>

Table 5.2. Conditions for observed instabilities.
where \( p \) and \( u \) are the wave induced pressure and horizontal velocity. Using the lowest order relations for \( u \) and \( p \) (2.20b, c), (5.3) becomes

\[
W = \int_{t_1}^{t_2} \int_{-h}^{0} \left( c^3 \rho \phi'(z)^2 A^2(x,t) + \frac{1}{2} c^3 \rho \phi'(z)^3 A(x,t) \right) \, dz \, dt
\]

or

\[
W = c^3 I_0 \int_{t_1}^{t_2} A^2(x,t) \, dt + \frac{1}{2} c^3 I_1 \int_{t_1}^{t} A^3(x,t) \, dt \tag{5.4}
\]

where \( I_0 \) and \( I_1 \) are given by (2.45a, b). Thus, by evaluating (5.4) at two locations \( x_0 \) and \( x_1 \), the total dissipation \( D_T \) between the points is given by

\[
D_T = W_0 - W_1 = \Delta W. \tag{5.5}
\]

The total dissipation of the lowest mode is made up of viscous boundary and interfacial dissipation \( D_V \) and dissipation \( D_B \) due to any wave breaking between \( x_0 \) and \( x_1 \).

Taking \( x_0 \) at the base of the slope (\( x_0 = 0 \)) and \( x_1 \) as the last measuring station (\( \xi = 3.66, L = 244 \) cm; \( \xi = 2.33, L = 488 \) cm; \( \xi = 1.93, L = 701 \) cm) estimates of \( D_B \) were constructed as follows. For a given experimental run \( W_0 \) and \( W_1 \) were calculated using (5.4) and
<table>
<thead>
<tr>
<th>$\frac{Ap}{\rho}$</th>
<th>No Instabilities</th>
<th></th>
<th>Strong Overturning Instability</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Average</td>
<td>No. Data Pts.</td>
<td>Average</td>
</tr>
<tr>
<td>0.017</td>
<td>0.05</td>
<td>3</td>
<td>0.13</td>
</tr>
<tr>
<td>0.035</td>
<td>0.00</td>
<td>5</td>
<td>0.10</td>
</tr>
<tr>
<td>0.053</td>
<td>-0.02</td>
<td>6</td>
<td>0.09</td>
</tr>
<tr>
<td>All Runs</td>
<td>0.00</td>
<td>14</td>
<td>0.10</td>
</tr>
</tbody>
</table>

Table 5.3. Values of $\text{D}_0/\text{W}_0$ for cases with (a) no wave instabilities and (b) strong overturning instabilities.
the experimental measurements of $A(x,t)$ or the numerical simulations for $A(x,t)$. The times $t_1$ and $t_2$ were chosen so that the complete incident wave or scattered wave group was included. Ideally, if no breaking occurred, values of (5.5) from the experimental data $D^E_T$ and the numerical simulation $D^N_T$ would be equal. With breaking, the difference would be equal to the energy lost from the lowest mode during the breaking event. Thus,

$$D^E_B - D^N_T = W^N_W - W^E_W \quad (5.6)$$

since $W^N_W = W^E_W = W_0$.

Table 5.3 lists average values of (5.6), normalized by $W_0$, for cases with no wave instabilities and cases with strong overturning instabilities and second-mode generation for all three density differences. The results are fairly consistent and show that about 10% of the incident wave energy will be lost from the lowest mode during a strong overturning instability. On average, about 40-50% of the incident wave is lost to combined dissipation and instabilities between $\xi = 0$ and the last measuring station so that when instabilities occur they constitute about 20% of the total energy lost from the lowest mode.
CHAPTER VI

DISCUSSION AND CONCLUSION

The objective of this thesis has been to study the evolution and stability of long nonlinear internal waves over slowly varying topography using a combination of theoretical and experimental methods. The outcome is an improved understanding of the important processes governing wave evolution and an improved ability in the prediction and interpretation of field data. The specific problem addressed has been the propagation of single and multiple (two) rank-ordered solitary waves over slope/shelf topography in which a turning point (point of equal layer depths) is encountered.

A KdV model incorporating continuous stratification, slowly varying topography, boundary damping and cubic nonlinearity was formulated. Comparisons of the theoretical model with the experimental data showed very good agreement over a range of stratifications, topography and wave amplitudes. The results illustrated that the extended KdV theory was very robust, giving good predictions for situations which were outside the expected range of validity of the model. In particular, cases with relatively fast slopes, exhibiting a significant reflected wave, were modeled accurately even though the theory only accounts for unidirectional propagation. Comparisons were very good for $\lambda$ as large as $10\alpha$. 

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Poorest agreement was found as the lower-layer depth on the shelf was decreased (topographic scattering increased). Errors in the amplitude and frequency of the tail of the scattered packet became significant, although the leading trough and first 2 or 3 transmitted crests were modeled accurately. This error carried over into the modeling of two rank-ordered waves. When the tail of the lead wave was modeled inaccurately the interaction of the trailing wave with the scattered lead wave was poorly predicted. Given this limitation, practical application of the theory may be limited to cases for waves incident on a shelf with $d_\alpha/d_\gamma > 0.5-0.6$ or for widely spaced incident waves. This, however, is still a broad range, encompassing all situations except the shallowest lower-layer depths on the shelf.

In addition to illustrating the importance of cubic nonlinearity when waves propagate in regions of nearly equal layer depths, the study demonstrated the need for the continuous stratification model when topographic scattering occurs. Though the two-layer model gives acceptable results in uniform depth, errors in phase speed and dispersion of the scattered waves accumulate and lead to significant discrepancies. Recall from Table 1.1 that $h_p/D = 0.3-0.4$ in the Massachusetts Bay and Andaman Sea. Thus, departure from the two-layer model is even greater than for the laboratory experiments.

Dissipation by boundary shear also is an important feature of the wave evolution. Neglect of this process would have resulted in considerable error. At oceanic scales dissipation will probably
play a weaker role than in the laboratory, but will still be an important process as the measurements of Liu et al. (1984) and Sandström and Elliott (1984) indicate. Application of the model to field scale will be dependent upon specification of an effective eddy viscosity. This poses a problem which would probably require model calibration with field data. The eddy viscosity is the only free parameter in the model, thus calibration should be straightforward.

Wave instabilities were observed in the neighborhood of the shelf break under certain conditions. These instabilities were categorized as weak shearing or strong overturning instabilities. Slight shearing of the transmitted wave crests just past the shelf break was also observed. The important parameter in determining the type and extent of instability was found to be \((-a_c/d_s)\). Mixing increased as this parameter increased. Dependence on topography or stratification was weak. The weak instability at the shelf break was associated with a dislocation of the interface. The strong instabilities were apparently due to a combination of interfacial shearing and kinematic considerations. Very high velocities, \(0(c)\), were measured at the shelf break. The strong overturning instabilities resulted in significant horizontal and vertical mixing in the immediate region of the shelf break. Moreover, these strong events resulted in the generation of a second-mode solitary wave. Second-mode solitary waves will have closed streamlines, and therefore mass transport, if the amplitude is large (Davis and Acrivos 1967). Although entrapment of fluid in the waves was not positively
identified in this study, this mechanism of lowest mode instability and possible mass transport might have implications in coastal mixing processes. Clearly, the strong mixing in the shelf break region is important regardless of any second-mode generation. The disposition of energy lost in breaking is unresolved. Some is radiated as higher mode waves, some causes local mixing and some may cause mean flow generation. Clarification of the division might be possible with an experimental program concentrating on the breaking events. A desired outcome would be a quantitative estimate of the amount of horizontal and vertical mixing.

The influence of higher modal structure was also apparent in the shadowgraphs of the weaker instabilities. The question arises as to the role of higher modes in long internal wave propagation over topography. These features seemed to have little influence on the far field comparisons, but are probably important in the dynamics of the near field motion. Pursuit of this topic was outside the scope of this thesis but is suggested for future research.

In conclusion, an advancement in the understanding and modeling of long nonlinear internal waves over bottom topography has been made. The findings should give confidence to the application of the KdV model to a wide variety of conditions. The next step should be the application of the model to oceanic data. Factors such as slowly varying mean currents or background density field could be incorporated into the theory and would be an extension of the wave evolution in inhomogeneous media studied here. Other important
features such as radial spreading (Miles 1978) or rotation could also be added to the model without difficulty.
REFERENCES


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Haurey, L. R., Briscoe, M. G. and Orr, M. H. 1979, Tidally Generated Internal Wave Packets in the Massachusetts Bay, Nature 278, 312-317.


APPENDIX 1

COEFFICIENT OF CUBIC TERM

In this Appendix the extended KdV (EKdV) equation for continuous stratification is derived. The coefficient of the cubic nonlinear term found below differs from the results of Miles (1979) (denoted M) and Gear and Grimshaw (1983) (denoted GG) who assumed steady flow. The differences are discussed.

In order to simplify the following analysis an inviscid fluid of constant depth is assumed. We also consider waves in a region where the coefficient of the quadratic nonlinear term $3I_1/(2I_0c)$ of (2.46) is $0(\alpha)$. The appropriate scaling between dissipation and nonlinearity is then $\beta = 0(\alpha^2)$.

We start from the governing equation (2.2) - (2.5) and take the same nondimensionalization as used in the KdV equation derivation (§3.2). However, instead of $(D/k)^2 = \beta = 0(\alpha)$ we now take $\beta = 0(\alpha^2)$. The slow space variable now becomes

$$X = \alpha^2 x \quad .$$  \hspace{1cm} (A1.1)

Retaining the time variable $s$ (2.10) the governing equations are

$$\left( \frac{1}{c} \frac{\partial}{\partial s} + \alpha^2 \frac{\partial}{\partial X} \right) U + \frac{\partial W}{\partial Z} = 0$$  \hspace{1cm} (A1.2)
\[-\frac{\partial \rho}{\partial s} + \alpha u \left(\frac{1}{c^2} \frac{\partial}{\partial s} + \alpha^2 \frac{\partial}{\partial x}\right) \rho + \alpha w \frac{\partial \rho}{\partial z} + \rho w \rho = 0 \quad (A1.3)\]

\[\left(-\frac{\partial}{\partial s} + \alpha u \left(\frac{1}{c^2} \frac{\partial}{\partial s} + \alpha^2 \frac{\partial}{\partial x}\right) u + \alpha \frac{\partial u}{\partial z}\right) = -\left(\frac{1}{c^2} \frac{\partial}{\partial s} + \alpha^2 \frac{\partial}{\partial x}\right) p \quad (A1.4)\]

\[\alpha^2 \left(-\frac{\partial w}{\partial s} + \alpha u \left(\frac{1}{c^2} \frac{\partial}{\partial s} + \alpha^2 \frac{\partial}{\partial x}\right) w + \alpha \frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} - \rho \quad (A1.5)\]

subject to \( w = 0 \) at \( z = 0, -h \). \quad (A1.6)

Expanding

\[
\begin{pmatrix}
  u \\
  w \\
  p \\
  \rho \\
\end{pmatrix} = \begin{pmatrix}
  u \\
  w \\
  p \\
  \rho \\
\end{pmatrix}^{(0)} + \alpha \begin{pmatrix}
  u \\
  w \\
  p \\
  \rho \\
\end{pmatrix}^{(1)} + \alpha^2 \begin{pmatrix}
  u \\
  w \\
  p \\
  \rho \\
\end{pmatrix}^{(2)} + O(\alpha^3),
\]

collecting terms of like order in \( \alpha \) and rearranging each set of equations gives an equation for each \( w^{(n)} \) \((n=0,1,2,\ldots)\):

\[
0(\alpha^0) \quad L(w^{(0)}) = 0, \quad (A1.7)
\]

\[
0(\alpha) \quad L(w^{(1)}) = \frac{1}{c} \left\{ \rho^{(0)} \frac{\partial u^{(0)}}{\partial s} - \frac{1}{c} \rho u^{(0)} \frac{\partial u^{(0)}}{\partial s} - \rho w^{(0)} \frac{\partial u^{(0)}}{\partial z} \right\} z + \frac{1}{c^2} \left\{ \frac{1}{c} u^{(0)} \frac{\partial \rho^{(0)}}{\partial s} + w^{(0)} \frac{\partial \rho^{(0)}}{\partial z} \right\} \quad (A1.8)
\]

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\[ L(w^{(2)}) = -\left(\frac{\partial u^{(0)}}{\partial x}\right)_z + \frac{1}{c} \left\{ -\frac{\partial p^{(0)}}{\partial x} + p^{(0)} \frac{\partial u^{(1)}}{\partial s} + p^{(1)} \frac{\partial u^{(0)}}{\partial s} - \rho \frac{u^{(0)}}{c} \frac{\partial u^{(1)}}{\partial s} \right\} \\
\quad - \rho \frac{u^{(1)}}{c} \frac{\partial u^{(0)}}{\partial s} - p^{(0)} \frac{u^{(0)}}{c} \frac{\partial u^{(0)}}{\partial s} - p^{(1)} w^{(0)} \frac{\partial u^{(0)}}{\partial z} \\
\quad - \rho w^{(0)} \frac{\partial u^{(1)}}{\partial z} - \rho w^{(1)} \frac{\partial u^{(0)}}{\partial z}_z + \frac{1}{c^2} \left\{ \frac{u^{(0)}}{c} \frac{\partial p^{(1)}}{\partial s} + \frac{u^{(1)}}{c} \frac{\partial p^{(0)}}{\partial s} + w^{(0)} \frac{\partial p^{(1)}}{\partial z} + w^{(1)} \frac{\partial p^{(0)}}{\partial z} \right\} \\
\quad - \frac{\rho}{c^2} w^{(0)}_{ss} \tag{Al.9} \]

where the linear operator \( L \) is given by (2.15). All the equations above must satisfy the boundary conditions

\[ w^{(n)} = 0 \text{ at } z = 0, -h, n = 0, 1, 2, \ldots \tag{Al.10} \]

The \( O(\alpha^0) \) problem is identical to (2.13) and has the same solutions (2.20a,b,c,d). Employing these results on the right-hand side of (Al.8) and multiplying (Al.8) by \( \phi(z) \) then integrating for \( z = -h \) to \( z = 0 \) gives after suitable integration by parts the \( O(\alpha) \) solvability condition
\[ 0 = 3I_1 A_A \quad \text{(Al.11)} \]

where \( I_1 \) is given by (2.44b)

Before proceeding to the \( O(\alpha^2) \) problem the solutions for \( u(1), w(1), p(1) \) and \( \rho(1) \) must be found. From (Al.8) using (2.20a,b,c,d) we find

\[
L(w(1)) = A_A \{ (\overline{\rho \phi''})' - (\overline{\rho_z \phi'})' - (\overline{\rho \phi'})^2 + \frac{1}{c^2} \overline{\rho_{zz} \phi^2} \}
\]

or

\[
L(w(1)) = g(z) A_A \quad \text{(Al.12)}
\]

subject to (Al.10). This inhomogeneous boundary value problem has the general solution

\[
w(1) = w_h(1) + w_p(1) \quad \text{(Al.13)}
\]

where \( w_h(1) \) is the homogeneous solution and \( w_p(1) \) is the particular solution. The homogeneous solution is written as

\[
w_h(1) = - \phi A_s^{(1)} \quad \text{(Al.14)}
\]

where \( A^{(1)} \) is the \( O(\alpha) \) correction to the amplitude function.
The particular solution is written as

\[ w_p^{(1)} = f(z) AA_s \]  

(A1.15)

where

\[ L(f(z)) = g(z) \]  

(A1.16a,b)

\[ f = 0 \text{ at } z = 0, -h \ . \]

The function \( f(z) \) may be constructed by variation of parameters (GG) or from the theory of inhomogeneous boundary value problems (Boyce and DiPrima 1969, §11.4) it may be constructed as an expansion in the eigenfunctions of the Sturm–Liouville problem for \( \phi_n(z) \) and \( c_n (n=1,2,\ldots) \) (Eqns. (2.18) and (2.19)). Since we are concerned with the evolution of the mode 1 wave \( f(z) \) is an expansion in terms of the higher modes \( n = 2,3,\ldots \)

\[ f(z) = \sum_{n=2}^{\infty} a_n \left( \frac{1}{c_n^2} - \frac{1}{c_1^2} \right) \phi_n(z) \]  

(A1.17a)

where

\[ a_n = \int_{-h}^{0} g_1(z) \phi_n(z) \, dz , \]  

(A1.17b)

and \( g_1(z) \) is \( g(z) \) in (A1.12) evaluated with \( \phi_1(z) \).

Thus from (A1.13) using (A1.14) and (A1.15) we have

\[ w^{(1)} = -c\phi A_s^{(1)} + f(z) A_s^{(0)} A_s^{(0)} \]  

(A1.18)
where \( \phi \) is understood to be the mode \( n = 1 \) vertical structure function and the notation \( A^{(0)} \) has been introduced to identify the amplitude function from the \( O(\alpha^0) \) problem (Al.7). Now (Al.2), (Al.3) and (Al.4) at \( O(\alpha) \) and (Al.18) give

\[
\begin{align*}
\mathbf{u}^{(1)} &= c\phi'A^{(1)} - \frac{c}{2} f' A^{(0)^2} \\
\rho^{(1)} &= -\rho_z \phi'A^{(1)} + \frac{1}{2} \rho_z f A^{(0)^2} + \frac{1}{2c^2} (\overline{\phi''} A^{(0)^2} \\
\mathbf{p}^{(1)} &= \rho_c \phi'A^{(1)} - c^2 \rho \frac{1}{2} f' A^{(0)^2} \\
&\quad + \frac{1}{2} A^{(0)^2} \left( \overline{\phi''} - \frac{c}{2} \rho \phi' - \rho \phi'' \right). \quad \text{(Al.19a,b,c)}
\end{align*}
\]

Substituting these equations and (2.20a,b,c,d) into (Al.9) and invoking solvability gives

\[
0 = 2cI_0 A^{(0)} + \frac{1}{c^2} I_2 A^{(0)} + 3 I_1 A^{(0)} (A^{(1)}_s)
\]

\[
+ C A^{(0)^2} A^{(0)}_s, \quad \text{(Al.20)}
\]

where \( I_n \) (n=0,1,2) are given by (2.44a,b,c) and

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\[ C = 6I_3 + 12 \int_{-h}^{0} \overline{\rho \phi' \phi''} dz - 2 \int_{-h}^{0} \overline{\rho \phi^2} dz - \frac{9}{2} c \int_{-h}^{0} \overline{\rho \phi'^2 f'} dz + c \int_{-h}^{0} \overline{\rho \phi' \phi''} dz - c \int_{-h}^{0} \overline{\rho f \phi' \phi''} dz \quad (A.21) \]

where \( I_3 \) is given by (2.47).

The coefficient (A.21) can be simplified by integrating the fifth term in (A.21) by parts

\[ \int_{-h}^{0} \overline{\rho f' \phi''} \, dz = - \int_{-h}^{0} \overline{(\rho f')'} (\phi'^* + \phi' \phi'') \, dz . \]

Substituting

\[ (\overline{\rho f'})' = \frac{1}{c^2} \overline{\rho_2 f + g(z)} \]

from (A.16a) into the equation above, carrying out the integrations and then substituting the result back into (A.21) gives

\[ C = 3I_3 + 9 \int_{-h}^{0} \overline{\rho \phi' \phi''} \, dz - \frac{9}{2} c \int_{-h}^{0} \overline{\rho \phi'^2 f'} \, dz . \quad (A.22) \]

Returning to the 0(\( \alpha \)) solvability condition (A.11) we see that either \( I_1 = 0 \) or \( A^{(0)} \Lambda_s^{(0)} = 0 \). Since we assume
the latter is not true \( I_1 = 0 \). Correct to this order we could say \( I_1 = 0(\alpha) \). Thus the term \((A^{(0)}A^{(1)})_s\) in (Al.20) is \(0(\alpha)\) relative to (Al.20) and drops out of the equation. At the same time (Al.11) belongs in (Al.20) giving the extended extended KdV equation

\[
A^{(0)}_x + \frac{3}{2c} \frac{I_1}{I_0} A^{(0)} A^{(0)}_s + \frac{c}{2c I_0} A^{(0)}_s (0) + \frac{I_2}{2c^3 I_0} A_{sss} = 0. \quad (Al.23)
\]

The coefficient of the cubic nonlinear term is \( C/2c I_0 \).

Disregarding for the moment, the third term in (Al.22) involving \( f(z) \) (i.e., higher modes), we have

\[
\frac{C}{2c I_0} = \frac{1}{2c I_0} \left\{ 3 I_3 + 9 \int_{-h}^{0} \rho \phi r^2 \phi'' dz \right\} \quad (Al.24)
\]

as the coefficient of the cubic term. The discrepancy between this result and the coefficient

\[
\frac{C}{2c I_0} = - \frac{3 I_3}{I_0 c} \quad (Al.25)
\]

found by M and GG, who started with the assumption of steady flow, is unresolved. Extending the perturbation method of Benney (1966) to higher order also gives (Al.24). As discussed in Chapter II (Al.25) was chosen as the proper result because it collapses to the two-layer system coefficient derived independently.
by Kakutani and Yamasaki (1978) and Helfrich, Melville and Miles (1984).

GG, unlike M, retained the inhomogeneous term $f(z)$ in their derivation and found

$$\frac{C}{2I_0 c} = -\frac{3I_3}{cI_0} - \frac{9}{2} \frac{1}{I_0} \int_0^0 \rho \phi' z^2 f' \, dz$$

indicating that the discrepancy between the steady and unsteady derivations is not contained in the inhomogenous term. This term, however, was deleted from the cubic coefficient. It involves the influences of higher modes on the evolution of the lowest mode. In the two-layer system it is identically zero. M points out that (A1.25) gives the leading correction in situations when the coefficient of the quadratic term is small.
APPENDIX 2

INTERFACIAL SHEAR

In this Appendix the derivation of two-layer KdV equation incorporating interfacial shear is described. As in the bottom boundary layer analysis of \$3.2.1\$ the boundary layer height $\delta \ll D$, allowing the problem to be broken into an outer inviscid region and thin inner viscous regions on either side of the interface which are used to correct the inviscid results. For simplicity viscous effects at the bottom boundary and variable depth are ignored in this analysis, but may easily be incorporated.

Consider the system shown in Figure A2.1. The governing equations in the inviscid regions are

\[
\alpha \frac{\partial u_\pm}{\partial x} + \frac{1}{c} \frac{\partial u_\pm}{\partial s} + \frac{\partial w_\pm}{\partial z} = 0
\]  
(A2.1)

\[
- \frac{\partial u_\pm}{\partial s} + au_\pm (\alpha \frac{\partial}{\partial x} + \frac{1}{c} \frac{\partial}{\partial s}) u_\pm + aw_\pm \frac{\partial u_\pm}{\partial z} = - \frac{1}{\rho_\pm} (\alpha \frac{\partial}{\partial x} + \frac{1}{c} \frac{\partial}{\partial s}) p_\pm
\]  
(A2.2)

\[
\alpha(- \frac{\partial w_\pm}{\partial s} + au_\pm (\alpha \frac{\partial}{\partial x} + \frac{1}{c} \frac{\partial}{\partial s}) w_\pm + aw_\pm \frac{\partial w_\pm}{\partial z}) = - \frac{1}{\rho_\pm} \frac{\partial p_\pm}{\partial z}
\]  
(A2.3)

where the subscripts $+/-$ signify the upper/lower layers. The same nondimensionalization used in \$3.2\$ is employed along with the time variable $s$ (2.10) and the slow space scale $\chi = \alpha x$.

In the equations above $p$ is the perturbation pressure. The
Figure A2.1. Definition sketch for interfacial boundary layer problem.
hydrostatic contribution has been removed. The boundary conditions for the outer problem are

\[ \omega_\pm = 0 \quad \text{at} \quad z = \pm h_\pm \]  \hspace{1cm} (A2.4)

where \( h_\pm = d_\pm / D \). The boundary conditions at the interface are as yet unspecified.

Within the boundary layers on either side of the interface the boundary layer variables \( (\_)_{bl\pm} \) are governed by (A2.1) - (A2.3) with \( z \) replaced by the stretched variable \( \hat{z} = z/\gamma \) and (A2.2) and (A2.3) augmented by including viscosity on the right-hand side:

\[ (A2.2) \quad - \frac{\gamma^2 \alpha}{R_\pm^*} \left[ \left( \alpha \frac{\partial}{\partial x} + \frac{1}{c} \frac{\partial}{\partial \hat{z}} \right)^2 + \frac{1}{\gamma^2 \alpha} \frac{\partial^2}{\partial \hat{z}^2} \right] u_{bl\pm} \]

\[ (A2.3) \quad - \frac{\gamma^2 \alpha^2}{R_\pm^*} \left[ \left( \alpha \frac{\partial}{\partial x} + \frac{1}{c} \frac{\partial}{\partial \hat{z}} \right)^2 + \frac{1}{\gamma^2 \alpha} \frac{\partial^2}{\partial \hat{z}^2} \right] \omega_{bl\pm} . \]

Here \( R_\pm^* \) are defined by (2.3i). The boundary layer variables are the sum of the outer solution and a correction

\[ (u,w,p)_{bl\pm} = (u,w,p)_{o\pm} + (u,w,p)_{c\pm} . \]  \hspace{1cm} (A2.5)

The correction quantities must satisfy
\begin{align}
\left( \right)_{c^\pm} & \to 0 \quad \text{as} \quad \hat{z} \to \pm \infty , \quad (A2.6)

\text{The boundary conditions at the interface are}

- \frac{\partial A}{\partial s} + \alpha u_\pm \left( \alpha \frac{\partial}{\partial x} + \frac{1}{c} \frac{\partial}{\partial s} \right) A = w_\pm \quad (A2.7)_\pm

\sigma z + p_+ + 2 \frac{\gamma \alpha}{\mu^*_+} \frac{\partial w_+}{\partial z} = p_- + 2 \frac{\gamma \alpha}{\mu^*_-} \frac{\partial w_-}{\partial z} \quad (A2.8)

\frac{\mu^*_+}{\mu^*_-} \left[ \left( \frac{\alpha}{\gamma} \right) \frac{\partial u_+}{\partial z} + \alpha^2 \frac{\partial w_+}{\partial x} + \frac{\alpha}{c} \frac{\partial w_+}{\partial s} \right] = \left[ \left( \frac{\alpha}{\gamma} \right) \frac{\partial u_-}{\partial z} + \alpha^2 \frac{\partial w_-}{\partial x} + \frac{\alpha}{c} \frac{\partial w_-}{\partial s} \right] \quad (A2.9)

\text{and}

u_+ = u_- , \quad (A2.10)

\text{all evaluated at}

z = \alpha A

\text{or}

\hat{z} = \frac{\alpha}{\gamma} A . \quad (A2.11)

Equation (A2.7)_\pm \text{ is the kinematic condition, (A2.8) states the continuity of normal stress, (A2.9) states the continuity of tangential shear stress and (A2.10) gives the matching of}
tangential velocities. \( A \) is the interfacial displacement,

\[
A = A^{(0)} + \gamma \frac{\partial}{\partial \gamma} + \ldots \tag{A2.12}
\]

and \( \sigma = \Delta \rho/\rho_o \). Equations (A2.7) - (A2.10) are written in the coordinate set \((x, z)\), rather than a set at the interface \(z = \alpha A\), because the curvature of the interface \(\gamma 1/l\) is very much less than \(1/\delta\). The error in \((A2.7 - A2.10)\) is \(O(\frac{\delta}{\gamma})\) or \(O(\gamma^{1/2})\).

The dependent variables are expanded as power series in \(\alpha\),

\[
(u, w, p)_{\pm} = (u, w, p)^{(0)}_{\pm} + \alpha (u, w, p)^{(1)}_{\pm} + O(\alpha^2),
\]

and substituted into the governing equations (A2.1) - (A2.3) and the boundary conditions (A2.7) - (A2.10). The boundary conditions are then Taylor expanded about \(\hat{z} = 0\). Note that these conditions are applied within the boundary layers.

From the kinematic condition (A2.7), using (A2.12), we find at \(O(\alpha^0)\)

\[
- \frac{\partial A^{(0)}}{\partial s} = w^{(0)}_{b1 \pm} + \left(\frac{\alpha}{\gamma}\right) A^{(0)} \frac{\partial w^{(0)}_{b1 \pm}}{\partial \gamma} + \left(\frac{\alpha}{\gamma}\right)^2 A^{(0)} \frac{\partial^2 w^{(0)}_{b1 \pm}}{\partial \gamma^2} + \ldots \text{ at } \hat{z} = 0^\pm \tag{A2.13}
\]

After using (A2.5) and recalling

\[
\frac{\partial}{\partial \gamma} (\ldots)_o = \gamma \frac{\partial}{\partial z} (\ldots)_o
\]
(A2.13) becomes
\[
-\frac{\partial A^{(0)}}{\partial s} + w^{(0)} + w_{c}^{(0)} + \left(\frac{\alpha}{\gamma}\right) A^{(0)} \frac{\partial w_{c}^{(0)}}{\partial z} + \frac{1}{2} \left(\frac{\alpha}{\gamma}\right)^2 A^{(0)} \frac{\partial^2 w_{c}^{(0)}}{\partial z^2} + \cdots
\]

at \( z = 0^+ \). \hspace{1cm} (A2.14)

This equation and the other boundary conditions are nonlinear at the lowest order. This arises because relative to the boundary layer height, the unknown location of the interface is \( O(1) \). All the nonlinear terms involve boundary layer correction quantities and \( A^{(0)} \). To avoid this nonlinearity we are forced to neglect all these terms. In effect, we are considering the boundary to be flat and located at \( z = 0 \) as far as the boundary layer analysis is concerned. Leone, Segur and Hammack (1982) had to make the same assumption when they derived an equation for the damping of a single solitary wave in a two-layer system. They did not derive an evolution equation. Instead they constructed an adiabatic approximation by calculating the energy lost in boundary dissipation and equated it the loss of energy of the original wave.

The boundary conditions necessary for the solution then become
Kinematic (A2.7):

\[ 0(\alpha^0) - \frac{\partial A^{(0)}}{\partial s} = w^{(0)} + w^{(0)}_{c^+} \quad (A2.15a) \]

\[ 0(\alpha) - \frac{\partial A^{(1)}}{\partial s} + \frac{1}{c} u^{(0)}_{o^+} \frac{\partial A^{(0)}}{\partial s} = A^{(0)} \frac{\partial w^{(0)}}{\partial z} + w^{(1)}_{o^+} + w^{(1)}_{c^+} \quad (A2.15b) \]

Dynamic (A2.8):

\[ 0(\alpha^0) \quad \sigma_{A^{(0)}} + p^{(0)}_{o^+} + p^{(0)}_{c^+} = p^{(0)}_{o^-} + p^{(0)}_{c^-} \quad (A2.16a) \]

\[ 0(\alpha) \quad \alpha A^{(1)} + p^{(1)}_{o^+} + p^{(1)}_{c^+} + A^{(0)} \frac{\partial p^{(0)}_{o^+}}{\partial z} = \]

\[ p^{(1)}_{o^-} - p^{(1)}_{c^-} + A^{(0)} \frac{\partial p^{(0)}_{o^-}}{\partial z} \quad (A2.16b) \]

Shear Stress (A2.9):

\[ 0(\alpha^0) \quad \left( \frac{\mu_+}{\mu_-} \right) \frac{\partial u^{(0)}_{c^+}}{\partial z} = \frac{\partial u^{(0)}_{c^-}}{\partial z} \quad (A2.17) \]

Velocity (A2.10):

\[ 0(\alpha^0) \quad u^{(0)}_{o^+} + u^{(0)}_{c^+} = u^{(0)}_{o^-} + u^{(0)}_{c^-} \quad (A2.18) \]

where all quantities are evaluated at \( z = \hat{z} = 0 \).
The lowest order problem is solved as follows. Just as in the boundary layer analysis of §2.2.1 the $O(\alpha^0)$ continuity and vertical momentum equations in the boundary layers give

$$w^{(0)}_{c^\pm} = p^{(0)}_{c^\pm} = 0$$  \hspace{1cm} (A2.19)

The inviscid problem in each layer is, from (A2.1) - (A2.3), governed by

$$\frac{1}{c} \frac{\partial u^{(0)}_{o^\pm}}{\partial s} + \frac{\partial w^{(0)}_{o^\pm}}{\partial z} = 0$$

$$- \frac{\partial u^{(0)}_{o^\pm}}{\partial s} = - \frac{1}{\rho_{o^\pm} c} \frac{\partial p^{(0)}_{o^\pm}}{\partial s}$$

and

$$0 = \frac{\partial p^{(0)}_{o^\pm}}{\partial z},$$  \hspace{1cm} (A2.20a,b,c)

subject to the boundary conditions (A2.4), (A2.15a) and (A.216a).

Using (A2.19) the solutions for the complete $O(\alpha^0)$ problem in each layer are

$$u^{(0)}_{o^\pm} = \pm \frac{c}{h_{o^\pm}} A^{(0)}_{o^\pm}$$

$$p^{(0)}_{o^\pm} = \pm \rho_{o^\pm} c^2 \frac{h_{o^\pm}}{h_{o^\pm}} A^{(0)}_{o^\pm}$$

and

$$w^{(0)}_{o^\pm} = \pm \frac{c}{h_{o^\pm}} A^{(0)}_s (z + h_{o^\pm}).$$  \hspace{1cm} (A2.21a,b,c)
Here
\[ c^2 = \frac{\sigma h_+ h_-}{(\rho_+ h_+ + \rho_- h_-)} \] (A2.22)

or dimensionally
\[ c^2 = \sigma g \frac{d_+ d_-}{d_+ + d_-} \]

after employing the Boussinesq approximation.

The corrections to the horizontal velocity at \( O(a^0) \)
within each boundary layer are governed by the x-momentum
equation which can be reduced to
\[ \mathbb{R}^* \frac{\partial u_{c^\pm}(0)}{\partial \xi} + \frac{\partial^2 u_{c^\pm}(0)}{\partial z^2} = 0 . \] (A2.23)

This is identical to the equation (2.35) derived for the bottom
boundary layers. The boundary conditions, however, are different
and given by (A2.6), (A2.17) and (A2.18). The solution
procedure is identical to that used in §2.2.1 and results in
\[ u_{c^\pm}(0) = \pm \frac{c}{2\pi} B^* \int_{-\infty}^{\infty} \hat{A}^{(0)}(0) e^{+i\sigma \pm \hat{z}} e^{ik\xi} \, dk \] (A2.24)

where
\[ \sigma_{\pm} = (-ik \mathbb{R}^*)^{1/2} \] (A2.25)
\[ B_+ = \left[ 1 + \left( \frac{\mu_+ \rho_+}{\mu_- \rho_-} \right)^{1/2} \right] \left( \frac{h_+ + h_-}{h_+ h_-} \right) \]

\[ B_- = \left( \frac{\mu_+ \rho_+}{\mu_- \rho_-} \right)^{1/2} \left[ 1 + \left( \frac{\mu_+ \rho_+}{\mu_- \rho_-} \right)^{1/2} \right] \left( \frac{h_+ + h_-}{h_+ h_-} \right) \]  

and \( \hat{A}(0) \) is the Fourier transform (2.37) of \( A(0) \). In the Boussinesq limit and assuming \( \mu_+ = \mu_- = \mu \) we find

\[ B_+ = B_- = \frac{1}{2} \left( \frac{h_+ + h_-}{h_+ h_-} \right). \]  

(A2.27)

Following the procedure used in the bottom boundary layer analysis we find from the \( 0(\alpha) \) boundary layer continuity equation,

\[ w_{c_\pm}^{(1)} \bigg|_{k=0} = \frac{c}{2\pi (2Re_\tau)^{1/2}} \left( \frac{\chi}{\alpha} \right) \int_{-\infty}^{\infty} \hat{A}(0)(-1+i \text{sgn } k)|k|^{1/2} e^{ik\frac{\alpha}{2}} \text{d}k. \]  

(A2.28)

From the vertical momentum equation in the boundary layer at \( 0(\alpha) \) we also have

\[ \frac{\partial p_{c_\pm}^{(1)}}{\partial z} + \frac{\alpha}{\gamma} = 0 \]

\[ p_{c_\pm}^{(1)} = 0. \]  

(A2.29)

With (A2.28) and (A2.29) all the information necessary to complete the problem is available. The algebra is lengthy but the procedure is straightforward and is just sketched. From
(A2.1) - (A2.3) and (A2.21) relations for the outer quantities

\[ p_{o \pm}^{(1)} \text{ and } w_{o \pm}^{(1)} \text{ are given by} \]

\[ p_{o \pm}^{(1)} = \pm \frac{c \rho_{\pm}}{h_{\pm}} A_{ss}^{(0)} \left( \frac{z^2}{2} + h_{\pm} z \right) + E_{\pm} \]  \hspace{1cm} (A2.30)

and

\[ w_{o \pm}^{(1)} = \pm \frac{2c}{h_{\pm}} A_{X}^{(0)} (z + h_{\pm}) - \frac{c}{h_{\pm}^2} A_{ss}^{(0)} A_{s}^{(0)} (z + h_{\pm}) \]
\[ - \frac{3}{6} \frac{h_{\pm}^2}{2} + \frac{3}{3} + \frac{h_{\pm}^3}{3} - \frac{1}{c^2 \rho_{\pm}} E_{s} (z + h_{\pm}) \]  \hspace{1cm} (A2.31)

where (A2.31) satisfies the boundary conditions (A2.4) and
\[ E_{\pm} = E_{\pm} (X, s) \text{ is an unknown function.} \]

Substitution of (A2.30) and (A2.31) evaluated at
\[ z = 0, (A2.21), (A2.28) \text{ and } (A2.29) \]
to the kinematic conditions (A2.15b) and the dynamic condition (A2.16b) gives, after elimination of \( E_{\pm} \), the evolution equation for \( A_{X}^{(0)} \) (in dimensional variables)

\[ A_{X}^{(0)} + \frac{3}{2c} \left( \frac{d_{+} + d_{-}}{d_{+} - d_{-}} \right) A_{X}^{(0)} A_{s}^{(0)} + \frac{d_{+} - d_{-}}{6c^2} A_{ss}^{(0)} = \]

\[ \frac{1}{4c} \left( \frac{\nu}{\mu} \right)^{1/2} \left\{ \frac{1}{2} \frac{d_{+} + d_{-}}{d_{+} - d_{-}} \right\} \int_{-\infty}^{\infty} \frac{\partial A_{s}^{(0)}}{\partial s'} \frac{1 - \text{sgn}(s-s')}{|s-s'|^{1/2}} \, ds' \]  \hspace{1cm} (A2.32)
The convolution theorem (see (2.46)) has been used to rewrite the integral operator.

The interfacial dissipation is governed by the same integral operator as the bottom boundary condition. The only difference is a change in the coefficient reflecting the different boundary for the interfacial shear problem. If an adiabatic approximation for the decay of a single solitary wave is constructed (§2.3) using (A2.32) the result of Leone, Segur and Hammack (1982) is recovered.
implicit double precision(a-h,o-z)
common/data/uk(512),a(512),b(512),&s(512),dt,alength,time0
common/coefs/ulp,u2p,u3p,u1,u2,u3,ff,ca,ilast
common/dm/reset,beta,alamda,depth,d1,mzeta
common/den/dh,deltroho,y0,ho(30),dens(30),nh
common/ai/a10(41),a11(41),a12(41),a13(41),c(41),phi(41),
&phi(41),tau(41)

*********************************************

Solves the extended KdV equation.

This version includes the effects of
continuous stratification and dissipation. Wall,
bottom and interfacial (two-layer model)
boundary layers are considered.

The linear eigenvalue problem for the vertical structure
and phase speed is solved by the subroutine vstr. Within
this subroutine another subroutine is called to evaluate
the integrals I0,11,12 and 13. The value of these integrals
and the linear phase speed are saved in arrays and printed
out at the end of the run. The vertical structure functions
are not saved.

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*********************************************
common/adjus/idadjus
common/fft/n,.mixup(512)
common/pf=wf(256),wb(256)
common/write/idata
dimension f(512)
complex u(512),uk,a,b,um(512),t,c1,f(512),wf,wb,dis
complex c11,c22
sinh(y)=(dexp(y)-dexp(-y))/2.0d0
cosh(y)=(dexp(y)+dexp(-y))/2.0d0

read initial data

write(6,)' data input file'
read(5,)' idf
read(idf,)alength,n,dt,time0,mmax,iprint,idata,g
read(idf,)a1,x1,depth,z0,d1,s1length
read(idf,),a1u,delrho,x2,x2p2
read(idf,),mzeta,ca
read(idf,)icont.input file,elg
c density profile data (measured format)
read(idf,),nh,dh,h(1)
read(idf,5)(dens(j),j=1,1)
5 format(10f7.4)

c normalize (by total depth) the depths for measured density
c
dh=dh/depth
h(1)*h(1)/depth
do 6 j=2,nh
6 h(j)=h(j-1)-dh
ab1=0.0
set up the problem

call setup(a1, el, slength, delta, anu, amu1)
uprat/depth/beta*3.0d0*dsqrt(a10(1)/c(1))
up2=6.0d0*(a1/depth)**2/beta*c(1)*a10(1)
up3=delta/depth/beta/dsqr(2.0d0/c(1))**2
write(idata,800)alength,n,dt,time0,mmax,iprint
write(idata,801)a1,x1,depth,z0,d1,slength,ca
write(idata,802)am1,el,alalpha,alangle,delrho,gamma,anu
800 format(9x,'alength=',f6.1,2x,'n=',i2,2x,'dt=',e12.5,2x,
&'t0=',f9.6,2x,'mmax=',15.2x,'iprint=',i5)
801 format(9x,'a1',f10.5,2x,'x1=',f7.2,2x,'depth=',f6.2,2x
&'z0=',f8.5,2x,'d1=',f8.5,2x,'slength=',f8.1,2x,'ca=',f5.1)
802 format(9x,'amu=',f7.4,2x,'el=',f8.3,2x,'alang=',e11.4,2x,
&'delta=',f7.4,2x,'delrho=',f10.6,2x,'gamma=',f6.3,2x,
&'anu=',f6.4)

! 2**n
last=1
zero=0.0d0
one=1.0d0
ct=compl(x(zero,one)
dx=alength/float(nb)
m=0
ladjus=0
ljus=40

initial conditions

either read in or computed solitary wave

if(icont.eq.2) go to 11

10 continue
go to 15
11 read(infile,12)(f1(j),j=1,nb)
12 format(1x,7f11.8)
do 13 j=1,nb
fhold=f1(j)
13 f1(j)=compl(x(fhold,zero)
15 continue

get started

! call prefft
call fft(f,uk,wf,2)
call print(nb,f,time0,time0,m)
call trans

first time step -- euler step

call coeff(time0,xx)
c1=ct+dt
do 20 j=1,nb
t=f(j)
  u(j)=t-c11*a(j)*u1*t-u2*t+t+ca)+c1*b(j)-dt*u3*dis(j)
um1(j)=t
20 continue
  m=m+1
  if((m/i)rint)*i)rint.ne.m) go to 21
  time=dt+time0
  call coeff(time,xx)
  call print(nb,u,time,xx,m)
21 continue
  c22=2,d+00+c1
  c11=c22/dt
  i125=0
  i126=0
  c330=2.0d0*dt
  leap frog integration now
  do 40 m=2,mmax
    if((m/i)juis)*i)just.eq.m) call adjust(u,um1,m)
    time=(float(m-1)+0.5*float(iclass))*dt+time0
    call fft(u,uk,wf,2)
    call trans
    call coeff(time,xx)
    c33=c330+d3
    do 50 j=1,nb
      t=uj(j)
      u(j)=um1(j)-c11*a(j)*(u1*t-u2*t+t+ca)+c22*b(j)-c33*dis(j)
      um1(j)=t
    50 continue
  check to see if time to print
    if(i125.ne.0) go to 31
    if(xx.lt.1.0) go to 32
    time=time+dt
    call coeff(time,xx)
    call print(nb,u,time,xx,m)
    i125=1
 31 continue
    if(i126.ne.0) go to 33
    if(xx.lt.xp) go to 32
    time=time+dt
    call coeff(time,xx)
    call print(nb,u,time,xx,m)
    i126=1
 33 continue
    if(xx.lt.xp2) go to 32
    time=time+dt
    call coeff(time,xx)
    call print(nb,u,time,xx,m)
    go to 100
32 continue
    if((m/i)rint)*i)rint.ne.m) go to 40
    time=time+dt
    call coeff(time,xx)
    call print(nb,u,time,xx,m)
40 continue
  print integrals and phase speeds
100 call intpr
999 stop 
end 
subroutine trans 
   this subroutine performs the fft's necessary 
to compute the derivatives: 
       du/dx = INV F(ik F(u)) 
       (d/dx)**3 * u = INV F(-ik**3 F(u)) 
and the dissipation integral: 
       I = INV F((1-sgn(k)) |k|**0.5 F(u)) 

implicit double precision(a-h,o-z) 
real*8 twopi 
common/fft/n,mixup(512) 
common/pfft/wf(256),wb(256) 
common/data/uk(512),a(512),b(512),dis(512),dt,alength,tima0 
common/coefs/uip,u2p,u3p,ui,u2,u3,ff,ca,ilast 
complex uk,a,b,c(512),d(512),wf,wb,dis,e(512),q 
twopi=0.6283185307d+01 
dk=twopi/alength 
nb=2+n 
nb2=nb/2 
nb2p1=nb2+1 
zero=0.0d0 
aa=ca*dt 
q=cmp1x(1.0,-1.0) 
do 10 j=1,nb2 
     dk=dk*float(j-1) 
     c(j)=dk*uk(j) 
     d(j)=dsin(dk**3*dt+aa*dk)*uk(j) 
     e(j)=q*dsqrt(dk)*uk(j) 
10 continue 
q=cmp1x(1.0,1.0) 
do 15 j=nb2p1,nb 
     dk=dk*float(j-1-nb) 
     c(j)=dk*uk(j) 
     d(j)=dsin(dk**3*dt+aa*dk)*uk(j) 
     e(j)=q*dsqrt(-dk)*uk(j) 
15 continue 
c(nb2p1)=cmp1x(zero,zero) 
d(nb2p1)=c(nb2p1) 
ss=dk*float(nb2) 
*dsqrt(ss) 
e(nb2p1)=e*uk(nb2p1) 
call fft(c,a,wb,1) 
call fft(d,b,wb,1) 
call fft(e,dis,wb,1) 
return 
end 
subroutine fft(a,b,expt,inv) 

this subroutine performs the fft's according 
to the method of cooley, lewis and welch (1970)
implicit double precision(a-h,o-z)
c
common/fft/n,mixup(512)
c
complex b0,b1,a(1),b(1),expt(1)
c
nb=2**n
c
k1=1
c
k0=2
c
do 1 1=1,nb
c
lb=mixup(1)
c
b(l)=a(lb)
c
1 continue
c
do 6 merge=1,n
c
do 3 kp=1,k1
c
kheta=(kp-1)*nb/k0+1
c
do 4 j=kp,nb,k0
c
jp=j+k1
c
b0=b(j)
c
b1=b(jp)*expt(kheta)
c
b(j)=b0+b1
c
b(jp)=b0-b1

c
c
4 continue
c
3 continue
c
k1=k0
c
6 k0=2*k1
c
if(inv.ne.1) go to 8
c
anb=1.0d0/float(nb)
c
do 7 1=1,nb
c
7 b(1)=b(1)*anb
c
8 continue
c
return
c
end
c
subroutine prefft
c
sets up the fft calculations. called once prior to
c
the first fft subroutine call.
c
implicit double precision(a-h,o-z)
c
common/pfft/wf(256),wb(256)
c
common/fft/n,mixup(512)
c
complex wf,wb
c
pi=3.141592653589793d+00
c
k=2**n
c
k2=k/2
c
do 10 1=1,k
c
lb=1-1
c
11=0
c
do 11 j=1,n
c
jb=lb/2
c
kit=lb-2*jb
c
lb=jb
c
11=2*11+kit
11 continue
c
mixup(1)=11+1
c
10 continue
c
do 12 m=1,k2
c
c=dcos(pi*(m-1)/k2)
c
s=dsin(pi*(m-1)/k2)
c
wf(m)=cmplx(c,-s)
c
wb(m)=cmplx(c,s)
12 continue
return
end

subroutine print(nb,u,time,xx,m)

prints solution at present step

implicit double precision(a-h,o-z)
common/coeffs/ulp,u2p,u3p,u1,u2,u3,ff,ca,ilast
common/data/uk(512),a(512),b(512),dis(512),dt,alength,time0
common/pfft/wf(256),wb(256)
common/fft/n,mixup(512)
common/write/idata
complex u(1),uk,a,b,dis

c compute invariant M and the energy E along with
with the rhs of dE/Dt=dissip.

a1v1=0.0
a1v2=0.0
do 5 j=1,nb
 t=real(u(j))
a1v1=a1v1+t
a1v2=a1v2+t
5 continue

call fft(u,uk,wf,2)
dissip=0.0d0
dk=6.283185307d0/alength
do 10 j=1,(nb/2+1)
10 dissip=dissip+dsqrt(dk*(j-1))*(abs(uk(j)))**2

do 11 j=(nb/2+2),nb
11 dissip=dissip+dsqrt(-dk*(j-1-nb))*(abs(uk(j)))**2
a1v1=a1v1+alength/nb
a1v2=a1v2+alength/nb
dissip=dissip+dk
write(idata,6)time,xx,a1v1,a1v2,dissip
6 format(//,'time=',f10.6,','zeta=',f10.6,','inv 1 =',f10.6,','inv 2 =',&f10.6,',' dissip=',f10.6)
write(idata,9)u1,u2,u3,ff
9 format(1x,4(d12.6,5x))
write(idata,7)(real(u(j)),j=1,nb)
7 format((1x,7f11.8))
return
end

sub-routine adjust(u,um1,m)
implicit double precision(a-h,o-z)
common/coeffs/ulp,u2p,u3p,u1,u2,u3,ff,ca,ilast
common/a1,a0(41),a1(41),a2(41),a13(41),c(41),phi(41),&phi(41),tau(41)
common/alpha/beta,alambda,depth,d1,mzeta
common/data/uk(512),a(512),b(512),dis(512),dt,alength,time0

c this subroutine performs the time step
averaging suggested by Fornberg and
Whitham

common/fft/n,mixup(512)
common/pfft/wf(256),wb(256)

242
common/adjus/iadjus
complex u(1), um(1), uk, a, b, t, cl, wf, wb, dis
complex c1, c2
nb=2**n
zero=0.0d0
one=1.0d0
c1=cmpix(zero, one)
call fft(u, uk, wf, 2)
call trans
time=(float(m-1)+0.5*float(ladjus))*dt+time0
call coeff(time, xx)
c1=2.0d0*c1*dt
c2=2.0d0*c1
c3=2.0d0*dt*us
do 50 j=1, nb
   t=um(1)*c1+a(j)*(u1*u(j)-u2*u(j)**2+ca)+c2*b(j)-c3*dis(j)
   um(1)=u(j)+um(1)*0.5
   u(j)=(u(j)+t)*0.5
50 continue
ialdus=ialdus+1

      cut out highest frequency component of solution
      call fft(u, uk, wf, 2)
call fft(um, a, wf, 2)
nbd2p1=nb/2+1
uk(nbd2p1)=cmpix(0.0, 0.0, 0)
a(nbd2p1)=cmpix(0.0, 0.0, 0)
call fft(uk, u, wb, 1)
call fft(a, um, wb, 1)
return
end

subroutine setup(a1, el, sl, length, delta, anu, amu)
      find parameters λ, μ etc. and calculates τ vs. x/L.
      called once at beginning of program
      implicit double precision(a-h, o-z)
      common/ainte/beta, alamda, depth, dl, mzeta
      common/coefs/ulp, u2p, u3p, u1, u2, u3, ff, ca, ilast
      common/den, dethro, zo, h(30), dens(30), nh
      common/a1, a1O(41), a11(41), a12(41), a13(41), c(41), phipp(41), &
      phippm(41), tau(41)
p1=3.141592654d0
dzeta=1.0d0/float(mzeta)
mzetap1=mzeta+1
tau(1)=0.0d0
b=1.0d0-z0-dl
do 100 j=1, mzetap1
   x=float(j-1)*dzeta
   slope function
   d=1.0d0-0.5d0*b*(1.0d0-dcos(p1*x))
   d=1.0d0-b*x
call vstr(d, c(j), a1O(j), a11(j), a12(j), a13(j), phipp(j), &
   phippm(j), ifai1)
if(ifai1.ne.0) go to 998
100 continue
beta=a1/depth, 4.0d0/a12(1)+(a11(1)-a1/depth)*a13(1)
amu1=(a1/depth)**2/4.0d0/beta*a13(1)/a12(1)
el=depth/dsqrt(beta)
alamda=e1/slength
delta=dqrta(’anu*e1)/(980.0d0*depth)**0.25/dsqrta(c(1))
a=beta/alamda/2.0d0
do 200 j=2,mzetapi
200 tau(j)=tau(j-1)+0.5d0*dzeta*a+(c(1)**3)*
&((a12(j)/a10(j)/c(j)**3)+a12(j-1)/a10(j-1)/c(j-1)**3)
write(6, 998) ' done with setup'
return
998 write(6, 998) ' failure to converge in vstr'
end

subroutine coeff(time,xx)
c calculates coefficients U1 and U2 at location xx=x/L.
c implicit double precision(a-h,o-z)
common/coefs/ulp,u2p,u3p,u1,u2,u3,ff,ca,llast
common/aln,am,alamda,depth,d1,mzeta
common/a1i,a10(41),a1i(41),a12(41),a13(41),c(41),phipp(41),
&phippm(41),tau(41)
common/den/dh,delrho,z0,h(30),dens(30),nh
pi=3.141592654d0
w=38.0/depth
mzetapi=mzeta+1
a=beta/alamda/2.0d0
dzeta=1.0d0/float(mzeta)
if(time.ge.tau(mzetapi))go to 7
do 20 j=llast,mzeta
  ilastp=j
  if(time.ge.tau(j).and.time.lt.tau(j+1))go to 6
20 continue
6 ilast=ilastp
factor=(time-tau(llast))/(tau(llast)-tau(llast))
xx=+/float(llast)*dzeta+factor*+dzeta

c slope function
d=1.0d0-(1.0d0-z0-d1)*xx
c d=1.0d0-0.5d0*(1.0d0-z0-d1)*(1.0d0-dcos(pi*xx))
llastp=ilastp+1
al1i=a10(llast)+factor*(a10(llast1)-a1i(llast1))
a1i=a1i(llast)+factor*(a1i(llast1)-a1i(llast1))
a2i=a2i(llast)+factor*(a2i(llast1)-a2i(llast1))
a3i=a3i(llast)+factor*(a3i(llast1)-a3i(llast1))
c0=c(llast)+factor*(c(llast1)-c(llast))
pp=phipp(llast)+factor*(phipp(llast1)-phipp(llast))
ppm=phippm(llast)+factor*(phippmm(llast1)-phippm(llast))
u1=ulp+dsqrta(c0/a0)*a1/a2
u2=u2p+a3/a2/a0/c0
u3=u3p+c0**2/a2*+((1.0d0+2.0d0*(d-z0)/w)*pm**2+2.0d0*z0/w*pp**2
&+d**2/2.0d0*z0**2/(d-z0)**2)

c bottom boundary only
c u3=u3p+c0**2/a2*pm**2
ff=d
return
7 xx=1.0d0+(time-tau(mzetapi))+(c(mzetapi1)/c(1))**3*a/
&al10(mzetapi)/a12(mzetapi)
u1=ulp+dsqrta(c(mzetapi1)/a10(mzetapi1)+a1i(mzetapi1)/a12(mzetapi1)
u2=u2p+a3(mzetapi1)/a2(mzetapi1)/a12(mzetapi1)/a10(mzetapi1))/c(mzetapi1)
u3=u3p+c(mzetapi1)**2/a12(mzetapi1)*((1.0d0+2.0d0*d1/w)*
&phipp(mzetapi)**2+2.0d0*z0/w*phipp(mzetapi)**2
&+(z0+d1)**2/2.0d0*z0**2/d1**2)
bottom boundary only

ff = z0 + d1
return
end
subroutine vstr(d, ci, ai0, ai1, ai2, ai3, phi0, phi1, ifail)

computes vertical structure function and phase speed by
shooting method using a 4th order Runge-Kutta integrating
subroutine runge

implicit double precision(a-h,o-z)
dimension phi(201), phi(201)
common/den/dh, delrho, z0, h(30), dens(30), nh
nz = 200
epsilon = 1.0d-6
ifail = 0

initial guesses for phase speed from two-layer model

c0 = sqrt(delrho*z0*(d-z0)/d)
c1 = 0.80*c0
call runge(nz, a, c0, phi, phi)

10 fc0 = phi(nz+1)
call runge(nz, d, c1, phi, phi)
fc1 = phi(nz+1)
c1 = c1 - fc1*(c1 - c0)/(fc1 - fc0)
c0 = c1
c1 = c2
fc0 = fc1
if(dabs(c1 - c0).le.epsilon) go to 200
do 100 i = 1, 50
   call runge(nz, d, c1, phi, phi)
   fc1 = phi(nz+1)
c2 = c1 - fc1*(c1 - c0)/(fc1 - fc0)
c0 = c1
c1 = c2
   fc0 = fc1
   if(dabs(c1 - c0).le.epsilon) go to 200
100 continue
   ifail = 1
   go to 999
200 amax = 0.0
   do 205 j = 1, nz + 1
      if(phi(j).gt.amax) amax = phi(j)
      do 210 j = 1, nz + 1
         phi(j) = phi(j)/amax
      210 phi(j) = phi(j)/amax
      phi0 = phi0 + phi(nz + 1)
      phi1 = phi1 + phi(1)
call integrals(nz, d, phi, phi, ai0, ai1, ai2, ai3)
999 return
end
subroutine runge(nz, d, c, phi, phi)

4th order runge-kutta routine for solving the eigenvalue
problem

implicit double precision(a-h,o-z)
dimension phi(1), phi(1)
common/den/dh,delrho,z0,h(30),dens(30),nh
csq=c+c
dz=d/float(nz)
y2=1.0d0
y1=0.0d0
phi(1)=0.0d0
phi(p)=1.0d0
do 100 j=2,nz+1
  z=float(j-1)*dz-d
  ak1=dz*y2
  call density(z,rho,rhoz)
  a11=dz*(-rhoz*y2+rhoz/csq*y1)
  ak2=dz*(y2+0.5d0*a11)
  call density(z+0.5d0*dz,rho,rhoz)
  a12=dz*(-rhoz*(y2+0.5d0*a11)+rhoz/csq*(y1+0.5d0*ak1))
  ak3=dz*(y2+0.5d0*a12)
  a13=dz*(-rhoz*(y2+0.5d0*a12)+rhoz/csq*(y1+0.5d0*ak2))
  ak4=dz*(y2+a13)
  call density(z+dz,rho,rhoz)
  a14=dz*(-rhoz*(y2+a13)+rhoz/csq*(y1+a3))
  y1=y1+(ak1+2.0d0*(ak2+ak3)+ak4)/6.0d0
  y2=y2+(a11+2.0d0*(a12+a13)+a14)/6.0d0
  phi(j)=y1
  phi(p)=y2
  write(50,101)z,rhoz,csq,y1,y2
101  format(1x,5(d15.8,2x))
continue
return
end

subroutine density(z,rho,rhoz)
computes density gradient from measured density profile
implicit double precision(a-h,o-z)
common/den/dh,delrho,z0,h(30),dens(30),nh
if(z.le.h(nh))go to 20
if(z.ge.h(1))go to 30
n=(h(1)-z)/dh+1
fact=(h(n)-z)/dh
rho=dens(n)+fact*(dens(n+1)-dens(n))
rho=dens(n+1)+fact*(dens(n+2)-dens(n+1))
rho=dens(n)-fact*(dens(n)-dens(n-1))
if(n.eq.1)r=1.0d0
if(n.eq.nh-1)r2=1.0d0+delrho
rhoz=(r-r2)/2.0d0/dh:-ho
return
20  rho=dens(nh)
rho=0.0d0
return
30  rho=dens(1)
rho=0.0d0
return
end
subroutine integrals(nz,d,phi,p,ai0,ai1,ai2,ai3)
computes integrals 10,11,12,13 by Simpson's rule
implicit double precision(a-h,o-z)
dimension phi(1),phi(p)
common/den/dh,delrho,z0,h(30),dens(30),nh
nzp1=nz+1
dz=d/float(nz)
a0=phi(1)**2+phi(nzp1)**2
a1=phi(1)**3+phi(nzp1)**3
a2=phi(1)**2+phi(nzp1)**2
a3=phi(1)**4+phi(nzp1)**4
nzm2=nz-2
do 100 j=2,nzm2,2
   a0=a0+4.0d0*phi(j)**2+2.0d0*phi(j+1)**2
   a1=a1+4.0d0*phi(j)**3+2.0d0*phi(j+1)**3
   a2=a2+4.0d0*phi(j)**2+2.0d0*phi(j+1)**2
   a3=a3+4.0d0*phi(j)**4+2.0d0*phi(j+1)**4
100 continue
a0=(a0+4.0d0*phi(nz)**2)*dz/3.0d0
a1=(a1+4.0d0*phi(nz)**3)*dz/3.0d0
a2=(a2+4.0d0*phi(nz)**2)*dz/3.0d0
a3=(a3+4.0d0*phi(nz)**4)*dz/3.0d0
return
end

print integrals, etc. at locations on the slope

implicit double precision(a-h,o-z)
common/a1,a10(41),a11(41),a12(41),a13(41),c(41),phipp(41),
&phipm(41),tau(41)
common/write,idata
common/aintu/beta,alambda,depth,d1,mzeta
mzetap1=mzeta+1
write(idata,800)
800 format(ix,/' phase speed/sqrt(g*h0)'
   write(idata,801)(c(i),i=1,mzetap1)
801 format(ix,8(f11.6,1x))
write(idata,802)
802 format(ix,/' 10 * h0/rho0'
   write(idata,801)(a10(i),i=1,mzetap1)
   write(idata,803)
803 format(ix,/' 11 * h0**2/rho0'
   write(idata,801)(a11(i),i=1,mzetap1)
   write(idata,804)
804 format(ix,/' 12/h0/rho0'
   write(idata,801)(a12(i),i=1,mzetap1)
   write(idata,808)
808 format(ix,/' 13 * h0**3/rho0'
   write(idata,801)(a13(i),i=1,mzetap1)
   write(idata,805)
805 format(ix,/' phipp'
   write(idata,801)(phipp(i),i=1,mzetap1)
   write(idata,806)
806 format(ix,/' phipp'
   write(idata,801)(phipp(i),i=1,mzetap1)
   write(idata,807)
807 format(ix,/' tau'
   write(idata,801)(tau(i),i=1,mzetap1)
return
end
APPENDIX 4

ON INTERFACIAL SOLITARY WAVES OVER SLOWLY VARYING TOPOGRAPHY
On interfacial solitary waves over slowly varying topography

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The propagation of long, weakly nonlinear interfacial waves in a two-layer fluid of slowly varying depth is studied. The governing equations are formulated to include cubic nonlinearity, which dominates quadratic nonlinearity in some parametric configuration of equal layer depths. Numerical solutions are obtained for an initial profile corresponding to either a single solitary wave or a rank-ordered pair of such waves incident in a monotonic transition between two regions of constant depth. The numerical solutions, supplemented by inverse-scattering theory, are used to investigate the change of polarity of the incident waves as they pass through a turning point of approximately equal layer depths. Our results exhibit significant differences from those reported by Knickerbocker & Newell (1980), which were based on a model equation. In particular, we find that more than one wave of reversed polarity may emerge.

1. Introduction

Recent field observations have provided evidence of packets of long, first-mode internal waves in marginal seas and coastal waters (Osborne & Burch 1980; Apel et al. 1975). Such waves may evolve from disturbances caused by tidal flow over topography (e.g., sills and shelf breaks) and propagate for large distances before encountering any further significant variation in bottom topography. Solitary waves may emerge if the propagation distance is large enough. The generation process has been studied in detail (Lee & Beardsley 1974; Maxworthy 1979), but little is known about the ultimate fate of the waves as they propagate towards shore or in regions wherein the depth of the upper layer is a significant fraction of the changing depth of the water column. We consider here one aspect of this problem, the scattering of a solitary wave by a gradually varying change in depth.

Internal solitary waves appear to have been studied originally by Keulegan (1953); (see also Long 1956; Kakutani & Yamasaki 1973), who considered a two-layer liquid with a small discontinuity in density and upper and lower depths of \( d_+ \) and \( d_- \) and found that the interfacial displacement is positive/negative for \( d_+ \geq d_- \). This led Kaup & Newell (1978) to suggest that an interfacial solitary wave in water of variable depth could reverse polarity on passing through a transition region in which \( d_+ - d_- \) changes sign. Djordjevic & Redekopp (1978) and Miles (1980) have argued that this reversal is impossible if \( a, d_+ < l, L \leq 1 \), where \( a \) and \( l \) are the amplitude and
characteristic length of the wave and \( L \) is the length of the transition, but their arguments are inapplicable if \( l/L \ll a/d_- \). On the other hand, Knickerbocker & Newell (1980) have shown that such a reversal is possible in the latter case for a model \( \text{KdV} \) equation in which the coefficient of the quadratic term varies linearly over \( L \); however, their conclusion needs testing for the internal-wave problem in consequence of their neglect of cubic nonlinearity, which dominates quadratic nonlinearity and limits the attainable amplitude of a solitary wave in some neighbourhood of \( d_- = d_- \) (Long 1956).

We consider here the formulation and solution of the equations that govern internal solitary waves in a two-layer fluid of gradually varying depth. Significant dimensionless parameters are

\[
\begin{align*}
\alpha &= \frac{a}{d_-}, & \beta &= \frac{d^2}{l^2}, & \delta &= \frac{\rho_0 - \rho_-}{\rho_0 + \rho_-}, & \lambda &= \frac{l}{L},
\end{align*}
\]

(1.1 a, b, c, d)

where \( \alpha \) is a characteristic amplitude (which may be either positive or negative), \( d \) is a characteristic depth, \( l \) is a characteristic length of the wave, \( \rho_0 \) is the density of the upper/lower layer and \( L \) is a characteristic length of the depth variation. The parameters \( \alpha, \beta \) and \( \lambda \) are small by hypothesis, and \( \alpha \), which is a measure of nonlinearity, serves as the basic perturbation parameter; the parameter \( \delta \) may be in \((0, 1)\) but is ultimately assumed to be small.

The parameter \( \beta \), which measures dispersion, is \( O(\alpha) \) for a Boussinesq solitary wave (which represents a balance between quadratic nonlinearity and dispersion); however, \( \beta = O(\alpha^2) \) if \( |d_- - d_-|/(d_+ + d_-) = O(\alpha) \), in which regime cubic nonlinearity is comparable to, or dominates, quadratic nonlinearity.

The parameter \( \lambda \) is assumed to be \( O(\beta) \) in the derivation of the generalized \( \text{KdV} \) equation in §2. If \( \lambda \gg \beta \) the effects of variable depth dominate those of nonlinearity and dispersion, and (see §2) a generalization of Green’s law holds over the transition region.

We obtain numerical solutions of the resulting evolution equation for a transition between two layers of constant depth (in §4). We also determine asymptotic solutions through the use of inverse-scattering theory (IST), which uses intermediate numerical solutions as initial data, in §5.

2. Evolution equation

Let \( y = 0 \) be the equilibrium interface, \( y = \pm d_\pm(z) \) the upper and lower boundaries, \( y = y_i(x, t) \) the interfacial displacement, and \( \phi(x, y, t) \) the velocity potential (irrotational motion being assumed except at the interface, across which \( \phi \) may be discontinuous). The governing equations, based on the assumptions of incompressible, inviscid flow and continuity of pressure across the interface, are then

\[
\begin{align*}
\nabla^2 \phi &= 0 \quad (y = y_i), \\
\phi_y &= \pm d_\pm \phi_x \quad (y = \pm d_\pm), \\
\phi_y &= y_i + \phi_x y_i \quad (y = y_i^\pm), \\
\rho_- g y_i - \phi_t + \frac{1}{2} \nabla \phi \cdot \nabla \phi &+ y_i = 0, \quad \rho_- g y_i - \phi_t + \frac{1}{2} \nabla \phi \cdot \nabla \phi &+ y_i = 0,
\end{align*}
\]

(2.1) \hspace{1cm} (2.2) \hspace{1cm} (2.3) \hspace{1cm} (2.4)

where the subscripts \( x, y, t \) signify partial differentiation, the prime signifies differentiation with respect to \( x \), and, here and subsequently, alternative signs and subscripts are vertically ordered.
The reduction of (2.1) to an evolution equation for $y_1$ may be effected by substituting the expansions

$$\phi \equiv \frac{1}{2} y(d_1 \zeta_x + d_2 \zeta_y + \frac{1}{2} d_3 \zeta_x^2 - \frac{1}{2} y^2 \zeta_x^2 + \ldots) \phi_1(x,t) \quad (y \gg y_1),$$

which satisfy (2.1) and (2.2) into (2.3) and (2.4), and then proceeding as in the derivation of the Korteweg-de Vries equation for surface waves (Whitham 1974, §13.11). but retaining both quadratic and cubic nonlinear terms, invoking $ld'_x/d_x = C(\lambda)$, and introducing the characteristic variable

$$s = \int \frac{dx}{c} - t.$$  

The end result is (after dropping the subscript i)

$$d_{-1} c^4 (c^4 y)_x + d_1 y_{x} + \frac{1}{2} c^2 (d_{-2} y - 2 d_{-1} y^2) y_x = 0.$$  

where

$$d_n = \frac{\rho_- d^n + \rho_- d^{n-1}}{\rho_- - \rho_-},$$

$$c = \frac{2 \rho_- \rho_- d_{n-1} + \rho_- d_{n-1}^{-1}}{\rho_- d_{n-1} + \rho_- d_{n-1}^{-1}}.$$  

The wave speed $c$ may be identified as that of infinitesimal, non-dispersive disturbances: cf. Lamb (1932), §231 (11) in the limit $k \rightarrow 0$.

The parameter

$$d_n = \frac{\rho_- - \rho_-}{\rho_- - \rho_-} \equiv \delta$$

has the admissible range $(0, 1)$, but typically is small: the boundary condition (2.2) is a valid approximation for a free surface if and only if $\delta \ll 1$.

Accordingly, we assume that $\delta \ll 1$ throughout the subsequent development. The corresponding approximations to $d_n^2$ and $c$ are

$$d_n \approx \frac{1}{2} d_{n} + \frac{1}{2} \frac{2 \rho_- \rho_- d_{n-1}}{d_n + \rho_-} (\delta \ll 0).$$

It is convenient to introduce the dimensionless variables

$$\sigma = \frac{\delta}{\rho_-}, \quad \tau = \frac{1}{\rho_-} \int_{-\infty}^{x} d_0 d_x c \, dx, \quad \zeta = \frac{c}{c_0}, \quad \xi = \frac{c}{c_0},$$

where $c_0$ is a reference value of $c$, and transform (2.7) to

$$\zeta_\sigma + \zeta_{\sigma \sigma} + 12 (L_1 \zeta - 2 L_2 \zeta^2) \zeta_\sigma = 0.$$  

where

$$\mathcal{U}_1 = \frac{1}{3} a^2 (d_0 d_x^2 d_x - d_0 d_x d_x^2) d_0,$$

$$\mathcal{U}_2 = \frac{1}{3} a^2 (d_0 d_x^2 d_x - d_0 d_x d_x^2) d_0.$$  

are measures of quadratic and cubic nonlinearity respectively, relative to dispersion ($\mathcal{U}_1$ reduces to Ursell’s parameter $3 a^2 d_x^2$ in the limit $\rho_- / \rho_- = 0$ with $d_x$ fixed). If $\lambda \gg \sigma$, both the dispersion $\zeta_{\sigma \sigma}$ and nonlinear terms $\zeta_\sigma$ and $\zeta_\sigma^2$ may be neglected in (2.13), which then yields $\zeta = \text{constant}$ or, equivalently,

$$y \approx c^{-1}.$$  

which is equivalent to Green’s law.

* The omitted terms within the brackets are $(\nu d_x^2 / \rho d_x^2 / \rho d_x^2 / \rho d_x^2 / \rho d_x^2),$
3. Solitary-wave solution

Let \( d \) be the total depth.

\[ d_\pm = d d' \quad (0 < d < 1). \tag{3.1 a, b} \]

and choose \( a \) (note that \( a < 0 \) if \( d > \frac{1}{3} \)) and \( l \) according to

\[ x = \frac{a}{d} = \frac{2\mu}{1 + \mu} \frac{|d' (1 - d') (1 - d)|}{d^2 + (1 - d')^2}, \quad \beta = \frac{|\xi|}{d^2} = \frac{(1 - \mu)^2 d' (1 - d') (1 - d')^2 + (1 - d')^4}{3\mu (1 - d')^2}. \tag{3.2 a, b} \]

where \( d' \) is constant and \( 0 < \mu < 1 \); then (2.14) reduce to (with \( \epsilon \equiv 1 \))

\[ \mathcal{U}_1 = 1 + \mu. \quad \mathcal{U}_2 = \mu. \tag{3.3 a, b} \]

and (2.13) admits the solution (Miles 1979)

\[ \eta = \zeta = (\cosh^2 \theta - \mu \sinh^2 \theta)^{-1}. \tag{3.4} \]

where

\[ \theta = \sigma - 4t = \left[ 1 - \frac{(1 + \mu) (1 - d')^2}{(1 + \mu) (1 + d')^2} \right] \frac{x - ct}{l}. \tag{3.5} \]

\[ c^2 = 2dg d' (1 - d'). \tag{3.6} \]

Note that \( 0 < \mu < 1 \) implies \( \mathcal{U}_1^2 > 4 \mathcal{U}_2. \)

4. Numerical solutions

Equation (2.13) was solved numerically using the pseudospectral scheme of Fornberg & Whitham (1978). The use of this method is in principle straightforward; however, considerable difficulty was experienced initially with numerical instabilities triggered in the neighbourhood of \( d_\pm = d_\mp \). The numerical method and its linear stability criterion are presented in the Appendix.

The majority of runs were made for initial data corresponding to a single solitary wave in deep water of depth \( d \) propagating towards a cosine-shaped transition to shallow water:

\[ d_\pm = d d' \quad (\xi \equiv x/L < 0). \tag{4.1 a} \]

\[ = dL d' + \frac{1}{2} (d - d') \cos \pi \xi - 1 \quad (0 < \xi < 1). \tag{4.1 b} \]

\[ = dL d' \quad (\xi > 0). \tag{4.1 c} \]

where \( d > d' \). A limited number of runs were conducted for the corresponding transition from shallow to deep water, for linearly varying topography, and for initial data corresponding to a pair of rank-ordered solitary waves.

Figure 1 illustrates the scattering of a single solitary wave moving from deep to shallow water when \( \lambda \) is comparable to \( x \). No solitary waves of reversed polarity emerge.

The example described in figure 2 differs from that of figure 1 only in that \( \lambda \ll x \).

The plots are presented in a frame moving with the local linear wave speed, and show a leading wave of reversed polarity emerging from the scattered packet on the shelf (\( \xi = 1.59 \)). The profile on the shelf comprises several waves of reversed polarity travelling faster than the linear wave speed. Subsequent IST solutions (§5), using the profile at \( \xi = 1.02 \) as initial data, show 12 solitary waves emerging on the shelf.

Solitary waves in nature usually occur in a rank-ordered sequence, whereas it is of interest to determine whether the evolution of waves of reversed polarity is specific to initial data corresponding to a single solitary wave. The significant dispersion
Figure 1. Evolution of a single wave of depression over a transition of decreasing depth for $(x_n, \lambda, a, d') = (-0.0333, 0.041, 0.6, 0.15)$. Profiles of $\eta = y/a$ are shown as functions of the dimensionless characteristic time $(c_a/L)^2 \int_0^\infty \frac{dz}{\sigma - t}$ at four locations: $\xi = 0.07, 1.02, 1.23$. No solitary waves of elevation emerge.

Figure 2. Evolution of a single wave of depression over a transition of decreasing depth for $(x_n, \eta, a, d') = (-0.0067, 0.0041, 0.6, 0.15)$. Profiles of $\eta$ are shown at $\xi = 0.04, 0.06, 0.08, 0.10$. Note the separation of the leading waves from the scattered packet. IST shows 12 waves of reversed polarity emerging in this case.

evident in figures 1 and 2 implies that two rank-ordered solitary waves separated by a distance $O(t)$ at the bottom of the slope may produce significant interaction in the transition. Figure 3 shows the profiles for such a numerical experiment. Waves of reversed polarity are seen to emerge. Comparison of the results of figure 3 with the linear superposition of the solutions for each wave taken separately as initial data.
shows that significant interaction remains in the region in which waves of reversed polarity are emerging.

Figure 4 describes a transition from shallow to deep water, with $\lambda \ll x$. There is no evidence of waves of reversed polarity.

For cases in which waves of reversed polarity emerge, our numerical solutions display qualitative differences from those of Knickerbocker & Newell. They attribute the emergence of the wave of reversed polarity to the generation of a shelf behind
the incident solitary wave as it propagates up the slope: however, our results show no significant shelf development with the slope scaling used here. Other numerical experiments (not reported here) show that a very gradual slope is required to obtain a significant shelf: we estimate \( \lambda = O(x^n) \), \( n = 3-4 \), for figure 2 of Knickerbocker \\& Newell. We conclude that the emergence of the waves of reversed polarity is not attributable solely to the development of a shelf.

5. Asymptotic solutions

The relatively long times required to compute a clear separation of the solitary waves from the scattered packet preclude an extensive study of the asymptotic state on the shelf through numerical integration. If, however, \( d_\pm \) and \( d_- \) are constant, (2.13) can be solved by inverse-scattering theory, and the asymptotic solution on the shelf then can be evaluated using the numerical solution at the top of the slope as initial data.

Following Miles (1981), we set \( \xi_\tau = 2 \) and \( \xi_\sigma = 1 \) in (2.13), which, by virtue of (2.14), is equivalent to choosing appropriate values of \( \alpha \) and \( \eta \). The resulting evolution equation

\[
\zeta_\tau + \zeta_\sigma + 24\zeta(1-\zeta)\zeta_\tau = 0
\]

has solitary-wave solutions

\[
\zeta = \frac{2\gamma(1+\gamma)^{-1}}{\cosh^2 \chi - \gamma \sinh^2 \chi} \quad (0 < \gamma < 1),
\] (5.2a)

where

\[
\chi = \kappa \sigma - 4\kappa^3 \tau + \nu, \quad \kappa = 2\gamma^{1/3}(1+\gamma)^{-1}.
\] (5.2b, c)

\( \nu \) is a phase constant, and \( \gamma \) is a family parameter.

Equation (5.1) is reduced to the KdV equation

\[
B_\tau + 12BB_\tau + B_\sigma = 0
\]

through the Miura transformation (Miles 1979).

\[
B = \zeta_\tau + 2\zeta(1-\zeta).
\] (5.3)

The asymptotic solution on the shelf is dominated by a discrete set of solitary waves

\[
B \sim \sum_{n=1}^{\infty} \kappa_n^4 \text{sech}^2 \chi_n \quad \tau \to \infty,
\] (5.4)

where the \( \kappa_n \) are the discrete eigenvalues of

\[
\dot{\psi}(\sigma) + i - \kappa^2 + 2B_n(\sigma) \dot{\psi}(\sigma) = 0 \quad (-\infty < \sigma < \infty),
\] (5.5)

subject to

\[
\psi_n \sim e^{i \kappa_n \sigma} \quad \sigma \to -\infty,
\] (5.6)

and

\[
B_n(\sigma) \equiv B(\sigma; \tau_n) = \zeta_\tau + 2\zeta(1-\zeta)|_{\tau = \tau_n}
\] (5.7)

provides the initial data at the top of the shelf (\( \tau = \tau_n \)).

This eigenvalue problem was solved by expressing (5.6) in centred finite-difference form and evaluating the eigenvalues and eigenvectors by standard methods (Wilkinson 1965; Dalquist \\& Björck 1974).

According to inverse-scattering theory, the eigenvalues of (5.6) and (5.7) should be independent of \( \tau \), which is a parameter in the scattering problem. As a check on the solution of the eigenvalue problem, we compared the magnitude and number of discrete eigenvalues computed from initial data taken at two positions on the shelf, see table 1 for some typical cases. For transmitted waves comparable in amplitude.
\[ z = 1.02 \quad z = 1.23 \]

<table>
<thead>
<tr>
<th>Case</th>
<th>( N )</th>
<th>(-a_0^2 \lambda_1 )</th>
<th>( N )</th>
<th>(-a_0^2 \lambda_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>1</td>
<td>0.100</td>
<td>2</td>
<td>0.133</td>
</tr>
<tr>
<td>(b)</td>
<td>2</td>
<td>0.216</td>
<td>2</td>
<td>0.201</td>
</tr>
<tr>
<td>(c)</td>
<td>3</td>
<td>0.369</td>
<td>4</td>
<td>0.354</td>
</tr>
<tr>
<td>(d)</td>
<td>4</td>
<td>0.012</td>
<td>1</td>
<td>0.012</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.225</td>
<td>2</td>
<td>0.217</td>
</tr>
</tbody>
</table>

Table 1. Number \( N \) of transmitted waves and their respective amplitudes relative to the incident-wave amplitude \( a_0^2 \lambda_1 \), computed by IST from initial data at \( z = 1.02 \) and \( 1.23 \), for the following parameters: \( z, \lambda, \lambda_1, \lambda_2 \): (a) \(-0.04, 0.00246, 0.55, 0.368\); (b) \(-0.033, 0.0082, 0.6, 0.2\); (c) \(-0.05, 0.0082, 0.6, 0.2\); (d) \(-0.0667, 0.0041, 0.6, 0.3\). Note the weak dependence of the results on \( z \).

<table>
<thead>
<tr>
<th>Case</th>
<th>IST</th>
<th>Numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0.941</td>
<td>0.911</td>
</tr>
<tr>
<td>(b)</td>
<td>0.725</td>
<td>0.685</td>
</tr>
<tr>
<td>(c)</td>
<td>0.369</td>
<td>0.371</td>
</tr>
</tbody>
</table>

Table 2. Asymptotic relative amplitude of first lead wave \( (-a_0^2 \lambda_1) \) from direct numerical solution and from IST using solutions at \( z = 1.02 \) as initial data. The three cases correspond to the following values of parameters \( z, \lambda, \lambda_1, \lambda_2 \): (a) \(-0.0667, 0.0041, 0.6, 0.15\); (b) \(-0.0833, 0.0041, 0.6, 0.2\); (c) \(-0.05, 0.0082, 0.6, 0.2\).

To the incident wave, differences of a few percent in \( x_0^2 \), and hence the wave amplitude, are observed, with the absolute error remaining of the same order as the wave amplitude decreases. Thus the computed number \( N \) of solitary waves may be in error when the amplitude of the smallest eigenvalue approaches that of the error. We were not able to isolate the source of this error, but suspect that it is due in part to the finite resolution of the numerical solution used for initial data.

As a further check on the solution procedures, the numerical solution was run for an extended period in a few cases to provide a direct comparison with the results of the inverse-scattering theory. An example of such a comparison of the largest waves evolving in three separate cases is shown in Table 2. The differences between the numerical solutions and the analytical values obtained from IST are less than 5%, for lead waves whose amplitudes are comparable to those of the incident waves. These differences are comparable with those cited above for the errors in the implementation of the IST.

Numerical solutions through the transition region, supplemented by IST, were used to explore a larger parameter space than was practicable with the direct numerical solutions alone. Figure 5 shows the number of transmitted solitary waves versus \( \lambda \), the transition-length parameter, for fixed values of the incident-wave amplitude, and displays an increase in the number of waves as \( \lambda \) decreases (the length of the transition increases). The results of figure 5 were used to determine the wave amplitude at transition between \( N = 1.0 \) and \( N = 1.2 \), as shown in figure 6. The transition from

* Some of the results in this section were obtained with a higher-order contribution to \( \lambda \), as described in section 2.13. With amplitudes of the transmitted waves differing by \( 1-2^\alpha \), this additional term made no significant difference to the asymptotic results.
Figure 5. Number $N$ of transmitted solitary waves versus $\lambda$ for fixed values of $d' = 0.6$ and $d_1 = 0.2$, and incident amplitudes $Z_0 = -0.0833 (\ldots), -0.0867 (\ldots), -0.05 (\ldots), -0.0333 (\ldots), -0.0167 (\ldots)$.

Figure 6. Incident-wave amplitudes $Z_0$ at transitions between $N = 0$, 1 and $N = 1$, 2 versus the slope-length parameter $\lambda$, obtained from the data in figure 5. $\exists$, $N = 0$; $\bullet$, 1; $\Xi$, 2.

$N = 0, 1$ occurs for $|Z_0|$ increasing through a value comparable with $\lambda$. Similar results in figures 7 and 8 show that for a smaller depth change a larger incident wave is required to initiate a wave of reversed polarity.

Figure 9 shows the amplitude of the first transmitted solitary wave versus the amplitude of the incident wave for $\lambda$ in the range $[0.0041, 0.041]$. Our results show that the amplitude of the first transmitted wave increases as $\lambda$ decreases. This contrasts with the results of Knickerbocker & Newell (1980), which show an
asymptotic regime, independent of the transition lengthscale, in which $z_T = \frac{1}{2}z_0$, where $z_0$ and $z_T$ are the dimensionless amplitudes of the incident and transmitted solitary waves respectively (note that Knickerbocker & Newell found only one transmitted wave in their numerical solutions).

A limited study of the sensitivity of the results to the topographic shape was conducted. The results for the same depth change are shown in Table 3 for a cosine transition of half-wavelength $L$, a linear transition of length $L$ and a linear transition of length $2L$. The last corresponds to a slope equal to the maximum slope of the cosine transition. The differences in amplitude of the lead wave are greater than the
Figure 9. Amplitude $\sigma_1$ of first transmitted solitary wave versus the amplitude $\sigma_0$ of the incident wave for $d = 0.6$, $d_0 = 0.2$. $\lambda = 0.00041$, $\lambda = 0.0082$, $\lambda = 0.0123$, $\lambda = 0.0225$. $\lambda = 0.041$.

<table>
<thead>
<tr>
<th>Shape</th>
<th>N</th>
<th>$-\sigma_1^2/\sigma_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Half-cosine length $L$</td>
<td>3</td>
<td>0.23, 0.09, 0.02</td>
</tr>
<tr>
<td>(b) Linear length $L$</td>
<td>2</td>
<td>0.39, 0.11</td>
</tr>
<tr>
<td>(c) Linear length $2L/\pi$</td>
<td>2</td>
<td>0.22, 0.01</td>
</tr>
</tbody>
</table>

Table 3. Dependence of transmitted-wave amplitude on transition geometry ($\sigma_0, \lambda, d, d_0$) $\approx (0.0087, 0.016, 0.6, 0.2)$ from IST using initial data at $\xi = 1.02$

errors discussed above and suggest that the details of the topography may be significant in the application of these results.

A number of runs were conducted without the cubic term. Over the range of parameters studied we found quantitative rather than qualitative differences from those solutions which included the effects of cubic nonlinearity. The differences were most pronounced in the neighbourhood of the slope and less so in the asymptotic results on the shelf. Laboratory experiments (which will be reported later) show that the inclusion of cubic nonlinearity provides a significantly improved prediction of the wave profiles over gradual slopes.

The authors wish to thank Dr Daniel Meiron for a helpful conversation on eigenvalue solvers. This work was supported by the National Science Foundation with an award of computer time at the National Center for Atmospheric Research and Grants OCE 77-24005 (UCSD) and OCE-81-17339 (UCSD), and by contracts (at both MIT and UCSD) with the Office of Naval Research.
Appendix

Equation (2.13) was solved using the explicit pseudospectral method of Fornberg & Whitham (1978). The method evaluates the \( \sigma \)-derivatives of \( \zeta(\sigma, \tau) \) in Fourier space, and steps forward in \( \tau \) using a leapfrog procedure. For the constant-coefficient KdV equation

\[ u_\sigma + uu_\sigma + u_{\sigma \sigma \sigma} = 0 \]  

\( \text{(A 1)} \)

this gives

\[ u_j^{n+1} - u_j^n - 2i \Delta \tau u_j^n \mathcal{F}^{-1} \{ k \mathcal{F} \{ u_j^n \} \} - 2i \mathcal{F}^{-1} \{ \sin(k^2 \Delta \tau) \mathcal{F} \{ u_j^n \} \} = 0 \]  

\( \text{(A 2)} \)

where

\[ k = \frac{2\pi}{N \Delta \sigma} \quad (\nu = 0, \pm 1, \ldots, \pm \frac{1}{2} N). \]

and \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) represent the forward and inverse Fourier transforms respectively.

Using the same procedure for (2.13), we obtain

\[ \zeta_j^{n+1} - \zeta_j^n + 2i \Delta \tau (\mathcal{F}^{-1} \{ \alpha \mathcal{F} \{ \zeta_j^n \} - 2i \mathcal{F}^{-1} \{ \sin(k^2 \Delta \tau) \mathcal{F} \{ \zeta_j^n \} \} = 0. \]  

\( \text{(A 3)} \)

A numerical stability criterion for (A 3) may be approximated by considering the linear equation

\[ \zeta_j + \Delta \sigma \zeta_j + \zeta_{\sigma \sigma \sigma} = 0 \]  

\( \text{(A 4)} \)

with \( \Delta \sigma \) constant. Following FW, (A 4) has the stability condition

\[ \frac{1}{\Delta \sigma} \sin \left( \frac{\pi}{\Delta \sigma} \Delta \tau \right) - \frac{\pi}{\Delta \sigma} \Delta \tau \mathcal{A} \leq 1. \]  

\( \text{(A 5)} \)

This condition is generalized for (2.13) by setting

\[ \mathcal{A} = \frac{12}{|\zeta_j - 2\zeta_j|}, \quad \text{or} \quad \mathcal{A} \approx \frac{12}{|\zeta_j - 2\zeta_j|} \]  

since \( \zeta \leq 1 \). The maximum of \( |\zeta_j - 2\zeta_j| \) occurs on the shelf when \( \zeta_j < 0 \), so that (A 5) becomes

\[ \frac{1}{\Delta \sigma} \sin \left( \frac{\pi}{\Delta \sigma} \Delta \tau \right) - \frac{\pi}{\Delta \sigma} \Delta \tau |12(\mathcal{A} - 2)\} \leq 1. \]  

\( \text{(A 6)} \)

The condition (A 6) may be very restrictive, since it is determined from \( \zeta_j - 2\zeta_j \) and \( \zeta_j \) which in some cases may be an order of magnitude larger than \( \zeta_j - 2\zeta_j \) prior to the slope. To alleviate this problem, we first optimize the time step \( \Delta \tau \), and then add and subtract a dummy advective term \( C_3 \zeta_j \) in (A 3), where \( C_3 \) is a constant. The numerical approximation then becomes

\[ \zeta_j^{n+1} - \zeta_j^n + 2i \Delta \tau (\mathcal{F}^{-1} \{ \alpha \mathcal{F} \{ \zeta_j^n \} - 2i \mathcal{F}^{-1} \{ \sin(k^2 \Delta \tau) \mathcal{F} \{ \zeta_j^n \} \} = 0. \]  

\( \text{(A 7)} \)

and the stability condition is modified to

\[ \frac{1}{\Delta \sigma} \sin \left( \frac{\pi}{\Delta \sigma} \Delta \tau - \frac{\pi}{\Delta \sigma} C_3 \Delta \tau \right) - \frac{\pi}{\Delta \sigma} \Delta \tau \mathcal{A} \leq 1. \]  

\( \text{(A 8)} \)

The constant \( C_3 \) is chosen so that \( \Delta \tau \) is maximized and (A 8) is satisfied during the entire run.

* Abbreviated to FW in this appendix.
Interfacial waves over slowly varying topography

As a check of the numerical solutions, the integral invariants of (2.13).

\[ \int_{-\infty}^{\infty} \xi^2 \, d\sigma = \sum_{i=1}^{N} \xi_i^2 = \text{constant} \]

and

\[ \int_{-\infty}^{\infty} \xi^4 \, d\sigma = \sum_{i=1}^{N} (\xi_i^2)^2 = \text{constant}. \]

were calculated. Variations in these quantities over the duration of a run were within \( \pm 0.1\% \) of the initial values for most cases and within \( \pm 1\% \) in the worst case.

In order to compute solutions at large values of \( \xi \), it was necessary in some cases to expand the \( \sigma \)-domain from 256 to 512 grid points. This prevented the tail of the scattered packet from wrapping around (in consequence of the periodic boundary conditions). For a few runs, repeated grid expansion would have required excessive computational time: in these situations the tail of the scattered packet was truncated with a tanh window. The effect of truncation on the asymptotic solutions was tested using IST theory. IST solutions for the asymptotic conditions were performed on untruncated and truncated initial data. No effect on the IST results were observed until the initial data were truncated within \( \gamma > 2 \) waves from the front of the packet, where \( \gamma \) is the number of discrete eigenvalues found from the truncated solution.

Figure 2 is an example of a run in which the grid was expanded and the tail truncated. Only the central 256 grid points are shown in the figure; the truncation is not shown.

REFERENCES


APPENDIX 6

EXPERIMENT COMPUTER PROGRAMS
MOVMOV.MAC

Assembly language code for wave generation.
C
C PROGRAM TO GENERATE TIMING INFORMATION FOR
C RUNNING THE WAVEMAKER WITH THE FALCON SBC
C
C PROGRAM PULTIM
C
C LOGICAL*1 INPSTR(15),ERRFLG
C IMPLICIT REAL*8 (A-H,O-Z)
C DIMENSION IBUF(4000)
C
TH(Y)=((DEXP(Y)-DEXP(-Y))/(DEXP(Y)+DEXP(-Y)))
C
C SECH(Y)=2.0/(DEXP(Y)+DEXP(-Y))
C
C GET SET-UP INFORMATION
C
C WRITE(5,5)
C
$ 5
C FORMAT(1H , 'PROGRAM TO GENERATE TIMING DATA FOR PULSING THE',/
C      ' WAVE MAKER. CRATES A SINGLE SOLITARY WAVE. ')
C
C WRITE(5,10)
C
$ 10
C FORMAT(1H , 'ENTER DZ,HZ,DP,DM,DELRHO,AO - CGS UNITS')
C READ(S,*)DZ,HZ,DP,DM,DELRHO,AO
C
C CALCULATE PARAMETERS
C
AL=120.0
C AK=SQR(0.75*(DP-DM)*AO)/DM/DP
C C=SQR(0.980.0*DELRHO*DP*DM/(DP+DM))
C SV=4.0*ABS(AO)/AL/AX
C SVN=0.75*SV
C MMAX=SVN/DZ
C CAX=C*AK
C A3=SV+1.90
C A2=SV/2.0
C A1=0.999*A2
C T0=0.0
C T0=0.0
C SUM=0.0
C
C USE NEWTON-RAPHESON ITERATION FOR SOLUTION
C
C DO 20 J=1,MMAX-5
C Z=FLOAT(J)*DZ+1.33333
C DO 15 I=1,200
C T=3.8*(1.0-T)
C IT=TT.LT.-3.8)IT=-3.8
C F=A1-A2*TH(TT)-Z
C FP=A3*SECH(TT)**2
C TP=T-1./FP
C IF(ABS(TP-T).LE.0.0004)GO TO 39
C T=TP
C 15 CONTINUE
C GO TO 998
C
C GET NUMBER OF CLOCK TICKS THIS STEP
C
C 39
C TC=(TP-TO)*3.8/CAK+HZ
C T=TP
C TO=TP
C IF(TC.GT.32000.0)TC=32000.0
C IBUF(J)=IFIX(TC)
C SUM=SUM+IBUF(J)
C
C 20 CONTINUE
C SUM=SUM/HZ
C WRITE(5,*)SUM
PULTIM.FOR (con't)

C
C OUTPUT DATA FILE NAME(WILL BE MERGED WITH MVMOT.MAC)
C
21 WRITE(5,30)
30 FORMAT(1H, 'ENTER THE OUTPUT FILE NAME')
CALL GETSTR(5, INPSTR, 14, ERRFLG)
IF(ERRFLG)GO TO 21
OPEN(UNIT=2, NAME=INPSTR, TYPE='NEW')
C
C OUTPUT DATA HEADER FIRST THEN DATA
C
40 WRITE(2,40)
40 FORMAT(1X, 'TIMING DATA FOR ONE SOLITARY WAVE')
WRITE(2,45)A0, DELRHO, DM, DP
45 FORMAT(1X, 'AMP='F5.2, ' DELRHO=' F5.3, ' DM='F5.2, ' DP='F5.2)
WRITE(2,50)HZ, DZ, SV
50 FORMAT(1X, 'CLOCK RATE(HZ)=', F8.1, ' DZ=', F6.4, ' PADDLE STROKE='
$ ,F5.2)
51 WRITE(2,51)(MMAX-5)
51 FORMAT(1X, 'NDATA: ', I5, '.')
JS=0
J1=1
JF=8
WRITE(2,55)(IBUF(J), J=1, 8)
55 FORMAT(1X, 'DATA: ', I5, '.', 7('.' ,I5, '.'))
56 J1=J1+8
JF=JF+8
IF(JF.GE.(MMAX-5)) JF=MMAX-5
IF(JF.GE.(MMAX-5)) JS=1
WRITE(2,60)(IBUF(J), J=J1, JF)
60 FORMAT(9X, 'WORD ', I5, ' ', 7('.' ,I5, '.'))
IF(JS.EQ.1) GO TO 999
GO TO 56
999 CLOSE(UNIT=2)
STOP 'GOOD'
998 STOP 'BAD'
END
PROGRAM TO MOVE PROBES AND SAMPLE THE CONDUCTIVITY PROFILE. FROM 1 TO 4 PROBES CAN BE SPECIFIED.

PROGRAM CALLS MP.MAC, WAIT2.MAC AND THE DTLIB.

C

PROGRAM PRCALM
DIMENSION IDATE(3), ICHAN(4), NUNIT(4), VOLT(4)
LOGICAL*1 INPTR(15)
LOGICAL*1 ERRFLG
SCALE=20./4096.
IUNIT=0

C

READ THE PROBE INFORMATION

C

WRITE(5,*)' ENTER THE NUMBER OF PROBES'
READ(5,*) NPROME
IF(NPROBE.EQ.1)NPP=1
IF(NPROBE.EQ.2)NPP=3
IF(NPROBE.EQ.3)NPP=7
IF(NPROBE.EQ.4)NPP=17
WRITE(5,1)
1 FORMAT(1H5, ' ENTER THE DATE- MM-DD-YR ')
READ(5,2)(IDATE(I), I=1,3)
2 FORMAT(I2,1X,I2,1X,I2)
WRITE(5,51)
51 FORMAT(1H1, ' ENTER THE TRIAL NUMBER ')
READ(5,*) NTRIAL
DO 200 NP=1,NPROBE
WRITE(5,*)' PROBE', NP, ' INFORMATION'
WRITE(5,3)
3 FORMAT(1HS,' ENTER THE PROBE NUMBER AND LOCATION ')
READ(5,*) IPN,PLOC
WRITE(5,5)
5 FORMAT(1HS, ' ENTER THE PROBE CHANNEL ')
READ(5,*) ICHAN(NP)
7 WRITE(5,8)
8 FORMAT(1HS,' ENTER NAME OF OUTPUT FILE ')
CALL GETSTR(5, INPTR, 14, ERRFLG)
IF(ERRFLG) GO TO 7
NUNIT(NP)=NPP+1
OPENUNIT=NUNIT(NP),NAME=INPTR, TYPE='NEW'
IU=NUNIT(NP)
WRITE(IU,9,ERR=12,END=12)
9 FORMAT(1H1, ' INITIAL PROFILE INFORMATION')
GO TO 15
12 WRITE(5,907)
907 FORMAT(1H1, ' ERROR IN OPENING/STARTING DATA FILE')
GO TO 7
15 CONTINUE
IU=NUNIT(NP)
WRITE(IU,20)(IDATE(I), I=1,3), IPN, PLOC, NTRIAL
20 FORMAT(1H1, ' DATE ', I2, ' ', I2, ' ', I2, ' ', I2, ' ', PROBE NO. ',
I2,5X, ' PROBE LOC. ',F7.2,5X, ' TRIAL ', I2)
CONTINUE
WRITE(5,*)' ENTER THE INITIAL AND FINAL DEPTHS'
READ(5,*) HO, HF
WRITE(5,*)' ENTER THE NUMBER OF STEPS BETWEEN READINGS'
WRITE(5,*)' EACH STEP = 0.02743 CM'
READ(5,*) NSTEP
ISTEP=INT((HF-HO)/0.02743)
IREAD=ISTEP/NSTEP

C INITIAL READING
C
DO 201 NP=1,NPROBE
IVAL=IADC(ICHAN(NP),IUNIT)
VOLT(NP)=SCALE*IVAL
IU=NUNIT(NP)
201 WRITE(IU,B5)HO,VOLT(NP)
85 FORMAT(1H ,F7.3,1X,F7.3)
H=HO
DH=NSTEP*0.02743
WRITE(5,*)' ENTER A NUMBER WHEN READY TO START'
WRITE(5,*)' REMEMBER TO SET SWITCHES TO FULL STEP AND CCW'
READ(5,*)IGC

C DO PROFILE
C
DO 100 J=1,IREAD
   CALL NP(NSTEP,NPP)
   CALL WAIT2(IGO)
   CALL WAIT2(IGO)
   H=H+DH
   DO 202 NP=1,NPROBE
      IVAL=IADC(ICHAN(NP),IUNIT)
      VOLT(NP)=SCALE*IVAL
      IU=NUNIT(NP)
      WRITE(IU,B5)H,VOLT(NP)
   202 CONTINUE
100 CONTINUE
DO 203 NP=1,NPROBE
CLOSE(UNIT=NUNIT(NP))
STOP ' DONE WITH PROFILE'
END

267
IWDATI.FOR

Main data acquisition program.

C PROGRAM FOR ANALOG DATA ACQUISITION.
C UP TO 8 CHANNELS MAY BE SAMPLED SIMULTANEOUSLY.
C CALLS ROUTINES FROM THE DATA TRANSLATION LIBRARY.
C ALSO USES THE COMPLETION ROUTINE SAVEIW.FOR
C TO WRITE DATA TO DISK AND CALLS THE SUBROUTINE
C MOTGO.MAC TO START THE FALCON SBC (WAVE GENERATION).
C
C KARL HELFRICH
C
C external saveiw
common/saveiw/libf,nbuf,ncnt,nsusz,nline,
scale,nout,read,dbuf(4000)
logical=1 lnstr(15)
logical=1 errflag
data ncnt,nout,nline/0,0,20/
scale=20./4096.
C
C GET GENERAL INFORMATION ON RING BUFFER
C CHARACTERISTICS ETC.
C
C write(5,10)
10 format(1h , ' date - dd/mm/yr '
read(5,20)id,im,ly
read(5,30)
20 format('1h , trial number - ')
write(5,40)itrial
write(5,50)
30 format('1h , number of input channels - ')
read(5,40)ncch
write(5,60)
40 format('1h , enter NBUF - ')
read(5,70)nbuf
write(5,80)
50 format('1h , enter number of scans per buffer - ')
read(5,70)nsans
isiz=nbuf*ncch*nsans
nsusz=ncch*nsans
C
C READ OUTPUT FILE NAME
C
85 write(5,90)
90 format(1h , ' enter the name of the output file - ')
call getstr(5,lnstr,14,errflag)
if(errflag)go to 85
open(unit=2,name=lnstr,type='NEW')
C
C CLOCK SET-UP
C
100 format(1h , ' set clock features ')
write(5,110)
110 format(1h , ' enter IRATE, IRATE =5 *100hz ')
read(5,40)irate
write(5,120)
120 format(1h , ' enter PRESET ')
write(5,130)
read(5,130)prest
130 format(f5.1)
IWDATI.FOR (con't)

C
C START REAL TIME CLOCK OPERATION

C

call sysr(irate,unit1,mode,preset,icmf)
write(5,170)
read(5,70)nread
write(2,140)td,im,iy
format(1h,,'IW expt. raw data date - ',
12,'-',12,'-',12)
write(2,150)trial,nch
format(1h,,'trial no. = ',12,5x,'no. channels = ',12)
clock=10.0*(7-irate)/preset
write(2,160)clock,nread
format(1h,,'data acquisition rate - ',5x,'total scans ',15)
170 format(1h$,'enter number of data points desired - ')

C CHECK THAT NUMBER OF DATA READS FILLS AN INTEGRAL
C NUMBER OF SUBBUFFERS.

C
	nread=(nread/nscans)*nscans
if(nread.gt.1read)1read=1read+nscans
nread=nread*nch
istchn=0
unit2=0
icmf=0
igain=1
mode=2
write(5,180)
format(1h$,'enter an integer when data acquisition is to start')
read(5,40)ique

C SEND QUE TO FALCON SBC TO START WAVE GENERATION

call motyo(ique)

C START DATA ACQUISITION

call rts,ibus,lsiz,nbuf,1read,istchn,nch,
      unit2,igain,mode,icmf,iben.,savelu)
call 1wait(icmf,0)
if(icmf.eq.-1) go to 21
write(5,200)
format(1h,,'data acquisition complete ')
close(unit=2)
stop 'successful collection'

21 write(5,210)
format(1h,,'error in data acquisition')
close(unit=2)
stop 'error exit - close file as written'
end

269
IWDATI.FOR (con't)

Completion routine for continuous writing data to disk.

```
subroutine saveiw

   COMMON/saveiw/ibef,nbuf,ncnt,nsbsz,nline,
   scale,nout,nread,ibuf(1)
   dimension buf(20)
   initad=nsbsz+ncnt
   ifinad=initad+nsbsz-1
   do 20 i=initad,ifinad,nline
       nlimit=min0(nline,(nread-nout))
       if(nlimit.eq.0) return
       write(2,820,err=800,end=810)(ibuf(i+j),j=1,nlimit)
       nout=nout+nlimit
   20 continue
   ncnt=ncnt+1
   ibef=ibef+1
   if(ncnt.gt.nbuf) return
   ncnt=0
   return

800 close(unit=2)
810 stop 'error during write to data file'
```

270
VOLDIP.FOR

Program for calculation of displacements.

program voidl
  PROGRAM FOR CONVERTING RAW PROBE VOLTAGE DATA
  TO DISPLACEMENT DATA ASSUMING LOWEST MOTION
  OF THE INTERFACE. MEASURED VOLTAGES ARE CONVERTED
  TO DISPLACEMENT USING THE STATIC PROFILES AND
  LINEAR INTERPOLATION.

  KARL HELFRICH

  virtual vv(10000)
  dimension v(100),h(100)
  logical=1 inpro(15)
  logical=1 inpstr(15)
  logical=1 lostr(15)
  logical=1 errflag

  READ GEN'L INFO

  write(5,800)
  format(1h , ' enter the name of the input file ')
  call getstr(5,inpstr,14,errflag)
  if(errflag)go to 5
  write(5,810)
  format(1h , ' enter the name of the output file ')
  call getstr(5,lostr,14,errflag)
  if(errflag)go to 6

  continue

  write(5,811)
  format(1h , ' enter initial profile file name ')
  call getstr(5,inpro,14,errflag)
  if(errflag)go to 7
  open(unit=4,name=inpro,type='OLD')

  READ STATIC CONDUCTIVITY PROFILE DATA

  read(4,890)
  format(1h ,/)
  do 50 j=1,100
  read(4,891,err=998,end=55)h(j),v(j)
  format(1h ,2(f7.4,lx))
  50 continue

  FIND LIMITS OF INTERFACE REGION

  n=j-1
  nd2=n/2
  do 30 j=1,nd2
   jp=nd2+j-1
   if(v(jp).ge.v(jp+1))go to 31
  30 mmax=jp
  do 32 j=1,nd2-1
   jm=nd2-j+1
   if(v(jm).le.v(jm-1))go to 33
  32 min=jm

  READ MEASURED PROBE DATA AND OPEN OUTPUT
  FILE FOR COMPUTED DISPLACEMENT DATA.

  open(unit=2,name=inpstr,type='OLD')
  open(unit=3,name=lostr,type='NEW')
VOLDIP.FOR (con't)

301       read(2,805) im, id, iy, itrial, ichan
305       format(1h ,5x,12,1x,12,1x,12,1x,12,1x,12,1x,12,1x,12,1x,12,1x,12)
310       read(2,820) clock, idata
315       format(1h ,19x,f5.1,23x,f5.1,15)
320       write(3,830) im, id, iy, itrial, ichan
325       format(1h ,'date ','12,'''-''',''12,'''-''',''12,5x,''trial no. ','12,
330       5x,''data channel no. ','12)
335       format(1h ,''sampling rate - Hz ','f5.1,5x,''displmnt output '','
340       'of ',15,' points',2x,'calculated h0='','f5.2)
345       read(2,860)(vv(i),i=1,idata)
350       format(1h ,10f7.3)
355       
360       FIND h0, THE INITIAL PROBE DEPTH (CALCULATED)
365       
370       sum=0.0
375       do 40 i=1,20
380       sum=sum+vv(i)
385       sum=sum/20.0
390       if(sum.lt.v(nmin))sum=v(nmin)
395       if(sum.gt.v(nmax))sum=v(nmax)
400       do 34 j=nmin,nmax-1
405       if(sum.lt.v(j+1).and.sum.ge.v(j))go to 35
410       continue
415       jj=j+1
420       h0=h(j)+(v(j)-sum)/(v(j)-v(jj))*(h(jj)-h(j))
425       if(h0.gt.h(nmax))h0=h(nmax)
430       
435       MORE OUTPUT
440       
445       write(3,840) clock, idata, h0
450       
455       START COMPUTATIONS
460       
465       nmi=nmax-1
470       do 51 k=1,idata
475       if(vv(k).lt.v(nmin))vv(k)=v(nmin)
480       if(vv(k).gt.v(nmax))vv(k)=v(nmax)
485       do 20 j=nmin,nmi
490       jh=j
495       jp1=j+1
500       if(vv(k).lt.v(jp1).and.vv(k).ge.v(jh)) go to 21
505       continue
510       fact=(v(jh)-vv(k))/(v(jh)-v(jp1))
515       hh=h(jh)*fact*(h(jp1)-h(jh))
520       if(hh.gt.h(nmax))hh=h(nmax)
525       vv(k)=hh-h0
530       continue
535       
540       WRITE COMPUTED DISPLACEMENTS
545       
550       write(3,860)(vv(k),k=1,idata)
555       close(unit=2)
560       close(unit=3)
565       close(unit=4)
570       STOP ' successful completion or error in reading probe data'
575       END