Combinatorial Decompositions of Characters of $\tilde{S}\tilde{L}(n,C)$

by

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Abstract

This thesis gives an explanation of many of the mysterious but beautiful
enumerative properties of certain characters of the complex special linear group,
$SL(n, C)$. Previous work by R. K. Gupta, P. Hanlon, and R. P. Stanley has
suggested that the decompositions of these characters into irreducibles should
exhibit combinatorial structure.

The characters studied are constructed from the exterior, symmetric and
tensor algebras of the adjoint representation of $SL(n, C)$. A number of results
concerning the decompositions of these characters are presented which are ob-
tained primarily via the theory of symmetric functions.

A combinatorial interpretation of the decomposition of the character of the
exterior algebra is given. Some necessary conditions, which are conjectured to be
sufficient, are given for deciding whether a particular irreducible representation
of $SL(n, C)$ occurs in the exterior algebra. It is shown that in extremal and near-
extremal cases, the decompositions exhibit a multiplicative structure. A proof
of Gupta and Hanlon's conjectured "first layer formula" for the multiplicities of
first layer representations in the exterior powers of the adjoint representation is
given.

The expansion of the symmetric function

$$\psi_n(z, q) = \prod_{i \leq j \leq n} \prod_{k \geq 1} \frac{1 - q^k x_i x_j^{-1}}{1 - z q^{k-1} x_i x_j^{-1}}$$

in terms of the Schur functions is studied. It is shown that the generalized
exponents of $SL(n, C)$ and the decompositions of the characters of the exterior
powers and Hanlon's "Macdonald complex" can all be obtained as special cases of
the Schur function expansion of $\psi_n(z, q)$. A technique of D. Stanton is ex-
tended and used to give a recursion for computing this expansion. The recursion is used
to generalize the first layer formula from the exterior algebra to the Macdonald
complex. It is also used to give a combinatorial reformulation of Macdonald's affine analogue of the Weyl denominator formula for the root system $A_n$.

It is known that the $q$-Dyson Theorem, which is a collection of formal power series constant-term identities, can be used to find the multiplicity of the trivial character in the Macdonald complex. The first layer formula for the Macdonald complex and some elementary properties of the Hall-Littlewood symmetric functions are used to extend the $q$-Dyson Theorem from identities involving constant terms to identities involving the coefficients of some nonconstant monomials.

Thesis Supervisor: Richard P. Stanley
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Combinatorial Decompositions of Characters of $SL(n, C)$
Preface

Combinatorics has been present throughout the development of many branches of algebra. However, it is only recently that combinatorially minded people have begun to systematically distill the beautiful combinatorial structures which are hidden in such areas as invariant theory, lattice theory, root systems, Coxeter groups, and group representation theory. One of the more successful instances of this is the representation theory of the special linear group $SL(n, \mathbb{C})$ (or equivalently, the general linear group, $GL(n, \mathbb{C})$). The theory of Schur functions and semistandard tableaux effectively allow one to "reduce" the problem of computing multiplicities in irreducible decompositions of representations of $SL_n$ to combinatorial problems.

The goal of this thesis is to elucidate some of the combinatorial structure which is present in some particular representations of $SL_n$ which have been the subject of much recent interest among both combinatorists and algebraists. This structure is found in the decomposition of representations of $SL_n$ which are built out of the exterior, symmetric and tensor algebras of the adjoint representation.

The manner in which I have chosen to present this material has been a compromise. It would be more convenient for myself (and the specialist) to simply assume that the reader has sufficient background in both combinatorics and representation theory, and immediately state and prove these results with no development of context or motivation. This would, on the other hand, severely limit the potential audience for this work and require a significant effort for the nonspecialist to read it. I have therefore attempted to give an introduction to the necessary combinatorics and algebra so that this thesis can be read profitably by the combinatorist who is interested in representation theory and the algebraist who is interested in combinatorics. It would be absurd however, to give such an introduction which is wholly "from scratch"; a balance has been struck between giving proofs of known results and giving the reader references to the literature.

The only background required of the reader is a basic familiarity with the most elementary aspects of algebra, such as groups, rings, and modules. Occasional reference will be made to root systems, Lie groups, and Lie algebras, but familiarity with them will not be essential for the purposes of this thesis; they merely serve to place this work in a broader context.
In the first chapter, we develop the theory of symmetric functions. When possible, we have tried to follow the notation used by Macdonald in [32]. Most of what we present in this chapter can be found elsewhere (e.g., [28], [32], and [41]), and we have conscientiously tried to correctly attribute the proofs we have given. However, our point of view differs occasionally from these articles; one finds that what are definitions for one become theorems for others; proofs given for one purpose can be refashioned for another. In any case, the reader who is familiar with the theory of symmetric functions (including the Schur functions) may wish to skip Sections 1, 2 and 3, perhaps regarding them as a notational appendix. In Sections 4 and 5 we describe the basic tools needed from the theory of symmetric functions which facilitate the computation of Schur function expansions of symmetric functions. The two most crucial ones are (1) the use of alternating functions to reduce Schur function expansions to the extraction of the coefficients of monomials, and (2) the Littlewood-Richardson rule.

In the second chapter, the representation theory of $GL_n$ and $SL_n$ is developed, with an emphasis on the characters of their representations and the combinatorics of such characters. Sufficient references to the literature are given so that the interested reader can find more detailed accounts of this theory. The book by Boerner [3] and the article by Stanley [39] are good places to start. The reader who is already familiar with this theory may wish to skip Sections 6 and 7.

In Sections 8 and 9, the theories of symmetric functions and representations of $SL_n$ and $GL_n$ are unified, and the basic problems which will be studied in detail are identified. These problems amount to the study of the decomposition into irreducibles of the characters of the exterior, symmetric and tensor algebras of the adjoint representation (i.e., the action of $GL_n$ on $gl_n$ and $SL_n$ on $sl_n$). For convenience, we modify the usual sense of the adjoint representation in that we actually study the action of $SL_n$ on $gl_n$. We also refine the exterior, symmetric and tensor algebras in several directions. In the exterior and symmetric cases, we consider the character decomposition of the so called "exterior powers" $\text{Ext}^k(gl_n)$ and "symmetric powers" $\text{Sym}^k(gl_n)$. In the case of the tensor algebra, we allow the simultaneous action of the symmetric group $S_k$ on the $k$th tensor power $T^k(gl_n)$, and show how the decomposition of the resulting representation of $S_k \times GL_n$ into $S_k$-isotypic components can be reduced to a Schur function expansion problem. One other type of action which we study is P. Hanlon's so-called "Macdonald complex", which amounts to the study of the decomposition of the character of $T^k(\text{Ext}(gl_n))$ in which the submodules

$$\text{Ext}^{a_1}(gl_n) \otimes \cdots \otimes \text{Ext}^{a_k}(gl_n)$$

are given special weightings. We also discuss the previous work of R. K. Gupta, P. Hanlon and R. Stanley which shows that the decompositions of the characters described above exhibit striking behavior in the limit $n \to \infty$. 
In the third chapter, we study in detail the decomposition of the characters of the exterior algebra $\text{Ext}(gl_n)$, the exterior powers $\text{Ext}^k(gl_n)$, and to a lesser extent, the Macdonald complex of $gl_n$. We give a combinatorial interpretation for the decomposition of $\text{Ext}(gl_n)$. Using this interpretation, we give some necessary conditions, which we conjecture to be sufficient, for a given irreducible representation of $SL_n$ to appear in the decomposition of $\text{Ext}(gl_n)$. We show that the multiplicities of the irreducible representations which correspond to the extremal and near-extremal cases of these conditions exhibit a multiplicative structure which respects each of the exterior powers. These conditions and extremal structures are generalized to the Macdonald complex in Section 13. In Section 12, we give a proof of Gupta and Hanlon's conjectured "first layer formula", which is an explicit formula for the multiplicities of the first layer representations in the exterior powers $\text{Ext}^k(gl_n)$.

In the fourth chapter we study the Schur function expansion of the symmetric function

$$\psi_n(z, q) = \prod_{1 \leq i < j \leq n} \prod_{k \geq 1} \frac{1 - q^k x_i x_j^{-1}}{1 - z q^{k-1} x_i x_j^{-1}}.$$

The decomposition of the characters of the exterior algebra and the Macdonald complex, and the generalized exponents of $SL_n$ can all be obtained as special cases of the Schur function expansion of $\psi_n(z, q)$. Macdonald's affine analogue of the Weyl denominator formula for the root system $A_{n-1}$ can also be regarded as a special case (namely, $z = 0$) of this expansion. We use a proof technique essentially equivalent to one used by D. Stanton [42] to give a more combinatorial formulation of this identity than those previously known. We extend Stanton's technique to give a recursion for computing the coefficients of the full expansion of $\psi_n(z, q)$. We give an application of this recursion in which we generalize the first layer formula from the exterior algebra to the expansion of $\psi_n(z, q)$, thus yielding first layer formulas for the Macdonald complex of $gl_n$ and the generalized exponents.

It is known that the $q$-Dyson Theorem for equal parameters, which amounts to a formula for the constant term (with respect to $x_1, \ldots, x_n$) in the formal power series

$$\prod_{1 \leq i < j \leq n} \prod_{i=1}^k (1 - q^{l-1} x_j x_i^{-1})(1 - q^l x_i x_j^{-1}),$$

can be used to give a formula for the coefficient of the Schur function $s_\lambda(x_1, \ldots, x_n)$ in the expansion of $\psi_n(z, q)$. We show that the first layer formula for $\psi_n(z, q)$ can be used to express coefficients of monomials of the form $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha_i \geq -1$ as specializations of a certain class of symmetric functions which are closely related to the Hall-Littlewood symmetric functions. In the case in which all of the exponents with $\alpha_i = -1$ occur consecutively, we give an explicit evaluation
of this substitution, thus giving a generalization of the $q$-Dyson Theorem in the case of equal parameters.

**Notations and conventions**

We assemble here some standard notation which will be used throughout this thesis. An index for additional notation which is of more than transient use can be found in the appendix.

$\mathbb{Z}$ the ring of integers

$\mathbb{Z}/(n)$ the ring of integers modulo $n$

$\mathbb{N}$ the nonnegative integers

$\mathbb{P}$ the positive integers

$[n]$ the set of integers $\{1, \ldots, n\}$.

$\mathcal{S}_n$ the symmetric group of permutations of $[n]$

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ the rational, real and complex fields

If $S$ is any of the above sets, $S^k$ denotes the Cartesian product of $k$ copies of $S$.

If $A$ is any commutative ring with identity,

$A[x_1, \ldots, x_n]$ denotes the polynomial ring in the variables $x_1, \ldots, x_n$.

$A[[x_1, \ldots, x_n]]$ denotes the formal power series ring in the variables $x_1, \ldots, x_n$.

The following miscellaneous notations will be used:

$\bigoplus$, $\oplus$ direct sum of vector spaces or modules

$|S|$ cardinality of the finite set $S$

$\lfloor x \rfloor$ the largest integer not exceeding the real number $x$.

$\delta_{ij}$ the Kronecker delta: $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$A^t$ transpose of the matrix $A$

$\text{tr}A$ trace of the matrix $A$

$\det A$ determinant of the matrix $A$

At the risk of being too obvious, we mention that equations have been numbered consecutively within each section, so that for example, a reference to equation '(8.6)' refers to the equation labelled '(6)' in Section 8, while a reference to equation '(6)' refers to the current section.
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John Stembridge
Cambridge, Massachusetts
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Chapter I.

The Theory of Symmetric Functions

1. Partitions and shapes

In this section we introduce a mostly standard system of notation for working with partitions, which are indispensable in the theory of symmetric functions.

Definition 1.1: A partition is a sequence of nonnegative integers

\[ \lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots) \]

such that:

1. The terms are weakly decreasing; i.e., \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \).
2. Only a finite number of the terms are nonzero.

The nonzero terms are called the parts of \( \lambda \). The number of parts is called the length of \( \lambda \), and is written \( \ell(\lambda) \). The sum of the parts is called the weight, and we write

\[ |\lambda| = \sum_{i \geq 1} \lambda_i. \]

More generally, if \( \alpha = (\alpha_1, \alpha_2, \ldots) \) is any sequence of integers with only finitely many nonzero terms, we define

\[ |\alpha| = \sum_{i \geq 1} \alpha_i. \]

The unique partition of weight 0 is denoted by \( \emptyset \). Let \( \lambda \) be a partition. If \( \lambda \) is of weight \( k \), then \( \lambda \) is said to be a partition of \( k \). If \( n \geq \ell(\lambda) \), we will sometimes prefer to delete all but the first \( n \) terms of \( \lambda \) and regard \( \lambda \in \mathbb{N}^n \).

Example 1.2: In practice, one suppresses the commas and trailing zeroes in a partition. With such a convention, the partitions of 5 are

\( (5), (41), (32), (311), (221), (2111), (11111) \).
Even more briefly, they are

5, 41, 32, 311, 221, 2111, 11111.

It is frequently useful to describe partitions according to the exponential notation. We write

$$\lambda = 1^{n_1}2^{n_2}3^{n_3} \ldots$$

where $n_i = n_i(\lambda)$ is the number of times that $i$ occurs as a part of $\lambda$; i.e.,

$$n_i(\lambda) = |\{j \geq 1 : \lambda_j = i\}|.$$

Conventionally, the parts of multiplicity zero and the exponents for those of multiplicity one are suppressed, so that $1^23^4$ is the exponential notation for the partition 444311.

**Definition 1.3:** The diagram or shape of a partition $\lambda$ is the set

$$D_\lambda = \{(i, j) \in \mathbb{Z}^2 : 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}.$$

We will have a tendency to identify a partition with its diagram, so that, for example, the notation $x \in \lambda$ should not be taken literally; it should be interpreted as $x \in D_\lambda$. It is convenient to view the diagram not as an abstract set, but as a collection of cells or boxes in the plane with matrix-style coordinates. One puts a cell in the $(i, j)$ position of an imaginary matrix for each element $(i, j) \in D_\lambda$; thus, there are $\lambda_i$ cells in the $i$th row of the diagram. See Figure 1.1.

The conjugate of a partition $\lambda$ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ whose diagram is given by

$$D_{\lambda'} = \{(i, j) \in \mathbb{Z}^2 : (j, i) \in D_\lambda\}.$$

In other words, the diagram of $\lambda'$ is obtained from the diagram of $\lambda$ by exchanging rows and columns. The conjugate can also be characterized by the fact that

$$n_i(\lambda') = \lambda_i - \lambda_{i+1}.$$
The conjugate of the partition $\lambda = 5332$ in Figure 1.1 is $\lambda' = 44311$.

The content $c(x)$ of a cell $x = (i, j) \in \lambda$ is defined to be
\[
c(x) = j - i. \tag{1}\]

The hooklength $h(x)$ of a cell $x = (i, j) \in \lambda$ is defined to be
\[
h(x) = \lambda_i - i + \lambda_j' - j + 1. \tag{2}\]

Geometrically, $h(x)$ counts the number of cells directly below or directly to the right of $x$, including $x$. For any sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ of integers with only finitely many nonzero terms, let
\[
n(\alpha) = \sum_{i \geq 1} (i - 1) \alpha_i. \tag{3}\]

If one assigns the number $i - 1$ to each cell of the $i$th row of a diagram $D_\lambda$, the sum of the numbers so defined is $n(\lambda)$.

**Definition 1.4:** The rank of a partition is the number of cells on the main diagonal of $\lambda$; namely,
\[
|\{i \in \mathbb{Z} : (i, i) \in D_\lambda\}|.
\]

A partition of rank 1 is said to be a hook.

Another useful notation for describing partitions is the Frobenius notation, which can be defined as follows. Let $\lambda$ be a partition of rank $r$, and define sequences $\alpha, \beta \in \mathbb{N}^r$ where
\[
\alpha_i = \lambda_i - i, \quad \beta_i = \lambda_i' - i \quad (1 \leq i \leq r). \tag{4}\]

Geometrically, $\alpha_i$ counts the cells of $\lambda$ to the right of $(i, i)$ and $\beta_i$ counts the cells of $\lambda$ below $(i, i)$. Notice that $\alpha$ and $\beta$ uniquely determine $\lambda$, and
\[
\alpha_1 > \alpha_2 > \cdots > \alpha_r \geq 0 \quad \beta_1 > \beta_2 > \cdots > \beta_r \geq 0.
\]

The Frobenius notation for $\lambda$ is written
\[
\lambda = (\alpha|\beta) = (\alpha_1, \ldots, \alpha_r|\beta_1, \ldots, \beta_r).
\]

For example, $54332111 = (420|731)$. The utility of this notation will become more apparent in Section 17.
Remark 1.5: A partition $\lambda$ for which $\lambda = \lambda'$ is said to be self-conjugate. There is a well known weight-preserving bijection, due to Sylvester, between self-conjugate partitions and partitions into distinct odd parts. If $\lambda$ is self-conjugate and of rank $r$, then $\lambda$ is necessarily of the form $(\alpha | \alpha)$ for some $\alpha \in \mathbb{N}^r$. Sylvester's bijection can be described as

$$\lambda \mapsto (2\alpha_1 + 1, \ldots, 2\alpha_r + 1).$$

There are two operations on pairs $\lambda, \mu$ of partitions which occur frequently in the theory of symmetric functions; namely, $\lambda \cup \mu$ (partwise union) and $\lambda + \mu$ (partwise sum). They are defined as follows:

$$n_i(\lambda \cup \mu) = n_i(\lambda) + n_i(\mu)$$

$$\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots).$$

(5)

(6)

Also, we will use the notation $k\lambda$ for the $k$-fold sum $(k\lambda_1, k\lambda_2, \ldots)$.

We may define a partial order $\subseteq$ on partitions by insisting that $\mu \subseteq \lambda$ if and only if $D_{\mu} \subseteq D_{\lambda}$. Equivalently, we have

$$\mu \subseteq \lambda \iff \mu_i \leq \lambda_i \text{ for all } i \geq 1.$$ 

This partial order is easily seen to be a distributive lattice; it is sometimes known as Young's lattice. Whenever $\mu \subseteq \lambda$ we let $\lambda/\mu$ denote the set-theoretic difference

$$D_{\lambda} \setminus D_{\mu} = \{(i, j) \in \mathbb{Z}^2 : 1 \leq i \leq \ell(\lambda), \mu_i < j \leq \lambda_i\},$$

and refer to $\lambda/\mu$ as a skew diagram or a skew shape. As with ordinary diagrams, we prefer to think of a skew diagram as a collection of cells in the plane with matrix-style coordinates. See Figure 1.2. A skew shape $\lambda/\mu$ with at most one cell in each row ($\lambda_i - \mu_i \leq 1$) is said to be a vertical strip, while a skew shape with at most one cell in each column is said to be a horizontal
strip. The notions of weight and conjugate are extended to skew shapes in the
natural way:

\[ |\lambda/\mu| = |\lambda| - |\mu| \]
\[ (\lambda/\mu)' = \lambda'/\mu'. \]

Another partial order which occurs in the theory of symmetric functions is
the dominance order. Let \( \lambda, \mu \) be partitions of the same weight. If for all \( i \geq 1 \),

\[ \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i, \quad (7) \]

then \( \lambda \) is said to dominate \( \mu \). One writes \( \lambda \geq \mu \).

A lowering of a diagram is an operation in which a cell is removed from
the diagram and reinserted into a lower row, thereby creating a new diagram.
It is well known that the dominance order can be described in terms of these
lowerings. In fact,

**Proposition 1.6:** Let \( \lambda, \mu \) be partitions of the same weight. We have \( \lambda > \mu \)
if and only if one can obtain the diagram of \( \mu \) from that of \( \lambda \) by a sequence of
lowerings.

**Proof:** If one can obtain \( \mu \) from \( \lambda \) by a sequence of lowerings it is clear that
\( \lambda > \mu \), since a sequence of lowerings describes a chain in the partial order.

On the other hand, suppose that \( \lambda > \mu \). There must exist integers \( i \) for which
\( \lambda_i > \mu_i \). In fact, there must exist a pair \( i < j \) for which

\[ \lambda_i > \mu_i, \; \lambda_{i+1} = \mu_{i+1}, \ldots, \lambda_{j-1} = \mu_{j-1}, \; \lambda_j < \mu_j. \]

In this case, we may remove a cell from the \( i \)th row of \( \lambda \) and insert it into the
\( j \)th row, thereby creating a partition \( \nu \) with \( \nu \geq \mu \). We may repeat this process
inductively until we obtain the partition \( \mu \).

2. Symmetric functions defined and exhibited

The results presented in this section concern some of the classical, elementary
facts about symmetric functions. Our exposition follows the presentations given
by Macdonald [32; Chp. I] and Stanley [41; Part 1].

Consider an arbitrary monomial in the variables \( x_1, x_2, x_3 \ldots \); say

\[ x^a = x_1^{a_1} x_2^{a_2} x_3^{a_3} \ldots. \]
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To say that such an expression is a monomial is tantamount to saying that
the exponent sequence \( \alpha = (\alpha_1, \alpha_2, \ldots) \) is a nonnegative integer sequence with
finitely many nonzero terms. The degree of the monomial \( x^\alpha \) is \( |\alpha| \). The type
of a monomial is the partition obtained by sorting the terms of the exponent
sequence into decreasing order. Thus, the type of the monomial \( x_1^2x_3^3x_5^2 \) is the
partition 4221.

Recall that a formal power series in the variables \( x_1, x_2, x_3, \ldots \) is a formal sum
\[
  f(x_1, x_2, \ldots) = \sum_\alpha a_\alpha x^\alpha
\]
over monomials \( x^\alpha \), with coefficients \( a_\alpha \) chosen from a suitable ring. Such formal
power series have an obvious ring structure analogous to polynomial rings. Recall
that the formal power series \( f \) is said to be homogeneous of degree \( k \) if all
the monomials which occur in \( f \) are of degree \( k \).

**Definition 2.1:** A formal power series \( f \) in the variables \( x_1, x_2, x_3, \ldots \) is said to
be symmetric if \( f \) is invariant under bijections of the variables; i.e.,
\[
  f(x_1, x_2, x_3, \ldots) = f(\pi(x_1), \pi(x_2), \pi(x_3), \ldots)
\]
for any bijection \( \pi \) of \( \{x_1, x_2, x_3, \ldots\} \).

A symmetric formal power series \( f \) is said to be a symmetric function if it is
of bounded degree; i.e., if the degrees of the monomials which occur in \( f \) are bounded. Thus,
\[
  \prod_{i \geq 1} (1 + x_i)
\]
is a symmetric formal power series, but not a symmetric function.

Note that the symmetric functions form a subring of the ring of formal
power series. Let \( \Lambda(x) \), or simply \( \Lambda \), denote this ring of symmetric functions
of \( x_1, x_2, x_3, \ldots \). If we need to augment the coefficient ring of \( \Lambda \) to some ring
\( A \supseteq \mathbb{Z} \), we will usually do so without further comment, however if we want to
emphasize this fact we will denote the resulting ring by \( \Lambda_A \).

Define \( \mathbb{Z} \)-modules \( \Lambda^k \) via
\[
  \Lambda^k = \Lambda^k(x) = \{ f \in \Lambda : f \text{ homogeneous of degree } k \}.
\]
Observe that we have the following graded ring decomposition:
\[
  \Lambda = \prod_{k \geq 0} \Lambda^k.
\]
Particularly in the representation theory of $GL_n$ and $SL_n$ it is convenient to restrict the variable set to $x_1, \ldots, x_n$. Thus we define $\Lambda_n = \Lambda(x_1, \ldots, x_n)$ to be the ring of symmetric functions of the variables $x_1, \ldots, x_n$, and

$$\Lambda_n^k = \Lambda_n^k(x_1, \ldots, x_n) = \{ f \in \Lambda_n : f \text{ homogeneous of degree } k \}.$$  

Notice that the members of $\Lambda_n$ are actually polynomials, and we still have the graded ring decomposition

$$\Lambda_n = \prod_{k \geq 0} \Lambda_n^k.$$  

In fact, the ring $\Lambda_n$ is merely a homomorphic image of $\Lambda$.

If $f \in \Lambda(x)$ is any symmetric function of the indeterminates $x_1, x_2, \ldots$, we will frequently abbreviate the dependence of $f$ on the indeterminates by writing $f(x)$. This should not be confused as a special instance of the notation $f(x_1, \ldots, x_n)$ (with $n = 1$), which denotes the image of $f$ in $\Lambda_n$; i.e., the result of substituting $x_{n+1}, x_{n+2}, \ldots = 0$.

**Definition 2.2:** Let $\lambda$ be a partition. The monomial symmetric function $m_\lambda = m_\lambda(x_1, x_2, \ldots)$ corresponding to $\lambda$ is defined by

$$m_\lambda(x) = \sum x^\alpha : \text{$x^\alpha$ is of type } \lambda;$$  

i.e., $m_\lambda$ is the sum of all monomials of type $\lambda$.

The monomial symmetric functions are obviously symmetric, as any bijection of the indeterminates $x_1, x_2, \ldots$ merely permutes monomials of a given type.

**Proposition 2.3:**

(a) A $Z$-basis of $\Lambda^k$ is given by $\{ m_\lambda : |\lambda| = k \}$.

(b) A $Z$-basis of $\Lambda_n^k$ is given by $\{ m_\lambda(x_1, \ldots, x_n) : |\lambda| = k, \ell(\lambda) \leq n \}$.

**Proof:** To say that $f$ is a symmetric function is equivalent to saying that the coefficient $a_\alpha$ of the monomial $x^\alpha$ in $f$ depends only on the type. Thus, if $f \in \Lambda^k$, we have the unique decomposition

$$f = \sum_{|\lambda| = k} a_\lambda m_\lambda.$$  

If we restrict the indeterminates to $x_1, \ldots, x_n$, then the only monomials which survive are those whose type is of length at most $n$.

**Definition 2.4:** Let $r$ be a positive integer.
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(a) The rth elementary symmetric function \( e_r \) is given by
\[
e_r(x) = m_{1^r}(x) = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}.
\]

(b) The rth complete homogeneous symmetric function \( h_r \) is given by
\[
h_r(x) = \sum_{|\lambda|=r} m_\lambda(x) = \sum_{i_1 \leq \cdots \leq i_r} x_{i_1} \cdots x_{i_r}.
\]

(c) The rth power-sum symmetric function \( p_r \) is given by
\[
p_r(x) = m_r(x) = \sum_{i \geq 1} x_i^r.
\]

Extend the definitions of these symmetric functions to all partitions \( \lambda \) by defining
\[
e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots
\]
\[
h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots.
\]
\[
p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots
\]

Notice that \( e_1 = h_1 = p_1 = m_1 = \sum x_i \) and that \( e_\lambda, h_\lambda, p_\lambda \) and \( m_\lambda \) are all homogeneous of degree \(|\lambda|\). By convention, we define \( e_0 = h_0 = p_0 = 1 \).

Proposition 2.5: Let \( n \) be a positive integer. We have

\[\sum_{0 \leq r \leq n} (-1)^r e_r h_{n-r} = 0\]

\[\sum_{1 \leq r \leq n} p_r h_{n-r} = nh_n.\]

Proof: (Macdonald [32; I.(2.6'),(2.11)], Stanley [41; (14)])

Consider the generating functions
\[
E(t) = \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t)
\]
\[
H(t) = \sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} \frac{1}{1 - x_i t}.
\]

From (2) and (3) we deduce that
\[
\left[\sum_{r \geq 0} e_r (-t)^r\right] \left[\sum_{r \geq 0} h_r t^r\right] = 1.
\]
Extraction of the coefficient of $t^n$ in (4) yields part (a).

To prove (b), notice from (3) that

$$\log H(t) = \sum_{i,r \geq 1} \frac{1}{r!} x_i^r t^r = \sum_{r \geq 1} p_r \frac{t^r}{r},$$

so

$$\frac{dH}{dt} = H(t) \sum_{r \geq 1} p_r t^{r-1}. \quad (5)$$

Extraction of the coefficient of $t^{n-1}$ in (5) yields part (b). ♦

Note that Proposition 2.5(a) tells us that the $e_r$'s (and hence, the $e_\lambda$'s) can be expressed as $\mathbb{Z}$-linear combinations of the $h_\lambda$'s and conversely. Similarly, Proposition 2.5(b) tells us that the $h_\lambda$'s can be expressed as $\mathbb{Q}$-linear combinations of the $p_\lambda$'s. Therefore, the $e_\lambda$'s, $h_\lambda$'s and $p_\lambda$'s all generate the same ring. More precisely, we have

**Theorem 2.6:**

(a) $\{e_\lambda : |\lambda| = k\}$ is a $\mathbb{Z}$-basis of $\Lambda^k$. In particular, the $e_r$'s are algebraically independent and

$$\Lambda = \mathbb{Z}[e_1, e_2, \ldots].$$

(b) $\{h_\lambda : |\lambda| = k\}$ is a $\mathbb{Z}$-basis of $\Lambda^k$. In particular, the $h_r$'s are algebraically independent and

$$\Lambda = \mathbb{Z}[h_1, h_2, \ldots].$$

(c) $\{p_\lambda : |\lambda| = k\}$ is a $\mathbb{Q}$-basis of $\Lambda_{\mathbb{Q}}^k$. In particular, the $p_r$'s are algebraically independent and

$$\Lambda_{\mathbb{Q}} = \mathbb{Q}[p_1, p_2, \ldots].$$

**Proof:** (cf. Macdonald [32; I.2.I.6])

In view of Proposition 2.5 and the preceding remarks, it suffices to prove part (a); the other parts would follow immediately.

Let $\lambda$ be a partition of $k$. By Proposition 2.3, there are integers $a_{\lambda \mu}$ such that

$$e_\lambda = \sum_{|\mu|=k} a_{\lambda \mu} m_\mu. \quad (6)$$

In fact, if one expands the product

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$$
into monomials with brute force and Definition 2.4, one can see that $a_{\lambda\mu}$ is the number of 0,1 matrices with row sums $\lambda$ and column sums $\mu$.

Observe that the matrix with 1's in each of the positions corresponding to the cells of $\lambda$ and 0's elsewhere has row sums $\lambda$ and column sums $\lambda'$. All other 0,1 matrices with row sums $\lambda$ are obtained by shifting some 1's of this matrix to the right. Their column sums are all later than $\lambda'$ in the sense of reverse lexicographic order. It follows that there is a suitable ordering of the partitions $\mu$ so that the matrix $[a_{\lambda\mu}]$ is upper triangular with 1's on the main diagonal. We conclude that the linear relations (6) are invertible and that the inverse relations are also integral. Hence, $\{e_{\lambda} : |\lambda| = k\}$ is a $\mathbb{Z}$-basis of $\Lambda^k$, by Proposition 2.3(a).

It is interesting to note that a result due independently to Gale and Ryser [35] characterizes exactly when there exist 0,1 matrices with row sums $\lambda$ and column sums $\mu$. Such matrices exist precisely when $\lambda' \geq \mu$ in the dominance order (1.7).

Define a ring homomorphism $\omega : \Lambda \rightarrow \Lambda$ via $\omega(e_r) = h_r$. In view of Theorem 2.6, $\omega$ is certainly well-defined. Moreover,

**Proposition 2.7:** We have:

(a) $\omega(h_r) = e_r$.

(b) $\omega^2 = 1$ (so $\omega$ is an automorphism of $\Lambda$).

(c) $\omega(p_r) = (-1)^{r-1}p_r$.

**Proof:** (Macdonald [32; I.(2.7)], Stanley [41; Prop.3.3])

For (a), apply $\omega$ to the identity in Proposition 2.5(a), and proceed with the obvious induction. Part (b) follows immediately from part (a) and Theorem 2.6.

To prove (c), recall from (5) that

$$\log \left( \sum_{r \geq 0} h_r t^r \right) = \sum_{r \geq 1} \frac{1}{r} p_r t^r.$$  (7)

By similar reasoning, one may deduce from (2) that

$$\log \left( \sum_{r \geq 0} e_r t^r \right) = \sum_{r \geq 1} \frac{(-1)^{r-1}}{r} p_r t^r.$$  (8)

Applying $\omega$ to (7) and comparing with (8) yields the desired result.

**Remark 2.8:** By the multiplicativity (1) of the symmetric functions $p_\lambda$, it follows that

$$\omega(p_\lambda) = e_\lambda p_\lambda,$$
where \( \varepsilon_\lambda = (-1)^{|\lambda|-d(\lambda)} \) is the sign character of the symmetric group. In other words, if \( w \) is a permutation with \( c_1 \) 1-cycles, \( c_2 \) 2-cycles, etc., then the sign of \( w \) is \( \varepsilon_\lambda \), where \( \lambda = 1^{c_1}2^{c_2}\ldots \). This is only a superficial instance of a much deeper connection between the theory of symmetric functions and the symmetric group, which we will have more to say about in Chapter II.

3. Schur functions

We now consider another \( \mathbb{Z} \)-basis of the ring of symmetric functions, the Schur functions (also known as S-functions), which have a number of beautiful properties both combinatorial and algebraic. Indeed, the richness of their theory makes it difficult to give them a fully satisfactory definition; choosing any particular characterization as a definition simultaneously makes some of their aspects seem natural, while others become less transparent.

As our purpose is to apply the methods of combinatorics to problems in representation theory, we will use a combinatorial definition of the Schur functions. This is the same point of view adopted by Stanley [39], [41].

**Definition 3.1:**

Let \( \lambda \) be a partition. A tableau \( T \) of shape \( \lambda \) is an assignment

\[
T : D_\lambda \to \mathbb{P}
\]

of positive integers to the cells of \( \lambda \) such that

1. The numbers assigned increase weakly along the rows of \( \lambda \) from left to right; i.e., \( T(i, j) \leq T(i, j + 1) \).
2. The numbers assigned increase strictly along the columns of \( \lambda \) from top to bottom; i.e., \( T(i, j) < T(i + 1, j) \).

Similarly, a skew tableau \( T \) of shape \( \lambda/\mu \) is an assignment

\[
T : \lambda/\mu \to \mathbb{P}
\]

of positive integers to the cells of the skew diagram \( \lambda/\mu \) satisfying 1 and 2.

The weight of a (possibly skew) tableau \( T \) is the sequence

\[
w(T) = (\alpha_1, \alpha_2, \alpha_3, \ldots)
\]

where \( \alpha_i \) is the number of cells assigned to the integer \( i \) by \( T \).

In practice, one does not think of a tableau as an abstract mapping, but as a way of filling in the diagram with positive integers subject to the constraints 1
and 2. See Figure 3.1. Thus, when we say "there is a 3 in the 4th row of $T$", the pedantic reader should interpret this as "there is an integer $j$ such that $(4, j) \in \lambda$ and $T(4, j) = 3."" Notice that in any tableau, the cells occupied by the integer $i$ form a horizontal strip.

A (possibly skew) tableau with $n$ cells is said to be standard if its weight is the partition $1^n$. Such tableaux are also sometimes called standard Young tableaux. Let

$$f^{\lambda/\mu} = \left| \{ \text{standard tableaux of shape } \lambda/\mu \} \right|.$$ 

More generally, let $K^{\lambda/\mu, \alpha}$ denote the number of skew tableaux of shape $\lambda/\mu$ and weight $\alpha = (\alpha_1, \alpha_2, \ldots)$ These numbers are sometimes referred to as Kostka numbers. One of the beautiful results connecting tableaux to algebra is the fact that $f^\lambda$ is the degree of an irreducible representation of the symmetric group.

**Definition 3.2:**

Let $\lambda$ be a partition. The Schur function $s_\lambda$ corresponding to $\lambda$ is the formal power series

$$s_\lambda(x) = \sum_T x^{w(T)},$$

where the sum ranges over all tableaux $T$ of shape $\lambda$.

Let $\lambda/\mu$ be a skew shape. The skew Schur function $s_{\lambda/\mu}$ corresponding to $\lambda/\mu$ is the formal power series

$$s_{\lambda/\mu} = \sum_T x^{w(T)},$$

where the sum ranges over all tableaux $T$ of shape $\lambda/\mu$.

**Remark 3.3:** The following observations follow directly from this definition.

(a) The Schur function $s_\lambda$ is homogeneous of degree $|\lambda|$; the skew Schur function $s_{\lambda/\mu}$ is homogeneous of degree $|\lambda/\mu|$.
(b) The Schur functions corresponding to a partition whose diagram is a single row or column were introduced in section 2. Specifically, we have

\[ s_{1^r} = e_r ; \quad s_r = h_r. \]

(c) The skew Schur function \( s_{\lambda/\mu} \) depends only on the relative positions of the cells of \( \lambda/\mu \) and not on their absolute coordinates. For example, the skew shapes \( 4331/1 \) and \( 75442/72111 \) are translates of each other, so their corresponding skew Schur functions are the same. See Figure 3.2.

(d) A subset \( \theta \) of the plane \( \mathbb{Z}^2 \) is connected if there is a path (moving along rows and columns) connecting any two cells of \( \theta \). The connected components of a skew shape \( \lambda/\mu \) are the maximal connected subsets of \( \lambda/\mu \). If \( \theta_1, \theta_2, \ldots \) are the connected components of \( \lambda/\mu \), then the tableau constraints may be imposed independently on each of \( \theta_1, \theta_2, \ldots \), and so we have the product formula

\[ s_{\lambda/\mu} = s_{\theta_1} s_{\theta_2} s_{\theta_3} \cdots. \]

For example, the skew shape \( 65521/322 \) depicted in Figure 1.2 has two connected components, yielding

\[ s_{65521/322} = s_{4331} s_{21}. \]

The following crucial fact is not so obvious from the combinatorial point of view:

**Theorem 3.4:** The Schur functions \( s_{\lambda} \) and \( s_{\lambda/\mu} \) are symmetric functions.

**Proof:** (Bender and Knuth [22])

Since we have

\[ s_{\lambda/\mu}(x) = \sum_{\alpha} K_{\lambda/\mu, \alpha} x^\alpha, \]

it suffices to show that the Kostka number \( K_{\lambda/\mu, \alpha} \) depends only on the multiset \( \{\alpha_1, \alpha_2, \ldots\} \) and not on the ordering of the \( \alpha_i \)'s.
Consider the special case in which \( \alpha = (a, b, 0, 0, \ldots) \); i.e., consider tableaux of shape \( \lambda/\mu \) filled with \( a \) 1's and \( b \) 2's. We want to show that these are equinumerous with the tableaux of shape \( \lambda/\mu \) filled with \( b \) 1's and \( a \) 2's. Clearly, there are no such tableaux in either case unless there are at most two cells in every column of \( \lambda/\mu \). Therefore, let us assume that this is the case.

Call a cell of \( \lambda/\mu \) free if it is the only cell in its column. Notice that a skew tableau of a given shape which only uses 1's and 2's can be uniquely determined by specifying the number of free cells in each row which are to be filled with 1's. Suppose that there are \( f_i \) free cells in the \( i \)th row of \( \lambda/\mu \). Let \( T \) be a tableau of shape \( \lambda/\mu \), and suppose that there are \( a_i \) free cells in the \( i \)th row of \( T \) which are occupied by 1's. Let \( T' \) denote the tableau obtained by assigning 1's to exactly \( f_i - a_i \) of the free cells in the \( i \)th row. It follows that the operation

\[
T \mapsto T'
\]

is an involution with the property that if \( T \) has \( a \) 1's and \( b \) 2's then \( T' \) has \( b \) 1's and \( a \) 2's.

This construction applies more generally in that it allows us to swap \( \alpha_i \) and \( \alpha_{i+1} \). Assume that \( \lambda/\mu \) is an arbitrary skew shape, and let \( T \) be a skew tableau of shape \( \lambda/\mu \) and weight \( \alpha \). Identify the skew shape occupied by the integers \( i \) and \( i + 1 \), and apply the above operation on that shape, with the identification \( 1 \leftrightarrow i, 2 \leftrightarrow i + 1 \). Since the operation is an involution, we deduce that

\[
K_{\lambda/\mu, (\ldots, \alpha_i, \alpha_{i+1}, \ldots)} = K_{\lambda/\mu, (\ldots, \alpha_{i+1}, \alpha_i, \ldots)}.
\]

By repeated application of this identity it follows that \( K_{\lambda/\mu, \alpha} \) does not depend on the order of the \( \alpha_i \)'s.

Corollary 3.5: We have

\[
s_{\lambda/\mu}(x) = \sum_{\nu} K_{\lambda/\mu, \nu} m_{\nu}(x),
\]

where the sum runs through partitions \( \nu \).

Remark 3.6:

(a) The Schur functions are sometimes defined (e.g., [41]) in terms of column strict plane partitions, or c.s.p.p. A c.s.p.p. of shape \( \lambda \) is an assignment

\[
T : D_\lambda \to P
\]

dependent on the order of the \( \lambda \)'s. 

1. The numbers assigned to any row of \( \lambda \) decrease weakly from left to right.
2. The numbers assigned to any column of $\lambda$ decrease strictly from top to bottom.

A skew c.s.p.p. is defined analogously. Informally, c.s.p.p.'s are tableaux with their inequalities reversed. The weight of a c.s.p.p. is defined in the same way as the weight of a tableau.

Note that there is an obvious weight reversing bijection between tableaux and c.s.p.p. of the same shape: if $T$ is a tableau in which the smallest integer used is $r$ and the largest integer used is $s$, replace every occurrence of $i$ in $T$ by $r + s - i$. Thus, in view of Theorem 3.4, we see that if Schur functions were defined in terms of c.s.p.p.'s instead of tableaux, we would still obtain the same formal power series.

(b) For any skew shape $\lambda/\mu$, let $(\lambda/\mu)^\triangleright$ denote the skew shape obtained by rotating the diagram of $\lambda/\mu$ by $180^\circ$. The alert reader will complain that $(\lambda/\mu)^\triangleright$ is not a well-defined diagram, however we will be content to note that $(\lambda/\mu)^\triangleright$ is well-defined up to translation in $\mathbb{Z}^2$. Note that if one rotates a skew tableau by $180^\circ$, the result is a skew c.s.p.p. Therefore, by Remarks 3.6(a) and 3.3(c), we deduce that

$$s_{\lambda/\mu}(x) = s_{(\lambda/\mu)^\triangleright}(x).$$

Now that we know that the Schur functions are indeed symmetric, we may easily deduce

**Theorem 3.7**: The Schur functions $\{s_\lambda : |\lambda| = k\}$ form a $\mathbb{Z}$-basis of $\Lambda^k$.

**Proof**: (essentially Macdonald [32; I.6])

Let $\lambda, \mu$ range over partitions of $k$. Since the symbols $1, 2, \ldots, i$ may only appear in the first $i$ rows of a tableau, it follows that $K_{\lambda, \mu} = 0$ unless $\lambda \geq \mu$ in the dominance order (1.7). Furthermore, if $\lambda = \mu$, we are forced to put only $i$'s in the $i$th row of any tableau of shape $\lambda$, so $K_{\lambda, \lambda} = 1$. Therefore, the matrix $[K_{\lambda, \mu} : |\lambda| = |\mu| = k]$, if its columns are suitably ordered, is upper triangular with 1's on the diagonal. Hence, the linear relations

$$s_\lambda = \sum_{|\mu| = k} K_{\lambda, \mu} m_\mu$$

implied by Corollary 3.5 are invertible and have an integer inverse. Apply Proposition 2.3.

Any description of the combinatorial aspects of Schur functions would be remiss if it did not at least pay homage to a combinatorial correspondence known variously as Schensted's correspondence, Knuth's correspondence, or the
Robinson-Schensted-Knuth correspondence, or.... Generalizations, analogues and applications of this correspondence have become a cottage industry among combinatorists. Our use of the correspondence, however, is rather limited; it will be felt indirectly through the identities in Corollary 3.9 below. For an introduction to the correspondence, the reader is referred to Knuth [22; 5.1.4].

For our purposes, the following will suffice:

Theorem 3.8: Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$ be integer sequences with only finitely many nonzero terms.

(a) The number of nonnegative integer matrices with row sums $\alpha$ and column sums $\beta$ is the same as the number of pairs $(S, T)$ of tableaux of the same shape with weights $w(S) = \alpha$ and $w(T) = \beta$.

(b) The number of 0,1 matrices with row sums $\alpha$ and column sums $\beta$ is the same as the number of pairs $(S, T)$ of tableaux of conjugate shape with weights $w(S) = \alpha$ and $w(T) = \beta$.

Knuth’s correspondence [23] amounts to a combinatorial proof of Theorem 3.8. His construction is a generalization of an algorithm due to Schensted [36], which can be viewed as a combinatorial proof of Theorem 3.8 in the case of permutation matrices and standard tableaux; i.e., $\alpha = \beta = 1^n$. A vague statement of an algorithm essentially equivalent to Schensted’s algorithm was given in an earlier paper by G. de B. Robinson [34].

In Remark 5.10 we show how Theorem 3.8 (but not an actual bijection) can be deduced directly from the Littlewood-Richardson rule. That such a deduction is possible is not surprising, as a number of authors (e.g. White [45], Zelevinsky [49]) have found connections between the two algorithms.

Let $y = (y_1, y_2, \ldots)$ denote a collection of indeterminates in addition to the usual ones $x = (x_1, x_2, \ldots)$.

Corollary 3.9: We have:

(a) \[
\sum_{\lambda} s_\lambda(x)s_\lambda(y) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j}
\]

(b) \[
\sum_{\lambda} s_\lambda(x)s_{\lambda'}(y) = \prod_{i,j \geq 1} (1 + x_i y_j).
\]

Proof: On the right side of (a), the coefficient of $x^\alpha y^\beta$ can be identified with the number of nonnegative integer matrices with row sums $\alpha$ and column sums $\beta$. The coefficient of $x^\alpha y^\beta$ on the left side of (a) can be identified with the number of pairs of tableaux $(S, T)$ of the same shape with weights $w(S) = \alpha$ and $w(T) = \beta$. 
The identity (b) can be proved analogously.

**Corollary 3.10:**

\[ \omega(s_\lambda) = s_{\lambda'}. \]

**Proof:** Let \( \omega_y \) denote the automorphism obtained by letting \( \omega \) act on symmetric formal power series in the indeterminates \( y_1, y_2, \ldots \) with coefficients taken from the ring of formal power series in the indeterminates \( x_1, x_2, \ldots \). From the generating functions (2.2) and (2.3) we find

\[ \omega_y \prod_{i,j \geq 1} \frac{1}{1-x_i y_j} = \prod_{i,j \geq 1} (1 + x_i y_j). \]

If we apply \( \omega_y \) to the identity in Corollary 3.9(a) and compare it with the identity in Corollary 3.9(b), the result follows as the Schur functions are a \( \mathbb{Z} \)-basis of \( \Lambda(x) \) (Theorem 3.7).

We mention that Corollaries 3.9 and 3.10 can also be proved without recourse to the Robinson-Schensted-Knuth correspondence. See [32; I.3,4].

### 4. Decompositions in the rings \( \Lambda, \Lambda_n, \) and \( \Omega_n \)

The family of representation-theoretic problems we will consider can all be expressed in terms of computing the coefficients that occur when one decomposes a symmetric function into a linear combination of Schur functions. However, this decomposition is usually taken in the ring \( \Lambda_n \) or the related ring \( \Omega_n \), which is the quotient

\[ \Omega_n = \Lambda_n / (x_1 \cdots x_n - 1). \] (1)

We therefore assemble in this and the following section some tools from the theory of symmetric functions which will facilitate such decomposition problems.

**Definition 4.1:** Let \( \langle \cdot, \cdot \rangle \) denote the inner product on \( \Lambda \) obtained by insisting that the Schur functions \( s_\lambda \) form an orthonormal basis; i.e.,

\[ \langle s_\lambda, s_\mu \rangle = \delta_{\lambda \mu}. \]

Note that Corollary 3.10 tells us that \( \omega \) is an isometry; i.e., for any \( f, g \in \Lambda \),

\[ \langle \omega f, \omega g \rangle = \langle f, g \rangle. \]

Some authors prefer to define the inner product \( \langle \cdot, \cdot \rangle \) by insisting that the \( h_\lambda \)'s and \( m_\mu \)'s form dual bases; i.e.,

\[ \langle h_\lambda, m_\mu \rangle = \delta_{\lambda \mu}. \] (2)
A proof of the equivalence of these two definitions can be found, for example, in [32; I.4] or [41]. Their equivalence can also be regarded as a restatement of

**Proposition 4.2:** (Kostka's Theorem) We have

\[ h_\lambda = \sum_\mu K_{\mu,\lambda} s_\mu. \]

**Proof:** Recall from Corollary 3.5 that

\[ s_\mu = \sum_\lambda K_{\mu,\lambda} m_\lambda. \]

Therefore, since \( h_\lambda \) and \( m_\mu \) are dual,

\[ \langle h_\lambda, s_\mu \rangle = K_{\mu,\lambda}, \]

which must be the coefficient of \( s_\mu \) in the decomposition of \( h_\lambda \).

We know (Theorem 3.7) that the Schur functions are a \( \mathbb{Z} \)-basis of the ring of symmetric functions \( \Lambda \). We have yet to investigate their behavior in \( \Lambda_n \). Since \( \Lambda_n \) is merely a homomorphic image of \( \Lambda \), it certainly must be the case that \( s_\lambda(x_1, \ldots, x_n) \) will \( \mathbb{Z} \)-span all of \( \Lambda_n \). However, directly from Definition 3.2 it is easy to see that \( s_\lambda(x_1, \ldots, x_n) \) vanishes whenever \( \ell(\lambda) > n \): any tableau of shape \( \lambda \) must use at least \( \ell(\lambda) \) distinct integers. Fortunately, this is the only complication.

**Proposition 4.3:** The Schur functions

\[ \{ s_\lambda(x_1, \ldots, x_n) : |\lambda| = k, \ell(\lambda) \leq n \} \]

form a \( \mathbb{Z} \)-basis of \( \Lambda_n^k \).

**Proof:** By the preceding remarks, we certainly know that the \( \mathbb{Z} \)-span of

\[ \{ s_\lambda(x_1, \ldots, x_n) : |\lambda| = k, \ell(\lambda) \leq n \} \]

exhausts all of \( \Lambda_n^k \). On the other hand, by Proposition 2.3(b), we know that \( \Lambda_n^k \) is a free abelian group on \( \{ \lambda : |\lambda| = k, \ell(\lambda) \leq n \} \) generators.

Thus the inner product \( \langle \cdot, \cdot \rangle \) we defined for \( \Lambda \) will suit us perfectly well for \( \Lambda_n \), provided we exercise care in avoiding partitions of length more than \( n \).

**Definition 4.4:** Let \( f(x_1, x_2, \ldots) \) be a formal power series or formal Laurent series. The notation

\[ [x^a] f(x) \]
denotes the coefficient of the monomial $x^\alpha$ in $f(x)$.

Decomposing a symmetric function into a linear combination of monomial symmetric functions is conceptually quite simple. Given $f \in \Lambda$, we have

$$f = \sum_\lambda c_\lambda m_\lambda$$

where $c_\lambda = [x^\lambda] f(x)$. Fortunately, given a suitably constructed framework, the problem of decomposing a symmetric function into a linear combination of Schur functions can also be considered a coefficient extraction problem in the sense of monomials in formal power series. That framework is provided by the notion of alternating functions.

Recall that a formal power series or formal Laurent series $f(x_1, \ldots, x_n)$ in $n$ indeterminates is said to be alternating if

$$f(x_{w(1)}, \ldots, x_{w(n)}) = \varepsilon_w f(x_1, \ldots, x_n)$$

for all permutations $w \in S_n$, where $\varepsilon_w$ denotes the sign of the permutation $w$. As with symmetric functions, we say that $f$ is an alternating function if it is of bounded degree; i.e., a polynomial.

**Definition 4.5:** Let $\alpha \in \mathbb{N}^n$. The monomial alternating function $a_\alpha(x_1, \ldots, x_n)$ corresponding to $\alpha$ is defined by

$$a_\alpha(x_1, \ldots, x_n) = \sum_{w \in S_n} \varepsilon_w x_{w(1)}^{\alpha_1} \cdots x_{w(n)}^{\alpha_n}.$$  

It is clear that these polynomials actually are alternating functions.

**Remark 4.6:**

(a) Note that $a_\alpha$ vanishes unless the $\alpha_i$'s are distinct. Furthermore, permuting the order of the $\alpha_i$'s will at worst change the sign of $a_\alpha$, so we don't lose anything by presuming

$$\alpha_1 > \alpha_2 > \cdots > \alpha_n \geq 0.$$  

In other words, we may assume that $\alpha$ is a partition of length at most $n$, with distinct parts. Such partitions are of the form $\alpha = \lambda + \delta$, where $\lambda$ is a partition with $\ell(\lambda) \leq n$, and

$$\delta = (n-1, n-2, \ldots, 1, 0).$$  

(b) By analogy with the case of monomial symmetric functions and $\Lambda$, it is easy to see that

$$\{a_{\lambda+\delta}(x_1, \ldots, x_n) : \ell(\lambda) \leq n\}$$
will freely generate all of the alternating functions. In particular, if \( f(x_1, \ldots, x_n) \) is any alternating function, then

\[
[x^{\lambda+\delta}]f(x)
\]
is the coefficient of \( a_{\lambda+\delta} \) in \( f \).

(c) The definition we gave for \( a_\alpha \) can also be expressed as a determinant:

\[
a_\lambda(x_1, \ldots, x_n) = \det[x_i^{a_j}].
\]

In the special case \( a_\delta \), we have the well-known Vandermonde determinant:

\[
a_\delta(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j).
\]  \( \tag{4} \)

Since any alternating function \( f \) must vanish under the substitution \( x_i = x_j \), it follows that \( f \) must be divisible by \( a_\delta \) and the quotient \( f/a_\delta \) is therefore symmetric.

**Theorem 4.7:** If \( \lambda \) is a partition with \( \ell(\lambda) \leq n \), then

\[
s_\lambda(x_1, \ldots, x_n) = \frac{a_{\lambda+\delta}(x_1, \ldots, x_n)}{a_\delta(x_1, \ldots, x_n)}.
\]

This is a remarkable result. More accurately, it is remarkable from the way we have chosen to develop Schur functions. For others, (e.g., Macdonald [32; I.3]), this is not remarkable at all; it is merely the definition of \( s_\lambda \). Indeed, this is the classical definition due to Jacobi, who called them "bilateralternants." A derivation of Theorem 4.7 directly from Definition 3.2 has been given by Stanley [41; sec. 10].

**Remark 4.8:** We have now completed the framework which allows us to view decomposition of symmetric functions as coefficient extraction. Let \( f \in \Lambda_n \); say

\[
f(x_1, \ldots, x_n) = \sum_{\ell(\lambda) \leq n} c_\lambda s_\lambda(x_1, \ldots, x_n).
\]

By Theorem 4.7 and Remark 4.6(b) we may deduce

\[
c_\lambda = [x^{\lambda+\delta}]f(x)a_\delta(x).
\]

Next we turn our attention to the complications that arise when one attempts decompositions in the ring \( \Omega_n \) (see (1)), which we defined to be the ring obtained by viewing \( \Lambda_n \) modulo the relation \( x_1 \cdots x_n = 1 \). In order to emphasize when we
are talking about $\Omega_n$ or $\Lambda_n$, let us introduce the notation $\bar{f}$ to denote the image in $\Omega_n$ of the symmetric function $f \in \Lambda_n$. More generally, if $S \subseteq \Lambda_n$, let $\bar{S}$ denote the image of $S$ in $\Omega_n$.

**Remark 4.9:**

(a) We can give $\Omega_n$ a $\mathbb{Z}/(n)$-grading

$$\Omega_n = \Omega_n^0 \oplus \cdots \oplus \Omega_n^{n-1}, \quad (5)$$

where

$$\Omega_n^k = \prod_{l \equiv k \mod n} \Lambda_l^l.$$  

To say that (5) is a $\mathbb{Z}/(n)$-grading means that

$$\Omega^i_n \cdot \Omega^j_n \subseteq \Omega^{i+j}_n$$

for any $i, j \in \mathbb{Z}/(n)$.

(b) Since $\Omega_n$ is a homomorphic image of $\Lambda_n$, we certainly know that

$$\{\bar{s}_\lambda(x_1, \ldots, x_n) : \ell(\lambda) \leq n\}$$

will $\mathbb{Z}$-span all of $\Omega_n$. However, notice that if $\lambda$ and $\mu$ are partitions which differ only in the number of columns of length $n$, say $\lambda = \mu + l^n$, then we have

$$s_\lambda(x_1, \ldots, x_n) = (x_1 \cdots x_n)^l s_\mu(x_1, \ldots, x_n),$$

since any tableau which uses only the integers $1, \ldots, n$ must fill any column of length $n$ with one each of $1, \ldots, n$. Hence, in this case we have $\bar{s}_\lambda = \bar{s}_\mu$. Fortunately, this is the only complication that arises in passing from $\Lambda_n$ to $\Omega_n$:

**Proposition 4.10:** The Schur functions

$$\{\bar{s}_\lambda(x_1, \ldots, x_n) : |\lambda| \equiv k \mod n, \ell(\lambda) < n\}$$

form a $\mathbb{Z}$-basis of $\Omega_n^k$.

**Proof:** In view of the preceding remarks, we certainly know that the $\mathbb{Z}$-span of the claimed basis exhausts all of $\Omega_n^k$. Suppose that there was a dependence relation among the $\bar{s}_\lambda$'s. This would yield a relation

$$\sum_{\ell(\lambda) < n} c_\lambda \bar{s}_\lambda(x_1, \ldots, x_n) = (x_1 \cdots x_n - 1) \cdot f(x_1, \ldots, x_n) \quad (6)$$
in $\Lambda_n$ for some $f \in \Lambda_n$. Let $\mu$ be a partition with $\ell(\mu) \leq n$ such that $s_\mu(x_1, \ldots, x_n)$ appears in the decomposition of $f$ into Schur functions, with $\mu$ chosen so that $|\mu|$ is maximized. It must be that $s_{\mu+1^n}(x_1, \ldots, x_n)$ appears on the right side of (6), contrary to the fact that only Schur functions $s_\lambda$ with $\ell(\lambda) < n$ appear on the left side of (6). Therefore, $f = 0$ and the only dependence relations are trivial.

**Remark 4.11:**

(a) One immediate consequence of Proposition 4.10 and Remark 4.9 is that for homogeneous $f \in \Lambda_n$, the decomposition of $\bar{f}$ into Schur functions in $\Omega_n$ contains all of the information necessary to recover the decomposition of $f$ in $\Lambda_n$. Specifically, suppose that $f \in \Lambda_n^k$ and that

$$\bar{f}(x_1, \ldots, x_n) = \sum_{\ell(\lambda) < n} c_\lambda \bar{s}_\lambda(x_1, \ldots, x_n)$$

is the decomposition of $\bar{f}$ in $\Omega_n$. The decomposition of $f$ in $\Lambda_n$ must be

$$f(x_1, \ldots, x_n) = \sum_{\ell(\lambda) < n} c_\lambda s_{\lambda+r^n}(x_1, \ldots, x_n)$$

where $r$ is the integer (depending on $\lambda$) such that $|\lambda| + rn = k$.

(b) Proposition 4.10 also tells us that we may index a $\mathbb{Z}$-basis of $\Omega_n$ by partitions $\lambda$ with $\ell(\lambda) < n$. We will be especially interested in decompositions of symmetric functions in $\Omega_n^0$. In this case, the basis is indexed by partitions with $\ell(\lambda) < n$ and $|\lambda|$ divisible by $n$. When $|\lambda| = ln$ we will say that $\lambda$ belongs to the $l$th layer.

There is another indexing scheme for $\Omega_n^0$ which we will use extensively:

**Definition 4.12:** Call any integer sequence $\alpha \in \mathbb{Z}^n$ a dominant weight if $|\alpha| = 0$ and

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n.$$

This terminology is nonstandard. It is derived from the representation theory of semisimple Lie algebras, where dominant weights are used to index irreducible representations. In that context, a dominant weight is a more general object than those we consider here. What we have called a dominant weight corresponds to the dominant weights of the Lie algebra $\mathfrak{sl}_n$ which lie in the root lattice. A detailed discussion of dominant weights is given by Humphreys [20].

The correspondence $\alpha \leftrightarrow \lambda$ between partitions and dominant weights is clear:

$$(\alpha_1, \ldots, \alpha_n) \leftrightarrow (\alpha_1 - \alpha_n, \alpha_2 - \alpha_n, \ldots, 0).$$

The dominant weight $\alpha$ corresponds to a partition in the $(-\alpha_n)$th layer.
It is also possible to describe a dominant weight by a pair of partitions \( \alpha, \beta \) of the same weight with \( \ell(\alpha) + \ell(\beta) \leq n \) as follows. Suppose that \( \ell(\alpha) = r \) and \( \ell(\beta) = s \). The pair \( \alpha, \beta \) correspond to the dominant weight \([\alpha, \beta)_n\) which we define by

\[
[\alpha, \beta)_n = (\alpha_1, \ldots, \alpha_r, 0, \ldots, 0, -\beta_s, \ldots, -\beta_1).
\]

The partition corresponding to \([\alpha, \beta)_n\) belongs to the \((\beta_1)\)th layer.

**Example 4.13:** Let \( n = 7 \) and consider the 3rd layer partition \( \lambda = 853311 \). The dominant weight corresponding to \( \lambda \) is \((5, 2, 0, 0, -2, -2, -3)\). The corresponding pair of partitions \( \alpha, \beta \) is \(52, 322\).

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5. The Littlewood-Richardson rule

The Littlewood-Richardson rule (or LR rule), which is one of the cornerstones of combinatorial representation theory, can be viewed as an elegant solution to the following problems:

1. Decompose the skew Schur function \( s_{\lambda/\mu} \) into a linear combination of Schur functions.

2. Decompose the product \( s_{\lambda}s_{\mu} \) into a linear combination of Schur functions.

First stated by Littlewood and Richardson [29] in 1934, the LR rule was not given a complete, error-free proof until the 1970's. The first proofs were apparently published by Thomas [43] and Lascoux and Schützenberger [38]. Valid proofs have also been given by Macdonald [32] and James and Kerber [21].

In order to state the Littlewood-Richardson rule we need some additional notation and terminology:

**Definition 5.1:** A sequence \( a_1, a_2, \ldots, a_m \) of positive integers is a lattice permutation provided that for all \( j \in \mathbb{P} \) and \( 1 \leq i \leq m \)

\[
(\# \text{ of } j \text{'s among } a_1, \ldots, a_i) \geq (\# \text{ of } (j+1) \text{'s among } a_1, \ldots, a_i).
\]

For example, 1123123213 is a lattice permutation, but 1123123321 is not.
The Theory of Symmetric Functions

Figure 5.1: word(T) = 42115225451.

Definition 5.2: If T is a (possibly skew) tableau, the word of T, or simply word(T), is the sequence obtained by reading the integers assigned to the cells from right to left and top to bottom.

An example is given in Figure 5.1.

Theorem 5.3: (LR rule, first version)

Let λ/μ be a skew shape and let ν be a partition. The inner product 〈s_{λ/μ}, s_ν〉 is the number of skew tableaux T of shape λ/μ such that T is of weight ν and word(T) is a lattice permutation.

Example 5.4: Consider the decomposition of s_{3321/21}. We find

\[ s_{3321/21} = s_{33} + 2s_{321} + s_{311} + s_{22} + s_{221}, \]

corresponding to the tableaux

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 2 & 2 \\
2 & 3 & 3 \\
\end{array}
\quad
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 2 & 2 \\
3 & 3 & 3 \\
\end{array}
\quad
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 3 & 3 \\
2 & 4 & 4 \\
\end{array}
\quad
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 2 \\
2 & 3 & 4 \\
\end{array}
\quad
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & 3 \\
3 & 3 & 4 \\
\end{array}
\]

In practice, the lattice permutation conditions can be unwieldy to apply. However, there is a well-known bijection between lattice permutations and standard tableaux which can be useful. If a = (a_1, \ldots, a_m) is a lattice permutation, then a corresponds to the standard tableau T on a shape with m cells in which the integer i (1 ≤ i ≤ m) occurs in the a_i-th row. That this defines a bijection is easily proved and is left to the reader. For example, if we denote this correspondence by a ↔ T, then we have

\[
121123324 \leftrightarrow
\begin{array}{ccc}
1 & 3 & 4 \\
2 & 5 & 8 \\
6 & 7 & \\
9 & \\
\end{array}
\]
Notice that the length of the $i$th row of $T$ is the number of $i$'s which occur in $a$.

This correspondence allows us to easily restate the LR rule in terms of standard tableaux, rather than lattice permutations. This was observed by White [45], but the actual rules that result were apparently first explicitly stated by Remmel and Whitney [33]. In order to give this reformulation of the LR rule, we need the following

**Definition 5.5:** Let $\lambda/\mu$ be a skew shape. The canonical labelling of $\lambda/\mu$ is obtained by numbering the cells of $\lambda/\mu$ from 1 to $|\lambda/\mu|$, right to left and top to bottom.

An example of the canonical labelling of a skew shape is given in Figure 5.2.

**Theorem 5.6:** (LR rule, second version)

Let $\lambda/\mu$ be a skew shape and let $\nu$ be a partition. The inner product $\langle s_{\lambda/\mu}, s_{\nu} \rangle$ is the number of standard tableaux $T$ of shape $\nu$ such that:

1. If the integers $a$ and $a+1$ occur in the same row of the canonical labelling of $\lambda/\mu$, then $a+1$ occurs in a row at least as high\(^1\) as $a$ in $T$.

2. If the integer $b$ occurs directly below $a$ in the canonical labelling of $\lambda/\mu$, then $b$ occurs in a lower row of $T$ than $a$.

**Proof:** By Theorem 5.3, we know that $\langle s_{\lambda/\mu}, s_{\nu} \rangle$ is the number of skew tableaux $T$ of shape $\lambda/\mu$ and weight $\nu$ such that $\text{word}(T)$ is a lattice permutation. If we apply the above correspondence to the collection of words, we obtain a set of standard tableaux of shape $\nu$. We claim that these tableaux are characterized by rules 1 and 2.

Let $T$ be a skew tableau of shape $\lambda/\mu$ as described in Theorem 5.3, and let $S$ be the standard tableau corresponding to $T$. First observe that the integers assigned to the cells of $\lambda/\mu$ by $T$ are the row assignments in the tableau $S$. The integer which is assigned to a cell of $S$ is the canonical label of the corresponding cell of $\lambda/\mu$.

Let $x$ and $y$ be two adjacent cells of $\lambda/\mu$. If they occur in the same row, their

\(^1\)The first row is considered the highest.
canonical labels may be assumed to be $a$ and $a + 1$, respectively, presuming that $x$ occurs to the right of $y$. The fact that $T$ must be non-decreasing along the rows means that the row assignments of $x$ and $y$ must satisfy rule 1. Suppose that $x$ and $y$ occur in the same column of $\lambda/\mu$ and that their canonical labels are $a$ and $b$, respectively. Presuming that $x$ occurs above $y$, the fact that $T$ must be strictly increasing along the columns means that the row assignments of $x$ and $y$ must satisfy rule 2.

It is easy to see that these steps are reversible: given the standard tableau $S$, one may reconstruct $T$ secure in the knowledge that rules 1 and 2 will guarantee that the result will actually be a skew tableau.

As previously claimed, the LR rule also gives a combinatorial solution to the problem of decomposing $s_\lambda s_\mu$ into Schur functions. In fact, some authors prefer to call this rule the Littlewood-Richardson rule, rather than Theorems 5.3 or 5.6.

**Theorem 5.7:** (LR rule, third version)

Let $\lambda$, $\mu$, $\nu$ be partitions. The inner product $\langle s_\lambda s_\mu, s_\nu \rangle$ will vanish unless $\lambda, \mu \subseteq \nu$. In that case, we have

$$\langle s_\lambda s_\mu, s_\nu \rangle = \langle s_\mu, s_{\nu/\lambda} \rangle;$$

i.e., $\langle s_\lambda s_\mu, s_\nu \rangle$ is the number of skew tableaux $T$ of shape $\nu/\lambda$ and weight $\mu$ such that $\text{word}(T)$ is a lattice permutation.

**Sketch of proof:** (Remmel and Whitney [33])

Let $\theta$ denote a skew shape whose connected components are the diagrams of $\lambda$ and $\mu$. Recall (Remark 3.3(d)) that $s_\theta = s_\lambda s_\mu$, and so the inner product

$$\langle s_\lambda s_\mu, s_\nu \rangle = \langle s_\theta, s_\nu \rangle.$$

may be computed from the previous versions of the LR rule. A simple bijection may be given between the objects enumerated by $\langle s_\theta, s_\nu \rangle$ and the objects enumerated by $\langle s_\mu, s_{\nu/\lambda} \rangle$ using Theorem 5.6.

**Remark 5.8:** If $\lambda$ and $\mu$ are partitions such that $\mu \not\subseteq \lambda$ (so that $\lambda/\mu$ is not a skew shape), it is convenient to formally define $s_{\lambda/\mu} = 0$. With such a convention, Theorem 5.7 may be succinctly stated as

$$\langle s_{\lambda/\mu}, s_\nu \rangle = \langle s_\mu, s_{\nu/\lambda} \rangle \tag{1}$$

for all partitions $\lambda$, $\mu$ and $\nu$. Since the Schur functions are a $\mathbb{Z}$-basis for $\Lambda$, it follows more generally that

$$\langle \bar{f} s_{\lambda}, s_\nu \rangle = \langle \bar{f}, s_{\nu/\lambda} \rangle.$$
for any $f \in \Lambda$. This identity says that 'skewing by $\lambda$' is adjoint to 'multiplication by $s_\lambda$', a fact which was apparently first noticed by Littlewood [28; p.110]. Macdonald [32; I.5] takes the identity (1) as the definition of the skew Schur function $s_{\nu/\lambda}$.

With only a modest amount of additional work, we can extend the Littlewood-Richardson rule to the computation of the inner product of an arbitrary pair of skew Schur functions.

**Corollary 5.9:** (Zelevinsky [49])

Let $\lambda/\mu$ and $\nu/\rho$ be skew shapes. The inner product $\langle s_{\lambda/\mu}, s_{\nu/\rho} \rangle$ is the number of standard skew tableaux $T$ of shape $\nu/\rho$ such that:

1. If $a$ and $a + 1$ occur in the same row of the canonical labelling of $\lambda/\mu$, then $a + 1$ occurs in a row at least as high as $a$ in $T$.
2. If the integer $b$ occurs directly below $a$ in the canonical labelling of $\lambda/\mu$, then $b$ occurs in a lower row of $T$ than $a$.

**Proof:** Zelevinsky's purpose was to give a generalization of the Robinson-Schensted-Knuth correspondence, and to deduce this rule (in particular, the LR rule) as a special case. We are content to deduce this directly from the LR rule.

By Remarks 5.8 and 3.3(d) we know that

$$\langle s_{\lambda/\mu}, s_{\nu/\rho} \rangle = \langle s_{\lambda/\mu}s_{\rho}, s_{\nu} \rangle = \langle s_{\theta}, s_{\nu} \rangle,$$

where $\theta$ denotes a skew shape whose connected components are the diagram of $\rho$ and the connected components of $\lambda/\mu$. The skew shape $\theta = ((\rho+r^*) \cup \lambda)/(r^* \cup \mu)$, which has the diagram of $\rho$ in its northeast corner, achieves this purpose if $r$ and $s$ are chosen sufficiently large.

By Theorem 5.6 (LR rule, second version), we know that $\langle s_{\theta}, s_{\nu} \rangle$ is the number of standard tableaux of shape $\nu$ which satisfy rules 1 and 2. It is easy to see that these rules force the integers $1, \ldots, |\rho|$ to occupy the northwest $\rho$-shaped corner of $\nu$, with the integers assigned consecutively from left to right and top to bottom. For example, if $\rho = 533$, then all of the standard tableaux enumerated by $\langle s_{\theta}, s_{\nu} \rangle$ must contain

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in their northwest corner. If this corner is erased, we see that the remaining $|\lambda/\mu|$ integers are free to be distributed into the skew shape $\nu/\rho$ according to the usual rules.
Remark 5.10: As previously promised, we can show that the existence of the Robinson-Schensted-Knuth correspondence follows from Corollary 5.9.

Let $\alpha \in \mathbb{N}^n$. As we did for partitions in (2.1), extend the definitions of the $h_r$'s and $e_r$'s so that

$$h_\alpha = h_{\alpha_1}h_{\alpha_2}\ldots$$
$$e_\alpha = e_{\alpha_1}e_{\alpha_2}\ldots$$

Note that $h_\alpha = s_\theta$ and $e_\alpha = s_\phi$, where $\theta$ is a skew shape whose connected components are single rows of lengths $\alpha_1, \alpha_2, \ldots$.

Let $\alpha, \beta \in \mathbb{N}^n$ and let $\theta$ and $\phi$ be corresponding skew shapes for which $h_\alpha = s_\theta$ and $h_\beta = s_\phi$. Consider the problem of computing the inner product $\langle h_\alpha, h_\beta \rangle$.

By Kostka's theorem (Proposition 4.2), we know that

$$\langle h_\alpha, h_\beta \rangle = \langle \sum_\lambda K_{\lambda,\alpha} s_\lambda, \sum_\mu K_{\mu,\beta} s_\mu \rangle = \sum_\lambda K_{\lambda,\alpha} K_{\lambda,\beta},$$

which is the number of pairs $(S, T)$ of tableaux of the same shape with $w(S) = \alpha$ and $w(T) = \beta$.

On the other hand, by Corollary 5.9, we know that $\langle h_\alpha, h_\beta \rangle = \langle s_\theta, s_\phi \rangle$ is the number of standard skew tableaux which satisfy rules 1 and 2. Let $T$ be such a tableau, and let $a_{ij}$ be the number of integers from the $i$th row of the canonical labelling of $\phi$ which are in the $j$th row of $T$. Clearly, $[a_{ij}]$ is a nonnegative integer matrix with row sums $\alpha$ and column sums $\beta$. Since the rows of $T$ must be weakly increasing and obey rule 1, there is a unique $T$ corresponding to the matrix $[a_{ij}]$. See Figure 5.3. This proves Theorem 3.8(a).

The existence of the dual correspondence (Theorem 3.8(b)) can be similarly obtained by computing $\langle h_\alpha, e_\beta \rangle$ in two ways.

Identifying $s_\lambda s_\mu$ as a skew Schur function has proved to be a useful trick. In the following, we will exhibit an apparently new way to identify $s_\lambda s_\mu$ as a skew Schur function in the ring $\Lambda_n$, which will give another Littlewood-Richardson-style interpretation to the decomposition coefficients $\langle s_\lambda s_\mu, s_\nu \rangle$. 
Definition 5.11: Let $\lambda$ be a partition of length at most $n$. Let $\tilde{\lambda}$ denote the partition defined via

$$\tilde{\lambda} = (\lambda_1 - \lambda_n, \lambda_1 - \lambda_{n-1}, \ldots, \lambda_1 - \lambda_2, 0, 0, \ldots).$$

Geometrically, $\tilde{\lambda}$ is obtained by removing $\lambda$ from the northwest corner of an $n \times \lambda_1$ rectangle and rotating the result by $180^\circ$. The notation ' ~ ' is perhaps ill-conceived since the operation also depends on the integer $n$. However, the value of $n$ should be clear from the context.

Proposition 5.12: Let $\lambda, \mu, \nu$ be partitions of length at most $n$, and let $k = \lambda_1$. We have

$$\langle s_\lambda s_\mu, s_\nu \rangle = \langle s_{(\mu+k^n)/\tilde{\lambda}}, s_\nu \rangle.$$

Geometrically, the skew shape $(\mu + k^n)/\tilde{\lambda}$ can be obtained by rotating the diagram of $\lambda$ by $180^\circ$ and adjoining the result to the left of the diagram of $\mu$. See Figure 5.4.

Proof: Since the Schur functions $\{s_\nu(x_1, \ldots, x_n) : \ell(\nu) \leq n\}$ form a $\mathbb{Z}$-basis of $\Lambda_n$ (Proposition 4.3), we see that it suffices to show

$$s_\lambda s_\mu(x_1, \ldots, x_n) = s_{(\mu+k^n)/\tilde{\lambda}}(x_1, \ldots, x_n)$$

holds in the ring $\Lambda_n$. To prove this, recall that $s_\lambda$ can be viewed as the generating function for c.s.p.p.'s of shape $\lambda$ (Remark 3.6(a)). Hence, $s_\lambda s_\mu(x_1, \ldots, x_n)$ is the generating function for pairs $(S, T)$, where $S$ is a c.s.p.p. with parts $\leq n$ and shape $\lambda$, and $T$ is a tableau with parts $\leq n$ and shape $\mu$.

Let $S^\nu$ denote the result of rotating $S$ by $180^\circ$, putting the first row of $S$ into the $n$th row of $S^\nu$. Note that $S^\nu$ is a skew tableau of shape $k^n/\tilde{\lambda}$. Adjoin the tableau $S^\nu$ to the left of $T$, and call the result $R$. See Figure 5.4.

We claim that $R$ is a skew tableau, necessarily of shape $(\mu + k^n)/\tilde{\lambda}$. To verify this, we only need to check that weakly increasing rows are maintained across the
junction between $S^\gamma$ and $T$. The smallest integer that can appear in cell $(i, 1)$ of $T$ is $i$, while the largest integer that can appear in cell $(i, 1)$ of $S$ is $n - i + 1$. Since the cells $(i, 1)$ of $\mu$ and $(n - i + 1, 1)$ of $\lambda$ become adjacent in $(\mu + \lambda)/\lambda$, the claim follows.

We may therefore conclude that (2) holds.
Chapter II.

The Theory of Characters of $GL_n$ and $SL_n$

6. Group representations

Let $V$ be a finite dimensional complex vector space. Recall that the general linear group $GL(V)$ is the group of all invertible linear transformations $X : V \to V$. If $V$ is $n$-dimensional, one may choose a basis of $V$ and identify $GL(V)$ as the group $GL_n = GL(n, \mathbb{C})$ of all nonsingular $n \times n$ matrices.

The special linear group $SL(V)$ is the subgroup of linear transformations of determinant 1. Again, we may choose a basis of $V$ and identify $SL(V)$ as the group $SL_n = SL(n, \mathbb{C})$ of $n \times n$ matrices of determinant 1.

Definition 6.1: Let $G$ be a group. A (linear, complex, finite dimensional) representation of $G$ is a group homomorphism

$$\rho : G \to GL(V).$$

If $V$ is $m$-dimensional, $\rho$ is said to be of degree $m$.

For our purposes $V$ will always be a finite dimensional complex vector space, while the group $G$ will usually be $GL_n$ or $SL_n$. However, we will also need to consider the representations of other groups, such as the symmetric group $S_n$.

The study of representations of $G$ is equivalent to the study of $G$-modules, which are linear actions of the group $G$ on a vector space $V$. More precisely, a $G$-module is a vector space $V$ endowed with an operation $G \times V \to V$, which we may write

$$(g, v) \mapsto g \circ v,$$

such that:

1. The operation is linear in $V$.
2. $g \circ (h \circ v) = (gh) \circ v$ for all $g, h \in G, v \in V$. 

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3. \(1 \circ v = v\) for all \(v \in V\).

We have used '1' to denote the identity element of the group \(G\).

The equivalence of \(G\)-modules and representations is clear. We will henceforth use the words "representation" and "\(G\)-module" interchangeably as the occasion demands.

**Definition 6.2:** Two representations \(\rho_1 : G \to GL(V)\) and \(\rho_2 : G \to GL(W)\) of \(G\) are said to be equivalent if there is a linear isomorphism \(S : V \to W\) such that:

\[ S \rho_1(g) S^{-1} = \rho_2(g) \]

for all \(g \in G\).

In other words, \(\rho_1\) and \(\rho_2\) are equivalent if they differ only by a change of basis.

Let \(V\) be a \(G\)-module. If \(W \subseteq V\) is a subspace on which the action of \(G\) is invariant (i.e., \(G \circ W \subseteq W\)), then \(W\) is said to be a \(G\)-submodule of \(V\). If the only submodules of \(V\) are the trivial ones; namely, \(W = \{0\}\) and \(W = V\), then \(V\) is said to be irreducible; otherwise \(V\) is reducible. By convention, the trivial \(G\)-module \(\{0\}\) is not considered to be irreducible.

If \(V\) and \(W\) are \(G\)-modules, it is easy to see that the vector space \(V \oplus W\) can be endowed with a \(G\)-module structure in a natural way:

\[ g \circ (v, w) = (g \circ v, g \circ w). \]

If \(\rho_1 : G \to GL(V)\) and \(\rho_2 : G \to GL(W)\) are the representations corresponding to \(V\) and \(W\), let \(\rho_1 \oplus \rho_2 : G \to GL(V \oplus W)\) denote the representation corresponding to \(V \oplus W\).

If the \(G\)-module \(V\) has a decomposition

\[ V = V_1 \oplus \cdots \oplus V_m, \]

where each \(V_i\) is an irreducible \(G\)-module, then \(V\) is said to be completely reducible. In particular, any irreducible \(G\)-module is considered to be completely reducible. By convention, \(\{0\}\) is considered to be completely reducible. Some authors prefer to define \(V\) to be completely reducible if each \(G\)-submodule \(W\) of \(V\) possesses an invariant complement; i.e., a \(G\)-submodule \(W'\) of \(V\) such that \(V = W \oplus W'\). A proof of the equivalence of these two definitions can be found in [3; III.1].

The following key result, which is basically the Jordan-Hölder theorem, says that there is essentially only one way to decompose a completely reducible representation. A proof can be found in [16; Chp. 8].
Theorem 6.3: The decomposition $V = V_1 \oplus \cdots \oplus V_m$ of a completely reducible $G$-module $V$ into irreducibles is unique up to order and equivalence of the factors.

Unfortunately, $G$-modules are not always completely reducible. However, two fundamental results which hold for complex representations are the following:

Theorem 6.4:

(a) Representations of finite groups are completely reducible.

(b) Continuous representations of compact Lie groups are completely reducible.

A proof of Theorem 6.4(a) can be found in [16; Chp. 16]. A proof of Theorem 6.4(b) can be found in [5; Chp. 6].

Theorems on complete reducibility are extremely important. Once a class of representations is known to be completely reducible, then it suffices to determine all of the irreducible representations, since all others are merely direct sums of these. Once the irreducible representations have been identified, then the game of representation theory turns to the study of how particular representations decompose into irreducibles.

A vital technique for solving decomposition problems is provided by the notion of the character of a representation.

Definition 6.5: Let $\rho : G \to GL(V)$ be a representation of the group $G$. The character of $\rho$ is the map $\chi : G \to \mathbb{C}$ defined by

$$\chi(g) = \text{tr} \rho(g).$$

A character is said to be reducible or irreducible according to the nature of its corresponding representation.

The following facts are immediate consequences of this definition, but are nonetheless worth mentioning:

Proposition 6.6:

(a) Equivalent representations have the same character.

(b) Characters are class functions; i.e., $\chi(g)$ depends only on the conjugacy class of $g$.

(c) Let $\rho_1$ and $\rho_2$ be representations of $G$ and let $\chi_1$ and $\chi_2$ be their corresponding characters. The character of $\rho_1 \oplus \rho_2$ is $\chi_1 + \chi_2$.

The converse of Proposition 6.6(a) is not always valid; it is possible to have two inequivalent representations with the same character. For example, the representations of $Z$ defined by

$$i \mapsto \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad i \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
have the same character but are clearly inequivalent. However, under suitable restrictions, characters do determine representations up to equivalence. A very general result in this direction can be deduced from the following:

**Theorem 6.7:** (Frobenius and Schur [9])

Let \( \rho_1, \ldots, \rho_m \) be irreducible, inequivalent representations of a group \( G \). Choose bases for each \( \rho_k \), and let \( \rho_{ij}^k : G \to \mathbb{C} \) denote the \( ij \)-entry of the matrix function \( \rho_k \). The entire collection of matrix entries

\[
\{ \rho_{ij}^k : 1 \leq k \leq m, 1 \leq i, j \leq \deg \rho_k \}
\]

is a linearly independent set.

**Corollary 6.8:** The completely reducible representations of a group \( G \) are determined up to equivalence by their character.

**Proof of Corollary:** If there were inequivalent, completely reducible \( G \)-modules with the same character, then by Proposition 6.6(c), their decompositions into irreducible representations would yield a nontrivial dependence relation among the corresponding irreducible characters. Since a character is a sum of matrix entries, this would contradict Theorem 6.7.

In the following we will give an elementary derivation of Theorem 6.7 which has the additional virtue that it illustrates some of the standard techniques in the theory of group representations.

**Lemma 6.9:** (Schur's Lemma [16; Chp. 16], [3; Chp. I])

Let \( V \) and \( W \) be irreducible \( G \)-modules, and let \( \phi : V \to W \) be a \( G \)-module homomorphism. Either \( \phi = 0 \), or \( V \) and \( W \) are equivalent and \( \phi \) is an isomorphism. Furthermore, the only \( G \)-module homomorphisms \( V \to V \) are scalar multiples of the identity.

**Proof:** The kernel of \( \phi \) would be a \( G \)-submodule of \( V \), and the image of \( V \) under \( \phi \) would be a \( G \)-submodule of \( W \). If \( \phi \neq 0 \), the irreducibility of \( V \) thus forces \( \phi \) to be injective and the irreducibility of \( W \) thus forces \( \phi \) to be surjective.

Let \( 1 \) denote the identity map on \( V \), and consider the case \( \phi : V \to V \). Let \( c \in \mathbb{C} \) be an eigenvalue of the linear transformation \( \phi \). Note that \( \phi - c 1 \) is also a \( G \)-module homomorphism of \( V \), and that it has a nontrivial kernel. By the irreducibility of \( V \), it must be the case that \( \phi = c 1 \).

Let \( G \) be an arbitrary group. Recall that the group algebra \( \mathbb{C}G \) is a complex algebra which, as a vector space, has a basis in one-to-one correspondence with the group \( G \), and has a multiplicative structure inherited in the obvious way from the multiplicative structure of \( G \). Recall that the algebra \( \text{End}(V) = \text{End}_\mathbb{C}(V) \) consists of the linear transformations \( V \to V \) of a vector space \( V \). It is easy
to see that a representation \( \rho : G \to GL(V) \) induces an algebra homomorphism \( CG \to \text{End}(V) \). Equivalently, a \( G \)-module structure on \( V \) induces a \( CG \)-module structure on \( V \).

The following result is a straightforward consequence of the theory of associative algebras (cf. the “Density Theorem” [25; Chp. 17.§3]), but an elementary proof may also be given:

**Lemma 6.10:** (Burnside’s Theorem)

Let \( V \) be an irreducible \( G \)-module. The homomorphism \( CG \to \text{End}(V) \) induced by \( V \) is surjective.

**Proof:** For any \( a \in CG \) and \( v \in V \), let \( a \circ v \) denote the action induced by the action of \( G \) on \( V \). Let \( W \) be a subspace of \( V \) and \( v \in V \setminus W \). We claim that there exists \( a \in CG \) such that \( a \circ W = (0) \) and \( a \circ v \neq 0 \). The proof proceeds by induction on \( \dim W \).

Suppose that \( \dim W = 0 \). Since the identity of \( G \) must act as the identity on \( V \), the choice \( a = 1 \) will suffice to prove the claim for any nonzero \( v \in V \).

Suppose that \( \dim W > 0 \). Let \( U \) be any maximal proper subspace of \( W \), and choose \( w \in W \setminus U \). Thus, we have \( W = U \oplus Cw \). Notice that the annihilator \( A(U) = \{ a \in CG : a \circ U = (0) \} \) is a left ideal of \( CG \), so \( A(U) \circ w \) is a \( G \)-submodule of \( V \). By the induction hypothesis, there exists \( a \in A(U) \) such that \( a \circ w \neq 0 \), so \( A(U) \circ w \) is a nonzero submodule of \( V \). Since \( V \) is assumed to be irreducible, we must have \( V = A(U) \circ w \).

Suppose that there is no suitable \( a \in CG \) such that \( a \circ W = (0) \) and \( a \circ v \neq 0 \); i.e., suppose that \( a \circ W = (0) \Rightarrow a \circ v = 0 \). Define a map \( \phi : V \to V \) by insisting that \( \phi(a \circ w) = a \circ v \) for all \( a \in A(U) \). This is well-defined since \( W = U \oplus Cw \) implies

\[
a \circ w = a' \circ w \Rightarrow (a - a') \circ w = 0 \Rightarrow (a - a') \circ W = 0 \Rightarrow (a - a') \circ v = 0.
\]

In fact, it is easy to see that \( \phi \) is a \( G \)-module homomorphism. Note that we must have \( \phi(w) - v \in U \), since \( \phi(w) - v \notin U \) and the induction hypothesis would imply that there exists \( a \in A(U) \) such that \( a \circ (\phi(w) - v) \neq 0 \), contradicting the fact that \( a \circ v = \phi(a \circ w) = a \circ \phi(w) \). On the other hand, by Schur’s Lemma, \( \phi \) must act on \( V \) as a scalar multiple of the identity, so the fact that \( \phi(w) - v \in U \) would imply \( v \in W \), which is also a contradiction. This completes the induction and proves our claim.

To prove that the homomorphism \( CG \to \text{End}(V) \) is surjective, let \( v_1, \ldots, v_r \) be a basis of \( V \), and choose arbitrary \( u_1, \ldots, u_r \in V \). By the above claim, there exists \( a_i \in CG \) such that \( a_i \circ v_j = 0 \) if and only if \( i \neq j \). Since \( V \) is irreducible, it follows that \( CG \circ v = V \) for any nonzero \( v \in V \). Thus, there exists \( b_i \in CG \) such that \( b_i a_i \circ v_i = u_i \). Therefore, the action of \( b_1 a_1 + \cdots + b_r a_r \) on \( V \) is the linear transformation which maps \( v_i \to u_i \). \( \bullet \)
Proof of Theorem 6.7: The following argument was suggested by M. Artin.

Let \( \rho_k : G \rightarrow GL(V_k) \) \((1 \leq k \leq m)\) be irreducible, inequivalent representations of \( G \), and let \( I_k \) be the kernels of the induced homomorphisms \( CG \rightarrow \text{End}(V_k) \). By Lemma 6.10, we know that these homomorphisms are surjective, and so \( \text{End}(V_k) \cong CG/I_k \). Observe that each kernel \( I_k \) is a maximal two-sided ideal, since any ideal \( J \supset I_k \) would induce an ideal \( J/I_k \) in the endomorphism ring \( \text{End}(V_k) \), which has no nontrivial two-sided ideals. Since the representations \( \rho_k \) are assumed to be inequivalent, it follows that \( I_k + I_l = CG \) whenever \( k \neq l \).

By the Chinese Remainder Theorem [25; Chp. 2.§2], we may deduce that the natural homomorphism

\[
CG/I_1 \cap \cdots \cap I_m \rightarrow CG/I_1 \times \cdots \times CG/I_m
\]

is an isomorphism. Thus, the image of the natural homomorphism

\[
CG \rightarrow \text{End}(V_1 \oplus \cdots \oplus V_m)
\]

is \( \text{End}(V_1) \oplus \cdots \oplus \text{End}(V_m) \), which is of dimension \( \Sigma (\deg \rho_k)^2 \). If there was a dependence relation among the matrix entries \( \rho_{ij}^k \), this maximum possible dimension could not be attained. ●

Let \( V \) be a completely reducible \( G \)-module. Suppose that

\[
V = V_1 \oplus \cdots \oplus V_m
\]

is a decomposition into irreducible representations, and let \( \chi \) be an irreducible character of \( G \). In general, there may be several factors \( V_i \) whose character is \( \chi \). The \( \chi \)th isotypic component of \( V \) is the \( G \)-submodule \( W \) defined by

\[
W = \bigsqcup V_i : V_i \text{ having character } \chi.
\]

A priori, it would seem that the isotypic components of \( V \) should depend on the decomposition (1) that is chosen, even though the number of summands in (2) is invariant. However,

Proposition 6.11: The isotypic components of a completely reducible representation are independent of the decomposition (1).

Although this fact is well-known and not difficult to prove, some of the standard studies of group representations (e.g., [16], [3]) seem to neglect this observation. It can be proved as a straightforward application of Schur's Lemma:

Proof: Let \( \chi \) be an irreducible character of \( G \) and suppose that

\[
V = V_1 \oplus \cdots \oplus V_m = W_1 \oplus \cdots \oplus W_m
\]
are two decompositions of a $G$-module $V$ into irreducible representations. Let $V_X$ and $W_X$ denote the isotypic components obtained from these decompositions. For each $v_i \in V_i$ there is a unique decomposition

$$v_i = w_1 + \cdots + w_m ; \ w_j \in W_j.$$  

Let $\phi_{ij} : V_i \to W_j$ denote the $G$-module homomorphism defined by

$$\phi_{ij}(v_i) = w_j.$$  

By Schur's lemma, we see that if $V_i$ is a summand of $V_X$ then $\phi_{ij} = 0$ unless $W_j$ is a summand of $W_X$. Therefore, we must have $V_X \subseteq W_X$. Analogously, we must have $W_X \subseteq V_X$, and so the result follows. 

**Remark 6.12:**

(a) To describe a (completely reducible) representation $\rho$ of a group $G$ up to equivalence, it suffices to determine the multiplicity of each irreducible representation of $G$ in $\rho$, by Theorem 6.3. In view of Corollary 6.8, we see that this is equivalent to the problem of determining the unique integers $a_i \geq 0$ such that

$$\chi = \sum_i a_i \chi_i,$$

where $\chi_i$ runs through the distinct irreducible characters of $G$. This is usually a much simpler problem, and as we shall see, sometimes amenable to the methods of combinatorics.

(b) By Proposition 6.6(c) we know that sums of characters are characters. It can be convenient to consider differences of characters too. Such objects are called virtual characters and thus form an abelian group.

(c) The tensor product of representations demonstrates that the (pointwise) product of two characters is also a character:

Let $V$ and $W$ be $G$-modules, and let $V \otimes W$ denote the tensor product over the vector spaces $V$ and $W$. Recall that if $v_1, \ldots, v_r$ is a basis of $V$ and $w_1, \ldots, w_s$ is a basis of $W$, then $\{ v_i \otimes w_j : 1 \leq i \leq r, 1 \leq j \leq s \}$ is a basis of $V \otimes W$. Also recall that the tensor '$\otimes$' is linear in both $V$ and $W$. One gives $V \otimes W$ a $G$-module structure via the action

$$g \circ (v \otimes w) = (g \circ v) \otimes (g \circ w),$$

extended to all of $V \otimes W$ by linearity. An easy calculation will convince the reader that if $\chi_1$ and $\chi_2$ are the characters of $V$ and $W$, then $\chi_1 \chi_2$ is the character of $V \otimes W$. 

7. Representations of $GL_n$ and $SL_n$

Now we turn from the general to the particular; our goal in this section is to describe those aspects of the representations of $GL_n$ and $SL_n$ which are especially combinatorial. The reader may consult the article by Stanley [39] for more details.

**Representations of $GL_n$**

Not all representations of $GL_n$ are completely reducible. Indeed, if the representation of $GL_n$ of degree 2 defined by

$$X \mapsto \begin{bmatrix} 1 & \log |\det X| \\ 0 & 1 \end{bmatrix}$$

was completely reducible, it would have to be equivalent to the identity representation ($X \mapsto 1$) of degree 2, which it clearly is not.

If we restrict our attention to rational representations, the situation becomes much more pleasant.

**Definition 7.1:** A representation $\rho : GL_n \rightarrow GL(V)$ is said to be rational if, after choosing a basis for $V$, the matrix entries of $\rho(X)$ are rational functions of the entries of $X \in GL_n$. If these rational functions are polynomials, then $\rho$ is said to be a polynomial representation of $GL_n$.

All representations of $GL_n$ that we will consider will be rational.

**Example 7.2:** The following are some simple examples of rational representations of $GL_n$. More substantial examples will be encountered later.

(a) $X \mapsto X$

(b) $X \mapsto (X^t)^{-1}$

(c) $X \mapsto \det X$

Example (a) is called the defining representation. The $GL_n$-module corresponding to it is the $n$-dimensional vector space on which the $n \times n$ matrices of $GL_n$ act; we will refer to this action as the natural action of $GL_n$.

The rational representations of $GL_n$ have a beautiful theory which was originally developed by Schur [37] in his dissertation. Indeed, all of the theorems in this section are either due to Schur or can be deduced easily from his work.

**Theorem 7.3:** Rational representations of $GL_n$ are completely reducible. Moreover, (cf. Corollary 6.8) the rational representations of $GL_n$ are determined up to equivalence by their character.
Given sufficient background in the theory of Lie groups, one could derive Theorem 7.3 directly from Theorem 6.4(b). A direct proof can be found, for example, in [28; Chp. 10].

Armed with this result, we may safely restrict our attention to the characters of \( GL_n \), secure in the knowledge that any decomposition of a character into irreducible characters will lift up to a corresponding implication about representations.

Let \( \rho : GL_n \to GL(V) \) be a (rational) representation, and let \( \chi \) be its character. Let \( X \) be a diagonalizable matrix with eigenvalues \( x_1, \ldots, x_n \). Since \( \chi \) is a class function, it follows that \( \chi(X) \) depends only on \( x_1, \ldots, x_n \). In fact, since \( \rho \) must be continuous, and diagonalizable matrices are dense in \( GL_n \), it follows that \( \chi(X) \) only depends on the eigenvalues of \( X \) even if \( X \) is not diagonalizable. Thus, we are led to think of \( \chi \) as a function of \( n \) variables.

Define \( \chi(x_1, \ldots, x_n) \) to be the value of \( \chi \) at any matrix whose eigenvalues are \( x_1, \ldots, x_n \); e.g., \( X = \text{diag}(x_1, \ldots, x_n) \). Since permuting the \( x_i \)'s will not change the conjugacy class of \( X \), we see that \( \chi \) is a symmetric rational function of \( x_1, \ldots, x_n \).

A more refined statement about characters of \( GL_n \) is possible:

**Theorem 7.4:** Let \( \rho \) be a rational representation of \( GL_n \). There exists a multiset \( M \) of \( \mathbb{Z}^n \) with the property that if the eigenvalues of \( X \in GL_n \) are \( x_1, \ldots, x_n \), then the eigenvalues of \( \rho(X) \) are the Laurent monomials \( x^\alpha : \alpha \in M \).

**Example 7.5:** The characters of the representations in Example 7.2 are easy to compute:

<table>
<thead>
<tr>
<th>representation</th>
<th>character</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) ( X \mapsto X )</td>
<td>( \chi(x_1, \ldots, x_n) = x_1 + \cdots + x_n )</td>
</tr>
<tr>
<td>(b) ( X \mapsto (X^t)^{-1} )</td>
<td>( \chi(x_1, \ldots, x_n) = x_1^{-1} + \cdots + x_n^{-1} )</td>
</tr>
<tr>
<td>(c) ( X \mapsto \det X )</td>
<td>( \chi(x_1, \ldots, x_n) = x_1 \cdots x_n )</td>
</tr>
</tbody>
</table>

**Remark 7.6:**

(a) *A priori*, we only know that the character \( \chi \) of a rational representation \( \rho \) is a symmetric rational function with (possibly) complex coefficients. Theorem 7.4 tells us that in fact, \( \chi \) is a symmetric function in the polynomial ring \( \mathbb{Z}[x_1^{\pm1}, \ldots, x_n^{\pm1}] \). Furthermore, if \( \rho \) is a polynomial representation, then \( \chi \) is a member of \( \Lambda_n \).

(b) Let \( r \) be an integer. If \( \rho : GL_n \to GL(V) \) is a representation, then so is the map \( \rho^r : GL_n \to GL(V) \) which is defined via

\[
\rho^r(X) = (\det X)^r \rho(X).
\]
It is clear that $\rho^r$ is irreducible if and only if $\rho$ is irreducible. If $\chi$ is the character of $\rho$, then the character $\chi^r$ of $\rho^r$ is easily seen to be

$$\chi^r(x_1, \ldots, x_n) = (x_1 \cdots x_n)^r \chi(x_1, \ldots, x_n).$$

In view of Theorem 7.4, we see that if $r$ is sufficiently large, then $\chi^r$ is a polynomial character. Thus, it suffices to determine the irreducible polynomial characters.

This brings us to the key result which connects the representation theory of $GL_n$ to combinatorics:

**Theorem 7.7:** The characters of the irreducible polynomial representations of $GL_n$ are the Schur functions $\{s_\lambda(x_1, \ldots, x_n) : \ell(\lambda) \leq n\}$.

For any partition $\lambda$ of length at most $n$, let $V_\lambda$ denote a $GL_n$-module whose character is $s_\lambda(x_1, \ldots, x_n)$, and let $\rho_\lambda : GL_n \to GL(V_\lambda)$ denote the associated representation. Let $V_\lambda^r$ denote the $GL_n$-module associated with the representation $\rho_\lambda^r$. Theorem 7.7 implies that every irreducible polynomial representation of $GL_n$ is equivalent to $V_\lambda$ for some partition $\lambda$ with $\ell(\lambda) \leq n$. An explicit description of these irreducible representations can be found, e.g., in [28; 10.1].

**Remark 7.8:**

(a) Theorem 7.7 is quite powerful in that it allows us to apply all of the machinery developed in Chapter I to these decomposition problems. We see that the multiplicity of $V_\lambda$ in a polynomial $GL_n$-module $V$ is the coefficient of $s_\lambda(x_1, \ldots, x_n)$ in the character of $V$.

(b) We may dispense with the rational representations of $GL_n$ without any additional work. In view of Remark 7.6(b), we know that the representations $\rho_\lambda^r$ must exhaust all of the rational representations of $GL_n$. However, they are not all inequivalent, since their characters

$$(x_1 \cdots x_n)^r s_\lambda(x_1, \ldots, x_n)$$

are not all distinct. See Remark 4.9(b). By Proposition 4.10, we see that this is remedied by restricting the length of $\lambda$ so that $\ell(\lambda) < n$.

Hence, to compute the multiplicities of the irreducible $GL_n$-modules $V_\lambda^r$ in a given rational representation with character $\chi$, it suffices to find the unique integers $a_\lambda^r \geq 0$ such that:

$$\chi(x_1, \ldots, x_n) = \sum_{r \in \mathbb{Z}, \ell(\lambda) < n} a_\lambda^r \cdot (x_1 \cdots x_n)^r s_\lambda(x_1, \ldots, x_n).$$
(c) The irreducible $GL_n$-modules corresponding to a diagram with exactly one row of $k$ cells can be given a fairly simple description. Consider the vector space

$$V_k = \{ p \in C[z_1, \ldots, z_n] : p \text{ homogeneous of degree } k \}.$$ 

The group $GL_n$ acts on $V_k$ in a straightforward way. Take $z_1, \ldots, z_n$ to be a basis for the natural action of $GL_n$. The action of $X \in GL_n$ on a polynomial $p \in V_k$ is given by

$$X \circ p(z_1, \ldots, z_n) = p(Xz_1, \ldots, Xz_n),$$

which is clearly a polynomial action. Notice that if $X = \text{diag}(x_1, \ldots, x_n)$ and we consider a monomial $z^\alpha$, then

$$X \circ z^\alpha = x^\alpha z^\alpha.$$ 

Hence, the character of $V_k$ is the sum of all monomials of degree $k$, which is the $k$th complete homogeneous symmetric function $h_k$ (Definition 2.4). Since $h_k = s_k$, we see that we have indeed constructed the $GL_n$-module corresponding to the partition $(k)$.

## Representations of $SL_n$

Now we describe the modifications which can be made for the group $SL_n$.

Any representation of $GL_n$ yields a representation of $SL_n$ upon restriction. The converse is also true. To see this, let $\rho : SL_n \to GL(V)$ be a representation of $SL_n$ and define

$$\rho^*(X) = \rho \left( \frac{1}{\det X} X \right)$$

for $X \in GL_n$. It is easy to see that $\rho^*$ is a representation of $GL_n$ for which $\rho$ is the restriction to $SL_n$. Notice that $\rho$ is rational if and only if $\rho^*$ is rational, and $\rho$ is irreducible if and only if $\rho^*$ is irreducible.

Hence, the irreducible rational representations of $SL_n$ are just the restrictions of the representations $\rho^*_\lambda$. However, it need not be the case that these are all inequivalent as representations of $SL_n$. Observe that the character $\chi$ of a rational representation must still be a symmetric rational function of $x_1, \ldots, x_n$, but the domain of $\chi$ is now restricted to eigenvalues for which $x_1 \cdots x_n = 1$. In view of Remarks 7.8(b) and 4.9(b) we see that the character of a $SL_n$-module should be regarded as an element of $\Omega_n$. By Proposition 4.10, it follows that the character of $V_\lambda$ is $s_\lambda(x_1, \ldots, x_n)$, and that $V_\lambda$ and $V_\mu$ are equivalent as $SL_n$-modules if and only if $\lambda$ and $\mu$ differ only in the number of columns of length $n$.

We summarize the preceding discussion by the following

**Theorem 7.9:**

(a) Rational representations of $SL_n$ are completely reducible. Hence, any rational representation of $SL_n$ is completely determined by its character.
(b) A complete set of inequivalent, irreducible rational $SL_n$-modules is given by \( \{ V_\lambda : \ell(\lambda) < n \} \). Therefore, rational representations of $SL_n$ are polynomial representations.

(c) The characters of rational representations of $SL_n$ belong to the ring $\Omega_n$. In particular, the character of $V_\lambda$ is $\bar{s}_\lambda(x_1, \ldots, x_n)$.

Thus, we see that finding the multiplicities of the irreducible representations in a rational representation of $SL_n$ is equivalent to decomposing a symmetric function in $\Omega_n$ into Schur functions.

We illustrate the methods of this chapter with two results which will be of use in later sections.

Let $\rho : G \to GL(V)$ be a representation of an arbitrary group $G$. Let $\hat{\rho} : G \to GL(V)$ denote the representation defined via

$$\hat{\rho}(g) = \rho(g^{-1})^t.$$ 

Note that $\hat{\rho}$ is irreducible if and only if $\rho$ is irreducible.

In the case of (rational) representations of $GL_n$, we know (Theorem 7.4) that there is a multisubset $M$ of $\mathbb{Z}^n$ such that if $X$ has eigenvalues $x_1, \ldots, x_n$, then $\rho(X)$ has eigenvalues $x^\alpha : \alpha \in M$. Hence, $\hat{\rho}$ has eigenvalues $x^{-\alpha} : \alpha \in M$, so if the character of $\rho$ is $\chi$, then the character of $\hat{\rho}$ is $\chi(x_1^{-1}, \ldots, x_n^{-1})$.

Since $\rho_\lambda$ is irreducible, then so must be $\hat{\rho}_\lambda$. By Remark 7.8(b), we deduce that $\hat{\rho}_\lambda$ must be of the form $\rho_\mu^r$ for some integer $r$ and partition $\mu$. In other words, it must be the case that $\bar{s}_\lambda(x_1^{-1}, \ldots, x_n^{-1})$ is a Schur function $\bar{s}_\mu(x_1, \ldots, x_n)$. That such a relation exists among Schur functions is well-known and not very difficult to prove:

**Proposition 7.10:** Let $\lambda$ be a partition of length at most $n$. We have

$$s_\lambda(x_1^{-1}, \ldots, x_n^{-1}) = (x_1 \cdots x_n)^{\lambda_1} s_{\bar{\lambda}}(x_1, \ldots, x_n),$$

where $\bar{\lambda}$ is the partition described in Definition 5.11.

**Proof:** (Stanley [39])

Let $T$ be an arbitrary tableau with parts $\leq n$ and shape $\lambda$. Regard $T$ as a partial filling of the rectangular diagram of size $n \times \lambda_1$. The $i$th column contains some subset $C_i$ of the integers $[n]$. Fill out the $i$th column by adding the remaining integers $[n] \setminus C_i$, sorted into decreasing order from top to bottom. Let $\hat{T}$ denote the tableau of shape $\bar{\lambda}$ obtained by removing $T$ from this filled
rectangle and rotating the remainder by 180°. See Figure 7.1. Notice that \( T \mapsto \hat{T} \) is an involution, and that under the substitution \( x_i \to x_i^{-1} \), we have
\[
w(T) \to (x_1 \cdots x_n)^{\lambda_1}w(\hat{T}).
\]

The adjoint representation of a Lie group is basic to the study of Lie groups and their algebras; it is an action of a Lie group on its Lie algebra. The Lie algebra of \( GL_n \) is denoted by \( gl_n = gl(n, \mathbb{C}) \) and is called the general linear Lie algebra. The Lie algebra of \( SL_n \) is denoted by \( sl_n = sl(n, \mathbb{C}) \) and is called the special linear Lie algebra.

As vector spaces, \( gl_n \) consists of all \( n \times n \) complex matrices, and \( sl_n \) consists of all \( n \times n \) matrices of trace 0. The Lie algebras \( gl_n \) and \( sl_n \) are endowed with \( GL_n \)- and \( SL_n \)-module structures via the adjoint representation. The adjoint action of \( X \in GL_n \) or \( SL_n \) on a matrix \( A \in gl_n \) or \( sl_n \) is
\[
X \circ A = XAX^{-1},
\]
which is clearly rational. Strictly speaking, the adjoint representation of \( GL_n \) is the adjoint action on \( gl_n \), while the adjoint representation of \( SL_n \) is the adjoint action on \( sl_n \). However, we will find it more convenient to abuse the language and refer to any of the four possible actions we have defined as an adjoint action.

To compute the characters of these actions, we introduce the standard basis of \( gl_n \); namely, \( \{e_{ij} : 1 \leq i, j \leq n\} \), where \( e_{ij} \) is the matrix with a 1 in the \( ij \)-position and 0's elsewhere. Let \( X = \text{diag}(x_1, \ldots, x_n) \). Observe that
\[
X \circ e_{ij} = (x_i x_j^{-1})e_{ij},
\]
so that \( e_{ij} \) is an eigenvector for this action with eigenvalue \( x_i x_j^{-1} \). Therefore, the character of the action of \( GL_n \) on \( gl_n \) is
\[
\sum_{1 \leq i,j \leq n} x_i x_j^{-1}, \tag{1}
\]
and the character of the action of $GL_n$ on $sl_n$ is

$$ (n - 1) + \sum_{1 \leq i \neq j \leq n} x_i x_j^{-1}. \quad (2) $$

Proceeding as in Remark 7.8(b), we would like to decompose (1) and (2) into Schur functions. Consider the partition $\lambda = 1^{n-2}2$, which is of length $n - 1$. A tableau of shape $\lambda$ must assign $n - 1$ distinct elements to the first column of $\lambda$. If we restrict the integers to the set $[n]$, this means that the first column will omit precisely one integer $i \in [n]$. The remaining cell $(1, 2)$ may be assigned arbitrarily, with the sole exception that if $i = 1$ is omitted from the first column, then the integer assigned to $(1, 2)$ cannot be 1. We conclude that

$$ s_{1^{n-2}}(x_1, \ldots, x_n) = (n - 1)x_1 \cdots x_n + \sum_{1 \leq i \neq j \leq n} (x_1 \cdots x_n)x_i x_j^{-1}. \quad (3) $$

Comparing (1), (2), and (3) we see that the adjoint actions decompose as follows:

$$ \begin{align*}
& \begin{array}{c}
& \text{gl}_n \\
& GL_n \quad s_{\emptyset}(x) + (x_1 \cdots x_n)^{-1}s_{1^{n-2}}(x) \\
& SL_n \quad 3_{\emptyset}(x) + 3_{1^{n-2}}(x)
\end{array} \\
& \begin{array}{c}
& \text{sl}_n \\
& SL_n \quad 3_{\emptyset}(x) + 3_{1^{n-2}}(x)
\end{array}
\end{align*} $$

where we have abbreviated $(x_1, \ldots, x_n)$ by $(x)$.

The $0$th isotypic component of $\text{gl}_n$ is readily identified as the submodule of scalar multiples of the identity matrix. Also note that $sl_n$ must be an irreducible $GL_n$- or $SL_n$-module as its character is a Schur function. By (2), we see that the character of $sl_n$ as an $SL_n$-module is a member of the subring $\Omega^0_n$, and we may use the dominant weight notation introduced in section 4. In this context, $sl_n$ is the irreducible representation corresponding to the dominant weight $(1, 0, \ldots, 0, -1) = [1,1]^n$.

The adjoint representation illustrates a phenomenon that will occur repeatedly amongst the character decomposition problems that we will consider. Given a multiplicity decomposition for an $SL_n$-module, one can easily recover the decomposition for the corresponding $GL_n$-module, and conversely, provided that the characters involved are homogeneous. See Remark 4.11(a). In some cases, the decomposition can be stated more elegantly for $SL_n$, while in other cases $GL_n$ is preferred. We will frequently state decomposition theorems for either $GL_n$ or $SL_n$, leaving to the reader the simple task of translating for the other group.
8. Exterior, symmetric, and tensor algebras

In this section, we introduce several important types of $GL_n$- and $SL_n$-modules, compute their characters, and discuss the decomposition problems which result. Our presentation of the exterior and symmetric algebras does not differ substantially from that given by Stanley in [39].

The exterior algebra

Let $V$ be a $G$-module for an arbitrary group $G$. Let $\text{Ext}(V)$ denote the exterior algebra of the complex vector space $V$. Recall that if $v_1, \ldots, v_m$ is a basis of $V$, then

$$\{v_{i_1} \wedge \cdots \wedge v_{i_k} : 0 \leq k \leq m, 1 \leq i_1 < \cdots < i_k \leq m\}$$

is a basis of $\text{Ext}(V)$. Also recall that the wedge `$\wedge$' is an antisymmetric, associative, bilinear product on $\text{Ext}(V)$. The exterior algebra is given a $G$-module structure via the action

$$g \circ (v_{i_1} \wedge \cdots \wedge v_{i_k}) = (g \circ v_{i_1}) \wedge \cdots \wedge (g \circ v_{i_k}),$$

(1)

extended to all of $\text{Ext}(V)$ by linearity.

The $k$th exterior power of $V$ is the $G$-submodule of $\text{Ext}(V)$ defined by

$$\text{Ext}^k(V) = \text{C-Span}\{v_{i_1} \wedge \cdots \wedge v_{i_k} : 1 \leq i_1 < \cdots < i_k \leq m\}.$$

Thus, we have the decomposition

$$\text{Ext}(V) = \text{Ext}^0(V) \oplus \text{Ext}^1(V) \oplus \cdots \oplus \text{Ext}^m(V).$$

(2)

Given a (rational) $GL_n$- or $SL_n$-module $V$, a natural question to ask is how $\text{Ext}(V)$ decomposes into irreducibles. In view of (2), we may ask the more refined question of how $\text{Ext}^k(V)$ decomposes into irreducibles for each integer $k$. Thus we are led to compute the characters of these modules.

Let $Y : V \to V$ be a linear transformation of the $GL_n$-module $V$, and suppose that $Y$ has eigenvalues $y_1, \ldots, y_m$. If $Y$ is diagonalizable, with eigenvectors $v_1, \ldots, v_m$, then the induced transformation $Y : \text{Ext}(V) \to \text{Ext}(V)$ defined by (1) is also diagonalizable, since

$$Y \circ (v_{i_1} \wedge \cdots \wedge v_{i_k}) = y_{i_1} \cdots y_{i_k} (v_{i_1} \wedge \cdots \wedge v_{i_k}).$$

However, the exterior algebra of a rational representation is also rational; in particular it is continuous. Therefore, even if $Y$ is not diagonalizable, the eigenvalues of $Y$ must be

$$\{y_{i_1} \cdots y_{i_k} : 0 \leq k \leq m, 1 \leq i_1 < \cdots < i_k \leq m\}.$$  

(3)
Let $M \subset \mathbb{Z}^n$ be the multiset (Theorem 7.4) such that if $X \in GL_n$ has eigenvalues $x_1, \ldots, x_n$ then the action of $X$ on $V$ has eigenvalues $x^x : \alpha \in M$. By (3), we see that the eigenvalues of $X$ acting on $\text{Ext}(V)$ are obtained by taking products of all possible sets of $x^x$'s, so that the character of $\text{Ext}(V)$ is

$$\prod_{\alpha \in M} (1 + x^\alpha),$$

and the character of $\text{Ext}^k(V)$ is the coefficient of $q^k$ in

$$\prod_{\alpha \in M} (1 + q x^\alpha).$$  \hfill (4)

Consider the special case $V = V_{(1)}$ of the natural action of $GL_n$. In this case, the character of $\text{Ext}^k(V_{(1)})$ is the elementary symmetric function $e_k(x_1, \ldots, x_n)$ (Definition 2.4). Since $e_k = s_{1^k}$, we see that $\text{Ext}^k(V_{(1)})$ yields a construction of the irreducible representation $V_{1^k}$.

The symmetric algebra

Next we consider the symmetric algebra $\text{Sym}(V)$ of a $G$-module $V$. Recall that if $v_1, \ldots, v_m$ is a basis of $V$, then $\text{Sym}(V)$ can be described as the algebra of polynomials $\mathbb{C}[v_1, \ldots, v_m]$, where $v_1, \ldots, v_m$ are considered indeterminates. In particular,

$$\{v_{i_1} \cdots v_{i_k} : k \geq 0, 1 \leq i_1 \leq \cdots \leq i_k \leq m\}$$

forms a basis of $\text{Sym}(V)$. The symmetric algebra is given a $G$-module structure via the action

$$g \circ p(v_1, \ldots, v_m) \equiv p(g \circ v_1, \ldots, g \circ v_m)$$  \hfill (5)

for any $p \in \mathbb{C}[v_1, \ldots, v_m]$.

The pedantic reader will complain that we have violated our promise to consider only finite dimensional representations. This violation is not very serious, since $\text{Sym}(V)$ can be decomposed into the finite dimensional submodules called the symmetric powers; namely,

$$\text{Sym}(V) = \prod_{k \geq 0} \text{Sym}^k(V),$$

where

$$\text{Sym}^k(V) = \{p \in \text{Sym}(V) : p \text{ homogeneous of degree } k\}.$$

We previously encountered the symmetric powers of the natural action of $GL_n$ in Remark 7.8(c); they are the irreducible $GL_n$-modules $V_{(k)}$. 
As with the exterior powers, the decomposition of the symmetric powers of (rational) $GL_n$- and $SL_n$-modules is of considerable interest. Using an argument analogous to the one given for the exterior algebra, it is not difficult to show that if $Y : V \rightarrow V$ is a linear transformation of $V$ with eigenvalues $y_1, \ldots, y_m$, then the linear transformation $Y : \text{Sym}(V) \rightarrow \text{Sym}(V)$ induced by (5) has eigenvalues

$$\{y_{i_1} \cdots y_{i_k} : 1 \leq i_1 \leq \cdots \leq i_k \leq m\};$$

i.e., the set of all monomials $y^\alpha$ of degree $k$. Hence, let $M \subset \mathbb{Z}^n$ be the usual multiset such that if $X \in GL_n$ has eigenvalues $x_1, \ldots, x_n$, then the action of $X$ on $V$ has eigenvalues $x^\alpha : \alpha \in M$. It follows from (6) that the character of $\text{Sym}^k(V)$ is the coefficient of $q^k$ in

$$\prod_{\alpha \in M} \frac{1}{1 - q x^\alpha}.$$  

(7)

### The tensor algebra

Now we consider the tensor algebra $T(V)$ of a $G$-module $V$, and the $k$th tensor powers $T^k(V)$. Recall that if $v_1, \ldots, v_m$ form a basis of $V$, then as vector spaces, we have

$$T^k(V) = \text{C-Span}\{v_{i_1} \otimes \cdots \otimes v_{i_k} : 1 \leq i_1, \ldots, i_k \leq m\}.$$  

Furthermore, we have the decomposition

$$T(V) = \prod_{k \geq 0} T^k(V).$$

Recall that the tensor `$\otimes$' is an associative, bilinear product on $T(V)$. We endow the tensor powers $T^k(V)$ with $G$-module structures via the action

$$g \circ (v_{i_1} \otimes \cdots \otimes v_{i_k}) = (g \circ v_{i_1}) \otimes \cdots \otimes (g \circ v_{i_k}),$$

(8)

extended to all of $T^k(V)$ by linearity.

As we noted earlier (Remark 6.12(c)), it is easy to see that if $\chi$ is the character of the $G$-module $V$, then $\chi^k$ is the character of $T^k(V)$. In the case of $GL_n$-modules, we see that the character of $T^k(V_\lambda)$ is

$$\delta^k_\lambda(x_1, \ldots, x_n)$$

and so the Littlewood-Richardson rule (Theorem 5.7) could be used to compute the multiplicities of the irreducible $GL_n$-modules $V_\mu$ in the decomposition of $T^k(V_\lambda)$. 
The Theory of Characters of $GL_n$ and $SL_n$

The tensor powers $T^k(V)$ of $GL_n$- or $SL_n$-modules exhibit a more interesting structure than the rather pedestrian action in (8). Observe that the symmetric group $S_k$ also acts on $T^k(V)$ via

$$w \circ (v_{i_1} \otimes \cdots \otimes v_{i_k}) = (v_{i_{w(1)}}) \otimes \cdots \otimes (v_{i_{w(k)}}),$$

and that this action commutes with the action of $GL_n$. This is equivalent to an action of $S_k \times GL_n$ on $T^k(V)$; namely,

$$(w, X) \circ (v_{i_1} \otimes \cdots \otimes v_{i_k}) = (X \circ v_{i_{w(1)}}) \otimes \cdots \otimes (X \circ v_{i_{w(k)}}). \quad (9)$$

Thus, it would be of interest to decompose $T^k(V)$ into irreducible $S_k \times GL_n$-modules, or at least compute the multiplicities involved. In order to appreciate the combinatorial nature of such a problem and its connection to the theory of symmetric functions, we digress briefly and review some of the fundamental aspects of the representation theory of $S_k$. For a thorough account of the representation theory of the symmetric group, the reader is referred to [21]. The connection between characters of the symmetric group and symmetric functions can also be found in [28; Chp. 5] and [32; I.7].

Since $S_k$ is a finite group, all representations of $S_k$ are completely reducible (Theorem 6.4), and the character determines a representation up to equivalence. Recall that the conjugacy classes of $S_k$ may be indexed by partitions of $k$; the partition $1^{c_1}2^{c_2}\cdots$ corresponds to the class consisting of permutations with $c_1$ 1-cycles, $c_2$ 2-cycles, etc. Thus we may regard a character $\chi$ of a representation $\rho : S_k \rightarrow GL(V)$ as a function whose domain is either $S_k$, or the partitions of $k$. We will do so interchangeably as the need arises.

In the representation theory of finite groups, it is known that the irreducible representations are in one-to-one correspondence with the conjugacy classes. Hence, for the symmetric group, there exist irreducible $S_k$-modules $S^\lambda$ for each partition $\lambda$ of $k$. In fact, an explicit construction of the module $S^\lambda$ (called the Specht module) can be given in which there is a natural basis in one-to-one correspondence with the standard tableaux of shape $\lambda$. Therefore, the degree of $S^\lambda$ is $f^\lambda$, the number of standard tableaux of shape $\lambda$. See [21; Sec. 7.2] for details.

Let $\chi^\lambda$ be the character of $S^\lambda$. We mention that the character corresponding to $\lambda = (k)$ is the trivial character, while the character corresponding to $\lambda = 1^k$ is the sign character. Let $Cl(S_k)$ denote the space of class functions; i.e.,

$$Cl(S_k) = \{ f : S_k \rightarrow C : f(w) = f(uwu^{-1}) \text{ for all } u, w \in S_k \}.$$ 

Since irreducible characters are linearly independent, it follows that the $\chi^\lambda$'s form a basis of $Cl(S_k)$. For any permutation $w \in S_k$, let $\lambda = \lambda(w)$ be the partition corresponding to the conjugacy class of $w$. The basic connection between symmetric
functions and characters of $S_k$ is the characteristic map $\text{ch} : \text{Cl}(S_k) \to \Lambda^k_C$, which is the vector space isomorphism defined by

$$\text{ch}(f) = \frac{1}{k!} \sum_{w \in S_k} f(w)p_{\lambda(w)}.$$  \hspace{1cm} (10)

**Theorem 8.1:** (Frobenius [8]) We have $\text{ch}(\chi^\lambda) = s_\lambda$.

Thus, the decomposition of a character $\chi$ of $S_k$ into irreducible characters can be viewed as the problem of finding the coefficients of the Schur functions $s_\lambda$ in the symmetric function $\text{ch}(\chi) \in \Lambda^k$.

Consider the problem of decomposing representations of $S_k \times \text{GL}_n$ into irreducibles. Notice that if $V$ is an $S_k$-module and $W$ is a $\text{GL}_n$-module, then the tensor product $V \otimes W$ can be given a $S_k \times \text{GL}_n$-module structure via the action

$$(w, X) \circ (u \otimes v) = (w \circ u) \otimes (X \circ v)$$

for any $w \in S_k$ and $X \in \text{GL}_n$. To avoid confusion with the previous tensor products we have considered, we will denote this module by $V \times W$ to emphasize the fact that it was constructed from representations of different groups.

**Proposition 8.2:**

(a) Representations of $S_k \times \text{GL}_n$ which are rational with respect to $\text{GL}_n$ are completely reducible.

(b) The representations

$$\{S^\lambda \times V^r_\mu : |\lambda| = k, \ell(\mu) < n, r \in \mathbb{Z}\}$$

are a complete set of inequivalent, irreducible representations of $S_k \times \text{GL}_n$ which are rational with respect to $\text{GL}_n$. The representations

$$\{S^\lambda \times V_\mu : |\lambda| = k, \ell(\mu) < n\}$$

are a complete set for those which are polynomial with respect to $\text{GL}_n$.

(c) The character of $S^\lambda \times V^r_\mu$ is

$$(w, X) \mapsto \chi^\lambda(w)(x_1 \cdots x_n)^r s_{\mu}(x_1, \ldots, x_n)$$

and the character of $S^\lambda \times V_\mu$ is

$$(w, X) \mapsto \chi^\lambda(w)s_{\mu}(x_1, \ldots, x_n).$$
Proof:
(a) Let $V$ be an $S_k \times GL_n$-module, rational with respect to $GL_n$. We will show that $V$ is completely reducible by finding an invariant complement for each $S_k \times GL_n$-submodule of $V$.

Let $U$ be a submodule of $V$. Since $U, V$ are $GL_n$-modules, then there is a $GL_n$-complement for $U$, say $V = U \oplus U'$. Choose bases for $U$ and $U'$. With respect to these bases, $V$ corresponds to a representation $\rho$ of the form

$$\rho(w, X) = \begin{bmatrix} \pi(w, X) & \theta(w, X) \\ 0 & \sigma(w, X) \end{bmatrix}. \tag{11}$$

The map $\pi$ is the representation corresponding to $U$, while $\sigma$ is the representation corresponding to the quotient $V/U$. Since $U'$ is a $GL_n$-module, it follows that

$$\theta(1, X) = 0. \tag{12}$$

By (11) and the fact that $\rho$ is a homomorphism, it follows that

$$\pi(w_1, X_1)\theta(w_2, X_2) + \theta(w_1, X_1)\sigma(w_2, X_2) = \theta(w_1w_2, X_1X_2).$$

From (12) we deduce

$$\pi(X)\theta(w) = \theta(w)\sigma(X) = \theta(w, X), \tag{13}$$

where we have abbreviated $(1, X)$ by $(X)$ and $(w, 1)$ by $(w)$.

The following trick can be used to prove that finite groups have completely reducible representations (see [16; Chp. 16] or [3; III.1]). Observe that since

$$\begin{bmatrix} \pi(w) & \theta(w) \\ 0 & \sigma(w) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \pi(w) & 0 \\ 0 & \sigma(w) \end{bmatrix} \tag{14}$$

have the same character, they must be equivalent representations of $S_k$. Define a matrix $\phi$ via

$$\phi = \frac{1}{k!} \sum_{w \in S_k} \theta(w)\sigma(w^{-1}).$$

An easy exercise will verify that

$$\pi(w)\phi + \theta(w) = \phi\sigma(w), \tag{15}$$

which is equivalent to the identity

$$\begin{bmatrix} \pi(w) & \theta(w) \\ 0 & \sigma(w) \end{bmatrix} \begin{bmatrix} 1 & \phi \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \phi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \pi(w) & 0 \\ 0 & \sigma(w) \end{bmatrix}.$$
This provides explicit evidence that the representations in (14) are equivalent. From (13) and the definition of $\phi$ it follows that 

$$\pi(X)\phi = \phi\sigma(X),$$

and therefore, (15) must hold for the full group $S_k \times GL_n$; i.e., 

$$\pi(w, X)\phi + \theta(w, X) = \phi\sigma(w, X).$$

Hence, we deduce that 

$$\begin{bmatrix} \pi(w, X) & \theta(w, X) \\ 0 & \sigma(w, X) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \pi(w, X) & 0 \\ 0 & \sigma(w, X) \end{bmatrix}$$

are equivalent representations of $S_k \times GL_n$, thus providing the desired invariant complement for $U$.

(b),(c) Consider the case of representations which are polynomial with respect to $GL_n$. The representations $\rho_\mu : GL_n \to GL(V_\mu)$ and (say) $\phi_\lambda : S_k \to GL(S^\lambda)$ must have matrix entries $\rho_{ij}^\mu$ and $\phi_{ij}^\lambda$ which are respectively linearly independent (Theorem 6.7). With respect to a suitable basis, the matrix entries of 

$$\phi_\lambda \otimes \rho_\mu : S_k \times GL_n \to GL(S^\lambda \times V_\mu)$$

are of the form 

$$(w, X) \mapsto \phi_{ij}^\lambda(w)\rho_{ki}^\mu(X) \quad (16)$$

and are therefore linearly independent. It follows that the $S_k \times GL_n$-module $S^\lambda \times V_\mu$ must be irreducible, since reducibility would imply that, upon suitable choice of basis, some of the matrix entries of $\phi_\lambda \otimes \rho_\mu$ vanish, yielding a dependence relation among the terms in (16).

Directly from (16) it follows that the character of $S^\lambda \times V_\mu$ agrees with the formula claimed in part (c). In particular, the representations $S^\lambda \times V_\mu$ are all inequivalent since the characters are all distinct. Since the Schur functions span $\Lambda_n$ and the characters $\chi^\lambda$ span $Cl(S_k)$, we deduce that the characters of $S^\lambda \times V_\mu$ span the space $Cl(S_k) \otimes \Lambda_n$. Therefore, there can be no other irreducible polynomial characters.

The case of representations which are rational with respect to $GL_n$ can be established by a similar argument.

Remark 8.3:

(a) Let $y = (y_1, y_2, \ldots)$ be an additional set of indeterminates. Extend the characteristic map in the obvious way to characters of $S_k \times GL_n$, regarding the characteristic as a map of the form 

$$\text{ch} : Cl(S_k) \otimes \Lambda_n(x) \to \Lambda(y) \otimes \Lambda_n(x).$$
We may thus compute the multiplicity of $S^\lambda \times V_\mu$ in the decomposition of a representation into irreducibles by computing the coefficient of $s_\lambda(y)s_\mu(x_1, \ldots, x_n)$ in the characteristic of its character.

(b) Let $W$ be a (rational) $S_k \times GL_n$-module. Since $W$ must also be an $S_k$-module, we may decompose $W$ into its $S_k$-isotypic components; say

$$W = \coprod_{|\lambda|=k} W_\lambda.$$ 

For any $X \in GL_n$, we have

$$W = X \circ W = \coprod_{|\lambda|=k} X \circ W_\lambda,$$

which is another $S_k$-isotypic decomposition of $W$. Since these decompositions are unique (Proposition 6.11), it follows that

$$W_\lambda = X \circ W_\lambda;$$

i.e., each component $W_\lambda$ is also a $GL_n$-module.

Let $\chi \in CL(S_k) \otimes \Lambda_n$ be the character of $W$. By Proposition 8.2 and the previous remark, we see that the coefficient of $s_\lambda(y)$ in $f^\lambda \chi(\chi)$ is the character of $W_\lambda$ as a $GL_n$-module.

Finally, we are ready to discuss the problem which motivated this digression into the symmetric group and $S_k \times GL_n$-modules; namely, the tensor powers $T^k(V)$ of a (rational) $GL_n$-module $V$. Let $M \subset \mathbb{Z}^n$ be the multiset such that if $x_1, \ldots, x_n$ are the eigenvalues of $X \in GL_n$, then the eigenvalues of $X$ acting on $V$ are $x^\alpha : \alpha \in M$.

**Theorem 8.4:** Let $\lambda$ be a partition of $k$ and $\mu$ a partition of length at most $n$. The multiplicity of $S^\lambda \times V_\mu^*$ in the decomposition of $T^k(V)$ is the coefficient of $(x_1 \cdots x_n)^*s_\mu(x_1, \ldots, x_n)s_\lambda(y)$ in the formal power series

$$\prod_{i \geq 1} \prod_{\alpha \in M} \frac{1}{1 - y_i x^\alpha}.$$ 

Equivalently, the character (as a $GL_n$-module) of the $\lambda$th $S_k$-isotypic component of $T^k(V)$ is the coefficient of $s_\lambda(y)$ in

$$f^\lambda \prod_{i \geq 1} \prod_{\alpha \in M} \frac{1}{1 - y_i x^\alpha}.$$
Proof: Let \(v_1, \ldots, v_m\) be a basis of \(V\), and let \(a_{ij} = a_{ij}(X)\) denote the matrix entries of the action of \(GL_n\) on \(V\) with respect to this basis; i.e.,

\[
X \circ v_i = \sum_{j} a_{ij}(X) v_j.
\]

By (9), we have

\[
(w, X) \circ (v_{i_1} \otimes \cdots \otimes v_{i_k}) = \sum_{j_1, \ldots, j_k} a_{i_1 j_1} \cdots a_{i_k j_k} v_{j_{w(1)}} \otimes \cdots \otimes v_{j_{w(k)}}.
\]

The trace of this transformation is therefore

\[
\sum_{i_1, \ldots, i_k} a_{i_1 i_{w^{-1}(1)}} \cdots a_{i_k i_{w^{-1}(k)}}.
\] (17)

Closer inspection of (17) reveals that for each cycle of length \(l\) in \(w\), the terms

\[
\text{tr}([a_{ij}]^l) = \sum_{j_{i_1, \ldots, j_{il}}} a_{j_{il}i_{l-1}} \cdots a_{j_{i_1} i_1} a_{j_{i_1}i_1}
\]

may be factored out of (17).

Since \([a_{ij}(X)]\) has eigenvalues \(x^\alpha : \alpha \in M\), then \([a_{ij}(X)]^l\) has eigenvalues \(x^\alpha : \alpha \in M\). Therefore, the character \(\chi_k\) of \(T_k(V)\) satisfies

\[
\chi_k(w, X) = p_{\lambda(w)}(x^\alpha : \alpha \in M).
\]

Application of the characteristic map yields

\[
\text{ch}(\chi_k) = \frac{1}{k!} \sum_{w \in S_k} p_{\lambda(w)}(y) p_{\lambda(w)}(x^\alpha : \alpha \in M).
\]

By a well-known identity (e.g., [32; I.(2.14)]), or equivalently, the fact that the characteristic of the trivial character is \(h_k\) (Theorem 8.1), it follows that

\[
\text{ch}(\chi_k) = h_k(y \alpha x^\alpha : \alpha \in M),
\]

so that (cf. (2.3))

\[
\sum_k \text{ch}(\chi_k) = \prod_{i \geq 1} \prod_{\alpha \in M} \frac{1}{1 - y_i x^\alpha}.
\]

The theorem is now a consequence of Remark 8.3.
Remark 8.5:

(a) Consider the special case \( V = V_{(1)} \) of the natural action of \( GL_n \). By Corollary 3.9, we have

\[
\prod_{i \geq 1} \prod_{j \leq n} \frac{1}{1 - y_i x_j} = \sum_{\lambda \leq n} s_\lambda(y) s_\lambda(x_1, \ldots, x_n).
\]

This tells us that for each partition \( \lambda \) of length at most \( n \), the \( \lambda \)-th \( S_k \)-isotypic component of \( T^k(V_{(1)}) \) is also isotypic as a \( GL_n \)-module. In fact, we must have

\[
T(V_{(1)}) \cong \prod_{\lambda \leq n} S^\lambda \times V_\lambda.
\]

That such a simple decomposition exists is no mere happy accident. Schur derived the classification of the irreducible representations of \( GL_n \) [37] by decomposing \( T^n(V_{(1)}) \) into irreducibles.

(b) Finally, we note that the exterior and symmetric algebras can be viewed as submodules of the tensor algebra. If \( \chi \) is a linear character of a group \( G \) (i.e., the character of a representation of degree 1), and \( V \) is any \( G \)-module, then

\[
\{ v \in V : g \circ v = \chi(g)v \text{ for all } g \in G \}
\]

is the \( \chi \)-th isotypic component of \( V \).

In the case of the symmetric group \( S_k \), there are two linear characters: the trivial character and the sign character. An easy exercise will convince the reader that the isotypic components of \( T^k(V) \) corresponding to these characters are the exterior and symmetric powers:

\[
\text{Ext}^k(V) \cong \{ t \in T^k(V) : w \circ t = \varepsilon_w t \text{ for all } w \in S_k \}
\]
\[
\text{Sym}^k(V) \cong \{ t \in T^k(V) : w \circ t = t \text{ for all } w \in S_k \}.
\]

The expressions (4) and (7) we obtained for the characters of \( \text{Ext}^k(V) \) and \( \text{Sym}^k(V) \) are different from those implied by Theorem 8.4. These differences can be easily reconciled. Theorem 8.4 says that the character of \( \text{Sym}^k(V) \) is the coefficient of \( s_k(y) \) in

\[
\prod_{j \geq 1} \prod_{\alpha \in M} \frac{1}{1 - x^\alpha y_j}.
\]

By Corollary 3.9 we have

\[
\prod_{j \geq 1} \prod_{\alpha \in M} \frac{1}{1 - x^\alpha y_j} = \sum_{\lambda} s_\lambda(y) s_\lambda(x^\alpha : \alpha \in M),
\]
so that the coefficient of \( s_k(y) \) is in fact

\[
s_k(x^\alpha : \alpha \in M) = h_k(x^\alpha : \alpha \in M),
\]

which does agree with (7). The reconciliation for \( \text{Ext}^k(V) \) can be similarly described.

As a final remark, we mention that the coefficient extraction problems suggested by Theorem 8.4 can be translated into the language of plethysm\(^1\). However, it is not clear whether such a reformulation would provide any substantial new insight.

9. Problems arising from the adjoint action

The problems which we will study in detail concern the decompositions of the exterior, symmetric and tensor algebras of \( gl_n \) and \( sl_n \) as \( GL_n \)- or \( SL_n \)-modules. In this section, we describe these problems more precisely and discuss some of what is already known about them, particularly through the work of R. K. Gupta, P. Hanlon, and R. Stanley.

As we remarked in Section 7, the adjoint action should be thought of as the action of \( GL_n \) on \( gl_n \) or \( SL_n \) on \( sl_n \). For our purposes, we will find it most convenient, and the statement of many results to be more elegant, if we restrict our attention to the action of \( SL_n \) on \( gl_n \). This requires no great sacrifice, as it is easy to recover (if the reader desires) the corresponding results for the actions of \( GL_n \) on \( gl_n \) or \( SL_n \) on \( sl_n \).

Recall from the discussion of the adjoint representation in Section 7 that if \( X \in SL_n \) has eigenvalues \( x_1, \ldots, x_n \), then the action of \( X \) on \( gl_n \) has eigenvalues \( x_i x_j^{-1} : 1 \leq i, j \leq n \). Therefore, by (8.4), the character of \( SL_n \) acting on \( \text{Ext}^k(gl_n) \) is the coefficient of \( q^k \) in

\[
\prod_{1 \leq i, j \leq n} \frac{1}{1 - qx_i x_j^{-1}},
\]

or rather, the image of (1) in the ring \( \Omega_n[q] \). In fact, throughout this section, all formal power series in \( x_1, \ldots, x_n \) should be viewed modulo the relation \( x_1 \cdots x_n = 1 \), unless stated otherwise.

Similarly, by (8.7), the character of \( SL_n \) acting on \( \text{Sym}^k(gl_n) \) is the coefficient of \( q^k \) in

\[
\prod_{1 \leq i, j \leq n} \frac{1}{1 - qx_i x_j^{-1}}.
\]

\(^{1}\)A discussion of plethysm can be found, e.g., in [32; I.8].
If we desire the characters of all the $S_k$-isotypic components of $T^k(gl_n)$, Theorem 8.4 tells us that we should compute the coefficients of the Schur functions $s_\lambda(y)$ for partitions $\lambda$ of $k$ in the formal series

$$\prod_{1 \leq i, j \leq n} \prod_{r \geq 1} \frac{1}{1 - y_r x_i x_j^{-1}}. \quad (3)$$

One further $SL_n$-module that we will consider is the Macdonald complex of $gl_n$, which we denote by $M_k(gl_n)$. This complex is a structure devised by P. Hanlon [17], in the more general context of an arbitrary semisimple Lie algebra, to aid in the study of Macdonald's root system conjectures [31]. The connection between the Macdonald complex and the Macdonald root system conjectures will be discussed in Chapter IV. To define the Macdonald complex, first consider the tensor powers $T^k(\text{Ext}(gl_n))$. Notice that this $SL_n$-module has a decomposition

$$T^k(\text{Ext}(gl_n)) = \prod_{a \in \mathbb{N}^k} \text{Ext}^{a_1}(gl_n) \otimes \cdots \otimes \text{Ext}^{a_k}(gl_n).$$

We will refer to the submodule

$$\text{Ext}^{a_1}(gl_n) \otimes \cdots \otimes \text{Ext}^{a_k}(gl_n)$$

as the $a$th graded submodule of $T^k(\text{Ext}(gl_n))$. Notice that the character of the $a$th graded submodule is the coefficient

$$[y^a] \prod_{1 \leq i, j \leq n} (1 + y_i x_i x_j^{-1}) \cdots (1 + y_k x_i x_j^{-1}).$$

For our purposes, we shall regard the Macdonald complex $M_k(gl_n)$ as the $SL_n$-module $T^k(\text{Ext}(gl_n))$, endowed with a special grading which assigns the weight

$$(-1)^{|a|} q^{a_1 + n(a)} = (-1)^{\sum a_i q^{\sum i a_i}}$$

to the $a$th graded submodule of $T^k(\text{Ext}(gl_n))$. Therefore, if we sum the characters of the $a$th graded submodules according to their weights, we obtain the weighted character

$$\prod_{1 \leq i, j \leq n} (1 - q x_i x_j^{-1}) \cdots (1 - q^k x_i x_j^{-1}). \quad (4)$$

Notice that each of the characters $(1), (2), (3),$ and $(4)$ belong to the subring $\Omega_0^n$. Thus, we may use dominant weights (Definition 4.12) and the notation $[\alpha, \beta]_n$, as well as the usual partitions $\lambda$ with $\ell(\lambda) < n$ and $|\lambda|$ divisible by $n$, to index the irreducible characters in their decompositions.
For each partition \( \lambda \) with \( \ell(\lambda) < n \) and \( |\lambda| \) divisible by \( n \), define formal power series \( c_\lambda = c_\lambda(q) \) via

\[
\prod_{1 \leq i, j \leq n} \frac{1}{1 - qx_i x_j^{-1}} = \sum_\lambda c_\lambda(q) \overline{s}_\lambda(x_1, \ldots, x_n).
\]  

(5)

Thus, \( c_\lambda(q) \) tells us how the symmetric algebra decomposes. R. K. Gupta [13] has studied the behavior of \( c_\lambda \) in the limit as \( n \) tends to infinity. A certain amount of delicacy is required to do this, since for each \( n \), the collection of partitions for which \( c_\lambda \) is defined must have weights divisible by \( n \). In terms of the dominant weight notation, Gupta has shown that for any pair of partitions \( \alpha \) and \( \beta \) of the same weight, the limit

\[
c_{\alpha \beta}(q) = \lim_{n \to \infty} c_{[\alpha, \beta]}(q)
\]

exists as a formal power series, and she conjectured that the \( c_{\alpha \beta} \)'s satisfy a number of remarkable properties.

In view of Gupta's results, Stanley [40] was led to consider the more general decomposition problem

\[
\prod_{1 \leq i, j \leq n} \prod_{r \geq 1} \frac{1}{1 - y_r x_i x_j^{-1}} = \sum_\lambda c_\lambda(y) \overline{s}_\lambda(x_1, \ldots, x_n)
\]

(6)

in the limit as \( n \) tends to infinity. The series \( c_\lambda(y) \) is a symmetric formal power series in the variables \( y = (y_1, y_2, \ldots) \); the series \( c_\lambda(q) \) in (5) can be recovered by the specialization \( y_1 \to q, y_2, y_3, \ldots \to 0 \).

We mention that Stanley also introduced the indeterminates \( z = (z_1, z_2, \ldots) \) and showed that the decomposition of

\[
\prod_{1 \leq i, j \leq n} \prod_{r \geq 1} \frac{1 - z_r x_i x_j^{-1}}{1 - y_r x_i x_j^{-1}}
\]

(7)

into Schur functions \( \overline{s}_\lambda(x_1, \ldots, x_n) \) can be obtained from the decomposition of (6) without any extra work. As Gupta did for the symmetric algebra, Stanley showed that the limit

\[
c_{\alpha \beta}(y) = \lim_{n \to \infty} c_{[\alpha, \beta]}(y)
\]

exists as a formal power series, and found an explicit formula for the \( c_{\alpha \beta} \)'s. In order to state Stanley's formula, we need to introduce the internal product, which is an operation on symmetric functions invented by D. E. Littlewood.

**Definition 9.1:** Let \( f, g \in \Lambda^k \), and let \( \phi, \chi \in Cl(S_k) \) denote the corresponding class functions; i.e.,

\[
f = \text{ch}(\phi) \quad \text{and} \quad g = \text{ch}(\chi).
\]
The internal product of \( f \) and \( g \) is the symmetric function \( f \ast g \in \Lambda^k \) defined by

\[
f \ast g = \text{ch}(\phi \chi).
\]

Notice that the internal product is commutative, associative, and bilinear. It is sometimes convenient to extend the definition of the internal product to all of \( \Lambda \) by insisting that the product remain bilinear, and that if \( f \) and \( g \) are homogeneous, then \( f \ast g = 0 \) unless \( f \) and \( g \) are of the same degree.

Let \( \lambda \) and \( \mu \) be partitions of \( k \). A well-known problem among those who study combinatorial representation theory is the problem of finding a reasonable (i.e., combinatorial) description of the multiplicities of the irreducible \( S_k \)-modules \( S^\nu \) in the decomposition of \( S^\lambda \otimes S^\mu \). In other words, the problem is to describe the integers \( g_{\lambda \mu \nu} \) for which

\[
\chi^\lambda \chi^\mu = \sum_{|\nu| = k} g_{\lambda \mu \nu} \chi^\nu. \tag{8}
\]

The internal product allows us to translate this problem into a symmetric function problem: \( g_{\lambda \mu \nu} \) is the coefficient of \( s_\nu \) in the decomposition of \( s_\lambda \ast s_\mu \) into Schur functions.

A notational convenience which will be useful now and in subsequent sections is the following. Recall that the power sums \( p_r \) are algebraically independent generators of \( \Lambda^Q \) (Theorem 2.6). Thus, we may specify a unique algebra homomorphism of \( \Lambda^Q \) by choosing elements \( a_1, a_2, \ldots \) arbitrarily from some desired algebra (e.g., an algebra of formal power series) and insisting that \( p_r \mapsto a_r \). Therefore, for any \( f \in \Lambda^Q \), the notation

\[
f(p_r \mapsto a_r) \tag{9}
\]

shall be an abbreviation for the image of \( f \) under this homomorphism. Equivalently, this notation can be viewed as shorthand for the result obtained by writing \( f \) as a polynomial in the power sums, and substituting each occurrence of \( p_r \) with \( a_r \).

We are now in a position to state Stanley's result:

**Theorem 9.2:** (Stanley [40; Theorem 6.2])

Let \( \alpha \) and \( \beta \) be partitions of the same weight. We have

\[
e_{\alpha \beta}(y) = \left[ \prod_{r \geq 1} \frac{1}{1 - p_r(y)} \right] \ s_\alpha \ast s_\beta \left( p_r \mapsto \frac{p_r(y)}{1 - p_r(y)} \right).
\]

In other words, if \( \lambda \) is a partition of \( k \) and \( n \) is sufficiently large, the multiplicity of \( S^\lambda \times V_{[\alpha, \beta]_n} \) in \( T^k(gl_n) \) is the inner product

\[
\left< \left[ \prod_{r \geq 1} \frac{1}{1 - p_r} \right] \ s_\alpha \ast s_\beta \left( p_r \mapsto \frac{p_r}{1 - p_r} \right), s_\lambda \right>.
\]
By the definitions of the internal product and the characteristic map \((8.10)\) we see that
\[
s_\lambda \ast s_\mu = \frac{1}{k!} \sum_{w \in S_k} \chi^\lambda(w) \chi^\mu(w) p_\lambda(w).
\]
Therefore, Stanley's formula for \(c_{\alpha\beta}\) can be more explicitly expressed in terms of the power sums \(p_\lambda\), if desired. By specializing the indeterminates \(y\) in \(c_{\alpha\beta}(y)\) and by using properties of the internal product and characters of the symmetric group, Stanley [40] deduces numerous limiting-case properties of the decomposition of the exterior and symmetric algebras and the Macdonald complex. He does not explicitly mention the fact that the “unspecialized” decomposition formula in Theorem 9.2 can be given the representation-theoretic interpretation implied by Theorem 8.4.

The work of Gupta and Stanley led P. Hanlon [17] to study the limiting-case decompositions of the \(S_k\)-isotypic components of the tensor algebra \(T^k(L)\) for all of the classical Lie algebras \(L\) (types \(A_n\), \(B_n\), \(C_n\) and \(D_n\)). His methods, which are completely different from those of Stanley, rely on a clever mixture of combinatorial ideas and tools from the representation theory of semisimple Lie algebras. Hanlon was able to show that for each classical type \((A, B, C, D)\), one may pass to a limit in a carefully chosen way and obtain a stable decomposition analogous to Theorem 9.2 for the exterior and symmetric algebras and the Macdonald complex of these Lie algebras.

We will not expend much effort to specifically study the characters of the symmetric powers \(\text{Sym}^k(gl_n)\) because their decompositions are known, or at least, better understood from algebraic and combinatorial points of view, than are the characters of \(\text{Ext}^k(gl_n)\) and \(M_k(gl_n)\).

Let \(I_n \subset \text{Sym}(gl_n)\) denote the invariants of the symmetric algebra; i.e., the \(\emptyset\)th isotypic component of \(\text{Sym}(gl_n)\), which is given by
\[
I_n = \{ t \in \text{Sym}(gl_n) : X \circ t = t \text{ for all } X \in SL_n \}.
\]
It is well-known that the invariants \(I_n\) form a polynomial ring generated by the coefficients \(t_1, \ldots, t_n\) of the characteristic polynomial of a generic matrix \(A \in gl_n\); i.e.,
\[
\det(xI - A) = x^n + t_1(A)x^{n-1} + \cdots + t_n(A).
\]
Note that \(t_k \in \text{Sym}^k(gl_n)\). It follows that (see (5))
\[
c_\emptyset(q) = \frac{1}{(1-q)(1-q^2) \cdots (1-q^n)}.
\]
A combinatorial proof of this has been given by Stanley [39].
Kostant has shown [24; Theorem 0.2] that \( \text{Sym}(gl_n) \) has a decomposition of the form
\[
\text{Sym}(gl_n) = I_n \otimes H_n, \tag{11}
\]
where \( H_n \) is a certain graded \( SL_n \)-submodule of \( \text{Sym}(gl_n) \) whose members are polynomials on \( gl_n \) which are called \( SL_n \)-harmonic. In other words, we have
\[
H_n = \prod_{k \geq 0} H^k_n; \quad H^k_n = H_n \cap \text{Sym}^k(gl_n).
\]

For each partition \( \lambda \) with \( \ell(\lambda) < n \) and \( |\lambda| \) divisible by \( n \), define a formal power series \( G^n[\lambda](q) \) in the indeterminate \( q \) via
\[
G^n[\lambda](q) = (1 - q)(1 - q^2) \cdots (1 - q^n)c_\lambda(q). \tag{12}
\]

It follows from (5), (10), and (11) that \( G^n[\lambda] \) is the multiplicity-enumerator for the \( SL_n \)-harmonic polynomials \( H_n \); i.e., the multiplicity of \( V_\lambda \) in the decomposition of \( H^k_n \) into irreducible \( SL_n \)-modules is the coefficient of \( q^k \) in \( G^n[\lambda](q) \). The various integers \( k \) for which the multiplicity of \( V_\lambda \) in \( H^k_n \) is nonzero (counted according to their multiplicity) are called the generalized exponents of \( SL_n \) corresponding to \( \lambda \). Kostant’s work implies that there are only finitely many exponents corresponding to any particular \( \lambda \), so that \( G^n[\lambda] \) is in fact a polynomial.

W. Hesselink [19] and D. Peterson\(^2\) independently found a recursion for the generalized exponents for any semisimple (complex) Lie algebra. Recently, R. K. Gupta noticed that in the case of \( sl_n \), this recursion is formally the same as a recursion given by Macdonald [32; III.ex.6.4] for computing the charge of certain tableaux of given shape and weight. A discussion and definition of charge can be found in [32; III.6]. Thus, the generalized exponents of \( SL_n \) can be expressed as charges of certain tableaux. The details of this combinatorial interpretation of the generalized exponents will appear in a forthcoming paper by Gupta.

In the following sections we will prove a number of results about the character decompositions of the exterior and symmetric algebras and the Macdonald complex of \( gl_n \), without passing to the limit \( n \to \infty \). Although the results we obtain will not apply to as great a level of generality as those of Hanlon, Stanley and Gupta, a complete solution would be considerably more difficult without the stability provided by the assumption that \( n \) is sufficiently large.

The decomposition of the exterior algebra, and to a lesser extent, the Macdonald complex, will be the subject of Chapter III. The decomposition of
\[
\prod_{1 \leq i,j \leq n} \prod_{k \geq 1} \frac{1 - q^k x_i x_j^{-1}}{1 - z q^{k-1} x_i x_j^{-1}},
\]

\(^2\)Private communication.
into Schur functions \( \overline{s}_\lambda(x_1, \ldots, x_n) \), where \( z \) and \( q \) are indeterminates, will be the subject of Chapter IV. Notice that as special cases,

- The decompositions of the exterior powers (1) are obtained via the specialization \( z \rightarrow q^2 \), \( q \rightarrow -q \).
- The decompositions of the symmetric powers (2) are obtained via the specialization \( q \rightarrow 0 \).
- The decomposition of the Macdonald complex (4) is obtained via the specialization \( z \rightarrow q^{k+1} \).
Chapter III.

The Exterior Algebra of $gl_n$

10. Preliminaries

Let $\lambda$ vary over partitions with length less than $n$ and weight divisible by $n$. Define integer polynomials $E^n[\lambda]$ in the indeterminate $q$ so that

$$\prod_{1 \leq i,j \leq n} (1 + qx_i x_j^{-1}) = \sum_{\lambda} E^n[\lambda](q) \overline{s}_\lambda(x_1, \ldots, x_n)$$

holds in $\Omega_n[q]$. These polynomials have nonnegative coefficients, since we have seen (9.1) that the coefficient of $q^r$ in $E^r[\lambda](q)$ is the multiplicity of $V_\lambda$ in the $SL_n$-module $\text{Ext}^r(gl_n)$. For the Macdonald complex $M_k(gl_n)$, define integer polynomials $M^n_k[\lambda]$ in the indeterminate $q$ so that

$$\prod_{1 \leq i,j \leq n} (1 - qx_i x_j^{-1}) \cdots (1 - q^k x_i x_j^{-1}) = \sum_{\lambda} M^n_k[\lambda](q) \overline{s}_\lambda(x_1, \ldots, x_n)$$

holds in $\Omega_n[q]$. Notice that $E^n[\lambda](q) = M^n_k[\lambda](-q)$.

In this section we begin our study of the polynomials $E^n[\lambda]$ and $M^n_k[\lambda]$, collecting here some of the more elementary properties which they can be shown to satisfy. Among them are a combinatorial interpretation of $E^n[\lambda](1)$ (Theorem 10.4), and a collection of necessary conditions (Theorem 10.7), which we conjecture to be sufficient, for the polynomial $E^n[\lambda]$ to be nonzero.

Since the characters of $\text{Ext}(gl_n)$ and $M_k(gl_n)$ belong to $\Omega^0_n$, we are justified in restricting the partitions $\lambda$ to those whose weights are divisible by $n$. We may also use dominant weights to index their decompositions; if $\gamma \in \mathbb{Z}^n$ is the dominant weight corresponding to $\lambda$, regard the polynomials $E^n[\gamma]$, $E^n[\lambda]$ and $M^n_k[\gamma]$, $M^n_k[\lambda]$ as synonymous.

Recall that we say that $\lambda$ belongs to the $l$th layer if $|\lambda| = ln$. It is easy to show that the only layers for which there are nonzero polynomials $E^n[\lambda]$ are the
layers 0, 1, ..., \(n - 1\): since

\[(x_1 \cdots x_n)^{n-1} \prod_{i,j}(1 + qx_ix_j^{-1}) = (1 + q)^n \prod_{i \neq j}(x_j + qx_i)\]  \hspace{1cm} (1)

is a homogeneous polynomial in \(x_1, \ldots, x_n\) of degree \(n(n - 1)\), the decomposition of (1) into Schur functions is indexed by partitions of \(n(n - 1)\) with at most \(n\) rows. When this is projected down to \(\Omega_n\), we see that the highest possible layer is \(n - 1\). See Remark 4.11(a).

More generally, since

\[
\prod_{1 \leq i, j \leq n} (1 + y_1x_ix_j^{-1}) \cdots (1 + y_kx_ix_j^{-1})
\]

is equivalent modulo \(x_1 \cdots x_n = 1\) to a homogeneous polynomial in \(x_1, \ldots, x_n\) of degree \(kn(n - 1)\), it follows that the partitions \(\lambda\) for which the multiplicities of \(V_\lambda\) in

\[
\text{Ext}^{a_1}(gl_n) \otimes \cdots \otimes \text{Ext}^{a_k}(gl_n)
\]

are nonzero must come from the layers \(0, 1, \ldots, k(n - 1)\). In particular, if \(M^a_\lambda[\lambda](q) \neq 0\), then the layer of \(\lambda\) can be at most \(k(n - 1)\).

The following result says that the coefficients of \(E^a[\lambda]\) and \(M^a_\lambda[\lambda]\) are symmetric about the middle:

**Proposition 10.1**: We have:

(a) \(E^a[\lambda](q) = q^{n^2} E^a[\lambda](q^{-1})\)

(b) \(M^a_\lambda[\lambda](q) = (-1)^{kn^2} q^{n^2\binom{k+1}{2}} M^a_\lambda[\lambda](q^{-1})\)

**Proof**: Part (a) follows from the identity

\[q^{n^2} \prod_{i,j}(1 + q^{-1}x_ix_j^{-1}) = \prod_{i,j}(q + x_ix_j^{-1}) = \prod_{i,j}(1 + qx_jx_i^{-1}),\]

which holds in \(\Omega_n[q]\). Part (b) follows similarly.

The polynomials \(E^a[\lambda]\) and \(M^a_\lambda[\lambda]\) exhibit a second kind of symmetry:

**Proposition 10.2**:

(a) \(E^a[\lambda] = E^a[\lambda^\dagger]\) or equivalently, \(E[\alpha, \beta]_n = E[\beta, \alpha]_n\).

(b) \(M^a_\lambda[\lambda] = M^a_\lambda[\lambda^\dagger]\) or equivalently, \(M_k[\alpha, \beta]_n = M_k[\beta, \alpha]_n\).

(c) The multiplicities of \(V_\lambda\) and \(V_{\lambda^\dagger}\) in the \(a\)th graded submodule

\[
\text{Ext}^{a_1}(gl_n) \otimes \cdots \otimes \text{Ext}^{a_k}(gl_n)
\]
of \( T^k(\text{Ext}(gl_n)) \) are the same.

**Proof:** Observe that parts (a) and (b) follow directly from part (c). To prove part (c), let \( b_\lambda \) denote the multiplicity of \( V_\lambda \) in the \( \alpha \)th graded submodule, so that
\[
[y^\alpha] \prod_{1 \leq i, j \leq n} (1 + y_1 x_i x_j^{-1}) \cdots (1 + y_n x_i x_j^{-1}) = \sum_\lambda b_\lambda \overline{s}_\lambda(x_1, \ldots, x_n) \quad (2)
\]
holds in \( \Omega_n \). Under the substitution \( x_i \rightarrow x_i^{-1} \), the left side of (2) is invariant, so we obtain
\[
\sum_\lambda b_\lambda \overline{s}_\lambda(x_1^{-1}, \ldots, x_n^{-1}) = \sum_\lambda b_\lambda \overline{s}_\lambda(x_1, \ldots, x_n).
\]
However, we know that \( \overline{s}_\lambda(x_1^{-1}, \ldots, x_n^{-1}) = \overline{s}_\lambda(x_1, \ldots, x_n) \) by Proposition 7.10, and so the result follows.

The following lemma connects the character of \( \text{Ext}(gl_n) \) to the product of certain Schur functions. It appears in an equivalent form in [32; I .ex. 3.7].

**Lemma 10.3:** In \( \Omega_n \) we have
\[
\prod_{1 \leq i, j \leq n} (1 + x_i x_j^{-1}) = 2^n \cdot \overline{s}_\delta^2(x_1, \ldots, x_n).
\]

A definition of the partition \( \delta \) was given in (4.3).

**Proof:** By (1) we see that
\[
\prod_{i \neq j} (1 + x_i x_j^{-1}) = 2^n \prod_{i \neq j} (x_i + x_j) = 2^n \prod_{i < j} (x_i + x_j)^2
\]
holds in \( \Omega_n \). On the other hand,
\[
s_\delta(x_1, \ldots, x_n) = \frac{a_{2\delta}(x_1, \ldots, x_n)}{a_\delta(x_1, \ldots, x_n)} = \prod_{i < j} \frac{x_i^2 - x_j^2}{x_i - x_j} = \prod_{i < j} (x_i + x_j)
\]
by Theorem 4.7.

As an application of Lemma 10.3, we may give a combinatorial description of the integers \( E^n[\lambda](1) \); i.e., the multiplicities of the irreducible \( SL_n \)-modules \( V_\lambda \) in the decomposition of \( \text{Ext}(gl_n) \).

**Theorem 10.4:** Let \( \lambda \) be a partition in the \( l \)th layer with \( \ell(\lambda) < n \) and \( l < n \). Let \( \mu \) be the partition \( \lambda + (n - l - 1)^n \). The following numbers are equal:

(a) \( 2^{-n} E^n[\lambda](1) \)

(b) The number of skew tableaux \( T \) of shape \( \mu/\delta \) and weight \( \delta \) such that \( \text{word}(T) \) is a lattice permutation.
(c) The number of standard tableaux \( S \) of shape \( \lambda \) such that for all integers \( i \):
1. If \( i+1 \) occurs in a lower row of \( T \) than \( i \), then \( i \equiv 0 \mod l \).
2. If \( i \not\equiv 0 \mod l \), then \( i-l+1 \) occurs in a higher row of \( T \) than \( i \).

**Proof:** By Lemma 10.3 we know that
\[
2^n \cdot \bar{\sigma}_\delta^2(x_1, \ldots, x_n) = \sum_\lambda E^n[\lambda](1) \bar{\sigma}_\lambda(x_1, \ldots, x_n).
\]

Therefore,
\[
E^n[\lambda](1) = 2^n \langle s_\delta^2, s_\mu \rangle, \tag{3}
\]

where \( \mu \) is the partition we defined above. By version three of the Littlewood-Richardson rule (Theorem 5.7), the description in part (b) follows. To prove that the numbers (b) and (c) are equal, notice that by Proposition 5.12 and (3), it follows that
\[
E^n[\lambda](1) = 2^n \langle s_{\delta+(n-1)n/\delta}, s_{\lambda+(n-l-1)n} \rangle.
\]

By repeated application of the adjoint identity (Remark 5.8), we discover that
\[
E^n[\lambda](1) = 2^n \langle s_{\delta+(n-1)n/\lambda+(n-1)n}, s_\delta \rangle
= 2^n \langle s_{\delta+l\lambda}, s_\delta \rangle
= 2^n \langle s_{\delta+l\lambda/\delta}, s_\lambda \rangle.
\]

If we apply version two of the Littlewood-Richardson rule (Theorem 5.5) to the inner product \( \langle s_{\delta+l\lambda/\delta}, s_\lambda \rangle \), the description in part (c) results.

**Example 10.5:** Let \( n = 4 \) and consider the second layer partition \( \lambda = 422 \). There are three standard tableaux of shape 422, pictured below, which satisfy the constraints 1 and 2. We may conclude that \( E^4[422](1) = 16 \cdot 3 \).

\[
\begin{array}{cccc}
1 & 2 & 6 & 8 \\
3 & 4 & & \\
5 & 7 & & \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 4 & 8 \\
3 & 6 & & \\
5 & 7 & & \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 4 & 6 \\
3 & 5 & & \\
7 & 8 & & \\
\end{array}
\]

For lower layer partitions, the description in part (c) is quite useful, since it involves shapes with only \( ln \) cells instead of roughly \( n^2/2 \). Indeed, since the conditions 1 and 2 above are vacuous when \( l = 1 \), we see that \( E^n[\lambda](1) \) is in one-to-one correspondence with the standard tableaux of shape \( \lambda \). We record this as

**Corollary 10.6:** Let \( \lambda \) be a partition in the first layer (or \( \lambda = 0 \)). We have \( E^n[\lambda](1) = 2^n f^\lambda \).
In section 12 we will refine this result and give an explicit formula for the polynomials \( E^n[\lambda](q) \) for every first layer partition \( \lambda \).

Next, we give necessary conditions which must be satisfied by any partition \( \lambda \) with \( E^n[\lambda] \neq 0 \). We also conjecture that these conditions are sufficient.

**Theorem 10.7:** Let \( \lambda \) be a partition in the \( l \)th layer with \( \ell(\lambda) < n \). If \( E^n[\lambda] \neq 0 \) (i.e., \( V_\lambda \) occurs in the decomposition of \( \text{Ext}(gl_n) \)) then \( l < n \) and

\[
\lambda_1 + \cdots + \lambda_i \leq i(n + l - i) \quad \text{for } 1 \leq i \leq l.
\]

(4)

If \( \mu = \lambda + (n - l - 1)^n \), then (4) is equivalent to having

\[
\mu \leq 2 \cdot \delta
\]

(5)

in the dominance order. If \( \alpha \in \mathbb{Z}^n \) is the dominant weight corresponding to \( \lambda \), then (4) is also equivalent to

\[
\alpha_1 + \cdots + \alpha_i \leq i(n - i) \quad \text{for } 1 \leq i \leq n.
\]

(6)

**Proof:** The equivalence of the conditions (4), (5) and (6) is elementary.

Let \( \lambda \) be a partition for which \( E^n[\lambda] \neq 0 \). As we saw earlier, it certainly must be the case that the layer \( l \) of \( \lambda \) must not exceed \( n - 1 \). Since the coefficients of \( E^n[\lambda] \) are nonnegative, then \( E^n[\lambda] \neq 0 \) implies \( E^n[\lambda](1) \neq 0 \). By Theorem 10.4, there must exist a skew tableau \( T \) of shape \( \mu/\delta = \lambda + (n - l - 1)^n/\delta \) and weight \( \delta \) such that word\( (T) \) is a lattice permutation.

Notice that if \( T \) is a skew tableau of arbitrary shape and weight such that word\( (T) \) is a lattice permutation, then the first \( i \) rows of \( T \) may only be filled with the integers \( 1, \ldots, i \). For example, if there is a 3 in the second row, then this 3 would occur before all of the 2's in word\( (T) \), violating the lattice permutation condition. In the situation at hand, there are

\[
\sum_{1 \leq j \leq i} (\lambda_j + n - l - 1) - (n - j)
\]

cells in the first \( i \) rows of \( \mu/\delta \), which must vie among the

\( (n - 1) + \cdots + (n - i) \)

integers \( \leq i \) available to fill \( T \). Hence,

\[
\sum_{1 \leq j \leq i} (\lambda_j + n - l - 1) \leq \sum_{1 \leq j \leq i} 2(n - j),
\]

from which the inequalities in (4) follow. These constraints are redundant when \( l \leq i \leq n \), since in that case

\[
\lambda_1 + \cdots + \lambda_i \leq ln \leq i(n + l - i)
\]

must hold, regardless of \( \lambda \).\( \bullet \)
11. A splitting theorem

Our goal in this section is to bring more of the combinatorial machinery of Chapter I to bear on the problem of computing the polynomials $E^n[\lambda]$. As an application, we will prove that under certain extremal conditions, $E^n[\lambda]$ can be expressed as a product of polynomials $E^r[\mu]$ for certain partitions $\mu$ and integers $r < n$.

Let $\lambda$ be any partition whose diagram fits inside an $n \times n$ square; i.e., $\lambda \subseteq n^n$. Define a partition $\lambda^*$ via

$$\lambda^* = (n - \lambda'_n, n - \lambda'_{n-1}, \ldots, n - \lambda'_1),$$

which is another partition which fits inside an $n \times n$ square. Pictorially, the rows of $\lambda^*$ are the columns that remain when $\lambda$ is removed from the diagram $n^n$. In fact, one may obtain the diagram of $\lambda^*$ by reflecting the skew shape $n^n/\lambda$ across the line $i + j = n + 1$. For example, if $n = 5$ and $\lambda = 43311$, then $\lambda^* = 5422$. See Figure 11.1.

Proposition 11.1: Let $\lambda$ be a partition in the $l$th layer with $\ell(\lambda) < n$. Let $\nu$ denote the partition $\lambda + (n - l)^n$. We have

$$E^n[\lambda](q) = \sum_{\mu \subseteq n^n} q^{\mu} \langle s_\mu s_{\mu^*}, s_\nu \rangle.$$

Proof: Recall that the identity in Corollary 3.9(b) tells us that

$$\prod_{i,j} (1 + x_i y_j) = \sum_{\lambda} s_\lambda(x)s_{\lambda'}(y).$$

Since the Schur function $s_\lambda$ is homogeneous of degree $|\lambda|$ it follows that

$$\prod_{i,j} (1 + qx_i y_j) = \sum_{\lambda} q^{|\lambda|} s_\lambda(x)s_{\lambda'}(y).$$
The Schur function $s_\lambda$ will vanish when restricted to $x_1, \ldots, x_n$, unless $\ell(\lambda) \leq n$. If we apply the substitution $y_i \to x_i^{-1}$ and Proposition 7.10, we deduce that
\[
\prod_{1 \leq i, j \leq n} (1 + qx_i x_j^{-1}) = \sum_{\mu \subseteq n^n} q^{\mu_1} \bar{s}_\mu(x_1, \ldots, x_n) \bar{s}_{\mu^*}(x_1, \ldots, x_n).
\]
holds in $\Omega_n[q]$. Since $\mu^*$ and $\mu'$ differ only in the number of columns of length $n$, we find that
\[
\prod_{1 \leq i, j \leq n} (1 + qx_i x_j^{-1}) = \sum_{\mu \subseteq n^n} q^{\mu_1} \bar{s}_\mu(x_1, \ldots, x_n) \bar{s}_{\mu^*}(x_1, \ldots, x_n),
\]
also holds in $\Omega_n[q]$. Since $s_\mu s_{\mu^*}$ is homogeneous of degree $n^2$, it follows that
\[
\langle s_\mu s_{\mu^*}, s_{\nu} \rangle
\]
is the coefficient of $\bar{s}_\lambda(x_1, \ldots, x_n)$ in $\bar{s}_\mu \bar{s}_{\mu^*}(x_1, \ldots, x_n)$. The proposition now follows directly from (2).

Another formula similar in spirit to Proposition 11.1 is the following:

**Proposition 11.2:** Let $\lambda$ be a partition in the $l$th layer with $\ell(\lambda) < n$. We have
\[
E^n[\lambda](q) = \sum_{\mu \subseteq n^n} q^{\mu_1} \langle s_{\mu+n^l/\mu'}, s_\lambda \rangle.
\]

**Proof:** Let $\nu$ be the partition described in Proposition 11.1, and let $\mu$ be any partition such that $\mu \subseteq n^n$. We claim that
\[
\langle s_\mu s_{\mu^*}, s_{\nu} \rangle = \langle s_{\mu+n^l/\mu'}, s_\lambda \rangle.
\]
To see this, recall that by Proposition 5.12, we have
\[
\langle s_\mu s_{\mu^*}, s_{\nu} \rangle = \langle s_{\mu+n^n/\mu'}, s_{\nu} \rangle.
\]
By repeated application of the adjoint identity (Remark 5.8), it follows that
\[
\langle s_{\mu+n^n/\mu'}, s_{\nu} \rangle = \langle s_{\mu+n^n/\lambda+(n-l)^n}, s_{\mu'} \rangle
\]
\[
= \langle s_{\mu+l^n/\lambda}, s_{\mu'} \rangle
\]
\[
= \langle s_{\mu+l^n/\mu'}, s_{\lambda} \rangle.
\]
This proves the claim. Apply Proposition 11.1.

**Corollary 11.3:** We have
\[
E^n[\emptyset](q) = (1 + q)(1 + q^2) \cdots (1 + q^{2n-1}).
\]
Proof: Apply Proposition 11.2 to the unique partition of the 0th layer; i.e., let \( \lambda = \emptyset \). Since \( s_{\mu/\mu'} = 0 \) unless \( \mu' \subseteq \mu \), it follows that only self-conjugate partitions \( \mu \) can contribute to the sum in (3). If \( \mu \) is self-conjugate, then we have \( s_{\mu/\mu'} = s_\emptyset \), in which case the contribution is \( q^{s_\emptyset} \).

However, recall that there is a bijection between self-conjugate partitions and partitions into distinct odd parts (Remark 1.5). The restriction \( \mu \subseteq n^n \) on this bijection has the effect of restricting the distinct odd parts to \( 1, 3, \ldots, 2n - 1 \).

We remark that this is a well-known result; both Stanley [39] and D. E. Littlewood [26] have given proofs which are roughly equivalent to ours in that they use Corollary 3.9(b) to show that \( E^n[\emptyset](q) \) is the weight enumerator for self-conjugate partitions.

Consider the invariants of the exterior algebra; i.e., the \( \emptyset \)th isotypic component of the \( SL_n \)-module \( \text{Ext}(gl_n) \), which is given by

\[
\{ t \in \text{Ext}(gl_n) : X \circ t = t \text{ for all } X \in SL_n \}.
\]

Corollary 11.3 suggests that there should exist invariants \( t_i \in \text{Ext}^{2i-1}(gl_n) \) for \( i = 1, \ldots, n \) which freely generate (as an exterior algebra) all of the invariants. This is indeed the case, and is in fact well-known among those who study the classical groups. The polynomial \( E^n[\emptyset] \) is known in those circles as the Poincaré polynomial of \( gl_n \).

Recall from Theorem 10.7 that we know \( E^n[\lambda] = 0 \) unless the \( l \)th layer partition \( \lambda \) satisfies

\[
\lambda_1 + \cdots + \lambda_i \leq i(n + l - i) \quad 1 \leq i \leq l.
\]  \( \text{(5)} \)

The following remarkable result shows that if equality should occur in any of these constraints, then \( E^n[\lambda] \) factors into a product of smaller polynomials \( E^r[\mu] \). It has been brought to my attention that this theorem can also be derived from a result due to Vogan and Zuckerman [44; Prop. 3.6].

**Theorem 11.4**: Let \( \lambda \) be a partition of \( In \) with \( \ell(\lambda) < n \) and \( l < n \). If for some \( s (1 \leq s \leq l) \) we have

\[
\lambda_1 + \cdots + \lambda_s = s(n + l - s)
\]

then

\[
E^n[\lambda](q) = q^{s(n-s)} E^s[\alpha](q) \cdot E^{n-s}[\beta](q)
\]

where \( \alpha \) and \( \beta \) are the partitions defined by

\[
\alpha = (\lambda_1 - \lambda_s, \ldots, \lambda_{s-1} - \lambda_s)
\]

\[
\beta = (\lambda_{s+1}, \ldots, \lambda_{n-1} )
\]
Note that $\alpha$ and $\beta$ belong to the layers $n + l - s - \lambda_s$ and $l - s$, respectively.

Proof: Let $\lambda$, $\alpha$ and $\beta$ be as described above. Note that the theorem is vacuously true if any of the constraints (5) are violated. In such a case, Theorem 10.7 tells us that $E^n[\lambda] = 0$, and it is routine to verify that either $\alpha$ or $\beta$ must violate the $s$- or $n - s$-analogues of (5). We will henceforth assume that the constraints in (5) are satisfied.

Let $\nu$ denote the partition $\lambda + (n - l)^n$ and let $\nu_A$ and $\nu_B$ denote the partitions defined by

$$\nu_A = (\nu_1 - 2(n - s), \ldots, \nu_s - 2(n - s))$$
$$\nu_B = (\nu_{s+1}, \ldots, \nu_n).$$

The fact that $\nu_A$ is a well-defined partition (i.e., $\nu_s \geq 2(n - s)$) follows from (5):

$$\lambda_1 + \cdots + \lambda_{s-1} \leq (s - 1)(n + l - s + 1)$$

and

$$\lambda_1 + \cdots + \lambda_s = s(n + l - s)$$

imply

$$\nu_s = \lambda_s + n - l \geq 2(n - s) + 1.$$

Our proof will proceed by constructing a simple bijection between those partitions $\mu \subseteq n^n$ for which $\langle s_\mu s_\mu^*, s_\nu \rangle \neq 0$ and pairs $(\mu_A, \mu_B)$ of partitions with $\mu_A \subseteq n^n$ and $\mu_B \subseteq (n - s)^{n-s}$ for which $\langle s_{\mu_A} s_{\mu_A^*}, s_{\nu_A} \rangle \neq 0$ and $\langle s_{\mu_B} s_{\mu_B^*}, s_{\nu_B} \rangle \neq 0$. This bijection will have the property that

$$q^{|\mu|} \langle s_{\mu} s_{\mu^*}, s_\nu \rangle = q^{s(n-s)+|\mu_A|+|\mu_B|} \langle s_{\mu_A} s_{\mu_A^*}, s_{\nu_A} \rangle \langle s_{\mu_B} s_{\mu_B^*}, s_{\nu_B} \rangle.$$

Once this has been established, the theorem will follow from the obvious application of Proposition 11.1.

Partition the cells of $n^n$ into four regions as indicated in the following diagram.

$$\begin{array}{c|c|c}
    & n-s & s \\
  \hline
  s & X & A \\
  n-s & B & Y \\
\end{array}$$

Let $\mu \subseteq n^n$ be a partition such that $\langle -s_{\mu} s_{\mu^*}, s_\nu \rangle \neq 0$. By version three of the Littlewood-Richardson rule (Theorem 5.7), we know that there must exist skew tableaux $T$ of shape $\nu/\mu$ and weight $\mu^*$ such that $T$ is a lattice permutation;
\( \langle s_\mu s_{\mu^*}, s_\nu \rangle \) is the number of such tableaux. As we argued in the proof of Theorem 10.7, we know that only the integers 1, \ldots, s may occur in the first s rows of \( \nu/\mu \). There are
\[
\sum_{1 \leq i \leq s} \nu_i - \mu_i = s(n + l - s) + s(n - l) - (\mu_1 + \cdots + \mu_s)
\]
cells in the first s rows, so it must be the case that
\[
\mu_1^* + \cdots + \mu_s^* \geq 2sn - s^2 - (\mu_1 + \cdots + \mu_s) \tag{6}
\]
However, notice that if one uses the pictorial representation of \( \mu \) and \( \mu^* \), the quantity \( (\mu_1 + \cdots + \mu_s) + (\mu_1^* + \cdots + \mu_s^*) \) can be interpreted as the area of a certain subregion of \( X \cup A \cup Y \). The entire area of \( X \cup A \cup Y \) is \( 2sn - s^2 \). Therefore, there must be equality in (6), \( \mu \) must include all of \( X \), and any skew tableau \( T \) as described above may assign the symbols 1, \ldots, s to only the first s rows and the symbols \( s + 1, \ldots, n \) to only the last \( n - s \) rows.

The pair of partitions \( (\mu_A, \mu_B) \) that correspond to \( \mu \) are now easy to describe: \( \mu_A \) consists of those cells of \( \mu \) in region A and \( \mu_B \) consists of those cells of \( \mu \) in region B. Clearly, we have
\[
|\mu| = s(n - s) + |\mu_A| + |\mu_B|.
\]
Let \( T \) be a skew tableau of shape \( \nu/\mu \) with weight \( \mu^* \) such that \textit{word}(\( T \)) is a lattice permutation. Since the symbols assigned to the first s rows are uniformly smaller than those in the last \( n - s \) rows, the tableau constraints and lattice permutation constraints may each be imposed independently on the two sets of rows. That is, the tableaux \( T \) correspond bijectively to the Cartesian product of the subtableaux appearing in the first s rows with the subtableaux appearing in the last \( n - s \) rows.

Clearly, the number of tableaux which appear in the last \( n - s \) rows is \( \langle s_{\mu_B} s_{\mu_B^*}, s_{\nu_B} \rangle \). The number of tableaux appearing in the first s rows is the number of skew tableaux \( S \) of shape \( (\nu_1, \ldots, \nu_s)/(\mu_1, \ldots, \mu_s) \) and type \( (\mu_1^*, \ldots, \mu_s^*) \) such that \textit{word}(\( S \)) is a lattice permutation. Since
\[
(\nu_1, \ldots, \nu_s) = \nu_A + (2n - 2s)^* \\
(\mu_1, \ldots, \mu_s) = \mu_A + (n - s)^* \\
(\mu_1^*, \ldots, \mu_s^*) = \mu_A^* + (n - s)^*,
\]
then this number is, by the Littlewood-Richardson rule and the adjoint identity (Remark 5.8), the inner product
\[
\langle s_{\nu_A + (n-s)^*}/\mu_A, s_{\mu_A^* + (n-s)^*} \rangle = \langle s_{\nu_A}, s_{\mu_A} s_{\mu_A^*} \rangle.
\]
This correspondence between $\mu$ and $(\mu_A, \mu_B)$ is clearly invertible and so the proof is complete.

**Remark 11.5:** If desired, one may reverse the point of view of Theorem 11.4. Given partitions $\alpha$ of $as$ and $\beta$ of $br$ such that $\ell(\alpha) < s$ and $\ell(\beta) < r$, the product

$$q^n E^s[\alpha](q) \cdot E^r[\beta](q)$$

is of the form $E^n[\lambda](q)$, where $n = s + r$ and $\lambda$ is the partition

$$\lambda = (\alpha_1 + n + b - a, \ldots, \alpha_s + n + b - a, \beta_1, \ldots, \beta_{n-s}),$$

which belongs to the layer $b + s$.

**Example 11.6:**

(a) By the previous remark, we can multiply together any of the polynomials $E^n[\lambda]$ which are known and obtain new results. For example, the polynomial $E^n[n^k]$ corresponding to the rectangular partition $n^k$ decomposes as follows:

$$E^n[n^k](q) = q^{k(n-k)} E^k[\emptyset](q) E^{n-k}[\emptyset](q)$$

$$= q^{k(n-k)} \prod_{1 \leq i \leq k} (1 + q^{2i-1}) \cdot \prod_{1 \leq i \leq n-k} (1 + q^{2i-1}).$$

This suggests that the $n^k$-isotypic component of $\text{Ext}(gl_n)$ should have a structure isomorphic to the tensor product of the invariants of $\text{Ext}(gl_k)$ and $\text{Ext}(gl_{n-k})$. Similarly, if we apply Theorem 11.4 repeatedly to the partition $\lambda = 2\delta$, we find

$$E^n[2\delta](q) = q^\binom{n}{2} (1 + q)^n.$$

(b) Let $\lambda$ be a partition of $n(n-1)$ with $\ell(\lambda) < n$. Thus, $\lambda$ belongs to the $(n-1)$th layer. Certainly $\lambda_1 + \cdots + \lambda_{n-1} = n(n-1)$, which is the extreme case of the constraint (5) at $i = n-1$. Applying Theorem 11.4, yields

$$E^n[\lambda](q) = q^{n-1}(1 + q) E^{n-1}[\mu](q),$$

where $\mu$ is the partition defined by

$$\mu = (\lambda_1 - \lambda_{n-1}, \lambda_2 - \lambda_{n-1}, \ldots, \lambda_{n-2} - \lambda_{n-1}).$$

Conversely, given any partition $\mu$ in the $l$th layer with $\ell(\mu) < n-1$ and $l < n-1$, one can multiply $E^{n-1}[\mu]$ and $E^l[\emptyset]$ to obtain $E^n[\lambda]$ for any $(n-1)$th layer partition $\lambda$. Hence, the polynomials $E^n[\lambda]$ corresponding to the $(n-1)$th layer contain all of the information needed to determine all of the polynomials $E^{n-1}[\mu]$ and vice-versa.

To conclude this section, we mention that in Section 13, we will generalize this splitting theorem to the Macdonald complex $M_k(gl_n)$ by utilizing a completely different technique. As a byproduct of this technique, we will also show that if any of the bounds in (5) are "nearly" achieved by a partition $\lambda$, then a similar splitting theorem can be demonstrated.
12. The first layer formula for \( \text{Ext}(gl_n) \)

In this section we will derive a beautiful explicit formula, conjectured by Gupta and Hanlon [15], for the polynomials \( E^n[\lambda] \) corresponding to first layer partitions \( \lambda \). In order to understand the statement of the formula, the reader may wish to review the definitions of the partition statistics \( c(x) \), \( h(x) \) and \( n(\lambda) \) given in Section 1. See (1.1), (1.2), and (1.3). Also, if \( m \) is any nonnegative integer, let

\[
[m!]_q = (1 - q)(1 - q^2) \cdots (1 - q^m).
\]

The first layer formula for the exterior algebra is the following:

**Theorem 12.1:** Let \( \lambda \) be a partition of \( n \). We have

\[
E^n[\lambda](q) = q^{2n(\lambda)}[n!]_q^2 \prod_{x \in \lambda} \frac{q + q^{2c(x)}}{1 - q^{2h(x)}},
\]

or equivalently,

\[
E^n[\lambda](q) = [n!]_q^2 \prod_{x = (i,j) \in \lambda} q^{2i-1} + q^{2j-2} \frac{1}{1 - q^{2h(x)}}.
\]

In other words, the multiplicity of \( V_\lambda \) in the decomposition of \( \text{Ext}^r(gl_n) \) into irreducibles is the coefficient of \( q^r \) in (1) or (2).

Although we would normally forbid \( \lambda = 1^n \) in Theorem 12.1 since \( 1^n \) is a partition of length \( n \), notice that (1) and (2) are still valid, as they yield the formula for \( E^n[\emptyset](q) \) we derived in Corollary 11.3.

If we let \( q \to 1 \) in (1) or (2), we obtain a formula for \( E^n[\lambda](1) \), the ungraded multiplicity of \( V_\lambda \) in \( \text{Ext}(gl_n) \):

\[
E^n[\lambda](1) = 2^n n! \prod_{x \in \lambda} \frac{1}{h(x)}.
\]

On the other hand, we already know that \( 2^{-n} E^n[\lambda](1) \), for a first layer partition, is the number of standard tableaux of shape \( \lambda \) (Corollary 10.6). Hence, it follows that

\[
f^\lambda = n! \prod_{x \in \lambda} \frac{1}{h(x)},
\]

which is the famous hooklength formula of Frame, Robinson and Thrall [7]. However, this circuitous route would hardly qualify as a reasonable proof of the hooklength formula. A short proof can be found in [32; I. ex. 3.4].
Proof: Let $\lambda$ be a partition of $n$. By Proposition 11.2, we know that
\[ E^n[\lambda](q) = \sum_{\mu \subseteq n^n} q^{\mu} \langle s_{\mu + 1^n/\mu'}, s_\lambda \rangle. \] (3)
This is valid even for the partition $\lambda = 1^n$. By version two of the Littlewood-Richardson rule (Theorem 5.6), we know that the coefficients appearing in (3) can be expressed as the number of standard tableaux of shape $\lambda$ satisfying certain conditions. Our strategy is to use this combinatorial interpretation of (3) to derive a recursion for the polynomials $E^n[\lambda]$ and then to use the theory of symmetric functions to show that the claimed formula (1) satisfies the same recursion.

The first step is to realize that the partitions $\mu$ and $\mu^*$ (see (11.1)) make similar contributions to (3). More precisely, recall from (11.4) that we have
\[ \langle s_{\mu + 1^n/\mu'}, s_\lambda \rangle = \langle s_{\mu} s_{\mu^*}, s_{\lambda + (n-1)n} \rangle, \]
which is clearly invariant if we replace $\mu$ by $\mu^*$. This suggests that we may cut the range of study in (3) by half if we adopt a scheme which picks out one partition from each pair $\{\mu, \mu^*\}$ for $\mu \subseteq n^n$. The scheme we will use is based on the fact that for all such $\mu$, either $(1, n) \in \mu$ or $(1, n) \in \mu^*$, but not both. This is readily apparent from the definition of $\mu^*$. Thus, we are led to define polynomials $A^n[\lambda](q)$ as follows:
\[ A^n[\lambda](q) = \sum_{\mu \subseteq n^n, \mu^1 < n} q^{\mu} \langle s_{\mu + 1^n/\mu'}, s_\lambda \rangle. \] (4)
By the previous discussion and the fact that $|\mu| + |\mu^*| = n^2$, it follows that
\[ E^n[\lambda](q) = A^n[\lambda](q) + q^{n^2} A^n[\lambda](q^{-1}). \] (5)
Thus, it suffices to find a recursion for $A^n[\lambda]$. To do this will require a careful study of the Littlewood-Richardson coefficients in (4).

Let us define
\[ A_\lambda = \left\{ (\mu, T) : \mu \subseteq n^n, \mu^1 \subseteq \mu + 1^n, \text{ and } T \text{ is a standard tableau of shape } \lambda \text{ satisfying rules 1 and 2 below} \right\}. \]

1. If the integers $a$ and $a + 1$ occur in the same row of the canonical labelling of $\mu + 1^n/\mu'$, then $a + 1$ occurs in a row at least as high as $a$ in $T$.
2. If the integer $b$ occurs directly below $a$ in the canonical labelling of $\mu + 1^n/\mu'$, then $b$ occurs in a lower row of $T$ than $a$. 

By version two of the Littlewood-Richardson rule (Theorem 5.6), formulas (3) and (4) may be rewritten as
\[
E^n[\lambda](q) = \sum_{(\mu,T) \in A_\lambda} q^{[\mu]} \quad \text{and} \quad A^n[\lambda](q) = \sum_{(\mu,T) \in A_\lambda, \mu_1 < n} q^{[\mu]}.
\] (6)

The recursion we give will be found by monitoring the row to which the integer \(n\) is assigned by the tableaux in \(A_\lambda\). Thus, we define polynomials \(A^n_i[\lambda]\) so that
\[
A^n_i[\lambda](q) = \sum_{(\mu,T) \in A_\lambda} q^{[\mu]} : \mu_1 < n \text{ and } T \text{ assigns } n \text{ to the } i\text{th row}.
\] (7)

Therefore,
\[
A^n[\lambda] = \sum_i A^n_i[\lambda].
\]

Notice that in any standard tableau of shape \(\lambda\), the cell which is assigned to \(n\) must be a corner cell; i.e., a cell of \(\lambda\) which has no other cells below or to the right of it. Thus, there is a corner cell in the \(i\)th row of \(\lambda\) if and only if \(\lambda_i > \lambda_{i+1}\). In particular, we have \(A^n_i[\lambda] = 0\) unless there is a corner cell in the \(i\)th row.

**Lemma 12.2:** Assume that there is a corner cell in the \(i\)th row of \(\lambda\). Let \(\lambda^{(i)}\) denote the partition obtained by deleting this cell from \(\lambda\). We have
\[
A^n_i[\lambda](q) = E^{n-1}[\lambda^{(i)}](q) - (1 - q^{2n-2}) \sum_{j \geq i} A_{j}^{n-1}[\lambda^{(i)}](q).
\]

**Proof:** Let \(\mu\) be a partition for which \(\mu \subseteq n^n\) and \(\mu' \subseteq \mu + 1^n\). Geometrically, one can see that the skew diagram \(\mu + 1^n/\mu'\) can be obtained by sliding the diagram of \(\mu\) one column to the right, reflecting the original diagram across the diagonal \(i = j\), and removing this diagram from the former. For example, take \(n = 6\) and let \(\mu = 54311\). The skew diagram \(\mu + 1^n/\mu'\) and its canonical labelling are depicted in Figure 12.1. When \(\mu_1 < n\), notice that the cell \((n, 1)\) belongs to \(\mu + 1^n/\mu'\); its canonical label will always be \(n\).

We will analyze the sum (7) by focusing on the relative placement of the cells which are labelled \(n\) and \(n - 1\) in the canonical labelling.

**Case 1:** Suppose that \(\mu_1 < n\) and \(\mu'_1 = n\).

In this case we must have \(\mu_i = n - 1\); otherwise \(\mu' \not\subseteq \mu + 1^n\). We have outlined below a typical example of such a partition.
Figure 12.1: The canonical labelling of $\mu + 1^n/\mu'$.

Note that $\nu = (\lambda_2 - 1, \ldots, \lambda_n - 1)$ is some partition such that $\nu \subseteq (n - 1)^{n-1}$ and $\nu_1 < n - 1$. Clearly, we have

$$|\mu| = 2n - 2 + |\nu|.$$

Observe that $\mu' \subseteq \mu + 1^n$ if and only if $\nu' \subseteq \nu + 1^{n-1}$. Furthermore, the canonical labelling of $\mu + 1^n/\mu'$ is obtained from that of $\nu + 1^{n-1}/\nu'$ by adding a cell labelled $n$ to the immediate left of the cell labelled $n - 1$ in $\nu + 1^{n-1}/\nu'$. For example, take $n = 5$ and $\mu = 43311$. The canonical labelling of $\mu + 1^n/\mu'$ is depicted in Figure 12.2.

Let $T$ be a standard tableau such that $(\mu, T) \in A_\lambda$, and assume that $n$ is assigned to the $i$th row of $T$. Let $S$ denote the standard tableau of shape $\lambda^{(i)}$ obtained by deleting the cell containing $n$. We have $(\nu, S) \in A_{\lambda^{(i)}}$, and rule 1 tells us that the integer $n - 1$ must be assigned to some row $j$ with $j \geq i$. These conditions characterize $(\nu, S)$. Therefore,

$$\sum_{(\mu, T) \in A_\lambda} q^{|\mu|} : \mu_1 < n, \mu'_1 = n \quad T \text{ assigns } n \text{ to } i \text{th row}$$

$$= q^{2n-2} \sum_{(\nu, S) \in A_{\lambda^{(i)}}} q^{|\nu|} : \nu_1 < n - 1, j \geq i \quad S \text{ assigns } n - 1 \text{ to } j \text{th row}$$

$$= q^{2n-2} \sum_{j \geq i} A_j^{n-1}[\lambda^{(i)}](q), \quad (8)$$

by the definition (7).

Case 2: Suppose that $\mu_1 < n$ and $\mu'_1 < n$.

In this case, $\mu$ itself fits inside the square diagram $(n - 1)^{n-1}$, and we have $\mu' \subseteq \mu + 1^n$ if and only if $\mu' \subseteq \mu + 1^{n-1}$. The canonical labelling of $\mu + 1^n/\mu'$ can
be obtained from $\mu + 1^{n-1}/\mu'$ by labelling the cell $(n, 1)$ with the integer $n$. This is the only cell of $\mu + 1^n/\mu'$ in the $n$th row.

Let $T$ be a standard tableau with $n$ assigned to the $i$th row and so that $(\mu, T) \in A_{\lambda}$. Let $S$ denote the standard tableau of shape $\lambda^{(i)}$ obtained by deleting the cell containing $n$. If it should happen that $\mu_1 = n - 1$, then $(n, 1)$ is the only cell of $\mu + 1^n/\mu'$ in the first column. The example in Figure 12.1 illustrates this phenomenon. Thus, there are no constraints implied by rules 1 and 2 regarding the placement of the integer $n$ relative to the integers $1, \ldots, n - 1$, aside from the usual ones associated with standard tableaux.

Otherwise, if it should happen that $\mu_1 < n - 1$, then the cell $(n - 1, 1)$ will also belong to $\mu + 1^n/\mu'$; its canonical label is the integer $n - 1$. By rule 2, we see that $n - 1$ must be assigned to a row $j$ with $j < i$.

To summarize, the only pairs $(\mu, S) \in A_{\lambda^{(i)}}$ with $\mu_1, \mu'_1 < n$ which are forbidden are those with $\mu_1 < n - 1$ and $n - 1$ assigned to the $j$th with $j \geq i$. Therefore, we have

$$\sum_{(\mu, T) \in A_{\lambda}} q^{[\mu]} : \quad \mu_1, \mu'_1 < n \quad T \text{ assigns } n \text{ to } i \text{th row}$$

$$= \sum_{(\mu, S) \in A_{\lambda^{(i)}}} q^{[\mu]} - \sum_{(\mu, S) \in A_{\lambda^{(i)}}} q^{[\mu]} : \quad \mu_1 < n - 1, \quad j \geq i \quad S \text{ assigns } n - 1 \text{ to } j \text{th row}$$

$$= E^{n-1}[\lambda^{(i)}](q) - \sum_{j \geq i} A_i^{n-1}[\lambda^{(i)}](q), \quad (9)$$

by the identities (6) and (7).

If we add the contributions (8) and (9) from Cases 1 and 2, we find

$$A_i^n[\lambda](q) = \sum_{(\mu, T) \in A_{\lambda}} q^{[\mu]} : \quad \mu_1 < n, \quad T \text{ assigns } n \text{ to the } i \text{th row}$$
\[ E^{n-1}(\lambda^{(i)})(q) - (1 - q^{2n-2}) \sum_{j \geq i} A^{n-1}_j(\lambda^{(i)})(q). \]

This completes the proof of the lemma.●

Recall that a horizontal \( r \)-strip is a skew diagram of \( r \) cells with at most one cell in each column. If \( \lambda \) is any partition, let

\[ H_r(\lambda) = \{ \mu \subseteq \lambda : \lambda / \mu \text{ is a horizontal } r \text{-strip} \}. \]

**Lemma 12.3:** We have

\[ A^n(\lambda)(q) = \sum_{r>0} \sum_{\mu \in H_r(\lambda)} (-1)^{r-1}(1 - q^{2n-2})(1 - q^{2n-4}) \cdots (1 - q^{2n-2r+2}) E^{n-r}[\mu](q). \]

**Proof:** By Lemma 12.2, we have

\[ A^n(\lambda)(q) = \sum_i E^{n-1}[\lambda^{(i)}](q) - (1 - q^{2n-2}) \sum_{j \geq i} A^{n-1}_j(\lambda^{(i)})(q), \]

where \( i \) ranges over those rows for which there is a corner cell in the diagram of \( \lambda \). If we apply Lemma 12.2 again, we obtain

\[ A^n(\lambda)(q) = \sum_i E^{n-1}[\lambda^{(i)}](q) - (1 - q^{2n-2}) \sum_{j \geq i} E^{n-2}[\lambda^{(i,j)}](q) \]

\[ + (1 - q^{2n-2})(1 - q^{2n-4}) \sum_{k \geq j \geq i} A^{n-2}_k[\lambda^{(i,j)}](q), \]

where \( j \) ranges over those rows for which there is a corner cell in the diagram of \( \lambda^{(i)} \), and \( \lambda^{(i,j)} \) denotes the result of deleting that cell from \( \lambda^{(i)} \). More generally, we have

\[ A^n(\lambda)(q) = \sum_{r>0} \sum_{i_1 \leq \cdots \leq i_r} (-1)^{r-1}(1 - q^{2n-2}) \cdots (1 - q^{2n-2r+2}) E^{n-r}[\lambda^{(i_1,\ldots,i_r)}](q), \]

where the indices \( i_1 \leq \cdots \leq i_r \) are restricted to those integers for which there is a corner cell in row \( i_k \) of \( \lambda^{(i_1,\ldots,i_{k-1})} \) for \( k = 2, \ldots, r \). This is a finite sum as eventually there will be no more cells to remove.

Let \( \mu = \lambda^{(i_1,\ldots,i_r)} \). Note that the conditions on the indices \( i_1, \ldots, i_r \) are equivalent to the condition that \( \lambda / \mu \) forms a horizontal \( r \)-strip. This is the case because if \( \lambda / \mu \) had a column with two cells in it; say rows \( j \) and \( j + 1 \), then the cell corresponding to row \( j + 1 \) would have been removed before the cell corresponding to row \( j \). This would violate the restriction that \( i_1 \leq \cdots \leq i_r \). Conversely, given
a horizontal strip \( \lambda/\mu \), it is easy to recover the integers \( i_1 \leq \cdots \leq i_r \) for which 
\( \mu = \lambda^{(i_1, \ldots, i_r)} \).

In view of (5), we are now prepared to describe a recursion for the polynomials 
\( E^n[\lambda] \). By Proposition 10.1 and Lemma 12.3, it follows that

\[
q^n A^n[\lambda](q^{-1}) = \sum_{r > 0} \sum_{\mu \in H_r(\lambda)} (-1)^{r-1} q^n (1 - \frac{1}{q^{2n-2r+2}}) \cdots (1 - \frac{1}{q^{2n-2r+2}}) E^{n-r}[\mu](q^{-1})
\]

\[
= \sum_{r > 0} \sum_{\mu \in H_r(\lambda)} q^{2n-r} (1 - q^{2n-2}) \cdots (1 - q^{2n-2r+2}) E^{n-r}[\mu](q). \quad (10)
\]

Hence, if we add the contributions of Lemma 12.3 and (10), we obtain

\[
E^n[\lambda](q) = \sum_{r > 0} \sum_{\mu \in H_r(\lambda)} (q^{2n-r} + (-1)^{r-1})(1 - q^{2n-2}) \cdots (1 - q^{2n-2r+2}) E^{n-r}[\mu](q).
\]

Notice that this recursion uniquely determines the polynomials \( E^n[\lambda] \) up to the initial conditions on \( E^0[\emptyset] \).

This recursion may be recast into a slightly more pleasant form. If we multiply by \( 1 - q^{2n} \) and add the term corresponding to \( \mu = \lambda \) (the empty horizontal strip), we find

\[
\sum_{r \geq 0} \sum_{\mu \in H_r(\lambda)} ((-1)^r - q^{2n-r})(1 - q^{2n}) \cdots (1 - q^{2n-2r+2}) E^{n-r}[\mu](q) = 0,
\]

or equivalently,

\[
\sum_{r \geq 0} \sum_{\mu \in H_r(\lambda)} ((-1)^r - q^{2n-r}) \frac{E^{n-r}[\mu](q)}{[(n-r)!]q^s} = 0. \quad (11)
\]

These relations, along with the fact that \( E^0[\emptyset] = 1 \), uniquely determine the first layer polynomials \( E^n[\lambda] \).

Now we will apply the theory of symmetric functions to solve this recursion. Recall the notation \( f(p_r \rightarrow a_r) \) which was defined in formula (9.9).

**Lemma 12.4:** Let \( \{\psi_\lambda(q)\} \) be a collection of formal power series with \( \psi_\emptyset = 1 \). Suppose that for each partition \( \lambda \),

\[
\sum_{r \geq 0} \sum_{\mu \in H_r(\lambda)} ((-1)^r - q^{2n-r}) \psi_\mu(q) = 0. \quad (12)
\]

One may then conclude that

\[
\psi_\lambda(q) = s_\lambda \left( p_r \rightarrow \frac{q^r - (-1)^r}{1 - q^{2r}} \right). \quad (13)
\]
Proof: It is not difficult to see that the relations (12) uniquely determine the \( \psi_\lambda \)'s, so it suffices to show that the claimed solution (13) satisfies these relations.

Let \( a \) be an indeterminate. We claim that

\[
\sum_{r \geq 0} \sum_{\mu \in \mathcal{H}_r(\lambda)} a^r s_\mu(x_1, x_2, \ldots) = s_\lambda(a, x_1, x_2, \ldots).
\]

This is essentially a special case of a well-known identity (e.g., [32; I.(5.7)]), but it is easy also easy to see directly:

By the definition of the Schur functions, and the fact that they are symmetric, we have

\[
s_\lambda(a, x_1, \ldots, x_m) = s_\lambda(x_1, \ldots, x_m, a) = \sum_T x_1^{\alpha_1(T)} \cdots x_m^{\alpha_m(T)} a^{\alpha_{m+1}(T)},
\]

where the sum ranges over all tableaux \( T \) with parts \( \leq m + 1 \) and shape \( \lambda \), and \( \alpha_i(T) \) denotes the number of \( i \)'s which occur in \( T \). Notice that we can partition \( T \) into a tableau of shape \( \mu \) consisting of the parts \( \leq m \) and a horizontal strip of shape \( \lambda / \mu \) consisting of the parts equal to \( m + 1 \). These tableaux are otherwise arbitrary. Therefore, formula (14) asserts that

\[
s_\lambda(a, x_1, \ldots, x_m) = \sum_{r \geq 0} \sum_{\mu \in \mathcal{H}_r(\lambda)} a^r \sum_T x_1^{\alpha_1(T)} \cdots x_m^{\alpha_m(T)},
\]

where \( T \) ranges over all tableaux with parts \( \leq m \) and shape \( \mu \). The claim now follows by taking \( m \) sufficiently large.

In particular, it follows that

\[
\sum_{r \geq 0} \sum_{\mu \in \mathcal{H}_r(\lambda)} (-1)^r s_\mu(x_1, x_2, \ldots) = s_\lambda(-1, x_1, x_2, \ldots)
\]

and

\[
\sum_{r \geq 0} \sum_{\mu \in \mathcal{H}_r(\lambda)} q^{2n-r} s_\mu(x_1, x_2, \ldots) = s_\lambda(q, q^2 x_1, q^2 x_2, \ldots).
\]

Hence, we have reduced the problem to showing that the images of

\[
s_\lambda(-1, x_1, x_2, \ldots) \quad \text{and} \quad s_\lambda(q, q^2 x_1, q^2 x_2, \ldots)
\]

under the substitution

\[
p_r(x) \rightarrow \frac{q^r - (-1)^r}{1 - q^{2r}}
\]

are the same. To do this, it suffices to show that

\[
p_r(-1, x_1, x_2, \ldots) = p_r(q, q^2 x_1, q^2 x_2, \ldots)
\]
under this substitution, since the power sums \( p_r \) generate the algebra \( \Lambda_Q \). However this is obvious, since

\[
p_r(x) = \frac{q^r - (-1)^r}{1 - q^{2r}}
\]

implies

\[
(-1)^r + p_r(x) = q^r + q^{2r}p_r(x).
\]

We remark that the specialization of Schur functions, of which (13) is a typical example, have been studied by D. E. Littlewood [28; Chp. VII]. The following formula for specializing Schur functions is implicit in his work. An explicit proof can be found in [32; I. exs. 2.5, 3.3].

**Proposition 12.5:** Let \( a, b, q \) be indeterminates. We have

\[
s_\lambda \left( p_r \rightarrow \frac{a^r - b^r}{1 - q^r} \right) = q^{n(\lambda)} \prod_{x \in \lambda} \frac{a - bq^{c(x)}}{1 - q^{h(x)}}.
\]

At last, we have the tools to finish the proof of the first layer formula. By comparing (11) and Lemma 12.4, we deduce that

\[
E^n[\lambda](q) = [n!]q^a \cdot s_\lambda \left( p_r \rightarrow \frac{q^r - (-1)^r}{1 - q^{2r}} \right).
\]

If we apply Proposition 12.5 in the special case \( a \rightarrow q, b \rightarrow -1, q \rightarrow q^2 \), we obtain the first layer formula.

A generalization of the first layer formula to higher layers would be desirable. Indeed, if we combine Proposition 10.2, the splitting theorem (Theorem 11.4), and the first layer formula, we can obtain "nice" formulas for many of the polynomials \( E^n[\lambda] \) in higher layers. However, there still seem to be many other partitions \( \lambda \) for which \( E^n[\lambda] \) are nice, but for which the above techniques shed no light. Although we prefer to leave to the reader a characterization of "nice", we do mention that many of the polynomials \( E^n[\lambda] \) are of the form

\[
q^a \frac{(1 \pm q^{b_1}) \cdots (1 \pm q^{b_r})}{(1 \pm q^{a_1}) \cdots (1 \pm q^{a_s})}
\]

for some integers \( a, b_1, \ldots, b_r, c_1, \ldots, c_s \) and choice of signs.

For integers \( a_1, \ldots, a_r, b_1, \ldots, b_r \) and an indeterminate \( q \), introduce the notation

\[
\left[ \frac{a_1 \cdots a_r}{b_1 \cdots b_r} \right]_q = \frac{(1 - q^{a_1}) \cdots (1 - q^{a_r})}{(1 - q^{b_1}) \cdots (1 - q^{b_r})}.
\]
Using the techniques of chapter IV, it is easy to show that the family of second layer partitions corresponding to the dominant weights \([2, 2]_n (n \geq 3)\) satisfy:

\[
E[2, 2]_n = q^3(1 + q)^2E^{n-2}[\emptyset](q) \cdot \left[ \frac{(n - 1)(n - 2)}{1 \cdot 2} \right]_{q^2}, \tag{16}
\]

Similarly, the polynomials \(E[\alpha, \beta]_n\) with \(|\alpha| = |\beta| \leq 3\) can all be shown to be of the form in (15). All of the polynomials \(E^n[\lambda]\) with \(n \leq 4\) are also of this form; however, there is a single exception in the case \(n = 5\), corresponding to the partition \(\lambda = 442\). Indeed, any attempted generalization of the first layer formula cannot be too successful if it should take into account \(\lambda = 442\) or any of the family of partitions indexed by the dominant weights \([22, 22]_n (n \geq 5)\). Given the methods developed in Chapter IV and sufficient patience, one can show that

\[
E[22, 22]_n(q) = q^7(1 + q)^3(1 + q^3)E^{n-4}[\emptyset](q) \left[ \frac{4 \cdot (n - 2)(n - 3)(n - 3)(n - 4)}{1 \cdot 2 \cdot 2 \cdot 3} \right]_{q^2} + q^7(1 + q)^2(1 + q^3)E^{n-3}[\emptyset](q) \left[ \frac{(n - 2)(n - 3)(n - 4)}{1 \cdot 1 \cdot 3} \right]_{q^3} + q^4(1 + q)(1 + q^3)E^{n-2}[\emptyset](q) \left[ \frac{(n - 2)(n - 3)}{1 \cdot 2} \right]_{q^2}, \tag{17}
\]

which does not seem to simplify in any meaningful way.

The formulas (16) and (17) were found by using a recursion which will be described in Chapter IV for decomposing the Macdonald complex (Theorem 16.1). It is of a rather different nature than (11) in that it gives linear relations satisfied by the polynomials \(M^*_n[\lambda]\) for any layer. In section 17, we will use this recursion to generalize the first layer formula to the Macdonald complex.

### 13. More about splitting

In this section, we will apply a technique developed in Section 4 which we have so far neglected to utilize; namely, computing coefficients of Schur functions via extracting coefficients of monomials. See Remark 4.8. We will use this idea to extend the splitting theorem (Theorem 11.4) to the Macdonald complex, and derive conditions analogous to those in Theorem 10.7 on the partitions \(\lambda\) which are necessary for \(M^*_n[\lambda] \neq 0\). As a byproduct of this technique, we will show that even if the conditions for splitting are not met, but are nearly achieved, a similar factorization can be given which yields more information about the structure of the polynomials \(M^*_n[\lambda]\) and \(E^n[\lambda]\).
Let \( y_1, \ldots, y_k \) be indeterminates, and let \( \lambda \) vary over partitions with \( \ell(\lambda) < n \) and \( |\lambda| \) divisible by \( n \). Define polynomials \( b_\lambda^n \in \mathbb{Z}[y_1, \ldots, y_k] \) so that
\[
\prod_{1 \leq i, j \leq n} (1 + y_i x_i x_j^{-1}) \cdots (1 + y_k x_i x_j^{-1}) = \sum_\lambda b_\lambda^n(y_1, \ldots, y_k) \varphi_\lambda(x_1, \ldots, x_n)
\]
holds in \( \Omega_n[y_1, \ldots, y_k] \). As we have done in previous instances, we may at times elect to index these polynomials by dominant weights rather than partitions. Recall from Section 9 that these polynomials are essentially weight enumerators for the decomposition of \( T^k(\text{Ext}(gl_n)) \), since the coefficient
\[
[y^\alpha] b_\lambda^n(y_1, \ldots, y_k)
\]
is the multiplicity of the irreducible \( SL_n \)-module \( V_\lambda \) in the \( \alpha \)th graded submodule of \( T^k(\text{Ext}(gl_n)) \). Observe that by the definition of the polynomials \( M_\alpha^n[\lambda] \), we have
\[
M_\alpha^n[\lambda](q) = b_\lambda^n(-q, -q^2, \ldots, -q^k).
\] (1)

Let \( z = (z_1, z_2, \ldots) \) be indeterminates, and for each \( \alpha \in \mathbb{Z}^n \) with \( |\alpha| = 0 \), let \( F_\alpha(z) \) be the symmetric formal power series defined by
\[
F_\alpha(z_1, z_2, \ldots) = [x^\alpha] \prod_{r \geq 1} \prod_{1 \leq i < j \leq n} (1 + z_r x_i x_j^{-1}).
\]

**Lemma 13.1:** Let \( \alpha \in \mathbb{Z}^n \) be a dominant weight. We have
\[
b_\alpha^n(y_1, \ldots, y_k) = \prod_{1 \leq i \leq k} y_i^{(2)} (1 + y_i)^n \cdot F_\beta(-1, y_1, y_1^{-1}, \ldots, y_k, y_k^{-1}),
\]
where \( \beta \in \mathbb{Z}^n \) is given by \( \beta_i = \alpha_i - k(n - 2i + 1) \).

**Proof:** Let \( \gamma_i = k(n - 2i + 1) \). We have
\[
\prod_{1 \leq i, j \leq n} (1 + y_1 x_i x_j^{-1}) \cdots (1 + y_k x_i x_j^{-1})
\]
\[
= \prod_{1 \leq i \leq k} (1 + y_i)^n \prod_{1 \leq r \leq k} \prod_{i < j} y_r x_i x_j^{-1} (1 + y_r x_j x_i^{-1})(1 + y_r^{-1} x_j x_i^{-1})
\]
\[
= \prod_{1 \leq i \leq k} y_i^{(2)} (1 + y_i)^n \cdot x^\gamma \prod_{1 \leq r \leq k} \prod_{i < j} (1 + y_r x_j x_i^{-1})(1 + y_r^{-1} x_j x_i^{-1}).
\] (2)
On the other hand, by Remarks 4.8 and Remark 4.11(a), we have

\[ b_\alpha^n(y_1, \ldots, y_k) = [x^{\alpha+\delta}]a_\delta(x) \prod_{1 \leq r \leq k} \prod_{1 \leq i, j \leq n} (1 + y_r x_i x_j^{-1}) \]

\[ = [x^{\alpha}] \prod_{i < j} (1 - x_i x_j^{-1}) \prod_{1 \leq r \leq k} \prod_{1 \leq i, j \leq n} (1 + y_r x_i x_j^{-1}). \quad (3) \]

The lemma follows upon comparison of (2) and (3). •

Notice that the vectors \( \beta \in \mathbb{Z}^n \) for which \( F_\beta(z) \neq 0 \) can be described as the \( \mathbb{N} \)-span of the vectors \( \varepsilon_i - \varepsilon_j \) \((i > j)\), where \( \varepsilon_1, \ldots, \varepsilon_n \) denote the standard basis of \( \mathbb{Z}^n \). One can easily show, by induction for example, that these vectors can also be described as the set

\[ \Phi_n = \{ \beta \in \mathbb{Z}^n : |\beta| = 0, \beta_1 + \cdots + \beta_n \geq 0 \text{ for } 1 \leq i \leq n \}, \]

or equivalently,

\[ \Phi_n = \{ \beta \in \mathbb{Z}^n : |\beta| = 0, \beta_1 + \cdots + \beta_i \leq 0 \text{ for } 1 \leq i \leq n \}. \]

This observation allows us to give necessary conditions which must be satisfied by any partition \( \lambda \) with \( M^\alpha_\beta[\lambda] \neq 0 \) (cf. Theorem 10.7).

**Proposition 13.2:** Let \( \lambda \) be a partition of \( ln \) with \( \ell(\lambda) < n \). The polynomial \( b_\alpha^n(y_1, \ldots, y_k) \) vanishes unless \( l \leq k(n - 1) \) and

\[ \lambda_1 + \cdots + \lambda_i \leq i(l + k(n - i)) \quad \text{for all } i \leq l/k. \quad (4) \]

In particular, \( M^\alpha_\beta[\lambda] = 0 \) unless (4) is satisfied. Furthermore, if \( \alpha \in \mathbb{Z}^n \) is the dominant weight corresponding to \( \lambda \), then (4) is equivalent to

\[ \alpha_1 + \cdots + \alpha_i \leq ik(n - i) \quad \text{for all } 1 \leq i \leq n. \quad (5) \]

If \( \mu = \lambda + (kn - k - l)n \), then (4) is also equivalent to the condition

\[ \mu \leq 2k \cdot \delta. \quad (6) \]

**Proof:** The equivalence of (4), (5) and (6) is immediate, although we should mention that we are allowed to restrict the inequalities in (4) to \( i \leq l/k \) because \( i(l + k(n - i)) \geq ln \) whenever \( l/k \leq i \leq n \).

By Lemma 13.1, we see that if \( \alpha \in \mathbb{Z}^n \) is the dominant weight corresponding to \( \lambda \), and \( \beta_i = \alpha_i - k(n - 2i + 1) \), then

\[ b_\alpha^n \neq 0 \quad \text{if and only if} \quad F_\beta(-1, y_1, y_1^{-1}, \ldots, y_k, y_k^{-1}) \neq 0. \]
By the previous discussion, we know that $F_{\beta}(z) \neq 0$ implies $\beta \in \Phi_n$. Since

$$\beta_1 + \cdots + \beta_i = \alpha_1 + \cdots + \alpha_i - ik(n - i),$$

then the inequalities in (5) follow.

Whether these inequalities characterize the partitions $\lambda$ for which $M_k^\lambda \neq 0$ is unknown but seems plausible.

If $A = [a_{ij}]$ is any matrix, let $r(A)$ denote the row sum vector of $A$. If $A$ is skew-symmetric, call $A$ nonnegative if $a_{ij} \geq 0$ for $i > j$. Extend the definition of the elementary symmetric functions $e_r$ to nonnegative skew-symmetric matrices by defining

$$e_A(z) = \prod_{i > j} e_{a_{ij}}(z).$$

**Lemma 13.3:** Let $\beta \in \mathbb{Z}^n$. We have

$$F_{\beta}(z) = \sum_{r(A) = \beta} e_A(z)$$

where the sum ranges over nonnegative skew-symmetric $n \times n$ matrices $A$.

**Proof:** Observe that

$$\prod_{1 \leq i < j \leq n} (1 + z_j x_j x_i^{-1}) = \prod_{i > j} \sum_{r \geq 0} e_r(z)(x_i x_j^{-1})^r = \sum_A e_A(z)x^{r(A)},$$

summed over all nonnegative skew-symmetric $n \times n$ matrices $A$. Choosing $a_{ij} = r$ in $A$ corresponds to choosing the term $e_r(z)(x_i x_j^{-1})^r$. Extract the coefficient of $x^{\beta}$.

Since $2 \times 2$ skew symmetric matrices have a rather trivial structure, we may immediately write down the polynomials $M_k^2[\lambda]$.

**Corollary 13.4:** For dominant weights of the form $(r, -r)$ ($0 \leq r \leq k$) we have

$$M_k^2((r, -r))(q) = (-1)^r q^{r+1} \frac{(1 - q^k) \cdots (1 - q^{k-r+1})}{(1 - q^{k+1}) \cdots (1 - q^{k+r+1})} \cdot \frac{[2k + 1]_q!}{1 - q^{k+1}}.$$

**Proof:** By Lemma 13.1, we have

$$b^2_{(r, -r)}(y_1, \ldots, y_k) = \prod_{1 \leq i \leq k} y_i(1 + y_i)^2 \cdot F_{(r-k, k-r)}(-1, y_1, y_1^{-1}, \ldots, y_k, y_k^{-1}).$$

Lemma 13.3 implies that $F_{(r-k, k-r)}(z) = e_{k-r}(z)$. Applying (1) yields

$$M_k^2((r, -r))(q) = (-1)^k q^{k+1} [k]_q! \cdot (-1)^{k-r} q^{-k(r-k)} e_{k-r}(1, q, q^2, \ldots, q^{2k}).$$
It is well-known and easy to show (e.g., [32; I. ex. 2.3]) that
\[ e_r(1, q, \ldots, q^{m-1}) = q^\binom{r}{2} \frac{[m!_q]}{[r!_q][(m - r)!_q]} . \]

The claimed formula now follows easily.

We are now in a position to extend the splitting theorem to the Macdonald complex \( M_k(gl_n) \) (cf. Theorem 11.4). By analogy with the case of the exterior algebra, this result is concerned with the consequences of equality in any of the constraints in (4), or equivalently, (5) or (6).

**Theorem 13.5:** Let \( \lambda \) be a partition of \( ln \) with \( \ell(\lambda) < n \). If
\[ \lambda_1 + \cdots + \lambda_s = s(l + k(n - s)) \tag{7} \]
for some \( s \leq n \), then
\[ M^a_k[\lambda](q) = (-1)^{ks(n-s)}q^{s(n-s)}(^{k+1}_k)M^a_k[\alpha](q) \cdot M^{a-s}_k[\beta](q), \]
where \( \alpha \) and \( \beta \) are the partitions defined by
\[ \alpha = (\lambda_1 - \lambda_s, \ldots, \lambda_{s-1} - \lambda_s) \]
\[ \beta = (\lambda_{s+1}, \ldots, \lambda_{n-1}). \]

Note that \( \alpha \) and \( \beta \) belong to the layers \( l + k(n - s) - \lambda_s \) and \( l - ks \), respectively.

**Proof:** Let \( \lambda, \alpha, \beta \) be as described above. By Lemma 13.1, we know
\[ b^a_k(y_1, \ldots y_k) = \prod_{1 \leq i \leq k} y_i^{(2)}(1 + y_i)^n \cdot F_\gamma(-1, y_1, y_1^{-1}, \ldots, y_k, y_k^{-1}), \tag{8} \]
where \( \gamma_i = \lambda_i - l - k(n - 2i + 1) \). The fact that (7) holds is equivalent to
\[ \gamma_1 + \cdots + \gamma_s = 0. \]

By Lemma 13.3, we have
\[ F_\gamma(z) = \sum_{r(A) = \gamma} e_A(z), \]
summed over nonnegative skew-symmetric \( n \times n \) matrices \( A \). We may partition such matrices \( A \) into blocks of the form indicated in Figure 13.1. Note that \( B \) and \( C \) are themselves nonnegative skew-symmetric matrices of sizes \( s \times s \) and \( n - s \times n - s \), respectively. The matrix \( D \) is nonnegative in the ordinary sense. Since \( B \) is skew-symmetric, the sum of its entries is zero. Since \( \gamma_1 + \cdots + \gamma_s = 0 \),
then the sum of the entries of the first $s$ rows of $A$ must also be zero. Hence, it must be the case that $D = 0$. We may conclude that

$$F_\gamma(x) = F_{(\gamma_1, \ldots, \gamma_s)}(x) \cdot F_{(\gamma_{s+1}, \ldots, \gamma_n)}(x).$$

On the other hand, if we apply Lemma 13.1 to $\alpha$ and $\beta$ we obtain

$$b_\alpha^s(y_1, \ldots, y_k) = \prod_{1 \leq i \leq k} y_i^{(z)} (1 + y_i)^s \cdot F_{(\gamma_1, \ldots, \gamma_s)}(-1, y_1, y_1^{-1}, \ldots, y_k, y_k^{-1})$$

(9)

and

$$b_\beta^{n-s}(y_1, \ldots, y_k) = \prod_{1 \leq i \leq k} y_i^{(n-s)} (1 + y_i)^{n-s} \cdot F_{(\gamma_{s+1}, \ldots, \gamma_n)}(-1, y_1, y_1^{-1}, \ldots, y_k, y_k^{-1}).$$

(10)

Comparison of (9) and (10) with (8) yields

$$b_\alpha^s(y_1, \ldots, y_k) = (y_1 \cdots y_k)^s(n-s) b_\alpha^s(y_1, \ldots, y_k) b_\beta^{n-s}(y_1, \ldots, y_k).$$

The theorem now follows upon the the substitution $y_i \rightarrow -q^i$.  

As a simple example, notice that for the partition $\lambda = 2k \cdot \delta$, we have equality in all of the constraints in (4). Repeated application of Theorem 13.5 shows

$$M^\alpha_k[2k \cdot \delta](q) = (-1)^{k\binom{2}{2}} q^{\binom{k+1}{2}} [k!]_q^n. \quad (11)$$

The structure of the nonnegative skew-symmetric matrices $A$ in Figure 13.1 shows that if equality is nearly achieved in the constraints in (4), then the corresponding matrix $D$ must be sparse. By exploiting this behavior, we may extract more information about the polynomials $M^\alpha_k[\lambda]$ and $E^n[\lambda]$. The following application of this observation describes in excruciating detail the consequences of

$$\lambda_1 + \cdots + \lambda_s + 1 = s(l + k(n - s)).$$
Theorem 13.6: Let \( \alpha \in \mathbb{Z}^n \) be a dominant weight satisfying (5).

(a) If \( \alpha_1 + 1 = k(n - 1) \) then

\[
M_k^n[\alpha](q) = (-1)^n q^{(n-1)(\frac{k+1}{2})} k! \frac{1 - q^{2k+1}}{1 - q} \sum_i M_k^{n-1}[\alpha^{(i)}](q),
\]

where

\[
\alpha^{(i)} = (\alpha_2 + k, \ldots, \alpha_i + k - 1, \ldots, \alpha_n + k)
\]

and the index \( i \) ranges over the integers \( 2 \leq i \leq n \) such that \( \alpha^{(i)} \) is actually a dominant weight; i.e., either \( \alpha_i > \alpha_{i+1} \) or \( i = n \).

(b) If \( \alpha_1 + \cdots + \alpha_s + 1 = ks(n - s) \) for some integer \( s \) \((1 \leq s \leq n)\), then

\[
(-1)^n q^{(n-1)(\frac{k+1}{2})} k! \frac{1 - q^{2k+1}}{1 - q} M_k^n[\alpha](q)
\]

\[
= (-1)^{ks(n-s)} q^{s(n-s)(\frac{k+1}{2})} M_k^{s+1}[\beta](q) \cdot M_k^{n-s+1}[\gamma](q),
\]

where \( \beta \in \mathbb{Z}^{s+1} \) and \( \gamma \in \mathbb{Z}^{n-s+1} \) are the dominant weights defined by

\[
\beta = (\alpha_1 - k(n - s - 1), \ldots, \alpha_s - k(n - s - 1), -(ks - 1))
\]

\[
\gamma = (k(n - s) - 1, \alpha_{s+1} + k(s - 1), \ldots, \alpha_n + k(s - 1)).
\]

Notice that if Proposition 10.2 is applied to part (a), we obtain a similar decomposition for those dominant weights with \( \alpha_n - 1 = -k(n - 1) \). These weights correspond to the partitions in the layer \( l = k(n - 1) - 1 \). Furthermore, note that in part (b), the weight \( \gamma \in \mathbb{Z}^{n-s+1} \) satisfies the hypothesis of (a), while \( \beta \in \mathbb{Z}^{s+1} \) corresponds to a partition in the layer \( l = ks - 1 \). Part (b) is itself vacuous when \( s = 1 \) or \( s = n - 1 \).

Proof: Let \( \alpha \in \mathbb{Z}^n \) be a dominant weight satisfying the constraints in (5), and assume that

\[
\alpha_1 + \cdots + \alpha_s + 1 = ks(n - s).
\]

(12)

By Lemma 13.1, we know that

\[
b^n_\alpha(y_1, \ldots, y_k) = \prod_{1 \leq i \leq k} y_i^{(2)} (1 + y_i)^n \cdot F_n(-1, y_1, y_1^{-1}, \ldots, y_k, y_k^{-1}),
\]

where \( \eta_i = \alpha_i - k(n - 2i + 1) \). The fact that (12) holds is equivalent to

\[
\eta_1 + \cdots + \eta_s + 1 = 0.
\]
Let $A$ be a nonnegative skew-symmetric matrix with row sums $\eta$. By the same reasoning and notation used to prove Theorem 13.5, it follows that the $n - s \times s$ submatrix $D$ must be nonnegative in the ordinary sense, and the sum of its entries must be one. Thus, $D$ is a 0,1-matrix with exactly one entry equal to 1. Therefore, by Lemma 13.3,

$$F_{\eta}(z) = e_1(z) \left[ \sum_i F_{(\eta_1, \ldots, \eta_s)} + \epsilon_i(z) \right] \left[ \sum_j F_{(\eta_{s+1}, \ldots, \eta_n)} - \epsilon_j(z) \right].$$

Consider the special case $s = 1$. In this case,

$$F_{\eta}(z) = e_1(z) \sum_j F_{(\eta_2, \ldots, \eta_n)} - \epsilon_j(z). \quad (13)$$

By the definition of $\alpha^{(j)},$

$$(\eta_2, \ldots, \eta_n) - \epsilon_j = (\alpha_1^{(j)} - k(n - 2), \ldots, \alpha_{n-1}^{(j)} + k(n - 2))$$

and so by Lemma 13.1,

$$b_{\alpha}^n(y_1, \ldots, y_k) = \prod_{1 \leq i \leq k} y_i^{n-1} (1 + y_i) \cdot e_1(-1, y_1, y_1^{-1}, \ldots, y_k, y_k^{-1}) \sum_j b_{\alpha^{(j)}}^{n-1}(y_1, \ldots, y_k). \quad (14)$$

The alert reader will accuse us of making a cavalier deduction: it is possible for some $j$ that $\alpha^{(j)}$ is not a dominant weight. This happens when $\alpha_j = \alpha_{j+1}$, so that $\ldots, \alpha_j - 1, \alpha_{j+1}, \ldots$ is no longer a decreasing sequence. However, Lemma 13.1 is still formally valid even if $\alpha$ is not a dominant weight, provided that we regard $b_{\alpha}^n$ as the coefficient of $x^{\alpha + \delta}$ in a certain formal power series. Thus, we have $b_{\alpha^{(j)}}^n = 0$ unless all of the terms in $\alpha^{(j)} + \delta$ are distinct. If $\alpha_j = \alpha_{j+1}$, then the terms are not distinct and we are justified in restricting (14) to those integers $j$ for which $\alpha^{(j)}$ is a dominant weight. Part (a) now follows from (14) upon the substitution $y_i \to -q^i$.

In the case of arbitrary $s$, notice that

$$F_{(\eta_1, \ldots, \eta_s, 1)}(z) = e_1(z) \sum_i F_{(\eta_1, \ldots, \eta_s) + \epsilon_i(z)}$$

and

$$F_{(-1, \eta_{s+1}, \ldots, \eta_n)}(z) = e_1(z) \sum_j F_{(\eta_{s+1}, \ldots, \eta_n) - \epsilon_j(z)}.$$ 

Hence,

$$e_1(z)F_{\eta}(z) = F_{(\eta_1, \ldots, \eta_s, 1)}(z)F_{(-1, \eta_{s+1}, \ldots, \eta_n)}(z). \quad (15)$$
Moreover, by the definitions of $\beta$ and $\gamma$, we have

$$(\eta_1, \ldots, \eta_s, 1) = (\beta_1 - ks, \beta_2 - k(s - 2), \ldots, \beta_{s+1} + ks)$$

and

$$(-1, \eta_{s+1}, \ldots, \eta_n) = (\gamma_1 - k(n - s), \gamma_2 - k(n - s - 2), \ldots, \gamma_{n-s+1} + k(n - s)).$$

Hence, by Lemma 13.1 and (15),

$$\prod_{1 \leq i \leq k} y_i^{-s(n-s)+n}(1 + y_i)^2 \cdot e_1(-1, y_1, y_i^{-1}, \ldots, y_k, y_k^{-1}) b_\alpha^n(y_1, \ldots, y_k)$$

$$= b_\beta^{n+1}(y_1, \ldots, y_k) b_\gamma^{n-s+1}(y_1, \ldots, y_k),$$

from which part (b) follows.

Theorem 13.6 applies equally well to the polynomials $E^n[\lambda]$:

**Corollary 13.7:** Let $\alpha \in \mathbb{Z}^n$ be a dominant weight satisfying (10.6).

(a) If $\alpha_1 = n - 2$, then

$$E^n[\alpha](q) = q^{n-2}(1 + q^3) \sum_i E^{n-1}[\alpha^{(i)}](q),$$

where

$$\alpha^{(i)} = (\alpha_2 + 1, \ldots, \alpha_i, \ldots, \alpha_n + 1)$$

and the index $i$ ranges over those integers $2 \leq i \leq n$ such that $\alpha_i > \alpha_{i+1}$ or $i = n$.

(b) If $\alpha_1 + \cdots + \alpha_s + 1 = s(n - s)$ for some integer $s$ $(1 \leq s < n)$, then

$$q^{n-1}(1 + q)(1 + q^3) E^n[\lambda](q) = q^{s(n-s)} E^{s+1}[\beta](q) \cdot E^{n-s+1}[\gamma](q),$$

where $\beta \in \mathbb{Z}^{s+1}$ and $\gamma \in \mathbb{Z}^{n-s+1}$ are the dominant weights defined by

$$\beta = (\alpha_1 - n + s + 1, \ldots, \alpha_s - (n + s + 1), -(s - 1))$$

$$\gamma = (n - s - 1, \alpha_{s+1} + s - 1, \ldots, \alpha_n + s - 1).$$

**Example 13.8:** Let $n = 5$ and consider the second layer partition $\lambda = 541$, which corresponds to the dominant weight $\alpha = (3, 2, -1, -2, -2)$. If we apply Corollary 13.7(b) with $s = 2$, we obtain

$$q^4(1 + q)(1 + q^3) E^5[541](q) = q^6 E^3[21](q) \cdot E^4[31](q).$$
The first layer formula (Theorem 12.1) can be used to evaluate $E^3[21]$ and $E^4[31]$, so we deduce

$$E^5[541](q) = q^5(1 + q)^3(1 + q^3)^2(1 + q^2)(1 + q^2 + q^4).$$

Clearly, the techniques developed here could be extended to prove more complicated results about the polynomials $M^n_\lambda[\lambda]$, although one quickly is confronted by the economics of diminishing returns. Indeed, as we have seen, these techniques are best suited for those partitions $\lambda$ which are "close" to $2\lambda \cdot \delta$. This is unfortunate, as one of the primary motivations for studying the Macdonald complex is to obtain a formula for $M^n_\emptyset[\emptyset]$, which is at the other extreme.

Evaluating the polynomial $M^n_\emptyset[\emptyset]$ is known to be equivalent to Macdonald's root system conjecture for the root system $A_{n-1}$. This conjecture for $A_{n-1}$ was recently solved, via the $q$-Dyson Theorem, by Zeilberger and Bressoud [48]. The details of these connections and some generalizations will be the subject of the next chapter.
Chapter IV.

The Macdonald complex of \( \mathfrak{gl}_n \) and the \( q \)-Dyson Theorem

14. The \( q \)-Dyson Theorem

For any nonnegative integer \( k \) and indeterminate \( z \), let

\[
(z)_k = \prod_{0 \leq i < k} (1 - q^i z).
\]

By convention, \((z)_0 = 1\) and

\[
(z)_\infty = \prod_{i \geq 0} (1 - q^i z),
\]

regarded as a formal power series. Notice that the \( q \)-dependence of these expressions is tacitly assumed.

Let \( \lambda \) range over partitions with \( \ell(\lambda) < n \), and \(|\lambda|\) divisible by \( n \). Define formal power series \( C^n[\lambda](z, q) \) so that

\[
\prod_{1 \leq i, j \leq n} \frac{(qx_i x_j^{-1})_\infty}{(zx_i x_j^{-1})_\infty} = \sum_{\lambda} C^n[\lambda](z, q)^{\mathfrak{g}}(x_1, \ldots, x_n)
\]

\[(1)\]

is a formal power series identity in the ring \( \mathbb{Z}[[z, q]] \otimes \Omega_n(x) \). Since the terms in this identity belong to \( \Omega_n^0 \), we may also index the coefficients by dominant weights. Thus, if \( \alpha \in \mathbb{Z}^n \) is the dominant weight corresponding to \( \lambda \), then \( C^n[\alpha] \) shall be considered synonymous with \( C^n[\lambda] \).

This chapter will consist of a detailed study of these series \( C^n[\lambda] \), which have the virtue that any result about them gives simultaneous results about the decompositions of the exterior algebra and Macdonald complex of \( \mathfrak{gl}_n \) and the
The Macdonald complex of $gl_n$ and the $q$-Dyson Theorem

generalized exponents of $SL_n$. In fact, by the discussion in Section 9 and the
definitions of the polynomials $E^n[\lambda]$, $M_k^n[\lambda]$ and $G^n[\lambda]$, we have

$$E^n[\lambda](q) = C^n[\lambda](q^2, -q),$$
$$M_k^n[\lambda](q) = C^n[\lambda](q^{k+1}, q),$$
$$G^n[\lambda](z) = [n!]_z \cdot C^n[\lambda](z, 0).$$

It is also worth mentioning that Macdonald's affine analogue of the Weyl
denominator formula for the root system $A_{n-1}$ can be obtained from the series
$C^n[\lambda](0, q)$. We shall have more to say about this aspect of the series $C^n[\lambda]$ in
Section 15.

Of course, the series $C^n[\lambda]$ are similar in many respects to the polynomials
$M_k^n[\lambda]$ and $E^n[\lambda]$ which we have already studied. For example, it is easy to see
that $C^n[\lambda] = C^n[\lambda]$ for any partition $\lambda$. Although the $C^n[\lambda]$'s are not polynomials,
we will see later that they are always of the form $f \cdot C^n[0]$ where $f$ is a rational
function of $z$ and $q$ depending on $\lambda$.

At first glance, the problem of determining $C^n[\lambda]$ seems harder than determin-
ing $M_k^n[\lambda]$, but these problems should actually be considered equivalent. If
one can find a formula for $M_k^n[\lambda](q)$ for all integers $k \geq 1$, then a formula for
$C^n[\lambda](z, q)$ can be obtained by replacing every occurrence of $q^k$ in this formula
by $z/q$. This observation can be made rigorous by the following:

**Lemma 14.1:** Let $F(x, y)$ be a formal power series in the indeterminates $x, y$. If
$F(y^k, y) = 0$ for infinitely many nonnegative integers $k$, then $F(x, y) = 0$.

**Proof:** Assume towards a contradiction that $F(x, y) \neq 0$ but $F(y^k, y) = 0$ for
arbitrarily large $k$. Certainly $F$ has an expansion

$$F(x, y) = F_0(y) + xF_1(y) + x^2F_2(y) + \cdots \tag{2}$$

for suitable formal power series $F_r(y)$. We may assume that $F_0 \neq 0$ by rescaling
our counterexample $F$ by a suitable power of $x$. Suppose that $y^r$ is the smallest
power of $y$ for which $[y^r]F_0(y) \neq 0$. Choose $k > r$ so that $F(y^k, y) = 0$. By (2),

$$[y^r]F(y^k, y) = [y^r]F_0(y) + [y^r]y^k(F_1(y) + y^kF_2(y) + \cdots) = [y^r]F_0(y),$$

which contradicts our choice of $r$.●

There is a constant term identity known as the $q$-Dyson Theorem which can be
used to give an explicit formula for the series $C^n[0]$. The original motivation
for this identity came from a study in statistical mechanics by Freeman Dyson
in 1962 [6]. In this study, Dyson was led to conjecture that

$$[x^n] \prod_{1 \leq i \neq j \leq n} (1 - x_ix_j^{-1})^k = \frac{(kn)!}{k!n!}. \tag{3}$$
Later that year, independent proofs of (3) were given by Gunson [12] and Wilson [46], who showed more generally that

$$[x^0] \prod_{1 \leq i < j \leq n} (1 - x_i x_j^{-1})^{a_i} = \frac{(a_1 + \cdots + a_n)!}{a_1! \cdots a_n!}$$

(4)

for nonnegative integers $a_1, \ldots, a_n$. This identity had also been conjectured by Dyson [6]. An elegant one-line proof of (4) was given by Good in 1970 [11]. A combinatorial proof was given by Zeilberger in 1982 [47].

In 1975, G. Andrews conjectured a $q$-analogue of this result and proved the conjecture for $n = 1, 2, 3$ [1]. Subsequently, this conjecture came to be known as the “$q$-Dyson Conjecture”. In 1983, Zeilberger and Bressoud [48] proved Andrews’ conjecture, which may now be called the “$q$-Dyson Theorem”.

**Theorem 14.2:** ($q$-Dyson Theorem)
Let $a_1, \ldots, a_n$ be nonnegative integers. We have

$$[x^0] \prod_{1 \leq i < j \leq n} (x_i x_j^{-1})_{a_i} (q x_i x_j^{-1})_{a_j} = \frac{[(a_1 + \cdots + a_n)!]_q}{[a_1!]_q \cdots [a_n!]_q}.$$  

Dyson’s original conjecture (4) can be obtained by taking the limit $q \to 1$.

Using the same techniques developed by Zeilberger and Bressoud to prove the $q$-Dyson Theorem, Bressoud and Goulden [4] found a number of constant term identities of a similar flavor. Among the identities they proved, the following is of particular interest:

**Theorem 14.3:** (Bressoud and Goulden [4; Theorem 2.2])
Let $a_1, \ldots, a_n$ be nonnegative integers. We have

$$[x^0] \prod_{1 \leq i < j \leq n} (x_i x_j^{-1})_{a_i} (q x_i x_j^{-1})_{a_j-1} = \frac{[(a_1 + \cdots + a_n)!]_q}{[a_1!]_q \cdots [a_n!]_q} \prod_{1 \leq i \leq n} \frac{1 - q^{a_i}}{1 - q^{a_n + \cdots + a_i}}.$$  

This result makes a direct evaluation of $C^n[\emptyset]$ possible:

**Corollary 14.4:** We have

$$C^n[\emptyset](z, q) = \frac{(q)_\infty}{(q z^n)_\infty} \cdot \frac{1}{[n!]_z}.$$  

In particular,

$$M^n_k[\emptyset](q) = \frac{[n(k + 1)!]_q}{[n!]_{q^{k+1}}}.$$
Proof: Consider the special case $z = q^k$. In this situation, the identity (1) becomes
\[
[(k - 1)!!]_q^n \cdot \prod_{i \neq j} (q x_i x_j^{-1})_{k-1} = \sum_{\lambda} C^n[\lambda](q^k, q) \cdot \bar{s}_\lambda(x_1, \ldots, x_n).
\]

Therefore, by Remark 4.8, we have
\[
C^n(\emptyset)(q^k, q) = [x^k]a_\emptyset(x) \cdot [(k - 1)!!]_q^n \cdot \prod_{i \neq j} (q x_i x_j^{-1})_{k-1}
\]
\[= [x^0][(k - 1)!!]_q^n \cdot \prod_{i < j} (x_j x_i^{-1})_{k} (q x_i x_j^{-1})_{k-1}.
\]

If we apply Theorem 14.3 in the case $a_1 = \cdots = a_n = k$, we find
\[
C^n(q^k, q) = [(k - 1)!!]_q^n \cdot \frac{[(nk)!]_q}{[k!]^n} \cdot \frac{(1 - q^k)_n}{[n!]}.
\]

Therefore, the formal power series
\[
C^n[\emptyset](z, q) \quad \text{and} \quad (q)_\infty \cdot \frac{1}{(qz^n)_\infty} \cdot \frac{[n!]}{[n!]}.
\]

agree for infinitely many of the special cases $z = q^k$. Apply Lemma 14.1.°

While the $q$-Dyson Theorem was still a conjecture, Macdonald formulated a series of conjectures for arbitrary root systems in which Andrews’ conjectures would correspond to the root system $A_{n-1}$. To appreciate Macdonald’s conjectures will require on the reader’s part at least a basic understanding of the nature of root systems. Rather than digress and survey their theory, we refer the reader to the excellent introduction by Humphreys [20; Chp. 3].

If one thinks of $x_i = e^{s_i}$ as a formal exponential, then the monomials $x_i x_j^{-1} = e^{s_i - s_j} : i < j$ correspond to a system of positive roots for the root system $A_{n-1}$. Using this identification, Macdonald was led to conjecture the following:

Conjecture 14.5: (Macdonald’s root system conjecture [31])

Let $R$ be a (reduced) root system, $R^+$ a system of positive roots, and let $m_1, \ldots, m_n$ be the exponents of $R$. For each $\alpha \in R^+$, let $e^\alpha$ denote a formal exponential. The constant term with respect to the exponentials $e^\alpha$ in
\[
\prod_{\alpha \in R^+} (e^{-\alpha})_k(qe^\alpha)_k \quad \text{should be} \quad \prod_{1 \leq i \leq n} \frac{[k(m_i + 1)]_q}{[k!]_q \cdot [(km_i)]_q}.
\]
Since the exponents of the root system $A_{n-1}$ are $1, 2, \ldots, n - 1$, it is easily verified that Macdonald's conjecture reduces to the $q$-Dyson Theorem for equal parameters (i.e., $a_1 = \cdots = a_n = k$) in this case. This conjecture remains open for the other root systems, however we should mention that Bressoud has announced a proof of the conjecture for the root system $G_2$. Macdonald [31] has proved the conjecture for $k = 1, 2, \infty$ or $q = 1$, and Hanlon [18] has proved the conjecture for any of the classical root systems $R = A_n, B_n, C_n$ or $D_n$ in the limit as $n$ tends to infinity.

We remark that in [31], Macdonald shows that evaluating the constant term in (5) is equivalent to finding the multiplicity of the trivial character in a certain weighted character of a compact, connected Lie group whose root system is $R$. In particular, it follows that the formula for $M_k^n[0]$ (Corollary 14.4) could have been deduced directly from the $q$-Dyson Theorem. Conversely, the formula for $M_k^n[0]$ could be used to recover the equal-parameter version of the $q$-Dyson Theorem.

Such possibilities can also be seen without reference to representation theory. Bressoud and Goulden's main result in [4] is a constant term identity similar to Theorem 14.3 in which the indices $(i, j)$ are allowed to vary over an arbitrary tournament on $n$ vertices. They deduced the $q$-Dyson Theorem directly from this result. In Section 18 we will show how formulas for the polynomials $M_k^n$ can be used to extract coefficients of (possibly) nonconstant monomials in

$$\prod_{1 \leq i < j \leq n} (x_j x_i^{-1})_k(q x_i x_j^{-1})_k,$$

thus generalizing the equal parameter version of the $q$-Dyson Theorem.

15. Macdonald's identity for $A_{n-1}$

Ordinary root systems can be viewed as finite subsets of a finite dimensional real Euclidean space satisfying certain axioms. Macdonald [30] generalized this notion to certain (infinite) collections of affine-linear functionals on a real Euclidean space, and dubbed them affine root systems. In addition to classifying these root systems, Macdonald proved a generalization of Weyl's denominator formula for affine root systems.

For an ordinary root system $R$, let $R^+$ denote a system of positive roots, and $W$ the Weyl group of $R$. Weyl's denominator formula [32; L. ex.5.9] states that

$$\sum_{w \in W} \varepsilon(w) \varepsilon^{w \rho} = \prod_{\alpha \in R^+} (e^{\alpha/2} - e^{-\alpha/2}),$$

where $e^\alpha$ denotes a formal exponential, $\varepsilon(w)$ denotes the sign of $w \in W$, and $\rho$ is half the sum of the positive roots. The reader can easily verify that for the
root system $A_{n-1}$, this identity is equivalent to the Vandermonde determinant formula (4.4).

Macdonald's identities for the affine root systems of type $A$ are essentially equivalent to a decomposition of the symmetric formal Laurent series

$$\prod_{1 \leq i, j \leq n} (qx_i x_j^{-1})_\infty$$

into Schur functions $s_\lambda(x_1, \ldots, x_n)$ over $\Omega_n$. In view of our definition (14.1) of the series $C^n[\lambda]$, we see that Macdonald's identities yield formulas for

$$C^n[\lambda](0, q) = \lim_{k \to \infty} M_k^n[\lambda|(q).$$

In the following, we will give not only a precise statement of the formulas for $C^n[\lambda](0, q)$, but also a proof. We emphasize, however, that the proof we shall give is modelled on the technique developed by D. Stanton in [42], where he gives short, elementary proofs of Macdonald's identities. Our purpose in giving the proof is twofold. The statement of the formulas we give are not readily identifiable as the form of the identities given by Macdonald [30; (8.1)] or Stanton [42; (5.2)]; some sort of justification is warranted. Secondly, the proof technique will motivate the methods used in subsequent sections in our study of the full series $C^n[\lambda](z, q)$.

In order state the formulas in a reasonable fashion, we introduce some notation. Let $\lambda$ denote a partition with $\ell(\lambda) < n$ and $|\lambda|$ divisible by $n$. Let $v = (v_1, \ldots, v_n)$ denote the increasing sequence of integers defined by

$$v_i = \lambda_{n+1-i} + i - 1.$$ We shall refer to $v$ as the vertical sequence of $\lambda$. This terminology is derived from the following geometric interpretation. View the diagram of $\lambda$ as a lattice path from southwest to northeast, and number the horizontal and vertical steps consecutively from 0 to $n + \lambda_1 - 1$. The vertical steps are labelled $v_1, \ldots, v_n$. An example in which $n = 8$ and $\lambda = 764322$ is given in Figure 15.1. Notice that the terms of the vertical sequence are merely the terms of $\lambda + \delta$ taken in reverse order. Also notice that the initial term of the vertical sequence is always zero.

If $\sigma = (\sigma_1, \ldots, \sigma_n)$ is any integer sequence, recall that the inversion number of $\sigma$ is defined by

$$inv(\sigma) = |\{(i, j) : 1 \leq i < j \leq n, \sigma_i > \sigma_j\}|.$$

We are now ready to state the formula for $C^n[\lambda](0, q)$ implied by Macdonald's identities [30].
Theorem 15.1: Let $\lambda$ be a partition with $\ell(\lambda) < n$ and $|\lambda|$ divisible by $n$.

(a) We have $C_n[\lambda](0, q) = 0$ unless the vertical sequence of $\lambda$, taken modulo $n$, is a permutation of $\mathbb{Z}/(n)$.

(b) If the vertical sequence $v$ does form a permutation of $\mathbb{Z}/(n)$, associate to $v$ the sequence $\sigma$, where

$$\sigma_i = \lfloor v_j/n \rfloor \quad \text{if} \quad v_j \equiv i - 1 \mod n.$$ 

We have

$$C_n[\lambda](0, q) = (-1)^{|\sigma| + \text{inv}(\sigma)} q^{\eta(\sigma)} (q)_\infty$$

where

$$\eta(\sigma) = \sum_{1 \leq i < j \leq n} \left( \frac{\sigma_i - \sigma_j}{2} \right).$$

When $v$ does form a permutation of $\mathbb{Z}/(n)$, we shall refer to the sequence $\sigma$ defined above as the associated sequence for $\lambda$.

Example 15.2:

(a) In Figure 15.1, we have $n = 8$ and $\lambda = 764322$. Taken modulo 8, the vertical sequence is $(0, 1, 4, 5, 7, 1, 4, 6)$, which is evidently not a permutation of $\mathbb{Z}/(8)$. We conclude that

$$C_8[764322](0, q) = 0.$$ 

(b) Consider $n = 8$ and let $\lambda$ denote the partition corresponding to the dominant weight $[775, 8632]_8$. Taken modulo 8, the vertical sequence of $\lambda$ is $(0, 3, 7, 1, 4, 2, 5, 6)$, which is a permutation of $\mathbb{Z}/(8)$. The associated sequence for
$\lambda$ is $\sigma = (0, 1, 2, 0, 1, 2, 2, 0)$. One can easily verify that $\text{inv}(\sigma) = 8$ and $\eta(\sigma) = 27$. We conclude that

$$C[775, 8632]_8(0, q) = q^{27} (q)_\infty.$$

(c) The partitions $\lambda$ with $C^n[\lambda](0, q) \neq 0$ and for which the associated sequence $\sigma$ consists only of 0's and 1's have an interesting characterization. In this case, the vertical sequence must be of the form

$$(w_1, \ldots, w_k, n + w_{k+1}, \ldots, n + w_n)$$

where $w$ is a permutation of $0, 1, \ldots, n - 1$ such that $0 = w_1 < \cdots < w_k$ and $w_{k+1} < \cdots < w_n$.

If the lattice path corresponding to $\lambda$ is numbered as described in the definition of the vertical sequence, we may break the path into two complementary pieces: the part numbered from 0 to $n - 1$, and the part numbered from $n$ to $n + \lambda_1 - 1$. It follows that the vertical steps in the first part correspond to horizontal steps in the second part, and conversely. Hence, there must be a partition $\alpha$ for which $[\alpha, \alpha']_n$ is the dominant weight corresponding to $\lambda$.

Since the associated sequence consists only of 0's and 1's, it follows that $\eta(\sigma)$ counts the number of increases in $\sigma$, i.e.,

$$\eta(\sigma) = |\{(i, j) : 1 \leq i < j \leq n, \sigma_i = 0, \sigma_j = 1\}|.$$

Moreover,

$$\text{inv}(\sigma) = |\sigma|(n - |\sigma|) - \eta(\sigma),$$

so

$$(n - 1)|\sigma| - \text{inv}(\sigma) \equiv \eta(\sigma) \mod 2.$$

Suppose that $n + j$ ($0 < j < n$) is the label attached to a vertical step in the lattice path of $\lambda$. Note that the number of cells of $\alpha$ to the left of this vertical step is the number of horizontal steps of the form $n + i$ ($0 \leq i < j$); i.e., the number of pairs $(i, j)$ with $\sigma_i = 0, \sigma_j = 1$ is $\alpha_j$.

In summary we may conclude that the only partitions $\lambda$ with $\lambda_1 \leq n$ and $C^n[\lambda](0, q) \neq 0$ correspond to dominant weights of the form $[\alpha, \alpha']_n$, and in that case,

$$C[\alpha, \alpha']_n(0, q) = (-q)^{|\alpha|} (q)_\infty.$$

**Proof of Theorem 15.1:**

Let $\lambda$ be a partition in the $l$th layer with $\ell(\lambda) < n$, and $\alpha \in \mathbb{Z}^n$ the dominant weight corresponding to $\lambda$. Let the symmetric group $S_n$ act on $\mathbb{Z}^n$ in the natural way:

$$w \circ (\gamma_1, \ldots, \gamma_n) = (\gamma_{w(1)}, \ldots, \gamma_{w(n)}).$$
Our proof will be preceded by three technical lemmas:

**Lemma 15.3:** If $C^n[\alpha](0, q) \neq 0$ then the terms in the sequence $\alpha + \delta - 1^n + n \epsilon_n$ are distinct. If the terms are distinct, then there is a unique permutation $w \in S_n$ and dominant weight $\beta \in \mathbb{Z}^n$ such that

$$\alpha + \delta - 1^n + n \epsilon_n = w \circ (\beta + \delta).$$

Moreover,

$$C^n[\alpha](0, q) = \epsilon_w(-1)^{n-1} q' \cdot C^n[\beta](0, q).$$

**Proof:** Let

$$F_n(x_1, \ldots, x_n) = a_\delta(x) \prod_{1 \leq i,j \leq n} (qx_i x_j^{-1})_\infty.$$

Recall that by the definition (14.1) of $C^n[\lambda]$, we have

$$\prod_{1 \leq i,j \leq n} (qx_i x_j^{-1})_\infty = \sum_\lambda \lambda(0, q) \cdot \overline{s}_\lambda(x_1, \ldots, x_n)$$

in $\Omega_n[[q]]$. Therefore, by Remark 4.8,

$$C^n[\alpha](0, q) = [x^{\alpha + \delta}] F_n(x_1, \ldots, x_n). \tag{1}$$

The key to this lemma is to find a functional equation satisfied by $F_n$; this is the idea which we have borrowed most explicitly from Stanton. Observe that

$$\frac{F_n(x_1, \ldots, x_{n-1}, qx_n)}{F_{n-1}(x_1, \ldots, x_{n-1})} = \prod_{1 \leq i < n} (x_i - qx_n) \cdot (q^2 x_n x_i^{-1})_\infty \cdot (x_i x_n^{-1})_\infty$$

$$= (x_1 \cdots x_{n-1}) \prod_{1 \leq i < n} (1 - x_i x_n^{-1}) \cdot (q x_n x_i^{-1})_\infty \cdot (qx_i x_n^{-1})_\infty$$

$$= (-1)^{n-1} x_1 \cdots x_n \frac{F_n(x_1, \ldots, x_n)}{x_n^{n-1} F_{n-1}(x_1, \ldots, x_{n-1})}.$$\]

Therefore,

$$F_n(x_1, \ldots, x_{n-1}, qx_n) = (-1)^{n-1} (x_1 \cdots x_n) x_n^{-n} \cdot F_n(x_1, \ldots, x_n).$$

Extracting the coefficient of $x^{\alpha + \delta}$ and comparing with (1), we find

$$q^n C^n[\alpha](0, q) = (-1)^{n-1} [x^{\alpha + \delta - 1^n + n \epsilon_n}] F_n(x_1, \ldots, x_n). \tag{2}$$

Note that $F_n(x_1, \ldots, x_n)$ is an alternating series. Therefore, any coefficient we care to extract, say $x^7$, will vanish unless the terms in $\gamma$ are distinct. If the
terms are distinct, then there is a unique permutation \( w \in S_n \) and dominant weight \( \beta \in \mathbb{Z}^n \) so that \( \gamma = w \circ (\beta + \delta) \), and in that case, we have

\[
[x^\gamma] F_n = \varepsilon_w [x^{\beta + \delta}] F_n = \varepsilon_w C^n[\beta](0,q).
\]

The lemma now follows by applying this observation to (2) and using the fact that the layer \( l \) of \( \lambda \) is \(-\alpha_n\).

**Lemma 15.4:** The recursion given in Lemma 15.3 uniquely determines the coefficients \( C^n[\lambda](0,q) \) up to the initial condition \( C^n[\emptyset](0,q) \). Moreover, \( C^n[\lambda](0,q) \) is nonzero if and only if the vertical sequence of \( \lambda \), taken modulo \( n \), is a permutation of \( \mathbb{Z}/(n) \).

**Proof:** Partially order dominant weights by insisting that \( \alpha \geq \beta \) if and only if

\[
\alpha_1 + \cdots + \alpha_i \geq \beta_1 + \cdots + \beta_i \quad \text{for } 1 \leq i \leq n.
\]

This is essentially the dominance order on partitions, translated to dominant weights. The reader familiar with root systems will recognize this as the usual partial order on the root lattice of \( A_{n-1} \) : \( \alpha > \beta \) if and only if \( \alpha - \beta \) is a sum of positive roots. Notice that the dominant weight \( \emptyset \) is at the bottom of this partial order.

Let \( \alpha > \emptyset \) be a dominant weight, and assume that the terms of \( \alpha + \delta - 1^n + n \varepsilon_n \) are distinct. We must have

\[
\alpha + \delta - 1^n + n \varepsilon_n = w \circ (\beta + \delta)
\]

where \( \beta \) is the dominant weight given by

\[
\beta = (\alpha_1 - 1, \ldots, \alpha_k - 1, \alpha_n + k, \alpha_{k+1}, \ldots, \alpha_{n-1}),
\]

\( w \in S_n \) is the permutation for which

\[
w \circ (\gamma_1, \ldots, \gamma_n) = (\gamma_1, \ldots, \gamma_k, \gamma_{k+2}, \ldots, \gamma_n, \gamma_{k+1}),
\]

and \( k \) is the unique integer \( 1 \leq k < n \) for which

\[
\alpha_k + n - k - 1 > \alpha_n + n - 1 > \alpha_{k+1} + n - k - 2.
\]

Such an integer \( k \) exists since \( \alpha_n + n - 1 > \alpha_1 + n - 2 \) would imply \( \alpha = \emptyset \). Notice that the sign of \( w \) is \((-1)^{n-k-1}\). It is clear from (3) that \( \alpha > \beta \). Therefore, the recursion in Lemma 15.3 expresses \( C^n[\alpha](0,q) \) in terms of \( C^n[\beta](0,q) \) for some \( \beta < \alpha \). We may conclude that the recursion uniquely determines \( C^n[\lambda](0,q) \) up to the initial condition \( C^n[\emptyset](0,q) \).
To complete the proof of the lemma, it suffices to show that the vertical sequence of $\alpha$ is a permutation of $\mathbb{Z}/(n)$ if and only if the same is true for the vertical sequence of $\beta$. Of course, here we are making implicit use of the fact that the vertical sequence of $\emptyset$, which is $(0, 1, \ldots, n - 1)$, does form a permutation of $\mathbb{Z}/(n)$.

If the vertical sequence $v$ corresponding to $\lambda$, taken modulo $n$, is a permutation of $\mathbb{Z}/(n)$, then the same is true for $\lambda + \delta$, as its terms are those of $v$ taken in reverse order. Since each of the operations

$$
\gamma \mapsto \gamma \pm 1^n \\
\gamma \mapsto \gamma \pm n\epsilon_n \\
\gamma \mapsto \omega \circ \gamma
$$

preserve the property of being a permutation of $\mathbb{Z}/(n)$, the lemma follows.

Thus, we have proved part (a) of the theorem, and we also may conclude that if $C^n[\lambda](0, q) \neq 0$, then

$$
C^n[\lambda](0, q) = (-1)^a q^b C^n[\emptyset](0, q)
$$

for suitable integers $a$ and $b$. Henceforth, let us assume that $\lambda$ is a partition whose vertical sequence $v$, taken modulo $n$, is a permutation of $\mathbb{Z}/(n)$. Let $\alpha$ be the dominant weight corresponding to $\lambda$, and let $\sigma$ denote its associated sequence. Notice that $\sigma_i$ is characterized by the fact that $v_j = n\sigma_i + i - 1$, where $v_j$ is the unique term of $v$ for which $v_j \equiv i - 1 \mod n$.

For example, if $n = 7$ and $\sigma = (0, 1, 1, 0, 2, 0, 1)$, this means that the terms of $v$ whose residues mod 7 are 0, 3, 5 must be 0, 3, 5; the terms of $v$ whose residues mod 7 are 1, 2, 6 must be 8, 9, 13; and the term of $v$ whose residue mod 7 is 4 must be 18. Thus $v = (0, 3, 5, 8, 9, 13, 18)$.

Consider the effect of the recursion in Lemma 15.3 on $\sigma$. Let $\beta$ be the dominant weight defined in (3), and $w$ the permutation defined in (4). Notice that the integer $k$, defined in (5), is the number of terms of the form $\lambda_i + n - i$ which exceed $n$, since

$$
\lambda_i + n - i = \alpha_i - \alpha_n + n - i \geq n
$$

is equivalent to

$$
\alpha_i + n - i - 1 \geq \alpha_n + n - 1.
$$

But the terms $\lambda_i + n - i$ are those of $v$ taken in reverse order, so we conclude that

$$
k = |\{i : 1 \leq i \leq n, \sigma_i > 0\}|.
As we remarked earlier, the sign of \( w \) is \((-1)^{n-k-1}\), so we have
\[
\varepsilon_w \cdot (-1)^{n-1} = (-1)^k.
\]

Furthermore, observe that
\[
\binom{n}{2} + \ln = |\lambda + \delta| = |v| = \binom{n}{2} + n \sum \sigma_i,
\]
so we have \( l = |\sigma| = \sum \sigma_i \).

Let \( u \) be the vertical sequence corresponding to the dominant weight \( \beta \), and let \( r \) be its associated sequence. Assume for the moment that \( k < n - 1 \). Since
\[
\beta + \delta = (\alpha_1 + n - 2, \ldots, \alpha_k + n - k - 1, \alpha_n + n - 1, \alpha_{k+1} + n - k - 2, \ldots, \alpha_{n-1}), \quad (6)
\]

it follows that \( u \) is obtained from (6) by subtracting \( \alpha_{n-1} \cdot 1^n \) and reversing the elements. Hence, the unsorted terms of \( u \) are
\[
\{n - v_2, 0, v_3 - v_2, \ldots, v_n - v_2\},
\]

since \( v_2 = \alpha_{n-1} + 1 - \alpha_n \).

Let \( v_j \equiv i - 1 \mod n \), where \( 1 \leq i \leq n \). Since \( v_j = n\sigma_i + (i - 1) \), we have
\[
v_j - v_2 = \begin{cases} \n \sigma_i + (i - 1 - v_2) & \text{if } i \geq v_2 + 1 \\ n(\sigma_i - 1) + (n - v_2 + i - 1) & \text{if } i < v_2 + 1. \end{cases}
\]

Therefore, we may conclude that
\[
r = (\sigma_r, \sigma_{r+1}, \ldots, \sigma_n, \sigma_1, \sigma_2 - 1, \ldots, \sigma_{r-1} - 1),
\]

where \( r = v_2 + 1 \). Note that \( r \) is the least integer (\( > 1 \)) such that \( \sigma_r = 0 \).

The analysis for the case \( k = n - 1 \) is even easier. Since
\[
\beta + \delta = (\alpha_1 + n - 2, \ldots, \alpha_{n-1}, \alpha_n + n - 1),
\]

the vertical sequence of \( \beta \) must be
\[
u = (0, v_2 - n, \ldots, v_n - n)
\]

and the associated sequence for \( \beta \) must be
\[
r = (\sigma_1, \sigma_2 - 1, \ldots, \sigma_n - 1).
We summarize the previous discussion by the following:

**Lemma 15.5**: We have

\[ C^n[\alpha](0, q) = (-1)^k q' \cdot C^n[\beta](0, q), \]

where \( k \) is the number of positive terms in \( \sigma \), \( l = |\sigma| \), and the associated sequence for \( \beta \) is given by

\[ \tau = (\sigma_r, \sigma_{r+1}, \ldots, \sigma_n, \sigma_1, \sigma_2 - 1, \ldots, \sigma_{r-1} - 1), \]

where \( r \) is the least integer \((> 1)\) for which \( \sigma_r = 0 \). If no such integer exists, then

\[ \tau = (\sigma_1, \sigma_2 - 1, \ldots, \sigma_n - 1). \]

This lemma gives us an algorithm for computing \( C^n[\alpha](0, q) \) in terms of the associated sequence \( \sigma \). If we iterate the algorithm, we will obtain a series \( \sigma = \sigma^0, \sigma^1, \sigma^2, \ldots \) of associated sequences. The iterations cease as soon as \( \sigma^m = \emptyset \) for some integer \( m \). Let \( k_p \) and \( l_p \) denote the \( k \)- and \( l \)-statistics corresponding to the \( p \)th iterate \( \sigma^p \). Lemma 15.5 tells us that

\[ C^n[\alpha](0, q) = (-1)\sum k_p q\sum l_p C^n[\emptyset](0, q). \]

In view of Corollary 14.4, it suffices to show that

\[ (n - 1)|\sigma| + \text{inv}(\sigma) \equiv \sum k_p \mod 2 \quad \text{and} \quad \eta(\sigma) = \sum l_p. \]

To prove this, it is conceptually easier to think of the operation \( \sigma \mapsto \tau \) as being performed on an \( n \)-tuple of variables \( (\sigma_1, \ldots, \sigma_n) \). We obtain \( \sigma^p \) from \( \sigma^{p-1} \) by cyclically shifting the initial variable in \( \sigma^{p-1} \) to the end of the \( n \)-tuple repeatedly, until the initial variable is 0. All of the variables which were cyclically shifted to the end are decremented by 1, except for the previous initial 0.

In this scheme, suppose that we have the initial values \( \sigma_i = a \) and \( \sigma_j = b \), where \( i < j \) and \( a < b \). We want to examine the contributions of \( \sigma_j \) to the statistics \( k_p \) and \( l_p \) when the initial variable of \( \sigma^p \) is \( \sigma_i \). Of course, \( \sigma_i \) will never occur as an initial variable until it has been decremented \( a \) times, and hence we have \( \sigma_i = 0 \) and \( \sigma_j = b - a \). Thus, the first time that \( \sigma_i \) is an initial variable, \( \sigma_j \) will contribute \( b - a \) to \( l_p \) and 1 to \( k_p \). In subsequent rounds, \( \sigma_j \) will contribute \( b - a - 1, b - a - 2, \ldots \) to \( l_p \) and 1's to \( k_p \). As soon as \( \sigma_j = 0 \), it will cease to contribute. Hence, \( \sigma_j \) contributes a total of \( \binom{b-a+1}{2} = \binom{a+b}{2} \) to \( \sum l_p \) and a total of \( b - a \) to \( \sum k_p \).

The analysis when \( a > b \) is nearly the same. This time, we want to examine the contributions of \( \sigma_i \) to the statistics \( k_p \) and \( l_p \) when \( \sigma_j \) is the initial term of
\( \sigma^p \). Since the larger term \( a \) precedes the smaller term \( b \), \( \sigma_i \) will be decremented before \( \sigma_j \). The only rounds with \( \sigma_j = 0 \) will have \( \sigma_i = a - b - 1, \ldots, 2, 1, 0 \). Hence, \( \sigma_i \) contributes a total of \( \binom{a-b}{2} \) to \( \sum l_p \) and \( a - b - 1 \) to \( \sum k_p \).

In summary, we have shown

\[
\sum l_p = \sum_{i<j} \left( \binom{\sigma_i - \sigma_j}{2} \right) = \eta(\sigma)
\]

and

\[
\sum k_p = \sum_{i<j} \left\{ \begin{array}{ll}
\sigma_j - \sigma_i & \text{if } \sigma_i < \sigma_j \\
\sigma_i - \sigma_j - 1 & \text{if } \sigma_i > \sigma_j
\end{array} \right. \equiv \text{inv}(\sigma) + \sum_{i<j} (\sigma_i + \sigma_j) \mod 2
\equiv \text{inv}(\sigma) + (n - 1)|\sigma| \mod 2,
\]

which completes the proof.

16. A recursion for computing \( C^n[\lambda](z, q) \)

In the previous section we found a functional equation satisfied by the formal Laurent series

\[
F_n(x_1, \ldots, x_n) = a^e(x) \prod_{1 \leq i, j \leq n} (qx_i x_j^{-1})_\infty,
\]

and used it find a recursion for the series \( C^n[\lambda](0, q) \). In this section, we will extend this technique to obtain a functional equation for the formal Laurent series

\[
G_n(x_1, \ldots, x_n) = a^e(x) \prod_{1 \leq i, j \leq n} \frac{(qx_i x_j^{-1})_\infty}{(zx_i x_j^{-1})_\infty}.
\]

(1)

As before, the functional equation will yield a recursion for the series \( C^n[\lambda](z, q) \). Of course, this recursion will simultaneously yield recursions for decomposing the characters of the exterior and symmetric algebras of \( gl_n \), as well as the Macdonald complex, by specialization. In the case of the symmetric algebra, this amounts to a recursion for computing the generalized exponents of \( SL_n \). Surprisingly, this recursion appears to be different from the Hesselink-Peterson-Macdonald recursion we mentioned in Section 9.

Let us introduce additional notation and terminology. Assume that \( \lambda \) is a partition with \( \ell(\lambda) < n \) and \( |\lambda| \) divisible by \( n \). The horizontal sequence \( h = (h_0, h_1, h_2, \ldots) \) of \( \lambda \) is the infinite sequence of integers defined by

\[
h_i = \begin{cases} 
0 & \text{if } i = 0 \\
n - \lambda_i' + i - 1 & \text{if } i > 0
\end{cases}
\]
This terminology is derived from the same geometric considerations which motivated the introduction of the vertical sequence in Section 15. View the diagram of $\lambda$ as a lattice path from southwest to northeast, and number the vertical and horizontal steps consecutively from 0, starting with a vertical step in the $n$th row. The horizontal steps are labelled $h_1, h_2, h_3, \ldots$. Although this notation conflicts with the complete homogeneous symmetric functions $h_i$, there shall be no occasion in which the use of $h_i$ would be ambiguous. In the example given in Figure 15.1, we have $n = 8$, $\lambda = 764322$, and the corresponding horizontal sequence is $h = (0, 2, 3, 6, 8, 10, 11, 13, 15, 16, \ldots)$.

Recall that the rank of a partition is the size of the largest square diagram $r^r$ contained in $\lambda$ (Definition 1.4). Let us call the largest integer $r$ for which the rectangular diagram $(r+1)^r$ is contained in $\lambda$ the rectangular rank of $\lambda$. For example, the rectangular rank of 764322 is 3. Notice that among partitions $\lambda$ with $\ell(\lambda) < n$ and $|\lambda|$ divisible by $n$, only $\emptyset$ has rectangular rank 0.

Fix an integer $k \geq 0$. For any partition $\nu$, let $\nu_L$ denote the partition whose diagram is the first $k$ columns of $\nu$, and let $\nu_R$ denote the partition whose diagram is the remainder; i.e.,

$$\nu'_L = (\nu'_1, \ldots, \nu'_k) \quad \text{and} \quad \nu'_R = (\nu'_{k+1}, \nu'_{k+2}, \ldots).$$

Recall the notation $\lambda \cup \mu$ (1.5). Define

$$B_k(\lambda) = \{ \mu : \mu \text{ satisfies rules 1, 2 and 3} \}$$

1. $\ell(\mu) \leq n$, $|\mu| = |\lambda|$.
2. $\mu_R \subseteq (\lambda \cup (k))_R$ and the skew diagram $(\lambda \cup (k))_R/\mu_R$ is a vertical strip.
3. $(\lambda \cup (k))_L \subseteq \mu_L$ and the skew diagram $\mu_L/(\lambda \cup (k))_L$ is a vertical strip.

Thus, rules 2 and 3 say that $\mu$ must be obtained from $\lambda \cup (k)$ by adding a vertical strip to the first $k$ columns and deleting a vertical strip from the columns to the right of the first $k$. Notice that $B_0(\lambda) = \{ \lambda \}$ and $B_k(\lambda)$ is nonempty if and only if $0 \leq k \leq r$, where $r$ is the rectangular rank of $\lambda$. For example, let $n = 6$, $k = 2$ and $\lambda = 5553$. We have

$$B_2(5553) = \{54432, 55422, 54421, 444321\}.$$ 

The diagrams of these partitions are given in Figure 16.1. The cells of $\lambda \cup (k)$ have been marked with the symbol $\cdot$.

On occasion, we will violate our conventions and speak of the series $C^n[\mu]$ when $\mu$ is a partition of length $n$. In such circumstances, it should be considered
The Macdonald complex of $gl_n$ and the $q$-Dyson Theorem

Figure 16.1: An illustration of the construction of $B_k(\lambda)$.

synonymous with $C^n[\lambda]$, where $\lambda = (\mu_1 - \mu_n, \ldots, \mu_{n-1} - \mu_n)$ is the partition obtained by removing all of the columns of length $n$ from $\mu$.

We are now ready to state a recursion for the computation of $C^n[\lambda](z, q)$. Although it can be stated in many forms, the following will suffice for the present.

**Theorem 16.1:**

(a) Let $\lambda$ be a partition of $ln$ with $\ell(\lambda) < n$. Let $h$ be the horizontal sequence and $r$ the rectangular rank of $\lambda$. We have

$$
\sum_{0 \leq k \leq r} \sum_{\mu \in B_k(\lambda)} (-1)^k (z^{h_k} - q^l z^{n-h_{k+1}})(1 - z^{h_{k+1}-h_k}) C^n[\mu](z, q) = 0. \quad (2)
$$

(b) Moreover, the linear relations in (2) uniquely determine the series $C^n[\lambda]$ for all $\lambda$ up to the initial value $C^n[\emptyset](z, q)$.

**Example 16.2:**

(a) In the case $\lambda = \emptyset$, we have $r = l = h_0 = 0$, $h_1 = n$ and Theorem 16.1 becomes vacuous. A more typical example occurs if we take $n = 5$ and let $\lambda$ be the second layer partition 4411. Its horizontal sequence is $\{0, 1, 4, 5, 6, 9, \ldots\}$ and $\lambda$ has rectangular rank 2. One may verify that $B_1(\lambda) = \{43111\}$ and $B_2(\lambda) = \{33211\}$. By applying Theorem 16.1, we may deduce

$$
C^5[4411](z, q) = \frac{(z - q^2 z^4)(1 - z^3)}{(1 - q^2 z^4)(1 - z)} C^5[32](z, q) - \frac{z^4 - q^2}{1 - q^2 z^4} C^5[221](z, q).
$$

(b) Consider the first layer partition $\lambda = 1^{n-2}2$. The rectangular rank of $\lambda$ is 1, and we have $B_1(\lambda) = \{1^n\}$. Theorem 16.1 implies that

$$(1 - qz^{n-1})(1 - z) C^n[1^{n-2}2](z, q) = (z - q)(1 - z^{n-1}) C^n[\emptyset](z, q).$$

Therefore, by Corollary 14.4,

$$
C^n[1^{n-2}2](z, q) = \frac{(z - q)(1 - z^{n-1})}{(1 - qz^{n-1})(1 - z)[n]_z} \cdot \frac{(q)_{\infty}}{(qz^n)_{\infty}}.
$$
In Section 17 we will use Theorem 16.1 to find a formula for \( C^n[\lambda](z, q) \) for all first layer partitions, thus generalizing the first layer formula for the exterior algebra (Theorem 12.1).

**Proof of Theorem 16.1:** To deduce part (b) from part (a) is straightforward. All of the partitions in \( B_k(\lambda) \) can be obtained by removing at least \( k \) cells from the diagram of \( \lambda \) and inserting them into lower rows. By Proposition 1.6, it follows that \( \lambda \geq \mu \) for any \( \mu \in B_k(\lambda) \), with equality only when \( k = 0 \). Since the coefficient

\[
(z^{h_k} - q^{1} z^{n-h_k+1})(1 - z^{h_{k+1}-h_k})
\]

of \( C^n[\mu](z, q) \) in (2) can vanish only when \( \lambda = \emptyset \), it follows that when \( \lambda > \emptyset \), one can solve for \( C^n[\lambda](z, q) \) in terms of \( C^n[\mu](z, q) \) for some partitions \( \mu < \lambda \). Part (b) thus follows by induction.

We remark that it is now clear, as we claimed earlier, that each of the series

\[
C^n[\lambda](z, q)
\]

is of the form \( f(z, q) \cdot C^n[\emptyset](z, q) \) for some rational function \( f \).

To prove part (a), we proceed by a series of lemmas, the first of which concerns a functional equation for the formal Laurent series \( G_n \) we defined in (1).

**Lemma 16.3:** We have

\[
G_n(x_1, \ldots, x_{n-1}, qx_n) \cdot \prod_{1 \leq i < n} \left( 1 - \frac{z}{q} x_i x_n^{-1} \right) = G_n(x_1, \ldots, x_n) \cdot \prod_{1 \leq i < n} \left( z - x_i x_n^{-1} \right).
\]

**Proof:** By the definition of \( G_n \),

\[
\frac{G_n(x_1, \ldots, x_{n-1}, qx_n)}{G_{n-1}(x_1, \ldots, x_{n-1})} = \prod_{1 \leq i < n} \left( x_i - qx_n \right) \frac{(q^2 x_n x_i^{-1})_\infty}{(q x_n x_i^{-1})_\infty} \frac{(x_i x_n^{-1})_\infty}{(z q^{-1} x_i x_n^{-1})_\infty}
\]

\[
= (x_1 \cdots x_{n-1}) \cdot \prod_{1 \leq i < n} \left( 1 - x_i x_n^{-1} \right) \cdot \frac{(q x_n x_i^{-1})_\infty}{(z x_n x_i^{-1})_\infty} \cdot \frac{(q x_i x_n^{-1})_\infty}{(z x_i x_n^{-1})_\infty} \cdot \frac{1 - x_n x_i^{-1}}{1 - z q^{-1} x_i x_n^{-1}}
\]

\[
= \frac{G_n(x_1, \ldots, x_n)}{G_{n-1}(x_1, \ldots, x_{n-1})} \cdot \prod_{1 \leq i < n} \frac{z - x_i x_n^{-1}}{1 - z q^{-1} x_i x_n^{-1}}.
\]

The lemma now follows immediately.

Let \( \kappa(S) \in \mathbb{Z}^n \) denote the characteristic vector of any subset \( S \subseteq [n] \); i.e.,

\[
\kappa(S) = (\kappa_1, \ldots, \kappa_n),
\]

where

\[
\kappa_i = \begin{cases} 
1 & \text{if } i \in S \\
0 & \text{if } i \notin S
\end{cases}
\]

Let \( s \) denote the cardinality of \( S \).
Lemma 16.4: Let \( \lambda \) be a partition of \( n \) with \( \ell(\lambda) < n \). We have

\[
\sum_{S \subseteq [n-1]} \epsilon_{w(S)}(-z)^s C^n[\mu(S)](z, q) = q^l z^{n-1} \sum_{S \subseteq [n-1]} \epsilon_{w(S)}(-z)^{-s} C^n[\mu(S)](z, q),
\]

where \( S \) ranges over those subsets of \([n-1]\) for which the terms of \( \lambda + \delta + s \varepsilon_n - \kappa(S) \) are distinct\(^1\). For such \( S \), \( w(S) \) denotes the unique permutation and \( \mu(S) \) the unique partition such that

\[
\lambda + \delta + s \varepsilon_n - \kappa(S) = w(S) \circ (\mu(S) + \delta).
\]

**Proof:** By Remark 4.8 and the definition (14.1) of \( C^n[\lambda] \), we know that

\[
C^n[\lambda](z, q) = [x^{\alpha+\delta}] G_n(x_1, \ldots, x_n)
\]  

(3)

where \( \alpha \in \mathbb{Z}^n \) is the dominant weight corresponding to \( \lambda \). By the functional equation in Lemma 16.3, we have

\[
\sum_{S \subseteq [n-1]} \left( -\frac{z}{q} \right)^s \frac{x_S}{x_n^s} G_n(x_1, \ldots, x_{n-1}, qz_n)
\]

\[
= z^{n-1} \sum_{S \subseteq [n-1]} (-z)^{-s} \frac{x_S}{x_n^s} G_n(x_1, \ldots, x_n),
\]  

(4)

where we have abbreviated \( \prod_{i \in S} x_i \) by \( x_S \). If we apply the fact that for any formal power series \( F \)

\[
[x^k]F(ax) = a^k [x^k] G(x),
\]

and extract the coefficient of \( x^{\alpha+\delta} \) in (4), we obtain

\[
q^\alpha \sum_{S \subseteq [n-1]} (-z)^s [x^{\alpha+\delta}] \frac{x_n^s}{x_S} G_n(x_1, \ldots, x_n)
\]

\[
= z^{n-1} \sum_{S \subseteq [n-1]} (-z)^{-s} [x^{\alpha+\delta}] \frac{x_n^s}{x_S} G_n(x_1, \ldots, x_n).
\]  

(5)

We have yet to exploit the fact, which is evident from (1), that \( G_n \) is an alternating series. This means that any coefficient we care to extract, say \( [x^\gamma] G_n \), will vanish unless the terms of \( \gamma \) are distinct. If they are distinct, then there is a unique permutation \( w \in S_n \) and dominant weight \( \beta \in \mathbb{Z}^n \) for which \( \gamma = w \circ (\beta + \delta) \). Furthermore, by (3),

\[
[x^\gamma] G_n = \epsilon_w [x^{\beta+\delta}] G_n = \epsilon_w C^n[\beta](z, q).
\]

---

\(^1\) Even though \( S \subseteq [n-1] \), we still regard \( \kappa(S) \in \mathbb{Z}^n \).
The lemma now follows from this observation and (5).

**Lemma 16.5:** Let $h$ be the horizontal sequence of $\lambda$.

(a) If $h_k \leq s < h_{k+1}$ then $B_k(\lambda) = \{\mu(S) : |S| = s\}$.

(b) If $\mu(S) \in B_k(\lambda)$ then $\varepsilon_{w(S)} = (-1)^{s-k}$.

**Proof:** Let $S$ be an $s$-subset of $[n-1]$ such that the terms of $\lambda + \delta + s\varepsilon_n - \kappa(S)$ are distinct.

First, we consider the possibility that $s = h_k$ for some $k \geq 0$. Since $s$ is one of the horizontal steps in the lattice path of $\lambda$ (or $s = 0$), then the terms of $\lambda + \delta + s\varepsilon_n$ are distinct, and their relative order cannot be affected by $\kappa(S)$. The sorted elements of $\lambda + \delta + s\varepsilon_n$ are

$$(\lambda_1 + n - 1, \ldots, \lambda_{n-j-1} + j + 1, s, \lambda_{n-j} + j, \ldots, \lambda_{n-1} + 1),$$

where $j$ is the unique integer such that

$$\lambda_{n-j-1} + j + 1 > s > \lambda_{n-j} + j.$$  

If $s = 0$, define $j = 0$. Notice that $w(S)$ is therefore a $(j+1)$-cycle, and so $\varepsilon_{w(S)} = (-1)^j$. By the definition of the horizontal sequence, $s = h_k$ is the $k$th smallest positive integer not among the terms of $\lambda + \delta$. Therefore,

$$h_k = j + k,$$

and in particular, $\varepsilon_{w(S)} = (-1)^{s-k}$ in this case.

Let $E_j = \{n - 1, \ldots, n - j\}$. Notice that if we choose to include all of $E_j$ in $S$, we find

$$\lambda + \delta + s\varepsilon_n - \kappa(E_j) = w(S) \circ ((\lambda \cup (k)) + \delta).$$

This is clear from (6) if one decrements the last $j$ terms by 1 and subtracts $\delta$. The remaining $k$ members of $S$ must be chosen from the first $n - j - 1$ integers.

Clearly, in order to maintain distinct terms in (6), the rows of $\lambda \cup (k)$ which are chosen must be a collection of rows from which one can remove a vertical strip. Furthermore, the cells in such a vertical strip must occupy the columns $k+1, k+2, \ldots$.

More generally, if we do not include all of $E_j$ in $S$, we must add cells to the first $k$ columns (or equivalently, the last $j$ rows) of $\lambda \cup (k)$ in such a way that distinct terms are maintained; i.e., we must add a vertical strip. We conclude that $B_k(\lambda) = \{\mu(S) : |S| = h_k\}$.

The analysis for the case when $|S| = s = h_k + i$, where $h_k < s < h_{k+1}$, introduces only slight complications. Notice that each of the integers $h_k + 1, \ldots, h_k + i$ must appear among the terms of $\lambda + \delta$; say,

$$\lambda + \delta = (\lambda_1 + n - 1, \ldots, \lambda_{n-j-i-1} + j + i + 1,$$
where $j$ is the integer defined in (7). In order to avoid having equal terms in $\lambda + \delta + s\varepsilon_n - \kappa(S)$, we are forced to include all of

$$T = \{n - j - i, \ldots, n - j - 1\}$$

in $S$. Notice that

$$\lambda + \delta + s\varepsilon_n - \kappa(T) = (\lambda_1 + n - 1, \ldots, \lambda_{n-j-i-1} + j + i + 1, \lambda_{b} + i - 1, \ldots, \lambda_{h}, \lambda_{n-j} + j, \ldots, \lambda_{n-1} + 1, h_{b} + i).$$

Therefore, $w(S)$ must be a $(j + i + 1)$-cycle, and so

$$\varepsilon_{w(S)} = (-1)^{i+j} = (-1)^{h_{b} + i - k} = (-1)^{s - k},$$

as desired. Furthermore, we see from (9) that the terms of $\lambda + \delta + s\varepsilon_n - \kappa(T)$ are those of $\lambda + \delta + h_{b}\varepsilon_n$, and so the partitions $\mu(S)$ obtained for the $\varepsilon$-subsets $S$ coincide with those obtained for $h_{b}$-subsets. Here we have made use of the fact, which is evident from (8), that if $|S| = h_{b}$ and the terms of $\lambda + \delta + h_{b}\varepsilon_n - \kappa(S)$ are distinct, then $S \cap T = \emptyset$. This completes the proof of the lemma.

We may now complete the proof of the theorem. Let $S$ range over subsets of $[n - 1]$ for which the terms of $\lambda + \delta + s\varepsilon_n - \kappa(S)$ are distinct, where $s$ denotes $|S|$. Let $r$ be the rectangular rank of $\lambda$. Using the notation defined in Lemma 16.4 and applying Lemma 16.5 yields

$$(1 - z) \sum_{S \subseteq [n-1]} \varepsilon_{w(S)}(-z)^{s}C^{n}[\mu(S)](z, q)$$

$$= (1 - z) \sum_{0 \leq k \leq r} \sum_{\mu \in B_{k}(\lambda)} (-1)^{k}(z^{h_{b} + \ldots + h_{b+1-1}})C^{n}[\mu](z, q)$$

$$= \sum_{0 \leq k \leq r} \sum_{\mu \in B_{k}(\lambda)} (-1)^{k}z^{h_{b}}(1 - z^{h_{b+1}-h_{b}})C^{n}[\mu](z, q),$$

and

$$(1 - z)q^{t}z^{n-1} \sum_{S \subseteq [n-1]} \varepsilon_{w(S)}(-z)^{s}C^{n}[\mu(S)](z, q)$$

$$= (1 - z)q^{t} \sum_{0 \leq k \leq r} \sum_{\mu \in B_{k}(\lambda)} (-1)^{k}(z^{n-1-h_{b} + \ldots + z^{n-h_{b+1}}})C^{n}[\mu](z, q)$$

$$= \sum_{0 \leq k \leq r} \sum_{\mu \in B_{k}(\lambda)} (-1)^{k}q^{t}z^{n-h_{b}}(1 - z^{h_{b+1}-h_{b}})C^{n}[\mu](z, q).$$

Compare (10) and (11) and apply Lemma 16.4.
17. The first layer formula for $C^n[\lambda](z, q)$

We will now apply the recursion developed in the previous section to find an explicit formula for $C^n[\lambda](z, q)$ for first layer partitions $\lambda$. Recall the definition of the hook-length $h(i, j)$ associated with the cell $(i, j)$ of a partition (1.2). Let $t$ be an indeterminate. Extend the notation $(z)_k$, which was introduced in Section 14, as follows:

$$(z; t)_k = (1 - z)(1 - tz) \cdots (1 - t^{k-1}z).$$

Thus, $(z)_k = (z; q)_k$.

**Theorem 17.1:** Let $\lambda$ be a partition of $n$. We have

$$C^n[\lambda](z, q) = \frac{(q)_\infty}{(qz^n)_\infty(q; z)_n} \cdot \prod_{(i, j) \in \lambda} \frac{z^{i-1} - qz^{i-1}}{1 - z^{h(i, j)}}.$$

The reader can easily verify that we recover the formula (Corollary 14.4) for $C^n[\emptyset](z, q)$ in the special case $\lambda = 1^n$. Also, we recover the first layer formula for the exterior algebra (Theorem 12.1) via the specialization $z \to q^2$, $q \to -q$.

If we specialize to the Macdonald complex, we find

**Corollary 17.2:** Let $\lambda$ be a partition of $n$. We have

$$M^\lambda_k(q) = \frac{[n(k + 1)!]_q}{(q; q^{k+1})_n} \cdot \prod_{(i, j) \in \lambda} \frac{q^{(k+1)(j-1)} - q^{(k+1)(i-1)+1}}{1 - q^{(k+1)h(i, j)}}.$$

If we specialize to the symmetric algebra, we obtain an explicit formula for the first-layer generalized exponents of $SL_n$. This formula can also be derived from the Hesselink-Peterson-Macdonald recursion we mentioned in Section 9.

**Corollary 17.3:** Let $\lambda$ be a partition of $n$. We have

$$G^n[\lambda](z) = z^{n(\lambda^n)}[n!], \prod_{(i, j) \in \lambda} \frac{1}{1 - z^{h(i, j)}}.$$

The first step in proving the first layer formula is to examine more carefully the recursion in Theorem 16.1. Notice that the partitions $\mu \in B_k(\lambda)$ are obtained by adding and removing vertical strips from the partition $\lambda \cup (k)$. This is formally similar to the process involved in certain instances of the Littlewood-Richardson
rule. Specifically, we remark that by versions one and three of the LR rule (Theorems 5.3 and 5.7), it follows that

\begin{equation}
\begin{aligned}
    s_\lambda s_{1^k} &= \sum_{\mu} s_\mu : \mu/\lambda \text{ is a vertical } k\text{-strip} \\
    s_{\lambda/1^k} &= \sum_{\mu} s_\mu : \lambda/\mu \text{ is a vertical } k\text{-strip}.
\end{aligned}
\end{equation}

Figure 17.1: An illustration of the construction of $\lambda \ast k$.

In order to exploit this situation, let $\psi_n : \Lambda \to \mathbb{Z}[[z,q]]$ be an arbitrary $\mathbb{Z}$-linear transformation for which

1. $\psi_n(s_\lambda)$ vanishes unless $\ell(\lambda) \leq n$.
2. $\psi_n(s_\lambda) = \psi_n(s_\mu)$ if $\mu = \lambda + c^n$.

The practical reader will complain that we have given an overly elaborate presentation of what should be considered a group homomorphism $\Omega_n \to \mathbb{Z}[[z,q]]$. However, we will find it more convenient to allow an arbitrary symmetric function in the domain of $\psi_n$. Our goal is to impose constraints on $\psi_n$ in such a way that to have $\psi_n(s_\lambda) = C^n[\lambda](z,q)$ is the only way to meet those constraints.

Let $\lambda$ vary over partitions with $\ell(\lambda) < n$ and $|\lambda|$ divisible by $n$. Let $h$ be the horizontal sequence of $\lambda$. For any nonnegative integer $k$, define a partition $\lambda \ast k$ via

\begin{equation}
    (\lambda \ast k)' = \begin{cases} 
    (n, \lambda'_1 + 1, \ldots, \lambda'_{k-1} + 1, \lambda_{k+1}, \lambda_{k+2} \ldots) & \text{if } k \geq 0 \\
    \lambda' & \text{if } k = 0.
    \end{cases}
\end{equation}

For example, if $n = 7$, $k = 3$, and $\lambda = 76422$, then $\lambda \ast k = 7643331$. See Figure 17.1, in which the diagram of $\lambda \ast k$ is given, and the cells of $\lambda$ have been marked.

Notice that the partition $\lambda \ast k$ can also be defined as the partition obtained by adding a cell to each row of $\lambda \cup (k)$ which has fewer than $k$ cells. Also note
that $\lambda \ast k/\lambda$ is a border strip of $h_k$ cells; i.e., a connected skew diagram which contains no $2 \times 2$ square as a subdiagram.

Using this this new notation, we can reformulate the recursion we gave in Theorem 16.1:

**Theorem 17.4.** Let $\psi_n : \Lambda \to \mathbb{Z}[[z, q]]$ be a $\mathbb{Z}$-linear transformation as described above. Let $\lambda$ vary over partitions with $\ell(\lambda) < n$ and $|\lambda|$ divisible by $n$. Let $l$ denote the layer of $\lambda$ and $r$ the rectangular rank of $\lambda$. If

$$
\sum_{0 \leq k \leq r} (-1)^k (z^{h_k} - q^lz^{n-h_r+1})(1 - z^{h_{r+1}-h_k}) \psi_n(s_{\lambda \ast k/1^{h_k}}) = 0
$$

for all such $\lambda$, then $\psi_n(s_{\lambda})$ must be, apart from multiplicative factors, $C^n[\lambda](z, q)$; i.e., there must exist a formal power series $c(z, q)$, independent of $\lambda$, such that

$$
\psi_n(s_{\lambda}) = c(z, q)C^n[\lambda](z, q).
$$

**Proof:** For $k \geq 0$, define

$$
g_k = \sum_{\mu \in B_k(\lambda)} s_\mu \quad \text{and} \quad f_k = s_{\lambda \ast k/1^{h_k}}.
$$

Let $k \geq 1$ and assume that the lowest row of $\lambda \cup (k)$ of length $k$ is the $j$th row. Recall that the partitions $\mu \in B_k(\lambda)$ are obtained by adding a vertical strip below the $j$th row, and deleting a vertical strip from the part of $\lambda$ occupying columns $k + 1, k + 2, \ldots$. Since $\lambda \ast k$ is obtained by adding a cell to every row of $\lambda \cup (k)$ below the $j$th, we see that $\mu \in B_k(\lambda)$ if and only if $\mu$ can be obtained by deleting a vertical $h_k$-strip from $\lambda \cup (k)$, provided that this strip does not include the cell $(j, k)$. If one removes a vertical $h_k$-strip from $\lambda \cup (k)$ which does include the cell $(j, k)$, one obtains a partition in $B_{k-1}(\lambda)$, and conversely. By our previous discussion of the LR rule (see (1)), we conclude that

$$
f_0 = g_0 \quad \text{and} \quad f_k = g_k + g_{k-1}.
$$

Since

$$(1 - z^{h_{k+1}-h_k})(z^{h_k} - q^lz^{n-h_k+1})
$$

$$
= (1 - z^{h_{r+1}-h_k})(z^{h_k} - q^lz^{n-h_r+1}) - (1 - z^{h_{r+1}-h_k+1})(z^{h_k+1} - q^lz^{n-h_r+1}),
$$

it follows from (4) that

$$
\sum_{0 \leq k \leq r} (-1)^k (z^{h_k} - q^lz^{n-h_r+1})(1 - z^{h_{r+1}-h_k})\psi_n(f_k)
$$
\[
\sum_{0 \leq k \leq r} (-1)^k (z^{h_k} - q^i z^{n-h_k+1}) (1 - z^{h_{k+1} - h_k}) \psi_n(g_k).
\]

Therefore, we see that (3) is satisfied if and only if \(\psi_n(s_\lambda)\) satisfies the recursion in Theorem 16.1. The uniqueness of the solution is guaranteed by part (b) of Theorem 16.1.\(*\)

**Remark 17.5:** Notice that a similar result holds even if we restrict our attention to the first layer. If \(\lambda\) is in the first layer, then only partitions of \(n\) appear in \(B_k(\lambda)\). Therefore, if we have a \(\mathbb{Z}\)-linear map \(\psi_n : \Lambda^n \to \mathbb{Z}[[z, q]]\) which satisfies (3) for all partitions \(\lambda\) of \(n\), then the series \(\psi_n(s_\lambda)\) satisfy (16.2) for all such \(\lambda\). We may conclude that \(\psi_n(s_\lambda)\) and \(C^n[\lambda](z, q)\) differ by at most a multiplicative factor.

Recall that in the Frobenius notation for partitions, if \(\lambda\) has rank \(r\), we write \(\lambda = (\alpha | \beta)\), where \(\alpha, \beta \in \mathbb{N}^r\) are the strictly decreasing \(r\)-tuples defined in (1.4).

**Lemma 17.6:** Let \(\lambda\) be a first layer partition of rectangular rank \(r\), and let \(h\) be the horizontal sequence of \(\lambda\). Define \(\alpha \in \mathbb{N}^r, \beta \in \mathbb{N}^{r+1}\) via

\[
\begin{align*}
\alpha_i &= \lambda_i - i - 1 & 1 \leq i \leq r \\
\beta_i &= n - h_i = \lambda'_i - i + 1 & 1 \leq i \leq r + 1.
\end{align*}
\]

If \(1 \leq k \leq r + 1\), then

\[
s_{\lambda* k/1^h_k} = s_{1^h_k} \cdot s_{(\alpha | \beta_1, ..., \beta_{k-1}, \beta_{k+1}, ..., \beta_{r+1})}.
\]

**Proof:** We claim that the skew shape \(\lambda* k/1^h_k\) is disconnected. Since the diagram of \(1^h_k\) is a single column, this is equivalent to showing that \(h_k \geq (\lambda * k)_2\). By the definition of \(\lambda * k\), we see that we must show

\[
h_k \geq \begin{cases} 
\lambda'_k & \text{if } k = 1 \\
\lambda'_k + 1 & \text{if } k > 1.
\end{cases}
\]

However, these are immediate from the fact that \(h_k = n - \lambda'_k + k - 1\) and \(\lambda\) is a partition of \(n\). Therefore \(\lambda * k/1^h_k\) is indeed disconnected.

Since the first column of \(\lambda * k/1^h_k\) has length \(n - h_k = \beta_k\), we have

\[
s_{\lambda* k/1^h_k} = s_{1^h_k} \cdot s_\mu,
\]

where \(\mu\) is the partition obtained by deleting the first column of \(\lambda * k\). Notice that \(\mu\) is of rank \(r\). Inspection of (2) reveals that \(\mu = (\alpha | \beta_1, ..., \beta_{k-1}, \beta_{k+1}, ..., \beta_{r+1})\) is the Frobenius notation for \(\mu\).\(*\)

The following beautiful identity expresses the Schur function \(s_\lambda\) as a determinant of Schur functions corresponding to hooks; i.e., Schur functions of the form
$s_{(a|b)}$ with $a,b \in \mathbb{N}$. An elementary proof is given by Macdonald [23: I. ex.3.9], who attributes the result to Giambelli [10].

**Proposition 17.7:** Let $(\alpha|\beta)$ be the Frobenius notation for a partition $\lambda$ of rank $r$. We have

$$s_{\lambda} = \det[s_{(\alpha_i|\beta_j)} : 1 \leq i, j \leq r].$$

**Lemma 17.8:** Let $\lambda$ be a first layer partition, with notation as in Lemma 17.6. Let $M_{\lambda}$ denote the matrix of order $r+1$ defined by:

$$M_{\lambda} = \begin{bmatrix}
(1 - z^{n-\beta_1})(1 - qz^{\beta_1})s_{1,}\sigma_1 & s_{(\alpha_1|\beta_1)} & \cdots & s_{(\alpha_r|\beta_1)} \\
\vdots & \ddots & \ddots & \vdots \\
(1 - z^{n-\beta_r+1})(1 - qz^{\beta_r+1})s_{r,}\sigma_{r+1} & s_{(\alpha_1|\beta_{r+1})} & \cdots & s_{(\alpha_r|\beta_{r+1})}
\end{bmatrix}. $$

The identity $\psi_n(\det M_{\lambda}) = 0$ is equivalent to (3).

The pedantic reader may wish to enhance the coefficient ring of $A$ to include all of $\mathbb{Z}[[z, q]]$ before applying $\psi_n$ to $\det(M_{\lambda})$.

**Proof:** If (3) holds, then

$$(1 - qz^{n-h_{r+1}})(1 - z^{h_{r+1}})\psi_n(s_{\lambda})$$

$$= \sum_{1 \leq k \leq r} (-1)^{k-1}(z^{h_k} - qz^{n-h_{r+1}})(1 - z^{h_{r+1} - h_k})\psi_n(s_{\lambda + k/1 h_k}).$$

By Lemma 17.6, it follows that

$$(1 - qz^{\beta_{r+1}})(1 - z^{n-\beta_{r+1}})\psi_n(s_{\lambda})$$

$$= \sum_{k=1}^{r+1} (-1)^{k-1}(z^{n-\beta_k} - qz^{\beta_{r+1}})(1 - z^{\beta_k - \beta_{r+1}})\psi_n(s_{1,}\sigma_k s_{\alpha|\beta_{1},\beta_{k-1},\beta_{k+1},\ldots,\beta_{r+1}}). \quad (5)$$

Let $N_{\lambda}$ denote the matrix of order $r+1$ whose $k$th row ($1 \leq k \leq r+1$) is:

$$((z^{n-\beta_k} - qz^{\beta_{r+1}})(1 - z^{\beta_k - \beta_{r+1}}), s_{(\alpha_1|\beta_k)}, \ldots, s_{(\alpha_r|\beta_k)}).$$

If we expand the determinant of $N_{\lambda}$ by minors along the first column and apply Proposition 17.7, we see that (5) is equivalent to

$$(1 - qz^{\beta_{r+1}})(1 - z^{n-\beta_{r+1}})\psi_n(s_{\lambda}) = \psi_n(\det N_{\lambda}). \quad (6)$$
On the other hand, suppose for the moment that $\beta_{r+1} > 0$; i.e., $\lambda'_{r+1} > r$. If so, then $\lambda$ is of rank $r + 1$ and the Frobenius notation for $\lambda$ must be

$$(\alpha_1 + 1, \ldots, \alpha_r + 1, 0 | \beta_1 - 1, \ldots, \beta_{r+1} - 1).$$

By Proposition 17.7, it follows that $s_\lambda$ is the determinant of the matrix of order $r + 1$ whose $k$th row is

$$(s_{(\alpha_1+1|\beta_1-1)}, \ldots, s_{(\alpha_r+1|\beta_r-1)}, s_{1^k}). \tag{7}$$

If it should happen that $\beta_{r+1} = 0$, then $\lambda$ is only of rank $r$, but we may still assert that $s_\lambda$ is the determinant of the matrix defined by (7), provided that we define $s_{(a|-1)} = 0$ for $a > 0$, since the last row of the matrix would become $(0, \ldots, 0, 1)$ in this case.

Notice that by (1), or simply the LR rule, we have

$$s_{(a+1)} \cdot s_{1^k} = s_{(a|b)} + s_{(a+1|b-1)}.$$ 

Therefore, by subtracting suitable multiples of the $(r + 1)$th column of (7) from the other columns, we deduce that $s_\lambda$ is the determinant of the matrix whose $k$th row is

$$(-s_{(\alpha_1|\beta_1)}, \ldots, -s_{(\alpha_r|\beta_r)}, s_{2^k})$$

or equivalently,

$$(s_{2^k}, s_{(\alpha_1|\beta_1)}, \ldots, s_{(\alpha_r|\beta_r)}). \tag{8}$$

This remains valid even when $\beta_{r+1} = 0$.

It is now clear that $\psi_n(\det M_\lambda) = 0$ is equivalent to (3); replace $s_\lambda$ in (6) by the determinant of the matrix in (8) and use the linearity of the determinant in the first column.

We are now sufficiently prepared to give the

**Proof of Theorem 17.1:** Recall the notation $f(p_k \to a_k)$ which was defined in (9.9). We claim that if we define

$$\psi_n(s_\mu) = s_\mu \left( p_k \to (-1)^{k-1} \frac{1 - q^k}{1 - z^k} \right)$$

for every partition $\mu$ of $n$, then all of the identities $\psi_n(\det M_\lambda) = 0$ are satisfied for first layer partitions $\lambda$. By Proposition 12.5, we can give an explicit formula for $\psi_n(s_\mu)$; namely,

$$\psi_n(s_\mu) = z^{n(\mu)} \cdot \prod_{z \in \mu} \frac{-q + z^i(z)}{1 - z^h(z)} = \prod_{(i,j) \in \mu} \frac{z^{j-1} - qz^{i-1}}{1 - z^{h(i,j)}}. \tag{9}$$
In particular, notice that the substitution
\[ p_k \mapsto (-1)^{k-1} \frac{1 - q^k}{1 - z^k} \]
defines a ring homomorphism \( \Lambda \to \mathbb{Z}[[z,q]] \). Therefore, to verify that \( \psi_n(\det M_\lambda) \) vanishes, it suffices to show that
\[
\det \left[ M_\lambda \left( p_k \mapsto (-1)^{k-1} \frac{1 - q^k}{1 - z^k} \right) \right] = 0. \tag{10}
\]
By Proposition 12.5, we have
\[
g_{(a|b)} \left( p_k \mapsto (-1)^{k-1} \frac{1 - q^k}{1 - z^k} \right) = \frac{(q;z)_{b+1} f_a(z,q)}{[a]_s [b]_s} \cdot \frac{1}{1 - z^{a+b+1}},
\]
where
\[
f_a(z,q) = (z - q)(z^2 - q) \cdots (z^a - q).
\]
Therefore, the \( k \)th row of the matrix in (10) is:
\[
\frac{(q;z)_{\beta_k+1}}{[\beta_k]_s} \left( 1 - z^{n-\beta_k}, \frac{f_a(z,q)}{[\alpha_1]_s}, \frac{1}{1 - z^{\alpha_1+\beta_k+1}}, \ldots, \frac{f_a(z,q)}{[\alpha_r]_s}, \frac{1}{1 - z^{\alpha_r+\beta_k+1}} \right),
\]
where \( \alpha \) and \( \beta \) are as defined in Lemma 17.6. By rescaling the rows and columns of this matrix, we deduce that the vanishing of \( \psi_n(\det M_\lambda) \) is equivalent to the vanishing of the determinant of the matrix whose \( k \)th row is
\[
\left( 1 - z^{n-\beta_k}, \frac{1}{1 - z^{\alpha_1+\beta_k+1}}, \ldots, \frac{1}{1 - z^{\alpha_r+\beta_k+1}} \right). \tag{11}
\]
Consider the determinant of the matrix whose \( k \)th row is
\[
\left( 1, \frac{1}{1 - z^{\alpha_1+\beta_k+1}}, \ldots, \frac{1}{1 - z^{\alpha_r+\beta_k+1}} \right). \tag{12}
\]
Subtracting the first column from the remaining \( r \) columns yields
\[
\left( 1, \frac{z^{\alpha_1+\beta_k+1}}{1 - z^{\alpha_1+\beta_k+1}}, \ldots, \frac{z^{\alpha_r+\beta_k+1}}{1 - z^{\alpha_r+\beta_k+1}} \right).
\]
Extracting common factors from the rows and columns of this matrix shows that the determinant of (12) is the same as the determinant of the matrix whose \( k \)th row is
\[
\left( z^{n-\beta_k}, \frac{1}{1 - z^{\alpha_1+\beta_k+1}}, \ldots, \frac{1}{1 - z^{\alpha_r+\beta_k+1}} \right). \tag{13}
\]
Comparison of (12) and (13) shows that the matrix defined in (11) must indeed be singular, so our claim is verified.

Now we are virtually finished. By Lemma 17.8, we deduce that the choices of \( \psi_n(s_\mu) \) for partitions \( \mu \) of \( n \) we made in (9) actually satisfy the linear relations in (3). By Theorem 17.4 (see Remark 17.5), it follows that there must be a formal power series \( c(z, q) \) such that

\[
C^n[\lambda](z, q) = c(z, q) \cdot \prod_{(i,j) \in \lambda} \frac{z^{i-1} - qz^{i-1}}{1 - z^{h(i,j)}}
\]

for all partitions \( \lambda \) of \( n \). The series \( c(z, q) \) can be determined by taking \( \lambda = 1^n \) and applying Corollary 14.4.

18. An extension of the \( q \)-Dyson Theorem

The first layer formula (Theorem 17.1) has essentially provided us with an explicit formula for the coefficients of the monomials \( x^{\alpha+\delta} \) in the formal Laurent series

\[
G_n(x_1, \ldots, x_n) = a_\delta(x) \prod_{1 \leq i, j \leq n} \frac{(qx_i x_j^{-1})_\infty}{(x_i x_j^{-1})_\infty},
\]

for those dominant weights \( \alpha \in \mathbb{Z}^n \) corresponding to first layer partitions; i.e., dominant weights with \( \alpha_n = -1 \). If we take \( z = q^k \) (which suffers no genuine loss of generality) and delete those factors corresponding to terms with \( i = j \), we may equivalently view the first layer formula as a formula for the coefficients of certain monomials in

\[
\prod_{1 \leq i < j \leq n} (x_j x_i^{-1})_k (qx_i x_j^{-1})_{k-1}.
\]

Specifically, we have

\[
[x^\alpha] \prod_{i < j} (x_j x_i^{-1})_k (qx_i x_j^{-1})_{k-1} = [(k - 1)!]_q^{-n} \cdot C^n[\alpha](q^k, q).
\]

Thus, the first layer formula gives a generalization of Bressoud and Goulden’s result (Theorem 14.3) in the case of equal parameters \( a_1 = \cdots = a_n = k \). It should be pointed out, however, that we used their identity (via Corollary 14.4) to prove this generalization.

As we noted earlier, Bressoud and Goulden deduce the \( q \)-Dyson Theorem (Theorem 14.2) by evaluating a constant term in which ‘\( i < j \)’ is replaced in Theorem 14.3 by ‘(\( i, j \) \( \in \) \( T \)’, where \( T \) is an arbitrary tournament. In this section,
we will show that the first layer formula can be used to find some coefficients of (not necessarily constant) monomials in the formal Laurent series
\[
H_n(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} \frac{\left( x_j x_i^{-1} \right)_\infty}{\left( z x_j x_i^{-1} \right)_\infty} \cdot \frac{\left( q x_i x_j^{-1} \right)_\infty}{(q z x_i x_j^{-1})_\infty},
\]
and in particular \((z = q^k)\), some coefficients of monomials in
\[
\prod_{1 \leq i < j \leq n} (x_j x_i^{-1})_k (q x_i x_j^{-1})_k.
\]
Thus, we will find a generalization of the equal-parameter version of the \(q\)-Dyson Theorem. In order to compute these new coefficients, we need to introduce some additional tools from the theory of symmetric functions.

For each \(\alpha \in \mathbb{N}^n\), define a symmetric function \(R^n_\alpha(x_1, \ldots, x_n; q) \in \Lambda_n[q]\) via
\[
R^n_\alpha(x_1, \ldots, x_n; q) = \sum_{w \in S_n} w \left( x^\alpha \prod_{1 \leq i < j \leq n} \frac{x_i - q x_j}{x_i - x_j} \right) = \frac{1}{a_\delta(x)} \sum_{w \in S_n} \varepsilon_w \left( x^\alpha \prod_{1 \leq i < j \leq n} (x_i - q x_j) \right).
\]
Notice that \(R^n_\alpha\) is homogeneous of degree \(|\alpha|\) with respect to \(x_1, \ldots, x_n\). It is clear from (2) and Theorem 4.7 that there must exist polynomials \(a_{\alpha \lambda}(q) \in \mathbb{Z}[q]\) such that
\[
R^n_\alpha(x_1, \ldots, x_n; q) = \sum_{|\lambda| = k} a_{\alpha \lambda}(q) s_\lambda(x_1, \ldots, x_n),
\]
where \(|\alpha| = k\). We mention that these functions \(R^n_\alpha\) are only slightly more general than the symmetric functions \(R_\lambda\) considered by Macdonald [32; III.1].

If \(|\alpha| \leq n\), then the partitions \(\lambda\) which appear in (3) all have length at most \(n\), and so by Proposition 4.3, we conclude that there is a unique symmetric function in \(\Lambda[q]\), which we denote by \(R_\alpha\), whose image in \(\Lambda_n[q]\) is \(R^n_\alpha\). In fact, from (3), we see that
\[
R_\alpha(x_1, x_2, \ldots; q) = \sum_{|\lambda| = k} a_{\alpha \lambda}(q) s_\lambda(x_1, x_2, \ldots).
\]

These symmetric functions are closely related to the Hall-Littlewood symmetric functions, which have a number of remarkable algebraic and combinatorial properties. If \(\lambda\) is a partition, let \(m\) be any integer for which \(m \geq \ell(\lambda)\). Define an integer polynomial
\[
\nu^m_\lambda(q) = \frac{[(m - \ell(\lambda))!]_q}{(1 - q)^{m - \ell(\lambda)}} \cdot \frac{[n_1!]_q}{(1 - q)^{n_1}} \cdot \frac{[n_2!]_q}{(1 - q)^{n_2}} \cdots,
\]
where \( n_i = n_i(\lambda) \) is the multiplicity of \( i \) in \( \lambda \). It can be shown [32; III.1(5.1), (2.5)] that the coefficients \( a_{\lambda \mu}(q) \) of \( R_{\lambda}^m \) are all divisible by \( v_{\lambda}(q) \) in \( \mathbb{Z}[q] \). Moreover, there is a unique symmetric function \( P_{\lambda}(x; q) \in \Lambda[q] \), called the Hall-Littlewood symmetric function corresponding to \( \lambda \), whose image in \( \Lambda_m[q] \) for any \( m \geq \ell(\lambda) \) satisfies

\[
P_{\lambda}(x_1, \ldots, x_m) = \frac{R_{\lambda}^m(x_1, \ldots, x_m; q)}{v_{\lambda}(q)}.
\]

These functions were originally devised by P. Hall to study the combinatorics of finite abelian \( p \)-groups, and later by Littlewood [27]. For a detailed account of the properties of Hall-Littlewood symmetric functions, the reader is referred to [32; III].

In the case that \( \lambda \in \mathbb{N}^n \) is a partition, we see that \( R_{\lambda}(x; q) \) and \( P_{\lambda}(x; q) \) differ only by a polynomial in \( q \). However, it should be emphasized that even though we regard as identical two representations of a partition which differ only in the number of trailing zeroes, the symmetric function \( R_{\alpha}(x; q) \) is defined only for finite sequences of nonnegative integers. In particular \( R_{\lambda}(x; q) \) depends not only on the partition \( \lambda \) but on the chosen integer \( n \geq \ell(\lambda) \) for which \( \lambda \in \mathbb{N}^n \).

The coefficients we will extract from \( H_n(x_1, \ldots, x_n) \) are the coefficients of the monomials \( x^\alpha \) where \( \alpha \in \mathbb{Z}^n \) and \( \alpha_i \geq -1 \). Of course, one could apply the transformation \( x_i \to x_i^{-1} \) and thus obtain identical results about the coefficients of \( x^\alpha \) with \( \alpha_i \leq 1 \). The following result connects these coefficient extraction problems to the symmetric functions \( R_\beta \).

**Theorem 18.1:** Let \( \alpha \in \mathbb{Z}^n \) and assume \( |\alpha| = 0, \alpha_i \geq -1 \) (1 \( \leq i \leq n \)). We have

\[
[x^\alpha] H_n(x_1, \ldots, x_n) = \frac{(x)_n}{(q)^{n-1}(qz^n)_n} \cdot R_\beta \left( p_r \to (-1)^{r-1} \frac{1 - q^r}{1 - q^r}; z \right),
\]

where \( \beta = \alpha + 1^n = (\alpha_1 + 1, \ldots, \alpha_n + 1) \).

In the special case \( z = q^k \), we find

**Corollary 18.2:** Let \( \alpha, \beta \) be as described above. We have

\[
[x^\alpha] \prod_{1 < j} (x_j x_i^{-1})_k (q x_i x_j^{-1}) = \frac{[(kn)!]_q}{[(k-1)!]_q (q; q^k)_n} \cdot R_\beta \left( p_r \to (-1)^{r-1} \frac{1 - q^r}{1 - q^r}; q^k \right).
\]

**Proof of Theorem 18.1:** By the definitions of \( H_n \) and \( G_n \), we have

\[
H_n(x_1, \ldots, x_n) = x^{-\ell} (x)_n \frac{G_n(x_1, \ldots, x_n)}{(q)_n} \prod_{1 < j} (1 - zz_i x_j^{-1}).
\]
Let $S = \{(i, j) : 1 \leq i < j \leq n\}$. For each subset $T \subseteq S$ define a sequence $\gamma = \gamma(T) \in \mathbb{Z}^n$ via
\[
\gamma_i = |\{j : 1 \leq j \leq n, (j, i) \in T\}| - |\{j : 1 \leq j \leq n, (i, j) \in T\}|.
\]
Let $\alpha \in \mathbb{Z}^n$ and assume that $|\alpha| = 0$ and $\alpha_i \geq -1$. It follows from (5) that
\[
[x^\alpha]H_n(x_1, \ldots, x_n) = \frac{(x)_{n!}}{(q)_{n!}} \sum_{T \subseteq S} (-1)^{|T|}[x^{\alpha + \gamma(T)}]G_n(x_1, \ldots, x_n). \tag{6}
\]
If $T \subseteq S$ has the property that the terms of $\alpha + \delta + \gamma(T)$ are all distinct, let $\alpha(T)$ denote the unique dominant weight and $w(T) \in S_n$ the unique permutation such that
\[
\alpha + \delta + \gamma(T) = w(T) \circ (\alpha(T) + \delta).
\]
Since $G_n$ is an alternating function, it follows from (6) and (16.3) that
\[
[x^\alpha]H_n(x_1, \ldots, x_n) = \frac{(x)_{n!}}{(q)_{n!}} \sum_{T \subseteq S} \varepsilon_{w(T)}(-1)^{|T|}C^n[\alpha(T)](z, q), \tag{7}
\]
summed over those $T \subseteq S$ for which the terms of $\alpha + \delta + \gamma(T)$ are distinct. Notice that for any $T \subseteq S$, the terms of $\delta + \gamma(T)$ are nonnegative. Therefore, the terms of $\alpha + \delta + \gamma(T)$ are all at least $-1$, and so the dominant weights $\alpha(T)$ which appear in (7) correspond to partitions in the 0th or first layers. Recall that when we proved the first layer formula, we showed that
\[
C^n[\lambda](z, q) = \frac{(q)_{n!}}{(q z^n)_{\infty}} s_{\lambda}(p_r \rightarrow (-1)^{r-1} \frac{1 - q^r}{1 - z^r})
\]
for partitions $\lambda$ of $n$. Using this information in (7), yields
\[
[x^\alpha]H_n(x_1, \ldots, x_n)
\]
\[
= \frac{(x)_{n!}}{(q)_{n!}} \sum_{T \subseteq S} \varepsilon_{w(T)}(-1)^{|T|}s_{\alpha(T)}(p_r \rightarrow (-1)^{r-1} \frac{1 - q^r}{1 - z^r}), \tag{8}
\]
where $\lambda(T) = \alpha(T) + 1^n$.

On the other hand, consider the symmetric function $R_\beta(x; q)$, where $\beta = \alpha + 1^n$. It follows from (2) that
\[
a_\delta(x)R_\beta(x_1, \ldots, x_n; q) = \sum_{w \in S_n} \varepsilon_w \cdot w \left( x^{\beta + \delta} \prod_{1 \leq i < j \leq n} (1 - qx_jx_i^{-1}) \right)
\]
The Macdonald complex of $gl_n$ and the $q$-Dyson Theorem

$$= \sum_{T \subseteq S} (-q)^{|T|} a_{\beta+\delta+\gamma(T)}(x_1, \ldots, x_n).$$

Application of Theorem 4.7 yields

$$R_{\beta}(x_1, \ldots, x_n; q) = \sum_{T \subseteq S} \varepsilon_{w(T)}(-q)^{|T|} s_{\lambda(T)}(x_1, \ldots, x_n),$$

so

$$R_{\beta}(x; q) = \sum_{T \subseteq S} \varepsilon_{w(T)}(-q)^{|T|} s_{\lambda(T)}(x). \quad (9)$$

The theorem follows upon comparison of (8) and (9). \qed

We remark that (9) is essentially equivalent to an identity given by Macdonald [32; I. ex.2.3] for Hall-Littlewood symmetric functions.

Theorem 18.1 compels us to study the effect of substitutions of the form

$$p_k \to \frac{a^k - b^k}{1 - q^k}$$

on the symmetric functions $R_{\alpha}$ where $\alpha \in \mathbb{N}^n$ and $|\alpha| = n$. We know of no general result which gives explicit formulas for such substitutions. However, we will be able to give an explicit formula in the case where the 0's of $\alpha$ occur consecutively. The resulting formula is the following:

**Theorem 18.3:** Let $\gamma \in \mathbb{Z}^n$, $|\gamma| = 0$ and assume that there exist $\alpha \in \mathbb{N}^r$, $\beta \in \mathbb{N}^s$ such that $\gamma$ is of the form

$$\gamma = (\alpha_1, \ldots, \alpha_r, -1, \ldots, -1, \beta_s, \ldots, \beta_1).$$

Let $t = n - r - s$ be the number of $-1$'s in $\gamma$. We have

$$[x^\gamma] H_n(x_1, \ldots, x_n) = (-1)^t q^{m(\alpha, \beta)} \cdot \frac{(qz)^n_{\infty} [t!]_q (q; z)^{r+s}}{(q)_{n-1}^\infty (qz^n)^{\infty} (q; z)_n},$$

where

$$m(\alpha, \beta) = n(\alpha) - n(\beta) - |\beta| + st.$$ 

The special case $z = q^k$ yields

**Corollary 18.4:** Let $\alpha, \beta, \gamma$ be as described above. We have

$$[x^\gamma] \prod_{1 \leq i < j \leq n} (x_j x_i^{-1})_k (qx_i x_j^{-1})_k = (-1)^t q^{m(\alpha, \beta)} \cdot \frac{[(kn)!]_q [t!]_q (q; q^k)^{r+s}}{[k!]_q^n (q; q^k)_n}.$$


In particular, when \( \gamma = \emptyset \), we recover the \( q \)-Dyson Theorem for equal parameters. Also it is interesting to consider the limit \( q \to 1 \) and thus obtain a generalization of the "1-Dyson" Theorem (14.3). In this case we need not worry about where the \(-1\)'s occur in \( \gamma \) since the limiting series

\[
\prod_{i \neq j} (1 - x_i x_j^{-1})^k
\]

is a symmetric function of \( x_1, \ldots, x_n \).

**Corollary 18.5:** Let \( \gamma \in \mathbb{Z}^n \), \( |\gamma| = 0 \) and suppose that \( \gamma_i \geq -1 \) (1 \( \leq i \leq n \)). If there are \( t \) \(-1\)'s among the terms of \( \gamma \), then

\[
[x^\gamma] \prod_{i \neq j} (1 - x_i x_j^{-1})^k = (-1)^t \frac{(nk)!}{(k!)^n} \frac{k^t \cdot t!}{(1 + k(n - 1)) \cdots (1 + k(n - t))}.
\]

As a first step towards proving Theorem 18.3, we show that the computation of formulas for

\[
R_\theta \left( p_r \to (-1)^{r-1} \frac{1 - q^r}{1 - z^r}; z \right)
\]

is no more difficult than the computation of formulas for

\[
R_\theta(1, q, \ldots, q^{m-1}; q).
\]

**Proposition 18.6:** Let \( f \in \Lambda^k \) be a homogeneous symmetric function, and \( F(x, y) \) a formal power series such that

\[
f(1, q, \ldots, q^{m-1}) = F(q^m, q)
\]

for sufficiently large integers \( m \). In that case,

\[
f \left( p_r \to \frac{a^r - b^r}{1 - q^r} \right) = a^k F(ba^{-1}, q).
\]

**Proof:** Notice that

\[
p_r(a, aq, \ldots, aq^{m-1}) = \frac{a^r - (aq^m)^r}{1 - q^r}.
\]

Therefore, since \( f \) is homogeneous,

\[
f \left( p_r \to \frac{a^r - (aq^m)^r}{1 - q^r} \right) = f(a, aq, \ldots, aq^{m-1}) = a^k F(q^m, q).
\]
By Lemma 14.1, it follows that
\[
f \left( p_r \to \frac{a^r - (ab)^r}{1 - q^r} \right) = a^k F(b, q),
\]
and the proposition is now obvious.

The computation of \( R^n_\alpha(1, q, \ldots, q^{n-1}; q) \) is comparatively easy:

**Lemma 18.7:** If \( \alpha \in \mathbb{N}^n \), then
\[
R^n_\alpha(1, q, \ldots, q^{n-1}; q) = q^{n(\alpha)} \frac{[n]_q}{(1 - q)^n}.
\]

**Proof:** The argument we present here is the same as the one given by Macdonald [32; III. ex.2.1]. (He makes no use of the fact, but presumes, that \( \alpha \) is a partition.)

Let \( w \in S_n \) and consider the product
\[
\prod_{1 \leq i < j \leq n} (q^{w(i)-1} - q^{w(j)}).
\]
This will vanish if (say) \( i + 1 \) occurs before \( i \) among \( w(1), \ldots, w(n) \). In other words, this product vanishes unless \( w \) is the identity. Thus, the only term in the definition (1) of \( R^n_\alpha(x_1, \ldots, x_n; q) \) which survives under the substitution \( x_i \to q^{i-1} \) is the term corresponding to \( w = 1 \). The result is now immediate.

Unfortunately, Lemma 18.7 gives us no direct information about
\[
R_\alpha(1, q, \ldots, q^{m-1}; q)
\]
when \( \alpha \in \mathbb{N}^n \) and \( m > n \). In order to compute formulas in such cases we need the following result, which relates the symmetric functions \( R_\alpha \) with \( \alpha \in \mathbb{N}^n \) to \( R_\beta \) with \( \beta \in \mathbb{N}^{n+1} \).

**Lemma 18.8:** Let \( \alpha \in \mathbb{N}^{n+1} \) and assume that \( |\alpha| \leq n \). Let \( Z \) be the set of zeroes of \( \alpha \); i.e.,
\[
Z = \{ i : 1 \leq i \leq n + 1, \ \alpha_i = 0 \}.
\]
We have
\[
R_\alpha(x; q) = \sum_{i \in Z} q^{n+1-i} R_{\alpha/i}(x; q),
\]
where
\[
\alpha/i = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{n+1}) \in \mathbb{N}^n.
\]
Proof: By the definition (1) of \( R^m_{\alpha+1} \), we see that

\[
R^m_{\alpha+1}(x_1, \ldots, x_{n+1}; q) = \sum_{w \in S_{n+1}} x^{\alpha_1}_{w(1)} \cdots x^{\alpha_{n+1}}_{w(n+1)} \prod_{1 \leq i < j \leq n+1} \frac{x_w(i) - qx_w(j)}{x_w(i) - x_w(j)}.
\]

In particular, observe that when we substitute \( x_{n+1} = 0 \), the only terms which survive correspond to permutations \( w \in S_{n+1} \) with \( w(i) = n+1 \) for some \( i \in \mathbb{Z} \). The remainder of \( w \),

\[
w(1), \ldots, w(i-1), w(i+1), \ldots, w(n+1)
\]

constitutes a permutation of \( 1, \ldots, n \). Since

\[
\prod_{j<i} \frac{x_w(j) - qx_{n+1}}{x_w(j) - x_{n+1}} \cdot \prod_{i<j \leq n+1} \frac{x_{n+1} - qx_w(j)}{x_{n+1} - x_w(j)} \to q^{n+1-i}
\]

under the substitution \( x_{n+1} = 0 \), we conclude that

\[
R^m_{\alpha+1}(x_1, \ldots, x_n, 0; q) = \sum_{i \in \mathbb{Z}} q^{n+1-i} R^n_{\alpha/i}(x_1, \ldots, x_n; q).
\]

The Lemma now follows from the definition (4) of the symmetric functions \( R_{\alpha} \) and the fact that \( |\alpha| \leq n \).

For any \( \alpha \in \mathbb{N}^r, \beta \in \mathbb{N}^s \) and \( m > r + s \), let

\[
\langle \alpha, \beta \rangle_m = \langle \alpha_1, \ldots, \alpha_r, 0, \ldots, 0, \beta_s, \ldots, \beta_1 \rangle \in \mathbb{N}^m.
\]

We are now sufficiently prepared to give the

**Proof of Theorem 18.3:** Let \( \gamma \in \mathbb{N}^n, \alpha \in \mathbb{N}^r, \beta \in \mathbb{N}^s \) be as defined in the statement of the theorem, and let \( t = n - r - s \). Define \( \alpha^+ = \alpha + 1^r \) and \( \beta^+ = \beta + 1^s \). Notice that

\[
\langle \alpha^+, \beta^+ \rangle_m = \gamma + 1^n.
\]

The zeroes of \( \langle \alpha^+, \beta^+ \rangle_m \) occur in the positions \( i \) for which \( r + 1 \leq i \leq m - s \). For any such \( i \), \( \langle \alpha^+, \beta^+ \rangle_m / i = \langle \alpha^+, \beta^+ \rangle_{m-1} \). Hence, for any \( m > n \), Lemma 18.8 implies

\[
R_{\langle \alpha^+, \beta^+ \rangle_m}(x; q) = (q^s + \cdots + q^{m-r-1}) R_{\langle \alpha^+, \beta^+ \rangle_{m-1}}(x; q).
\]

Successive applications of this recursion yield

\[
R_{\langle \alpha^+, \beta^+ \rangle_m}(x; q) = q^{(m-n)} \frac{(1-q^{m-r-s}) \cdots (1-q^{t+1})}{(1-q)^{m-n}} R_{\langle \alpha^+, \beta^+ \rangle_n}(x; q).
\]
If we substitute $x_i \rightarrow q^{i-1}$ ($1 \leq i \leq m$) and apply Lemma 18.7, we obtain

\[
R_{(\alpha^+, \beta^+)}(1, q; \ldots, q^{m-1}; q) = q^{m((\alpha^+, \beta^+)_m) - s(m-n)} \frac{[m!]_q (1 - q)^{-n}}{(1 - q^{m-r-s}) \cdots (1 - q^{i+1})}
\]

\[
= q^{m(\alpha, \beta) + (r^+)} q^m \beta(1 - q^m) \cdots (1 - q^{m-r-s+1}) \frac{[\ell!]_q}{(1 - q)^n}.
\]

Hence, by Proposition 18.6

\[
R_{(\alpha^+, \beta^+)} \left( p_k \rightarrow \frac{a^k - b^k}{1 - q^k} ; q \right) = q^{m(\alpha, \beta) + (r^+)} a^n (ba^{-1})^{\beta} \frac{[\ell!]_q}{(1 - q)^n} \prod_{i=0}^{r+s-1} (1 - ba^{-1} q^{-i}).
\]

If we substitute $q \rightarrow z$, $a \rightarrow -q$, $b \rightarrow -1$, we find

\[
R_{(\alpha^+, \beta^+)} \left( p_k \rightarrow (-1)^k - \frac{1 - q^k}{1 - z^k} ; z \right) = (-1)^{\ell} q^{[\ell]_z} z^{m(\alpha, \beta)} (z)_n \frac{[\ell!]_z}{(1 - z)^n}.
\]

Apply Theorem 18.1.\(\ast\)
Open Problems

We conclude with a discussion of some unsolved problems suggested by or related to the combinatorial decomposition problems we have studied.

Problem A: Let $\lambda$ be a partition in the $l$th layer with $\ell(\lambda) < n$. We know (Theorem 10.7) that if $V_\lambda$ actually occurs in the decomposition of $\text{Ext}(gl_n)$ into irreducible $SL_n$-modules, then $l \leq n - 1$ and

$$\lambda_1 + \cdots + \lambda_i \leq i(n + l - i) \quad (1 \leq i \leq l).$$

Are these constraints actually sufficient? They are known to be sufficient for $n \leq 5$. A combinatorial proof of this fact, which might proceed by constructing a tableau $T$ of the sort described in Theorem 10.4(b) or (c) for any such $\lambda$, would be desirable.

Similarly, we know (Proposition 13.2) that if $V_\lambda$ actually occurs in the decomposition of $T^k(\text{Ext}(gl_n))$ into irreducible $SL_n$-modules, then $l \leq k(n - 1)$ and

$$\lambda_1 + \cdots + \lambda_i \leq i(l + k(n - i)) \quad (1 \leq ki \leq l).$$

(1)

Are these constraints actually sufficient? This seems to be the case, but there is less evidence than for the exterior algebra. This too can be viewed as a purely combinatorial problem: show that if $\mu$ is a partition of $kn(n - 1)$ with $\ell(\mu) \leq n$ and $\mu \leq 2k \cdot \delta$, then $s_\mu$ actually appears in the decomposition of $s_\delta^{2k}$ into Schur functions. A stronger result would be to show that (1) is sufficient to imply that $M^n_k[\lambda] \neq 0$.

Problem B: Theorem 10.4 gives us a combinatorial interpretation for the multiplicity of $V_\lambda$ in the decomposition of $\text{Ext}(gl_n)$ into irreducibles. If we choose the interpretation in Theorem 10.4(c), we have:

$$E^n[\lambda](1) = 2^n \cdot |T_\lambda|$$

where $T_\lambda$ is a certain collection of standard tableaux of shape $\lambda$. Is there a combinatorially natural statistic which refines this interpretation for each of the exterior powers $\text{Ext}^k(gl_n)$? This would amount to assigning an integer $k = k(A,T)$ for each pair $(A,T)$, where $A \subseteq [n]$ and $T \in T_\lambda$ with the property that

$$E^n[\lambda](q) = \sum_{A,T} q^{k(A,T)}.$$
We mention that such a statistic can easily be given in the first layer case, as pointed out by R. Stanley. Let $\lambda$ be a partition of $n$. In this case, $T_\lambda$ is the set of all standard tableaux of shape $\lambda$, and the first layer formula (Theorem 12.1) tells us that

$$E^n[\lambda](q) = f_\lambda(q^2) \cdot \prod_{z \in \lambda} (q + q^{2c(z)}),$$

where

$$f_\lambda(q) = \frac{q^{c(\lambda)}[n!]}{\prod_{z \in \lambda} (1 - q^{h(z)})}.$$

On the other hand, Stanley has shown that $f_\lambda(q)$ is the generating function for standard tableaux of shape $\lambda$ according to their greater index. If $T$ is a standard tableau, its greater index is the integer $g(T)$ defined by

$$g(T) = \sum i : i \text{ is in a higher row than } i + 1 \text{ in } T.$$

For example, the following standard tableau has greater index 23.

```
  1 3 7 10 11
  2 4 8
  5 6
  9
```

Stanley's result implies that

$$f_\lambda(q) = \sum_T q^{g(T)},$$

where $T$ runs through all standard tableaux of shape $\lambda$. By using this combinatorial description in (2), the reader may easily supply a suitable combinatorial statistic $k = k(A,T)$ for the first layer.

**Problem C:** The polynomials $E^n[\lambda]$, $M^n[\lambda]$, $G^n[\lambda]$ and the formal power series $G^n[\lambda]$ are still not completely understood. Although an explicit formula for them which is reasonably nice is probably out of the question (cf. (12.17)), it is still conceivable that they can be described in more explicit ways than we have given.

In particular, it would also be of interest to find some explanation, either combinatorial or algebraic, for so many of the nice factorizations which $E^n[\lambda]$, $M^n[\lambda]$ and $G^n[\lambda]$ possess. Of course, many of the results we have given, such as the splitting theorems, first layer formulas, and recursions provide some explanation, empirical evidence suggests that there is still more structure waiting to be uncovered.
**Problem D:** Let $\alpha, \beta$ be partitions of the same weight, and define $G_{\alpha\beta}(q)$ to be the generating function for the limiting generalized exponents of $SL_n$; i.e.,

$$G_{\alpha\beta}(q) = \lim_{n \to \infty} G[\alpha, \beta]_n(q).$$

By Stanley’s stability theorem (Theorem 9.2) and (9.12), it follows that

$$G_{\alpha\beta}(q) = s_\alpha \ast s_\beta(q, q^2, q^3, \ldots).$$

By studying properties of the internal product, Stanley has shown [40; Prop. 8.2]

$$G_{\alpha\beta}(q) = \frac{P_{\alpha\beta}(q)}{\prod_{z \in \alpha} (1 - q^{h(z)})},$$

where $P_{\alpha\beta} \in \mathbb{Z}[q]$ is a polynomial for which $P_{\alpha\beta}(1) = f^\beta$, i.e., the sum of its coefficients is the number of standard tableaux of shape $\beta$. These results had been conjectured by Stanley and Gupta. Additionally, they have conjectured [40; Conjecture 8.3], that the coefficients of $P_{\alpha\beta}(q)$ are nonnegative. If this is true, it is natural to ask if one can find a combinatorially meaningful statistic $c(T)$ defined for standard tableaux $T$ of shape $\beta$ such that

$$P_{\alpha\beta}(q) = \sum_T q^{c(T)}.$$

More generally, one may derive from Stanley’s stability theorem that

$$\lim_{n \to \infty} C[\alpha, \beta]_n(z, q) = \left[ \prod_{k \geq 1} \frac{1 - q^k}{1 - z^k} \right] s_\alpha \ast s_\beta \left( p_r \rightarrow \frac{z^r - q^r}{1 - z^r} \right),$$

which brings a whole family of specializations of the internal product under scrutiny. Along these lines Hanlon\textsuperscript{2} and Gupta [14] have independently conjectured formulas for certain specializations of $s_\alpha \ast s_\beta$. Their conjectures would amount to showing that there exist integers $c(T)$ and $d(x, T)$ for each standard tableau $T$ of shape $\beta$ and each cell $x \in \alpha$ such that

$$s_\alpha \ast s_\beta \left( p_r \rightarrow \frac{z^r - q^r}{1 - z^r} \right) = \sum_T q^{c(T)} \prod_{x \in \alpha} \frac{a - bq^{d(x, T)}}{1 - q^{h(x)}}.$$

Notice that this would imply that the coefficients of $P_{\alpha\beta}$ are nonnegative: take $a \to q, b \to 0$. Again, if this conjecture is true, it is natural to ask if the integers $c(T)$ and $d(x, T)$ can be described in a combinatorially meaningful fashion.

\textsuperscript{2}private communication
**Problem E:** Although their idea is combinatorially elegant, Zeilberger and Bressoud's proof [48] of the $q$-Dyson Theorem is very long. Is there a short proof? A short proof of the equal parameter version of the $q$-Dyson Theorem would also be interesting, and possibly easier, since this can be viewed as the problem of computing $M_k^n(\emptyset)(q)$; i.e., a problem concerning the decomposition of a certain symmetric function into Schur functions.

One conceivable approach to this problem would be to use the recursion (Theorem 16.1) to find an explicit formula for the rational function $f$ for which

$$M_k^n(2k \cdot \delta)(q) = f(q)M_k^n(\emptyset)(q).$$

Since a formula for $M_k^n(2k \cdot \delta)(q)$ is easily found (13.11), one could thereby deduce a formula for $M_k^n(\emptyset)$.

**Problem F:** In Theorem 18.1, we showed that evaluating the coefficients

$$[x^n] \prod_{i<j}(x_j x_i^{-1})_k(q x_i x_j^{-1})_k,$$

in which the exponent sequence $\gamma$ satisfies $\gamma_i \geq -1$, hinges on the study of specializations of the symmetric functions $R_\beta$ of the form

$$R_\beta(1, q, \ldots, q^{m-1}; q) \ (\beta \in \mathbb{N}^n, |\beta| = n).$$

Can reasonable, explicit formulas be given for all such $\beta$? Of course, one could mimic the technique suggested by our proof of Theorem 18.3, but when the zeroes of $\beta$ are not consecutive, there will be further complications.

**Problem G:** Lastly, and perhaps most importantly, is the problem of extending the combinatorial decompositions we have presented to other classical groups. If one prefers, this problem may also be regarded as the problem of extending these results to other classical Lie algebras or other irreducible root systems.

It is far from clear if there is a first layer analogue for the exterior or symmetric algebras, or the Macdonald complex, of the classical Lie algebras. In fact, it is not even clear what a first layer dominant weight should be in these cases, if there is such a notion. On the positive side, at least Macdonald's root system conjecture (Conjecture 14.5) predicts that the multiplicity of the trivial character in the Macdonald complex of a semisimple Lie algebra should be nice.

A further impediment to any such investigation from a combinatorial point of view is the fact that there is nothing nearly as well-developed as the theory of symmetric functions for investigating the representations of other classical groups as there is for $SL_n$ and $GL_n$. 

## TABLES

1. The polynomials $E^n[\lambda]$

The polynomials $E^n[\lambda]$ were defined in Section 10. The following is a list of all the nonzero polynomials with $2 \leq n \leq 5$. The fact that $E^n[\lambda] = E^n[\bar{\lambda}]$ has been used to shorten the list.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\bar{\lambda}$</th>
<th>$E^n<a href="q">\lambda</a>$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>3</td>
<td>$(1 + q) \quad (1 + q^3)$</td>
</tr>
<tr>
<td>2</td>
<td>33</td>
<td>$q \quad (1 + q)^2$</td>
</tr>
<tr>
<td>$\emptyset$</td>
<td>3</td>
<td>$(1 + q) \quad (1 + q^3) \quad (1 + q^5)$</td>
</tr>
<tr>
<td>3</td>
<td>33</td>
<td>$q^2(1 + q)^2(1 + q^3)$</td>
</tr>
<tr>
<td>21</td>
<td>21</td>
<td>$q \quad (1 + q)^2(1 + q^3) \quad (1 + q^5)$</td>
</tr>
<tr>
<td>42</td>
<td>21</td>
<td>$q^2(1 + q)^3$</td>
</tr>
<tr>
<td>$\emptyset$</td>
<td>4</td>
<td>$(1 + q) \quad (1 + q^3) \quad (1 + q^5) \quad (1 + q^7)$</td>
</tr>
<tr>
<td>4</td>
<td>444</td>
<td>$q^3(1 + q)^2(1 + q^3) \quad (1 + q^5)$</td>
</tr>
<tr>
<td>31</td>
<td>332</td>
<td>$q^2(1 + q)^2(1 + q^3)^2(1 + q^2 + q^4)$</td>
</tr>
<tr>
<td>22</td>
<td>22</td>
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</tr>
<tr>
<td>211</td>
<td>211</td>
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<td>$q^2(1 + q)^3(1 + q^3)$</td>
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<tr>
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<td>543</td>
<td>$q^4(1 + q)^3(1 + q^3) \quad (1 + q^5)$</td>
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<td>44</td>
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<td>431</td>
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<td>422</td>
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</tr>
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<td>642</td>
<td>642</td>
<td>$q^4(1 + q)^4$</td>
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<td>633</td>
<td>$q^5(1 + q)^5(1 + q^3)$</td>
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<tr>
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<td>$(1 + q) \quad (1 + q^3) \quad (1 + q^5) \quad (1 + q^7) \quad (1 + q^9)$</td>
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<tr>
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<tr>
<td>$\lambda$</td>
<td>$\tilde{\lambda}$</td>
<td>$E^n<a href="q">\lambda</a>$</td>
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<td>--------------------</td>
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<td>6653</td>
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<tr>
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<tr>
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<td>442</td>
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<td>$q^5(1 + q)^3(1 + q^3)^2(1 + q^2)(1 + q^4)$</td>
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</table>
The series $C_n[\lambda](z, q)$

The series $C_n[\lambda](z, q)$ were defined in Section 14. The following is a list of all of the rational functions $C_n[\gamma]/C_n[\emptyset]$ for dominant weights $\gamma$ of the form $[\alpha, \beta]_n$, where $\alpha$ and $\beta$ are partitions of weight at most 3. The fact that $C[\alpha, \beta]_n = C[\beta, \alpha]_n$ has been used to shorten the list. (The series $C_n[\emptyset]$ was evaluated in Corollary 14.4.)

<table>
<thead>
<tr>
<th>$\alpha, \beta$</th>
<th>$\frac{C[\alpha, \beta]_n(z, q)}{C_n[\emptyset](z, q)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 1)$</td>
<td>$\frac{z - q}{1 - qz^{n-1}} \left[ \frac{n - 1}{1} \right]_s$</td>
</tr>
<tr>
<td>$(11, 11)$</td>
<td>$z \frac{(1 - q)(z - q)}{(1 - qz^{n-1})(1 - qz^{n-2})} \left[ \frac{n \cdot (n - 3)}{1 \cdot 2} \right]_s$</td>
</tr>
<tr>
<td>$(2, 11)$</td>
<td>$z \frac{(z - q)(z^2 - q)}{(1 - qz^{n-1})(1 - qz^{n-2})} \left[ \frac{(n - 1) \cdot (n - 2)}{1 \cdot 2} \right]_s$</td>
</tr>
<tr>
<td>$(2, 2)$</td>
<td>$z \frac{(1 - q)(z - q)(1 - q^2z^{n-3})}{(1 - qz^{n-1})(1 - qz^{n-2})(1 - q^2z^{n-1})} \left[ \frac{n \cdot (n - 1)}{1 \cdot 2} \right]_s$</td>
</tr>
<tr>
<td>$(111, 111)$</td>
<td>$z^2 \frac{(1 - q)(z - q)(1 - qz)}{(1 - qz^{n-1})(1 - qz^{n-2})(1 - qz^{n-3})} \left[ \frac{n \cdot (n - 1) \cdot (n - 5)}{1 \cdot 2 \cdot 3} \right]_s$</td>
</tr>
<tr>
<td>$(21, 111)$</td>
<td>$z \frac{(1 - q)(z - q)(z^2 - q)}{(1 - qz^{n-1})(1 - qz^{n-2})(1 - qz^{n-3})} \left[ \frac{n \cdot (n - 2) \cdot (n - 4)}{1 \cdot 1 \cdot 3} \right]_s$</td>
</tr>
<tr>
<td>$(3, 111)$</td>
<td>$z \frac{(z - q)(z^2 - q)(z^3 - q)}{(1 - qz^{n-1})(1 - qz^{n-2})(1 - qz^{n-3})} \left[ \frac{(n - 1) \cdot (n - 2) \cdot (n - 3)}{1 \cdot 2 \cdot 3} \right]_s$</td>
</tr>
</tbody>
</table>
\[ \frac{C[\alpha, \beta|n(z, q)}{C^n[0]}(z, q) \]

21, 21
\[
z \frac{(1-q)(z-q)(1-qz)(1-q^2z^{n-4})}{(1-qz^{n-1})(1-qz^{n-2})(1-qz^{n-3})(1-q^2z^{n-1})} \left[ \frac{2 \cdot n \cdot (n-1) \cdot (n-2)}{1 \cdot 1 \cdot 1 \cdot 3} \right]_z
\]

\[ - \frac{(z-q^2)(z-q)}{(1-qz^{n-1})(1-q^2z^{n-1})} \left[ \frac{(n-1) \cdot (n-1)}{1 \cdot 1} \right]_z \]

3, 21
\[
z \frac{(1-q)(z-q)(z^2-q)(1-q^2z^{n-4})}{(1-qz^{n-1})(1-qz^{n-2})(1-qz^{n-3})(1-q^2z^{n-1})} \left[ \frac{n \cdot (n-1) \cdot (n-2)}{1 \cdot 1 \cdot 3} \right]_z
\]

3, 3
\[
z^2 \frac{(1-q)(z-q)(1-q^3z^{n-3})}{(1-qz^{n-1})(1-qz^{n-2})(1-q^3z^{n-1})} \left[ \frac{n \cdot (n-1) \cdot (n-2)}{1 \cdot 2 \cdot 3} \right]_z \times
\]
\[
\left\{ \frac{(1+z)(z^2-q)(1-q^2z^{n-4})}{(1-qz^{n-3})(1-q^2z^{n-1})} + \frac{(1-z^3)(1-q^2z^{n-3})}{(1-z^{n-2})(1-q^2z^{n-1})} - \frac{(z^2-q)(1-q^3z^{n-5})}{(1-qz^{n-3})(1-q^3z^{n-3})} \right\}
\]
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NOTATION

Chapter I

\( \lambda, \lambda_i \) 
\( \ell(\lambda) \) 
\( |\lambda|, |\alpha| \) 
\( \emptyset \) 
\( n_i(\lambda) \) 
\( D_\lambda \) 
\( \lambda' \) 
\( c(x), h(x) \) 
\( n(\lambda), n(\alpha) \) 
\( (\alpha|\beta) \) 
\( \lambda \cup \mu \) 
\( \lambda + \mu \) 
\( \mu \subseteq \lambda \) 
\( |\lambda/\mu| \) 
\( (\lambda/\mu)' \) 
\( \lambda \geq \mu \) 
\( x^\alpha \) 
\( \Lambda, \Lambda(x), \Lambda_\lambda \) 
\( \Lambda^k, \Lambda^k(x) \) 
\( \Lambda_n, \Lambda^k_n \) 
\( m_\lambda \) 
\( e_\alpha, h_\alpha, p_\alpha \) 
\( e_\lambda, h_\lambda, p_\lambda \) 
\( w \) 
\( e_\lambda \) 
\( \omega \) 
\( w(T) \) 
\( f^{\lambda/\mu} \) 
\( K_{\lambda/\mu, \alpha} \) 
\( (\lambda/\mu)^\nu \) 
\( \Omega_n \) 
\( \langle, \rangle \) 
\( [x^\alpha]f(x) \) 
\( \varepsilon_w \) 
\( a_\alpha(x_1, \ldots, x_n) \) 
\( \delta \) 
\( f \) 
\( \Omega_n^j \) 
\( [\alpha, \beta]_n \) 

Chapter II

\( \text{word}(T) \) 
\( h_\alpha, e_\alpha \) 
\( \lambda \) 
\( GL(V), GL(n, \mathbb{C}), GL_n \) 
\( SL(V), SL(n, \mathbb{C}), SL_n \) 
\( \rho \) 
\( g \circ v \) 
\( \rho_1 \oplus \rho_2 \) 
\( \chi \) 
\( CG \) 
\( \text{End}(V) \) 
\( V \otimes W \) 
\( \rho^f \) 
\( V_\lambda, V^\lambda_\lambda \) 
\( \rho_\lambda, \rho^f_\lambda \) 
\( gl_n, gl(n, \mathbb{C}) \) 
\( sl_n, sl(n, \mathbb{C}) \) 
\( \varepsilon_{ij} \) 
\( \text{Ext}(V), \text{Ext}^k(V) \) 
\( v \wedge w \) 
\( \text{Sym}(V), \text{Sym}^k(V) \) 
\( T(V), T^k(V) \) 
\( S^\lambda \) 
\( \chi^\lambda \) 
\( Cl(S_k) \) 
\( \lambda(\omega) \) 
\( \text{ch}(f) \) 
\( V \times W \) 
\( \text{Sym}_k(gl_n) \) 
\( c_\lambda(q), c_\lambda(y_1, y_2, \ldots) \) 
\( c_{\alpha\beta}(q), c_{\alpha\beta}(y_1, y_2, \ldots) \) 
\( f * g \) 
\( f(p_\alpha \rightarrow a_\alpha) \) 
\( G^n[\lambda](q) \) 

Chapter III

\( E^n[\lambda](q), E^n[\gamma](q) \) 
\( M^k[\lambda](q), M^k[\gamma](q) \)
### Notation

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