THE LINEAR MATROID PARITY PROBLEM

by

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(1979)

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS OF THE
DEGREE OF

DOCTOR OF PHILOSOPHY
IN ELECTRICAL ENGINEERING AND COMPUTER SCIENCE

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
February, 1985

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Submitted to the Department of Electrical Engineering and Computer Science in October, 1984 in partial fulfillment of the requirements for the Degree of Doctor of Philosophy in Electrical Engineering and Computer Science.

ABSTRACT

Consider an \( m \times 2n \) matrix \( A \) with columns \( a_1, a_2, \ldots, a_{2n} \) where for each \( i \in [1..2n] \) column \( a_{2i-1} \) is 'paired' to column \( a_{2i} \). We refer to \( A \) as a matching matrix and we call a linearly independent set of paired columns a matching. The linear matroid parity problem is to find a matching of maximum cardinality.

This problem, henceforth referred to as the 'parity problem', is of theoretical interest as a common generalization of diverse problems in Combinatorial Optimization. These include the matching problem in graphs, the matroid intersection problem in representable matroids, and Giles' (1982) maximum cardinality matching forest problem.

Lawler (1971) originally posed the parity problem in general matroids. However, Lovasz (1979) and, independently, Jensen and Korte (1981) showed that any algorithm for the general matroid parity problem requires an exponentially large number of steps in the worst case. Lovasz (1979) developed both a duality theory and a polynomial time
algorithm for the linear matroid parity problem.

In this thesis we extend Lovasz's duality theory and develop a more efficient and conceptually simpler polynomial time algorithm for the parity problem. Whereas other algorithms proved (Lovasz (1979)) or improved (Gabow and Stallman (1984)) a polynomial bound for the parity problem, our algorithm generalizes and unifies known algorithms for a broad class of problems in Combinatorial Optimization.

Thesis Supervisor: Dr. James B. Orlin
ACKNOWLEDGEMENTS

It is impossible to adequately express either the enormity of my debt or the depth of my gratitude to my committee chairman, Professor James B. Crlin. He has contributed substantially to both the formulation and presentation of the concepts in this thesis. Simply put, this dissertation would not have been possible without him.

I take pleasure in thanking Professor J. H. Shapiro and Bill Northup for their patience and support throughout. Further thanks are due to Professors Tom Magnanti, Favi Kanan and Gary Miller for serving on my committee. I would especially like to thank Elie Gugenheim and Jan Hammond for their contributions.

It is impossible to acknowledge all who have contributed. I hope they will accept my humble thanks.

Lastly, Surya Jagannathan has been a continuing source of encouragement and support.
CHAPTER 1:

INTRODUCTION

Consider an \( m \times 2n \) matrix \( A \) with columns \( a_1, a_2, \ldots, a_{2n} \) where for each \( i \in [1..2n] \) column \( a_{2i-1} \) is 'paired' to column \( a_{2i} \). We refer to \( A \) as a matching matrix and we call a linearly independent set of paired columns a matching. The linear matroid parity problem is to find a matching of maximum cardinality.

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Our algorithm generalizes Edmonds’ blossom algorithm in the following way. The blossom algorithm reduces the matching problem in a non-bipartite graph to a matching problem in a related bipartite graph by shrinking certain odd cycles. Our algorithm reduces the parity problem to a sequence of related matroid intersection problems by ‘shrinking’ certain subspaces with odd rank. This is a natural generalization in light of the following relationships between the problems.

The graphic matching problem is the special case of the parity problem in which each column of the matching matrix is a unit vector. The linear matroid intersection problem is the special case of the parity problem in which the rows of the matching matrix can be partitioned into two subsets such that one column of each pair has all of its non-zero entries in one subset and the other column has all its non-zero entries in the other subset. Finally, the bipartite matching problem is the special case of the parity problem which is both a graphic matching problem and a matroid intersection problem.

In the remainder of this chapter we review some fundamental definitions of matroid theory. More specific background is given in the introduction to each of the next three chapters. In Chapter 2, we introduce and discuss the matroidal constructs used in our algorithm for the parity problem. In Chapter 3, we present our algorithm and develop a duality theory which generalizes both Lovasz’s duality theory for the parity problem and Edmonds’ duality theory for the graphic matching problem. Finally, in Chapter 4, we present Lawler’s ‘primal’ algorithm for the weighted matroid intersection problem and give an elementary ‘dual’ proof of the algorithm’s correctness. Our motivation for discussing the weighted matroid intersection problem arises from our interest in the weighted version of the parity problem. In
particular, we are ultimately interested in developing a polynomial time algorithm for the problem of determining a maximum weight matching where the weight of a matching \( M \) is the sum of the weights assigned to the paired columns in \( M \). As our algorithm for the cardinality parity problem relies heavily on the cardinality matroid intersection procedure, we conjecture that an algorithm for the weighted parity problem might rely on the weighted matroid intersection procedure. Moreover, we feel that in order to develop a duality theory for the weighted parity problem, it will be necessary to understand thoroughly weighted matroid intersection duality. Chapter 3 provides insight into weighted matroid intersection duality.

1.1 An introduction to matroid theory

A matroid \( M \) is a pair \((E, I)\), where \( E \) is a finite set of elements and \( I \) is the collection of independent subsets of \( E \). Whitney (1935) first introduced and investigated matroids as a generalization of the properties of linear independence. In particular, \( M = (E, I) \) is a matroid if

(i) \( \emptyset \in I \),

(ii) for each \( I \in I \), every proper subset of \( I \) is in \( I \), and

(iii) for each pair \( I \) and \( I' \) in \( I \) with \( |I| > |I'| \), there is an element \( e \in I - I' \) such that \( I' + e \in I \).

**EXAMPLE 1.** Let \( A \) be a matrix with columns \( a_1, a_2, \ldots, a_m \). Then \( M = (E, I) \) is a matroid where \( E = [1..n] \) and a subset \( I \subseteq E \) is in \( I \) if the set \( \{a_i : i \in I\} \) is linearly independent. The matrix \( A \) is called a representation of the matroid \( M \). Any matroid which admits a representation in this way is called representable. \( \Box \)

**EXAMPLE 2.** Let \( E = \{E_i : i \in S\} \) be a partition of the finite set \( E \) into disjoint subsets. Then \( M = (E, I) \) is a matroid where a subset \( I \subseteq E \) is in \( I \) if \( |I \cap E_i| \leq 1 \) for each \( i \in S \). The matroid \( M \) is called the partition matroid induced by \( E \). \( \Box \)
EXAMPLE 3. Let $A$ be a matrix with columns $a_1, a_2, \ldots, a_n$. Then $M = (E, I)$ is a matroid where $E = \{1..n\}$ and a subset $I \subseteq E$ is in $I$ if the set $\{a_i : i \in I\}$ contains a basis of $A$. □

The rank function $r$ of a matroid $M = (E, I)$ is defined as follows. For a subset $X \subseteq E$, $r(X) = \max\{\|i\| : I \subseteq X$ and $I \in I\}$, i.e., the rank of $X$ is the maximum cardinality of an independent subset of $X$.

The span of a subset $X \subseteq E$ in a matroid $M = (E, I)$ is the largest set $Y \subseteq E$ such that $X \subseteq Y$ and $r(X) = r(Y)$. A base of a subset $X \subseteq E$ in a matroid $M = (E, I)$ is a maximal independent subset of $X$. A set $X \subseteq E$ is dependent if $X \notin I$. A minimal dependent set in $M$ is called a circuit of $M$.

For further discussion of matroids, see Welsh (1976), Lawler (1976) and Tutte (1971). In addition, we discuss specific properties in the introduction to each of the next three chapters.
CHAPTER 2:

THE LINEAR MATROID PARITY PROBLEM:

PART I
Introduction

Consider an $m \times 2n$ matrix $A$ with columns $a_1, a_2, \ldots, a_{2n}$, where column $a_{2i-1}$ is 'paired' to column $a_{2i}$ for $i \in [1..n]$. We call a subset of $k$ pairs a matching if the corresponding set of $2k$ columns is linearly independent. The linear matroid parity problem is to find a matching of maximum cardinality.

This problem generalizes the matching problem on graphs, the matroid intersection problem in representable matroids and Giles' (1982) maximum cardinality matching forest problem.

Lovasz (1979) developed a duality theory and a polynomial time algorithm for the linear matroid parity problem. Although Lovasz's algorithm is good in the sense of Edmonds, it is not (nor was it intended to be) efficient in practice.

In this, the first of two papers in which we develop a more efficient and conceptually simpler polynomial time algorithm for the linear matroid parity problem, we introduce and discuss the matroid constructions used in our algorithm. We present the algorithm for the linear matroid parity problem in the second paper. For further discussion on the background of the matroid parity problem, see the second paper.

In sections 2 and 3, we review the relevant properties of matroids and matroid intersections. In Section 4, we discuss "extra-transversal matroids", a generalization of transversal matroids, and we introduce "$P\subseteq$ matroids", a related construction based on matroid partitions. In Section 5, we introduce "shrinking matroids", a third and more general construction. We show that the problem of
determining independence in a shrinking matroid requires an exponentially large number of steps if one is restricted to the use of oracles for testing independence. We prove, however, that independence in representable shrinking matroids is equivalent to independence in related \( P^3 \) matroids and hence can be determined using Edmonds' (1969) matroid intersection algorithm.

In our algorithm, we reduce the linear matroid parity problem to a sequence of related matroid intersection problems. In each of these matroid intersection problems, the first matroid is an extra-transversal matroid and the second is a shrinking matroid. In Section 6 of this paper, we describe an efficient procedure for solving the matroid intersection problem in these special matroids.

Although we review some of the basic properties of matroids and matroid intersections in sections 2 and 3, we assume that the reader is familiar with all of these properties as described, for example, in Lawler (1976).

2. Preliminaries

In this section we review some of the basic properties of matroids. A matroid \( M \) is a pair \((E, I)\), where \( E \) is a finite set of elements and \( I \) is the collection of independent subsets of \( E \).

In general, for a matroid \( M = (E, I) \), a base is a maximal independent subset; a circuit is a minimal dependent set; if \( I \in I \) and \( I + x \notin I \), then \( C(I, x) \) is the unique circuit in \( I + x \). The rank of a subset \( X \subseteq E \) with respect to \( M \), denoted \( r(X) \), is defined as \( r(X) = \max\{|I|: I \subseteq X \text{ and } I \in I\} \). The span of a subset \( X \subseteq E \) with respect to \( M \), denoted \( \text{span}(X) \), is the largest set \( Y \) with \( X \subseteq Y \subseteq E \) and \( r(Y) = r(X) \). The dual of the matroid \( M \), denoted \( M^* = (E, I^*) \), is the matroid in which
a subset \( I \subseteq E \) is a base of \( M^* \) if and only if \( E - I \) is a base of \( M \). The rank of a subset \( X \subseteq E \) with respect to \( M^* \), denoted \( r^*(X) \), is \( |X| - (r(E) - r(E-X)) \). For a subset \( K \subseteq E \), we define the matroid \( M \) contract \( K \), denoted \( M/K = (E/K, I/K) \), as follows: a subset \( I \subseteq E/K = E - \text{span}(K) \) is in \( I/K \) if and only if \( I \in I \) and \( r(IUK) = r(I) + r(K) \). The rank of a subset \( X \subseteq E/K \) with respect to \( M/K \), denoted \( r(X/K) \), is \( r(XUK) - r(K) \). For further details on these definitions, see Welsh (1976).

Let \( A \) be an \( m \times n \) matrix with columns \( a_1, a_2, \ldots, a_n \), and let \( E = [1..n] \). A subset \( I \subseteq E \) is independent in the matroid induced by \( A \), denoted \( M(A) \), if the set \( \{a_i : i \in I\} \) is linearly independent. Any matroid that may be associated with a matrix in this way is said to be representable and the matrix \( A \) is called a representation of the matroid \( M(A) \).

We refer to a number of matroids \( M_1 = (E_1, I_1) \), \( M_2 = (E_2, I_2) \), etc., each distinguished by its subscript. Thus, for example, the span and rank of a subset \( X \subseteq E_1 \) with respect to the matroid \( M_1 = (E_1, I_1) \) are \( \text{span}_1(X) \) and \( r_1(X) \), respectively.
3. Matroid Intersections

A subset $I \subseteq E$ is an intersection in two matroids $M_1 = (E, I_1)$ and $M_2 = (E, I_2)$ if $I \subseteq I_1 \cap I_2$. If in addition $|I| = k$, then $I$ is a $k$-intersection.

A pair $(E_1, E_2)$ of disjoint subsets of $E$ is a cover of $E$ if $E_1 \cup E_2 = E$. The rank of the cover $(E_1, E_2)$ with respect to the matroids $M_1$ and $M_2$ is $r_1(E_1) + r_2(E_2)$. Edmonds (1971) proved the following strong duality theorem for matroid intersections.

**Theorem 1.** (Edmonds) The maximum cardinality of an intersection in the matroids $M_1 = (E, I_1)$ and $M_2 = (E, I_2)$ is the minimum rank of a cover of $E$ with respect to $M_1$ and $M_2$.  \\

Let $I$ be an intersection in $M_1 = (E, I_1)$ and $M_2 = (E, I_2)$. As in Lawler (1976), we define the auxiliary digraph $G(I)$ as follows. The vertex set $V$ of $G(I)$ consists of the elements of $E$ together with the distinguished vertices $v_1$ and $v_2$. The edge set of $G(I)$ consists of the following edges:

(i) the edge $(v_1, x)$ for each $x \in E - \text{span}_1(I)$,

(ii) the edge $(x, v_2)$ for each $x \in E - \text{span}_2(I)$,

(iii) the edge $(x, y)$ for each $x \in E - I$ and each $y \in C_2(I, x) - x$, and

(iv) the edge $(y, x)$ for each $y \in I$ and each $x \in E - I$ such that $y \in C_1(I, x)$.

A path $P = \{v_1, x_1, y_1, ..., x_n, y_n\}$ in $G(I)$ is short cut-free if for each $1 \leq i < j \leq n$, $y_j \not\in C_2(I, x_i)$ and $y_i \not\in C_1(I, x_{j+1})$.

**Lemma 1.** (Edmonds) Let $I$ be an intersection in $M_1$ and $M_2$. If $P = \{v_1, x_1, y_1, ..., x_n, y_n\}$ is a short cut-free path in
If \( B(I) \) from \( v_1 \) to \( y_k \epsilon I \), then \( I' = I - \{ y_1, y_2, ..., y_k \} U \{ x_1, x_2, ..., x_k \} \) is an intersection in \( M_1 \) and \( M_2 \) such that \( y_k \epsilon \text{span}_1(I') \). Moreover, for each \( x \epsilon E - I \) such that \( P' = \{ v_1, x_1, y_1, ..., x_k, y_k, x \} \) is a short cut-free path in \( B(I) \), \( x \epsilon \text{span}_1(I') \). □

Edmonds (1971) developed an efficient algorithm for determining the maximum cardinality of an intersection based on Lemma 1. Briefly, given an intersection \( I \), Edmonds' algorithm employs a breadth-first labeling procedure to determine any short cut-free path in \( B(I) \) from \( v_1 \) to \( v_2 \). If such a path exists, then "augmenting" along it yields a \( (|I|+1) \)-intersection \( I' \). If, on the other hand, no such path exists, the labels applied by the procedure lead to a minimum rank cover as follows. Let \( E_2 \) be the set of elements \( e \epsilon E \) for which there is a path in \( B(I) \) from \( v_1 \) to \( e \). We refer to \( E_2 \) as the "labeled" elements and to \( E_1 = E - E_2 \) as the "unlabeled" elements. See Lawler (1976) for a proof that \( (E_1, E_2) \) is a cover of \( E \) whose rank equals the cardinality of \( I \).

An intersection \( I' \) is reachable from an intersection \( I \) in \( M_1 \) and \( M_2 \) if \( I' \) can be obtained by "augmenting" \( I \) along a short cut-free path in \( B(I) \) from \( v_1 \) to some element \( y \epsilon I \). The following lemma is a consequence of Lemma 1.

**Lemma 2.** Let \( I \) be a maximum cardinality intersection in \( M_1 \) and \( M_2 \) and let \( (E_1, E_2) \) be the minimum rank cover determined by Edmonds' intersection procedure. Then \( E_1 = \cap(\text{span}_1(I')) = I' \) reachable from \( I \).

**Proof.** It follows from Lemma 1 that \( \cap(\text{span}_1(I')) = I' \) is reachable from \( I \) \( \subseteq E_1 \). By the definition of \( E_1 \), we have that \( E_1 \subseteq \cap(\text{span}_1(I')) = I' \) is reachable from \( I \). □

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4. Extra-transversal and $P^3$ Matroids

Let $M_1 = (S, I_1)$ be the partition matroid induced by the partition $S = \{S_i: i \in E\}$ of $S$ as follows: a subset $I \subseteq S$ is in $I_1$ if $|S_i \cap I| \leq 1$ for each $i \in E$. For each subset $X \subseteq S$ let $IND(X; S) = \{i: S_i \cap X \neq \emptyset\}$. For any matroid $M_2 = (S, I_2)$ defined on $S$, we define $ET(S, M_2)$, the extra-transversal matroid induced by $S$ and $M_2$, as follows. A subset $I \subseteq E$ is independent in $ET(S, M_2)$ if there is an intersection $X \in I_1 \cap I_2$ such that $IND(X; S) = I$.

In addition, we define $P^3(S, M_2)$ as follows: a subset $I \subseteq E$ is independent in $P^3(S, M_2)$ if there is a subset $X \subseteq S$ such that $X \in I_1$, $S - X \in I_2$ and $IND(X; S) = E - I$.

The independent sets in $P^3(S, M_2)$ are derived from partitions of $S$ into sets $X$ in $I_1$ and $S - X$ in $I_2$, as projected onto the partition indices. Hence, $P^3(S, M_2)$ may be described as the projection of the matroid partitions in $M_1$ and $M_2$ onto the partition matroid $M_1$.

**EXAMPLE 1.** Let $A$ be a matrix with columns $a_1, a_2, \ldots, a_t$ and for each subset $I \subseteq [1..t]$, let $A(I)$ denote the set $\{a_i: i \in I\}$. Finally, let $S = \{S_i: i \in E\}$ be a partition of $[1..t]$. A subset $I \subseteq E$ is independent in $ET(S, M(A))$ if there is a linearly independent subset $A(J)$ of columns with $|J \cap S_i| = 1$ for $i \in I$.

A subset $I \subseteq E$ is independent in $P^3(S, M(A))$ if there is a basis $A(J)$ of $A$ with
(i) $S_i \subseteq J$ for $i \in I$ and
(ii) $|S_i \cap J| \geq |S_i| - 1$ for $i \in E - I$. □

**EXAMPLE 2.** Let $M$ be the matroid induced by a directed graph $G = (V, E)$, i.e., a subset of edges is independent if it contains no (undirected) cycles. Let $S = \{S_i: i \in V\}$, where
for each vertex \( v \in V \), \( S_i = \{ e \in E : e \text{ is directed into vertex } i \} \). A directed tree \( T \) is a tree with a root node \( v \) such that the path from \( v \) to each vertex is a directed path in \( T \) from \( v \). A directed forest is a forest of directed trees. A subset \( I \subseteq V \) is independent in \( ET(S, M) \) if there is a directed forest \( F \) of \( G \) such that no root vertex is in \( I \). A subset \( I \subseteq V \) is independent in \( F^*(S, M) \) if there is a directed forest \( F \) in \( G \) such that

(i) for each \( i \in I \), \( S_i \subseteq F \), and

(ii) for each \( i \in E \setminus I \), \( |F \cap S_i| = |S_i| - 1 \).  

**Example 3.** Let \( S = \{S_i : i \in \{1, \ldots, k\} \} \) be a partition of the vertices of a graph \( G = (V, E) \) and let \( M \) be the matching matroid of \( G \), i.e., a subset \( X \subseteq V \) is independent in \( M \) if there is a matching incident to each vertex of \( X \). A subset \( I \subseteq \{1, \ldots, k\} \) is independent in \( F^*(S, M) \) if there is a matching \( M \subseteq E \) that meets each vertex of \( S_i \) if \( i \in I \) and meets all but one vertex of \( S_i \) if \( i \notin I \).  

In Lemma 3 and Lemma 4, we prove that \( ET(S, M) \) and, under certain conditions, \( F^*(S, M) \) are matroids.

**Lemma 3.** (Nash-Williams) Let \( M = (E, I) \) be a matroid and let \( f : S \rightarrow E \) be a surjection of \( S \) onto \( E \). Then \( f(M) = (E, f(I)) \) is a matroid where \( f(I) = \{ f(I) : I \in I \} \).  

**Lemma 4.** Let \( M_1 = (S, I_1) \) be the partition matroid induced by the partition \( S = \{S_i : i \in E\} \) of \( S \) and let \( M_2 = (S, I_2) \) be any matroid defined on \( S \). Then:

(i) \( ET(S, M_2) \) is a matroid.

(ii) \( F^*(S, M_2) \) is a matroid if and only if

\[
\forall \{U(S_i : i \in I) \geq \sum_{i \in I} |S_i| - 1 : i \in I \}
\]

for each \( I \subseteq E \).

(iii) If \( F^*(S, M_2) \) is a matroid, then it is the dual of \( ET(S, (M_2)^*) \).
PROOF. First, define the surjection \( f : S \to E \) so that for each \( x \in S \), \( f(x) = \text{IND}(x; S) \). Then \( ET(S, M_2) = f(M_2) \) and, by Lemma 3, \( ET(S, M_2) \) is a matroid.

Next, we prove that each \( \text{P}^3 \) matroid is the dual of an extra-transversal matroid. A subset \( I \subseteq E \) is a base of \( \text{P}^3(S, M_2) \) if and only if there is a subset \( X \subseteq S \) such that \( \text{IND}(X; S) = E - I \), \( x \in I \), and \( S - X \) is a base of \( M_2 \). Equivalently, \( I \) is a base of \( \text{P}^3(S, M_2) \) if and only if there is a \( (|S| - r_2(S)) \)-intersection \( X \) in \( M_1 \) and \( (M_2)^* \) with \( \text{IND}(X) = E - I \). In particular, if there is a \( (|S| - r_2(S)) \)-intersection in \( M_1 \) and \( (M_2)^* \), then \( \text{P}^3(S, M_2) \) is the dual of \( ET(S, (M_2)^*) \) and hence a matroid.

On the other hand, if the maximum cardinality of an intersection in \( M_1 \) and \( (M_2)^* \) is less than \( |S| - r_2(S) \), then there is a cover \( (S^1, S^2) \) of \( S \) such that \( r_1(S^1) + r_2(S^2) < |S| - r_2(S) \). Let \( I = \text{IND}(S^1; S) \). Since

\[
r_1(S^1) = r_1(U(S_1: i \in I)) = |I|,
\]

and

\[
r_2^*(S^2) \geq r_2^*(U(S_1: i \in E - I))
\]

\[
= |U(S_1: i \in E - I)| - (r_2(S) - r_2(U(S_1: i \in I))),(\)

we have that

\[
|I| + \sum(|S_1|: i \in E - I) + r_2(U(S_1: i \in I)) < \sum(|S_1|: i \in E),
\]

and hence that

\[
(1) \quad r_2(U(S_1: i \in I)) < \sum(|S_1|-1: i \in I).
\]

Thus, either there is a subset \( I \subseteq E \) for which \( (1) \) is true,
or \( P^+(S, M_2) \) is the dual of \( ET(S, (M_2)^* \) and hence a matroid. Moreover, it is easy to see that if (1) is true for some \( I \subseteq E \), then \( P^+(S, M_2) \) has no independent sets and is not a matroid; proving (ii) and (iii). \( \square \)

The following lemma characterizes the independent sets of extra-transversal and \( P^+ \) matroids.

**Lemma 5.** Let \( S = \{S_i : i \in E \} \) be a partition of the set \( S \) and let \( M_2 = (S, I_2) \) be any matroid defined on \( S \). Then

(i) A subset \( I \subseteq E \) is independent in \( ET(S, M_2) \) if and only if

\[
\tau_2(U(S_i : i \in I')) \geq |I'| \quad \text{for each subset} \quad I' \subseteq I.
\]

(ii) A subset \( I \subseteq E \) is independent in \( P^+(S, M_2) \) if and only if

\[
\tau_2(U(S_i : i \in IUI')) \geq \sum \{|S_i| : i \in IUI'\} - |I'|
\]

for each \( I' \subseteq E - I \).

**Proof.** Let \( M_1 = (S, I_1) \) be the partition matroid induced by \( S \). A subset \( I \subseteq E \) is independent in \( ET(S, M_2) \) if and only if there is a \(|I|\)-intersection in \( M_1 \) and \( M_2 \), i.e., \( I \subseteq E \) is independent in \( ET(S, M_2) \) if and only if \( \tau_1(S^1) + \tau_2(S^2) \geq |I| \) for each partition \( (S^1, S^2) \) of \( U(S_i : i \in I) \). Since \( M_1 \) is a partition matroid, \( I \) is independent in \( ET(S, M_2) \) if and only if \(|I - I'| + \tau_2(U(S_i : i \in I')) \geq |I'| \) for each \( I' \subseteq I \); from which (i) follows.

Similarly, a subset \( I \subseteq E \) is independent in \( P^+(S, M_2) \) if and only if \( I \) is a subset of a base \( B \) of \( P^+(S, M_2) \). Now, \( B \subseteq E \) is a base of \( P^+(S, M_2) \) if and only if there is a \(|B| - \tau_2(S)|\)-intersection \( X \subseteq U(S_i : i \in B) \) in \( M_1 \) and \((M_2)^* \). We conclude then that \( I \) is independent in \( P^+(S, M_2) \) if and only if
\[ r_2 \left( U(S_i : i \in \mathcal{I} \cap \mathcal{I}') \right) \geq \sum_{i \in \mathcal{I}'} \left| S_i \right| - \left| \mathcal{I}' \right| \text{ for each } \mathcal{I}' \subseteq \mathcal{E} - \mathcal{I}. \quad \square \]

Observe that extra-transversal and \( P^3 \) matroids do not constitute special classes of matroids. In fact, every matroid may be described as an extra-transversal or \( P^3 \) matroid in the following way. Let \( M = (E, I) \) be any matroid and let \( S \) be the partition of \( E \) into subsets each containing one element. Then \( M = ET(S, M) \) and \( M = P^3(S, M) \).

Rather than viewing extra-transversal and \( P^3 \) matroids as descriptions of the matroids, we view them as descriptions of the procedures used to construct the matroids. Both extra-transversal and \( P^3 \) matroids play a key role in the solution of the linear matroid parity problem.
5. Shrinking Matroids

Let $M = (S, I)$ be the partition matroid induced by the partition $S = \{S_i : i \in E\}$ of $S$ and let $M = (S, I)$ be any matroid defined on $S$. Let $D = \{D_i : i \in E\}$, where for each $i \in E$, $D_i = \{S_{i,j} : j \in [0..m_i]\}$ is a collection of subsets of $S_i$ satisfying the following properties:

1. $m_i > 0$,
2. $S_{i,j} \in I$ for each $j \in [0..m_i]$,
3. $r(S_{i,0}) = r(S_i)$ and
4. $r(S_{i,j}) = r(S_i)-1$ for each $j \in [1..m_i]$.

We refer to each subset $S_{i,j}$ for $j \in [1..m_i]$ as a deficient independent subset of $S_i$ (and we say that $S_{i,j}$ is deficient in $S_i$) since each such $S_{i,j}$ can be extended -- by the addition of a single element -- to a base of $S_i$. Thus, with the exception of the base $S_{i,0}$, the subsets in the collection $D_i$ are all deficient in $S_i$.

Let $D = \{D_i : i \in E\}$ be a collection satisfying (2) - (5). We define the shrinking matroid induced by $M$ and $D$, denoted $SM(M, D)$, as follows. A subset $I \subseteq E$ is independent in $SM(M, D)$ if there is a subcollection $F = \{S_{i,j} : i \in E\}$ such that

1. $U(S_{i,j}) : i \in E \in I$ and
2. $f^{-1}(0) = I$.

Thus, $F$ is composed of a base of $S_i$ for each $i \in I$ and a deficient independent subset of $S_i$ for each $i \notin I$.

We say that $D$ satisfies the shrinking property on $M$ if $D$ satisfies conditions (2) - (5) and if, in addition, the empty set is independent in $SM(M, D)$, i.e., if it is possible to select exactly one deficient independent subset $S_{i,j}$ from each $D_i$ so that $U(S_{i,j}) : i \in E$ is independent in $M$.

**Lemma 6.** If $D = \{D_i : i \in E\}$ satisfies the shrinking property on the matroid $M$, then $SM(M, D)$ is a matroid.
PROOF. First, observe that \( \emptyset \) is independent in \( SM(M, D) \) by assumption. Since every proper subset of a set \( I \) independent in \( SM(M, D) \) is likewise independent in \( SM(M, D) \), it remains to show that for any pair \( I, I' \) of independent sets in \( SM(M, D) \) with \( |I| > |I'| \), there is an element \( i \in I - I' \) such that \( I' + i \) is independent in \( SM(M, D) \).

Suppose that \( I \) and \( I' \) are independent in \( SM(M, D) \) and that \( |I| > |I'| \). Let \( F = \{ S_i \in E : i \in E \} \) and \( G = \{ S_i \in E : i \in E \} \) be subcollections of \( D \) satisfying (6) and (7) with \( f^{-1}(0) = I \) and \( g^{-1}(0) = I' \). Moreover, assume that among all such subcollections, \( |F| \geq |G| \) is maximum. Let \( X = U(S_i \in E : i \in E) \) and let \( X' = U(S_i \in E : i \in E) \).

Since \( r(X) > r(X') \), there is an element \( x \in X - \text{span}(X') \). Let \( i \in E \) be the index for which \( x \in S_i \). Since \( x \notin \text{span}(X') \), we have that \( X' + x \in I \) and \( i \notin I' \). In particular, \( X' = X' - S_i \cup S_i \) is independent in \( M \) and hence \( I' + i \) is independent in \( SM(M, D) \).

It remains to show that \( i \in I \). If \( i \notin I \), then \( X'' = X' - S_i \cup S_i \) is independent in \( M \) and hence \( H = \{ S_j \in E : j \in E \} \), where

\[
\begin{align*}
g(j) & \quad \text{for } j \neq i, \\
h(j) & = f(i) \quad \text{for } j = i,
\end{align*}
\]

is a subcollection satisfying (6) and (7) with \( h^{-1}(0) = I' \) and \( |H| \geq |G| \); contradicting our choice of \( F \) and \( G \). Thus, \( i \in I - I' \), completing the proof. \( \Box \)

Our motivation for referring to the matroid \( SM(M, D) \) as a shrinking matroid stems from the following example. (See Edmonds (1965) for all relevant definitions).

**EXAMPLE 4.** Let \( G \) be a graph and let \( S = \{ S_i : i \in E \} \) be a
partition of the vertex set \( V \) into subsets such that for each \( i \in E \) the subgraph \( G[S_i] \) induced by \( S_i \) is hypomatchable, i.e., for each vertex \( v \in S_i \), the subgraph \( G[S_i - v] \) admits a perfect matching.

Let the collection \( D \) be determined so that for each \( i \in E \), \( D_i = \{ S_{i,v} : v \in S_i \} \cup \{ S_{i,0} \} \), where \( S_{i,0} = S_i \) and for each \( v \in S_i \), \( S_{i,v} \) is a perfect matching of \( G[S_i - v] \). Finally, let \( M = (V, I) \) be the matching matroid of \( G \). In order to determine whether a subset \( I \subseteq E \) is independent in \( SM(M, D) \), it suffices to shrink each subset \( S_i \) of vertices to a single pseudo-vertex as per Edmonds’ blossom algorithm and then search for a matching incident to the \( |I| \) pseudo-vertices corresponding to the set \( I \). \( \square \)

In the remainder of this section we address the problem of determining whether a subset \( I \) is independent in \( SM(M, D) \) and, in particular, whether the collection \( D \) satisfies the shrinking property on the matroid \( M \). As the following example illustrates, if there is only an oracle function for evaluating the rank in matroid \( M \), any algorithm for this problem requires an exponential number of function evaluations in the worst case. Our example is a variant of the examples of Lovász (1979) and of Jensen and Korte (1981) which showed that the general matroid parity problem requires an exponential number of function evaluations in the worst case.

**Example 5.** Let \( F = \{ f_1, \ldots, f_{4n} \} \), let \( S = \{ g_1, \ldots, g_{4n} \} \) and let \( D = \{ D_i: i \in \{ 1, \ldots, 2n \} \} \), where for each \( i \in \{ 1, \ldots, 2n \} \), \( S_i = \{ f_{2i-1}, f_{2i}, g_{2i-1}, g_{2i} \} \) and \( D_i = \{ (f_{2i-1}, f_{2i}, g_{2i-1}), (f_{2i-1}, f_{2i}), (g_{2i-1}, g_{2i}) \} \). Let \( T \) be the set of \( 2^n \) subsets formed by taking one deficient from each of the \( 2n \) sets \( D_i \) and let \( T' \) be a subset of \( T \). Now, define \( M = (F \cup S, I) \) as follows. A subset \( X \subseteq F \cup S \) is independent in \( M \) if

(i) \( |X \cap S_i| \leq 3 \) for each \( i \in \{ 1, \ldots, 2n \} \),
(ii) $|X| \leq 4n$ and
(iii) $x \notin T'$.

We claim the following:
(a) $M$ is a matroid.
(b) The collection $D$ satisfies conditions (2) - (5) with respect to $M$.
(c) The empty set is independent in $SM(M, D)$ if and only if $T' \neq T$.
(d) If we are allowed only queries of the form: "What is $r(X)$?", then to determine if $T' = T$ requires as many as $2^{2n}$ queries.

Combining (a) - (d), we conclude that determining independence in $SM(M, D)$ requires an exponential number of function evaluations. (In our example, we wish to determine whether $\emptyset$ is independent in $SM(M, D)$. It is easy to modify our example to show that determining whether an arbitrary given subset $I \subseteq [1..2n]$ is independent in $SM(M, D)$ requires an exponential number of function evaluations.)

We first verify that $M$ is a matroid. Suppose $X, X' \in I$ and that $|X'| > |X|$. We wish to find an element $y \in X' - X$ such that $X + y \in I$. Select index $j \in [1..2n]$ such that $|X' \cap S_j| > |X \cap S_j|$ and choose $y \in (X' - X) \cap S_j$. If $X + y \in I$, then we are through. Otherwise, since $X + y$ satisfies (i) and (ii), $X + y \in T'$. In this case choose any other element $y' \in X'$ (such an element exists, since $X' \neq X + y$). Since $|X \setminus S_j| = 2$ for $i \neq j$ and $|X \setminus S_j| = 1$, we have that $|(X + y') \setminus S_j| \leq 3$ for $i \in [1..2n]$. Moreover, we claim that $X + y' \in T'$. If $y' \in S_i$ with $i \neq j$, then $|(X + y') \setminus S_j| = 3$, proving that $X + y' \in T'$. On the other hand, if $y' \notin S_j$, then either $y' \in F$ and $y \in G$ or $y \in F$ and $y \in G$. In either case, we cannot have both $X + y'$ and $X + y$ in $T'$.

It is easy to verify that $D$ satisfies properties (2) - (5) with respect to $M$ and that $\emptyset$ is independent in $SM(M, D)$.
if and only if $T' \neq T$. Finally, we demonstrate that (d) is true.

Suppose some algorithm determines that $T' = T$ without having queried the oracle for $r(U(S_{1,i}: S_{1,i} \in H))$ for some member $H \in T$. Replacing $T'$ with $T' - \{H\}$, we see that for every other element $F \in T$, $r(U(S_{1,i}: S_{1,i} \in F))$ remains unchanged. Thus, the algorithm must likewise conclude that $T - \{H\} = T$. This conclusion is obviously incorrect. Any algorithm must ask for $r(U(S_{1,i}: S_{1,i} \in F))$ for each member $F \in T$. □

Example 5 illustrates that, in general, determining independence in $SM(M, D)$ requires an exponential number of function evaluations. Nevertheless, as stated in Lemma 7 and Theorem 2, there are at least two significant classes of instances $(M, D)$ for which independence in $SM(M, D)$ can be determined in polynomial time.

**Remark 1.** Let $S = \{S_i: i \in E\}$ be a partition of $S$ such that $S_i$ is independent in $M = (S, I)$ for each $i \in E$. If $D = \{D_i: i \in E\}$ satisfies the shrinking property on $M$ and if for each $i \in E$ and each $e \in S_i$, $S_i - e \notin D_i$, then $SM(M, D) = \mathbb{F}^E(S, M)$. □

Remark 1 is a direct consequence of our definitions of shrinking and $\mathbb{F}^E$ matroids. Observe, moreover, that the condition that $S_i - e \notin D_i$ for each $i \in E$ and each $e \in S_i$ is not as restrictive as it might appear. Suppose, for example, that $S_i - e \notin D_i$ for some element $e \in S_i$. Then, for any subcollection $F = \{S_{1,i}: i \in E\}$ of $D$, $e \notin U(S_{1,i}: i \in E)$. With this in mind, let $K_i = \cup(S_{1,i}: S_{1,i} \in D_i)$ for each $i \in E$ and let $K = U(K_i: i \in E)$. Note that if $D$ satisfies the shrinking property on $M$, then $K$ is independent in $M$.

Let $S' = \{S_i: i \in E\}$, where for each $i \in E$, $S_i = S_i - K$. The following lemma is a consequence of Remark 1 together with the observation that for any subcollection $F =$
\[ \langle S_i, \ast : i \in E \rangle \text{ of } D, \quad K \subseteq U(S_i, \ast : i \in E). \]

**Lemma 7.** Let \( S = \langle S_i : i \in E \rangle \) be a partition of \( S \) such that \( S_i \) is independent in \( M = (S, I) \) for each \( i \in E \). If \( D \) satisfies (2) - (5) with respect to \( M \), then \( SM(M, D) = P^S(M/K, S') \) where \( S' \) and \( K \) are defined as above. \[ \Box \]

The second class of instances \( (M, D) \) for which we can readily determine independence in \( SM(M, D) \), is the class in which \( M \) is representable. In this case, we refine our definitions of \( K \) and \( S' \) as follows:

1. \( K = \text{span}(U(K_i : i \in E)), \) where for each \( i \in E, \quad K_i = \cap \text{span}(S_{i, \ast : i \in D_i}), \) and

2. \( B' = \langle S_i' : i \in E \rangle, \) where for each \( i \in E, \quad S_i' \) is a base of \( S_i \) in \( M/K \).

**Theorem 2.** Let \( S = \langle S_i : i \in E \rangle \) be a partition of the elements of the representable matroid \( M = (S, I) \) and suppose that \( D = \langle D_i : i \in E \rangle \) is a collection satisfying conditions (2) - (5) with respect to \( M \). Then \( SM(M, D) = P^S(S', M/K), \) where \( K \) and \( S' \) are defined as in (8) and (9).

**Proof.** We first show that if \( I \subseteq E \) is a base of \( P^S(S', M/K), \) then \( I \) is a base of \( SM(M, D). \) Assume, without loss of generality, that the indices \( E - I \) are labeled 1, 2, ... , \( t. \)

We construct a subcollection \( F = \langle S_i, \ast : i \in E \rangle \) of \( D \) satisfying (6) and (7) as follows. Since \( I \) is a base of \( P^S(S', M/K), \) there is a subset \( X = \{ x_i : i \in [1..t] \} \) of \( S' = U(S_i : i \in E) \) such that

- \( x_i \in S_i \) for each \( i \in [1..t] \)
- \( B = S' - X \) is a base of \( M/K. \)

We determine the deficient \( S_i, \ast : i \) of \( F \) as follows. Since \( B \) is a base of \( M/K, \) \( B + x_i \) is dependent in \( M/K. \) Thus, there is an element \( y_i \in S_i \) in the span of \( B - S_i \) with respect to \( M/K. \) Moreover, since \( y_i \notin K, \) there is a deficient \( S_i, \ast : i \in D_i, \) such that \( y_i \notin \text{span}(S_i, \ast : i \). Finally, \( B' = B - S_i \).
(S_i \cup (1) - K) is a base of M/K.

Proceeding iteratively, we obtain a subcollection F = {S_i \cup (1): i \in E} of D such that
(10) f^{-1}(0) = I and
(11) \cup (S_i \cup (1): i \in E) is a base of M;
thereby proving that I is a base of SM(M, D).

We next show that if I \subseteq E is a base of SM(M, D), then
I is a base of P^3(S', M/K). We determine a subset X = \{x_i: i \in E - I\} of S' such that
(i) for each i \in E - I, x_i \in S_i and
(ii) S' - X is a base of M/K,
as follows. Since I is a base of SM(M, D), there is a
subcollection F = {S_i \cup (1): i \in E} of D satisfying (10) and
(11). Let X = \{x_i: i \in E - I\} where for each i \in E - I, x_i is an
element of S_i not contained in the span of S_i \cup (1) - K with
respect to M/K. Since |S' - X| = |\cup (S_i \cup (1): i \in E) - K| =
r(\cup (S_i \cup (1): i \in E)/K) = r(S/K), we see that I is a base of
P^3(S', M/K). \Box

Note that Theorem 2 implies that if M is representable,
then the matroid SM(M, D) does not depend on the specific
sets of deficient independent subsets in D, but instead
depends only on the sets K_i defined in (8).
6. **Intersections in Extra-transversal and Shrinking Matroids**

In our algorithm for the linear matroid parity problem, we reduce the problem of finding a maximum cardinality matching to a sequence of related matroid intersection problems. In each of these intersection problems, the first matroid is an extra-transversal matroid and the second matroid is a shrinking matroid.

In this section we introduce an efficient procedure for finding a maximum cardinality intersection in $\text{ET}(Q, M_1)$ and $\text{SM}(M_2, D)$ when $M_1$ and $M_2$ are representable.

To avoid computing circuits in the derived matroids $\text{ET}(Q, M_1)$ and $\text{SM}(M_2, D)$, we transform the problem into an equivalent matroid intersection problem in the two matroids $M_3$ and $M_4$.

Let $Q = \{Q_i: i \in E\}$ be a partition of the elements of the representable matroid $M_1 = (Q, I_1)$ and let $D = \{D_i: i \in E\}$ be a collection satisfying (2) - (5) with respect to the partition $S = \{S_i: i \in E\}$ of the elements of the representable matroid $M_2 = (S, I_2)$. Finally, let $K$ and $S'$ be defined as in (8) and (9). We define the matroid $M_3 = (T, I_3)$ on $T = QUS'$, where $S' = U(S_i: i \in E)$, as follows. A subset $XUY \subseteq T$, where $X \subseteq Q$ and $Y \subseteq S'$ is independent in $M_3$ if $X$ is independent in $M_1$ and $Y$ is independent in the dual of $M_2/K$. Thus, $M_3$ is the matroid sum of $M_1$ and the dual of $M_2/K$. We define the matroid $M_4 = (T, I_4)$ as follows. A subset $XUY \subseteq T$, where $X \subseteq Q$ and $Y \subseteq S'$, is independent in $M_4$ if $|XQ_i| + |YN_S'i| \leq 1$ for each $i \in E$, i.e., $M_4$ is the partition matroid induced by the partition $\{Q_iUS_i: i \in E\}$ of $T$.

**Lemma 8.** A subset $I \subseteq E$ is an intersection in $\text{ET}(Q, M_1)$ and $\text{SM}(M_2, D)$ if and only if there is an intersection $XUY$ in $M_3$ and $M_4$, with $X \subseteq Q$ and $Y \subseteq S'$, satisfying
(12) \( \text{IND}(X; Q) = I \) and
(13) \( \|Y\| = \|S'\| - r_2(S/K) \).

**Proof.** Suppose \( I \subseteq E \) is an intersection in \( \text{ET}(Q, M_1) \) and \( \text{SM}(M_2, D) \). By Theorem 2, \( \text{SM}(M_2, D) = P^3(S', M_2/K) \). Hence, \( I \) is an intersection in \( \text{ET}(Q, M_1) \) and \( P^3(S', M_2/K) \). In particular, since \( I \) is independent in \( \text{ET}(Q, M_1) \), there is a subset \( X \subseteq Q \) such that
(14) \( \|X \cap Q\| \leq 1 \) for each \( i \in E \),
(15) \( X \) is independent in \( M_1 \), and
(16) \( \text{IND}(X; Q) = I \).

Likewise, since \( I \) is independent in \( P^3(S', M_2/K) \), there is a subset \( Y \subseteq S' \) such that
(17) \( \|Y \cap S'\| \leq 1 \) for each \( i \in E \),
(18) \( S' - Y \) is a base of \( M_2/K \), and
(19) \( \text{IND}(Y; S') \subseteq E - I \).

Combining (15) and (18), we see that \( X \cup Y \subseteq I_3 \). Combining (16) and (19), we see that \( X \cup Y \subseteq I_4 \). Finally, combining (16) and (18), we see that \( X \) and \( Y \) satisfy (12) and (13).

Suppose, on the other hand, that \( X \cup Y \subseteq T \), where \( X \subseteq Q \) and \( Y \subseteq S' \), satisfy (12) and (13). Since \( Y \) is independent in the dual of \( M_2/K \) and \( \|Y\| = \|S'\| - r_2(S/K) \), we conclude that \( S' - Y \) is a base of \( M_2/K \). Moreover, since \( \|Y \cap S'\| \leq 1 \) for each \( i \in E \), we conclude that \( E - \text{IND}(Y; S') \) is independent in \( P^3(S', M_2/K) \). Hence, by Theorem 5, \( I \subseteq E - \text{IND}(Y; S') \) is independent in \( \text{SM}(M_2, D) \). Finally, since \( X \subseteq I_1 \) and \( \|X \cap Q\| \leq 1 \) for each \( i \in E \), \( I = \text{IND}(X; Q) \) is independent in \( \text{ET}(Q, M_1) \); proving that \( I \) is an intersection in \( \text{ET}(Q, M_1) \) and \( \text{SM}(M_2, D) \). \( \square \)

Lemma 8 suggests the following procedure for determining a maximum cardinality intersection in \( \text{ET}(Q, M_1) \) and \( \text{SM}(M_2, D) \).

**Procedure 1.**
Step 0. (Start) Compute a maximum cardinality intersection $Y$ in the dual of $M_2/K$ and the matroid induced by the partition $S'$. If $|Y| < |S'| - r_2(S/K)$, then stop; $SM(M_2, D)$ contains no independent sets. Otherwise, set $X = \emptyset$ and proceed to Step 1.

Step 1. (Labeling and Augmentation)

(1.0) Construct the auxiliary digraph $G(XUY)$ with respect to $M_3$ and $M_4$.
(1.1) Determine any short cut-free path in $G(XUY)$ from $v_1$ to $v_2$. If no such path exists, stop; $I = IND(X; S)$ is a maximum cardinality intersection. Otherwise, augment the intersection $XUY$ and repeat Step 1.

In light of Lemma 8, the only argument needed to establish the correctness of this procedure is the following.

**Lemma 9.** Let $X Y$ be an intersection in $M_3$ and $M_4$ such that $X \subseteq Q$, $Y \subseteq S'$ and $|Y| = |S'|-r_2(S/K)$. For any augmentation $X'Y'$ of $XUY$, $|Y'| = |Y|$.

**Proof.** Since $M_3$ is the direct sum of $M_1$ and the dual of $M_2/K$,

(i) $C_3(XUY, e) \cap Y = \emptyset$ for $e \in Q$ and

(ii) $C_3(X, Y, e) \subseteq Y+e$ for $e \in S'$.

In particular, for any path $P$ from $v_1$ to $v_2$ in $G(XUY)$, if $y \in Y$ is a vertex on $P$, then the vertex immediately following $y$ on $P$ is an element of $S'$. Moreover, if $e \in S' - Y$ is a vertex on $P$, then the vertex immediately preceding $e$ on $P$ is either $v_1$ or an element of $Y$. Finally, since $S' - Y$ is a base of $M_2/K$, there is no edge $(v_1, e)$ in $G(XUY)$ for any element $e \in S'$. We conclude then that $|Y'| = |Y|$.

Finally, we interpret the minimum rank cover $(T_3, T_4)$ of $T$ constructed by Edmonds' matroid intersection procedure
in terms of the original matroids ET(Q, M₁) and SM(M₂, D).

**Lemma 10.** Let (T₃, T₄) be the minimum rank cover of T constructed by Edmonds' matroid intersection procedure where we are adding with respect to M₃. Then (E₁, E₂) is a minimum rank cover of E with respect to ET(Q, M₁) and SM(M₂, D), where E₁ = {i ∈ E: Qⱼ \(\subseteq\) T₃}. Moreover,

(i) \(r_{E}(E₁) = r₁(U(Q₁: i ∈ E₁)),\) where \(r_{E}\) is the rank function of ET(Q, M₁),

(ii) \(r_{M}(E₂) = r₂(U(S₁: i ∈ E₂)) - \sum(r₂(S₁) - 1; i ∈ E₂),\) where \(r_{M}\) is the rank function of SM(M₂, D), and

(iii) for each element \(q \in \text{span}_E(U(Q₁: i ∈ E₁)),\) there is a maximum cardinality intersection XUY in M₃ and M₄ such that

(a) \(X \subseteq Q, Y \subseteq S',\)

(b) \(|Y| = |S'| - r₂(S/K)\) and

(c) \(q \in \text{span}_E(X).\)

**Proof.** Let XUY be the maximum cardinality intersection in M₃ and M₄ determined by Procedure 1 and let \(T₄ = \{e ∈ T: \text{there is a path in } G(XUY) \text{ from } v₁ \text{ to } e\}\) and \(T₃ = T - T₄.\) In particular, (T₃, T₄) is the minimum rank cover of T constructed by Edmonds' matroid intersection procedure.

First, let \(X₁ = XNT₃\) and let \(X₂ = XNT₄.\) We show that \(r_{E}(E₁) = r₁(X₁) = r₁(U(Q₁: i ∈ E₁))\) as follows. Consider an element \(q ∈ U(Q₁: i ∈ E₁).\) Since there is no path in \(G(XUY)\) from \(v₁\) to \(q,\) but there is a path from \(v₁\) to each \(x ∈ X₂,\) we conclude that there cannot be an edge \((x, q)\) in \(G(XUY)\) for any \(x ∈ X₂.\) In particular, \(C₁(X, q) \cap X₂ = \emptyset.\) Thus, \(q ∈ \text{span}_E(X₁).\) Since this is true for each \(q ∈ U(Q₁: i ∈ E₁),\) we conclude that \(r₁(U(Q₁: i ∈ E₁)) = r₁(X₁) = |X₁|.\) Finally, by the definition of \(E₁, \text{IND}(X₁; Q) \subseteq E₁\) and hence

\(|X₁| = r₁(X₁) \leq r_{E}(E₁) \leq r₁(U(Q₁: i ∈ E₁)) = r₁(X₁).\)

Now, let \(Y₂ = YNT₄.\) We show that

\(\)
\[ r_{se}(E_2) = r_2(U(S_1; i \in E_2)/K) - \Sigma(r_2(S_1/K)-1; i \in E_2) = |X_2| \]

as follows. First, since XUY is a maximum cardinality intersection in M_3 and M_4, we have that \( E_2 \subseteq \text{IND}(X;Q) \cup \text{IND}(Y;Q') \). Moreover, by the definition of \( E_1 \), \( E_2 = \text{IND}(X_2;Q) \cup \text{IND}(Y_2;Q') \).

Next, we show that \( \text{IND}(X_2;Q) \) is a base of \( E_2 \) in \( P^3(S', M_3/K) \). Since, \( Y \) is independent in \( M_3 \), and \(|Y| = |S'| = r_2(S/K)\), we see that \( S' \setminus Y \) is independent in \( M_3/K \). Thus, \( U(S_1; i \in E_2) \setminus Y_2 \) is independent in \( M_3/K \). Now, consider an element \( y \in Y_2 \). Since there is an edge \((y, s)\) in \( B(XUY) \) for each \( s \) in the circuit of \( S' \setminus Y + y \) with respect to \( M_3/K \), we conclude that \( y \) is in the span of \( U(S_1; i \in E_2) \setminus Y_2 \) with respect to \( M_3/K \). Since this is true for each \( y \in Y_2 \), we conclude that \( U(S_1; i \in E_2) \setminus Y_2 \) is a base of \( U(S_1; i \in E_2) \) in \( M_3/K \). In particular, since

\[ r_2(U(S_1; i \in E_2)/K) = |U(S_1; i \in E_2) \setminus Y_2| \\
= |U(S_1; i \in E_2)| + |E_2| - |X_2|, \]

we conclude that

\[ |X_2| = r_{se}(E_2) = r_2(U(S_1; i \in E_2)/K) - \Sigma(r_2(S_1/K)-1; i \in E_2). \]

Finally, we see that if \( q \not\in \text{span}_1(U(Q, i \in E_1)) \), then there is a maximum cardinality intersection XUY in M_3 and M_4 satisfying (a), (b) and (c) as follows. Since \( q \not\in \text{span}_1(U(Q_1; i \in E_1)) \) = \( \text{span}_1(X_1) \), \( C_1(X, q) \cap X_2 = \emptyset \) and hence some element \( x \in X_2 \) precedes \( q \) on the short cut-free path in \( B(XUY) \) from \( v_1 \) to \( q \). By Lemma 1, augmenting along this short cut-free path from \( v_1 \) to \( x \) gives the desired intersection. \( \Box \)

In the second part of our discussion leading to an efficient algorithm for the linear matroid parity problem,
we reduce the parity problem to a sequence of related matroid intersection problems. In each of these intersection problems, the first matroid is an extra-transversal matroid and the second matroid is a shrinking matroid.
CHAPTER 3:

THE LINEAR MATROID PARITY PROBLEM:

PART II
1. Introduction

Consider an \( m \times 2n \) matrix \( A \) with columns \( a_1, a_2, \ldots, a_{2n} \), where column \( a_{2i-1} \) is "paired" to column \( a_{2i} \) for \( i \in \{1..n\} \). We refer to \( A \) as a matching matrix and we call a subset of \( k \) pairs a matching if the set corresponding set of \( 2k \) columns is linearly independent. The linear matroid parity problem, henceforth referred to as the 'parity problem', is to find a matching of maximum cardinality.

In this, the second of two papers on the subject, we develop a polynomial time algorithm for the parity problem. Just as the parity problem generalizes a number of diverse problems in Combinatorial Optimization -- including the matching problem on graphs, the matroid intersection problem in representable matroids and Giles' (1982) maximum cardinality matching forest problem -- our algorithm generalizes standard algorithms for these special cases. In particular, our algorithm generalizes both the blossom algorithm and Edmonds' algorithm for the matroid intersection problem. Moreover, our approach to the parity problem provides new insight into linear matroid parity duality. Thus, we not only provide a common, efficient algorithm for a broad class of problems, we also develop a general theory for these diverse problems.

The graphic matching problem is to find a maximum cardinality subset of edges in a graph \( G \) no two of which are incident to a common vertex. To formulate the matching problem in the graph \( G \) as a parity problem, construct the matching matrix \( A \) of \( G \) as follows. With each edge \((i,j)\) in \( G \) associate the pair of columns in \( A \) consisting of the \( i^{th} \) and \( j^{th} \) unit vectors. It is easy to see that a set of paired columns in the matching matrix of \( G \) is linearly independent if and only if no single unit vector appears more than once, i.e., if and only if the corresponding set of edges is a
matching in $G$. For further discussion of the graphic matching problem, see Edmonds (1965), Even and Kariv (1975), Lawler (1976), Papadimitriou and Steiglitz (1982), and Cornuejols and Hartvigsen (1983).

Let $B = (b_{ik})$ and $C = (c_{ik})$ be two $m \times n$ matrices. A subset of indices $I \subseteq [1..n]$ is an intersection if both the corresponding columns of $B$ and the corresponding columns of $C$ are linearly independent. The linear matroid intersection problem is to find an intersection of maximum cardinality. For further discussion on the matroid intersection problem, see Edmonds (1969), Lawler (1976) and Frank (1981).

To formulate the linear matroid intersection problem as a parity problem construct the matching matrix $A$ with a pair of columns for each index $i \in [1..n]$. With the index $i$ associate the pair of columns whose first member consists of the $i^{th}$ column of $B$ extended to a $2m$-vector with trailing zeros and whose second member consists of the $i^{th}$ column of $C$ extended to a $2m$-vector with leading zeros.

For further applications of the parity problem, see Lawler (1976), Lovasz (1980) and Stallman (1982).

Lawler (1971) first introduced the matroid parity problem in general matroids. However, Lovasz (1978) and, independently, Jensen and Korte (1982) showed that if one is restricted to the use of oracles, then in the worst case, the general matroid parity problem requires an exponentially large number of steps.

Lovasz (1980) developed both a duality theory and a polynomial time algorithm for the (unweighted) parity problem. This theory generalizes Edmonds' theories for the unweighted non-bipartite matching problem (1965) and the unweighted (linear) matroid intersection problem (1969). As of this time there is no duality theory for the weighted
parity problem.

Although Lovasz's algorithm is 'good' in the sense of Edmonds, it is not (nor was it intended to be) efficient in practice. Here, we develop a more efficient and conceptually simpler polynomial time algorithm for the parity problem. In particular, our algorithm (first described in Orlin, Hammond, Gugenheim and Vande Vate (1983)) solves the parity problem in \(O(m^2n)\) operations. Independently, Gabow and Stallman (1984) have developed an even more efficient algorithm requiring only \(O(m^3n)\) operations.

In the following section, we introduce the notation used in this paper. In Section 3, we review Lovasz's duality theory for the parity problem and develop the generalization which motivates our approach to the problem. This more general duality theory reduces the parity problem to a sequence of matroid intersection problems described in Section 4. In Section 5, we show how to determine a matching from an intersection in these matroids. Finally, in Section 6, we describe in detail both our algorithm and the computational effort it requires.
2. Preliminaries

Throughout this paper we employ the terminology introduced in Part 1. In this section, we review that terminology and introduce new notation for the parity problem.

A matroid \( M \) is a pair \( (E, I) \), where \( E \) is a finite set of elements and \( I \) is the collection of independent subsets of \( E \).

We reserve \( M \) to denote either a generic representable matroid or the specific matroid induced by the matching matrix \( A \). Moreover, we reserve \( V \) to denote either a generic vector space or the specific vector space spanned by the columns of the matching matrix.

Let the matrix \( A \) with columns from the vector space \( V \) be a representation of the matroid \( M = (E, I) \) and, for each subset \( X \subseteq E \), let \( A(X) \) be the corresponding set of columns of the matrix \( A \). We let \( r(X) \) denote the linear rank of a subset \( X \) of vectors in \( V \) and, hence, the matroidal rank of a subset \( X \) of elements in \( E \); i.e., for any subset \( X \) of elements in \( E \), \( r(X) = r(A(X)) \). Similarly, we let \( \text{span}(X) \) denote the linear span of a subset \( X \) of vectors in \( V \) and the matroidal span of a subset \( X \) of elements in \( E \). Finally to avoid cumbersome notation, for any subset \( X \) of elements in \( E \), we define \( \text{Lspan}(X) \) to be \( \text{span}(A(X)) \) and, for any subspace \( K \) of \( V \), we define \( \text{Lspan}(XK) \) to be \( \text{span}(A(X)K) \). Note that \( \text{Lspan}(X) \) as defined here depends on the representation.

We sometimes refer to a number of matroids \( M_1 = (E_1, I_1) \), \( M_2 = (E_2, I_2) \), etc.. In general, for a matroid \( M_1 = (E_1, I_1) \) and a representation \( A_1 \) with columns from the vector space \( V_1 \) we let \( \text{span}_1(X) \) and \( r_1(X) \) denote the linear span and linear rank, respectively, of a subset \( X \) of vectors.
in \( V_4 \) and the matroidal span and matroidal rank, respectively, of a subset \( X \) of elements in \( E_4 \). Finally, for a set \( X \) of elements in \( E_4 \), we define \( \text{Lspan}_4(X) \) to be \( \text{span}_4(A_4(X)) \).

For a subset \( X \) of elements in a matroid \( M = (E, I) \), we define \( M \) contract \( X \), denoted \( M/X = (E/X, I/X) \) as follows: a subset \( I \subseteq E/X = E - \text{span}(X) \) is in \( I/X \) if \( I \) is in \( I \) and \( r(IUX) = r(I) + r(X) \). The rank of a subset \( I \subseteq E/X \) with respect to \( M/X \), denoted \( r(I/X) \), is \( r(IUX) - r(X) \).

Let the \( m \times n \) matrix \( A \) with columns from the vector space \( V \) be a representation of the matroid \( M = (E, I) \). For any subspace \( K \) of \( V \), we define \( M \) contract \( K \), denoted \( M/K \), in an analogous way. First, embed \( M \) in a larger matroid \( M' = (EUF, I') \) as follows. Let the \( m \times r(K) \) matrix \( C \) with columns indexed by \( F \) be a basis of the subspace \( K \) and let \( M' \) be the matroid induced by the \( m \times (n+r(K)) \) matrix \( A' = [A, C] \). We define \( M/K \) to be the matroid \( M'/F \). Moreover, we may obtain a representation of \( M/K \) as follows. Extend the matrix \( C \) to a basis \( B \) of the matrix \( A' \). Premultiplying \( A' \) by the basis inverse, we obtain:

\[
\begin{bmatrix}
[B^{-1}A]_1 & 0 \\
B^{-1}A' & 1 - - - - - - - - - - \\
[B^{-1}A]_2 & I \
\end{bmatrix}
\]

where \( I \) is the \( r(K) \times r(K) \) identity matrix. The non-zero columns of the \( m \times n \) matrix

\[
\begin{bmatrix}
[B^{-1}A]_1 \\
1 - - - - - - - - - - \\
0 
\end{bmatrix}
\]

form a representation of \( M/K \).

For example, suppose \( M = (E, I) \) is the matroid induced
by the matching matrix A for the graphic matching problem defined on the graph G. Suppose V is the vector space spanned by the columns of A. Then a subspace K usually corresponds to a subset of the vertices V. In this case, for a subset X of elements in E, $r(X)$ is the number of distinct unit vectors in $A(X)$ and $r(X/K)$ is the number of distinct unit vectors in $A(X)$ which are not in the subspace K.

For a subset X of elements in the matroid $M = (E, I)$, we define $M$ shrink X, denoted $M_{X} = (E, I_{X})$, as follows. A subset $I \subseteq E$ is in $I_{X}$ if I is in I and $r(\text{span}(I) \cap X) \leq 1$. The above definition of independence induces the following equivalent definition of rank. The rank of a subset $I \subseteq E$ with respect to $M_{X}$, denoted $r(I_{X})$, is:

$$r(I_{X}) = \min\{r(I), r(I/X) + 1\}.$$ 

Equivalently, $r(I_{X}) = r(I/X) + 1$ unless $\text{span}(I) \cap X = \emptyset$. In this latter case, $r(I_{X}) = r(I) = r(I/X)$. We leave it as an exercise for the reader to show that $M/X$ is indeed a matroid.

Let the $m \times n$ matrix $A$ with columns from the vector space V be a representation of the matroid $M = (E, I)$. For a subspace $K$ of $V$, we define $M$ shrink K, denoted $M_{K} = (E, I_{K})$, in an analogous way to $M/K$. In particular, a subset $X \subseteq E$ is in $I_{K}$ if $X \subseteq I$ and $r(\text{span}(X) \cap K) \leq 1$.

For example, if $M$ is the matroid induced by the matching matrix $A$ of Table 1, whose associated graph is given in Figure 1, and $K$ corresponds to the subspace spanned by columns 2, 3 and 4, then $M_{K}$ is the matroid induced by the matching matrix of Table 2 whose associated graph is given in Figure 2.

For further discussion of matroids, see Whitney (1935), Tutte (1971) or Welsh (1976).
TABLE 1

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

THE MATCHING MATRIX A

FIGURE 1

THE GRAPH 6
TABLE 2

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

THE MATCHING MATRIX AFTER SHRINKING

FIGURE 2

THE GRAPH G AFTER SHRINKING
This is the most complete text of the thesis available. The following page(s) were not included in the copy of the thesis deposited in the Institute Archives by the author:

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matching, we say that \( L \) is a 2-circuit. In a graphic matching problem, every 2-circuit corresponds to two edges incident to a common vertex. (Lovasz called near-matchings and 2-circuits 'flowers' and 'circuits').

Suppose that \( L = \{(1,2), (3,4), \ldots, (2k-1,2k)\} \) is a near-matching. It is easy to see that \( L \) contains a unique 2-circuit determined as follows. Since \( A(L) \) does not have full rank, there is a non-zero vector \( x = (x_i) \) such that

\[
\sum (A(i)x_i : i \in [1..2k]) = 0.
\]

The unique 2-circuit \( C \) in \( L \) consists of those lines \( l_i = (2i-1, 2i) \) for which either \( x_{2i-1} \) or \( x_{2i} \) is non-zero. Moreover, it is easy to see that for each \( l_i = (2i-1, 2i) \) in \( C \), the point

\[
p = A(2i-1)x_{2i-1} + A(2i)x_{2i},
\]

denoted by \( C(l_i) \), is the unique point in \( L\text{span}(l_i) \cap L\text{span}(C-l_i) \). In a graph, the point \( C(l_i) \) corresponds to the vertex at which the two edges of the 2-circuit are adjacent.
3. **Linear Matroid Parity Duality**

As is the case with matroid intersection and non-bipartite matching, weak and strong duality also play an important role in solving the parity problem. Here, we review Lovasz's duality theory and develop the generalization which motivates our approach to the parity problem.

We assume that the reader is familiar with Edmonds' blossom algorithm for non-bipartite matchings. In fact, throughout our discussion we rely on examples from matching theory to illustrate the more general concepts of the parity problem. These examples also serve to demonstrate the close connection between the two theories.

Let \( L \) be a subset of the lines of the matroid \( M \). We say that a pair \( (K, P) \), where \( K \) is a subspace of \( V \) and \( P = \{ L_i : i \in E \} \) is a partition of \( L \), is a **cover** of \( L \) and we define the **capacity** of \( (K, P) \), denoted \( c(K, P) \), as follows:

\[
c(K, P) = r(K) + \sum (r(L_i / K) / 2) \text{ if } L_i \in P.
\]

Here, for a real number \( x \), \( \lfloor x \rfloor \) denotes the greatest integer less than or equal to \( x \). Observe that the cover \( (K, P) \) depends on the representation since \( K \) is a subspace of \( V \).

We refer to the members \( L_0, L_1, \ldots, L_\ell \) of the partition \( P \) in the cover \( (K, P) \) as **components**. (In general, we index the components from 0 to \( \ell \)). If \( r(L_i / K) \) is odd, \( L_i \) is an **odd** component. Otherwise, \( L_i \) is an **even** component. If \( r(L_i / K) = \lvert L_i \rvert = 1 \), then \( L_i \) is a **trivial** component.

**Lemma 1.** (Weak Duality) Let \( L \) be a subset of the lines of \( M \). For any matching \( M \subseteq L \) and any cover \( (K, P) \) of \( L \), the cardinality of \( M \) is at most the capacity of \( (K, P) \), i.e., \( \lvert M \rvert \leq c(K, P) \).
\[ \leq c(K, P). \]

**Proof.** Let \( M = \{(1, 2), \ldots, (2k-1, 2k)\} \) be a matching in \( L \) and let \((K, P)\) be a cover of \( L \). For any point \( p \in K \), if \( p \in \text{Lspan}(M) \), there is a non-zero vector \( x = (x_i) \) such that

\[ p = \sum(A(i)x_i; i \in [1..2k]). \]

Deleting a pair \((2i-1, 2i)\) from \( M \) for which either \( x_{2i-1} \) or \( x_{2i} \) is non-zero, we obtain a matching \( M' \) such that \( r(M'/p) = r(M') \). (If \( p \in \text{Lspan}(M) \), we may delete any line from \( M \) to obtain \( M' \)). Proceeding iteratively, we determine a matching \( M^* \) in \( L \) such that \( r(M^*/K) = r(M^*) \) and

\[ |M| = |M^*| + r(K) \leq v(L/K) + r(K). \]

We have thus shown that \( v(L) \leq r(K) + v(L/K) \).

It is also clear that

\[ v(L/K) \leq \sum(v(L_i/K); L_i \in P) \leq \sum(Lr(L_i/K)/2; L_i \in P). \]

Combining (1) and (2) proves the lemma. \( \Box \)

We call a triple \( L = \{L, K, P\} \), where \((K, P)\) is a cover of \( L \), a **strong duality set** if \( v(L) = c(K, P) \). Lovasz's duality theory for the parity problem states that for each set \( L \) of lines there is a strong duality set \( L = \{L, K, P\} \). Our algorithm provides a new constructive proof of this theorem.

**Theorem 1.** (Lovasz) The maximum cardinality of a matching in a set \( L \) of lines is equal to the minimum capacity of a cover of \( L \), i.e., \( v(L) = \min\{c(K, P)\} \).

Theorem 1 generalizes duality theories for the graphic matching problem and the linear matroid intersection
EXAMPLE 1. Consider the matching problem in the graph \( G \) of Figure 3. It is easy to verify that \((K, P)\) is a minimum capacity cover of the edge set of \( G \) where \( K \) corresponds to the vertex set \( \{1, 2\} \) and \( P \) has the non-trivial components circled in Figure 4. In this example, our algorithm would identify the alternative minimum capacity cover \((K', P')\) where \( K' \) corresponds to the vertex set \( \{1, 2\} \) and \( P' \) has the non-trivial components circled in Figure 5. \( \square \)

EXAMPLE 2. Suppose the \( m \times n \) matrices \( B \) and \( C \) are representations of the matroids \( M_1 = (E, I_1) \) and \( M_2 = (E, I_2) \), respectively. For any minimum rank cover \((E_1, E_2)\) of \( E = \{1..n\} \), in the matroid intersection problem, we construct a minimum capacity cover for the corresponding parity problem as follows. Let \( K = \text{span}\{A(2i-1): i \in E_1\} \cup \{A(2i): i \in E_2\}\), where \( A \) is the \( 2m \times 2n \) matching matrix for the parity problem, and let \( P \) be the partition in which each component contains exactly one line. Clearly, for any line \( l_1 = (2i-1, 2i) \), \( r(l_1/K) \leq 1 \) since if \( i \in E_1 \), then \( A(2i-1) \in K \) and if \( i \in E_2 \), then \( A(2i) \in K \). Thus,

\[
c(K, P) = r(K) = r_1(E_1) + r_2(E_2)
\]

and \((K, P)\) is a minimum capacity cover. \( \square \)

In the remainder of this section we develop a generalization of Lovasz's duality theory which motivates our approach to the parity problem.

Let \( L \) be a subset of the lines of \( M \) and let \( K \) be a subspace of \( V \). We say that the matching \( M \) in \( L \) is a \( K \)-hyper-matching of \( L \) if

\[
r(M) = r(M/K) = r(L/K) - 1.
\]
FIGURE 3

THE GRAPH 6

FIGURE 4

THE COVER \((K, P)\)

FIGURE 5

THE COVER \((K', P')\)
If \( M \) is a \( K \)-hypomatching of \( L \), then \( r(L/K) \) is odd and \( M/K \) is a maximum cardinality matching in \( L/K \). We say that \( L \) is **\( K \)-hypomatchable** if there is a non-empty collection \( B \) of \( K \)-hypomatchings of \( L \) such that \( \cap(L \text{span}(M); M \in B) \) is the null space, henceforth denoted by \( 0 \). Note that if \( L \) is \( K \)-hypomatchable, then for each point \( p \in K \) in \( L \text{span}(L) \) there is a \( K \)-hypomatching \( M \) of \( L \) such that \( p \not\in L \text{span}(M) \).

**EXAMPLE 3.** Let \( K \) be a subset of the vertices \( V \) in a graph \( G \). A **\( K \)-hypomatching** of \( G \) is a matching of the subgraph \( G(V-K) \) which leaves exactly one vertex \( v \in V-K \) unmatched. The graph \( G \) is **\( K \)-hypomatchable** if for each vertex \( v \in V-K \), there is a matching \( M \) of the subgraph \( G(V-K) \) which leaves only the vertex \( v \) unmatched. \( \square \)

We say that a matching \( M \) in \( L \) is a **\( K \)-hypermatching** of \( L \) if

\[
r(M) = r(M \times K) = r(L/K) + 1.
\]

If \( M \) is a \( K \)-hypermatching of \( L \) then the following are true:
(i) \( r(L/K) \) is odd
(ii) \( M \times K \) is a maximum cardinality matching in \( L \times K \), and
(iii) \( \text{span}(L) \cap K \) is a unique point denoted by \( K(M) \).

We say that \( L \) is **\( K \)-hypermatchable** if there is a collection \( B \) of \( K \)-hypermatchings of \( L \) such that

\[
\text{span}(\{K(M); M \in B\}) = L \text{span}(L) \cap K.
\]

**EXAMPLE 4.** Let \( K \) be a subset of vertices \( V \) in a graph \( G \). Then a **\( K \)-hypermatching** in \( G \) is a perfect matching in the graph \( G' \) obtained by shrinking the vertex set \( K \) to a single vertex. Equivalently, if \( M \) is a \( K \)-hypomatching in \( G \) and \( e \) is an edge incident to the unmatched vertex of \( V-K \) and a vertex \( v \) in \( K \), then \( M+e \) is a \( K \)-hypermatching and \( K(M+e) = v \). \( \square \)
We say that a matching \( M \) in \( L \) is a \( K \)-perfect matching of \( L \) if

\[
  r(M) = r(M/K) = r(L/K).
\]

If \( M \) is a \( K \)-perfect matching of \( L \) then \( r(L/K) \) is even and \( M/K \) is a maximum cardinality matching in \( L/K \).

Finally, corresponding to \( K \)-hypomatchings in a set \( L \) of lines, we call an independent set \( X \) of elements in \( \text{span}(L) \) a \( K \)-deficient of \( L \) if

\[
  r(X) = r(X/K) = r(L/K) - 1.
\]

By convention, we refer to the empty set as a \( K \)-deficient of \( L \) if \( r(L/K) \leq 1 \). For example, if \( B \) is a base of \( L \) in \( M/K \), then \( B - e \) is a \( K \)-deficient of \( L \) for each \( e \in B \). Note that a \( K \)-hypomatching of \( L \) is a \( K \)-deficient of \( L \), but not every \( K \)-deficient of \( L \) is a \( K \)-hypomatching.

The following lemma illustrates the significance of \( K \)-hypomatchings, \( K \)-hypermatchings and \( K \)-perfect matchings.

**Lemma 2.** Let \( L = \{L, K, P\} \) be a strong duality set for a subset \( L \) of the lines of \( M \). Then a matching \( M \) in \( L \) is a maximum cardinality matching if and only if each of the following is true:

(a) for each even component \( L_1 \) in \( P \), the submatching \( M_{L_1} = M \cap L_1 \) is a \( K \)-perfect matching of \( L_1 \),

(b) for each odd component \( L_1 \) in \( P \), the submatching \( M_{L_1} = M \cap L_1 \) is either a \( K \)-hypomatching or a \( K \)-hypermatching of \( L_1 \), and

(c) exactly \( r(K) \) of the submatchings \( M_{L_1} \) are \( K \)-hypermatchings.

**Proof.** First, suppose the matching \( M \) in \( L \) satisfies (a), (b) and (c) and let \( I = \{i: M_{L_i} \text{ is a } K \text{-hypermatching of } L_1\} \).
Then, since
\[ |M'| = \sum (|M'_i|: L_i \in \mathcal{P}) = |I'| + \sum (\mu_r(L_i/K)/2J: L_i \in \mathcal{P}) \]
\[ = r(K) + \sum (\mu_r(L_i/K)/2J: L_i \in \mathcal{P}) = c(K, \mathcal{P}) = v(L), \]

M is a maximum cardinality matching.

Now, suppose M is a maximum cardinality matching in L and for each L_i in \mathcal{P}, let M_i = M \cap L_i. Construct the matching M* from M so that r(M*/K) = r(M*) = 2|M*| as in the proof of Lemma 1, and for each L_i in \mathcal{P}, let M'_i = M* \cap L_i.

First, |M*'| = v(L/K) because
\[ |M*'| = |M'| - r(K) = \sum (\mu_r(L_i/K)/2J: \mathcal{I} \in \mathcal{E}) \geq v(L/K). \]

Second, |M'_i'| = v(L_i/K) = \mu_r(L_i/K)/2J for each L_i in \mathcal{P} since
\[ |M*'| = \sum (|M'_i'|: L_i \in \mathcal{P}) \]
\[ \leq \sum (\mu_r(L_i/K)/2J: L_i \in \mathcal{P}) \]
\[ \leq \sum (v(L_i/K): L_i \in \mathcal{P}) = |M*|. \]

In particular, M'_i is a K-perfect matching of L_i for each even component L_i in \mathcal{P} and M'_i is a K-hypomatching of L_i for each odd component L_i in \mathcal{P}.

Now, let I be the set of indices for which |M'_i| > |M'_i'|. We show that |I| = r(K). First, suppose that |I| < r(K). Since
\[ |U(M_i: i \in I)| = r(K) + \sum (\mu_r(L_i/K)/2J: i \in I), \]
it follows that
\[ |U(M_k : i \in I)| > r(K)/2 + \sum(L_k(L_k/K)/2) + 1/2; i \in I), \]

\[ \geq 1/2(r(K) + \sum(r(L_k/K); i \in I)), \]

\[ \geq 1/2(r(UL_k; i \in I)) \geq |U(M_k : i \in I)|, \]

a contradiction. Hence \(|I| \geq r(K)\).

Second, since \(|M| = |M*| + r(K)\), it is clear that \(|I| \leq r(K)\). Thus, we conclude that for each even component \(L_k\) in \(P\), \(M_k\) is a \(K\)-perfect matching of \(L_k\) and for each odd component \(L_k\) in \(P\), \(M_k\) is a \(K\)-hypermatching or a \(K\)-hypomatching of \(L_k\) according as \(i \in I\) or \(i \notin I\). 

**COROLLARY 1.** Let \(L = \{L, K, P\}\) be a strong duality set. For any maximum cardinality matching \(M\) in \(L\),

(i) \(K \subseteq \text{Lspan}(M)\) and

(ii) for each even component \(L_k\) in \(P\), \(L_k \subseteq \text{span}(M)\). 

**EXAMPLE 5.** Consider the matching \(M\) illustrated in Figure 6. In the minimum capacity cover of Figure 4, \(M\) contains the \(K\)-hypermatching \(\{(1,3), (4,5)\}\) in the non-trivial odd component consisting of edges \(\{(1,3), (1,4), (2,4), (3,4), (3,5), (4,5)\}\). The edge \((8,9)\) is a \(K\)-perfect matching of the even component and \(M\) contains a \(K\)-hypomatching of each of the remaining components.

Similarly, in the minimum capacity cover of Figure 5, \(M\) contains the \(K\)-hypomatching \(\{(4,5)\}\) in the non-trivial odd component \(\{(3,4), (3,5), (4,5)\}\). The edge \((8,9)\) is again a \(K\)-perfect matching of the even component. The edge \((1,3)\) is a \(K\)-hypermatching of the component containing it and the edge \((2,6)\) is a \(K\)-hypermatching of the component containing it. 

**COROLLARY 2.** Let \(L = \{L, K, P\}\) be a strong duality set and
FIGURE 6

THE MATCHING M
let \( u \) be a line not in \( L \). If \( M \) is a \((v(L)+1)\)-matching in \( L+u \), then:

(i) \( u \in M \),

(ii) \( M_\mathbf{1} = M \cap L_\mathbf{1} \) is either a \( K \)-hypomatching, a \( K \)- hypermatching or a \( K \)-perfect matching of \( L_\mathbf{1} \) for each component \( L_\mathbf{1} \) in \( P \), and

(iii) \( K = \text{span}(\{K(M_\mathbf{1}): M_\mathbf{1} \text{ is a } K \text{-hypermatching of } L_\mathbf{1}\}) \). \( \square \)

Corollary 2 is a direct consequence of Lemma 2 since \( M-u \) is a maximum cardinality matching in \( L \). Moreover, this corollary suggests that given a strong duality set \( L = \{L, K, P\} \) and a line \( u \in L \), we can construct a maximum cardinality matching in \( L+u \) from \( K \)-hypomatchings, \( K \)-hypermatchings and \( K \)-perfect matchings.
4. An Induced Matroid Intersection Problem

Whereas the blossom algorithm may be viewed as reducing the matching problem in a non-bipartite graph to the matching problem in a related bipartite graph, our algorithm reduces the parity problem to a sequence of related matroid intersection problems. In this section, given a strong duality set \( L = \{L, K, P\} \) and a line \( u \notin L \), we determine \( v(L+u) \) by solving a sequence of matroid intersection problems.

Conceptually, each matroid intersection problem may be viewed as selecting the \( K \)-hypomatchings and \( K \)-hypermatchings of a maximum cardinality matching. In particular, Matroid 1, the first matroid in each intersection problem, selects the \( K \)-hypermatchings and Matroid 2, the second matroid in each intersection problem, selects the \( K \)-hypomatchings. Thus, given a strong duality set \( L = \{L, K, P\} \) and a line \( u \notin L \), our algorithm constructs matchings in \( L+u \) as the union of \( K \)-hypomatchings and \( K \)-hypermatchings from each odd component of \( L \).

Each intersection problem is induced by a pair \( \{L, u\} \) satisfying certain conditions. In particular, we say that a strong duality set \( L = \{L, K, P\} \) is a primary duality result if the following are true.

(3) \( P \) contains exactly one (possibly empty) even component \( L_0 \) and \( L_0 \) is a matching.

(4) Each odd component \( L_1 \) in \( P \) is both \( K \)-hypomatchable and \( K \)-hypermatchable.

(5) For each point \( p \notin L\text{span}(L_0UK) \), there is a matching \( M \in B \) such that \( p \notin L\text{span}(M) \).

We say that \( u \) is exposed with respect to \( L \) if

(6) \( r(u/K) = 2 \) and

(7) \( r((L_1+u)/K) > r(L_1/K) \) for each \( L_1 \) in \( P \).
It is easy to verify that if \( u \) is not exposed with respect to \( L \), then \( v(L+u) = v(L) \). In particular, if \( r(u/K) < 2 \), then \( Lr(u/K)/2J = 0 \) and adding \( u \) to \( P \) as a component does not increase the capacity of the cover. Similarly, if \( r((L_1+u)/K) = r(L_1/K) \) for some \( L_1 \) in \( P \), then \( Lr((L_1+u)/K)/2J = Lr(L_1/K)/2J \) and adding \( u \) to \( L_1 \) does not increase the capacity of the cover. In fact, it is easy to see that if no line in a set \( L' \) is exposed with respect to \( L \), then \( v(LUL') = v(L) \).

Let \( L = \{L, K, P\} \) be a strong duality set and suppose that \( u \) is exposed with respect to \( L \). We say that the pair \( \{L, u\} \) is an intermediate duality result if \( L \) satisfies (3) and (4) and if there is a nonempty collection \( B \) of \( v(L) \)-matchings in \( L+u \) such that

(8) For each point \( p \in L\text{span}(L_0\text{UK}) \), there is a matching \( M \in B \) such that \( p \in L\text{span}(M) \), and

(9) For each matching \( M \in B \) and each component \( L_1 \) in \( L \), the submatching \( M \mid L_1 \) is either a \( K \)-hypomatching, a \( K \)-hypermatching or a \( K \)-perfect matching of \( L_1 \).

We refer to the line \( u \) as an even component, denoted \( L_u \), of the intermediate duality result \( \{L, u\} \) and we refer to the line \( u \) as a \( K \)-perfect matching of \( L_u \).

It is easy to see that for any matching \( M \), \( L(M) = \{M, 0, \{M\}\} \) is a primary duality result. Moreover, for any line \( u \not\in \text{span}(M) \), the pair \( \{L(M), u\} \) is an intermediate duality result. More generally, if \( L = \{L, K, P\} \) is a primary duality result and \( u \) is exposed with respect to \( L \), then \( \{L, u\} \) is an intermediate duality result. However, it is not in general true that if \( \{L, u\} \) is an intermediate duality result, then \( L \) is a primary duality result. Finally, observe that if \( \{L, u\} \) is an intermediate duality result, there is a maximum cardinality matching \( M \) in \( L \) such that \( u \not\in \text{span}(M) \).

Our algorithm starts with a trivial matching and develops larger matchings by means of an "augmenting path.
procedure". Suppose, for example, that at some stage in our algorithm we have found a matching $M$, and we wish to find a matching of cardinality $|M|+1$ or prove that none exists. We first observe that the strong duality set $L(M) = \{M, 0, \{M\}\}$ is a primary duality result. Our algorithm then sequentially considers lines $u_1, u_2, \ldots$ not in $M$. Prior to considering the line $u_1$ (and assuming that no larger matching has been found), the algorithm has constructed a primary duality result $L = M \cup \{u_1, u_2, \ldots, u_{i-1}\}$. Subsequently, the algorithm considers a sequence of at most $|M|$ matroid intersection problems. Each of these intersection problems leads to either a $(|M|+1)$-matching or an intermediate duality result $\{L, u_i\}$. If no larger matching is found then the last of these intersection problems leads to a primary duality result for $M \cup \{u_1, u_2, \ldots, u_i\}$. (Incidentally, although the procedure in this section does determine a primary duality result, the proof of this fact is deferred to the next section.) At this point the line $u_{i+1}$ is considered, and we iterate until we either find a $(|M|+1)$-matching or construct a primary duality result for the parity problem.

Matroid 1

Let $\{L, u\}$ be an intermediate duality result where $L = \{L, K, P\}$. Suppose $M$ is a $(n(L)+1)$-matching in $L+u$ and for each $L_i$ in $P = \{L_i : i \in E\}$, let $M_i = MNL_i$. In Corollary 2 we observed that that the set $\{K(M_i) : M_i \text{ is a } K\text{-hypermatching of } L_i\}$ is linearly independent and spans $K$. With this in mind, we define Matroid 1 as follows. A subset $I$ of component indices is independent in Matroid 1 if there is a collection $X$ of $K$-hypermatchings, one from each odd component $L_i$ with $i \in I$ such that the set $\{K(M_i) : M_i \in X\}$ of points is linearly independent. In addition, for each subset $I$ of component indices independent in Matroid 1, $I+u$ is independent in Matroid 1.
Equivalently, Matroid 1, denoted $M_1(L, u)$, may be described as an extra-transversal matroid in the following way. Let $Q = \{Q_k: i \in E + u\}$ where $Q_0$ consists of any point not in $K$, $Q_\lambda$ consists of any point not in $\text{span}(K Q_0)$, and for each odd component $L_\lambda$ in $P$, $Q_\lambda = \{K(M_{Q_\lambda}): M_{Q_\lambda}$ is a $K$-hypermatching of $L_\lambda\}$. Then $M_1(L, u)$ is the extra-transversal matroid $ET(Q, M)$. Note that since $\{L, u\}$ is an intermediate duality result, each odd component $L_\lambda$ in $P$ is $K$-hypermatchable. Thus, for each odd component $L_\lambda$ in $P$, $\text{span}(Q_\lambda) = L\text{span}(L_\lambda) \cap K$. Moreover, by Lemma 5 of Part 1, we see that a subset $I$ of component indices is dependent in $M_1(L, u)$ if and only if there is a subset $I' \subseteq I$ such that $r(U(L\text{span}(L_\lambda) \cap K; i \in I')) < |I'|$. In this case, it is easy to see that there can be no matching $M$ containing a $K$-hypermatching of each component $L_\lambda$ with $i \in I'$.

**EXAMPLE 6.** Consider the set $L$ of lines corresponding to the edges of the graph $G$ in Figure 3 and suppose that $u$ corresponds to the edge $(6, 7)$. Then, $\{L, u\}$ is an intermediate duality result where $L = \{L, K', P'\}$ and $(K', P')$ is the minimum capacity cover described in Example 1. In this case it is easy to verify that the set $\{1, 2\}$ of component indices is dependent in $M_1(L, u)$, where $L_1$ corresponds to the edge $(2, 6)$ and $L_2$ corresponds to the edge $(2, 7)$. On the other hand, the set $\{1, 3\}$ of component indices is independent in $M_1(L, u)$ where $L_3$ corresponds to the edge $(1, 3)$. Note that since the component $L_4$ corresponding to the edges $\{(3, 4), (3, 5), (4, 5)\}$ admits no $K$-hypermatching, the index 4 is itself dependent in $M_1(L, u)$. □

Matroid 2

Again, let $\{L, u\}$ be an intermediate duality result, where $L = \{L, K, P\}$. Suppose $M$ is a $(v(L)+1)$-matching in $L+u$ and for each $L_\lambda$ in $P = \{L_\lambda: i \in E\}$, let $M_\lambda = MN L_\lambda$. In Corollary 2 we observed that for each odd component $L_\lambda$ in $P$,
$M_1$ is either a $K$-hypomatching or a $K$-hypermatching of $L_1$. With this in mind, we define Matroid 2 as follows. A subset $I \subseteq E+u$ is independent in Matroid 2 if there is a collection $\{M_i : i \in E+u\}$ such that $r(U(M_i : i \in E+u)/K) = \sum r(M_i/K : i \in E+u)$, where:

10. $M_i$ a $K$-hypermatching of $L_1$ for each odd component $L_i$ with $i \in I$,
11. $M_i$ a $K$-hypomatching of $L_1$ for each odd component $L_i$ with $i \in I$,
12. $M_i$ a $K$-perfect matching of $L_1$ for each even component $L_i$ with $i \in I$,
13. $M_i$ a $K$-deficient of $L_1$ for each even component $L_i$ with $i \in I$.

Matroid 2, denoted $N_2(L, u)$, may be described as a shrinking matroid in the following way. Let $D = \{D_i : i \in E+u\}$, where $D_0$ consists of the matching $L_0$ together with the collection of all $K$-deficients of $L_0$, $D_u$ consists of the line $u$ together with the collection of all $K$-deficients of $u$, and for each odd component $L_1$ in $P$, $D_i$ consists of a base $M_{i0}$ of $L_1$ in $M/K$ together with the set of all $K$-hypomatchings of $L_1$. Then $M_2(L, u)$ is the shrinking matroid $SM(M/K, D)$. By Lemma 5 of Part 1, we have that a subset $I \subseteq E+u$ is dependent in $M_2(L, u)$ if and only if there is a subset $I' \subseteq E+u - I$ such that

$$r(U(L_i : i \in I'\cup I)/K) < \sum r(L_i/K : i \in I) + \sum r(L_i/K-1 : i \in I').$$

In this case it is easy to see that no maximum cardinality matching $M$ can contain a $K$-hypermatching of each odd component $L_i$ and a $K$-perfect matching of each even component $L_i$ with $i \in I$.

**Example 7.** Consider the set $L$ of lines corresponding to the edges of the graph $G$ in Figure 3 and suppose that $u$ corresponds to the edge $(6, 7)$. We observed in Example 6 that $(L, u)$ is an intermediate duality result where $L = \{L,$
K', P'). It is easy to verify that the set \{1, 2\} of component indices is dependent in M_2(L, u), where L_1 corresponds to the edge (2, 6) and L_2 corresponds to the edge (2, 7). On the other hand, the set \{1, 3\} of component indices is independent in M_2(L, u) where L_3 corresponds to the edge (1, 3).

In Lemma 4 we show that if \{L, u\} is an intermediate duality result, then M_2(L, u) is a matroid. Moreover, we observe that if there is a \(r(K)+2\)-intersection in M_1(L, u) and M_2(L, u), there is a \(v(L)+1\)-matching in L+u.

**Lemma 4.** Suppose \{L, u\} is an intermediate duality result, where L = \{L, K, P\}, then:

(i) M_2(L, u) is a matroid,

(ii) the maximum cardinality of an intersection in M_1(L, u) and M_2(L, u) is either r(K)+1 or r(K)+2;

(iii) if there is a \(r(K)+2\)-intersection in M_1(L, u) and M_2(L, u), there is a \(v(L)+1\)-matching in L+u;

(iv) if there is a maximum cardinality intersection I in M_1(L, u) and M_2(L, u) with 0 \(\not\subseteq\) I, there is a \(v(L)+1\)-near-matching N with 2-circuit C such that CNL_0 \(\not=\) \(\emptyset\).

**Proof.** First, by Theorem 2 and Lemma 4 of Part 1, to show that M_2(L, u) is a matroid, we need only show that some set (possibly \(\emptyset\)) is independent in M_2(L, u). In fact, we construct a \(r(K)+1\)-intersection I in M_1(L, u) and M_2(L, u), thereby proving not only that M_2(L, u) is a matroid, but also that the maximum cardinality of an intersection in M_1(L, u) and M_2(L, u) is at least r(K)+1.

Since \{L, u\} is an intermediate duality result, there is a maximum cardinality matching M \(\subseteq\) L such that u \(\not\subseteq\) \(\text{span}(M)\). Let M_4 be an element of u not contained in \(\text{span}(M)\) and for each component L_4 in P, let M_4 = MeL_4. We argue that I, the set of component indices for which M_4 is either a K-hypermatching or a K-perfect matching of L_4, is a \(r(K)+1\)-
intersection in \( M_1(L, u) \) and \( M_2(L, u) \).

First, by Lemma 2, \( M_0 \) is a \( K \)-perfect matching of \( L_0 \) and exactly \( r(K) \) of the submatchings \( M_k \) are \( K \)-hypermatchings. Therefore, \( |I| = r(K)+1 \). Moreover, since \( K \subseteq \text{Lspan}(M) \),

\[
r(U(M_i : i \in E+u) / K) = r(M+u) - r(K) = r(M) + 1 - r(K)
\]

\[
= \Sigma (r(M_i / K) : i \in E+u).
\]

Hence I is independent in \( M_2(L, u) \) and therefore \( M_2(L, u) \) is a matroid. Moreover, since \( M \) is a matching, the set \{ \( K(M_i) : i \in I, i \neq 0 \) \} is linearly independent. Hence I is independent in \( M_1(L, u) \); proving that the maximum cardinality of an intersection in \( M_1(L, u) \) and \( M_2(L, u) \) is at least \( r(K)+1 \). Since \( r_1(E+u) \leq r(K)+2 \), we conclude that the maximum cardinality of an intersection in \( M_1(L, u) \) and \( M_2(L, u) \) is either \( r(K)+1 \) or \( r(K)+2 \).

Now, suppose I is a \((r(K)+2)\)-intersection in \( M_1(L, u) \) and \( M_2(L, u) \). We construct a \((\nu(L)+1)\)-matching \( M \subseteq L+u \) from \( I \) as follows. First, since \(|I| = r(K)+2 \), \( u \) and \( 0 \) are in \( I \). Next, since I is independent in \( M_1(L, u) \), there is a collection \( X \) of \( K \)-hypermatchings, one from each odd component \( L_i \) with \( i \in I \), such that the set \{ \( K(M_i) : M_i \in X \) \} is linearly independent. Moreover, since I is independent in \( M_2(L, u) \), there is a collection \{ \( M_i : i \in E+u \) \} satisfying (10) - (13) such that

\[
r(U(M_i : i \in E+u) / K) = \Sigma (r(M_i / K) : i \in E+u).
\]

For each \( i \in I \), let \( M_i^* = M_i \) and for each \( i \in E-I \), let \( M_i^* = M_i \). Since \( M = U(M_i^* : i \in E+u) \) is independent in \( M \), \( M \) is a \((\nu(L)+1)\)-matching in \( L+u \).

Finally, suppose I is a maximum cardinality intersection in \( M_1(L, u) \) and \( M_2(L, u) \) with \( 0 \in I \). We construct
the \((v(L)+1)\)-near-matching \(N\) as follows. Since \(0 \leq I, \quad |I| < r(K)+2\) and hence by (ii), \(|I| = r(K)+1\).

Since \(I\) is independent in \(M_1(L, u)\), there is a collection \(X\) of \(K\)-hypermatchings, one from each odd component \(L_i\) with \(i \in I\), such that the set \(\{K(M_i) : M_i \in X\}\) is linearly independent.

Since \(I\) is independent in \(M_2(L, u)\), there is a collection \(\{M_i : i \in E+u\}\) satisfying (10) - (13) such that

\[
S(r(U(M_i : i \in E+u)/K) = \Sigma r(M_i/K) : i \in E+u).
\]

For each \(i \in I\), let \(M_i^* = M_i\) and for each \(i \in E-I, i \neq 0\), let \(M_i^* = M_i\). Since \(M^* = U(M_i^* : i \in E+u, i \neq 0)\) is independent in \(M\), \(M^*\) is a matching. Moreover, since \(I\) is a maximum cardinality intersection in \(M_1(L, u)\) and \(M_2(L, u)\) and \(0 \leq I\),

\[
r(M^*U_L) = r(M^*U_{L_0}/K) + r(K)
= \Sigma r(M_i'/K) : i \in E+u) + r(K)
= 2!M^*U_L+1.
\]

We conclude then that \(N = M^*U_L\) is a near-matching and, since \(M^*\) is a matching, that the 2-circuit \(C\) of \(N\) contains at least one line of \(L_0\). \(\square\)

In Lemma 4 we showed that if there is a \((r(K)+2)\)-intersection in \(M_1(L, u)\) and \(M_2(L, u)\), then \(v(L+u) = v(L) + 1\).

In Lemma 5 we consider the case in which the maximum cardinality of an intersection in \(M_1(L, u)\) and \(M_2(L, u)\) is \(r(K)+1\).

**Lemma 5.** Suppose \(\{L, u\}\) is an intermediate duality result where \(L = \{L, K, P\}\). If the maximum cardinality of an intersection in \(M_1(L, u)\) and \(M_2(L, u)\) is \(r(K)+1\), then either
(a) there is a strong duality set $L' = \{L+u, K', P'\}$ proving that $\nu(L+u) = \nu(L)$ or

(b) there is an intermediate duality result $\{L', u\}$, where $L = \{L, K', P'\}$, such that $L_0$, the even component of $L'$, is a proper subset of $L_0$, the even component of $L$.

PROOF. Suppose that the maximum cardinality of an intersection in $M_1(L, u)$ and $M_2(L, u)$ is $r(K)+1$. First, we show that if the index of the even component in $P$ is in every maximum cardinality intersection, then $\nu(L+u) = \nu(L)$.

Let $(E_1, E_2+u)$ be the minimum rank cover constructed by the matroid intersection procedure where we are adding with respect to Matroid 1. (We know that the element $u$ gets labeled by our definition of Matroid 1. Thus, we rule out the alternative possibility that the cover is $(E_1+u, E_2)$.) By Theorem 1 of Part 1,

$$r_1(E_1) + r_2(E_2+u) = r(K)+1$$

and by Lemma 2 of Part 1, $0 \in E_1$. Moreover, by Lemma 10 in Part 1 we have that

$$r_1(E_1) = r(K')+1$$

where $K' = \text{span}(U(L\text{span}(L_1) \cap K; i \in E_1))$, and that

$$r_2(E_2+u) = r(U(L_1; i \in E_2+u)/K) - \Sigma (r(L_1/K)-1; i \in E_2+u).$$

Combining (14), (15) and (16), we have that

$$r(U(L_1; i \in E_2+u)/K)$$

$$= r(K) - r(K') + \Sigma (r(L_1/K)-1; i \in E_2+u).$$

and hence that
\[ r(L_i/K') \leq 2r(K) - 2r(K') + \sum (r(L_i/K) - 1; i \in \mathbb{E}_3^u) \]

where \( L_i = U(L_i; i \in \mathbb{E}_3^u) \).

Thus, since \( r(L_i/K) - 1 \) is even for each \( i \in \mathbb{E}_2 \),

\[ L_r(L_i/K')/2 \leq r(K) - r(K') + \sum (L_r(L_i/K)/2; i \in \mathbb{E}_3) \].

Finally, by the definition of \( K' \) we have that for each odd component \( L_i \) with \( i \in \mathbb{E}_1 \)

\[ r(L_i/K') = r(L_i/K). \]

Combining (17) and (18) we conclude that

\[ v(L+u) \leq r(K') + \sum (L_r(L_i/K')/2; i \in \mathbb{E}_3) + L_r(L_i/K')/2 \]

\[ = r(K) + \sum (L_r(L_i/K)/2; i \in \mathbb{E}) = v(L). \]

Next, suppose \( I \) is a maximum cardinality intersection with \( O \cap I \). By Lemma 4, there is a \((v(L)+1)\)-near-matching \( N \) with 2-circuit \( C \) such that \( C \cap L_0 \neq \emptyset \).

We construct a strong duality set \( L' = \{L, M, K', P'\} \) with \( L_0 \), the even component of \( P' \), a proper subset of \( L_0 \), the even component of \( P \), as follows.

Let \( K' = \text{span}(K \cup \{C(l); l \in C \cap L_0\}) \) and let \( P' = \{L_i; i \in \mathbb{E}'\} \) be the partition of \( L \) in which

(i) \( L_i' = L_i \) for each odd component \( L_i \) in \( P \),
(ii) \( L_0' = L_0 - C \) and
(iii) for each line \( l \in C \cap L_0 \), the singleton \( \{l\} \) is a component in \( P' \).

Now, we have that each odd component \( L_i \) in the original partition \( P \) is both hypomatchable and hypermatchable since

\[ r(L_i/K') = r(L_i/K). \] Likewise, each newly formed odd
component $L'$ is a trivial component and hence is both hypomatchable and hypermatchable.

Thus, it is easy to see that $L' = \{L, K', P'\}$ is a strong duality set satisfying (3) - (4) and the collection $B' = BU(N-l: i \in CNL_0)$ satisfies (8) and (9). Thus, if $u$ is exposed with respect to $L'$, the pair $\{L', u\}$ is an intermediate duality result. Otherwise, $r(u/K') = 1$ and adding $u$ to $P'$ as a trivial component yields a strong duality result $L'$ proving that $v(L+u) = v(L)$. $\square$

Together, Lemma 4 and Lemma 5 suggest the following procedure for determining $v(L+u)$ given a primary duality result $L = \{L, K, P\}$ and a line $u$ exposed with respect to $L$. Procedure 2 determines $v(L+u)$ by solving a sequence of related matroid intersection problems -- each induced by an intermediate strong duality result $\{L, u\}$. Initially, the two matroids are defined on $u$ together with the set $E$ of indices of the components in $P$. The solution of each matroid intersection problem leads to either:

(a) a $(r(K)+2)$-intersection $I$ in $M_1(L, u)$ and $M_2(L, u)$,
(b) a minimum rank cover $(E_1, E_2+u)$ of $E+u$ proving that no such intersection exists. (The 'element' $u$ is always labeled by the matroid intersection procedure.)

As we proved in Lemma 4, the $(r(K)+2)$-intersection identified in case (a) leads to a $(v(L)+1)$-matching $M'$ in $L+u$. In a graph, this corresponds to the blossom algorithm identifying an augmentation leading to a larger matching.

If there is no $(r(K)+2)$-intersection in $M_1(L, u)$ and $M_2(L, u)$, the matroid intersection procedure provides a cover $(E_1, E_2+u)$ with rank $r(K)+1$. If the index of the even component in $P$ is in every maximum cardinality intersection, we construct a strong duality set $L' = \{L+u, K', P'\}$ by combining the components $\{L_i: i \in E_2+u\}$ to obtain the new component $L_i$, thereby proving that $v(L+u) = v(L)$. In a
graph, this case corresponds to blossom formation.

Finally, if there is a \( r(K)+1 \)-intersection \( I \) which does not contain the index of the even component in \( P \), we use the resulting \( (v(L)+1) \)-near-matching to construct a new intermediate duality result \( \{L', u\} \), where \( L = \{L, K', P'\} \) and \( L_0 \), the even component in \( P' \), is a proper subset of \( L_0 \), the even component of \( P \). In a graph, this case corresponds to propagating an alternating tree rooted at the exposed vertex of the edge \( u \). Clearly, this case can occur at most \(|L_0|\) times before we either find a larger matching or prove none exists.

**PROCEDURE 2.**

**Input:** a primary duality result \( L = \{L, K, P\} \)

a line \( u \) exposed with respect to \( L \).

**Output:** \( v(L+u) \).

Step 1. (Labeling).

(1.0) Determine the maximum cardinality of an intersection in \( M_1(L, u) \) and \( M_2(L, u) \) using Procedure 1 of Part 1. If there is a \( (r(K)+1) \)-intersection \( I \), go to Step 2. Otherwise, the maximum cardinality of an intersection is \( r(K)+1 \); construct the minimum rank cover \( \{E_1, E_2+u\} \) as in Lemma 10 of Part 1. If \( 0 \in E_1 \), then go to Step 3. Otherwise, there is a maximum cardinality intersection \( I \) with \( 0 \notin I \); go to Step 1.1.

(1.1) Construct the \( (v(L)+1) \)-near-matching \( N \) from \( I \) and determine the 2-circuit \( C \). Revise the strong duality set \( L = \{L, K, P\} \) as follows. Set \( K = \text{span}(K \cup \{C(1): l \in CNL_0\}) \). Add each line \( l \in CNL_0 \) to \( P \) as a trivial component and set \( L_0 = L_0 - C \). If \( r(u/K) = 2 \), then
return to Step 1.0. Otherwise, add \( u \) to \( P \) as a trivial component. Stop, \( v(L+u) = v(L) \).

Step 2. (Augment). Stop, \( v(L+u) = v(L)+1 \).

Step 3. (Component Formation). Construct the strong duality set \( L' = \{L+u, K', P'\} \) as follows. Set \( K' = \text{span}(U(L_{\text{span}}(L_i) \cap K: 1 \leq i \leq 1)) \) and let \( P' \) be the partition of \( L+u \) with \( L_i \) in \( P' \) for each \( i \leq 1 \) and with \( L'_1 = U(L_i: 1 \leq i \leq 2) + u \) in \( P \). Stop, \( v(L+u) = v(L) \). \( \square \)

Given a primary duality result \( L = \{L, K, P\} \) and a line \( u \) exposed with respect to \( L \), Procedure 2 determines the maximum cardinality of a matching in \( L+u \) by solving \( O(m) \) matroid intersection problems. In particular, if there is a \( (r(K)+2) \)-intersection in \( M_1(L, u) \) and \( M_2(L, u) \), then Lemma 4 tells us that there is a \( (v(L)+1) \)-matching \( M \) in \( L+u \). However, Procedure 2 supplies no mechanism for actually determining \( M \). In the next section we develop an efficient procedure for actually determining the matching \( M \) in this case. Moreover, we show that the strong duality set \( L' = \{L+u, K', P'\} \) constructed in Step 3 is a primary duality result. These observations lead to the polynomial time algorithm described in Section 6.
5. Augmentation and Component Formation

Let \( \{L, u\} \) be an intermediate duality result, where \( L = \{L, K, P\} \). In Section 4, we proved that if there is a \((r(K)+2)\)-intersection in \( M_1(L, u) \) and \( M_2(L, u) \), then there is a \((v(L)+1)\)-matching \( M \subseteq L+u \). Here, we develop an efficient procedure for actually determining the matching \( M \). Moreover, we prove that the strong duality sets constructed in Step 3 of Procedure 2 are primary duality results. Together, these observations prove the correctness of our algorithm described in Section 6.

In the previous section, we described Matroid 1 and Matroid 2 as selecting the \( K \)-hypermarchings and \( K \)-hypomarchings, respectively, of a maximum cardinality matching. A more precise interpretation of the intersection problems formulated in the previous section is as follows. A maximum cardinality intersection \( I \) in Matroid 1 and Matroid 2 may be viewed as partitioning the space in such a way that we may determine a maximum cardinality matching by selecting a maximum cardinality submatching from each subspace. Moreover, this partition is such that there is a natural one-to-one correspondence between the subspaces of the partition and the components of the strong duality set. In fact, any maximum cardinality submatching in a subspace of the partition corresponding to an odd component \( L_i \) with \( i \in I \), is a \( K \)-hypermarching of \( L_i \). Similarly, any maximum cardinality submatching in a subspace corresponding to an odd component \( L_i \) with \( i \in E - I \), is a \( K \)-hypomarching of \( L_i \).

In this section we develop an efficient procedure for determining the 'appropriate' submatchings of each odd component \( L_i \) in \( L \). Given an \((r(K)+2)\)-intersection \( I \) in Matroid 1 and Matroid 2, our 'augmentation' procedure determines a maximum cardinality matching in the following way. First, the procedure sequentially determines \( K-\)
hypermatchings $M_i$ of each odd component $L_i$ with $i \in I$ such the 
points \{K(M_i)\}: $L_i$ an odd component with $i \in E$ are linearly 
independent. Thus, each successive K-hypermatching $M_i$ is 
chosen so that the point $K(M_i)$ is not in the subspace 
spanned by the preceding K-hypermatchings. Second, the 
procedure sequentially selects K-hypomatchings $M_i$ of each 
odd component $L_i$ with $i \notin I$. The linear span of the matching 
$M'$ consisting of the union of the K-hypomatchings together 
with the first $k-1$ K-hypomatchings meets the linear span of 
each remaining odd component in at most one point. The K- 
hypomatching $M_i$ of the next odd component $L_i$ is chosen so 
that the point $p = L\text{span}(M') \cap L\text{span}(L_i)$ is not in $L\text{span}(M_i)$.

Consider the odd component $L_i$ in $L$. If $L_i = \{l\}$ is a 
trivial component, the problem of determining an appropriate 
K-hypomatching or K-hypermatching of $L_i$ is indeed trivial -- 
the only K-hypermatching of $L_i = \{l\}$ is the line $l$ itself, 
and the only K-hypomatching of $L_i$ is the empty set. The 
problem of determining an appropriate K-hypomatching or K- 
hypermatching of a non-trivial odd component is more 
difficult. Conceptually, we determine a maximum cardinality 
submatching in the subspace corresponding to a non-trivial 
odd component by employing a sequence of matroid 
intersection problems (related to those described in Section 
4) to further partition the subspace. Ultimately, this 
recursion reduces the problem to one of selecting 
appropriate submatchings of a number of trivial components.

Specifically, each non-trivial odd component 
encountered by the algorithm is constructed as the newly 
formed component in Step 3 of Procedure 2. Thus, in the 
remainder of this section we consider the problem of determining an appropriate submatching of the non-trivial odd component $L_i = U(L_i; \forall i \in E_{\neq u})$ of $L' = \{L+u, K', P'\}$, formed from the intermediate duality result $\{L, u\}$ where $L = \{L, K, P\}$.
We are interested in two kinds of submatchings of the non-trivial odd component \(L_i\). First, we may wish to find a \(K'\)-hypomatching \(M_i\) of \(L_i\) such that the point \(K'(M_i)\) is not in the subspace \(K_i\) spanned by the points \(K'(M_i)\) of the \(K'\)-hypomatchings chosen in other odd components. Note that such a matching \(M_i\) is a perfect matching of \(L_i/K_i \times K'\).

Second, we may wish to find a \(K'\)-hypomatching \(M_i\) of \(L_i\) such that the point \(p \in L\text{span}(M_i)\). Note that such a matching \(M_i\) is a perfect matching of \(L_i/(K'+p)\).

We construct these submatchings as the union of appropriate \(K\)-hypomatchings and \(K\)-hypomatchings from each component \(L_i\) with \(i \in E_2\). In turn, we determine the appropriate \(K\)-hypomatchings and \(K\)-hypomatchings in each component \(L_i\) with \(i \in E_2\) by again applying the same procedure. Namely, we solve a sequence of matroid intersection problems related to those described in the previous section, partitioning the subspace corresponding to each component \(L_i\) with \(i \in E_2\) into yet smaller subspaces.

\(K'\)-hypomatchings of \(L_i\)

We first consider the problem of determining a \(K'\)-hypomatching \(M_i\) of \(L_i\) such that the point \(p \in L\text{span}(M_i)\).

If the point \(p\) is in the subspace \(K\) (but not in \(K'\)), then \((L^*, u)\) is an intermediate duality result where \(L^* = (L^*, K^*, P^*)\), \(L^* = (L_i-u)/(K'+p)\), \(K^*\) is a subspace of \(K\) orthogonal to \(K'+p\) such that \(K^*U(K'+p)\) spans \(K\), and \(P^* = (L_i/(K'+p); i \in E_2)\). In particular, \((L^*, u)\) is an intermediate duality result for a subset of the lines of \(M/(K'+p)\) and \(M_i \subseteq L^*+u\) is a \((v(L^*+1))-\text{matching in } L^*+u\) if and only if \(M_i\) is a \(K'\)-hypomatching of \(L_i\) such that \(p \in L\text{span}(M_i)\). Thus, we can apply Procedure 2 to determine a \((r(K^*)+2))-\text{intersection in } M_i(L^*, u/(K'+p))\) and \(M_2(L^*, u/(K'+p))\) and apply our 'augmentation' procedure to determine \(M_i\).
If the point \( p \) is not in the subspace \( K \), then we partition the subspace corresponding to \( L_i \) by solving a single matroid intersection problem. In this intersection problem, the first matroid, Matroid 5, is closely related to Matroid 1. In fact Matroid 5 is the restriction of \( M_5(L, u) \) to the indices in \( E_2 \). In particular, a subset \( I \subset E_2 \) is independent in Matroid 5, denoted \( M_5(L, E_2, p) \), if there is a collection \( \{ M_i : i \in I \} \) of \( K \)-hypomatchings, one from each component \( L_i \) with \( i \in I \), such that \( r(\{K(M_i) : i \in I\}/K') = |I| \).

The second matroid in this intersection problem, Matroid 6, is likewise closely related to Matroid 2. In particular, a subset \( I \subset E_2 \) is independent in Matroid 6, denoted \( M_6(L, E_2, p) \), if there is a collection \( \{ M_i : i \in E_2 \} \) with

1. \( M_i \) a \( K \)-hypomatching of \( L_i \) for each \( i \in I \), and
2. \( M_i \) a \( K \)-hypermatching of \( L_i \) for each \( i \in E_2 - I \)

such that \( r(U(M_i : i \in E_2)/K+p) = \Sigma r(M_i/K): i \in E_2 \).

The following lemma illustrates the strong connection between \( r(K/K') \)-intersections in Matroid 5 and Matroid 6 and \( K' \)-hypomatchings \( M_i \) of \( L_i \) such that \( p \notin \text{Lspan}(M_i) \).

**Lemma 6.** Let \( p \notin K \) be a point in \( \text{Lspan}(L_i) \). There is a \( K' \)-hypomatching \( M_i \) of \( L_i \) such that

(i) \( u \notin M_i \),

(ii) \( p \notin \text{Lspan}(M_i) \), and

(iii) for each \( i \in E_2 \), \( M_i \cap L_i \) is either a \( K \)-hypomatching or a \( K \)-hypermatching of \( L_i \),

if and only if there is a \( r(K/K') \)-intersection in \( M_5(L, E_2, p) \) and \( M_6(L, E_2, p) \).

**Proof.** It is easy to verify that if \( M_i \) is a \( K' \)-hypomatching of \( L_i \) satisfying (i) - (iii), then the set \( I = \{ i \in E_2 : M_i \cap L_i \}

is a \( K \)-hypermatching of \( L_i \}) \) is a \( r(K/K') \)-intersection in \( M_5(L, E_2, p) \) and \( M_6(L, E_2, p) \).

On the other hand, suppose that the subset \( I \subset E_2 \) is a \( r(K/K') \)-intersection in \( M_5(L, E_2, p) \) and \( M_6(L, E_2, p) \). Since
I is independent in $M_{\mathbf{r}}(L, E_2, p)$, there is a collection $\{M_i : i \in I\}$ of $K$-hypermatings, one from each component $L_i$ with $i \in I$, such that $r(\{K(M_i) : i \in I\}/K') = |I|$. Similarly, since $I$ is independent in $M_0(L, E_2, p)$, there is a collection $\{M_i : i \in E_2\}$ with

(i) $M_i$ a $K$-hypermating of $L_i$ for each $i \in I$ and
(ii) $M_i$ a $K$-hypomatching of $L_i$ for each $i \in E_2 - I$ such that $r(U(M_i : i \in E_2)/K) = \Sigma r(M_i/K) : i \in E_2)$.

For each $i \in I$, let $M_i^* = M_i$, and for each $i \in E_2 - I$, let $M_i^* = M_i^*$. It is easy to verify that $M_i^* = U(M_i^* : i \in E_2)$ is a $K'$-hypomatching of $L_i$ such that $p \in \text{Lspan}(M_i^*)$. $\square$

$K'$-hypermatings of $L_i$

We next consider the problem of determining a $K'$-hypermating $M_i$ of $L_i$ such that the point $K'(M_i)$ is not contained in the subspace $K_1$ of $K'$. Here, we consider separately those $K'$-hypermatings which contain the line $u$, and those which do not.

First, consider those $K'$-hypermatings $M_i$ which do not contain the line $u$. Here again, we partition the subspace corresponding to the component $L_i$ by solving a single matroid intersection problem. Again, the first matroid in this intersection problem, Matroid 7, is closely related to Matroid 1. In particular, a subset $I \subseteq E_2$ is independent in Matroid 7, denoted $M_7(L, E_2, K_1)$, if there is a collection $\{M_i : i \in I\}$ of $K$-hypermatings, one from each component $L_i$ with $i \in I$, such that $r(\{K(M_i) : i \in I\}/K x K') = |I|$. The second matroid in this intersection problem, Matroid 8, is likewise closely related to Matroid 2. In particular, a subset $I \subseteq E_2$ is independent in Matroid 8, denoted $M_8(L, E_2, K_1)$, if there is a collection $\{M_i : i \in E_2\}$ with

(i) $M_i$ a $K$-hypermating of $L_i$ for each $i \in I$, and
(ii) $M_i$ a $K$-hypomatching of $L_i$ for each $i \in E_2 - I$, such that $r(U(M_i : i \in E_2)/K) = \Sigma r(M_i/K) : i \in E_2)$. 

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The following lemma illustrates the strong connection between \( r(K \times K') \)-intersections in Matroid 7 and Matroid 8 and \( K' \)-hypomatchings \( M_i \) of \( L_i \) such that \( K'(M_i) \not\in K_1 \).

**Lemma 7.** There is a \( K' \)-hypomatching \( M_i \) of \( L_i \) such that

(i) \( u \in M_i \),

(ii) \( K'(M_i) \not\in K_1 \), and

(iii) for each \( i \in E_2 \), \( M_i \cap L_4 \) is either a \( K \)-hypomatching or a \( K \)-hypomatching of \( L_4 \),

if and only if there is a \( r(K \times K') \)-intersection in \( M_7(L, E_2, K_1) \) and \( M_8(L, E_2, K_1) \).

**Proof.** It is easy to verify that if \( M_i \) is a \( K' \)-hypomatching of \( L_i \) satisfying (i) - (iii), then the subset \( I = \{i \in E_2: M_i \cap L_4 \text{ is a } K \text{-hypomatching of } L_4\} \) is a \( r(K \times K') \)-intersection in \( M_7(L, E_2, K_1) \) and \( M_8(L, E_2, K_1) \).

Now, consider a \( r(K \times K') \)-intersection \( I \) in \( M_7(L, E_2, K_1) \) and \( M_8(L, E_2, K_1) \). Since \( I \) is independent in \( M_7(L, E_2, K_1) \), there is a collection \( \{M_i: i \in I\} \) of \( K \)-hypomatchings, one from each component \( L_4 \) with \( i \in I \), such that \( r(\{K(M_i): i \in I\}/K \times K') = \|I\| \). Similarly, since \( I \) is independent in \( M_8(L, E_2, K_1) \), there is a collection \( \{M_i: i \in E_2 \} \) with

(i) \( M_i \) a \( K \)-hypomatching of \( L_4 \) for each \( i \in I \) and

(ii) \( M_i \) a \( K' \)-hypomatching of \( L_4 \) for each \( i \in E_2 - I \)

such that \( r(U(M_i: i \in E_2)/K) = \Sigma(r(M_i/K): i \in E_2) \).

For each \( i \in I \), let \( M_i^* = M_i \), and for each \( i \in E_2 - I \), let \( M_i^* = M_i^* \). It is easy to verify that \( M_i^* = U(M_i^*: i \in E_2) \) is a \( K' \)-hypomatching of \( L_i \) such that \( K'(M_i^*) \not\in K_1 \).

On the other hand, Lemma 8 shows that if the maximum cardinality of an intersection in Matroid 7 and Matroid 8 is less than \( r(K \times K') \), then there is an intermediate duality result \( \{L^*, u\} \) where \( L^* = \{L^*, K^*, P^*\} \) and \( L^* = (L_i-u)/K_1 \times K' \). In particular, this implies that if the line \( u \) is
in the $K'$-hypermatching $M_i$ of $L_i$, then we may partition the subspace corresponding to the component $L_i$ by again using Procedure 2. Before stating Lemma 8, we observe that this lemma relies on the fact that the intermediate duality result $\{L, u\}$ which gave rise to $L'$, was itself constructed from a primary duality result $L^0 = \{L, K^0, P^0\}$ via some number of applications of Step 1.1 in Procedure 2.

**Lemma 8.** If the maximum cardinality of an intersection in $M_7(L, E_2, K_1)$ and $M_8(L, E_2, K_1)$ is less than $r(KxK')$, then there is a primary duality result $L^* = \{L^*, K^*, P^*\}$, where $L^* = (L_i - u) / K_1 x K'$ is a subset of the lines of $M/K_1 x K'$, such that $u$ is exposed with respect to $L^*$.

**Proof.** Suppose that the maximum cardinality of an intersection in $M_7(L, E_2, K_1)$ and $M_8(L, E_2, K_1)$ is less than $r(KxK')$, and let $(E_{21}, E_{22})$ be the minimum rank cover constructed as in Lemma 9 of Part 1. We construct the strong duality result $L^*$ from $(E_{21}, E_{22})$ as follows. By Lemma 9 of Part 1,

$$r_7(\text{E}_{21}) = r(K^*/K_1 x K'), \tag{19}$$

where $K^* = \text{span}(U(L\text{span}(L_1) \cap K; i \in E_{21}))$ and

$$r_8(\text{E}_{22}) = r(L_0^*/K) - \Sigma(r(L_1/K) - 1; i \in E_{22}) \tag{20},$$

where $L_0^* = U(L_1; i \in E_{22})$. Combining (19) and (20), we conclude that

$$r'(L_0^*/K^*) \leq 2r(K/K') - 2r'(K^*) + d(L\text{span}(L_0^*UK^*)) + \Sigma(r(L_1/K) - 1; i \in E_{22}),$$

where $r'$ is the rank function of $M/K_1 x K'$ and where for any subspace $X$ of $V$,
0 if \( X\mathcal{N}_K \subseteq K_1 \), and
\[ d(X) = \]
1 otherwise.

Thus, since for each \( i \in \mathcal{E}_2 \) \( r(L_1/K) - 1 \) is even,
\[
\text{Lr}'(L_0^*/K^*)/2J \leq r(K/K') - r'(K^*) + \xi(\text{Lr}(L_1/K)/2J; i \in \mathcal{E}_2).
\]

Finally, by the definition of \( K^* \), we have that \( L^* = \{ L^*, K^*, P^* \} \) is a strong duality set where \( P^* \) is the partition of \( L^* \) with \( L_0^* \) in \( P^* \) and \( L_1^* = L_1/K_1 x K' \) in \( P^* \) for each \( i \in \mathcal{E}_2 \).

It remains then only to show that \( \{ L^*, u \} \) is an intermediate duality result. In fact, we show that \( L^* \) is a primary duality result and that \( u \) is exposed with respect to \( L^* \).

We first prove that \( L_0^* \) is the unique even component of \( L^* \) and \( L_0^* \) is a matching. To accomplish this, we show that \( r'(K^*) = r(K^*/K') \) as follows. Combining (19) and (20) we see that
\[
r(L_0^*/K^*UK') \leq
2r(K/K') - 2r(K^*/K') - d(K^*) + \xi(r(L_1/K) - 1; i \in \mathcal{E}_2),
\]
and hence that
\[
\text{Lr}(L_0^*/K^*UK')/2J
\]
\[
\leq r(K) - r(K^*UK') - d(K^*) + \xi(\text{Lr}(L_1/K)/2J; i \in \mathcal{E}_2).
\]

Moreover, by the definitions of \( K^* \) and \( K' \), we see that for each \( i \in \mathcal{E}_1 \cup \mathcal{E}_2 \), \( r(L_1/K^*UK') = r(L_1/K) \). Thus,
\[
v(L)
\]

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\[ \leq r(K^{*}UK') + \sum (L_i/K^{*}UK')/2L: i \in E_1UE_{21} + LR(L_0*/K^{*}UK')/2L \]

\[ \leq r(K) + \sum (L_i/K)/2L: i \in E - d(K^*) = v(L) - d(K^*), \]

from which we conclude that \( d(K^*) = 0 \) and hence that \( K^* \cap K_1 \). We see that \( L_0^* \) is an even component in \( L^* \) and that \( L_0^* \) is a matching as follows. Since \( d(K^*) = 0 \), \( L_0^* \) is an even component in the strong duality set \( \{L, K^*UK', P''\} \), where \( L_0^* \) is in \( P'' \) and for each \( i \in E_1UE_{21}, L_i \) is in \( P'' \). Thus, by Corollary 1, we conclude that \( L_{\text{span}}(L_0^*) \subseteq L_{\text{span}}(M) \) for each maximum cardinality matching \( M \) in \( L \). Thus, since \( L_0 \) is a primary duality result, we conclude that \( L_0^* \subseteq L_0^* \), the even component of \( L_0^* \). In particular, \( L_0^* \) is a matching and since \( L_{\text{span}}(L_0^*UK^*) \cap K' = L_{\text{span}}(K^*) \cap K' \subseteq K_1 \), we conclude that \( L_0^* \) is an even component of \( L^* \).

That each odd component of \( L^* \) is both \( K^* \)-hypomatchable and \( K^* \)-hypermatchable follows from the fact that \( \{L, u\} \) is an intermediate duality result. That for each point \( p \in L_{\text{span}}(L^*UK^*) \), there is a maximum cardinality matching \( M \subseteq L \) such that \( p \in L_{\text{span}}(M) \) follows from the fact that \( L_0^* \) is a primary duality result. Finally, that \( u \) is exposed with respect to \( L^* \) follows from the fact that \( u \) is exposed with respect to \( L_0^* \). Q.E.D.

**Component Formation**

We are now prepared to prove that the strong duality set \( L' = \{L+u, K', P'\} \) constructed in Step 3 of Procedure 2 is a primary duality result. Again, we presume that after constructing a sequence of intermediate duality results \( \{L^1, u\}, ..., \{L, u\} \) from the primary duality result \( L_0^* \), Procedure 2 terminates with the strong duality set \( L' \) constructed in Step 3.

**Lemma 9.** The strong duality set \( L' = \{L+u, K', P'\} \) constructed in Step 3 of Procedure 2 is a primary duality
result.

PROOF. We see that $L'$ has exactly one even component $L_0$ and that $L_0$ is a matching as follows. First, since $0 \in E_1$ and $K' \subset K$, the even component of $L$ is an even component of $L'$ which is also a matching. Moreover, by the definition of $K'$, each odd component $L_1$ of $L$ with $i \in E_1$ is an odd component of $L'$. Finally, since the line $u$ is not contained in the span of every maximum cardinality matching in $L+u$, we have by Corollary 1 that $L_i$ is an odd component in $L'$. Thus, the matching $L_0 = L_0$ is the unique even component of $L'$.

We see that there is a collection $B'$ of $v(L+u)$-matchings in $L+u$ satisfying (5) as follows. Since $(L, u)$ is an intermediate duality result, there is a collection $B$ of $v(L+u)$-matchings in $L+u$ such that for each point $p \in \text{Lspan}(L_\Omega K)$ there is a matching $M \in B$ such that $p \in \text{Lspan}(M)$. Moreover, by Lemma 4 and by Lemma 9 and Lemma 10 of Part 1, for each point $p \in \text{Lspan}(L_\Omega K)$ with $p \in \text{Lspan}(L_\Omega K')$ there is a maximum cardinality matching $M$ in $L+u$ such that $p \in \text{Lspan}(M)$. Adding these matchings to $B$ gives the desired collection $B'$.

By the definition of $K'$, each $K$-hypomatching of an odd component $L_1$ with $i \in E_1$ is also a $K'$-hypomatching of $L_1$. Likewise, each $K$-hypermatching of an odd component $L_1$ with $i \in E_1$ is also a $K'$-hypermatching of $L_1$. Thus, for each odd component $L_1$ with $i \in E_1$, $L_1$ is both $K'$-hypomatchable and $K'$-hypermatchable. It remains to show that $L_i$ is both $K'$-hypomatchable and $K'$-hypermatchable.

We see that $L_i$ is $K'$-hypomatchable as follows. We already proved that there is a collection $B'$ of $v(L+u)$-matchings in $L+u$ satisfying (5). Thus, for each point $p \in K'$ in $\text{Lspan}(L_i)$, there is a matching $M$ in $B'$ such that $p \in \text{Lspan}(M)$. By Lemma 2, $M_1 = M \cap L_i$ is either a $K'$-hypomatching or a $K'$-hypermatching of $L_i$. However, since $p \in \text{Lspan}(M)$, we conclude that $M_1$ is a $K'$-hypomatching of $L_i$.
and \( p \in \text{span}(M_i) \). As this is true for each such point \( p \), \( L_i \) is \( K' \)-hypomatchable.

We see that \( L_i \) is \( K' \)-hypermatchable as follows. Let \( B_i \) be the collection of all \( K' \)-hypermatchings of \( L_i \) and let \( K_i = \text{span}(\{K' \}(M_i): \text{Me}B_i) \). By Lemma 8, there is a primary duality result \( L^* = (L^*, K^*, P^*) \) for the set \( L^* = (L_i - u)/K_i \times K' \) of lines in \( M/K_i \times K' \) such that \( u \) is exposed with respect to \( L^* \). Moreover, by definition of \( K_i \), we have that Procedure 2 will terminate in Step 3 after at most \( |L_0^*| \) iterations. We argue that the strong duality set constructed by Procedure 2 is exactly \( L^{**} = (L^{**}, K^{**}, P^{**}) \) where \( K^{**} = 0 \) and \( P^{**} \) contains exactly one component. First, since \( L_i \) is \( K' \)-hypomatchable, we conclude that any strong duality set \( L^{***} = (L^{***}, K^{***}, P^{***}) \) must have \( K^{***} = 0 \). Next, we see that the strong duality set constructed by Procedure 2 has exactly one component as follows. Consider a maximum cardinality intersection \( I \) in \( M_1(L, u) \) and \( M_2(L, u) \). By assumption, \( |I| = r(K)+1 \) and \( |I \cap \text{Me}_1| = r(K')+1 \). In particular, \( I - E_1 \) is a \( r(K/K') \)-intersection in \( M_1(L^*, u) \) and \( M_2(L^*, u) \). We conclude then that since each element of \( E_2 \) was labeled by the intersection procedure when determining the maximum cardinality of an intersection in \( M_1(L, u) \) and \( M_2(L, u) \), each element will again be labeled by the intersection procedure when determining the maximum cardinality of an intersection in \( M_1(L^*, u) \) and \( M_2(L^*, u) \). Thus, we see that the strong duality set constructed in Step 3 of Procedure 2 has exactly one component. Finally, since \( L_i \) is \( K' \)-hypomatchable, this component is odd. We conclude that \( r(L_i/K_i \times K') = r(L_i/K') \) and hence that \( L_i \in \text{span}(L_i)(K_i \times K') \subset K' \), i.e., \( L_i \) is \( K' \)-hypermatchable.

In the next section we extend Procedure 2 to an efficient algorithm for the parity problem.
6. An algorithm for the parity problem

In this section we describe both our algorithm and the computational effort it requires.

Linear Matroid Parity Algorithm

Step 0. (Start) Let M be any (possibly empty) matching and let \( L = L(M) \).

Step 1. (Labeling)

1.0 Select a line \( u \) exposed with respect to \( L = \{L, K, P\} \). If no such line exists, go to Step 4. Otherwise, go to Step 1.1.

1.1 Determine the maximum cardinality of an intersection in \( M_1(L, u) \) and \( M_2(L, u) \). If there is a \( (r(K)+2) \)-intersection in \( M_1(L, u) \) and \( M_2(L, u) \), go to Step 2. Otherwise, if the index of the even component in \( L \) is labeled, go to Step 1.2. If it is unlabeled, go to Step 3.

1.2 Let I be a maximum cardinality intersection in \( M_1(L, u) \) and \( M_2(L, u) \) with \( 0 \in I \). Augment I to obtain a \( (V(L)+1) \)-near-matching \( N \) with 2-circuit C. Revise the strong duality set \( L \) as follows. Set \( K = \text{span}(KU(C(l); l \in CNL_0)) \). Add the lines in \( CNL_0 \) to \( P \) as trivial components and set \( L_0 = L_0-C \). If \( r(u/K) = 2 \), return to Step 1.1. Otherwise, add \( u \) to \( P \) as a trivial component and return to Step 1.0.

Step 2. (Augmentation) Let I be a \( (r(K)+2) \)-intersection in \( M_1(L, u) \) and \( M_2(L, u) \). Augment I to obtain a \( (V(L)+1) \)-matching M in \( L+u \). Set \( L = L(M) \) and return to Step 1.0.

Step 3. (Component Formation) Let \( (E_1, E_2+u) \) be the minimum rank cover of \( E+u \) determined by the intersection
procedure, where we are adding with respect to \( M_i(L, u) \). Revise the strong duality set \( L \) as follows. Set \( K = \text{span}(U(\text{span}(L_i) \setminus \{e_i\})) \) and combine \( u \) together with the components with indices in \( E_1 \) into a new non-trivial odd component \( L_i \) in \( P \). Add each line \( l \in L \) such that \( r(l/K) = 2 \) and \( r((L_i+l)/K) = r(L_i/K) \) to \( L_i \), and return to Step 1.0.

**Step 4. (Maximality)** The matching \( M \) is a maximum cardinality matching. Construct the strong duality set \( L \) as follows. For each line \( l \in L \), if \( r((L_0+l)/K) = r(L_0/K) \), then add \( l \) to \( L_0 \). Otherwise, add \( l \) to \( P \) as a trivial component. 0

Let us consider the complexity of the algorithm. For an \( m \times 2n \) matching matrix \( A \), there can be at most \( O(m) \) augmentations. Moreover, between successive augmentations, each application of Step 1.1 leads to either a \((v(L)+1)\)-near-matching \( N \) or to component formation. In the former case, the cardinality of \( L_0 \) is reduced in Step 1.2. In the latter case, the rank of \( K \) is reduced in Step 3. Thus, overall, there can be no more than \( O(m^2) \) applications of Step 1. As implemented here, each application of Step 1 requires \( O(m^2 n) \) steps to determine the maximum cardinality of an intersection and \( O(m^2 n) \) steps each time Step 1.2 is invoked. Hence altogether Step 1 requires \( O(m^2 n) \) steps.

Each augmentation requires \( O(m^2 n) \) steps and as there can be at most \( O(m) \) augmentations, Step 2 contributes \( O(m^2 n) \) steps overall.

Finally, each component formation requires \( O(mn) \) operations both to determine the labeled elements and determine the lines in the span of the new component. As there can be at most \( O(m^2) \) components formed, Step 3 contributes \( O(m^2 n) \) operations overall.
Thus, the total complexity of the algorithm as described here is $O(m^n)$ operations. This bound is determined the effort required for $O(m^2)$ applications of Step 1.2. We note that it is possible determine the new intermediate duality result without constructing the $(v(L)+1)$-near-matching N. In particular, it is possible to determine CNLo directly from the intersection procedure. Thus, we can reduce the overall complexity of the algorithm to $O(m^n)$. 
CHAPTER 4:

On a 'Primal' Matroid Intersection Algorithm
Abstract

Given two matroids $M_1 = (S, I_1)$ and $M_2 = (S, I_2)$ and a weight function $s$ on $S$, the weighted matroid intersection problem is to find a common independent set of maximum weight. In this paper, we present Lawler's 'primal' algorithm for matroid intersection, and we give an elementary 'dual' proof of the algorithm's correctness.
1. Introduction

Given two matroids $M_1 = (S, I_1)$ and $M_2 = (S, I_2)$ and a
weight function $w$ on $S$, the weighted matroid intersection
problem is to find a common independent set of maximum
weight. As is well known (see, for example, Lawler (1976)),
the weighted matroid intersection problem generalizes the
optimal assignment problem and the maximum weight directed
spanning tree problem. Edmonds (1969) developed an efficient
algorithm and an elegant duality theory for this problem.
Since then, other authors including Lawler (1975), Krogdahl
(1975), Iri and Tomizawa (1976), and Frank (1981) have
developed other algorithms and alternative proofs of this
duality theorem.

In this note we consider Lawler's (1976) primal
parametric algorithm and Frank's (1981) primal-dual approach
for solving the weighted matroid intersection problem. In
particular, we reinterpret Lawler's augmenting path
procedure as a longest path procedure on an auxiliary graph
of the type developed by Fujishige (1977) for solving a
generalization of matroid intersection. We then use Frank's
method of 'weight-splitting' to obtain the dual variables
for the matroid intersection problem from the dual variables
of the longest path problem.

As in Lawler's algorithm, the dual variables are not
needed to compute the augmenting paths. In this sense,
Lawler's algorithm is 'primal' rather than 'primal-dual'.
However, we do exploit properties of the dual variables to
give a shorter and perhaps more intuitive proof of the
correctness of Lawler's algorithm.

Our results rely on some well known properties of
shortest-path algorithms and on some elementary results
concerning the greedy algorithm for matroids. (See, for
example, Edmonds (1971)). We do not assume any previous knowledge of matroid intersection. However, to make this note self-contained we have included some lemmas of Frank and one of Frank's proofs. Moreover, our notation and definitions closely follow that of Frank.
2. Preliminaries

For each subset $X \subseteq S$, the weight of $X$ is $s(X) = \sum_{x \in X} s(x)$. If $F$ is a family of subsets of $S$, we say that $F \subseteq F$ is $s$-maximal in $F$ if $s(F) \geq s(X)$ for all $X \subseteq F$.

For a given matroid $M = (S, I)$, let $I^k = \{I : I \subseteq I, |I| = k\}$. Thus, $I^k$ is the collection of independent sets of cardinality $k$. If $I \subseteq I$ and $I + x \in I$, then we let $C(I, x)$ denote the unique circuit in $I + x$. For any subset $X \subseteq S$, we let $r(X)$ denote the rank of $X$, i.e., $r(X) = \max\{|I| : I \subseteq S, I \subseteq I\}$.

The first lemma below is Frank's restatement of Edmonds' greedy algorithm.

**Lemma 1.** $I \subseteq I^k$ is $s$-maximal in $I^k$ if and only if both (i) and (ii) are true:

(i) If $x \in I$ and $I + x \in I$, then $s(x) \leq s(y)$ for all $y \in C(I, x)$.

(ii) If $x \in I$ and $I + x \in I$, then $s(x) \leq s(y)$ for all $y \in I$.

We say that $(s_1, s_2)$ is a weight-splitting of $s$ if $s_1(x) + s_2(x) = s(x)$ for all $x \in S$. Frank introduced the concept of weight-splitting to help simplify the matroidal duality theory. The motivation for weight-splitting is inherent in Lemma 2 below.

Let $I_{12}^k = I_1^k \cap I_2^k$.

**Lemma 2.** Suppose that $I \subseteq I_{12}^k$. Then $I$ is $s$-maximal in $I_{12}^k$ if and only if there is a weight-splitting $(s_1, s_2)$ of $s$ so that:

(i) $I$ is $s_1$-maximal in $I_1^k$, and

(ii) $I$ is $s_2$-maximal in $I_2^k$.

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PROOF. The 'if' part is elementary. The 'only if' part is a corollary of Theorem 1 below, whose proof is based on the augmenting path algorithm. □

As one final preliminary, we state another of Frank’s lemmas and include his original proof. Hereafter, for I∈I, we let \( \text{span}(I) = \{x \in S : I + x \notin I\} \).

**Lemma 3.** Let I be s-maximal in \( I^k \). Suppose that \( x_1, \ldots, x_t \) are distinct elements of \( S - I \) and that \( y_1, \ldots, y_t \) are distinct elements of I such that:

(i) Either \( I + x_i \notin I \) or \( y_i \notin C(I, x_i) \),
(ii) \( s(x_i) = s(y_i) \), and
(iii) if \( s(y_i) = s(y_j) \) and \( i < j \), then \( y_i \notin C(I, x_j) \).

Then \( I' = I - \{y_1, \ldots, y_t\} U \{x_1, \ldots, x_t\} \) is also s-maximal in \( I^k \). Moreover, if \( I + x_i \notin I \) for each \( i \), then \( \text{span}(I) = \text{span}(I') \).

**Proof.** Since \( s(I') = s(I) \) it suffices to prove that \( I' \in I \). We prove this inductively on t. The case \( t = 1 \) is trivial, so we assume that \( t > 1 \). Let \( y_r \) be the least index element with the property \( s(y_r) \leq s(y_j) \) for \( 1 \leq j \leq t \). Thus, by our choice of \( r \) and by (iii) above it follows that \( y_r \notin C(I, x_j) \) for \( j \neq r \). Otherwise, either \( s(x_j) = s(y_j) = s(y_r) \) and (iii) is violated, or else \( s(x_j) = s(y_j) > s(y_r) \) in which case (i) of Lemma 1 is violated.

Now, the inductive hypothesis holds for \( I_r = I - y_r + x_r \) and \( x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_t, y_1, \ldots, y_{r-1}, y_{r+1}, \ldots, y_t \), from which the lemma follows. □
3. The Augmenting Path Algorithm

The 'augmenting path' procedure below is initialized with any set \( I \) that is s-maximal in \( I_{i} \). (A suitable first choice is \( I = \emptyset \) and \( k = 0 \)). In the case that \( I \) is not maximum cardinality in \( I_{i} \), the procedure then determines a set \( I' \in I_{i} \) and a weight-splitting \((s_1, s_2)\) of \( s \) such that:

1. \( I \) is \( s_i \)-maximal in \( I_i \) for \( i = 1, 2 \), and
2. \( I' \) is \( s_i \)-maximal in \( I_{i+1} \) for \( i = 1, 2 \).

By Lemma 2, it follows that \( I' \) is \( s \)-maximal for \( I_{i+1} \).

To determine a maximal augmentation for an intersection \( I \), we first construct an auxiliary digraph \( G(I) \) of the type developed by Fujishige (1977). The vertex set of \( G(I) \) is \( V = \{x^1 : x \in S \text{ and } i = 1, 2\} \cup \{u^1, u^2\} \), where \( u^1 \) and \( u^2 \) are distinguished vertices. The edge set \( E(I) \) of \( G(I) \) and the associated edge distances \( d \) are as follows.

1. For each \( y \in I \), \((y^2, y^1) \in E(I)\) with length \( d(y^2, y^1) = -s(y) \).
2. For each \( x \in S - I \), \((x^1, x^2) \in E(I)\) with length \( d(x^1, x^2) = s(x) \).

The remaining edges all have length 0.

3. For each \( y \in I \), \((y^1, u^1) \in E(I)\) and \((u^2, y^2) \in E(I)\).
4. For each \( x \in S - I \), if \( I + x \in I_1 \) then \((u^1, x^1) \in E(I)\); otherwise, \((y^1, x^1) \in E(I)\) for each \( y \in C_1(I, x) \).
5. For each \( x \in S - I \), if \( I + x \in I_2 \) then \((x^2, u^2) \in E(I)\); otherwise, \((x^2, y^2) \in E(I)\) for each \( y \in C_2(I, x) \).

This auxiliary digraph (illustrated in Fig. 1) is different from the graph defined by Lawler. We leave it to the reader to verify that the maximum distance path from \( u^1 \) to \( u^2 \) as determined by a dynamic programming recursion algorithm leads to exactly the same construction as Lawler's algorithm.
FIGURE 1

THE AUXILIARY DIGRAPH $G(I)$
Let \( P = \{u^1, x_0^1, x_0^2, y_1^2, y_1^1, x_1^1, x_1^2, \ldots, y_n^1, x_n^2, u^2\} \) be the maximum distance path from \( u^1 \) to \( u^2 \) in \( G(I) \), and among all such paths let \( P \) have the fewest number of edges. (We may determine \( P \) as per Lawler (1976)). Let \( \Delta(v) \) be the maximum distance of a path from \( u^1 \) to \( v \) in \( G(I) \) for all \( v \in V \). If there is no path from \( u^1 \) to \( u^2 \) in \( G(I) \), we let \((S_1, S_2)\) be an element splitting of \( S \), where \( S_2 \) is the set elements \( x \in S \) such that there is a directed path from \( u^1 \) to \( x \) in \( G(I) \), and \( S_1 = S - S_2 \).

If there is a path from \( u^1 \) to \( u^2 \) in \( G(I) \), we define the weight-splitting \((s_1, s_2)\) and the new intersection \( I' \) in terms of \( P \) and \( \Delta \) as follows:

\[
\begin{align*}
    s_1(x) &= -\Delta(x^1) \text{ for } x \in S, \\
    s_2(x) &= \Delta(x^2) \text{ for } x \in S, \\
    I' &= IV\{x_0, \ldots, x_n\} - \{y_1, \ldots, y_n\}.
\end{align*}
\]

**THEOREM 1.** Let \( I \) be \( s \)-maximal in \( I_{1^k} \). If there is no path from \( u^1 \) to \( u^2 \) in \( G(I) \), then \((S_1, S_2)\) is an element splitting of \( S \) for which \( r_1(S_1) + r_2(S_2) = |I| \), and thus \( I \) is a maximum cardinality intersection. If, on the other hand, there is a path from \( u^1 \) to \( u^2 \) in \( G(I) \), then:

\[
\begin{enumerate}
    \item \((s_1, s_2)\) is a weight-splitting,
    \item \( I \) is \( s_i \)-maximal in \( I_{k} \) for \( i = 1, 2 \) and
    \item \( I' \) is \( s_i \)-maximal in \( I_{k+1} \) for \( i = 1, 2 \).
\end{enumerate}
\]

**PROOF.** Suppose there is no path from \( u^1 \) to \( u^2 \) in \( G(I) \). In this case the theorem is a restatement of Edmonds' (1971) duality result for the maximum cardinality matroid intersection problem, and our proof is essentially the same. First, we observe that for any intersection \( I' \), \( r_1(S_1) + r_2(S_2) \geq r_1(S_1 \cap I') + r_2(S_2 \cap I') = |I'| \). Now, let \( I_1 = S_1 \cap I \) and let \( I_2 = S_2 \cap I \). Clearly, \( r_1(I_1) + r_2(I_2) = |I_1| + |I_2| = |I| \).

Consider first \( x \in S_1 - I_1 \) and \( y \in S_2 - I_2 \). Since there is no path from \( u^1 \) to \( x^1 \), but there is a path from \( u^1 \) to \( y^1 \) in
\( G(I) \), it follows that \((y^1, x^1) \in E(I)\). Thus, \( y \in C_1(I, x) \). We conclude that \( C_1(I, x) \cap N_{I_2} = \emptyset \), and thus \( x \in \text{span}_1(I_1) \). It follows that \( r_1(S_1) = r_1(I_1) \).

Consider next \( x \in S_2 - I_2 \) and \( y \in I_1 \). Since \((x^1, x^2) \in E(I)\), and \((y^2, y^1) \in E(I)\), it follows that there is a path from \( u^1 \) to \( x^2 \), but no path from \( u^1 \) to \( y^2 \) in \( G(I) \). Thus, \((x^2, y^2) \notin E(I)\). We conclude that \( C_2(I, x) \cap N_{I_1} = \emptyset \), and thus \( x \in \text{span}_2(I_2) \). It follows that \( r_2(S_2) = r_2(I_2) \). We have thus demonstrated that \( I \) is a maximum cardinality intersection.

Suppose on the other hand that there is a path from \( u^1 \) to \( u^2 \) in \( G(I) \). Then it is easy to see that there is a path in \( G(I) \) from \( u^1 \) to every vertex in \( V \). Let us now assume that there are no positive length directed circuits in \( G(I) \). We shall later prove that this is in fact true.

We see that \((s_1, s_2)\) is a weight-splitting as follows. For all \( y \in I_1 \), \( y^2 \) immediately precedes \( y^1 \) on the maximum length path from \( u^1 \) to \( y^1 \). Thus, \( \Delta(y^1) = \Delta(y^2) - s(y) \), and \( s(y) = s_1(y) + s_2(y) \). For each \( x \in S - I \), \( x^1 \) immediately precedes \( x^2 \) on the maximum length path from \( u^1 \) to \( x^2 \). Thus, \( \Delta(x^2) = \Delta(x^1) + s(x) \), and \( s_1(x) + s_2(x) = s(x) \).

We see that \( I \) is \( s_1 \)-maximal in \( I_{1^*} \) as follows. If \( x \in S - I \) and \( y \in C_1(I, x) \), then \( d(y^1, x^1) = 0 \) and thus \( \Delta(y^1) \leq \Delta(x^1) \). Therefore, \( s_1(x) \leq s_1(y) \), and condition (i) of Lemma 1 is satisfied. If \( x \in S - I \), \( I + x \in I_1 \) and \( y \in I_* \), then \( d(y^1, u^1) = d(u^1, x^1) = 0 \). Therefore, \( \Delta(y^1) \leq \Delta(u^1) \leq \Delta(x^1) \), and \( s_1(x) \leq s_1(y) \). In this case condition (ii) of Lemma 1 is satisfied.

Similarly, it is easy to see that \( I \) is \( s_2 \)-maximal in \( I_{2^*} \).

We see that \( I' \) is \( s_1 \)-maximal in \( I_{1^{k+1}} \) as follows. First of all, \( I + x_0 \) is \( s_1 \)-maximal in \( I_{1^{k+1}} \) because \( I \) is
$s_1$-maximal in $I_1^k$ and $s_1(x_0) = 0 \geq s_1(x')$ for all $x' \in S - I$. Moreover, $s_1(I') = s_1(I + x_0)$. Thus it suffices to show that $I' \in I_1$. It is easy to verify that the sequence $x_1, y_1, \ldots, x_n, y_n$ satisfies the conditions of Lemma 3 with respect to $I + x_0$. Condition (iii) must be satisfied because $P$ has a minimum number of edges among maximum length paths from $u^1$ to $u^2$. Thus, $I' \in I_1$.

Similarly, it is easy to see that $I'$ is $s_2$-maximal in $I_2^{k+1}$.

We conclude not only that $I'$ is $s$-maximal in $I_1^{2k+1}$, but also that any $s$-maximal intersection in $I_1^{2k+1}$ is $s_i$-maximal in $I_i^{k+1}$ for $i = 1, 2$.

Now, we prove that if $I$ is $s$-maximal in $I_1^k$, then there is no positive length directed circuit in the auxiliary digraph $G(I)$.

First, we initialize the algorithm with $I = \emptyset$. It is easy to see that $G(\emptyset)$ contains no positive length directed circuit; in fact it contains no directed circuits at all.

Now, suppose for some integer $k$, if $I^k$ is $s$-maximal in $I_1^{2k}$ then $G(I^k)$ contains no positive length directed circuit. We show that if $I^k$ is not a maximum cardinality intersection, then $G(I^{k+1})$ contains no positive length directed circuit, for any $I^{k+1}$ $s$-maximal in $I_1^{2k+1}$.

For each $x \in S$ and $i = 1, 2$, let $l(x^i) = -\Delta(x^i)$. In addition, let $l(u^1) = \min\{l(y^1) : y \in I^{k+1}\}$ and let $l(u^2) = \max\{l(y^2) : y \in I^{k+1}\}$. It is easy to verify that $l$ is a feasible solution to the dual of the maximum length path problem (between any pair of vertices) in $G(I^{k+1})$, i.e., $l$ satisfies:

$$l(v) - l(v') \geq d(v, v') \text{ for each edge } (v, v') \in E(I^{k+1}).$$
Hence, every primal maximum length path problem in $G(I^{k+1})$ is either infeasible (i.e., there is no path between the two distinguished vertices) or has a bounded optimal solution (i.e., the maximum length path between the two vertices has finite length). We conclude, then, that $G(I^{k+1})$ can have no positive length directed circuit. □
4. Summary

In this paper, we have extended Frank's technique of weight-splitting to develop 'dual variables' for Lawler's primal-parametric weighted matroid intersection algorithm. (Actually, we did not produce the dual variables; instead, they are implicit in that they can be obtained from the two maximum weight independent set problems using weights $s_1$ and $s_2$. See Frank (1981) for more details). The algorithm can also be used to give a proof of Edmonds' matroid polyhedral intersection theorem. (Once again we refer the reader to Frank (1981)).

We believe that the use of Fujishige's auxiliary graph in Lawler's algorithm has several advantages. Most significantly, from our perspective, the proof of correctness is straightforward, albeit somewhat involved. In particular, our proof relies only on some elementary properties of shortest path procedures and of the greedy algorithm for matroids.

Secondly, this auxiliary graph helps highlight the differences between the two matroids. For example, our construction of the weight-splitting has a natural interpretation in terms of the auxiliary graph.

Thirdly, this auxiliary graph also may be an appropriate framework from which to analyze other algorithms for the weighted matroid intersection problem. For example, one can describe Edmonds' (1968) algorithm for maximum weight branchings as a primal parametric simplex pivoting procedure, where pivots are carried out on the auxiliary graph. We leave the details to the interested reader.
References


Gabow, H. and M. Stallman, (1994), An Augmenting Path Algorithm for the Parity Problem on Linear Matroids, to
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