FREGE'S PROBLEM ABOUT CONCEPTS

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ABSTRACT

Singular terms refer to objects, but what, if anything, do general terms refer to? Frege's celebrated distinction between concepts and objects gives rise to a difficult problem. According to Frege's theory, predicates refer to concepts, no concept is an object, and singular terms like 'the president of the United States' refer to objects. It is thus a consequence of Frege's theory that the concept horse is not a concept.

In Chapter I I introduce the problem and argue that Frege was wrong to dismiss it as a mere "awkwardness of language". In Chapter II I examine solutions proposed by Frege, Michael Dummett, Edwin Martin, and Montgomery Furth and conclude that none works. In Chapter III I examine Frege's Begriffsschrift, written before he developed his theory about concepts. In the course of my discussion I demonstrate the consistency of its logical system, which has been considered problematic. I also argue that problems with its interpretation foreshadow the problem about concepts.

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CHAPTER I  INTRODUCTION: THE PROBLEM

Frege's philosophy of language may be viewed as consisting of two intimately related theories: a theory of syntax and a theory of semantics. The syntactic theory concerns the expressions of language and provides an analysis of the structure of sentences and their parts. It describes how expressions fit together to form more complex expressions. The semantic theory concerns the meanings of expressions and describes how the meanings of sentence parts determine the meanings and truth-values of a sentence.

A striking feature of Frege's philosophy of language is the way in which the semantic theory parallels the syntactic theory. The syntactic theory divides expressions into two mutually exclusive categories and the semantic theory divides the meanings of expressions into two corresponding categories. Expressions are either complete or incomplete. Complete expressions have complete meanings and incomplete expressions have incomplete meanings. The distinction between complete and incomplete expressions and the corresponding distinction for their meanings are responsible for an interesting problem in Frege's philosophy of language. Before I state the problem I shall sketch the two theories and then develop those aspects of the theories that are relevant to my discussion of the problem about concepts.

Meaningful expressions are either proper names or
function names. Proper names are complete expressions and function names are incomplete expressions. Function names are incomplete in that they have holes or gaps for names of arguments. When these gaps are filled by appropriate argument names the result is a complete expression, or proper name. Among function names are predicates or "concept words". The completion of a concept word or predicate by a proper name is a sentence. Thus sentences are counted as proper names.

The function name '( )+3', for example, is completed by the proper name '2' to yield the proper name '2+3'. The predicate '( ) is a city' is completed by the proper name 'Paris' to yield the sentence 'Paris is a city'.

Frege distinguishes different levels of function names. A function name is of the first level if its argument place accepts proper names. A function name is of the \((n+1)st\) level if its argument place accepts \(n\)th level function names.

Frege makes two important distinctions in his semantic theory. One is between sense and reference. Generally, each sentence part and each sentence has a sense and a reference. The sense and reference of a sentence are determined by the senses and references of its parts. The senses and references of sentence parts fit together in a
way that parallels the way in which sentence parts fit together to yield a sentence. ¹

The other distinction is between object and function and corresponds to the syntactic distinction between complete and incomplete expressions. The theory divides the senses and references of expressions into two mutually exclusive categories. Just as expressions are complete or incomplete, so are the senses and references of expressions. Complete expressions have complete senses and references and incomplete expressions have incomplete senses and references. The senses and references of complete expressions are objects and the senses and references of incomplete expressions are functions. Thus functions are incomplete and objects are complete.

Proper names refer to objects and function names refer to functions. Moreover, objects are referred to only by proper names and functions are referred to only by function names. No complete expression refers to any incomplete entity and no incomplete expression refers to any complete entity. I am primarily concerned with the distinction between functions and objects in the realm of reference. I am especially interested in Frege's doctrines that predicates refer to concepts and concepts are incomplete or "unsaturated".

¹. In the body of the thesis I say little about sense. This is not the distinction with which I am concerned.
There are levels of functions corresponding to the levels of function names. A function is of the first level if it takes objects as arguments. A function is of the (n+1)st level if it takes nth level functions as arguments. Functions yield values for all arguments of the appropriate type and the value of a function for a given argument of the appropriate type is always an object. Among functions are concepts and among objects are truth-values. Frege argues that truth-values are objects as follows:

A statement contains no empty place, and therefore we must regard what it stands for as an object. But what a statement stands for is a truth-value. Thus the two truth-values are objects.  

Frege argues here that truth-values are objects because they are referred to by complete expressions. It seems that an object, for Frege, is anything referred to by a proper name.

The notion of object and function are correlative notions. In the following passage, Frege attempts to define 'object':

...the question arises what it is that we are here calling an object. I regard a regular definition as impossible, since we have something too simple to admit of logical analysis. It is only possible to indicate what is meant. Here I can only say briefly: An object is anything that is not a function, so that an expression for it does not contain an empty place.  

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3. Ibid.
Objects, then, are entities referred to by complete expressions, or proper names. At the same time, 'proper name' is defined as follows:

I call anything a proper name if it is a sign for an object.

Frege's explanation of what a concept is is not much more enlightening:

Kerry contests what he calls my definition of 'concept'. I would remark, in the first place, that my explanation is not meant as a proper definition. One cannot require that everything shall be defined, anymore than one can require that a chemist shall decompose every substance. What is simple cannot be decomposed, and what is logically simple cannot have a proper definition.

He later says:

The concept is predicative. It is, in fact, the reference of a grammatical predicate.

Thus the notions of concept and object are primitive, correlative notions. In what follows I attempt to clarify Frege's characterizations of objects, functions, and concepts.

We have seen that the sentence 'Paris is a city' is the completion of the predicate '( ) is a city' by the

4. Geach and Black, p. 47.
5. Ibid., p. 43.
6. Ibid.
proper name 'Paris'. The reference of the sentence is determined by the references of its parts in the following sense. The reference of '( ) is a city' is the concept city and the reference of 'Paris' is Paris. The value of the concept city for the argument Paris is the True just in case Paris falls under the concept city. The reference of the sentence is the value of the concept city for the argument Paris. A first-level concept is a first-level function that takes objects as arguments and yields the True or the False as value. The value of a concept for a given object as argument is the True if that object falls under the concept; otherwise it is the False. Hence, since Paris does fall under the concept city, the reference of the sentence is the True.

To each function there corresponds an object called its "value-range". The value of a function for an argument is, according to Frege, "the result of completing the function with the argument". 7

If we write

\[ x^2 - 4x = x(x-4) \]

we have not put one function equal to another, but only the values of one equal to another... But we can also say: 'the value range of the function \( x(x-4) \) is equal to that of the function \( x^2-4x \)', and here we have an equality between ranges of values. 8

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7. Ibid., p. 25.
He goes on to say that "value ranges of functions are objects, whereas functions themselves are not." A concept is a function whose value is always a truth-value and the value-range of a concept is its extension. The extension of a concept may be thought of as the collection of all and only those objects that fall under the concept. Thus, the extension of the concept city, for example, is the class of all cities. Since extensions are objects, the extension of a concept is not to be identified with the concept.

Objects and concepts are fundamentally different and hence concepts cannot stand in the same relations as objects. Identity is a relation that may only hold between objects; however, there is an analogous relation that may hold between concepts. A concept A is the same as a concept B if and only if every object that falls under A falls under B and every object that falls under B falls under A. That is, two concepts are the same if and only if their extensions are identical.

Frege provides a principle of substitutivity for predicates or concept-words. If two predicates have the same

9. Ibid., p. 32.

10. The is a somewhat oversimplified interpretation of Frege's notion of extension. I give a more detailed explanation later in the section on Furth but for my present purpose this will suffice.
reference, they may be substituted for one another preserving truth-value:

Just as proper names can replace one another salva veritate, so can two concept-words, if their extensions are the same. 11

So far I have said little about Frege's distinctions between complete and incomplete entities. The corresponding distinction for expressions is clearer. An incomplete expression contains one or more gaps as argument places, while complete expressions have no gaps. So, 'Frege', 'the even prime number', and 'Massachusetts' are complete expressions, while '( ) is happy', 'the capital of ( )', and '( )^2' are incomplete. It should be noted that Frege counts any expression of the form 'the so-and-so' as a proper name. He says, "The singular definite article always indicates an object, whereas the indefinite article accompanies a concept-word". 12 This distinction between complete and incomplete entities is not as clear and Frege's discussion of it tends to be metaphorical. I shall not examine these metaphors at this point. It suffices, for my present purpose, to point out that Frege does make a strict division between two kinds of entities and that this division corresponds to the syntactic division. Objects are referred to by

12. Geach and Black, p. 45.
proper names and are called "complete" or "saturated". Functions and concepts are referred to by function names and predicates and are called "incomplete" or "unsaturated". Frege does talk as if these latter entities contain gaps that correspond to the argument-places of their names. Thus this semantic division appears to be derived from the syntactic division. It is important to keep in mind that no gapless, complete expression has an incomplete reference. The theory precludes the possibility of a proper name referring to a function or concept.

I shall now give a preliminary statement of the problem with which I am concerned. According to Frege's theory, the predicate 'is a horse' refers to a concept. If asked, "Which concept?", one's answer would be, "the concept horse". Frege would say that

Secretariat is a horse

says of Secretariat that he falls under the concept horse.

However, by Frege's criteria, 'the concept horse' is a proper name. If it has a reference, its reference is an object, not a concept. It follows, then, that

The concept horse is not a concept, but rather an object.

This is paradoxical. Given any predicate or concept word, it seems to be impossible to specify which concept it refers
Attempts to name a concept result in naming an object. This is a problem that Frege discusses in "On Concept and Object". In that article he replies:

It must be recognized that here we are confronted by an awkwardness of language, which I admit cannot be avoided, if we say that the concept horse is not a concept, whereas, e.g., the city of Berlin is a city, and the volcano Vesuvius is a volcano. Language is here in a predicament that justifies the departure from custom.13

The problem, of course, can be avoided if the references of function names and predicates are taken to be objects. Why did Frege think that the reference of a predicate must be incomplete or "unsaturated". Frege never really answers this question, but his writings do provide a few clues. In "On Concept and Object" he writes:

Somebody may think that this is an artificially created difficulty; that there is no need at all to take account of what I call a concept; that one might, like Kerry, regard an object's falling under a concept as a relation in which the same thing could occur now as object, now as concept. The words 'object' and 'concept' would then serve only to indicate the different position in the relation. This may be done, but anybody who thinks the difficulty is avoided in this way is very much mistaken; it is only shifted. For not all the parts of a thought can be complete; at least one must be 'unsaturated' or predicative; otherwise they would not hold together.14

13. Ibid., p. 46.
Here Frege starts out to explain why the reference of a predicate must be unsaturated, yet he shifts, instead, to talk about the sense of a predicate. He argues that at least one part of the sense of a sentence must be unsaturated, for otherwise the thought does not "hold together". He continues:

For example the sense of the phrase 'the number 2' does not hold together with that of the expression 'the concept prime number' without a link. We apply such a link in the sentence 'the number 2 falls under the concept prime number'; it is contained in the words 'falls under', which need to be completed in two ways — by a subject and an accusative; and only because their sense is thus 'unsaturated' are they capable of serving as a link. Only when they have been supplemented in this two-fold respect do we get a complete sense, a thought. I say that such words or phrases stand for a relation. 15

It does not follow from anything Frege says here that the reference of a predicate must be unsaturated. According to Frege, being unsaturated or incomplete is essential to the function, but he has not told us why. He has argued that the sense of a sentence has senses as parts and that not all of the parts of a thought can be complete. At least one part of a complete thought must be unsaturated in order for the parts to adhere to one another. Does Frege intend a similar explanation for references of sentence parts? There is some evidence that Frege does believe that the

15. Ibid.
reference of part of a sentence must be unsaturated in order for the reference of the whole to be complete. For example, when discussing the references of concept words he writes:

This predicative component of our sentence...is also meaningful...We call it a concept-word... Just as it itself appears unsaturated, there is also something in the realm of [references] corresponding to it; we call this a concept. This unsaturatedness of one of the components is necessary, since otherwise the parts do not hold together. Of course two complete wholes can stand in a relation to one and another, but then this relation is a third element - and one that is doubly unsaturated. 16

Here Frege does offer the same sort of explanation for the unsaturatedness of the references of incomplete expressions that he offers for the senses of those expressions. He argues that two complete entities cannot"adhere" or "hold together" to form a complete, whole reference. This talk is, however, metaphorical and not a wholly satisfying response to our question. Frege's argument here is also puzzling, for the reference of a sentence is a truth-value. Truth-values are not composed of parts in the way that thoughts are. Thus, talk about the references of sentence parts adhering to one another to yield a complete reference seems irrelevant.

Frege offers another sort of explanation for the incompleteness of references of function names:

16. Frege, Posthumous Writings, p. 177.
I am concerned to show that the argument does not belong with the function, but goes together with the function to make a complete whole for the function by itself must be called incomplete in need of supplementation, or 'unsaturated'. And in this respect functions differ fundamentally from numbers.17

This passage suggests that the incompleteness of the function is derived from the incompleteness of the expressions that refer to them. There is a strain of this throughout Frege's writings about concepts and functions. At times it is explicit:

Accordingly I call the function itself unsaturated or in need of supplementation because its name has first to be completed with the sign of an argument if we are to obtain a reference that is complete in itself.18

We have seen that Frege's semantic theory strictly parallels his syntactic theory. Here it looks as if the syntax, in a sense, "determines" the semantics. In particular, since predicates and function names have gaps or holes, so do their references. On Frege's view, the function and argument fit together in just the same way the function name fits together with the argument name. While Frege never really explains why the references of predicates must be incomplete, the doctrine of unsaturated reference is central to his theory about concepts. The following passage provides a

17. Geach and Black, p. 24.
18. Frege, Posthumous Writings, p. 119.
good statement of his view:

... the unsaturatedness of the concept brings it about that the object, in effecting the saturation, engages immediately with the concept, without need of any special cement. Object and concept are fundamentally made for each other...\textsuperscript{19}

Given Frege's characterization of functions and objects, the question arises whether function names and predicates are functions or objects. Frege regards the sentence

> John met Mary

as the completion of the predicate expression '\((\ )\) met Mary' by the proper name 'John'. On Frege's view, are we to regard the expression 'met Mary' as a function or an object?

P.T. Geach\textsuperscript{20} maintains that Frege regards function names as linguistic functions. Accordingly, the sentence 'John met Mary' is taken to be the value of the function 'met Mary' for the proper name 'John' as argument. Thus the sentences

> John met Mary
> Smith met Mary
> Jones met Mary

are values of the same function for the arguments 'John', 'Smith', and 'Jones', respectively. Geach's view is an

\textsuperscript{19} Ibid., p. 178.

attractive one when one considers sentences like

John met Mary at the theater.

One way in which this sentence may be viewed by Frege is as the completion of the predicate expression

John met ____ at the theater

by the proper name 'Mary'. According to Frege's theory, this incomplete expression cannot be counted as an object, for it is not a complete or "saturated" entity. It is not simply a sequence of words; it contains a gap. Hence predicate expressions and function names cannot generally be taken to be objects. It is in keeping with Frege's characterization of functions and objects to view function names and predicates as functions. Of course, if predicates are functions, they cannot be referred to by proper names.

I can now state the problem in which I am interested in a fuller and more precise way. According to Frege, 'is a horse' refers to a concept. Which concept? The concept horse. But 'the concept horse' is a proper name and hence it does not refer to a concept. It follows, then, that

(1) The concept horse is not a concept

and
(2) 'Is a horse' does not refer to the concept horse.

A parallel argument yields

(3) The reference of 'is a horse' is not a concept

and

(4) 'Is a horse' does not refer to the reference of 'is a horse'.

Each of (1)-(4) follows from yet conflicts with Frege's theory. It seems that any attempt to specify the reference of a predicate leads to a contradiction.

The problem is worse yet. Not only are we unable to talk about particular concepts other than by using predicate expressions, but it seems that we are unable to make general statements about concepts and functions. Consider:

(5) ________ is a concept.

For (5) to be completed so that the result is grammatical, it must be completed with a proper name. That is, 'is a concept' is a first-level function name that refers to a concept that only accepts objects as arguments. All grammatical completions of (5) are false. Hence we cannot truthfully say of any entity that it is a concept. Now consider, for instance,
(6) All concepts are functions
(7) All concepts are unsaturated

and

(8) No concept is an object.

We cannot express (6)-(3) as

(6') (\forall x) (x is a concept \Rightarrow x is a function)
(7') (\forall x) (x is a concept \Rightarrow x is saturated)

and

(8') (\forall x) (x is a concept \Rightarrow x is an object).

(6')-(8') are trivially true on Frege's theory because every grammatical completion of the predicates 'is a concept', 'is a function', and 'is unsaturated' is false. If we add the claim, '(\exists x) (x is a concept)', to each of (6')-(8') then due to the peculiarity of the predicate 'is a concept', the results are false. Since 'is a concept' is a first-level predicate, general talk about concepts requires first-order quantification; however, because concepts are not objects, first-order quantificational logic is not adequate for the expression of (6)-(8). It seems that Frege's theory does not allow the formulation of its own doctrines.

Frege recognizes that his doctrines about functions, concepts, and objects are problematic, yet he writes:
I admit that there is quite a peculiar obstacle in the way of an understanding with my reader. By a kind of necessity of language, my expressions, taken literally, sometimes miss my thought; I mention an object, when what I intend is a concept. I fully realize that in such cases I was relying upon a reader who would be ready to meet me half-way — who does not begrudge me a pinch of salt.\textsuperscript{21}

In what follows I examine the solutions to this problem offered by several readers of Frege's work who do not "begrudge a pinch of salt". I maintain, however, that Frege was wrong to dismiss the problem as a mere "awkwardness of language" and I argue that none of the solutions proposed by his sympathetic readers works.

\textsuperscript{21} Geach and Black, p. 54.
CHAPTER II: FOUR SOLUTIONS TO THE PROBLEM

1. Frege's solution

Frege must have realized that his published response to the problem is not a satisfactory one, for in a later essay he again deals with the problem and proposes what is taken to be a solution. In that unpublished essay, which has been translated as "Comments on Sense and Meaning", Frege recognizes that we seem to be barred from specifying or naming the concept that corresponds to a given concept word or predicate. He says that, properly, the expression 'the reference of the concept word A' should be rejected as inadequate, and he proposes:

It would be better to say "what the concept-word A refers to"; then this is (if need be) to be used predicatively: "Jesus is what the concept-word 'man' refers to" in the sense of "Jesus is a man".¹

What is Frege's proposal? He is attempting to find an expression that has the same reference as, say, 'man', and yet retains the predicative nature of the predicate expression 'is a man'.² We have seen that 'the

¹ Frege, Posthumous Writings, p. 122.
² According to Frege, the copula is a mere sign of predication. The copula does not accompany a concept word. See Frege, Posthumous Writings, p. 177 and Geach and Black, p. 43.
concept *man* and "the reference of 'man'" fail to meet the requirements, so Frege proposes that the expression

(a) what the concept-word 'man' refers to

has the requisite incompleteness characteristic of predicate expressions. Furthermore, (a) has the same reference as

(b) is a man.

Thus, given Frege's principle of substitutivity for concept words,

(3) Jesus is what 'man' refers to

has the same reference as

(4) Jesus is a man.

Now, we are instructed, we can avoid the use of inadequate, misleading expressions such as 'the reference of 'horse'" by using 'what 'horse' refers to' in its stead. When asked "Which concept does the concept word A refer to?" we may respond, "What the concept word A refers to".

Sentence (3), however, requires further examination; for its truth appears to conflict with certain doctrines of Frege's theory. In his essay "Schroeder's Vorlesungen Ueber die Algebra der Logik", Frege is careful to point out that those objects that fall under a given concept are not named
by the corresponding concept word.

The word 'common name' is confusing here, for it makes it look as though the common name stood in the same, or much the same, relation to the objects that fall under the concept as the proper name does to a single object. Nothing could be more false!  

This point is amplified later:

The word 'planet' has no direct relation at all to the Earth, but only to a concept that the Earth, among other things, falls under; thus its relation to the Earth is an indirect one, by way of the concept.

Hence, 'man' bears no direct relation to Jesus or any other man. Any relation that does obtain between the word 'man' and any particular man is mediated by a certain concept.

I have said that these passages appear to conflict with the truth of (3). What does the conflict consist in? It is clear that Frege would affirm

(5) 'Man' does not refer to Jesus.

'Man' does have a reference, but its reference is not an object and Jesus is an object. To put this another way,

(6) Jesus is not the reference of 'man'

3. Geach and Black, p. 105.
4. Ibid.
is true. Yet (6) contains the misleading expression 'the reference of 'man'' . If (6) is well-formed, it denies an identity between two objects.

According to Frege's prescription, the occurrence of 'the reference of 'man'' in (6) should be replaced by the expression 'what 'man' refers to'. Thus we obtain

(7) Jesus is not what 'man' refers to.

But (7) contradicts (3). What has happened? If (5) is equivalent to (7) then Frege has been caught in a contradiction.

There is a way out for Frege. He may deny the equivalence of (5) and (7). He might claim that (7) is ambiguous. If it is, then what are its various readings? The 'is' in (7) may be either the 'is' of identity or the 'is' of predication. Suppose, first, that it is the 'is' of identity. Then (7) amounts to

(8) Jesus ≠ what 'man refers to.

If (8) is well-formed, then, since identity can only be meaningfully asserted of objects, 'what 'man' refers to' must be a proper name and refer to an object. But if 'what 'man' refers to' refers to an object, then

(9) What 'man' refers to is an object

must be true. Yet on Frege's theory, (9) is false. Hence,
Frege must regard (8) as false.

Suppose, then, that the 'is' in (7) is the 'is' of predication. Then (7) says of Jesus that he does not fall under a certain concept. Which concept? The concept referred to by 'what 'man' refers to'. In this case, 'what 'man' refers to' is to be taken as a predicate. Frege would say that it refers to, what we improperly call, the concept man. That is, Frege would argue that (7) just means

(10) Jesus is not a man.

So, on this construal, (7) is false.

Therefore, either 'what 'man' refers to' does not refer to a concept, or (7) is false. But we have seen that if 'what 'man' refers to' is taken to be a proper name (7) is false. Thus there is no contradiction. Frege can argue that (7) is not equivalent to (5) and that (7) is the negation of (3).

So, Frege's proposal does not conflict with the doctrines of his theory. The expression 'what 'man' refers to' can be used predicatively, and, in sentence (3), it has the same reference as 'is a man'. Yet, has Frege accomplished anything? If he is merely looking for an expression to replace 'the concept man' and 'the reference of 'man'' that refers to a concept and retains the appropriate incompleteness of a predicate, then he already has one. The predicate 'is a man'
satisfies those criteria. How does the introduction of 'what man' refers to' constitute an advance toward a solution to the problem of specifying the references of predicates? We can now say

(11) Smith is what 'man' refers to

and

(12) Jones is what 'man' refers to

where the "what"-phrase occurs predicatively in (11) and (12). The 'is' that occurs in (11) and (12) is the 'is' of predication. But we already had the means to express (11) and (12). Why should those sentences be preferable to

(13) Smith is a man

and

(14) Jones is a man?

Frege must have thought that his introduction of "what"-phrases accomplishes more than this. After making his proposal to use "what"-phrases, he goes on to say:

If we keep all this in mind, we are in a good position to assert, "What two concept-words refer to is the same if and only if the associated extensions coincide."

without being led astray into errors by the improper use of the word "same".

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5. Frege, *Posthumous Writings*, p. 122
I gather, then, that Frege believes that the introduction of these "what"-phrases enables us to more precisely formulate definitions and principles concerning concepts. Remember that identity is a relation that obtains only between objects. No relation that obtains between objects obtains between concepts. The expression 'is the same as' refers to a relation that holds between concepts. Frege has given a shorthand definition of 'is the same as' in the passage cited above. I shall attempt to state the definition more fully, using the prescribed "what"-phrases.

(S) What concept word A refers to is the same as what concept word B refers to if and only if the extension of what concept word A refers to is identical with the extension of what concept word B refers to.

If we examine (S), it appears that the expression 'what the concept word A refers to' and 'what the concept word B refers to' occur as proper names. Otherwise (S) would fail to be complete and well-formed. Consider a particular instance of (S):

(15) What 'even' refers to is the same as what 'divisible by two' refers to.

Now, if 'what 'even' refers to' and 'what 'divisible by two' refers to' occur predicatively in (15), then (15) would not be a well-formed sentence. The expression 'is the same as' would be flanked by incomplete expressions and hence the
whole would fail to be complete. Roughly, (15) would have the form of:

\[ \phi(\ ) \text{ is the same as } \psi(\ ). \]

The problem is a general one, and I believe that Frege's introduction of "what"-phrases is of no help. We are told that 'what 'horse' refers to' is to be used predicatively; it refers to a concept. That is,

(16) 'What 'horse' refers to' refers to a concept.

Equivalently,

(17) What 'horse' refers to is a concept.

But 'what 'horse' refers to' occurs as a proper name in (17). Consider:

(18) _______ is a concept.

For (18) to be completed so that the result is grammatical, it must be completed with a proper name. That is, 'is a concept' is a first-level function name and it refers to a concept that only accepts objects as arguments. Thus if (17) is well-formed it is false. I conclude from this that 'what 'horse' refers to' is ambiguous. It may occur in a sentence either as a proper name or as a concept name.

According to Frege 'what 'horse' refers to' has
the same reference as 'is a horse', and so may be substituted for it preserving truth-value. Yet it is not the case that we may always substitute 'is a horse' for 'what 'horse' refers to'. For example, (17) is false but it is a well-formed sentence. If we substitute 'is a horse' for 'what 'horse' refers to' in (17) the result is

(19) Is a horse is a concept.

(19) is nonsensical. Thus the expressions are not generally co-referential. I am assuming a certain principle of substitutivity here that Frege never explicitly states. Namely that if two expressions are co-referential, they may be substituted for one another in a sentence preserving grammaticality. Frege does, of course, state that co-referential expressions may be substituted for one another preserving truth-value, and the analogue for grammaticality is, I believe, a weaker principle. For how could an ungrammatical sentence have any truth-value?

A completion of (18) is grammatical just in case the blank is filled by a proper name. Yet if the blank is filled by a proper name the result is always false. This is, of course, just another version of the problem with the concept _horse_. It seems that we cannot say truthfully of any entity that is a concept.

The problem has not been solved by Frege's proposal. Frege has exploited the fact that these "what"-phrases are
ambiguous. He was aware that proper names may occur as concept words, and concept words as proper names. We have seen that "what"-phrases may occur either as predicates or as proper names in sentences.

For each predicate expression, 'is an A', there is a corresponding "what"-phrase, 'what the concept word 'A' refers to'. If the corresponding "what"-phrases can occur predicatively only in those contexts that will grammatically accept the straightforward predicate expression, 'is an A', then there does not appear to be any advantage to using the corresponding "what"-phrase in its place. Furthermore, it does seem to be the case that 'what the concept word A refers to' may occur predicatively only in those contexts that grammatically accept 'is an A'. In those contexts that accept the "what"-phrase but do not accept 'is an A', 'what A refers to' occurs as a proper name. (S) and (17) are examples of this.
2. Dummett's solution

Frege recognizes that his theory seems to prevent us from naming concepts, and I have found his attempt to solve the problem to be unsatisfactory, for the problem merely emerges in a different guise. It remains that we are forced to use proper names to refer to concepts. Frege's theory does not provide an alternative.

Frege also recognizes the problem with 'is a concept' and 'is a function'. They are inappropriate for their use because they refer to first-level concepts. Suitable predicates must be of the second-level, for they must accept first-level function names in their argument places. He says:

In the proposition 'something is an object', the word 'something' takes an argument of the first kind and stands for a proper name. Thus whatever we put in place of 'something' we always get a true proposition; for a function name cannot take the place of 'something'. Here we find ourselves in a situation where the nature of language forces us to make use of imprecise expressions. The proposition 'A is a function' is such an expression: it is always imprecise; for 'A' stands for a proper name. The concept of a function must be a second-level concept. While I am writing this I am well aware of having again expressed myself imprecisely. Sometimes this is just unavoidable. All that matters is that we know that we are doing it, and how it happens.¹

Thus Frege is aware of the problem yet he offers no solution; while he suggests that the suitable predicates must be of the

second level, he does not provide any alternatives to the unsuitable predicates.

Michael Dummett believes that Frege does resolve the paradox. He concentrates on two of the symptoms I have mentioned. First, that we cannot truthfully say of any entity that it is a concept, and second, that we cannot specify the reference of any given predicate. He does not deal with the difficulty of expressing generalities about concepts, but I presume that he believes that his solution to the general problem treats all the symptoms. His discussion is an elaboration and development of Frege's suggestions.

Dummett begins by noting that Frege is aware of the peculiar feature of the predicate 'is a concept'. In accordance with Frege's remarks, Dummett proposes to expel the word 'concept' and he plans to construct a suitable predicate to replace it. He does not argue in favor of its expulsion; he merely says that since its use leads to difficulties, we should not use it. Whereas Frege says that its use is sometimes unavoidable, Dummett believes that it can be uniformly replaced by a second-level predicate. There are other predicates that cause trouble for Frege's theory and presumably Dummett wants to expel these, too. Thus Dummett must both formulate a general way to name first-level concepts so that their names may fill the argument place of a second-level predicate, and construct a replacement for
'is a concept'. For this project Dummett relies heavily on Frege's proposal to use "what"-phrases in place of the "misleading" proper names as concept names.

I have already shown that "what"-phrases may be used predicatively, yet I have argued that they are ambiguous and may occur as proper names. Dummett holds that "what"-phrases of the form 'what A refers to' where, A is a concept word, may only occur predicatively. In support of this he implicitly invokes two principles, one for proper names and an analogous one for concept names. He notes that all instances of the following principle are true:

(PN) If \( \alpha \) is a referring proper name, then \( \alpha \) and \( \text{what } \alpha \text{ refers to} \) are co-referential.

For example, 'Paris' and 'what 'Paris' refers to' both refer to Paris. By analogy, Dummett argues, all instances of the corresponding principle for concept names are true:

(CN) if \( \phi \) is a referring concept name, the \( \phi \) and \( \text{what } \phi \text{ refers to} \) are co-referential.

Hence, Dummett concludes, 'is a horse' and 'what 'is a horse' refers to' are co-referential and completely interchangeable with one another. My counterexample to (CN) was

(1) What 'is a horse' refers to is a concept.

(1) is a counterexample only if we allow 'is a concept' as a legitimate predicate. It is not difficult, however, to find
other counterexamples, for just as proper names may occur as concept names, so "what"-phrases may occur as proper names. Consider, again, for example,

(2) What 'is even' refers to is the same as what 'is divisible by two' refers to.

If (2) is a grammatical, complete sentence, then the "what"-phrases occur there as proper names; for if they do not occur as proper names (2) would contain gaps and hence would not be a well-formed sentence. Perhaps Dummett would argue that 'is the same as' is "misleading" in the same way that 'is a concept' and 'is a function' are. However, then it would look as though he wants to say that wherever "what"-phrases occur as proper names there is something "wrong" with the predicate that occurs in that context.

So far Dummett has just followed Frege. As Frege prescribes, predicates such as 'is a concept' should not be used and "what"-phrases are to be used predicatively to replace the problematic proper names such as 'the concept horse' and 'the reference of 'is a horse'''. Yet this does not constitute a solution to the problem of naming concepts. Even if we allow Frege and Dummett these steps, we still lack the means for specifying the reference of a given predicate. Though "what"-phrases may indeed be used predicatively it is not obvious that the introduction of these expressions is an advance toward a solution.
It remains for Dummett to replace 'is a concept' with a suitable second-level predicate and construct a means by which we may name concepts. Frege does not provide a replacement for 'is a concept', though he does claim that the problem does not exist in his symbolic language:

In a conceptual notation we can introduce a precise expression for what we mean when we call something a function (of the first level with one argument), e.g.: 'f(ε)'. Accordingly, 'f(ε • 3 + 4)' would express precisely what is expressed imprecisely in the proposition 'x • 3 + 4 is a function'. Whatever we put in place of f( ), we always get a true proposition because we can only put names of functions of the first level with one arguments for the argument place here of the second kind.  

To devise such an expression of natural language, Dummett examines the use of "what"-clauses (relative clauses). He says that we may use relative clauses such as 'what I am not' to form first-level predicates. We merely supply an argument-place and copula. Thus, for example, '... is what I am not' is a first-level predicate that takes proper names as arguments. The predicate may be completed with the proper name 'Smith' to form the sentence 'Smith is what I am not'. Dummett plans to use this feature of relative clauses to form second-level predicates. He says:

There is, however, another use of relative clauses which represents quantification over concepts. On this use, which also employs the verb 'to be', what is formed is a second-level predicate.  

2. Ibid.

If we are to form second-level predicates which accept first-level concept names as arguments, then in order to complete these predicates we must be able to construct expressions that are appropriate for their argument places. Dummett gives detailed instructions for this. In the simplest case, a first-level predicate is formed from a sentence by omitting an occurrence of a proper name. For instance, from the sentences 'Smith is a doctor' and 'Jones is sad' we form the first-level predicates 'is a doctor' and 'is sad' by omitting the occurrences of 'Smith' and 'Jones' respectively. Now we can form what Dummett calls "predicative expressions" from the predicates by merely dropping the copula. The results are 'a doctor' and 'sad'. These, Dummett claims, are predicative and are suitable arguments for second-level predicates.

One might object here that Dummett ignores Frege's dictum that predicate expressions are incomplete and come equipped with gaps, and that 'a doctor' and 'sad' are not predicative, in Frege's sense, for they are not supplied with argument places. Although Dummett does not recognize this, he could respond that Frege would count these expressions as predicative and that they do refer to concepts. Frege provides a sort of test for concepthood:

I regard it as essential for a concept that the question whether something falls under it have a sense..."  

Dummett could argue that it does make sense to ask of an object whether it is a doctor or whether it is sad, so for the time being we shall count 'a doctor' and 'sad', and the like, as predicative expressions where the argument places are implicit. There are clear cases where such expressions are predicative. Consider, for example.

(3) A whale is a mammal.

'A whale' occurs predicatively in (3) and refers to a first-level concept. The sentence may be symbolized as:

(3') (x) (x is a whale ⊃ x is a mammal)

This rendering of (3) makes the argument-place that is implicit in the English sentence explicit.

Dummett's next step, then, is to construct sentences that use relative clauses as second-level predicates. These predicates are to accept first-level predicative expressions. The second-level predicates are relative clauses prefixed with a form of 'to be'. Dummett provides several examples of sentences which are supposedly second-level predicates completed by first-level predicative expressions. I shall give his analysis of these examples. The first three sentences are the following:

(4) A poet is what Blake was but Hayley was not
(5) Unhappy is what all Rumanians seem to be
(6) Underpaid is what Peter does not want to be.
In each of these sentences Dummett says that the "what"-clause is a second-level predicate, the 'is' represents the relation of a first-level concept falling under a second-level concept, and the expressions 'a poet', 'unhappy', and 'underpaid' refer to first-level concepts. Thus Dummett represents the "what"-clauses

(4a) is what Blake is but Hayley was not
(5a) is what all Rumanians seem to be
(6a) is what Peter does not want to be

as the open sentences

(4b) \( \phi (\text{Blake}) \) but not \( \phi (\text{Hayley}) \)
(5b) \( (x) (x \text{ is a Rumanian} \supset \text{it seems that} \phi (x)) \)
(6b) Peter does not want to be \( \phi (\text{Peter}) \)

respectively. (4b)-(6b) contain argument places that are to be filled by first-level predicates. Thus Dummett regards (4)-(6) as completions of (4b)-(6b).

Sentences (4)-(6) are a bit misleading. Dummett does not want to say that being a poet is the only property that Blake and Hayley fail to share, or that being underpaid is the only thing Peter dislikes. The confusion might be lessened if Dummett had used the word 'something' instead of 'what'. Thus
(4*) A poet is something Blake was but Hayley was not
(5*) Unhappy is something all Rumanians seem to be
(6*) Underpaid is something Peter does not want to be
are clearer than (4)-(6).

Dummett has here attempted to give examples of second-level predicates that have been constructed according to the formula he has provided. As we have seen, first-level quantification is not generally adequate for talk about concepts, for if our quantifiers range over just objects, then since objects and concepts are such different sorts of entities, it is not surprising that we cannot talk about concepts. So Dummett concludes that the answer is to use second-level predication to talk about concepts, functions, and their properties. Frege has suggested that using "what"-phrases might be the solution to the problem of talking about concepts and Dummett has tried to develop Frege's suggestion to use "what"-clauses predicatively. We have seen that "what"-clauses may be used in place of first-level predicates. Dummett argues that they may also be used for second-level predication. However, the examples he has supplied do not essentially involve second-level predication. That is, each of (4)-(6) is equivalent to a sentence which only involves first-level predication and does not contain a "what"-clause.

Sentences (4)-(6) are equivalent to
Blake was a poet but Hayley was not
All Rumanians seem to be unhappy
Peter does not want to be underpaid

respectively. Dummett says that the difference between (4) and (4') is "simply that the former emphasizes that the sentence is being regarded as the result of filling the second-level predicate 'φ (Blake) and not φ (Hayley)' by the first-level predicate 'ξ was a poet'."

What is gained by regarding (4)-(6) as involving second-level predication? Dummett's aim is to formulate an appropriate predicate to replace 'is a concept. He chooses:

(C) ...is something that everything either is or is not.

The gap is to be filled by a first-level predicative expression. Dummett represents (C) as the open sentence:

(C') (x) ( φ (x) or ¬φ(x))

where 'φ' represents the argument place. Thus, he argues that (C) is a second-level predicate, a completion of which is:

(7) A horse is something that everything either is or is not.

5. Dummett, p. 216.
(7), then, replaces the problematic 'The concept horse is a concept' and may be symbolized as

\[(7') (x) (x \text{ is a horse or } -x \text{ is a horse})\].

But (7') need not be viewed as involving second-level predication. (7) is just a cumbersome way of saying:

\[(8) \text{ Everything either is a horse or is not.}\]

Just as with (4)-(6), the relative clause may be eliminated in favor of straightforward first-level predication. If (7) and (8) are ways of saying that the concept horse is a concept, then we do not need second-level predication to do so. We may easily regard \((C')\) as a schema of which (7') is an instance. Rather than regarding '\(\phi\)' as representing an argument place we may regard \(\phi\) as a schematic letter. (7) is no more or less about a concept than (8) is. According to Frege we refer to concepts by predicates. If one regards (7) as being about a concept, then he would regard (8) as being about a concept, too.

The other half of Dummett's project is to show that his new terminology provides a way to specify the references of particular predicates. The example he provides is:

\[(9) \text{ A philosopher is what 'is a philosopher' refers to.}\]

I assume that Dummett must regard 'a philosopher' as a first-level predicative expression and 'is what 'is a philosopher'
refers to' as a first-level predicate. The 'is', then, must represent the relation of coincidence of extension, and hence if (9) is true, the two expressions are co-referential. However, if we substitute one for the other the result is:

(10) What 'is a philosopher' refers to is a philosopher.

Sentence (10) gives rise to the question, "Which philosopher?" while Dummett claims that (9) does not.

Dummett says that (9) may be expressed more informatively as:

(11) What 'is a philosopher' refers to is what Socrates and Plato both were

or

(12) 'Is a philosopher' refers to what Plato and Socrates both were.

Again, (12) is clearer if we substitute 'something' for 'what'. But (11) and (12) are roundabout ways of saying:

(13) Plato and Socrates were both philosophers

just as

(14) Plato is what 'is a philosopher' refers to

is a roundabout way of saying

(15) Plato is a philosopher.
Thus the "what"-clause in (11) and (12) can be eliminated.

A solution to the problem of talking about concepts and functions might involve the use of second-level predication; however, Dummett's proposal to use "what"-clauses as second-level predicates fares no better than Frege's suggestion to use "what"-phrases predicatively. All of the examples Dummett has provided of sentences with second-level "what"-clauses turn out to be equivalent to sentences that involve only first-level predication. Even the predicate he offers as a replacement for 'is a concept' need not be regarded as being of the second level. What can Dummett think is gained by what he calls his "reconstructed terminology"? Furthermore, Dummett gives no clue as to how he would handle the other "unsuitable" predicates such as 'is a function' and 'is unsaturated'. If we are to regard them as illegitimate, then, unless the predicates are replaced by other language, we lack the means to express generalities about concepts and functions.

I have examined Dummett's proposal to use "what"-clauses as second-level predicates and I have found it to be unsatisfactory for several reasons. First, the examples he provides of sentences containing occurrences of second-level "what"-clauses do not essentially involve second-level predication.
What I mean by this is that in each case the "what"-clause can be eliminated in favor of first-level predication. Each of these sentences can be construed as the completion of a second-level predicate by a first-level predicate, but it need not be. Second, sentences containing an occurrence of Dummett's replacement for the unsuitable 'is a concept' can be construed as first-order sentences. In fact, one could argue that it is more natural to do so. Finally, Dummett's view of the problem is far too narrow. He sees the problem primarily as a problem with the specification of the references of predicates, yet the specification problem is merely one symptom of a more general problem. A critical symptom of the problem is that we seem to be unable to state grammatically generalities about concepts and functions. The result is that there seems to be no way to formulate the general metaphysical doctrines of Frege's theory. Dunnett does not appear to recognize this and his analysis neither deals with this symptom nor does it lend itself to an expansion which might treat it.

I would like, however, to look more closely at Dunnett's proposal to use "what"-clauses as second-level predicates in an attempt to determine what Dunnett believes he has gained by their introduction. Dunnett is aware that the examples he discusses can be paraphrased so to eliminate second-level predication, for he writes:
The difference between the form 'A poet is what Blake was but Hayley was not' and the simpler 'Blake was a poet and Hayley was not' is thus simply that the former emphasizes that the sentence is being regarded as the filling of the second-level predicate \( \phi(\text{Blake}) \) and not \( \phi(\text{Hayley}) \) by the first-level predicate 'is a poet'.

Frege does hold that sentences admit various readings. In "On Concept and Object" he states:

...a thought can be split up in many ways so that now one thing, now another, appears as subject and predicate...It need not then surprise us that the same sentence may be conceived as an assertion about a concept and also about an object; only we must observe that what is asserted is different.

Thus while I do not wish to argue with Dummett on the point that sentences (4)-(6) can be viewed as second-level predicates completed by first-level predicates, it is important to ask: What do we gain by doing so and what role do the "what"-clauses play? I think that this question can be, at least partially, answered by an examination of a passage from the Begriffsschrift. Frege says there:

Since the sign \( \phi \) occurs in the expression \( \phi(A) \) and since we can imagine that it is replaced by other signs \( \psi \) and \( \chi \), which would express other

6. Ibid.

7. Geach and Black, p. 49.
functions of the argument \( A \), we can also regard \( \phi(A) \) as a function of the argument \( \phi \).

A little later, in a section on generality, Frege says:

In the expression of a judgment we can always regard the combination of signs to the right of \( \vdash \) as a function of the signs occurring in it. If we replace this argument by a German letter and if in the content stroke we introduce a concavity with this German letter in it, as in

\[
\vdash \phi(a),
\]

This stands for the judgment that whatever we may take for its argument, the function is a fact. Since a letter used as a sign for a function such as \( \phi \) in \( \phi(A) \), can itself be regarded as the argument of a function, its place can be taken, in the manner just specified by a German letter.

Frege's position can best be understood by considering a particular sentence, say,

(1) Socrates is a philosopher.

According to Frege, (1) can be regarded in two different ways, and depending upon how it is regarded it will lead to different generalizations. First, the sentence can be seen as the completion of a first-level predicate by a proper name.

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9. Ibid.
The predicate is 'is a philosopher' and the proper name is 'Socrates'. In this case Socrates is the argument. The sentence, then, is taken to be a completion of the open sentence

\[ x \text{ is a philosopher} \]

where 'Socrates' fills the argument place. Viewed in this way the sentence leads to the existential generalization

\[(2) \ (\exists x) (x \text{ is a philosopher}),\]

that is, something is a philosopher, or there is something that falls under the concept philosopher.

Frege maintains that an alternative way of regarding sentence (1) is to take the concept as the argument. He says that the concept may be viewed as the argument because we can imagine its name replaced by other signs, say 'is wise' or 'is a man'. Taken in this way the sentence is seen as a completion of the open sentence.

Socrates \( F \)

where 'is a philosopher' fills the argument place. On this view, the sentence leads to the existential generalization

\[(3) \ (\exists F) (\text{Socrates} \ F).\]

The English translation of (3) might be 'Socrates is something' or 'There is some concept under which Socrates falls'. There are some problems with finding a suitable translation in
natural language, but I will return to this point later.

Thus Frege maintains that the sentence may be regarded in either way, and, depending upon what is taken to be the argument, the sentence will lead to different generalizations. While the first reading of sentence (1) leads to a standard and uncontroversial existential quantification, the second leads to what is called a "second-order" quantification. Although Frege does not question the use of a second-order quantification, its use is controversial.

Given Frege's view that a sentence such as (1) may be regarded either as being about an object or about a concept, it is a bit clearer what Dummett intends the introduction of "what"-clauses to accomplish. While 'A poet is what Blake was but Hayley was not' merely means 'Blake was a poet and Hayley was not', the former is phrased so as to indicate that the concept poet is to be taken as the argument. Thus the sentence is viewed as the completion of a second-level predicate by a first-level predicate. In other words, the former is to be viewed as a completion of the open sentence

\[ F(\text{Blake}) \text{ and not } F(\text{Hayley}) \]

where 'F' is not viewed as a schematic letter but rather as a quantifiable variable. Thus this form leads us to the existential generalization
\( (\exists F) (F(\text{Blake}) \text{ and not } F(\text{Hayley})) \).

This form of the sentence emphasizes that the concept poet is taken to be the argument. That is, the sentence is about the concept. The simpler form, 'Blake was a poet and Hayley was not', does not have this emphasis.

But I believe that Dummett's "what"-phrases accomplish no more than providing emphasis. Sentences containing occurrences of "what"-clauses are merely stylistic variations of equivalent sentences without them. We did not have to rephrase sentence (1) as

\[ (4) \text{ A philosopher is what Socrates is} \]

in order to view the sentence as being about a concept. For Frege, (3) is as direct a generalization of (1) as (2) is. We do not have to mediate the transition to (3) by first rephrasing (1) as (4). Thus while Dummett may have provided a way to emphasize the considerations involved in moves like the one from (1) to (3) the emphasis provided by the use of "what"-clauses is certainly not necessary. According to Frege the simpler sentence 'Blake was a poet and Hayley was not' implies the existential generalization no less than the complicated form does. Emphasis is the only thing that recommends one form over another. On Frege's view, predicates refer to concepts and can be regarded as names of arguments.
We may regard sentence (1) as having the concept philosopher as subject without moving the predicative expression 'a philosopher' into the grammatical subject position.

Before I continue my examination of Dummett's analysis I think that it would be instructive to take a brief look at the controversy concerning the use of second-order quantification. My purpose is not to question the legitimacy of its use, but rather to examine the controversy in order to gain some insight into the implications of and the underlying assumptions involved in the use of second-order quantification. In particular it would be useful to look at W.V. Quine's objections to second-order quantification.

What I mean by "second-order" quantification is quantification involving predicate variables. The controversy is over whether predicate positions are accessible to quantified variables. Predicate variables are generally taken to range over attributes or properties. Quine's position is that while the move from sentence (1) to sentence (2) is a legitimate one, the move from (1) to (3) is not. The latter move involves viewing the predicate as being in a position that is accessible to a quantified variable. For Frege the move from (1) to (3) is made for exactly the same considerations as the move from (1) to (2). If we view the concept philosopher as argument, generalization yields (3).

Quine's objections to second-order quantification are
made on grammatical grounds. According to Quine, substituting a quantifiable variable for an expression is tantamount to taking that expression to be a name, that is to assuming that it has a reference. I am not interested here in questioning the existence of Fregean concepts. Rather, I am interested in the role that second-order quantification plays in Frege's theory and in the grammaticality of second-order existence claims.

In *Philosophy of Logic* Quine argues that only names can stand in place of a variable in an open sentence and that predicates are not names. Hence, we should not use predicate letters as variables. Quine points out that some philosophers who use second-order quantification do so out of confusion between use and mention. The confused second-order logician takes the quantifiers '(\exists F)' and '(F)' to range over predicate expressions rather than over those entities that predicates name, if predicates name anything at all. However, Frege, whom Quine calls the "prodigal logician", considers predicates to be names; they name concepts. Frege takes '(\exists F)' and '(F)' to range over concepts and functions and accordingly, they are read as 'There is a concept such that...' and 'All concepts are such that...'.

Later Quine argues that these entities over which second-order quantifiers are taken to range cannot be adequately individuated. The quantifiers '(\exists F)' and '(F)' range over attributes and properties and while sets are individuated by
the law of extensionality, this is not so for attributes and properties. He claims "open sentences never determine two sets, but may determine two attributes". This is not true, though, of Fregean concepts. Concepts are, indeed, individuated by a law of extensionality. Identity is a relation that may hold only between objects; however, there is an analogous relation that may hold between concepts. We say that concept A is the same as concept B if and only if every object that falls under A falls under B, and every object that falls under B falls under A. That is, two concepts are the same if and only if their extensions are identical. Two predicate expressions may refer to the same concept yet differ in sense. Thus Frege is not injured by Quine's objections so far. For Frege, predicates are names, in Quine's sense.

In *Methods of Logic* Quine distinguishes between concrete and abstract terms and singular and general terms. He says:

Those who draw a distinction between classes and attributes will see in 'humanity' a name of an attribute and in 'mankind' a name of a class, the class of all objects that partake of the attribute humanity. Both terms are abstract singular terms as opposed to the concrete general term 'man' or 'human'. This general term has the class mankind as its extension.


Although Frege does not explicitly draw these distinctions, any singular term, whether it be concrete or abstract, counts as a Fregean proper name. What Quine calls a "general term" is what I have called a "concept name". A difference between Quine and Frege is that while Frege considers concept words or predicates to be names, Quine does not. Quine argues that singular terms occur in positions that are accessible to variables while general terms do not. We have seen that Frege considers both proper names and predicates to be referring expressions, and hence either can be replaced by a variable. Quine's position is that since predicates do not refer to one thing at a time in the way that singular terms do, we cannot make sense of open sentences such as

Socrates is an x

or

All x are mortal

and therefore it is improper to imbed such open sentences in quantifications. Quine suggests that such open sentences be rephrased in terms of class membership, where the variable of the open sentence stands in place of an abstract singular term. For instance, in the open sentence

Socrates is a member of x

'x' is in a position accessible to the abstract singular term
'mankind' or 'the class of philosophers'. Quine insists, however, that the predicate letter 'F' just is not a variable, but rather a schematic letter. If the values of 'F' are taken to be classes or sets, then Quine urges switching over to the language of set theory, for otherwise what is being said is obscured. If one wants to admit classes as values for variables of quantification, he should write 'xey' rather than 'Fx'.

The major point of disagreement between Quine and Frege is on the syntactic and semantic role of the predicate. Frege views predicates as referring expressions and Quine does not. For Frege, predicates do occur in "name" positions, and they name concepts. A predicate is not a singular term but it does refer to one thing at a time. For Quine, predicates attach to singular terms and yield truths and falsehoods. Frege wants to quantify over those entities over which second-order quantifiers range, while Quine denies their existence. Indeed, in order for Frege to formulate his doctrines about concepts and functions he needs the machinery that will allow him to quantify over them. Rephrasing sentences in terms of class membership is not adequate for Frege; class names are Fregean proper names and hence refer to objects rather than concepts. No singular term, or proper name, refers to a concept.

Dummett's attempt to resolve the problem of specifying the reference of a given predicate is a digression from his
discussion of Frege's ascription of reference to predicates.

On page 245 of his book Frege: The Philosophy of Language, Dummett states:

There can be no reservation whatever about the existence of concepts, relations, and functions provided that we are prepared to admit second-level quantification.

This is the conclusion he reaches after examining Frege's thesis that incomplete expressions have reference. For the time being, I am concerned with neither the legitimacy of second-order quantification nor with the question of whether concepts exist. Instead I am concerned with whether there is a solution to the problem of talking about concepts; for if the theory bars us from talking about concepts, the theory is rendered untenable. Dummett believes that the answer is to be found in the relation between Frege's thesis that incomplete expressions refer and second-order quantification. When Dummett returns to the scrutiny of Frege's thesis that predicates have reference, he begins by analyzing the ascription of reference to a particular predicate. The example he chooses is:

(5) There is something that 'is a philosopher' refers to.

Dummett claims that if second-order quantification is accepted as legitimate then (5) is true. He argues as follows. He points out that 'something' in (5) must be construed as
a second-order quantifier since 'is a philosopher' does not refer to any object. Thus, Dummett continues, (5) is equivalent to

(6) There is such a thing as what 'is a philosopher' refers to,

where the "what"-clause in (6) is a predicate that refers to a concept. He then argues that (6) merely means

(7) There is such a thing as being a philosopher.

He goes on to say that (7) "must be admitted as both intelligible and true by anyone who allows any form of second-level quantification." Hence, since (7) is equivalent to (5), Dummett concludes, (5) is true provided that second-order quantification is accepted as legitimate. Later he draws the following conclusion from this argument:

We have reduced the assertion that a given predicate has a reference to a banality exactly parallel to that to which the ascription of a reference to a proper name amounts.

I should explain that Dummett maintains that to ascribe a reference to a given proper name, say 'Mount Everest', is to affirm

(8) There is such a thing as Mount Everest.

Analogously, to ascribe a reference to the predicate 'is a philosopher' is to affirm (7).

Before I continue to discuss Dummett's views on the relation between second-order quantification and Frege's thesis that predicates refer, I would like to take a closer look at the argument I have just outlined. I believe that Dummett is correct in saying that 'something' in (5) must be construed as a second-order quantifier, for, according to Frege, no predicate refers to any object, and first-order quantifiers range over objects. However, consider (6). Dummett has previously argued that any "what"-clause of the form "what φ refers to", where φ is a predicate, is "completely interchangeable" with φ. I take it that by "completely interchangeable" Dummett means substitutable preserving grammaticality as well as truth. Accordingly, (6) should be equivalent to

(6') There is such a thing as is a philosopher

However, (6') is ungrammatical. If the copula is dropped from the predicate 'is a philosopher' the result,

(6'') There is such a thing as a philosopher,

is grammatical. However, (6'') affirms the existence of at least one philosopher, not of a concept. It says that the
concept \textit{philosopher} is not empty. This is not the same thing as asserting the existence of a concept, for, according to Frege, there are empty concepts. Thus if we leave out the copula the result is grammatical but is equivalent to the first-order sentence \'(\exists x) (x \text{ is a philosopher})'. If we leave in the copula, the result, (6') is ungrammatical. Thus if we accept Dummett's principle of substitutivity for "what"-clauses and if the "what"-clause in (6) is a predicate, then (6) is also ungrammatical. Yet (6) certainly appears to be grammatical. It seems, then, that Dummett is using 'what 'is a philosopher' refers to' as a proper name in (6) and not as a predicate. As I have noted before, such "what"-clauses are ambiguous. They may occur either as proper names or as predicates. Consider, now:

(a) There is such a thing as _____.

While Dummett claims that some completions of (a) are second-order sentences (a) has the peculiar feature that any grammatical completion of it is a first-order sentence. That is, the blank in (a) is grammatically accessible only to proper names. Thus it is a problem for Frege, and Dummett, to make second-order existence claims, and hence to express grammatically the existence of concepts and functions. Now consider (7). If (7) is grammatical, 'being a philosopher' must be a proper name. Yet if it is a proper name, (7) is not equivalent to the ascription of a reference to the predicate
'is a philosopher'.

Where has Dummett gone wrong? After having so carefully analyzed the problems associated with the predicate 'is a concept', he does not appear to realize that a similar problem is associated with (a). It is helpful here to invoke Quine's distinctions between concrete and abstract terms and singular and general terms. Dummett seems to be claiming that each of 'is a philosopher', 'being a philosopher', and 'what 'is a philosopher' refers to' is coreferential, and hence interchangeable, with the others. Yet they are not all interchangeable with one another. To use Quine's terminology, 'is a philosopher' and 'a philosopher' are concrete general terms, while 'being a philosopher' is an abstract singular term. The expression 'what 'is a philosopher' refers to' is ambiguous and may occur either as a concrete general term or as an abstract singular term. With this in mind, we can see that in (6) the "what"-clause occurs as an abstract singular term, and (7) is the completion of (a) by the abstract singular term 'being a philosopher'. Hence if (6) and (7) assert the existence of anything, they assert the existence of an abstract object and not a concept.

Again, (a) may be grammatically completed only with a singular term, abstract or concrete, and no singular term refers to any concept or function. I consider this feature of (a) to be yet another symptom of the general problem of talking about concepts. While Frege needs second-order
quantification to make existential claims about concepts and functions, the quantifier $(a)$ is not an adequate one.

So far Dummett has argued that the ascription of reference to a particular predicate is a banality provided that we admit second-order quantification. However, the argument breaks down due to our inability to talk about concepts other than by using predicates, or what Quine calls "general terms". Dummett also provides a general argument, the conclusion of which is:

...the existence of concepts -- that is, of referents for predicates -- required for its proper expression, the use of second-level quantification, and was beyond doubt provided that the employment of that device is allowed as legitimate and intelligible.\(^\text{14}\)

Dummett's argument is as follows. If we can substitute a variable for an expression then that expression refers. To say that predicates refer is to say that concepts exist. Thus if we can substitute variables for predicates, concepts exist. Substituting variables for predicates is using second-order quantification. Therefore, Dummett concludes, concepts exist provided that we admit second-order quantification. I wish to argue, however, that we can accept second-order quantification as legitimate without accepting the existence of Fregean concepts.

As we have seen, Frege does not question the legitimacy

\[ \text{14. Dummett, p. 223.} \]
of second-order quantification; he uses it freely. However, a second-order theory adequate for Frege must differ significantly from a standard second-order theory of the sort suggested by Quine. In standard second-order theories two types of variables are used: first-order and second-order variables. First-order variables range over objects and second-order variables range over all subsets of the range of first-order variables. Such a theory would not be adequate for Frege, for according to Frege, concept and function variables must range over incomplete entities, not extensions or any other sort of object. Thus Frege's second-order theory cannot be a set theory.

Hence while Dummett may be correct in saying that the acceptance of Frege's thesis that predicates refer to concepts requires the acceptance of some theory of second-order quantification, he has not talked at all about what theory of second-order quantification he has in mind. Certainly in order for that theory to accommodate the peculiarities of Frege's theory of concepts and functions it must be appropriately "non-standard". In the process of examining Dummett's analysis I have discovered another serious symptom of the problem. It is clear that Frege needs a second-order quantifier in natural language in order to express the existence of concepts. However, the natural language quantifier (a) will accept only proper names. Again we are in a position where we are forced to use proper names to talk about concepts.
If we attempt to complete (a) with an incomplete expression, the result is rendered incomplete itself. We have seen that Dummett does not discuss the difficulties that exist, for Frege, in making particular second-order existence claims. The use of standard second-order quantification theory will give rise to similar problems.

Thus not only must the language be extended in an appropriate way so that we may talk about concepts and functions, but also a non-standard second-order theory is required to accommodate Frege's doctrines about concepts and objects. If Dummett is correct about the relation between Frege's thesis that incomplete expressions have reference and second-order quantification, then these are not separate projects.
3. **Martin's solution**

In his paper "Frege's Problems with 'The Concept Horse'"¹ Edwin Martin examines a problem associated with Frege's thesis that predicates refer to concepts, and proposes a solution. Martin sees the problem as a problem with the specification of references of predicates. While Frege maintains that predicates refer to, or stand for, concepts, we are prevented, by Frege's theory, from truthfully completing

(1) 'ξ is a horse' stands for _____.

No matter what expression is put in the blank of (1) the result will not be a true sentence. The blank of (1) is grammatically accessible only to proper names, yet if (1) is completed with a proper name, the result is false. Proper names refer to objects and no predicate refers to any object. On the other hand, if we attempt to complete (1) with an incomplete expression, the result inherits the incompleteness of that expression and hence is rendered ungrammatical. There is no corresponding problem with the specification of reference for proper names.

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Martin approaches the problem by examining the relation referred to by 'stands for' in (1). He compares it to the relation that holds between a proper name and its reference. Are the two relations the same? His answer is that technically they are not. The relation that obtains between 'is a horse' and its reference is of the second level because it holds between a predicate and a first-level function. The relation that obtains between 'Socrates' and Socrates is of the first level because it holds between a proper name and an object. Martin's approach to the problem with (1) is reminiscent of Dummett's treatment of the problem with the predicate 'is a concept'. Remember that the predicate 'is a concept' cannot be completed truthfully because it will accept grammatically only proper names and no proper name refers to a concept. If we attempt to complete the predicate with an incomplete expression, the result is rendered incomplete. Dummett's strategy was to construct a suitable second-level replacement for the first-level predicate 'is a concept'.

Martin's strategy in the case of the relation referred to by 'stands for' in (1) is similar to Dummett's. Martin argues that the name for that relation must be a second-level function name, for the relation take a predicate and a first-level function to a truth value. However, whatever that relation may be, it cannot be adequately schematized as
for any instance of (2) contains a gap that it inherits from
the first-level function name substituted for 'φ( )' and hence
cannot refer to a truth value.

Martin asks if this problem is associated with all
second-level function names. The argument place of a second­
level function name is filled with a first-level function
name. Yet a first-level function name must be incomplete.
We cannot represent the form of a second-level function name as

(3) M(φ( ))

for (3) is rendered incomplete by the first-level function
name 'φ( )'. Frege's most often used example of a second­
level function name, Martin notes, is the universal quantifier.
When we apply the universal quantifier to a first-level
function name the resulting expression does not have the
incompleteness that (3) does. The universal quantifier

(4) ∀x φ(x)

carries bound variables. The bound variables of the universal
quantifier fill the argument place of the first-level function
name 'φ( )'. The argument place of the quantifier is in
turn filled by the first-level function name. A mesh is
achieved when the quantifier is completed by a function name
of the appropriate type.
Martin uses the universal quantifier as a model in constructing an appropriate representation of the second-level reference relation. His insight is that the problem associated with (2) and (3) is dissolved in the case of the universal quantifier by having the argument place of the first-level function name filled by a bound variable carried by the second-level function name. What Martin is attempting to construct is a representation of the logical form of a suitable second-level function name for the second-level reference relation. What he provides is:

(5) \( \text{St } x('\xi \text{ is a horse}', \phi(x)) \).

What Martin intends is a second-level function name with a bound variable that fills the gap of the first-level function name. (5) is, of course, an abbreviation. The form of the name for the reference relation for predicates should not be represented by (2) but rather by (5).

Martin anticipates an objection to the use of bound variables in (5). The objection is that his solution is \textit{ad hoc}. He answers the objection by arguing that the use of bound variables is as appropriate in (5) as it is in the case of the universal quantifier. While it may not be evident that bound variables are needed in (5), bound variables are needed in those cases when the first-level function is polyadic. In those cases the bound variables serve the
same purpose they serve in complex quantifications. Consider, for example

(6) $\forall x \exists y F_{xy}$.

The bound variables in (6) keep track of references. We need some way to indicate that the universal quantifier binds the first argument-place and that the existential quantifier binds the second argument-place of the function. Now consider the reference relation applied to a binary function name, as in:

(7) $St \ x, y ('\xi \text{ is bigger than } \xi', \psi(y,x))$.

It should be clear that (7) differs in meaning from

(8) $St \ x, y ('\xi \text{ is bigger than } \xi', \psi(y,x))$.

Thus Martin maintains that (1) should have the form of (5) and should be completed as in

(9) $St \ x('\xi \text{ is a horse}', \ x \text{ is a horse})$.

The result is a true sentence. In order to avoid the problem with (2), the logical form of the specification of reference for predicates must be something like (5). That is, the relation of reference for predicates must be a second-level function and its name must have a gap that accommodates a first-level function name yet also provide a bound variable
to fill the gap of the first-level function name.

Martin has not told us what relation this second-level relation is. He has merely given us an abbreviation for the logical form of something that is suitable, in light of Frege's theory. What, then, is a likely candidate for something that has the form of (5)? The following has the appropriate form:

\[(10) \ (\forall x) \ ('x \text{ is a horse'} \text{ is true of } x \equiv \phi (x))\]

where \(\phi(x)\) is a first-level function name. It carries a bound variable that fills the argument-place of the first-level function name. Similarly the following has the form of (7):

\[(11) \ (\forall x) \ (\forall y) \ ('x \text{ is bigger than } y' \text{ is true of } <x,y> \equiv \psi (x,y))\]

The following, then, would be the specifications of the references of 'is a horse' and 'is bigger than', respectively:

\[(12) \ (\forall x) \ ('x \text{ is a horse'} \text{ is true of } x \equiv x \text{ is a horse})\]

and

\[(13) \ (\forall x) \ (\forall y) \ ('x \text{ is bigger than } y' \text{ is true of } <x,y> \equiv x \text{ is bigger than } y)\]

Sentences (12) and (13) do not have the problem associated with (1) and (2), but do they really specify the references of
the expressions 'is a horse' and 'is bigger than'? I believe not. If this is the complex second-level function that Martin takes (5) to represent, then little progress has been made toward a solution to the problem. It seems that he has replaced the notion of reference for predicates by the notion of a predicate being true of an object. The relation of being true of is a relation that holds between a predicate and an object. According to Frege, a predicate is true of an object just in case that object falls under the concept to which the predicate refers. Thus to define 'is true of' we need the notion of reference for predicates. It appears that Martin holds that the reference relation for predicates is different from the reference relation for proper names. He says "Presumably there is an unproblematic relation holding between proper names and their bearers. The situation here perhaps may be pictures as: St ('Aristotle', Aristotle)."²

What is the reference relation for predicates and how does it differ, if at all, from the reference relation for proper names? The relation that holds between predicates and their references is technically a different relation from that which holds between proper names and their references. The difference between the relations is akin to the difference between the identity relation for objects and the analogous

². Martin, p. 61.
relation for concepts. Identity is a relation that can only be asserted or denied as holding between objects; it is a first-level function. Thus, strictly speaking, according to Frege, we cannot say that concepts are identical. However, the analogous relation of sameness for concepts is very much like identity. When we say that two objects are identical we mean that they are indistinguishable. When we say that two concepts are the same we mean that they are indistinguishable.

Similarly, the relation of reference that holds between proper names and objects is a different relation from the relation that holds between predicates and concepts. The former is a first-level function and the latter is a second-level function. The two relations can indeed be distinguished, yet Frege talks as if the relations are closely analogous, if not the same. Frege talks as if he intends a "generic" relation of identity and a "generic" relation of reference. There is a technical difference between identity for objects and identity for concepts imposed by Frege's hierarchy of objects and functions; however, both relations are, in a sense, the relation of identity. Similarly, there is a technical difference between reference for proper names and reference for predicates, yet they are, in a sense, both the reference relation.

It should be noted that if the two reference relations are different, then there must be an infinite number of reference relations corresponding to the infinite hierarchy
of levels of functions. Yet it should be clear that Frege does not intend any significant or interesting difference between the relation that obtains between a first-level predicate and a first-level concept and the relation that obtains between a second-level predicate and a second level concept. The difference between these relations is merely a technical one dictated by the theory of syntax. I maintain that the difference between the reference relation for proper names and the reference relation for predicates is a similar difference. According to Frege, just as proper names refer, so do predicates. To reduce the second-level reference relation to the relation of a predicate being true of an object is to weaken Frege's theory. If (10) is not what Martin has in mind as the relation represented by (5), then it is not at all clear what Martin intends.

After Martin introduced (5) as a representation of the second-level reference relation, he goes on to say:

And other higher-level function names -- like 'is a (first-level) concept' -- also carry along with them bound variables. Thus:

\[ FCx \text{ (x is a horse).} \]

Martin has given a representation of the logical form of a second-level predicate to replace the unsuitable first-level predicate 'is a concept'. Dummett's suggestion for a suitable replacement has the appropriate form. Remember that Dummett suggests

\[ (14) \ (\forall x) \ (x \text{ is a horse or } - x \text{ is a horse}). \]

3. Martin, p. 60.
Sentence (14) is a completion of the second-level function name

\[(\forall x) (\phi(x) \text{ or } \neg \phi(x)).\]

by the first-level predicate 'is a horse'. Again, Martin's insight that second-level function names should carry bound variables to fill the argument places of first-level function names resembles Dummett's. Yet Dummett's proposal turned out to be inadequate because it could not treat other related problems with Frege's philosophy of language. Martin's proposal is similarly inadequate. How can we deny that Socrates is a first-level concept or that any given concept is an object?

Given Frege's theory, it is clear that a first-level predicate is required to assert grammatically of any entity that it is an object. The predicate 'is an object' does the job. It is accessible to proper names and proper names refer to objects. However, it is not the case, according to Frege, that everything is an object. That is, one might be tempted to express what Frege wants to deny as:

\[(\forall x) (x \text{ is an object}).\]

Thus he wants to affirm:

\[(\exists x) \neg (x \text{ is an object}).\]
However, Frege is prevented from saying truthfully of any entity that it fails to be an object. Any grammatical instantiation of (17) is false. Consider the open sentence:

\[(18) \quad x \text{ is an object.}\]

The variable is in a position that is accessible only to proper names, and hence any grammatical substitution will yield a falsehood. Given that 'is an object' is a first-level predicate, it cannot be completed with any expression other than a proper name. In particular we cannot deny that any first-level concept is an object. The best we can do, given an appropriate second-level replacement for 'is a concept', is affirm both the existence of objects and concepts. Thus:

\[(19) \quad (\exists x) (x \text{ is an object}) \land (\exists \phi) (\forall x) (\phi(x) \lor - \phi(x)).\]

Yet we still lack the means for expressing Frege's doctrine that no concept is an object. That we are unable to complete (18) truthfully is only part of the problem. It is clear that Frege cannot express the doctrine in question as:

\[(20) \quad (\forall x) (x \text{ is a concept} \supset - x \text{ is an object}).\]

As we have seen, (18), which occurs in (20), cannot be truthfully completed. Also, 'is a concept' is a first-level predicate and hence must be replaced by a suitable second-level predicate. But then attempts to make comparisons between
objects and concepts are blocked. If we can truthfully say of an entity that it is a concept then the name of that entity is of the first level. A first-level expression cannot grammatically complete the first-level predicate 'is an object', and hence we cannot deny that that entity is an object. Furthermore, generalization on a first-level name requires second-order quantification. Thus (20), which is a first-order quantification, is not adequate.

We need a second-level predicate to affirm that any entity is a concept, but we need a first-level predicate to deny that that entity is an object. Yet second-level predicates are accessible only to first-level expressions and first-level-predicates are accessible only to proper names. Thus it seems that we need both a proper name and a first-level predicate to name this entity. But this is just what Frege says cannot be done. There are many metaphysical doctrines that Frege wants to affirm yet is barred from expression grammatically. For any n, Frege has no way of saying that nth-level functions are distinct from (n+1)st-level functions and that not everything is of the nth level. The situation looks hopeless. If Frege's theory is true it cannot be expressed.

Martin's proposal is an enlightening one but, in the end, does not bring us any closer to a solution. The idea of constructing second-level function names on the model of the universal quantifier is interesting. However, it has several
defects. While the representation Martin gives us of an appropriate form for the name for the second-level reference relation does succeed in avoiding the problems associated with the open sentence:

'Is a horse' refers to ____

Martin has not told us what relation he has in mind. The candidate I have considered is not adequate. While the name for the second-level reference relation must be of the second level and the name for the first-level reference relation must be of the first level, I believe that Frege intends the two relations to be more intimately related than Martin proposes. Furthermore the situation is still graver than is suggested by Martin's treatment. Even if we can find an adequate name for the relation that obtains between a predicate and its reference, and thereby solve the "problem with 'the concept horse'", there are further problems with Frege's philosophy of language. Frege's theory prevents us from expressing its metaphysical doctrines. While the problem with specifying the references of predicates is a serious one for Frege, it is merely one symptom of a theory in trouble. Any adequate treatment of the problem of talking about concepts and functions must take care of the related problems I have discussed as well as the specification problem.
4. **Furth's solution**

In his paper "Two Types of Denotation" Montgomery Furth examines Frege's doctrine that incomplete expressions have reference and attempts to construct a modification of that doctrine that avoids the associated problems. He maintains that while there are problems with the doctrine, the problems can be solved and a version of the doctrine saved.

According to Frege, expressions can be divided into those that are complete and those that are incomplete. Expressions in both categories have reference, or are names. Complete expressions, or proper names, refer to objects and incomplete expressions refer to functions. The distinction between function and object is an important one for Frege, yet when attempting to explain the distinction he appeals to metaphors. He draws the distinction in terms of being "saturated" or "complete" and "unsaturated" or "in need of completion". Such metaphorical talk is unsatisfactory. We have seen that Frege's notion of a function is an obscure one. While the notion of incompleteness as applied to expression can be understood in terms of an expression containing a gap or argument place, Frege talks as if the references of these expressions have a corresponding incompleteness or unsaturatedness. A function is not an object yet it yields an object when "saturated" by an

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object. This is all figurative and rather obscure. The metaphors Frege offers by way of an explanation of what it is for an entity to be unsaturated or in need of completion are not enlightening. For instance:

Metaphorical expressions, if used cautiously, may after all help towards an elucidation. I compare that which needs completion to a wrapping, e.g. a coat, which cannot stand upright by itself; in order to do that, it must be wrapped round somebody. The man whom it is wrapped round may put on another wrapping e.g. a cloak. The two wrappings unite to form a single wrapping. There are thus two possible ways of looking at the matter; we may say either that a man who already wore a coat was now dressed in a second wrapping, a cloak, or that his clothing consists of two wrappings -- coat and cloak.

In addition to the obscurity of the distinction in the realm of reference between saturated and unsaturated, Furth is concerned with other problems associated with Frege's doctrine that incomplete expressions have reference. He is particularly concerned with the problem that arises when we attempt to specify the reference of a given incomplete expression. He states the problem in the following way. In the case of a proper name, or complete expression, A, we can ascribe and specify reference using the forms:

(I) \( (\exists x) (A \text{ refers to } x) \)

(II) A refers to x

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2. Geach and Black, p. 134.
where (I) ascribes reference to A and (II) specifies the reference of A. We cannot, however, use forms (I) and (II) where A is an incomplete expression. We cannot truthfully complete

(I) 'F( )' refers to ____

for if we fill the blank of (I) with a complete expression the result is false and if we fill the blank of (I) with an incomplete expression the result is not a sentence and hence is neither true nor false. Thus while Frege maintains that incomplete expressions have reference, his doctrine is shrouded in obscurity and gives rise to the problem with specification.

Furth asks: what does it mean to say that incomplete expressions have reference? In an attempt to answer this question he first reformulates Frege's doctrine. Furth holds that Frege's doctrine of "unsaturated" reference for incomplete expressions consists of two different theses. First, incomplete expressions do not refer to objects and second, incomplete expressions do have reference. Before defining 'incomplete expression', Furth distinguishes between simple and complex proper names. A complex proper name is a proper name that has a proper name as proper part. For instance, 'Massachusetts' and '2' are simple proper names while 'the capital of Massachusetts' and '2+3' are complex proper names. Sentences are, according to Frege, complex
proper names. Incomplete expressions, then, result from removing one or more proper names from a complex proper name. Thus, 'the capital of ( )', '( ) + 3', and '( ) is a horse' are incomplete expressions. Although proper names, as a class, "have reference" and refer to objects, some proper names refer and some fail to refer. For example, 'Pegasus' and 'the largest prime number' are proper names but they fail to refer. Frege holds that if a complex proper name contains a part that lacks reference then the whole lacks reference. Accordingly, 'Pegasus is a horse' is neither true nor false.

Furth takes the fact that the reference of an incomplete expression cannot be specified using the specification form for proper names as evidence that incomplete expressions do not have reference "in isolation"; that is, if they do have reference they do so in the context of a complex proper name. Hence, his reformulation of Frege's doctrine is put in terms of referring complex proper names. His formulation is as follows:

(R) A referring complex proper name must contain at least one incomplete part which (i) does not refer to an object, but (ii) does have reference.

Furth's aim is to establish (R) and to that end he discusses (i) and (ii) separately.

In support of (i), Furth argues that in every complex proper name that refers, there must be a part that plays a semantical role that is different from that of referring to
an object. A referring complex proper name refers to a unique object. An enumeration or list of the references of its complete parts does not suffice as a semantic account of that complex proper name, for the reference of a complex proper name may be distinct from the references of its complete constituents. Consider, for example, the complex proper name, '2+3'. Supposing that '2' and '3' exhaust its complete parts, there is "something else" that, within the complex name '2+3', combines the references of '2' and '3' so that the complex name refers to five. Furth argues that the incomplete expression that remains when the complete parts are removed plays a semantical role in yielding the reference, five. His account of this is that the complete parts of a complex proper name refer to objects while the incomplete part yields a referring proper name upon completion by referring proper names. This, then, is the semantic role of the incomplete part of a complex proper name. I shall not examine this argument, which Furth offers as an antidote to the Fregean metaphors, in greater detail. I am primarily interested in Furth's argument for (ii), for contained in his argument that incomplete expressions have reference is Furth's proposed solution to the specification problem.

Basically, Furth's claim in support of (ii) is that a distinction can be made for incomplete expressions that corresponds to the distinction for proper names between
having reference and lacking reference. By developing this analogy, Furth argues that incomplete expressions can properly be said to have reference. His project is to strengthen this analogy to the point that he can conclude that incomplete expressions do refer. His conclusion, then, is that incomplete expressions refer, but not in the same manner as do proper names. There are two types of reference.

The distinction Furth draws for incomplete expressions turns on a certain property which Furth calls "property Z".

\( (Z) \) An incomplete expression has property Z if and only if every result of completing that expression with any referring proper name has reference.

We are to assume that we know what it is for a proper name to refer. Furth's aim, then, is to argue that for an incomplete expression to have property Z is for that expression to have reference. His argument by analogy has two parts. First, he argues that a proper name's having reference and an incomplete expression's having property Z contribute in exactly the same way to the possession of reference by the complex proper name in which they occur. The second part of the analogy consists in showing that having property Z, for incomplete expressions, plays the same role as having reference, for proper names, in quantification.

Before I examine Furth's evidence in support of this symmetry, I should point out that he finds the following
passage from the *Grundgesetze* quite suggestive:

A name of a first-level function of one argument has a denotation (denotes something, succeeds in denoting) if the proper name that results from this function name by its argument-places' being filled by a proper name always has denotation.

This is, of course, the criterion that Dummett had in mind when he suggested the predicate 'is something that everything either is or is not' as a replacement for the unsuitable 'is a concept'. Concept words, or predicates, for which the law of excluded middle does not hold do not succeed in referring. While Furth recognizes that Frege does not offer the above passage as a definition of reference for incomplete expressions, Furth believes that, in practice, Frege does take the condition of an incomplete expression's having \( z \) as corresponding to the condition of a proper name's having reference. With respect to non-referring concept words, Frege writes:

> These are not such as, say, contain a contradiction -- for there is nothing wrong at all in a concept's being empty -- but such as have vague boundaries. It must be determinate for every object whether it falls under a concept or not; a concept word which

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does not meet this requirement on its [reference] is [referenceless].

I grant that if an incomplete expression lacks property Z then Frege would say that it does not refer. An incomplete expression lacks Z if some completion of it by a referring proper name lacks reference.

An example of what Frege has in mind here is the concept word 'bald'. Any predicate, or concept word, that is neither true nor false of a given object lacks Z. The complete expression that results from completing a predicate is a sentence. The expression '( ) is bald' does not have Z since there is at least one proper name α such that "α is bald" has no reference, or truth value.

Furth argues that having Z is a necessary and sufficient condition for an incomplete expression's having reference. His view is that an incomplete expression contributes to the reference of a complex proper name in which it occurs, and yet while he wants to call this contribution "reference", he holds that reference for incomplete expressions is different from reference for proper names.

Before I assess Furth's claim, I shall examine the analogy he draws between an incomplete expression's having Z and a proper name's having reference. It is on the basis

4. Frege, Posthumous Writings, p. 122.
of this analogy that Furth proposes that having \( Z \) is having reference.

I have said that the first half of the analogy consists in the claim that the possession of a reference by a proper name and the possession of property \( Z \) by an incomplete expression contribute to the possession of a reference by the complex proper name of which they are constituents. This can be put more clearly as follows. According to Frege, if any part of a complex proper name, complete or incomplete, lacks reference then the whole fails to refer. Since Furth is, for the time being, leaving open the question of whether incomplete expressions have reference, his plan is to establish a symmetry, with respect to this principle, between a proper name's having reference and an incomplete expression's having property \( Z \).

Consider the complex proper name 'a is an F'. If the incomplete expression '() is an F' has property \( Z \) and 'a is an F' does not refer, then it follows that 'a' does not refer. If 'a' does refer and 'a is an F' does not refer then it follows that '() is an F' does not have property \( Z \). Hence, Furth concludes, having property \( Z \) for '() is an F' and having reference for 'a' play analogous roles with respect to having reference for the complex proper name 'a is an F'. In fact, for a complex proper name to have reference it is sufficient that its complete parts have reference and its incomplete part have property \( Z \).
To strengthen the analogy, Furth argues that it is necessary, too. That is, for a complex proper name to have reference it is necessary that its complete parts have reference and that its incomplete parts have property Z. His claim is that if 'a is an F' has a reference then 'a' refers and '( ) is an F' has Z. He points out that Frege requires that functions be everywhere defined and that the law of excluded middle hold for concepts. Thus, in Furth's terminology, Frege requires that incomplete expressions have property Z. Those that lack Z are to be expelled from the language along with non-referring proper names.

The second half of the analogy concerns the link between quantification and reference. Since the Fregean terms 'first-level function name' and 'second-level function name' are not available to Furth at this juncture, he distinguishes different types of incomplete expressions. A type-1 incomplete expression is an incomplete expression that is completed by a proper name. A type-2 incomplete expression is an incomplete expression that is completed by a type-1 incomplete expression, and a type-3 incomplete expression is completed by a type-2 incomplete expression. A proper name is not a type-2 incomplete expression because it is not an incomplete expression. Among type-2 incomplete expressions are those whose behavior resembles the behavior of first-order quantifiers. He calls these new expressions "virtual" quantifiers. Furth examines the role of having reference by
proper names and having Z by type-1 incomplete expressions in these type-2 expressions.

The virtual quantification '[[x] Fx]' refers to the True if every completion of 'F ( )' by a referring proper name refers to the True. Thus the virtual quantifications are quantifications with respect to those objects that are nameable in the language. Furth's interest is in the role of referring proper names in the truth conditions for quantification. If a universal quantification '[(x)Fx]' refers to the True and if some instance 'Fa' does not refer to the True, then 'a' must lack reference. The reference of a virtual quantification '[[x] Fx]' depends on the reference of 'Fa' only for referring proper names 'a'. Consider, for example, the virtual quantification '[[x] (x=x)'] which is the completion of the type-2 expression '[[x] (ϕ(x))]' by the type-1 expression '(ξ) = (ξ)'. On Furth's account of the virtual quantifier, this refers to the True, for every completion of '(ξ) = (ξ)' by a referring name 'a' refers to the True. Note that completions 'a=a' by non-referring names 'a' do not refer to the True, for they do not refer at all.

Furth introduces certain type-3 incomplete expressions whose behavior resembles the behavior of second-order quantifiers. Second-order quantifiers are those that bind predicate variables. Of course Furth cannot here assume that predicate variables have reference. Again, Furth calls
these type-3 expressions "virtual" quantifiers. Furth strengthens the analogy between having property Z and having reference through an examination of these new expressions. He constructs an expression in which having Z plays the same role for type-1 incomplete expressions that having reference played for proper names in first-order virtual quantification. The expression:

\[(2) \ [f] (M_\delta f(\delta))\]

is a completion of a type-3 incomplete expression. It refers to the True if every completion of 'M_\delta f(\delta)' by a type-1 incomplete expression with property Z refers to the True. It refers to the False if any completion by a type-1 expression with Z does not refer to the True. Why does Furth stipulate that the type-1 expression must have Z?

If we complete the type-3 expression in (2) with the type-2 expression

\[(3) \ [x] (\phi(x) \supset \phi(x))\]

the result is

\[(4) \ [f] [x] (f(x) \supset f(x))\]

According to Furth's definition, (4) refers to the True if every completion of (3) by a type-1 incomplete expression with Z refers to the True. To see the role of property Z,
suppose that 'F( )' does not have property Z. Then the instance:

(5) \([x] (F(x) \supset \neg F(x))\)

does not refer to the True since not every completion of 'F( ) \supset \neg F( )' by a referring proper name refers to the True. Hence, if type-1 expressions lacking Z were allowed, (4) would not refer to the True. But obviously (4) is true.

This is analogous to Furth's discussion of '[x](x=x)'. Furth explains the reference of '[[x](F(x))]' in terms of the completion of 'F( )' by referring complete names. Otherwise expressions such as '[[x](x=x) which should refer to the True would not refer at all. Similarly, if the reference of (2) were not explained in terms of the completion of 'M_\beta \phi(\beta)' by type-1 expressions having Z, then expressions such as (4) would not refer to the True.

The analogy is now complete. Property Z plays the same role in second-order quantification that reference plays in first-order quantification.

On the basis of these analogies and symmetries Furth proposes that having property Z is having reference for incomplete expressions. Incomplete expressions, including predicates, can properly be regarded as referring, yet they do not refer to objects. Reference for predicates is not a two-place relation between an expression and an object. Proper names have "saturated reference" and predicates have
"unsaturated reference". For any predicate to have unsaturated
reference is for it to have property Z. There are not two
types of entities. A predicate does not refer in virtue of
bearing a certain relation to a peculiar kind of entity; it
refers in virtue of having the property Z.

But if we can properly call having Z "having reference",
how do we accordingly ascribe and specify reference for
incomplete expressions? According to Furth, just as there
are two ways of referring, there are two corresponding ways
of ascribing and specifying reference. Thus he offers the
following as forms for ascribing and specifying reference for
an incomplete expression B, respectively:

(III) (a) (β) (x) (a refers to x and β refers to x ⊃
(∃y) (the completion of B with a
refers to y and the completion of
B with β refers to y))

(IV) (a) (x) (a refers to x ⊃ the completion of B
with a refers to F(x))

The problem with specification, claims Furth, has been resolved.
We can specify the reference of 'is a horse' as follows:

(a) (x) (a refers to x ⊃ the completion of 'is a horse'
with a is true if x is a horse and false if
x is not a horse).

Furth's treatment of reference is a modification of Frege's
theory. Is it an acceptable one for Frege? It is clear
that Frege's theory requires quantification over functions
and concepts. If Furth's proposal cannot meet Frege's needs for second-order quantification, then this modification must be rejected as too radical.

Furth proposes that a predicate has "reference" in virtue of yielding referring proper names upon completion by referring proper names. He says that his proposal "consists in regarding the matter not in terms of two types of denoted "entities", but rather in terms of two types of denoting... On this basis, an incomplete name's meeting the Z condition is not a sign that it therefore denotes (in the manner of a complete name) something mysterious (unsaturated): it is itself a second manner of denoting." 5

Reference for predicates, then, is not a binary relation between an expression and an entity. Thus it does not follow from Furth's definition of reference for incomplete expressions that concepts and functions exist. It appears that Furth's proposal does not license quantification with respect to predicates and function-names. This criticism might, however, be based on a superficial reading of Furth.

Furth claims that it does make sense, on his proposal, to say of a given predicate that it refers to something, and in recognition of Frege's need for second-order quantification he writes:

5. Furth, p. 40.
We have a powerful motivation to seek a rationale for speaking of denoting in connection with [incomplete expressions]: in the need to introduce quantification in connection with them.\[6\]

We have seen that Frege requires the means to express existential and universal claims about concepts and functions, and this is to be done by replacing predicates by quantifiable variables. Concerning this, Furth says:

...the question whether a term is replaceable with a bindable variable and the question whether it may be regarded as denoting are two sides of the same question.\[7\]

I gather, then, that it is Furth's view that to say that predicates have reference is to say that predicates occur in positions accessible to variables of quantification. Furth has argued that while predicates do not refer to objects, they do, nevertheless, refer. Yet it is a problem for Furth to reconcile his treatment of unsaturated reference with the use of second-order quantification. When we use second-order quantification, the second-order variables must, for Frege, range over functions and concepts -- the references of predicates and function names. If predicates refer in the manner Furth has proposed, over what do the variables that

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6. Ibid., p. 28.
7. Ibid.
Frege maintains that "a concept is the reference of a predicate". Thus to ascribe reference to a predicate is to assert the existence of a concept. Furth proposes (III) as the form for ascription of reference to a predicate, and I think that from it we can determine what, on Furth's view, a concept is. I believe that Furth holds that when we claim that a concept exists, or that a predicate has reference, we are claiming that corresponding to a given predicate there is a certain assignment of truth values to all nameable objects. That is, if the predicate yields a true sentence upon completion by a proper name, then the object to which that proper name refers is assigned the value True. If the predicate yields a false sentence upon completion by a proper name then the object to which that proper name refers is assigned the value False. If the predicate yields a sentence that is neither true nor false upon completion by a proper name, then that predicate does not "refer"; the assignment is not complete. Hence, given (III), to ascribe reference to a predicate is to assert that there is such a complete assignment of truth values corresponding to that predicate. To specify the reference of a predicate is to specify what

8. Geach and Black, p. 48.

the assignment is. Specification consists in saying for which objects it is the case that the completion of the predicate with their names results in a true sentence.

This seems to be what Furth means by 'concept'. However, this explanation looks very much like an explanation of Frege's notion of the value-range or extension of a concept. Frege says "we designate as an extension the value-range of a function whose value for every argument is a truth value". An extension can be understood as a set of ordered pairs, the first member of which is an argument or object, and the second member of which is a truth value. Understood in this way, the extension of the concept philosopher, for example, is the set containing the pairs <Socrates, True>, <Frege, True>, and <Reagan, False>, among others.

Using Furth's form for specification, thus is what we specify when we specify the reference of the predicate 'is a philosopher'. However, Frege does not identify the extension of a concept with a concept. He says that the "concept takes logical precedence of its extension". Moreover, extensions are objects and concepts are not. If my explanation of Furth's view of concepts is correct, then it looks as if he has identified extension with concept. This is, of course, one avenue open to those who seek a solution.

10. Geach and Black, p. 31.
11. Ibid., p. 105.
to Frege's problem. For, one way for Frege to avoid the problem with the concept horse is just to give up the idea of unsaturated reference, and let predicates refer to extensions. However, this is not the avenue Furth has taken. He has attempted to retain Frege's thesis that predicates do not refer to objects and he argues that predicates do, nonetheless, refer. Hence Furth does not intentionally identify extension with concept.

There are two possible readings of Furth. On one reading predicates "have reference" but can't be said to refer to entities. In this case Furth's proposal does not allow for the quantification over functions and concepts. If this is the correct reading then his modification of Frege's doctrine must be rejected, for second-order quantification is crucial to the project of the Grundgesetze. However, Furth's proposal just is not Fregean in flavor. Frege really does intend for predicates to refer to entities, and Furth's proposal does not capture this at all. Furth fails to give an adequate characterization of concepts, for it appears that if he holds that predicates refer in the manner captured by (III) and (IV) and that a predicate does refer to something, then the references of predicates turn out to be Fregean objects. Thus on either reading Furth has not succeeded in constructing an acceptable modification of Frege's theory.

Furth has succeeded in developing an interesting analogy
between having reference and having property Z. But in the end, having Z is not having reference. I believe that Frege intends only one type of reference. Unless Furth can develop a "new" kind of quantification to correspond to the "new" kind of reference, then it seems that if incomplete expressions refer they must do so in the same way as proper names.
CHAPTER III: AN ANTICIPATION OF THE PROBLEM?

In his introduction to the translation of Frege's Begriffsschrift published in From Frege to Gödel: A Source Book in Mathematical Logic, Jean van Heijenoort writes:

...If we also observe that in the derivation of formula (91) he substitutes \( \exists \) for f, we see that [Frege] is on the brink of a paradox. 1

What this means may, at the moment, be obscure, but it is my aim to illuminate this passage and as I continue its meaning will become clear. Van Heijenoort's claim that Frege is "on the brink of a paradox" is, of course, metaphorical, for Frege's system either leads to a paradox or it does not. Terrell Bynum, in his edition to the Begriffsschrift, maintains that no paradox can be generated in Frege's system. He writes:

Van Heijenoort is in error in supposing that any paradox can arise from the deductive procedure Frege uses here. 2


In this chapter I attempt to resolve this dispute. In the first section of the chapter I reconstruct the system of Frege's Begriffsschrift and show that it is equivalent to a standard second-order predicate calculus, and then demonstrate the consistency of the system. I conclude, then, that if van Heijenoort is claiming that the system leads to a paradox or inconsistency, the dispute is settled on the side of Bynum. In the second part of the chapter I consider the interpretation of the system of the Begriffsschrift. Frege is not clear about how the system is to be interpreted. In light of Frege's later writings on the distinction between function and object, the interpretation of second-order quantifications presents some difficulties for Frege. These difficulties may be seen as an anticipation of the problem with the concept horse. The nature of the anticipation will be detailed later. Moreover, certain claims implicit in Frege's way of interpreting his system lead to Russell's paradox. Hence, I shall argue that while the system is demonstrably consistent, the problems that arise in its interpretation may provide a basis for van Heijenoort's concern.

Frege's Begriffsschrift presents an axiomatized system of logic. One fragment of the system is the propositional calculus and a larger fragment is the first-order predicate calculus. In Part I of the book Frege gives an explanation of the language of his system. He distinguishes two kinds of symbols. First, "those by which we may understand different
objects", and second, "those that have a completely determinate meaning". He calls the former "letters" and says that "no matter how indeterminate the meaning of a letter, we must insist that throughout a given context, the letter retain the meaning once given to it". Frege's "letters" are what we call "variables". Frege puts different types of letters to different uses and the typography of the Begriffsschrift is a bit confusing. Roughly, the capital Greek letters

\[ A, B, \Gamma, \Phi, \Psi, \text{ and } \chi \]

are used to stand for expressions. They might be called "syntactic" or "schematic" variables. Frege writes, "I use Greek letters as an abbreviation, and to each of these letters the reader should attach an appropriate meaning when I do not expressly give them a definition". The italic Latin letters

\[ a, b, c, d, e \ldots x, y, z \]

are used as propositional letters, as free variables and also to indicate generality when the scope of the universal quantifier is not limited. The Roman letter 'F' is used as a property or function variable. The lower case Greek letters

3. van Heijenoort, p. 11.

4. Ibid.
\(a, \beta, \gamma, \delta \ldots\)

are used to indicate argument places of functions. The German letters

\(a, b, \ldots, \epsilon\)

are used as bound variables of quantification.

Frege introduces the judgment sign and content stroke by saying that the judgment sign

\[\vdash\]

"stands to the left of the sign, or combination of signs, indicating the content of the judgment. If we omit the small vertical stroke at the left end of the horizontal one, the judgment will be transformed into a mere combination of ideas of which the writer does not state whether he acknowledges it to be true or not".\(^5\) According to Frege,

\[-A\]

is to be read "The proposition that A", and

\[\vdash A\]

is to be read "the judgment that A". There are problems with this distinction between the content of a judgment and a judgment, but it is not my intention to discuss the

\[5. \text{ Ibid.}\]
distinction here.

Frege's two primitive connectives are the conditional and negation. These are defined truth functionally. Frege's conditional is the material conditional. Conjunction and disjunction are defined in terms of the conditional and negation. Frege writes 'If B then A' as

\[ \top \quad \text{A} \quad \bot \quad \text{B} \]

and 'not A' as

\[ \top \quad \text{A} \]

His truth functional analysis of the conditional leads Frege to the rule, *modus ponens*. This rule of inference allows the derivation of a new judgment from two other judgments. From the judgments

\[ \top \quad \text{A} \quad \bot \quad \text{B} \quad \bot \quad \text{B} \]

the judgment

\[ \top \quad \text{A} \]

follows. The inference can be written as follows:
Frege's other primitive symbol is the identity sign.

\[ A \equiv B \]

is to mean that "the sign A and the sign B have the same conceptual content, so that we can everywhere put B for A and conversely".\(^6\)

Frege will later draw an important and celebrated distinction between function and object, but in the *Begriffsschrift* the distinction is between function and argument. He writes:

If in an expression, whose content need not be capable of becoming a judgment, a simple or compound sign has one or more occurrences and if we regard that sign as replaceable in all or some of these occurrences by something else (but everywhere by the same thing), then we call the part that remains invariant in the expression a function, and the replaceable part the argument of the function.\(^7\)

Frege's way of drawing the distinction is syntactic. It should be noted that his terminology here is not standard. At this point he takes sentence parts to be functions, yet later he takes sentence parts to stand for functions. In the *Begriffsschrift* Frege regards sentences as functions of the expressions occurring in them. He goes on to distinguish one-place from two-place functions and introduces appropriate notation. Predicates are treated as one-place functions and

\(^6\) Ibid., p. 21.  
\(^7\) Ibid., p. 22.
relational expressions as two-place functions.

In order to express an indeterminate function of the argument $A$, we write $A$, enclosed in parentheses, to the right of a letter, for example

$$\phi(A)$$

likewise,

$$\psi(A,B)$$

means a function of the two arguments $A$ and $B$ that is not determined any further...In general

$$\psi(A,B)$$

differs from

$$\psi(B,A).$$

Frege's distinction between function and argument leads him to treat "functions" as arguments. As we shall see shortly, this seems to provide the basis for Frege's use of second-order quantification, and may be responsible for van Heijenoort's worry.

Following his discussion of the function, Frege introduces his notion of generality as follows:

In the expression of a judgment we can always regard the combination of signs to the right of $\top$ as a function of the signs occurring in it. If we replace this argument by a German letter and if in the content stroke we introduce a concavity with this German letter in it, as in

$$\top\phi(A)$$

this stands for the judgment that whatever we may take for its argument, the function is a fact.9

8. Ibid., p. 23
Frege goes on to introduce what is, in effect, the second-order quantifier. He writes:

Since a letter used as a sign for a function, such as φ in φ(A) can itself be regarded as the argument of a function, its place can be taken in the manner just specified by a German letter. 10

Thus φ(A) yields both

$$\overline{\phi(A)}$$

and

$$\overline{\exists F(A)}.$$

Frege defines the existential quantifier in terms of the universal quantifier and negation in the usual manner.

The concavity in the content stroke indicates the scope of the German letter in it. For instance, the scope of 'a' in

$$\overline{\overline{A(a)} \overline{B(a,b)}}$$

is the entire content of the judgment while in

$$\overline{\overline{X(a)}}$$

the scope is confined to the antecedent of the conditional.

10. Ibid.
Italic Latin letters are used when the scope of the generality is the entire content. Given this, Frege introduces a principle of universal generalization which I shall discuss a bit later.

In Part II of the *Begriffsschrift* Frege begins to formulate his axiomatic system and derive some of its theorems. He draws a distinction between his rules or "modes" of inference and his axioms, or "rules of pure thought". He presents a total of nine axioms and derives more complex judgments in accordance with his rules of inference. At the start of Part II Frege writes:

Merely to know the laws is obviously not the same as to know them together with the connection that some have to others. In this way we arrive at a small number of laws in which, if we add those contained in the rules, the content of all the laws is included, albeit in an undeveloped state.\[11\]

The judgments of Parts II and III are numbered. The nine laws, or axioms, are the following:

\[\begin{align*}
(1) & \quad \quad \quad \quad a \\
& \quad \quad \quad b \\
& \quad \quad a \\
(2) & \quad \quad \quad a \\
& \quad c \quad b \\
& \quad d \quad c \\
& \quad \quad \quad a \quad b \quad c
\end{align*}\]

11. Ibid., pp. 28-29.
Axioms (1), (2), (8), (31), and (41) together with modus ponens and Frege's unstated substitution rule constitute a complete axiomatization of the propositional calculus.\textsuperscript{12} The nine axioms together with modus ponens, universal generalization, and confinement constitute a complete first-order predicate calculus.\textsuperscript{13} I shall show that Frege's nine axioms together with modus ponens, universal generalization, confinement, and his rules for substitution constitute a consistent second-order predicate calculus. Frege's substitution rules are not explicitly stated. I shall provide a formulation of these rules later.

The system to which I shall show Frege's to be equivalent is Joel Robbin's as presented in his Mathematical Logic.\textsuperscript{14} His system is given by six axioms and two rules of inference. The symbols of Robbin's system are the individual variables $x, y, z, x_1, y_1, z_1, \ldots$

the $n$-place relation variables\textsuperscript{15}

\textsuperscript{12} proved by Łukasiewicz, 1931.

\textsuperscript{13} shown by Kneale.


\textsuperscript{15} Robbin uses lower case Greek letters as relation variables.
and the symbols

\[ \cup \setminus \{ \forall \ [ \ ] ( ) \} \]

The class of well-formed formulas is defined inductively as follows:

(i) If \( \phi \) is an \( n \)-place relation variable or an \( n \)-place constant and \( a_1, a_2, \ldots, a_n \) is a sequence of symbols each of which is either an individual variable or individual constant, then \( \phi(a_1, a_2, \ldots, a_n) \) is a wff.

(ii) \( f \) is a wff.

(iii) If \( A \) and \( B \) are wffs, so is \( A \supset B \).

(iv) If \( A \) is a wff and \( v \) is any variable, then \( \forall v A \) is a wff.

(v) A formula is a wff only as required by (i)-(iv).

Robbins six axioms are the following:

(A1) \( A \supset (B \supset A) \)

(A2) \( (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)) \)

(A3) \( \neg \neg A \supset A \)

(A4) \( \forall v A \supset S^v_a \) where \( v \) is any variable and \( a \) is a variable or constant of the same type (that is, an individual variable or constant if \( v \) is an individual variable and an \( n \)-place relation variable or constant if \( v \) is an \( n \)-place variable) and \( a \) is free for \( v \) in \( A \)

(A5) \( \forall v (A \supset B) \supset (A \supset \forall v B) \) where \( v \) has no free occurrence in \( A \)

(A6) \( \exists \forall x_1 \forall x_2 \ldots \forall x_n (F(x_1, x_2, \ldots, x_n) \equiv A) \) where \( F \) is any \( n \)-place relation variable not occurring free in \( A \) and \( x_1, x_2, \ldots, x_n \) are individual variables.
(A6) is the comprehension axiom schema and says that every well-formed formula defines a relation of objects. The two rules of inference are **modus ponens** and **universal generalization**:

(R1) B can be inferred from A and A ⊃ B

(R2) ∀vA can be inferred from A if v is any variable.

Robbin's first three axioms are Frege's formulas (1), (2), and (31), respectively. To show that Frege's system is equivalent to Robbin's involves showing that Frege's substitution rules are equivalent to Robbin's comprehension axiom schema.

But first I shall discuss Robbin's rule of universal generalization and his (A4) and (A5) which are his principles of universal instantiation and confinement.

Robbin's principle of universal instantiation is given by the schema:

(A4) ∀vA ⊃ Sv

where v is any variable of the same type as a.

This formulation allows instantiation of both first-order and second-order quantifications. Frege's principle of universal instantiation is not stated in the same way. His formula (58) is a first-order principle of instantiation:

(58) f(c)
which together with substitution gives rise to an infinite number of instances. Although Frege does not enunciate a second-order principle of universal instantiation, it is clear from the text that he intends his system to include one.

Frege treats first-order and second-order quantification in a parallel fashion. At the time of the writing of the *Begriffsschrift* Frege had not yet distinguished the different levels of functions. Later he would say that the first-order quantifier stands for a second-level function and the second-order quantifier stands for a third-level function. In the *Begriffsschrift*, however, after introducing his notation for the first-order quantifier, he draws on his notions of function and argument to introduce second-order quantification. Since \( F(A) \) can be seen as a function of the argument \( F \), \( F \) can be replaced by a German letter.

At this point in the text Frege first introduces a principle of universal instantiation. He says:

> From such a judgment, therefore, we can always derive an arbitrary number of judgments of less general content by substituting each time something else for the German letter and then removing the concavity in the content stroke.\(^{16}\)

It is clear that Frege intends this principle to apply to both first-order and second-order quantifications. When he says that we can substitute "something else for the German letter", this German letter may be either upper case or lower case;

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\(^{16}\) van Heijenoort, p. 24.
that is, it may be either an individual bound variable or a functional bound variable. Thus from

\[ \vdash \phi(a) \]

we can derive each of

\[ \phi(a) \]
\[ \phi(b) \]
\[ \phi(c) \]

and from

\[ \vdash \exists f(a) \]

we are supposed to be able to derive

\[ \vdash f(a) \]
\[ \vdash g(a) \]
\[ \vdash h(a) \]

The same considerations apply to Frege's rules of universal generalization and confinement.

Frege gives his rule of universal generalization as follows:

A Latin letter may always be replaced by a German one that does not yet occur in the judgment; then the concavity must be introduced immediately after the judgment stroke. For example, instead of

\[ \vdash X(a) \]

we can write

\[ \vdash \exists X(a). \]

17. Ibid., p. 25.
Given Frege's remarks concerning the replacement of a function letter by a German letter, it is clear that he intends his rule of universal generalization to hold for second-order as well as first-order variables. Thus from

\[ \forall x (a) \]

we can derive

\[ \exists F(a) \]

The same can be said of Frege's principle of confinement, which Frege states as follows:

It is clear that from

\[ \phi(a) \]

we can derive

\[ \forall a \phi(a) \]

if \( a \) is an expression in which \( a \) does not occur.\(^{18}\)

The justification Frege provides for this principle can easily be adapted for the second-order case. Furthermore, it is clear from the text that Frege intends a second-order principle of confinement, for he uses a second-order principle in several derivations.\(^{19}\)

\(^{18}\) Ibid., p. 26.

\(^{19}\) See the derivations of formulas (91), (93), and (95).
Thus while Frege does intend his principles of universal instantiation, universal generalization, and confinement to be general and not restricted to first-order quantification, some modification of his formulations is required. This is not a serious difficulty. Frege does act as if these principles are general ones, and at this stage he saw no need to distinguish different orders of quantification. This may, in part, be what concerns van Heijenoort. Yet an appropriate modification is consistent with Frege's remarks about the quantifier and generality.

One possible modification is suggested by Bynum in his introduction to the Begriffsschrift. Bynum points out that Frege occasionally cites a first-order principle in a derivation when, strictly, he needs a corresponding second-order principle. In fact, it is these slips that worry van Heijenoort. In particular, van Heijenoort is concerned about the derivation of formulas (77) and (91). He writes:

In the derivation of (77) [Frege] substitutes $\mathcal{F}$ for $a$ in $f(a)$, at least as an intermediate step. If we also observe that in the derivation of formula (91) he substitutes $\mathcal{F}$ for $f$, we see that he is on the brink of a paradox. 20

Bynum agrees that there is a problem with Frege's derivations of (77) and (91) but he offers a solution. He does not view the difficulty as symptomatic of any inconsistency in Frege's

20. van Heijenoort, p. 3.
system. His solution is to provide, in each case, an appropriate corresponding second-order principle.

In the derivation of (77), for instance, Frege cites (68) and indicates the substitutions by a table. The table does indicate that $F$ is to be substituted for $a$ in (68), yet Bynum points out that Frege really should have cited an analogous second-order principle that involves quantification over functions. Bynum provides the appropriate second-order principle that Frege should have cited instead of the first-order principle that he does cite. Bynum treats the derivation of (91) in the same way. In a footnote to formula (77) he says:

The idea of treating $F(y)$ as a function of the function $F$ is in no way contrary to Frege's later thought. To state his thought precisely, however, required notational machinery (which he had not yet devised) to distinguish first- from second-level functions. With that available the difficulty can easily be resolved. Van Heijenoort is in error in supposing that any paradox can arise in the system. In the Begriffsschrift Frege never confuses first- and second-level functions, though he does not yet have separate terms for them.\(^{21}\)

I agree with Bynum here, yet his treatment has certain drawbacks. Frege's formula (68) is:

\[
(68) \quad \frac{f(c)}{b} \quad \frac{[\neg f(a) = b]}{}
\]

\(^{21}\) Bynum, p. 175.
The formula Bynum supplies to be used in its stead in the derivation of (77) is:

\[(68') \quad M_\beta f(\beta) \quad b \quad [M_\beta f(\beta) \equiv b]

Together with the substitution table he provides, (68') does yield (77). While Frege's (68) is a first-order formula, Bynum's (68') is a second-order formula. Yet the notation of (68') is not the notation of the Begriffsschrift. In formula (68), 'f' is a first-level function letter, and 'c' stands in its argument-place in 'f(c)'. In formula (68'), 'M' is a second-level function letter and 'f' is a first-level function letter that stands in its argument place in 'M_\beta f(\beta)'. Thus 'f(\xi)' is a first-level function name where '\xi' indicates the argument place and 'M_\beta \phi( )' is a second-level function name where '\phi( )' indicates the argument place. The first-order quantifier of (68) is a second-level function name whose argument place is filled by a first-level function name. The second-order quantifier of (68') is a third-level function name whose argument place is filled by a second-level function name.

Bynum's solution invokes a distinction Frege later makes between different levels of functions and accordingly involves
expanding the syntax of the Begriffsschrift to include higher level function names. Bynum's treatment of (68) suggests introducing second-order formulas which, together with Frege's substitution rules, will provide second-order principles of universal instantiation, universal generalization, and confinement. In particular, a second-order principle of universal instantiation would be:

\[(58') \quad \frac{Mβf(β)}{ββ Mβf(β)} \]

With substitution (58') gives rise to all the desired instances of universal instantiation. Universal generalization and confinement may be treated in a parallel manner.

Yet there is a problem with this solution, for, as we have seen, it entails expanding the notation of the Begriffsschrift to include second-level function letters as well as first-level function letters. If the notation is so expanded, the original problem is merely removed to a higher level. If we allow second-level function letters and the replacement of these letters by German letters, then third-order quantification is introduced. Hence third-order principles of universal instantiation, universal generalization, and confinement must be formulated. Yet if these third-order principles are introduced in a parallel way, they will involve third-level function letters, and the original problem recurs on a different
level.

A modification that avoids the defect of Bynum's treatment is to offer schematic formulations of second-order principles of universal instantiation, universal generalization, and confinement. In the case of universal instantiation, a schema is needed that will allow the derivation of

... $F$ ...

from

$\exists \ldots \exists \ldots$.

For universal generalization a schematic rule is needed that allows the derivation of

$\exists \ldots \exists \ldots$

from

... $F \ldots$,

and for confinement the needed schema will allow the derivation of

$\exists \ldots \exists \ldots$

from

$\exists \ldots \exists \ldots$.
where

\[ \ldots \phi \ldots \]

does not contain free occurrences of \( \mathcal{F} \). The principles may be stated after those of Robbin as follows:

\[(UI) \quad \forall v A \supset S^v_a \mid \quad \text{where } v \text{ and } a \text{ are function variables}\]

\[(UG) \quad \forall v A \text{ can be inferred from } A, \text{ if } v \text{ is a function variable}\]

\[(Conf) \quad \forall (A \supset B) \supset (A \supset \forall v B) \quad \text{where } v \text{ is a function variable not free in } A.\]

In this way, Frege's principles are equivalent to Robbin's.

The remaining discrepancy between Robbin's and Frege's systems can be eliminated by showing that Frege's rules for substitution are equivalent to the comprehension axiom schema.

As I have said, Frege does not clearly enunciate his rules for substitution, but by observing his practice the rules can be formulated.

Frege does state a principle for the alphabetic change of variables:

Replacing a German letter everywhere in its scope by some other one is, of course, permitted, so long as in places where different letters initially stood different ones also stand afterward. This has no effect on content.\(^{22}\)

Of course Frege must mean that a bound variable may be replaced by another variable of the same type. In this way we avoid

\[\text{22. van Heijenoort, p. 51.}\]
the substitution of a function letter for an individual variable and *vice versa*. Frege's remark that this "has no effect on the content" hints at an important restriction that must be placed on substitution. Roughly, we want to ensure that variables that are free or bound in the original formula remain free or bound in the formula that results from substitution. Frege's substitution rules are to preserve validity, implication, and equivalence. There are two kinds of substitution that must be regulated -- substitution for propositional letters and substitution for predicate and function letters.

Frege does permit substitution of formulas for propositional letters. He derives a formula or demonstrates its validity and uses substitution instances of the formula to derive other formulas. He treats propositional letters as variables and substitutes more complicated propositional and quantificational formulas for them. He says that expressions that give way to their various substitution instances "contain" those instances as "special cases".\(^{23}\)

In most derivations Frege cites the number of a judgment to be used in the derivation and indicates the substitutions to be made by a table under the citation. It is left to the reader to construct the appropriate substitution instance of the cited formula. The table serves as an abbreviation. For example,

23. Ibid., p. 16.
(1):

```
  a
  +---+---+
  |   |   |
  c   b   a
  +---+---+
  |   |   |
  a   b   c
```

is an abbreviation for:

in judgment (1) put

```
  a
  +---+---+
  |   |   |
  c   b   a
  +---+---+
  |   |   |
  a   b   c
```

for 'a', and

```
  a
  +---+---+
  |   |   |
  a   b
```

for 'b'.

The rule that governs the derivation of quantificational formulas from truth functional formulas, and more complex formulas from simpler formulas can be stated as follows:

(S1) A formula A may be substituted for a propositional letter in a formula B provided that no free variable of A is captured by a quantifier of B.


Frege also allows substitution for functional letters or predicate letters. For example, consider the table:

(53):
\[
F(A) \quad (A \equiv C).
\]

Judgment (53) is the following:

\[
\begin{array}{c}
\text{(53)} \\
\frac{f(d)}{c \equiv d} \\
\frac{c \equiv d}{f(c)} \\
\end{array}
\]

Hence the substitution instance constructed from the table is:

\[
\begin{array}{c}
d \equiv c \\
\frac{c \equiv d}{c \equiv c} \\
\end{array}
\]

Frege's use of the capital Greek letters 'A', 'B', and 'r' in such tables is similar to Quine's use of circled numerals '1', '2', '3', ... as place holders in schematic predicates. A schematic predicate, for Quine, is like an open sentence except that it contains place-holders instead of some or all of the free variables. For example,

'F 1', 'G 1 2', and '(\exists x)(F 1 \ x \ & \ Gx 1)'

are schematic predicates. Monadic schematic predicates may be substituted for monadic predicates and, in general, n-place schematic predicates may be substituted for n-place predicates.
In the table

\[ f(A) \mid (A \equiv C) \]

'A' acts like Quine's '[1]'. The table indicates that

'[1] \equiv c)' is to be substituted for the monadic 'f'.

Appropriate restrictions must be placed on the substitution of schematic predicates for predicates or function letters. It must be ensured that no variable of the substituted schematic predicate becomes captured by a quantifier of the formula in which it is substituted, and no variable of the formula is captured by a quantifier of the predicate. The rule may be stated as follows:

(S2) A schematic predicate may be substitute for a function letter or predicate in a formula A provided that no free variable of the predicate is captured by a quantifier of A and no free variable of A is captured by a quantifier of the predicate.

It remains to show that the rules of substitution and the comprehension axiom schema of Robbin's system are equivalent. In the proofs that follow I take for granted the rules of existential generalization, existential instantiation, and other basic logical rules that are common to first-order and second-order quantification. The justifications of these rules do not involve either substitution or the comprehension

axiom.

First, to show that substitution implies the comprehension axiom schema, note that

\[(1) \forall x \ (G(x) \leftrightarrow G(x))\]

is a theorem. By existential generalization we can derive

\[(2) \exists F \ (\forall x \ (F(x) \leftrightarrow G(x))).\]

By substituting \(\phi\) for \(G\) in (2) we get

\[(3) \exists F \ (\forall x \ (F(x) \leftrightarrow \phi(x)))\]

which is the comprehension axiom.

The other direction is the derivation of substitution from the comprehension axiom. Given some theorem

\[(1) \ldots G( ) \ldots\]

where \(G\) is a functional variable, we want to show

\[(2) \ldots \phi( ) \ldots\]

to be a theorem as well, where \(\phi\) is a formula substituted for \(G\) in (1). Frege's substitution table would look like:

\[
\begin{array}{c|c}
G(r) & \phi(r). \\
\end{array}
\]

In broad outline, the proof goes as follows. First we show
that

\[(3) \ (\forall x)(F(x) \iff \phi(x)) \supset (...)F(\ldots) \iff \ldots \phi(\ldots)\]

holds for all formula contexts.

The proof is by mathematical induction on the complexity of the formula context in which 'F' occurs. An application of universal generalization to (3) yields

\[(4) \ (\forall F) [(\forall x)(F(x) \iff \phi(x)) \supset (...)F(\ldots) \iff \ldots \phi(\ldots)]\]

Since 'F' does not occur free in 'ϕ', (4) is equivalent to

\[(5) \ (\exists F)(\forall x)(F(x) \iff \phi(x)) \supset (...)F(\ldots) \iff \phi(\ldots)\]

which is equivalent to

\[(6) \ (\exists F)(\forall x) (F(x) \iff \phi(x)) \& (...)F(\ldots) \supset \ldots \phi(\ldots)\]

Formula (1) is a theorem, hence we may apply universal generalization to get

\[(7) \ (\forall F) (...)F(\ldots)\]

and by universal instantiation we get

\[(8) \ \ldots F(\ldots)\]

By logic, the comprehension axiom,
(9) \((\exists F)(\forall x)(F(x) \leftrightarrow \phi(x))\),

together with (8) yields

(10) \((\exists F)(\forall x)(F(x) \leftrightarrow \phi(x)) \land \ldots F(\ldots)\ldots\).

Modus ponens applied to (6) and (10) yields

(2) \ldots \phi(\ldots)\ldots

which is what we set out to derive. Hence substitution follows from the comprehension axiom.

I have shown that Frege's substitution rules are equivalent to Robbin's comprehension axiom schema. The reconstruction of the Begriffsschrift is now complete. Frege's system is equivalent to Robbin's second-order predicate calculus. Thus in order to show that Frege's system is consistent, all that remains to show is that Robbin's second-order logic is consistent.

The function \(e\) is defined inductively as follows:

(i) If \(A\) is an atomic formula, \(e(A)\) is the result of erasing all the individual variables and constants of \(A\).

(ii) If \(A\) is a quantification \(\forall yB\), \(e(A)\) is \(e(B)\).

(iii) If \(A\) is a negation \(\neg B\), \(e(A)\) is \(\neg e(B)\).

(iv) If \(A\) is a conditional \(B \supset C\), \(e(A)\) is \(e(B) \supset e(C)\).

(v) If \(A\) is a quantification \(\forall yB\), \(e(A)\) is the conjunction of the result of substituting 'T' for 'V' in \(e(B)\) and the result of substituting 'L' for 'V' in \(e(B)\).
(vi) If $A$ is a quantification $\forall (\exists V)B$, $e(A)$ is the disjunction of the result of substituting 'T' for 'V' in $e(B)$ and the result of substituting 'F' for 'V' in $e(B)$.

We want to show that for each axiom $A$, $e(A)$ is a tautology and furthermore that the rules of inference preserve tautologousness. It will follow, then, that if $A$ is a theorem, $e(A)$ is a tautology.

For the first three axioms, (A1)-(A3), of Robbin's system, it is clear that an application of $e$ yields a tautology.

(A4) is universal instantiation:

$$e[\forall (\forall v)(A(v) \supset A(a))]$$

is a tautology of the form:

$$p \supset p$$

(A5) is confinement:

$$e[\forall (\forall v)(A \supset B) \supset (A \supset (\forall v)B)]$$

is a tautology of the form: $(p \supset q) \supset (p \supset q)$

(A6) is the comprehension axiom schema:

$$e[\forall (\exists F)(\forall x)(F(x) \leftrightarrow \phi(x))]$$

is $T' \supset e(\phi) \supset e(\phi)'$

is a tautology

(R1) is modus ponens:

If $e(A)$ and $e(A \supset B)$ are tautologies, then so is $e(B)$

(R2) is universal generalization:

If $e(A)$ is a tautology, then so is $e[(\forall v)A]'$, which is just $e(A)'$.

Hence for all formulas $A$, if $A$ is a theorem, $e(A)$ is a tautology. If $B$ and $-B'$ were both theorems, then $e(B)$ and
Frege's *Begriffsschrift* is consistent. It does not lead to a paradox. What, then, is behind van Heijenoort's worry? In an attempt to answer this question, I turn now to the interpretation of Frege's theory.

Frege says very little concerning the interpretation of the system presented in the *Begriffsschrift*. In particular, he says little about how we are to interpret the second-order formulas of the third part of the work. It is these formulas that concern van Heijenoort. While the inclusion of second-order quantifications does not render the theory inconsistent, there are problems associated with the interpretation of these formulas that can be seen as anticipating problems that arise in Frege's later writings. In particular, Frege seems dangerously close to Russell's paradox and the problem with the concept *horse*. Before I discuss the interpretation of Frege's theory I shall take a brief look at the various sorts of semantics for the second-order predicate calculus.

There are three sorts of structures for second-order logic. I call these "pre-models", "general models", and
"standard models". A "pre-model" is an infinite sequence

\[ M = \langle D_1, D_2, D_3, \ldots \rangle \]

where \( D \) is a non-empty set of objects over which the
individual variables range, and each \( D_n \) is an arbitrary non-empty set of \( n \)-place relations on \( D \). Thus each element of
\( D_n \) is a subset of \( D^n \). All the axioms of the system we have
considered hold in these structures except for the comprehension axiom schema. Not every instance of the comprehension schema
holds in all pre-models, for each \( D_n \) need not contain all
the \( n \)-place relations on \( D \). For example, consider the following
instance of the comprehension schema:

\[ (\exists F)(\forall x)(F(x) \leftrightarrow G(x) \land H(x)). \]

This will not hold in a structure of this sort if the set
does not contain the intersection of each pair of its elements.

"General models" are just like pre-models except that they
are constructed so that each instance of the comprehension axiom schema holds in the model. Thus it is stipulated that
these models contain every relation that can be defined in
the structure by a formula of second-order logic. There is
a completeness proof for second-order logic relative to
models of this second kind. The proof is essentially the
same as the completeness proof for first-order logic,
for this version of second-order logic is, in a sense, reducible to first-order logic. Thus it can be shown that every formula true in all general models is provable.

The third sort of model is a "standard model". These are the intended models for full second-order logic. The other sorts of models are non-standard and give something less than full second-order logic. The standard models specify a non-empty set of objects over which the individual variables range. The domain for the second-order variables is the set of all n-place relations on the domain of individuals. It is well known that full second-order logic is incomplete in the sense that the set of valid formulas is not axiomatizable. Hence there are valid formulas of full second-order logic that are not provable in the system we have shown to be consistent.

Frege was not concerned with truth in models, nor did he ask whether his system is complete. It is not clear how Frege chose his axioms, but it is clear that he did not have in mind any of the three sorts of models I have described. I shall describe the semantics I believe Frege did have in mind for the system of his Begriffsschrift.

The universe of discourse for Frege's individual variables is not restricted; it is the entire universe. Although Frege never explicitly states this, it is implicit in the text of

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the Begriffsschrift. Frege translates the quantification:

\[ \forall a \, \psi(a) \]

as "the judgment that whatever we may take for its argument, the function is a fact"\(^{27}\), and

\[ \forall a \, \phi(a) \]

as "whatever \( a \) may be, \( \phi(a) \) must always be denied".\(^{28}\) In another place Frege writes, "\( \forall a \, f(a) \) means that \( f(a) \) takes place whatever we may understand by \( a \)".\(^{29}\) It is clear that Frege does not restrict the range of the individual variables; they range of "whatever we may take as value", that is, everything that there is. As a result, any formula

\[ \phi(x) \]

expresses a generality about everything in the universe. The notation of the universal quantifier is introduced both as a way to confine the scope of the generality to the antecedent of a formula and, in conjunction with negation, to express particular existential claims. We can assume Frege to have a certain interpretation in mind. I see no reason to think that

\[ \ldots \]

28. Ibid., p. 27.
29. Ibid., p. 51.
functional and predicate letters do not range over Fregean functions and concepts. Since the range of the individual variables is the entire universe, the range of the functional variables is included in the range of the individual variables. Frege does not distinguish different universes of discourse for different types of variables. As we have seen, Frege makes no formal distinction between first-order and second-order quantification. Second-order quantification is introduced for the same considerations as is first-order quantification. Both follow naturally from his discussion of the function and argument.

It should be noted that although Frege's notions of function and argument are presented syntactically in the Begriffsschrift, in view of his later characterization of functions and concepts, function variables cannot range over sets of objects or individuals. Sets are "complete", or "saturated", and hence lack the requisite incompleteness of the function.

How, then, are we to interpret the second-order formulas of the Begriffsschrift? While Frege says little about quantificational formulas under interpretation, the text does provide a few clues. In his section on functions, Frege writes that the formula

$$\phi(A)$$
can be read as 'A has the property $\phi$'\textsuperscript{30}, and the formula

\[ \psi(A, B) \]

as 'B stands in the relation $\psi$ to A'.\textsuperscript{31} Thus one-place functional letters stand for properties and two-place functional letters for two-place relations. Frege translates

\[ \exists x \Lambda(x) \]

as 'there are $\Lambda$', where we call something that has the property $\Lambda$ a $\Lambda$. Hence when functional letters occur in quantifiers we can presume that they range over properties and relations. For Frege, an object has a property $\phi$ just in case that object falls under the concept $\phi$. In "On Concept and Object" Frege writes:

I call the concepts under which an object falls its properties; thus 'to be $\phi$ is a property of $\tau$' is just another way of saying '$\tau$ falls under the concept of a $\phi$'.\textsuperscript{32}

Frege does talk of the relation $\psi$ and the property $\phi$, yet we may assume that properties are Fregean concepts and that a suitable interpretation of the system must take care to

\textsuperscript{30} Ibid., pp. 23-24.
\textsuperscript{31} Ibid., p. 24.
\textsuperscript{32} Geach and Black, p. 51.
avoid running into the problem with the concept horse. In light of Frege's doctrines concerning concepts and objects, it is clear that the standard interpretation that allows functional variables to range over sets of objects is inadequate for Frege's theory.

In Part III of the *Begriffsschrift* Frege begins to use second-order quantification to develop his definition of the ancestral of a relation. Frege translates certain of the formulas of Part III into words. On examination of these translations it becomes evident that the functional letters are intended to range over properties and relations. In particular, formula (76), which is Frege's definition of the proper ancestral of a relation, contains a universal second-order quantifier. Frege says that (76) "can be rendered into words somewhat as follows":

If from the two propositions that every result of an application of the procedure f to x has property F and that F is hereditary in the f-sequence, it can be inferred, whatever F may be, that y has the property F, then I say: 'y follows x in the f-sequence'.

Here Frege translates the universal quantifier as "whatever F may be" and talks of "property F" and "the procedure f". Procedures are just relations and properties are concepts. Frege does not, however, give us any clear characterization of properties and procedures. He does, as I have pointed out,

33. van Heijenoort, p. 60.
lapse into using the definite article, he talks of "the property ..." and "the procedure...". Are these properties and relations saturated objects? Frege does say that the translation is a rough one, and I shall assume that his use of the definite article is due to an "awkwardness of language".

No matter how these properties and relations are characterized, if they exist then they must be contained in the range of the first-order variables; for the range of the first-order variables is the entire universe. However, if the range of the second-order variables is included in the range of the first-order variables, we find Frege very close to Russell's paradox. Consider:

\[(\exists F)(\forall x)(F(x) \iff \neg A(x,x))\]

The formula \((\ast)\) is an instance of the comprehension axiom schema and hence is a theorem of Frege's theory. Frege does not tell us how to interpret formulas for the form

\[(\exists F)(\ldots F(\ldots))\]

yet from what he does say it seems that such formulas are to be read as

There is a property such that ...

Let us say that an object has a property \(\phi\) just in case \(\phi\) applies to that object. If we suppose that properties exist, it follows that they are in the range of the individual variables. If
we let \( A(x,x) \) stand for 'x is a property that does not apply to itself' then (*) becomes:

\[
\text{There is a property that applies to an object if and only if that object is a property that does not apply to itself.}
\]

On the supposition that properties are themselves things in the universe, this interpretation of (*) gives us Russell's paradox. The odious interpretation can be avoided if it is stipulated that the range of \( F \) is not included in the range of the individual variables. Yet then it would follow that some values of functional variables are not in the universe. This way out of the paradox is not suggested by anything Frege writes. He certainly talks as if properties and relations exist, and hence are in the universe. When introducing second-order quantifications he states no qualifications about the range of their variables. Just as first-order quantifications express generalities about things in the universe, so second-order quantifications express generalities about properties and relations. Any restrictions on the range of the second-order variables must be seen as ad hoc.

Frege says nothing to exclude properties and relations from the range of the individual variables, and he never considers alternative interpretations or models for his system. All this lends support to van Heijenoort's claim that Frege is "on the brink of a paradox". Although the system is a consistent one, the interpretation of the system that is
suggested by the text does lead to Russell's paradox.

Another problem that arises in connection with the interpretation of the system presented in the Begriffsschrift is one associated with the problem with the concept horse. If the values of the second-order variables are not saturated objects and have the required incompleteness of the Fregean function, the how do we accordingly interpret second-order existence claims? We have seen that the quantifier

There is such a thing as __________

has the peculiar feature that all its grammatical completions assert the existence of objects and not concepts or functions. The problem is: If properties and relations do exist, how can we name these values of second-order variables without naming objects?

Frege needs the means to express grammatically second-order existence claims. If properties and relations cannot be named by proper names then the natural language quantifier above is an inadequate translation of the second-order existential quantifier. Yet there seems to be no other way to interpret formulas such as

\[ \overline{\mathcal{F}} \overline{\mathcal{F}(a)} \].

There is another related problem. Consider the formula:

\((\forall x)(F(x) \lor -F(x))\).
This would be interpreted as meaning:

Everything is such that either it has property F or it does not have property F.

If properties are in the universe then one instance of this formula is:

Property F either has property F or it does not.

However, given Frege's doctrines concerning concepts and objects, this is ungrammatical. We cannot grammatically predicate having property F or failing to have property F of property F. Thus in light of the Fregean hierarchy of objects, concepts, and functions, neither disjunct is grammatical.

This would be avoided if the universe of discourse for the different types of variables were distinct. Yet they cannot be distinct if the individual variables range over everything that there is. Even if this problem were resolved, Frege would be left with the problem of grammatically asserting the existence of properties and relations.

Thus while van Heijenoort is incorrect in claiming that the system presented in the *Begriffsschrift* leads to an inconsistency or paradox, the interpretation suggested by Frege's text does anticipate some serious problems that arise in Frege's later writings.
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