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DYNAMICAL EQUATIONS OF MULTI-BODY SYSTEMS WITH APPLICATION TO SPACE STRUCTURE DEPLOYMENT

by

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DYNAMICAL EQUATIONS OF MULTI-BODY SYSTEMS WITH
APPLICATION TO SPACE STRUCTURE DEPLOYMENT

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ABSTRACT

This thesis develops a formulation of the dynamical equations of interconnected bodies. A primary goal of the work was that the formulation be useful for simulating the deployment dynamics of large space structures (LSS). The thesis is a continuation of the line of research which began in the 1960's with investigations by Hooker and Margulies and by Roberson and Wittenburg. The interfaces between bodies are modeled using "joints" which each can have between 0 and 6 degrees of freedom. The bodies can be rigid or nonrigid. The graph of the system can have closed loops. Active control of degrees of freedom of the system is permitted. The formulation combines the best features of ones set up previously by Bodley and his associates and by Willems and his students. It is developed using the transformation operator approach introduced by Jerkovsky. There are three main steps in the derivation. First, all the joints are cut, and the dynamical equations for the individual bodies are developed. Second, a selected subset of the joints are reconnected, and the dynamical equations for this new system are derived directly from those of the previous cut-joint system by a transformation of variables. Third, equations for the constraints at the joints which are not reconnected are specified and introduced into the formulation. The final chapter of the thesis applies the formulation in a computer simulation of the deployment of a simplified version of a large space-based radar antenna proposed by the Grumman Aerospace Corporation.

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James E. Keat

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CHAPTER 1

INTRODUCTION

1.1 Summary
This thesis develops a new formulation of dynamical equations of interconnected bodies. The work is a continuation of the line of research which began in the 1960's with investigations by Hooker and Margulies and by Roberson and Wittenburg. A primary goal of the present study was that the formulation be applicable to computer simulation of the deployment dynamics of large space structures.

The interfaces between bodies are modeled using "joints" which each can have between 0 and 6 degrees of freedom. The constraints of the joints can be holonomic or nonholonomic. The motions at the joints can be large. The individual bodies can be rigid or nonrigid. The graph of the system can have closed loops. Active control of degrees of freedom of the system is permitted.

The formulation combines the best features of ones set up previously by Bodley and his associates and by Willems and his students. In particular, it permits a selected blend of absolute velocities and relative velocities to be employed. The development utilizes the velocity transformation approach introduced by Jerkovsky.

There are three main steps in developing the formulation. First, all the joints are cut, and the dynamical equations for the individual bodies are derived. Second, a selected subset of the joints are
reconnected, and the dynamical equations for this new system are derived
directly from those of the cut-joint system by a transformation of vari-
ables. Third, equations for the constraints at the joints which are not
reconnected are specified and introduced into the formulation.

The technique is applied to two examples in the final two chapt-
ers. The first is a relatively simple system comprised of five rigid
bodies and five joints; this system is considered in the thesis only to
illustrate the basic features of the method. The second system is a much
more complex one: namely, a simplified version of a large deployable
antenna proposed by the Grumman Aerospace Corporation. A computer pro-
gram to simulate the dynamics of this second system during deployment was
developed, checked out, and run successfully.

1.2 Motivation

A large fraction of the space vehicles of the past and present
have consisted of a main body with attached secondary bodies or append-
dages. Space vehicles which do not fit this pattern, however, will
become more prevalent in the near future. Foremost among these are the
so-called large space structures (LSS). In addition to their large dim-
ensions, LSS will be characterized by considerable mechanical complexity
and by distributed flexibility throughout the entire vehicle.

The dynamics of space vehicles during deployment has always been
an area of concern. This has been true even when the deployments con-
sisted of relatively benign operations such as extending booms. Fre-
quently, the question of dynamics during deployment has been considered
serious enough to warrant extensive pre-launch analytical studies and
dynamics simulation work.

The post-launch operations which will be employed to make LSS
vehicles operational often will include partially automatic mechan-
ical "deployment" operations. On LSS missions, however, these
deployments usually will not consist merely of extending or otherwise erecting secondary bodies from a main body. Rather, they usually will be much more complicated mechanically and sometimes will involve the entire vehicle. This makes deployment of LSS an area of much present concern. Consequently, the capability of studying deployment dynamics by analytical and simulation means is even more essential to mission success on LSS than on satellites of the past.

The research described in this document was supported, in part, under funding provided by the Air Force for a broader task of developing a general purpose computer program to simulate the deployment dynamics of LSS vehicles. This made it essential that the dynamics formulation developed during the research encompass at least the most common anticipated classes of LSS deployments.

Multibody formulations provide the most suitable approach for modeling the dynamics of LSS during deployment. However, multibody theory currently is weak is several areas which are essential for such problems. In particular, current multibody theory is weak in its capability of handling systems with (1) closed topological loops and/or (2) nonrigid individual bodies. It was concluded, therefore, that the formulation developed in the present work must encompass systems with closed topological loops and with some or all bodies (highly) nonrigid and that, in fact, the effort should focus largely upon these two issues.

1.3 Synopsis of Previous Research

The term "multibody theory" is used in this document in reference to a technical discipline which had its main origin in the 1960's in response to needs of the aerospace industry. The two pioneering studies on the topic were performed by Hooker and Margulies (1) and by Roberson and Wittenburg (2). These two papers were, in large part, responsible for a small sunburst of subsequent research in the area which has continued to the present. References (3) to (32) constitute a partial bibliography of papers and documents on the subject.
The underlying common feature of the studies in the multibody area has been the use of the "joint" concept for modeling the interactions between the bodies. Emphasis has been given to (1) handling the constraints at the joints in a rigorous manner, and (2) including in a rigorous manner the mathematical nonlinearities induced by large relative motions at the joints and by large rotational motion and velocity of the system as a whole. Most researchers have been interested primarily in establishing nonlinear equations of motion to be solved by numerical integration on a digital computer.

The kinetics equations for multibody systems can be derived by Newton-Euler methods or by analytical dynamics methods. In the two original papers\(^{(1),(2)}\) in the area, both pairs of researchers chose the Newton-Euler method. However, the majority of recent multibody dynamics investigators have employed analytical dynamics techniques. Lagrange's equation was used by Bodley et al.\(^{(3)}\) and by H\(_{2}\)\(^{(8),(9)}\). D'Alembert's principle of virtual work was used by Boland et al.\(^{(4)-(6),(26)}\) by Lilov and Wittenburg\(^{(20)}\), and by Wittenburg\(^{(30)-(32)}\). Kane's technique\(^{(33)-(36)}\) was used by Levinson\(^{(17)}\) and by Huston et al.\(^{(14),(15)}\).

Both pairs of original researchers\(^{(1),(2)}\) limited the scope of their work to a rather restricted class of multibody systems. In particular, they considered only systems

1. with joints that have no degrees of freedom in translation,
2. with rigid individual bodies, and
3. with tree structure.

Also, their work was devoted almost entirely to systems whose joints possess the full three degrees of freedom in rotation.

More general joint models that can handle between 0 and 6 degrees of freedom were introduced in subsequent studies. Such models were used by Bodley et al.\(^{(3)}\), Boland et al.\(^{(4)-(6),(26)}\), Lilov and Wittenburg\(^{(20)}\), Roberson\(^{(23)}\), and Wittenburg\(^{(30)-(33)}\).
A multibody formulation which assumes the individual bodies to be rigid is handicapped in its application to space vehicles with secondary appendages because the appendages, often are quite flimsy and hence highly deformable. An alternate approach, called the hybrid coordinate method, is often used to model such systems more efficiently. Likins (37) usually is credited as being the originator of this alternate technique. The multibody and hybrid coordinate approaches merge in multibody problems in which some of the individual bodies are nonrigid and this nonrigidity is modeled using continuous generalized coordinates.

A number of researchers (3)-(6), (8)-(10), (18), (22), (26), (28) in the multibody discipline have developed formulations in which individual bodies are nonrigid. However, only a fraction of these formulations are intended for systems in which all the bodies are nonrigid or in which arbitrary bodies can be rigid or nonrigid. Instead, emphasis has been placed on systems in which only the outermost bodies of the tree are nonrigid. Researchers who developed formulations in which any body can be rigid or nonrigid are Bodley et al (3), Boland et al (4)-(6), (26), Ho et al (8), and Singh and Likins (28). The last two of these studies, however, were restricted to the simple case in which the bodies are connected in an open chain.

A system of bodies and joints is said to have tree structure if it is possible to pass, via the joints, from any body to any other body by exactly one path. If more than one such path exists between one or more pairs of bodies, the system is said to have closed loops. The multibody dynamics researchers have been particularly sluggish in eliminating the restriction in Refs. 1 and 2 to the systems with tree structure. Bodley et al (3), Boland et al (5), Lilov and Wittenburg (20), and Wittenburg (30)-(32) are the only aerospace researchers to develop formulations which encompass systems with closed loops.
Bodley et al\(^{(3)}\), Boland et al\(^{(4)-(6),(26)}\), Lilov and Wittenburg\(^{(20),(30)-(32)}\), and Jerkovsky\(^{(16),(38)}\) are the researchers whose work was of most value for the study described in this document. All four of these groups of investigators used joint models which can handle 0 to 6 degrees of freedom. As noted above, Bodley, Boland, and Wittenburg developed formulations which encompass systems with closed loops. The work of Bodley and Boland encompasses systems in which arbitrary bodies can be rigid or nonrigid; that of Wittenburg is limited to systems of rigid bodies. Jerkovsky dropped a few remarks on the application of his methods to systems with closed loops and nonrigid bodies; in his multi-body paper\(^{(16)}\), however, the bulk of his mathematics was limited to systems of rigid bodies and tree structure.

With Bodley's approach, one starts, in effect, by drawing a free body diagram of the system with all joints being cut. Kinetics equations for the individual bodies are established using absolute velocities; that is, the velocity state of each individual body is defined relative to an inertial frame. Constraints at the joints are introduced by the Lagrange multiplier method. Holonomic joint constraints are handled by differentiating them and treating them as constraints on the velocities. Bodley developed a general purpose computer program using this approach, thereby verifying its feasibility and usefulness for realistic aerospace problems.

With Boland's technique, one cuts only enough joints to eliminate all closed loops. The configuration of the system is specified by (1) the location and orientation of a selected reference body relative to an inertial frame, (2) the relative coordinates at the uncut joints, and (3) coordinates which specify the deformations of the nonrigid bodies. The kinetics equations are established using this set of coordinates; Boland employed the principle of virtual work to develop these equations. Boland's mathematics is limited to systems in which the constraints at the uncut joints are holonomic. The Lagrange multiplier
approach could be used to handle the constraints at the cut joints. Boland, however, did not use Lagrange multipliers. Instead, he introduced a specialized approach which involves making joint cuts only at joints which possess no degrees of freedom; he was willing to introduce fictitious zero degree of freedom joints into the problem, if necessary, in order to be able to accomplish this. One of the weaknesses of Boland's approach to systems with closed loops is that he did not fully develop this specialized technique nor show how it could be implemented in a dynamic simulation. Boland left his equations of motion in an abstract form which is not at all immediately applicable for use in computer programs. In particular, he did not introduce generalized coordinates to specify body nonrigidity; rather, his final equations of motion are a hybrid set which includes partial differential equations for body deformations and ordinary differential equations for the other variables.

Wittenburg's approach basically is almost identical to that of Boland. As noted earlier, however, Wittenburg presented mathematics only for systems in which all individual bodies are rigid. Also, Wittenburg handled constraints at the cut joints by the Lagrange multiplier method. One of the drawbacks of the Lagrange multiplier method is that it can introduce numerical problems when used in computer programs. Wittenburg recommends a technique devised by Baumgarte\(^{39}\) as a means of alleviating these numerical difficulties. Wittenburg's formulation is described in considerable detail in Ref. 30; Likins' critique\(^{40}\) of this reference is interesting.

Jerkovsky has done some study on the dynamic modeling of single deformable bodies\(^{41}\). However, in his paper\(^{16}\) on the multibody problem, he presented mathematical details only for systems in which all bodies are rigid. Jerkovsky's work is dominated by his interest in a special technique for deriving equations of motion of dynamic systems. He calls this technique the transformation operator method. His main expose of it is presented in Ref. 38. The heart of Jerkovsky's method is
a velocity transformation. With Jerkovsky's basic method one can, for example, start out with equations of motion in absolute velocities a la Bodley and convert these to a set in relative velocities a la Boland by means of a transformation of variables. This can be a simpler technique for obtaining equations of motion in relative velocities than proceeding directly (a la Boland) would be.

In summary, the main features of previous multibody studies which, from the present point of view, are regarded unfavorably are as follows. Bodley's formulation does not permit relative velocities at the joints to be employed. Conversely, Boland's formulations do not permit absolute velocities to be used. Also, Boland did not work out the full details of his technique for handling constraints at cut joints. Wittenburg's work is limited to systems of rigid bodies. Jerkovsky did not investigate systems with closed loops or nonrigid bodies. Neither Jerkovsky nor Boland developed their formulations to the level of detail which would make them readily implementable in a dynamics simulation. The other multibody formulations presented in the aerospace literature do not encompass systems with closed loops.

The study described in this document constitutes, in large part, an attempt to enhance current multibody theory by unifying and extending the methods and results of the researchers named in the above paragraph.
CHAPTER 2

SYNOPSIS OF PRESENT STUDY

The formulation of multibody system dynamics which is developed in this document is viewed, in part, as an amalgam of the best features of the formulations developed by Bodley and by Boland. In particular, it permits an arbitrary blend of absolute velocities and relative velocities to be employed.

After spending considerable time developing the kinetics equations directly by D'Alembert's principle of virtual work a la Boland and Wittenburg, it was concluded that Jerkovsky's transformation operator approach is a simpler and superior way of introducing relative velocities at the joints. Therefore, this is the approach described in this document. The formulation presented here thus can also be viewed, in part, as a generalization and extension of the mathematics presented by Jerkovsky in Ref. (16) to make it applicable to systems with nonrigid bodies and/or closed loops.

Considerable emphasis is given in the formulation to incorporating body nonrigidity velocities and relative joint velocities which are driven by on-board control systems. The technique employed here to introduce the commanded velocities at cut joints is based on a method used by Bodley.

The modeling of body nonrigidity kinetics is considered in more detail in this document than in any of the aforementioned multibody system references. The kinetics equations presented here, in principle
at least, are applicable when the motions between particles within individual bodies are large. In particular, the generalized coordinates $\mathbf{q}_Z$ which model the nonrigidity of the individual bodies enter the kinetics equations directly only through integrals $\int f(\mathbf{q}_Z)dm$ that are taken over the bodies. The individual-body kinetics equations which are developed here are viewed as an extension and generalization of results presented in Ref. 42.

In developing the formulation, we start, in effect, by cutting all the joints of the system. The kinetics equations for the individual bodies are then derived, in absolute variables, as in Bodley's formulation. D'Alembert's principle of virtual work is used in the derivation. The kinetics equations for each body next are stacked one on top of the other and a "system notation" is introduced, thereby generating the kinetics equation of the so-called primitive system. This equation is of the form

$$M\dot{\mathbf{v}} = \mathbf{f} + f_C$$  \hspace{1cm} (2-1)

The term $f_C$ above is the vector of the unknown constraint forces at the joints. The velocity vector $\mathbf{v}$ is obtained by stacking the generalized velocity vectors of the individual bodies. (Note that Eq. (2-1) and the subsequent equations in this section are simplified versions of those which are developed in the following chapters.)

In addition, a constraint equation is established for every joint which has constraints. The constraints at each joint are modeled as constraints on the velocities at that joint. These equations for the individual joints then are stacked and manipulated into a composite-system constraint equation of the form

$$\mathbf{c} = \mathbf{Cy} - \mathbf{b} = \mathbf{0}$$  \hspace{1cm} (2-2)
A number of supplementary equations for the system also must be established: namely, those for kinematics and, when necessary, those for the on-board control systems. The system kinematics equation can be written as

\[ \ddot{u} = Rv \]  

(2-3)

where \( u \) is a configuration vector of the primitive system.

A velocity transformation now is introduced a la Jerkovsky. In the present document, this transformation is viewed as reconnecting selected joints of the system. The velocity transformation equation is of the form

\[ \dot{v} = T\dot{v}' \]  

(2-4)

\( \dot{v}' \) above can be composed of both absolute velocities and relative velocities, the relative velocities being those at the reconnected joints.

Inserting Eq. (2-4) into (2-1) and (2-2) and premultiplying the first of these by \( T^T \) produces the kinetics and constraint equations for the transformed system

\[ T^TMT\dot{v}' = -T^TMT\dot{v} + T^Tf + T^TE_c \]  

(2-5)

\[ c = CT\dot{v}' - b = 0 \]  

(2-6)

As is discussed later, \( \dot{v}' \) is specified in such a manner that constraints and constraint forces at reconnected joints drop out of the mathematics. The term \( T^TE_c \) in Eq. (2-5) hence will result solely from the constraints at the unreconnected joints; also, Eq. (2-6) will be comprised only of constraints at unreconnected joints.
A new kinematics equation also is needed. In the present work, a configuration vector \( \underline{u}' \) for the transformed system is established, and a kinematics equation of the form

\[
\underline{u}' = R' \underline{v}'
\]  

(2-7)

is developed.

Two techniques are presented for handling the transformed constraint force vector \( T^r_{f_c} \) of Eq. (2-5) for those problems in which it is not identically \( 0 \). The first of these is the traditional Lagrange multiplier approach. The material presented on this approach includes the minor extension needed for incorporating Baumgarte's technique for constraint stabilization. The second technique involves partitioning \( \underline{v}' \) into

\[
\underline{v}' = \begin{bmatrix}
\underline{v}'_r \\
\underline{v}'_s
\end{bmatrix}
\]  

(2-8)

and solving Eq. (2-6) for \( \underline{v}'_s \). One can then write an equation of the form

\[
\underline{v}' = \Pi \underline{v}'_r + \underline{\xi}
\]  

(2-9)

Eq. (2-9) is differentiated and inserted into Eq. (2-5) to eliminate \( \underline{v}'_s \); the constraint force term falls out of the resultant equations.

Section 4 of this document describes the modeling approach which is employed to specify the dynamics of the individual bodies of the system. The derivation of the kinetics equation for a single nonrigid body is sufficiently lengthy that it has been relegated to an appendix: namely, Appendix A. The main outputs from Chapter 4 are the kinetics and kinematics equations of the primitive system in absolute velocities.
Chapter 5 presents the modeling approach which is used for the joints. The main output from this chapter is the constraint equation for the composite system of joints.

The transformation of variables is introduced in Chapter 6. Some of the lengthier derivations (such as establishing the terms $T$ and $Tv'$ of Eq. (2-5)) have been relegated to Appendices B to E. The main output from Chapter 6 is the set of dynamical equations for the transformed system. This set of equations is considered to be the main result of the research described in this document.

The formulation is applied to a relatively simple illustrative example in Chapter 7. The system contains 5 rigid bodies, 5 single degree of freedom joints, and one closed loop. Equations of motion for this system are set up using the methods of Sections 4 to 6.

In Chapter 8, the approach is applied to a more complex problem: namely, the deployment of a large space structure. The LSS which is considered is a simplified version of a bicycle wheel configuration antenna proposed by the Grumman Aerospace Corporation. Equations of motion and a computer simulation of this problem were set up.

Appendix F considers the problem of altering the formulation to make it applicable to multibody systems in which there is mass flow between bodies. The material in Appendix F was necessary for the sample deployment problem discussed in Chapter 8. Apparently, the only previous multibody work which encompasses systems with mass flow between bodies is an unpublished study done by Roberson\textsuperscript{43}. 

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CHAPTER 3

NOTATIONAL TECHNIQUES

In the following chapters, most of the symbols that are not standard or obvious are defined the first place they appear. Also, most of the symbols are defined in the Nomenclature section at the end of the document. A few of the more significant notational methods are indicated in the following paragraphs.

An underbar is used to denote all column matrices and vectors except Gibbs vectors and Gibbs vector components. Gibbs vectors (i.e. vectors in three-dimensional "physical" space) are denoted by an overbar. A distinction is made between Gibbs vectors and their components. 3 × 1 matrices of Gibbs vector components are indicated by an overhead bar with the resolution coordinate frame indicated, in parenthesis, as a superscript. The skew-symmetric dyadic or matrix representations of Gibbs vectors and their components are indicated by the use of a tilde in place of the overbar. Unit Gibbs vectors are indicated via ê with identifying labels. A triad of orthogonal unit vectors is denoted by E; as a specific example, if ê_i^i, i = 1, 2, 3 are the unit vectors along the axes of frame v, then

\[ E_v \triangleq [ê^1_v \ ê^2_v \ ê^3_v] \]  

(3-1)
Gibbs displacement vectors are denoted by \( \bar{\mathbf{x}} \) with identifying labels; e.g. \( \bar{\mathbf{x}}_{\mu \nu} \) is the vector from the origin of frame \( \mu \) to the origin of frame \( \nu \). Similarly, all angular velocity vectors are indicated as \( \bar{\omega} \), with \( \bar{\omega}_{\nu \mu} \) being the angular velocity of Frame \( \nu \) relative to Frame \( \mu \).

Time derivatives of all terms except Gibbs vectors always are indicated by overhead dots. Overhead dots also are used to denote the time derivatives of Gibbs vectors relative to the inertial frame \( \mathbf{N} \). An overhead asterisk is employed to denote the time derivatives of Gibbs vectors relative to their base frame; e.g. \( \bar{\mathbf{x}}_{\nu \mu}^{(\xi)} \) is the time derivative of \( \bar{\mathbf{x}}_{\nu \mu} \) relative to frame \( \mu \). A more general notation for Gibbs vector time derivatives is used in a few places; for example \( \bar{\mathbf{x}}_{\nu \mu}^{(\xi)} \) is the time derivative of \( \bar{\mathbf{x}}_{\mu \nu} \) relative to frame \( \xi \).

Superscript label \( \mathbf{T} \) denotes transposition. This label usually is not introduced with the pre-multiplier term in dot products or inner products.

Partial derivatives with respect to a scalar are denoted via the usual comma technique. Partial derivatives with respect to a vector, say \( \mathbf{q} \), are indicated via \( (,\mathbf{q}) \) and are arranged in a row array; for example

\[
\left[ \bar{x}, \bar{x}_1, \bar{x}_2, \bar{x}_3 \right]
\]

Column arrays which can be comprised of a mixture of vectors, Gibbs vectors, and scalars are used in the work. For example

\[
\mathbf{v} = \left\{ \begin{array}{c} \bar{x}_{\mathbf{ba}} \\ \bar{\omega}_{\mathbf{ba}} \\ \mathbf{\bar{q}} \end{array} \right\}
\]
In this document, such quantities are called vectors. Similarly, the work employs rectangular matrix-like arrays whose elements are Gibbs vectors, dyadics, or gradient operators. For example

\[ T = \begin{bmatrix} \mathbf{I} - \mathbf{x} & \mathbf{x} \mathbf{q}_1 & \mathbf{x} \mathbf{q}_2 \end{bmatrix} \]  

(3-4)

In this document, such quantities are called arrays or, sometimes, operators. The word "matrix" is reserved for the more standard rectangular arrays whose elements are scalers.

In this document, equations which involve the so-called arrays and vectors noted in the previous paragraph are always set up in such a manner that they can be converted easily into the more common representation involving rectangular and column matrices with no change in the structure of the equation. All that needs to be done to accomplish this is to introduce a compatible coordinate frame resolution on the Gibbs vectors and dyadics. For example, the equation

\[
\mathbf{v} = \begin{bmatrix} \mathbf{I} - \mathbf{x} & \mathbf{x} \mathbf{q}_1 & \mathbf{x} \mathbf{q}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \omega \\ \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix}
\]  

(3-5)

can be converted to the usual matrix-equation form by introducing a coordinate frame resolution to produce
\[ \mathbf{v}(b) = [I - x(b) \ x, q_1 \ x, q_2] \begin{bmatrix} \mathbf{v}(b) \\ \mathbf{w} \\ \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} \] (3-6)

A self-defining notation is used with most operators and arrays. To wit, such quantities are denoted by the symbol \( T \) with identifying subscript labels. For example

\[ \mathbf{v} = T_{vw} \mathbf{w} \] (3-7)

The symbol \( S \) denotes column block subarrays of \( T \). For example, if

\[ \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \] (3-8)

then Eq. (3-7) can be written in the form

\[ \mathbf{v} = [S_{vw} \ S_{vw}] \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} = S_{vw} \mathbf{w}_1 + S_{vw} \mathbf{w}_2 \] (3-9)
CHAPTER 4

BODY MODELING AND DYNAMICAL EQUATIONS IN ABSOLUTE VARIABLES

4.1  Geometry of Single Body

The main geometric variables are shown in Fig. 4-1. The upper case Greek letter \( \zeta \) denotes an arbitrary body of the system. \( \zeta \) denotes the dynamic reference frame of body \( Z \). It is assumed that a frame \( \zeta \) has been specified for each body \( Z \). It is also assumed that an inertial frame, frame \( N \), for the composite system has been prescribed.

The manner in which each frame \( \zeta \) is attached to its body \( Z \) is not prescribed explicitly in the work. \( \zeta \) can be attached to a mass particle of \( Z \). Also, \( \zeta \) can be permitted to float relative to \( Z \) subject to the restriction that constraint equations are not needed to specify the motion of \( \zeta \) with respect to \( Z \). (An example of a floating frame is one which is attached to the c.m. of a nonrigid body with its axes continuously aligned with the principal inertia axes of that body). The topic of floating coordinate frames and the conditions under which their constraint equations drop out of the modeling has been discussed in recent aerospace literature \((19),(28),(44)-(46)\).

In order to explain the terms on Fig. 4-1 which have the subscript label \( \alpha_{n\zeta} \), it is necessary to briefly describe the "joint" technique used to model the interactions between bodies. (The topic is discussed in more detail in Chapter 5). The inter-body reactions are modeled using
Figure 4-1. Single body geometry.
elements which are called joints and designated by the upper case Greek letter $A$. Each joint $A$ has an input end ($n = 1$) and an output end ($n = 0$). As is discussed in Chapter 5, the input end is the end on the path back to the selected reference body of the tree. A coordinate frame $\alpha_n = (\alpha_1, \alpha_0)$ is prescribed for each of the two ends of every joint.

Bodies which react directly with one another are said to be "contiguous". There is at least one joint for each pair of contiguous bodies. The two bodies which are associated with a given joint $A$ are designated as $Z_{\alpha_n}$, $n = 1, 0$. Each joint coordinate frame $\alpha_n$ is "attached" to its body $Z_{\alpha_n}$ in some prescribed manner. The notation $\alpha_{n2}$ denotes a joint coordinate frame $\alpha_n$ which is attached to a given body $Z$.

The reactions between bodies are assumed to occur at discrete points on those bodies: namely, at the origins $O_{\alpha_n}$ of the frames $\alpha_n$. Cases in which the reactions between a pair of bodies $Z_{\alpha_1}$ and $Z_{\alpha_0}$ cannot be adequately modeled by assuming that they occur at a discrete point $O_{\alpha_1}$ and $O_{\alpha_0}$ on each body can be approached by introducing additional joints between the two bodies.

The terms $\overline{F}_{\alpha_{n2}}$ and $\overline{G}_{\alpha_{n2}}$ in Fig. 4-1 are the force and moment vectors respectively which are applied to body $Z$, through joint $A$, at $O_{\alpha_{n2}}$.

It is assumed that a set of generalized coordinates $q_Z$ has been established for each nonrigid body $Z$. Bodies which are rigid do not require a $q_Z$. $q_Z$ specifies the instantaneous configuration (i.e., the mass distribution) of body $Z$ relative to frame $\xi$. Thus, the location vector $\overline{x}_{P\xi}$ of an arbitrary differential mass element $P$ can be specified via
\[-x_{p\zeta} = \sum_{i=1}^{3} e_{\zeta}^i x_{p\zeta}^{i}(q_{Z}) = E_{\zeta} x_{p\zeta}^{\zeta}(q_{Z})\]  \hspace{1cm} (4.1-1)

(It is noted that the notational technique on the far right side of Eq. (4.1-1) is used frequently in this document). The work in this document does not restrict the functional dependency of \(-x_{p\zeta}^{\zeta}\) on \(q_{Z}\) to be linear.

The study assumes that the locations and orientation, relative to \(\zeta\), of the joint coordinate frames \(\alpha_{nZ}\) vary only with \(q_{Z}\). Thus

\[-x_{\alpha_{nZ}\zeta} = E_{\zeta} x_{\alpha_{nZ}\zeta}^{\zeta}(q_{Z})\]  \hspace{1cm} (4.1-2)

\[-\psi_{\alpha_{nZ}\zeta} = \psi_{\alpha_{nZ}\zeta}(q_{Z})\]  \hspace{1cm} (4.1-3)

The \(\psi\) term in Eq. (4.1-3) is a vector of attitude parameters, such as Euler angles or Euler symmetric parameters, which specifies the orientation of frame \(\alpha_{nZ}\) relative to frame \(\zeta\).

In the study, \(q_{Z}\) is partitioned into two subvectors

\[q_{Z} = \begin{pmatrix} q_{Z1} \\ q_{Z2} \end{pmatrix}\]  \hspace{1cm} (4.1-4)

The elements \(q_{Z1}\) of \(q_{Z}\) are called passive generalized coordinates. Kinetics equations for their second derivatives can be developed using solely the methods of mechanics as is done in Appendix A. Thus, the \(q_{Z1}\) are the usual generalized coordinates employed in analytical dynamics. An example of a \(q_{Z1}\) vector in, say, a cantilever beam problem is one whose
elements are the amplitudes of a set of assumed bending modes.

The elements $q_{zz}^1$ of $q_{zz}$ are called active generalized coordinates. The $q_{zz}$ are those generalized coordinates whose dynamical equations cannot be developed conveniently using the methods of mechanics. The responses of the $q_{zz}$ can be a priori known time functions. Alternatively, they can be specified as a function of other variables through control laws. The $q_{zz}$ concept is the technique which is used in this document to introduce rheonomic (i.e. explicit time-dependent) phenomena of the individual bodies. The common approach in abstract dynamics studies is to indicate such phenomena merely by denoting an unspecified dependency on time $t$ (e.g. $\bar{r} = \bar{r}(\mathbf{q}, t)$); this method, however, is regarded as being not sufficiently explicit for the present purposes. An example of an active generalized coordinate $q_{zz}^i$ is the length of a boom which is being deployed from a satellite by a servo motor.

The case in which the dynamics of $q_{zz}$ are governed by on-board control systems is of particular current interest. The work will assume that the response $\ddot{q}_{zz}$ either is an a priori known time function or else is specified directly by a control law.

The following vector $\mathbf{u}_Z$ will be introduced to specify the configuration of body $Z$ relative to frame $N$

$$\mathbf{u}_Z = \begin{bmatrix} \bar{x}_{\mathcal{CN}} \\ \bar{\psi}_{\mathcal{CN}} \\ q_Z \end{bmatrix}$$  \hspace{1cm} (4.1-5)

In subsequent sections, it will be convenient to coordinatize $\bar{x}_{\mathcal{CN}}$ along the axes of $N$. In this case, the following vector definition will be employed.
\[
\mathbf{u}_Z^{(N)} = \begin{Bmatrix} 
-x_{\zeta N} \\
\psi_{\zeta N} \\
\alpha_{Z_1}
\end{Bmatrix}
\] (4.1-6)

Also, subsequent work will require that \(u_Z\) be partitioned into passive and active subvectors as follows:

\[
\mathbf{u}_Z = \begin{Bmatrix} 
\mathbf{u}_{Z_1} \\
\alpha_{Z_2}
\end{Bmatrix}
\] (4.1-7)

where

\[
\mathbf{u}_{Z_1} = \begin{Bmatrix} 
-x_{\zeta N} \\
\psi_{\zeta N} \\
\alpha_{Z_1}
\end{Bmatrix}
\] (4.1-8)

A similar partitioning will be used with \(u_Z^{(N)}\) when necessary.

4.2 Kinematics of a Single Body

This section develops the formalism for specifying the velocities of point \(P\) and frame \(n_Z\) relative to frame \(N\). A composite velocity vector \(v_Z\) which specifies the velocity state of \(Z\) is introduced, and the relation between \(v_Z\) and \(\dot{u}_Z\) is specified.

Start with the kinematics of point \(P\). It is evident from Fig. 4-1 that

\[
\dot{x}_{PN} = \dot{x}_{\zeta N} + \dot{x}_{P\zeta}
\] (4.2-1)

Differentiating and employing the theorem of Coriolis produces
\[ \vec{x}_{PN} = \vec{v}_{\zeta N} + \omega_{\zeta N} \vec{x}_{P \zeta} + \vec{x}_{P \zeta} \]  

(4.2-2)

The vector \( \vec{v}_{\zeta N} = \vec{x}_{\zeta N} \) has been introduced in Eq. (4.2-2). Later work, however, will make a subtle distinction between \( \vec{x}_{\zeta N} \) and \( \vec{v}_{\zeta N} \): namely, \( \vec{x}_{\zeta N} \) will be considered to be expressed using frame N basis vectors, while \( \vec{v}_{\zeta N} \) will be considered to be expressed using the basis vectors of frame \( \zeta \).

\[ \vec{x}_{\zeta N} = B_\zeta \vec{x}_{N \zeta} \]  

(4.2-3a)

\[ \vec{v}_{\zeta N} = B_\zeta \vec{v}_{N \zeta} \]  

(4.2-3b)

An expression for the final term in Eq. (4.2-2) can be obtained by differentiating Eq. (4.1-1) to produce

\[ \vec{x}_{P \zeta} = \sum_i q_i \vec{x}_{P \zeta, i} = \vec{x}_{P \zeta, q^i} \]  

(4.2-4)

where

\[ \vec{x}_{P \zeta, i} = \frac{\partial}{\partial q^i} \vec{x}_{P \zeta} \]

Inserting Eq. (4.2-4) into (4.2-2) leads to

\[ \vec{x}_{PN} = T_{\zeta} \vec{v}_{P \zeta} \]  

(4.2-5)
where

\[
\mathbf{v}_Z = \begin{bmatrix} \mathbf{v}_{\xi N} \\ \omega_{\xi N} \\ \mathbf{g}_Z \end{bmatrix}
\]

and

\[
\mathbf{T}_{x_{PN}v_Z} = \begin{bmatrix} I & -\mathbf{x}_P \mathbf{\tau} & \mathbf{x}_P \mathbf{\tau} \mathbf{g}_Z \end{bmatrix}
\]

The vector \(\mathbf{v}_Z\) above is sufficient to specify the velocity of any particle of body \(Z\) relative to frame \(N\). In conjunction with the configuration vector \(\mathbf{u}_Z\) of Eq. (4.1-5), it specifies the complete state, \(\mathbf{s}_Z\), of body \(Z\) relative to \(N\).

\[
\mathbf{s}_Z = \begin{bmatrix} \mathbf{v}_Z \\ \mathbf{u}_Z \end{bmatrix}
\]

In later work it proves necessary to partition \(\mathbf{v}_Z\) into passive and active subvectors, \(\mathbf{v}_{Z1}\) and \(\mathbf{v}_{Z2}\) respectively, as was previously done with \(\mathbf{u}_Z\). To wit

\[
\mathbf{v}_Z = \begin{bmatrix} \mathbf{v}_{Z1} \\ \mathbf{v}_{Z2} \end{bmatrix}
\]

where
\[ \begin{pmatrix} \nu_{Z1} \\ \omega_{CH} \\ \varphi \end{pmatrix} \]

\[ \nu_{Z2} = \varphi_{Z2} \quad (4.2-11) \]

The partitioning of the \( T \) array of Eq. (4.2-7) which is compatible with the partitioning of \( \nu \) is as follows

\[
\begin{pmatrix} T_{\times} \\ x_{PN-Z} \end{pmatrix} = \begin{bmatrix} S_{\times} & S_{\varphi} \\ x_{PN-Z1} & x_{PN-Z2} \end{bmatrix}
\]

\[
= \begin{bmatrix} I & -x_{P\xi} \\ -x_{P\xi, Z1} & -x_{P\xi, Z2} \end{bmatrix} \quad (4.2-12)
\]

The second part of Eq. (4.2-12) defines the two \( S \) arrays which are listed in the first part.

Now consider the velocity of frame \( \alpha_{nZ} \) relative to frame \( N \). The equation for the translational velocity can be written immediately by analogy with Eq. (4.2-5).

\[
\begin{pmatrix} x_{\alpha \ N} \\ N \end{pmatrix} = \begin{pmatrix} T_{\times} \\ x_{\alpha \ N} \end{pmatrix} \quad (4.2-13)
\]

The \( T \) array in Eq. (4.2-13) is specified by Eq. (4.2-7 or 12) with \( P \) replaced by \( \alpha_{nZ} \).
To develop an expression for the rotational velocity of $\alpha_{nZ}$ start with

$$\overline{\omega}_{nZ}^N = \overline{\omega}_{N} + \overline{\omega}_{nZ} \zeta$$  \hspace{1cm} (4.2-14)

The final term in Eq. (4.2-14) is a linear function of $\dot{q}_Z$. In the present study this relation is denoted by

$$\overline{\omega}_{nZ} \zeta = \overline{\theta}_{nZ} \xi_\alpha \dot{q}_Z$$  \hspace{1cm} (4.2-15)

The $\theta$ array above is, in general, a function of $q_Z$.

Substituting Eq (4.2-15) into (4.2-14) and manipulating in order to introduce the vector $v_Z$ now produces

$$\overline{\omega}_{nZ}^N = T_{\overline{\omega}_{nZ}^N v_Z}$$  \hspace{1cm} (4.2-16)

where

$$T_{\overline{\omega}_{nZ}^N v_Z} = \begin{bmatrix} 0 & I_3 & \overline{\theta}_{nZ} \xi_\alpha \dot{q}_Z \\ \end{bmatrix}$$  \hspace{1cm} (4.2-17)

The following dual-vector which prescribes the translational and angular velocity of $\alpha_{nZ}$ relative to $N$ turns out to be useful
\[ \begin{pmatrix} x_a \, n_Z^N \vspace{1cm} \\ w_a \, n_Z^N \vspace{1cm} \\ -w_a \, n_Z^N \end{pmatrix} = \begin{pmatrix} v_{Z1} \vspace{1cm} \\ v_{Z1} \vspace{1cm} \\ v_{Z2} \end{pmatrix} \] (4.2-18)

The above vector can be expressed as a function of \( v_z \) by inserting Eqs. (4.2-13) and (4.2-17) into (4.2-18). For convenience in future reference, the result will be written using the partitioned form of \( v_z \). It is

\[
\begin{pmatrix} w_a \, n_Z^N \vspace{1cm} \\ \end{pmatrix} = \begin{pmatrix} S \, w_a \, n_Z^N \vspace{1cm} & S \, w_a \, n_Z^N \vspace{1cm} \\ \end{pmatrix} \begin{pmatrix} v_{Z1} \vspace{1cm} \\ v_{Z2} \end{pmatrix}
\]

\[
= \begin{pmatrix} I & -x_a \, n_Z \vspace{1cm} & -x_a \, n_Z \vspace{1cm} & \frac{x_a \, n_Z \vspace{1cm}}{a} \, \xi_g \, g_{Z1} \vspace{1cm} & \frac{x_a \, n_Z \vspace{1cm}}{a} \, \xi_g \, g_{Z2} \vspace{1cm} \\ 0 & I & \frac{\theta_a \, n_Z \vspace{1cm}}{a} \, \xi_g \, g_{Z1} \vspace{1cm} & \frac{\theta_a \, n_Z \vspace{1cm}}{a} \, \xi_g \, g_{Z2} \vspace{1cm} \\ \end{pmatrix} \begin{pmatrix} v_{Z1} \vspace{1cm} \\ v_{Z2} \end{pmatrix} \] (4.2-19)

The final portion of Eq. (4.2-19) defines the two S arrays in the middle portion.

Mathematics for transforming the passive and active subvectors, \( v_{Z1} \) and \( v_{Z2} \), of \( v_z \) into the corresponding subvectors, \( u_{Z1}^{(N)} \) and \( u_{Z2} \), of \( u_z^{(N)} \) will be needed later. The transformation equations can be deduced by comparing Eqs. (4.1-7 and 8) with Eqs. (4.2-9 to 11). The result is
\[ u_{Z1}^{N} = T_{*}^{N} v_{Z2} \]

\[ \dot{q}_{Z2} = v_{Z2} \]

where

\[ T_{*}^{N} = \text{Diag} \begin{bmatrix} E_{N}^{T} & T_{*}^{N} \omega_{N} \end{bmatrix} \]

4.3 Kinetics Equations of Single Body

The derivation of the kinetics equation for a single nonrigid body \( Z \) of a multibody system is sufficiently lengthy that it has been placed in Appendix A. The present section merely summarizes the results from Appendix A.

The kinetic equation, Eq. (A.4-1), for body \( Z \) which is derived in Appendix A is the following

\[ M_{Z1}^{*} v_{Z1} = -M_{Z2}^{*} v_{Z2} + f_{DZ}^{*} + f_{DZ}^{*} + f_{K}^{Z} + f_{C}^{Z} \]

The passive and active velocity vectors, \( v_{Z1} \) and \( v_{Z2} \), above were defined in Eqs. (4.2-9 to 11). In establishing \( v_{Z1}' \), the time derivatives of the two Gibbs vectors in \( v_{Z1} \) are taken with respect to Frame \( \zeta \).

\[ v_{Z1}' = \begin{bmatrix} (\xi) \\ (\xi) \\ (\omega) \\ \dot{\omega} \end{bmatrix} \]

\[ (\xi)_{\zeta N} \]

\[ (\omega)_{\zeta N} \]

\[ \dot{\omega}_{Z1} \]
The terms $M_{z1}$ and $M_{z2}$ in Eq. (4.3-1) are generalized mass arrays which are associated with $v_{z1}$ and $v_{z2}$. The equations which define them are Eqs. (A.3-16 and 17).

The term $f_{EIZ}^Z$ is the generalized force vector on body $Z$ due to all true external (E) and internal (I) forces on $Z$ except those at $Z$'s joints. $f_{EIZ}^Z$ is specified in Eq. (A.3-12).

$f_{DZ}^Z$ is the D'Alembert force on body $Z$. It consists predominately of the so called gyroscopic and Coriolis forces on $Z$ due to the acceleration of $Z$'s particles relative to frame $N$. More specifically, it is generated by the translational and angular motion of $Z$'s reference frame $\zeta$ relative to frame $N$. $f_{DZ}^Z$ also includes components due to second order terms in $\zeta$. $f_{DZ}^Z$ is specified in Eq. (A.3-18).

The vectors $f_{DZ}^{K}$ and $f_{DZ}^{C}$ in Eq. (4.3-1) are the generalized reaction forces applied to body $Z$ by its joints. $f_{DZ}^{K}$ consists of the "known" forces, and $f_{DZ}^{C}$ consists of the constraint forces. The so-called known forces at a joint are those which can be specified directly as functions of the relative displacements and velocities at that joint. Force and torsional springs and dampers across the joints are the most common devices that generate known joint forces. The constraint forces on the other hand are those which cause the geometrical or kinematical constraints at a joint to be satisfied. For example, at a pin-joint the constraint forces and torques (1) cause the two ends I and O of the joint to remain congruent and (2) prevent angular motion orthogonal to the axis of rotation of the joint.

The following equation for the two types of joint forces is developed in Appendix A.
\[ f_j^m = \sum_{n} \sum_{\alpha} S^\top_{n\alpha} L_{n\alpha}^m ; \ m = K, C \] (4.3-3)

In this equation, \( n = (I, O) \); the two summations thus encompass all joints with an end (end I or end O) attached to body Z. The \( S \) array is specified in Eq. (4.2-19). The \( L_{n\alpha}^m \) terms above contain both force and moment elements and, for this reason, are called dual-force vectors in the present document. They are obtained by separating the \( F \) and \( G \) vectors into known and constraint elements

\[
\begin{align*}
\bar{F}_\alpha &= \bar{F}_\alpha^K + \bar{F}_\alpha^C \\
\bar{G}_\alpha &= \bar{G}_\alpha^K + \bar{G}_\alpha^C
\end{align*}
\] (4.3-4)

\[
\begin{align*}
\bar{G}_\alpha &= \bar{G}_\alpha^K + \bar{G}_\alpha^C
\end{align*}
\] (4.3-5)

which are stacked to produce

\[
\frac{L_{n\alpha}^m}{L_{n\alpha}^m} = \begin{pmatrix}
\bar{F}_{n\alpha}^m \\
\bar{G}_{n\alpha}^m
\end{pmatrix} ; \ m = K, C
\] (4.3-6)

4.4 Summary of Multibody System Dynamical Equations in Absolute Variables

The main dynamical equations, in absolute variables, for the multibody system will now be summarized. For every body Z there is the kinetics equation previously listed as Eq. (4.3-1)

\[
\begin{align*}
\dot{M}_Z \ddot{v}_Z &= -M_{ZZ} \ddot{v}_Z + \frac{f_D}{D_Z} + \frac{f_{EI}}{E_I Z} + \frac{f_K}{k_Z} + \frac{f_C}{C_Z}
\end{align*}
\] (4.4-1)
For every Z there also are the kinematics equations previously listed as Eq. (4.2-20 and 21)

\[ v_{z1}^{*} = T_{0}^{*}(N) \cdot \frac{v_{z1}}{u_{z1}} \cdot \frac{v_{z1}}{v_{z1}} \quad (4.4-2) \]

\[ q_{z2}^{*} = \frac{v_{z2}^{*}}{v_{z2}} \quad (4.4-3) \]

If the system includes the active terms \( v_{z2}', v_{z2}, \) and \( q_{z2} \) then equations specifying \( v_{z2}^{*} \) also are required. These will be denoted as

\[ v_{z2}^{*} = g_{z} \quad (4.4-4) \]

Note that the present formalism assumes that \( v_{z2}^{*} \) is obtained by integrating \( v_{z2}^{*} \) and that \( q_{z2} \) is obtained by transforming (if necessary) and integrating \( v_{z2}' \). However, this is not the only possible approach. For example, in open loop control problems the responses of \( v_{z2}', v_{z2}, \) and \( q_{z2} \) sometimes are explicit functions only of time and thus known a priori.

Additional differential equations commonly are needed to supplement the above set in specific applications. In particular, in many problems, the control system, sensors and actuators possess significant dynamics. In order to include such dynamics in the formalism, the following equation will be included.

\[ x_{CS}^{*} = q_{z} \quad (4.4-5) \]

where \( x_{CS} \) is the state vector of the control system.
4.5 Multibody System Dynamical Equations in Absolute Variables Using Unified Notation

In this section each of the Eqs. (4.4-1 to 4.4-4) for the individual bodies \( Z \) will be rewritten as a single composite equation for the complete system of bodies. The resulting set of equations constitutes the dynamical equations of the primitive system.

It is assumed that an ordering scheme for the bodies has been established. Form a single kinetics equation for the composite system by stacking the kinetic equations, Eq. (4.4-1), for the individual bodies according to this ordering.

\[
\begin{bmatrix}
\dot{v}_{z_1} \\
\vdots \\
\dot{v}_{z_m}
\end{bmatrix} =
\begin{bmatrix}
-M_{z_1} \ddot{v}_{z_1} + \frac{f_{D_{z_1}}}{I_{z_1}} + \frac{f_{E_{I_{z_1}}}}{J_{z_1}} + \frac{f_{C}}{J_{z_1}} \\
-M_{z_2} \ddot{v}_{z_2} + \frac{f_{D_{z_2}}}{I_{z_2}} + \frac{f_{E_{I_{z_2}}}}{J_{z_2}} + \frac{f_{C}}{J_{z_2}} \\
\vdots \\
-M_{z_m} \ddot{v}_{z_m} + \frac{f_{D_{z_m}}}{I_{z_m}} + \frac{f_{E_{I_{z_m}}}}{J_{z_m}} + \frac{f_{C}}{J_{z_m}}
\end{bmatrix}
\]

Proceed similarly with Eqs. (4.4-2 to 4.4-4).

Now define composite-system variables to be introduced into the equations established in the above paragraph. Stack the vectors \( \dot{v}_{z_1} \) according to the body ordering scheme to produce a composite system velocity vector \( \dot{v}_z \).

\[
\dot{v}_z = \begin{bmatrix}
\dot{v}_{z_1} \\
\vdots \\
\dot{v}_{z_m}
\end{bmatrix}
\]

Proceed similarly with all other vectors which appear in Eqs. (4.4-1 to 4.4-4). For later use, define also
\[ u^{(N)} = \begin{cases} u_1^{(N)} \\ u_2 \end{cases} \]  
\hspace{1cm} (4.5-1a)

and

\[ v = \begin{cases} v_1 \\ v_2 \end{cases} \]  
\hspace{1cm} (4.5-1b)

Also arrange the \( M_{Z1} \) in a diagonal array according to the body ordering scheme and call the resultant system mass matrix \( M_1 \).

\[ M_1 = \text{Diag}[M_{Z1}]_{Z} = \begin{bmatrix} M_{Z1}^{1} \\ \vdots \\ \vdots \\ M_{Z1}^{m} \end{bmatrix} \]  
\hspace{1cm} (4.5-1c)

Do the same thing with the \( M_{Z2} \) and the \( T^{(N)} \).

Now introduce the system-variables defined in the above paragraph into the system-dynamical equations specified in the paragraph which preceded it. The result is

\[ M_{1} \dot{u}_1 = -M_{2} \dot{v}_2 + \dot{f}_D + \dot{f}_{EI} + \dot{f}_{KJ} + \dot{f}_C \]  
\hspace{1cm} (4.5-2a)

\[ \dot{u}_1 = T^{(N)} \begin{cases} v_1 \\ \dot{u}_1 \end{cases} v_1 \]  
\hspace{1cm} (4.5-2b)

\[ \dot{u}_2 = v_2 \]  
\hspace{1cm} (4.5-2c)
\[ \dot{v}_2 = g_{v_2} \]  \hspace{1cm} (4.5-2d)

\[ \dot{x}_{CS} = g_{x_{CS}} \]  \hspace{1cm} (4.5-2e)

Note that we have chosen to employ a subscript label \( J \) (for "Joint") with \( f_{KJ} \), but not with \( f_C \).

Eqs. (4.5-2) are the end result of this section.
CHAPTER 5

MODELING OF BODY INTERFACES AND CONSTRAINTS

5.1 Geometry of the Interfaces

5.1.1 Introduction

An introductory discussion of the body interface modeling technique used in this study was presented in Section 4.1. As noted there, the so-called "joint" concept is employed. The word "joint" is not especially appropriate here, since translational degrees of freedom, as well as rotational ones, are permitted at the interfaces.

The main geometric variables which are employed in this chapter are shown in Figure 5-1.

In the present study, each joint A is treated somewhat as a nonrigid body, of zero mass, which is subjected to no external forces or moments except the reaction ones with the pair of bodies, \( Z_{a_1} \) and \( Z_{a_0} \) at its input (I) and output (O) ends.

The configuration of joint A is specified completely by the location and orientation of its output frame \( a_0 \) relative to its input frame \( a_1 \). Thus each joint A has, at most, six degrees of freedom. Usually, however, joint A will have less than six degrees of freedom due to constraints which are imposed on the motion of frame \( a_0 \) relative to frame \( a_1 \).

The configuration of joint A will be specified by the following dual-displacement vector
Figure 5-1. Body interface geometry.
\[
Y_A = \begin{bmatrix}
-x_{A_0} \alpha_I \\
\psi_{A_0} \alpha_I
\end{bmatrix}
\quad (5.1.1-1)
\]

The \( \psi \) term above is a set of attitude variables, such as Euler angles or Euler symmetric parameters, which specify the orientation of \( A_0 \) relative to \( A_I \). In much of the subsequent work, it will be convenient to coordinatize \( x_{A_0} \alpha_I \) along the axes of coordinate frame \( A_I \). In this case, the notation

\[
Y_A = \begin{bmatrix}
\alpha_I \\
-x_{A_0} \alpha_I \\
\psi_{A_0} \alpha_I
\end{bmatrix}
\quad (5.1.1-2)
\]

will be used.

When developing the dynamical equations of Chapter 4, the point of view was taken that all joints of the system were cut. The transformation of variables which will be introduced in Chapter 6 will be viewed as reconnecting a selected set of the joints. Mathematically, the distinction between a cut joint and a reconnected one is based on the variables which are used to specify the velocity state of the reference frame \( A_0 \) of the body \( Z_{A_0} \) at the output end of the joint. For a cut joint, the velocity of \( A_0 \) is specified by absolute variables; that is by the vector \( v_{Z_{A_0}} \) which was defined in Eq. (4.2-6). Reconnecting the joint means introducing a new velocity formalism in which the elements \( \bar{v}_{Z_{A_0}} \bar{N} \) and \( \bar{\omega}_{Z_{A_0}} \bar{N} \) of \( v_{Z_{A_0}} \) are replaced by the relative velocities across joint \( A \).
In portions of the present chapter it will be desirable to distinguish between reconnected joints and cut joints. In such cases, the reconnected ones will be denoted by the upper case Greek letter B, and the cut ones will be denoted by \( \Gamma \). The coordinate frames which are associated with B and \( \Gamma \) will be indicated by the symbols \( \beta \) and \( \gamma \) respectively. The symbol \( A \) will be reserved for material which is applicable to both reconnected joints and cut joints. When referencing equations, the distinction between the labels A, B, \( \Gamma \) will be ignored, when possible.

5.1.2 Geometry of Reconnected Joints

It is assumed that a set of generalized coordinates \( g_B \) has been defined to specify the configuration of each reconnected joint B. Thus

\[
\begin{align*}
-x^{(\beta)}_{\beta O I} &= E_{\beta I} x^{(\beta)}_{\beta O I} (g_B) \\
\psi_{\beta O I} &= \psi_{\beta O I} (g_B)
\end{align*}
\]  

and hence

\[
\begin{align*}
(\beta)_{\beta I} &= (\beta)_{I} (g_B) \\
\chi_B &= \chi_B (g_B)
\end{align*}
\]  

The vector \( g_B \) must be chosen such that \( \dim g_B > h' \) where \( h' \) is six minus the number of holonomic constraints at joint B. In the majority of problems, it should be convenient to use \( \dim g_B = h' \). The modeling of three-dimensional attitude by Euler symmetric parameters or direction cosines is the most apparent example of a case which involves \( \dim g_B > h' \). When \( \dim g_B > h' \), the elements of \( g_B \) must satisfy certain mathematical constraints. For example, the Euler symmetric parameters \( \xi_I \) must satisfy
the constraint $\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 = 1$. The joint constraint formulation which is developed in Section 5.4 does not encompass constraints of this type. This type of constraint, however, generally is easier to handle in dynamic simulations than the type which is encompassed, since it is violated only as a result of imperfections, such as numerical round-off, in the simulation. The alleviation of such violations by ad hoc methods—such as the well known technique of normalizing the Euler symmetric parameter vector—usually is adequate.

5.1.3 Geometry of Cut Joints

The formulation which is developed in Chapter 6 is not readily amenable to the use of generalized coordinate vectors $q_I$ to specify the dual-displacement vector $y_I$ of cut joints $\Gamma$. Instead, it is more natural to specify $y_I$ as a function of the configurations of the two bodies, $z_I$ and $z_O$, which are adjacent to $\Gamma$. Therefore

$$y_I = y_I(z_I, z_O) \quad (5.1.3-1)$$

Utilizing the definitions of $y_I$ and $z_I$ which were indicated in Eq. (5.1.1-1) and (4.1-5), the more-detailed form of the above functional relationship can be determined to be

$$\bar{x}_{y_I} = \bar{x}_{y_I}(x_{y_I}, \psi_{y_I}, \kappa_{y_I}, g_{y_I}, x_{y_I}, \psi_{y_I}, \kappa_{y_I}, g_{y_I}) \quad (5.1.3-2)$$

$$\psi_{y_I} = \psi_{y_I}(x_{y_I}, \psi_{y_I}, \kappa_{y_I}, g_{y_I}, x_{y_I}, \psi_{y_I}, \kappa_{y_I}, g_{y_I}) \quad (5.1.3-3)$$

The present subsection presents mathematics for specifying $\bar{x}_{y_I}$ and
\( \Psi_{YI} \) as functions of the elements of \( u_{YI} \) and \( u_{YO} \).

Starting with the \( x \) term, the following equation can be developed with the aid of Figure 5-1

\[
\bar{x}_{YOI} = \sum_{n} \epsilon_n \left[ \bar{x}_{\xi_n} + \bar{x}_{\zeta_n} \right] \tag{5.1.3-4}
\]

In the above equation, the summation is taken over \( n = (I, 0) \). The term \( \epsilon_n \) is

\[
\epsilon_n = -1 \quad ; \quad n = I \tag{5.1.3-5}
\]
\[
= 1 \quad ; \quad n = 0
\]

The term \( x_{\xi_n} \) in Eq. (5.1.3-4) is an element of \( u_{\xi_n} \). The second \( x \) term on the RHS of the equation is specified by

\[
\bar{x}_{\xi_n} = E_{\xi_n} x_{\xi_n} \zeta_n \tag{5.1.3-6}
\]

\( q_{\xi_n} \) is an element of \( u_{\xi_n} \), and \( E_{\xi_n} \) is a function of \( \Psi_{YI} \) which also is an element of \( u_{\xi_n} \). Therefore, Eq. (5.1.3-4) specifies \( \bar{x}_{YOI} \) as a function of the configurations of \( Z_{YI} \) and \( ZYO \) as desired.
Now consider \( \Psi_{YO} \). Let the symbol \( T_{UV} \) denote the direction cosine matrix which transforms vector components from some frame \( U \) to some frame \( V \). With the aid of Figure 5-1, it can be deduced that

\[
T_{YO} = [T_{Y_O} \ z_{Y_O} \cdot T_{Y N} \cdot T_{Y_I} \ z_{Y_I}]^T
\]  

(5.1.3-7)

The matrices \( T_{Y N} \) above are determinable functions of the element \( \Psi_{N} \) of \( u_{Z \gamma} \). The matrices \( T_{Y_I} \) are functions of \( \Psi_{Y_I} \) which, in turn is a function of the element \( \zeta_{Z \gamma} \) of \( u_{Z \gamma} \). Therefore the RHS of Eq. (5.1.3-7) is a function of \( u_{Z \gamma} \), and \( u_{Z \gamma} \) as desired. The matrix on the LHS of Eq. (5.1.3-7) is a function of \( \Psi_{YO} \), and this relation can be inverted to yield

\[
\Psi_{YO} = \Psi_{YO} (T_{YO})
\]  

(5.1.3-8)

Equations (5.1.3-7) and (5.1.3-8) constitute the formalism for determining \( \Psi_{YO} \) from the \( u_{Z \gamma} \).

5.2 Kinematics of the Interfaces

5.2.1 Introduction

The concept of the state of a joint is useful. The present study will specify the state \( s_A \) of a joint \( A \) via

\[
s_A = \begin{pmatrix} u_A \\ y_A \end{pmatrix}
\]  

(5.2.1-1)
with the velocity element $w_A$ being the dual-vector

$$w_A = \begin{Bmatrix} x_{\alpha_0 I} \\ \bar{w}_{\alpha_0 I} \end{Bmatrix}$$  

(5.2.1-2)

The velocity vector

$$\dot{y}_A = \begin{Bmatrix} \dot{x}_{\alpha_0 I} \\ \dot{\bar{w}}_{\alpha_0 I} \end{Bmatrix}$$  

(5.2.1-3)

also will be used. The two velocity vectors are related through a transformation of the form

$$\dot{y}_A = T^{y}_{y_A}(y_I) w_A w_A$$

$$= \begin{bmatrix} E^T & 0 \\ 0 & T^{y}_{y_A}(y_I) \end{bmatrix} \begin{bmatrix} \dot{x}_{\alpha_0 I} \\ \dot{\bar{w}}_{\alpha_0 I} \end{bmatrix} w_A$$  

(5.2.1-4)

The following two subsections will develop mathematics for specifying $w_A$, with the reconnected joint and cut joint conditions being considered separately.
5.2.2 Kinematics of Reconnected Joints

For each reconnected joint $B$, it is necessary to introduce a generalized velocity vector, such as $\dot{q}_B$, to specify the translational and angular velocity of frame $\beta_0$ relative to $\beta_I$. However, the present study makes only minimal use of $\dot{q}_B$. Instead, a more general velocity vector, designated as $\rho_B$, is employed. The $\rho_B$ vector concept is applicable both to reconnected joints and to cut joints; but is used mainly with reconnected ones in the present work. The elements of $\rho_B$ are scalars, and $\rho_B$ in fact is treated in the mathematics more as a column matrix than as a vector per se.

Unlike $\dot{q}_B$, the vector $\rho_B$ can include elements, such as angular velocity components $\bar{\omega}_{OB_I}$, which are not the time derivatives of true coordinates. In fact, the main reason for introducing the $\rho_B$ concept is to enable the formalism to encompass such elements.

As a special case $\rho_B$ can be specified to be identical to $\dot{q}_B$. More generally, when dealing with joints for which $q_B$ is defined, the velocity $\dot{q}_B$ is a linear function of $\rho_B$

$$\dot{q}_B = T_{q_B} \rho_B \quad (5.2.2-1)$$

The $T$ operator above is a function of $q_B$.

In general, $\dim \rho_B \neq \dim q_B$. Instead, $\rho_B$ is defined in such a manner that $\dim \rho_B \leq \dim q_B$ with equality occurring when $B$ has no non-holonomic passive constraints and $q_B$ has no redundant elements. The dimensions and other restrictions on $\rho_B$ are tailored to fit the joint constraint formulation, and they will be specified in Subsection 5.4.5.
The dual-velocity vector $\omega_B^*$ which was introduced in Subsection 5.2.1 and its two elements are related linearly to $\omega_B^*$. The relations will be expressed as follows

$$\begin{align*}
\frac{*}{x_{\beta_0\beta_I}} &= \frac{T}{x_{\beta_0\beta_I}} \omega_B^* \\
\frac{-}{\omega_{\beta_0\beta_I}} &= \frac{T}{\omega_{\beta_0\beta_I}} \omega_B^* \\
\omega_B &= \frac{T}{\omega_B} \omega_B^*
\end{align*}$$

(5.2.2-2) (5.2.2-3) (5.2.2-4)

$$
\begin{bmatrix}
\frac{T}{x_{\beta_0\beta_I}} \\
\frac{-}{\omega_{\beta_0\beta_I}} \\
\frac{T}{\omega_{\beta_0\beta_I}} \\
\frac{-}{\omega_{\beta_0\beta_I}}
\end{bmatrix}
\begin{bmatrix}
\omega_B \\
\omega_B^*
\end{bmatrix}
$$

The $T$ operators on the RHS of the above equations are functions of $\omega_B^*$ or, more generally of $\omega_B^*(\beta_I)$.

A component form of Eq. (5.2.2-4) proves to be useful in subsequent work. Define the dual unit operator

$$
U_{\beta_I} =
\begin{bmatrix}
E_{\beta_I} & 0 \\
0 & E_{\beta_I}
\end{bmatrix}
$$

(5.2.2-5)
Then

\[
\begin{align*}
\begin{bmatrix} \beta_i \end{bmatrix}_B &= U_{\beta_i B}^T w_B \\
&= \begin{bmatrix} \pi_i \end{bmatrix}_B \\
&= \begin{bmatrix} \pi_i \end{bmatrix}_B w_B
\end{align*}
\]  

(5.2.2-6)

and it is a minor operation to transform Eq. (5.2.2-4) into

\[
\begin{align*}
\begin{bmatrix} \beta_i \end{bmatrix}_B &= T \begin{bmatrix} \beta_i \end{bmatrix}_B \rho_B \\
&= \begin{bmatrix} \pi_i \end{bmatrix}_B w_B \rho_B \\
&= \begin{bmatrix} \pi_i \end{bmatrix}_B w_B \\
&= \begin{bmatrix} \pi_i \end{bmatrix}_B \rho_B
\end{align*}
\]  

(5.2.2-7)

where

\[
\begin{align*}
T \begin{bmatrix} \beta_i \end{bmatrix}_B &= U_{\beta_i B}^T T \begin{bmatrix} \pi_i \end{bmatrix}_B w_B \rho_B \\
&= \begin{bmatrix} \pi_i \end{bmatrix}_B \rho_B
\end{align*}
\]  

(5.2.2-8)

5.2.3 Kinematics of Cut Joints

The formulation which is developed for joint constraints in Section 5.4 assumes that velocity vectors \( \rho_T \) are defined for the cut joints \( T \) just as \( \rho_B \) is defined for reconnected joints \( B \). However, \( \rho_T \) (a sub-vector designated as \( \rho_T^a \)) is employed only as an input command applied to \( T \) by a control system. The formulation which is developed in Chapter 6 does not permit \( \rho_T \) to be used directly for specifying the dual-velocity vector \( w_T \) of \( T \). Instead \( w_T \) must be specified in terms of the states of the two bodies \( T \) which are contiguous to \( T \).

\[
\begin{align*}
w_T &= w_T \begin{bmatrix} s_{Z_1} & s_{Z_2} \end{bmatrix} \gamma_T \gamma_I \\
&= \begin{bmatrix} s_{Z_1} & s_{Z_2} \end{bmatrix} \gamma_T \gamma_I \\
&= \begin{bmatrix} s_{Z_1} & s_{Z_2} \end{bmatrix} \gamma_T \gamma_I
\end{align*}
\]  

(5.2.3-1)

The present subsection develops the mathematical details of this relationship.
Recall from Eq. (5.2.1-2) that the elements of \( \mathbf{w}_n \) are \( x_{y_{0y}} \) and \( \omega_{y_{0y}} \). The elements of the \( \mathbf{s} \) can be ascertained from Eqs. (4.1-5), (4.2-5), and (4.2-7) by replacing \( Z \) by \( Z_{y_n} \) and \( \zeta \) by \( \xi_{y_n} \).

Consider the term \( x_{y_{0y}} \) first. Start with

\[
\frac{x_{y_{0y}}}{x_{y_{0y}}} = \frac{\sum x_{y_{n}}} {x_{y_{n}}} \frac{\sum x_{y_{n}}}{x_{y_{n}}}
\]

Employ the theorem of Coriolis to replace the LHS of the above equation by the corresponding expression involving \( x_{y_{0y}} \). Introduce Eq. (4.2-12) (with \( a_{nz} \) replaced by \( y_n \) and \( Z \) replaced by \( Z_{y_n} \)) into the RHS. Perform a minor rearrangement. The result is

\[
\frac{x_{y_{0y}}}{x_{y_{0y}}} = \frac{\sum x_{y_{n}}}{x_{y_{n}}} \frac{\omega_{y_{n}}}{\gamma_n} + \sum x_{y_{n}} \frac{\sum x_{y_{n}}}{x_{y_{n}}} \frac{\gamma_{y_{n}}}{y_n}
\]

The \( T \) array above is the same basic array which was used in Eq. (4.2-13). Specifically

\[
\begin{align*}
T_{y_{0y}} &= \begin{bmatrix}
S_{y_{0y}} & S_{y_{0y}} \\
-x_{y_{n}} - y_{z_{y_{n}}} & x_{y_{n}} - y_{z_{y_{n}}}
\end{bmatrix} \\
&= \begin{bmatrix}
I & -x_{y_{n}} \xi_{y_{n}} & x_{y_{n}} \xi_{y_{n}} & q_{z_{y_{n}}} & q_{z_{y_{n}}}
\end{bmatrix}
\end{align*}
\]

(5.2.3-4)
The passive and active terms in $T$ are denoted separately above for convenience in future reference.

Now consider the equation for the element $\hat{w}_{\gamma_0^\beta \gamma_1}^\gamma$ of $\hat{w}_T$. Start with

$$
\hat{w}_{\gamma_0^\beta \gamma_1}^\gamma = \sum_n \varepsilon_n \hat{w}_{\gamma_n^N}^\gamma
$$

(5.2.3-5)

Introduce Eq. (4.2-16) (with $\alpha_{nZ}$ replaced by $\gamma_n$ and $Z$ replaced by $Z_{\gamma_n}$) into Eq. (5.2.3-5) to obtain

$$
\hat{w}_{\gamma_0^\beta \gamma_1}^\gamma = \sum_n \varepsilon_n \frac{T}{\hat{w}_{\gamma_n^N}^\gamma} \frac{v_{\gamma_n^Z}}{\gamma_n^N}
$$

(5.2.3-6)

where

$$
\frac{T}{\hat{w}_{\gamma_n^N}^\gamma} \frac{v_{\gamma_n^Z}}{\gamma_n^N} = \begin{bmatrix}
S_{\hat{w}_{\gamma_n^N}^\gamma}^{v_{\gamma_n^Z}} & S_{\hat{w}_{\gamma_n^N}^\gamma}^{v_{\gamma_n^Z}} \\
\frac{v_{\gamma_n^Z}}{\gamma_n^N} & \frac{v_{\gamma_n^Z}}{\gamma_n^N}
\end{bmatrix}
$$

= \begin{bmatrix}
0 & I & \delta_{\gamma_n^N} & \delta_{\gamma_n^N}^2 \\
\delta_{\gamma_n^N} & \delta_{\gamma_n^N}^2 & 0 & I
\end{bmatrix}

(5.2.3-7)

The final step is to stack Eq. (5.2.3-3) and (5.2.3-6). For future reference, it is convenient to separate the result into passive ($v_{Z1}$) and active ($v_{Z2}$) terms. After manipulation, the following result can be obtained.
\[ w_r = \sum_n \epsilon_n s_w y_n v_z y_n \]

\[ = \sum_n \epsilon_n \sum_{i=1}^2 s_w y_n v_z y_i \]

\[ \text{(5.2.3-8)} \]

where

\[ s_w y_n v_z y_i = s_w y_n w_y n s_y n v_z y_i \]

\[ \text{(5.2.3-9)} \]

with

\[ s_w y_n w_y n = \begin{bmatrix} 1 & -y_n y_n \\ 0 & I \end{bmatrix} \]

\[ \text{(5.2.3-10)} \]

and

\[ s_w y_n v_z y_i = \begin{bmatrix} s_w y_n v_z y_i \\ s_w y_n v_z y_i \end{bmatrix} \]

\[ \text{(5.2.3-11)} \]

In establishing the above arrays, the following dual-velocity vector

57
\[ \dot{w}_{nN} = \begin{bmatrix} \dot{x}_{nN} \\ \dot{y}_{nN} \\ \dot{z}_{nN} \end{bmatrix} \quad (5.2.3-12) \]

was defined. The \( S \) array which is defined in Eq. (5.2.3-10) was introduced in order to isolate the effect of noncongruence of the origins of frames \( \gamma_I \) and \( \gamma_O \); this array reduces to an identity for \( n = 0 \).

Equation (5.2.3-8) and the equations which support it constitute the end result of this subsection.

5.3 Forces and Torques at the Interfaces

This section considers the reaction forces and torques which each joint \( A \) exerts on its neighboring bodies \( Z_{\alpha I} \) and \( Z_{\alpha O} \). A free body diagram is shown as Fig. 5-2. The reaction forces and torques on this figure are denoted as \( \overline{F}_{\alpha n} \) and \( \overline{G}_{\alpha n} \). Specifically, \( \overline{F}_{\alpha n} \) is the force which joint \( A \) exerts on body \( Z_{\alpha n} \), and \( \overline{G}_{\alpha n} \) is the torque which joint \( A \) exerts on body \( Z_{\alpha n} \) at point \( 0_{\alpha n} \). These vectors, \( \overline{F}_{\alpha n} \) and \( \overline{G}_{\alpha n} \), are the same quantities which were designated as \( \overline{F}_{\alpha n1} \) and \( \overline{G}_{\alpha n2} \) in Chapter 4 and Appendix A; the subscript label \( 2 \) is unnecessary in the present chapter and, therefore, is omitted.

As was noted in Chapter 4, it is convenient to introduce dual-force vectors \( \Phi \) which are formed by stacking the \( \overline{F} \) and \( \overline{G} \).
The two points labeled $O_{a_I}$ coincide, as do the two points labeled $O_{a_O}$.

Figure 5-2. Reaction forces and torques at interfaces.
\[ L_{\alpha n} = \begin{cases} \bar{F}_{\alpha n} \\ \bar{G}_{\alpha n} \end{cases} \]  \hspace{1cm} (5.3-1)

Since the joints are massless, force and moment balance are required. With the aid of Figure 5-2 it can be determined that the \( L \) vectors at the two ends of joint A are related through

\[ L_{\alpha I} = -S^{T} \begin{vmatrix} \bar{\omega}_{\alpha N} & \bar{\omega}_{\alpha N} \end{vmatrix} L_{\alpha O} \]  \hspace{1cm} (5.3-2)

The \( S \) array above was defined in Eq. (5.2.3-10).

As noted in Chapter 4, each dual force vector \( L \) can be separated into a "known" part \( K \) and a constraint part \( C \).

\[ L_{\alpha n} = L_{\alpha n}^{K} + L_{\alpha n}^{C} \]  \hspace{1cm} (5.3-3)

While little or no use of the concept will be made in this document, the "known" vector above can be separated into a "passive" part \( p \) and an "active" part \( a \)

\[ L_{\alpha n}^{K} = L_{\alpha n}^{Kp} + L_{\alpha n}^{Ka} \]  \hspace{1cm} (5.3-4)

The \( L_{\alpha n}^{Kp} \) vector above can, perhaps, be regarded best as resulting from translational and torsional springs and dampers connected between \( \alpha \) and \( \alpha \). More specifically, \( L_{\alpha n}^{Kp} \) is assumed to be a known function of \( s_{A} \) and hence computable when \( s_{A} \) is known. The \( L_{\alpha n}^{Ka} \) vector is generated by one
or more actuators, such as torque motors, which are located at joint A. Thus, control systems at the joints can be incorporated into the model through this term.

5.4 Constraints at the Interfaces

5.4.1 The Concept of Passive and Active Constraints

The formulation which is developed in this document separates the joint constraints into "passive constraints" and "active constraints"; these two terms are used here in a completely different sense from that often employed\(^{(21)}\) in studies involving constraints. The two classes of constraints are denoted by the subscript labels p and a respectively.

The term "passive constraints" refers to the usual type of constraints which ordinarily result from the basic structure of the joint. Joints which have passive constraints are said to be passively constrained.

The active constraint concept is introduced as a technique for modeling the effect of control systems at the joints. With this approach, the commands generated by the control system are velocity commands, and the joints are constrained to follow them perfectly. A similar approach was used by Bodely, et al,\(^{(3)}\) in a study which dealt only with cut joints.

It is noted that the active constraint concept is one of two methods which the formulation developed in this document provides for introducing control systems at joints. The other approach, in which the control actuators are introduced through the forces or torques which they apply at the joint, was noted briefly at the end of Section 5.3.

5.4.2 The Basic Constraint Equation

Consider the passive constraints first. In aerospace problems, passive joint constraints usually are holonomic. It will be recalled that a holonomic constraint is a constraint on or among the variables
which specify the configuration (geometric condition) of a system. For example, a rotational joint has the three scalar constraints \( x_{\alpha_0} = 0 \). A nonholonomic constraint is any constraint which is not holonomic. The most common type of nonholonomic constraint is the so-called simple type which is a linear non-integrable constraint among the velocity elements. The formulation developed in this document for joint constraints will be applicable not only to holonomic constraints but also to simple nonholonomic ones. To wit, holonomic constraints will be handled by differentiating them (in effect or in fact) and treating them as velocity constraints. This is a convenient and not uncommon approach. The mathematical details are presented below.

Let the symbols \( C_{NH}^{AP} \) and \( C_{H}^{AP} \) denote column matrices of the scalar passive nonholonomic (NH) and holonomic (H) constraints at a joint A. These two types of passive joint constraints will be represented by the following equations

\[
C_{NH}^{AP} = C_{NH}^{AP} (\alpha_i) (y_A)^T \alpha_i^T w_A = 0 \quad (5.4.2-1)
\]

\[
C_{H}^{AP} = C_{H}^{AP} (\alpha_i) = 0 \quad (5.4.2-2)
\]

The \( C(\alpha_i) \) term in Eq. (5.9.2-1) is a rectangular matrix. The \( U \) term was defined in Eq. (5.2.2-5); the function of this term is to coordinatize the two Gibbs vectors which comprise \( w_A \) onto the axes of \( \alpha_i \) as indicated in Eq. (5.2.2-6).

Differentiating Eq. (5.4.2-2) with respect to time produces

\[
C_{H}^{AP} = C_{H}^{AP} (\alpha_i)^T \alpha_i^T w_A = 0 \quad (5.4.2-3)
\]
where

\[
C_{Ap} = C_{Ap} \cdot (\alpha_I)_{T} - (\alpha_I)_{T} - (\alpha_I)
\]  \hspace{1cm} (5.4.2-4)

Equation (5.4.2-3) is identical in form to the nonholonomic constraint equation, Eq. (5.4.2-1).

In the remainder of the document there will be little need to distinguish between holonomic and nonholonomic passive constraints. Therefore, the following equation will be used to denote all passive constraints at joint A

\[
C_{Ap} = C_{Ap} \cdot \omega_A = 0
\]  \hspace{1cm} (5.4.2-5)

The operator

\[
C_{Ap} = C_{Ap} \cdot \Omega_A = 0
\]  \hspace{1cm} (5.4.2-6)

has been introduced above for notational compactness.

Now consider the active constraints. Let \(C_{Aa}\) be the column matrix of the scalar active constraints on A. The following equation will be employed for the active constraints

\[
C_{Aa} = C_{Aa} \cdot \omega_A - B_{Aa} \cdot \rho_{Aa}^* = 0
\]  \hspace{1cm} (5.4.2-7)

where

\[
C_{Aa} = C_{Aa} \cdot \Omega_A = 0
\]  \hspace{1cm} (5.4.2-8)
The B and $C^{(d_i)}$ terms above are matrices; the problem of specifying these two terms will be considered in Subsection 5.4.5. The $\rho^*$ term in Eq. (5.4.2-7) is the velocity command generated by the control system at joint A; this term will be defined in more detail in Subsections 5.4.4 and 5.4.5.

The equation for the complete set of constraints at joint A can be obtained by stacking Eq. (5.4.2-5) and (5.4.2-7)

$$\begin{bmatrix} C_{Ap} \\ C_{Aa} \end{bmatrix} = \begin{bmatrix} C_{Ap} \\ C_{Aa} \end{bmatrix} \begin{bmatrix} \mathbf{w}_A \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ B_{Aa} \end{bmatrix} \rho^*_A = 0 \quad (5.4.2-9)$$

In much of the subsequent work, it will not be necessary to distinguish between passive and active constraints. In such cases, Eq. (5.4.2-9) will be notationally compacted into

$$C_{A} = C_{A-A} - B_{A-A} \rho^*_A = 0 \quad (5.4.2-10)$$

The constraint equations which will be used subsequently with cut joints and also those which will be used with reconnected joints are not quite identical to Eq. (5.4.2-9) or (5.4.2-10); neither employs the vector $\mathbf{w}_A$ directly. The modified constraint equations for the two joint conditions will be considered separately in the following two subsections.

5.4.3 The Constraint Equation for Cut Joints

When dealing with cut joints $\Gamma$, the velocity vector $\mathbf{w}_\Gamma$ is not specified directly from relative conditions at $\Gamma$. Instead, it is established as a function of the states of the two bodies at the ends of $\Gamma$ as indicated in Eq. (5.2.3-8). In work to follow it is necessary that
the constraint equation for $\Gamma$ be written using the states of the two bodies adjacent to $\Gamma$. The required form can be obtained by inserting Eq. (5.2.3-8) into (5.4.2-10) to produce

$$
C_T = C_T \sum \varepsilon_n S \gamma_o \gamma_n - B_p \rho_a^* = 0 \quad (5.4.3-1)
$$

5.4.4 The Constraint Equations For Reconnected Joints

When dealing with reconnected joints $B$ in work to follow, the velocity condition of $B$ will be specified using the vector $\rho_B$ which was introduced in Subsection 5.2.2. The constraint equation to be used with reconnected joints can be obtained easily by inserting Eq. (5.2.2-4) into (5.4.2-10) to produce

$$
C_B = C_B T_{w_B} \rho_B - B_B \rho_{Ba}^* = 0 \quad (5.4.4-1)
$$

The remainder of this subsection will deal with the passive constraint portion of Eq. (5.4.4-1); to wit, with

$$
C_{Bp} = C_{Bp} T_{w_B} \rho_B = 0 \quad (5.4.4-2)
$$

Up to this point, no major restrictions on $\rho_B$ have been introduced. The interests of the study can be best served by establishing the $\rho_B$ in such a manner that the reconnected joint constraint equations and constraint forces fall out of the formalism for the so-called transformed system which will be developed in Chapter 6. In order to accomplish this, the dimension of $\rho_B$ henceforth will be restricted to being $\dim \rho_B \Delta s = 6 - m_p$ where $m_p$ is the number of scaler passive constraints at $B$.

It turns out to be necessary also to define $\rho_B$ in such a manner that its elements can be selected arbitrarily without violating the
passive constraint equation, Eq. (5.4.4-2). Perusal of Eq. (5.4.4-2) should show that this requirement implies

$$C_B \begin{bmatrix} T \end{bmatrix} w_B \rho_B = 0 \quad (5.4.4-3a)$$

The matrix representation of the above expression can be obtained by inserting Eq. (5.4.2-6) and the inverse of Eq. (5.2.2-8) into it to produce

$$C_B \begin{bmatrix} T \end{bmatrix} (\beta I) w_B \rho_B = 0 \quad (5.4.4-3b)$$

Assuming that the number of scalar passive constraints on B does not exceed 6 and that the C matrix of Eq. (5.4.4-3b) has full rank, it can be shown that full rank matrices T which satisfy Eq. (5.4.4-3b) exist; the basic requirement is that their columns span the null space of $C_B^{\beta I}$.

Eq. (5.4.4-3) should be interpreted as the formal requirement which the transformation T must satisfy in order that the values of the elements of $\rho_B$ can be chosen arbitrarily without violating the passive constraints at joint B. Thus $\rho_B$ must be chosen such that Eq. (5.4.4-3) is satisfied "automatically". In actual applications, suitable $\rho_B$ which cause Eq. (5.4.4-3) to be satisfied usually are obvious. For example, at a pin joint, the most practical approach is to let $\rho_B$ be the scalar angular velocity $\dot{\theta}_B$ across the joint.

5.4.5 Specification of Single Joint Velocity Vector and Active Constraint Matrices

This subsection deals with the active constraint portion of Eq. (5.4.4-1) written in the form
\[ c_{Aa} = c_{Aa}^{(a_I)} T_{(a_I)}^{(a_I)} \rho_A - B_{Aa} \rho_{Aa}^* = 0 \]  \hspace{1cm} (5.4.5-1)

The development presented in this subsection is applicable to both cut joints and reconnected joints. However, it turns out that the material is needed only for cut joints.

For passive constraints, the constraint matrix \( C_{Aa}^{(a_I)} \) is determined primarily by the basic characteristics of the joint; this does not present any theoretical difficulties which need to be delved into in the present study. When dealing with active constraints, however, a technique for establishing the matrices \( C_{Aa}^{(a_I)} \) and \( B_{Aa} \) is needed, and this is the topic addressed in this subsection.

In the interest of clarity in the following paragraphs, it should be useful to keep track of dimensions. Let \( m_p \) and \( m_a \) respectively be the number of scaler passive and active constraints on \( A \). Let \( m = m_p + m_a \). Then \( C_{Aa}^{(a_I)} \) is \( m_a \times 6 \). As noted in Subsection 5.4.4, \( \dim \rho_A^A = s = 6 - m_p \). Thus \( T_{(a_I)}^{(a_I)} \) is \( 6 \times s \).

As a first step, some restrictions on \( \rho_A \) will be prescribed. Partition \( \rho_A \) into

\[ \rho_A = \begin{bmatrix} \rho_{Af} \\ \rho_{Aa} \end{bmatrix} \]  \hspace{1cm} (5.4.5-2)

where \( \rho_{Aa} \) is \( m_a \times 1 \), and \( \rho_{Af} \) is \( m_f \times 1 \) with \( m_f = s - m_a = 6 - m \). The elements \( \rho_{Af} \) are the "free" (i.e., uncontrolled) velocities, and the elements of \( \rho_{Aa} \) are the "active" (i.e., controlled) velocities.
Partition the T matrix in Eq. (5.4.5-1) into two submatrices S which are compatible with the partitioning of \( \rho_a \). Eq. (5.4.5-1) then can be written as

\[
C_{AA} \left[ \begin{array}{c|c}
S(a_I) & \rho_Af \\
\frac{w_A}{A} \rho_{Af} & \frac{w_A}{A} \rho_{AA}
\end{array} \right] = B_{AA} \rho_{AA}^* \quad (5.4.5-3)
\]

\[
m \times 6 \quad 6 \times m_f \quad 6 \times m \quad m \times m_a
\]

The approach assumes that the vector \( \rho_{AA} \) is tailored to fit the features of the joint control system. \( \rho_{AA}^* \) is comprised of the velocities which are commanded by the control system. The vector \( \rho_{AA} \) must be comprised of these same velocities. The elements of \( \rho_Af \) can be selected arbitrarily subject to the restriction imposed by Eq. (5.4.4-3). The intent of the formalism is that (1) \( \rho_{AA} \) track \( \rho_{AA}^* \) perfectly and (2) \( \rho_{AA}^* \) have no direct influence on \( \rho_Af \). The problem to be considered below is to specify \( C_{AA}(a_I) \) and \( B_{AA} \) in such a manner that these two requirements are met.

The simplest approach involves making \( B_{AA} \) the identity matrix

\[
B_{AA} = I 
\]

\[
m \times m_a
\]

It then can be deduced from Eq. (5.4.5-3) that the two requirements will be met if \( C_{AA}(a_I) \) is chosen such that

\[
C_{AA} \left[ \begin{array}{c|c}
T(a_I) & \frac{w_A}{A} \rho_A \\
\frac{w_A}{A} \rho_{AA}
\end{array} \right] = \left[ \begin{array}{c|c}
0 & I
\end{array} \right] \quad (5.4.5-5)
\]

\[
m \times 6 \quad 6 \times m_a \quad m \times m_f \quad m \times m_a
\]
It can be shown that $C$ matrices which satisfy Eq. (5.4.5-5) exist if $T$ has full rank ($s$), but in general they are not unique. It also can be shown that these solutions will cause the composite constraint matrix $C_{A}^{(a_I)}$ at $A$ to have full rank, assuming that $C_{Ap}^{(a_I)}$ has full rank.

One algorithm for establishing a $C$ matrix which satisfies Eq. (5.4.5-5) is

$$C_{A}^{(a_I)} = [ \begin{array}{cc} 0 & I \\ \end{array} ] [ P_1 T_{\frac{w_A}{P_A}}^{(a_I)} ]^{-1} P_1$$

$$\begin{array}{cccc} m_A \times 6 & m_A \times s & s \times 6 & 6 \times s \end{array}$$

The term $P_1$ above can be regarded best as a submatrix of a permutation matrix

$$P = \begin{bmatrix} P_1 \\ 6 \times 6 \\ P_2 \end{bmatrix}$$

which permutes the rows of $T$ such that the first $s$ rows are linearly independent. The validity of Eq. (5.4.5-6) can be demonstrated by substituting it back into Eq. (5.4.5-5) to produce an identity.

5.5 Interface Constraint Equations in Absolute Variables Using Unified Notation

In this section, the constraint equations for the individual joints will be stacked into a composite equation which contains all the joint constraints of the system. This operation is analogous to that done in Section 4.5 for the dynamical equations of the bodies. In the present section, all joints of the system are regarded as being cut. Therefore, the $\Gamma$ notation will be used, and the absolute velocity vectors $\frac{v_z}{w_z}$ will be employed in the formulation.
Equation (5.2.3-8) expressed \( \mathbf{v}_n \) as a function of the velocity vectors \( \mathbf{v}_{\gamma_n} \) of the two bodies at the I and O ends of \( \Gamma \). As a first step this equation will be rewritten using the composite system velocity vector \( \mathbf{v} \) which was introduced in Section 4.5. The relation is

\[
\mathbf{v}_n = \mathbf{T}_{\mathbf{w}_n^\mathbf{v}} \mathbf{v} = \left[ \begin{array}{c} S_{\mathbf{w}_n^\mathbf{v}_1} \\ S_{\mathbf{w}_n^\mathbf{v}_2} \end{array} \right] \left( \begin{array}{c} \mathbf{v}_1 \\ \mathbf{v}_2 \end{array} \right)
\]  

(5.5-1)

The details of the \( T \) array above and its two subarrays \( S \) can be deduced from Eq. (5.2.3-8) and its auxiliary equations. The row array \( S_{\mathbf{w}_n^\mathbf{v}_1} \) has exactly two nonzero elements \( S_{\mathbf{w}_n^\mathbf{v}_1} \), \( n = I, O \). These elements are located in conformance with the locations of the two vectors \( \mathbf{v}_{\gamma_n} \) in \( \mathbf{v}_1 \). The second row array, \( S_{\mathbf{w}_n^\mathbf{v}_2} \), in Eq. (5.5-1) will have 0, 1, or 2 nonzero elements depending upon whether either or both of the \( Z_{\gamma_n} \) have active velocity elements; these nonzero elements will be located in conformance with the locations of the \( \mathbf{v}_{\gamma_n} \) in \( \mathbf{v}_2 \). Equation (5.2.3-8) indicates that the equations for the nonzero elements of the two \( S \) arrays in Eq. (5.5-1) are

\[
S_{\mathbf{w}_n^\mathbf{v}_{\gamma_n}^i} = \varepsilon_n S_{\mathbf{w}_n^\mathbf{v}_{\gamma_n}^i} \quad (5.5-2)
\]

where \( i = 1, 2 \)

Insert Eq. (5.5-1) into Eq. (5.4.2-9) to produce

\[
\mathbf{c}_\Gamma = \mathbf{c}_\Gamma \mathbf{T}_{\mathbf{w}_n^\mathbf{v}} \mathbf{v} = 0 \quad (5.5-3a)
\]

\[
\mathbf{c}_\Gamma = \mathbf{c}_\Gamma \mathbf{T}_{\mathbf{w}_n^\mathbf{v}} \mathbf{v} - \mathbf{\beta}_\Gamma \mathbf{c}_\Gamma^* = 0 \quad (5.5-3b)
\]
Note that Eqs. (5.5-3a) and (5.5-3b) are basically an expanded form of Eq. (5.4.3-1).

Equations (5.5-3a) and (5.5-3b) for the individual joints \( \Gamma \) will now be stacked into a unified constraint equation which contains all the joint constraints of the system. It was decided to take the approach of stacking all the passive constraints together in the upper block of this equation and all the active constraints together in the lower block.

Assume that an ordering scheme has been specified for the joints. Let \( \dot{w}_p \) be the vector of vectors which is formed by stacking, according to this ordering, the \( \dot{w}_r \) of all passively-constrained joints. Let \( T_{\dot{w}_p} \) be the rectangular array, compatible with \( \dot{w}_p \), which is formed by stacking the \( T_{\dot{w}_p} \) of all such joints. Let \( C_p \) be the array which is formed by stacking the \( C_{\Gamma_p} \) in block diagonal form. For the active constraints, define \( \dot{w}_a, T_{\dot{w}_a}, \) and \( C_a \) analogously. Also, stack the \( \rho_{a}^* \) into a composite vector \( \rho_a^* \) and arrange the \( B_{\Gamma_a} \) compatibly in a block diagonal array \( B_a \).

Using the terms defined in the above paragraph, Eq. (5.5-3) for the individual \( \Gamma \) can be combined into

\[
\begin{pmatrix}
C_p \\
C_a
\end{pmatrix}
= \begin{bmatrix}
C_p & 0 \\
0 & C_a
\end{bmatrix}
\begin{bmatrix}
T_{\dot{w}_p} \\
T_{\dot{w}_a}
\end{bmatrix}
\begin{bmatrix}
\dot{v} \\
B_a
\end{bmatrix}
- \begin{bmatrix}
0 \\
\rho_a^*
\end{bmatrix}
\]

\[
(5.5-4)
\]

or, in more concise notation

\[
C = C_{T_{\dot{w}_p}} - B_{\Gamma_a} \rho_a^* = 0
\]

\[
(5.5-5)
\]
In establishing Eq. (5.5-5), the vector

\[ \mathbf{w} = \begin{pmatrix} w_p \\ w_a \end{pmatrix} \]  

(5.5-6)

was defined implicitly.

Eq. (5.5-5) is the end result of this chapter.
CHAPTER 6

THE TRANSFORMED SYSTEM

6.1 Introduction

The main results thus far are Eqs. (4.5-2) which are the primitive equations for a set of bodies and Eq. (5.5-5) which is an equation for the constraints at the body interfaces. The configuration and velocity vectors, \( u \) and \( v \), that are used in these equations are said to be "absolute" variables because they specify the states of the bodies relative to an inertial frame \( N \). In generating these equations, the point of view was taken that all joints of the system were cut. For convenience, this system of bodies with cut joints will be called system \( \Sigma \) in this chapter.

A transformation of variables will be introduced in this chapter. In particular, the relative velocity vectors \( \rho_B \) of selected joints \( B \) will be employed in place of the absolute velocities \( x_{z_B}N \) and \( \omega_{z_B}N \) of \( Z_B \) at the output end of each \( B \). This reduces the dimension of the kinetics equations because \( \dim \rho_B < 6 \) for every \( B \) which is passively constrained. In addition, the constraint equations and constraint forces of the \( B \) drop out of the mathematics, and, as a result, the numerical accuracy problems, encountered in dynamic simulations, in maintaining the constraints at cut joints disappear at the \( B \).

The transformed configuration and velocity vectors will be denoted as \( u' \) and \( v' \) respectively. In general, a prime will be used to indicate variables that are associated with the transformed equations.
In generating the transformation of variables, the point of view will be taken that selected joints of the system are being reconnected. This new system of bodies and reconnected joints will be called system $\Sigma'$.

One restriction is made in choosing the joints which are to be reconnected: namely, $\Sigma'$ must contain no closed loops. $\Sigma'$ thus will consist of one or more trees, possibly many more. Note that there can be cut joints between bodies within a tree, and there also can be cut joints between bodies in different trees.

In applications, the joints which are reconnected should be selected to best fit the needs of the particular problem. In particular, they should be chosen such that the structure of $\Sigma'$ is physically meaningful and/or mathematically convenient.

The kinetics and constraint equations of $\Sigma'$ will be developed directly from those of $\Sigma$ using the velocity transformation

$$\mathbf{v} = T_{\mathbf{v}'} \mathbf{v}'$$

(6.1-1)

As noted previously, this is the transformation operator approach advocated by Jerkovsky. The kinematics equation for $\Sigma'$ will be developed directly using the definitions of $\mathbf{u}'$ and $\mathbf{v}'$.

Equations (4.5-1) and (5.5-5) which were developed for $\Sigma$ are not in a form that is suitable for computations. The essential ingredient which has not been introduced is a formalism which utilizes the constraint equation to either compute or eliminate the unknown constraint force vector $\mathbf{f}_c$. The topic of the determination and/or elimination of the constraint forces will be considered in this chapter as part of the work on $\Sigma'$.
6.2 Velocity Vector of the Transformed System

A precise specification of the velocity vector, $\mathbf{v}'$, of $\Sigma'$ is presented in this section.

Choose one of the bodies in each tree of $\Sigma'$ as a reference body $R$. Let its body reference frame be denoted as $r$. For mathematical convenience, define end $I$ of every reconnected joint $B$ in every tree to be the end which leads back to the $R$ of that tree.

In the study described in this document, the elements of $\mathbf{v}'$ were chosen to be the following:

1. the $\mathbf{v}_{rN}$ and $\mathbf{\omega}_{rN}$ of every $R$
2. the $\rho_{Bf}$ of every reconnected joint $B$
3. the $\mathbf{q}_{z1}$ of every nonrigid body
4. the $\rho_{Ba}$ of every actively constrained $B$
5. the $\mathbf{v}_{z2}$ of every actively controlled nonrigid body

The vector $\mathbf{v}'$ can be formed directly from $\mathbf{v}$. All that is required basically is to replace the $\mathbf{v}_{\zeta N}$ and $\mathbf{\omega}_{\zeta N}$ of every $Z$ that is attached to the output end of a $B$ by $\rho_{Bf}$ and $\rho_{Ba}$. However, it also is desirable to arrange the elements of $\mathbf{v}'$ in a sequence which has some useful properties. In the present work, it was found to be convenient to group the passive elements of $\mathbf{v}'$ into one block and the active elements into separate blocks. Thus, the following arrangement of $\mathbf{v}'$ is employed.

$$\mathbf{v}' = \begin{bmatrix} \mathbf{v}_{1}' \\ \rho_{Ba} \\ \mathbf{v}_{2}' \end{bmatrix}$$  \hspace{1cm} (6.2-1)
In the above expression, \( v_1' \) contains the passive elements of \( v' \); these are the elements listed as (1), (2), and (3) in the preceding paragraph. The term \( \rho_{ba} \) is formed by stacking the \( \rho_{Ba} \) of every B that has active control. The label \( b \) will be used henceforth to denote variables which are associated with the set of reconnected joints; this symbol was chosen because of its correspondence with B and \( \beta \). The term \( v_2' \) is identical to \( v_2 \)

\[
v_2' = v_2
\]

\( v_2 \) was specified in Section 4.5. To wit, it is a vector formed by stacking the active velocity vectors \( v_{22} \) of all bodies which have active control.

6.3 The Velocity Transformation Equation

The previously-listed expression for the velocity transformation is

\[
v = T v \mathbf{v}'
\]

(6.3-1)

However, for subsequent work, it proves convenient to partition \( v \) and \( v' \) into their subvectors and to write the transformation in the more explicit form

\[
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} = T v \begin{bmatrix}
v_1' \\
v_1 v' \\
0 0 1
\end{bmatrix} \begin{bmatrix}
\rho_{ba} \\
v_1 v_2'
\end{bmatrix}
\]

(6.3-2)

where

\[
T v_1 v' = \begin{bmatrix}
S_{v_1 v_1} & S_{v_1} & \rho_{ba} & S_{v_1} v_2'
\end{bmatrix}
\]

(6.3-3)

The development of the details of \( T v_1 v' \) is lengthy, and therefore it has been placed in Appendix B.
6.4 Configuration Vector of the Transformed System

In the study described in this document, the elements of the configuration vector, \( u' \), of \( \Sigma' \) were chosen to be the following:

1. the \( \bar{x}_{RN} \) and \( \bar{\psi}_{RN} \) of every reference body \( R \)
2. the \( \bar{q}_B \) of every reconnected joint \( B \)
3. the \( \bar{q}_{Z1} \) of every nonrigid body \( Z \)
4. the \( \bar{q}_{Z2} \) of every nonrigid \( Z \) which has active control.

However, it turned out to be convenient to work with the components, \( \bar{x}^{(N)}_{RN} \), of \( \bar{x}_{RN} \) rather than with the vector \( \bar{x}_{RN} \) itself. The configuration vector in which \( \bar{x}^{(N)}_{RN} \) is used in place of \( \bar{x}_{RN} \) is designated as \( u'(N) \).

The elements of \( u'(N) \) will be arranged in blocks of passive and active elements, analogous to the sequence used with the elements of \( v' \). To wit

\[
 u'(N) = \begin{bmatrix} u_1' (N) \\ q_2 \end{bmatrix}
\]

(6.4-1)

where \( u_1' (N) \) contains \( \bar{x}^{(N)}_{RN} \), \( \bar{\psi}_{RN} \), and the \( q_B \) and \( q_{Z1} \) vectors of the \( B \)'s and \( Z \)'s. The term \( q_2 \) of course is still the vector which is obtained by stacking the \( q_{Z2} \). Note that, unlike \( \bar{q}_B \), the vectors \( \bar{q}_B \) are not partitioned into passive and active subvectors.

6.5 Kinematics Equations of the Transformed System

The kinematics equation for \( \Sigma' \) is of the general form

\[
 u'_1 (N) = T'_u (N), v'
\]

(6.5-1)

However, for explicitness, \( u'_1 (N) \) and \( v' \) will be partitioned into their passive and active subvectors, and the kinematics equations for \( \Sigma' \) will be written as
\[
\begin{align*}
\dot{\mathbf{u}}_1^{(N)} &= T_\star^{(N)} \mathbf{v}' = \left[ S_\star^{(N)} \quad S_\star^{(N)} \quad 0 \right] \begin{pmatrix} \dot{\mathbf{v}}_1' \\ \dot{\mathbf{v}}_2' \\ \mathbf{p}_{ba} \end{pmatrix} \\
\dot{\mathbf{q}}_2 &= \mathbf{v}_2'
\end{align*}
\] (6.5-2a)

Comparison of the elements in \( \dot{\mathbf{u}}_1^{(N)} \) with those in \( \mathbf{v}' \) will show that the individual transformations in Eq. (6.5-2a) are quite simple, the non-identity ones being of the form

\[
\begin{align*}
\dot{\mathbf{\psi}}_R &= T_\star^{\psi} \mathbf{\psi}_R' \\
\dot{\mathbf{w}}_R &= \mathbf{w}_R \\
\mathbf{\omega}_R &= \mathbf{\omega}_R
\end{align*}
\] (6.5-3)

for all \( R \) and

\[
\begin{align*}
\dot{\mathbf{q}}_B &= T_\star^{q_B} (\mathbf{q}_B) \mathbf{p}_B \\
\mathbf{q}_B &= \mathbf{q}_B \mathbf{p}_B
\end{align*}
\] (6.5-4)

for all \( B \).

6.6 Constraint Equation of the Transformed System

6.6.1 Introduction

To develop the constraint equation for \( \Sigma' \), start by substituting Eq. (6.3-1) into (5.5-5) to produce

\[
c = c \mathbf{T}_W \mathbf{v}' - B \mathbf{p}_a^* = 0
\] (6.6.1-1)

where

\[
\mathbf{T}_W \mathbf{v}' = \mathbf{T}_W \mathbf{v} \mathbf{T}_V \mathbf{v}'
\] (6.6.1-2)
It is, perhaps, apparent from general principles that the constraint equations for reconnected joints should drop out of the formulation. In Subsection 6.6.2, it will be verified that this does in fact occur, and the formulation will be modified as necessary to eliminate the reconnected-joint constraint equations. In Subsection 6.6.3 the new constraint equation will be rearranged into a form more suitable for later work.

6.6.2 Elimination of Constraint Equations at Reconnected Joints

Start with the passive constraints. The portion of Eq. (6.6.1-1) which pertains to the passive constraints is

\[ c_p = C_p \frac{T_w}{w_p} v' = 0 \]  

(6.6.2-1)

Let \( c_{bp} \) be an element of \( c_p \) which pertains to some reconnected joint B which has passive constraints. Whether or not B also has active constraints is irrelevant. The equation in Eq. (6.6.2-1) which pertains to \( c_{bp} \) is

\[ c_{bp} = C_{bp} \frac{T_w}{w_p} v' = 0 \]  

(6.6.2-2)

The two subvectors \( \rho_{bf} \) and \( \rho_{ba} \) of \( \rho_B \) are elements of \( v' \). With the aid of Eq. (5.2.2-3) and (5.4.5-2) it can be deduced that

\[ \frac{w}{w} = T_{w} \begin{bmatrix} \rho_{bf} \\ \rho_{ba} \end{bmatrix} = \begin{bmatrix} S_{w} & S_{w} \\ w & w \end{bmatrix} \begin{bmatrix} \rho_{bf} \\ \rho_{ba} \end{bmatrix} = T_{w} v' \]  

(6.6.2-3)

Thus, the only nonzero elements of \( T_{w} v' \) are \( S_{w} \rho_{bf} \) and \( S_{w} \rho_{ba} \). However, Eq. (5.4.4-3a) indicates that premultiplying these two elements by \( c_{bp} \) produces 0. Hence

\[ c_{bp} \frac{T_w}{w} v' = 0 \]  

(6.6.2-4)

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Equation (6.6.2-4) indicates that Eq. (6.6.2-2) is satisfied for for all \( v' \). Thus, Eq. (6.6.2-2) is not a constraint on \( v' \). This conclusion holds for the passive constraints at all reconnected joints. Therefore, the passive constraints at all reconnected joints can be discarded from the set of constraint equations. In fact, it is essential that they be discarded, because the formulations of the dynamical equations which will be developed later require that the constraint matrix have full rank.

Equation (6.6.2-1) for the set of passive constraints henceforth will be written as

\[
\mathbf{C}_p = \mathbf{C}_p \mathbf{T}_{\mathbf{w}_p} \mathbf{v}' = 0
\]  

(6.6.2-5)

The new subscript label \( c \) above is intended to denote that only terms which pertain to cut joints are to be included. The symbol \( c \) was chosen because of its correspondence to \( \Gamma \) and \( \gamma \). Thus, for example, \( \mathbf{w}_c \) is the vector which is obtained by stacking the \( \mathbf{w}_p \) of all cut joints which have passive constraints.

Now consider the active constraints. The portion of Eq. (6.6.1-1) which pertains to the active constraints can be written as

\[
\mathbf{C}_a = \mathbf{C}_a \mathbf{T}_{\mathbf{w}_a} \mathbf{v}' - \mathbf{B}_a \mathbf{\rho}_a = 0
\]  

(6.6.2-6)

Let \( \mathbf{C}_{Ba} \) be an element of \( \mathbf{C}_a \) which pertains to some reconnected joint \( B \) which has active constraints. The equation in Eq. (6.6.2-6) which pertains to \( \mathbf{C}_{Ba} \) is

\[
\mathbf{C}_{Ba} = \mathbf{C}_{Ba} \mathbf{T}_{\mathbf{w}_B} \mathbf{v}' - \mathbf{B}_{Ba} \mathbf{\rho}_{Ba} = 0
\]  

(6.6.2-7)
Introducing Eq. (6.6.2-3) into the above equation produces

\[ c_{Ba} T_{\omega_B} \cdot \rho_B - B_{Ba} \rho_{Ba}^* = 0 \]  

(6.6.2-8)

Now coordinatize the C and T terms above as prescribed by Eq. (5.4.2-8) and the inverse of Eq. (5.2.2-7) to produce

\[ c_{Ba} \beta_B^{(1)} T_{\omega_B} \cdot \beta_B^{(1)} \rho_B - B_{Ba} \rho_{Ba}^* = 0 \]  

(6.6.2-9)

Equation (6.6.2-9) is identical to Eq. (5.4.5-1). Assuming that its C and B terms are established via Eqs. (5.4.5-4) and (5.4.5-6), it reduces to merely

\[ \rho_{Ba} = \rho_{Ba}^* \]  

(6.6.2-10)

Equation (6.6.2-10) makes possible a simpler method of handling the active constraints at reconnected joints than through the explicit use of constraint equations. To wit, wherever the response variable \( \rho_{Ba} \) appears, it can be replaced by its commanded value \( \rho_{Ba}^* \). In particular, the previous specification, Eq. (6.2-1), for \( \mathbf{v} \) can be replaced by

\[ \mathbf{v}' = \begin{pmatrix} v_1' \\ \rho_{Ba}^* \\ v_2' \end{pmatrix} \]  

(6.6.2-11)

where \( \rho_{Ba}^* \) is formed by stacking the \( \rho_{Ba}^* \).

Since the active constraints at reconnected joints will be incorporated into the formulation through the technique noted in the above paragraphs, the corresponding constraint equations must be removed from Eq. (6.6.2-6). Henceforth this reduced equation will be written as
\[ \frac{c}{c_a} = C \frac{T_{w_c^v}}{c_a} v' - B \frac{\rho^*_c}{c_a} = 0 \] (6.6.2-12)

The significance of the subscript label \( c \) was noted previously below Eq. (6.6.2-5).

The constraint equation, which includes both the passive and active constraints, for \( \Gamma' \) is formed by stacking Eqs. (6.6.2-5 and 6.6.2-12). It will be written as

\[ \frac{c}{c} = C \frac{T_{w_c^v}}{c} v' - B \frac{\rho^*_c}{c_a} = 0 \] (6.6.2-13)

In defining the \( T \) array above, the velocity vector

\[ \mathbf{w}_c = \begin{pmatrix} w_{cp} \\ w_{ca} \end{pmatrix} \] (6.6.2-14)

was introduced.

6.6.3 A Modified Form of the Constraint Equation

Equation (6.6.2-13) is in a form which proves to be unsuitable for the studies which will be performed in Sections 6.8 and 6.9. For this work, it turns out to be necessary to manipulate Eq. (6.6.2-13) so that the passive velocity vector \( v'_{1} \) appears explicitly. Also, it is convenient to introduce some simpler notation.

Define the following constraint operator, \( D' \), and suboperators \( D'_{1}, D'_{p}, \) and \( D'_{2} \)

\[ D' = C \frac{T_{w_c^v}}{c} = C \left[ \begin{array}{ccc} w_{c^v} & S_{w_c^v} & S_{w_c^v} \\ w_{c-1} & \rho_{ba} & w_{c-2} \end{array} \right] = [D'_{1} \ D'_{p} \ D'_{2}] \] (6.6.3-1)

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It is then easy to manipulate Eq. (6.6.2-13) into

\[ D'_1 \frac{v'_1}{v_1} = q' \]  

(6.6.3-2)

where

\[ q' = B_c \rho_{ca}^* - D'_1 \rho_{ba}^* - D'_2 \frac{v'_1}{v_2} \]  

(6.6.3-3)

Eq. (6.6.3-2) is the form that will be used in the remainder of this chapter.

6.7 **Kinetics Equation of the Transformed System**

The kinetics equation for \( \Sigma' \) can be obtained by substituting Eq. (6.3-2) into (4.5-2a) and premultiplying by \( S_{v_1} \). The result can be expressed as follows:

\[ M'_1 \frac{v'_1}{v_1} = -M'_2 \frac{v'_2}{v_1} + F'_D + \frac{F'_E}{v_{v_1}} + \frac{F'_K}{v_{v_1}} + \frac{F'_C}{v_{v_1}} \]  

(6.7-1)

where

\[ M'_1 = S_{v_1}^{v_1} \frac{M_1 S_{v_1}}{v_1} \]  

(6.7-2)

\[ M'_2 = S_{v_1}^{v_1} \frac{M_2 S_{v_1}}{v_2} \]  

(6.7-3a)

\[ M'_2 = S_{v_1}^{v_1} \left[ M_2 + M_1 \frac{v_1}{v_2} \right] \]  

(6.7-3b)

\[ \frac{F'_D}{v_{v_1}} = S_{v_1}^{v_1} \left\{ \frac{F_D}{v_{v_1}} - M_1 \frac{v_1}{v_2} \right\} \]  

(6.7-4a)

\[ \frac{F'_E}{v_{v_1}} = S_{v_1}^{v_1} \frac{F_E}{v_{v_1}} \]  

(6.7-4b)

\[ \frac{F'_K}{v_{v_1}} = S_{v_1}^{v_1} \frac{F_K}{v_{v_1}} \]  

(6.7-4c)

\[ \frac{F'_C}{v_{v_1}} = S_{v_1}^{v_1} \frac{F_C}{v_{v_1}} \]  

(6.7-4d)
For notational simplicity, Eqs. (6.7-4) henceforth will be combined into the single expression

\[ \mathbf{f}_K' = \mathbf{f}_D' + \mathbf{f}_{EI}' + \mathbf{f}_{KJ}' \]  \hspace{1cm} (6.7-6)\]

the term \( \mathbf{f}_K' \) above is the "known" generalized force vector on \( \Sigma' \).

The dynamic force vector, \( \mathbf{f}_D' \), on \( \Sigma' \) includes a term \( \mathbf{T}_{\Sigma\Sigma}' \mathbf{v}' \). The development of the details of this term are sufficiently lengthy that it has been placed in Appendix C.

It is, perhaps, apparent from general principles that constraint forces at reconnected joints should make no contribution to the constraint force vector \( \mathbf{f}_C' \). That this is indeed true is verified as part of a supplementary study of the joint constraint formulation which is presented as Appendix E. It thus follows that \( \mathbf{f}_C' \) is zero if all joints which contain constraints are reconnected; in this case, the joint constraints and joint constraint forces drop out of the dynamical equations for \( \Sigma' \). As was noted in Chapter 6.1, \( \Sigma' \) must contain no closed loops. Therefore, if the original system contains closed loops, one may or may not have the option of reconnecting all the constrained joints when establishing \( \Sigma' \).

Some material on the determination of \( \mathbf{f}_{KJ}' \) will be introduced at this point. Let \( \mathbf{L}_K^T \) be the composite force vector obtained by stacking \( \mathbf{l}_{\Sigma_0} \) of every joint (cut or reconnected) which has known forces. Similarly, let \( \mathbf{w}_K^T \) be the corresponding composite velocity vector which is formed by stacking the \( \mathbf{w}_A \) of these joints. It then can be shown that

\[ \mathbf{f}_{KJ}' = \mathbf{S}_{\mathbf{w}_K' \mathbf{L}_K^T} \]  \hspace{1cm} (6.7-7)\]
The above equation can be proven by duplicating the proof given in Appendix E for the corresponding equation, Eq. (6-3-3), for constraint forces. Introducing Eq. (6.7-7) into (6.7-4c) produces the following more explicit equation for $f_{KJ}'$

$$f_{KJ}' = S^T_{-K-1} \frac{L^K}{v_{-1}}$$

(6.7-8)

where

$$S^T_{-K-1} = S^T_{-K-1} S_{-K-1} S_{-K-1} v_{-1} v_{-1}$$

(6.7-9)

In applications, however, it usually is easier to establish the equations for $f_{KJ}'$ "directly" rather than through the use of the formal transformation denoted by Eqs. (6.7-8 and 6.7-9).

One can, perhaps, deduce that the sets of kinetics equations for the trees of $\Sigma'$ are coupled together only through the forces at the cut joints between bodies in different trees. In the following several paragraphs, Eq. (6.7-1) will be rewritten to show this explicitly. This more explicit formulation is useful in applications, such as the sample problem in Chapter 7.

Let the ordering scheme for the bodies $Z$ be such that the bodies of each individual tree $T_j; j = 1, \ldots, m$ are grouped together. One can then write

$$v_1 = \begin{pmatrix} v_{T_1}' \\ v_{T_2}' \\ \vdots \\ v_{T_m}' \end{pmatrix}$$

(6.7-10)
The term $v_{Tj1}$ above is the vector obtained by stacking the $v_{Zi}$ of all $Z$'s in $T_j$. Write similar expression for $v_{Tj2}$, $f_D$, $f_{EI}$, $f_{KJ}$, and $f_C$. Also, let the ordering scheme for the joints be such that the reconnected joints of $\xi'$ are grouped together by trees and write

$$
\begin{pmatrix}
\rho^*_T_{1,ba} \\
\vdots \\
\vdots \\
\rho^*_T_{m,ba}
\end{pmatrix}
$$

Finally, specify the elements of $v_{\xi'}$ to be arranged by trees and write

$$
\begin{pmatrix}
v_{T1}' \\
v_{T2}' \\
\vdots \\
v_{Tm}'
\end{pmatrix}
$$

It can be deduced that the ordering techniques denoted in the above paragraph yield arrays $S_{v_{\xi'}v_{\xi'}}$, $S_{v_{\xi'}\rho_{ba}}$, and $S_{v_{\xi'}v_{\xi2}}$ which are block diagonal. Thus

$$
v_{Tj1}' = S_{v_{Tj1}v_{Tj1}} v_{Tj1} + S_{v_{Tj1}\rho_{Tj,ba}} \rho^*_{Tj,ba} + S_{v_{Tj1}v_{Tj2}} v_{Tj2} (6.7-13)
$$

Since $M_1$ and $M_2$ are block diagonal by definition, it can be shown from Eqs. (6.7-2 and 3) that $M_1'$, $M_\rho'$, and $M_2'$ also are block diagonal. In addition, it can be shown from Eqs. (6.7-4a and b) that the $T_j$ force vectors $f_{TjD}'$ and $f_{TjEI}'$ are dependent only on the forces, $f_{TjD}$ and $f_{TjEI}$, acting on bodies in $T_j$. The following expression for $f_{TjKJ}'$ can be obtained from Eq. (6.7-8)
\[ f'_{T,J,K} = S^T_{K'}v_{T,J} \]  

(6.7-14)

The desired more-explicit form of the kinetics equations for \( L' \) now can be obtained by applying the material in the above two paragraphs to Eq. (6.7-1). The result is

\[ M'_{T,j} \dot{v}_{T,j} = -M'_{T,j} \rho \frac{\partial x_{T,j}}{\partial x_{T,j}} - M'_{T,j} \dot{v}_{T,j} + f'_{T,j} + f'_{T,j,C} \]  

(6.7-15)

where \( j = 1 \) to \( m \).

The main theoretical issue which still needs to be addressed in this chapter is the computation, or elimination, of \( f'_{C} \) for those problems in which it is not identically 0. Two methods of dealing with \( f'_{C} \) are presented in the following two sections. In the first, \( f'_{C} \) is computed using the traditional Lagrange multiplier approach. In the second, the constraint and kinetics equations for \( L' \) are manipulated in such a manner that \( f'_{C} \) drops out.

6.8 The Lagrange Multiplier Method for Handling the Constraints

6.8.1 Introduction

The equations from previous sections which will be used in this section are

\[ M'_{1} \dot{v}'_{1} = -M'_{1} \rho \frac{\partial x_{2}}{\partial x_{2}} - M'_{2} \dot{v}'_{2} + f'_{K} + f'_{C} \]  

[Reference (6.7-1)]

\[ D'_{1} \dot{v}'_{1} = g' \]  

[Reference (6.6.3-2)]
With the Lagrange multiplier method, $f'_C$ is established via

$$f'_C = D_1^T \lambda$$  \hspace{1cm} (6.8.1-1)$$

where $\lambda$ is the vector of Lagrange multipliers. The validity of the general operation of replacing constraint force vectors with the product of the transposed constraint operator and the so-called Lagrange multiplier vector, as is done in Eq. (6.8.1-1), is well known. A derivation of Eq. (6.8.1-1) is included in this document as part of the material in Appendix E.

6.8.2 Development of the Algorithm

The problem to be addressed in this subsection is the manipulation of Eqs. (6.7-1), (6.6.3-2), and (6.8.1-1) into a form which enables $\lambda$ to be computed. With the algorithm for establishing $\lambda$ in hand, $f'_C$ then can be computed via Eq. (6.8.1-1) and employed, as indicated, in Eq. (6.7-1).

For simplicity, it should be convenient, in the following paragraphs, to take the point of view that coordinate frame resolution has been introduced into Eqs. (6.7-1), (6.6.3-2), and (6.8.1-1). The terms which appear in the equations then are matrices with scalar elements.

It is assumed that $M_1'$ is nonsingular. Insert Eq. (6.8.1-1) into Eq. (6.7-1), and solve the resulting equation to produce an expression for $\dot{v}_1'$. Differentiate Eq. (6.6.3-2) with respect to time and insert the expression for $\dot{v}_1'$ into it. Rearrange to place the term involving $\lambda$ on the LHS. The result is

$$A_2 \lambda = \ddot{q}' - D_1 \dot{v}_1' - D_1 M_1'^{-1} \left[ f'_K - M_2 \dot{v}_2' \right]$$  \hspace{1cm} (6.8.2-1)$$
where

\[ A_\lambda = D_1' M_1'^{-1} D_1'^T \]  

(6.8.2-2)

The above derivation was based on the method used by Bodley\(^{(3)}\).

Some additional work on the equation for \( \lambda \) will be performed. Differentiate Eq. (6.6.3-3) for \( \varphi \) and insert into Eq. (6.8.2-1) to produce

\[
A_\lambda \lambda = D_1' M_1'^{-1} \hat{f}_k - D' v' + B_c \hat{p}^*_c - [D'_{\rho} - D_1' M_1'^{-1} M_1' \hat{s}] \hat{p}_{\rho} \\
- [D_2' - D_1' M_1'^{-1} M_2'] \hat{v}_2
\]

(6.8.2-3)

The \( D' \) terms above were defined previously in Eq. (6.6.3-1).

It is assumed that \( D_1' \) is long-rectangular. If, in addition, \( D_1' \) has full rank, it follows from Eq. (6.8.2-2) that \( A_\lambda \) will be nonsingular. When \( A_\lambda \) is nonsingular, Eq. (6.8.2-3) will have a unique solution for \( \lambda \); numerous algorithms are available for solving Eq. (6.8.2-3) to obtain this solution.

Unfortunately, problems in which \( D_1' \) has less than full rank are not uncommon. For example, both sample problems described in Chapters 7 and Chapter 8 encounter such a condition.

When \( D_1' \) has less than full rank, the vector \( \varphi' \) (when not identically zero) still will be in the column space of \( D_1' \) if the constraints have been modeled adequately, and Eq. (6.6.3-2) thus will still possess solutions for \( \lambda' \).

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When $D'_1$ has less than full rank, it signifies that the scalar constraint equations in Eq. (6.6.3-2) are not all independent. This can occur in specific problems if geometrical conditions are encountered which cause some of the constraints within one (or more) individual joints to become redundant. More importantly, the condition also can occur in some problems in which the original system contains closed topological loops. In this second case, the constraints within each individual joint are all independent, but the combined set of constraint equations for the overall system contains redundant members. This second case is rare in problems in which the individual bodies are flexible. However, it is very common in closed loop problems in which the individual bodies are all rigid.

It can be shown that when $D'_1$ has less than full rank, Lagrange multiplier vectors $\lambda$ which yield $f'_c$ through Eq. (6.8.1-1) still exist, but are not unique. The problem can be rectified by deleting equations from Eq. (6.6.3-2) to produce a $D'_1$ which does have full rank. This is equivalent to setting the corresponding elements of $\lambda$ to zero.

It is not claimed that in a dynamic simulation, it is an easy task to determine the number and correct rows of $D'_1$ to delete in order to produce a full rank $D'_1$ as prescribed in the above paragraph. The problem in particularly acute in general purpose computer programs in which little or no a priori information on the constraint problem is available for usage by the algorithm. This practical topic, however, was considered to fall outside the intended scope of the present work and hence was not pursued.

Equation (6.8.2-3) includes a term $D'y'$ which should be specified in more detail. The development, however, is sufficiently lengthy that it has been placed in Appendix D.
6.8.3 Summary of Dynamical Equations

To summarize, when the Lagrange multiplier method is used, the set of dynamical equations for \( \Sigma' \) is as follows:

1. Eq. (6.7-1) for kinetics,
2. Eq. (6.8.1-1) for the constraint force vector,
3. Eq. (6.8.2-3) for the Lagrange multiplier vector,
4. Eq. (6.5-2) for kinematics
5. Eq. (4.5-1d) for active velocities of nonrigid bodies, and
6. Eq. (4.5-1e) for control system dynamics

6.8.4 Constraint Stabilization

Dynamic simulations which employ the Lagrange multiplier approach can encounter inaccuracy or even instability problems in maintaining the constraint conditions. The problem results in part from the fact that this approach does not utilize the cut joint constraint equation, Eq. (6.6.3-2), directly. Rather, it employs the time derivative of this equation. (Recall, from Subsection 6.8.2, that the derivation of Eq. (6.8.2-3) entailed the differentiation of Eq. (6.6.3-2)). Thus, letting \( \dot{c}^{NH}_c = 0 \) denote the equation for the nonholonomic constraints at the cut joints, Eq. (6.8.2-3) actually utilizes

\[
\dot{c}^{NH}_c = 0
\]  
(6.8.4-1a)

The problem is doubly difficult for holonomic constraints, since they already are differentiated once just to get them into the form indicated in Eq. (6.6.3-2). Thus letting \( \dot{c}^H_c = 0 \) denote the equation for the holonomic constraints at the cut joints, Eq. (6.8.2-3) actually uses

\[
\dot{c}^H_c = 0
\]  
(6.8.4-1b)
A constraint stabilization technique which alleviates the problem has been proposed by Baumgarte\(^{(30),(39)}\). With Baumgarte's approach, Eqs. (6.8.4-1) are replaced by

\[
\begin{pmatrix}
\dot{C}^N \\
\dot{C}^H
\end{pmatrix} + A_1 \begin{pmatrix}
C^N \\
C^H
\end{pmatrix} + \begin{bmatrix}
0 & 0 \\
0 & A_2
\end{bmatrix} \begin{pmatrix}
C^N \\
C^H
\end{pmatrix} = 0 \quad (6.8.4-2)
\]

where \(A_1\) and \(A_2\) are (diagonal) matrices of suitably chosen weighting factors. References 30 and 39 discuss the approach in detail.

Baumgarte's technique can be implemented into the formulation developed in the present document by adding the terms

\[
K_1 \left[ D_1 \dot{\gamma}_1 - q' \right] + K_2 \frac{H}{C} \quad (6.8.4-3)
\]

to the RHS of Eq. (6.8.2-3). In this expression, \(K_1\) is a (diagonal) matrix which weights the errors in satisfying the velocity constraints. \(K_2\) is a non-diagonal matrix which (1) weights the errors in satisfying the holonomic constraints and also (2) permutes the elements of \(\frac{H}{C}\) for compatibility with the arrangement of the \(\frac{H}{C}\) terms in \(D_1 \dot{\gamma}_1 - q'\).

6.9 The Partitioned Velocity Vector Method for Handling the Constraints

6.9.1 Introduction

An alternative to the Lagrange multiplier method for handling the constraints will be developed in this section. The work again will utilize Eqs. (6.7-1) and (6.6.3-2). Again, the point of view will be that coordinate frame resolution has been introduced into these two equations so that the terms in them are matrices with scalar elements.
It will be informative to keep track of dimensions. Refer to Eqs. (6.7-1 and 6.6.3-2). Let $v'_1$ be $n \times 1$, and let $q'_1$ be $m \times 1$. Then $M'_1$ is $n \times n$ and $D'_1$ is $m \times n$.

It will be assumed that $m < n$; thus $D'_1$ is long-rectangular.

6.9.2 Development of the Algorithm

As a first step, we wish to partition $v'_1$ into a pair of subvectors and manipulate Eq. (6.6.3-2) to solve for one of these subvectors as a function of $q'_1$ and the other. The work will be limited to the case where $D'_1$ has full rank; i.e. rank $D'_1 = m$.

Since rank $D'_1 = m$, exactly $m$ of the $n$ columns of $D'_1$ will be linearly independent. Regretably, however, neither the first $m$ columns of $D'_1$ nor the last $m$ columns will be independent in all cases. This is inconvenient. The difficulty is regarded as being of sufficient importance that it should be taken into account in the formulation. Therefore, let $P$ be an $n \times n$ permutation matrix which permutes the columns of $D'_1$ in such a manner that the last $m$ of them are linearly independent.

$$D'_1 P = D'_1 \begin{bmatrix} P_r & P_s \end{bmatrix} = \begin{bmatrix} D'_1 P_r & D'_1 P_s \end{bmatrix} = \begin{bmatrix} D'_{1r} & D'_{1s} \end{bmatrix} $$

(6.9.2-1)

where $d = n - m$

Permute the elements of $v'_1$ analogously

$$\begin{bmatrix} v'_{1r} \\ v'_{1s} \end{bmatrix} = P^T \begin{bmatrix} v'_1 \end{bmatrix}$$

(6.9.2-2)
Since $P^{-1} = P^T$, the inverse of Eq. (6.9.2-2) is merely

$$v'_1 = P \left\{ \begin{array}{c} v'_r \\ \\ \end{array} \right\}$$  \hspace{1cm} (6.9.2-3)

Now insert Eq. (6.9.2-3) and the inverse of Eq. (6.9.2-1) into Eq. (6.6.3-2) and solve for $v'_{1s}$.

$$v'_{1s} = D'_{1s}^{-1} \left\{ g' - D'_{1r} v'_r \right\}$$ \hspace{1cm} (6.9.2-4)

Also, insert Eq. (6.9.2-4) into Eq. (6.9.2-3) to produce

$$v'_1 = \xi + \Pi v'_{1r}$$ \hspace{1cm} (6.9.2-5)

where

$$\xi = P_s D'_{1s}^{-1} q'$$ \hspace{1cm} (6.9.2-6)

$$\Pi = P_r - P_s D'_{1s}^{-1} D'_{1r}$$ \hspace{1cm} (6.9.2-7)

The next step is to obtain the modified kinetics equation. Start by differentiating Eq. (6.9.2-5), substituting this result into Eq. (6.7-1), and premultiplying by $\Pi^T$ to produce

$$\Pi^T M'_1 \Pi v'_{1r} = \Pi^T \left\{ \frac{\xi}{T} - M'_1 \frac{P_{bas}}{P_{ba}} - M'_2 \frac{v'_r}{v'_s} - M'_1 \left[ \frac{\xi}{T} + \Pi v'_{1r} \right] \right\} + \Pi^T f'_C$$ \hspace{1cm} (6.9.2-8)

The matrix $M'_1$ is assumed to be nonsingular. The matrix $\Pi$ is tail-rectangular, since $d = n - m < n$. It is not difficult to show that $\Pi$ has full rank; a proof can be produced by working solely with the
definition of \( \Pi \) given in Eq. (6.9.2-7). It follows, therefore, that 
\( \Pi^T M_1' \Pi \) is nonsingular. Hence Eq. (6.9.2-8) always can be solved for 
\( \dot{\nu}'_1 \).

We wish to show that the term \( \Pi^T f'_C \) in Eq. (6.9.2-9) is 0. One procedure is as follows:

1. employ Eq. (6.9.2-7) and the inverse of (6.9.2-1) to produce

\[
D'_1 \Pi = 0 \quad (6.9.2-9)
\]

2. deduce trivially from Eq. (6.9.2-9) that

\[
\Pi^T D'_1 \lambda = 0 \quad (6.9.2-10)
\]

for all m-vectors \( \lambda \)

3. complete the proof by inserting Eq. (6.8.1-1) into (6.9.2-10).

Since the constraint force vector drops out of Eq. (6.9.2-8), this equation in conjunction with Eq. (6.9.2-5), constitutes an algorithm for establishing \( \dot{\nu}'_1 \). However, it will be useful to do some additional work on the remaining terms on the RHS of Eq. (6.9.2-8). Specifically, the terms inside [ ] will be written out more explicitly.

Start by differentiating Eqs. (6.9.2-6) and (6.9.2-7) and manipulating to generate

\[
\dot{\xi} = P_s D'^{-1}_1 s [ \dot{\xi}' - D'_1 \dot{\xi} ] \quad (6.9.2-11)
\]

\[
\dot{\Pi} = -P_s D'^{-1}_1 s D'_1 \Pi \quad (6.9.2-12)
\]
Use of the above two equations and Eq. (6.9.2-5) produces

$$\ddot{\mathbf{r}} + \dot{\mathbf{r}} \cdot \mathbf{v}'_{1r} = \mathbf{P}_s \cdot \mathbf{D}_1^{-1} \left( \mathbf{g}' - \mathbf{D}_1 \cdot \mathbf{v}'_{1} \right) \quad (6.9.2-13)$$

The revised kinetics equation now can be obtained by differentiating Eq. (6.6.3-3) to form $\mathbf{q}'$, inserting this result into Eq. (6.9.2-13), and then introducing this new expression into Eq. (6.9.2-8) (with the $f'_C$ term deleted). The final kinetics equation is

$$\Pi^T \mathbf{M}_1' \Pi \mathbf{v}'_{1r} = \Pi^T \left[ \mathbf{f}'_k + \Xi \left[ \mathbf{D}' \cdot \mathbf{v}' - \mathbf{B}_c \cdot \mathbf{D}'_{ca} \right] 
- \left[ \mathbf{M}'_\rho - \Xi \mathbf{D}'_{\rho} \right] \mathbf{D}'_{ba} 
- \left[ \mathbf{M}'_2 - \Xi \mathbf{D}'_{2} \right] \mathbf{v}'_{2} \right] \quad (6.9.2-14a)$$

where

$$\Xi = \mathbf{M}_1' \cdot \mathbf{v}_s \cdot \mathbf{D}_1^{-1} \quad (6.9.2-14b)$$

Note the term $\mathbf{D}' \cdot \mathbf{v}'$ above. This term also appeared in Eq. (6.8.2-3) for the Lagrange multiplier method. As noted there, the detailed equations which specify this term are lengthy, and their development has been placed in Appendix D.

6.9.3 Summary of Dynamical Equations

To summarize, when the partitioned velocity vector method is used, the set of dynamical equations for $\mathbf{v}'$ is as follows.

1. Eq. (6.9.2-14) for kinetics,
2. Eq. (6.9.2-5) which establishes $\mathbf{v}'$ after Eq. (6.9.2-14) has been integrated,
3. Eq. (6.5-2) for kinematics,
4. Eq. (4.5-1d) for active velocities of nonrigid bodies and
5. Eq. (4.5-1e) for control system dynamics.
6.9.4 Constraint Stabilization

Unlike the Lagrange multiplier approach, the partitioned velocity vector method is not subject to the cut joint constraint instability problem, noted in Subsection 6.8.4, which results from employing differentiated velocity constraints. The fact that a subvector, \( \dot{v}^{\prime}_{18} \), of \( \dot{v}^{\prime} \) is computed directly via Eqs. (6.9.2-4 or 5) assures that the velocity constraints will be satisfied in dynamics simulations, regardless of whatever other inaccuracies the program may have. As noted in Subsection 6.9.4, however, the partitioned velocity vector approach does have a stability problem with holonomic constraints at the cut joints because such constraints must be differentiated just to get them into the velocity constraint form indicated in Eq. (6.6.3-2).

The above-noted holonomic constraint problem can be attacked in the spirit of Baumgarte. Basically, the technique involves replacing the equation \( \dot{c}^H = 0 \) for holonomic constraint derivatives at cut joints by

\[
\dot{c}_c^H + A c_c^H = 0 \quad (6.9.4-1)
\]

where \( A \) is a diagonal matrix of weighting factors. More specifically, the technique can be implemented into the formalism developed in Subsection 6.9.2 by adding the term \(-Kc_c^H\) to the expression, Eq. (6.6.3-3), for \( q^\prime \). The matrix \( K \) here (1) weights the errors in satisfying the holonomic constraints and (2) permutes them as necessary. It can be shown that this approach requires adding the term \( \dot{\Gamma}^T K c_c^H \) to the RHS of Eq. (6.9.2-14); in establishing this term, it was assumed that \( K \) is constant.
CHAPTER 7

APPLICATION TO SIMPLE EXAMPLE

7.1 Description of the Problem

This chapter presents an illustrative example of the application of the technique developed in Chapters 4 to 6. The method is employed to develop equations of motion for a relatively simple system which is comprised of 5 rigid bodies, 5 single degree of freedom joints, and one closed loop. Fig. 7-1 shows the basic system. A body-joint graph of the system is given as Fig. 7-2. A diagram which portrays the orientation relationships among the coordinate frames that will be used in the problem is presented as Fig. 7-3.

The bodies will be labeled 1, 2, 3, 4, and 5. The joints will be labeled α, β, γ, δ, and ε.

Joints α, β, γ, and δ are pin joints. Therefore, each has a single axis of rotation which is fixed relative to the two bodies that it connects. These 4 axes of rotation are assumed to be parallel to one another.

Body 5 can slide linearly relative to body 4. This motion is modeled as joint ε. The motion is assumed to be frictionless. Rotation of body 5 relative to body 4 is not permitted. The mass centers of 4 and 5 both lie on the axis of translation. The axis of translation is orthogonal to the axes of rotation of α, β, γ, and δ.
Figure 7-1. Bodies and joints of sample problem
Figure 7-2. Body-joint graph for sample problem.
Figure 7-3. Coordinate frame flow diagram for sample problem.
The system is not subjected to external forces or torques. Motion at joints \( \alpha, \beta, \) and \( \epsilon \) is restrained by springs. Motion at \( \gamma \) and \( \delta \) is not spring-restrained. The attach point of the translational spring to body 5 is at the center of mass of body 5; the force vector which this spring generates lies along the axis of translation.

The simple problem was motivated by one studied by Kane and Levinson\(^{(35)} \). The main difference is that Kane and Levinson assumed bodies 2, 3, and 4 to be massless. They also assumed body 5 to be a particle. Because of these differences, it is not possible to compare the methods and results of this chapter with those of Ref. 35.

7.2 Geometric Variables and Coordinate Frames

Let \( \hat{e}_r \) be a unit vector which is parallel to the axes of rotation of \( \alpha, \beta, \gamma, \delta \). Let \( \hat{e}_t \) be a unit vector which lies along the axis of translation at \( \epsilon \). \( \hat{e}_t \) is directed from the spring attach point toward body 5. Since body 5 always remains on the same side of the spring attach point, the polarity of \( \hat{e}_t \) is constant. The vectors \( \hat{e}_r \) and \( \hat{e}_t \) are orthogonal to one another.

Let the body reference frames for the 5 bodies be designated merely as frames 1, 2, 3, 4, and 5. Let the unit vectors along their axes be designated at \( \hat{e}_{1x}^i, \hat{e}_{1y}^i, \hat{e}_{1z}^i, i = 1 \) to 5. Let the origin of each frame \( i \) be located at the center of mass of the corresponding body \( i \). Choose the orientations of frames \( i \) to be such that all \( \hat{e}_{iy}^i \) are directed along \( \hat{e}_r \). Let the orientations of frames 4 and 5 be identical with \( \hat{e}_{4x} \) and \( \hat{e}_{5x} \) along \( \hat{e}_t \).

Figure 7-2 shows the body-joint graph including the joint polarity convention that will be used. Recall that every joint has an input frame and an output frame. The input frames are located at the bases of the arrows in the graph; the output frames are located at the tips.
The origins of the input and output frames of each pin joint (α, β, γ, δ) coincide and are located on the axis of rotation of that joint. The origins of the input and output frames (ε_I and ε_O) of ε do not coincide. Choose ε_I and ε_O to be located at the attach point of the translational spring to bodies 4 and 5 respectively. Designate the displacement vector from ε_I to ε_O as $\bar{x}_\varepsilon$. Therefore

$$\bar{x}_\varepsilon = x_{\varepsilon_I} - x_{\varepsilon_O}$$  \hspace{1cm} (7.2-1)

It will be convenient to choose the orientation of each joint frame to be parallel to the reference frame of the body upon which the joint frame is mounted. Thus, frames α_I, β_I, and 1 have the same orientation; frames α_O, 2, and γ_I have the same orientation; etc. This designation of joint frame orientation will enable the joint frames to be replaced by the corresponding body frames in most of the ensuing mathematics.

Let the rotation angles between the input and output frames of joints α, β, γ, δ be designated as $\phi_\alpha$, $\phi_\beta$, $\phi_\gamma$, and $\phi_\delta$ respectively.

Fig. 7-3 presents a flow diagram which summarizes the orientation relationships between all body and joint coordinate frames.

7.3 Kinetics Equation of the Primitive System

The following kinetics equations can be developed for the individual bodies i:

$$\begin{bmatrix}
    m_i & 0 \\
    0 & J_i
  \end{bmatrix} \begin{bmatrix}
    \dot{v}_i \\
    \dot{\omega}_i
  \end{bmatrix} = \begin{bmatrix}
    f_i^D + f_i^K + f_i^C
  \end{bmatrix}$$  \hspace{1cm} (7.3-1)
where

\[ f_1^D = - \{ m_1 \bar{\omega}_1, \bar{\omega}_1 J_1 \bar{\omega}_1 \} \quad (7.3-2) \]

\[ f_1^K = - \{ f_2^K + f_3^K \} \quad (7.3-3a) \]

\[ f_2^K = \{ \bar{\theta}, \bar{G}_0^K \} = - \{ \bar{\theta}, \varepsilon_r^k_\alpha \left[ \phi_\alpha - \phi_\alpha^g \right] \} \quad (7.3-3b) \]

\[ f_3^K = \{ \bar{\theta}, \bar{G}_0^K \} = - \{ \bar{\theta}, \varepsilon_r^k_\beta \left[ \phi_\beta - \phi_\beta^g \right] \} \quad (7.3-3c) \]

\[ f_4^K = - f_5^K \quad (7.3-3d) \]

\[ f_5^K = \{ F^K_\varepsilon, \bar{\theta} \} = - \{ \varepsilon_r^k_\varepsilon [x_\varepsilon - x_\varepsilon^g], \bar{\theta} \} \quad (7.3-3e) \]

\[ f_1^C = - L_\alpha_0^C - L_{\beta_0}^C \quad (7.3-4a) \]

\[ f_2^C = L_\alpha_0^C - L_{\gamma_0}^C \quad (7.3-4b) \]

\[ f_3^C = L_{\beta_0}^C - L_{\delta_0}^C \quad (7.3-4c) \]

\[ f_4^C = L_{\gamma_0}^C + L_{\delta_0}^C - S_{54}^T L_{\varepsilon_0}^C \quad (7.3-4d) \]

\[ f_5^C = L_{\varepsilon_0}^C \quad (7.3-4e) \]
\( \bar{v}_i \) and \( \bar{w}_i \) above are the translational and angular velocities of frame \( i \) relative to the inertial frame \( N \); the subscript label \( N \) used previously with these quantities is omitted in this chapter, for simplicity. For spatial compactness, the elements of column vectors often will be written in a row, as in Eqs. (7.3-2) and (7.3-3).

For any of the 5 joints (\( \sigma = \alpha, \beta, \gamma, \delta, \) and \( \varepsilon \)), the term \( \bar{F}^{m}_{\sigma_0} (m = K \text{ or } C) \) is the force which joint \( \sigma \) exerts on the body at its output end. Similarly, \( \bar{G}^{m}_{\sigma_0} \) is the torque which \( \sigma \) exerts on the body at its output end, specified at the origin of \( \sigma \)'s output frame. The terms \( \phi^s_{\alpha}, \phi^s_{\beta}, x^s_{\varepsilon} \) are the values of \( \phi_{\alpha}, \phi_{\beta}, x_{\varepsilon} \) at which the joint spring forces and torques are zero.

\( \bar{L}^{m}_{\sigma_0} \) is a dual vector which was defined previously in Section 5.3. Therefore:

\[
\bar{L}^{m}_{\sigma_0} = \begin{bmatrix}
\bar{F}^{m}_{\sigma_0} \\
\bar{G}^{m}_{\sigma_0}
\end{bmatrix}
\quad (7.3-5)
\]

The \( S \) array for joint \( \varepsilon \) in Eq. (7.3-4d) is analogous to the \( S \) term specified in Eq. (5.2.3-10). Specifically

\[
S^T_{54} = \begin{bmatrix}
I & 0 \\
-x_{54} & I
\end{bmatrix}
\quad (7.3-6)
\]

Corresponding \( S \) arrays are not required for the other 4 joints because the two ends of each of these joints are congruent.

The single kinematics equation of the so-called primitive system now can be obtained by stacking the 5 equations (\( i = 1, 5 \)) which comprise Eq. (7.3-1) and introducing some unified notation. The result is
\[ M \dot{\mathbf{v}} = f_D + f_K + f_C \]  
(7.3-7)

where

\[ M = \text{Diag} \left[ m_1 I, J_1, m_2 I, \ldots, J_5 \right] \]  
(7.3-8)

\[ \mathbf{v} = \left[ \dot{v}_1, \dot{\omega}_1, \dot{v}_2, \ldots, \dot{\omega}_5 \right] \]  
(7.3-9)

\[ \dot{\mathbf{v}} = \left[ \begin{array}{c} \dot{v}_1 \\ \dot{\omega}_1 \\ \dot{v}_2 \\ \vdots \\ \dot{\omega}_5 \end{array} \right] \]  
(7.3-10)

\[ \dot{f}_v = \left[ \begin{array}{c} \dot{f}_{v_1} \\ \vdots \\ \dot{f}_{v_5} \end{array} \right]; \mathbf{v} = D, K, C \]  
(7.3.11)

7.4 Joint Constraint Equations

Application of the general formulation to the present example requires developing velocity constraint equations of the form

\[ \dot{c}_\sigma = c_\sigma \mathbf{v} = 0 \]  
(7.4-1)

for each of the 5 joints (\( \sigma = \alpha, \beta, \gamma, \delta, \) and \( \varepsilon \)) and stacking them to produce

\[ \dot{c} = \dot{c}_\mathbf{v} = 0 \]  
(7.4.2)

where

\[ c = \left[ c_\alpha', c_\beta', c_\gamma', c_\delta', c_\varepsilon \right] \]  
(7.4-3)
\[
\begin{bmatrix}
C_u \\
C_v \\
C_w \\
\vdots \\
C_\epsilon
\end{bmatrix}
\]

(7.4-4)

Note that the quantities designed as \( C \) in this chapter are analogous to terms designated as CT previously; see, for example, Eq. (5.5-5).

The constraint equations for each of the 4 pin joints are basically identical. Each of these joints has 2 constraints on rotation and 3 on translation. Consider joint \( \alpha \) first.

The rotational constraints at \( \alpha \) can be specified by the following two equations

\[
\hat{\omega}_x \cdot [\bar{\omega}_2 - \bar{\omega}_1] = 0 \tag{7.4-5a}
\]

\[
\hat{\omega}_z \cdot [\bar{\omega}_2 - \bar{\omega}_1] = 0 \tag{7.4-5b}
\]

These equations merely indicate that the relative velocity vector at \( \alpha \) must lie along \( \hat{\omega}_x \).

The translational velocity constraint equation for \( \alpha \) can be developed by employing the requirement that the translational velocity of frame \( \alpha_0 \) be identical to that of \( \alpha_1 \). Thus

\[
\bar{v}_{\alpha_1} - \bar{v}_{\alpha_0} = 0 \tag{7.4-6}
\]
Expressing the above equation using the elements of \( \dot{y} \) produces

\[
\ddot{v}_1 - \dot{x}_{a_1} \omega_1 - \ddot{v}_2 + \dot{x}_{a_2} \omega_2 = 0
\]  

Equations (7.4-5) and (7.4-7) are the basic velocity constraint equations of \( \alpha \). It is a minor operation to manipulate them into the form required for Eq. (7.4-1). The resulting constraint operator \( C_\alpha \) is

\[
C_\alpha = \begin{bmatrix}
-I & \dot{x}_{a_1} & I & -\dot{x}_{a_2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \dot{e}_{1x}^T & 0 & -\dot{e}_{1x}^T & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \dot{e}_{1z}^T & 0 & -\dot{e}_{1z}^T & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]  

(7.4-8a)

Proceeding similarly with the other 3 rotational joints produces

\[
C_\beta = \begin{bmatrix}
-I & \dot{x}_{\beta_1} & 0 & 0 & I & -\dot{x}_{\beta_3} & 0 & 0 & 0 & 0 \\
0 & \dot{e}_{1x}^T & 0 & 0 & 0 & -\dot{e}_{1x}^T & 0 & 0 & 0 & 0 \\
0 & \dot{e}_{1z}^T & 0 & 0 & 0 & -\dot{e}_{1z}^T & 0 & 0 & 0 & 0
\end{bmatrix}
\]  

(7.4-8b)

\[
C_\gamma = \begin{bmatrix}
0 & 0 & -I & \dot{x}_{\gamma_2} & 0 & 0 & I & -\dot{x}_{\gamma_4} & 0 & 0 \\
0 & 0 & 0 & \dot{e}_{4x}^T & 0 & 0 & 0 & -\dot{e}_{4x}^T & 0 & 0 \\
0 & 0 & 0 & \dot{e}_{4z}^T & 0 & 0 & 0 & -\dot{e}_{4z}^T & 0 & 0
\end{bmatrix}
\]  

(7.4-8c)
Now consider ε. This joint has 3 constraints in rotation and 2 in translation. The rotation constraints can be written as

\[
\bar{\omega}_5 - \bar{\omega}_4 = \bar{0}
\]  
(7.4-9)

To obtain the translational constraint equations, note first that the translational velocity vector of \( \varepsilon_0 \) relative to \( \varepsilon_I \) (as seen by an observer whose orientation is fixed relative to frame 4) can be written as

\[
\varepsilon_0^{(4)} = \bar{v}_5 - \bar{v}_4 + \bar{x}_{54} \bar{w}_4
\]  
(7.4-10)

the translation constraint at \( \varepsilon \) implies that \( \varepsilon_0^{(4)} \) possesses no components along the y and z axes of frame 4. Therefore, the translation constraint can be specified as follows:

\[
\begin{bmatrix}
\nu_{54} \\
\nu_{54} \\
\nu_{54} \\
\nu_{54}
\end{bmatrix} = \begin{bmatrix}
\bar{v}_5 - \bar{v}_4 + \bar{x}_{54} \bar{w}_4
\end{bmatrix} = \bar{0}
\]  
(7.4-11)

The constraint equations at \( \varepsilon \) are Eqs. (7.4-9) and (7.4-11). Manipulating them into the form required by Eq. (7.4-1) produces the following expression for \( C_\varepsilon \)
7.5 The Velocity Transformation

Fig. 7-2 shows that the original system has one closed loop. Therefore, when reconnecting joints to form the transformed system, at least one joint (α, β, γ, or δ) must be left broken. Any or all of the remaining joints can also be left broken at the discretion of the analyst. For the present work, the decision was made to employ two broken joints: namely, γ and δ; it was believed that this approach would illustrate the basic features of the velocity transformation method better than working with a larger or smaller number of broken joints would have. The transformed system then consists of two trees. These will be designated as tree I and tree II. The bodies and reconnected joints in each of the two trees are as follows

Tree I: 1, 2, 3, α, β
Tree II: 4, 5, ε

Bodies 1 and 4 will be chosen as the reference bodies for the two trees. The velocity vector \( \mathbf{v}' \) of the transformed system will be selected to be the following

\[
\mathbf{v}' = \begin{bmatrix} v'_I \\ v'_{II} \end{bmatrix}
\]  

(7.5-1a)

with

\[
v'_I = \begin{bmatrix} v_1 \\ \omega_1 \\ \phi_\alpha \\ \phi_\beta \end{bmatrix}
\]  

(7.5-1b)

\[
v'_{II} = \begin{bmatrix} v_4 \\ \omega_4 \\ \dot{x}_\epsilon \end{bmatrix}
\]  

(7.5-1c)
It is useful to partition \( \mathbf{v} \) similarly.

\[
\mathbf{v} = \begin{bmatrix} \mathbf{v}_I & \mathbf{v}_{II} \end{bmatrix}
\]  

(7.5-2a)

with

\[
\mathbf{v}_I = \begin{bmatrix} \bar{v}_1 & \bar{\omega}_1 & \bar{v}_2 & \bar{\omega}_2 & \bar{v}_3 & \bar{\omega}_3 \end{bmatrix}
\]  

(7.5-2b)

\[
\mathbf{v}_{II} = \begin{bmatrix} \bar{v}_4 & \bar{\omega}_4 & \bar{v}_5 & \bar{\omega}_5 \end{bmatrix}
\]  

(7.5-2c)

The velocity transformation

\[
\mathbf{v} = T \mathbf{v}'
\]

(7.5-3)

can then be written in the partitioned form

\[
\begin{bmatrix} \mathbf{v}_I \\ \mathbf{v}_{II} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_I & 0 \\ 0 & \mathbf{T}_{II} \end{bmatrix} \begin{bmatrix} \mathbf{v}_I' \\ \mathbf{v}_{II}' \end{bmatrix}
\]  

(7.5-4)

It is an exercise to develop the following expressions for \( \mathbf{T}_I \) and \( \mathbf{T}_{II} \).

\[
\mathbf{T}_I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ I \bar{x}_{12} & \bar{\alpha}_2 \bar{\epsilon}_r & 0 \\ 0 & I & \bar{\epsilon}_r & 0 \end{bmatrix}
\]  

(7.5-5a)

\[
\mathbf{T}_{II} = \begin{bmatrix} 1 & \bar{x}_{13} & \bar{\alpha}_3 \bar{\epsilon}_r & 0 \\ 0 & I & 0 & \bar{\epsilon}_r \end{bmatrix}
\]
The basic kinetics equations for the transformed system are developed by premultiplying Eq. (7.3-7) by $T_T^T$; also, Eq. (7.5-3) must be differentiated and introduced. The result is

$$M' \ddot{v}' = f_D' + f_K' + f_C'$$  \hspace{1cm} (7.6-1)

where

$$M' = \begin{bmatrix} M'_I & 0 \\ 0 & M''_I \end{bmatrix} = \begin{bmatrix} T^T_T M_I T_T \\ 0 \\ 0 \\ T^T_T M''_I T''_T \end{bmatrix}$$  \hspace{1cm} (7.6-2)

$$f_D' = T^T_T \{f_D - M' \dot{v}' \}$$  \hspace{1cm} (7.6-3)

$$f_K' = T^T_T f_K$$  \hspace{1cm} (7.6-4)

$$f_C' = T^T_T f_C$$  \hspace{1cm} (7.6-5)

The explicit equation for the constraint force $f_C'$ on the transformed system can be obtained by introducing Eqs. (7.3-4) and (7.5-5) into (7.6-2c) and multiplying out. The result reduces to
\[
\mathbf{f}_C' = \begin{bmatrix}
-S^T_{\gamma 1} \mathbf{L}_C^T & S^T_{\delta 1} \mathbf{L}_O^T \\
-S^T_{\gamma 1} \mathbf{L}_O^T & -S^T_{\delta 1} \mathbf{L}_O^T \\
-e_r^T \cdot x_{\gamma a} \cdot \mathbf{L}_C^T & -e_r^T \cdot x_{\delta a} \cdot \mathbf{L}_O^T \\
-e_r^T \cdot x_{\delta b} \cdot \mathbf{L}_O^T & -e_r^T \cdot x_{\delta b} \cdot \mathbf{L}_O^T \\
S^T_{\gamma 4} \cdot \mathbf{L}_O^T & S^T_{\delta 4} \cdot \mathbf{L}_O^T \\
0 & 0
\end{bmatrix}
\] (7.6-6)

As predicted by the development presented in the preceding chapters, the constraint forces and torques at the reconnected joints (\(\alpha, \beta, \epsilon\)) do not appear in the above equation for \(\mathbf{f}_C'\).

The explicit equation for the known force vector \(\mathbf{f}_K'\) can be obtained similarly by introducing Eqs. (7.3-3) and (7.5-5) into (7.6-2c) and multiplying out. This result is

\[
\mathbf{f}_K' = [0, 0, -k_\alpha [\phi - \phi^S_\alpha], -k_\beta [\phi - \phi^S_\beta], 0, 0, -k_\epsilon [x - x^S_\epsilon]]
\] (7.6-7)

As indicated by Eq. (7.6-2b), the expression for the dynamic force \(\mathbf{f}_D'\) on the transformed system includes a term \(M \mathbf{\dot{v}}_v'.\) To derive the explicit equations for this term, start with \(\mathbf{v} = \mathbf{T} \mathbf{v}'\) where \(\mathbf{v}, \mathbf{T},\) and \(\mathbf{v}'\) are defined in Eqs. (7.3-8), (7.6-5), and (7.5-1). Multiply out and differentiate to form \(\mathbf{\dot{v}}\) as specified in Eq. (7.3-10). Manipulate and discard the terms pertaining to \(\mathbf{\dot{v}}'.\) Premultiply by \(M.\) The result is
\[
\begin{bmatrix}
0 \\
0 \\
-M_2 \hat{\phi}_a \left[ v_{11} - x_{21} \omega_1 + \omega_1 x_{a1} \right] \hat{e}_r \\
-J_2 \cdot e_\omega \hat{\omega}_1 \phi_a \\
-M_3 \hat{\phi}_\beta \left[ v_{11} - x_{31} \omega_1 + \omega_1 x_{\beta1} \right] \hat{e}_r \\
-J_3 \cdot e_\omega \hat{\omega}_1 \phi_\beta \\
0 \\
0 \\
m_5 \omega_4 \hat{e}_{x_t} \\
0
\end{bmatrix}
\]

(7.6-8)

The equations for \( f' \) will not be written out in full detail, since they are complicated and not particularly informative.

To obtain the explicit equations for \( M'_I \) and \( M''_I \) insert Eqs. (7.5-5) and (7.3-8) into (7.6-2a) and multiply out. The result is

\[
M'_I = \begin{bmatrix}
(m_1+m_2) & -m_2 x_{21} & m_2 x_a \hat{e}_r & m_3 x_\beta \hat{e}_r \\
+m_3 \hat{e}_r \\
-m_2 x_{21} & m_2 x_a \hat{e}_r & m_2 x_a \hat{e}_r & m_3 x_\beta \hat{e}_r \\
-m_3 x_31 & m_3 x_31 & m_3 x_31 & m_3 x_31 \\
J_1 + J_2 + J_3 & [J_2 + m_3 x_{21} x_a] \hat{e}_r & [J_2 + m_3 x_{21} x_a] \hat{e}_r & [J_3 + m_3 x_{31} x_\beta] \hat{e}_r \\
-m_2 x_{21} x_{21} & m_2 x_{21} x_{21} & m_2 x_{21} x_{21} & m_2 x_{21} x_{21} \\
-m_3 x_{31} x_{31} & m_3 x_{31} x_{31} & m_3 x_{31} x_{31} & m_3 x_{31} x_{31} \\
\text{SYM} & \hat{e}_r^T \left[ J_{21} - m_2 x_a x_a \right] \hat{e}_r & \hat{e}_r^T \left[ J_{21} - m_2 x_a x_a \right] \hat{e}_r & 0 \\
& \hat{e}_r^T \left[ J_{31} - m_3 x_\beta x_\beta \right] \hat{e}_r & \hat{e}_r^T \left[ J_{31} - m_3 x_\beta x_\beta \right] \hat{e}_r & \hat{e}_r^T \left[ J_{31} - m_3 x_\beta x_\beta \right] \hat{e}_r
\end{bmatrix}
\]

(7.6-9a)
\begin{align*}
M''_{II} &= 
\begin{array}{ccc}
(m_4 + m_5)I & -m_5 \xi_{54} & m_5 \theta_t \\
J_4 + J_5 & -m_5 \xi_{54} \xi_{54} & m_5 \xi_{54} \theta_t \\
\text{SYMMETRIC} & -m_5 \xi_{54} \xi_{54} & m_5 \\
\end{array} \\
&= \begin{array}{ccc}
(m_4 + m_5)I & -m_5 \xi_{54} & m_5 \theta_t \\
J_4 + J_5 & -m_5 \xi_{54} \xi_{54} & m_5 \xi_{54} \theta_t \\
\text{SYMMETRIC} & -m_5 \xi_{54} \xi_{54} & m_5 \\
\end{array} 
\tag{7.6-9b}
\end{align*}

### 7.7 Constraint Equations of the Transformed System

Substituting Eq. (7.5-3) into (7.4-1) produces

\[ C_g = C'_g \dot{y}' = 0 \tag{7.7-1} \]

where

\[ \sigma = \alpha, \beta, \gamma, \delta, \epsilon \]

and

\[ C'_g = C_g^T \tag{7.7-2} \]

Note that the terms designated here as \( C'_g \) are analogous to ones designated previously as \( D'_g \); see, for example, Eq. (6.6.3-2).

The elements of the five \( C_g \) operators were defined in Eqs. (7.4-8) and (7.4-12). The elements of \( T \) were defined in Eqs. (7.5-5). Substituting these results into (7.7-2) and multiplying out for the five separate cases produces

\[ C'_g = 0 \tag{7.7-3a} \]

\[ C'_\beta = 0 \tag{7.7-3b} \]

\[ C'_\gamma = \begin{bmatrix}
-I & -x_{y1} & -x_{y4} & 0 & I & -x_{y4} & 0 \\
0 & \dot{\theta}_{4x} & 0 & 0 & 0 & -\dot{\theta}_{4x} & 0 \\
0 & \dot{\theta}_{4z} & 0 & 0 & 0 & -\dot{\theta}_{4z} & 0 \\
\end{bmatrix} \tag{7.7-3c} \]
\[
C'_\delta = \begin{bmatrix}
-I & x^{\delta_1} & 0 & x^{\delta_2} e^T_\tau & I & -x^{\delta_4} & 0 \\
0 & e^T_{4x} & 0 & 0 & 0 & -e^T_{4x} & 0 \\
0 & e^T_{4z} & 0 & 0 & 0 & -e^T_{4z} & 0 
\end{bmatrix}
\] (7.7-3d)

\[
C'_\epsilon = 0
\] (7.7-3e)

As predicted by the material presented in the preceding chapters, the above constraint arrays of the three reconnected joints \( \alpha, \beta, \epsilon \) are 0. The techniques for utilizing the constraints which are presented in this document require that the constraint matrix have full rank. Therefore, when stacking the \( C'_\sigma \) to form the full constraint operator \( C' \) of the transformed system, the null operators \( C'_{\alpha}, C'_{\beta}, \) and \( C'_\epsilon \) should be excluded. In actual problems, there is no need to deal with constraints at joints which are to be reconnected.

Now consider the operators \( C'_\gamma \) and \( C'_\delta \) for the unreconnected joints \( \gamma \) and \( \delta \). As indicated by Eqs. (7.7-3c) and (7.7-3d) the last two rows of these two quantities are identical. Therefore when stacking \( C'_\gamma \) and \( C'_\delta \) to form \( C' \), the final two rows of \( C'_\delta \) should be excluded; otherwise \( C' \) will not have full rank.

\( C'_\gamma \) and \( C'_\delta \) possess a second, less obvious degeneracy. For convenience in describing it, let \( C_\gamma \) and \( C_\delta \) be partitioned into their translation and rotation elements

\[
C_\sigma = \begin{bmatrix}
-\sigma T \\
\sigma R 
\end{bmatrix}
\] (7.7-4)
Partition \( C' \) and \( C'' \) analogously. Coordinatize \( c_{YT} \) and \( c_{\delta T} \) along the axes of Frame 4 by premultiplying both by \( E_T^T = [e_{4x} \ e_{4y} \ e_{4z}]^T \). The translation constraint equations then are

\[
\begin{align*}
\overline{c}^{(4)}_{YT} &= E_T^T c_{YT} \dot{v}' = 0 \quad (7.7-5a) \\
\overline{c}^{(4)}_{\delta T} &= E_T^T c_{\delta T} \dot{v}' = 0 \quad (7.7-5b)
\end{align*}
\]

Consider the \( y \)-component of the above two equations. Recall that \( e_{1y} = e_{1r} \). Upon multiplying out by \( \dot{v}' \), we obtain

\[
\begin{align*}
c^{(4)y}_{YT} &= e_{1r} \cdot \left[ -\dot{v}_1 + x_{1y} \dot{w}_1 + x_{\gamma \alpha} e_{\gamma \alpha} \dot{\phi}_\alpha + \dot{v}_4 - x_{\gamma 4} \dot{w}_4 \right] = 0 \quad (7.7-6a) \\
c^{(4)y}_{\delta T} &= e_{1r} \cdot \left[ -\dot{v}_1 + x_{\delta 1} \dot{w}_1 + x_{\delta \beta} e_{\delta \beta} \dot{\phi}_\beta + \dot{v}_4 - x_{\delta 4} \dot{w}_4 \right] = 0 \quad (7.7-6b)
\end{align*}
\]

It can be deduced that the rotation constraint \( c_{\sigma R} \) implies

\[
\dot{w}_4 = \dot{w}_1 + e_{1r} \dot{w}_4 \quad (7.7-7)
\]

Inserting Eq. (7.7-7) into (7.7-6) and performing some algebra yields

\[
\overline{c}^{(4)y}_{YT} = \overline{c}^{(4)y}_{\delta T} = e_{1r} \cdot \left[ -\dot{v}_1 + x_{41} \dot{w}_1 + \dot{v}_4 \right] = 0 \quad (7.7-8)
\]

Since \( c^{(4)y}_{YT} \) and \( c^{(H)y}_{\delta T} \) are identical, only one should be included when forming the full constraint equation for the transformed system.

In summary, the constraint equation for the transformed system can be written as

\[
c' = C' \dot{v}' = 0 \quad (7.7-9a)
\]
where

\[
C' = \begin{bmatrix}
-W_4^T & W_4^T \gamma_1 & W_4^T \gamma_4 & 0 & W_4^T & -W_4^T \\
-W_4^T & W_4^T \delta_1 & 0 & W_4^T \delta_4 & 0 & W_4^T \\
-e_r^T & e_r^T x_41 & 0 & 0 & e_r^T & 0 \\
0 & W_4^T & 0 & 0 & 0 & -W_4^T
\end{bmatrix}
\]

(7.7-9b)

The symbol

\[
W_4^T = [\hat{e}_{4x} \; \hat{e}_{4z}]^T
\]

(7-7-10)

has been introduced above for compactness.

The problem of eliminating redundant constraint equations in the general case was discussed briefly in Section 6.8.

7.8 Computation of the Constraint Force Vector

This section summarizes the Lagrange multiplier method for dealing with the constraint force vector \( f_C \) which appears in the kinetics Eq. (7.6-1). As is indicated previously in Section 6.8, \( f_C \) can be computed via

\[
f_C' = C'T\lambda
\]

(7.8-1)

where \( \lambda \) is the Lagrange multiplier vector.

The equation for computing \( \lambda \) can be derived by applying the technique presented in Section 6.8 to Eqs. (7.1-1) and (7.7-9a). The result is

\[
c'M^{-1}C'T\lambda = -[c'v' + c'M^{-1}(f_D' + f_K')]
\]

(7.8-2)
The detailed equations for the term $\mathbf{C}^{'} \mathbf{v}^{'}$ which appears in Eq. (7.8-2) can be derived, by differentiation, from Eqs. (7.7-9). After some algebra, the following result was obtained

$$
\mathbf{C}^{'} \mathbf{v}^{'} = 
\begin{align*}
  &\mathbf{w}^{T}_{4} \left[ \mathbf{w}^{'}_{41} \mathbf{v}^{'}_{1} + \mathbf{w}^{'}_{41} \mathbf{v}^{'}_{1} \mathbf{y}^{'}_{1} - \mathbf{w}^{'}_{41} \mathbf{v}^{'}_{1} \mathbf{y}^{'}_{1} \mathbf{\phi} \mathbf{a} \right] \\
  &+ \mathbf{e}_{x} \left[ \mathbf{w}^{'}_{41} - \mathbf{\phi} \mathbf{e}_{x} \right] \mathbf{x}^{'}_{y} \mathbf{\phi} \mathbf{a} \\
  &\mathbf{w}^{T}_{4} \left[ \mathbf{w}^{'}_{41} + \mathbf{w}^{'}_{41} \mathbf{z}^{'}_{1} - \mathbf{w}^{'}_{41} \mathbf{z}^{'}_{1} \mathbf{\phi} \mathbf{b} \right] \\
  &+ \mathbf{e}_{x} \left[ \mathbf{w}^{'}_{41} - \mathbf{\phi} \mathbf{e}_{x} \right] \mathbf{x}^{'}_{y} \mathbf{\phi} \mathbf{b} \\
  &\mathbf{e}^{T}_{x} \mathbf{x}^{'}_{4} - \mathbf{e}_{x} \mathbf{x}^{'}_{4} \mathbf{x}^{'}_{1} + \mathbf{w}^{'}_{41} \mathbf{x}^{'}_{41} \\
  &\mathbf{w}^{T}_{4} \mathbf{w}^{'}_{41} \mathbf{w}^{'}_{41}
\end{align*}
$$

(7.8-3)

where

$$
\mathbf{w}^{'}_{41} = \mathbf{w}^{'}_{4} - \mathbf{w}^{'}_{1}
$$

7.9 **Kinematics Equation of the Transformed System**

As discussed in Section 6.5, a kinematics equation of the form

$$
\mathbf{u}^{'} = \mathbf{T} \mathbf{v}^{'}
$$

(7.9-1)

is needed. Several choices for the configuration vector $\mathbf{u}^{'}$ are possible for the present problem. One is

$$
\mathbf{u}^{'} = \begin{bmatrix} \mathbf{x}^{(N)}_{1} & \mathbf{\xi}^{(N)}_{1} & \mathbf{\phi}_{a} & \mathbf{\phi}_{b} & \mathbf{x}^{(N)}_{4} & \mathbf{\xi}^{(N)}_{4} & \mathbf{x}^{(N)}_{e} \end{bmatrix}
$$

(7.9-2)
In the above expression $\bar{x}_1^{(N)}$ denotes the components of the origin of Frame 1 relative to the inertial frame N, with frame N resolution. $\bar{\xi}_1$ denotes the Euler symmetric parameters which specify the orientation of Frame 1 relative to N. The terms $\bar{x}_4^{(N)}$ and $\bar{\xi}_4$ are defined similarly.

When the configuration vector denoted in Eq. (7.9-2) is used, Eq. (7.9-1) becomes

$$
\begin{align*}
\begin{bmatrix}
\dot{\bar{x}}_1 \\
\dot{\bar{\xi}}_1 \\
\dot{\phi}_\alpha \\
\dot{\phi}_\beta \\
\dot{\bar{x}}_4 \\
\dot{\bar{\xi}}_4 \\
\dot{x}_\epsilon
\end{bmatrix}
&= 
\begin{bmatrix}
T_{1N}^T E_1^T v_1 \\
R_i E_1^T \omega_1 \\
\phi_\alpha \\
\phi_\beta \\
T_{4N}^T E_4^T v_4 \\
R_i E_4^T \omega_4 \\
x_\epsilon
\end{bmatrix}
\end{align*}
$$

(7.9-3)

In the above expression, the $T_{1N}$ are direction cosine matrices. The $R_i$ are the matrices which transform $\omega_1^{(i)}$ to $\bar{\xi}_4$; the expressions for $R_i$ are well known.

7.10 **Summary**

To summarize, the main final equations for the sample problem are as follows
(1) Eq. (7.6-1) for kinetics
(2) Eq. (7.9-1) for kinematics
(3) Eqs. (7.8-1) and (7.8-2) for the constraint force vector $f'_{C}$

The velocity transformation matrix $T$ is specified by Eqs. (7.5-5). The transformed constraint matrix $C'$ is denoted in Eq. (7.7-9). The transformed velocity vector $v'$ is prescribed by Eq. (7.5-1). The remaining necessary "auxiliary" equations are scattered throughout the chapter.

As noted previously, the purpose of this chapter was to illustrate the technique for developing equations of motion of multi-body systems that was described in Chapters 4 to 6.
CHAPTER 8

APPLICATION IN LSS DEPLOYMENT SIMULATION

8.1 Introduction

In the final part of the research, the formalism described in Chapters 4 to 6 was used to develop a computer program to simulate the deployment dynamics of a large space structure (LSS). The LSS which was considered was a simplified version of a space-based radar antenna proposed to the U.S. Air Force by the Grumman Aerospace Corporation. The work consisted of employing the general formulation given in Chapters 4 to 6 to develop equations of motion for the deployment, coding these equations on a digital computer, debugging the program, and making runs. The central computer facility at the Draper Laboratory was used for this work.

The full mathematics for this problem and the resulting computer program are sufficiently lengthy so that it is neither practical nor desirable to describe them in detail in this document. The present chapter, therefore, only summarizes their main features.

8.2 Description of the Spacecraft

Fig. 8-1, from a Grumman report, shows the proposed vehicle after deployment. The main structural elements are the following:

1. An outer rim consisting of 32 tubes (Fig. 8-1 portrays a configuration with only 16 tubes.)

2. A central portion consisting of (a) a central hub and (b) a drum which surrounds this hub.
Figure 8-1. Post-deployment configuration of spacecraft
(3) 32 "gores" which extend from the drum to the rim.
(4) 16 upper stays and 16 lower stays which extend from the hub to the rim.

A small structural element called a bridge is situated between each pair of rim tubes. Each bridge permits each of the tubes at its two extremities to rotate with one degree of freedom during deployment. The rotation axes are locked after deployment.

The drum and central hub can rotate relative to one another with one degree of freedom about their common longitudinal axis. During deployment, the rotation is controlled by a motor. After deployment, this rotational degree of freedom also is locked.

The gores serve as the radar reflector. They possess very little inherent stiffness in bending and torsion. During and after deployment they are maintained in tension.

The function of the 32 stays is to rigidize the structure. They also are maintained in tension during and after deployment.

The upper left portion of Fig. 8-2 shows the system prior to deployment. At this time, each of the gores is wrapped around the drum like tape on a spool. The tubes are rotated 90° from their post deployment orientations relative to the bridges; they are stowed vertically against the drum as shown.

The lower right portion of Fig. 8-2 shows the system during deployment. Torsional springs between the bridges and tubes are the main sources of energy for deployment. (An alternate technique proposed by Grumman uses motors instead of springs.) The torque which is generated by the springs causes the tubes to rotate relative to the bridges. This rotation causes the tubes and bridges to move outward from the drum, and this, in turn, causes the gores to unwind from the drum as shown on the
Figure 8-2. Spacecraft deployment
figure. The stays are played out simultaneously with the rim and gores. However, the stay playout rate is sufficiently low that the stays slow down the deployment of the rim. Thus, it is the stay playout rate, not spring stiffness, which determines the rate at which the deployment proceeds.

In the present study, the structure was assumed to be free in space. (In some deployment operations proposed by Grumman, the hub would be attached to the space shuttle.)

8.3 Basic Modeling Methods

8.3.1 Rim

The rim model which was used in the work contained only one-fourth as many tubes and bridges as the system proposed by Grumman. That is, it consisted of 8 tubes and 8 bridges. The main reason for making this reduction was to produce a faster-running program.

The program included a running mode in which all 8 tubes were rigid and also a mode in which tube flexibility was included. The flexible tube running mode employed three generalized coordinates per tube - one in torsion and one each in bending about a pair of orthogonal axes. The bridges were assumed to be rigid.

8.3.2 Central Hub and Drum

The central hub was modeled as a rigid, symmetrical cylindrical with identical transverse moments of inertia. Similarly, the drum (excluding the gores) was modeled as a rigid, symmetrical tube with identical transverse moments of inertia. The longitudinal axes of the hub and drum were assumed to coincide with each other and with the axis of their rotation relative to one another. The mass centers of the hub and drum were assumed to coincide and to be located at their common geometric center.
8.3.3 Gores

The gores are the most difficult part of the spacecraft to model. The main difficulties are as follows:

1. The size of the gores (the portion of the gores in the space between the drum and the rim, that is) is not constant, but, rather, increases immensely during deployment.

2. The shape of the deployed gore material is not flat. Rather, as a perusal of Figs. 8-1 and 8-2 should reveal, each gore undergoes a large time-varying twist angle during deployment.

3. The gores are sufficiently massive relative to the other elements of the spacecraft that omission of their mass and moments of inertia would be difficult to justify.

As a practical necessity, a simple model of the gores was used. To wit, each gore was represented as two cables. These cables were located at the two edges of the gore. Each cable was considered to be a body possessing mass and moments of inertia.

During deployment, the portion of each cable which was already unwound from the drum was treated as being rigid and straight. While this is a major approximation, it is regarded as not being untenable as long as the cable is in tension.

In the modeling, the outer end of each cable was attached to a bridge by a translational spring. The springs were permitted to exert only tensile loads on the cables, not compressive ones.

8.3.4 Stays

The light weight of the 8 stays made it possible to approximate them as being massless. Thus, the force interfaces which the stays provide between the hub and the bridges were modeled as joints. Each of these 8 joints was given the full 6 degrees of freedom.
The physical point of view in the modeling was that the stays are massless inextensible cables which are played out from the central hub during deployment. The outer end of each stay was attached to its bridge through a translational spring. These springs were permitted to exert only tensile loads on the stays, not compressive ones.

8.3.5 Springs

As is indicated in the preceding sections, three sets of springs were included in the modeling: (1) the tube-bridge springs, (2) the stay-bridge springs, and (3) the cable-bridge springs. For simplicity, the force-versus-displacement or torque-versus-rotation relationships of all springs were assumed to be linear. Two running modes were provided with each of the three sets of springs: (1) a mode in which they were assumed to possess no damping and (2) a mode in which they were viscously damped.

8.3.6 Non-Nominal Conditions

Deployment is a relatively simple process in the ideal case in which (1) the system's initial angular velocity is zero; (2) the bridges, tubes, cables, bridge-tube springs, bridge-stay springs, bridge-cable springs, and stay rates are identical; (3) the tubes and bridges are perfectly aligned; and (4) external forces and torques are negligible. The main potential value of the simulation, however, is to investigate non-nominal deployments in which one or more of these ideal conditions are not met. Therefore, the following capabilities were built into the simulation.

(1) The angular velocity of the spacecraft at the start of deployment need not be zero, but instead can be selected arbitrarily.

(2) The spring constants, damping constants, and null offset angles of each of the 8 bridge-tube springs need not be identical, but instead can be selected arbitrarily.
The spring constants and damping constants of each of the 8 stay-bridge springs and of each of the 16 cable-bridge springs can be selected arbitrarily.

The 8 stay playout rates can each be selected arbitrarily.

The mass densities of each of the 16 cables can be selected arbitrarily.

In the flexible tube running mode, the EI and GJ values of each of the 8 tubes can be selected arbitrarily.

In addition, the program included a running mode which permitted selected small misalignment angles of the tubes relative to the bridges to be employed. The selection of these angles was not completely arbitrary, but instead was constrained by the requirement that the resulting rim be closed (i.e. unbroken) at the start of deployment.

8.4 More-Detailed Description of Modeling

8.4.1 Body-Joint Structure

A body-joint graph of the system is shown as Fig. 8-3. On this graph the nodes represent the bodies and the arcs represent the joints.

As was discussed in earlier chapters, the formalism developed in this thesis requires that joint cuts be made to eliminate all closed paths. For the simulation, it was decided to cut (1) all 8 stay joints between the central hub and the bridges, (2) all 16 joints between the gore cables and the bridges, and (3) one of 16 joints between the bridges and the rim tubes. These cuts break the system into two trees, as an inspection of Fig. 8-3 should reveal. These are called the rim tree and the hub tree.

The rim tree contains 16 bodies: namely, the 8 bridges and the 8 tubes. It has 15 uncut joints and one cut joint. As can be seen from Fig. 8-3, the so called rim tree is actually only an open chain. The reference body for the rim tree was chosen to be a bridge: namely, the bridge which is adjacent to the cut rim-joint.
Figure 8-3. Body-joint graph.
The hub tree contains 18 bodies: namely, the central hub, the drum, and the 16 cables. It has 17 uncut joints and no cut joints. The hub tree has a particularly simple structure in that the drum is "adjacent" to all other bodies of the tree; in other words, the hub tree consists of 17 one-joint paths. Nevertheless, the central hub - not the drum - was selected as the reference body for the hub tree.

The rim tree and the hub tree are interconnected through the 8 cut stay joints and 16 cut bridge-cable joints as shown on the figure.

8.4.2 State Vector

The state vector \( \mathbf{x} \) which was used in the study is comprised of 4 subvectors: namely, the configuration vector \( \mathbf{u}_R' \) and velocity vector \( \mathbf{v}_R' \) of the rim tree and the analogous vectors \( \mathbf{u}_H' \) and \( \mathbf{v}_H' \) of the hub tree. \( \mathbf{u}_R' \) and \( \mathbf{v}_R' \) contain 22 and 21 scaler elements respectively in the rigid tube running mode. These numbers increase to 46 and 45 respectively for the flexible tube running mode. The numbers of scaler elements in \( \mathbf{u}_H' \) and \( \mathbf{v}_H' \) are 40 and 39 respectively. Thus \( \mathbf{x} \) contains 122 elements in the rigid tube mode and 170 elements in the flexible tube mode.

The elements in \( \mathbf{u}_R' \) are the following:

1. The Gibbs vector components \( x_{\zeta_1}^{N} \) which specify the location of the frame \( \zeta_1 \) of the rim tree reference body \( Z_1 \) relative to the inertial frame \( N \).

2. The 4-vector \( \xi_{\zeta_1}^{N} \) of the Euler symmetric parameters which specify the orientation of \( \zeta_1 \) relative to frame \( N \).

3. The tube bridge rotation angles \( \psi_{B_1} \ldots \psi_{B_{15}} \) at the 15 uncut rim joints.

4. (In the flexible tube mode) the 8 generalized coordinate vectors \( q_{Z_1} \ldots q_{Z_8} \) which define the nonrigidity conditions of the 8 tubes relative to their reference frames. Each \( q_{Z_j} \) contains 3 elements - one to specify torsion and two to specify bending about a pair of orthogonal transverse axes.
$v_R'$ is comprised of $v_{\xi_1 N}'$, $\omega_{\xi_1 N}'$, and the $\dot{\psi}_i$ and $\dot{\theta}_i$.

The elements of $u_H'$ are

1. The components $x_{\eta_1 N}^{(N)}$ which specify the location of the frame $\eta_1$ of the hub tree reference body $H_1$ relative to frame $N$.
2. The Euler symmetric parameter vector $\xi_{\eta_1 N}$ which specifies the orientation of $\eta_1$ relative to frame $N$.
3. The angle $\theta_{21}$ between the drum and the hub.
4. 16 pairs of angles ($\theta_i$, $\psi_1$) which define the location, orientation, and lengths of the deployed cables relative to the drum.

$v_H'$ is comprised of $v_{\eta_1 N}'$, $\omega_{\eta_1 N}'$, $\dot{\theta}_{21}'$, and the ($\dot{\psi}_i$, $\dot{\theta}_i$).

### 8.4.3 Cable Deployment

The above-noted angles ($\theta_i$, $\psi_i$) and the cable deployment modeling will be described briefly at this point. The basic geometry is shown on Fig. 8-4. (Fig. 8-4 also includes some material on stay-geometry which will be referred to in Subsection 8.4.5.) It is assumed that prior to deployment the cables are wrapped around the drum like stripes on a barber pole. The attach points $P_i$ of the ends of the cables to the drum are located around the circumference of the center plane of the drum. The "pitch" angle $\phi_C$ of the wrapping is assumed to be constant.

Let $B_i$ denote the uncut joint between the drum ($H_2$) and the deployed portion ($H_1$) of cable $i$. $B_i$ is located at the point where cable $i$ unwraps from the drum. $H_1$ is tangent to $H_2$ at $B_i$. As deployment proceeds, $B_i$ moves around the circumference of $H_2$. $B_i$ also moves vertically (up or down) toward the center plane of $H_2$ during deployment; the instantaneous vertical distance of $B_i$ from this centerplane is the term denoted on Fig. 8-4 as $y_i$. 

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Figure 8-4. Cable deployment geometry.
As Fig. 8-4 illustrates, the angle $\theta_1$ specifies the instantaneous location of $B_1$ on the circumference of $H_2$. The geometry is such that $\theta_1$ is also the azimuth angle between the body reference frames, $\eta_2$ and $\eta_1$, of $H_2$ and $H_1$. The angle $\psi_1$ is the elevation angle of $\eta_1$ relative to $\eta_2$.

The instantaneous length $l_1$ of $H_1$ is a function of $\theta_1$, the relation being

$$l_1 = \frac{R_D}{\cos \phi_C} \left[ \theta_1 - \theta_{1IC} \right] + l_{1IC}$$  \hspace{1cm} (8.4.3-1)

In the above expression, $R_D$ is the drum radius, $\theta_{1IC}$ is the value of $\theta_1$ at the start of deployment, and $l_{1IC}$ is the value of $l_1$ at the start of deployment.

The following expression specifies the time-varying vertical location of $B_1$ on the drum

$$y_i = 0.5 \sigma_i h_D \left[ 1 - \frac{l_i}{l_C} \right]$$  \hspace{1cm} (8.4.3-2)

In this expression, $h_D$ is the length of the drum, and $l_C$ is the total fully-deployed length of each cable. $\sigma_i$ is +1 for the 8 upper cables and -1 for the 8 lower cables. The constant terms $\phi_C$, $l_C$, and $h_D$ can be shown to be related through

$$\phi_C = \arcsin \left( 0.5 \frac{h_D}{l_C} \right)$$  \hspace{1cm} (8.4.3-3)

The velocity transformation from $v_{H1}'$ to $v_{H1}$ requires that equations be developed to specify the cable c.m. velocities $\tilde{v}_{\eta_1N}$ and $\tilde{\omega}_{\eta_1N}$ as functions of the elements of $v_{H1}'$. The basic equation for $\tilde{\omega}_{\eta_1N}$ is merely

$$\tilde{\omega}_{\eta_1N} = \frac{\tilde{\omega}_{\eta_1N}}{\eta_1N} + (\dot{\theta}_2 - \dot{\theta}_1) \dot{\eta}_2 + \dot{\psi}_1 \dot{\eta}_1$$  \hspace{1cm} (8.4.3-4)
It is an exercise to develop the following basic equation for \( \vec{v}_{\eta_1 N} \)

\[
\vec{v}_{\eta_1 N} = \vec{v}_{\eta_1 N} - x_{\eta_1} n_2 \left[ \omega_{\eta_1 N} + \theta_2 \dot{\theta}_2 \right] + \dot{\theta}_1.5L_1 \cos \psi_1 \dot{\theta}_1 \dot{\eta}_1 + \psi_1.5l_1 \dot{\eta}_1^2
\]

(8.4.3-5)

Deployment actually consists of two phases: (1) the main phase in which the cables unwind from the drum and (2) a short terminal phase in which the cables swing outward from the drum to their post deployment locations. As can be deduced from the preceding discussion, only the main phase, not the short terminal phase, was simulated.

8.4.4 Constraints

As was discussed in earlier chapters, explicit attention need be given only to the constraints at cut joints, not to those at uncut ones. The 8 cut stay joints and the 16 cut cable-bridge joints all are 6 degree of freedom ones. Thus, they do not possess constraints. Therefore, the only constraints which appear explicitly are those at the cut rim joint. This joint will be designated as joint \( \Gamma \) in the following paragraphs.

The rim joints each have one degree of freedom. Thus 5 scaler constraint equations for \( \Gamma \) are required.

The pertinent geometry is shown on Fig. 8-5. On this figure, \( \gamma_I \) and \( \gamma_O \) are the input and output coordinate frames respectively of \( \Gamma \). The constraints at \( \Gamma \) can be expressed as follows

\[
\bar{x}_{\gamma_O \gamma_I} = 0
\]

(8.4.4-1)

\[
\bar{\epsilon}_i^1 \cdot \bar{\epsilon}_i^3 = 0; \ i = 1, 2
\]

(8.4.4-2)
Coordinate frame origins $O_Y$ and $O_0$ coincide

Figure 8-5. Geometry of cut rim joint $\Gamma$. 
However, as discussed in previous chapters, the formalism being employed in the present work requires velocity constraints, not position ones. The basic velocity constraints which were used are

\[
\begin{pmatrix}
\frac{\zeta_{16}}{X} \\
\gamma_0 \gamma_I \\
\omega \\
\gamma_0 \gamma_I
\end{pmatrix}
\] = \begin{pmatrix}
C \\
T \\
\dot{v}_R
\end{pmatrix} = 0
\] (8.4.4-3)

where

\[
C = \begin{bmatrix}
I & 0 \\
0 & \dot{\epsilon}_1^{(\zeta_{16})} \\
0 & \dot{\epsilon}_2^{(\zeta_{16})}
\end{bmatrix}
\] (8.4.4-4)

The term \( T \) in Eq. (8.4.4-3) is the operator which transforms \( \dot{v}_R \) into the \( \{ \} \) term in the middle portion of the equation. As indicated above, it was decided to use the reference frame, \( \zeta_{16} \), of \( \dot{Z}_{16} \) as the resolution frame.

The constraints at \( \Gamma \) were handled in the simulation by the Lagrange multiplier method which was discussed in Section 6.8. Baumgarte's constraint stabilization method, which was described in Section 6.8.4, was implemented.

8.4.5 Drum-Hub Motor

The motor between the drum and the hub was modeled as a torque generator which applied equal and opposite torques on these two bodies.
A number of different control strategies for utilizing the motor are possible. In the first version of the simulation, the strategy was to employ the motor to keep the azimuth (re: torsion) angle between the hub tree and the rim tree small. The control law was

$$G = -K_\theta \left[ \theta_{S1} - \theta_r \right] - K_\delta \left[ \dot{\theta}_{S1} - \dot{\theta}_r \right] \quad (8.4.5-1)$$

In the above equation, $G$ is the torque which the motor applies to the drum. $K_\theta$ and $K_\delta$ are control system gains. $\theta_{S1}$ is the azimuth angle of stay no. 1 as depicted previously in Figure 8-4. $\theta_{S1}$ lies in a plane perpendicular to the longitudinal axis $\eta_1$ of the hub and is nominally zero after deployment. The variable $\theta_r$ in Eq. (4.8.5-1) is a (possibly) time-varying reference response of $\theta_{S1}$ during deployment. Eq. (8.4.5-1) of course assumes that the system being simulated contains sensors to measure $\theta_{S1}$ and $\dot{\theta}_{S1}$.

8.5 Description of Computer Simulation

8.5.1 Introduction

A computer program for the mathematical model summarized in the preceding sections was developed and run at the central computer facility of The Charles Stark Draper Laboratory. The computer facility is an Amdahl with IBM software.

The program was developed to be applicable solely to the sample problem of interest. That is, no attempt was made to make it general enough to be applicable to other spacecraft. The program was a time running simulation which employed numerical integration. It was designed to start operating at the beginning of deployment and to end at or before the end of the main phase of deployment.
The Fortran IV programming language was used. The program consisted of a main routine and approximately 75 subroutines. It contained approximately 3.5 kilolines of code.

All computations were done in single precision except for a small number of double precision computations in the routines for inverting matrices and solving sets of linear algebraic equations.

8.5.2 Program Overview

A flow diagram which summarizes the operations performed by the program is presented as Fig. 8-6.

The Runge-Kutta-Gill (RKG) algorithm was chosen as the numerical integration technique. This choice was made because of its simplicity and because of the author's prior experience with it. The actual subroutine was a modified version of routine RKGS from the IBM Scientific Subroutines Package. Automatic integration step size control was not used.

The RKG algorithm requires that the state vector time derivative \( \mathbf{a}(x, t) \) in Fig. 8-6 be calculated four times in each integration step. The great bulk of the operations performed by the program were devoted to the calculations of \( \mathbf{a} \). In fact, the CPU time required for all other operations and computations was virtually negligible in comparison to that required for \( \mathbf{a} \). The computation of \( \mathbf{a} \) is discussed in the next subsection.

8.5.3 State Vector Derivative Computation

A flow diagram which summarizes the computation of the state vector time derivative \( \mathbf{a}(x, t) \) is shown as Fig. 8-7.

One of the practical difficulties which the general formulation developed in the preceding chapters presents is that several of the large matrices which appear in it are sparse - sufficiently so that one can ill
Figure 8-6. Flow diagram of overall computations
Figure 8-7. Flow diagram of state vector time derivative computations
Figure 8-7. Flow diagram of state vector time derivative computations (Continued)
afford not to take advantage of the sparseness in matrix storage and computations. While utilization of the matrix sparseness produces a faster-running and (possibly) a smaller program, the time required to develop and check out the program is significantly increased.

Continuing the discussion of matrix sparseness, in the present example, the large square matrices $M_R$ and $M_H$ were block diagonal and hence sparse. The large velocity transformation matrices $T_R$ and $T_H$ also were quite sparse. For this reason, $M_R$, $M_H$, $T_R$, and $T_H$ were stored in a compressed form. That is, the blocks of elements that were identically zero were not stored.

When performing matrix operations involving premultiplication by $T_R^T$ or $T_H^T$ and/or post multiplication by $T_R$ or $T_H$, the matrices were broken up into blocks so that multiplication by blocks of zeros could be avoided.

As noted previously, the program design included a running mode in which the rim tubes were rigid and also a running mode which included rim tube flexibility. The flexible tube running mode added greatly to the complexity of the program and to the time required to develop it. The difficulty here was not only the additional, more-complicated algorithms which the flexible tube mode required, but also the fact that these computations had to be eliminated and the program modified in other ways for the rigid tube mode. For example, the state vector $x$ and all matrices involving the rim tree have different dimensions in the two running modes.

As can be seen from Fig. 8-7, each computation of $\lambda$ required the inversion of two large matrices ($M_R^t$ and $M_H^t$) and also the solution of a smaller-dimensioned set of linear equations $A\lambda = b$. These operations were done in the program via the Cholesky triangular decomposition method. The matrix inversions were performed using modified versions of the routines MFSQ and SINV from the IBM Scientific Subroutine Package.
(SSP). While the SSP routines have a bad reputation, no difficulties were encountered with them in this application. The equation \( A\lambda = b \) was solved using the modified version of MFSD and a modified version of a routine listed in Ref. 24, page 254. The large symmetric matrices \( M_R' \) and \( M_R'^{-1} \) were stored in upper triangular form and overlaid, as were \( M_H' \) and \( M_H'^{-1} \).

Since none of the bodies of the hub tree are attached to cut joints with constraints, the matrix \( M_H'^{-1} \) actually is not required. In a later version of the program, the equation \( M_H'v_H' = f_H' \) was solved directly using the same approach as for \( A\lambda = b \), rather than by inverting \( M_H' \). This modification produced a substantial savings in CPU time - close to 15%.

In the equation \( A\lambda = b \), the symmetric matrix \( A \) is only \( 5 \times 5 \); hence solving this equation for \( \lambda \) requires a relatively small number of calculations. The solution, however, does present one complication: namely, \( A \) does not always have full rank. In particular, in rigid tube/zero misalignment runs, \( A \) has rank of 3 at the start of deployment and switches to rank 4 immediately thereafter. This is because under these conditions the 5 constraints at the cut rim joint are not all independent. The difficulty was handled in the program by automatically eliminating rows and columns from \( A \) (and solving a smaller set of equations) whenever the factorization routine was unable to perform the triangular decomposition \( A = RT_R \).

No external forces on the spacecraft were included in the program. The forces which contribute to the generalized rim force vector \( f_R \) on Fig. 8-7 are those from the stay and cable springs and also, in the flexible tube mode, the internal stiffness and damping force. The D'Alembert force on the rim due to the acceleration of the body reference frames also is included as is the peculiar dynamic force - \( T^T R M R T^T R v_R \), which results from the velocity transformation. It was found convenient in the program to add the generalized force contributions from the bridge-tube springs directly to \( f_R' \) rather than to include them in \( f_R \).
The forces which contribute to the generalized hub force vector $f_H$ in Fig. 8-7 are similar to those for the rim: namely, the stay and cable spring forces, the D'Alembert force, and $-T_H^T M_H T_H v_H'$. In addition, there is a torque due to the motor between the hub and the drum.

The last operation which is indicated on Fig. 8-7 is to form the state derivative vector $a$. Specifically, we have

$$
\mathbf{a} = \begin{bmatrix}
\mathbf{M}_R^{-1} \left( f_R' + f_C' \right) \\
v_R'' \\
M_H^{-1} f_H' \\
v_H''
\end{bmatrix}
$$

(8.5.3-1)

In the above expression, $v_R''$ is $\dot{v}_R$ with $\omega_{\zeta_1N}$ and $\omega_{\zeta_1N}'$ replaced by $\ddot{\zeta}_1N$ and $\ddot{\zeta}_1N'$. The second of these terms is the time derivative of the Euler symmetric parameter vector. $v_H''$ is defined similarly. The term $f_C'$ is the generalized force vector due to the constraints at the cut rim joint.

8.6 Discussion of Computer Simulation Results

8.6.1 Introduction

When choosing the linear dimensions, weights, and moments of inertia of the spacecraft to be simulated, information from Grumman was used wherever available and possible. However, it was not possible to match the simulation spacecraft closely to the Grumman one, the main reason being that the simulation spacecraft has only 8 sides, while the Grumman spacecraft has 32. Also, in most of the runs, the spring constants of the bridge-stay springs and bridge-cable springs were made very low in order to keep the cost of the runs down. (Large spring constants necessitate short integration steps and hence costly computer runs.)
a result, the results generated in the simulation work are not, in gen-
eral, quantitatively applicable to the Grumman spacecraft. It should be
realized in this regard that the fundamental purpose of the simulation
was to test the basic approach (i.e. the formulation of the equations of
motion) and not necessarily to generate numerical information on the
deployment of Grumman's spacecraft.

In perusing the material in this subsection, it should be realized
that the problem under investigation is a very difficult one because of
the large number of bodies, the large number of forces between them, and
the closed topological loop. It should be realized also that multibody
computer programs are infamous for slow running speed.

When debugging the program, it became apparent early in the work
that the program's speed would be slow - so slow in fact that the flex-
ible tube running mode could only be of very limited use for studying de-
ployment. Therefore, debugging work on this mode was discontinued, and
only the much faster rigid tube mode was fully checked out and utilized.

During the final phases of program checkout, several special pre-
liminary studies were made of the deployment dynamics of (1) the rim
alone and (2) the hub tree alone. The results of these preliminary
studies plus those which were then obtained for the combined rim/hub tree
system are described in the following subsections.

8.6.2 Rim Alone

In these studies, runs were made in which the rim alone was de-
ployed. That is, the stay-bridge springs and cable-bridge springs were
broken. The calculations pertaining to the hub tree dynamics were delet-
ed from the program in order to reduce the cost of the runs. One of the
reasons that runs of this type are useful is that they provide several
checks on the validity of the simulation. In particular:
(a) violations of the five position constraints and the five velocity constraints should remain acceptably small throughout the entire deployment,

(b) mechanical energy of the rim should be conserved if damping of the bridge-tube springs is set to zero, and

(c) deployment should be very nearly symmetrical if all tubes, bridges, and springs are made identical and the initial angular momentum of the rim is made zero.

In the early portions of the work, some problems were encountered in computing the Lagrange multiplier vector \( \lambda \). The main difficulty is that the \( 5 \times 5 \) matrix \( A \) which is used in the computation \( (A\lambda = b) \) never has rank 5 in rigid tube runs without misalignment. Instead, the rank is three at the exact start of deployment (rim angle \( \beta_1 = 90^\circ \)), two at the exact end of deployment (\( \beta_1 = 0^\circ \)), and four in between \( (90^\circ \geq \beta_1 \geq 0^\circ) \). A technique for handling the singular \( A \) during the interval \( 90^\circ \geq \beta_1 \geq 0^\circ \) was devised, and after this the constraints at the cut rim joint were maintained quite well; the position constraint violations usually were less than \(.01 \) ft. and \(.5^\circ \). Because of the small size of these violations, an attempt to reduce them through use of the Baumgarte method was deemed unnecessary.

In the runs, the program usually blew up when passing through \( \beta_1 = 0 \) at the end of deployment. No attempt was made to eliminate the blow up, since the program was not intended to operate after the end of the deployment. The phenomenon is attributed to the special singularity of \( A \) at \( \beta_1 = 0 \). Also, the program often experienced some slight deterioration in performance in the latter portion of deployment. It is believed that this deterioration was not due to the singular \( A \) problem, but instead was caused mainly by the fact that we did not introduce a sufficiently shorter integration step length to compensate for the speed up in dynamic motion which is encountered as deployment proceeds.
After the above-noted fix on \( A \) in the interval \( 90 < \beta_1 < 0 \) was made, the program conserved mechanical energy \( E \) quite well throughout deployment. The variations in \( E \) from its value at the start of deployment usually did not exceed \( .5\% \), and deviations this large were encountered only close to the end of deployment.

In an ideal symmetrical deployment of the rim, the orientation of the eight bridges would not change, and the responses \( (\beta_i(t); i = 1 \) to 16) of the 16 angles between the tubes and the bridges would be identical. In the early runs, some difficulty was encountered in generating a symmetrical deployment. In particular, high frequency natural vibration modes among the \( \dot{\beta}_1 \) and, more noticeably, among the \( \ddot{\beta}_1 \) were excited. After several bugs in the program were located and removed, however, the excitation of these modes in the rim-only runs was greatly reduced, and near-symmetrical deployments were obtained.

Except for the above-noted small variations, the responses of the \( \beta_1 \) resembled cosine waves with the full deployment comprising one-quarter of a cycle. (The resemblance was not perfect, however, since the slope around \( \beta = 0 \) was steeper than that of a perfect cosine wave.) This result is what would be predicted by linear single degree of freedom theory. The runs showed that the time \( T_D \) required to fully deploy the rim varies inversely with the square of the spring constant of the bridge-tube springs; this result also concurs with linear single degree of freedom theory.

With the spacecraft configuration values that were used in most of the runs, the deployment duration \( T_D \) was approximately 370 seconds. The studies of integration step size \( H \) showed that \( H = 20 \) seconds was sufficiently accurate, but that \( H = 10 \) or 5 were better. An efficient compromise was determined to be to start with \( H = 20 \) and later switch to \( H = 10 \) and then \( H = 5 \) as deployment speeded up (i.e. as the value of \( \dot{\beta}_1 \),
increased.) With the hub tree calculations being deleted, the CPU (i.e. 
computer) time per integration step was approximately .75 seconds. CPU 
time at CSDL costs approximately $.42 per second. Thus, the cost per run 
(considering only CPU time to perform integration steps) was $6 to $24.

8.6.3 Rim and Stays

In these runs the hub, in effect, was fixed rigidly in space. The 
bridge-cable springs still were cut. The lengths of the eight stays were 
given a specified time response \( l_8(t) \) which controlled the deployment 
of the rim. The response \( l_8(t) \) which was used in most of the runs was

\[
l_8(t) = 126.81 - 123.55 \cos (0.00349t); \quad t \leq 300
\]

\[
l_8(t) = 65.035 + 75 \sin (0.0046t - 1.380); \quad t > 300
\]

This response was chosen to slow down the deployment - particularly near 
the end - so that it would keep the stays in tension and dissipate rim 
energy. It produces a deployment time duration \( T_D \) of approximately 530 
seconds.

The runs showed that the simulation still performed properly in 
the presence of the compressive loads which the stays applied to the rim. 
The spacecraft response "tracked" the inputs \( l_8(t) \) well. Rim energy 
decreased monotonically in the presence of these loads as it should. The 
constraint violations at the cut rim joint did not increase noticeably 
over those seen in the previous runs without stays.

Most of the runs used a very low stay spring constant, \( K_8 \), of 
.005 lbs. per ft. For this value, an integration step length, \( H \), of 10 
seconds still was adequate, but an \( H \) of 20 encountered difficulties. An 
efficient way of running the program was determined to be to use \( H = 20 \) 
up to about 220 seconds and then switch to \( H = 10 \). Runs with larger \( K_8 \)
values required a smaller \( H \). The main problem in runs in which \( H \) was slightly too large often was that the solution would suddenly blow up in the middle of the run.

Some difficulty in maintaining the stays continuously in tension throughout the entire deployment was experienced during the runs. The problem appears to be due in part to the excitation of vibration modes of the spacecraft during deployment. The introduction of a reasonable amount of damping into the bridge-tube springs did little or nothing to alleviate the difficulty. It is believed that the problem could have been partially alleviated through the introduction of a large amount of damping into the stay-bridge springs; this theory was not tested, however, because the subroutine which computes the velocity of the bridges relative to the stay attachments on the hub had not been fully checked out.

It is hypothesized that damping the vibration modes which are associated predominately with the stays as noted in the above paragraph would have alleviated, but not fully solved, the problem of maintaining the stays continuously in tension during deployment. To be more specific, the rather small amount of work which was done on the problem in the present study appears to indicate that establishing an a priori time response \( \mathbf{i}_s(t) \) of the stays which will maintain them continuously in tension throughout deployment is not as trivial a task as it appears to be at first glance, even in the ideal case in which vibration excitation is negligible. Therefore, it is believed that a closed loop control approach which uses actual measurements of the stay tensions to establish the stay velocity commands \( \mathbf{i}_s \) might be superior to the above open loop technique in which the full trajectory \( \mathbf{i}_s(t) \) is completely established a priori.
8.6.4 Hub Tree Alone

These runs considered the dynamics of the hub tree (i.e. the hub, drum, and 16 cables) with the stay-bridge springs and cable-bridge springs cut. Two types of runs were made. In the first, cable deployment was produced by means of a torque pulse from the motor between the hub and the drum. In the second, constant forces were applied to the ends of the cables. Both axial and transverse forces were employed in the second type of run.

These runs turned out to be crucial for program debugging since they showed the presence of several Fortran errors and of an error in the equations for the hub tree velocity transformation.

8.6.5 Combined Rim/Hub Tree System

The runs which were made on the combined rim/hub tree system (i.e. on the complete spacecraft) were devoted to (1) looking for remaining errors in the mathematical model and Fortran coding, (2) establishing the techniques for running the program when simulating the complete spacecraft, and (3) investigating the actual dynamic performance of the spacecraft. The work on spacecraft performance was devoted largely to studying techniques for controlling its deployment. Particular consideration was given to control of the motor between the hub and the drum.

The runs showed that the forces which the cables exert on the bridges greatly increase the time $T_D$ required to deploy the rim. In the preceding work in which the stay-bridge and cable-bridge springs were cut, $T_D$ had a value of approximately 370 seconds. Connecting the cable-bridge springs increases $T_D$ to roughly 1700 seconds.

An integration step size, $H$, of 20 seconds proved to be too large for simulation runs of the complete spacecraft. Useful results were obtained with $H = 10$, and this is the value used in most of the runs. An $H$ of 5 seconds also was used occasionally when higher accuracy was desired or when difficulty was encountered with $H = 10$. 
The CPU time per integration step for the combined rim/hub tree system turned out to be approximately 1.12 seconds. Assuming \( H = 10 \) and raising \( T_D \) to, say, 2000 to allow for stay tension, it can be determined that a simulation run of the complete deployment would require approximately 200 integration steps and 225 CPU seconds; the cost would be about $95. Runs this expensive would be acceptable when needed to support an actual spacecraft project which is crucially dependent on them. However, they are extremely dear for the present purposes, and, therefore, only one was attempted. Almost all other runs were limited to the first 600 seconds or less of deployment.

The one full deployment run using \( H = 10 \) which was attempted did not reach the end. (The \( T_D \) value of 1700 seconds which was quoted above is an extrapolated value.) The simulation's performance in this run was satisfactory up to about 1000 seconds. By this time, however, the constraint violations had become large enough to cause serious concern. The run started to deteriorate badly around 1100 seconds. It failed completely at 1330 seconds. It is believed that the main reason for the deterioration and failure is that even \( H = 10 \) is not small enough during the second half of deployment. It is believed, therefore, that the full deployment could be simulated successfully if one would switch to smaller \( H \) value, say \( H = 5 \), around \( T_D = 800 \) seconds or less. Also the use of Baumgarte's technique to stabilize the constraint violations might be useful or necessary. A simulation study to test this approach was not undertaken, however, because of the expense.

Figure 8-8 shows the translational motion in inertial space of one of the rim bridges during the first 1200 seconds of deployment. The motions of the other seven bridges are virtually identical. In addition to the motion shown on this figure, there is also a vertical component toward the rim's post deployment plane. The figure is a hand plot of data generated in the long run noted in the above paragraph. In this run the bridge-cable springs were connected but the stay-bridge springs were
Figure 8-8. Bridge translation: No stay forces or motor torques.
not; the torque from the hub/drum servo motor was set to zero. Thus Fig. 8-8 is the trajectory predicted by the simulation if motor control is turned off and the stays are permitted to be pulled out freely by the bridges without exerting loads on the bridges.

The spiraling motion of the bridge on Fig. 8-8 is caused by the forces which the cables exert on it. In the runs discussed in subsections 8.6.2 and 8.6.3 in which the cable-bridge springs were cut, the bridges moved out radially; thus in these runs the motion of the bridge shown in Fig. 8-8 was along the +z axis. When the bridge-cable springs are connected, as in the Fig. 8-8 run, the drum rotates counterclockwise in compensation for the clockwise motion of the rim thereby causing the system angular momentum to remain zero. In the run shown on Fig. 8-8, the hub did not translate or rotate significantly, since the net force and torque on it were close to zero.

In the run shown on Fig. 8-8, all cables remained continuously in tension up to the time (shortly before 1100 seconds) at which the run started to go bad.

As was discussed in Subsection 8.4.5, the motor control strategy which was implemented into the simulation involved using it to control the azimuth angle component $\theta_{S_1}$ of the line of sight vector from one of the stay attachment points on the hub to the corresponding bridge. ($\theta_{S_1}$ is illustrated in Fig. 8-4.) The angle $\theta_{S_1}$ specifies the relative rotation, in azimuth, of the rim relative to the hub. Thus it specifies the "torsion" of the spacecraft. $\theta_{S_1}$ is nominally zero after deployment, and it is desirable, therefore, that $\theta_{S_1}$ be kept reasonably small during deployment. In the run shown in Fig. 8-8, which included neither motor torque nor stay forces, the magnitude of $\theta_{S_1}$ increased monotonically since the rim rotated continuously in azimuth while the orientation of the hub was constant; the first portion of this response is included as one of the three plots on Fig. 8-9.
The motor control strategy noted in the above paragraph assumes that the spacecraft contains a sensor system to measure $\theta_{S_1}$ and $\dot{\theta}_{S_1}$. Simulation runs indicated that this is an extremely effective technique for keeping $\theta_{S_1}$ small. Fig. 8-9 includes the results from one such run. Basically, the control problem is merely the simple one of controlling a double-integrator plant

$$G_p = \frac{K_p}{s^2}$$

by use of position and rate feedback

$$G_c = K_c[1+as]$$

The computer work indicates, however, that the above technique for controlling the motor creates a problem. To wit, the runs which were made using it experienced long time intervals in which the cables were not in tension. Typically, all cables would be in tension up to about 250 seconds and none would be in tension thereafter throughout the remainder of the run. It is our understanding that on the Grumman spacecraft proper deployment of the gores (which are actually rather complex structural systems) requires that they be kept continuously in tension. Therefore, it was taken as a groundrule for the present study that the cables should be maintained continuously in tension during deployment.

It is our belief that the difficulty in maintaining the cables (or gores) in tension during deployment is a real problem, not a fictitious one generated by shortcomings of the simulation or spacecraft-model. The problem was not alleviated in runs in which the spring constants of the cable-bridge springs were increased. It is believed that it also could not be alleviated significantly by the use of large damping in these springs. Rather, it is generated by the basic difficulty in maintaining
the cable deployment rate adequately synchronized with the rim deployment rate. Adjusting the stay deployment rates can provide almost no help, since stay forces can only slow the rim deployment down, not speed it up.

The surest way of assuring that the cables remain within allowable bounds in tension during deployment evidently is to measure the tensile loads on them and employ this data in the motor control law. The task of designing a motor control law to do this would be simplified if the motor could be relieved of the separate job of maintaining the torsion angle $\theta_{S_1}$ acceptably small. Obviously, the stays will prevent $\theta_{S_1}$ from building up monotonically. However, the computer runs indicate that the stiffness which the stays provide against $\theta_{S_1}$ motion is low. In runs which included stay forces but no motor torque, $\theta_{S_1}$ oscillations of approximately $\pm 100^\circ$ were encountered. The result of one such run is included on Fig. 8-9. Increasing the values of the spring constants of the stay-bridge springs did not reduce the amplitude of the $\theta_{S_1}$ motion. We believe that a deployment scheme which permits $\theta_{S_1}$ motion anywhere near as large as $\pm 100^\circ$ even under ideal nominal conditions would be risky and unacceptable.

To summarize our interpretation of the simulation results, we believe that they indicate clearly that even under ideal nominal conditions control of the deployment is not as simple a task as it may appear to be at first glance. Closed loop control of the motor and possibly of the stay deployment rates appears necessary. The motor must perform the dual and disparate functions of keeping all cable tensions within acceptable bounds and also keeping the torsion angle within acceptable bounds. A control law of considerable sophistication would be needed to accomplish this. Investigation of such a law was deemed to be outside the scope of the present effort which basically was intended to verify the feasibility of employing the equations of motion formulation in computer simulations of multibody dynamics.
CHAPTER 9

SYNTHESIS

Due, in part, to the advent of the space shuttle, a new generation of spacecraft, called large space structures (LSS), is just beyond the horizon. LSS are characterized by larger physical dimensions and greater structural complexity than spacecraft of the past. Like spacecraft of the past, many LSS will require a fully or partially automatic "deployment" operation in-orbit to get them into their final structural configuration. Because of their increased dimensions and structural complexity, deployment of LSS raises new questions and problems. Therefore, it is an area of considerable concern at the present time.

Among the tools which the still-youthful technology of LSS deployment requires are computer simulations of their deployment dynamics. The discipline of multibody dynamics theory which originated in the aerospace industry with the work of Hooker and Margulies and of Roberson and Wittenburg provides the most natural methods for developing the equations of motion to be used in these simulations. However, up to the present time, multibody dynamics theory in the aerospace field has emphasized systems with tree structure and (to a lesser extent) rigid individual bodies. This handicaps its application to LSS deployment problems.

The research described in this thesis was an attempt to develop a formalism for multibody dynamics which includes the basic features needed to model deployment of all currently-anticipated LSS. In fact, the aim was to develop a new formalism which is superior in this application to all previously published ones. As noted earlier, the formulation pre-
sented here is viewed in part as an amalgam of the best features of ones presented previously by Bodley et al, Boland et al, Wittenburg, and Jerkowsky. An attempt was made to make the formalism as explicit and detail-ed as possible without sacrificing generality and mathematical concise-ness. The key attributes of the formulation are (1) the use of a trans-formation of velocity variables as the means for introducing relative velocities at selected joints, (2) its applicability to systems with closed topological loops, (3) its applicability to systems in which arbitrary bodies can be rigid or nonrigid, (4) the use of a very general model of the joints, and (5) its capability of handling control systems at the joints.

The only general multibody computer simulation described in the aerospace literature which can handle closed topological loops is the one developed by Bodley and his associates at Martin Marietta. To the best of our knowledge, it is the only such program in existence. A comparison of the multibody formulation used by Bodley with the one presented in this thesis, therefore, appears desirable. The fundamental difference between the two methods is that Bodley's approach restricts one to the use of absolute velocities of the individual bodies to specify the system velocity vector, while the one presented here enables relative velocities at selected joints to be introduced.

The limitation of Bodley's approach to absolute velocities is both its strongest point and its weakest point. It is a strong point because it produces a formulation which is relatively simple and relatively easy to program. In this regard, it appears certain that programming the approach developed in this thesis as a general multibody simulation will present a more challenging task to the programmer and analyst than did Bodley's. However, unlike the method presented in this thesis, the absolute velocity approach has the major drawback of requiring one to deal directly with all constraints at all joints. This is a considerable handicap because the computation of the Lagrange multiplier vector which
is used to specify the system constraint forces is cumbersome. Also, the fact that the formulation employs differentiated constraint equations rather than the actual constraint equations introduces significant problems in keeping the constraints satisfied during numerical integrations; this difficulty does not occur with joints that are handled by the relative velocity approach. Our experience with even the small number (five) of constraint equations in the computer program described in Chapter 8 has convinced us that constraint equations are a very major headache, and their use should be avoided through the introduction of relative velocities at the joints whenever possible.

As an indication of the relative size of the matrices to be inverted and of the sets of linear algebraic equations to be solved by the two approaches, consider the sample problem described in Chapter 8. The approach presented in this thesis required the following operations at each computation of the state vector time derivative: (1) the inversion of a 21 x 21 generalized mass matrix to establish the rim acceleration vector \( \dot{x}_R \), (2) the solution of 39 linear algebraic equations to establish the hub tree acceleration vector \( \dot{x}_H \), and (3) the solution of 5 linear algebraic equations to establish the Lagrange multiplier vector \( \lambda \). In contrast, the absolute velocity vector method would require (1) the solution of 80 linear algebraic equations for the Lagrange multiplier vector of the rim and (2) the solution of 69 linear algebraic equations for the Lagrange multiplier vector of the hub tree. (The absolute velocity vector method would not require the inversion of generalized mass matrices or the corresponding solution of algebraic equations, since the generalized mass matrices could be made diagonal through the use of principal inertia axes.)

The use of relative velocities at the joints in multibody formulations and computer simulations of course is not new. For example, such a computer program reportedly has been developed recently at Minneapolis-Honeywell in Florida. However, to our knowledge, no such existing simulation can handle systems with closed topological loops, and therefore none could have handled the sample problem presented in Chapter 8.
Some possibilities for future work in the area pursued in the thesis will be noted now. In regard to the general formulation of multi-body equations of motion presented in Chapters 4 to 6, the possibilities include the following.

(1) The formulation made no use of the center of mass of the composite system nor of the centers of mass of the individual trees. However, there are applications where the use of centers of mass would appear to be advantageous, and their introduction into the formulation therefore could be investigated. (Problems in which one wishes to simulate gravity gradient forces and the orbital motion of the system are the most obvious such application).

(2) In the sample problem of Chapter 8, the flexible tube running mode was concluded to be not a useful feature because of its obvious disastrous effect on program speed. It is believed, however, that a "quasi-static" mode which simulates their static deformations of the tubes under external loads but omitted the dynamic vibratory motion would have been useful. Modification of the general formulation to encompass such quasi-static deformations of the individual bodies could be investigated.

It is planned that the next major CSDL effort in the area will be an investigation of methods of implementing the general formulation into a computer program which can simulate a wide variety of LSS deployment problems. Design of a general computer program such as this presents a multitude of problems and questions not encountered in the simulation described in Chapter 8 which was devoted to one specific spacecraft. The work will emphasize (1) making the program sufficiently general and (2) the ever-present problem of running speed. It will include, of necessity, investigation of efficient methods of handling the sparse matrices which appear in the general formulation.
CHAPTER 10

CONCLUSIONS

A new formalism for the equations of motion of multibody systems has been developed in the thesis. Its key features are (1) the use of a transformation of velocity variables as the means of introducing relative velocities at selected joints into the state vector, (2) its applicability to systems with closed topological loops, (3) its applicability to systems in which arbitrary bodies can be rigid or nonrigid, (4) the use of a very general model of the joints, and (5) its capability of handling control systems at the joints.

The feasibility of employing the approach in computer simulations of the dynamics of realistic spacecraft has been demonstrated by applying it successfully to a large and difficult sample problem.
APPENDIX A

DERIVATION OF KINETICS EQUATION OF SINGLE BODY

A.1 Introduction

This appendix develops an equation for the kinetics of a single body of a multibody system. The formalism includes body nonrigidity; it can be made applicable to a rigid body, however, by merely deleting the equations and variables which pertain to body nonrigidity.

The geometric and kinematic variables and relationships which are needed were presented in Sections 4.1 and 4.2. The main geometric variables were illustrated in Fig. 4-1.

To put the material in the following sections into context, consider a hypothetical multibody system. Regard all joints of the system as being cut. That is, replace the joints by the forces and torques which they exert on the bodies. A kinetics equation for any body Z of the system then can be developed by treating the forces and torques, $\overrightarrow{F}^n_z$ and $\overrightarrow{G}^n_z$, exerted on it by the joints as being external in nature. By proceeding in this manner a kinetics equation for each body of the system can be derived. These equations are coupled together through the joint forces and torques. A technique for computing or eliminating these joint forces and torques, of course, is needed to unify the set of kinetic equations; this topic is treated in the main body of the document.
The derivation employs D'Alembert's principle of virtual work. The first step is to obtain an expression for the virtual work $\delta W_Z$ done by body Z during a virtual displacement of its particles. This expression then must be transformed into a form in which all elements of the vector $(\delta \mathbf{w}_Z)$ which specifies the virtual displacement can be selected arbitrarily. The kinetics equation for body Z then can be written down directly from this expression for $\delta W_Z$.

A.2 The Virtual Work Expression

Let Z be an arbitrary body of a multibody system. Let the forces on Z be considered to include the usual body, surface, and internal forces and also (1) the forces and torques exerted on body Z by its joints and (2) the apparent dynamic forces due to the accelerations of the particles of Z relative to N. Thus body Z is in force-equilibrium.

Let body Z have some arbitrary configuration and velocity distribution relative to an inertial coordinate frame N. Introduce a virtual displacement of the particles of body Z relative to N. Let $\delta W_Z$ be the resulting virtual work done by body Z.

The following expression for $\delta W_Z$ can be deduced.

$$
\delta W_Z = \int_{V_Z} \delta - \lambda_{PN} \cdot \frac{\mathbf{F}_p}{\mathbf{F}_p} \rho dV + \int_{S_Z} \delta - \lambda_{PN} \cdot \frac{\mathbf{F}_p}{\mathbf{F}_p} dS
$$

$$
- \int_{V_Z} \delta - \lambda_{PN} \cdot \frac{\mathbf{x}_{PN}}{\mathbf{x}_{PN}} \rho dV - \int_{V_Z} \delta \mathbf{e}_p \cdot \frac{\mathbf{g}_{PN}}{\mathbf{g}_{PN}} dV
$$

$$
+ \sum_{n} \sum_{\alpha} \left[ \mathbf{N}_{n_{Z}} \cdot \mathbf{F}_{\alpha} + \delta \mathbf{N}_{n_{Z}} \cdot \mathbf{F}_{\alpha} \right] \left[ \mathbf{N}_{n_{Z}} \cdot \mathbf{G}_{\alpha} + \delta \mathbf{N}_{n_{Z}} \cdot \mathbf{G}_{\alpha} \right]
$$

(A.2-1)
The notation $\delta_{N}^{X}$ above denotes virtual displacement relative to N. The first three integrals are the virtual work due to the body, surface, and dynamic forces respectively. $\bar{F}_{P}^{B}$ is the body force per unit mass at an arbitrary point P, and $\bar{F}_{P}^{S}$ is the surface force per unit area at a surface point P. For simplicity, pure torques and point forces such as might be used when modeling control actuators have not been included in Eq. (A.2-1).

The fourth integral in Eq. (A.2-1) is the virtual work which results from the internal forces among the particles of body Z. The variables $\varepsilon_{P}$ and $\sigma_{P}$ are the 6-vector representations of the symmetric strain and stress tensors at P. More precisely, for large deformations $\varepsilon_{P}$ is defined to be the vector representation of Green's strain tensor and (for negligible internal damping) $\sigma_{P}$ is defined to be the vector representation of the second Piola-Kirchhoff stress tensor. The $\varepsilon_{P}$ and $\sigma_{P}$ concept is being used in the present formulation because of its fundamental significance. In application, however, one frequently does not employ stress and strain tensors or vectors directly in the modeling, but instead uses stiffness matrices obtained with the aid of a computer program. When structural damping is included in applications, it is commonly assumed to be viscous and introduced through a damping matrix or else is added directly onto the final kinetics equations in an ad hoc manner.

The final term in Eq. (A.2-1) is the virtual work due to the forces and torques exerted on Z by the joints. The outer summation in this term is taken over $n = 1, 0$; the two summations together encompass all joints with an end (end I or end 0) attached to Z.

A.3 Transformation of the Virtual Work Expression
Define the following virtual displacement vector

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\[
\begin{align*}
\delta \pi_{v_{Z1}} &= \begin{pmatrix}
\delta \bar{x}_{N} \\
\delta \theta_{CN} \\
\delta q_{Z1}
\end{pmatrix} \\
&= (A.3-1)
\end{align*}
\]

In the present work, \( \delta \pi \) is used as a catchall symbol with virtual displacements. The subscript label \( v_{Z1} \) denotes that this is the virtual displacement which corresponds to the velocity vector \( v_{Z1} \). The correspondence can be seen by comparing Eq. (A.3-1) with (4.2-10) for \( v_{Z1} \). The term \( \delta \theta_{CN} \) in Eq. (A.3-1) is the virtual rotation vector of frame \( \zeta \) relative to frame \( N \).

The goal of obtaining a kinetics equation for \( Z \) can be achieved by expressing all virtual displacement terms in Eq. (A.2-1) as functions of \( \delta \bar{v}_{v_{Z1}} \). (In this regard, it should be realized that the active generalized coordinate vector \( q_{Z1} \) should not be given a virtual displacement. Thus \( \delta \pi_{v_{Z1}} \) is used, not \( \delta \pi_{v_{Z}} \).)

Start with the term \( \delta \bar{x}_{PN} \) in Eq. (A.2-1). As a result of the analogy between \( v_{Z1} \) and \( \delta \bar{v}_{v_{Z1}} \), the following relation can be written down immediately.

\[
\delta \bar{x}_{PN} = S_{x} \delta \pi_{v_{Z1}} \hspace{1cm} (A.3-2)
\]

The S term in Eq. (A.3-2) was defined in Eq. (4.2-12).

It is assumed that a strain model \( \varepsilon_{p} = \varepsilon_{p}(q_{Z}) \) has been established a priori. Hence

\[
\delta \varepsilon_{p} = \varepsilon_{p,q_{Z1}} \delta q_{Z1} \hspace{1cm} (A.3-3)
\]
Eq. (A.3-3) can be written in terms of $\delta \pi_{vZ1}$ as follows:

$$\delta \varepsilon_p = \varepsilon_p E_Z \delta \pi_{vZ1} \quad \text{(A.3-4)}$$

where

$$E_Z = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad \text{(A.3-5)}$$

Now consider the final term in Eq. (A.2-1). For notational compactness it is convenient to introduce the following dual-vectors

$$\mathbf{L}_{\alpha nZ} = \begin{bmatrix} \overline{F}_{\alpha nZ} \\ \overline{G}_{\alpha nZ} \end{bmatrix} \quad \text{(A.3-6)}$$

$$= \delta \pi_{\overline{v}_{\alpha nZ}} = \begin{bmatrix} \delta_{NX \alpha nZ} \\ \delta_{N \alpha nZ} \end{bmatrix} \quad \text{(A.3-7)}$$

Also, $\mathbf{L}_{\alpha nZ}$ will be separated into known forces $\mathbf{L}^K_{\alpha nZ}$ and constraint forces $\mathbf{L}^C_{\alpha nZ}$

$$\mathbf{L}_{\alpha nZ} = \mathbf{L}^K_{\alpha nZ} + \mathbf{L}^C_{\alpha nZ} \quad \text{(A.3-8)}$$

The $\delta \pi$ term in Eq. (A.3-7) is the virtual displacement which corresponds to the velocity vector $\mathbf{v}$ which was introduced in Eq. (4.2-18). The following relation can be written down immediately.
\[
\delta \pi \left[ \begin{array}{c}
\alpha_{nZ} \\cdot S_{nZ}^{-1} \cdot \frac{\delta \pi}{nZ} \\
\end{array} \right] = S_{nZ}^{-1} \cdot \frac{\delta \pi}{nZ} \cdot \frac{v}{nZ} \cdot V_{Z1} \quad (A.3-9)
\]

The \( S \) array above was defined in Eq. (4.2-19).

The material introduced in the above paragraph enables the final term in Eq. (A.2-1) to be written in the form

\[
\frac{\psi}{nZ} \cdot \frac{L^C}{nZ} \cdot \frac{L^N}{nZ} \cdot \frac{S}{nZ} \cdot \frac{v}{nZ} \cdot \frac{v}{nZ} \cdot \frac{\delta \pi}{nZ} \left( \begin{array}{c}
\end{array} \right) (A.3-10)
\]

Now introduce Eqs. (A.3-2, -4, and -10) into (A.2-1) to produce

\[
\delta W = \delta \pi \left[ \begin{array}{c}
\frac{\psi}{nZ} \cdot \frac{L^C}{nZ} \cdot \frac{L^N}{nZ} \cdot \frac{S}{nZ} \cdot \frac{v}{nZ} \cdot \frac{v}{nZ} \cdot \frac{\delta \pi}{nZ} \left( \begin{array}{c}
\end{array} \right) (A.3-11)
\]

where

\[
\frac{\psi}{nZ} \cdot \frac{L^C}{nZ} \cdot \frac{L^N}{nZ} \cdot \frac{S}{nZ} \cdot \frac{v}{nZ} \cdot \frac{v}{nZ} \cdot \frac{\delta \pi}{nZ} \left( \begin{array}{c}
\end{array} \right) (A.3-12)
\]

and

\[
\frac{\psi}{nZ} \cdot \frac{L^C}{nZ} \cdot \frac{L^N}{nZ} \cdot \frac{S}{nZ} \cdot \frac{v}{nZ} \cdot \frac{v}{nZ} \cdot \frac{\delta \pi}{nZ} \left( \begin{array}{c}
\end{array} \right) (A.3-13)
\]

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The term \( f_{EIZ} \) is the generalized force on body Z due to all true external and internal forces except those at the joints. \( f_{JZ}^K \) is the generalized force on body Z due to the known joint forces. \( f_{JZ}^C \) is the generalized force on body Z due to the joint constraint forces.

The apparent dynamic forces on body Z are included as the integral term in Eq. (A.3-11). Clearly some additional work on this term is required. The first thing which is needed is an expression for \( \ddot{x}_{PN} \). A satisfactory expression can be obtained by differentiating Eq. (4.2-2), employing the theorem of Coriolis, and manipulating. The form of the result which is most convenient for the present work is

\[
\ddot{x}_{PN} = S_z \left( \dddot{v}_{Z1} + \dddot{x}_{PN} \dddot{v}_{Z2} + \dddot{\omega}_C \dddot{v}_C \dddot{\omega}_C - \dddot{\omega}_C \dddot{\omega}_C \dddot{x}_{PN} \dddot{\omega}_C \dddot{\omega}_C \right)
- 2 \sum_{i} q_i x_{PN} \dddot{\omega}_i + \sum_{i,j} q_i q_j x_{PN}
\]  

(A.3-14)

The \( S \) term in Eq. (A.3-14) was defined in Eq. (4.2-12).

Now substitute Eq. (A.3-14) into the integral term in Eq. (A.3-11). After considerable manipulation, the following equation can be produced.

\[
\int_{V}^{S} \dddot{x}_{PN} \rho dV = M_{Z1} \dddot{v}_{Z1} + M_{Z2} \dddot{v}_{Z2} - f_{DZ}
\]  

(A.3-15)

where

\[
M_{Z1} = \int_{V}^{S} \frac{s}{x_{PN-Z1}} \rho dV
\]  

(A.3-16)
\[ M_{Z2} = \int_{V_Z} s^T \nabla_{x_{PN-Z1}} q_{Z2} \rho dv \quad (A.3.17) \]

\[ f_{DZ} = \int_{V_Z} s^T \rho dv \nabla_{x_{CN}} w_{CN} + 2 \sum_{i} q_i \int_{V_Z} s^T \nabla_{x_{PN-Z1}} x_{PNZ1} v_{Zi} \rho dv \nabla_{\omega_{CN}} + D(\omega_{CN}) \int_{V_Z} f_{DZ} \rho dv \nabla_{\omega_{CN}} \quad (A.3.18) \]

The variables \( D(\omega_{CN}) \) and \( U_{PN} \) in the final expression in Eq. (A.3-18) are defined as follows:

\[ D(\omega_{CN}) = \text{Diag}[\omega_{CN}, \omega_{CN}, \omega_{CN}, \ldots, \omega_{CN}] \quad (A.3-19) \]

\[ U_{PNZ1} = [I_3, -x_{PNZ1}, x_{PNZ1,1}, x_{PNZ1,2}, \ldots] \quad (A.3-20) \]

The terms \( M_{Z1} \) and \( M_{Z2} \) are generalized mass arrays. \( f_{DZ} \) is the generalized dynamic force on body \( Z \).

The final expression for virtual work now can be produced by substituting Eq. (A.3-15) into (A.3-11),

\[ \delta W_Z = \delta_{V_Z} s^T \left[ -M_{Z1} v_{Z1} - M_{Z2} v_{Z2} + f_{DZ} + f_{EIZ} + f_{KZ} + f_{CZ} \right] \quad (A.3-21) \]
A.4 The Single Body Kinetics Equation

All forces acting on body $Z$, including the dynamic and constraint forces, were included when developing the equations in the preceding subsection for the virtual work $\delta W_Z$. Therefore, it is known from basic principles that $\delta W_Z = 0$. Also, all elements of the virtual displacement vector $\delta \pi_{V1}$, which is used in Eq. (A.3-21) can be chosen arbitrarily; that is, there are no constraints on or among them. Thus, the RHS of Eq. (A.3-21) is zero for all $\delta \pi_{V1}$. This implies that the term inside [ ] is identically zero. Therefore

\[ M_{1Z} \ddot{v} = -M_{2Z} \ddot{v}_{Z2} + \dot{f}_D + \dot{f}_{EI} + \dot{f}^K + \dot{f}^C \]  

(A.4-1)

Eq. (A.4-1) is the basic kinetics equation for body $Z$. The main supplementary equations which specify the variables that appear in it are Eqs. (A.3-12 and 13) and (A.3-16 to 20).

Eq. (A.4-1) actually consists of three sets of equations: translational kinetics, rotational kinetics, and internal kinetics. These three sets of equations have been combined into a single equation, Eq. (A.4-1), because the development in the main body of the document can be performed most easily if a single equation for the kinetics of $Z$ is used. The separate equations for translational, rotational, and internal dynamics can, of course, be obtained by substituting into Eq. (A.4-1) the complete expressions for the $M$, $v$, and $f$ terms which were listed as Eqs. (A.3-12 and 14) and (A.3-16 to 20).

The $M_Z$, $f_D$, and $f_{EI}$ terms in Eq. (A.4-1) involve integrals taken over the volume of $Z$. These integrals are functions of $g_Z$. In applications, their values must be calculated. This, in general, is a big task. The use of truncated series expansions in $g_Z$ about a nominal configuration is the most evident approach. Truncation to first order for internal stiffness and to zeroth order for the other integrals is adequate in most problems.
APPENDIX B

DERIVATION OF EQUATIONS FOR THE ELEMENTS OF THE VELOCITY
TRANSFORMATION OPERATOR

B.1 Introduction

This appendix develops equations for determining the elements of
the operator $T_{\nu_1 \nu}$, which transforms $\nu'$ to $\nu_1$

$$\nu_1 = T_{\nu_1 \nu} \nu'$$  \hspace{1cm} \text{[Reference Eq. (4.3-2)]}

Recall from Chapter 4 that the elements of $\nu_1$ are the $\nu_{\zeta N'}$, $\omega_{\zeta N'}$, and $\dot{q}_{Z1}$ of the individual bodies $Z$. As discussed in Chapter 6, the elements of $\nu'$ are (1) the $\nu_{rN'}$, $\omega_{rN'}$ of the reference body $R$ of every tree of the transformed system, (2) the passive and active velocities $\rho_{Bf}$ and $\rho_{Ba}$ of every reconnected joint $B$, and (3) the passive and active generalized coordinate velocities $\dot{q}_{Z1}$ and $\dot{q}_{Z2}$ of every nonrigid body.

Both $\nu'$ and $\nu_1$ contain the $\dot{q}_{Z1}$ of every nonrigid body. Therefore, the transformations necessary to produce the $\dot{q}_{Z1}$ elements in $\nu_1$ are merely identity transformations. The elements in $\nu_1$ of present interest thus are the $\nu_{\zeta N}$ and $\omega_{\zeta N}$.

The row of Eq. (4.3.2) which pertains to an arbitrary $\nu_{\zeta N}$ can be denoted as

$$\nu_{\zeta N} = T_{\nu_{\zeta N} \nu'}$$  \hspace{1cm} \text{(B.1-1a)}

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Similarly, the row which pertains to an arbitrary \( \bar{\omega}_{ZN} \) can be written as

\[
\bar{\omega}_{ZN} = \bar{v}_{\frac{\bar{v}_{Z}}{\bar{\omega}_{YN}}} \quad (B.1-1b)
\]

The \( \bar{v}_{\frac{\bar{v}_{Z}}{\bar{\omega}_{YN}}} \) and \( \bar{\omega}_{\frac{\bar{\omega}_{Z}}{\bar{\omega}_{YN}}} \) of any Z are functions only of those elements in \( \bar{v}' \) which pertain to Z itself (\( \bar{q}_{Z1} \), \( \bar{q}_{Z2} \) that is) or which pertain to bodies and joints that lie inboard of Z on the same tree as Z. Upon introducing some new notational methods that will be explained below, this enables Eq. (B.1-1) to be written in the more explicit form

\[
\bar{v}_{\frac{\bar{v}_{Z}}{\bar{\omega}_{YN}}} = S_{\bar{v}_{\frac{\bar{v}_{Z}}{\bar{\omega}_{YN}}} - \bar{v}_{\bar{\omega}_{YN}} - \bar{w}_{\bar{\omega}_{YN}} + \bar{q}_{\bar{H}} + \bar{q}_{\bar{B}}}
\]

\[
\bar{\omega}_{\frac{\bar{\omega}_{Z}}{\bar{\omega}_{YN}}} = S_{\bar{\omega}_{\frac{\bar{\omega}_{Z}}{\bar{\omega}_{YN}}} - \bar{w}_{\bar{\omega}_{YN}} - \bar{w}_{\bar{\omega}_{YN}} + \bar{q}_{\bar{H}} + \bar{q}_{\bar{B}}}
\]

\[
\bar{v}_{\frac{\bar{v}_{Z}}{\bar{\omega}_{YN}}} = \bar{v}_{\frac{\bar{v}_{R}}{\bar{\omega}_{YN}}} + \bar{w}_{\frac{\bar{w}_{Z}}{\bar{\omega}_{YN}}} + \bar{w}_{\frac{\bar{w}_{Z}}{\bar{\omega}_{YN}}} + \bar{q}_{\bar{H}} + \bar{q}_{\bar{B}}
\]

\[
\bar{\omega}_{\frac{\bar{\omega}_{Z}}{\bar{\omega}_{YN}}} = \bar{\omega}_{\frac{\bar{\omega}_{R}}{\bar{\omega}_{YN}}} + \bar{w}_{\frac{\bar{w}_{Z}}{\bar{\omega}_{YN}}} + \bar{w}_{\frac{\bar{w}_{Z}}{\bar{\omega}_{YN}}} + \bar{q}_{\bar{H}} + \bar{q}_{\bar{B}}
\]

In the above equations, the upper case Greek letter \( \bar{H} \) denotes an arbitrary body. The symbol \( \bar{B} \) still denotes an arbitrary reconnected joint. The notation \([R,Z]\) denotes the set of bodies and joints on the path between R and Z with R and Z themselves being included. The conventional notation from set theory will be used later to denote the sets in which R or Z are excluded. The notation implies that R is the reference body of Z's tree. Thus \( \bar{H}e[R,Z] \) signifies that \( \bar{H} \) is a body on the path between R and Z, while \( \bar{B}e[R,Z] \) signifies that \( \bar{B} \) is a joint on the path between R and Z. For simplicity, the passive and active elements of \( \bar{q}_{\bar{H}} \) and \( \bar{q}_{\bar{B}} \) have not been denoted separately in the above equations.
The problem to be addressed in this Appendix is the development of equations which specify the \( S \) arrays in Eqs. (B.1-2). This will specify \( T_{\gamma \gamma} \), as explicitly as is possible at the level of generality being employed in this document. The \( S \) arrays which appear in the transformation to \( \bar{\gamma} \), will be developed in Section B.2, while those which appear in the transformation to \( \bar{w} \), will be considered in Section B.3.

Before proceeding, it will be convenient to introduce some new terminology and notation. Fig. B-1 shows the pertinent geometry. On this figure \( H \) is an arbitrary body on the path between \( R \) and \( Z \) with the possibilities \( H = R \) and \( H = Z \) being included. Similarly, \( B \) is an arbitrary reconnected joint on this path.

Consider the reconnected joint frames which are attached to \( H \). In the notation introduced in Chapter 4, these are designated as frames \( \beta_{nH} \). Recall that \( n = I, O \) with \( \beta_{IH} \) being the input frame of \( B \) and \( \beta_{OH} \) being the output frame of \( B \). In general, one is free to choose which of the two frames of any joint to call the "input" frame and which call the "output" frame. However, for mathematical simplicity the work in this appendix and the following one will designate the input frame of each joint to be the one at the end leading back to the reference body \( R \).

One can also speak of body input and output frames as well as of joint input and output frames. Note that body input frames are joint output frames and vice versa. Every body \( H \) except \( R \) has exactly one body input frame \( \beta_{IH} \). (\( R \) has none). This frame also will be designated as Frame \( \eta_{I} \). \( \eta_{I} \) is of course the frame on body \( H \) which leads back to body \( R \). The remaining reconnected joint frames \( \beta_{nH} \) which are attached to body \( H \) are all body output frames. Terminal bodies of a tree have no body output frames. Non-terminal bodies can have an arbitrary nonzero number
Figure B-1. Geometry for velocity transformation operator study.
depending on the structure of the tree. The body output frames of \( H \) will be designated by the symbol \( \eta_0 \). If \( H \in [R, Z] \) exactly one \( \eta_0 \) leads to body \( Z \); this frame will be designated as \( \eta_G(\zeta) \). In later work it will be necessary to define

\[
\zeta_0(\zeta) = \zeta \quad (B.1-3a)
\]

and also

\[
r_I = r \quad (B.1-3b)
\]

in order to make things work out correctly at the end points in summations.

B.2 The Transformation to \( \vec{v}_{\zeta N} \)

It is evident that

\[
x_{\zeta N} = x_{rN} + x_{\zeta r} \quad (B.2-1)
\]

and therefore

\[
\vec{v}_{\zeta N} = \vec{v}_{rN} + \vec{x}_{\zeta r} \quad (B.2-2)
\]

Applying the theorem of Coriolis to the second term on the RHS above yields

\[
\vec{v}_{\zeta N} = \vec{v}_{rN} - \vec{x}_{\zeta r} \omega_{\zeta rN} + \vec{x}_{\zeta r} \quad (B.2-3)
\]
Consider the final term on the RHS in Eq. (B.2-3). Realize that the time derivative here is specified relative to Frame r. With the aid of Fig. B-1, if necessary, one should be able to deduce that this term is a function of the velocities \( \dot{q}_H \) and \( \dot{\rho}_B \) of all bodies H and joints B on the interval \([R, Z]\). It can, further, be deduced that the relation is linear and of the form

\[
\dot{x}_r = \sum_{H \in [R, Z]} x_\zeta H \dot{q}_H \dot{q}_H + \sum_{B \in [R, Z]} S^{*}_{x \zeta B} \dot{\rho}_B
\]  

(B.2-4)

It is an exercise to show that

\[
\dot{x}_\zeta H = \dot{x}_\zeta H = \dot{x}_\zeta H = -\dot{x}_\zeta H \quad \text{for } H \in [R, Z]
\]

(B.2-5a)

To develop an analogous equation for the S term on the RHS of Eq. (B.2-4), let \( dH_B \) be the differential displacement vector corresponding to \( \rho_B \). That is, \( dH_B = \rho_B \, dt \). Then one can develop

\[
S^{*}_{x \zeta B} = \dot{x}_\zeta B = \dot{x}_\zeta B = \dot{x}_\zeta B \quad \text{for } B \in [R, Z]
\]

(B.2-5b)

Substituting Eq. (B.2-4) into (B.2-3) and comparing this result with Eq. (B.1-2a) shows that the S terms for \( v_{CQ} \) are
\[ S_{\mu}^N = I \] \hspace{1cm} (B.2-6a)

\[ S_{\mu}^N \omega_B^N = -x \zeta r \] \hspace{1cm} (B.2-6b)

\[ S_{\mu}^N \zeta^I = x \zeta^I \eta^I \] \hspace{1cm} (B.2-6c)

\[ S_{\mu}^N \beta_B^I = x \zeta^I \eta^I \beta_B^I \] \hspace{1cm} (B.2-6d)

with the terms on the RHS of (B.2-6c and d) being specified by Eq. (B.2-5).

**B.3 The Transformation to \( \overline{\omega}^N \)**

It is evident that

\[ \overline{\omega}^N = \overline{\omega}^N + \overline{\omega}^r \] \hspace{1cm} (B.3-1)

Using Fig. B-1 as a guide, if necessary, one can deduce that

\[ \overline{\omega}^r = \sum_{H \in [R,Z]} \overline{\omega}_O(\zeta) \eta^I_{B \in [R,Z]} \] \hspace{1cm} (B.3-2)

The \( \overline{\omega} \) terms on the RHS of Eq. (B.3-2) can be specified via

\[ \overline{\omega}_O(\zeta) \eta^I = \overline{\theta}_O(\zeta) \eta^I \zeta^I \eta^I \] \hspace{1cm} (B.3-3a)
\[ \bar{\omega}_{\alpha \beta} \frac{\Theta}{\iota} = \bar{\Theta}_{\alpha \beta} \frac{\Theta}{\iota} \cdot \bar{\omega} \]  

(B.3-3b)

Therefore

\[ \bar{\omega}_{\zeta} = \sum_{\mathcal{H}[R,Z]} \bar{\eta}_0(\zeta) \eta_I \mathcal{G}_H \mathcal{G}_H^* + \sum_{\mathcal{B}[R,Z]} \bar{\Theta}_{\alpha \beta} \frac{\Theta}{\iota} \mathcal{G}_B \mathcal{G}_B^* \]  

(B.3-4)

Comparison of Eqs. (B.3-1 and 4) with (B.1-3b) shows that the S terms for \( \bar{\omega}_{\zeta N} \) are

\[ \bar{\omega}_{\zeta N} = 0 \]  

(B.3-5a)

\[ \bar{\omega}_{\zeta N} = \frac{\Theta}{\iota} \]  

(B.3-5b)

\[ \bar{\omega}_{\zeta N} \mathcal{G}_H = \bar{\Theta}_0(\zeta) \frac{\Theta}{\iota} \mathcal{G}_H \]  

for \( \mathcal{H}[R,Z] \)

(B.3-5c)

\[ = 0; \text{for } \mathcal{H} \text{ not } \mathcal{E}[R,Z] \]  

(B.3-5d)

\[ \bar{\omega}_{\zeta N} \mathcal{G}_B = \bar{\Theta}_{\alpha \beta} \frac{\Theta}{\iota} \mathcal{G}_B \]  

for \( \mathcal{B}[R,Z] \)

(B.3-5e)

\[ = 0; \text{for } \mathcal{B} \text{ not } \mathcal{E}[R,Z] \]  

(B.3-5f)
APPENDIX C

DERIVATION OF EQUATIONS FOR THE GYROSCOPIC FORCE ON THE TRANSFORMED SYSTEM

C.1 Introduction

As is shown in Eq. (6.7-4a), the dynamic force \( \mathbf{f}_D^' \) on \( E' \) includes a "gyroscopic" term

\[
-\mathbf{S}^T \mathbf{v}_{1}^' \mathbf{M}_1 \mathbf{T} \mathbf{v}_{1}^' \mathbf{v}_{1}^' \mathbf{v}
\]

which has no counterpart in the dynamic force \( \mathbf{f}_D \) on \( E \). The present appendix develops a formalism for specifying the \( \mathbf{T}_{v, v'} v' \) portion of this new gyroscopic force.

The development will utilize material presented in Appendix B. For purposes of explanation, start with the basic equation

\[
\mathbf{v}_1 = \mathbf{T} \mathbf{v}_{1}^' \mathbf{v}
\]

[Ref. 6.3-1]

As was discussed in Appendix B, the elements of \( \mathbf{v}_1 \) for which the above transformation is nontrivial are the \( \mathbf{v}_{QN} \) and \( \mathbf{\omega}_{QN} \) of the individual bodies \( E \). For an arbitrary body, the two rows of interest in Eq. (6.3-1) can be written in the form

\[
\mathbf{v}_{QN} = \mathbf{T} \mathbf{v}_{QN} \mathbf{v} \]

[Ref. B.1-1a]
\[
\overline{\omega}_N = \frac{T_{\overline{\omega}_N \overline{\omega}_N}}{\frac{\partial}{\partial t}}
\]  
[Ref. B.1-1b]

Taking the time derivatives of Eqs (B.1-1) relative to \( \zeta \) produces

\[
\overline{\omega}_N = \frac{T_{\overline{\omega}_N \overline{\omega}_N}}{\frac{\partial}{\partial t}} \cdot \overline{\omega}_N + \frac{T_{\overline{\omega}_N \overline{\omega}_N}}{\frac{\partial}{\partial t}} \cdot \overline{\omega}_N
\]  
(C.1-1a)

\[
\overline{\omega}_N = \frac{T_{\overline{\omega}_N \overline{\omega}_N}}{\frac{\partial}{\partial t}} \cdot \overline{\omega}_N + \frac{T_{\overline{\omega}_N \overline{\omega}_N}}{\frac{\partial}{\partial t}} \cdot \overline{\omega}_N
\]  
(C.1-1b)

the notation employed above for the time derivatives of \( T \) and \( \overline{\omega}_N \) is a slight generalization of that used previously in this document. It indicates that the time derivatives of the Gibbs vectors in \( T \) and \( \overline{\omega}_N \) are to be taken with respect to frame \( \zeta \).

In the vector \( \overline{\omega}_N \) which is used in the main body of this document, the time derivatives of the Gibbs vectors (\( \overline{\omega}_r \) and \( \overline{\omega}_N \)) in \( \overline{\omega}_N \) are taken with respect to frame \( r \). Thus

\[
\overline{\omega}_N = \frac{T_{\overline{\omega}_N \overline{\omega}_N}}{\frac{\partial}{\partial t}} \cdot \overline{\omega}_N
\]  
(C.1-2)

It can be shown that

\[
\overline{\omega}_N = \overline{\omega}_N - \Omega_{\zeta r} \overline{\omega}_N
\]  
(C.1-3)

where

\[
\Omega_{\zeta r} = \text{Diag} \left[ \begin{array}{c}
\overline{\omega}_r \\
\overline{\omega}_r \\
0
\end{array} \right]
\]  
(C.1-4)
Inserting Eq. (C.1-3) into (C.1-1) produces

\[ \begin{align*}
\frac{\ddot{v}_{\zeta N}}{v_{\zeta N}} &= \frac{T_{\zeta N}}{v_{\zeta N}} \cdot \frac{v'}{v_{\zeta N}} + \frac{T_{\zeta N}}{v_{\zeta N}} \cdot \frac{v''}{v_{\zeta N}} \quad \text{(C.1-5a)} \\
\frac{\ddot{\omega}_{\zeta N}}{\omega_{\zeta N}} &= \frac{T_{\zeta N}}{\omega_{\zeta N}} \cdot \frac{v'}{\omega_{\zeta N}} + \frac{T_{\zeta N}}{\omega_{\zeta N}} \cdot \frac{\omega''}{\omega_{\zeta N}} \quad \text{(C.1-5b)}
\end{align*} \]

where

\[ \begin{align*}
\frac{T_{\zeta N}}{v_{\zeta N}} &= \frac{T_{\zeta N}}{v_{\zeta N}} \cdot \frac{v'}{v_{\zeta N}} + \frac{T_{\zeta N}}{v_{\zeta N}} \cdot \frac{v''}{v_{\zeta N}} \quad \text{(C.1-6a)} \\
\frac{T_{\zeta N}}{\omega_{\zeta N}} &= \frac{T_{\zeta N}}{\omega_{\zeta N}} \cdot \frac{v'}{\omega_{\zeta N}} + \frac{T_{\zeta N}}{\omega_{\zeta N}} \cdot \frac{\omega''}{\omega_{\zeta N}} \quad \text{(C.1-6b)}
\end{align*} \]

Our present concern is with the first term on the RHS of Eqs. (C.1-5a and b). A formalism which establishes the mathematical details of these two terms constitutes a specification of \( \frac{T_{\zeta N}}{v_{\zeta N}} \cdot \frac{v'}{v_{\zeta N}} \) as explicitly as is possible at the level of generality being employed in this document.

Eqs. (B.2-3 to 5) specify \( v_{\zeta N} \). Similarly, Eqs. (B.3-1 and 4) specify \( \omega_{\zeta N} \). Taking the time derivatives of these two sets of equations with respect to \( \zeta \), manipulating, and comparing with Eqs. (C.1-5) shows that

\[ \frac{\ddot{v}}{v'} = \frac{v_{\zeta r}}{v_{\zeta r}} + \frac{\omega_{\zeta r}}{v_{\zeta r}} + \frac{\omega_{\zeta r}}{v_{\zeta r}} + \frac{x_{\zeta r}}{v_{\zeta r}} + \frac{\ddot{v}}{v_{\zeta r}} \text{ (C.1-5)} \]
\[
\mathbf{v}' = \mathbf{v} + \sum_{H \in \{R, Z\}} \mathbf{\xi}_H \mathbf{a}_H + \sum_{B \in \{R, Z\}} \mathbf{\xi}_B \mathbf{a}_B - \mathbf{\rho}_B \quad (C.1-7)
\]

\[
\mathbf{v} = \mathbf{v}' - \sum_{B \in \{R, Z\}} \mathbf{\xi}_B \mathbf{a}_B \quad (C.1-8)
\]

The remaining task is to work out the details of the RHS of Eqs. (C.1-7) and (C.1-8). This will be done separately for the two equations in the following two sections.

### C.2 Translational Acceleration

Applying the theorem of Coriolis to the third term on the RHS of Eq. (C.1-7) produces

\[
\mathbf{v}_r = \mathbf{v} + \sum_{H \in \{R, Z\}} \mathbf{\xi}_H \mathbf{a}_H + \sum_{B \in \{R, Z\}} \mathbf{\xi}_B \mathbf{a}_B - \mathbf{\rho}_B \quad (C.2-1)
\]

the equation for \( \mathbf{v}_r \) was given as Eq. (B.2-4)

Now apply the theorem of Coriolis to the terms inside the two summations on the RHS of Eq. (C.1-7) to produce

\[
\mathbf{v}_r = \mathbf{v} + \sum_{H \in \{R, Z\}} \mathbf{\xi}_H \mathbf{a}_H + \sum_{B \in \{R, Z\}} \mathbf{\xi}_B \mathbf{a}_B - \mathbf{\rho}_B \quad (C.2-2a)
\]

\[
\mathbf{v}_r = \mathbf{v} + \sum_{H \in \{R, Z\}} \mathbf{\xi}_H \mathbf{a}_H + \sum_{B \in \{R, Z\}} \mathbf{\xi}_B \mathbf{a}_B - \mathbf{\rho}_B \quad (C.2-2b)
\]
To specify the first terms on the RHS of Eq. (C.2-2), first differentiate Eq. (B.2-5) to produce

\[
\frac{(\eta_I)}{x_\zeta \eta_I, q_H} \cdot q_H = \left\{ \frac{(\eta_I)}{x_\zeta \eta_O(\zeta) \eta_I, q_H} - x_\zeta \eta_O(\zeta) \frac{\theta}{\eta_O(\zeta)} \eta_I, q_H - x_\zeta \eta_O(\zeta) \frac{\theta}{\eta_O(\zeta)} \eta_I, q_H \right\} q_H
\]

(C.2-3a)

\[
\frac{(\beta_I)}{x_\zeta \beta_I, q_B} \cdot q_B = \left\{ \frac{(\beta_I)}{x_\zeta \beta_O \beta_I, q_B} - x_\zeta \beta_O \frac{\beta}{\beta_O \beta_I, q_B} - x_\zeta \beta_O \frac{\beta}{\beta_O \beta_I, q_B} \right\} q_B
\]

(C.2-3b)

Expressions for four of the six derivatives on the RHS of Eq. (C.2-3) can be written down immediately. To wit

\[
\frac{(\eta_I)}{x_\zeta \eta_O(\zeta) \eta_I, q_H} = \sum_i q_H \frac{x_i}{\eta_O(\zeta) \eta_I, q_H^i}
\]

(C.2-4a)

\[
\frac{(\eta_I)}{x_\zeta \beta_O \beta_I, q_B} = \sum_i q_B \frac{x_i}{\beta_O \beta_I, q_B^i}
\]

(C.2-4b)

\[
\frac{(\beta_I)}{x_\zeta \beta_O \beta_I, q_B} = \sum_i q_B \frac{x_i}{\beta_O \beta_I, q_B^i}
\]

(C.2-4c)

\[
\frac{(\beta_I)}{x_\zeta \beta_O \beta_I, q_B} = \sum_i q_B \frac{x_i}{\beta_O \beta_I, q_B^i}
\]

(C.2-4d)

It is noted that the x term on the right side of Eq. (C.2-4a) is a second derivative term; that is, the partial derivative is taken with respect to both the vector \(q_H^i\) and the scalar \(q_H^i\).
To obtain expressions for the remaining two time derivative terms in Eqs. (C.2-3) write

\[
-\dot{x} \zeta \eta_0(\zeta) \frac{\partial}{\partial \eta_0(\zeta)} \eta_l \frac{\partial}{\partial \eta_0(\zeta)} = - \dot{\eta}_l \eta_0(\zeta) \eta_1 \frac{\partial}{\partial \eta_0(\zeta)} \eta_0(\zeta)
\]

\[
= - \dot{\eta}_l \eta_0(\zeta) \eta_1 \left[ \frac{\partial}{\partial \eta_0(\zeta)} \eta_0(\zeta) + \frac{\partial}{\partial \eta_0(\zeta)} \eta_1 \right]
\]

(C.2-5a)

\[
-\dot{x} \zeta \beta_0(\zeta) \frac{\partial}{\partial \beta_0(\zeta)} \beta_I \sigma_B = - \dot{\beta}_I \beta_0(\zeta) \frac{\partial}{\partial \beta_0(\zeta)} \beta_0(\zeta)
\]

\[
= - \dot{\beta}_I \beta_0(\zeta) \beta_I \left[ \frac{\partial}{\partial \beta_0(\zeta)} \beta_0(\zeta) + \frac{\partial}{\partial \beta_0(\zeta)} \beta_I \right]
\]

(C.2-5b)

An equation for the complete translational acceleration term now can be obtained by collecting the pertinent derivative terms above and substituting them into Eq. (C.1-7). A variety of forms can be generated depending upon the manipulations which one chooses to employ. One form is the following

\[
\dot{T} \frac{\partial}{\partial \zeta} \nu = - \dot{\nu} \left[ \nu N + \dot{\nu} \frac{\partial}{\partial \zeta} \tau \right]
\]

\[
+ \sum_{\eta \in [R, Z]} \left\{ \dot{\eta}_0(\zeta) \eta_1 \dot{\eta}_0(\zeta) \eta_1 \frac{\partial}{\partial \eta_0(\zeta)} \eta_0(\zeta) \right\}
\]

\[
+ \sum_{j} \left\{ \zeta H \eta_0(\zeta) \eta_1 \frac{\partial}{\partial \eta_0(\zeta)} \eta_1 \right\}
\]

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\[ + \sum_{B \in \{R,Z\}} \left\{ \tilde{\omega}_B \tilde{\omega}_I \tilde{\omega}_B \tilde{\omega}_I \overline{\zeta}_B \right\} \]

\[ + \left\{ \sum_j q^j_B \overline{\zeta}_B I \overline{\beta}_I \right\} q^j_B + \left( \tilde{\omega}_B I N - \tilde{\omega}_B \right) \overline{\zeta}_B I \overline{\beta}_B \right\} P_B \]

where

\[ \overline{\zeta}_B I \overline{\beta}_B \overline{q}^j_B = \overline{\zeta}_B I \overline{\beta}_B q^j_B - \overline{\zeta}_B I \overline{\beta}_B \overline{q}^j_B \]

In developing Eq. (C.2-6), the following useful relation was used.

\[ \sum_{H \in \{R,Z\}} \tilde{\omega}_I H \overline{\zeta}_I H q^j_H + \sum_{B \in \{R,Z\}} \tilde{\omega}_B I \overline{\zeta}_B I \overline{\beta}_B \]

\[ = \sum_{H \in \{R,Z\}} \tilde{\omega}_I H \overline{\zeta}_I H \overline{\eta}_I (\zeta) + \sum_{B \in \{R,Z\}} \tilde{\omega}_B I \overline{\zeta}_B I \overline{\beta}_B \overline{\eta}_I (\zeta) \]

Eq. (C.2-8) can be verified by substituting the series expressions for \( \tilde{\omega}_I H \) and \( \tilde{\omega}_B I \) that will be listed later as Eqs. (C.2-10e) and (C.2-10f) into the left side, manipulating, and eventually producing the right side. The manipulations include interchanging the order of double summations, a task that can best be accomplished by the brute force procedure of writing out enough terms of the series to see how they must be regrouped. Introducing, temporarily, a unified notation for the bodies and joints is very helpful. However, the detailed mathematics of the proof is considered too laborious to be presented here.
The variables $q^i_B$ in Eq. (C.2-6) can be replaced by the $\rho_B$ variables employed more commonly in this document through the transformation

$$
\dot{q}^i_B = T^*_{\rho B} \rho_B
$$

(C.2-9)

The $\bar{\omega}$ terms which appear in Eq. (C.2-6) are the skew-symmetric dyadic representatives of corresponding angular velocity vectors $\bar{\omega}$. Equations for several of these angular velocity vectors were listed in Section B.3. The additional equations which are needed to specify the remaining ones are as follows:

$$
\bar{\omega}^\eta_{IN} = \bar{\omega}^\eta_{IR} + \bar{\omega}^\eta_{IN}
$$

(C.2-10a)

$$
\bar{\omega}^\beta_{IN} = \bar{\omega}^\beta_{IR} + \bar{\omega}^\beta_{IN}
$$

(C.2-10b)

$$
\bar{\omega}^\zeta_{I} = \sum_{K \in [H, Z]} \bar{\omega}^\zeta_{K} I + \sum_{A \in [H, Z]} \bar{\omega}^\alpha_{A} I
$$

(C.2-10c)

$$
\bar{\omega}^\zeta_{I} = \sum_{K \in [B, Z]} \bar{\omega}^\zeta_{K} I + \sum_{A \in [B, Z]} \bar{\omega}^\alpha_{A} I
$$

(C.2-10d)

$$
\bar{\omega}^\eta_{IR} = \sum_{K \in [R, H]} \bar{\omega}^\eta_{K} I + \sum_{A \in [R, H]} \bar{\omega}^\alpha_{A} I
$$

(C.2-10e)

$$
\bar{\omega}^\beta_{IR} = \sum_{K \in [R, B]} \bar{\omega}^\beta_{K} I + \sum_{A \in [R, B]} \bar{\omega}^\alpha_{A} I
$$

(C.2-10f)
C.3 Angular Acceleration

The detailed form of Eq. (C.1-8) will be developed in this section. Applying the theorem of Coriolis to the terms inside the two summations produces

\[
\frac{\ddot{\theta}_{\ell o}(\zeta)}{\dot{\theta}_{\eta o}(\zeta) n_I q_H} = \left[ \frac{\ddot{\theta}_{\ell o}(\zeta) n_I q_H}{\dot{\theta}_{\eta o}(\zeta) n_I q_H} - \omega_{\zeta \eta_I} \frac{\ddot{\theta}_{\ell o}(\zeta) n_I q_H}{\dot{\theta}_{\eta o}(\zeta) n_I q_H} \right] \dot{q}_H \tag{C.3-1a}
\]

\[
\frac{\ddot{\theta}_{\ell o}(\zeta) n_I q_B}{\dot{\theta}_{\eta o}(\zeta) n_I q_B} = \left[ \frac{\ddot{\theta}_{\ell o}(\zeta) n_I q_B}{\dot{\theta}_{\eta o}(\zeta) n_I q_B} - \omega_{\zeta \eta_I} \frac{\ddot{\theta}_{\ell o}(\zeta) n_I q_B}{\dot{\theta}_{\eta o}(\zeta) n_I q_B} \right] \dot{q}_B \tag{C.3-1b}
\]

Eqs. (C.3-1) can be manipulated into a variety of forms. One approach is to employ

\[
\frac{\ddot{\theta}_{\eta o}(\zeta) n_I q_H}{\dot{\theta}_{\eta o}(\zeta) n_I q_H} = \sum_{i} \left[ \frac{\ddot{q}_H}{q_H} \right] n_{\eta o}(\zeta) n_I q_H, q^i_H \tag{C.3-2a}
\]

\[
\frac{\ddot{\theta}_{\eta o}(\zeta) n_I q_B}{\dot{\theta}_{\eta o}(\zeta) n_I q_B} = \sum_{j} \left[ \frac{\ddot{q}_B}{q_B} \right] n_{\eta o}(\zeta) n_I q_B, q^j_B \tag{C.3-2b}
\]

and Eqs. (B.3-3) to produce

\[
\frac{\ddot{\theta}_{\eta o}(\zeta) n_I q_H}{\dot{\theta}_{\eta o}(\zeta) n_I q_H} = \left[ \sum_{i} \left[ \frac{\ddot{q}_H}{q_H} \right] n_{\eta o}(\zeta) n_I q_H, q^i_H \right] \dot{q}_H + \omega_{\eta o}(\zeta) n_I \omega_{\zeta \eta_I} \tag{C.3-3a}
\]

\[
\frac{\ddot{\theta}_{\eta o}(\zeta) n_I q_B}{\dot{\theta}_{\eta o}(\zeta) n_I q_B} = \left[ \sum_{j} \left[ \frac{\ddot{q}_B}{q_B} \right] n_{\eta o}(\zeta) n_I q_B, q^j_B \right] \dot{q}_B + \omega_{\eta o}(\zeta) \omega_{\zeta \eta_I} \tag{C.3-3b}
\]

The desired equation for the angular acceleration term now can be obtained by substituting Eq. (C.3-3) into (C.1-8) to yield
\[
\frac{\bar{\omega}}{\omega} \tilde{v}' = \tilde{\omega} \tilde{\zeta} \tilde{v}'
\]

\[
+ \sum_{H \in \{R, z\}} \left[ \sum_{i} \tilde{\theta} \eta_{O}(\zeta) \eta_{I} q_{H}^{i} q_{H}^{i} \right] q_{H}^{i} + \tilde{\omega} \eta_{O}(\zeta) \eta_{I} \tilde{\omega} \eta_{I}
\]

\[
+ \sum_{B \in \{R, z\}} \left[ \sum_{j} \tilde{\theta} \beta_{O}^{j} \beta_{B}^{j} q_{B}^{j} q_{B}^{j} \right] \beta_{B}^{j} + \tilde{\omega} \beta_{O}^{j} \tilde{\omega} \beta_{B}^{j}
\]

(C.3-4)
APPENDIX D

DERIVATION OF EQUATIONS FOR THE CONSTRAINT DERIVATIVE VECTOR

D.1  Introduction

This appendix develops a formalism to specify the vector $D'v'$ which appears in both the Lagrange multiplier and the partitioned velocity vector formulations.

Using the definition of $D'$ indicated by Eq. (6.6.3-1) we have

$$D'v' = C_{T_w}^{T_c} v'$$  \hspace{1cm} (D.1-1)

The above equation can be partitioned into a set whose members each pertain to either the active or the passive constraints at one of the cut joints $\Gamma$. Each of these equations can be written in the form

$$D_d^{T_d} v' = C_{T_w}^{T_d} T_{w_d} v'$$  \hspace{1cm} (D.1-2)

In the above expression, $d = p$ or $a$ to denote whether the constraints are passive or active.

For the present purposes, it is easier to work with a single member, Eq. (D.1-2), than with the full set of equations indicated as Eq. (D.1-1). However, the following more-detailed formulation turns out to be convenient.
\[ D_{\Gamma d} \dot{v} = C_{\Gamma d} \dot{w} \]  
(D.1-3a)

with

\[ w = \sum_{n} S_{n} N_{n} v_{n} = \sum_{n} w_{n} \gamma_{n}^{2} v_{n}^{2} \]  
(D.1-3b)

and

\[ \begin{align*}
\dot{v}_{n} &= \begin{pmatrix}
\dot{v}_{\gamma n} \\
\dot{w}_{\gamma n} \\
\dot{S}_{\gamma n}
\end{pmatrix} \\
\dot{w}_{n} &= \begin{pmatrix}
\dot{v}_{\gamma n} \\
\dot{w}_{\gamma n} \\
\dot{S}_{\gamma n}
\end{pmatrix}
\end{align*} \]  
(D.1-3c)

The \( S \) array in Eq. (D.1-3b) was specified in Subsection 5.2.3. The upper two \( T \) arrays in Eq. (D.1-3c) are defined by the material presented in Appendix B. Since all vectors \( \dot{S}_{\gamma n} \) are included in \( v' \), the lower \( T \) array in Eq. (D.1-3c) is merely an identity transformation.

The derivation procedure which will be used in this appendix to establish \( D_{\Gamma d} \dot{v} \) will consist of differentiating Eqs. (D.1-3) and discarding the \( D_{\Gamma d} \dot{w} \) terms. In order to avoid major blunders, it is necessary to pay careful attention to the coordinate frames which the time derivatives of Gibbs vectors are specified relative to. Recall, from Appendix C, that
\[ \dot{\mathbf{v}} \Delta (r) \]

[Reference (C.1-2)]

Therefore, when differentiating Eqs. (D.1-3), coordinate frame \( r \) should be used as the reference frame.

Differentiating Eqs. (D.1-3) as indicated above, manipulating and discarding the \( D' \mathbf{v}' \) terms produces

\[
D_{Td} \mathbf{v}' = \frac{\partial (r)}{\partial (r)} = [ C_{Td} \mathbf{w}_n + C_{Td} \sum_{\gamma} S_{\gamma} \gamma_{O N} \mathbf{v}_n \gamma_{\gamma} n ] \left\{ \begin{array}{c}
\omega_{\gamma} r \gamma_{n} + \frac{\mathbf{t}}{\gamma_{n}} \gamma_{n} \\
\omega_{\gamma} r \gamma_{n} + \frac{\mathbf{t}}{\gamma_{n}} \gamma_{n} \\
0
\end{array} \right\}
\]

\[ (D.1-4) \]

The final term in Eq. (D.1-4) was set up in such a manner that the material in Appendix C would be applicable. In generating it, the expressions

\[
\frac{(r)}{\mathbf{v}_n} = \frac{(Q)}{\mathbf{v}_n} + \omega_{\mathbf{v}_n} \mathbf{v}_n \quad (D.1-5a)
\]

\[
\frac{(r)}{\mathbf{v}_n} = \frac{(Q)}{\mathbf{v}_n} + \omega_{\mathbf{r}_n} \mathbf{r}_n \quad (D.1-5b)
\]

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and Eqs. (C.1-5) were used. The two $\dot{T}v'$ terms in Eq. (D.1-4) are the same ones which were specified in Appendix C. Therefore, they need not be considered further in the present appendix.

The remaining task is to work out the details of the first two terms on the RHS of Eq. (D.1-4). The two terms will be considered separately in the following two sections.

D.2 The First Term in Eq. (D.1-4)

Introducing Eqs. (5.4.2-b and/or 8) into the first term on the RHS of Eq. (D.1-4) produces

$$
C_{\text{td}} w_I = C_{\text{td}} y_I T U_{\text{tr}} + C_1 \dot{I} d T U_{\text{tr}} \quad (D.2-1)
$$

The first term on the RHS of Eq. (D.2-1) can be disposed of quickly. Differentiating Eq. (5.2.2-4) for $U$ and introducing the result into the first term yields

$$
C_{\text{td}} y_I T U_{\text{tr}} = - C_{\text{td}} y_I U_{\text{tr}} \left[ \begin{array}{cc}
\omega_{y_I^T} & 0 \\
0 & \omega_{y_I^T}
\end{array} \right] \quad (D.2-2)
$$

The remaining task in this subsection is to specify the $\dot{C}$ matrix in the final term of Eq. (D.2-1). The passive ($d = p$) and active ($d = a$) cases will be investigated separately, with the passive one being considered first.
As was discussed in Subsection 5.4.2,

\[ C_{\Gamma P} = C_{\Gamma P} \left( Y_{\Gamma} \right) \]  \hspace{1cm} (D.2-3)

with \( Y_{\Gamma} \) being the dual-displacement vector specified in Eq. (5.1.1-2). Since \( C_{\Gamma P} \) is a matrix of scalers, coordinate frame questions do not arise when it is differentiated. Therefore

\[ \tilde{C}_{\Gamma P} = \sum C_{\Gamma P, k} \tilde{Y}_{\Gamma} \]  \hspace{1cm} (D.3-4)

where \( \tilde{Y}_{\Gamma} \) is the kth scaler element of \( \tilde{Y}_{\Gamma} \). The \( \tilde{Y}_{\Gamma} \) terms in Eq. (D.2-4) can be replaced by the \( \tilde{w}_{\Gamma} \) vector used more commonly in this document through the transformation indicated in Eq. (5.2.1-4).

Now consider the active constraint case. The equations for \( C_{\Gamma a} \) can be obtained by differentiating Eq. (5.4.5-6). The result is

\[ \tilde{C}_{\Gamma a} = -C_{\Gamma a} \tilde{w}_{\Gamma} \left( Y_{\Gamma} \right) \tilde{w}_{\Gamma} \left( Y_{\Gamma} \right) + \frac{\partial_{\Gamma}}{\partial_{\Gamma}} \left[ \left( \begin{array}{c} p_{\Gamma} \left( Y_{\Gamma} \right) \left( Y_{\Gamma} \right) \\ \tilde{w}_{\Gamma} \left( Y_{\Gamma} \right) \tilde{w}_{\Gamma} \left( Y_{\Gamma} \right) \end{array} \right) \right]^{-1} \]  \hspace{1cm} (D.2-5a)

The matrix in Eq. (D.2-5a) is a function of \( Y_{\Gamma} \). Therefore

\[ \tilde{w}_{\Gamma} \left( Y_{\Gamma} \right) = \sum \left[ T \tilde{w}_{\Gamma} \left( Y_{\Gamma} \right) \tilde{w}_{\Gamma} \left( Y_{\Gamma} \right) \right] \]  \hspace{1cm} (D.2-5b)

In summary, the equations which specify the first term on the RHS of Eq. (D.1-4) are Eq. (D.2-1, 2, 4 and 5).
D.3 The Second Term in Eq. (D.1-4)

As noted previously, the two \( S \) arrays (one for \( n = 0 \) and one for \( n = 1 \)) in the second term on the RHS of Eq. (D.1-4) were specified in Subsections 5.2.3. In the present work, there is no need to separate \( \frac{v_Z}{\gamma_n} \) into its passive and active subvectors. Therefore, the first form of Eq. (5.2.3-8) is applicable; the subscript labels \( i = 1, 2 \) which denote passive and active elements are not needed; and the \( S \) arrays in Eq. (5.2.3-11) are, in fact, \( T \) ones.

Introducing Eqs. (5.2-9 to 11) into the second term on the RHS of Eq. (D.1-4), performing the indicated differentiation, and utilizing Eqs. (5.2.3-4 and 7) produces

\[
C_{T_d} \sum_{n} \epsilon_{n} S_{\omega_{\gamma}, n} \frac{v_Z}{\gamma_n} \gamma_n
\]

\[
= C_{T_d} \left\{ \begin{array}{c}
- \omega_{\gamma, n} \\
0
\end{array} \right\} + \sum_{n} \epsilon_{n} \left\{ \begin{array}{c}
I \\
0
\end{array} \right\}
\]

\[
= C_{T_d} \left\{ \begin{array}{c}
- \omega_{\gamma, n} \\
0
\end{array} \right\} + \sum_{n} \epsilon_{n} \left\{ \begin{array}{c}
I \\
0
\end{array} \right\}
\]

\[
= C_{T_d} \left\{ \begin{array}{c}
- \frac{(r)}{x_{\gamma, n}} \\
\frac{(r)}{x_{\gamma, n}}
\end{array} \right\} + \sum_{n} \epsilon_{n} \left\{ \begin{array}{c}
I \\
0
\end{array} \right\}
\]

\[
\frac{(r)}{x_{\gamma, n}} \frac{v_Z}{\gamma_n} \gamma_n
\]

where

\[
\frac{(r)}{x_{\gamma, n}} \frac{v_Z}{\gamma_n} \gamma_n = \omega_{\gamma, n} \frac{(r)}{x_{\gamma, n}} \frac{v_Z}{\gamma_n} \gamma_n + \sum_{n} \epsilon_{n} \frac{(r)}{x_{\gamma, n}} \frac{v_Z}{\gamma_n} \gamma_n
\]

\[
(D.3-2a)
\]
\[ (r) \quad \frac{\partial}{\partial x_n} v_n^{\gamma} = \Sigma q^j_n \frac{\partial}{\partial q^j_n} v_n^{\gamma} \quad (D.3-2b) \]

The remaining task is to establish expressions to specify the four time-derivatives in the above equations. It is an exercise to obtain the following results for these derivatives:

\[ (r) \quad \frac{\partial}{\partial x_n} v_n^{\gamma} = \frac{\partial}{\partial x_n} v_n^{\gamma} + \frac{\partial}{\partial T_n} v_n^{\gamma} \quad (D.3-3) \]

\[ (r) \quad x_n^{\gamma} = x_n^{\gamma} - x_n^{\gamma} + \Sigma q^j_n \frac{\partial}{\partial q^j_n} g_n^{\gamma} \quad (D.3-4) \]

\[ (r) \quad \frac{\partial}{\partial x_n} q^j_n = \frac{\partial}{\partial x_n} q^j_n + \frac{\partial}{\partial q^j_n} g_n^{\gamma} \quad (D.3-5) \]

\[ (r) \quad \frac{\partial}{\partial \theta} q^j_n = \frac{\partial}{\partial \theta} q^j_n + \frac{\partial}{\partial q^j_n} g_n^{\gamma} \quad (D.3-6) \]

The T term in Eq. (D.3-3) was specified in Eq. (5.2.3-4).
APPENDIX E

SUPPLEMENTARY ANALYSIS OF THE JOINT CONSTRAINT FORMULATION

E.1 Introduction

This appendix presents supplementary material on the joint constraint formulation. It emphasizes joint constraint forces and Lagrange multipliers. Section E.2 considers constraints at a single joint. Section E.3 then applies this material to the composite set of joints. The work is oriented largely toward developing Eq. (6.8.1-1) and to verifying that constraint forces at reconnected joints drop out of the dynamical equations for $\Sigma'$.

E.2 Single Joint

Start with the joint A constraint equation which was listed as Eq. (5.4.2-10). Let $c_A^i$ denote the ith scaler constraint at A; this constraint can be either a passive one or an active one. The row of Eq. (5.4.2-10) which pertains to $c_A^i$ can be written in the form

$$c_A^i = c_A^T w_A - b_A^i \cdot \delta_{aa} = 0 \quad (E.2-1)$$

Define the dual-vectors

$$c_A^i = \begin{pmatrix} i \\ c_A^i \\ F \end{pmatrix}$$

$$L = \begin{pmatrix} c_A^i \\ F \end{pmatrix}$$

(E.2-2)
and

\[
\delta \mathbf{w}_A = \begin{pmatrix}
\delta \mathbf{x}_A \\
\delta \mathbf{a}_A \\
\delta \mathbf{\theta}_A
\end{pmatrix}
\]

(E.2-3)

The \( F \) and \( G \) terms above are the force and moment vectors generated on \( Z_{\alpha_O} \) by constraint \( i \). The \( \delta \mathbf{w} \) term is the virtual displacement vector which corresponds to \( \mathbf{w}_A \).

The present study is limited to the usual type of constraints for which the constraint force does no virtual work in any virtual displacement which satisfies that constraint. In mathematical terms, this states that

\[
\mathbf{L}_{\alpha_O}^T \cdot \delta \mathbf{w}_A = 0
\]

(E.2-4)

for all \( \delta \mathbf{w}_A \) for which

\[
\mathbf{c}_A^T \cdot \delta \mathbf{w}_A = 0
\]

(E.2-5)

It can be proven that Eqs. (E.2-4) and (E.2-5) imply that the \( L \) and \( C \) terms in these two equations are linearly related. Let the constant of proportionality be denoted as \( \lambda^i_A \). Then

\[
\mathbf{L}_{\alpha_O}^T = \mathbf{c}_A^T \lambda^i_A
\]

(E.2-6)
Now define $L^C_{\alpha O P}$ and $L^C_{\alpha O a}$ to be the force vectors on $Z_{\alpha O}$ generated by all the passive and active constraints on A. That is

$$L^C_{\alpha O P} = \sum_{\text{passive constraints on A}} L^C_{\alpha O}$$

(E.2-7a)

$$L^C_{\alpha O a} = \sum_{\text{active constraints on A}} L^C_{\alpha O}$$

(E.2-7b)

Inserting Eq. (E.2-6) into (E.2-7) produces

$$L^C_{\alpha O P} = C^T_{\alpha P} \lambda_{\alpha P}$$

(E.2-8a)

$$L^C_{\alpha O a} = C^T_{\alpha A} \lambda_{\alpha A}$$

(E.2-8b)

The $C$ terms above are the constraint operators which were defined in Chapter 5. $\lambda_{\alpha P}$ and $\lambda_{\alpha A}$ are column matrices whose elements are the $\lambda^i_A$.

Defining $L^C_{\alpha O}$ as the total constraint force vector on $Z_{\alpha O}$ due to all constraints on A, one obtains also

$$L^C_{\alpha O} = L^C_{\alpha O P} + L^C_{\alpha O a} = \begin{bmatrix} C^T_{\alpha P} & C^T_{\alpha A} \end{bmatrix} \begin{bmatrix} \lambda_{\alpha P} \\ \lambda_{\alpha A} \end{bmatrix} = C^T_A \lambda_A$$

(E.2-9)

As a final step in this subsection, three important relations that will be needed later will be developed. First, premultiply Eq. (E.2-8a) by $T^T_A$ and call upon Eq. (5.4.4-3a) to produce

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\[ T_{\omega_{A-A}^p}^T \cdot L_{\omega_{A-A}^p}^{C} = 0 \]  \hspace{1cm} (E.2-10)

Also, premultiply Eq. (E.2-8b) by \( T^T \) similarly, introduce coordinate frame resolution on the RHS via Eqs. (5.2.2-7) and (5.4.2-8), employ Eq. (5.4.5-5), and partition the \( T^T \) term on the RHS into its \( \omega_{Af} \) and \( \omega_{Aa} \) elements to generate

\[ S_{\omega_{A-Af}^a}^T \cdot L_{\omega_{A-Af}^a}^{C} = 0 \]  \hspace{1cm} (E.2-11a)

\[ S_{\omega_{A-Aa}^a}^T \cdot L_{\omega_{A-Aa}^a}^{C} = \lambda_{\omega_{A-Aa}^a} \]  \hspace{1cm} (E.2-11b)

E.3 The System of Joints

The material developed in the preceding section now will be applied to the composite system of joints.

Assume that an ordering scheme has been defined for the joints. Stack the \( L_{\omega_{A-A}^p}^{C} \) and \( L_{\omega_{A-A}^a}^{C} \) separately according to this scheme to generate a pair of vectors that will be denoted as \( L_{\omega_{A-A}^p}^{C} \) and \( L_{\omega_{A-A}^a}^{C} \).

\[ L_{\omega_{A-A}^p}^{C} = \{ L_{\omega_{A-A}^p}^{C} \} \text{ joints with passive constraints} \]  \hspace{1cm} (E.3-1a)

\[ L_{\omega_{A-A}^a}^{C} = \{ L_{\omega_{A-A}^a}^{C} \} \text{ joints with active constraints} \]  \hspace{1cm} (E.3-1b)

Define also

\[ L_{\omega_{A-A}}^{C} = \left\{ \begin{array}{c} L_{\omega_{A-A}^p}^{C} \\ L_{\omega_{A-A}^a}^{C} \end{array} \right\} \]  \hspace{1cm} (E.3-2)
We now wish to prove that

$$f_C = s^T_w v_1 \cdot L_C$$  \hspace{1cm} (E.3-3)

where $f_C$ is the generalized force on $\Sigma$ as indicated in Eq. (4.5-2a). The vector $w$ was defined in Eq. (5.5-6).

The proof will utilize the principle of virtual work. Regard all joints of the system as being cut. Introduce an arbitrary virtual displacement of the bodies. Equation (A.3-11) shows that the virtual work $\delta w^C_Z$ done by any $Z$ due to all the joint constraint forces on it will be

$$\delta w^C_Z = \delta w^C_{v Z} \cdot f^C_Z$$  \hspace{1cm} (E.3-4)

It follows from Eq. (E.3-4) that the virtual work $\delta w^C$ done by all the $Z$ due to the joint constraint forces will be

$$\delta w^C = \sum_Z \delta w^C_Z = \delta w^C_{v_1} \cdot f_C$$  \hspace{1cm} (E.3-5)

However, $\delta w^C$ also can be expressed as a function of the vectors $L_C^\alpha_n$. To wit

$$\delta w^C = \sum_A \sum_n \delta w^C_{w_n} \cdot L_C^\alpha_n$$  \hspace{1cm} (E.3-6)

where $w_n^\alpha n$ was defined in Eq. (4.2-18). Introducing Eq. (5.3-2) and the relation

$$\delta w_{w_n^\alpha n} = \delta w_{w_n^\alpha} + s_{w_n^\alpha} w_{w_n^\alpha n} w^C_{w_n^\alpha n}$$  \hspace{1cm} (E.3-7)

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into Eq. (E.3-6) produces

\[ \delta W^C = \sum_A \delta \pi^A_w \cdot L^C_A = \delta \pi^w \cdot L^C \]  

(E.3-8)

Equating Eq. (E.3-5) to (E.3-8) and employing

\[ \delta \pi^w = \sum_{v_1} \delta \pi^w_{v_1} \]  

(E.3-9)

yields

\[ \delta \pi^w_{v_1} \cdot \left[ f^C - S^T_w L^C \right] = 0 \]  

(E.3-10)

But \( \delta \pi^w_{v_1} \) is arbitrary, and therefore Eq. (E.3-10) implies (E.3-3).

It will now be verified that constraint forces at reconnected joints make no contribution to the generalized constraint force vector \( f'_C \) on \( \Sigma' \).

Start by introducing Eq. (E.3-3) into Eq. (6.7-5) to obtain

\[ f'_C = S^T_w L^C \]  

(E.3-11)

Eq. (E.3-11) can be written in the form

\[ f'_C = S^T_w L^C + \sum \text{reconnected joints with passive constraints} + \sum \text{reconnected joints with active constraints} \]  

(E.3-12)
The first term on the RHS above is the contribution from the cut joints. \( w_c \) is formed by removing the reconnected joint velocity vectors \( w_b \) from \( w \). The vector \( L_C^C \) is formed by removing the reconnected joint force vectors \( L_B^C \) from \( L_C^C \).

Using an argument similar to that employed in the second paragraph of Subsection 6.6.2, it can be deduced that the only nonzero element of \( S_T \) is \( S_T^T v_1 \). However, Eqs. (E.2-10) and (E.2-11a) show that post multiplying this term by \( L_C^C \); \( m = (p, a) \) produces 0. Therefore, the final two terms on the RHS of Eq. (E.3-12) are 0. Hence, as asserted, the constraint forces at reconnected joints do not contribute to \( f'_C \). Thus

\[
 f'_C = S_T^T v_1 \cdot L_C^C \quad \text{(E.3-13)}
\]

The final operation in this appendix will be to verify Eq. (6.8.1-1). To begin, we work first with Eq. (E.2-8a). Stack the \( L_C^C \) of all cut joints with passive constraints to form

\[
 L_C^C = C_T \lambda \quad \text{(E.3-14a)}
\]

Do the same thing with Eq. (E.2-8b) to produce

\[
 L_C^C = C_T \lambda \quad \text{(E.3-14b)}
\]

Also, stack the above pair of equations to produce \( L_C^C \).
\[
\mathbf{L}_c^C = \begin{pmatrix} L_{cp}^C \\ L_{ca}^C \end{pmatrix} = \begin{bmatrix} c_{cp} & 0 \\ 0 & c_{ca} \end{bmatrix}^T \begin{pmatrix} \lambda_p \\ \lambda_a \end{pmatrix} = C_c^T \lambda \quad (E.3-15)
\]

Finally, insert (E.3-15) into (E.3-13) to obtain the desired expression, Eq. (6.8.1-1)

\[
\mathbf{f}_c' = (C_c S_{w,v}')^T \lambda = D_1^T \lambda \quad (6.8.1-1)
\]
APPENDIX F

EXTENSION OF THE FORMULATION TO INCLUDE VARIABLE-MASS BODIES

F.1 Introduction

The kinetics equations which were developed in Appendix A for a single body $Z$ assume that the mass of $Z$ is invariant. However, the sample problem in Chapter 8 included bodies whose mass was not invariant. Specifically, the so-called "drum" included not only the drum itself, but also the portions of the cables which, at instantaneous time $t$, were still wrapped around the drum. Similarly, the so-called "cables" included only the portions of the cables which had already been unwrapped from the drum at time $t$. Thus, mass flowed continuously from the "drum" to the "cables" as deployment proceeded.

This appendix considers briefly the extensions which must be made to the formulation in Appendix A so that it encompasses the mass flow phenomenon encountered in the sample problem. It is emphasized that we consider only those aspects of the mass flow phenomenon which are necessary in order to accomplish this. Neither the drum nor the cables in the sample problem included passive generalized coordinates $q_Z$. Therefore, the present appendix will not present mathematics for the effect of mass flow on the internal kinetics ($\dot{s}_Z$) of the individual bodies. What is developed are the expressions for the terms which must be added to the translational and angular kinetics equations of a single body which arise due to mass flow into or out of that body. The fact that we do not consider the internal kinetics equations makes it possible and expedient to use Newton-Euler methods.
Texts and papers on fluid dynamics (48), the mathematics of rocket propulsion (49)-(52), and the so-called jet damping phenomenon (52)-(55) are the main background references for the material presented in this appendix.

F.2 Development

For the present purpose, it will suffice to deal with only a single body, since the extension to multibody cases is straightforward.

Let \( Z \) be a body. Let there be mass flow into or out of \( Z \) across its surface. In the terminology of fluid dynamics, the region of space occupied by \( Z \) is called the control volume. Let \( V(t) \) denote the control volume of \( Z \) at some time \( t \). Let \( S(t) \) be the surface area of \( Z \) at time \( t \). Let \( S_M(t) \) be the portion of \( S(t) \) across which mass flows at time \( t \).

Introduce the term "instantaneous material system" (\( M \)) to denote the mass contained in \( Z \). Thus \( M(t) \) denotes the mass within \( V(t) \) at time \( t \). Note that by \( M(t) \) we mean the mass itself, not its magnitude. In particular, the symbol \( M(t) \) will be used with integrals to denote integration over all the mass particles in \( Z \) at time \( t \).

Let \( a \) denote some intensive scalar, vector, or tensor property of the mass particles of \( Z \). For example, \( a \) could be mass density. Our interest below will be in the cases in which \( a \) is linear momentum per unit mass (\( \bar{z} \)) and angular momentum per unit mass (\( \bar{\text{h}} \)). Let \( A \) be the extensive property of \( Z \) which corresponds to \( a \). That is,

\[
A(t) = \int a(P,t) \, dm / M(t) \tag{F.1}
\]

where \( P \) denotes a mass particle of \( Z \).
The work will utilize Reynolds' transport theorem which is well known in fluid dynamics. The equation which Reynolds' transport theorem specifies can be written in the form

\[ D_N^A(t) = \dot{\alpha}_N^A(t) - \int_{S_M} a(P,t) \dot{m}(P',t) dS \]  \hspace{1cm} (F.2)

The subscript label N above indicates that the time derivative is specified with respect to coordinate frame N. The label N is not needed when A is a scaler. The term \( D_N^A(t) \) is the rate of change of the A possessed by the instantaneous material system, \( M(t) \), at time \( t \). The term \( \dot{\alpha}_N^A(t) \) is the rate of change of the A inside the control volume, \( V(t) \), at time \( t \). It will be convenient below to use the notation \( \dot{A}(t) \) in place of \( \dot{\alpha}_N^A(t) \).

The second term on the right side of Eq. (F.2) is the rate of change of A due to the influx and efflux of mass across \( S_M \). The symbol \( P' \) is intended to denote the point on \( S_M \) instantaneously occupied by particle \( P \) which is crossing \( S_M \). The term \( \dot{m} \) is negative for mass flow out.

The reader is referred to texts on continuum mechanics or fluid dynamics, such as Ref. 48, for a derivation of Reynolds' transport theorem and for a fuller explanation of the three terms which appear in it. It is noted that the common derivation of the theorem assumes that the control volume \( V \) is invariant in shape and fixed in position and orientation relative to the reference frame \( N \). These two assumptions greatly simplify the mathematics of the derivation. It can be deduced, however, that neither is necessary for the validity of the final equation. This is significant for the material which is presented below.
Now let $\vec{l}$ denote linear momentum density. Let $\vec{h}$ denote angular momentum density about the origin $O_\zeta$ of $Z$'s dynamic reference frame $\zeta$. Let the reference frame $N$ be an inertial frame. Then

$$\vec{l}(P,t) = \vec{v}_{PN}(t) \quad (F.3)$$

$$\vec{h}(P,t) = \vec{\times}_{P_\zeta}(t)\vec{v}_{P_\zeta}(t) \quad (F.4)$$

$\vec{v}_{PN}$ and $\vec{v}_{P_\zeta}$ above are the velocities of mass particle $P$ relative to $O_N$ and $O_\zeta$ respectively. The observation frame for both these velocities is frame $N$. Let $\overline{L}$ and $\overline{h}$ be the extensive properties which correspond to $\vec{l}$ and $\vec{h}$. Then

$$\overline{L}(t) = \int \vec{l}(P,t)\,dm \quad (F.5) \quad \frac{M(t)}{M(t)}$$

$$\overline{h}(t) = \int \vec{h}(P,t)\,dm \quad (F.6) \quad \frac{M(t)}{M(t)}$$

Writing Eq. (F.2) in terms of $\overline{L}$ and $\overline{h}$ and making a trivial rearrangement produces

$$\dot{\overline{L}} = D_N \overline{L} + \int \vec{v}_{PN} \cdot \vec{m} \, dS \quad (F.7)$$

$$\dot{\overline{h}} = D_N \overline{h} + \int \vec{\times}_{P_\zeta}\vec{v}_{P_\zeta} \cdot \vec{m} \, dS \quad (F.8)$$

It is crucial to realize that the time derivatives which appear in the usual Newton-Euler kinetics equations pertain to the instantaneous material system $M(t)$. Thus

$$D_N \overline{L} = \overline{F} \quad (F.9)$$

$$D_N \overline{h} = \overline{G} - m \times_{P_\zeta} \vec{v}_{P_\zeta} \overline{C_\zeta} \overline{C_N} \quad (F.10)$$
In Eq. (F.10) the term \( \overline{G'} \) is the external torque on \( Z \) about \( O' \), and \( \overline{x_{C' \zeta}} \) is the skew symmetric form of the vector from \( O' \) to the center of mass, \( C' \), of \( Z \). \( m \) is the instantaneous value of \( M \), and \( \overline{F} \) is the external force of \( Z \). In problems where \( O' \) is not chosen to coincide with \( C' \), it frequently is convenient to use the following expression

\[
D_N \overline{H} = \overline{G'_{C'}} + m \overline{\times}_{C' \zeta} \overline{v}_{C' \zeta} \tag{F.11}
\]

as an alternative to Eq. (F.10).

The basic kinetics equations for the translational and angular motion of \( Z \) are obtained by inserting Eqs. (F.9) and (F.11) into (F.7) and (F.8).

\[
\overline{\dot{L}} = \overline{F} + \int \overline{v}_{pN} m \, dS \tag{F.12}
\]

\[
\overline{\dot{H}} = \overline{G'_{C'}} + m \overline{\times}_{C' \zeta} \overline{v}_{C' \zeta} + \int \overline{\times}_{p \zeta} \overline{v}_{p \zeta} m \, dS \tag{F.13}
\]

Equations (F.12) and (F.13) can be converted into the more common and more explicit form by differentiating the equations for \( \overline{L} \) and \( \overline{H} \),

\[
\overline{L} = \int \overline{v}_{pN} \, dm = m \overline{v}_{CN} \tag{F.14}
\]

\[
\overline{H} = \int \overline{x}_{p \zeta} \overline{v}_{p \zeta} \, dm = J \overline{\omega}_{CN} + \overline{H}_{Z \zeta} \tag{F.15}
\]

employing the theorem of Coriolis, and inserting into the LHS. These more explicit equations, however, will not be written out, since our current interest is mainly in the final terms on the RHS of Eqs. (F.12) and (F.13). In Eq. (F.15), \( J \) is the usual inertia dyadic and \( \overline{H}_{Z \zeta} \) is the angular momentum of \( Z \) relative to frame \( \zeta \).
\[ J = - \int \frac{\ddot{x}_{p\zeta}}{p_{p\zeta}} \frac{\dot{x}_{p\zeta}}{p_{p\zeta}} \, dm \quad (F.16) \]

\[ \frac{\Delta_{Z\zeta}}{Z\zeta} = \int \frac{\dot{x}}{p_{p\zeta}} \frac{\ddot{x}}{p_{p\zeta}} \, dm \quad (F.17) \]

In Eq. (F.17), the symbol * denotes differentiation with respect to frame \( \zeta \).

The final terms on the RHS of Eqs. (F.12) and (F.13) are the ones which should be added to the linear and angular momentum kinetics equations for \( Z \) to account for the effects of mass flow across the boundaries of \( Z \). These two terms can be written in the following more explicit form through a manipulation which includes the use of the theorem of Coriolis.

\[ \int \frac{-v_{PN} \, m \, ds}{m_{Z} \, v_{CN}} + \int \frac{-v_{PN} \, m \, ds}{m_{Z} \, v_{CN}} \quad (F.18) \]

\[ \int \frac{-v_{PN} \, m \, ds}{m_{Z} \, v_{CN}} + \int \frac{-v_{PN} \, m \, ds}{m_{Z} \, v_{CN}} \quad (F.19) \]

the term \( \dot{m}_{Z} \) above is the rate of change of \( M \)

\[ \dot{m}_{Z} = \int \dot{m} \, ds \quad (F.20) \]
NOMENCLATURE

The main symbols which are used in the document are defined in this section. A sizable number of minor symbols are omitted for the sake of brevity. Additional information on nomenclature and notational techniques is presented in Chapter 3.

\[ B_{\mu \nu} \]

The velocity command matrix for \( c_{\mu \nu} \) which uses the \( \omega_A \) as the inputs; see Eqs. (5.4.2-7) or (5.5-5) for example*

\[ b^T_A \]

The ith row of \( B_A \); see Eq. (D.2-1)**

\[ C_{\mu \nu} \]

The velocity constraint operator for \( c_{\mu \nu} \) which uses the \( w_A \) as the inputs; see Section 5.4.2

\[ (a_I) \]

The velocity constraint matrix for \( c_{\mu \nu} \) which uses the \( \omega_A \) as the inputs; see Section 5.4.2.

\[ c \]

Column matrix of all constraints at all joints; see Eq. (5.5-5)

*\( \mu \) and \( \nu \) are not symbols which are used in the text. Rather, they are general-purpose symbols that are employed in this section for brevity.

**Most of the symbols with A or \( a \) subscripts have counterparts, not listed in this section, with B or b and \( \Gamma \) or \( \gamma \) subscripts.
\( \mathbf{c}_A \)  
Column matrix of all constraints at joint A; see Eq. (5.4.2-10)

\( \mathbf{c}_A^i \)  
The ith scalar constraint at joint A; see Eq. (D.2-1)

\( \mathbf{c}_A^T \)  
The ith row of \( \mathbf{c}_A \); see Eq. (D.2-1)

\( \mathbf{C}_{Ap}^p \), \( \mathbf{C}_{Ap}^a \)  
Column matrices of all passive (p) and active (a) constraints respectively at joint A; see Eqs. (5.4.2-5 and 5.4.2-7)

\( \mathbf{H}_{Ap}^p \), \( \mathbf{H}_{Ap}^{NH} \)  
Column matrices of the holonomic (H) and nonholonomic (NH) passive constraints respectively at A; see Eqs. (5.4.2-1 and 5.4.2-2)

\( \mathbf{C}_c \)  
Column matrix of all constraints at all cut joints of system \( \Sigma' \); see Eq. (6.6.2-13)

\( \mathbf{C}_c^H \), \( \mathbf{C}_c^{NH} \)  
Column matrices of the holonomic (H) and nonholonomic (NH) constraints at all cut joints of \( \Sigma' \); see Subsections (6.9.4 and 6.9.4)

\( \mathbf{C}_p \), \( \mathbf{C}_a \)  
Column matrices of all passive (p) and active (a) constraints respectively at all joints; see Eqs. (5.5-4) and (6.6.2-1 and 6.6.2-6)

\( \mathbf{C}_{cp}^p \), \( \mathbf{C}_{cp}^a \)  
Column matrices of all passive (p) and active (a) constraints at all cut joints of system \( \Sigma' \); see Eqs. (6.6.2-5 and 6.6.2-12)

\( \mathbf{D}' \)  
The constraint operator for \( \mathbf{c}_c \) which uses \( \mathbf{v}' \) as the input; see Eq. (6.6.3-1)
$D'_1, D'_2$ \{ The subblocks of $D'$ which use $v'_1, a_{n_1}$, and $v_2$ respectively as the input; see Eq. (6.6.3-1) \\
D'_{1\Sigma}$ \ A matrix comprised of the columns of $D'_1$ which are not included is $D'_{1\Sigma}$; see Eq. (6.9.2-1) \\
D'_{1s}$ \ A nonsingular matrix comprised of columns of $D'$; see Eq. (6.9.2-1) \\
D'_{1d}$ \ The row block of $D'$ which pertains to the passive ($d = p$) or active ($d = a$) constraints at a cut joint $\Gamma$ of $\mathcal{L}'$; see Eq. (E.1-3) \\
$E_{\mu}$ \ Triad of unit vectors along the axes of frame $\mu$; see Chapter 3 \\
$e_{\mu}^i$ \ Unit vector along axis $i$ of frame $\mu$ \\
$\overline{F}_{\alpha n}$ \ The Gibbs force vector applied by joint $A$ on body $Z_{\alpha n}$; see Section 5.3. \\
$\overline{F}_{\alpha n Z}$ \ The Gibbs force vector applied on $Z$ by a joint $A$ whose n-end is attached to body $Z$; see Section A.2 \\
$\overline{C}_{\alpha n Z}$, $\overline{K}_{\alpha n Z}$ \ The constraint (C) and known (K) portions respectively of $\overline{F}_{\alpha n Z}$; see Eqs. (4.3-4) \\
$\overline{f}'_{K}$ \ Generalized force on $\mathcal{L}'$ due to all known forces on the bodies; see Eq. (6.7-6). \\
$\overline{f}_{D Z}$ \ Generalized dynamic (D) force on body $Z$; see Eqs. (4.3-1) and (A.3-18)
\[ \mathbf{f}_{EIZ} \] Generalized force on body Z due to all true external (E) and internal (I) forces except those at the joints; see Eqs. (4.3-1) and (A.3-12)

\[ \mathbf{f}_{CJZ}, \mathbf{f}_{KJZ} \] Generalized forces on body Z due to constraint (C) and known (K) reactions at Z's joints J_Z; see Eqs. (4.3-1) and (4.3-2)

\[ \begin{cases} \mathbf{f}_{D}, & \mathbf{f}_{EI} \\ \mathbf{f}_{D}, & \mathbf{f}_{KJ} \end{cases} \] Generalized forces on system Σ; see Section 4.5

\[ \begin{cases} \mathbf{f}_{C}, & \mathbf{f}_{EI} \\ \mathbf{f}_{C}, & \mathbf{f}_{KJ} \end{cases} \] Generalized forces on system Σ'; see Eq. (6.7-4)

\[ \mathbf{G}_{\alpha n} \] The Gibbs moment vector applied by joint A, at point O_{\alpha n}, on body Z_{\alpha n}; see Section 5.3

\[ \mathbf{G}_{\alpha nZ} \] The Gibbs moment vector applied by joint A, on body Z, at point O_{\alpha n}, by a joint A whose n-end is attached to Z; see Section A.2

\[ \mathbf{G}_{\alpha nZ}^{C}, \mathbf{G}_{\alpha nZ}^{K} \] The constraint (C) and known (K) portions respectively of \[ \mathbf{G}_{\alpha nZ} \]; see Eq. (4.3-5)

\[ \mathbf{g} \] The velocity command vector in the constraint equation for system Σ'; see Eqs. (6.6.3-2 and 6.6.3-3)

\[ \begin{cases} \mathbf{g}_{\nu_22}, & \mathbf{g}_{\nu_2} \\ \mathbf{g}_{\nu_2}, & \mathbf{g}_{\nu_2} \end{cases} \] The vectors which specify \[ \dot{\nu}_22, \dot{\nu}_2, \] and \[ \dot{\nu}_CS \]; see Eqs. (4.4-4), (4.5-2d), and (4.4-5)
I

(1) The identity operator or matrix

(2) The input end of a joint

$L^C_p$ Vector formed by stacking $L^C_\alpha$ and $L^C_\alpha$; see Eq. (D.3-2)

$L^K$ Vector formed by stacking $L^K_{a_0}$

$L^C_c$ Vector formed by stacking $L^C_{cp}$ and $L^C_{ca}$; see Eq. (D.3-15)

$L^C_{cp}, L^C_{ca}$ Vectors formed by stacking $L^C_{a_0p}$ and $L^C_{a_0a}$ respectively of the cut joints of $\Sigma'$; see Eq. (D.3-14)

$L^C_p, L^C_a$ Vector formed by stacking $L^C_{a_0p}$ and $L^C_{a_0a}$ respectively of all joints; see Eq. (D.3-1)

$L^a_n$ The dual-force vector applied by joint $A$ on body $Z^a_n$; see Eq. (5.3-1)

$L^a_n, L^a_{nz}$ The dual-force vector applied on body $Z$ by a joint $A$ whose n-end is attached to $Z$; see Eq. (A.3-6)

$L^\nu_n, L^\nu_{nz}$ The constraint ($\nu = \chi$) and known ($\nu = K$) portions of $L^\nu_n, L^\nu_{nz}$; see Eq. (5.3-3), (D.2-9), (A.3-8)

$L^C_\alpha A$ The dual-force vector applied by joint $A$ on body $Z^a_\alpha$ due to the $i$th scaler constraint at $A$; see Eq. (D.2-2)

$L^C_{a_0p}, L^C_{a_0a}$ The dual-force vectors applied by join $A$ on body $Z^a_\alpha$ due to all passive ($p$) and active ($a$) constraints respectively at $A$; see Eqs. (D.2-7)
$M'_1$  Generalized mass array for the passive velocities of system $\Sigma'$; see Eqs. (6.7-1) and (6.7-2)

$M'_p, M'_2$  Generalized mass arrays for the active velocities of system $\Sigma'$; see Eqs. (6.7-1) and (6.7-3)

$M_{Z1}, M_{Z2}$  Generalized mass arrays for the passive (1) and active (2) velocities respectively of body $Z$; see Eqs. (4.3-1), (A.3-16) and (A.3-17)

$M_1, M_2$  Generalized mass arrays for the passive (1) and active (2) velocities respectively of system $\Sigma$; see Section 4.5

$m$  Number of trees in system $\Sigma'$

$N$  The inertial reference coordinate frame

$O$  The output end of a joint

$O_\mu$  The origin of coordinate frame $\mu$

$P$  (1) A differential mass element
    (2) A permutation matrix

$P_r', P_s$  Submatrices of $P$; see Eq (6.9.2-1)

$P_1$  A matrix which extracts the maximum number of linearly independent rows from the $T$ matrix; see Eq. (5.4.5-6)

$g_\alpha$  The vector of generalized coordinates which specify the location and orientation of frame $\alpha_0$ relative to frame $\alpha_T$; see Section 5.1.2
$q_z$ The vector of generalized coordinates which specify the configuraton of body $Z$ relative to frame $\zeta$; see Section 4.1

$q_{z1}, q_{z2}$ Vectors comprised of the passive (1) and active (2) elements respectively of $q_z$; see Eq. (4.1-4)

$q_{Z2}$ Vector obtained by stacking the $q_{z2}$ of all bodies $Z$; see Section 4.5

$R$ The reference body for a tree of system $\Sigma'$; see Section 6.2

$r$ The dynamic reference frame of body $R$; see Section 6.2

$S_{\mu\nu}$ A block of columns of a $T$ operator; see Chapter 3

$s_A$ The state vector of joint $A$; see Eq. (5.2.1-1)

$s_Z$ The state vector of body $Z$; see Eq. (4.2-7)

$T_j$ The $j$th tree of system $\Sigma'$; Section 6.7

$T_{\mu\nu}$ The operator which transforms $\mu$ to $\nu$; see Chapter 3

$u, u'$ Configuration vectors of systems $\Sigma$ and $\Sigma'$ respectively; see Sections 4.5 and 6.4

$U_{\beta I}$ Defined in Eqs. (5.2.2-4)
\( \mathbf{u}_Z \) The configuration vector of body \( Z \) relative to frame \( N \); see Eq. (4.1-5)

\[ \mathbf{u}_Z', \mathbf{u}_1' \] Vectors comprised of the passive elements of \( \mathbf{u}_Z, \mathbf{u}, \mathbf{u}' \) respectively; see Eq. (4.1-8), Section 4.5, Eq. (6.4-1)

\( \mathbf{v}, \mathbf{v}' \) The velocity vectors of systems \( \Sigma \) and \( \Sigma' \) respectively; see Eq (4.5-1b) and Section 6.2.

\( \mathbf{v}_T, \mathbf{v}'_T \) The subvectors of \( \mathbf{v}_1, \mathbf{v}_1' \) which consist of the velocity vectors of \( T \); see Eqs. (6.7-10 and 12)

\( \mathbf{v}_Z \) Vector specifying the velocity condition of body \( Z \) relative to frame \( N \); see Eqs. (4.2-5)

\( \mathbf{v}_Z', \mathbf{v}_Z' \) Vectors comprised of the passive (1) and active (2) elements respectively of \( \mathbf{v}_Z \); see Eqs. (4.2-8 to 10)

\( \mathbf{v}_X, \mathbf{v}_N \) \( \mathbf{v}_X' \) \( \mathbf{v}_N' \) see Eq. (4.2-3)

\( \mathbf{v}_1, \mathbf{v}_2 \) Vectors comprised of the passive (1) and active (2) elements respectively of \( \mathbf{v} \); see Eq. (4.5-1b)

\( \mathbf{v}_1', \mathbf{v}_2' \) Vectors comprised of the passive and active elements or respectively of \( \mathbf{v}' \); see Section 6.2
$\vec{v}_r, \vec{v}_s$ Vectors comprised of the independent (r) and dependent (s) elements respectively of $\vec{v}_i$; see Eq. (6.9.2-2)

$\vec{w}$ Vector obtained by stacking $\vec{w}_p$ and $\vec{w}_a$; see Eq. (5.5-6)

$\vec{w}_A$ Dual-velocity vector of frame $\alpha_O$ relative to frame $\alpha_I$; see Eq. (5.2.1-2)

$\vec{w}_c$ Vector obtained by stacking $\vec{w}_{cp}$ and $\vec{w}_{ca}$; see Eq. (6.6.2-14)

$\vec{w}_{cp}, \vec{w}_{ca}$ Vectors obtained by stacking the $\vec{w}_r$ of cut joints $\Sigma$ which have passive (p) or active (a) constraints respectively; see Section 6.6.2

$\vec{w}_k$ Vector formed by stacking the $\vec{w}_A$ of every joint which has "known" forces $\vec{L}_n$

$\vec{w}_p, \vec{w}_a$ Vectors obtained by stacking the $\vec{w}_A$ of all joints which have passive (p) or active (a) constraints respectively; see Section 5.5

$\vec{w}_{uN}$ Dual-velocity vector of frame $\mu$ relative to frame $N$; see Eqs. (4.2-17) or (5.2.3-12)

$\vec{x}_{CS}$ Control system state-vector

$\vec{x}_{v\mu}$ The vector from the origin of Frame $\mu$ to the origin of Frame $v$

$\vec{y}_A$ Dual-displacement vector of frame $\alpha_O$ relative to frame $\alpha_I$; see Eq. (5.1.1-1)
A \hspace{2cm} An arbitrary joint; see Sections 4.1 and 5.1.1

\alpha_n \hspace{2cm} The coordinate frame at end n of joint A; see Subsection 5.1.1

\alpha_{nz} \hspace{2cm} The frame at the end n of joint A which is contiguous to body Z; see Section 4.1

\bar{B} \hspace{2cm} A reconnected joint of system \Sigma'; see Subsection 5.1.1

\Gamma \hspace{2cm} A cut joint of system \Sigma'; see Subsection 5.1.1

\delta_{\nu}\bar{u} \hspace{2cm} The Gibbs virtual rotation vector of frame \nu relative to frame \mu

\delta\Pi_{\nu} \hspace{2cm} The virtual displacement vector which corresponds to the velocity vector \nu; see Eq. (A.3-1).

\varepsilon_n \hspace{2cm} \varepsilon_n = 1 \text{ for } n = 2; \varepsilon_n = -1 \text{ for } n = 1; \text{ see Eq. (5.1.3-5)}

\varepsilon_P \hspace{2cm} Vector representation of the strain tensor at point P; see Eq. (A.2-1)

Z, H, K \hspace{2cm} Arbitrary bodies

Z_{\alpha_n} \hspace{2cm} The body at end n of joint A

\zeta, \eta, \kappa \hspace{2cm} The body reference frames for bodies Z, H, K respectively

\lambda \hspace{2cm} The Lagrange multiplier vector for all constraints at all cut joints of system \Sigma'; see Eqs. (6.8.1-1) and D.3-15)
\( \lambda_A \) The Lagrange multiplier vector for all constraints at joint \( A \); see Eq. (D.2-9)

\( \lambda^i_A \) The Lagrange multiplier for the \( i \)th scaler constraint at joint \( A \); see Eq. (D.2-1)

\( \lambda_{Ap}, \lambda_{Aa} \) Lagrange multiplier vectors for passive (p) and active (a) constraints at joint \( A \); see Eqs. (D.2-8)

\( \lambda_p, \lambda_a \) Lagrange multiplier vectors for all passive (p) and active (a) constraints respectively at all cut joints of system \( \Sigma' \); see Eqs. (D.3-14)

\( \xi \) The velocity command vector in the \( \vec{v}_{1r} \to \vec{v}_1 \) transformation; see Eqs. (6.9.2-5 and 6.9.2-6)

\( \Pi \) The transfer matrix in the \( \vec{v}_{1r} \to \vec{v}_1 \) transformation; see Eqs. (6.9.2-5 and 6.9.2-7)

\( \omega_A \) The generalized velocity vector of joint \( A \); see Subsections 5.2.2 and 5.4.5
\( \mathbf{Q}_{Af}, \mathbf{Q}_{Aa} \) Vectors comprised of the passive (re: "free" f) and active (a) elements of \( \mathbf{Q}_A \); see Eq. (5.4.5-2)

\( \mathbf{Q}_a^* \) The vector formed by stacking the \( \mathbf{Q}_{Aa}^* \) of every actively-controlled joint; see Eq. (5.5-4)

\( \mathbf{Q}_{ba}, \mathbf{Q}_{ca} \) Vectors formed by stacking the \( \mathbf{Q}_{Aa}^* \) of the actively-controlled reconnected (b) and cut (c) joints respectively of system \( \Sigma' \); see Eqs. (6.2-2) and (6.6.2-1 and 6.6.2-12)

\( \mathbf{Q}_{va}^* \) The commanded value of \( \mathbf{Q}_{va} \)

\( \mathbf{Q}_{ba}^* \) The subvector of \( \mathbf{Q}_{ba}^* \) which contains the \( \mathbf{Q}_{ba}^* \) of the reconnected joints in \( T_j \); see Eq. (6.7-11)

\( \Sigma \) The primitive system (all joints cut)

\( \Sigma' \) The transformed system (selected joints reconnected); see Section 6.1

\( \mathbf{Q}_p \) Vector representation of the stress tensor at point P; see Eq. (A.2-1)

\( \psi_{v\mu} \) A vector of attitude parameters which specify the orientation of frame \( v \) relative to frame \( \mu \)

\( \omega_{v\mu} \) The angular velocity vector of frame \( v \) relative to frame \( \mu \)
LIST OF REFERENCES


43. Roberson, R., A General Dynamical Formalism For Systems of Material Bodies, unpublished write-up generated in late 1960's.


