MULTIVARIABLE FLIGHT CONTROL WITH TIME-SCALE SEPARATION

by

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ABSTRACT

The design of a compensator to provide a multivariable closed-loop, feedback control system with bandwidth separated input-output groups is considered. The basic contribution is to analyze, improve, and illustrate the pragmatically used "closed-one-loop at-a-time" design procedure and to investigate the attainable performance and robustness properties. The design example used is the AV-8A Harrier aircraft. The underlying philosophy employed is the Linear Quadratic Gaussian with Loop Transfer Recovery (LQG/LTR) design methodology.

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CHAPTER 1
BACKGROUND AND OVERVIEW

1.1 Motivation

A common phenomenon in physical systems is the presence of widely separated modes. For example, it is well known that aircraft, spacecraft, and power systems all exhibit an appreciable modal separation reflected in different bandwidths.

In many applications, it is desirable that this modal separation be preserved in the closed-loop system and that different groups of output variables are characterized by an appreciable separation in their response times to their corresponding inputs (i.e. bandwidth separated input-output groups). In addition to providing a closed-loop system with bandwidth separated input-output groups, a well designed control system should still result in good command following, disturbance rejection, as well as make the system response insensitive to modeling errors while it remains robust and stable in the presence of plant uncertainties and unmodeled dynamics.

With the introduction of frequency domain methods via singular values for multivariable control system design, the control engineer gained a valuable tool for designing systems to have good command following, disturbance rejection, and robustness properties [1]. Unfortunately, existing singular value design techniques obscure the "directionality" information inherent in the problem, and thus provide little detailed information concerning input-output groups with different bandwidths. The scope of the research, therefore, is to investigate design procedures which can result in bandwidth separated multivariable designs.
1.2 Prior Research

The design of control systems for systems with widely separated, in time and frequency, modes has been investigated using singular perturbation theory [2,3]. Singular perturbation theory has been applied to the Linear Quadratic Gaussian (LQG) problem [4] to show that the optimal compensator for a system can be approximated by a sub-optimal compensator, which is a combination of a fast-mode compensator and slow-mode compensator designed separately. However, at present the use of frequency domain concepts in the design of the slow and fast mode compensators is not understood: for this reason singular perturbation techniques will not be explicitly used in this research.

The author is unaware of any other research which has dealt with the design of bandwidth separated input-output groups for truly multivariable systems exhibiting modal separation.

1.3 Research Contributions

This thesis contains a number of contributions. First, this thesis will document and illustrate the pragmatically used 'One-Group-At-a-Time' (OGAT) design procedure for designing multivariable systems to have bandwidth separated I/O groups. A linear model of the longitudinal dynamics of the AV-8A Harrier aircraft, at a medium speed flight condition, was used as the primary design example to illustrate the methodology. The multivariable control is provided through the integration of the aerodynamic and propulsion controls.

To develop a better understanding of the OGAT design procedure, this thesis will also generate a number of important properties of the OGAT design procedure for both
two-input-two-output systems and more general multivariable systems. The properties developed for two-input-two-output systems will then be shown to provide an improvement of the pragmatic OGAT design procedure, and this improved OGAT design procedure will be illustrated using an academic example.

All of these results enhance our understanding of the OGAT design procedure and demonstrate that the OGAT design procedure can produce closed-loop systems with bandwidth separated input-output groups which have good properties in the sense of command following, disturbance rejection, and robustness.

1.4 Outline of Thesis

Chapter 2 describes the bandwidth separated I/O group design problem and the pragmatic solution; the OGAT design procedure. This chapter also discusses the issues related to the use of the OGAT design procedure.

Chapter 3 develops the properties of the OGAT design procedure for two-input-two-output systems. These properties are then exploited to improve the available OGAT design procedure. An academic example, using the improved OGAT design procedure, is also presented in this chapter.

Chapter 4 generalizes some of the properties and concepts developed in chapter 3 to the more general multivariable case.

Chapter 5 presents the design of a 3-input-3-output feedback control system for the longitudinal dynamics of the AV-8A Harrier aircraft using the basic OGAT design procedure and utilizing a certain integration of aerodynamic and propulsion controls.

Chapter 6 contains a summary, conclusions, and suggestions for further research.
CHAPTER 2
PROBLEM DEFINITION

2.1 Introduction

The purpose of this chapter is to overview the MIMO time-scale separation control design problem and to discuss the commonly used but not well understood "One-Group-At-a-Time" (OGAT) design procedure. In addition, this chapter will provide the necessary framework for the analysis of the CGAT design procedure.

Section 2.2 presents the general MIMO control design problem and the general MIMO time-scale separation design problem. Section 2.3 presents the introduction to the OGAT design procedure, and section 2.4 discusses the specifics of the OGAT design procedure and related design issues. Sections 2.5 and 2.6 discuss the intuitive basis and the structure of the analysis needed to fill in the gaps in the understanding of the OGAT design procedure.

2.2 The General Design Problem and the Time-Scale Separation Design Problem

The general MIMO control design problem uses the standard feedback configuration of figure 2.1. This configuration consists of the interconnected plant \(G(s)\) and compensator \(K(s)\) driven by commands \(r(s)\), disturbances \(d(s)\), and sensor noise \(\eta(s)\). All disturbances are assumed to be reflected to the output of the plant \(y(s)\).

The design problem consists of finding a compensator \(K(s)\) which satisfies the absolute requirement of nominal closed-loop stability, stability robustness to unmodeled dynamics, as well as a number of performance (command following and
disturbance rejection) specifications. The input-output behavior of the system of figure 2.1, if it is stable, is given by the equations:

\[
\gamma(j\omega) = G(j\omega)K(j\omega)\left[I+G(j\omega)K(j\omega)\right]^{-1}[r(j\omega)-\eta(j\omega)]
+ \left[I+G(j\omega)K(j\omega)\right]^{-1}d(j\omega)
\]  

(2.1)

\[
e(j\omega) \triangleq r(j\omega) - \gamma(j\omega)
= \left[I+G(j\omega)K(j\omega)\right]^{-1}[r(j\omega)-d(j\omega)]
+ G(j\omega)K(j\omega)\left[I+G(j\omega)K(j\omega)\right]^{-1}\eta(j\omega)
\]

(2.2)

Using these equations, it has been established [1] that command following and disturbance rejection are quantified by the singular values of the loop transfer function matrix \(\sigma_1[T(j\omega)] = \sigma_1[G(j\omega)K(j\omega)]\) and the minimum singular value of the return difference function \(\sigma_{\min}[I+T(j\omega)]\), respectively. Additionally, it has been established [1] that stability robustness with respect to uncertainties at the plant outputs is quantified by the minimum singular value of the inverse return difference function \(\sigma_{\min}[I+T^{-1}(j\omega)]\). Thus, all command following, disturbance rejection, and stability robustness specifications can be translated to specifications on the singular values of the loop transfer, return difference, and inverse return difference functions, respectively.

After determining all of the specifications on the singular values of the loop transfer, return difference, and inverse return difference functions, a designer can proceed by finding a compensator which stabilizes the system (or just maintains stability) and which shapes the singular values of the three functions to meet their respective specifications. Alternatively, instead of dealing with all three functions
Fig. 2.1. A Standard Multivariable Control System

Fig. 2.2. Example of the Singular Values of the Loop Transfer Function of a System along with the System Specifications
(loop transfer, return difference, and inverse return difference function) in the
design of a compensator, a designer can use the inequalities:

\[
\sigma_{\min}[I+T(j\omega)] > \sigma_{\min}[T(j\omega)]^{-1}
\]  \hspace{1cm} (2.3)

\[
\sigma_{\min}[I+T^{-1}(j\omega)] > \sigma_{\max}[T(j\omega)]^{-1}
\]  \hspace{1cm} (2.4)

to translate all of the frequency domain specifications to the loop transfer function alone. Of course, such a translation is only reasonable for frequencies away from the crossover of the loop transfer function. In this case, the design problem has been reduced to finding a compensator which stabilizes the system (or just maintains stability) and which shapes the singular values of the loop transfer function to have a "nice" crossover range (so the return difference and inverse return difference functions do not become too small near crossover) and to meet the low frequency performance specifications and high frequency stability robustness specification (figure 2.2).

A number of design methods, including the Linear-Quadratic-Gaussian with Loop Transfer Recovery Design Methodology [LQG/LTR] [1,5,6], have been used to produce stabilizing compensators which also shape the singular values of the loop transfer function to meet the performance and robustness specifications and to have a "nice" crossover range. Most of these design methods will attempt to match the minimum and maximum singular values of the loop transfer function in the low frequency range (figure 2.3), the high frequency range, or both (figure 2.4) to aid in meeting the design specifications. One by-product of these design methods is that the width of the crossover range will inevitably be fairly small.
Fig. 2.3. Example of a System with Singular Values Matched at Low Frequency
Fig. 2.4. Example of a System with Singular Values Matched at All Frequencies
In many cases, due to inherent modal separation or other physical considerations, it is desirable that different input-output (I/O) groups of singular values within a system have an appreciable separation of bandwidths. Figure 2.5 shows the loop transfer function of such a system. Clearly, systems with bandwidth-separated (time-scale separated) I/O groups cannot have a small crossover range, so most existing singular value design methods are inappropriate for time-scale separation design. Fortunately, a design method does exist which deals with time-scale separation designs.

2.3 Introduction to the OGAT Design Procedure

For time-scale separation designs, the plant (called the global plant) will be assumed to have the structure of figure 2.6. A design procedure which uses the global plant structure of figure 2.6 and which produces a block diagonal compensator ($K_G(s)$) and bandwidth-separated I/O groups (figure 2.7) is the OGAT design procedure.

The OGAT design procedure consists of two design steps which essentially attempt to design the response of each I/O group separately. The first step of the OGAT design procedure (called the fast loop design) is to design the response of the high bandwidth I/O group while ignoring the low bandwidth inputs and outputs (figure 2.7). The second step of the OGAT design procedure (called the slow loop design) is to design the response of the low bandwidth I/O group with the fast loop closed (i.e. the loop inside of the broken line box in figure 2.7 is closed).

This separation of designs is based on the following intuitive argument: For a system with the structure of figure 2.7, the low bandwidth output response can be
Fig. 2.5. Example of a System with Bandwidth Separated I/O Groups
Fig. 2.6. Assumed Structure of the Global Plant

\[ r_1(s)/y_1(s) : \text{Low Bandwidth I/O Group} \]

\[ r_2(s)/y_2(s) : \text{High Bandwidth I/O Group} \]

Fig. 2.7. Global System Including Block Diagonal Global Compensator
considered as essentially "constant" when using the high bandwidth time-scale as a frame of reference. Thus, the response of the high bandwidth I/O group is essentially independent of the outer loop of figure 2.7 and can be designed accordingly. Since the response of the low bandwidth I/O group is clearly not independent of the high bandwidth outputs, the response of the low bandwidth I/O group should be designed with the fast loop closed to incorporate the effects of the high bandwidth outputs.

Due to its intuitive appeal, the OGAT design procedure is pragmatically used for designs with bandwidth separated I/O groups.* Unfortunately, the OGAT design procedure is not completely understood at a fundamental level. The major question surrounding the use of the OGAT design procedure is whether or not the global system has good loop shapes when the two individual designs have good loop shapes. The next section sets down the framework for the investigation of this question by discussing related issues as well as presenting the details of the OGAT design procedure.

2.4 Design Issues and the OGAT Design Procedure

This section discusses many of the design issues and the details of the OGAT design procedure so that a greater understanding of the relationships between the global system and the slow loop and fast loop designs can be established.

The global plant was assumed to have the structure of figure 2.6. This structure is very general. However, it is not clear how one should choose the controls which drive the low bandwidth and high bandwidth outputs. For some plants, such as the AV-8A aircraft in chapter 5, knowledge of the dynamics of the plant drives the choice of controls. For other systems where the possible controls outnumber the outputs and simple knowledge of the plant dynamics does not indicate a clear choice of controls, the singular value decomposition of the plant at low frequency can be

*The OGAT design procedure is an engineering generalization of minor loop classical design method [7].
used to choose the appropriate controls. Ideally, each control would drive a
single output and the choice of controls would be trivial. In the non-ideal case,
the singular vectors of a plant reveal which control has the most effect on each
output. Using this information, a control which is found to essentially effect
only one output is chosen to control that output. If more controls are needed than
the number which clearly only effect one output, then a control which effects only
the outputs within one of the I/O groups is chosen to control that I/O group.
Reference [8] contains a much more involved discussion of this technique as well
as an example (not a time-scale separated example).

Using the discussion of section 2.2, specifications on command following,
disturbance rejection, and stability robustness for the global system can be trans-
lated to specifications on the loop transfer, return difference, and inverse return
difference functions of the global system. In addition to these specifications,
bandwidth and command following specifications on the response of the high bandwidth
and low bandwidth I/O groups are needed to completely define the time-scale separation
design problem. Driven by the intuitive nature of the fast and slow loops, the
specifications on the high bandwidth I/O group and low bandwidth I/O group are used
as separate specifications for the fast loop design and slow loop design, respectively.
As mentioned in section 2.3, the relationships (if any) between the loop shapes of
the global system and the loop shapes of the two individual OGAT designs are unknown.
Hence, command following, disturbance rejection, and stability robustness specifica-
tions on the global system cannot be translated directly into specifications on the
two OGAT designs. This translation of specifications is one of the major issues of
OGAT design.
To complete an analysis of the OGAT design procedure, the notation of the specific details of the procedure need to be established. Using figures 2.6 and 2.7, the plant for the fast loop design is given by:

\[ G_f(s) = G_{22}(s) \]  

(2.5)

Thus, as shown in figure 2.8, the fast loop design step consists of choosing the fast loop compensator \((K_2(s))\) to shape the loop transfer function \((T_f(j\omega))\), return difference function \((I+T_f(j\omega))\), and inverse return difference function \((I+T_f^{-1}(j\omega))\) of the fast loop. With the fast loop closed, figure 2.9 shows the plant for the slow loop design \((G_s(s))\). Using figure 2.9, the following equations derive the mathematical representation of the slow loop plant \((G_s(s))\):

\[ y_1(s) \triangleq G_s(s)u_1(s) \]  

(2.6)

\[ y_1(s) = G_{11}(s)u_1(s) - G_{12}(s)K_2(s)y_2(s) \]  

(2.7)

\[ y_2(s) = -G_{22}(s)K_2(s)y_2(s) + G_{21}(s)u_1(s) \]  

(2.8)

\[ y_2(s) = [I+G_{22}(s)K_2(s)]^{-1}G_{21}(s)u_1(s) \]  

(2.9)

\[ y_1(s) = \{G_{11}(s) - G_{12}(s)K_2(s)[I+G_{22}(s)K_2(s)]^{-1}G_{21}(s)\}u_1(s) \]  

(2.10)

\[ \Rightarrow G_s(s) = G_{11}(s) - G_{12}(s)K_2(s)[I+G_{22}(s)K_2(s)]^{-1}G_{21}(s) \]  

(2.11)

Thus, as shown in figure 2.10, the slow loop design consists of choosing the slow loop compensator \((K_1(s))\) to shape the loop transfer function \((T_s(j\omega))\), return difference function \((I+T_s(j\omega))\), and inverse return difference function \((I+T_s^{-1}(j\omega))\) of the slow loop.
Fig. 2.8. Illustration of the Fast Loop Design Step.

Choose $K_2(s)$ to shape $T_f(j\omega) = G_{22}(j\omega)K_2(j\omega)$, $I + T_f(j\omega)$, and $I + T_f^{-1}(j\omega)$.

Fig. 2.9. Plant for the Slow Loop Design
Fig. 2.10. Illustration of the Slow Loop Design Step.

Choose $K_1(s)$ to shape $T_s(j\omega) = G_s(j\omega)K_1(j\omega)$, 
$I + T_s(j\omega)$, and $I + T_s^{-1}(j\omega)$. 
As previously mentioned, the result of the OGAT design procedure is a block diagonal global compensator. The global compensator \( K_G(s) \) and the loop transfer function of the global system \( T_G(j\omega) \) are given by:

\[
K_G(s) \triangleq \begin{bmatrix}
K_1(s) & 0 \\
0 & K_2(s)
\end{bmatrix}
\]  

\[
T_G(j\omega) = G(j\omega)K_G(j\omega)
\]

\[
= \begin{bmatrix}
G_{11}(j\omega)K_1(j\omega) & G_{12}(j\omega)K_2(j\omega) \\
G_{21}(j\omega)K_1(j\omega) & G_{22}(j\omega)K_2(j\omega)
\end{bmatrix}
\]  (2.13)

and the return difference function \((I+T_G(j\omega))\) and inverse return difference function \((I+T_G^{-1}(j\omega))\) of the global system are easily derived from equation (2.13).

2.5 Goals of the Analysis

Now that the notation and design issues of the OGAT design procedure have been established, the central question regarding the global system loop shapes can be considered. Since a successful OGAT design will always have acceptable shapes for the individual OGAT loops, the central question surrounding the use of the OGAT design procedure for time-scale separated designs is whether or not the loop properties of the individual OGAT designs are a good indication of the loop properties of the global system. Figure 2.11 graphically depicts this central question.
Fig. 2.11. Illustration of the Central Question Regarding the OGAT Design Procedure. When are the Loop Properties of the Individual OGAT Designs a Good Indication of the Loop Properties of the Global System?
A trivial example will provide insight into the answer to the central question. For a block diagonal global plant, each of the two individual OGAT designs is the design of a separate decoupled system. If

\[ \sigma_{\min}[T_f(j\omega)] > \sigma_{\max}[T_s(j\omega)] \quad \forall \omega \]  

(2.14)

for this block diagonal global system, then

\[ \sigma_{\max}[T_G(j\omega)] = \sigma_{\max}[T_f(j\omega)] \]  

(2.15)

and

\[ \sigma_{\min}[T_G(j\omega)] = \sigma_{\min}[T_s(j\omega)] \]  

(2.16)

Similar relationships hold for the return difference and inverse return difference functions of the global system. Hence, for block diagonal global systems, the loop properties of the individual OGAT designs exactly define the loop properties of the global system. Intuitively, a plant which has light coupling ("small" off-diagonal terms) should exhibit similar relationships.

By analyzing some examples of plants with light coupling, it becomes clear that relationships between the individual OGAT designs and the global system do exist for at least some global plants. Thus, it appears that a deeper analysis of the central question has an intuitive basis and is thus justified.

2.6 Two Aspects of the Analysis

The following analysis of the relationships between the individual OGAT designs and the global system appears in two cases: (1) Two-Input-Two-Output (TITO) systems which employ classical design methods in the individual OGAT designs, and (2) Several
Input-And-Output (SIAO) systems which must use multivariable design methods in the individual OGAT designs. The distinction between these two cases arises in the amount of information used in deriving the relationships between the individual OGAT designs and the global system. For TITO systems, each block element is assumed to be known exactly and thus directionality information in addition to magnitude information can be used in the analysis. For SIAO systems, only the singular values of the block elements are used in the analysis. Under these different amounts of information, the results for TITO systems would be expected to be sharper than those achievable for SIAO systems.

2.7 Concluding Remarks

Clearly, the time-scale separation design problem is important due to the large number of applications, and the commonly used OGAT design procedure is not well understood. If the following analysis provides relationships (including conditions) between the loop properties of the global system and the loop properties of the individual OGAT designs, then many of the gaps in the understanding of the OGAT design procedure will be filled. If these results can be tailored to use in the design phase (able to translate global specifications to specifications on the individual OGAT designs) then the analysis will have also produced an improvement of the basic OGAT design process.
CHAPTER 3
ANALYSIS OF THE OGAT DESIGN PROCEDURE FOR TITO SYSTEMS

3.1: Introduction

This chapter contains an analysis of the One-Group-At-a-Time (OGAT) design procedure for Two-Input-Two-Output (TITO) systems. Two major issues will be investigated. The first issue deals with determining a class of plants which can be stabilized with the OGAT design procedure. The second issue deals with determining the relationships (if any) between the global system and individual OGAT designs. Further, if some explicit relationships exist between the global system and the individual OGAT designs, they will be exploited to establish an improved step-by-step time-scale separation design procedure based on the OGAT design method.

Section 3.2 contains a discussion of the stability issues. Section 3.3 contains a linear algebra result used in section 3.4 to derive the relationships between the global system and the individual OGAT designs. Section 3.5 formalizes an improvement of the OGAT design procedure around these relationships and section 3.6 provides an illustration of the improved OGAT design procedure.

3.2 Stability of the Global System

This section will discuss the issues related to the stability of the global system. It is well known that a compensator (not necessarily block diagonal) can be found to stabilize a plant which is both stabilizable and detectable [9]. Unfortunately, every plant which is both stabilizable and detectable is not necessarily stabilized by a block diagonal compensator. Thus, using the OGAT design procedure, a stable global system may not be achievable for every stabilizable and detectable global plant.
Nonetheless, some global plants, which are stabilizable and detectable, can be stabilized using the OGAT design procedure, so it is reasonable to assume that an entire subclass of stabilizable and detectable global plants can be stabilized using the OGAT design procedure. As we discussed in chapter 2, the slow loop design is done assuming that the fast loop is closed (fig. 2.10); hence the stability of the closed-loop slow loop system is equivalent to the stability of the global system.

Consider the fairly general case when some of the unstable poles of the global system are both stabilizable from $u_2$ and detectable from $y_2$ and the rest of the unstable poles of the global system are both stabilizable from $u_1$ and detectable from $y_1$. In this case, the fast loop design can stabilize all of the poles which are stabilizable from $u_2$ and detectable from $y_2$ and thus produce a slow loop plant which is both stabilizable and detectable. The design of the slow loop can stabilize the remaining unstable poles of the global plant, and thus produce a global system with guaranteed closed-loop nominal stability.

Thus, a subclass of stabilizable and detectable global plants which can be stabilized by the OGAT design procedure is defined by the sufficient condition that each unstable pole of the global system must be both stabilizable from $u_1$ and detectable from $y_1$ or both stabilizable from $u_2$ and detectable from $y_2$.* One important fact brought out in the preceding discussion is that the fast loop need not be stable. If a global plant within the predefined subclass has unstable poles which are not both stabilizable from $u_2$ and detectable from $y_2$, then the fast loop design will be unstable (due to a lack of internal stability) although the global closed-loop system is stable.

*Note that the choice of control variables for the fast and slow systems (see section 2.4) can influence this property.
3.3 A Preliminary Result

This section contains a linear algebra result which forms the basis of the analysis relating the individual OGAT designs to the global system. The derivation of the following result is contained in Appendix A.

Let $M$ be a $2 \times 2$ complex-valued matrix

$$
M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
$$

(3.1)

Define the real scalar $\alpha$ by

$$
\alpha = \left[ 1 + \frac{|b|^2}{|d|^2} + \frac{|c|^2}{|d|^2} \right]^{1/2}
$$

(3.2)

Suppose that

$$
\frac{|d|}{|a-bd^{-1}c|} \geq \rho_1 >> 1
$$

(3.3)

and that

$$
\frac{|d|}{|a|} \geq \rho_2 >> 1
$$

(3.4)

Then, the following approximations hold (and become better and better as $\rho_1 \to \infty$, $\rho_2 \to \infty$)

$$
\sigma_{\text{max}}[M] \approx |d| \cdot \alpha
$$

(3.5)

$$
\sigma_{\text{min}}[M] \approx |a-bd^{-1}c| \cdot \frac{1}{\alpha}
$$

(3.6)
3.4 Main Results

Using the form of the loop transfer function of the global system established in chapter 2 (eqn. (2.13)), the loop transfer function of the global TITO system is:

\[
T_G(j\omega) = \begin{bmatrix}
g_{11}(j\omega)k_1(j\omega) & g_{12}(j\omega)k_2(j\omega) \\
g_{21}(j\omega)k_1(j\omega) & g_{22}(j\omega)k_2(j\omega)
\end{bmatrix}
\]  

(3.7)

Hereafter, the frequency dependence of the matrix elements will be suppressed. Pointwise in frequency, the loop transfer function of the global TITO system is a 2x2 complex valued matrix. Thus, the linear algebra results of section 3.2 can be applied to the loop transfer function of the global TITO system. This yields the following result.

Define the real scalar \( \beta(\omega) \) by

\[
\beta(\omega) \triangleq \left[ 1 + \frac{|g_{12}|^2}{|g_{22}|^2} + \frac{|g_{21}k_1|^2}{|g_{22}k_2|^2} \right]^{1/2}
\]  

(3.8)

Suppose that

\[
\frac{|g_{22}k_2|}{|g_{11}k_1 - g_{12}k_2(g_{22}k_2)^{-1}g_{21}k_1|} \geq \rho_1 >> 1
\]  

(3.9)

and that

\[
\frac{|g_{22}k_2|}{|g_{11}k_1|} \geq \rho_2 >> 1
\]  

(3.10)

then

\[
\sigma_{\text{max}}[T_G(j\omega)] \approx |g_{22}k_2| \cdot \beta(\omega)
\]  

(3.11)

\[
\sigma_{\text{min}}[T_G(j\omega)] \approx |g_{11}k_1 - g_{12}k_2(g_{22}k_2)^{-1}g_{21}k_1| \cdot \frac{1}{\beta(\omega)}
\]  

(3.12)
A number of simplifications in the above equations can be obtained by recognizing various terms as designed quantities and by considering different frequency ranges (and the approximations which hold within each frequency range). Using the notation of equation (3.7), the loop transfer functions of the fast loop \( t_f(j\omega) \) and slow loop \( t_s(j\omega) \) are:

\[
\begin{align*}
t_f(j\omega) &= g_{22}k_2 \\
t_s(j\omega) &= g_{11}k_1 - g_{12}k_2(1+g_{22}k_2)^{-1}g_{21}k_1
\end{align*}
\]

As defined in figure 3.1, the passband of the fast loop design is the frequency range where the gain of the fast loop is much larger than unity; the stopband of the slow loop design is the frequency range where the gain of the slow loop is much less than unity. Using this information, the approximation

\[
|g_{11}k_1 - g_{12}k_2(1+g_{22}k_2)^{-1}g_{21}k_1| \approx |t_s(j\omega)|
\]

must hold for the passband of the fast loop design.

Using equations (3.13) to (3.15), the results in equations (3.8) to (3.12) simplify as follows:

Define the real scalar \( \beta(\omega) \) by

\[
\beta(\omega) \triangleq \left( 1 + \frac{|g_{12}|^2}{|g_{22}|^2} + \frac{|g_{21}k_1|^2}{|g_{22}k_2|^2} \right)^{1/2}
\]

\[
\beta(\omega) \triangleq \left( 1 + \frac{|g_{12}|^2}{|g_{22}|^2} + \frac{|g_{21}k_1|^2}{|g_{22}k_2|^2} \right)^{1/2}
\]
Fig. 3.1. Definition of the Passband of the Fast Loop and Stopband of the Slow Loop for Bandwidth Separated TITO Systems.
Suppose that for the passband of the fast loop design

\[
\frac{|t_f(j\omega)|}{|t_s(j\omega)|} \geq \rho_1 \gg 1
\]  

and that

\[
\frac{|g_{22}k_2|}{|g_{11}k_1|} \geq \rho_2 \gg 1,
\]  

then for the passband of the fast loop

\[
\sigma_{\text{max}}[T_G(j\omega)] \approx |t_f(j\omega)| \cdot \beta(\omega)
\]  

\[
\sigma_{\text{min}}[T_G(j\omega)] \approx |t_s(j\omega)| \cdot \frac{1}{\beta(\omega)}
\]  

Suppose that for the stopband of the slow loop design

\[
\frac{|g_{22}k_2|}{|g_{11}^{-1}g_{12}g_{22}^{-1}g_{21}k_1|} \geq \rho_1 \gg 1
\]  

and that

\[
\frac{|g_{22}k_2|}{|g_{11}k_1|} \gg \rho_2 \gg 1
\]  

then for the stopband of the slow loop

\[
\sigma_{\text{max}}[T_G(j\omega)] \approx |t_f(j\omega)| \cdot \beta(\omega)
\]  

\[
\sigma_{\text{min}}[T_G(j\omega)] \approx |g_{11}^{-1}g_{12}k_2(g_{22}^{-1}g_{21}^{-1})k_1| \cdot \frac{1}{\beta(\omega)}
\]  

At least for the passband of the fast loop, these results (eqns. (3.19), (3.20), (3.23), (3.24), contain explicit relationships between the singular values
of the loop transfer function of the global system and the gain of the loop transfer functions of the individual OGAT designs.

Similar relationships for the return difference and inverse return difference functions of the global system are derived in Appendix B. The results of Appendix B which are of interest in the design of a time-scale separated system are given below:

Suppose that for the passband of the fast loop

$$\frac{|1 + t_f(j\omega)|}{|1 + t_s(j\omega)|} \geq \rho_1 \gg 1$$

(3.25)

and that

$$\frac{|1 + g_{22}k_2^2|}{|1 + g_{11}k_1|} \geq \rho_2 \gg 1$$

(3.26)

then for the passband of the fast loop

$$\sigma_{\text{max}}[I + T_G(j\omega)] \approx |1 + t_f(j\omega)| \cdot \beta(\omega)$$

(3.27)

$$\sigma_{\text{min}}[I + T_G(j\omega)] \approx |1 + t_s(j\omega)| \cdot \frac{1}{\beta(\omega)}$$

(3.28)

$$\sigma_{\text{min}}[I + T_G^{-1}(j\omega)] \approx \frac{|1 + t_s(j\omega)|}{|1 + t_s(j\omega)| + \beta(\omega)}$$

(3.29)

and suppose that for the stopband of the slow loop

$$\frac{|1+[(g_{11} - g_{12}(g_{22})^{-1}g_{21})k_1^{-1}]|}{|1+(g_{22}^{-1}k_2)^{-1}|} \geq \rho_1 \gg 1$$

(3.30)

and that

$$\frac{|1+[(g_{11}^{-1}k_1^{-1} - g_{12}k_2^{-1}g_{22}k_2^{-1}g_{21}^{-1})k_1^{-1}]|}{|1+(g_{11}^{-1}k_1^{-1})(g_{22}^{-1}k_2^{-1})^{-1}[(g_{11}^{-1}k_1^{-1} - g_{12}k_2^{-1}g_{22}k_2^{-1}g_{21}^{-1})^{-1}]|} \geq \rho_2 \gg 1$$

(3.31)
and that
\[ |g_{11}k_1 - g_{12}k_2 (g_{22}k_2)^{-1} g_{21}k_1| << 1 \]  \hspace{1cm} (3.32)
then for the stopband of the slow loop
\[ \sigma_{\min}[1 + T^{-1}_G(j\omega)] \approx \left|1 + t^{-1}_f(j\omega) \right| \cdot \frac{1}{\beta(\omega)} \]  \hspace{1cm} (3.33)
\[ \sigma_{\min}[1 + T_G(j\omega)] \geq \frac{\left|1 + t^{-1}_f(j\omega) \right|}{\left|1 + t_f(j\omega) + \beta(\omega) \right|} \]  \hspace{1cm} (3.34)
The above results also show a clear relationship between the return difference and inverse return difference functions of the individual OGAT designs and the return difference and inverse return difference functions of the global system.

The equations (3.16) to (3.34) provide an increased level of understanding of the OGAT design procedure. From these equations, it is reasonable to infer that if \( \beta(\omega) \) is not too large and the individual OGAT designs have good loop shapes, then the global system will also have good loop shapes. Indeed, if \( \beta(\omega) \) were compensator independent (and not too large) and conditions (3.17), (3.18), (3.21), (3.22), (3.25), (3.26), and (3.30) to (3.32) are satisfied, then the above results could be used to translate the global system specifications into specifications for the individual OGAT loops and thus to improve the OGAT design procedure.

3.5 Improved OGAT Design Procedure

This section will use the results of section 3.4 to improve the time-scale design procedure based on the OGAT design method. The issues involve choosing appropriate values of \( \rho_1 \) and \( \rho_2 \), reducing the nine validating conditions to five conditions, meeting
those five conditions which validate the results of section 3.4, determining any additional conditions which render $\beta(\omega)$ independent of the compensators, and determining the specifications on the two separate OGAT designs.

As discussed in the derivation of the linear algebra result of section 3.2 (Appendix A), reasonable numerical values of $\rho_1$ and $\rho_2$ are 10 and $\sqrt{50}$, respectively. These values of $\rho_1$ and $\rho_2$ are chosen since the approximations made using each of the equations involving $\rho_1$ and $\rho_2$ result in an approximation error of less than one percent.

The conditions of equations (3.25), (3.26), (3.30) and (3.31) all have the form
\[
\frac{|1+x|}{|1+z|} \geq \rho_1
\]
(3.35)
where $|x|$ is known to be much larger than unity and the equations (3.17), (3.18), (3.21), and (3.22) contain a corresponding ratio
\[
\frac{|x|}{|z|} \geq \rho_1
\]
(3.36)
Clearly, when $|z|$ is much larger than unity, equation (3.36) is equivalent to equation (3.35). Since the equations (3.25), (3.26), (3.30), and (3.31) are only used for approximations, equation (3.36) can be substituted for equation (3.35) for any value of $|z|$ if the designer is willing to tolerate the corresponding degradation in the approximations when $|z| \approx 1$. This is the approach taken here. Hence, the results of section 3.4 useful for design can be compacted to the following statements:

Define the real scalar $\beta(\omega)$ as
\[
\beta(\omega) \triangleq \left[ 1 + \frac{|g_{12}|^2}{|g_{22}|^2} + \frac{|g_{21}k_1|^2}{|g_{22}k_2|^2} \right]^{1/2}
\]
(3.37)
Suppose that for the passband of the fast loop

\[
\left| \frac{t_f(j\omega)}{t_s(j\omega)} \right| \geq \rho_1
\]  

(3.38)

and that

\[
\left| \frac{g_{22}k_2}{g_{11}k_1} \right| \geq \rho_2
\]  

(3.39)

then for the passband of the fast loop

\[
\sigma_{\min}[T_G(j\omega)] \approx \left| t_s(j\omega) \right| \cdot \frac{1}{\beta(\omega)}
\]  

(3.40)

\[
\sigma_{\min}[1+T_G(j\omega)] \approx \left| 1 + t_s(j\omega) \right| \frac{1}{\beta(\omega)}
\]  

(3.41)

\[
\sigma_{\min}[1+T_G^{-1}(j\omega)] \geq \left| 1 + t_s(j\omega) \right| \frac{1}{\beta(\omega)}
\]  

(3.42)

and suppose for the stopband of the slow loop

\[
\left| \frac{g_{22}k_2}{(g_{11} - g_{12}(g_{22})^{-1}g_{21})k_1} \right| \geq \rho_1
\]  

(3.43)

and that

\[
\left| \frac{g_{22}k_2}{g_{11}k_1} \right| \geq \rho_2
\]  

(3.44)

and that

\[
\left| (g_{11} - g_{12}(g_{22})^{-1}g_{21})k_1 \right| \ll 1
\]  

(3.45)

then for the stopband of the slow loop
\[
\sigma_{\text{max}}[T_G(j\omega)] \approx |t_f(j\omega)| \cdot \beta(\omega)
\]

(3.46)

\[
\sigma_{\text{min}}[I+T_{\text{G}}^{-1}(j\omega)] \approx |1+t_f^{-1}(j\omega)| \cdot \frac{1}{\beta(\omega)}
\]

(3.47)

\[
\sigma_{\text{min}}[I+T_{\text{G}}(j\omega)] \geq \frac{|1+t_f^{-1}(j\omega)|}{|1+t_f^{-1}(j\omega)| + \beta(\omega)}
\]

(3.48)

By examining \(\beta(\omega)\), it is clear that the condition

\[
\frac{|g_{22}k_2|}{|g_{21}k_1|} \geq \rho_2 \gg 1
\]

(3.49)

can be used to approximate \(\beta(\omega)\) and render it compensator independent. Equation (3.49) uses \(\rho_2\) since this equation results in the same approximation error produced when equations (3.39) and (3.44) were used in Appendix A. Under this condition, \(\beta(\omega)\) can be approximated as:

\[
\beta(\omega) \approx \left[1 + \frac{|g_{12}|^2}{|g_{22}|^2}\right]^{1/2}
\]

(3.50)

The next issue is how to guarantee that conditions (3.38), (3.39), (3.43) to (3.45), and (3.49) can be met. Meeting (3.38) is relatively simple and is dealt with directly in the design of the fast and slow loops. Conditions (3.39), (3.43), (3.44), and (3.49) all contain the multiplication of a ratio between terms dependent only on the global plant and a ratio between the gain of the two compensators (\(|k_2|/|k_1|\)). By calculating the plant dependent ratio, the compensators can be constrained to guarantee that equations (3.39), (3.43), (3.44), and (3.49) will hold. Although equation (3.45) does not contain any ratios it can be handled exactly as the other conditions.
Aggregating all of the discussions from this chapter as well as chapter 2 produces the following improved OGAT design procedure.

**Step 1:** Translate all of the specifications on the global system (not the individual I/O groups) to specifications on the loop transfer, return difference, and inverse return difference functions of the global system.

**Step 2:** Calculate $\beta(\omega)$ using equations (3.50).

**Step 3:** Use equations (3.40) to (3.42) and (3.46) to (3.48) to insure that the global system specifications are consistent with the value of $\beta(\omega)$ (e.g. For $\beta(\omega)=2$, a global system specification of $\sigma_{\min}[I+T_G(j\omega)] > .75 \forall \omega$ is inconsistent because the best level which can be achieved by OGAT design is $1/\beta(\omega)$). If the specifications are inconsistent, then attempt scaling, compromise the specifications, or abandon the OGAT design procedure.

**Step 4:** Check that for all frequencies

$$\frac{|g_{22}|}{|g_{11}|} > \sqrt{50} \quad (3.51)$$

$$\frac{|g_{22}|}{|g_{21}|} = \sqrt{50} \quad (3.52)$$

and for the design's best estimate of the stopband of the slow loop design

$$\frac{|g_{22}|}{|g_{11} - g_{12}(g_{22})^{-1}g_{21}|} \geq 10 \quad (3.53)$$

$$|g_{11} - g_{12}(g_{22})^{-1}g_{21}| < 1 \quad (3.54)$$

If any of these inequalities do not hold, then constrain the two compensators to guarantee that conditions (3.39) and (3.49) hold for
all frequencies and that conditions (3.43) and (3.45) hold for the stopband of the slow loop design.

**Step 5:** As discussed in chapter 2, translate the specifications on the low bandwidth and high bandwidth I/O groups to the slow and fast loop functions, respectively. Then use the specifications on the various functions of the global system from step 1, equations (3.40) to (3.42) and (3.46) to (3.48), and the value of $\beta(\omega)$ from step 2 to modify the specifications on the fast and slow loop designs to make them consistent with the global system specifications. This assures that one design iteration will satisfy all the global specifications.

**Step 6:** Complete the two OGAT designs to meet the specifications determined in step 5, any constraints from step 4, and equation (3.38) with $\rho_1$ equal to ten.

The next section provides an illustration of this improved OGAT design procedure.

### 3.6 A TITO Design

This section contains an "academic" design example of a TITO system using the step-by-step design procedure of section 3.5.

The global plant for the example is given by:

$$
G(s) = \begin{bmatrix}
\frac{.005}{(s+.01)(s+.05)} & \frac{10}{s(s+1.1667)} \\
\frac{.004}{(s+.01)(s+.05)} & \frac{10}{s(s+1)}
\end{bmatrix}
$$

(3.55)

The desired specifications for this system are:
G1: The low bandwidth I/O pair has a bandwidth of \( .1 \frac{\text{rad}}{\text{sec}} \).

G2: The high bandwidth I/O pair has a bandwidth of \( 10 \frac{\text{rad}}{\text{sec}} \).

G3: Command following for the high bandwidth I/O pair must be within 4\%
    for frequencies below \( 1 \frac{\text{rad}}{\text{sec}} \).

G4: Output disturbances must be attenuated by a factor of 10 for frequencies
    below \( .01 \frac{\text{rad}}{\text{sec}} \).

G5: The system must be robust to multiplicative plant uncertainties bounded
    by a gain of 10 for frequencies above 100 \( \frac{\text{rad}}{\text{sec}} \).

G6: The sensitivity of the system should never by greater than 2 for all
    frequencies.

The steps of the design procedure will now be followed in order. The first step
of the design procedure is to translate the global system specifications (not the I/O
groups) to the various functions of the global system. Translating specifications G4
through G6 into singular-value bounds produces:

G4: \( \sigma_{\min}[I+T_G(j\omega)] > 10 \) for \( \omega < .01 \frac{\text{rad}}{\text{sec}} \)

G5: \( \sigma_{\min}[I+T^{-1}_G(j\omega)] > 10 \) for \( \omega > 100 \frac{\text{rad}}{\text{sec}} \)

G6: \( \sigma_{\min}[I+T_G(j\omega)] > .5 \) for all \( \omega \)

The next step of the design procedure is to calculate \( \beta(\omega) \). For the global
plant of equation (3.55),

\[
\beta(\omega) = \left[ 1 + \frac{|1+j\omega|^2}{|1.667+j\omega|^2} \right]^{1/2} \approx \begin{cases} 
1.166 & \omega < 1 \text{ rad/s} \\
1.414 & \omega > 1.667 \text{ rad/s}
\end{cases} \tag{3.56}
\]

and \( \beta(\omega) \) is shown in figure 3.2.
Fig. 3.2. Plot of $\beta(\omega)$ from Equation (3.56)
After calculating $\beta(\omega)$, the consistency of the global specifications should be checked. Specifications G4 and G5 are clearly consistent. Since, the smallest numerical value of $1/\beta(\omega)$ (over all frequencies) is .707, the last global system specification, G6, is attainable through design and thus it is also a consistent specification.

Figure 3.3 shows the plant dependent ratios of equations (3.51), (3.52), and (3.53) as well as the chosen values of $\rho_1$ and $\rho_2$ (10 and $\sqrt{50}$, respectively). Clearly, equations (3.51) to (3.53) all hold. For an estimated slow loop stopband containing frequencies above 1 rad/s, figure 3.4 shows that equation (3.54) is also satisfied. Hence, the compensators will not be constrained.

The next step of the design procedure is to determine the specifications on the fast and slow loops. Translating the specifications on the I/O groups of the global system to the fast and slow loop functions produces:

S1: The slow loop has a bandwidth of .1 $\frac{\text{rad}}{\text{sec}}$

F1: The fast loop has a bandwidth of 10 $\frac{\text{rad}}{\text{sec}}$

F2: $|t_f(j\omega)| > 25$ for $\omega < 1 \frac{\text{rad}}{\text{sec}}$

Using the approximate values of $\beta(\omega)$ from equation (3.56) and equations (3.40) to (3.42) and (3.46) to (3.48), these specifications on the fast and slow loops are modified to be consistent with the global system specifications, G4 through G6. Specifications F1, F2, and S1 are unaffected by this modification, but the additional specifications given by:

S2: $|1+t^{-1}_s(j\omega)| > 11.66$ for $\omega < 0.01 \frac{\text{rad}}{\text{sec}}$

F3: $|1+t^{-1}_f(j\omega)| > 14.14$ for $\omega > 100 \frac{\text{rad}}{\text{sec}}$

S3: $|1+t_s(j\omega)| > 0.707$ for all $\omega$

are needed to make the OGAT designs consistent with the global system specifications.
Fig. 3.3. Illustration of the Ratios:

\[ A = \frac{|g_{22}|}{|g_{11}|}, \quad B = \frac{|g_{22}|}{|g_{21}|}, \quad C = \frac{|g_{22}|}{|g_{11}^{-1} - g_{12} g_{22} g_{21}|} \]
Fig. 3.4. Illustration of the Term: $|g_{11}g_{12}^{-1}g_{22}g_{21}|$

Notice that this term is much less than unity for frequencies within the estimated stopband of the slow loop ($\omega > 1 \frac{\text{rad}}{\text{sec}}$).
Using the three slow loop specifications (S1-S3) and the three fast loop specifications (F1-F3) and maintaining a gain separation of 10 between the two designs (to satisfy equation (3.38)), the two design steps of the OGAT design procedure are completed.

A compensator which satisfies the fast loop specifications and produces a stable fast loop is found using the LQG/LTR design methodology. This fast loop compensator is given by:

\[
k_2(s) = \frac{8136(s+4.61)}{s^2+149.14s+11121}
\]  

(3.57)

The loop transfer function of the fast loop is given by:

\[
t_f(s) = \frac{81360(s+4.61)}{s(s+10)(s^2+149.14s+11121)}
\]  

(3.58)

and the gain of the loop transfer function of the fast loop design is shown in figure 3.5. The plant for the slow loop design is then given by:

\[
g_s(s) = \frac{.00545(s+2.7)(s^2+1.323s+11.04)}{(s+.05)(s+.01)(s+1.667)(s^2+8.72s+37.5)}
\]  

(3.59)

and a compensator which results in a stable slow loop (and thus stable global system) that meets the slow loop specifications is found, again using the LQG/LTR design methodology, and is given by:

\[
k_1(s) = \frac{95.93(s+1.667)(s+.1014)}{(s^2+1.768s+2.356)(s+1.994)}
\]  

(3.60)
Fig. 3.5. Frequency Response of the Loop Transfer Function of the Fast Loop Design, from eqn. (3.58)
The loop transfer function of the slow loop is:

\[ t_s(s) = g_s(s)k_1(s) \]  \hspace{1cm} (3.61)

and the gain of the loop transfer function of the slow loop design is shown in figure 3.6.

After completing the two designs, the singular values of the loop transfer function of the global system should be checked to verify that all of the global system specifications are met.

Figure 3.7 shows the singular values of the loop transfer function of the global system along with the approximations of the singular values using \( \beta(\omega) \) from equation (3.50) and equations (3.19) and (3.20). Figures 3.8 and 3.9 show the singular values of the return difference function of the global system along with the approximations of the singular values using \( \beta(\omega) \) from equation (3.50) and equations (3.27) and (3.28). Similarly, figures 3.10 and 3.11 show the singular values of the inverse return difference function of the global system along with the approximation of the minimum value using \( \beta(\omega) \) from equation (3.50) and equation (3.33). From these figures, it is apparent that the approximations are very good for the appropriate frequency ranges. Figures 3.9 and 3.11 also show that the last three global system specifications (G4, G5, and G6) are met.

The other three global system specifications (G1, G2, and G3) concern the two I/O pairs. Figures 3.12 and 3.13 show the closed-loop frequency response between the high bandwidth input and output and the low bandwidth input and output, respectively. From figures 3.12 and 3.13, the \( \frac{1}{2} \) -power bandwidth of the low bandwidth I/O pair is \( .14 \text{ rad/sec} \), the bandwidth of the high bandwidth I/O pair is \( 10.2 \text{ rad/sec} \), and command
Fig. 3.6. Frequency Response of the Loop Transfer Function of the Slow Loop Design, from eqns. (3.59) to (3.61)
Fig. 3.7. Singular Values of the Loop Transfer Function of the Global System. Note the validity of the approximations over the appropriate frequency ranges.
Fig. 3.8. Singular Values of the Return Difference Function of the Global System. Notice the validity of the approximations over the appropriate frequency range.
Fig. 3.9. Singular Values of the Return Difference
Function of the Global System.
Fig. 3.10. Singular Values of the Inverse Return Difference Function of the Global System. Notice the validity of the approximation over the appropriate frequency range.
Fig. 3.11. Singular Values of the Inverse Return Difference
Function of the Global System
Fig. 3.12. Closed-loop Frequency Response Between the
High Bandwidth Input (r_2(j\omega)) and the High
Bandwidth Output (y_2(j\omega))
Fig. 3.13. Closed-loop Frequency Response Between the
Low Bandwidth Input ($r_1(j\omega)$) and the Low
Bandwidth Output ($y_1(j\omega)$)
following for the high bandwidth inputs is within 4% for frequencies below 1 rad/s. This verifies that the first three global system specifications (G1, G2, and G3) are met.

3.7 Concluding Remarks

This chapter provided an analysis of the OGAT design procedure for TITO systems. Section 3.2 defined a subclass of global system which can be stabilized using the OGAT design procedure. The results of section 3.4 demonstrate an explicit relationship between the global system and the individual OGAT designs. Under an additional constraint, the results of section 3.4 have been shown to be useful for design modification and an improved step-by-step design procedure was developed on this basis. Thus, for a global system within the predefined stabilizable class which does not have a very large \( \beta(\omega) \), a designer can use the OGAT design procedure and produce good global system loop shapes.

As this analysis has proved to be fruitful for the TITO case, the next chapter will attempt to extend these results to the more general case.
CHAPTER 4
ANALYSIS OF THE OGAT DESIGN PROCEDURE FOR SIAO SYSTEMS

4.1 Introduction

This chapter contains an analysis of the OGAT design procedure for SIAO systems. The objective is to determine relationships (analogous to (3.16) through (3.34)) between the global system and the individual OGAT designs for the general case. Unfortunately, the relationships that were derived cannot be used directly in the design phase. The results do provide valuable insight into the relationship between the global system and the individual OGAT designs, and it is the author's belief that any future developments involving time-scale separation design procedures should start with the results of this chapter as a basis.

Section 4.2 presents a linear algebra result used in section 4.3 to derive the relationships between the global system and the individual OGAT designs. Section 4.4 discusses the shortfalls of the results of section 4.3 from a designer's point of view.

4.2 A Preliminary Result

The derivation of the following result is contained in Appendix C.

Let $M$ be a general complex-valued square matrix defined by four block elements

$$ M \triangleq \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{4.1} $$

Define the real scalars $\gamma$ and $\delta$ by

$$ \gamma \triangleq \left[ 1 + \frac{\sigma_{\text{max}}^2 [B]}{\sigma_{\text{max}}^2 [D]} + \frac{\sigma_{\text{max}}^2 [C]}{\sigma_{\text{max}}^2 [D]} \right]^{1/2} \tag{4.2} $$
\[ \delta \Delta \left[ 1 + \frac{\sigma_{\max}^2[B]}{\sigma_{\min}^2[D]} + \frac{\sigma_{\max}^2[C]}{\sigma_{\min}^2[D]} \right]^{1/2} \]  

(4.3)

Suppose that

\[ \sigma_{\min}[A] - \sigma_{\max}[B]\sigma_{\min}^{-1}[D]\sigma_{\max}[C] \geq 0 \]  

(4.4)

and that

\[ \frac{\sigma_{\min}[D]}{\sigma_{\max}[A] + \sigma_{\max}[B]\sigma_{\min}^{-1}[D]\sigma_{\max}[C]} \geq \rho_1 \gg 1 \]  

(4.5)

and that

\[ \frac{\sigma_{\min}[D]}{\sigma_{\max}[A]} \geq \rho_2 \gg 1 \]  

(4.6)

Then the following approximate bounds hold and the approximations become better and better as \( \rho_1 \to \infty \), \( \rho_2 \to \infty \).

\[ \sigma_{\max}[M] \leq \sigma_{\max}[D] \cdot \gamma \]  

(4.7)

\[ \sigma_{\min}[M] \geq \{ \sigma_{\min}[A] - \sigma_{\max}[B]\sigma_{\min}^{-1}[D]\sigma_{\max}[C] \} \cdot \frac{1}{\delta} \]  

(4.8)

4.3 Main Results

As given in chapter 2 (eqn. (2.13)), the loop transfer function of the global SIAO system is:

\[ T_G(j\omega) = \begin{bmatrix} G_{11}(j\omega)K_1(j\omega) & G_{12}(j\omega)K_2(j\omega) \\ G_{21}(j\omega)K_1(j\omega) & G_{22}(j\omega)K_2(j\omega) \end{bmatrix} \]  

(4.9)
For simplicity, the frequency dependence of the block elements of the loop transfer function of the global system \( (G_{ij}(j\omega) \) and \( K_i(j\omega) \)) will be suppressed. Pointwise in frequency, the loop transfer function of the global SISO system is a complex-valued square matrix with the form of equation (4.1). Thus, the linear algebra result of section 4.2 can be applied to the loop transfer function of the global SISO system. This yields the following result.

Define the real scalars \( \xi(\omega) \) and \( \eta(\omega) \) by

\[
\xi(\omega) \triangleq \left[ 1 + \frac{\sigma^2_{\max}[G_{12}K_2] + \sigma^2_{\max}[G_{21}K_1]}{\sigma^2_{\max}[G_{22}K_2]} \right]^{1/2} \quad (4.10)
\]

\[
\eta(\omega) \triangleq \left[ 1 + \frac{\sigma^2_{\max}[G_{12}K_2] + \sigma^2_{\max}[G_{21}K_1]}{\sigma^2_{\min}[G_{22}K_2]} \right]^{1/2} \quad (4.11)
\]

Suppose that

\[
\sigma_{\min}[G_{11}K_1] - \sigma_{\max}[G_{12}K_2]\sigma_{\min}[G_{22}K_2]\sigma_{\max}[G_{21}K_1] \geq 0 \quad (4.12)
\]

and that

\[
\frac{\sigma_{\min}[G_{22}K_2]}{\sigma_{\max}[G_{11}K_1] + \sigma_{\max}[G_{12}K_2]\sigma_{\min}[G_{22}K_2]\sigma_{\max}[G_{21}K_1]} \geq \rho_1 \gg 1 \quad (4.13)
\]

and that

\[
\frac{\sigma_{\min}[G_{22}K_2]}{\sigma_{\max}[G_{11}K_1]} \geq \rho_2 \gg 1 \quad (4.14)
\]

then

\[
\sigma_{\max}[T_G(j\omega)] \leq \sigma_{\max}[G_{22}K_2] \cdot \xi(\omega) \quad (4.15)
\]

\[
\sigma_{\min}[T_G(j\omega)] \geq \sigma_{\min}[G_{11}K_1] - \sigma_{\max}[G_{12}K_2]^{-1}\sigma_{\min}[G_{22}K_2]\sigma_{\max}[G_{21}K_1] \cdot \frac{1}{\eta(\omega)} \quad (4.16)
\]
From strictly an analysis point of view, the above equations are in the simplest form. However, these equations can be simplified to provide insight into the relationships between the global SIAO system and the individual OGAT designs as well as to illustrate the problems in trying to use them in the design phase.

Simplification of equations (4.10) to (4.16) can be attained by recognizing designed quantities, bounds on the singular values of designed quantities, and by considering different frequency ranges (and the approximations which hold within each frequency range). From chapter 2, using the notation of equation (4.9), the loop transfer functions of the fast loop ($T_f(j\omega)$) and slow loop ($T_s(j\omega)$) are:

$$T_f(j\omega) = \frac{G_{22}K_2}{G_{12}K_2}$$  \hspace{1cm} (4.17)

$$T_s(j\omega) = \frac{G_{11}K_1}{G_{12}K_2 (1+G_{22}K_2)^{-1}G_{12}K_1}$$  \hspace{1cm} (4.18)

As defined in figure 4.1, the passband of the fast loop design is the frequency range where the minimum singular value of the loop transfer function of the fast loop is much larger than unity; the stopband of the slow loop design is the frequency range where the maximum singular value of the loop transfer function of the slow loop is much less than unity. Using this information, the approximation

$$T_s(j\omega) = \frac{G_{11}K_1}{G_{12}K_2 (1+G_{22}K_2)^{-1}G_{21}K_1}$$

$$\approx \frac{G_{11}K_1}{G_{12}K_2 (G_{22}K_2)^{-1}G_{21}K_1}$$  \hspace{1cm} (4.19)

must hold for the passband of the fast loop design. Hence, for the passband of the fast loop, the minimum and maximum singular values of the loop transfer function of the slow loop design are given by:
Fig. 4.1. Illustration of the Passband of the Fast Loop and Stopband of the Slow Loop for Bandwidth Separated SIAO Systems.
\[ \sigma_{\max} [T_s(j\omega)] \approx \sigma_{\max} [G_{11} K_1 - G_{12} K_2 (G_{22} K_2)^{-1} G_{21} K_1] \] (4.20)

\[ \sigma_{\min} [T_s(j\omega)] \approx \sigma_{\min} [G_{11} K_1 - G_{12} K_2 (G_{22} K_2)^{-1} G_{21} K_1] \] (4.21)

Using the singular value inequalities

\[ \sigma_{\max} [X-Y] \leq \sigma_{\max} [X] + \sigma_{\max} [Y] \] (4.22)

\[ \sigma_{\max} [X^{-1} Y] \leq \sigma_{\max} [X] \sigma_{\min} [Y] \sigma_{\max} [Z] \] (4.23)

and equation (4.20), an upper bound on the maximum singular value of the loop-transfer function of the slow loop can be calculated. This bound (denoted \( UB\{\sigma_{\max} [T_s(j\omega)]\} \)) is given by:

\[ UB\{\sigma_{\max} [T_s(j\omega)]\} \Delta \sigma_{\max} [G_{11} K_1] + \sigma_{\max} [G_{12} K_2] \sigma_{\min} [G_{22} K_2] \sigma_{\max} [G_{21} K_1] \] (4.24)

Similarly, using equations (4.21) and (4.23) and the singular value inequality

\[ \sigma_{\min} [X-Y] \geq \sigma_{\min} [X] - \sigma_{\max} [Y] \] (4.25)

a lower bound on the minimum singular value of the loop transfer function of the slow loop can be calculated. This bound (denoted \( LB\{\sigma_{\min} [T_s(j\omega)]\} \)) is given by:

\[ LB\{\sigma_{\min} [T_s(j\omega)]\} \Delta \sigma_{\min} [G_{11} K_1] - \sigma_{\max} [G_{12} K_2] \sigma_{\min} [G_{22} K_2] \sigma_{\max} [G_{21} K_1] \] (4.26)

Using equations (4.17), (4.24), and (4.26), the results in equations (4.10) to (4.16) simplify as follows:
Define the real scalars $\xi(\omega)$ and $\eta(\omega)$ by:

$$\xi(\omega) = \left[ 1 + \frac{\sigma_{\max}^2 [G_{12}]}{\sigma_{\max}^2 [G_{22}]} + \frac{\sigma_{\max}^2 [G_{21}K_1]}{\sigma_{\max}^2 [G_{22}K_2]} \right]^{1/2}$$

$$\eta(\omega) = \left[ 1 + \frac{\sigma_{\max}^2 [G_{12}K_2]}{\sigma_{\min}^2 [G_{22}K_2]} + \frac{\sigma_{\max}^2 [G_{21}K_1]}{\sigma_{\max}^2 [G_{22}K_2]} \right]^{1/2}$$

(4.27)  

(4.28)

Suppose that for the passband of the fast loop design

$$LB\{\sigma_{\min} [T_s(j\omega)]\} \geq 0$$

(4.29)

and that

$$\frac{\sigma_{\min} [T_f(j\omega)]}{UB\{\sigma_{\max} [T_s(j\omega)]\}} \geq \rho_1 >> 1$$

(4.30)

and that

$$\frac{\sigma_{\min} [G_{22}K_2]}{\sigma_{\max} [G_{11}K_1]} \geq \rho_2 >> 1$$

(4.31)

then for the passband of the fast loop

$$\sigma_{\max} [T_s(j\omega)] \leq \sigma_{\max} [T_f(j\omega)] \cdot \xi(\omega)$$

(4.32)

$$\sigma_{\min} [T_G(j\omega)] \geq LB\{\sigma_{\min} [T_s(j\omega)]\} \cdot \frac{1}{\eta(\omega)}$$

(4.33)

Suppose that for the stopband of the slow loop design

$$\sigma_{\min} [G_{11}K_1] - \sigma_{\max} [G_{12}K_2] \sigma_{\min}^{-1} [G_{22}K_2] \sigma_{\max} [G_{21}K_1] \geq 0$$

(4.34)

and that

$$\frac{\sigma_{\min} [G_{22}K_2]}{\sigma_{\max} [G_{11}K_1] + \sigma_{\max} [G_{12}K_2] \sigma_{\min}^{-1} [G_{22}K_2] \sigma_{\max} [G_{21}K_1]} \geq \rho_1 >> 1$$

(4.35)
and that

$$\frac{\sigma_{\min}[G_{22}K_2]}{\sigma_{\max}[G_{11}K_1]} \geq \rho_2 \gg 1$$  \hspace{1cm} (4.36)$$

then for the stopband of the slow loop.

$$\sigma_{\max}[T_G(j\omega)] \leq \sigma_{\max}[T_f(j\omega)] \cdot \xi(\omega)$$  \hspace{1cm} (4.37)$$

$$\sigma_{\min}[T_G(j\omega)] \geq \sigma_{\min}[G_{11}K_1] - \sigma_{\max}[G_{12}K_2] \frac{1}{\sigma_{\min}[G_{22}K_2] \sigma_{\max}[G_{21}K_1]}$$  \hspace{1cm} (4.38)$$

At least for the passband of the fast loop, these results (eqns. (4.32),(4.33), (4.37),(4.38)) show relationships between the global system and the individual OGAT designs. Unfortunately, rather than containing the actual singular values, these results contain bounds on the singular values of the loop transfer function of the slow loop. As will be discussed, it is for this reason that these results have not proven useful for improving the general OGAT design procedure.

4.4 Discussion of Limitations

In order to use the results of section 4.3 for design, an iterative method must be developed to satisfy the necessary conditions (4.29) to (4.31) and (4.34) to (4.36)), approximations must be developed which render \(\xi(\omega)\) and \(\eta(\omega)\) compensator independent, and a method must be developed to translate global system specifications to slow loop and fast loop specifications. Even if \(\eta(\omega)\) is compensator independent, translating global system command following and disturbance rejection specifications to equivalent specifications on the slow loop design is difficult because equation (4.33) contains a
lower bound on the singular values of the loop transfer function of the slow loop rather than the minimum singular value itself. Further, some of the necessary conditions (eqns. (4.29),(4.30),(4.34), and (4.35)) cannot be checked until after the OGAT designs have been completed, and it is not clear how to proceed if equations (4.29) and (4.34) do not hold for an initial design. All of these considerations indicate that the design of a global SIAO system using the OGAT design procedure would inevitably involve design iterations.

With this in mind, it seems rather wasteful to have a designer produce design iterations which attempt to meet necessary conditions ((4.29) to (4.31) and (4.34) to (4.36)), for the bounded relationships between the global system and the OGAT designs (eqns. (4.32),(4.33),(4.37),(4.38)), when the actual global system singular values are available at each design iteration. Thus, the results of section 4.3 will not be used to develop a modified step-by-step OGAT design procedure.

4.5 Concluding Remarks

This chapter provided an analysis of the OGAT design procedure for SIAO systems. The results of section 4.3 demonstrate a bounded relationship between the global system and the individual OGAT designs. As they are not explicit, these relationships only provide the designer with a qualitative understanding of the results of the OGAT design method rather than with a general step-by-step design process.

The limitations of the results of section 4.3 may stem from the fact that only singular value information was used from each of the block elements of the loop transfer function of the global system. It is the author's belief that further research which
is based on the results of section 4.3 and which incorporates the directional information of the block elements of the global system loop transfer function may lead to additional progress.

Even without such progress, time-scale separation designs for SIAO systems are possible using only the basic OGAT design procedure. Such a design is illustrated in chapter 5 for the AV-8A Harrier aircraft.
CHAPTER 5
INTEGRATED FLIGHT CONTROL FOR A AV-8A HARRIER AIRCRAFT MODEL

5.1 Introduction

This section presents an OGAT design study for the Marine Corps AV-8A Harrier Aircraft. Historically this design study began with investigations of different MIMO designs involving the thrust vectoring capability of the aircraft. Due to the inherent physical time-scale separation between flight path and attitude dynamics, these early investigations quickly exposed the need to design time-scale separated I/O groups and the limitations of the basic OGAT design procedure. Thus, the analysis of chapters 3 and 4 was motivated by our desire to complete good time-scale separated designs for the Harrier. As discussed in chapter 4, however our analysis results have only limited usefulness for general SIAO systems. Hence, the basic OGAT design procedure continued to be the tool for the design results presented here.

Section 5.2 presents a linear model of the Harrier and section 5.3 presents the choice of desired low bandwidth and high bandwidth outputs as well as the specifications used for the OGAT designs. Section 5.4 presents the LQG/LTR design methodology used for each of the OGAT designs and sections 5.5 and 5.6 present the slow and fast loop designs. The global system properties are the examined in section 5.7. These show that the basic OGAT design procedure can produce good global system loop shapes for systems, such as the AV-8A, which naturally exhibit the time-scale separation in their dynamics.
5.2 Linear Aircraft Model

This section presents a linear model of the longitudinal dynamics of the AV-8A Harrier aircraft at a medium speed (189kt) flight condition. For readers unfamiliar with aircraft and related terminology, reference [10] provides a good background. Hereafter, a certain basic familiarity with aircraft dynamics and will be assumed.

The longitudinal motion (wings-level motion in the vertical plane) of an aircraft subject to aerodynamic, propulsive, and gravitational forces is described by force equations (relating the motion of the center of mass to the external forces) and moment equations (relating the rotation about the center of mass to the external moments). These equations are highly non-linear functions of many variables. However, it is well known that linearized versions provide an excellent representation of the longitudinal motion of an aircraft about an operating point. It is this linearization method which was used to produce a linear model of the longitudinal dynamics of the Harrier.

The variables which describe the longitudinal dynamics of the Harrier are defined in figure 5.1, and the medium speed flight condition (operating point) of interest for this design is given in table 5.1. The non-linear longitudinal equations of motion of the Harrier as well as the linearization of these equations around the operating point of table 5.1 are given in reference [11]. As derived in reference [11], the state-space representation of the longitudinal dynamics of the Harrier is given by:

\[ \dot{x}_H(t) = A_H x_H(t) + B_H u_H(t) \]  \hspace{1cm} (5.1)
\( \theta(t) \): Pitch Attitude, rad
\( q(t) \): Pitch Rate = \( \dot{\theta}(t) \), \( \frac{\text{rad}}{\text{sec}} \)
\( \gamma(t) \): Flight Path Angle, rad
\( v(t) \): Velocity, \( \frac{\text{ft}}{\text{sec}} \)
\( \delta_n(t) \): Stabilizer Angle, deg
\( \theta_n(t) \): Engine Nozzle Angle, deg
\( N(t) \): Engine Fan Speed, %
\( T(N) \): Engine Thrust - a function of \( N(t) \)
\( N_c(t) \): Commanded Fan Speed - input to the fan actuator
\( \delta_{nc}(t) \): Commanded Stabilizer Angle - input to the stabilizer actuator
\( \theta_{nc}(t) \): Commanded Engine Nozzle Angle - input to the nozzle actuator

Fig. 5.1. Definition of Variables for the Longitudinal Dynamics of
the AV-8A Harrier Aircraft
where

\[ x_H(t) = \begin{bmatrix} \theta(t) \\ q(t) \\ \gamma(t) \\ v(t) \\ N(t) \\ \delta_n(t) \\ \theta_n(t) \end{bmatrix} \text{ (rad)} \]

\[ u_H(t) = \begin{bmatrix} N_c(t) \\ \theta_{nc}(t) \\ \delta_{nc}(t) \end{bmatrix} \text{ (%)} \]

and where \( A_H \) and \( B_H \) are matrices with numerical values given in Appendix D.1.

Of the many designs completed using the model of equation (5.1), this chapter will present the design which uses the three controls \((N_c(t), \theta_{nc}(t), \text{ and } \delta_{nc}(t))\) to control velocity \((v(t))\), flight path angle \((\gamma(t))\), and pitch attitude \((\theta(t))\). For each of these outputs, the corresponding control will be chosen using physical considerations. The stabilizer \((\delta_{nc}(t))\) will be used to control the pitch attitude since the stabilizer generates much more pitching moment than lift or drag. Fanspeed and nozzle angle, on the other hand, produce primarily forces on the aircraft with relatively lower moments. Hence, these two inputs will be used to control flight path angle and velocity. From this discussion, the state-space output equation is given by:

\[ y_H(t) = C_H x_H(t) \] \hspace{1cm} (5.1a)

where

\[ y_H(t) = \begin{bmatrix} v(t) \\ \gamma(t) \\ \theta(t) \end{bmatrix} \text{ (ft/sec)} \]

\[ \text{ (deg)} \]
### TABLE 5.1
OPERATING POINT VALUES OF LONGITUDINAL VARIABLES

<table>
<thead>
<tr>
<th>Variables</th>
<th>Operating point value</th>
</tr>
</thead>
<tbody>
<tr>
<td>θ  (pitch attitude)</td>
<td>.1398 rad</td>
</tr>
<tr>
<td>q  (pitch rate)</td>
<td>0 ( \frac{\text{rad}}{\text{sec}} )</td>
</tr>
<tr>
<td>γ  (flight path angle)</td>
<td>0 rad</td>
</tr>
<tr>
<td>v  (velocity)</td>
<td>325 ( \frac{\text{ft}}{\text{sec}} )</td>
</tr>
<tr>
<td>δ  (stabilizer angle)</td>
<td>-4.923 deg</td>
</tr>
<tr>
<td>( \theta_n ) (engine nozzle angle)</td>
<td>1.026 deg</td>
</tr>
<tr>
<td>N  (engine fan speed)</td>
<td>69.31%</td>
</tr>
<tr>
<td>( N_c ) (commanded fan speed)</td>
<td>69.31%</td>
</tr>
<tr>
<td>( \delta_{nc} ) (commanded stabilizer angle)</td>
<td>-4.923 deg</td>
</tr>
<tr>
<td>( \theta_{nc} ) (commanded engine nozzle angle)</td>
<td>1.026 deg</td>
</tr>
</tbody>
</table>

### TABLE 5.2
MODE CHARACTERISTICS

<table>
<thead>
<tr>
<th></th>
<th>Short-Period</th>
<th>Phugoid</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_n ) (natural frequency)</td>
<td>1.687 ( \frac{\text{rad}}{\text{sec}} )</td>
<td>.0973 ( \frac{\text{rad}}{\text{sec}} )</td>
</tr>
<tr>
<td>T (Period)</td>
<td>3.72 sec</td>
<td>64.6 sec</td>
</tr>
<tr>
<td>( \xi ) (Damping coefficient)</td>
<td>.72</td>
<td>.25</td>
</tr>
<tr>
<td>( t_{1/2} ) (Time to 1/2 magnitude)</td>
<td>.566 sec</td>
<td>28.95 sec</td>
</tr>
</tbody>
</table>
and where \( \Gamma_{H} \) is a matrix with numerical values given in Appendix D.1.

The open loop transfer function matrix is

\[
\Gamma_{H}(s) = \Gamma_{H}(sI-A_{H})^{-1}B_{H} \tag{5.2}
\]

As expected, the longitudinal dynamics of the linear model of the Harrier are composed of a fast (short-period) mode and a slow (phugoid) mode. The characteristics of these fast and slow modes are shown in table 5.2. Further, as expected, the response of velocity and flight path angle to any of the controls is dominated by the phugoid model while the pitch angle responses include both the short-period and phugoid modes. Thus, the linear model of the Harrier (eqns. (5.1) and (5.2)) has an inherent time-scale separation.

This time-scale separation also manifests itself in the singular values of the open-loop plant \( \Gamma_{H}(s) \) (figure 5.2). Since the linear model of the Harrier clearly has an inherent time-scale separation, the design of bandwidth separated I/O groups using the OGAT design procedure seems adequately justified.

The next section presents the specifications on the time-scale separation design.

5.3 Specifications

This section presents the choice of the desired high and low bandwidth outputs, a discussion of scaling issues, the global system specifications, and the specifications used for the fast and slow loop designs.

Driven by the inherent time-scale of each output as discussed in section 5.2, the chosen two low bandwidth outputs \( (\gamma_1) \) and high bandwidth scalar output \( (\gamma_2) \)
Fig. 5.2. Open-loop Singular Values of the Linear Model
of the Harrier, from eqn. (5.2)
(along with the corresponding controls chosen in section 5.2) are given by:

\[
\begin{align*}
\chi_1 &= \begin{bmatrix} v \\ \gamma \end{bmatrix} \\
u_1 &= \begin{bmatrix} N_c \\ \theta \\
nc \end{bmatrix} \\
y_2 &= \theta \\
u_2 &= \delta \\
\end{align*}
\]  

(5.3) (5.4) (5.5) (5.6)

Fortunately, the state-space and transfer function representations of the Harrier plant (eqns. (5.1) and (5.2)) are in the assumed global plant form (fig. 2.6) for the above choice of low and high bandwidth outputs. Thus, the following discussions will use the notation, terminology, and results of the previous chapters.

Although chapter 4 did not establish tight relationships between the global SIAO system loop shapes and the loop shapes of the fast and slow loops, the bounding results of chapter 4 (eqns. (4.32) and (4.33)) do contain singular value degradation factors \((\xi(\omega) \text{ and } \eta(\omega))\) analogous to the degradation factor of global TITO systems \((\beta(\omega))\). As discussed in chapter 3, the global TITO system will have good loop shapes when the individual OGAT designs have good loop shapes and the value of \(\beta(\omega)\) is not too large. Qualitatively, this same property applies to the degradation factors of chapter 4 with respect to the loop shapes of the global SIAO system.

Since large degradation factors are undesirable, we will consider the largest of the degradation factors of chapter 4. This degradation factor \((\eta(\omega))\) is given by:
\[ \eta(\omega) = \left[ 1 + \frac{\sigma_{\text{max}}^2[G_{12}K_2] + \sigma_{\text{max}}^2[G_{21}K_1]}{\sigma_{\text{min}}^2[G_{22}K_2]} \right]^{1/2} \]  

(5.7)

Using the definitions of the high bandwidth output and control (eqns. (5.5) and (5.6)) and the low bandwidth outputs and controls for the Harrier, this degradation factor reduces to:

\[ \eta(\omega) = \left[ 1 + \frac{\sigma_{\text{max}}^2[G_{12}]}{|g_{22}|^2} + \frac{\sigma_{\text{max}}^2[G_{21}K_1]}{|g_{22}K_2|^2} \right]^{1/2} \]  

(5.8)

Clearly, in order for \( \eta(\omega) \) to be small, the compensator independent ratio, \( rt_1(\omega) \),

\[ rt_1(\omega) \triangleq \frac{\sigma_{\text{max}}[G_{12}]}{|g_{22}|} \]  

(5.9)

must be small. The plant dependent ratio,

\[ rt_2(\omega) = \frac{\sigma_{\text{max}}[G_{21}]}{|g_{22}|} \]  

(5.10)

will also provide a priori information of the size of \( \eta(\omega) \). Figure 5.3 shows the curves of \( rt_1(\omega) \) and \( rt_2(\omega) \) vs frequency for the Harrier model defined by equations (5.1) to (5.6). Clearly, since \( rt_1(\omega) \) is large \( \eta(\omega) \) will be quite large regardless of the choice of compensators. Fortunately, as discussed in Appendix E, scaling the outputs and controls on the plant can alter the ratios \( rt_1(\omega) \) and \( rt_2(\omega) \) without changing the open-loop relationship between each control and output within an I/O group (i.e. \( G_{11}(j\omega) \) and \( g_{22}(j\omega) \) are unaffected by the scaling).
Fig. 5.3. Illustration of the Ratios $r_{t1}(\omega)$ and $r_{t2}(\omega)$, from eqns. (5.9) and (5.10), for the Linear Model of the Harrier Given by Equations (5.1) and (5.2)
Consider the scaled Harrier plant given by:

\[ \dot{x}_{\text{HS}}(t) = A_{\text{HS}} x_{\text{HS}}(t) + B_{\text{HS}} u_{\text{HS}}(t) \]

\[ y_{\text{HS}}(t) = C_{\text{HS}} x_{\text{HS}}(t) \]

\[ G_{\text{HS}}(s) = C_{\text{HS}} (sI - A_{\text{HS}})^{-1} B_{\text{HS}} \]  

(5.11)  

(5.12)

where \( x_{\text{HS}}(t) = x_H(t) \)

\[
\begin{bmatrix}
N_c(t) \\
\theta_{nc}(t) \\
\delta_{nc}(t)
\end{bmatrix}
\]

(tenths of pct) (tenths of deg) (deg)

\[ u_{\text{HS}}(t) \]

\[
\begin{bmatrix}

v(t) \\
\gamma(t) \\
\theta(t)
\end{bmatrix}
\]

(tenths of \( \text{ft/sec} \)) (tenths of deg) (deg)

and where \( A_{\text{HS}}, B_{\text{HS}}, \) and \( C_{\text{HS}} \) are matrices with numerical values given in Appendix D.2.

The only difference between the scaled plant (eqns. (5.11) and (5.12)) and the unscaled plant (eqns. (5.1) and (5.2)) is that the units on the low bandwidth outputs and controls of the scaled plant have been scaled down by a factor of 10. Figure 5.4 shows the ratios, \( r_{t_1}(\omega) \) and \( r_{t_2}(\omega) \), for this scaled plant. Clearly, scaling has reduced the value of \( \eta(\omega) \) to a reasonably small value (assuming that \( \sigma_{\max}[K_1] \) is not much greater than \( |k_{t_2}| \)). For this reason the global system should be expected to have good loop shapes whenever the individual OGAT designs have good loop shapes. Hereafter, the scaled Harrier plant of equations (5.11) and (5.12) will be used as the plant.
Fig. 5.4. Illustration of the Ratios $r_1(\omega)$ and $r_2(\omega)$, from eqns. (5.9) and (5.10), for the Scaled Linear Model of the Harrier Given by Equations (5.11) and (5.12)
for the OGAT designs and thus all references to the global plant will be interpreted as references to its scaled version.

As discussed earlier, the compensator design for the Harrier will follow the basic OGAT design procedure. This means that design specifications will be imposed directly on the fast and slow loops. The two separate designs will be completed and the global system loop properties will be checked to insure that the combination of the fast loop and slow loop has "nice" properties (i.e. good robustness, sensitivities that are not too large, etc.). Design iterations will then be conducted until the global system has the desired properties. Hence, the specifications include the chosen fast loop and slow loop specifications along with some global specifications.

**Fast Loop Specifications**

For this design, the fast loop specifications (chosen from perceived pilot preferences and physical considerations) are given by:

**F1:** The fast loop has a step response peak time of less than 1 second (i.e. a bandwidth greater than \( \frac{\pi \text{ rad}}{\text{sec}} \)).

**F2:** The fast loop is self-trimming in the face of dc disturbances (i.e. step responses have zero steady-state error).

**F3:** The fast loop is robust to modeling errors bounded by a gain of 20 due to structural resonances above \( 40 \frac{\text{rad}}{\text{sec}} \) (i.e. \( \sigma [1 + t_f^{-1}(j\omega)] > 20 \) for \( \omega > 40 \frac{\text{rad}}{\text{sec}} \)).

**Slow Loop Specifications**

The slow loop specifications are given by:

**S1:** The slow loop has a bandwidth of \( .1 \frac{\text{rad}}{\text{sec}} \).
S2: Command following for the slow loop is within 10% for frequencies below .02 rad/sec (i.e. $\sigma_{\min\left\{T_s(j\omega)\right\}} > 9$ for $\omega < .02 \text{ rad/sec}$).

Global Specifications

The desired global system properties are given by:

G1: The global system sensitivity to disturbances is less than 2 for all frequencies (i.e. $\sigma_{\min\left\{I + T_s(j\omega)\right\}} < .5$ for all $\omega$).

G2: Due to unmodeled bending modes, the global system must be robust to a multiplicative modeling error bounded by a gain of 20 for frequencies above 40 rad/sec (i.e. $\sigma_{\min\left\{I + T_s^{-1}(j\omega)\right\}} < 20$ for $\omega > 40 \text{ rad/sec}$).

Recapitulation

This section has defined the global plant for the Harrier by choosing the desired low bandwidth and high bandwidth outputs and by considering scaling issues. Further, this section stated the fast and slow loop specifications along with some desired properties of the global system. The next section presents the LQG/LTR design methodology used in the individual OGAT designs.

5.4 The LQG/LTR Methodology

A state-space representation of a design model is:

\[ \begin{aligned}
\dot{x}(t) &= A x(t) + B u(t) \\
\chi(t) &= C x(t) \\
G(s) &= C(sI-A)^{-1}B
\end{aligned} \] (5.13) (5.14)
The LQG compensator for this model is given by:

$$K(s) = G[sI-A+B \ G+H \ C]^{-1}H$$  \hspace{1cm} (5.15)

The gain matrices, $G$ and $H$, are given by:

Control Gain: $G = R^{-1}B^T K$  \hspace{1cm} (5.16)

Filter Gain: $H = \Sigma C^T \Theta^{-1}$  \hspace{1cm} (5.17)

where $K$ and $\Sigma$ are the solutions of the Riccati equations:

Control Algebraic Riccati Equations (CARE)

$$KA + A^TK + Q - KB R^{-1}B^TK = 0$$  \hspace{1cm} (5.18)

Filter Algebraic Riccati Equation (FARE)

$$\Sigma A + A^T\Sigma + \Xi - \Sigma C^T\Theta^{-1} C \Xi = 0$$  \hspace{1cm} (5.19)

and the matrices $Q$, $R$, $\Xi$, and $\Theta$ are chosen so that the closed-loop system meets the specifications.

The Loop Transfer Recovery (LTR) results [1,12] state that if $Q = QC^TC$ and $R = I$, then

$$\sigma_{\text{in}} [G(j\omega)K(j\omega)] + \sigma_{\text{in}} [C(j\omega I-A)^{-1}H] \text{ as } q^{\infty}$$  \hspace{1cm} (5.20)

provided that the design model (eqn. (5.14)) does not have any right-half plane zeroes.
Thus, if a designer can shape the singular values

\[ \sigma_1 [C(j\omega I - A)^{-1}H] \]

to meet the posed specifications, then he/she can use the loop transfer recovery method to recover those singular values.

It is well known [13] that if \( Z = L^T \) and \( \Theta = \mu I \) are used in the solution of the FARE (eqn. (5.19)), then

\[ \sigma_1 [C(sI-A)^{-1}H] \approx \frac{1}{\sqrt{\mu}} \sigma_1 [C(sI-A)^{-1}L] \]  \hspace{1cm} (5.21)

when

\[ \frac{1}{\sqrt{\mu}} \sigma_1 [C(sI-A)^{-1}L] \gg 1 \]

Thus, a designer can shape

\[ \sigma_1 [C(sI-A)^{-1}H] \]

by choosing \( L \) and \( \mu \) so that

\[ \frac{1}{\sqrt{\mu}} \sigma_1 [C(sI-A)^{-1}L] \]

meets the system specifications.

Summarizing the above discussion results in the following step-by-step description of the LQG/LTR methodology.

**Step 1:** Choose \( L \) and \( \mu \) so that

\[ \frac{1}{\sqrt{\mu}} \sigma_1 [C(sI-A)^{-1}L] \]

meets the system specifications.
Step 2: Solve the FARE (eqn. (5.19)) with $\Xi = L^T L$ and $\Theta = \mu I$ and check that $\sigma_1 [C(sI - A)^{-1} H]$ meets the system specifications.

Step 3: Solve the CARE (eqn. (5.18)) with $Q = qC^T C$ and $R = I$ using a "large" value of $q$ ($q \to \infty$).

Step 4: Calculate the resultant LQG compensator $K(s)$ (eqn. (5.15)) and check that the loop singular values $\sigma_1 [G(j\omega)K(j\omega)]$ meet the system specifications. If not, and the design model does not have any right half plane zeroes, then return to step 3 and increase $q$.

In general, the LQG/LTR compensator as defined above may not be sufficient to shape the singular values of the loop transfer function of a system to meet its specifications. In this case, the nominal plant ($G_p(s)$) is augmented by adding dynamics at the plant input ($G_a(s)$). The design model ($G_d(s)$) is then used in the LQG/LTR methodology where

$$G_d(s) = \frac{G_p(s)}{G_a(s)}$$  \hspace{1cm} (5.22)

Using this design model and the step-by-step description of the LQG/LTR design methodology, a compensator ($K(s)$) can be determined so that the singular values of the loop transfer function ($G_d(s)K(s)$) meet the system specifications. The overall compensator ($K_{LQG}(s)$) is then given by:

$$K_{LQG}(s) = G_a(s)K(s)$$  \hspace{1cm} (5.23)

This dynamic augmentation approach to loop shaping provides added flexibility for the designer.
The previous discussion provides a method for shaping the loop transfer function of a system. As discussed in chapter 2, if all the system specifications are translated to the loop transfer function, then the loop transfer function must have a "nice" crossover frequency behavior. Fortunately, LQG compensators provide every minimum phase system with a nice crossover range due to the asymptotic \(q \to \infty\) LQG/LTR properties:

1. \(\sigma_{\min}[I + G(j\omega)K(j\omega)] > 1\) for all \(\omega\)
2. \(\sigma_{\min}[I + (G(j\omega)K(j\omega))^{-1}] > 1/2\) for all \(\omega\)
   as \(q \to \infty\)

Now that the specifications on the fast and slow loops have been established and the design methodology has been described, the two designs can be presented.

5.5 Fast Loop Design

This section presents the design of the fast loop. The state-space representation of the SISO fast loop plant is given by:

\[
\begin{align*}
\dot{x}_f(t) &= A_f x_f(t) + B_f u_2(t) \\
y_2(t) &= C_f x_f(t) \\
g_f(s) &= C_f(sI - A_f)^{-1}B_f
\end{align*}
\]

where

\[
x_f(t) = \begin{bmatrix} \theta(t) \\ q(t) \\ \gamma(t) \\ v(t) \\ \delta_n(t) \end{bmatrix} \quad \text{rad, rad/sec, rad, ft/sec, deg}
\]
where $A_{fd}$, $B_{fd}$, and $C_{fd}$ are matrices with numerical values given in Appendix D.3. Since one of the fast loop specifications (F2) is for zero steady-state error to a step input, an integrator must be augmented to the input of the plant, i.e. at the stabilizer control channel. Including the appended integrator, the SISO design model for the fast loop is given by:

$$\begin{align*}
\dot{x}_{fd}(t) &= A_{fd}x_{fd}(t) + B_{fd}u_{fd}(t) \\
y_2(t) &= C_{fd}x_{fd}(t) \\
g_{fd}(s) &= C_{fd}(sI - A_{fd})^{-1}B_{fd}
\end{align*}$$ (5.26)

where $A_{fd}$, $B_{fd}$, and $C_{fd}$ are matrices with numerical values given in Appendix D.4.

Using the LQG/LTR design methodology, values of $L_{fd}$, $\mu_{fd}$, and $q_{fd}$ were chosen (numerical values in Appendix D.4) such that the resulting compensator shaped the loop transfer, return difference, and inverse return difference functions of the fast loop to meet the appropriate specifications. The resulting fast loop compensator (not including the appended dynamics) is given by:

$$\begin{align*}
\dot{x}_{fc}(t) &= A_{fc}x_{fc}(t) + B_{fc}e_{fc}(t) \\
u_{fd}(t) &= C_{fc}x_{fc}(t) \\
k_{fd}(s) &= C_{fc}(sI - A_{fc})^{-1}B_{fc}
\end{align*}$$ (5.28)

where $A_{fc}$, $B_{fc}$, and $C_{fc}$ are matrices with numerical values given in Appendix D.5, and the overall fast loop compensator (including appended dynamics) is given by:

$$\begin{align*}
\dot{x}_{LQGf}(t) &= A_{LQGf}x_{LQGf}(t) + B_{LQGf}e_{fc}(t) \\
u_2(t) &= C_{LQGf}x_{LQGf}(t)
\end{align*}$$ (5.30)
\begin{align}
    k_{LQGf}(s) &= C_{LQGf} \left( sI - A_{LQGf} \right)^{-1} B_{LQGf} 
    \tag{5.31}
\end{align}

where \( A_{LQGf} \), \( B_{LQGf} \), and \( C_{LQGf} \) are matrices with numerical values given in Appendix D.6. Figure 5.5 shows a block diagram of the fast loop which clearly illustrates the interconnections between the plant, appended dynamics, and compensator.

The loop transfer, return difference, and inverse return difference functions of the fast loop are given by:

\begin{align}
    t_f(s) &= g_f(s) k_{LQGf}(s) 
    \tag{5.32} \\
    1 + t_f(s) &= 1 + g_f(s) k_{LQGf}(s) 
    \tag{5.33} \\
    1 + t_f^{-1}(s) &= 1 + (g_f(s) k_{LQGf}(s))^{-1} 
    \tag{5.34}
\end{align}

Figure 5.6 shows the frequency response of the loop transfer function of the fast loop as well as the specification boundaries. Similarly, figure 5.7 shows the magnitude of the inverse return difference function of the fast loop and the frequency domain specification boundaries. Clearly, this fast loop design meets the fast loop specifications of section 5.2.

5.6 Slow Loop Design

With the fast loop closed, the TITO model of the slow loop plant is given by:

\begin{align}
    \dot{x}_s(t) &= A_s x_s(t) + B_s u_1(t) \\
    y_1(t) &= C_s x_s(t) \\
    G_s(j\omega) &= C_s (j\omega I - A_s)^{-1} B_s 
    \tag{5.36}
\end{align}
Fig. 5.5. Block Diagram of the Fast Loop Plant (given by equation (5.25)), Appended Dynamics, and Compensator Representations (k_FD(s) and k_LQG_F(s) given by eqns. (5.29) and (5.31))
Fig. 5.6. Frequency Response of the Loop Transfer

Function of the Fast Loop ($|t_f(j\omega)|$) given by equation (5.32)
Fig. 5.7. Magnitude of the Inverse Return Difference

Function of the Fast Loop, from eqn. (5.34)
where $A_{sd}$, $B_{sd}$, and $C_{sd}$ are matrices with numerical values given in Appendix D.7.

The two singular values of the slow loop plant ($G_{sd}(j\omega)$) are shown in figure 5.8.

From figure 5.8 it is fairly clear that dynamic augmentation is needed to meet the
command following and bandwidth specifications on the slow loop. Hence, a simple pole
at $0.01$ rad/s was augmented to each input channel of the slow loop plant (i.e. the
engine nozzle angle and fan speed control channels). Since the state-space repre-
sentation of the slow loop plant (eqn. (5.35)) is very high order, model reduction
techniques [14] were used to produce the reduced order slow loop design model
(including appended simple poles) given by:

\[
\begin{align*}
\dot{x}_{sd}(t) &= A_{sd} x_{sd}(t) + B_{sd} u_{sd}(t) \\
\dot{y}_1(t) &= C_{sd} x_{sd}(t) \\
G_{sd}(j\omega) &= C_{sd} (j\omega I - A_{sd})^{-1} B_{sd}
\end{align*}
\] (5.37)

where $A_{sd}$, $B_{sd}$, and $C_{sd}$ are given in Appendix D.8.

Using the LQG/LTR design methodology values of $L_{sd}$, $u_{sd}$, and $q_{sd}$ (given in Appen-
dix D.8) were chosen so that the resulting compensator shaped the loop transfer,
return difference, and inverse return difference functions of the slow loop to meet
the specifications of section 5.2. The resulting fast loop compensator (not
including appended dynamics) is given by:

\[
\begin{align*}
\dot{x}_{sc}(t) &= A_{sc} x_{sc}(t) + B_{sc} e_{sc}(t) \\
u_{sd}(t) &= C_{sc} x_{sc}(t) \\
K_{sc}(j\omega) &= C_{sc} (j\omega I - A_{sc})^{-1} B_{sc}
\end{align*}
\] (5.39)
Fig. 5.8. Singular Values of the Slow Loop Plant,
from eqn. (5.36)
where \( A_{LQG_s}, B_{LQG_s}, \) and \( C_{LQG_s} \) are given in Appendix D.9, and the overall slow compensator (including appended dynamics) is given by:

\[
\begin{align*}
\dot{x}_{LQG_s}(t) &= A_{LQG_s}x_{LQG_s}(t) + B_{LQG_s}e_{sc}(t) \\
    &\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
Fig. 5.9. Block Diagram of the Slow Loop Plant, from eqn. (5.36), Appended Dynamics, and Compensator

Representations $K_{sc}(s)$ and $K_{LQGs}(s)$ given by eqns. (5.40) and (5.42)
Fig. 5.10. Singular Values of the Loop Transfer Function of the Slow Loop, from eqn. (5.43)
From equations (5.30), (5.31), (5.41), and (5.42), the global compensator is given by:

\[
\begin{bmatrix}
\dot{x}_{LQG_s}(t) \\
\dot{x}_{LQG_f}(t)
\end{bmatrix}
= 
\begin{bmatrix}
A_{LQG_s} & 0 \\
0 & A_{LQG_f}
\end{bmatrix}
\begin{bmatrix}
x_{LQG_s}(t) \\
x_{LQG_f}(t)
\end{bmatrix}
+ 
\begin{bmatrix}
B_{LQG_s} \\
B_{LQG_f}
\end{bmatrix}
\begin{bmatrix}
e_{sc} \\
e_{fc}
\end{bmatrix}
\]

(5.46)

\[
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
= 
\begin{bmatrix}
C_{LQG_s} & 0 \\
0 & C_{LQG_f}
\end{bmatrix}
\begin{bmatrix}
x_{LQG_s}(t) \\
x_{LQG_f}(t)
\end{bmatrix}
\]

(5.47)

Using the global compensator (eqns. (5.46) and (5.47)) and the global plant (eqns. (5.11) and (5.12)), the loop transfer, return difference, and inverse return difference functions of the global system are given by:

\[
T_G(s) = G_{HS}(s)K_{LQG}(s)
\]

(5.48)

\[
\frac{1}{1 + T_G(s)} = \frac{I + G_{HS}(s)K_{LQG}(s)}{1 + G_{HS}(s)K_{LQG}(s)}
\]

(5.49)

\[
\frac{1}{1 + T_G^{-1}(s)} = \frac{I + [G_{HS}(s)K_{LQG}(s)]^{-1}}{1 + [G_{HS}(s)K_{LQG}(s)]^{-1}}
\]

(5.50)
Figure 5.11 shows the singular values of the loop transfer function of the global system (equation (5.48)) along with the singular values of the loop transfer functions of the slow loop and fast loop (equations (5.32) and (5.43)). Figure 5.12 shows the singular values of the return difference function of the global system, slow loop, and fast loop as well as the global system sensitivity boundary. Similarly, figure 5.13 shows the singular values of the inverse return difference function of the global system, slow loop, and fast loop and the global system robustness boundary. These figures show that the global system has all of the desired properties stated in section 5.2. Further, these figure illustrate that the global system singular value shapes are very near to the corresponding shapes of the slow and fast loops when the system has light coupling (i.e. when $\eta(\omega)$ of equation (5.8) is small).

Time responses to step inputs of the global closed-loop system and the fast and slow closed-loop systems confirm these properties. By definition, the closed-loop slow loop system is equivalent to the global closed-loop system. Thus, the time responses of these systems are identical. Figure 5.14 shows the time response of the global closed-loop system (and the closed-loop slow loop) to a step input in the velocity reference input ($r_v$). Similarly, figure 5.15 shows the response of the global closed-loop system to a step input in the flight path angle reference input ($r_\gamma$). Figure 5.16 shows the response of the global closed-loop system and the closed-loop fast loop to a step input in the pitch attitude reference input ($r_\theta$). These figures clearly illustrate the desired time-scale separation between the response times of the two I/O groups. Further, these figures show that the time responses of the global closed-loop system are highly correlated with the time responses of the fast and slow closed-loop systems.
Fig. 5.11. Singular Values of the Loop Transfer Function of the Global System, Fast Loop, and Slow Loop, from eqns. (5.32), (5.43), and (5.48)
Fig. 5.12. Singular Values of the Return Difference

Function of the Global System, Fast Loop, and Slow Loop, from eqns. (5.33), (5.44), and (5.49)
Fig. 5.13. Singular Values of the Inverse Return Difference Function of the Global System, Fast Loop, and Slow Loop, from eqns. (5.34), (5.45), and (5.50)
Fig. 5.14. Global System Response to a Step Input
in the Velocity Reference Input (\( r_v(t) \))
Fig. 5.15. Global System Response to a Step Input
in the Flight Path Angle Reference Input
\( r_y(t) \)
Fig. 5.16. Global System and Fast Loop Closed-loop
Responses to a Step Input in the Pitch
Attitude Reference Input ($r_\theta(t)$)
5.8 Concluding Remarks

This chapter provided an example of a design of a SIAO system with 3 inputs and 3 outputs which has time-scale separated I/O groups. Figures 5.11 to 5.13 clearly demonstrate that the basic OGAT design procedure is quite adequate for producing good global system singular value shapes for lightly coupled SIAO systems. Further, as shown by figure 5.16, the closed-loop response of the fast loop to the high bandwidth reference input, commanded pitch angle, \( r_0 \), is essentially unaffected by the closing of the slow loop around it (i.e. the global closed-loop system response to the same input is equivalent to the fast loop response). Since the global closed-loop system and closed-loop slow loop responses to low bandwidth reference inputs are also identical, the basic OGAT design procedure does provide a design tool for separately designing the time responses of the two I/O groups and resulting in good global system loop shapes.

The next chapter summarizes the contributions of this study as well as the areas where further research may expand upon these results.
CHAPTER 6

SUMMARY AND DIRECTIONS FOR FURTHER RESEARCH

6.1 Summary

This thesis developed a number of explicit (for TITO systems) and qualitative
(for SIAO systems) relationships between the loop shapes of the two OGAT designs
and the loop shapes of the global system. Along with the design examples of chapters
3 and 5, these results expand our understanding of the OGAT design procedure and
demonstrate that, for lightly coupled systems, the global system will have good loop
shapes whenever the individual OGAT designs have good loop shapes.

Further, the explicit relationships between the global TITO system and the two
SISO OGAT designs (derived in chapter 3) have been shown to provide an improvement
of the basic OGAT design process such that all consistent global system specifications
can be met in one OGAT design iteration. This one iteration design procedure is
illustrated in the "academic" design example of chapter 3.

6.2 Directions for Further Research

While the results of chapter 4 provide a qualitative understanding of the relationship
between the OGAT designs and the global system, they do not provide a generalization
of the improved OGAT design procedure developed for TITO systems. Further research
should attempt to use all the directionality information within the global system loop
shapes to hopefully find approximate relationships (rather than bounding relationships)
between the global system loop shapes and the loop shapes of the OGAT design. Such
relationships could then be used to generalize the improved OGAT design procedure de-
veloped for TITO systems.
APPENDIX A

This appendix contains the derivation of a linear algebra result which provides the basis for the analysis of chapter 3. The goal of this appendix is to prove the following linear algebra result.

Let $M$ be a 2x2 complex-valued matrix

$$
M = \begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix}
$$

Define the real scalar $\alpha$ by

$$
\alpha \triangleq \left[ 1 + \frac{|b|^2}{|d|^2} + \frac{|c|^2}{|d|^2} \right]^{1/2}
$$

Suppose that

$$
\frac{|d|}{|a-bd^{-1}c|} \geq \rho_1 \gg 1
$$

and that

$$
\frac{|d|}{|a|} \geq \rho_2 \gg 1
$$

Then the following approximations hold (and become better and better as $\rho_1 \rightarrow \infty$, $\rho_2 \rightarrow \infty$)

$$
\sigma_{\text{max}}[M] \approx |d|\alpha
$$

$$
\sigma_{\text{min}}[M] \approx |a-bd^{-1}c| \cdot \frac{1}{\alpha}
$$

(A.5)

(A.6)
The proof of the above linear algebra result is provided below.

By definition

$$\sigma_i[M] = \{\lambda_i[M^*M]\}^{1/2}$$ \hspace{1cm} (A.7)

so the problem is reduced to finding the eigenvalues of $M^*M$.

$$M^*M = \begin{bmatrix}
|a|^2 + |c|^2 & a^*b + c^*d \\
b^*a + d^*c & |b|^2 + |d|^2
\end{bmatrix}$$ \hspace{1cm} (A.8)

By definition, the eigenvalues of $M^*M$ are the values of $\lambda_i$ which satisfy equation (A.9).

$$|\lambda_i I - M^*M| = 0$$ \hspace{1cm} (A.9)

Since,

$$|\lambda_i I - M^*M| = \begin{vmatrix}
\lambda_i - |a|^2 - |c|^2 & -a^*b - c^*d \\
-b^*a - d^*c & \lambda_i - |b|^2 - |d|^2
\end{vmatrix}$$ \hspace{1cm} (A.10)

$$= \lambda_i^2 - (|a|^2 + |b|^2 + |c|^2 + |d|^2)\lambda_i + |ad-bc|^2$$ \hspace{1cm} (A.11)

the eigenvalues of $M^*M$ are just the zeros of the second-order polynomial of equation (A.11). Solving this polynomial yields:

$$\lambda_i[M^*M] = \frac{|a|^2 + |b|^2 + |c|^2 + |d|^2}{2} \left[ 1 \pm \sqrt{1 - \frac{4|ad-bc|^2}{(|a|^2 + |b|^2 + |c|^2 + |d|^2)^2}} \right]$$ \hspace{1cm} (A.12)
Using the assumption of equation (A.3), it is easy to show that

\[
\frac{4|ad-bc|^2}{(|a|^2+|b|^2+|c|^2+|d|^2)^2} < \frac{4}{\rho_1^2}.
\]  

(A.13)

Given the inequality of equation (A.13) and a large numerical value of \( \rho_1 \) (much larger than unity), the approximation

\[
\sqrt{1-\varepsilon} \approx 1 - \frac{1}{2} \varepsilon \quad \text{for small } \varepsilon
\]  

(A.14)

can be used to simplify equation (A.12) as follows:

\[
\lambda_1^{[M^*M]} \approx \left\{ \frac{2}{|a|^2+|b|^2+|c|^2+|d|^2} \right\} \left\{ 1 + \left( 1 - \frac{2|ad-bc|^2}{(|a|^2+|b|^2+|c|^2+|d|^2)^2} \right) \right\}
\]  

(A.15)

Hence

\[
\lambda_{\max}^{[M^*M]} \approx |a|^2+|b|^2+|c|^2+|d|^2 - \frac{|ad-bc|^2}{|a|^2+|b|^2+|c|^2+|d|^2}
\]  

(A.16)

\[
\lambda_{\min}^{[M^*M]} \approx \frac{|ad-bc|^2}{|a|^2+|b|^2+|c|^2+|d|^2}
\]  

(A.17)

The following simple steps outline the remaining derivation needed to obtain the approximation of \( \sigma_{\min}[M] \).

\[
\lambda_{\min}^{[M^*M]} \approx \frac{|ad-bc|^2}{|a|^2+|b|^2+|c|^2+|d|^2}
\]  

(A.18)

\[
\approx |a-bd^{-1}c|^2 \left[ 1 + \frac{|a|^2}{|d|^2} + \frac{|b|^2}{|d|^2} + \frac{|c|^2}{|d|^2} \right]^{-1}
\]  

(A.19)
Given the assumption of equation (A.4) and a large value of $\rho_2$ (much larger than unity),

$$\frac{|a|^2}{|b|} << 1$$  

(A.20)

Hence,

$$\lambda_{\text{min}}[M^*M] \approx |a-bd^{-1}c|^2 \left[ 1 + \frac{|b|^2}{|d|^2} + \frac{|c|^2}{|d|^2} \right]^{-1}$$  

(A.21)

$$\sigma_{\text{min}}[M] \approx |a-bd^{-1}c| \left[ 1 + \frac{|b|^2}{|d|^2} + \frac{|c|^2}{|d|^2} \right]^{-1/2}$$

$$\sigma_{\text{min}}[M] \approx |a-bd^{-1}c| \cdot \frac{1}{\alpha}$$  

(A.22)

The derivation needed to obtain the approximation for $\sigma_{\text{max}}[M]$ is a little more involved.

$$\lambda_{\text{max}}[M^*M] = |a|^2 + |b|^2 + |c|^2 + |d|^2 - \frac{|ad-bc|^2}{|a|^2 + |b|^2 + |c|^2 + |d|^2}$$  

(A.23)

Given the assumption of equation (A.3), it is easy to show that

$$\frac{|ad-bc|^2}{|a|^2 + |b|^2 + |c|^2 + |d|^2} \leq \rho_1^2 \left[ 1 + \frac{|a|^2}{|d|^2} + \frac{|b|^2}{|d|^2} + \frac{|c|^2}{|d|^2} \right]$$  

(A.24)

so for large values of $\rho_1$ (much larger than unity) equation (A.23) can be approximated as:

$$\lambda_{\text{max}}[M^*M] \approx |a|^2 + |b|^2 + |c|^2 + |d|^2$$  

(A.25)
Using equation (A.20), this equation (eqn. (A.25)) can also be approximated as:

$$\lambda_{\text{max}}[M^T M] \approx |b|^2 + |c|^2 + |d|^2$$  \hspace{1cm} (A.26)

and so

$$\sigma_{\text{max}}[M] \approx |d| \sqrt{1 + \frac{|b|^2}{|d|^2} + \frac{|c|^2}{|d|^2}}$$  \hspace{1cm} (A.27)

$$\sigma_{\text{max}}[M] \approx |d| \alpha$$  \hspace{1cm} (A.28)

Since only the assumptions of equations (A.3) and (A.4) were used to derive the preceding results, the numerical values of $\rho_1$ and $\rho_2$ determine the quality of the approximations in equations (A.22) and (A.28). From the preceding derivation, it is easy to see that the approximation become exact in the limit as $\rho_1$ and $\rho_2$ both approach infinity ($\rho_1 \to \infty, \rho_2 \to \infty$). The assumption of equation (A.3) was used to approximate equation (A.12) by equations (A.16) and (A.17) and then to approximate equation (A.16) by equation (A.25). By considering the worst case for these approximations, it is easy to show that a numerical value for $\rho_1$ of about $\rho_1 = 10$ will guarantee that equations (A.17) and (A.25) have less than 1% approximation error. Similarly, the assumption of equation (A.4) was used to approximate equation (A.19) by equation (A.21) and to approximate equation (A.25) by equation (A.26), and the approximation error introduce in equations (A.21) and (A.26) by assumption (A.3) can be held to less than 1% if $\rho_2 \geq \sqrt{50}$. 

APPENDIX B

This appendix derives the relationships between the return difference and inverse return difference functions of the global system and the return difference and inverse return difference functions of the individual OGAT designs.

From the loop transfer function of the global TITO system (eqn. (3.7)), the return difference and inverse return difference functions of the global TITO system are given by:

\[
1 + T_G(j\omega) = \begin{bmatrix}
1 + g_{11}(j\omega)k_1(j\omega) & g_{12}(j\omega)k_2(j\omega) \\
g_{21}(j\omega)k_1(j\omega) & 1 + g_{22}(j\omega)k_2(j\omega)
\end{bmatrix}
\]  \hspace{1cm} (B.1)

\[
1 + T_G^{-1}(j\omega) = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} + \begin{bmatrix}
g_{11}(j\omega)k_1(j\omega) & g_{12}(j\omega)k_2(j\omega) \\
g_{21}(j\omega)k_1(j\omega) & g_{22}(j\omega)k_2(j\omega)
\end{bmatrix}^{-1}
\]  \hspace{1cm} (B.2)

Hereafter, the frequency dependence of the matrix elements of equations (B.1) and (B.2) will be suppressed. Calculating the matrix inverse simplifies equations (B.2) to:

\[
1 + T_G^{-1}(j\omega) = 1 + \frac{1}{(g_{11}k_1)(g_{22}k_2)(g_{21}k_1)(g_{12}k_2)} \begin{bmatrix}
g_{22}k_2 & -g_{12}k_2 \\
-g_{21}k_1 & g_{11}k_1
\end{bmatrix}
\]  \hspace{1cm} (B.3)

From the loop transfer functions of the fast loop and slow loop (eqns. (3.13) and (3.14)), the return difference and inverse return difference functions of the fast and slow loop are given by:
\[ 1 + t_f(j\omega) = 1 + g_{22}k_2 \quad (B.4) \]

\[ 1 + t_f^{-1}(j\omega) = 1 + (g_{22}k_2)^{-1} \quad (B.5) \]

\[ 1 + t_s(j\omega) = 1 + g_{11}k_1 - g_{12}k_2(1+g_{22}k_2)^{-1}g_{21}k_1 \quad (B.6) \]

\[ 1 + t_s^{-1}(j\omega) = 1 + [g_{11}k_1 - g_{12}k_2(1+g_{22}k_2)^{-1}g_{21}k_1]^{-1} \quad (B.7) \]

Pointwise in frequency, the return difference function of the global TITO system is a 2x2 complex valued matrix. Thus, the linear algebra result of section 3.2 can be applied to the return difference function of the global TITO system. This yields the following result.

Define the real scalar \( \phi(\omega) \) by

\[ \phi(\omega) \triangleq \left[ \frac{|g_{12}k_2|^2 + |g_{21}k_1|^2}{|1+g_{22}k_2|^2} \right]^{1/2} \quad (B.8) \]

Suppose that

\[ \frac{|1 + g_{22}k_2|}{|1+g_{11}k_1 - g_{12}k_2(1+g_{22}k_2)^{-1}g_{21}k_1|} \geq \rho_1 \gg 1 \quad (B.9) \]

and that

\[ \frac{|1+g_{22}k_2|}{|1+g_{11}k_1|} \geq \rho_2 \gg 1 \quad (B.10) \]

then

\[ \sigma_{\max} [I+T_G(j\omega)] \approx |1+g_{22}k_2| \cdot \phi(\omega) \quad (B.11) \]

\[ \sigma_{\min} [I+T_G(j\omega)] \approx |1+g_{11}k_1 - g_{12}k_2(1+g_{22}k_2)^{-1}g_{21}k_1| \cdot \frac{1}{\phi(\omega)} \quad (B.12) \]
Clearly, equation (B.9) can only hold for the passband of the fast loop design, so the approximation

\[ |1 + g_{22} k_2| \approx |g_{22} k_2| \]  \hspace{1cm} (B.13)

can be used to simplify the above equations. Using equations (B.4), (B.6), and (B.13), the results in equations (B.8) to (B.12) simplify as follows:

Define the real scalar \( \beta(\omega) \) by

\[ \beta(\omega) \triangleq \left[ 1 + \frac{|g_{12}|^2}{|g_{22}|^2} + \frac{|g_{21} k_1|^2}{|g_{22} k_2|^2} \right]^{1/2} \]  \hspace{1cm} (B.14)

Suppose that for the passband of the fast loop

\[ \frac{|1 + t_f(j\omega)|}{|1 + t_s(j\omega)|} \geq \rho_1 \gg 1 \]  \hspace{1cm} (B.15)

and that

\[ \frac{|1 + g_{22} k_2|}{|1 + g_{11} k_1|} \geq \rho_2 \gg 1 \]  \hspace{1cm} (B.16)

then for the passband of the fast loop

\[ \sigma_{\text{max}}[I + T_G(j\omega)] \approx |1 + t_f(j\omega)| \cdot \beta(\omega) \]  \hspace{1cm} (B.17)

\[ \sigma_{\text{min}}[I + T_G(j\omega)] \approx |1 + t_s(j\omega)| \frac{1}{\beta(\omega)} \]  \hspace{1cm} (B.18)

Using the singular value inequality [15]

\[ \sigma_{\text{min}}[I + G^{-1}] \geq \frac{\sigma_{\text{min}}[I + G]}{\sigma_{\text{min}}[I + G] + 1} \]  \hspace{1cm} (B.19)
and equation (B.18), a lower bound on the minimum singular value of the stability robustness function of the global system for the passband of the fast loop design can be calculated. This lower bound is given by:

\[
\sigma_{\min}[\mathbf{I} + T_G^{-1}(j\omega)] \gtrsim \frac{|1 + t_s(j\omega)|}{|1 + t_s(j\omega)| + \beta(\omega)} \tag{B.20}
\]

Now consider the inverse return difference function of the global system. Pointwise in frequency, the inverse return difference function of the global TITO system is a 2x2 complex valued matrix. Thus, the linear algebra result of section 3.2 can be applied to the inverse return difference function of the global TITO system.

Unfortunately, applying the results of section 3.2 to equation (B.3) produces conditions which are never met. We can get around this problem by premultiplying and post-multiplying equation (B.3) by the unitary matrix \( \mathbf{P} \) where

\[
\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{B.21}
\]

The product of this pre- and post-multiplication denoted \( \{\mathbf{I} + T_G^{-1}(j\omega)\}' \) is given by:

\[
\{\mathbf{I} + T_G^{-1}(j\omega)\}' = \mathbf{I} + \frac{1}{(g_{11}k_1)(g_{22}k_2) - (g_{12}k_2)(g_{21}k_1)} \begin{bmatrix} g_{11}k_1 & -g_{21}k_1 \\ -g_{12}k_2 & g_{22}k_2 \end{bmatrix} \tag{B.22}
\]

Since the singular values of a matrix are unaffected when the matrix is multiplied by a unitary matrix, the singular values of the matrices given in equations (B.3) and (B.22) are the same. Using this fact when applying the results of section 3.2 to equation (B.22), results in:
Define the real scalar $\psi(\omega)$ by

$$
\psi(\omega) \triangleq \left[ 1 + \frac{|g_{12}k_2|^{-2} + |g_{21}k_1|^{-2}}{|g_{22}k_2|^{-2}|1+g_{11}k_1-g_{12}k_2(g_{22}k_2)^{-1}g_{21}k_1|^2} \right]^{1/2}
$$

(B.23)

Suppose that

$$
\frac{|1+[g_{11}k_1-g_{12}k_2(g_{22}k_2)^{-1}g_{21}k_1]^{-1}| \cdot |1+[g_{11}k_1-g_{12}k_2(g_{22}k_2)^{-1}g_{21}k_1]|}{|1+(g_{22}k_2)^{-1}||1+g_{11}k_1-g_{12}k_2(1+g_{22}k_2)^{-1}g_{21}k_1|} \geq \rho_1 \gg 1
$$

(B.24)

and that

$$
\frac{|1+[g_{11}k_1-g_{12}k_2(g_{22}k_2)^{-1}g_{21}k_1]^{-1}|}{|1+(g_{11}k_1)(g_{22}k_2)^{-1}[g_{11}k_1-g_{12}k_2(g_{22}k_2)^{-1}g_{21}k_1]^{-1}|} \geq \rho_2 \gg 1
$$

(B.25)

Then

$$
\sigma_{\text{max}}[I+T_{-G}^{-1}(j\omega)] \approx |1+[g_{11}k_1-g_{12}k_2(g_{22}k_2)^{-1}g_{21}k_1]^{-1}| \cdot \psi(\omega)
$$

(B.26)

$$
\sigma_{\text{min}}[I+T_{-G}^{-1}(j\omega)] \approx |1+(g_{22}k_2)^{-1}| \cdot \frac{1}{\psi(\omega)} \cdot \frac{|1+g_{11}k_1-g_{12}k_2(1+g_{22}k_2)^{-1}g_{22}k_1|}{|1+g_{11}k_1-g_{12}k_2(g_{22}k_2)^{-1}g_{21}k_1|}
$$

(B.27)

By examination, it is evident that the condition of equation (B.24) can only be satisfied for the stopband of the slow loop design (fig. 3.1). By definition,

$$
|g_{11}k_1-g_{12}k_2(1+g_{22}k_2)^{-1}g_{21}k_1| \ll 1
$$

(B.28)

for the stopband of the slow loop design. By imposing the additional constraint

$$
|g_{11}k_1-g_{12}k_2(g_{22}k_2)^{-1}g_{21}k_1| \ll 1
$$

(B.29)
and by using equations (B.5) and (B.28), the results of equations (B.23) to (B.27) simplify as follows:

Define the real scalar $\beta(\omega)$ by

$$\beta(\omega) \triangleq \left[ 1 + \frac{|g_{12}|^2}{|g_{22}|^2} + \frac{|g_{21}k_1|^2}{|g_{22}k_2|^2} \right]^{1/2}$$  \hspace{1cm} (B.30)

Suppose that for the stopband of the slow loop

$$\frac{|1+[(g_{11} - g_{12}(g_{22})^{-1}g_{21})k_1]^{-1}|}{|1+(g_{22}k_2)^{-1}|} \geq \rho_1 >> 1$$  \hspace{1cm} (B.31)

and that

$$\frac{|1+[g_{11}k_1^{-1} - g_{12}k_2^{-1}(g_{22}k_2)^{-1}g_{21}k_1]^{-1}|}{|1+(g_{11}k_1^{-1})(g_{22}k_2)^{-1}[(g_{11}k_1^{-1} - g_{12}k_2^{-1}(g_{22}k_2)^{-1}g_{21}k_1]^{-1}|} \geq \rho_2 >> 1$$  \hspace{1cm} (B.32)

and that

$$|(g_{11} - g_{12}(g_{22})^{-1}g_{21})k_1| << 1$$  \hspace{1cm} (B.33)

then for the stopband of the slow loop

$$\sigma_{\max}[I+T_{-G}^{-1}(j\omega)] \approx |1+[g_{11}k_1^{-1} - g_{12}k_2^{-1}(g_{22}k_2)^{-1}g_{21}k_1]^{-1}| \cdot \beta(\omega)$$  \hspace{1cm} (B.34)

$$\sigma_{\min}[I+T_{-G}^{-1}(j\omega)] \approx |1+t_f^{-1}(j\omega)| \cdot \frac{1}{\beta(\omega)}$$  \hspace{1cm} (B.35)

Using the singular value inequality [15]

$$\sigma_{\min}[I+G] > \frac{\sigma_{\min}[I+G^{-1}]}{\sigma_{\min}[I+G^{-1}] + 1}$$  \hspace{1cm} (B.36)

and equation (B.35), a lower bound on the minimum singular value of the return difference function of the global system for the stopband of the slow loop can be calculated. This lower bound is given by:

$$\sigma_{\min}[I+T_{-G}(j\omega)] > \frac{|1+t_f^{-1}(j\omega)|}{|1+t_f^{-1}(j\omega)| + \beta(\omega)}$$  \hspace{1cm} (B.37)
APPENDIX C

This appendix derives the following linear algebra result.

Let $M$ be a general complex-valued matrix defined by its four block matrix elements as follows:

$$
M = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
$$

(C.1)

Define the real scalars $\gamma$ and $\delta$ by:

$$
\gamma \triangleq \left[ 1 + \frac{\sigma_{\text{max}}^2[B]}{\sigma_{\text{max}}^2[D]} + \frac{\sigma_{\text{max}}^2[C]}{\sigma_{\text{max}}^2[D]} \right]^{1/2}
$$

(C.2)

$$
\delta \triangleq \left[ 1 + \frac{\sigma_{\text{max}}^2[B]}{\sigma_{\text{min}}^2[D]} + \frac{\sigma_{\text{max}}^2[C]}{\sigma_{\text{min}}^2[D]} \right]^{1/2}
$$

(C.3)

Suppose that

$$
\sigma_{\text{min}}[A] - \sigma_{\text{max}}[B] \sigma_{\text{min}}^{-1}[D] \sigma_{\text{max}}[C] > 0
$$

(C.4)

and that

$$
\frac{\sigma_{\text{min}}[D]}{\sigma_{\text{max}}[A] + \sigma_{\text{max}}[B] \sigma_{\text{min}}^{-1}[D] \sigma_{\text{max}}[C]} \geq \rho_1 \gg 1
$$

(C.5)

and that

$$
\frac{\sigma_{\text{min}}[D]}{\sigma_{\text{max}}[A]} \geq \rho_2 \gg 1
$$

(C.6)
Then the following approximate bounds hold and the approximations become better and better as \( \rho_1 \to \infty, \rho_2 \to \infty \).

\[
\sigma_{\text{max}}[M] \leq \sigma_{\text{max}}[D] \cdot \gamma \quad \text{(C.7)}
\]

\[
\sigma_{\text{min}}[M] \geq \{\sigma_{\text{min}}[A] - \sigma_{\text{max}}[B] \sigma_{\text{min}}^{-1}[D] \sigma_{\text{max}}[C]\} \cdot \frac{1}{\epsilon} \quad \text{(C.8)}
\]

The derivation of this result is given below. By definition

\[
\sigma_{\text{max}}[Q] = \max_{||z|| = 1} ||Qz|| \quad \text{(C.9)}
\]

\[
\sigma_{\text{min}}[Q] = \min_{||z|| = 1} ||Qz|| \quad \text{(C.10)}
\]

Hence, the maximum and minimum singular values of a matrix provide bounds on the range of magnification of a unit vector by that matrix. By determining the range of magnification of a general unit vector by \( M \), the absolute bounds on the singular values of \( M \) can be found by considering the "worst" cases of a unit vector lining up with the possible directions of \( A, B, C, \) and \( D \). A property which is crucial to this analysis is:

For any matrix \( Q \) and unit vector \( \mathbf{v} \), there exist a unit vector \( \mathbf{z} \) such that

\[
Q \mathbf{v} = q \mathbf{z} \quad \text{(C.11)}
\]

where

\[
\sigma_{\text{max}}[Q] \geq q \geq \sigma_{\text{min}}[Q]
\]
Consider a general unit vector $\mathbf{x}$ given by:

$$
\mathbf{x} = \begin{bmatrix}
    w_1 v_1 \\
    w_2 v_2 
\end{bmatrix}
$$

(C.12)

where $v_1$ and $v_2$ are unit vectors and $w_1$ and $w_2$ are "free" real scalars except for the constraint that $\mathbf{x}$ must be a unit vector. Using equation (C.11), the product $M \mathbf{x}$ can be expressed as:

$$
M \mathbf{x} = \begin{bmatrix}
    z_a q_a & z_b q_b \\
    z_c q_c & z_d q_d 
\end{bmatrix}
\begin{bmatrix}
    w_1 \\
    w_2 
\end{bmatrix}
$$

(C.13)

where $z_a$, $z_b$, $z_c$, and $z_d$ are all unknown unit vectors and

$$
\sigma_{\text{max}}[A] \geq q_a \geq \sigma_{\text{min}}[A] \tag{C.14}
$$

$$
\sigma_{\text{max}}[B] \geq q_b \geq \sigma_{\text{min}}[B] \tag{C.15}
$$

$$
\sigma_{\text{max}}[C] \geq q_c \geq \sigma_{\text{min}}[C] \tag{C.16}
$$

$$
\sigma_{\text{max}}[D] \geq q_d \geq \sigma_{\text{min}}[D] \tag{C.17}
$$

The variables $z_a$, $z_b$, $z_c$, $z_d$, $q_a$, $q_b$, $q_c$, and $q_d$ are all functions of the choice of $v_1$ and $v_2$. Define $\mathbf{w}$ and $\mathbf{z}$ by:

$$
\mathbf{w} = \begin{bmatrix}
    w_1 \\
    w_2 
\end{bmatrix}
$$

(C.18)

$$
\mathbf{z} = \begin{bmatrix}
    z_a q_a & z_b q_b \\
    z_c q_c & z_d q_d 
\end{bmatrix}
$$

(C.19)
Clearly, for any choice of $v_1$ and $v_2$, the range of magnification of $x$ by $M$ must
equal the range of magnification of $w$ by $Z$ (equation (C.13)), and thus,

$$\sigma_{\text{max}}[M] = \sigma_{\text{max}}[Z]$$  \hspace{1cm} (C.20)

$$\sigma_{\text{min}}[M] = \sigma_{\text{min}}[Z]$$  \hspace{1cm} (C.21)

The following steps derive the singular values of $Z$

$$Z^* Z = \begin{bmatrix}
q_a^2 + q_c^2 & q_a q_b (z^*_a - z^*_b) + q_c q_d (z^*_c - z^*_d) \\
q_a q_b (z^*_b - z^*_a) + q_c q_d (z^*_c - z^*_d) & q_b^2 + q_d^2
\end{bmatrix}$$  \hspace{1cm} (C.22)

$$\lambda_i [Z^* Z] = \frac{q_a^2 + q_b^2 + q_c^2 + q_d^2}{2}$$ \cdot \sqrt{1 - \frac{4[q_a^2 + q_b^2 + q_c^2 + q_d^2]^2}{[q_a^2 + q_b^2 + q_c^2 + q_d^2]^2}}$$  \hspace{1cm} (C.23)

Since, for any two unit vectors $k$ and $\lambda$, the projection of $k$ on $\lambda$ ($k^* \lambda$) has a real
part between -1 and 1, it is easy to show that

$$\frac{4(q_a^2 + q_b^2 + q_c^2 - 2q_a q_b q_c q_d \text{Re}[z^*_a z^*_b z^*_c z^*_d])}{[q_a^2 + q_b^2 + q_c^2 + q_d^2]^2}$$

$$\leq \frac{4(q_a q_d + q_b q_c)^2}{[q_a^2 + q_b^2 + q_c^2 + q_d^2]^2} < \frac{4(q_a + q_b + q_c - 1)^2}{q_d^2}$$  \hspace{1cm} (C.24)

This inequality and condition (C.5) guarantee that
\[
\frac{4(q_a^2 + q_b^2 + q_c^2 - 2q_a q_b q_c q_d \text{Re}[z^* z_{a,b,c,d}])}{[q_a^2 + q_b^2 + q_c^2 + q_d^2]^2} \leq \frac{1}{\rho_1^2}
\]  
(C.25)

for all values of \( q_a, q_b, q_c, q_d, z_a, z_b, z_c, \) and \( z_d \). Thus, for large values of \( \rho_1 \), equation (C.23) can be simplified by using the square root approximation of Appendix A (eqn. (A.14)). These simplified expressions for the eigenvalues of \( Z \) are given by:

\[
\lambda_{\text{max}}[Z^* Z] \approx q_a^2 + q_b^2 + q_c^2 + q_d^2 - \lambda_{\text{min}}[Z^* Z]
\]  
(C.26)

\[
\lambda_{\text{min}}[Z^* Z] \approx \frac{q_a^2 q_b^2 + q_b^2 q_c^2 - 2q_a q_b q_c q_d \text{Re}[z^* z_{a,b,c,d}]}{q_a^2 + q_b^2 + q_c^2 + q_d^2}
\]  
(C.27)

For any values of \( q_a, q_b, q_c, q_d, z_a, z_b, z_c \) and \( z_d \), using conditions (C.5) and (C.6), these equations can be simplified in a manner analogous to the simplifications in Appendix A (eqns. (A.16) to (A.28)). These simplified expressions are given by:

\[
\lambda_{\text{max}}[Z^* Z] \approx q_b^2 + q_c^2 + q_d^2
\]  
(C.28)

\[
\lambda_{\text{min}}[Z^* Z] \approx \frac{q_a^2 q_d^2 + q_b^2 q_c^2 - 2q_a q_b q_c q_d \text{Re}[z^* z_{a,b,c,d}]}{q_b^2 + q_c^2 + q_d^2}
\]  
(C.29)

Thus, for any choice of \( v_1 \) and \( v_2 \), the magnification of \( x \) by \( M \) is bounded by:

\[
\sigma_{\text{max}}[M] \approx q_d \sqrt{1 + \frac{q_b^2 + q_c^2}{q_d^2}}
\]  
(C.30)

and

\[
\sigma_{\text{min}}[M] \approx \sqrt{q_a^2 + q_b^2 + q_c^2 - 2q_a q_b q_c q_d - 1} \text{Re}[z^* z_{a,b,c,d}] \sqrt{1 + \frac{q_b^2 + q_c^2}{q_d^2}}
\]  
(C.31)
To find the absolute bounds on the singular values of $M$, the "worst" case of the terms on the right hand side of equations (C.30) and (C.31) will be considered by ignoring the dependence of those terms on the choice of $v_1$ and $v_2$. Since

$$\sigma_{\text{max}}[M] \approx [q_d^2 + q_b^2 + q_c^2]^{1/2}$$  \hspace{1cm} (C.32)

for any choice of $u_1$ and $u_2$, it is clear that the maximum singular value of $M$ is bounded above by using the maximum of $q_d, q_b,$ and $q_c$. Hence

$$\sigma_{\text{max}}[M] < \sigma_{\text{max}}[D] \left[ 1 + \frac{\sigma_{\text{max}}^2[D] + \sigma_{\text{max}}^2[C]}{\sigma_{\text{max}}^2[D]} \right]^{1/2}$$  \hspace{1cm} (C.33)

which can be expressed as:

$$\sigma_{\text{max}}[M] < \sigma_{\text{max}}[D] \cdot \gamma$$  \hspace{1cm} (C.34)

Clearly, the lower bound on the minimum singular value of $M$ is zero or it is achieved when

$$z^*_a z^*_b z^*_c z^*_d = 1$$  \hspace{1cm} (C.35)

Assuming that the lower bound is not zero, equation (C.31) can be expressed as:

$$\sigma_{\text{min}}[M] > \left| q_a - q_b q_d^{-1} q_c \right| \cdot \left[ 1 + \frac{q_b^2}{q_d^2} + \frac{q_c^2}{q_d^2} \right]^{-1/2}$$  \hspace{1cm} (C.36)

In light of condition (C.4), the minimization of the right hand side of equation (C.36) over $q_a, q_b, q_c,$ and $q_d$ produces the expression:

$$\sigma_{\text{min}}[M] > \left\{ \sigma_{\text{min}}[A] - \sigma_{\text{max}}[B] \sigma_{\text{min}}^{-1}[D] \sigma_{\text{max}}[C] \right\} \left[ 1 + \frac{\sigma_{\text{max}}^2[B] + \sigma_{\text{max}}^2[C]}{\sigma_{\text{min}}^2[D]} \right]^{-1/2}$$  \hspace{1cm} (C.37)
which can be written as:

\[
\sigma_{\min}[M] > \{\sigma_{\min}[A] - \sigma_{\max}[B]\sigma_{\min}[D]\sigma_{\max}[C]\} \cdot \frac{1}{\delta}
\]  \hspace{1cm} (3.8)

As all the approximations used in this chapter are analogous to those of Appendix A, all the approximations in this appendix get better and better as \(\rho_1 \rightarrow \infty\) and \(\rho_2 \rightarrow \infty\).
APPENDIX D.1: The state-space matrices for the linear model of the Harrier (eqn. (5.1)) are given by:

\[
A_H = \begin{bmatrix}
0.0000E+00 & 1.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
-1.8370E+00 & -1.8930E+00 & 1.8370E+00 & -4.0980E-04 & 6.1660E-03 & -1.2430E-01 \\
3.6420E-03 & 5.2950E-01 & 8.5080E-03 & -5.2950E-01 & 5.7390E-04 & 2.3490E-04 \\
6.9090E-04 & -3.4500E+01 & 0.0000E+00 & 2.3000E+00 & -6.2130E-02 & 4.2090E-01 \\
-3.5660E-02 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & -1.9660E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
-1.2000E+01 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00
\end{bmatrix}
\]

\[
B_H = \begin{bmatrix}
0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
1.9660E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 1.2000E+01 \\
0.0000E+00 & 1.2000E+01 & 0.0000E+00
\end{bmatrix}
\]

\[
C_H = \begin{bmatrix}
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 1.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 5.7300E+01 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
5.7300E+01 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00
\end{bmatrix}
\]
APPENDIX D.2: The state-space matrices for the scaled linear model of the Harrier (eqn. (5.11)) are given by:

\[
A_{HS} = \\
\begin{bmatrix}
0.0000E+00 & 1.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
-1.8370E+00 & -1.8930E+00 & 1.8370E+00 & -4.0980E-04 & 6.1660E-03 \\
3.6420E-03 & 8.5080E-03 & 5.2950E-01 & 5.7390E-04 & 2.3490E-04 \\
5.2950E-01 & 6.9090E-04 & 0.0000E+00 & -6.2130E-02 & 4.2090E-01 \\
-3.4500E+01 & 2.3000E+00 & -6.2130E-02 & 4.2090E-01 & -4.5190E-02 \\
-1.2000E+01 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
\end{bmatrix}
\]

\[
B_{HS} = \\
\begin{bmatrix}
0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
1.9660E+01 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 1.2000E+01 \\
0.0000E+00 & 1.2000E+02 & 0.0000E+00 \\
\end{bmatrix}
\]

\[
C_{HS} = \\
\begin{bmatrix}
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 1.0000E-01 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 5.7300E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 5.7300E+00 & 0.0000E+00 & 0.0000E+00 \\
5.7300E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
\end{bmatrix}
\]
APPENDIX D.3: The state-space matrices for the fast loop plant (eqn. (5.24)) are given by:

\[
A_f = \begin{bmatrix}
0.0000e+00 & 1.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 \\
-1.8370e+00 & -1.8930e+00 & 1.8370e+00 & -4.0980e-04 & -1.2430e-01 \\
5.2950e-01 & 8.5080e-03 & -5.2950e-01 & 5.7390e-04 & 1.7140e-03 \\
-3.4500e+01 & 0.0000e+00 & 2.3000e+00 & -6.2130e-02 & -4.5190e-02 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & -1.2000e+01
\end{bmatrix}
\]

\[
B_f = \begin{bmatrix}
0.0000e+00 \\
0.0000e+00 \\
0.0000e+00 \\
0.0000e+00 \\
1.2000e+01
\end{bmatrix}
\]

\[
C_f = \begin{bmatrix}
5.7300e+01 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00
\end{bmatrix}
\]
APPENDIX D.4: The state-space matrices for the fast loop design model including appended dynamics (eqn. (5.26)) and the choice of design parameters are given by:

\[
A_{fd} = \begin{bmatrix}
0.0000E+00 & 1.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
-1.8370E+00 & -1.8930E+00 & 1.8370E+00 & -4.0980E-04 & -1.2430E-01 & 0.0000E+00 \\
5.2950E-01 & 8.5080E-03 & -5.2950E-01 & 5.7390E-04 & 1.7140E-03 & 0.0000E+00 \\
-3.4500E+01 & 0.0000E+00 & 2.3000E+00 & -6.2130E-02 & -4.5190E-02 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & -1.2000E+01 & 1.2000E+01 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00
\end{bmatrix}
\]

\[
B_{fd} = \begin{bmatrix}
0.0000E+00 \\
0.0000E+00 \\
0.0000E+00 \\
0.0000E+00 \\
0.0000E+00 \\
1.0000E+00
\end{bmatrix}
\]

\[
C_{fd} = \begin{bmatrix}
5.7300E+01 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00
\end{bmatrix}
\]

\[
L_{fd} = \begin{bmatrix}
0.0000E+00 \\
0.0000E+00 \\
0.0000E+00 \\
0.0000E+00 \\
0.0000E+00 \\
5.0000E+00
\end{bmatrix}
\]

\[\mu_{fd} = 1\]

\[q_{fd} = 25 \times 10^6\]
APPENDIX D.5: The state-space matrices for the fast loop compensator (eqn. (5.28)) are given by:

\[
\begin{align*}
A_{fc} &= \begin{bmatrix}
-4.6576E+00 & 1.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
-1.2684E+01 & -1.8930E+00 & 1.8370E+00 & -4.0980E-04 & -1.2430E-01 & 0.0000E+00 \\
2.3047E-01 & 8.5080E-03 & -5.2950E-01 & 5.7390E-04 & 1.7140E-03 & 0.0000E+00 \\
-7.6327E+00 & 0.0000E+00 & 2.3000E+00 & -6.2130E-02 & -4.5190E-02 & 0.0000E+00 \\
2.8478E+02 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & -1.2000E+01 & 1.2000E+01 \\
2.8418E+05 & 2.6789E+04 & 2.6084E+03 & -5.4365E-01 & -1.2459E+02 & -5.4683E+01 \\
\end{bmatrix}
\end{align*}
\]

\[
B_{fc} = \begin{bmatrix}
8.1284E-02 \\
1.8929E-01 \\
5.2186E-03 \\
-4.6889E-01 \\
-4.9700E+00 \\
-5.0000E+00 \\
\end{bmatrix}
\]

\[
C_{sc} = \begin{bmatrix}
-2.8389E+05 & -2.6789E+04 & -2.6084E+03 & 5.4365E-01 & 1.2459E+02 & 5.4683E+01 \\
\end{bmatrix}
\]
APPENDIX D.6: The state-space matrices for the overall fast loop compensator including appended dynamics (eqn. (5.30)) are given by:

\[
A_{LQGf} = \begin{bmatrix}
0.0000E+00 & 1.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 1.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 1.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 1.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 1.0000E+00 \\
0.0000E+00 & -3.3321E+05 & -6.8629E+05 & -5.6852E+04 & -2.6486E+03 & -7.3766E+01 \\
\end{bmatrix}
\]

\[
B_{LQGf} = \begin{bmatrix}
0.0000E+00 \\
0.0000E+00 \\
0.0000E+00 \\
0.0000E+00 \\
0.0000E+00 \\
1.0000E+00 \\
\end{bmatrix}
\]

\[
C_{LQGf} = \begin{bmatrix}
-1.0786E+06 & -2.9940E+06 & -1.7719E+06 & -4.7613E+05 & -2.9053E+04 & 0.0000E+00 \\
\end{bmatrix}
\]
APPENDIX D.7: The state-space matrices for the slow loop plant (eqn. 5.35)) are given by:

\[
A_s =
\begin{bmatrix}
0.0000E+00 & 1.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 1.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 1.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 1.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 1.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & -3.3321E+05 & -6.8629E+05 & -5.6852E+04 & -2.6486E+03 & -7.3766E+01 \\
-5.7300E+01 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 1.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & -1.8370E+00 & 1.8370E+00 & -4.0980E-04 & 6.1660E-03 & -1.2430E-01 \\
-3.6420E-03 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 8.5080E-03 & -5.2950E-01 & 5.7390E-04 & 2.3490E-04 & 1.7140E-03 \\
6.9090E-04 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 2.4500E+01 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
-3.5660E-02 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
-1.2943E+07 & -3.5928E+07 & -2.1263E+07 & -5.7135E+06 & -3.4864E+05 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
-1.2000E+01 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00
\end{bmatrix}
\]
\[
B^s = \\
\begin{bmatrix}
0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 \\
0.9660E+01 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 1.2000E+02
\end{bmatrix}
\]

\[
C^s = \\
\begin{bmatrix}
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 1.0000E+01 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 5.7300E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00
\end{bmatrix}
\]
APPENDIX D.8: The state-space matrices for the slow loop design model including appended dynamics (eqn. (5.37)) and the choice of design parameters are given by:

\[
A_{sd} = \begin{bmatrix}
-1.0000E-02 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & -1.0000E-02 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 \\
-4.3025E+00 & 3.3588E-01 & -5.9727E-02 & 0.0000E+00 & 0.0000E+00 \\
1.8718E+00 & -5.2394E+00 & 0.0000E+00 & -5.0648E-01 & 2.2192E-04 \\
-5.7305E+00 & 1.6214E+01 & 0.0000E+00 & -2.2192E-04 & -5.0648E-01
\end{bmatrix}
\]

\[
B_{sd} = \begin{bmatrix}
1.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 1.0000E+00 \\
0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00
\end{bmatrix}
\]

\[
C_{sd} = \begin{bmatrix}
6.6397E-03 & 1.2273E-03 & -7.2913E-03 & -3.5758E-01 & -1.1267E-01
\end{bmatrix}
\]

\[
L_{sd} = \begin{bmatrix}
1.9982E-02 & -1.1680E-01 \\
-1.6296E-03 & -1.4314E+00 \\
0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00
\end{bmatrix}
\]

\[
\mu_{sd} = 1
\]

\[
q_{sd} = 10^6
\]
APPENDIX D.9: The state-space matrices for the slow loop compensator (eqn. (5.39)) are given by:

\[ A_{sc} = \begin{bmatrix}
-2.2993E+02 & 8.3081E-01 & 2.6912E+01 & 2.5405E+02 & 8.0075E+01 \\
7.6517E-01 & -1.0353E+01 & -9.2191E+01 & 1.4765E+02 & 4.6358E+01 \\
-4.4724E+00 & 3.3565E-01 & -1.3665E-01 & -8.3526E-02 & -2.6467E-02 \\
2.3721E+00 & -5.2238E+00 & 1.0031E-01 & -4.6831E+00 & -1.3154E+00 \\
-7.2803E+00 & 1.6165E+01 & -3.1055E-01 & 1.2945E+01 & 3.5712E+00
\end{bmatrix} \]

\[ B_{sc} = \begin{bmatrix}
-3.7732E-03 & 1.0706E-01 \\
-2.4978E-01 & 1.2767E+00 \\
-7.4875E-01 & -2.8209E-01 \\
1.8463E+00 & -1.1561E+01 \\
-5.7189E+00 & 3.5832E+01
\end{bmatrix} \]

\[ C_{sc} = \begin{bmatrix}
2.2991E+02 & -8.3094E-01 & -2.6911E+01 & -2.5401E+02 & -8.0063E+01 \\
-8.3094E-01 & 1.0342E+01 & 9.2175E+01 & -1.4719E+02 & -4.6212E+01
\end{bmatrix} \]
APPENDIX D.10: The state-space matrices for the overall slow loop compensator including appended dynamics (eqn. (5.41)) are given by:

\[
A_{\text{LQGS}} = \begin{bmatrix}
-2.2993E+02 & 8.3081E-01 & 2.6912E+01 & 2.5405E+02 & 8.0075E+01 & 0.0000E+00 \\
7.6517E-01 & -1.0353E+01 & -9.2191E+01 & 1.4765E+02 & 4.6358E+01 & 0.0000E+00 \\
-4.4724E+00 & 3.3565E-01 & -1.3665E+01 & -8.3526E-02 & -2.6467E-02 & 0.0000E+00 \\
2.3721E+00 & -5.2238E+00 & 1.0031E-01 & -4.6831E+00 & -1.3154E+00 & 0.0000E+00 \\
-7.2803E+00 & 1.6165E+00 & -3.1055E+01 & 1.2945E+01 & 3.5712E+00 & 0.0000E+00 \\
2.2991E+02 & -8.3094E+01 & -2.6911E+01 & -2.5401E+02 & -8.0063E+01 & -1.0000E-02 \\
-8.3094E-01 & 1.0342E+01 & 9.2175E+01 & -1.4719E+02 & -4.6212E+01 & 0.0000E+00 \\
-1.0000E-02 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00
\end{bmatrix}
\]

\[
B_{\text{LQGS}} = \begin{bmatrix}
-3.7732E-03 & 1.0706E-01 \\
-2.4978E-01 & 1.2767E+00 \\
-7.4875E-01 & -2.8209E-01 \\
1.8463E+00 & -1.1561E+01 \\
-5.7189E+00 & 3.5832E+01 \\
0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00
\end{bmatrix}
\]

\[
C_{\text{LQGS}} = \begin{bmatrix}
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 1.0000E+00 \\
0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 \\
1.0000E+00 & 0.0000E+00
\end{bmatrix}
\]
APPENDIX E

This appendix discusses scaling issues which can alter the ratios, \( r_{t_1}(\omega) \) and \( r_{t_2}(\omega) \), without changing the open-loop relationship between each control and output within an I/O group.

For a system

\[
\begin{bmatrix}
  y_1 \\
y_2
\end{bmatrix}
= G(j\omega)
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\]

(E.1)

where

\[
G(j\omega) =
\begin{bmatrix}
ge_{11}(j\omega) & g_{12}(j\omega) \\
ge_{21}(j\omega) & g_{22}(j\omega)
\end{bmatrix}
\]

Two ratios of interest are given by:

\[
r_{t_1}(\omega) = \frac{\sigma_{\text{max}}[g_{12}(j\omega)]}{\sigma_{\text{min}}[g_{22}(j\omega)]}
\]

(E.2)

\[
r_{t_2}(\omega) = \frac{\sigma_{\text{max}}[g_{21}(j\omega)]}{\sigma_{\text{min}}[g_{22}(j\omega)]}
\]

(E.3)

Using a scaling transformation on the input and output vectors, these ratios can be altered without changing the block diagonal elements of \( G(j\omega) \).

Define the input \( (N_u) \) and output \( (N_y) \) scaling transformations by:

\[
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix}
= N_y
\begin{bmatrix}
y_{1t} \\
y_{2t}
\end{bmatrix}
\]

(E.4)
\[
\begin{bmatrix}
u_1 \\ u_2
\end{bmatrix} = N_u \begin{bmatrix}
u_{1t} \\ u_{2t}
\end{bmatrix}
\] (E.5)

where the vectors

\[
\begin{bmatrix}
y_1 \\ y_2
\end{bmatrix} = \begin{bmatrix}
u_1 \\ u_2
\end{bmatrix}
\]

are the original vectors and the vectors

\[
\begin{bmatrix}
y_{1t} \\ y_{2t}
\end{bmatrix} = \begin{bmatrix}
u_{1t} \\ u_{2t}
\end{bmatrix}
\]

are the transformed vectors. Consider the special case of \( N_u \) and \( N_y \) given by:

\[
N_u = N_y = \begin{bmatrix}
c_1 I & 0 \\ 0 & I
\end{bmatrix}
\] (E.6)

Using these input and output scaling transformations, the scaled system is:

\[
\begin{bmatrix}
y_{1t} \\ y_{2t}
\end{bmatrix} = \begin{bmatrix}
G_{11t}(j\omega) & G_{12t}(j\omega) \\ G_{21t}(j\omega) & G_{22t}(j\omega)
\end{bmatrix} \begin{bmatrix}
u_{1t} \\ u_{2t}
\end{bmatrix}
\] (E.7)

where

\[
G_{11t}(j\omega) = G_{11}(j\omega)
\]
\[
G_{12t}(j\omega) = \frac{1}{c_1} G_{12}(j\omega)
\]
\[
G_{21t}(j\omega) = c_1 G_{21}(j\omega)
\]
\[
G_{22t}(j\omega) = G_{22}(j\omega)
\]
Clearly, this scaling transformation has not changed the open-loop relationship between each control and output within an I/O group (i.e. $G_{11t}(j\omega) = G_{11}(j\omega)$ and $G_{22t}(j\omega) = G_{22}(j\omega)$) although the units on $u_1$ and $y_1$ have been scaled down by a factor of $c_1$.

For the scaled system (eqn. (E.7)), the ratios $rt_{1t}(\omega)$ and $rt_{2t}(\omega)$ and their relationship to the ratios of the unscaled system ($rt_1(\omega)$ and $rt_2(\omega)$) are given by:

$$rt_{1t}(\omega) = \frac{\sigma_{\max}[G_{12t}(j\omega)]}{\sigma_{\min}[G_{22t}(j\omega)]}$$

$$= \frac{\sigma_{\max}\left[\frac{1}{c_1}G_{12}(j\omega)\right]}{\sigma_{\min}[G_{22}(j\omega)]}$$

$$= \frac{1}{c_1}rt_1(\omega)$$

$$rt_{2t}(\omega) = \frac{\sigma_{\max}[G_{21t}(j\omega)]}{\sigma_{\min}[G_{22t}(j\omega)]}$$

$$= \frac{\sigma_{\max}\left[c_1G_{21}(j\omega)\right]}{\sigma_{\min}[G_{22}(j\omega)]}$$

$$= c_1rt_2(\omega)$$

Thus, the scaling transformations of equation (E.6) can be used to tradeoff the size of one of the ratios for the other ratio.
REFERENCES


13. Ref. 830405, Section 3.3 from MIT 6.232 Class Notes, Spring 1983.
